MODELING AND PLANNING ACCELERATED LIFE TESTING WITH PROPORTIONAL ODDS

by

HAO ZHANG

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ABSTRACT OF THE DISSERTATION

Modeling and Planning Accelerated Life Testing with

Proportional Odds

By HAO ZHANG

Dissertation Director:

Elsayed A. Elsayed

Accelerated life testing (ALT) is a method for estimating the reliability of products at normal operating conditions from the failure data obtained at the severe conditions. We propose an ALT model based on the proportional odds (PO) assumption to analyze failure time data and investigate the optimum ALT plans for multiple-stress-type cases based on the PO assumption.

We present the PO-based ALT model and propose the parameter estimation procedures by approximating the general baseline odds function with a polynomial function. Numerical examples with experimental data and Monte Carlo simulation data verify that

the PO-based ALT model provides more accurate reliability estimate for the failure time data exhibiting PO properties.

The accuracy of the reliability estimates is directly affected by the reliability inference model and how the ALT is conducted. The latter is addressed in the literature as the design of ALT test plans. Design of ALT test plans under one type of stress may mask the effect of other critical types of stresses that could lead to the component's failure. The extended life of today's products makes it difficult to obtain "enough" failures in a reasonable amount of testing time using single stress type. Therefore, it is more realistic to consider multiple stress types. This is the first research that investigates the design of optimum ALT test plans with multiple stress types. We formulate nonlinear optimization problems to determine the optimum ALT plans. The optimization problem was solved with a numerical optimization method.

Reliability practitioners could choose different ALT plans in terms of the stress loading types. In this dissertation we conduct the first investigation of the equivalency of ALT plans, which enables reliability practitioners to choose the appropriate ALT plan according to resource restrictions. The results of this research show that one can indeed develop efficient test plans that can provide accurate reliability estimate at design conditions in much shorter test duration than the traditional test plans.

Finally, we conduct experimental testing using miniature light bulbs. The test units are subjected to different stress types. The results validate the applicability of the PO-based ALT models.

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CHAPTER 1

INTRODUCTION

1.1 Research Background

The developments of new technologies and global competition have emphasized the need for more accurate estimation of reliability of a product, system or component in a shorter time. Traditional life data analysis involves analyzing failure data in order to quantify the life characteristics of the product, system or component. In many situations it is very difficult, if not impossible, to obtain such failure data under normal operating conditions because of the long life of today's products, the short time period between design and release and the challenge of testing products that are used continuously under normal conditions. Given this difficulty, reliability practitioners have attempted to devise methods to induce failures quickly by subjecting the products to severer environmental conditions without introducing additional failure modes other than those observed under normal operating conditions. The failure data obtained under the severe conditions are used to estimate the life characteristics, and the reliability performance of products at normal operating conditions. The term *accelerated life testing* (ALT) has been used to describe those methods.

The accuracy of reliability estimation depends on the models that relate the failure data under severe conditions, or high stress, to that at normal operating conditions, or design stress. Elsayed (1996) classifies these models into three groups: *statistics models*, *physics-statistics models*, and *physics-experimental models*. Furthermore, he classifies the statistics models into two sub-categories: parametric and nonparametric models. With parametric models, failure times at each stress level are used to determine the most appropriate failure time distribution along with its parameters. Parametric models assume that the failure times at different stress levels are related to each other by a common failure time distribution with different parameters. Usually, the shape parameters of the failure time distribution remain unchanged for all stress levels, but the scale parameters may present a multiplicative relationship with the stress levels.

Nonparametric models relax the requirement of the common failure time distribution, i.e., no common failure time distribution is required. Cox's Proportional Hazards (PH) model (1972, 1975) is the most popular nonparametric model. It has become the standard nonparametric regression model for accelerated life testing in the past few years. This model usually produces "good" reliability estimation with failure data for which the proportional hazards assumption holds and even when it does not hold exactly. In many applications, however, it is often unreasonable to assume the effects of covariates (stresses) on the hazard rates remain fixed over time. Brass (1971) observes that the ratio of the death rates, or hazard rates, of two populations under different stress levels (for example, one population for smokers and the other for non-smokers) is not constant with age, or time, but follows a more complicated course, in particular converging closer to unity for older people. So the PH model is not suitable for this case. Brass (1974) proposes a more realistic model: the Proportional Odds (PO) model. The Proportional Odds model has been successfully used in categorical data analysis (McCullagh 1980, Agresti and Lang 1993) and survival analysis (Hannerz 2001) in the medical fields. The PO model has a distinct different assumption on proportionality, and is complementary to the PH model. It has not been used in reliability analysis of accelerated life testing so far.

The accuracy of reliability estimation is a major concern in accelerated life testing, since it is important for making appropriate subsequent decisions regarding preventive maintenance, replacement and warranty policies. In the accelerated life testing experiments, a comprehensive reliability estimation procedure includes an appropriate ALT model and a carefully designed test plan in order to achieve high accuracy of the reliability estimates.

An appropriate ALT model is important since it explains the influences of the stresses on the expected life of a product based on its physical properties and the related statistical properties. On the other hand, a carefully designed test plan improves the accuracy and efficiency of the reliability estimation. The design of an accelerated life testing plan consists of the formulation of objective function, the determination of constraints and the definition of the decision variables such as stress levels, sample size, allocation of test units to each stress level, stress level changing time and test termination time, and others. Inappropriate decision variable values result in inaccurate reliability estimates and/or unnecessary test resources. Thus it is important to design test plans to optimize the objective function under specific time and cost constraints.

Most of the previous work on ALT plans involves only a single stress type with two or three stress levels. However, as products become more reliable due to technological advances, it becomes more difficult to obtain significant amount of failure data within reasonable amount of time using single stress type only. Multiple-stress-type ALTs have been employed as a means to overcome such difficulties. For instance, Kobayashi *et al.* (1978), Minford (1982), Mogilevsky and Shirn (1988), and Munikoti and Dhar (1988) use two stress types to test certain types of capacitors, and Weis *et al.* (1988) employ two stress types to estimate the lifetime of silicon photodetectors.

1.2 Problem Definition

In this dissertation, we investigate two related problems. The first problem deals with a nonparametric accelerated life testing model, proportional odds model, and its applicability for reliability predication at normal operating condition. We also discuss its characteristics, robustness and its accuracy of reliability estimates. We begin by defining the odds function $\theta(t)$, which is the ratio of cumulative distribution function to the reliability function:

$$\theta(t) = \frac{F(t)}{R(t)} = \frac{F(t)}{1 - F(t)}, \tag{1.1}$$

or

$$\theta(t) = \frac{\Pr(T \le t)}{\Pr(T > t)} = \frac{\Pr(T \le t)}{1 - \Pr(T \le t)}, \tag{1.2}$$

where T is the failure time of a test unit.

Therefore the odds function $\theta(t)$ is literally the odds of failure of a unit at time t. The odds function $\theta(t)$ is related to the hazard rate $\lambda(t)$ at time t by

$$\lambda(t) = \frac{\theta'(t)}{\theta(t) + 1}, \tag{1.3}$$

or

$$\theta(t) = e^{\int_0^t \lambda(u)du} - 1. \tag{1.4}$$

The proportional hazards model is a widely used nonparametric accelerated life testing model, which assumes that the covariates, or stresses, act multiplicatively on the hazards rate function, and is expressed as:

$$\lambda(t;\mathbf{z}) = \lambda_0(t)\exp(\boldsymbol{\beta}^t\mathbf{z}) = \lambda_0(t)\exp\left(\sum_{j=1}^{k}\beta_j z_j\right), \tag{1.5}$$

where $\boldsymbol{z}$ is the vector of the applied stresses, $\boldsymbol{\beta}$ is the vector of unknown parameters and $\lambda_0(t)$ is an arbitrary baseline hazard rate. However proportional hazards model may not be the most appropriate model especially when the hazards rates at different stress levels converge to the baseline hazard rate over time. In this case, proportional odds model is more accurate for reliability estimation.

Failure times under two different stress levels are said to follow proportional odds model if

$$\frac{\theta_2(t;\mathbf{z}_2)}{\theta_1(t;\mathbf{z}_1)} = -\exp[\boldsymbol{\beta}^t(\mathbf{z}_2 - \mathbf{z}_1)]. \tag{1.6}$$

The current estimation procedures in literatures for the PO model are either too complicated for practical accelerated life testing use, or have no rigorous justification for large sample properties. In literatures, Dabrowska *et al.* (1988) and Wu (1995) propose estimation methods for only two-sample data. Pettitt (1984) estimates the parameters of the PO model using ranks, which ignores the actual observations, and results in corresponding inaccurate estimation. Murphy *et al.* (1997) propose the estimation method of the PO model based on profile likelihood. The likelihood is constructed from the conditional probability density. Because the number of unknown parameters in the profile likelihood is extremely large, the estimation calculation is also extremely computationally intensive.

In this dissertation, a new approach for the reliability estimation of accelerated life testing based on proportional odds model is proposed. The estimates obtained from this approach are verified using both simulation study and experimental failure time data. We may use this new approach to predict the point estimate of reliability of the product at any time t and any stress level.

The interval estimates of model parameters and reliability of the product are also interesting to us in this dissertation. We construct the confidence intervals through the covariance matrix obtained by taking the inverse of Fisher information matrix. In order to validate the assumption of the proposed accelerated life testing model and estimation procedures, we use both likelihood ratio test to test the model sufficiency and Cox-Snell residual to verify the proportionality.

The second major problem of this dissertation deals with the accuracy of reliability estimation. In order to increase the accuracy of reliability estimates in accelerated life testing problem, a carefully designed test plan is required. This test plan is designed to minimize a specified criterion, usually the variance of a reliability-related estimate, such as reliability function, mean time to failure and a percentile of failure time, under specific time and cost constraints. Most of the previous work on ALT plans involves only a single stress type with two or three stress levels. However, as products become more reliable due to technological advances, it becomes more difficult to obtain significant amount of failure data within reasonable amount of time using single stress type only. Multiple-stress-type ALTs have been employed as a means to overcome such difficulties.

For instance, Kobayashi *et al.* (1978), Minford (1982), Mogilevsky and Shirn (1988), and Munikoti and Dhar (1988) use two stress types to test certain types of capacitors, and Weis *et al.* (1988) employ two stress types to estimate the lifetime of silicon photodetectors. Unlike the case of the single-stress-type ALT, little work has been done on designing multiple-stress-type ALT plans. Escobar and Meeker (1995) develop statistically optimal and practical plans with two stress types with no interaction between them. However, if prior information does not support the nonexistence of interaction or if the so-called sliding level technique cannot be employed to avoid the potential interaction, then the analysis based on the main effects only could lead to serious bias in estimation. Park and Yum (1996) develop ALT plans in which two stresses are employed with possible interaction between them with exponential distribution assumption. Elsayed and Zhang (2005) consider optimum ALT plans with proportional hazard model, which involve only two stress types with two stress levels for each stress type.

In this dissertation, we also propose to design and develop optimum multiple-stress-type accelerated life testing plans based on the proportional odds model with both constant-stress loading and simple step-stress loading. The objective function is chosen to minimize the asymptotic variance of reliability function estimate at the design stress conditions. The plans determine the optimum stress levels, the number of test units allocated to each stress level, the stress level changes and the corresponding changing times, and/or the test termination time. We adopt the widely used cumulative exposure model to derive the cumulative distribution function of the failure time for a test unit experiencing step-stress loading. Since the model parameters are unknown before the test planning, we use the estimates of those parameters through a preliminary baseline experiments or through engineering experience to design the optimum test plans.

1.3 Organization of the Dissertation Proposal

The remainder of the dissertation proposal is organized as follows. Chapter 2 provides a thorough review of the current literature of accelerated life testing models and discusses the problems encountered in ALT modeling. In chapter 3, we propose the new approach for accelerated life testing based on proportional odds model after careful investigation of the properties of odds function. This new approach is verified by both a simulation study and an experimental data application. In chapter 4, we construct the confidence intervals for the model parameters and reliability estimate at design stress conditions based on proportional odds model. We also provide methods to investigate the model sufficiency based on likelihood ratio test and to validate the model assumption based on Cox-Snell residuals. In chapter 5, we design optimum multiple-stress-type accelerated testing plans based on proportional odds model with both constant-stress loading and simple step-stress loading. Those carefully designed test plans provide the most accurate reliability estimates at design stress conditions since the plans are achieved by minimizing the asymptotic variance of reliability estimates at design stress condition. A preliminary investigation of the equivalent ALT plans is presented in chapter 6. The results of this research enable reliability practitioners to choose the appropriate ALT plan according to their practice and available resources. In chapter 7 we conduct experimental testing using miniature light bulbs as test units and subject them to various test plans. We validate the reliability estimates at different stress conditions. In chapter 8, we give conclusive remarks of this research and present future work related to the PO-based accelerated life testing model and the proposed optimum multiple-stress-type accelerated life testing plans based on the proportional odds model.

CHAPTER 2

LITERATURE REVIEW

In this chapter, we present a detailed overview of the reliability models for accelerated life testing and the motivation of the new ALT method. We then discuss the review of the ALT test plans, objective constraints and those limitations. We begin by the review of the proportional hazards (PH) model and its parameter estimation methods in section 2.1.1. In section 2.1.2, we present a group of models, the accelerated failure time (AFT) models, and point out that the Weibull distribution is the only distribution that yields both a proportional hazards and an accelerated failure time model. After that, we introduce the proportional odds (PO) model as a complement to the PH model and several parameter estimation methods for the PO model in section 2.1.3. Since the current parameter estimation methods for the PO model are not suitable for accelerated life testing, we conclude that a new estimation method of the PO model is needed for accelerated life testing. In section 2.2, we present a thorough review of literature about ALT plans. The review indicates that there are no optimum plans for proportional odds cases when neither the AFT nor the PH assumptions hold for the failure time data. Furthermore, current existing ALT plans only consider single stress type situations. However, as products become more reliable thanks to technological advances, it becomes more difficult to induce significant amount of failure time data within the limited testing duration using a single stress type only. Multiple-stress-type ALTs and ALT plans have to be applied to overcome such difficulties.

2.1 ALT Models

2.1.1 Cox's Proportional Hazards Model

The most widely used model describing the influence of covariates on the failure time distribution is the proportional hazards (PH) or Cox model, introduced by D. Cox (1972). It was first used in biomedical applications. The PH model has also been used as an accelerated life testing model. Although it does not assume any form of the failure time distribution, it still allows us to quantify the relationship between the failure time and a set of explanatory variables, which are stresses in accelerated life testing area. A brief description of the PH model follows.

Let T denote the failure time, τ the right censoring time. The data, based on a sample of size n, consists of the triple $(t_i, I_i, \boldsymbol{z}_i)$, $i = 1, \ldots, n$ where t_i is the failure time of the *i*th test unit under study, I_i is the failure indicator for the *i*th unit ($I_i = 1$ if the failure has occurred, which means $t_i \leq \tau$, and $I_i = 0$ if the failure time is right-censored, or $t_i > \tau$), and $\boldsymbol{z}_i = (z_{1i}, \ldots, z_{ki})^t$ is the vector of covariates or stresses for the *i*th unit, which affects the reliability distribution of T, including temperature, voltage, humidity, etc.

Let $\lambda(t;\boldsymbol{z})$ be the hazard rate at time t for a unit with a vector of stresses, $\boldsymbol{z}$. The basic PH model is as follows:

$$\lambda(t;\boldsymbol{z}) = \lambda_0(t) c(\boldsymbol{\beta}^t \boldsymbol{z}), \tag{2.1}$$

where

$\lambda_0(t)$	an arbitrary baseline hazard rate;
$\boldsymbol{z} = (z_1, z_2, \ldots, z_k)^t$	a vector of the covariates or the applied stresses;
$\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_k)^t$	a vector of the unknown regression parameters;
$c(\boldsymbol{\beta}^t \boldsymbol{z})$	a known function
k	the number of the stresses.

Because hazard rate function $\lambda(t;\boldsymbol{z})$ must be positive, a common feasible function for $c(\boldsymbol{\beta}^t \boldsymbol{z})$ is

$$c(\boldsymbol{\beta}^t \boldsymbol{z}) = \exp(\boldsymbol{\beta}^t \boldsymbol{z}) = \exp\left(\sum_{j=1}^{k} \beta_j z_j\right),$$

yielding

$$\lambda(t;\boldsymbol{z}) = \lambda_0(t)\exp(\boldsymbol{\beta}^t \boldsymbol{z}) = \lambda_0(t)\exp\left(\sum_{j=1}^{k} \beta_j z_j\right). \tag{2.2}$$

The main assumption of the proportional hazards (PH) model is that the ratio of two hazard rates of two units under two stress levels $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ is constant over time. In other words:

$$\frac{\lambda(t;\mathbf{z}_1)}{\lambda(t;\mathbf{z}_2)} = \frac{\lambda_0(t)\exp(\boldsymbol{\beta}^t\mathbf{z}_1)}{\lambda_0(t)\exp(\boldsymbol{\beta}^t\mathbf{z}_2)} = \exp[\sum_{j=1}^{k}\beta_j(z_{1j} - z_{2j})].$$

This implies that the hazard rates are proportional to the applied stress levels.

Without the specification of the form of baseline hazard rate function $\lambda_0(t)$, the inference for $\boldsymbol{\beta}$ could be based on a partial or conditional likelihood rather than a full likelihood approach.

Suppose that there are no ties between the failure times. Let $t_1 < t_2 < \cdots < t_D$ denote the ordered failure times with corresponding stresses $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_D$. If $D < n$, then the remaining $n - D$ units are censored. Define the risk set at time t_i, $R(t_i)$, as the set of all units that are still surviving at a time just prior to t_i. Then the partial likelihood, based on the hazard function as specified by Eq. (2.1), is expressed by

$$L(\boldsymbol{\beta}) = \prod_{i=1}^{D}\frac{\exp(\boldsymbol{\beta}^t\mathbf{z}_i)}{\sum_{j\in R(t_i)}\exp(\boldsymbol{\beta}^t\mathbf{z}_j)}. \tag{2.3}$$

The log likelihood is obtained as

$$l(\boldsymbol{\beta}) = \sum_{i=1}^{D}\boldsymbol{\beta}^t\mathbf{z}_i - \sum_{i=1}^{D}\ln[\sum_{j\in R(t_i)}\exp(\boldsymbol{\beta}^t\mathbf{z}_j)]. \tag{2.4}$$

The partial maximum likelihood estimates are found by solving a set of nonlinear equations, which are obtained by setting the first derivates of the log likelihood with respect to the k unknown parameters $\beta_1, \beta_2, \ldots, \beta_k$ to zero. This can be done numerically using a Newton-Raphson method or some other iterative methods. Note that Eq. (2.4) does not depend on the baseline hazard rate $\lambda_0(t)$, so that inferences may be made on the effects of the covariates without knowing $\lambda_0(t)$.

Due to the way failure times are recorded, ties between failure times are often found in the data. Alternate partial likelihoods have been provided by a variety of authors (Breslow 1974, Efron 1977, Cox 1972) when there are ties between failure times. Among them, Efron suggests a partial likelihood as

$$L(\boldsymbol{\beta}) = \prod_{i=1}^{D} \frac{\exp(\boldsymbol{\beta}^t \boldsymbol{s}_i)}{\prod_{j=1}^{d_i} [\sum_{k \in R(t_i)} \exp(\boldsymbol{\beta}^t \boldsymbol{z}_k) - \frac{j-1}{d_i} \sum_{k \in D_i} \exp(\boldsymbol{\beta}^t \boldsymbol{z}_k)]},$$

where

$t_1 < t_2 < \cdots < t_D$ denote the D distinct, ordered, failure times,

d_i is the number of failures at t_i,

D_i is the set of all units that failure at time t_i,

$\boldsymbol{s}_i$ is the sum of the vector $\boldsymbol{z}_j$ over all units that failure at t_i, that is $\boldsymbol{s}_i = \sum_{j \in D_i} \boldsymbol{z}_j$,

$R(t_i)$ is the set of all units at risk just prior to t_i.

After obtaining the estimates of the unknown parameters $\boldsymbol{\beta}$, there are several ways to estimate the reliability of units at the design stress level $\boldsymbol{z}_D$. One way, without the explicit specification of baseline hazard rate function, is based on Breslow's estimator (Breslow 1975) of the baseline cumulative hazard rate. Breslow's estimator of the baseline cumulative hazard rate is given by

$$\hat{\Lambda}_0(t) = \sum_{t_i \le t} \frac{d_i}{\sum_{j \in R(t_i)} \exp(\hat{\boldsymbol{\beta}}^t \boldsymbol{z}_j)},$$

which is a step function with jumps at the observed failure times. Therefore the estimator of the baseline reliability function, $R_0(t) = \exp[-\Lambda_0(t)]$ is given by

$$\hat{R}_0(t) = \exp[-\hat{\Lambda}_0(t)].$$

This is an estimator of the reliability function of a unit with a baseline set of stresses, $\boldsymbol{z} = \boldsymbol{0}$. To estimate the reliability function for a unit at the design stresses level $\boldsymbol{z}_D$, we use the estimator

$$\hat{R}(t; \boldsymbol{z}_D) = \hat{R}_0(t)^{\exp(\hat{\boldsymbol{\beta}}^t \boldsymbol{z}_D)}.$$

Other ways to estimate the reliability function of units at design stress require estimating the baseline hazard rate function $\lambda_0(t)$ first. Anderson (1980) describes a piecewise smooth estimate of $\lambda_0(t)$ instead of step function of $\hat{\Lambda}_0(t)$. He assumes $\lambda_0(t)$ to be a quadratic spline. Other authors (Elsayed 2002) also suggest to use quadratic function to estimate $\lambda_0(t)$. Based on the estimator $\hat{\lambda}_0(t)$, the estimator of the reliability function is given by

$$\hat{R}(t;\boldsymbol{z}_D) = \exp[-\exp(\hat{\boldsymbol{\beta}}^t \boldsymbol{z}_D)\int_0^t \hat{\lambda}_0(u)du].$$

The PH model usually produces good reliability estimation with failure data for which the proportional hazards assumption does not even hold exactly.

2.1.2 Accelerated Failure Time Models

In the previous section, we present nonparametric methods for accelerated life testing which do not require any specific distributional assumptions about the shape of the reliability function. In this section, we present a class of parametric models for ALT, named *Accelerated Failure Time* (*AFT*) *models*, which have an *accelerated failure time model* representation and a *linear model* representation in log of failure time.

The accelerated failure time model representation assumes that the covariates act multiplicatively on the failure time, or linearly on the log of failure time. Let T denote the failure time and $\mathbf{z}$ a vector of covariates, the AFT model is defined by the relationship

$$R(t;\mathbf{z}) = R_0[\exp(\boldsymbol{\delta}^t\mathbf{z})t]. \tag{2.5}$$

The factor $\exp(\boldsymbol{\delta}^t\mathbf{z})$ is called an acceleration factor telling how a change in the covariate values changes the time scale from the baseline time scale. The model presented by Eq. (2.5) implies that hazard rate relationship is given by

$$\lambda(t;\mathbf{z}) = \lambda_0[\exp(\boldsymbol{\delta}^t\mathbf{z})t]\exp(\boldsymbol{\delta}^t\mathbf{z}). \tag{2.6}$$

The second representation of this group of models is the linear relationship between log failure time and the stress values, and is given by

$$\ln T = \mu + \boldsymbol{\omega}^t\mathbf{z} + \psi W ,$$

where $\boldsymbol{\omega}^t$ is a vector of regression coefficients and W is the random error.

The two representations are closely related. If we let $R_0(t)$ to be the reliability function of the random variable $\exp(\mu + \psi W)$, then the linear log failure time model is equivalent to the AFT model with $\boldsymbol{\delta} = -\boldsymbol{\omega}$.

A variety of distributions can be used for W or, equivalently, for $R_0(t)$. If $R_0(t)$ is assumed to be Weibull reliability function, or W has a standard extreme value distribution, we will then have a Weibull distributed AFT model. If $R_0(t)$ is assumed to be log-logistic reliability function, or W has a logistic distribution, we have a log-logistic distributed AFT model. If $R_0(t)$ is assumed to be log-normal reliability function, or W has a normal distribution, we have a log-normal distributed AFT model. Therefore the AFT model also implies that the failure times at different stress levels follow a common distribution with the same shape parameter but different scale parameter, which has some relationship with the applied stresses. For the AFT models, estimates usually must be found numerically. When these parametric AFT models provide a good fit to failure time data, they tend to give more precise estimates of the reliability. However, if a parametric model is chosen incorrectly, it may lead to consistent estimates with significant errors.

We note that the Weibull distribution is the only continuous distribution that yields both a proportional hazards and an accelerated failure time model. A PH model for T with a Weibull baseline hazard $\lambda_0(t) = \alpha\lambda t^{\alpha-1}$ is

$$\lambda(t;\mathbf{z}) = \lambda_0(t)\exp(\boldsymbol{\beta}^t\mathbf{z}) = (\alpha\lambda t^{\alpha-1})\exp(\boldsymbol{\beta}^t\mathbf{z})\,. \tag{2.7}$$

An AFT model for T with a Weibull baseline hazard $\lambda_0(t) = \alpha\lambda t^{\alpha-1}$ is

$$\lambda(t;\mathbf{z}) = \lambda_0[\exp(\boldsymbol{\delta}^t\mathbf{z})t]\exp(\boldsymbol{\delta}^t\mathbf{z}) = \alpha\lambda t^{\alpha-1}\exp(\alpha\boldsymbol{\delta}^t\mathbf{z})\,. \tag{2.8}$$

Comparing Eq. (2.7) and Eq. (2.8), we have $\boldsymbol{\beta} = \alpha\boldsymbol{\delta}$.

2.1.3 Proportional Odds Model and Inference

The PH model is a widely used nonparametric model for accelerated life testing because of its relative easiness of the estimating procedure of partial likelihood and its large sample inference properties demonstrated using martingale theory. Moreover, reliability practitioners have easy access to statistical software, including SAS, BMDP, and S-Plus, as described by Klein (1997), for this model. Therefore, there is a temptation to use the PH model to analyze failure time data, even when the model does not fit the data well. In many applications, however, it is often unreasonable to assume the effects of covariates on the hazard rates remain fixed over time. Brass (1971) observes that the ratio of the death rates, or hazard rates, of two populations under different stress levels (for example, one population for smokers and the other for non-smokers) is not constant with age, or time, but follows a more complicated course, in particular converging closer to unity for older people. So the PH model is not suitable for this case. Brass (1974) proposes a more realistic model as

$$\frac{F(t;\mathbf{z})}{1-F(t;\mathbf{z})} = e^{\boldsymbol{\beta}'\mathbf{z}} \frac{F_0(t)}{1-F_0(t)}. \tag{2.9}$$

This model is referred to as the Proportional Odds (PO) model since the odds functions, which are defined as $\theta(t) \equiv \dfrac{F(t)}{1-F(t)}$, under different stress levels are proportional to each other.

The proportional odds models have been widely used as a kind of ordinal logistic regression model for categorical data analysis, as described by McCullagh (1980) and Agresti (2002). Recently the PO model has received more attentions for survival data analysis. Dabrowska *et al.* (1988) and Wu (1995) propose estimation methods for only two-sample life time data based on PO model. Pettitt (1984) estimates the parameter in the PO model using ranks, which ignores the actual observations, and results in corresponding inaccurate estimation. Murphy *et al.* (1997) propose the estimation method of the PO model based on profile likelihood. The likelihood is constructed from the conditional probability density. Because the number of unknown parameters in the profile likelihood is extremely large, the estimation calculation is also extremely computationally intensive. The description of the PO model and the current inference procedures of parameter estimation are summarized as follows.

2.1.3.1 Description of PO model

Let T denote the failure time, τ the right censoring time. The data, based on a sample of size n, consists of the triple $(t_i, I_i, \mathbf{z}_i)$, $i = 1, \ldots, n$ where t_i is the failure time of the *i*th unit under study, I_i is the failure indicator for the *i*th unit ($I_i = 1$ if the failure has occurred, which means $t_i \le \tau$, and $I_i = 0$ if the failure time is right-censored, or $t_i > \tau$), and $\boldsymbol{z}_i = (z_{i1}, \ldots, z_{ip})^t$ is the vector of covariates, or stresses, for the ith unit.

Let $\theta(t;\boldsymbol{z}) = \dfrac{F(t;\boldsymbol{z})}{1-F(t;\boldsymbol{z})}$ be the odds function, which is the ratio of the probability of a unit's failure to the probability of its not failing, at time t for a unit with a vector of stresses, $\boldsymbol{z}$. The basic PO model is given:

$$\theta(t;\boldsymbol{z}) = \theta_0(t)\exp(\boldsymbol{\beta}^t\boldsymbol{z}), \tag{2.10}$$

where

$\theta_0(t)$	an arbitrary baseline odds function;
$\boldsymbol{z} = (z_1, z_2, \ldots, z_k)^t$	a vector of the covariates or the applied stresses;
$\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_k)^t$	a vector of the unknown regression parameters;
k	the number of the stresses.

For two failure time samples with stress levels $\mathbf{z}_1$ and $\mathbf{z}_2$, the difference between the respective log odds functions

$$\log[\theta(t;\mathbf{z}_1)] - \log[\theta(t;\mathbf{z}_2)] = \boldsymbol{\beta}^t(\mathbf{z}_1 - \mathbf{z}_2)$$

is independent of the baseline odds function $\theta_0(t)$, and furthermore of the time t, thus one odds function is constantly proportional to the other. The baseline odds function could be any monotone increasing function of time t with the property of $\theta_0(0) = 0$. When $\theta_0(t) = t^{\varphi}$, PO model described by Eq. (2.10) becomes the log-logistic accelerated failure time model (Bennett 1983), which is a special case of the general PO model.

To investigate the relation between the PO model and the PH model, we represent the PO model as described by Eq. (2.10) with hazard rate function. After mathematical transform, the PO model in Eq. (2.10) can be represented by

$$\lambda(t;\mathbf{z}) = \frac{e^{\boldsymbol{\beta}'\mathbf{z}}\lambda_0(t)}{1-(1-e^{\boldsymbol{\beta}'\mathbf{z}})F_0(t)} \quad , \tag{2.11}$$

where $F_0(t)$ is the baseline cumulative distribution function.

It is easy to see, from Eq. (2.11), that in two failure time sample setup, the model implies that the ratio of the hazard rates at different stress levels converges to 1 over time. Thus the PO model is useful when the covariate (stress) effect on the hazard rate diminishes over time.

2.1.3.2 Estimating the Parameter in a Two-sample PO Model

Dabrowska and Doksum (1988) and Wu (1995) give an estimation procedure for the so-called generalized odds-rate models, of which the PO model is a special case, in two-sample case. They show that the estimates are consistent and asymptotically normal. They first consider the generalized odds function for a random failure time T as defined by

$$\Lambda_T(t \mid c) = \begin{cases} \dfrac{1}{c}\left[\dfrac{1-(1-F(t))^c}{(1-F(t))^c}\right], & c > 0 \\ -\log[1-F(t)], & c = 0 \end{cases} \tag{2.12}$$

where $F(t)$ is the cumulative distribution function for failure time T.

Note that $\Lambda_T(t \mid 0)$ is the cumulative hazard rate, whereas $\Lambda_T(t \mid 1)$ is the odds of the failure before time t. For c other than 1, $\Lambda_T(t \mid c)$ also has an interpretation as an odds function for some situations. When considering experiments involving covariates that affect the distribution of the failure time, the generalized proportional odds-rate model is represented by

$$\Lambda_{T_1}(t; z_1 \mid c) = e^{\beta(z_1 - z_2)} \Lambda_{T_2}(t; z_2 \mid c), \tag{2.13}$$

where T_1 is a random failure time sample with size n_1 associated with covariate z_1, and T_2 is a random failure time sample with size n_2 associated with covariate z_2.

When $c=1$, the generalized proportional odds-rate model as in Eq. (2.13) is just the proportional odds model as described in Eq. (2.10).

By rewriting (2.13), they obtain

$$(1-F_{T_1}(t))^{c+1}dF_{T_2}(t)=e^{\beta(z_1-z_2)}(1-F_{T_2}(t))^{c+1}dF_{T_1}(t). \tag{2.14}$$

Based on Eq. (2.14), the estimator of $e^{\beta(z_1-z_2)}$, or $\hat{\sigma}(e^{\beta(z_1-z_2)})$, is given by

$$\hat{\sigma}(e^{\beta(z_1-z_2)})=\frac{\int_0^\infty \Psi(\hat{F}_{T_1}(t))[1-\hat{F}_{T_2}(t)]^{-(c+1)}d\hat{F}_{T_2}(t)}{\int_0^\infty \Psi(\hat{F}_{T_1}(t))[1-\hat{F}_{T_1}(t)]^{-(c+1)}d\hat{F}_{T_1}(t)}, \tag{2.15}$$

where $\hat{F}_{T_1}(t)$ and $\hat{F}_{T_2}(t)$ are left-continuous empirical distributions based on the failure time samples T_1 and T_2, respectively, and $\Psi(\cdot)$ is some score function. Then the estimator of β is given by

$$\hat{\beta}=\frac{\ln[\hat{\sigma}(e^{\beta(z_1-z_2)})]}{(z_1-z_2)}.$$

Under some mild regularity conditions, Dabrowska and Doksum show that $(n_1+n_2)^{1/2}(\hat{\sigma}(e^{\beta(z_1-z_2)})-e^{\beta(z_1-z_2)})$ has an asymptotically normal distribution and an efficient score function is given by the follow equation for $e^{\beta(z_1-z_2)}=1$,

$$\Psi(u) = (c+1)(1-u)^{2c+1},$$

However, for $e^{\beta(z_1-z_2)} \neq 1$, the efficient score function $\Psi(\cdot)$ remains unknown.

Wu (1995) consider the special case of the two-sample proportional odds model and show that an efficient estimator of $e^{\beta(z_1-z_2)}$ can be constructed based on the solution of a pair of integral equations. Because of the special structure of the proportional odds model, the solution of the equations can be obtained in a closed form. Wu further proves that, by selecting any suitable and convenient score function, the estimator of $e^{\beta(z_1-z_2)}$ is asymptotically normal.

Although the asymptotically efficient estimate of $e^{\beta(z_1-z_2)}$ could be constructed explicitly for this special two-sample proportional odds model, this estimator has limited usage for accelerated life testing. Firstly, in the accelerated life testing field, there are always more than two samples, which means there are always more than two stress levels. Furthermore this estimation method can be used for the situation where there is only one stress type. If there are two stress types, such at temperature and voltage, or more, we can not estimate the parameters vector $\boldsymbol{\beta}$ from the estimator of $e^{\boldsymbol{\beta}^t(\mathbf{z}_1-\mathbf{z}_2)}$. Also, an appropriate score function is difficult to determine. Finally, this estimation method doesn't give an approach to the reliability estimate at stress levels other than the two observed levels.

2.1.3.3 Estimates of the PO model Using Ranks

Pettitt (1984) proposes an approximate estimation method of proportional odds model for failure time data using the ranks of the failure times, instead of the true observations, without specification of the format of the baseline odds function. The approximation is based upon a Taylor series expansion of the logarithm of the marginal likelihood about zero, and can be used for censored failure time data. The approximate method is as follows.

Let $t_1, \ldots, t_D$ be observed uncensored failure times and $t'_{D+1}, \ldots, t'_n$ be right censored failure times. Order the $t_j,\ j = 1, \ldots, D$, in the usual way to obtain the ordered failure times $t_{(1)} < \ldots < t_{(D)}$. The rank r_j of t_j is defined in the usual way. The ranks, r_j, of the censored failure times $t'_j,\ j = D+1, \ldots, n$, are defined as follows: t'_j has rank r_j if and only if t'_j lies in the interval $(t_{(r_j)}, t_{(r_j+1)})$, with $t_{(0)} = 0$ and $t_{(D+1)} = \infty$.

Inference for $\boldsymbol{\beta}$ in the proportional odds model described by Eq. (2.10) is now based on the ranks estimate

$$\hat{\boldsymbol{\beta}}_R = [Z^t(B-A)Z]^{-1} Z^t \boldsymbol{a}, \tag{2.16}$$

where Z is the model matrix, and $\boldsymbol{a}$, A and B are defined as follows:

$\boldsymbol{a}$ is a $n \times 1$ vector with

$$a_j = 1 - (2 - c_j)\xi_{r_j} ,$$

A is a $n \times n$ symmetric matrix with

$$(A)_{ij} = (2 - c_i)(2 - c_j)(v_{r_i, r_j} - \xi_{r_i}\xi_{r_j}) ,$$

B is a $n \times n$ diagonal matrix with

$$(B)_{jj} = (2 - c_j)(\xi_{r_j} - \tau_{r_j}) ,$$

where $c_j = 0$ for $j = 1, \ldots, D$ and $c_j = 1$ for $j = D + 1, \ldots, n$,

$$\xi_j = \prod_{i=1}^{j} \left(\frac{u_i}{1 + u_i} \right), \qquad (j = 1, \ldots, D),$$

$$\tau_j = \prod_{i=1}^{j} \left(\frac{u_i}{2 + u_i} \right), \qquad (j = 1, \ldots, D),$$

and

$$v_{ij} = \prod_{k=1}^{i}\left(\frac{u_k}{2+u_k}\right)\prod_{k=1}^{j}\left(\frac{u_k}{1+u_k}\right), \qquad (1 \le i \le j = D).$$

The u_j's are defined as follows: Let m_j be the number of the t_i' $(i = D+1,\ldots,n)$ having their ranks, r_i, equal to j, $j = 1,\ldots,D$, then define $u_j = (m_j + 1) + \ldots (m_D + 1)$, $j = 1,\ldots,D$, so that u_j is the total number of failures, censored or uncensored, greater than or equal to $t_{(j)}$.

Pettitt (1984) also gives the variance-covariance matrix $M = [Z^t(B-A)Z]^{-1}$ for the estimator $\hat{\boldsymbol{\beta}}_R$ based on ranks without proof.

This estimator looks appealing to reliability practitioners since it gives a closed form of the estimation. But it actually has little usage for accelerated life testing because of the common drawback of rank estimator, inaccuracy, which is the corresponding result of the fact that these rank estimates are based on the ranks of the failure time observations instead of their actual values. Meanwhile the estimation procedure based on ranks does not provide any well-tested approach to the estimation of the reliability function of products at the design stress level nor does it produce residuals for model checking using the rank estimates, since no specification and inference about the baseline odds function are discussed in this procedure.

2.1.3.4 Profile Likelihood Estimates of PO Model

The estimation method of the PO model based on profile likelihood is proposed by Murphy *et al.* (1997). The likelihood is constructed from the conditional probability density. The profile likelihood estimation method of the PO model by Murphy *et al.* is summarized as follows.

From the PO model described by Eq. (2.10) and the relationship between odds function and reliability function, $\theta(t;\mathbf{z}) = \dfrac{1 - R(t;\mathbf{z})}{R(t;\mathbf{z})}$, we have

$$R(t;\mathbf{z}) = \frac{\exp(-\boldsymbol{\beta}^t \mathbf{z})}{\theta_0(t) + \exp(-\boldsymbol{\beta}^t \mathbf{z})}, \tag{2.17}$$

Then we specify the conditional probability density with respect to the sum of Lebesgue and the counting measure by

$$\frac{\exp(-\boldsymbol{\beta}^t \mathbf{z})}{[\theta_0(t) + \exp(-\boldsymbol{\beta}^t \mathbf{z})]} \cdot \frac{\vartheta(t)}{[\theta_0(t-) + \exp(-\boldsymbol{\beta}^t \mathbf{z})]}, \tag{2.18}$$

where $\theta_0(t-)$ is the left limit of $\theta_0(t)$ at t. If $\theta_0(t)$ is absolutely continuous, the $\vartheta(t)$ is the derivative of $\theta_0(t)$; if $\theta_0(t)$ is discrete, then $\vartheta(t) = \Delta\theta_0(t) = \theta_0(t) - \theta_0(t-)$. The Eq. (2.18) is the approximation of the continuous conditional probability density $f(t;\mathbf{z})$ defined as

$$
\begin{aligned}
f(t;\boldsymbol{z}) = R(t;\boldsymbol{z})\cdot\lambda(t;\boldsymbol{z}) &= R(t;\boldsymbol{z})\cdot\frac{\theta'(t;\boldsymbol{z})}{1+\theta(t;\boldsymbol{z})} \\
&= R(t;\boldsymbol{z})\cdot\frac{\exp(\boldsymbol{\beta}^t\mathbf{z})\theta_0'(t)}{1+\exp(\boldsymbol{\beta}^t\mathbf{z})\theta_0(t)} \\
&= R(t;\boldsymbol{z})\cdot\frac{\theta_0'(t)}{\exp(-\boldsymbol{\beta}^t\mathbf{z})+\theta_0(t)} \\
&= \frac{\exp(-\boldsymbol{\beta}^t\mathbf{z})}{[\theta_0(t)+\exp(-\boldsymbol{\beta}^t\mathbf{z})]}\cdot\frac{\theta_0'(t)}{[\exp(-\boldsymbol{\beta}^t\mathbf{z})+\theta_0(t)]},
\end{aligned}
$$

We consider the use of the conditional probability density as described in Eq. (2.18) for right-censored failure time data. If we let T denote the failure time, let τ to be the right censoring time. Then based on a sample of size n, the observations consist of the triple $(t_i, I_i, \boldsymbol{z}_i)$, $i = 1,\ldots,n$ where t_i is the failure time of the *i*th unit under study, I_i is the failure indicator for the *i*th unit ($I_i = 1$ if the failure has occurred, which means $t_i \le \tau$, and $I_i = 0$ if the failure time is right-censored, or $t_i > \tau$), and $\boldsymbol{z}_i = (z_{1i},\ldots,z_{ki})^t$ is the vector of covariates or stresses for the *i*th unit. Then the likelihood based on the conditional probability density of Eq. (2.18) for one observation is

$$
L(t;\theta_0,\boldsymbol{\beta}) = \left\{\frac{\exp(-\boldsymbol{\beta}^t\mathbf{z})}{[\theta_0(t)+\exp(-\boldsymbol{\beta}^t\mathbf{z})]}\cdot\frac{\vartheta(t)}{[\theta_0(t-)+\exp(-\boldsymbol{\beta}^t\mathbf{z})]}\right\}^I\cdot\left\{\frac{\exp(-\boldsymbol{\beta}^t\mathbf{z})}{[\theta_0(t)+\exp(-\boldsymbol{\beta}^t\mathbf{z})]}\right\}^{1-I}.
$$

Murphy *et al.* prove that if $\theta_0(t)$ is known to be absolutely continuous, then, as in the case of density estimation, there is no maximizer of the likelihood, but if $\theta_0(t)$ is known to be discrete, then a maximizer exists. The likelihood function for the whole failure time observations can be written as

$$L(\boldsymbol{t};\theta_0,\boldsymbol{\beta})=\prod_{i=1}^{n}\left\{\frac{\exp(-\boldsymbol{\beta}^t\mathbf{z}_i)}{[\theta_0(t_i)+\exp(-\boldsymbol{\beta}^t\mathbf{z}_i)]}\right\}\cdot\left\{\frac{\Delta\theta_0(t_i)}{[\theta_0(t_i-)+\exp(-\boldsymbol{\beta}^t\mathbf{z}_i)]}\right\}^{I_i}, \tag{2.19}$$

where we have replaced ϑ by $\Delta\theta_0$. The MLE of θ_0 will be a non-decreasing step function with steps at the observed failure times.

The profile log-likelihood for $\boldsymbol{\beta}$ is given by

$$\Pr lik_n(\boldsymbol{\beta})=\log L(\boldsymbol{t};\hat{\theta}_0(\boldsymbol{\beta}),\boldsymbol{\beta}),$$

where $\hat{\theta}_0(\boldsymbol{\beta})$ maximizes the log-likelihood for a fixed $\boldsymbol{\beta}$. The maximum profile likelihood estimator, $\hat{\boldsymbol{\beta}}$, maximizes $\Pr lik_n(\boldsymbol{\beta})$.

The presence of the terms $\Delta\theta_0(t_i)$ in the likelihood of Eq. (2.19) forces the estimator of θ_0 to have positive jumps at the observed failure times and have no jumps at any other points. Thus the number of unknown parameters is k, the number of covariates, plus the number of observed failure times.

Murphy *et al.* also prove that the maximum profile likelihood estimator of $\boldsymbol{\beta}$ is consistent, asymptotically normal, and efficient. Differentiation of the profile likelihood yields consistent estimators of the efficient information matrix. Additionally, a profile likelihood ratio statistic can be compared to percentiles of the chi-squared distribution to produce asymptotic hypothesis tests of the appropriate size.

The calculation of the maximum profile likelihood depends on the numerical algorithms, from which Murphy *et al.* choose a modification of the Newton-Raphson algorithm. The numerical algorithms can not always guarantee a convergent solution, especially when the size of the unknown parameter is large. As mentioned in the paper by Murphy *et al*, the number of the unknowns in the likelihood function of Eq. (2.19) is k, the number of covariates, plus the number of observed failure times. As the size of failure time sample increases, the maximization problem becomes extremely difficult. Another problem of this maximum profile likelihood estimator of $\boldsymbol{\beta}$ is the fact that the estimator is a function of the failure times only through their ranks. The profile likelihood is the same whether we use the actual failure time observations or replace them by their ranks. From the maximum profile likelihood estimation, the estimates of the baseline odds function can be only obtained at the observed failure times. As a result, the reliability estimates also can be only obtained at the observed failure times, only explicit reliability function estimate is available. This problem also makes the maximum profile likelihood estimation method not applicable to accelerated life testing which requires the reliability function estimation of the unit at the normal design stress level.

2.2 ALT Test Plans

In order to increase the accuracy of reliability estimates in accelerated life testing problem, a carefully designed ALT test plan is required. This test plan is designed to minimize a specified criterion, usually the variance of a reliability-related estimate, such as reliability function, mean time to failure and a percentile of failure time, under specific time and cost constraints.

For easy implementation, commonly used constant stress ALT test plans consist of several equally spaced constant stress levels, each is allocated a proportion of the total number of test units. Such standard test plans are usually inefficient for estimating reliability of the product at design stress. Earlier work by Meeker and Nelson (1978) propose optimal statistical plans for constant stress ALTs which include only two stress levels. Such plans lack robustness since the assumed life-stress relationship is difficult, if not impossible, to validate. Meeker and Hahn (1985) address this by proposing a 4:2:1 allocation ratio for low, middle, and high stress levels and give the optimal low level by assuming the middle stress to be the average of the high and the low stress levels. In recent years, by considering other test constraints and allowing non-constant shape parameter of the failure time distribution (Meeter and Meeker, 1994), the use of compromised ALT plans for three stress levels without optimizing the middle stress level and allocation of test units is advocated. The existing ALT plans could be classified into two categories: parametric ALT plans and nonparametric ALT plans. Parametric ALT plans are based on a pre-specified common reliability distribution and an assumed life-stress relationship. These plans are typically not robust to deviations from the model assumptions and the model parameters. Recently, Elsayed and Jiao (2002) consider optimum ALT plans based on nonparametric proportional hazards model. These nonparametric ALT plans are based on the minimization of the asymptotic variance of the hazard rate estimate at design stress. The plans determine the optimum three stress levels and the allocation of test units to each stress level. Most of the previous work on ALT plans involves only a single stress type with two or three stress levels. However, as products become more reliable due to technological advances, it becomes more difficult to obtain significant amount of failure data within reasonable amount of time using single stress type only. Multiple-stress-type ALTs have been employed as a means to overcome such difficulties. For instance, Kobayashi *et al.* (1978), Minford (1982), Mogilevsky and Shirn (1988), and Munikoti and Dhar (1988) use two stress types to test certain types of capacitors, and Weis *et al.* (1988) employ two stress types to estimate the lifetime of silicon photodetectors. Unlike the case of the single-stress-type ALT, little work has been done on designing multiple-stress-type ALT plans. Escobar and Meeker (1995) develop statistically optimal and practical plans with two stress types with no interaction between them. However, if prior information does not support the nonexistence of interaction or if the so-called sliding level technique cannot be employed to avoid the potential interaction, then the analysis based on the main effects only could lead to serious bias in estimation. Park and Yum (1996) develop ALT plans in which two stresses are employed with possible interaction between them with exponential distribution assumption. Elsayed and Zhang (2005) consider optimum ALT plans with proportional hazard model, which involve two stress types with two stress levels for each stress type.

Based on the employed accelerated life testing model in designing an ALT, we classify the existing ALT plans into two categories: *Accelerated failure time* model based (AFT-based) ALT plans, *Proportional hazards* model based (PH-based) ALT plans. There is extensive research concerning the AFT-based optimum ALT which involves the most widely used parametric accelerated failure time regression model with the exception of Elsayed and Jiao (2002) who consider optimum ALT plans based on nonparametric proportional hazards model. There is little research that investigates the design of optimum ALT plans using multiple stress types. We now describe these categories in details.

2.2.1 AFT-based ALT Plans

In general, AFT-based ALT plans use the parametric accelerated failure time (AFT) regression model. These ALT plans assume that the failure times follow a predetermined distribution, e.g. Weibull distribution, lognormal distribution, extreme value distribution, etc. Under this model, a higher stress level has the effect on reduced failure time through a scale factor. This can be expressed in terms of reliability function as

$$R(t;\boldsymbol{z}) = R_0[\exp(\boldsymbol{\delta}^t \boldsymbol{z})t]. \tag{2.20}$$

The factor $\exp(\boldsymbol{\delta}^t \boldsymbol{z})$ is called an acceleration factor. Since the performance of these ALT plans is highly dependent on the assumed failure time distribution, the optimum plans thus obtained are typically not robust to the deviations from the assumed distribution.

2.2.1.1 General Assumptions of AFT-based ALT Plans

The AFT-based ALT plans use the following general assumptions:

(1) The log-time-to-failure for each unit follows a location-scale distribution such that

$$\Pr(Y \le y) = \Phi\left(\frac{y-\mu}{\sigma}\right),$$

where $y = \log(t)$ is the logarithm of failure time t, μ and σ are location and scale parameters respectively, and $\Phi(\cdot)$ is the standard form, cumulative distribution function, of the location-scale distribution.

(2) Failure times for all test units, at all stress levels, are statistically independent.

(3) The location parameter μ is a linear function of stresses (z_1, z_2, ...). Specifically we assume that

$$\mu = \mu(z_1) = \gamma_0 + \gamma_1 z_1 \,, \qquad \text{for a one-stress ALT, or}$$

$$\mu = \mu(z_1, z_2) = \gamma_0 + \gamma_1 z_1 + \gamma_2 z_2 \,, \qquad \text{for a two-stress ALT}$$

(4) The scale parameter σ does not depend on the stress levels.

(5) All units are tested until time τ, a pre-specified censoring time.

The γ_i's and σ are unknown parameters to be estimated from the available ALT data.

2.2.1.2 Criteria for the Development of AFT-based ALT Plans

In developing preliminary test plans we assume constant stress levels during the ALT. The constant stress test plans in the literature can be further classified as: (1) ALT plans to estimate percentiles of the life distribution at specified design stress, and (2) ALT plans to estimate reliability at specified time and design stress based on different optimization criterion.

1. *ALT plans to estimate percentiles of the life distribution at specified design stress*

Chernoff (1962) considers maximum likelihood (ML) estimation of the failure rate of an exponential distribution at the design stress level. The relationship for the failure rate is assumed as a quadratic function of stress and an exponential function of stress. He gives optimum plans both for simultaneous testing with Type I censored data and for successive testing with complete data. Chernoff calls the plans "Locally optimum" because they depend on the true (unknown) parameter values. Most optimum designs associated with nonlinear estimation problems (including estimation with censored data) result in locally optimum designs.

Mann *et al.* (1974) consider linear estimation with order statistics to estimate a percentile of an extreme value (or Weibull) distribution at design stress and obtain optimal plans for failure data with censored observations.

Nelson and Kielpinski (1975, 1976) obtain optimum plans and best traditional plans (traditional plans use equally spaced levels of stress with equal allocation of test units to each stress level) for the median of a normal and lognormal distribution. Their model assumes that the normal distribution location parameter μ (also the mean) is a linear function of stress and the scale parameter σ (also the standard deviation) does not depend on stress. They also assume simultaneous testing of all test units and censoring at a pre-specified time.

Nelson and Meeker (1978) provide similar optimum test plans to estimate percentiles of Weibull and smallest extreme-value distributions at a specified design stress when test units are overstressed. They assume that the smallest extreme-value location parameter μ (also the 0.632 percentile) is a linear function of stress and that the scale parameter σ is constant.

Using similar assumptions, Meeker (1984) compares the statistically optimum test plans to more practical test plans that have three levels of stress. Meeker and Hahn (1985) provide extensive tables and practical guidelines for planning an ALT. They present a statistically optimum test plan and then suggest an alternative plan that meets practical constraints and has desirable statistical properties. The tables allow assessment of the effect of reducing the testing stress levels (thereby reducing the degree of extrapolation) on statistical precision. Jensen and Meeker (1990) provide a computer program that allows the user to develop and compare optimum and compromise ALT plans. The program also allows the user to modify or specify plans and evaluate their properties.

Previous work on planning ALTs assumes that the scale parameter σ for a location-scale distribution of log lifetime remains constant over all stress levels. This assumption is inappropriate for many applications including accelerated tests for metal fatigue and certain electronic components. Meeter and Meeker (1994) extend the existing maximum likelihood theory for test planning to the nonconstant scale parameter model and present test plans for a large range of practical testing. The test setup assumes simultaneous testing of test units with time censoring.

Barton (1991) proposes a variation of the optimum ALT plans described by Nelson and others. He shows how to minimize the maximum test-stress subject to meeting a certain standard-deviation of the reliability estimate at normal operating conditions.

Nelson, Meeker and others choose equal censoring times at each stress level. This practice does not completely cover field applications. A long censoring time that yields sufficient failure at the lowest stress level is considered too long at the highest stress level. Yang (1994) proposes an optimum design of 4-level constant-stress accelerated life test plans with various censoring times. The optimal plans choose the stress level, test units allocated to each stress, and censoring times to minimize the asymptotic variance of the MLE of the mean life at design stress and test duration. The test plans derived are proven to be more robust than the 3-level best-compromise test plans.

2. *ALT plans to estimate reliability at specified time and design stress*

Easterling (1975) considers the determination of a lower bound on reliability at design stress conditions. He utilizes the probit model to show that one can obtain an appreciably improved lower bound on reliability by allocating test units to stresses higher than the design stress, instead of testing all units at stresses close to or at the design stress.

Martz and Waterman (1977) use Bayesian methods for determining the optimal test stress for a single test unit to estimate the survival probability at a design stress.

Maxim *et al.* (1977) determine optimum accelerated test plans using the D-optimality criterion and assuming a bivariate exponential or Weibull model.

Meeker and Hahn (1977) consider the optimum allocation of test units to overstress conditions when it is desired to estimate the survival probability to a specified time at a design condition. The optimal criterion is to minimize the large sample variance under a logistic model assumption.

Based on the review of the literatures, we summarize the following guidelines for planning ALTs:

(1) Assuming a linear life-stress relationship for the statistically optimum plan, tests at only two levels of stress.

(2) Choosing the highest stress level to be as high as possible will increase precision and statistical efficiency. But the highest stress level should not be so high that new failure modes would be introduced and the assumed life-stress relationship would be invalid.

(3) Compromising test plans that use three or four levels of stress have somewhat reduced statistical efficiency but tend to be more robust to misspecification of the model and its parameters.

(4) More test units should be allocated to lower stress levels. There are two reasons for this: small portion of units will fail at the lower stress due to the limited test time and inferences at lower stress levels are closer to the extrapolation at design stress conditions. So more failures at lower stresses result in a more accurate estimate of reliability at normal conditions.

2.2.2 PH-based ALT Plans

Recently Elsayed and Jiao (2002) consider optimum ALT plans based on the nonparametric proportional hazards model. This ALT plan determines three optimum stress levels: high, medium and low levels, and optimum allocations of units to the three stress levels such that the asymptotic variance of the hazard rate estimate at the design stress level is minimized. The constraints include the maximum available test duration, total number of test units and the minimum number of failures at each stress level. The optimum ALT plans are based on the proportional hazards model. The procedure of the PH-based optimum ALT plans is summarized as follows.

The proportional hazards (PH) model is expressed as,

$$\lambda(t;z) = \lambda_0(t)\exp(\beta z). \tag{2.21}$$

The baseline hazard function $\lambda_0(t)$ is assumed to be linear with time,

$$\lambda_0(t) = \gamma_0 + \gamma_1 t. \tag{2.22}$$

Substituting Eq. (2.22) into Eq. (2.21) yields

$$\lambda(t;z) = (\gamma_0 + \gamma_1 t)\exp(\beta z). \tag{2.23}$$

Then the corresponding cumulative hazard function $\Lambda(t;z)$ and reliability function $R(t;z)$ are obtained respectively as,

$$\Lambda(t;z) = \int_0^t \lambda(u)du = (\gamma_0 t + \gamma_1 t^2/2)\exp(\beta z), \tag{2.24}$$

$$R(t;z) = \exp[-\Lambda(t;z)] = \exp[-(\gamma_0 t + \gamma_1 t^2/2)\exp(\beta z)]. \tag{2.25}$$

For a failure time sample (t_i, I_i, z_i), $i = 1,\ldots,n$, where t_i is the failure time of the *i*th unit under study, I_i is the failure indicator for the *i*th unit ($I_i = 1$ if the failure has occurred, which means $t_i \le \tau$, and $I_i = 0$ if the failure time is right-censored, or $t_i > \tau$), and z_i is the stress level for the *i*th unit (only a single stress type is considered in this situation). Based on the proportional hazard model and the baseline hazard function, the log likelihood for the failure time sample is constructed as

$$\begin{aligned} l &= \sum_{i=1}^{n} [I_i \ln \lambda(t_i; z_i) - \Lambda(t_i; z_i)] \\ &= \sum_{i=1}^{n} [I_i \ln(\gamma_0 + \gamma_1 t_i) + I_i \beta z_i - (\gamma_0 t_i + \frac{\gamma_1 t_i^2}{2}) \exp(\beta z)]. \end{aligned} \tag{2.26}$$

The maximum likelihood estimates $\hat{\gamma}_0$, $\hat{\gamma}_1$ and $\hat{\beta}$ are the parameter values that maximize the above sample log likelihood function and are obtained by setting the three first derivatives of the log likelihood function with respect to the parameters to zero, and solving the resultant equations simultaneously.

An accelerated life test is planned to obtain the most accurate hazard rate estimate of the products at the design stress level under given constraints (time, cost, test units, etc.). In order to achieve such objective, the optimal criterion chosen is to minimize the asymptotic variance of the hazard rate estimate at the design stress level over a pre-specified period of time T, that is to minimize

$$\int_0^T Var[\hat{\lambda}(t; z_D)]dt = \int_0^T Var[(\hat{\gamma}_0 + \hat{\gamma}_1 t)\exp(\hat{\beta} z_D)]dt\,, \tag{2.27}$$

where the variance $Var[\hat{\lambda}(t; z_D)]$ is obtained through delta method as

$$Var[\hat{\lambda}(t; z_D)] = Var[(\hat{\gamma}_0 + \hat{\gamma}_1 t)e^{\hat{\beta} z_D}] = \left[\frac{\partial\hat{\lambda}}{\partial\hat{\gamma}_0}, \frac{\partial\hat{\lambda}}{\partial\hat{\gamma}_1}, \frac{\partial\hat{\lambda}}{\partial\hat{\beta}}\right] \Sigma \left[\frac{\partial\hat{\lambda}}{\partial\hat{\gamma}_0}, \frac{\partial\hat{\lambda}}{\partial\hat{\gamma}_1}, \frac{\partial\hat{\lambda}}{\partial\hat{\beta}}\right]^T, \qquad (2.28)$$

and Σ is the variance-covariance matrix of the parameter estimates $(\hat{\gamma}_0, \hat{\gamma}_1, \hat{\beta})$, which is calculated as the inverse of the Fisher information matrix.

Let's consider the setup of an ALT plan, which is defined by

1. There is a single accelerating stress type z.
2. Test can be run at three or more stress levels.
3. Total test duration is limited to τ.
4. A total of n test units is available for testing.
5. The highest stress level z_H is defined at the highest level at which the failure mode will not change.
6. Initial values for the model parameter estimates are given by a baseline experiment.

Under the constraints of available test units, test time and specification of minimum number of failures at each stress level, the objective of the ALT plans is to optimally allocate stress levels and test units so that the asymptotic variance of the hazard rate estimate at the design stress is minimized over a pre-specified period of time T. The optimum decision variables $(z_L^*, z_M^*, p_1^*, p_2^*, p_3^*)$ are determined by solving the following nonlinear optimization problem.

$$\text{Min} \qquad f(\boldsymbol{x}) = \int_0^T Var[(\hat{\gamma}_0 + \hat{\gamma}_1 t)e^{\hat{\beta} z_D}]dt$$

Subject to

$$\Sigma = F^{-1}$$

$$0 < p_i < 1, \quad i = 1, 2, 3$$

$$\sum_{i=1}^{3} p_i = 1$$

$$np_i \Pr[t \le \tau \mid z_i] \ge MNF, \quad i = 1, 2, 3$$

where MNF is the minimum number of failures and Σ is the inverse of the Fisher's information matrix F .

The above nonlinear optimization problem can be solved by numerical methods.

2.3 Conclusions

A thorough review of the literature about ALT models indicates that widely-used nonparametric proportional hazards model can not be used for all failure time data, and proportional odds model is a good alternative to proportional hazards model when the failure time data present the property of proportionality of odds function instead of hazard rate function. Unfortunately the current parameter estimation methods of PO model are not applicable for accelerated life testing. Therefore we resort to a new parameter estimation method of PO model.

A thorough review of the literature about ALT plans indicates that there are no optimum plans for proportional odds cases when neither the AFT nor the PH assumptions hold for the failure time data. Furthermore, current existing ALT plans only consider single stress type situations. However, as products become more reliable thanks to technological advances, it becomes more difficult to induce significant amount of failure time data within the limited testing duration using a single stress type only. Multiple-stress-type ALTs have been applied to overcome such difficulties. But little work has been done on designing ALT plans with multiple stress types. In this dissertation proposal, we propose to design and develop the PO-based ALT plans with multiple stress types when neither the assumption of AFT model nor the PH model is valid.

CHAPTER 3

ALT MODEL BASED ON PROPORTIONAL ODDS

In accelerated life testing, it is more important to extrapolate the reliability performance of products at the design stress level from the failure time data at more severe stress conditions. Therefore reliability practitioners are interested in estimating the baseline function as well as the covariate parameters. Since the baseline odds function of the general PO models could be any monotonically increasing function, it is important to find a viable baseline odds function structure to approximate most, if not all, of the possible odds function. In order to find such a universal baseline odds function, we investigate the properties of odds function and its relation to the hazard rate function in this chapter. Based on these properties, we propose a general form of the baseline odds function to approximate the odds function. We construct the log-likelihood function for the PO-based ALT model with the proposed baseline odds function. The estimates of the unknown parameters are obtained through a numerical algorithm. We also demonstrate the PO-based ALT model by a simulation study and an experimental study.

3.1 Properties of Odds Function

The odds function $\theta(t)$ is defined as the odds on failure of a unit at time t, or the ratio between the probability of failure and the probability of survival, which is expressed as

$$\theta(t) \equiv \frac{F(t)}{1-F(t)} = \frac{1-R(t)}{R(t)} = \frac{1}{R(t)} - 1, \tag{3.1}$$

where $F(t)$ is cumulative probability function of the random failure time T, and $R(t) = 1 - F(t)$ is the corresponding reliability function. From Eq. (3.1), we could investigate the properties of odds function through the reliability function. Since most failure time processes are modeled on a continuous scale, all the failure time distributions considered in this dissertation proposal are continuous.

From the properties of reliability function and its relation to odds function shown in Eq. (3.1), we could easily derive the following properties of odds function $\theta(t)$:

(1) $\theta(0) = 0$, $\theta(\infty) = \infty$;

(2) $\theta(t) \geq 0$ for $t \geq 0$;

(3) $\theta(t)$ is a monotone non-decreasing function for $t \geq 0$, that is $\theta'(t) \geq 0$.

Proof:

(1) Since $R(0) = \Pr(T > 0) = 1$, then $\theta(0) = \dfrac{1}{R(0)} - 1 = 1 - 1 = 0$;

and $\operatorname*{Lim}_{t\to\infty} R(t) = \operatorname*{Lim}_{t\to\infty} \Pr(T > t) = 0$, then $\theta(\infty) = \operatorname*{Lim}_{t\to\infty} \theta(t) = \dfrac{1}{\operatorname*{Lim}_{t\to\infty} \Pr(T > t)} - 1 = \infty$.

(2) Since $0 \leq R(t) \leq 1$ for $t \geq 0$, then $\dfrac{1}{R(t)} \geq 1$, so $\theta(t) = \dfrac{1}{R(t)} - 1 \geq 0$ for $t \geq 0$.

(3) Since $R(t) = \Pr(T > t)$ is a monotone nonincreasing function, we have $R'(t) \leq 0$. Then $\theta'(t) = [\dfrac{1}{R(t)} - 1]' = -\dfrac{R'(t)}{[R(t)]^2} \geq 0$, so $\theta(t)$ is a monotone nondecreasing function.

We also summarize the relationship between odds function and cumulative hazard rate function $\Lambda(t)$, and hazard rate function $\lambda(t)$.

(4) $\theta(t) = e^{\Lambda(t)} - 1$, and $\Lambda(t) = \ln[\theta(t)+1]$;

(5) $\lambda(t) = \dfrac{\theta'(t)}{\theta(t)+1}$, and $\theta(t) = \exp(\int_0^t \lambda(u)du) - 1$.

Proof:

(4) $\theta(t) = \dfrac{1}{R(t)} - 1 = \dfrac{1}{e^{-\Lambda(t)}} - 1 = e^{\Lambda(t)} - 1$, $\Rightarrow$ $\Lambda(t) = \ln[\theta(t)+1]$.

(5) $\lambda(t) = \dfrac{f(t)}{R(t)} = -\dfrac{R'(t)}{R(t)} = -\dfrac{\{1/[\theta(t)+1]\}'}{\{1/[\theta(t)+1]\}} = \dfrac{\theta'(t)}{\theta(t)+1}$, and

$$\theta(t) = e^{\Lambda(t)} - 1 = \exp(\int_0^t \lambda(u)du) - 1.$$

3.2 Proposed Baseline Odds Function Approximation

Furthermore, plotting the odds functions of some common failure time distributions as shown in Figure 3.1 provides further clarification of the odds functions.

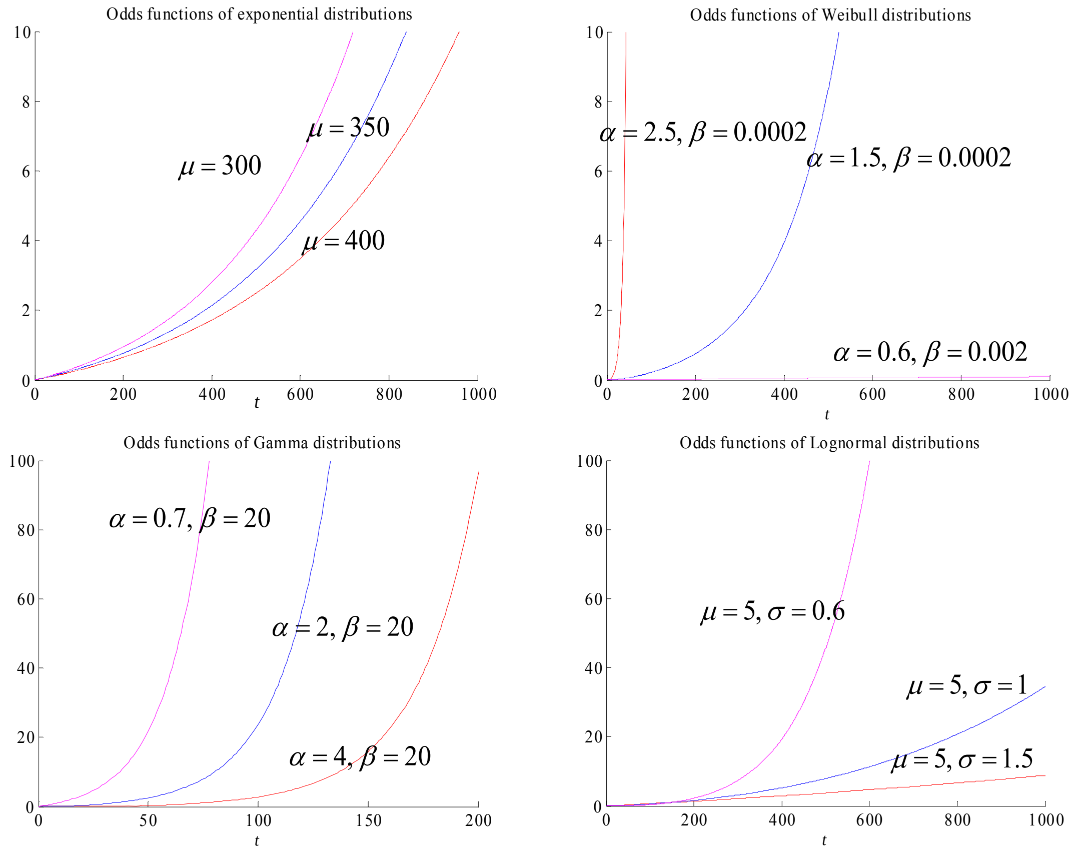

Figure 3.1 Odds functions of four common failure time distributions

Based on the properties of the odds function and the plots in Figure 3.1, we could use a polynomial function to approximate the general baseline odds function in the PO models.

The proposed general baseline odds function is

$$\theta_0(t) = \gamma_1 t + \gamma_2 t^2 + \gamma_3 t^3 + \ldots$$

Usually high orders are not necessary, second or third order polynomial function is sufficient to cover most of possible odds function.

If the baseline odds function is assumed to be quadratic form, that is

$$\theta_0(t) = \gamma_1 t + \gamma_2 t^2 \,. \tag{3.2}$$

we have the following theorem:

***Theorem*:** $\gamma_1 \geq 0$ and $\gamma_2 \geq 0$ in Eq. (3.2).

Proof: This theorem can be proven by contradiction.

The values of γ_1 and γ_2 must fall into one of the following four cases:

1. $\gamma_1 < 0$ and $\gamma_2 < 0$;

2. $\gamma_1 < 0$ and $\gamma_2 > 0$;

3. $\gamma_1 \geq 0$ and $\gamma_2 < 0$;

4. $\gamma_1 \geq 0$ and $\gamma_2 \geq 0$.

If $\gamma_1 < 0$ and $\gamma_2 < 0$, then $\theta_0(t) = \gamma_1 t + \gamma_2 t^2 \leq 0$ for $t \geq 0$. This is a contradiction to the property (2). So case 1 is not possible for baseline odds function.

If case 2 is true, that is $\gamma_1 < 0$ and $\gamma_2 > 0$, then the baseline odds function

$$\theta_0(t) = \gamma_1 t + \gamma_2 t^2 = t(\gamma_1 + \gamma_2 t)$$

has roots $r_1 = 0$ and $r_2 = -\gamma_1 / \gamma_2$.

Since $\gamma_1 < 0$ and $\gamma_2 > 0$, we have $r_2 = -\gamma_1 / \gamma_2 > 0$. Then we obtain the derivative of the baseline odds function at r_1, $\theta_0'(t = r_1 = 0) = \gamma_1 < 0$, which contradicts to the property (3) of odds function.

If case 3 is true, then the baseline odds function still has two roots $r_1 = 0$ and $r_2 = -\gamma_1 / \gamma_2$. Since $\gamma_1 \geq 0$ and $\gamma_2 < 0$, we have $r_2 = -\gamma_1 / \gamma_2 > 0$. Then we obtain the derivative of the baseline odds function at r_2, $\theta_0'(t = r_2 = -\gamma_1 / \gamma_2) = \gamma_1 + 2\gamma_2(-\gamma_1 / \gamma_2) = -\gamma_1 \leq 0$, which also contradicts to the property (3) of odds function.

So the values of γ_1 and γ_2 must satisfy $\gamma_1 \geq 0$ and $\gamma_2 \geq 0$. This theorem gives a bound for parameter estimation in the later sections of the dissertation proposal.

3.3 Log-Likelihood Function of the PO-based ALT Model

We assume that the baseline odds function is expressed in a quadratic form:

$$\theta_0(t) = \gamma_1 t + \gamma_2 t^2, \qquad \gamma_1 \geq 0, \gamma_2 \geq 0 \tag{3.3}$$

where γ_1 and γ_2 are unknown parameters which we are interested to estimate and the intercept parameter is zero since the odds function crosses the origin according to property (1) in section 3.1.

The proportional odds model is then represented by:

$$\theta(t; \mathbf{z}) = \exp(\boldsymbol{\beta}^t \mathbf{z})\theta_0(t) = \exp(\beta_1 z_1 + \beta_2 z_2 + \cdots \beta_k z_k)(\gamma_1 t + \gamma_2 t^2), \tag{3.4}$$

where

$\theta_0(t)$	an arbitrary baseline odds function;
$\boldsymbol{z} = (z_1, z_2, \ldots, z_k)^t$	a vector of the covariates or the applied stresses;
$\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_k)^t$	a vector of the unknown regression parameters;
k	the number of the stresses.

Therefore the hazard rate function $\lambda(t;\boldsymbol{z})$ and cumulative hazard rate function $\Lambda(t;\boldsymbol{z})$ are

$$\lambda(t;\boldsymbol{z}) = \frac{\theta'(t;\boldsymbol{z})}{\theta(t;\boldsymbol{z})+1} = \frac{\exp(\boldsymbol{\beta}^t\boldsymbol{z})(\gamma_1 + 2\gamma_2 t)}{\exp(\boldsymbol{\beta}^t\boldsymbol{z})(\gamma_1 t + \gamma_2 t^2)+1},$$

$$\Lambda(t;\boldsymbol{z}) = \ln[\theta(t;\boldsymbol{z})+1] = \ln[\exp(\boldsymbol{\beta}^t\boldsymbol{z})(\gamma_1 t + \gamma_2 t^2)+1].$$

The parameters of the PO model with the proposed baseline odds function for censored failure time data are obtained as follows.

Let $t_i, i = 1, \ldots, n$ represent the failure time of the *i*th testing unit, $\boldsymbol{z}_i = (z_{1i}, z_{2i}, \ldots, z_{ki})^t$ the stress vector of this unit, and I_i the indicator function, which is defined by

$$I_i = I(t_i \le \tau) = \begin{cases} 1 & \text{if } t_i \le \tau, \text{ failure observed before time } \tau, \\ 0 & \text{if } t_i > \tau, \text{ censored at time } \tau. \end{cases}$$

The log likelihood function of the proposed PO-based ALT model is

$$l = \sum_{i=1}^{n} I_i \ln[\lambda(t_i; \mathbf{z}_i)] - \sum_{i=1}^{n} \Lambda(t_i; \mathbf{z}_i)\,. \tag{3.5}$$

Substituting the hazard rate function and cumulative hazard function into Eq. (3.5) results in:

$$\begin{aligned} l &= \sum_{i=1}^{n} I_i \ln[\frac{e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 + 2\gamma_2 t_i)}{e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1}] - \sum_{i=1}^{n} \ln[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1] \\ &= \sum_{i=1}^{n} I_i \{\ln[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 + 2\gamma_2 t_i)] - \ln[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]\} - \sum_{i=1}^{n} \ln[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1] \\ &= \sum_{i=1}^{n} I_i \{\boldsymbol{\beta}^t \mathbf{z}_i + \ln(\gamma_1 + 2\gamma_2 t_i) - \ln[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]\} - \sum_{i=1}^{n} \ln[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1] \end{aligned} \tag{3.6}$$

Taking the derivatives of the log likelihood function with respect to the unknown parameters ($\boldsymbol{\beta}, \gamma_1, \gamma_2$) respectively, we obtain

$$\frac{\partial l}{\partial \beta_1} = \sum_{i=1}^{n} I_i z_{1i} - \sum_{i=1}^{n} I_i \frac{z_{1i} e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)}{e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1} - \sum_{i=1}^{n} \frac{z_{1i} e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)}{e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1}\,,$$

$$\vdots$$

$$\frac{\partial l}{\partial \beta_k} = \sum_{i=1}^{n} I_i z_{ki} - \sum_{i=1}^{n} I_i \frac{z_{ki} e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)}{e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1} - \sum_{i=1}^{n} \frac{z_{ki} e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)}{e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1}\,,$$

$$\frac{\partial l}{\partial \gamma_1} = \sum_{i=1}^{n} I_i \frac{1}{(\gamma_1 + 2\gamma_2 t_i)} - \sum_{i=1}^{n} I_i \frac{e^{\boldsymbol{\beta}^t \mathbf{z}_i} t_i}{e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1} - \sum_{i=1}^{n} \frac{e^{\boldsymbol{\beta}^t \mathbf{z}_i} t_i}{e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1},$$

$$\frac{\partial l}{\partial \gamma_2} = \sum_{i=1}^{n} I_i \frac{2t_i}{(\gamma_1 + 2\gamma_2 t_i)} - \sum_{i=1}^{n} I_i \frac{e^{\boldsymbol{\beta}^t \mathbf{z}_i} t_i^2}{e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1} - \sum_{i=1}^{n} \frac{e^{\boldsymbol{\beta}^t \mathbf{z}_i} t_i^2}{e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1}.$$

The estimates of the model parameters ($\boldsymbol{\beta}, \gamma_1, \gamma_2$) are obtained by setting the above derivatives to zero and solving the resultant equations simultaneously. Unfortunately there are no closed form solutions for these equations. Therefore the solutions can be only obtained by using numerical methods such as Newton-Raphson method.

To use Newton-Raphson method, we calculate the second derivatives of the log-likelihood function as follows.

$$\frac{\partial^2 l}{\partial \boldsymbol{\beta}^2} = -\sum_{i=1}^{n} I_i \frac{\mathbf{z}_i \mathbf{z}_i^t e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]^2} - \sum_{i=1}^{n} \frac{\mathbf{z}_i \mathbf{z}_i^t e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]^2},$$

$$\frac{\partial^2 l}{\partial \gamma_1^2} = -\sum_{i=1}^{n} I_i \frac{1}{(\gamma_1 + 2\gamma_2 t_i)^2} + \sum_{i=1}^{n} I_i \frac{e^{2\boldsymbol{\beta}^t \mathbf{z}_i} t_i^2}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]^2} + \sum_{i=1}^{n} \frac{e^{2\boldsymbol{\beta}^t \mathbf{z}_i} t_i^2}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]^2},$$

$$\frac{\partial^2 l}{\partial \gamma_2^2} = -\sum_{i=1}^{n} I_i \frac{4t_i^2}{(\gamma_1 + 2\gamma_2 t_i)^2} + \sum_{i=1}^{n} I_i \frac{e^{2\boldsymbol{\beta}^t \mathbf{z}_i} t_i^4}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]^2} + \sum_{i=1}^{n} \frac{e^{2\boldsymbol{\beta}^t \mathbf{z}_i} t_i^4}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]^2},$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\beta}\,\partial\gamma_1} = -\sum_{i=1}^{n} I_i \frac{\mathbf{z}_i\, e^{\boldsymbol{\beta}^t \mathbf{z}_i} t_i}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]^2} - \sum_{i=1}^{n} \frac{\mathbf{z}_i\, e^{\boldsymbol{\beta}^t \mathbf{z}_i} t_i}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]^2},$$

$$\frac{\partial^2 l}{\partial\gamma_1 \partial \boldsymbol{\beta}} = -\sum_{i=1}^{n} I_i \frac{\mathbf{z}_i^t e^{\boldsymbol{\beta}^t \mathbf{z}_i} t_i}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]^2} - \sum_{i=1}^{n} \frac{\mathbf{z}_i^t e^{\boldsymbol{\beta}^t \mathbf{z}_i} t_i}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]^2}$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\beta}\,\partial\gamma_2} = -\sum_{i=1}^{n} I_i \frac{\mathbf{z}_i\, e^{\boldsymbol{\beta}^t \mathbf{z}_i} t_i^2}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]^2} - \sum_{i=1}^{n} \frac{\mathbf{z}_i\, e^{\boldsymbol{\beta}^t \mathbf{z}_i} t_i^2}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]^2},$$

$$\frac{\partial^2 l}{\partial\gamma_2 \partial \boldsymbol{\beta}} = -\sum_{i=1}^{n} I_i \frac{\mathbf{z}_i^t e^{\boldsymbol{\beta}^t \mathbf{z}_i} t_i^2}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]^2} - \sum_{i=1}^{n} \frac{\mathbf{z}_i^t e^{\boldsymbol{\beta}^t \mathbf{z}_i} t_i^2}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]^2}$$

$$\frac{\partial^2 l}{\partial\gamma_1 \partial\gamma_2} = -\sum_{i=1}^{n} I_i \frac{2t_i}{(\gamma_1 + 2\gamma_2 t_i)^2} + \sum_{i=1}^{n} I_i \frac{e^{2\boldsymbol{\beta}^t \mathbf{z}_i} t_i^3}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]^2} + \sum_{i=1}^{n} \frac{e^{2\boldsymbol{\beta}^t \mathbf{z}_i} t_i^3}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]^2}.$$

We then construct the Hessian matrix as:

$$\boldsymbol{H}(\boldsymbol{\beta}, \gamma_1, \gamma_2) = \begin{bmatrix} \dfrac{\partial^2 l(\boldsymbol{\beta}, \gamma_1, \gamma_2)}{\partial \boldsymbol{\beta}^2} & \dfrac{\partial^2 l(\boldsymbol{\beta}, \gamma_1, \gamma_2)}{\partial \boldsymbol{\beta}\,\partial\gamma_1} & \dfrac{\partial^2 l(\boldsymbol{\beta}, \gamma_1, \gamma_2)}{\partial \boldsymbol{\beta}\,\partial\gamma_2} \\ \dfrac{\partial^2 l(\boldsymbol{\beta}, \gamma_1, \gamma_2)}{\partial\gamma_1 \partial \boldsymbol{\beta}} & \dfrac{\partial^2 l(\boldsymbol{\beta}, \gamma_1, \gamma_2)}{\partial\gamma_1^2} & \dfrac{\partial^2 l(\boldsymbol{\beta}, \gamma_1, \gamma_2)}{\partial\gamma_1\,\partial\gamma_2} \\ \dfrac{\partial^2 l(\boldsymbol{\beta}, \gamma_1, \gamma_2)}{\partial\gamma_2 \partial \boldsymbol{\beta}} & \dfrac{\partial^2 l(\boldsymbol{\beta}, \gamma_1, \gamma_2)}{\partial\gamma_1\,\partial\gamma_2} & \dfrac{\partial^2 l(\boldsymbol{\beta}, \gamma_1, \gamma_2)}{\partial\gamma_2^2} \end{bmatrix}$$

Let $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{\beta}^t & \gamma_1 & \gamma_2 \end{bmatrix}^t$, $\boldsymbol{x_0}$ is the initial value of the parameters. The Newton-Raphson method starts with this initial value, and after j steps of the algorithm, the updated estimate of the parameters is given by

$$\boldsymbol{x}_{j+1} = \boldsymbol{x}_j - \boldsymbol{H}(\boldsymbol{x}_j)^{-1}\boldsymbol{u}(\boldsymbol{x}_j),$$

where $\boldsymbol{u}(\boldsymbol{x}_j) = \begin{bmatrix} \dfrac{\partial l(\boldsymbol{\beta}, \gamma_1, \gamma_2)}{\partial \boldsymbol{\beta}} & \dfrac{\partial l(\boldsymbol{\beta}, \gamma_1, \gamma_2)}{\partial \gamma_1} & \dfrac{\partial l(\boldsymbol{\beta}, \gamma_1, \gamma_2)}{\partial \gamma_2} \end{bmatrix}^t |_{\boldsymbol{x}=\boldsymbol{x}_j}$.

The Newton-Raphson algorithm converges quite rapidly when the initial value is not too far from the true value. When the initial value is poor, the algorithm may move in the wrong direction or may take a step in the correct direction, but overshoot the root. The value of the log-likelihood function is computed at each step to ensure that the algorithm is moving in the correct direction. If $l(\boldsymbol{x}_k)$ is smaller than $l(\boldsymbol{x}_{k+1})$, one option is to cut the step size in half and try $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \boldsymbol{H}(\boldsymbol{x}_k)^{-1}\boldsymbol{u}(\boldsymbol{x}_k)/2$.

3.4 Simulation Study

In this section, we compare the performance of the PO models with that of the PH models using simulation data. To validate the proposed PO-based ALT model, we apply both the PO-based and PH-based ALT models to a group of simulated failure time data sets. Sets of random failure time data from mixed log-logistic distribution and Weibull distribution are generated by Monte Carlo simulation method. We now describe the details of the simulation study.

3.4.1 Generation of Failure Time Data

By definition, the reliability function of the mixed distribution, from which we generate the random failure time data, with stress level z is

$$R(t) = \frac{w}{1+(\gamma_1 t)e^{\beta z}} + (1-w)\exp(-\alpha t e^{\beta z}), \tag{3.7}$$

where α is a Weibull distribution parameter, γ_1 is a log-logistic distribution parameter, and $0 \le w \le 1$.

The reason for choosing the log-logistic distribution and the Weibull distribution to construct the mixed distribution is due to the fact that the log-logistic distribution and the Weibull distribution are special cases for the PO models and the PH models respectively as we discussed in Chapter 2. The first part of Eq. (3.7) is a special case of the general PO models with the baseline odds function $\theta_0(t) = \gamma_1 t$. While the second part, which is a Weibull distribution reliability function, is a special case of the general PH models. Equation (3.7) becomes a pure log-logistic distribution when w is 1, a pure Weibull distribution when w is 0, or a mixed distribution when w takes any value between 0 and 1. If w is close to 1, the generated failure time data from the mixed distribution are much like being drawn from a log-logistic distribution. On the other hand, if w is close to 0, the generated failure time data from the mixed distribution are much like being drawn from a Weibull distribution. We expect that the simulation study reveals the result that if w is equal to, or close to 1, the PH-based ALT model is much better than PO counterpart, and if w is equal to, or close to 0, the PO-based ALT model is much better than PH model.

When w is 1, solve t from Eq. (3.7), we have

$$t = \frac{e^{-\beta z}}{\gamma_1}[R(t)^{-1} - 1],$$

or

$$t = \frac{e^{-\beta z}}{\gamma_1}[(1 - F(t))^{-1} - 1]. \tag{3.8}$$

Based on Eq. (3.8), a random log-logistic failure time t_i' is generated by Monte Carlo simulation method as

$$t_i' = \frac{e^{-\beta z}}{\gamma_1}[(1 - rand1(i))^{-1} - 1], \tag{3.9}$$

where $rand1(i)$ is uniformly-distributed random variable between [0, 1].

Similarly, for $w = 0$, the Weibull distributed random failure time t_i'' is generated by

$$t_i'' = \frac{e^{-\beta z}}{\alpha} \ln[1 - rand2(i)] \tag{3.10}$$

For any $0 \le w \le 1$, the mixed random failure time t_i is generated by the following equation:

$$t_i = \begin{cases} t_i' & rand3(i) \le w \\ t_i'' & rand3(i) > w \end{cases} \tag{3.11}$$

where $rand1(i), rand2(i),$ and $rand3(i)$ are uniformly-distributed random variables bounded in the interval [0, 1].

Using the following parameters: $\beta = -30$, $\gamma_1 = 800$, and $\alpha = 0.2$, the Monte Carlo random failure time t_i ($n = 1, \ldots, 120$) is generated by Eq. (3.11) with $w = 1$ and four stress levels: 0.3, 0.35, 0.4, and 0.45. We generate thirty random failure times for each of the stress levels. Similarly we could generate several datasets with different w values as summarized in table 3.1.

Table 3.1 Monte Carlo simulation datasets

w	1	0.9	0.7	0.5	0.3	0.1	0
Dataset	I	II	III	IV	V	VI	VII

3.4.2 Simulation Results

After generating the random failure time data, we apply the PO-based ALT model and PH-based ALT model to the data sets respectively. The unknown parameters of the models are estimated by the Maximum Likelihood Estimation procedures as discussed in section 3.3 for the PO-based ALT and Chapter 2 for the PH-based ALT respectively. Then the reliabilities at a specific stress level can be estimated with those parameter estimates for proportional odds model and proportional hazards model respectively. Since the failure time data are generated from the mixed distribution given in Eq. (3.7), we could obtain the theoretical reliability at any time for any specified stress level using that equation. Then the reliability estimates obtained from the PO-based ALT and the PH-based ALT models are compared with the theoretical reliability respectively. The comparisons are based on graphical technique and the sum of squared reliability errors (SSE) at the simulated failure times for the specified stress level.

First, when $w=1$, we apply the PO-based ALT model to the simulated data, we obtain the following parameter estimates:

$$\hat{\beta} = \quad -31.25$$

$$\hat{\gamma}_1 = \quad 1294$$

$$\hat{\gamma}_2 = \quad 0.039$$

Thereafter we use the estimated parameters to predict the reliability at the design stress level 0.5. Figure 3.2 shows the theoretical reliability obtained from the original parameters $\beta = -30$, $\gamma_1 = 800$, and $\gamma_2 = 0$ and the predicted reliability obtained from the estimated parameters $\hat{\beta} = -31.25$, $\hat{\gamma}_1 = 1294$, and $\hat{\gamma}_2 = 0.039$ at stress level 0.5. The sum of squared errors is 0.0175.

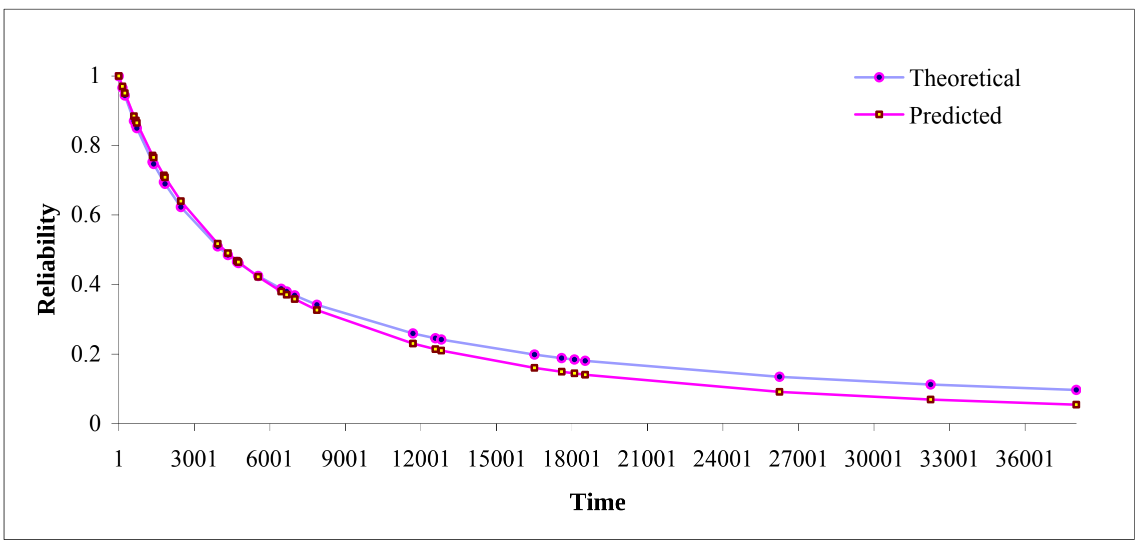

Figure 3.2 Theoretical reliability and predicted reliability at stress level 0.5 by the PO-based ALT model

Similarly we fit the same simulated data with the PH-based ALT model; the theoretical and predicted reliabilities are shown in Figure 3.3 with sum of squared errors 4.3783.

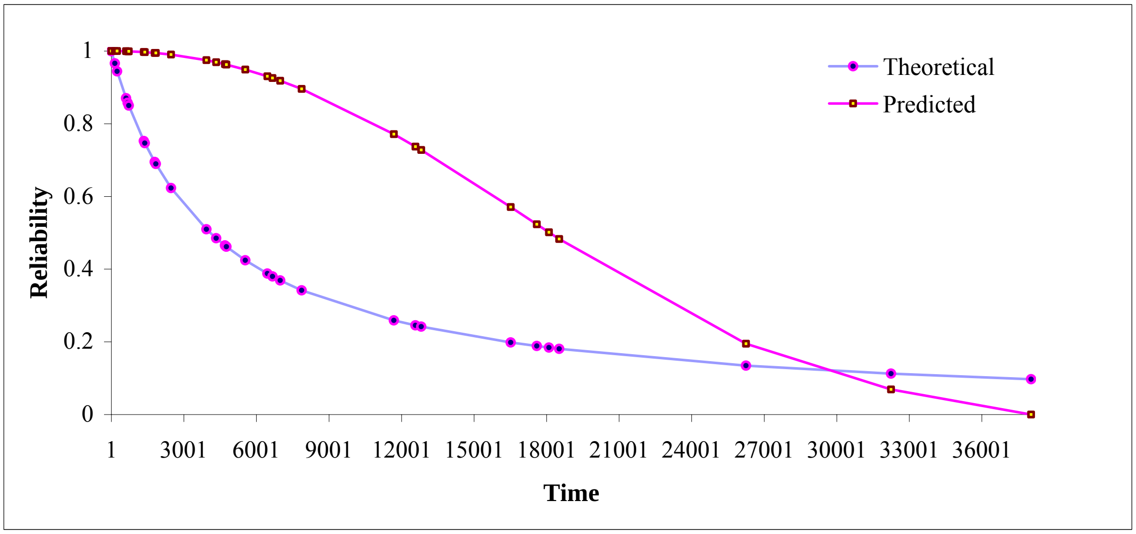

Figure 3.3 Theoretical reliability and predicted reliability at stress level 0.5 by the PH-based ALT model

The simulation results show that the PO-based ALT model provides more accurate reliability estimates than the PH-based ALT model when $w = 1$.

The reliability functions at other stress levels are plotted in Figure 3.4 through Figure 3.11 for the PO-based ALT and the PH-based ALT models respectively.

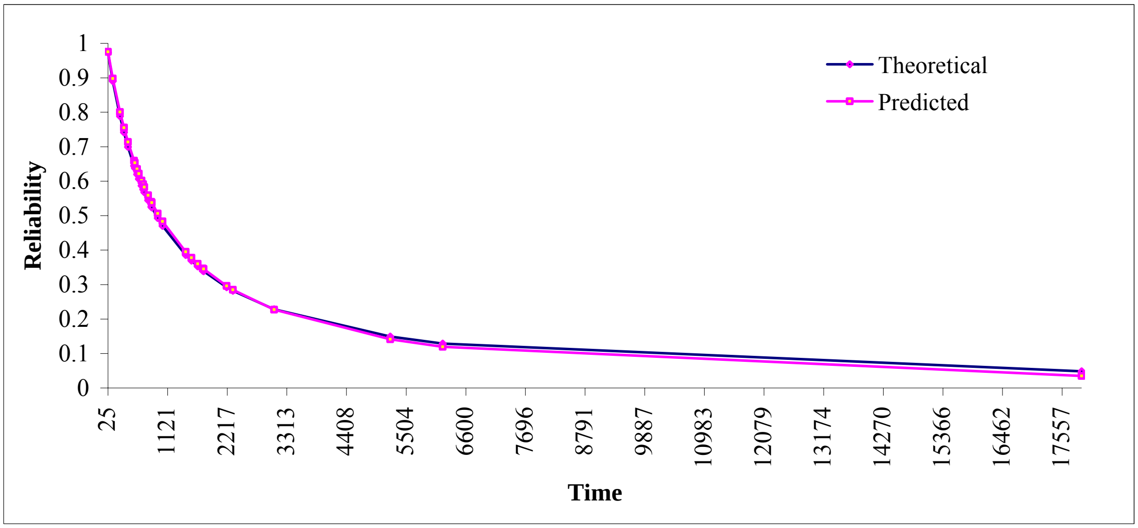

Figure 3.4 Reliability comparison for the PO-based ALT model

with $w = 1$ at stress level 0.45

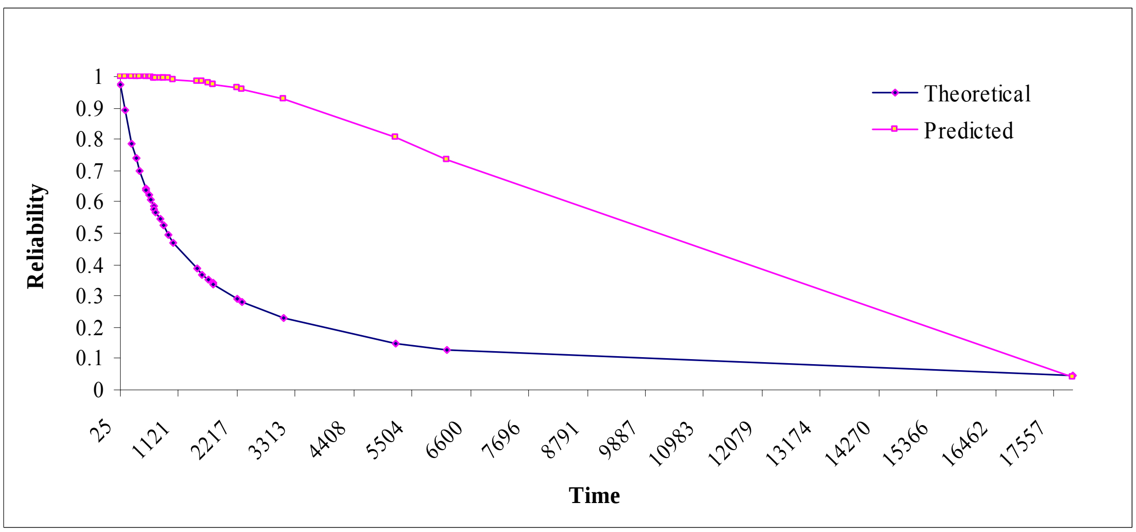

Figure 3.5 Reliability comparison for the PH-based ALT model

with $w = 1$ at stress level 0.45

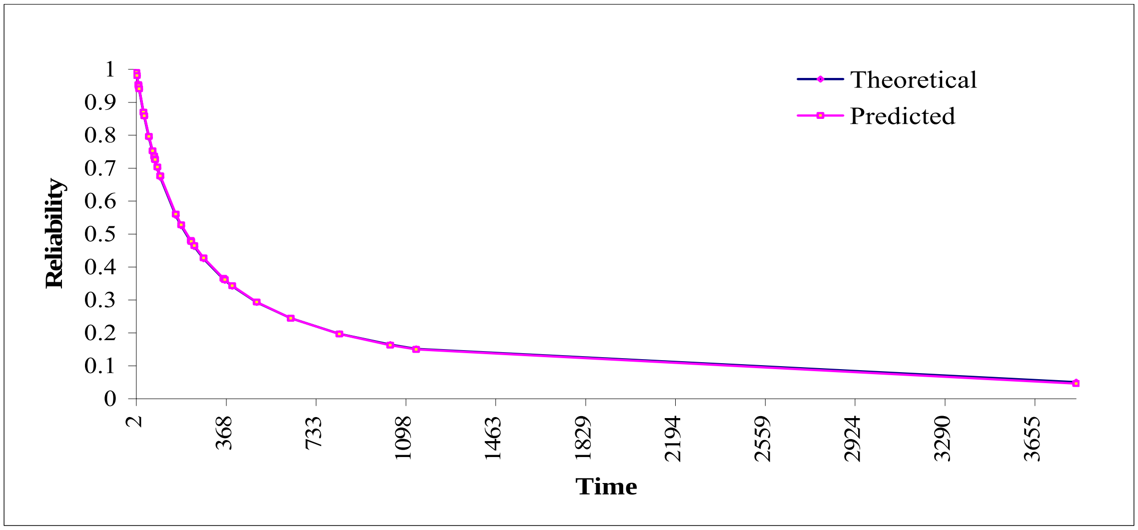

Figure 3.6 Reliability comparison for the PO-based ALT model

with $w = 1$ at stress level 0.4

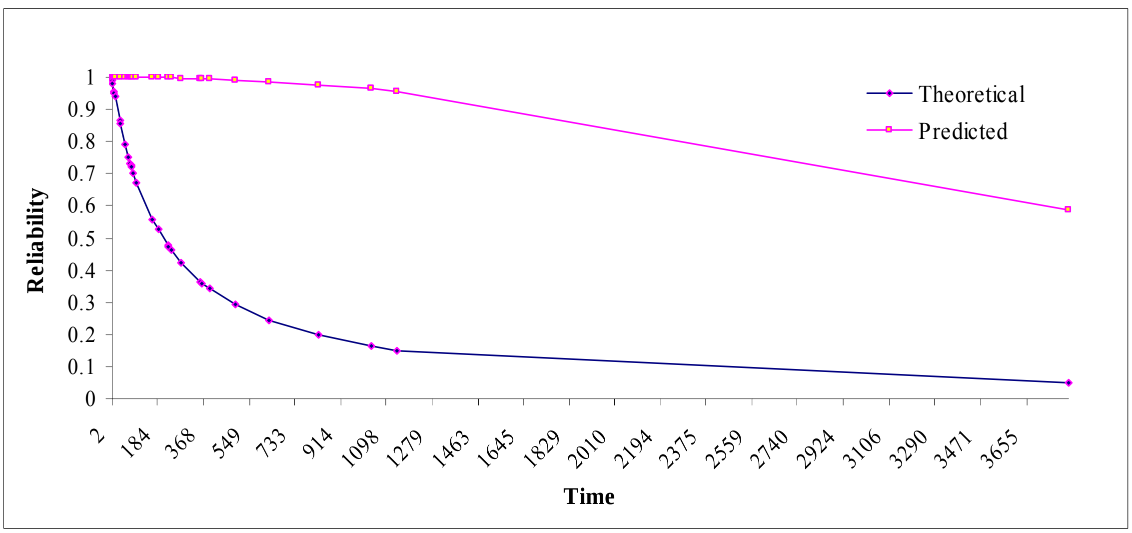

Figure 3.7 Reliability comparison for the PH-based ALT model

with $w = 1$ at stress level 0.4

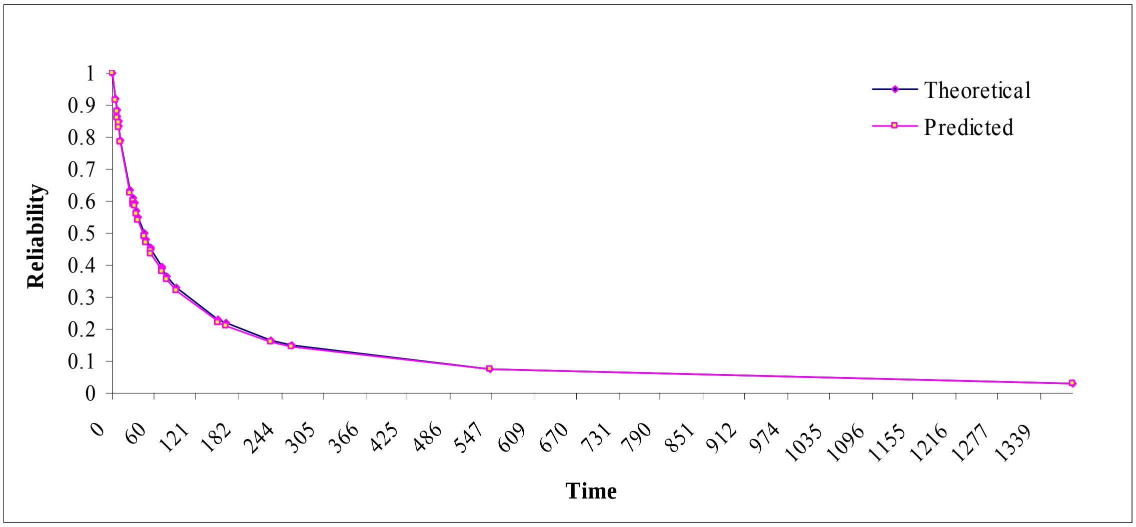

Figure 3.8 Reliability comparison for the PO-based ALT model

with $w = 1$ at stress level 0.35

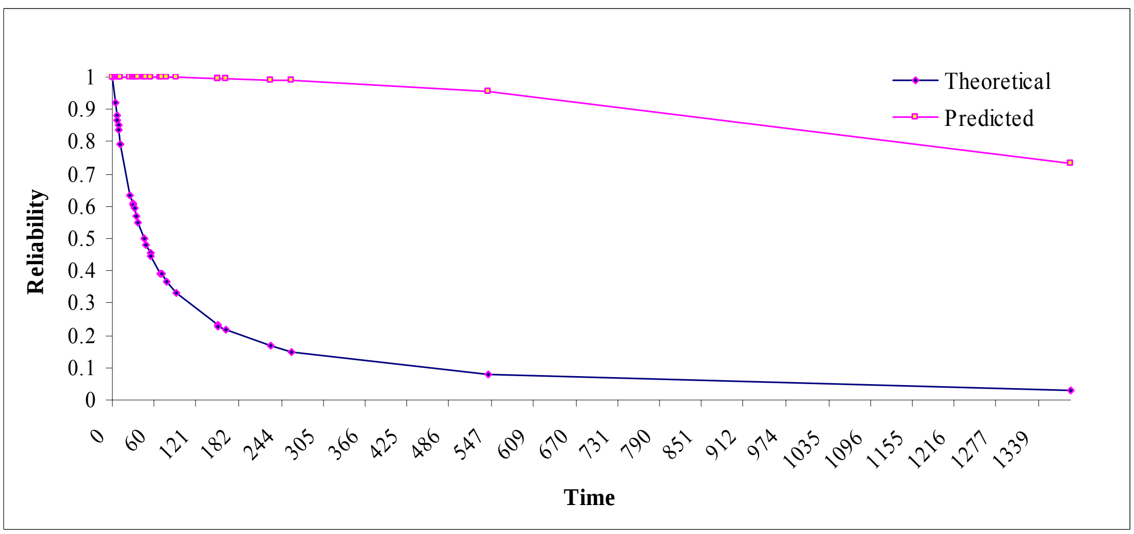

Figure 3.9 Reliability comparison for the PH-based ALT model

with $w = 1$ at stress level 0.35

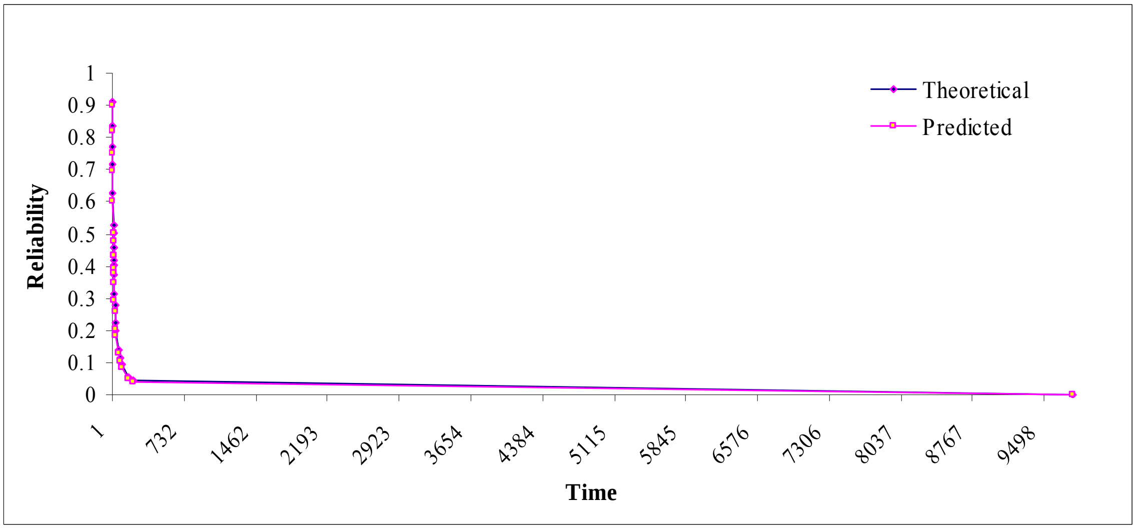

Figure 3.10 Reliability comparison for the PO-based ALT model with $w = 1$ at stress level 0.3

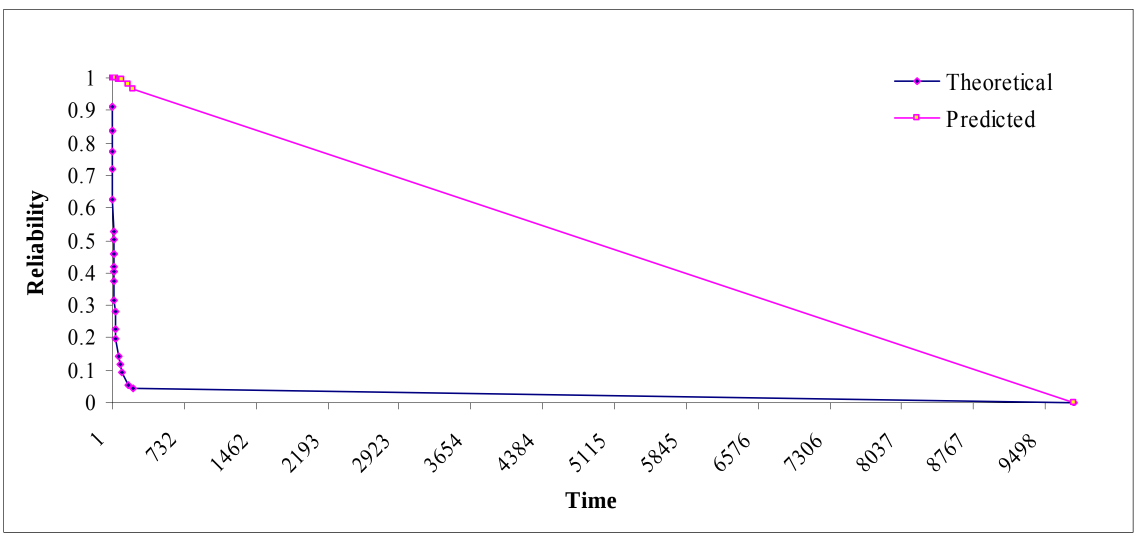

Figure 3.11 Reliability comparison for the PH-based ALT model with $w = 1$ at stress level 0.3

The sums of squared errors at different stress levels are summarized in Table 3.2 for dataset I using the PO-based ALT and the PH-based ALT models respectively.

Table 3.2 Sums of squared errors with different stress levels

Stress	PO-based ALT	PH-based ALT
0.5	0.0175	4.3783
0.45	0.0040	6.6690
0.4	0.0002	6.7100
0.35	0.0023	8.5391
0.3	0.0113	9.6553

Similarly, we repeat the above procedures with different w values. The sums of squared errors (SSE) for both the PO-based ALT and the PH-based ALT model are summarized in Table 3.3.

Table 3.3 Sums of squared errors with different w values

w	PO-based ALT	PH-based ALT
1	0.0175	4.3783
0.9	0.0319	3.0929
0.7	0.5800	2.3328
0.5	0.9411	1.5906
0.3	1.6377	0.6617
0.1	2.6617	0.2329
0	3.7889	0.0052

Table 3.3 indicates that the PO-based ALT model provides more accurate reliability estimates when w approaches 1; while the PH-based ALT model provides more accurate reliability estimates when w approaches 0.

As the results of the simulation study show, the choice of either the PO-based ALT model or the PH-based ALT model depends on the inherent property of the failure time data. Neither the PO-based ALT model nor the PH-based ALT model can be used to fit failure time data without checking the model assumption.

3.5 Example with Experimental Data

In this section we use a set of experimental data as an example. The failure time data of MOS devices with temperature as the applied stress are provided by Swartz (1986). The temperature has five levels: $225\,°C$, $200\,°C$, $125\,°C$, $50\,°C$, and $25\,°C$, where stress level $25\,°C$ is the design stress level of the devices. The failure times t_i ($i = 1,\ldots,199$) of unit i is associated with its stress level z_i. According to the Arrehenius model, we transform the Celsius temperature to 1/Kelvin as the stress z in the PO model. Therefore the transformed five temperature stress levels are: $0.201\,K^{-1}$, $0.211\,K^{-1}$, $0.251\,K^{-1}$, $0.309\,K^{-1}$, and $0.335\,K^{-1}$.

The validity of the PO-based ALT model relies heavily on the assumption of proportionality of the odds functions of units at different stress levels. Therefore, we plot the logarithm of the odds functions at different stress levels using product-limit, or Kaplan-Meier, reliability estimates shown in Figure 3.11.

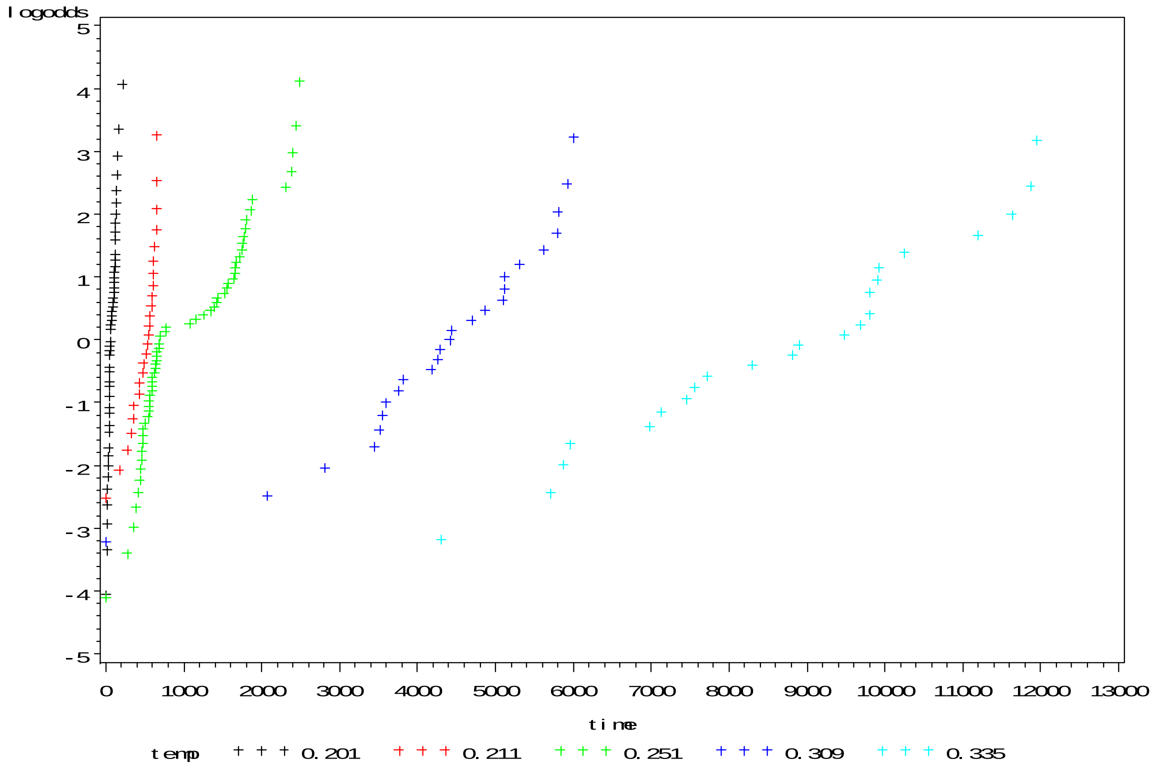

Figure 3.12 Plot of log odds functions versus time for Swartz's data

From Figure 3.12, there is no significant deviation from the proportional odds assumption since the curves do not cross. We also plot the cumulative hazards functions at different stress levels using Kaplan-Meier reliability estimates to validate the proportional hazard rate functions as shown in Figure 3.13 Similar to the conclusions from Figure 3.12, there is no significant violation of the proportional hazards assumption. Therefore it is difficult to determine whether the inherent proportionality of Swartz's data is closer to proportional odds assumption or proportional hazards assumption.

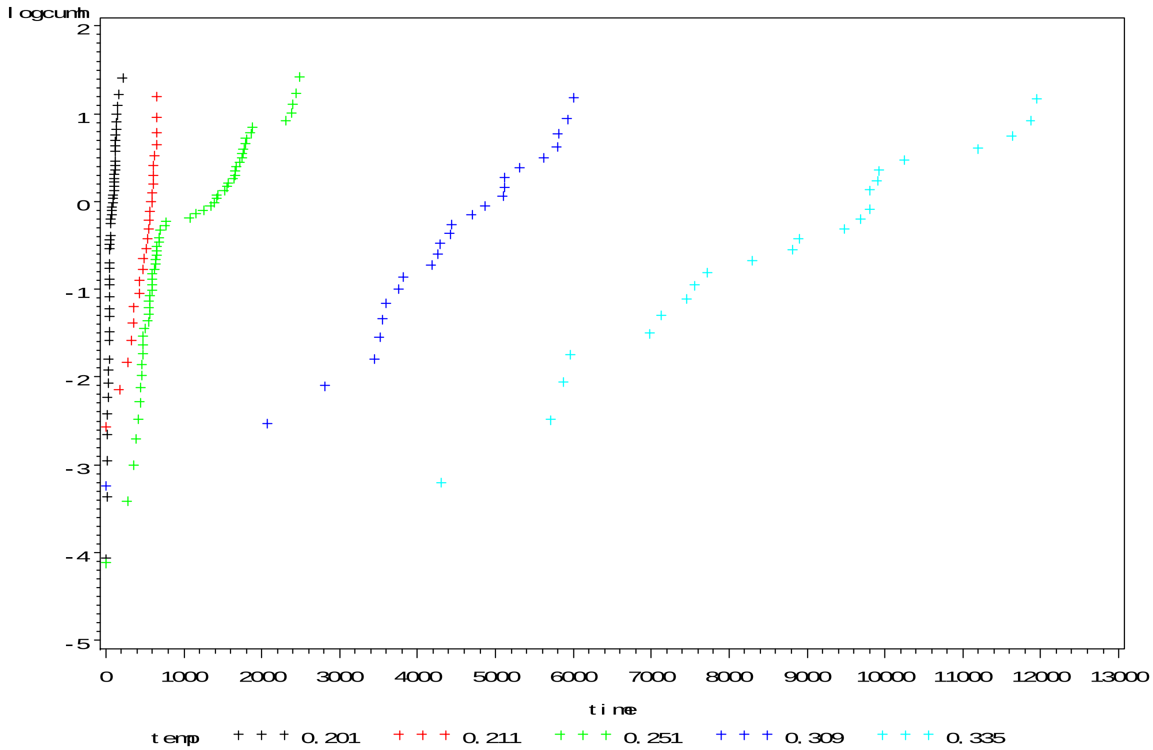

Figure 3.13 Plot of log cumulative hazards functions versus time for Swartz's data

In order to determine the appropriateness of the PO-based ALT model or the PH-based ALT model for the reliability predication at design stress level, we propose the following procedures:

1. We fit Swartz's data using only failure time data at stress levels $225\,°C$, $200\,°C$, $125\,°C$, and $50\,°C$, with the PO-based ALT model to estimate parameters of the model, and then predict the reliability at stress level $25\,°C$.

2. Compare the Kaplan-Meier reliability estimates obtained from actual failure time data at stress level $25\,°C$ with those reliability estimated by the PO-based ALT model and calculate the sum of squared errors (SSE).

3. Repeat steps 1 and 2 using the PH-based ALT model.

4. Choose the model with smaller SSE.

Applying the above procedures, we analyze the Swartz's data as follows:

Assuming that the failure time dataset follows the proportional odds model, the log likelihood function is then formulated as

$$l(\beta,\gamma_1,\gamma_2)=\sum_{i=1}^{174} I_i\{\beta z_i+\ln(\gamma_1+2\gamma_2 t_i)-\ln[e^{\beta z_i}(\gamma_1 t_i+\gamma_2 t_i^2)+1]\}-\sum_{i=1}^{174}\ln[e^{\beta z_i}(\gamma_1 t_i+\gamma_2 t_i^2)+1].$$

In the above log likelihood function, only the failure time data at stress levels $225\,°C$, $200\,°C$, $125\,°C$, and $50\,°C$ are used to estimate the three unknown parameters of the model, which are used to predict the reliability of the devices at design stress level $25\,°C$. The predicted reliability is then compared to the Kaplan-Meier reliability estimates based on the actual failure time data obtained at stress level $25\,°C$.

The MLE estimates of (β,γ_1,γ_2) are:

$$\hat{\beta}= \quad -64.93$$

$$\hat{\gamma}_1= \quad 833.73$$

$$\hat{\gamma}_2= \quad 5.18$$

Figure 3.14 shows the Kaplan-Meier reliability and the predicted reliability at temperature level $25\,°C$. The sum of squared errors (SSE), i.e. the sum of squared difference between the predicted reliability and Kaplan-Meier reliability estimate, is 0.2904.

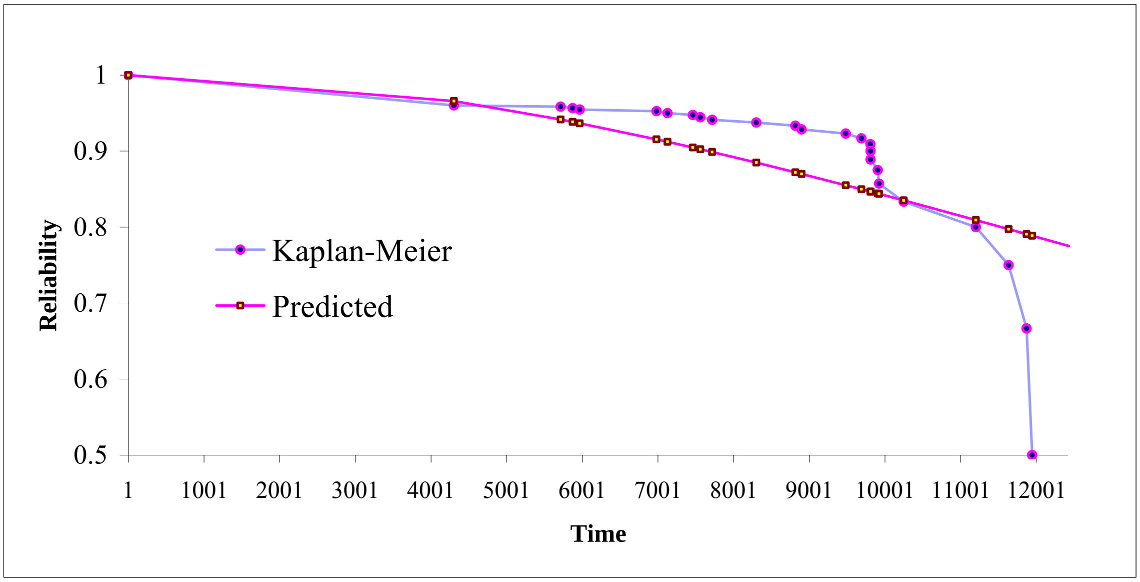

Figure 3.14 Kaplan-Meier reliability vs. predicted reliability at temperature $25\,°C$ by the PO-based ALT model

Similarly we fit the same failure time data with the PH-based ALT model and compare the predicted reliability estimates with the Kaplan-Meier reliability estimates obtained from the actual failure data at the stress level $25\,°C$ as shown in Figure 3.15.

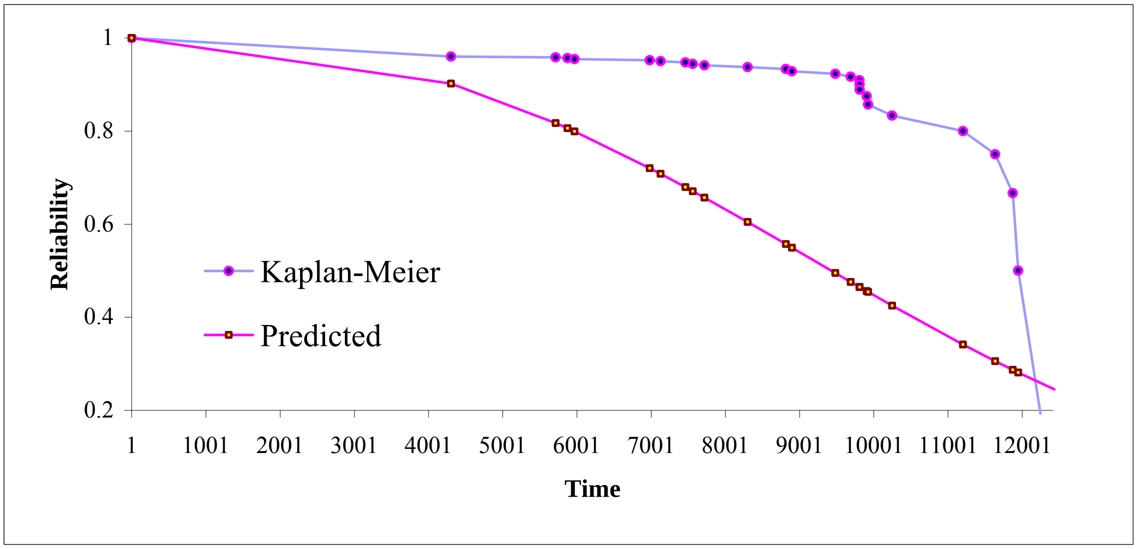

Figure 3.15 Kaplan-Meier Reliability vs. Predicted Reliability

at Temperature $25\,^{\circ}C$ by the PH-based ALT Model

The sum of squared errors of the PH-based ALT model is 2.9149, which is much larger than that of the PO-based ALT model. We conclude that the PO-based ALT model provides more accurate reliability estimates at normal operating conditions than the PH-based ALT model for the given failure time data.

CHAPTER 4

CONFIDENCE INTERVALS AND MODEL VALIDATION

Chapter 3 provides the point estimate of the unknown parameters of the PO-based ALT model and the point estimate of reliability at design stress conditions. Some reliability estimation problems in many systems require the interval estimate of the reliability at a specified stress level, especially the design stress level. In this chapter, we introduce a methodology for obtaining the confidence intervals for the unknown parameters and the estimated reliability at the design stress level through Fisher information matrix. We also describe the likelihood ratio test and the modified Cox-Snell residuals to validate the model. Numerical examples are also given in this chapter to display the interval estimation procedure and the validation procedures.

4.1 Confidence Intervals

4.1.1 Fisher Information Matrix

We begin by the calculation of Fisher information matrix.

Assume that the baseline odds function is expressed in a quadratic form:

$$\theta_0(t) = \gamma_1 t + \gamma_2 t^2 , \qquad \gamma_1 \geq 0, \gamma_2 \geq 0 \tag{4.1}$$

where γ_1 and γ_2 are constant parameters to be estimated, and the intercept parameter is zero since the odds function crosses the origin according to the first property of odds function as described in Chapter 3.

The proportional odds model is then represented by:

$$\theta(t;\boldsymbol{z}) = \exp(\boldsymbol{\beta}^t\boldsymbol{z})\theta_0(t) = \exp(\beta_1 z_1 + \beta_2 z_2 + \cdots \beta_k z_k)(\gamma_1 t + \gamma_2 t^2)\,, \tag{4.2}$$

and the hazard rate function $\lambda(t;\boldsymbol{z})$ and cumulative hazard rate function $\Lambda(t;\boldsymbol{z})$ are

$$\lambda(t;\boldsymbol{z}) = \frac{\theta'(t;\boldsymbol{z})}{\theta(t;\boldsymbol{z})+1} = \frac{\exp(\boldsymbol{\beta}^t\boldsymbol{z})(\gamma_1 + 2\gamma_2 t)}{\exp(\boldsymbol{\beta}^t\boldsymbol{z})(\gamma_1 t + \gamma_2 t^2)+1}\,,$$

$$\Lambda(t;\boldsymbol{z}) = \ln[\theta(t;\boldsymbol{z})+1] = \ln[\exp(\boldsymbol{\beta}^t\boldsymbol{z})(\gamma_1 t + \gamma_2 t^2)+1]\,.$$

For a censored failure time data set, let t_i represent the failure time of the ith unit, $\boldsymbol{z}_i = (z_{1i}, z_{2i}, \ldots, z_{ki})^t$ the stress vector of this unit, and I_i the indicator function, which is 1 if $t_i \le \tau$ (the censoring time), or 0 if $t_i > \tau$. The log likelihood function of the proposed PO-based ALT model is

$$l = \sum_{i=1}^{n} I_i \ln[\lambda(t_i;\boldsymbol{z}_i)] - \sum_{i=1}^{n} \Lambda(t_i;\boldsymbol{z}_i)\,, \tag{4.3}$$

Substituting hazard rate function and cumulative hazard rate function into the log likelihood function, we obtain:

$$l = \sum_{i=1}^{n} I_i \{\boldsymbol{\beta}^t \mathbf{z}_i + \ln(\gamma_1 + 2\gamma_2 t_i) - \ln[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]\} - \sum_{i=1}^{n} \ln[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1]\,. \quad (4.4)$$

The unknown parameters ($\boldsymbol{\beta}, \gamma_1, \gamma_2$) are obtained by the maximum likelihood estimation procedure. To determine the confidence intervals of the estimated parameters, we should first construct the Fisher information matrix, which requires the first and second derivatives of the log likelihood function l with respect to each of three parameters, respectively:

$$\frac{\partial l}{\partial \beta_1} = \sum_{i=1}^{n} I_i z_{1i} - \sum_{i=1}^{n} I_i \frac{z_{1i} e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)}{e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1} - \sum_{i=1}^{n} \frac{z_{1i} e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)}{e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1},$$

$$\frac{\partial l}{\partial \beta_k} = \sum_{i=1}^{n} I_i z_{ki} - \sum_{i=1}^{n} I_i \frac{z_{ki} e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)}{e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1} - \sum_{i=1}^{n} \frac{z_{ki} e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)}{e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1},$$

$$\frac{\partial l}{\partial \gamma_1} = \sum_{i=1}^{n} I_i \frac{1}{(\gamma_1 + 2\gamma_2 t_i)} - \sum_{i=1}^{n} I_i \frac{e^{\boldsymbol{\beta}^t \mathbf{z}_i} t_i}{e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1} - \sum_{i=1}^{n} \frac{e^{\boldsymbol{\beta}^t \mathbf{z}_i} t_i}{e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1},$$

$$\frac{\partial l}{\partial \gamma_2} = \sum_{i=1}^{n} I_i \frac{2t_i}{(\gamma_1 + 2\gamma_2 t_i)} - \sum_{i=1}^{n} I_i \frac{e^{\boldsymbol{\beta}^t \mathbf{z}_i} t_i^2}{e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1} - \sum_{i=1}^{n} \frac{e^{\boldsymbol{\beta}^t \mathbf{z}_i} t_i^2}{e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2) + 1},$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\beta}^2} = -\sum_{i=1}^{n} I_i \frac{\mathbf{z}_i \mathbf{z}_i^t e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)+1]^2} - \sum_{i=1}^{n} \frac{\mathbf{z}_i \mathbf{z}_i^t e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)+1]^2},$$

$$\frac{\partial^2 l}{\partial \gamma_1^2} = -\sum_{i=1}^{n} I_i \frac{1}{(\gamma_1 + 2\gamma_2 t_i)^2} + \sum_{i=1}^{n} I_i \frac{e^{2\boldsymbol{\beta}^t \mathbf{z}_i} t_i^2}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)+1]^2} + \sum_{i=1}^{n} \frac{e^{2\boldsymbol{\beta}^t \mathbf{z}_i} t_i^2}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)+1]^2},$$

$$\frac{\partial^2 l}{\partial \gamma_2^2} = -\sum_{i=1}^{n} I_i \frac{4t_i^2}{(\gamma_1 + 2\gamma_2 t_i)^2} + \sum_{i=1}^{n} I_i \frac{e^{2\boldsymbol{\beta}^t \mathbf{z}_i} t_i^4}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)+1]^2} + \sum_{i=1}^{n} \frac{e^{2\boldsymbol{\beta}^t \mathbf{z}_i} t_i^4}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)+1]^2},$$

$$\frac{\partial^2 l}{\partial \gamma_1 \partial \gamma_2} = -\sum_{i=1}^{n} I_i \frac{2t_i}{(\gamma_1 + 2\gamma_2 t_i)^2} + \sum_{i=1}^{n} I_i \frac{e^{2\boldsymbol{\beta}^t \mathbf{z}_i} t_i^3}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)+1]^2} + \sum_{i=1}^{n} \frac{e^{2\boldsymbol{\beta}^t \mathbf{z}_i} t_i^3}{[e^{\boldsymbol{\beta}^t \mathbf{z}_i}(\gamma_1 t_i + \gamma_2 t_i^2)+1]^2}.$$

The elements of the Fisher information matrix are the negative expectations of the second derivatives of the log likelihood function:

$$F = \begin{bmatrix} E[-\frac{\partial^2 l}{\partial \boldsymbol{\beta}^2}] & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & E[-\frac{\partial^2 l}{\partial \gamma_1^2}] & E[-\frac{\partial^2 l}{\partial \gamma_1 \partial \gamma_2}] \\ \boldsymbol{0} & E[-\frac{\partial^2 l}{\partial \gamma_1 \partial \gamma_2}] & E[-\frac{\partial^2 l}{\partial \gamma_2^2}] \end{bmatrix}, \tag{4.5}$$

where

$$E[g(t)] = \int_0^{\infty} f(t) g(t)\, dt \tag{4.6}$$

$g(t)$ is the negative of the second derivative of log likelihood function, or any element of the Fisher information matrix, and $f(t)$ is the *pdf* of the failure time distribution, given by

$$f(t) = -R'(t) = \frac{(\gamma_1 + 2\gamma_2 t)e^{\beta z}}{[1+(\gamma_1 t + \gamma_2 t^2)e^{\beta z}]^2}.$$

So the Fisher information matrix F is a 3×3 symmetric matrix.

Since the baseline odds function is assumed to be independent of the stress vector $\boldsymbol{z}$ in the PO model, the correlations between the stress coefficient $\boldsymbol{\beta}$ and baseline parameters γ_1, γ_2 are zero. So the cells (1, 2), (1, 3), (2, 1), and (3, 1) of the Fisher information matrix are zero as in Eq. (4.5).

The calculation of the Fisher information matrix is performed numerically to integrate Eq. (4.6) from time 0 to time T, which is large enough and more than the longest failure time of the failure time observations. In the numerical calculation procedure, the unknown parameters $(\boldsymbol{\beta}, \gamma_1, \gamma_2)$ of the proportional odds model are replaced by the estimated values obtained from the maximum likelihood estimation procedure mentioned in Chapter 3.

4.1.2 Variance-Covariance Matrix and Confidence Intervals of Model Parameters

The asymptotic variance-covariance matrix Σ of the maximum likelihood estimates of the parameters is the inverse of the Fisher information matrix F :

$$\mathbf{\Sigma} = F^{-1} = \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma_{22} & \sigma_{23} \\ \mathbf{0} & \sigma_{23} & \sigma_{33} \end{bmatrix}. \tag{4.7}$$

So the asymptotic variances of the parameter estimates $(\hat{\boldsymbol{\beta}}, \hat{\gamma}_1, \hat{\gamma}_2)$ for the PO-based ALT model are:

$$Var(\hat{\beta}_j) = \mathbf{\Sigma}_{11}(j), \qquad j = 1, \ldots, k ,$$

$$Var(\hat{\gamma}_1) = \sigma_{22} ,$$

$$Var(\hat{\gamma}_2) = \sigma_{33} .$$

where $\mathbf{\Sigma}_{11}(j)$ is the *j*th diagonal element of the matrix $\mathbf{\Sigma}_{11}$.

Therefore the $(1-\alpha)\%$ confidence intervals of the parameters are:

$$\boldsymbol{\beta}: \quad \hat{\beta}_j \pm Z_{\alpha/2}\sqrt{Var(\hat{\beta}_j)}, \qquad j = 1, \ldots, k ,$$

$$\gamma_1: \quad \hat{\gamma}_1 \pm Z_{\alpha/2}\sqrt{Var(\hat{\gamma}_1)}\ ,$$

$$\gamma_2: \quad \hat{\gamma}_2 \pm Z_{\alpha/2}\sqrt{Var(\hat{\gamma}_2)}\ .$$

4.1.3 Confidence Intervals of Reliability Estimates

We use the delta method to calculate the asymptotic variance of the estimated reliability at design stress level z_D with the PO-based ALT model. Under the PO model the reliability function of any unit at the design stress z_D is

$$R(t;\mathbf{z}_D) = \frac{1}{1+(\gamma_1 t + \gamma_2 t^2)e^{\boldsymbol{\beta}^t \mathbf{z}_D}}\ ,$$

The asymptotic variance of the estimated reliability is given by delta method as:

$$Var[\hat{R}(t;\mathbf{z}_D)] = [\frac{\partial R(t;\mathbf{z}_D)}{\partial \beta_1} \ldots \frac{\partial R(t;\mathbf{z}_D)}{\partial \beta_k}\ \frac{\partial R(t;\mathbf{z}_D)}{\partial \gamma_1}\ \frac{\partial R(t;\mathbf{z}_D)}{\partial \gamma_2}]\Sigma \begin{bmatrix} \frac{\partial R(t;\mathbf{z}_D)}{\partial \beta_1} \\ \vdots \\ \frac{\partial R(t;\mathbf{z}_D)}{\partial \beta_k} \\ \frac{\partial R(t;\mathbf{z}_D)}{\partial \gamma_1} \\ \frac{\partial R(t;\mathbf{z}_D)}{\partial \gamma_2} \end{bmatrix} \Big|_{(\boldsymbol{\beta},\, \gamma_1,\, \gamma_2)=(\hat{\boldsymbol{\beta}},\, \hat{\gamma}_1,\, \hat{\gamma}_2)} \tag{4.8}$$

Therefore, the $(1-\alpha)\%$ confidence intervals of the reliability at the design stress z_D is

$$\left[\hat{R}(t;\mathbf{z}_D)-Z_{\alpha/2}\sqrt{Var[\hat{R}(t;\mathbf{z}_D)]},\ \hat{R}(t;\mathbf{z}_D)+Z_{\alpha/2}\sqrt{Var[\hat{R}(t;\mathbf{z}_D)]}\right]. \tag{4.9}$$

4.1.4 Numerical Example of Reliability Confidence Intervals

We use Swartz's data (Swartz 1986) as an example to calculate the 95% confidence intervals of unknown parameters and reliability estimate at the design stress $z_D = 25°C$. The variance-covariance matrix is

$$\Sigma = \begin{bmatrix} 0.266 & 0 & 0 \\ 0 & 37030 & -36.24 \\ 0 & -36.24 & 0.314 \end{bmatrix},$$

Then the 95% confidence intervals of the parameters are:

$$\beta: \quad \hat{\beta} \pm Z_{\alpha/2}\sqrt{Var(\hat{\beta})} = -64.93 \pm 1.96\sqrt{0.266} = (-65.94, -63.92)$$

$$\gamma_1: \quad \hat{\gamma}_1 \pm Z_{\alpha/2}\sqrt{Var(\hat{\gamma}_1)} = 833.73 \pm 1.96\sqrt{37030} = (456.56, 1210.90)$$

$$\gamma_2: \quad \hat{\gamma}_2 \pm Z_{\alpha/2}\sqrt{Var(\hat{\gamma}_2)} = 5.18 \pm 1.96\sqrt{0.314} = (4.08, 6.28)$$

The 95% reliability confidence intervals at stress level $z_D = 25°C$, which are calculated with Eq. (4.9), are depicted in Figure 4.1. As shown, the intervals intend to large when time is larger. Therefore the reliability estimates are more accurate at the early stage.

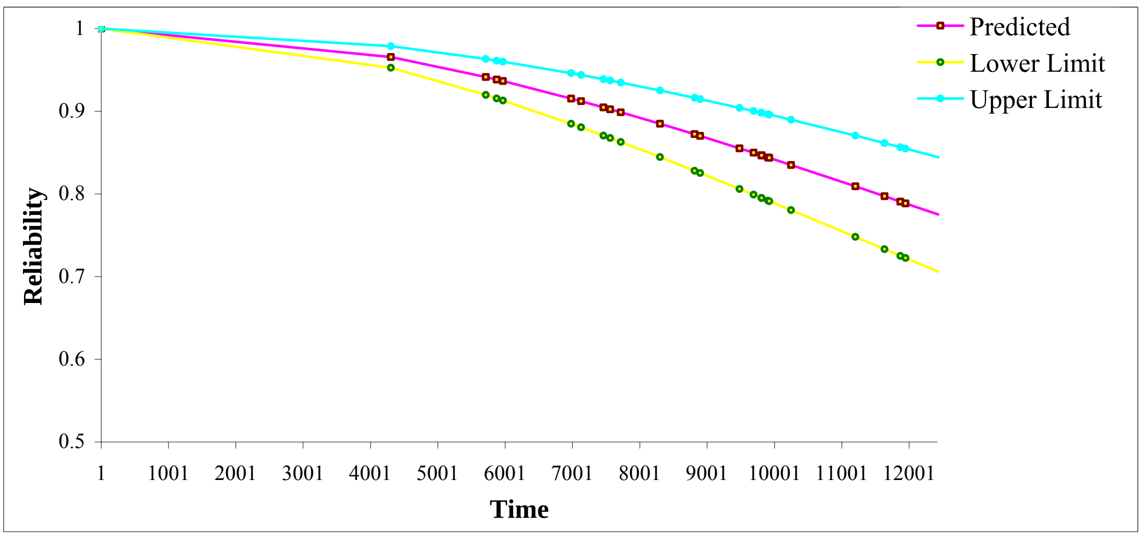

Figure 4.1 95% reliability confidence intervals at temperature $z_D = 25°C$ level

4.2 Model Validations

4.2.1 Model Sufficiency

In Chapter 3, a polynomial baseline odds function is proposed to approximate the general odds function,

$$\theta_0(t) = \gamma_1 t + \gamma_2 t^2 + \gamma_3 t^3 + \ldots \tag{4.10}$$

We also point out that higher order may not be necessary and a quadratic baseline odds function is already sufficient in most cases. In order to test the sufficiency of quadratic baseline odds function, we use the likelihood ratio test.

The hypothesis can be expressed as:

$$H_0 : \gamma_3 = 0, \text{ versus } H_1 : \gamma_3 \neq 0.$$

To test the null hypothesis H_0 against the alternative hypothesis H_1, we use the likelihood ratio statistic

$$\chi^2_{LR} = -2[\ln L(\beta', \gamma_1', \gamma_2') - \ln L(\beta'', \gamma_1'', \gamma_2'', \gamma_3'')], \tag{4.11}$$

where $(\beta', \gamma_1', \gamma_2')$ is found by maximizing the log likelihood function with consideration of only γ_1 and γ_2, and $(\beta'', \gamma_1'', \gamma_2'', \gamma_3'')$ is found by maximizing the log likelihood function with γ_3 in the model.

This statistic has an asymptotic chi-squared distribution with 1 degree of freedom. A large value of χ^2_{LR} gives the evidence against the null hypothesis H_0.

To test if we need higher order polynomial baseline odds function than cubic equation, we could use the similar likelihood ratio test.

4.2.2 Model Residuals

To test the validation of the assumption of proportional odds after the fitting of the regression model, we need to calculate the residual of model fitting. In the usual linear regression setup, it is quite easy to define a residual for the fitted regression model. In the proportional odds regression model, the definition of the residual is not straightforward. We could use Cox-Snell type residual to assess the fit of a PO-based ALT model. The Cox-Snell residuals (Cox and Snell 1968) could be used for assessing the goodness-of-fit of a Cox model. With some modifications, we could also use the similar type residual to assess the goodness-of-fit of a proportional odds model.

We suppose a PO-based ALT model was used to fit the dataset (t_i, I_i, z_i), $i = 1, \ldots, n$. We also suppose that the proportional odds model $\theta(t;z) = e^{\beta z}\theta_0(t) = e^{\beta z}(\gamma_1 t + \gamma_2 t^2)$ has been utilized. If the model is correct, then, it is well known that, if we make the probability integral transformation $F(t_i; z_i)$ on the true failure time t_i, the resulting random variable has a uniform distribution on the unit interval. Therefore we can prove that the random variable $Y = \Lambda(t_i; z_i)$ has an exponential distribution with hazard rate 1. Here, $y = \Lambda(t;z)$ is the cumulative hazard rate of failure time at stress level z after the parameter estimates have obtained.

Theorem: the random variable $Y = \Lambda(t_i; z_i)$ has an exponential distribution with hazard rate 1 if proportional odds model is correct.

Proof: If the model is correct, the random variable $x = F(t_i; z_i)$, which is obtained from the probability integral transformation of the failure time, follows a uniform distribution on the interval $[0, 1]$. Therefore the probability density function of x is given by

$$f(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \tag{4.12}$$

Then the random variable $Y = \Lambda(t_i; z_i)$, which is the cumulative hazard rate function of failure time, can be also expressed in terms of the random variable $x = F(t_i; z_i)$,

$$Y = \Lambda(t_i; z_i) = -\ln[1 - F(t_i; z_i)]\,. \tag{4.13}$$

Without loss of generality, Eq. (4.13) can be simplified as

$$y = -\ln(1 - x)\,. \tag{4.14}$$

So the random variable y is a function of the random variable x, which has a uniform distribution on the interval $[0, 1]$, and is described by the probability density function Eq.

(4.12). We can obtain the probability density function of random variable y (DeGroot and Schevish 2002):

If a random variable X has a continuous distribution; the probability density function of X is $f(x)$; and another random variable is defined as $Y = r(X)$. For each real number y, the cumulative distribution function of Y can be derived as follows:

$$F(y) = \Pr(Y \le y) = \Pr[r(X) \le y] = \int_{\{x: r(x) \le y\}} f(x)dx \,. \tag{4.15}$$

Since the random variable Y is defined by $Y = -\ln(1 - X)$ as Eq. (4.14) here, then Y must belong to the interval $0 \le Y < \infty$. Thus, for each value of Y such that $0 \le Y < \infty$, the cumulative distribution function $F(y)$ of Y is derived from Eq. (4.15) as follows:

$$\begin{aligned} F(y) &= \Pr(Y \le y) = \Pr[-\ln(1 - X) \le y] \\ &= \Pr[\ln(1 - X) \ge -y] \\ &= \Pr[(1 - X) \ge e^{-y}] \\ &= \Pr[X \le 1 - e^{-y}] \\ &= \int_0^{1-e^{-y}} f(x)dx \\ &= 1 - e^{-y} \end{aligned} \tag{4.16}$$

For $0 < Y < \infty$, the probability density function $f(y)$ of Y is

$$f(y) = \frac{dF(y)}{dy} = e^{-y} \,. \tag{4.17}$$

Eq. (4.17) is the probability density function of exponential distribution with hazard rate 1. Proof is complete. □

Therefore we could utilize the above theorem to generate the Cox-Snell type residuals for the PO-based ALT model to verify the assumption of proportional odds. If the estimates of the unknowns $(\beta,\ \gamma_1,\ \gamma_2)$ from the postulated model are $(\hat{\beta},\ \hat{\gamma}_1,\ \hat{\gamma}_2)$, then, the Cox-Snell type residuals for the PO-based ALT model are defined as

$$r_i = \hat{\Lambda}(t_i; z_i) = \ln[\hat{\theta}(t_i; z_i) + 1] = \ln[e^{\hat{\beta} z_i}(\hat{\gamma}_1 t_i + \hat{\gamma}_2 t_i^2) + 1],\ i = 1, \ldots, n. \tag{4.18}$$

If the proportional odds model is correct and the estimates $(\hat{\beta}, \hat{\gamma}_1, \hat{\gamma}_2)$ are close to the true values of $(\beta, \gamma_1, \gamma_2)$, then the r_i's should look like a censored sample from a unit exponential distribution.

To check whether the r_i's behave as a sample from a unit exponential distribution, we compute the Nelson-Aalen estimator (Klein and Moeschberger 1997) of the cumulative hazard rate of the r_i's. If the unit exponential distribution fits the data, then, the Nelson-Aalen estimator should be approximately equal to the cumulative hazard rate of the unit exponential distribution, $\Lambda_E(t) = t$. Thus, a plot of the estimated cumulative hazard rate of r_i's, or $\hat{\Lambda}_r(r_i)$, versus r_i should be a straight line through the origin.

The nonparametric Nelson-Aalen estimator of the cumulative hazard rate is obtained as follows. To allow for possible ties in the data, suppose that the failures occur at D distinct times $t_1 < t_2 < \cdots < t_D$, and that at time t_i there are d_i failures. Let Y_i be the number of units at risk at time t_i, or the number of units which are operating properly at time t_i. Then the Nelson-Aalen estimator of the cumulative hazard rate is defined as follows

$$\hat{\Lambda}(t) = \begin{cases} 0, & \text{if} \quad t < t_1 \\ \sum_{t_i \le t} d_i / Y_i, & \text{if} \quad t \ge t_1 \end{cases} \tag{4.19}$$

The Nelson-Aalen estimator has better small-size performance than the estimator based on the Kaplan-Meier estimator. The Nelson-Aalen estimator has two primary uses in analyzing failure time data. The first is selecting parametric models for the data. For our case, a plot of Nelson-Aalen estimators of r_i's, or $\hat{\Lambda}_r(r_i)$, versus r_i will be approximately linear if the exponential distribution fits the data r_i's. The second use of the Nelson-Aalen estimator is providing crude estimate of the hazard rate of the failure time data. This estimate is the slope of the plot obtained above. For our case, the slope of the plot of Nelson-Aalen estimators of r_i's versus r_i will be roughly 1 if the data are from an exponential distribution with hazard rate 1.

4.2.3 Numerical Example of Cox-Snell Residuals

We use the simulation dataset I in Table 3.1 as an example to demonstrate the performance of the Cox-Snell residuals.

We apply the PO-based ALT model to this data set. The unknown parameters of the models are estimated by the Maximum Likelihood Estimation procedures as discussed in section 3.3. We obtain the following parameter estimates:

$$\hat{\beta} = \quad -31.25$$

$$\hat{\gamma}_1 = \quad 1294$$

$$\hat{\gamma}_2 = \quad 0.039$$

Therefore the Cox-Snell type residuals for the PO-based ALT model are calculated by

$$r_i = \hat{\Lambda}(t_i; z_i) = \ln[\hat{\theta}(t_i; z_i) + 1] = \ln[e^{\hat{\beta} z_i}(\hat{\gamma}_1 t_i + \hat{\gamma}_2 t_i^2) + 1],\ i = 1, \ldots, n.$$

To check whether the r_i's behave as a sample from a unit exponential distribution, we treat them as failure times and compute the Nelson-Aalen estimators of the cumulative hazard rate of the r_i's. The plot of the cumulative hazard rate of the r_i's versus the r_i's is shown in Figure 4.2. As shown in this figure, the step function is the estimated cumulative hazard rate of the r_i's and the straight line is a line with the slope of 1. It is obvious that the estimated cumulative hazard rate of r_i's, or $\hat{\Lambda}_r(r_i)$, is approximately a straight line with the slope of 1. Therefore, we conclude that the proportional odds assumption is valid for the given data set.

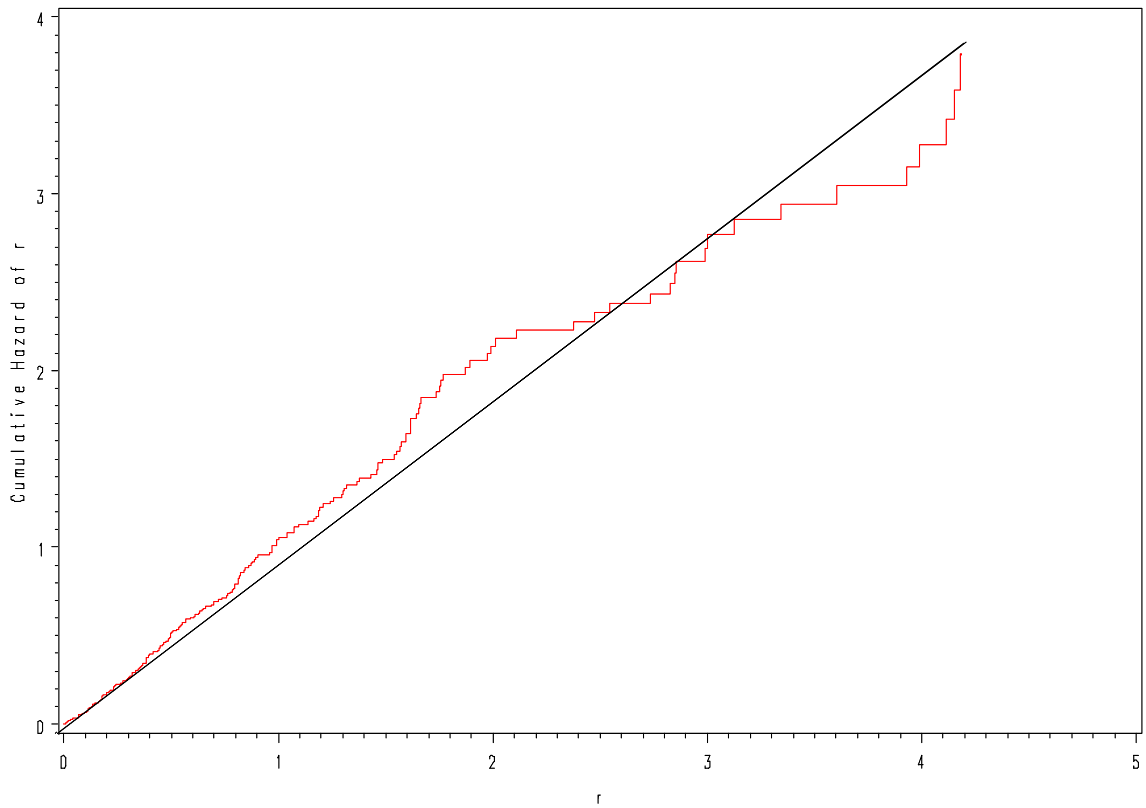

Figure 4.2 Nelson-Aalen estimators of Cox-Snell residuals

CHAPTER 5

PO-BASED MULTIPLE-STRESS-TYPE ALT PLANS

While accelerated life test saves time and expenses over testing at design stress conditions, the reliability estimates obtained via extrapolation in both stress and time are inevitably less accurate. One interesting measurement to obtain more accurate estimates is to devise a test plan that tests units at appropriately selected stress levels with proper allocation of test units to each level. In other words, an optimum accelerated life testing plan will result in more accurate estimates of reliability at design stress conditions. Design of ALT plans under one type of stress may mask the effect of other critical types of stresses that could lead to the component's failure. Therefore, it is more realistic to consider multiple stress types. In this chapter, we are investigating the design of optimum ALT plans based on the proportional odds model with multiple stress types.

5.1 PO-based ALT Plans with Constant Multiple Stress Types

In this section, we design constant multiple-stress-type ALT plans based on the proportional odds assumption.

5.1.1 The Assumptions

We assume the following conditions for the ALT plans with multiple stress types.

1. There are k types of stress $\mathbf{z} = (z_1, z_2, \ldots, z_k)$ with 3 levels for each stress type used in the test.
2. The proportional odds (PO) model is employed to relate the reliability at different accelerated stress levels and the design stress levels

$$\theta(t;\mathbf{z}) = \exp(\beta_1 z_1 + \beta_2 z_2 + \cdots \beta_k z_k)\theta_0(t) . \tag{5.1}$$

   where $\theta(t;\mathbf{z})$ is the odds function of test units at time t and stress vector $\mathbf{z}$; $\theta_0(t)$ is the baseline odds function of the PO model; $\beta_1, \beta_2, \ldots, \beta_k$ are unknown model parameters, which explain the effects of stresses on the failure time.
3. The baseline odds function is quadratic and given by

$$\theta_0(t) = \gamma_1 t + \gamma_2 t^2 , \tag{5.2}$$

   where γ_1 and γ_1 are unknown model parameters.
4. The number of stress level combination to determined is 3^k as shown in Figure 5.1 for $k = 2$. The upper bounds of stresses are pre-specified as the highest stress levels beyond which the failure mode will change. While the lower bounds of two stresses are specified as the design stress levels.

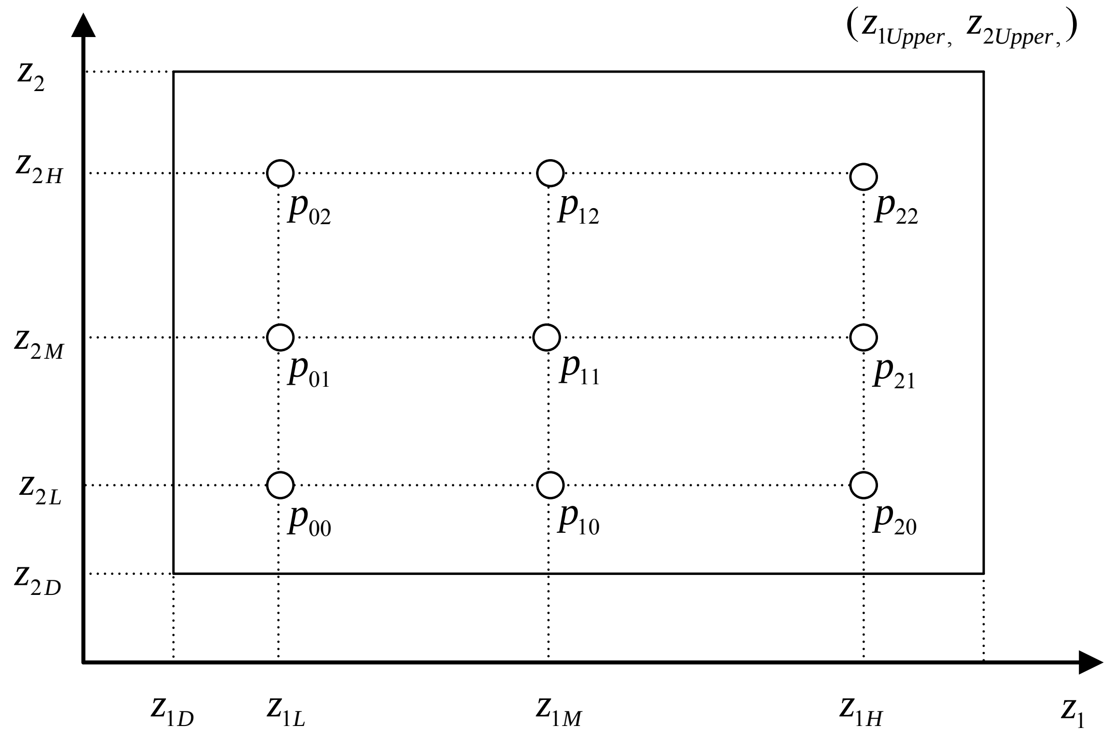

Figure 5.1 Design of ALT plan with two stress types and three levels for each stress type

5. A total of n units are available for testing. The proportion of testing units allocated to the testing point of the i_1th level of the first stress type z_1, i_2th level of the second stress type z_2, …, i_k th level of the last stress type z_k is denoted by $p_{i_1 i_2 \ldots i_k}$, $i_j = 0, 1, 2;\ j = 1, 2, \ldots, k$, where the notation of the levels $0, 1, 2$ are equivalent to the notation of the levels L, M, H, standing for lower level, medium level, and high level respectively. Throughout the thesis, both notations are interchangeable.
6. The test is terminated at the pre-specified censoring time τ.

As defined by the above assumptions, the decision variables are the stress levels (z_{1L}, z_{1M}, z_{1H}), (z_{2L}, z_{2M}, z_{2H}), …, (z_{kL}, z_{kM}, z_{kH}) and the proportions of test units allocated to the 3^k stress combinations $p_{i_1 i_2 \ldots i_k}$, $i_j = 0, 1, 2;\ j = 1, 2, \ldots, k$. The total number of decision variables is ($3k + 3^k$). This factorial arrangement of stress levels and test units shown in Figure 5.1 is not statistically optimal. However, this arrangement is motivated by the actual practice of reliability engineers. Since any other arrangement with the same number of testing points will result in more stress levels for each stress type, it enables reliability engineers to carry out the entire test simultaneously by utilizing the available equipment in an efficient manner and lead to substantial savings in test time and cost. In addition, it allows for testing of interactions after the data are collected.

5.1.2 The Log Likelihood Function

Using the assumption of proportional odds and the baseline odds function we obtain the odds function at stress combination $\boldsymbol{z} = (z_1, z_2, \ldots, z_k)^t$

$$\theta(t;\boldsymbol{z}) = \exp(\beta_1 z_1 + \beta_2 z_2 + \cdots \beta_k z_k)(\gamma_1 t + \gamma_2 t^2)\,. \tag{5.3}$$

Then the corresponding hazard rate function $\lambda(t;\boldsymbol{z})$, cumulative hazard function $\Lambda(t;\boldsymbol{z})$, reliability function $R(t;\boldsymbol{z})$, and probability density function $f(t;\boldsymbol{z})$ are obtained as follows:

$$\lambda(t;\boldsymbol{z}) = \frac{\theta'(t;\boldsymbol{z})}{\theta(t;\boldsymbol{z})+1} = \frac{\exp(\boldsymbol{\beta}^t\boldsymbol{z})(\gamma_1 + 2\gamma_2 t)}{\exp(\boldsymbol{\beta}^t\boldsymbol{z})(\gamma_1 t + \gamma_2 t^2)+1}\,, \tag{5.4}$$

$$\Lambda(t;\mathbf{z}) = \ln[\theta(t;\mathbf{z})+1] = \ln[\exp(\boldsymbol{\beta}^t\mathbf{z})(\gamma_1 t+\gamma_2 t^2)+1], \tag{5.5}$$

$$R(t;\mathbf{z}) = \frac{1}{\theta(t;\mathbf{z})+1} = \frac{1}{\exp(\boldsymbol{\beta}^t\mathbf{z})(\gamma_1 t+\gamma_2 t^2)+1}, \tag{5.6}$$

$$f(t;\mathbf{z}) = \frac{\theta'(t;\mathbf{z})}{[\theta(t;\mathbf{z})+1]^2} = \frac{\exp(\boldsymbol{\beta}^t\mathbf{z})(\gamma_1+2\gamma_2 t)}{[\exp(\boldsymbol{\beta}^t\mathbf{z})(\gamma_1 t+\gamma_2 t^2)+1]^2}. \tag{5.7}$$

Let $t_i, i=1,\ldots,n$ represent the failure time of the *i*th testing unit, $\mathbf{z}_i = (z_{1i}, z_{2i}, \ldots, z_{ki})^t$ the stress vector of this unit, and I_i the indicator function, which is defined by

$$I_i = I(t_i \le \tau) = \begin{cases} 1 & \text{if } t_i \le \tau, \text{ failure observed before time } \tau, \\ 0 & \text{if } t_i > \tau, \text{ censored at time } \tau. \end{cases}$$

The log likelihood of the proposed ALT based on proportional odds model for this unit is

$$l_i = I_i \ln[\lambda(t_i;\mathbf{z}_i)] - \Lambda(t_i;\mathbf{z}_i). \tag{5.8}$$

Substituting the hazard rate function in Eq. (5.4) and cumulative hazard function in Eq. (5.5) into above log likelihood results in:

$$l_i = I_i[(\boldsymbol{\beta}^t\mathbf{z}_i) + \ln(\gamma_1+2\gamma_2 t_i)] - I_i \ln[e^{(\boldsymbol{\beta}^t\mathbf{z}_i)}(\gamma_1 t_i+\gamma_2 t_i^2)+1] - \ln[e^{(\boldsymbol{\beta}^t\mathbf{z}_i)}(\gamma_1 t_i+\gamma_2 t_i^2)+1]. \tag{5.9}$$

Then the log likelihood function for the whole failure time sample is

$$l = l_1 + l_2 + \cdots + l_n .$$

5.1.3 The Fisher Information Matrix and Covariance Matrix

For a single observation triple $(t, I, \mathbf{z})$, where $\mathbf{z}$ is a vector of stress levels, the first partial derivatives of log likelihood of this observation with respect to the model parameters are:

$$\frac{\partial l}{\partial \beta_1} = Iz_1 - I_i \frac{z_1 e^{(\beta' \mathbf{z})}(\gamma_1 t + \gamma_2 t^2)}{e^{(\beta' \mathbf{z})}(\gamma_1 t + \gamma_2 t^2) + 1} - \frac{z_1 e^{(\beta' \mathbf{z})}(\gamma_1 t + \gamma_2 t^2)}{e^{(\beta' \mathbf{z})}(\gamma_1 t + \gamma_2 t^2) + 1}, \tag{5.10}$$

$$\vdots$$

$$\frac{\partial l}{\partial \beta_k} = Iz_k - I \frac{z_k e^{(\beta' \mathbf{z})}(\gamma_1 t + \gamma_2 t^2)}{e^{(\beta' \mathbf{z})}(\gamma_1 t + \gamma_2 t^2) + 1} - \frac{z_k e^{(\beta' \mathbf{z})}(\gamma_1 t + \gamma_2 t^2)}{e^{(\beta' \mathbf{z})}(\gamma_1 t + \gamma_2 t^2) + 1}, \tag{5.11}$$

$$\frac{\partial l}{\partial \gamma_1} = \frac{I}{(\gamma_1 + 2\gamma_2 t)} - \frac{Ie^{(\beta' \mathbf{z})} t}{e^{(\beta' \mathbf{z})}(\gamma_1 t + \gamma_2 t^2) + 1} - \frac{e^{(\beta' \mathbf{z})} t}{e^{(\beta' \mathbf{z})}(\gamma_1 t + \gamma_2 t^2) + 1}, \tag{5.12}$$

$$\frac{\partial l}{\partial \gamma_2} = \frac{2I_i t_i}{(\gamma_1 + 2\gamma_2 t_i)} - \frac{I_i e^{(\beta' \mathbf{z}_i)} t_i^2}{e^{(\beta' \mathbf{z}_i)}(\gamma_1 t_i + \gamma_2 t_i^2) + 1} - \frac{e^{(\beta' \mathbf{z}_i)} t_i^2}{e^{(\beta' \mathbf{z}_i)}(\gamma_1 t_i + \gamma_2 t_i^2) + 1}, \tag{5.13}$$

Summing the above first derivatives over all test units and setting them equal to zero will provide the equations for solving the maximum likelihood estimates of the model parameters $(\beta_1, \beta_2, \ldots, \beta_k, \gamma_1, \gamma_2)$.

Assuming the correlations only exist between γ_1 and γ_2, then, for a single observation triple $(t, I, \boldsymbol{z})$, the second partial derivatives of the log likelihood with respect to model parameters are

$$\frac{\partial^2 l}{\partial \beta_1^2} = -I \frac{z_1^2 e^{(\boldsymbol{\beta}'\boldsymbol{z})}(\gamma_1 t + \gamma_2 t^2)}{[e^{(\boldsymbol{\beta}'\boldsymbol{z})}(\gamma_1 t + \gamma_2 t^2) + 1]^2} - \frac{z_1^2 e^{(\boldsymbol{\beta}'\boldsymbol{z})}(\gamma_1 t + \gamma_2 t^2)}{[e^{(\boldsymbol{\beta}'\boldsymbol{z})}(\gamma_1 t + \gamma_2 t^2) + 1]^2}, \tag{5.14}$$

$$\vdots$$

$$\frac{\partial^2 l}{\partial \beta_k^2} = -I \frac{z_k^2 e^{(\boldsymbol{\beta}'\boldsymbol{z})}(\gamma_1 t + \gamma_2 t^2)}{[e^{(\boldsymbol{\beta}'\boldsymbol{z})}(\gamma_1 t + \gamma_2 t^2) + 1]^2} - \frac{z_k^2 e^{(\boldsymbol{\beta}'\boldsymbol{z})}(\gamma_1 t + \gamma_2 t^2)}{[e^{(\boldsymbol{\beta}'\boldsymbol{z})}(\gamma_1 t + \gamma_2 t^2) + 1]^2}, \tag{5.15}$$

$$\frac{\partial^2 l}{\partial \gamma_1^2} = -\frac{I}{(\gamma_1 + 2\gamma_2 t)^2} + \frac{I e^{2(\boldsymbol{\beta}'\boldsymbol{z})} t^2}{[e^{(\boldsymbol{\beta}'\boldsymbol{z})}(\gamma_1 t + \gamma_2 t^2) + 1]^2} + \frac{e^{2(\boldsymbol{\beta}'\boldsymbol{z})} t^2}{[e^{(\boldsymbol{\beta}'\boldsymbol{z})}(\gamma_1 t_i + \gamma_2 t^2) + 1]^2}, \tag{5.16}$$

$$\frac{\partial^2 l}{\partial \gamma_2^2} = -\frac{4 I t^2}{(\gamma_1 + 2\gamma_2 t)^2} + \frac{I e^{2(\boldsymbol{\beta}'\boldsymbol{z})} t^4}{[e^{(\boldsymbol{\beta}'\boldsymbol{z})}(\gamma_1 t + \gamma_2 t^2) + 1]^2} + \frac{e^{2(\boldsymbol{\beta}'\boldsymbol{z})} t^4}{[e^{(\boldsymbol{\beta}'\boldsymbol{z})}(\gamma_1 t + \gamma_2 t^2) + 1]^2}, \tag{5.17}$$

$$\frac{\partial^2 l}{\partial \gamma_1 \partial \gamma_2} = -\frac{2 I t}{(\gamma_1 + 2\gamma_2 t)^2} + \frac{I e^{2(\boldsymbol{\beta}'\boldsymbol{z})} t^3}{[e^{(\boldsymbol{\beta}'\boldsymbol{z})}(\gamma_1 t + \gamma_2 t^2) + 1]^2} + \frac{e^{2(\boldsymbol{\beta}'\boldsymbol{z})} t^3}{[e^{(\boldsymbol{\beta}'\boldsymbol{z})}(\gamma_1 t + \gamma_2 t^2) + 1]^2}. \tag{5.18}$$

The elements of the Fisher information matrix for an observation are the negative expectation of the above second partial derivatives:

$$E\left[-\frac{\partial^2 l}{\partial \beta_1^2}\right] = \int_0^{\tau} -\frac{\partial^2 l}{\partial \beta_1^2} f(t;\mathbf{z})dt\,, \tag{5.19}$$

$$\vdots$$

$$E\left[-\frac{\partial^2 l}{\partial \beta_k^2}\right] = \int_0^{\tau} -\frac{\partial^2 l}{\partial \beta_k^2} f(t;\mathbf{z})dt\,, \tag{5.20}$$

$$E\left[-\frac{\partial^2 l}{\partial \gamma_1^2}\right] = \int_0^{\tau} -\frac{\partial^2 l}{\partial \gamma_1^2} f(t;\mathbf{z})dt\,, \tag{5.21}$$

$$E\left[-\frac{\partial^2 l}{\partial \gamma_2^2}\right] = \int_0^{\tau} -\frac{\partial^2 l}{\partial \gamma_2^2} f(t;\mathbf{z})dt\,, \tag{5.22}$$

$$E\left[-\frac{\partial^2 l}{\partial \gamma_1 \partial \gamma_2}\right] = \int_0^{\tau} -\frac{\partial^2 l}{\partial \gamma_1 \partial \gamma_2} f(t;\mathbf{z})dt\,, \tag{5.23}$$

where probability density function is give by Eq. (5.7).

With layout of the ALT plan shown in Figure 5.1, the Fisher information matrix for the stress combination $\boldsymbol{z}_L$ is

$$F_{L\cdots L}=\begin{bmatrix} E\left[-\frac{\partial^2 l(\mathbf{z}_L)}{\partial\beta_1^2}\right] & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & 0 & \vdots & \vdots \\ 0 & 0 & E\left[-\frac{\partial^2 l(\mathbf{z}_L)}{\partial\beta_k^2}\right] & 0 & 0 \\ 0 & 0 & 0 & E\left[-\frac{\partial^2 l(\mathbf{z}_L)}{\partial\gamma_1^2}\right] & E\left[-\frac{\partial^2 l(\mathbf{z}_L)}{\partial\gamma_1\partial\gamma_2}\right] \\ 0 & 0 & 0 & E\left[-\frac{\partial^2 l(\mathbf{z}_L)}{\partial\gamma_1\partial\gamma_2}\right] & E\left[-\frac{\partial^2 l(\mathbf{z}_L)}{\partial\gamma_2^2}\right] \end{bmatrix}. \tag{5.24}$$

Similarly the Fisher information matrices for other stress combinations can be easily obtained.

Finally, the Fisher information matrix for all test units allocated in Figure 5.1 is obtained as

$$F=\sum_{i_1=0}^{2}\cdots\sum_{i_k}^{2} np_{i_1\cdots i_k}F_{i_1\cdots i_k}\ . \tag{5.25}$$

which is a function of the model parameters $(\beta_1,\beta_2,\ldots,\beta_k,\gamma_1,\gamma_2)$, stress levels ($z_{1L}$, z_{1M}, z_{1H}), (z_{2L}, z_{2M}, z_{2H}), …, (z_{kL}, z_{kM}, z_{kH}) and the proportions of test units allocated to the 3^k stress combinations $p_{i_1i_2\ldots i_k}$, $i_j=0,1,2;\ j=1,2,\ldots,k$. The asymptotic variance-covariance matrix Σ of the ML estimates $(\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_k, \hat{\gamma}_1, \hat{\gamma}_2)$ is the inverse of the Fisher information matrix F

$$\Sigma = \begin{bmatrix} Var(\hat{\beta}_1) & 0 & \cdots & 0 & 0 \\ 0 & \ddots & 0 & \vdots & \vdots \\ \vdots & 0 & Var(\hat{\beta}_k) & 0 & 0 \\ 0 & \cdots & 0 & Var(\hat{\gamma}_1) & Cov(\hat{\gamma}_1, \hat{\gamma}_2) \\ 0 & \cdots & 0 & Cov(\hat{\gamma}_1, \hat{\gamma}_2) & Var(\hat{\gamma}_2) \end{bmatrix} = F^{-1}, \tag{5.26}$$

which is a $(k+2)\times(k+2)$ symmetric matrix.

Under the standard regularity conditions, the ML estimates $(\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_k, \hat{\gamma}_1, \hat{\gamma}_2)$, which are calculated based on the values of failure time sample, are asymptotically normally distributed with mean $(\beta_1, \beta_2, \ldots, \beta_k, \gamma_1, \gamma_2)$ and variance-covariance matrix Σ. The asymptotic covariance matrix Σ depends on the inherent regression model, the layout of the ALT design and the parameters. Thus, for the POM-based ALT, if we have the initial baseline estimates for $(\beta_1, \beta_2, \ldots, \beta_k, \gamma_1, \gamma_2)$, it is straightforward to estimate the variance-covariance matrix Σ numerically based on the initial estimates.

5.1.4 Optimization Problem Formulation

The appropriate optimization criterion for ALT plans depends on the purpose of the accelerated life testing. As presented earlier, the possible criteria include the minimizations of:

(1) the variance of estimate of a percentile of the failure time distribution at the design stress conditions,

(2) the variance of reliability estimate or hazard rate estimate at design stress conditions over a pre-specified period of time,

(3) the variance of an reliability estimate or hazard rate estimate over a range of stress,

(4) the variance of the estimate of a particular parameter.

In this section, the optimization criterion is chosen to minimize the asymptotic variance of the reliability function estimate over a pre-specified period of time T at the design stress conditions, ie, to minimize $\int_0^T Var[\hat{R}(t;\mathbf{z}_D)]dt$.

The variance of the reliability function estimate at the design stress levels $Var[\hat{R}(t\,|\,\mathbf{z}_D)]$ is obtained by Delta method as

$$
\begin{aligned}
&Var[\hat{R}(t;\mathbf{z}_D)] \\
&=\begin{bmatrix} \dfrac{\partial R(t;\mathbf{z}_D)}{\partial \beta_1} & \cdots & \dfrac{\partial R(t;\mathbf{z}_D)}{\partial \beta_k} & \dfrac{\partial R(t;\mathbf{z}_D)}{\partial \gamma_1} & \dfrac{\partial R(t;\mathbf{z}_D)}{\partial \gamma_2} \end{bmatrix}\Sigma \\
&\times\begin{bmatrix} \dfrac{\partial R(t;\mathbf{z}_D)}{\partial \beta_1} & \cdots & \dfrac{\partial R(t;\mathbf{z}_D)}{\partial \beta_k} & \dfrac{\partial R(t;\mathbf{z}_D)}{\partial \gamma_1} & \dfrac{\partial R(t;\mathbf{z}_D)}{\partial \gamma_2} \end{bmatrix}^T \Bigg|_{(\beta_1,\beta_2,\ldots,\beta_k,\gamma_1,\gamma_2)=(\hat{\beta}_1,\hat{\beta}_2,\ldots,\hat{\beta}_k,\hat{\gamma}_1,\hat{\gamma}_2)}
\end{aligned}
\tag{5.27}
$$

With the limitations of available test units n, test time τ and specification of minimum failures at each stress combination, the objective of the ALT plan is to optimally allocate the stress levels (z_{1L}, z_{1M}, z_{1H}), (z_{2L}, z_{2M}, z_{2H}), …, (z_{kL}, z_{kM}, z_{kH}) and the proportions of test units allocated to the 3^k stress combinations $p_{i_1 i_2 \ldots i_k}$, $i_j = 0, 1, 2;\ j = 1, 2, \ldots, k$, so that the asymptotic variance of the reliability estimate at design stress levels is minimized over a pre-specified period of time T. The optimum decision variables [(z_{1L}, z_{1M}, z_{1H}), (z_{2L}, z_{2M}, z_{2H}), …, (z_{kL}, z_{kM}, z_{kH}), $p_{i_1 i_2 \ldots i_k}$, $i_j = 0, 1, 2;\ j = 1, 2, \ldots, k$] are determined by solving the following nonlinear optimization problem

Objective function

Min $$f(\mathbf{x}) = \int_0^T Var[\hat{R}(t; \mathbf{z}_D)]dt$$

Subject to

$$0 < p_{i_1 i_2 \ldots i_k} < 1, \qquad i_j = 0, 1, 2;\ j = 1, 2, \ldots, k$$

$$\sum p_{i_1 i_2 \ldots i_k} = 1$$

$$z_{1D} \le z_{1L} \le z_{1M} \le z_{1H} \le z_{1upper}$$

$$\vdots$$

$$z_{kD} \le z_{kL} \le z_{kM} \le z_{kH} \le z_{kupper}$$

$$np_{i_1 i_2 \ldots i_k}[1 - R(\tau \mid z_{1i_1}, z_{2i_2}, \ldots, z_{2i_k})] \ge MNF, \qquad i_1, i_2, \ldots i_k = 0, 1, 2$$

$$\Sigma = F^{-1}$$

where MNF is the required minimum number of failures.

5.1.5 Optimization Algorithm

The nonlinear optimization problem formulated in the above section is an optimization problem with nonlinear objective function, nonlinear constraints, and multivariable decision variables. Since the objective function is complicated, we use a version of the multivariable constrained search methods, where no derivatives are required, to solve this optimization problem.

The procedure is based on COBYLA (Constrained Optimization BY Linear Approximations) optimization method proposed by Powell (1992). This method is a sequential search technique which has proven effective in solving problems with nonlinear objective function subject to nonlinear constraints as well as linear and boundary constraints.

Without loss of generality, we use the following constrained multivariable nonlinear problem formulation to illustrate the COBYLA algorithm:

$$\left.\begin{array}{lll} \text{Min} & f(\boldsymbol{x}), & x \in R^n \\ \text{Subject to} & c_i(\boldsymbol{x}) \geq 0, & i = 1,2,\ldots,m \end{array}\right\}, \tag{5.28}$$

The COBYLA algorithm is based on the Nelder and Mead's method (Nelder and Mead 1965) and the idea of generating the next vector of variables from function values at the vertices $\{\boldsymbol{x}^j : j = 0,1,\ldots,n\}$ of a non-degenerate simplex in R^n. In this case there are unique linear functions, $\hat{f}$ and $\{\hat{c}_i : i = 1,2,\ldots,m\}$, that interpolate the nonlinear objective function f and nonlinear constraints $\{c_i : i = 1, 2, \ldots, m\}$ at the vertices, and we approximate the calculation (5.28) by the linear programming problem

$$\left.\begin{array}{lll} \text{Min} & \hat{f}(\boldsymbol{x}), & x \in R^n \\ \text{Subject to} & \hat{c}_i(\boldsymbol{x}) \geq 0, & i = 1, 2, \ldots, m \end{array}\right\} \tag{5.29}$$

Changes to the variables are restricted by a trust region bound, which gives the user some control over the steps that are taken automatically and which respond satisfactorily to the fact that there may be no finite solution to the linear programming problem (5.29). The trust region radius ρ remains constant until predicted improvements to the objective function and feasibility conditions fail to occur. Then the trust region radius is reduced until it reaches a final value that has to be set by the user. The COBYLA algorithm employs a merit function of the form

$$\Phi(\boldsymbol{x}) = f(\boldsymbol{x}) + \mu \max\{\max\{-c_i(\boldsymbol{x}) : i = 1, 2, \ldots, m\}, 0\} \tag{5.30}$$

in order to compare the goodness of two different vectors of variables. Here μ is a parameter that is adjusted automatically. So we have $\Phi(\boldsymbol{x}) = f(\boldsymbol{x})$ whenever $\boldsymbol{x}$ is feasible, and $\boldsymbol{x} \in R^n$ is better than $\boldsymbol{y} \in R^n$ if and only if the inequality $\Phi(\boldsymbol{x}) < \Phi(\boldsymbol{y})$ holds.

The algorithm includes several strategies, and is summarized with the aid of Figure 5.2.

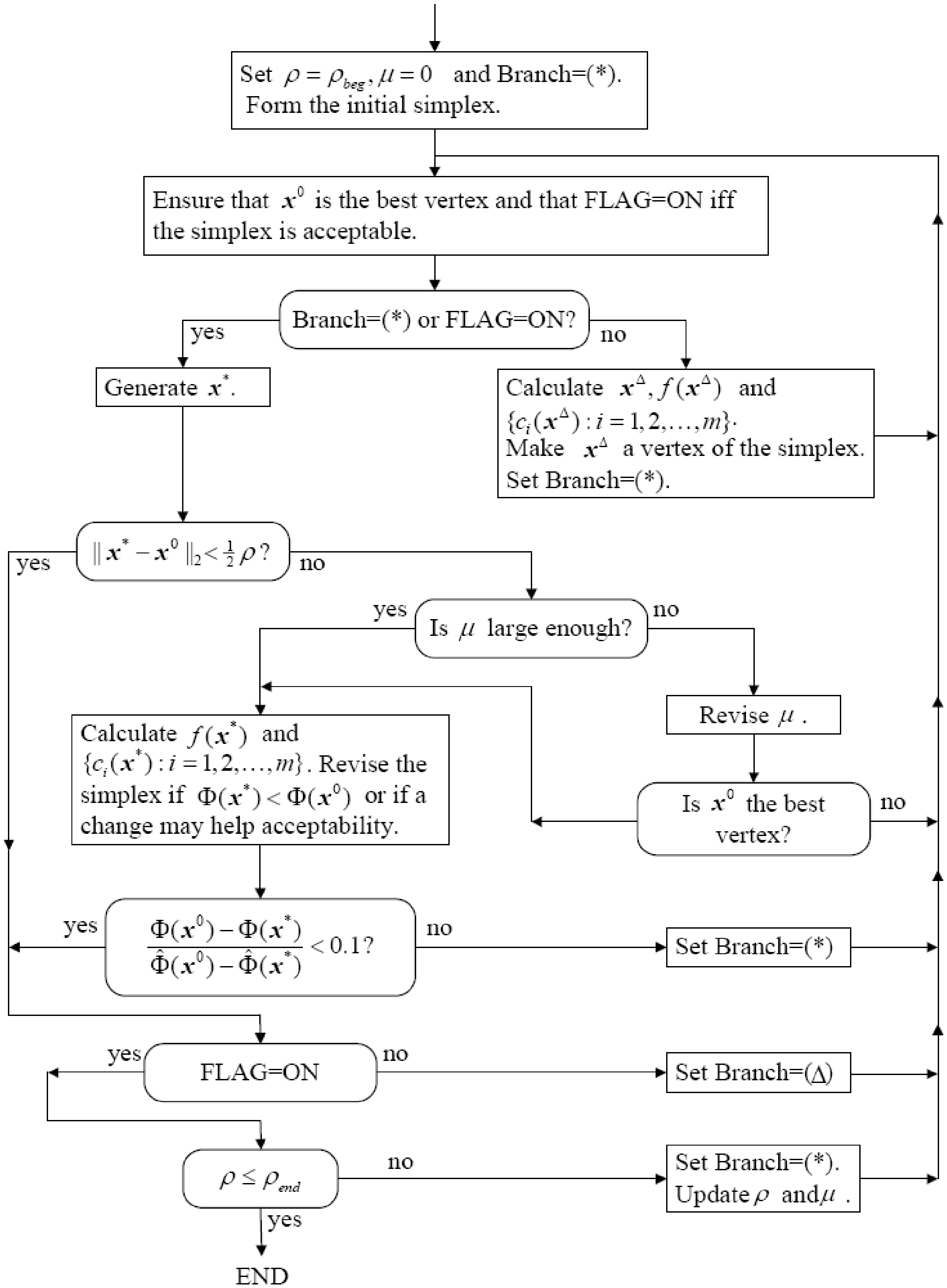

Figure 5.2 A summary of the COBYLA algorithm

Firstly we consider the "Generate $\boldsymbol{x}^*$". The vector of variables $\boldsymbol{x}^*$ is generated by solving the linear programming problem (5.29). If the resultant $\boldsymbol{x}^*$ is not in the trust region, we redefine $\boldsymbol{x}^*$ by minimizing the greatest of the constraint violations $\{-\hat{c}_i(\boldsymbol{x}^*) : i = 1, 2, \ldots, m\}$ subject to the trust region bound.

The branch Δ ensures that the simplex is acceptable. The definition of "acceptability" follows: For $j = 1, 2, \ldots, n$, let σ^j be the Euclidean distance from the vertex $\boldsymbol{x}^j$ to the opposite face of the current simplex, and let η^j be the length of the edge between $\boldsymbol{x}^j$ and $\boldsymbol{x}^0$. We say that the simplex is "acceptable" if and only if the inequalities

$$\left.\begin{array}{l} \sigma^j \ge \alpha\rho \\ \eta^j \le \beta\rho \end{array}\right\}, \quad j = 1, 2, \ldots, n, \tag{5.31}$$

hold, where α and β are constants that satisfy the conditions $0 < \alpha < 1 < \beta$.

The vector $\boldsymbol{x}^\Delta$ is defined as follows. If any of the numbers $\{\eta^j : j = 1, 2, \ldots, n\}$ of Eq. (5.31) is greater than $\beta\rho$, we let l be the least integer from $[1, n]$ that satisfies the equation

$$\eta^l = \max\{\eta^j : j = 1, 2, \ldots, n\} \ . \tag{5.32}$$

Otherwise we obtain l from the formula

$$\sigma^{l} = \min\{\sigma^{j} : j = 1, 2, \ldots, n\} . \tag{5.33}$$

The iteration replaces the vertex $\boldsymbol{x}^{l}$ by $\boldsymbol{x}^{\Delta}$, so we require $\boldsymbol{x}^{\Delta}$ to be well away from the face of the simplex that is opposite the vertex $\boldsymbol{x}^{l}$. Therefore we let $\boldsymbol{v}^{l}$ be the vector of unit length that is perpendicular to this face, and we define the vector $\boldsymbol{x}^{\Delta}$ by

$$\boldsymbol{x}^{\Delta} = \boldsymbol{x}^{0} \pm \gamma\rho\boldsymbol{v}^{l} , \tag{5.34}$$

where the sign is chosen to minimize the approximation $\hat{\Phi}(\boldsymbol{x}^{\Delta})$ to the new value of the merit function, and γ is a constant from the interval $(\alpha, 1)$. Then the next iteration is given the simplex that has the vertices $\{\boldsymbol{x}^{j} : j = 0, 1, \ldots, n, j \neq l\}$ and $\boldsymbol{x}^{\Delta}$.

5.1.6 Numerical Example

An accelerated life test is to be carried out at three temperature levels and three voltage levels for MOS devices in order to estimate its reliability function at design temperature level of $25\,^{\circ}C$ and voltage level of 5V. The layout of the test is shown as in Figure 5.1 where there are totally 9 stress combinations. The test needs to be completed in 300 hours. The total number of units available for testing is 200. To avoid the inducing of failure modes different from that expected at the design stress levels, it has been determined, through engineering judgment, that the temperature level should not exceed

$250\,°C$ and the voltage level should not exceed 10V. The minimum number of failures for each stress combination is specified as 15 units. Furthermore, we expect that the accelerated life testing provide the most accurate reliability estimate over a 10-year period of time. The test plan is designed through the following steps:

1. According to the Arrehenius model, we transform the Celsius temperature scale to 1/Kelvin scale as the covariate z_1 in the test plan. Then the design stress level of temperature z_{1D} is 1/(25+276.13) or 0.00335 K^{-1}, and the upper bound of stress level of temperature z_{1upper} is 1/(250+276.13) or 0.0019 K^{-1}.

2. A baseline experiment is conducted to obtain a set of initial values of the parameters for the PO model with quadratic odds function. These values are:

$$\beta_1 = -1600$$

$$\beta_2 = 0.02$$

$$\gamma_1 = 2$$

$$\gamma_2 = 0$$

3. Under the constraints of available test units and test time, the objective of the test plan is to optimally allocate the stress levels and test units so that the accelerated life test provides the most accurate reliability estimate of the product at design stress conditions, i.e. to ensure that the asymptotic variance of the reliability function

estimate at design conditions is minimized over a pre-specified time period of 10-year. The decision variables are the stress levels (z_{1L}, z_{1M}, z_{1H}, z_{2L}, z_{2M}, and z_{2H}) and the proportion of test units (p_{ij}, $i = 0,1,2;\ j = 0,1,2,$) at each stress level combination as shown in Figure 5.1. The values of the optimum decision variables are determined by solving the following nonlinear optimization problem with nonlinear constraints as well as linear and boundary constraints:

Objective function

Min $\quad f(\boldsymbol{x}) = \int_0^T Var[\hat{R}(t \mid \boldsymbol{z}_D)]dt$

Subject to

$$0 < p_{ij} < 1, \qquad i_j = 0, 1, 2;\ j = 1, 2, 3$$

$$\sum_{i,j} p_{ij} = 1,$$

$$25°C \le 1/z_{1L} - 273.16 \le 1/z_{1M} - 273.16 \le 1/z_{1H} - 273.16 \le 250°C,$$

$$5 \le z_{2L} \le z_{2M} \le z_{2H} \le 10,$$

$$np_{ij}[1 - R(\tau \mid z_{1i}, z_{2j})] \ge MNF, \qquad i, j = 0, 1, 2$$

$$\Sigma = F^{-1}$$

where $\quad MNF = 15$,

$T = 10$ years,

$n = 200$,

$\tau = 300$ hours,

$\boldsymbol{x} = (z_{1L}, z_{1M}, z_{1H}, z_{2L}, z_{2M}, z_{2H}, p_{00}, p_{01}, p_{02}, p_{10}, p_{11}, p_{12}, p_{20}, p_{21}, p_{22})^t$.

4. We use the algorithm as described in section 5.2.5 to solve this optimization problem. The algorithm is implemented using SAS/IML and SAS/OR. SAS/IML software is a powerful and flexible programming language (**I**nteractive **M**atrix **L**anguage) in a dynamic, interactive environment. The fundamental object of the language is a data matrix. The programming is dynamic because necessary activities such as memory allocation and dimensioning of matrices are performed automatically.

5. The optimum decision variables that minimize the objective function and meet the requirement of the constraints are:

   Stress levels:

   $$T_L = 76°C\ ,\ \ T_M = 165°C\ ,\ T_H = 244°C\ ;$$

   $$V_L = 5.35V\ ,\ \ V_M = 7.39V\ ,\ V_H = 9.25V\ .$$

   The corresponding proportions of units allocated to each stress level combination are are list in the following table:

Table 5.1 Proportions of units allocated to each stress level combination

p_{ij}	$T_L = 76°C$	$T_M = 165°C$	$T_H = 244°C$
$V_L = 5.35V$	0.247	0.141	0.085
$V_M = 7.39V$	0.128	0.089	0.077
$V_H = 9.25V$	0.080	0.077	0.076

The objective function value is 0.0975.

5.2 PO-based Multiple-Stress-Type ALT Plans with Simple Step-Stress Loading

Accelerated life testing (ALT) procedures are commonly used to evaluate the lifetime characteristics of highly reliable components. If a constant stress ALT is used and some selected stress levels are not high enough, there are many survived units by the limited testing period, thus reducing the effectiveness of ALT. To ensure enough failed units in a limited testing period, step-stress accelerated life testing (SSALT) has been developed.

Most of the previous work on designing ALT plans is focused on the application of a single stress type. Especially no previous work on step-stress ALT plans has investigated the use of multiple stress types. Alhadeed and Yang (2005) design optimal simple step-stress plan for cumulative exposure model with consideration of single stress type. Only optimal time of changing stress level using log-normal distribution is determined in their paper. Elsayed and Zhang (2005) propose optimum simple step-stress ALT plans based on nonparametric proportional hazards model. Alhadeed and Yang (2002) also design optimal simple step-stress plan for Khamis-Higgins model using Weibull distribution. Teng and Yeo (2002) present a Transformed Least-Squares (TLS) approach to drive optimum ALT plans. Khamis and Higgins (1996) design the optimum step-stress test using the exponential distribution. Bai *et al.* (1989) discuss an optimal plan for a simple step-stress ALT with censoring for exponential life data. Miller and Nelson (1983) obtain the optimum simple step-stress ALT plans where the test units have exponentially distributed life. All of the previous work has only considered a single stress type application. However, the mission time of today's products is extended so much that it becomes more difficult to obtain significant amount of failure data within reasonable amount of time using single stress type for simple step-stress test. Multiple-stress-type ALTs have been employed as a means of overcoming such difficulties. For instance, Kobayashi *et al.* (1978), Minford (1982), Mogilevsky and Shirn (1988), and Munikoti and Dhar (1988) use two stresses to test capacitors, and Weis *et al.* (1988) employ two stresses to estimate the lifetime of silicon photodetectors.

In this section, we propose multiple-stress-type ALT plans based on the proportional odds (PO) model with simple step-stress loading. We do not make assumptions about a common life distribution of the test units. The cumulative exposure model is used to derive the life distribution of test units after the stress level changing time. The plans are optimized such that the asymptotic variance of reliability prediction at design stress over a specified period of time is minimized. We present the proportional odds model and corresponding maximum likelihood estimation for accelerated life testing. We introduce a simple step-stress test based on PO model with multiple stress types and explain how the cumulative exposure model is applied to this case. A nonlinear optimization problem is formulated to design the optimum ALT plans in this section. Asymptotic variance of the reliability prediction at the design stress levels has been chosen as the objective function of the nonlinear optimization problem. This optimization problem could be solved by direct search algorithm, such as the COBYLA algorithm proposed by Powell (1992).

Step-stress loading of accelerated life testing is described as follows. A set of samples is subjected to test starting with specified low stresses. If we don't have enough failures by a specified time, the stress levels are increased and held constant for another period of time. We repeat this procedure until enough failures achieved. The step-stress ALT usually results in enough failures in a shorter period of time than the constant-stress ALT does.

Now we consider POM-based multiple-stress-type ALT plans with simple step-stress loading. In the testing scheme, test units are initially placed on test at a low stress level vector $\mathbf{z}_L$, where these low stress levels satisfy $\mathbf{z}_L \geq \mathbf{z}_D$, and run until stress level changing time τ_1. Then the stress levels are increased to the high level vector $\mathbf{z}_H$ and held constant until a predetermined censoring time τ_2.

5.2.1 Test Procedure

1. There are k types of stresses, z_1, z_2, … z_k, applied to the test units.
2. There are n test units initially placed under test at the low stress levels $\mathbf{z}_L = (z_{1L}, z_{2L}, \ldots, z_{kL})^t$ until changing time τ_1. Then surviving units at time τ_1 are subjected to higher stress levels $\mathbf{z}_H = (z_{1H}, z_{2H}, \ldots, z_{kH})^t$ until a pre-determined censoring time τ_2. Figure 5.3 depicts the test procedure for the case of $k = 2$. As shown in Figure 5.3. We start the test with all low stress levels and increase all the stress levels simultaneously at the stress changing time τ_1. As an alternative we

could increase the different stress levels following some arbitrary orders. How to determine the optimum order is beyond the coverage of this dissertation and will be investigated as a topic of *Equivalent ALT Plans* in the future work as mentioned in Chapter 6. In this dissertation we only consider increasing all the stress levels simultaneously since it is easy to carry out.

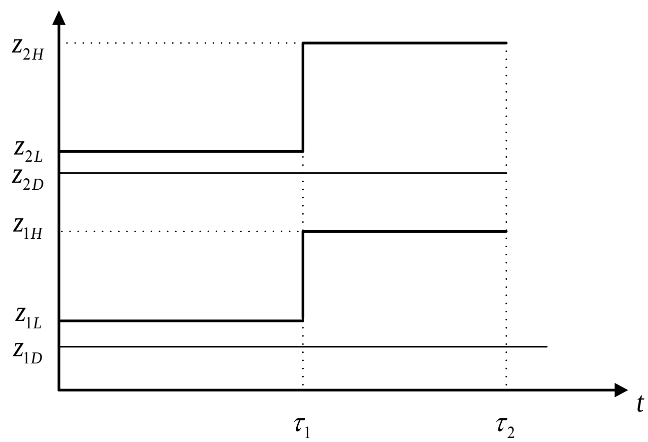

Figure 5.3 Step-Stress loading with multiple-stress-type

3. There are n_i failure times observed while testing at the low stress levels and high stress levels respectively, $i = L, H$, and $n_L + n_H \le n$.
4. The objective of the step-stress ALT (SSALT) plan with multiple stress types is to determine the optimum values of the stress levels $\mathbf{z}_L$ and $\mathbf{z}_H$, as well as the optimum changing time τ_1 so as to minimize the estimation error of the reliability estimates at the design stress levels $\mathbf{z}_D$.

5.2.2 Assumptions

1. The proportional odds (PO) model is assumed to fit the failure time data:

$$\theta(t;\mathbf{z}) = \exp(\boldsymbol{\beta}^t \mathbf{z})\theta_0(t)\,, \tag{5.35}$$

   where $\mathbf{z} = (z_1, z_2, \ldots, z_k)^t$, which is a column vector of stress levels; and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_k)^t$, which is a column vector of model parameters.

2. The baseline odds function $\theta_0(t)$ is quadratic and given by

$$\theta_0(t) = \gamma_1 t + \gamma_2 t^2\,, \tag{5.36}$$

   where γ_1 and γ_2 are unknown parameters, there is no intercept term because the odds function always crosses the origin.

3. The failure times of the test units are statistically independent.

5.2.3 Cumulative Exposure Models

To analyze the failure time data from a step stress test, we need to relate the life distribution under step-stresses to the distribution under constant stresses. We adopt the most widely used cumulative exposure model to derive the cumulative density function of the failure time for a test unit experiencing step-stress loading since the odds function changes immediately after τ_1, the stress level changing time under step-stress loading.

The cumulative exposure models assume that the remaining life of a test unit depends only on the "exposure" it has seen, and the unit does not remember how the exposure was accumulated. Figure 5.4 shows the relationship between constant-stress and step-stress distributions.

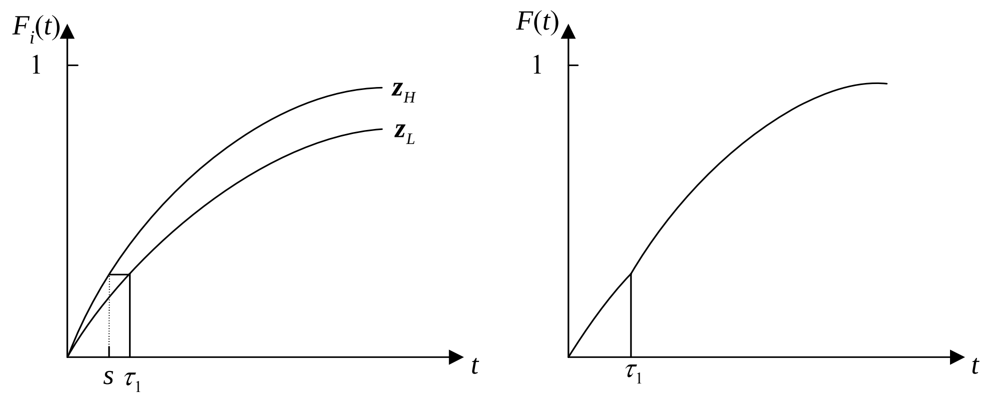

Figure 5.4 Relationship between constant-stress and step-stress distributions

Let $F_i(t)$ denote CDF of time to failure for units run at constant stress $\mathbf{z}_i$, $i = L, H$. Following proportional odds model, they are given by:

$$F_L(t) = 1 - \exp[-\ln\{\theta_0(t)e^{\beta' \mathbf{z}_L} + 1\}], \tag{5.37}$$

$$F_H(t) = 1 - \exp[-\ln\{\theta_0(t)e^{\beta' \mathbf{z}_H} + 1\}]. \tag{5.38}$$

The test runs under stress $\mathbf{z}_L$ to time τ_1 at step 1. The population CDF of units failed by time t in step 1 is:

$$F(t) = F_L(t) \quad t < \tau_1 \tag{5.39}$$

Test units at step 2 have an equivalent starting time s, which would have produced the same population cumulative failures. Thus, s is the solution to the following equation

$$F_H(s) = F_L(\tau_1), \tag{5.40}$$

or equivalently

$$1 - \exp[-\ln\{(\gamma_1 s + \gamma_2 s^2)e^{\boldsymbol{\beta}'\mathbf{z}_H} + 1\}] = 1 - \exp[-\ln\{(\gamma_1 \tau_1 + \gamma_2 \tau_1^2)e^{\boldsymbol{\beta}'\mathbf{z}_L} + 1\}], \tag{5.41}$$

The above equation is obtained by substituting Eq. (5.37) and Eq. (5.38) into Eq. (5.40).

Therefore the population CDF of units failing by time $t \geq \tau_1$ is

$$F(t) = F_H(t - \tau_1 + s), \qquad t \geq \tau_1 \ . \tag{5.42}$$

In summary, the equivalent population cumulative distribution function of units failing under multiple step-stresses is

$$F(t)=\begin{cases} F_L(t), & t<\tau_1 \\ F_H(t-\tau_1+s), & t\geq\tau_1 \end{cases} \tag{5.43}$$

5.2.4 Log Likelihood Function

A test unit experiences one of two possible types of failure patterns: (a) it either fails under stress level $\mathbf{z}_L$ before the stress is changed at time τ_1, or (b) it does not fail by time τ_1 and continues to run either to failure or to censoring time τ_2 at stress level $\mathbf{z}_H$. The following provides the log likelihood of a single observation *t* (time to failure). Firstly we define the indicator function $I_1 = I_1(t\leq\tau_1)$ in terms of the stress changing time τ_1 by:

$$I_1 = I_1(t\leq\tau_1)=\begin{cases} 1 & \text{if } t\leq\tau_1,\ \text{failure observed before time } \tau_1, \\ 0 & \text{if } t>\tau_1,\ \text{failure observed after time } \tau_1. \end{cases}$$

and the indicator function $I_2 = I_2(t\leq\tau_2)$ in terms of the censoring time τ_2 by:

$$I_2 = I_2(t\leq\tau_2)=\begin{cases} 1 & \text{if } t\leq\tau_2,\ \text{failure observed before time } \tau_2, \\ 0 & \text{if } t>\tau_2,\ \text{censored at time } \tau_2. \end{cases}$$

where $\tau_1\leq\tau_2$.

Following proportional odds model and Eq. (5.43), we have

$$f(t;\mathbf{z}_L) = \frac{(\gamma_1 + 2\gamma_2 t)e^{\boldsymbol{\beta}^t \mathbf{z}_L}}{[(\gamma_1 t + \gamma_2 t^2)e^{\boldsymbol{\beta}^t \mathbf{z}_L} + 1]^2}, \qquad \text{for } t < \tau_1$$

$$f(t;\mathbf{z}_H) = \frac{(\gamma_1 + 2\gamma_2 t')e^{\boldsymbol{\beta}^t \mathbf{z}_H}}{[(\gamma_1 t' + \gamma_2 t'^2)e^{\boldsymbol{\beta}^t \mathbf{z}_H} + 1]^2}, \qquad \text{for } t \geq \tau_1$$

$$\Lambda(t;\mathbf{z}_L) = \ln[(\gamma_1 t + \gamma_2 t^2)e^{\boldsymbol{\beta}^t \mathbf{z}_L} + 1], \qquad \text{for } t < \tau_1$$

$$\Lambda(t;\mathbf{z}_H) = \ln[(\gamma_1 t' + \gamma_2 t'^2)e^{\boldsymbol{\beta}^t \mathbf{z}_H} + 1], \qquad \text{for } t \geq \tau_1$$

where $t' = t - \tau_1 + s$.

Therefore the log likelihood of one single observation *t* can be expressed as

$$\begin{aligned} l = \ln L(t;\mathbf{z}_L,\mathbf{z}_H) &= I_2\{I_1[\ln f(t;\mathbf{z}_L)] + (1-I_1)[\ln f(t;\mathbf{z}_H)]\} - (1-I_2)\Lambda(t;\mathbf{z}_H) \\ &= I_1 I_2\{\ln(\gamma_1 + 2\gamma_2 t) + \boldsymbol{\beta}^t \mathbf{z}_L - 2\ln[(\gamma_1 t + \gamma_2 t^2)e^{\boldsymbol{\beta}^t \mathbf{z}_L} + 1]\} \\ &\quad + (1-I_1)I_2\{\ln(\gamma_1 + 2\gamma_2 t') + \boldsymbol{\beta}^t \mathbf{z}_H - 2\ln[(\gamma_1 t' + \gamma_2 t'^2)e^{\boldsymbol{\beta}^t \mathbf{z}_H} + 1]\} \\ &\quad - (1-I_2)\ln[(\gamma_1 t' + \gamma_2 t'^2)e^{\boldsymbol{\beta}^t \mathbf{z}_H} + 1] \end{aligned} \tag{5.44}$$

The first partial derivatives with respect to the model parameters are:

$$\frac{\partial \ln L}{\partial \gamma_1} = \frac{I_1 I_2}{\gamma_1 + 2\gamma_2 t} - \frac{2I_1 I_2 t e^{\boldsymbol{\beta}^t \mathbf{z}_L}}{(\gamma_1 t + \gamma_2 t^2) e^{\boldsymbol{\beta}^t \mathbf{z}_L} + 1} + \frac{(1-I_1) I_2}{\gamma_1 + 2\gamma_2 t'} - \frac{2(1-I_1) I_2 t' e^{\boldsymbol{\beta}^t \mathbf{z}_H}}{(\gamma_1 t' + \gamma_2 t'^2) e^{\boldsymbol{\beta}^t \mathbf{z}_H} + 1}$$
$$- \frac{(1-I_2) t' e^{\boldsymbol{\beta}^t \mathbf{z}_H}}{(\gamma_1 t' + \gamma_2 t'^2) e^{\boldsymbol{\beta}^t \mathbf{z}_H} + 1} \tag{5.45}$$

$$\frac{\partial \ln L}{\partial \gamma_2} = \frac{2I_1 I_2 t}{\gamma_1 + 2\gamma_2 t} - \frac{2I_1 I_2 t^2 e^{\boldsymbol{\beta}^t \mathbf{z}_L}}{(\gamma_1 t + \gamma_2 t^2) e^{\boldsymbol{\beta}^t \mathbf{z}_L} + 1} + \frac{2(1-I_1) I_2 t'}{\gamma_1 + 2\gamma_2 t'} - \frac{2(1-I_1) I_2 t'^2 e^{\boldsymbol{\beta}^t \mathbf{z}_H}}{(\gamma_1 t' + \gamma_2 t'^2) e^{\boldsymbol{\beta}^t \mathbf{z}_H} + 1}$$
$$- \frac{(1-I_2) t'^2 e^{\boldsymbol{\beta}^t \mathbf{z}_H}}{(\gamma_1 t' + \gamma_2 t'^2) e^{\boldsymbol{\beta}^t \mathbf{z}_H} + 1} \tag{5.46}$$

$$\frac{\partial \ln L}{\partial \boldsymbol{\beta}^t} = I_1 I_2 \mathbf{z}_L - \frac{2I_1 I_2 \mathbf{z}_L (\gamma_1 t + \gamma_2 t^2) e^{\boldsymbol{\beta}^t \mathbf{z}_L}}{(\gamma_1 t + \gamma_2 t^2) e^{\boldsymbol{\beta}^t \mathbf{z}_L} + 1} + (1-I_1) I_2 \mathbf{z}_H - \frac{2(1-I_1) I_2 \mathbf{z}_H (\gamma_1 t' + \gamma_2 t'^2) e^{\boldsymbol{\beta}^t \mathbf{z}_H}}{(\gamma_1 t' + \gamma_2 t'^2) e^{\boldsymbol{\beta}^t \mathbf{z}_H} + 1}$$
$$- \frac{(1-I_2) \mathbf{z}_H (\gamma_1 t' + \gamma_2 t'^2) e^{\boldsymbol{\beta}^t \mathbf{z}_H}}{(\gamma_1 t' + \gamma_2 t'^2) e^{\boldsymbol{\beta}^t \mathbf{z}_H} + 1} \tag{5.47}$$

The last equation could be decomposed into

$$\frac{\partial \ln L}{\partial \beta_1} = I_1 I_2 z_{L1} - \frac{2I_1 I_2 z_{1L} (\gamma_1 t + \gamma_2 t^2) e^{\boldsymbol{\beta}^t \mathbf{z}_L}}{(\gamma_1 t + \gamma_2 t^2) e^{\boldsymbol{\beta}^t \mathbf{z}_L} + 1} + (1-I_1) I_2 \mathbf{z}_{1H} - \frac{2(1-I_1) I_2 z_{1H} (\gamma_1 t' + \gamma_2 t'^2) e^{\boldsymbol{\beta}^t \mathbf{z}_H}}{(\gamma_1 t' + \gamma_2 t'^2) e^{\boldsymbol{\beta}^t \mathbf{z}_H} + 1}$$
$$- \frac{(1-I_2) z_{1H} (\gamma_1 t' + \gamma_2 t'^2) e^{\boldsymbol{\beta}^t \mathbf{z}_H}}{(\gamma_1 t' + \gamma_2 t'^2) e^{\boldsymbol{\beta}^t \mathbf{z}_H} + 1}$$

$$\vdots$$

$$\frac{\partial \ln L}{\partial \beta_k} = I_1 I_2 z_{Lp} - \frac{2I_1 I_2 z_{kL}(\gamma_1 t + \gamma_2 t^2)e^{\boldsymbol{\beta}^t \mathbf{z}_L}}{(\gamma_1 t + \gamma_2 t^2)e^{\boldsymbol{\beta}^t \mathbf{z}_L} + 1} + (1 - I_1) I_2 \mathbf{z}_{kH} - \frac{2(1 - I_1) I_2 z_{kH}(\gamma_1 t' + \gamma_2 t'^2)e^{\boldsymbol{\beta}^t \mathbf{z}_H}}{(\gamma_1 t' + \gamma_2 t'^2)e^{\boldsymbol{\beta}^t \mathbf{z}_H} + 1}$$
$$- \frac{(1 - I_2) z_{kH}(\gamma_1 t' + \gamma_2 t'^2)e^{\boldsymbol{\beta}^t \mathbf{z}_H}}{(\gamma_1 t' + \gamma_2 t'^2)e^{\boldsymbol{\beta}^t \mathbf{z}_H} + 1}$$

The above equations, when summed over all test units and set equal to zero, provide the maximum likelihood estimates for the model parameters.

5.2.5 *Fisher's Information Matrix and Variance-Covariance Matrix*

If we only consider the correlations among γ_1 and γ_2, the second partial derivatives with respect to the model parameters are as shown as:

$$\frac{\partial^2 \ln L}{\partial \gamma_1^2} = -\frac{I_1 I_2}{(\gamma_1 + 2\gamma_2 t)^2} + \frac{2I_1 I_2 t^2 e^{2\boldsymbol{\beta}^t \mathbf{z}_L}}{[(\gamma_1 t + \gamma_2 t^2)e^{\boldsymbol{\beta}^t \mathbf{z}_L} + 1]^2} - \frac{(1 - I_1) I_2}{(\gamma_1 + 2\gamma_2 t')^2}$$
$$+ \frac{2(1 - I_1) I_2 t'^2 e^{2\boldsymbol{\beta}^t \mathbf{z}_H}}{[(\gamma_1 t' + \gamma_2 t'^2)e^{\boldsymbol{\beta}^t \mathbf{z}_H} + 1]^2} + \frac{(1 - I_2) t'^2 e^{2\boldsymbol{\beta}^t \mathbf{z}_H}}{[(\gamma_1 t' + \gamma_2 t'^2)e^{\boldsymbol{\beta}^t \mathbf{z}_H} + 1]^2}, \tag{5.48}$$

$$\frac{\partial^2 \ln L}{\partial \gamma_2^2} = -\frac{4I_1 I_2 t^2}{(\gamma_1 + 2\gamma_2 t)^2} + \frac{2I_1 I_2 t^4 e^{2\boldsymbol{\beta}^t \mathbf{z}_L}}{[(\gamma_1 t + \gamma_2 t^2)e^{\boldsymbol{\beta}^t \mathbf{z}_L} + 1]^2} - \frac{4(1 - I_1) I_2 t'^2}{(\gamma_1 + 2\gamma_2 t')^2}$$
$$+ \frac{2(1 - I_1) I_2 t'^4 e^{2\boldsymbol{\beta}^t \mathbf{z}_H}}{[(\gamma_1 t' + \gamma_2 t'^2)e^{\boldsymbol{\beta}^t \mathbf{z}_H} + 1]^2} + \frac{(1 - I_2) t'^4 e^{2\boldsymbol{\beta}^t \mathbf{z}_H}}{[(\gamma_1 t' + \gamma_2 t'^2)e^{\boldsymbol{\beta}^t \mathbf{z}_H} + 1]^2}, \tag{5.49}$$

$$\frac{\partial^2 \ln L}{\partial\gamma_1\partial\gamma_2} = -\frac{2I_1I_2t}{(\gamma_1+2\gamma_2 t)^2} + \frac{2I_1I_2t^3e^{2\boldsymbol{\beta}^t\mathbf{z}_L}}{[(\gamma_1 t+\gamma_2 t^2)e^{\boldsymbol{\beta}^t\mathbf{z}_L}+1]^2} - \frac{2(1-I_1)I_2t'}{(\gamma_1+2\gamma_2 t')^2}$$
$$+\frac{2(1-I_1)I_2t'^3e^{2\boldsymbol{\beta}^t\mathbf{z}_H}}{[(\gamma_1 t'+\gamma_2 t'^2)e^{\boldsymbol{\beta}^t\mathbf{z}_H}+1]^2} + \frac{(1-I_2)t'^3e^{2\boldsymbol{\beta}^t\mathbf{z}_H}}{[(\gamma_1 t'+\gamma_2 t'^2)e^{\boldsymbol{\beta}^t\mathbf{z}_H}+1]^2}, \tag{5.50}$$

$$\frac{\partial^2 \ln L}{\partial\beta_1^2} = -\frac{2I_1I_2z_{1L}^2(\gamma_1 t+\gamma_2 t^2)e^{\boldsymbol{\beta}^t\mathbf{z}_L}}{[(\gamma_1 t+\gamma_2 t^2)e^{\boldsymbol{\beta}^t\mathbf{z}_L}+1]^2} - \frac{2(1-I_1)I_2z_{1H}^2(\gamma_1 t'+\gamma_2 t'^2)e^{\boldsymbol{\beta}^t\mathbf{z}_H}}{[(\gamma_1 t'+\gamma_2 t'^2)e^{\boldsymbol{\beta}^t\mathbf{z}_H}+1]^2}$$
$$-\frac{(1-I_2)z_{1H}^2(\gamma_1 t'+\gamma_2 t'^2)e^{\boldsymbol{\beta}^t\mathbf{z}_H}}{[(\gamma_1 t'+\gamma_2 t'^2)e^{\boldsymbol{\beta}^t\mathbf{z}_H}+1]^2}, \tag{5.51}$$

$$\vdots$$

$$\frac{\partial^2 \ln L}{\partial\beta_k^2} = -\frac{2I_1I_2z_{kL}^2(\gamma_1 t+\gamma_2 t^2)e^{\boldsymbol{\beta}^t\mathbf{z}_L}}{[(\gamma_1 t+\gamma_2 t^2)e^{\boldsymbol{\beta}^t\mathbf{z}_L}+1]^2} - \frac{2(1-I_1)I_2z_{kH}^2(\gamma_1 t'+\gamma_2 t'^2)e^{\boldsymbol{\beta}^t\mathbf{z}_H}}{[(\gamma_1 t'+\gamma_2 t'^2)e^{\boldsymbol{\beta}^t\mathbf{z}_H}+1]^2}$$
$$-\frac{(1-I_2)z_{kH}^2(\gamma_1 t'+\gamma_2 t'^2)e^{\boldsymbol{\beta}^t\mathbf{z}_H}}{[(\gamma_1 t'+\gamma_2 t'^2)e^{\boldsymbol{\beta}^t\mathbf{z}_H}+1]^2}. \tag{5.52}$$

The above derivatives are given in terms of the random quantities I_1, I_2 and stress levels $\mathbf{z}_L$, $\mathbf{z}_H$ as well as the model parameters. The elements of the Fisher information matrix for an observation are the negative expectations of the second partial derivatives:

$$E\left[-\frac{\partial^2 l}{\partial\beta_1^2}\right] = \int_0^{\tau_2} -\frac{\partial^2 l}{\partial\beta_1^2} f(t;\mathbf{z}_L,\mathbf{z}_H)dt, \tag{5.53}$$

$$\vdots$$

$$E\left[-\frac{\partial^2 l}{\partial \beta_k^2}\right] = \int_0^{\tau_2} -\frac{\partial^2 l}{\partial \beta_k^2} f(t;\mathbf{z}_L,\mathbf{z}_H)dt\,, \tag{5.54}$$

$$E\left[-\frac{\partial^2 l}{\partial \gamma_1^2}\right] = \int_0^{\tau_2} -\frac{\partial^2 l}{\partial \gamma_1^2} f(t;\mathbf{z}_L,\mathbf{z}_H)dt\,, \tag{5.55}$$

$$E\left[-\frac{\partial^2 l}{\partial \gamma_2^2}\right] = \int_0^{\tau_2} -\frac{\partial^2 l}{\partial \gamma_2^2} f(t;\mathbf{z}_L,\mathbf{z}_H)dt\,, \tag{5.56}$$

$$E\left[-\frac{\partial^2 l}{\partial \gamma_1 \partial \gamma_2}\right] = \int_0^{\tau_2} -\frac{\partial^2 l}{\partial \gamma_1 \partial \gamma_2} f(t;\mathbf{z}_L,\mathbf{z}_H)dt\,, \tag{5.57}$$

where the probability density function $f(t;\mathbf{z}_L,\mathbf{z}_H)$ is given as the derivative of the equivalent cumulative density function in Eq. (5.43).

The above equations show the components of the Fisher's information matrix for a single observation. Since all n test units placed under the step-stress test experience the same test conditions, the Fisher's information matrix for the sample is just the summation of all units:

$$F = n\begin{bmatrix} E\left[-\frac{\partial^2 l}{\partial\beta_1^2}\right] & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & 0 & \vdots & \vdots \\ 0 & 0 & E\left[-\frac{\partial^2 l}{\partial\beta_k^2}\right] & 0 & 0 \\ 0 & 0 & 0 & E\left[-\frac{\partial^2 l}{\partial\gamma_1^2}\right] & E\left[-\frac{\partial^2 l}{\partial\gamma_1\partial\gamma_2}\right] \\ 0 & 0 & 0 & E\left[-\frac{\partial^2 l}{\partial\gamma_1\partial\gamma_2}\right] & E\left[-\frac{\partial^2 l}{\partial\gamma_2^2}\right] \end{bmatrix}. \tag{5.58}$$

The Variance-Covariance matrix for maximum likelihood estimates ($\hat{\beta}_1, \ldots, \hat{\beta}_k, \hat{\gamma}_1, \hat{\gamma}_2,$) is defined as the inverse matrix of the Fisher's information matrix:

$$\Sigma = \begin{bmatrix} Var(\hat{\beta}_1) & 0 & \cdots & 0 & 0 \\ 0 & \ddots & 0 & \vdots & \vdots \\ \vdots & 0 & Var(\hat{\beta}_k) & 0 & 0 \\ 0 & \cdots & 0 & Var(\hat{\gamma}_1) & Cov(\hat{\gamma}_1, \hat{\gamma}_2) \\ 0 & \cdots & 0 & Cov(\hat{\gamma}_1, \hat{\gamma}_2) & Var(\hat{\gamma}_2) \end{bmatrix} = F^{-1}\ . \tag{5.59}$$

5.2.6 Optimization Criterion

In order to obtain the most accurate reliability estimate under the constraints of testing conditions, such as time, cost, number of available units, etc, we choose to minimize the asymptotic variance of the reliability estimates at the design stress levels over a pre-specified period of time, i.e., minimize

$$\int_0^T Var[\hat{R}(t;\mathbf{z}_D)]dt = \int_0^T Var[\frac{1}{(\hat{\gamma}_1 t + \hat{\gamma}_2 t^2)e^{\hat{\beta}'\mathbf{z}_D} + 1}]dt \,. \tag{5.60}$$

As shown earlier, the asymptotic variance of the reliability estimate at the design stress levels is derived as:

$$\begin{aligned} & Var[\hat{R}(t;\mathbf{z}_D)] \\ & = \begin{bmatrix} \frac{\partial R(t;\mathbf{z}_D)}{\partial \beta_1} & \cdots & \frac{\partial R(t;\mathbf{z}_D)}{\partial \beta_k} & \frac{\partial R(t;\mathbf{z}_D)}{\partial \gamma_1} & \frac{\partial R(t;\mathbf{z}_D)}{\partial \gamma_2} \end{bmatrix} \Sigma \\ & \times \begin{bmatrix} \frac{\partial R(t;\mathbf{z}_D)}{\partial \beta_1} & \cdots & \frac{\partial R(t;\mathbf{z}_D)}{\partial \beta_k} & \frac{\partial R(t;\mathbf{z}_D)}{\partial \gamma_1} & \frac{\partial R(t;\mathbf{z}_D)}{\partial \gamma_2} \end{bmatrix}^T \Bigg|_{(\beta_1, \beta_2, \ldots, \beta_k, \gamma_1, \gamma_2) = (\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_k, \hat{\gamma}_1, \hat{\gamma}_2)} \end{aligned}$$

5.2.7 Problem Formulation

The problem is to design an optimum multiple-stress-type ALT based on proportional odds model using simple step-stress loading with consideration of censoring and the constraints of test units, censoring time and specification of minimum number of failures at the low stress levels, such that the asymptotic variance of the reliability estimate at the design stress levels is minimized over a pre-specified period of time T. This optimum ALT plan gives the most accurate reliability prediction at the design stress levels over the pre-specified period of time T. The decision variables include the low stress levels $\mathbf{z}_L$ and stress changing time τ_1. The high stress levels $\mathbf{z}_H$ are given as the highest possible stress levels beyond which the failure mode will change. The optimum solution of $\mathbf{z}_L$ and τ_1 is determined by solving the following nonlinear optimization problem.

Objective function

Min $$f(\boldsymbol{x}) = \int_0^T Var[\hat{R}(t;\boldsymbol{z}_D)]dt = \int_0^T Var[\frac{1}{(\hat{\gamma}_1 t + \hat{\gamma}_2 t^2)e^{\hat{\boldsymbol{\beta}}^t \boldsymbol{z}_D} + 1}]dt$$

Subject to

$$n\Pr[t \le \tau_1; \boldsymbol{z}_L] \ge MNF \ ,$$

$$\Sigma = F^{-1}.$$

where

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{z}_L \\ \tau_1 \end{bmatrix} = \begin{bmatrix} z_{1L} \\ \vdots \\ z_{kL} \\ \tau_1 \end{bmatrix},$$

and

MNF is the minimum required number of failures at low stress levels.

The optimum solution depends on the values of model parameters $(\boldsymbol{\beta}^t, \gamma_1, \gamma_2)$. A design using the pre-estimates of the model parameter is called a locally optimum design (Chernoff, 1962) and is commonly adopted by Bai and Kim (1989), Bai and Chun (1991), and Nelson (1990). We also assume that the baseline estimates of $(\boldsymbol{\beta}^t, \gamma_1, \gamma_2)$ are available through either preliminary test or engineering experience obtained prior to the design of the optimum ALT plan.

5.2.8 Numerical Example

A multiple-stress-type accelerated life test with simple step-stress loading is to be carried out for MOS capacitors in order to estimate their life distribution at design stress levels: temperature of $50°C$ and voltage of $5V$. The test needs to be completed in 300 hours. The total number of available testing units is 200. To avoid failure modes other than those expected at the design stress levels, it has been determined, through engineering judgment, that the applied testing temperature level should not exceed $250°C$ and the voltage level should not exceed $10V$. The required minimum number of failures at the low stress levels is specified as 50. Furthermore, the objective of the accelerated life test is to provide the most accurate reliability predication at the design stress levels over a 10-year period of time. The optimum ALT plan is determined as follows:

1. According to the Arrehenius model, we transform the Celsius temperature scale to 1/Kelvin as the covariate in the ALT model, i.e., the design stress level of temperature is $z_{1D} = 1/(50+273.16) = 1/323.16\ K^{-1}$, and the highest stress level of temperature is $z_{1H} = 1/523.16\ K^{-1}$.
2. The PO model is used to fit the failure time data. The model is given by:

$$\theta(t; z_1, z_2) = \exp(\beta_1 z_1 + \beta_2 z_2)\theta_0(t),$$

   where z_1 is the stress level of temperature, z_2 is the stress level of voltage, β_1, β_2 are unknown parameters, and $\theta_0(t)$ is the baseline odds function, which is given by $\theta_0(t) = \gamma_1 t + \gamma_2 t^2$.

3. A baseline experiment is carried out to obtain the initial values for the model parameters. These values are listed below:

$\hat{\beta}_1$	-1600
$\hat{\beta}_2$	0.02
$\hat{\gamma}_1$	2
$\hat{\gamma}_2$	0.01

4. The problem is to optimally design a multiple step-stresses ALT to fit failure time data with type I censoring, under the constraints of available test units, censoring time and minimum required number of failure units at low stress levels, such that the asymptotic variance of the reliability estimate of the product at design stress levels is minimized over a pre-specified period of time T , which is 10 years in this case. The optimum decision variables, including the low stress levels z_{1L}^* , z_{2L}^* and stress changing time τ_1^* are determined by solving the following nonlinear optimization problem:

Objective function

Min $\quad f(\boldsymbol{x}) = \int_0^T Var[\frac{1}{(\hat{\gamma}_1 t + \hat{\gamma}_2 t^2)e^{\hat{\beta}' \boldsymbol{z}_D} + 1}]dt$

Subject to

$$n \Pr[t \le \tau_1; \boldsymbol{z}_L] \ge MNF \ ,$$

$$1/523.16 \le z_{1L} \le 1/323.16,$$

$$5 \le z_{2L} \le 10,$$

$$\tau_1 < \tau,$$

$$\Sigma = F^{-1}.$$

where

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{z}_L \\ \tau_1 \end{bmatrix} = \begin{bmatrix} z_{1L} \\ z_{2L} \\ \tau_1 \end{bmatrix},$$

$$\boldsymbol{z}_D = \begin{bmatrix} z_{1D} \\ z_{2D} \end{bmatrix} = \begin{bmatrix} 1/323.16 \\ 5 \end{bmatrix},$$

$$\boldsymbol{z}_H = \begin{bmatrix} z_{1H} \\ z_{2H} \end{bmatrix} = \begin{bmatrix} 1/523.16 \\ 10 \end{bmatrix}$$

$$T = 87600,$$

$$\tau = 300,$$

$$MNF = 50.$$

5. We use the algorithm as described in section 5.2.5 to solve this optimization problem. The algorithm is implemented using SAS/IML and SAS/OR. SAS/IML software is a powerful and flexible programming language (**I**nteractive **M**atrix **L**anguage) in a dynamic, interactive environment. The fundamental object of the language is a data matrix. The programming is dynamic because necessary activities such as memory allocation and dimensioning of matrices are performed automatically.
6. The optimum solutions that minimize the objective function and satisfy the constraints are

$z_{1L}^{*} = 132°C$, $z_{2L}^{*} = 8.7V$, and $\tau_{1}^{*} = 127$ hours.

CHAPTER 6

EQUIVALENT ALT PLANS

6.1 Introduction

The significant increase in the introduction of new products coupled with the significant reduction in time from product design to manufacturing, as well as the increasing customer's expectation for high reliability, have prompted industry to shorten its product test duration. In many cases, accelerated life testing (ALT) might be the only feasible approach to meet this requirement. The accuracy of the statistical inference procedure obtained using ALT data has a profound effect on the reliability estimates and the subsequent decisions regarding system configuration, warranties and preventive maintenance schedules. Specifically, the reliability estimate depends on two factors, the ALT model and the experimental design of test plans. Without an optimal test plan, it is likely that a sequence of expensive and time-consuming tests results in inaccurate reliability estimates and misleading the final product design requirements. That might also cause delays in product release, or the termination of the entire product.

Traditionally, ALT is conducted under constant stresses during the entire test. For example, a typical constant temperature test plan consists of defining three temperature levels (high, medium, and low) and test units are divided among these three levels where units at each level are exposed to the same temperature until failure or until the test is terminated. The test results are then used to extrapolate the product life at normal conditions. In practice, constant-stress tests are easy to carry out but need more test units and long time at low stress level to yield "enough" number of failures. However, in many cases the available number of test units and test duration are extremely limited. This has prompted industry to consider step-stress test where the test units are first subjected to a lower stress level for some time; if no failures or only a small number of failures occur, the stress is increased to a higher level and held constant for another amount of time; the steps are repeated until all units fail or the predetermined test time expires. Usually, step-stress tests yield failures in a much shorter time than constant-stress tests, but the statistical inference from the data is more difficult to make. Moreover, since the test duration is short and a large proportion of failures occur at high stress levels far from the design stress level, much extrapolation has to be made, which may lead to poor estimation accuracy. On the other hand, there could be other choices in stress loadings (e.g., cyclic-stress and ramp-stress) in conducting ALT experiments. Each stress loading has some advantages and drawbacks. This has raised many practical questions such as: Can accelerating test plans involving different stress loadings be designed such that they are equivalent? What are the measures of equivalency? Can such test plans and their equivalency be developed for multiple stresses especially in the setting of step-stress tests and other profiled stress tests? When and in which order should we change the stress levels in multi-stress multi-step tests?

Figure 6.1 shows various stress loading types as well as their adjustable parameters. These stress loadings have been widely utilized in ALT experiments. For instance, static-fatigue tests and cyclic-fatigue tests (Matthewson and Yuce, 1994) have been frequently performed on optical fibers to study their reliability; dielectric-breakdown of thermal oxides (Elsayed, Liao and Wang, 2006) have been studied under elevated constant electrical fields and temperatures; the lifetime of ceramic components subject to slow crack growth due to stress corrosion have been investigated under cyclic stress by NASA (Choi and Salem, 1997). These types of stress loading are selected based on the simplification of statistical analyses, and familiarity of existing analytical tools and industrial routines without following a systematic refinement procedure. Due to budget and time constraints, there is an increasing necessity to determine the best stress loading type and the associated parameters in order to shorten the test duration and reduce the total cost while achieving reliability estimate with equivalent accuracy to that of constant stress testing. Research on ALT Plans has been focused on the design of optimum test plans for given stress loading type. However, fundamental research on the equivalency of these tests has not yet been investigated in the reliability engineering. Without the understanding of such equivalency, it is difficult for a test engineer to determine the best experimental settings before conducting actual ALT.

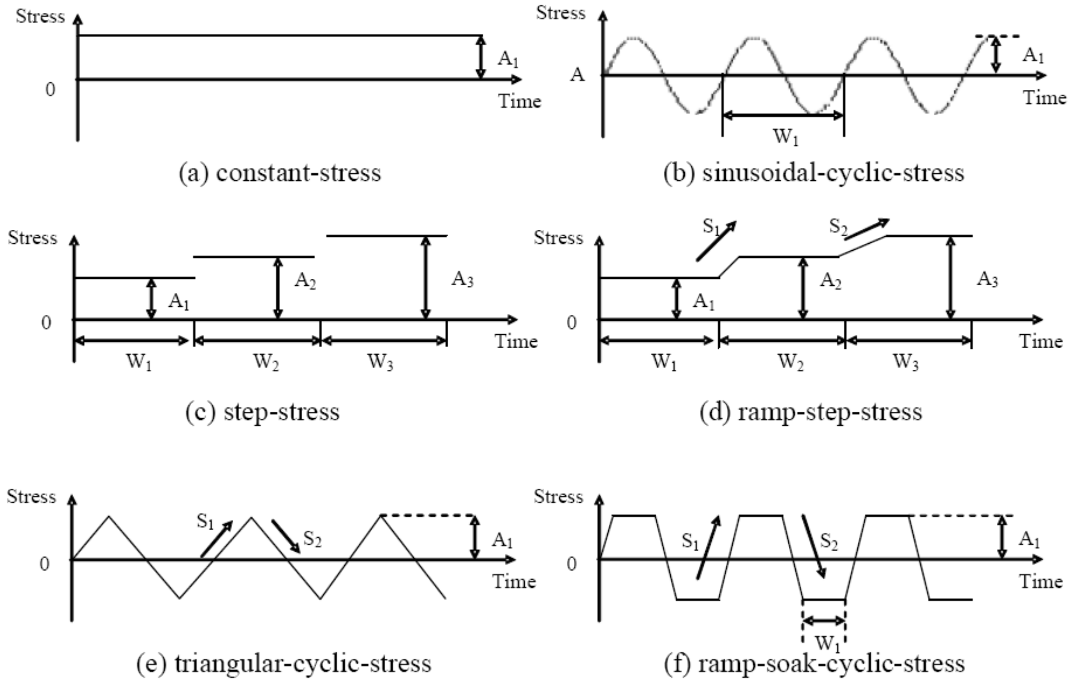

Figure 6.1 Various loadings of a single type of stress

Furthermore, as is often the case, products are usually exposed to multiple stress types in actual use such as temperature, humidity, electric current and various types of vibration. To study the reliability of such products, it is important to subject test units to multiple stress types simultaneously in ALT experiments. For constant-stress tests, it might not be difficult to extend the statistical methods in the design of optimum test plans for single stress to multiple stresses scenarios. However, the problem becomes complicated when time-varying stresses such as step-stresses are considered. For example, in a multiple-stress-type multi-step test, issues such as when and in which order the levels of the stresses should be changed are challenging and unsolved. Figure 6.2 illustrates two experimental settings out of thousands of choices as one can imagine in conducting a multiple-stress-type multi-step ALT. In general, an arbitrary selection from combinations of multiple stress profiles may not result in the most accurate reliability estimates, especially when the effects of the stresses on the reliability of the product are highly correlated. Therefore, optimization of test plans by tuning the high dimensional decision variables under time and cost constraints needs to be carefully investigated from the perspective of statistics, operations research and engineering physics.

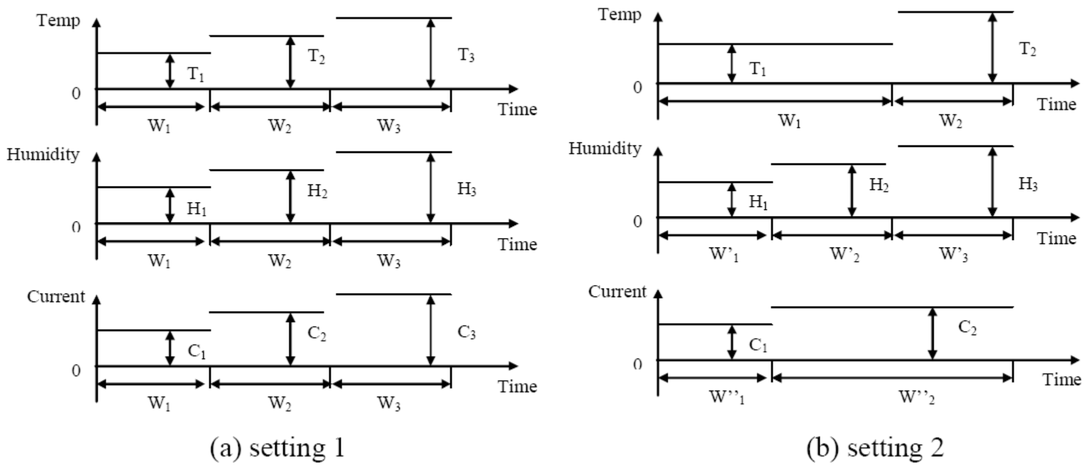

Figure 6.2 Two example settings of an ALT involving temperature, current, humidity and electric current

6.2 Definition of Equivalency

We will consider different optimization criteria depending on the type of stress loading and the objective of the test. The optimization criteria to be considered are the minimization of the asymptotic variance of the maximum likelihood estimate (MLE) of:

(1) the specified failure time quantile at normal operating conditions;

(2) the mean time to failure at normal operating conditions;

(3) the reliability function or (cumulative distribution function) at a given age of the product at normal operating conditions;

(4) the hazard function over a specified period of time at normal operating conditions;

(5) the model parameter(s) (for multiple parameters, consider D-optimality or A-optimality). D-optimality is the criterion that determines how well the coefficients of the design Approximation are estimated. It requires changes in the locations of the sampling points to maximize this criterion which in turn maximizes the confidence in the coefficients of the approximation model. A-optimality is based on the sum of the variances of the estimated parameters for the model

To study the equivalency among ALT plans involving different stress loadings, several definitions are explored. Some feasible definitions are:

Definition 1. Two ALT plans are equivalent if they generate the same values of the optimization criterion during the testing with Type I censoring.

Definition 2. Two ALT plans are equivalent if difference between the estimated times to failure by the two plans at normal operating conditions are within $\delta\%$, where δ is an acceptable level of deviation.

In the following sections, we investigate the equivalent ALT plans based on the first definition.

6.3 Equivalency of Step-stress ALT Plans and Constant-stress ALT Plans

The first scenario of equivalent ALT plans considered is the case of the equivalency of step-stress ALT plans and constant-stress ALT plans. Since constant-stress tests are the most commonly conducted accelerated life tests in industry and their statistical inference has been extensively investigated, we focus on determining the equivalent step-stress

ALT plan. The constant-stress ALT plan serves as the baseline test result for comparison with the step-stress plan. More importantly, the constant-stress ALT plan requires longer test duration when compared with other test plans. Therefore, the efficiency of equivalent plans will also be measured by the percent reduction in test time. In this section, we are interested in determining the minimized test duration of the step-stress ALT plan, which is equivalent to the baseline constant-stress ALT plan. .

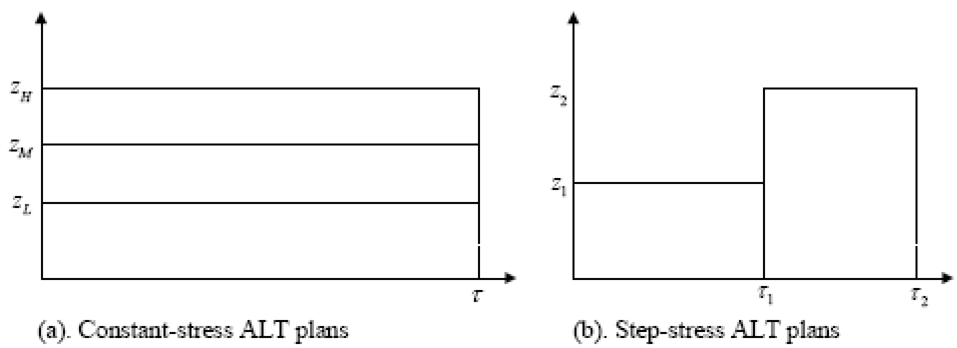

Figure 6.3 An example of equivalent constant ALT plan and step-stress ALT plan

As a preliminary investigation of the equivalent ALT plans, we consider the following simplified single stress type case. As shown in Figure 6.3a, the baseline optimum constant-stress ALT plan is determined with the pre-determined censoring time τ. The baseline constant-stress ALT plan is a good compromise optimum plan (Nelson 2004). The adjective "good compromise" is due to the fact that only the low stress level z_L is determined through the optimization process in order to minimize the asymptotic variance of the reliability estimates at the design stress level z_D. It assumes that the high stress level z_H is chosen to be the highest stress level beyond which another failure mode will be introduced. The intermediate stress level $z_M = (z_L + z_H)/2$ is midway between will be introduced. The intermediate stress level $z_M = (z_L + z_H)/2$ is midway between the low stress level z_L and the high stress level z_H. The optimum z_L is obtained such that the asymptotic variance of ML estimates of the reliability function at the design stress level z_D is minimized. The allocation of fraction of test units to the stress levels z_L, z_M and z_H are $p_L = 4/7$, $p_M = 2/7$, and $p_H = 1/7$ respectively. This unequal allocation is a compromise that extrapolates reasonably well. For a sample of n test units, the allocation is $n_L = 4n/7$, $n_M = 2n/7$, and $n_H = 2n/7$. Therefore the optimization problem can be formulated by following the same procedure described in Section 5.1.4 with $k = 1$:

Objective function

Min $$f_C(x) = \int_0^T Var[\hat{R}_C(t; z_D)]dt$$

Subject to

$$\Sigma_C = F_C^{-1}$$

$$p_L = 4/7,\ p_M = 2/7,\ p_H = 1/7$$

$$z_D \le z_L,\ z_H = z_{upper},\ z_M = (z_L + z_H)/2$$

$$np_L[1 - R(\tau; z_L)] \ge MNF$$

$$np_M[1 - R(\tau; z_M)] \ge MNF$$

$$np_H[1 - R(\tau; z_H)] \ge MNF$$

where

$$F_C = n_L \begin{bmatrix} E\left[-\frac{\partial^2 l_C(z_L)}{\partial \beta^2}\right] & 0 & 0 \\ 0 & E\left[-\frac{\partial^2 l_C(z_L)}{\partial \gamma_1^2}\right] & E\left[-\frac{\partial^2 l_C(z_L)}{\partial \gamma_1 \partial \gamma_2}\right] \\ 0 & E\left[-\frac{\partial^2 l_C(z_L)}{\partial \gamma_1 \partial \gamma_2}\right] & E\left[-\frac{\partial^2 l_C(z_L)}{\partial \gamma_2^2}\right] \end{bmatrix}$$

$$+ n_M \begin{bmatrix} E\left[-\frac{\partial^2 l_C(z_M)}{\partial \beta^2}\right] & 0 & 0 \\ 0 & E\left[-\frac{\partial^2 l_C(z_M)}{\partial \gamma_1^2}\right] & E\left[-\frac{\partial^2 l_C(z_M)}{\partial \gamma_1 \partial \gamma_2}\right] \\ 0 & E\left[-\frac{\partial^2 l_C(z_M)}{\partial \gamma_1 \partial \gamma_2}\right] & E\left[-\frac{\partial^2 l_C(z_M)}{\partial \gamma_2^2}\right] \end{bmatrix}$$

$$+ n_H \begin{bmatrix} E\left[-\frac{\partial^2 l_C(z_H)}{\partial \beta^2}\right] & 0 & 0 \\ 0 & E\left[-\frac{\partial^2 l_C(z_H)}{\partial \gamma_1^2}\right] & E\left[-\frac{\partial^2 l_C(z_H)}{\partial \gamma_1 \partial \gamma_2}\right] \\ 0 & E\left[-\frac{\partial^2 l_C(z_H)}{\partial \gamma_1 \partial \gamma_2}\right] & E\left[-\frac{\partial^2 l_C(z_H)}{\partial \gamma_2^2}\right] \end{bmatrix}$$

$$x = z_L$$

The only decision variable is the low stress level z_L in the above optimization problem. Let z_L^* denote the optimum solution, and $f_C(z_L^*)$ denote the minimum asymptotic variance of the reliability estimates given by the constant-stress ALT plan.

Compared with the constant-stress ALT plan, the step-stress ALT plan can substantially shorten the test duration. The censoring time τ_2 of the step-stress ALT plan shown in Figure 6.3b is less than the censoring time τ of the constant-stress ALT plan shown in Figure 6.3a. The objective of the equivalency of ALT plans is to determine the minimum

τ_2 which results in the equivalent asymptotic variances of the reliability estimates at the design stress level obtained from the two test plans.

To obtain the optimum asymptotic variance of the reliability estimates at the design stress level for any given τ_2 of the step-stress ALT plan, we formulate the optimization problem following the procedures described in section 5.2.7:

Objective function

Min $$f_S(\boldsymbol{x}) = \int_0^T Var[\hat{R}_S(t; z_D)]\,dt$$

Subject to

$$n \Pr[t \le \tau_1; z_1] \ge MNF_1$$

$$n \Pr[t \le \tau_2; z_1, z_2] \ge MNF_2$$

$$z_2 = z_{upper}$$

$$\Sigma_S = F_S^{-1}$$

where

$$\boldsymbol{x} = \begin{bmatrix} z_1 \\ \tau_1 \end{bmatrix}$$

$$F_S = n \begin{bmatrix} E\left[-\dfrac{\partial^2 l_S}{\partial \beta^2}\right] & 0 & 0 \\ 0 & E\left[-\dfrac{\partial^2 l_S}{\partial \gamma_1^2}\right] & E\left[-\dfrac{\partial^2 l_S}{\partial \gamma_1 \partial \gamma_2}\right] \\ 0 & E\left[-\dfrac{\partial^2 l_S}{\partial \gamma_1 \partial \gamma_2}\right] & E\left[-\dfrac{\partial^2 l_S}{\partial \gamma_2^2}\right] \end{bmatrix}$$

MNF_1 and MNF_2 are the minimum required number of failures at low stress level and high stress level respectively

Solving the above optimization problem, we obtain the optimum solution: z_1^*, τ_1^*, and the achieved minimized asymptotic variance $f_S(z_1^*, \tau_1^*; \tau_2)$ for given τ_2. Following the definition of the equivalent ALT plans in section 6.2, the minimum censoring time τ_2^* of the step-stress ALT plan is defined as:

$$\tau_2^* = \inf\{\tau_2 \mid f_S(z_1^*, \tau_1^*; \tau_2) \le f_C(z_L^*)\}. \tag{6.1}$$

We use the following bisectional search procedure to determine the minimum censoring time τ_2^*:

Step 1. Solve the nonlinear optimization problem defined in the constant-stress ALT plan, and find $f_C(z_L^*)$, choose a small value number ε arbitrarily.

Step 2. Let $i = 1$, $\tau_L = 0$, $\tau_H = \tau$, and $\tau_2^i = (\tau_L + \tau_H) = \tau / 2$.

Step 3. Determine $f_S(z_1^*, \tau_1^*; \tau_2^i)$ by solving the above nonlinear optimization problem defined in the step-stress ALT plan.

Step 4. Let $i = i + 1$. If $f_S(z_1^*, \tau_1^*; \tau_2^{i-1}) \le f_C(z_L^*)$, then let $\tau_L = \tau_2^{i-1}$ and $\tau_2^i = (\tau_L + \tau_H)/2$; otherwise let $\tau_H = \tau_2^{i-1}$ and $\tau_2^i = (\tau_L + \tau_H)/2$.

Step 5. Determine $f_S(z_1^*, \tau_1^*; \tau_2^i)$ by solving the above nonlinear optimization problem defined in the step-stress ALT plan. If $|\tau_2^i - \tau_2^{i-1}| \le \varepsilon$ and $f_S(z_1^*, \tau_1^*; \tau_2^i) \le f_C(z_L^*)$, then stop and let $\tau_2^* = \tau_2^i$; otherwise continue Step 4.

6.4 Numerical Example

A constant-stress accelerated life test is conducted at three temperature levels for MOS devices in order to estimate its reliability function at design temperature level $25\,°C$. The test needs to be completed in 100 hours. The total number of available test units is 200. The highest temperature level that test units can experience is $250\,°C$. It is expected that the ALT will provide the most accurate reliability estimate over a period of 10000 hours. A constant-stress ALT plan can be designed through the following steps:

1. According to Arrehenius model, the scaled stress $z = 1000/(temp + 276.13)$ is used. Then the design stress level $z_D = 3.35$ and the upper bound is $z_{upper} = 1.19$.

2. A baseline experiment is conducted to obtain a set of initial values of the parameters for the PO model with quadratic odds function. These values are: $\beta = -1.8$, $\gamma_1 = 10$, and $\gamma_2 = 0.001$.

3. Following the formulation in section 6.3 for the constant-stress ALT plan, the nonlinear optimization problem is expressed as follows:

Objective function

Min $\quad f_C(x) = \int_0^{10000} Var[\hat{R}_C(t;3.35)]dt$

Subject to

$\Sigma_C = F_C^{-1}$

$p_L = 4/7$, $p_M = 2/7$, $p_H = 1/7$

$3.35 = z_D \le z_L$, $z_H = z_{upper} = 1.91$, $z_M = (z_L + z_H)/2$

$np_L[1 - R(\tau; z_L)] \ge 90$

$np_M[1 - R(\tau; z_M)] \ge 50$

$np_L[1 - R(\tau; z_H)] \ge 40$

where

$$F_C = n_L \begin{bmatrix} E\left[-\frac{\partial^2 l_C(z_L)}{\partial \beta^2}\right] & 0 & 0 \\ 0 & E\left[-\frac{\partial^2 l_C(z_L)}{\partial \gamma_1^2}\right] & E\left[-\frac{\partial^2 l_C(z_L)}{\partial \gamma_1 \partial \gamma_2}\right] \\ 0 & E\left[-\frac{\partial^2 l_C(z_L)}{\partial \gamma_1 \partial \gamma_2}\right] & E\left[-\frac{\partial^2 l_C(z_L)}{\partial \gamma_2^2}\right] \end{bmatrix}$$

$$+ n_M \begin{bmatrix} E\left[-\frac{\partial^2 l_C(z_M)}{\partial \beta^2}\right] & 0 & 0 \\ 0 & E\left[-\frac{\partial^2 l_C(z_M)}{\partial \gamma_1^2}\right] & E\left[-\frac{\partial^2 l_C(z_M)}{\partial \gamma_1 \partial \gamma_2}\right] \\ 0 & E\left[-\frac{\partial^2 l_C(z_M)}{\partial \gamma_1 \partial \gamma_2}\right] & E\left[-\frac{\partial^2 l_C(z_M)}{\partial \gamma_2^2}\right] \end{bmatrix}$$

$$+ n_H \begin{bmatrix} E\left[-\frac{\partial^2 l_C(z_H)}{\partial \beta^2}\right] & 0 & 0 \\ 0 & E\left[-\frac{\partial^2 l_C(z_H)}{\partial \gamma_1^2}\right] & E\left[-\frac{\partial^2 l_C(z_H)}{\partial \gamma_1 \partial \gamma_2}\right] \\ 0 & E\left[-\frac{\partial^2 l_C(z_H)}{\partial \gamma_1 \partial \gamma_2}\right] & E\left[-\frac{\partial^2 l_C(z_H)}{\partial \gamma_2^2}\right] \end{bmatrix}$$

$$n = 200, \quad \tau = 100$$

$$x = z_L$$

4. The solution of the optimization problem is: $z_L^* = 3.1$ and $f_C(3.1) = 0.2823$.

In order to shorten the test period and still obtain the equivalent asymptotic variance, we consider the step-stress ALT plan. The nonlinear optimization problem of the step-stress ALT plan is formulated as follows:

Objective function

Min $\qquad f_S(\boldsymbol{x}) = \int_0^{10000} Var[\hat{R}_S(t; 3.35)]\,dt$

Subject to

$$n \Pr[t \le \tau_1; z_1] \ge 50$$

$$n \Pr[t \le \tau_2; z_1, z_2] \ge 140$$

$$z_2 = 1.19$$

$$\Sigma_S = F_S^{-1}$$

where

$$\boldsymbol{x} = \begin{bmatrix} z_1 \\ \tau_1 \end{bmatrix}$$

$$F_S = n\begin{bmatrix} E\left[-\frac{\partial^2 l_S}{\partial\beta^2}\right] & 0 & 0 \\ 0 & E\left[-\frac{\partial^2 l_S}{\partial\gamma_1^2}\right] & E\left[-\frac{\partial^2 l_S}{\partial\gamma_1\partial\gamma_2}\right] \\ 0 & E\left[-\frac{\partial^2 l_S}{\partial\gamma_1\partial\gamma_2}\right] & E\left[-\frac{\partial^2 l_S}{\partial\gamma_2^2}\right] \end{bmatrix}$$

$$n = 200$$

The minimum censoring time τ_2^* of the step-stress ALT plan is determined by:

$$\tau_2^* = \inf\{\tau_2 \mid f_S(z_1^*, \tau_1^*; \tau_2) \le f_C(z_L^*)\}.$$

Following the bisectional search algorithm (ε is arbitrarily chosen as 1) in Section 6.3, the results are listed in Table 6.1.

Table 6.1 Results of bisectional search

i	τ_2^i	$f_S(z_1^*, \tau_1^*; \tau_2^i)$
1	50	0.2884
2	75	0.2489
3	63	0.2648
4	56	0.2764
5	53*	0.2822
6	55	0.2782
7	54	0.2802

From Table 6.1, the minimum censoring time $\tau_2^* = 53$. The corresponding optimum low stress level is $z_1 = 3.15$; the optimum stress changing time is $\tau_1 = 46$.

6.5 Simulation Study

In this section, we conduct a simulation study to verify the resultant constant ALT plan and step-stress ALT plan in the previous section are equivalent indeed in terms of the estimation accuracy.

The sets of simulation data following both constant ALT plan and step-stress ALT plan are generated by Monte Carlo simulation method based on the following reliability function.

$$R(t) = \frac{1}{1 + (\gamma_1 t)\exp(\beta z)}. \tag{6.2}$$

It can be easily verified that the assumption of the PO model is valid for the failure time samples generated by the above reliability function. The corresponding PO model can be expressed as

$$\theta(t; z) = \theta_0(t)\exp(\beta z) = (\gamma_1 t)\exp(\beta z). \tag{6.3}$$

The total sample size is 200 and divided into three stress groups with constant ALT plan: $n_L = 200$, $p_L = 800/7 \approx 114$, $n_M = 57$, and $n_H = 29$.

The model parameters used to generate the Monte Carlo simulation failure times are summarized in Table 6.2 for the constant ALT plan.

Table 6.2 Model parameters for the constant-stress ALT simulation

Common parameters	Censoring time: $\tau = 100$; Design stress level: $z_D = 3.5$; Baseline parameters: $\beta = -1.8$, $\gamma_1 = 10$		
Stress	Low	Medium	High
Stress level	$z_L = 3.1$	$z_M = 2.15$	$z_H = 1.19$
Sample size	$n_L = 114$	$n_M = 57$	$n_H = 29$
Dataset	I	II	III

Rewriting Eq. (6.3) by solving for t, we obtain

$$t = \frac{F(t)\exp(-\beta z)}{\gamma_1[1 - F(t)]} \tag{6.4}$$

Therefore 114 simulated failure times are generated for low stress level by the following equation

$$t_i = \frac{rand(i)\exp(-\beta z_L)}{\gamma_1[1 - rand(i)]}, \qquad \text{for} \quad i = 1, \ldots, 114 \tag{6.5}$$

where $rand(i)$ is uniformly distributed random variables on the interval (0, 1).

And 57 simulated failure times are generated for medium stress level based on the equation

$$t_i = \frac{rand(i)\exp(-\beta z_M)}{\gamma_1[1 - rand(i)]}, \qquad \text{for} \quad i = 115, \ldots, 171 \tag{6.6}$$

As well there are 29 simulated failure times generated for high stress level derived from the equation

$$t_i = \frac{rand(i)\exp(-\beta z_H)}{\gamma_1[1 - rand(i)]}, \qquad \text{for} \quad i = 172, \ldots, 200 \tag{6.7}$$

Any simulated failure time greater than 100 will be censored.

Similarly, the model parameters for the step-stress ALT simulation are listed in Table 6.3.

Table 6.3 Model parameters for the step-stress ALT simulation

Common parameters	Censoring time: $\tau_2 = 53$; Design stress level: $z_D = 3.5$; Stress changing time $\tau_1 = 46$; Baseline parameters: $\beta = -1.8$, $\gamma_1 = 10$	
Stress	Low	High
Stress level	$z_1 = 3.15$	$z_2 = 1.19$
Sample size	$n = 200$	
Dataset	VI	V

Using the model parameters in Table 6.3, the Monte Carlo simulation data for the step-stress ALT are generated based on the cumulative exposure model, which is explained in Chapter 5 and shown in Figure 6.4. To be more specific, the procedure of generation of the Monte Carlo simulation data for the step-stress ALT is as follows

1. A total of 200 failure times are generated based on the equation

$$t_i = \frac{rand(i)\exp(-\beta z_1)}{\gamma_1[1-rand(i)]}, \qquad \text{for} \quad i = 1, \ldots, 200 \tag{6.8}$$

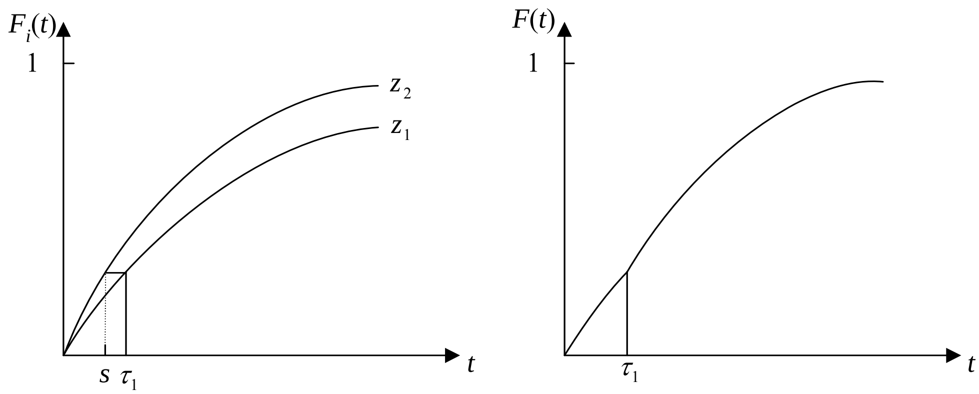

Figure 6.4 Cumulative exposure model

2. Among the 200 failure times simulated in step 1, all failure times greater than τ_1 are discarded.

3. Suppose that only n_1 failure times are kept in step 2. The remaining $200 - n_1$ failure time are generated based on the equation

$$t_i = \frac{rand(i)\exp(-\beta z_2)}{\gamma_1[1-rand(i)]} + \tau_1 - s, \qquad \text{for} \quad i = n_1 + 1, \ldots, 200 \tag{6.9}$$

4. Among the $200 - n_1$ failure times simulated in step 3, all failure times smaller than τ_1 are discarded and all failure times greater than τ_2 are censored.

The results of the reliability estimations from the simulated failure time data based on the constant-stress ALT plan and the step-stress ALT plan are shown in Figure 6.5.

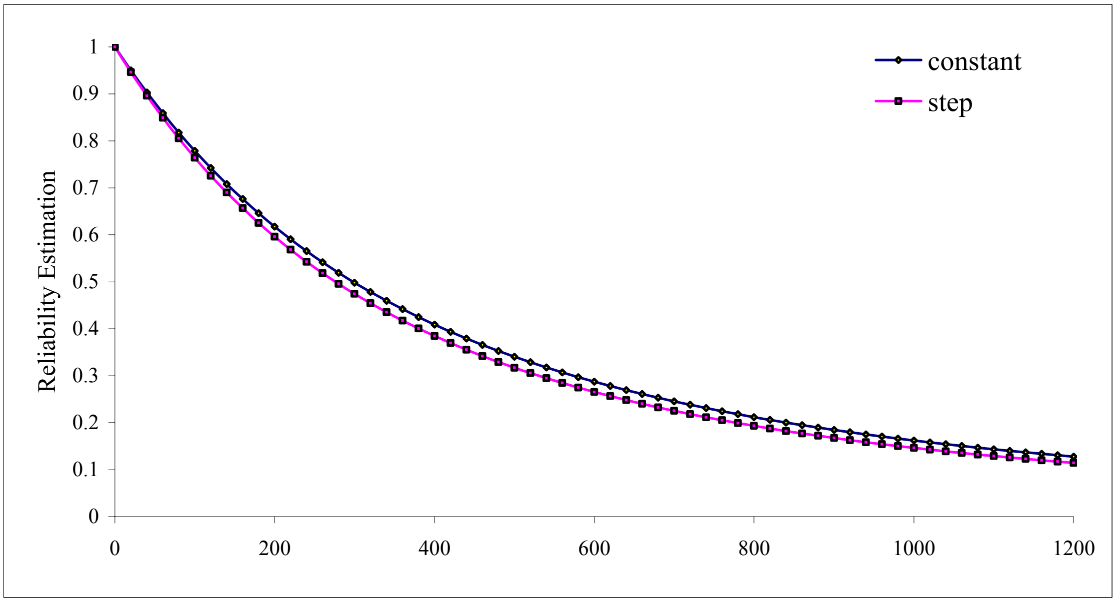

Figure 6.5 Estimated reliability functions from constant and step simulated data

As shown in Figure 6.5, the estimated reliability functions are so close that the equivalency of the constant-stress ALT plan and step-stress ALT plan are verified.

CHAPTER 7

EXPERIMENTAL VALIDATION

The objective of this chapter is to validate the PO model and the optimal ALT plans based on this model by conducting an accelerated life testing in the Quality and Reliability Engineering Laboratory of the Industrial and Systems Engineering Department.

7.1 Experimental Samples

The experiment is designed to study the effects of temperature and voltage on the lifetime distribution of the miniature light bulbs, to predict their reliability at normal operating conditions using the PO model, and to design optimum ALT plans based on the PO model. Each experimental set has a board that contains up to 32 miniature light bulbs depending on the applied stress levels. The set is placed in a temperature and humidity chamber where humidity is held constant.

The test units are Chicago Miniature 606-CM49 type miniature light bulbs. The design work conditions of this light bulb are: voltage is 2 volts, and current is 0.06 amps. Mechanically, light bulbs consist of a metal base, which itself consists of a screw thread contact (attached electrically to one side of the filament), insulating material and an electrical "foot" contact (the little brass bulge on the bottom which is electrically connected to the other end of the filament). The metal contacts at the base of the bulb are connected to two stiff wires that go to the center of the bulb and, in turn, hold the filament. The bulb itself is the glass housing that not only shields the filament from oxygen in the atmosphere but also holds in an inert gas, usually argon. The filament is the part of the bulb that does the work to create light. It is made up of a long, extremely thin (about .01 inch) length of tungsten, a very versatile metal. The typical filament in a household bulb is over six feet long and is tightly wound to form a double-coil.

When electricity is passed through the bulb, the electrons (current) vibrate through the filament, creating very high temperatures: up to 4,000 degrees F. This temperature is needed to cause the atoms in the filament to gain energy and then to emit photons in sufficient quantities to bathe the area in a useful amount of visible light. Most of the photons emitted by a bulb are infrared. Tungsten is one of the only widely available metals that can withstand such temperatures, but in the presence of oxygen it will catch fire and burn itself up. This is why the bulb is evacuated and filled with an inert gas.

Light bulbs eventually fail due to one of four modes:

1. Breakage of the glass bulb accounts for a small portion of failures, especially in motor vehicles.

2. Sealing Failure occurs when the bulb's atmospheric seal is broken and oxygen enters the bulb. The filament burns up instantly. Such failures occur when bulbs are screwed into sockets too tightly.

3. Long Term Failure occurs when the filament eventually becomes so fatigued that its electrical resistance increases to the point that current will not flow. The inside of the bulb gets very dark and the electrical contact on the base starts to burn.

4. Thermal shock is the most common failure mode of bulbs. As soon as the switch is turned on the bulb flashes brightly and then fails. The reason most bulbs fail in this manner is thermal shock. When the switch is turned on, full current suddenly hits the filament at the speed of light. This sudden, massive vibration causes the filament to wildly bounce. This mechanical movement causes metal fatigue (just like bending a paper clip until it breaks) that results in breakage of the filament.

The results of three experiments are used in estimating the parameter of the PO so that the reliability of miniature light bulbs is estimated using the PO at the same stress conditions of the fourth experiment. The reliability is then compared with the experimentally obtained reliability from the result of fourth experiment. The details of experiment are presented in the next section. Typical set up that shows all bulbs working is shown in Figure 7.1 while Figure 7.2 shows some failed bulbs.

Figure 7.1 Samples of the miniature light bulbs

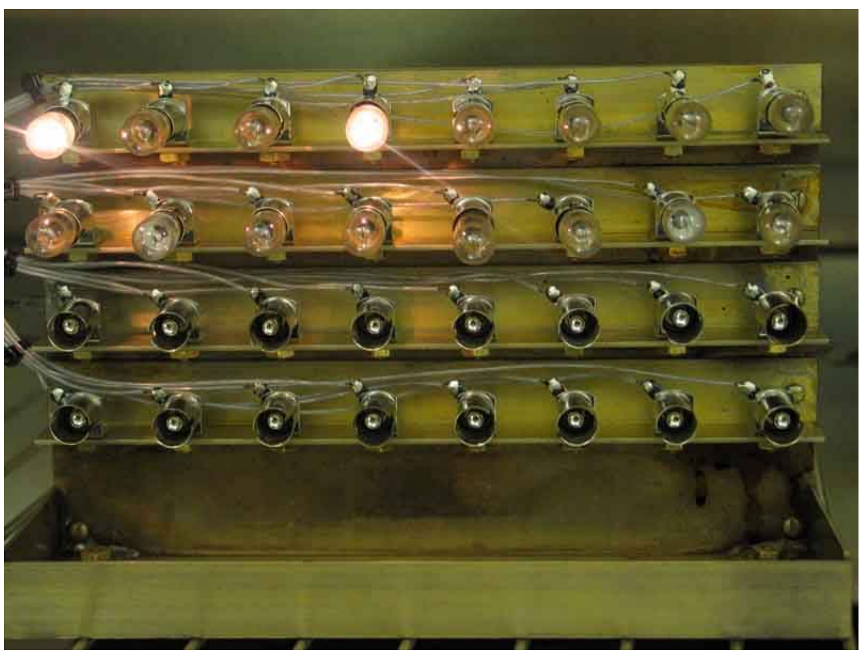

Figure 7.2 Miniature light bulbs testing set

7.2 Experiments Setup

In order to continuously monitor the failure times of testing components and to control the applied factors, an automatic accelerated life testing environment is designed. Figure 7.3 depicts the layout of the experimental equipment.

The NI PCI 6229 is a multifunction analog, digital and timing data acquisition board with 32 analog input, 48 digital I/O and . The SCB-68 is a shielded I/O connector block with 68 screw terminals for easy signal connection to a National Instruments 68- or 100-pin DAQ device. The SCB-68 features a general breadboard area for custom circuitry and sockets for interchanging electrical components. The two expansion boards for the computer are used to retrieve the information of the current status of testing components and the testing environment.

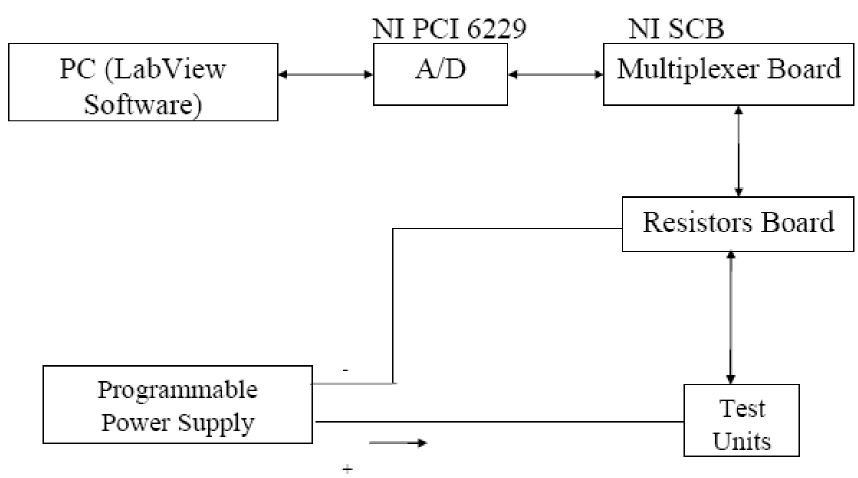

Figure 7.3 The layout of the accelerated life testing equipment

NI 6229 DAQ board is connected to SCB 68 I/O connector block, which is then wired to the resistor board and the test units.

Figure 7.4 shows the programmable power supply used to provide different voltage stress levels.

Figure 7.4 The programmable power supply

NI LabView software is used to develop the application used to continuously monitor the status of the test units. The failure time data are automatically saved as spreadsheet file by LabView application. Figure 7.5 shows the graphic user interface of the programmed LabView application.

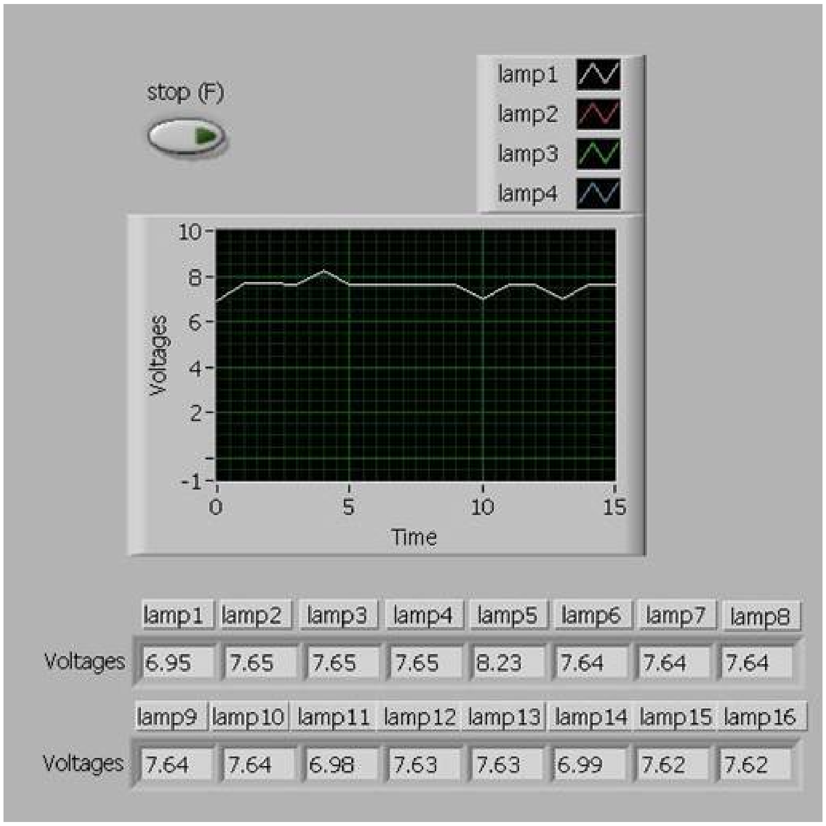

Figure 7.5 LabView application interface

7.3 Test Conditions

Accelerated tests are performed at DC voltage stress of $3.5V \sim 5V$ and temperature between 75^oC and 200^oC. Temperatures less than 75^oC and above 200^oC are not appropriate since it is difficult to observe the miniature light bulb failure in a reasonable test time and extrapolation can not be justified respectively. As a compromise, the range of $75^oC \sim 200^oC$ is known as the appropriated temperature range to observe failures of miniature light bulbs,

The experiment is conducted at four different stress levels (75^oC, $3.5V$), (75^oC, $5V$), (150^oC, $3.5V$), (150^oC, $5V$), and (50^oC, $2V$). Miniature light bulbs are tested at each test level. In each test a miniature light bulbs set is placed in a temperature chamber where both the temperature and applied voltage in the circuit are held constant. The working status of the light bulbs is automatically measured and recorded by LabView application.

7.4 Analysis of the Experimental Result

The test begins with up to 32 miniature light bulbs being tested at each of the first four experimental conditions, and it is conducted at four different stress levels as stated in section 7.3. Table 7.1 contains the summary of the failure time (in hours) data from the experiment. The last data set from experimental condition (50^oC, $2V$) is only used to verify the reliability prediction of the models.

Table 7.1 Summary of the experimental data

Level 1		**Level 2**		**Level 3**		**Level 4**		**Level 5**	
(75^oC, $3.5V$)		(75^oC, $5V$)		(150^oC, $3.5V$)		(150^oC, $5V$)		(50^oC, $2V$)	
Time	Failure	Time	Failure	Time	Failure	Time	Failure	Time	Failure
43.6	1	22.3	1	20.5	1	37.8	1	223.1	1
51.1	1	24.7	1	23.2	1	65.9	1	254.0	1
58.6	1	39.6	1	26	1	75.6	1	316.7	1
65.5	1	41.8	1	34.1	1	82.5	1	560.2	1
89	1	47.7	1	43.6	1	88.1	1	679.0	1
121.5	1	62.1	1	44.9	1	106.6	1	737.0	1
151.8	1	65.5	1	61.6	1	113.1	1	894.4	1
171.7	1	87.8	1	70.8	1	121.1	1	930.5	0
181	1	118.3	1	145.4	1	128.3	1	930.5	0
211.7	1	120.1	1	206.7	1	202.7	1	930.5	0

230.7	1	157.4	1	215.2	1	249.9	1	930.5	0
275.6	1	180.9	1	218.7	1	506.4	1	930.5	0
285	1	187.7	1	313.7	1	876.3	1	930.5	0
296.2	1	204	1	314.1	1	890.0	0	930.5	0
358.5	1	213.9	1	317.9	1	890.0	0	930.5	0
379.8	1	254.1	1	430.2	1	890.0	0	930.5	0
434.5	1	262.6	1					930.5	0
493.1	1	293	1					930.5	0
561.1	1	304	1					930.5	0
570	1	337.7	1					930.5	0
577.7	1							930.5	0
922	1							930.5	0
941	0							930.5	0
941	0							930.5	0

Figure 7.6 illustrates the log of cumulative hazard rate plot of the first four data sets. These four estimated functions are not well approximated by parallel lines. Therefore, the Proportional Hazard model is not an approximate failure time model for this data set.

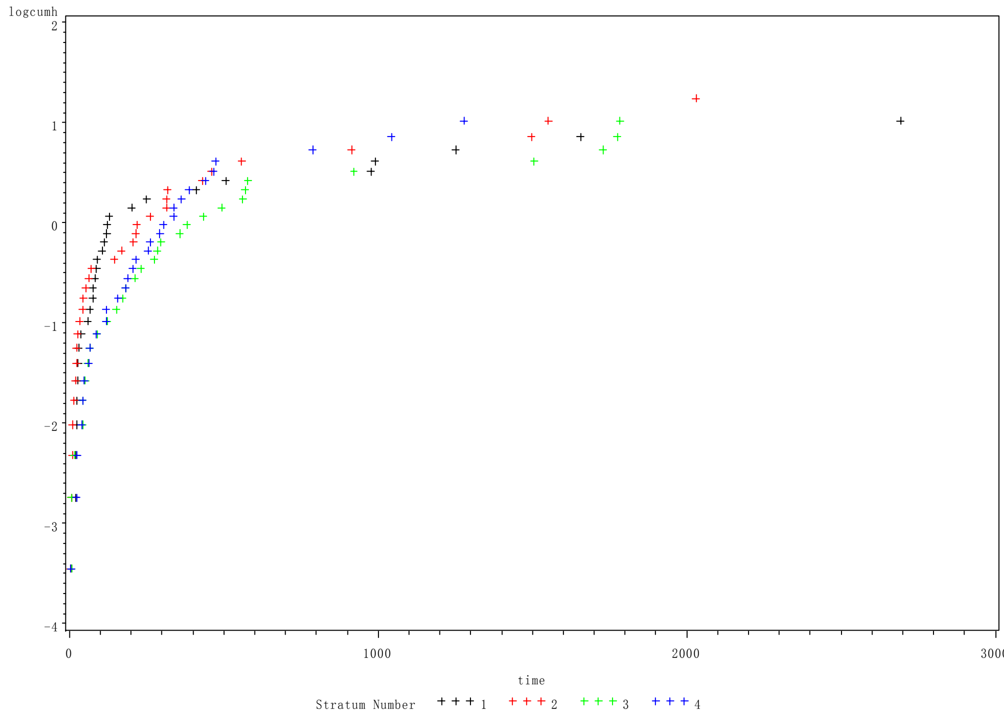

Figure 7.6 Log of cumulative hazard rate plot

Compared with the PH model, the PO model ($\theta(t\,|\,z) = \exp(\boldsymbol{\beta}^T \boldsymbol{z})\theta_0(t)$) where $\boldsymbol{\beta} = (\beta_1, \beta_2)$, and $\boldsymbol{z} = (z_1, z_2)$ correspond to absolute temperature and voltage in V provides a broader feasibility for analyzing the miniature light bulb failure time data. A simple quadratic polynomial function $\theta_0(t) = \gamma_1 t + \gamma_2 t^2$ is found sufficient to approximate the baseline odds function.

Table 7.2 gives the parameters of the PO model at constant stress level.

Table 7.2 Estimated parameters of the PO model

Parameter	Estimates
γ_1	0.1089
γ_2	0.0121
β_1	-1.1925
β_2	0.0498

It is more important to verify the reliability prediction of the model by comparing the estimated reliability at level 5 condition by using the model to its Kaplan-Meier reliability estimates. Figure 7.7 shows the results of three reliability curves estimated by Kaplan-Meier method or predicted by the PO model and the PH model, respectively. The results show that the PO model provides more accurate reliability predication than the PH model.

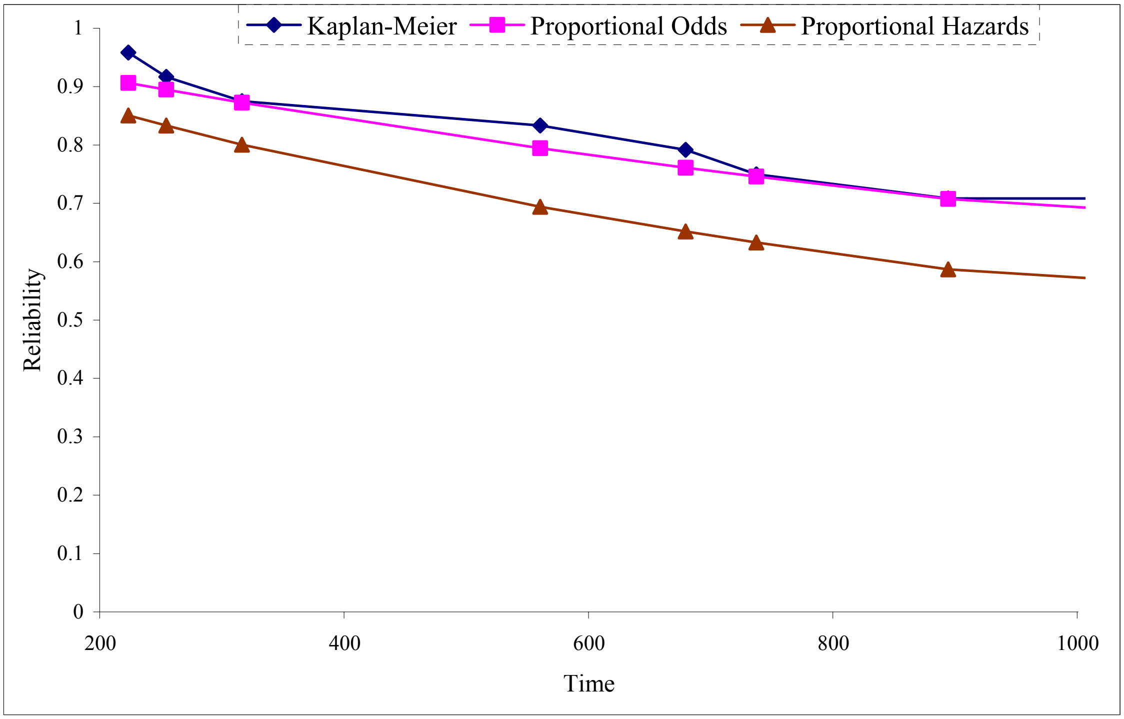

Figure 7.7 Result comparison

CHAPTER 8

CONCLUSIONS AND FUTURE WORK

We have investigated several challenging topics related the PO-based accelerated life testing models, including parameter estimation procedures for the PO-based ALT model, the construction of the confidence intervals, the validation of the PO-based ALT model, the constant-stress and step-stress PO-based multiple-stress-type ALT plans, and the equivalency of ALT plans. We summarize the main conclusions of these topics and future improvements.

8.1 Summary

8.1.1 The PO-based ALT Model and Estimation Procedures

In Chapter 3, we present the PO-based ALT model and propose relatively simple parameter estimation procedures by approximating the general baseline odds function with a polynomial function. The log-likelihood function is constructed to obtain the maximum likelihood estimates of the model parameters. This proposed PO-based ALT model can be used to not only estimate the heterogeneity described by the stresses and the corresponding regression parameters, but also to extrapolate reliability performance at the design stress level. Numerical examples with experimental data and Monte Carlo simulation data are used to verify that the applicability of the PO-based accelerated life testing. The results show that the PO-based ALT model provides more accurate reliability estimates for the failure time data exhibiting proportional odds properties than the PH-based ALT model.

8.1.2 Confidence Intervals and Model Validation

In Chapter 4, confidence intervals of the unknown parameters and the predicted reliability at the design stress level are obtained through Fisher information matrix. We also present the likelihood ratio test and modified Cox-Snell residuals to validate the model. The likelihood ratio test is used to verify the model sufficiency. The order of the baseline polynomial odds function is determined by the likelihood ratio test. The Cox-Snell residuals are constructed to check the assumption of proportional odds. Literally, the Cox-Snell residual $r_i = \hat{\Lambda}(t_i; z_i)$ is the cumulative hazard rate function of the failure time t_i estimated by the PO-based ALT model. If the model assumption holds, the Cox-Snell residuals exhibit an exponential distribution with hazard rate of 1 and the plot of the cumulative hazard rate of the Cox-Snell residuals versus the Cox-Snell residuals is a straight line with slope of 1. Therefore we check the model assumption by plotting the Nelson-Aalen estimators of the cumulative hazard rate of the Cox-Snell residuals.

8.1.3 The PO-based Multiple-stress-type ALT Plans

The accuracy of the reliability estimates obtained via extrapolation in both stress and time is a major issue in the accelerated life testing problem. One interesting measurement to obtain more accurate estimates is to devise a test plan that tests units at appropriately selected stress levels with proper allocation of test units to each level. Design of ALT plans under one type of stress may mask the effect of other critical types of stresses that could lead to the component's failure. Furthermore, due to the technological advances, the extended life of today's products makes it difficult to obtain enough failures in a reasonable testing duration using single stress type when carrying out an accelerated life test. Therefore, it is more realistic to consider multiple stress types. In the Chapter 5, we investigate the design of optimum ALT plans based on the proportional odds model with multiple stress types for both constant stress loading and simple step-stress loading. We formulate nonlinear optimization problems to determine the optimum ALT plans. Due to the highly nonlinear properties of those problems, we solve the optimization problems using a numerical method, COBYLA (Constrained Optimization BY Linear Approximations) optimization method. We show that applying multiple stresses in a systematic way results in a significant reduction in the test duration while obtaining "good" estimates of reliability at design conditions.

8.1.4 The Equivalency of ALT Plans

The investigation of the equivalency of ALT plans involving different stress-loading types is given in Chapter 6. The definitions of the equivalent ALT plans are discussed. The initiative research of equivalent ALT plans focuses on the equivalency of constant-stress ALT plan and step-stress ALT plan for single stress type problem. In this scenario, the baseline constant-stress ALT plan is given and the equivalent step-stress ALT plan is determined based on the definition of the equivalency of ALT plans. The research of the equivalent ALT plans enables reliability practitioners to choose the appropriate ALT plan under given constraints. The results show that an equivalent step-stress plan to the constant stress plan results in a significant reduction in the test period without sacrificing the accuracy of the reliability estimates at design stress conditions.

8.2 Future Work

The PO-based ALT model and its optimum ALT plans and corresponding equivalent ALT plans provide advantages for accurate reliability prediction. To further refine the results, we need to improve and extend our work as follows.

The PO model discussed in this dissertation only considers time-constant coefficients. It should be extended to incorporate time-varying coefficients for both baseline odds function and stress effects. This will accommodate situations when the applied stress is time-dependent.

In the dissertation, we assume that the effects of different stress types are not correlated for multiple-stress-type ALT and test plans. The correlations of different stress types should be considered in future research for more practical cases.

To approximate the general odds functions, the polynomial function is utilized due to the properties of odds functions. Exploring more general baseline odds functions will further extend the breadth of the application of the proposed ALT model and plans.

In this dissertation, only a simple case of equivalent ALT plans is investigated. Statistical and analytical models should be used to investigate the equivalency among ALT experiments involving different stress loadings; develop a general approach for the design of optimum test plans with flexibility of choosing stress loadings and adjusting their parameters and extend the results to the optimum design of test plans under multiple stresses scenarios.

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CURRICULUM VITA

HAO ZHANG

1995	B.S., Automatic Control, Harbin Institute of Technology, Harbin, China
1998	M.S., Automation, South China University of Technology, Guangzhou, China
1998-1999	Systems Engineer, Guangdong Planning and Designing Institute of Telecom, Guangzhou, China
2001	M.S., Industrial Engineering, Montana State University, Bozeman, MT
2002-2006	Graduate Assistant, Department of Industrial & Systems Engineering, Rutgers, The State University of New Jersey, New Brunswick, NJ
2005	M.S., Statistics, Rutgers, The State University of New Jersey, New Brunswick, NJ
2006	Statistician, Decision Science, Fidelity National Information Services, St Petersburg, FL

PUBLICATIONS RELATED TO THE DISSERTATION

1. Zhang, H., and Elsayed, E. A., (2006), "Nonparametric accelerated life testing based on proportional odds model," *International Journal of Reliability, Quality and Safety Engineering*, Vol. 13, No. 4, 365-378.

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