DIFFERENTIAL DELAY EQUATIONS WITH SEVERAL FIXED DELAYS

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ABSTRACT OF THE DISSERTATION

Differential Delay Equations with Several Fixed Delays

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We study nonlinear autonomous real-valued differential delay equations with several fixed delays

$$x'(t) = \sum_{i=1}^{D} F_i(x(t-d_i)), \tag{1}$$

where the F_i are continuous, have nonzero limits at $\pm\infty$, and are similar (in a sense we make precise) to step functions. Our focus is on periodic solutions of (1), and our approach is to link (1) to an appropriately related equation

$$y'(t) = \sum_{i=1}^{D} h_i(y(t-d_i)) \tag{2}$$

where the h_i are in fact step functions.

Given a periodic solution p of (2), we describe conditions under which this solution implies the existence of a similar periodic solution q of (1), and further conditions under which asymptotic stability of p implies asymptotic stability of q.

We also make a partial study of the global dynamics of (2).

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I finally thank Professor Stephen Greenfield for his friendly advice throughout my time at Rutgers, especially with my teaching and my job search.

Dedication

In gratitude for the joy it has brought me, I humbly dedicate this thesis to the glory of ALMIGHTY GOD.

> *Great are the works of the Lord, studied by all who have pleasure in them.*
> *Full of honor and majesty is his work, and his righteousness endures for ever.*
>
> – Psalm 111

Table of Contents

Chapter 1

Introduction

1.1 What and why

We study differential delay equations of the form

$$x'(t) = \sum_{i=1}^{D} F_i(x(t-d_i)),\ t > 0; \quad x(t) = x_0(t),\ t \in [-d_D, 0] \tag{1.1.1}$$

where $x(t)$ is real, the F_i are continuous functions, $0 < d_1 < \cdots < d_D$, and $x_0(\cdot)$ is a continuous function on $[-d_D, 0]$. (1.1.1) is a real-valued autonomous equation with several fixed delays.

A time-varying quantity might be modeled by an equation like (1.1.1) if it is governed by distinct feedback mechanisms with different time delays (the level of a protein regulated by multiple pathways); a particular feedback mechanism whose strength varies over time (the size of a population whose members' fecundity varies with age); or some delayed control effect that is later undone (the amount in a tank of a chemical that breaks down after a fixed period) . Here is an example of the third type, adapted from [5]. Suppose that a species is reproductive from ages d_1 to d_2, $p(t)$ is the population of reproductive individuals at time t, and the reproduction rate is given by $g(p(t))$. Under the assumption that all individuals live through reproductive age, the function p satisfies

$$p'(t) = g(p(t-d_1)) - g(p(t-d_2)). \tag{1.1.2}$$

The collection of proposed applications of (1.1.1) does not seem very rich; this is both cause and consequence of the fact that these equations (when $D \geq 2$ and the

F_i are nonlinear) have received relatively little attention. The equations (1.1.1) are, though, natural extensions of their intensively-studied single-delay counterparts. It is not surprising that introducing additional delays allows for new dynamical phenomena, and we will discuss some of these. Nevertheless, numerical studies of (1.1.1) with several delays leaves the same general impression as with one delay: that strongly attractive periodic solutions frequently play a dominant role in the global dynamics of the equations.

Our primary objective, therefore, is to explore the existence and stability of periodic solutions of (1.1.1). We focus on equations where the feedback functions F_i are similar in some suitable sense to step functions, and our approach is to link the behavior of (1.1.1) to the behavior of appropriately related equations where the feedback functions are in fact step. We call these latter equations *step differential delay equations*; they are of the form

$$y'(t) = \sum_{i=1}^{D} h_i(y(t-d_i)),\ t>0;\quad y(t) = y_0(t),\ t\in[-d_D, 0] \tag{$SDDE$}$$

$$h_i(y) = \begin{cases} b_i \neq 0, & y<0; \\ 0, & y=0; \\ -a_i \neq 0, & y>0. \end{cases}$$

The idea of using differential delay equations with step feedback to shed light on more classical equations is not new, and has ranged in its application from the heuristic to the rigorous. We will show, loosely speaking, that certain periodic solutions of ($SDDE$) imply the existence of similar periodic solutions of (1.1.1) if the feedback functions F_i are similar enough to the h_i. The following two examples illustrate and motivate our main results.

Example 1.1.3. Consider the function

$$F(x) = \frac{1-e^x}{e^x + \frac{1}{2}}$$

and the family of problems

$$x'(t) = rF(x(t-1)) + rF(x(t-5)),\quad r>0. \tag{E_r}$$

Numerical studies suggest that, for moderate to large values of r, equation (E_r) has a stable periodic solution q with four zeros per period, with one of the gaps between successive zeros of q greater than the longest delay 5. Accordingly we seek asymptotic theorems about the existence and stability of solutions like q as $r \to \infty$.

By recasting (E_r) we can clarify its asymptotic relationship to $(SDDE)$: if x is a solution of (E_r), then the function $y(t) = x(t)/r$ satisfies the equation

$$\begin{aligned} y'(t) = \frac{1}{r}\left(rF(x(t-1)) + rF(x(t-5))\right) &= F(ry(t-1)) + F(ry(t-5)) \qquad (\tilde{E}_r) \\ &= \tilde{F}(y(t-1)) + \tilde{F}(y(t-5)), \end{aligned}$$

where $\tilde{F}(y) = F(ry)$. Thus solutions of (E_r) correspond to solutions of $(\tilde{E}_r)$. If we replace $\tilde{F}$ with its pointwise limit as $r \to \infty$, we obtain the equation with step feedback

$$y'(t) = h(y(t-1)) + h(y(t-5)), \qquad (SE)$$

where

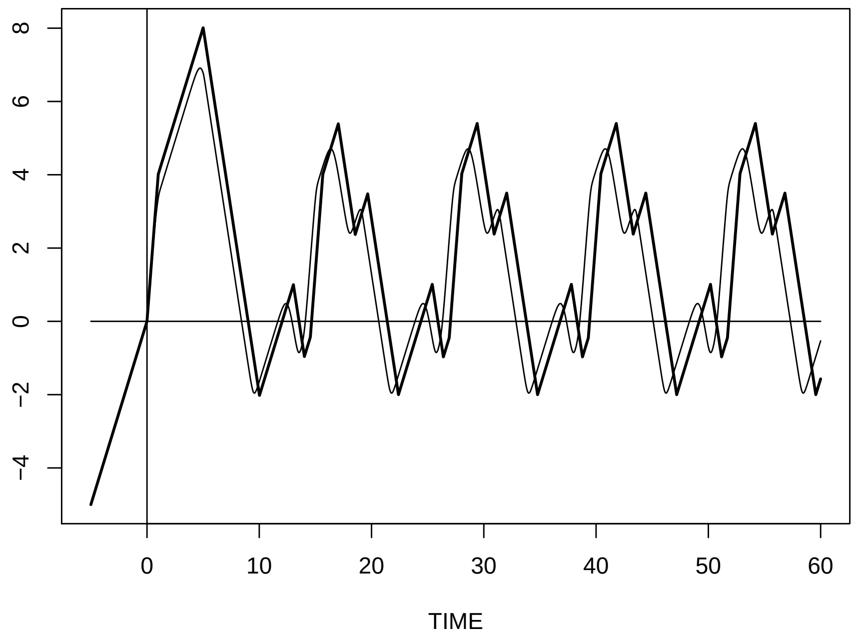
$$h(y) = \begin{cases} 2, & y < 0; \\ 0, & y = 0; \\ -1, & y > 0. \end{cases}$$

In Figure 1 we show simulated continuations of the initial condition

$$y_0(t) = t, \; t \in [-5, 0]$$

both as a solution of $(\tilde{E}_5)$ and as a solution of (SE) (the thicker curve, with the "corners," is the solution of (SE)). We see that the two solutions track each other fairly well and that both solutions apparently converge to periodic solutions with the features described above.

Figure 1



Given any continuous function $y_0 : [-5, 0] \to \mathbf{R}$ with $y_0(s) < 0$ for $s \in [-5, 0)$ and $y_0(0) = 0$, it is fairly simple to calculate, explicitly, its continuation y as a solution of (SE). Performing such a calculation shows that y coincides, after its first positive zero at time $t = 8$, with the periodic solution suggested in Figure 1. We shall prove that this easily obtained periodic solution implies the existence of a similar periodic solution of $(\tilde{E}_r)$ when r is sufficiently large.

Example 1.1.4. Consider the function

$$G(x) = \frac{1 - e^x}{1 + e^x}$$

and the family of problems

$$x'(t) = 2G(rx(t-1)) + G(rx(t-3)). \tag{E_r}$$

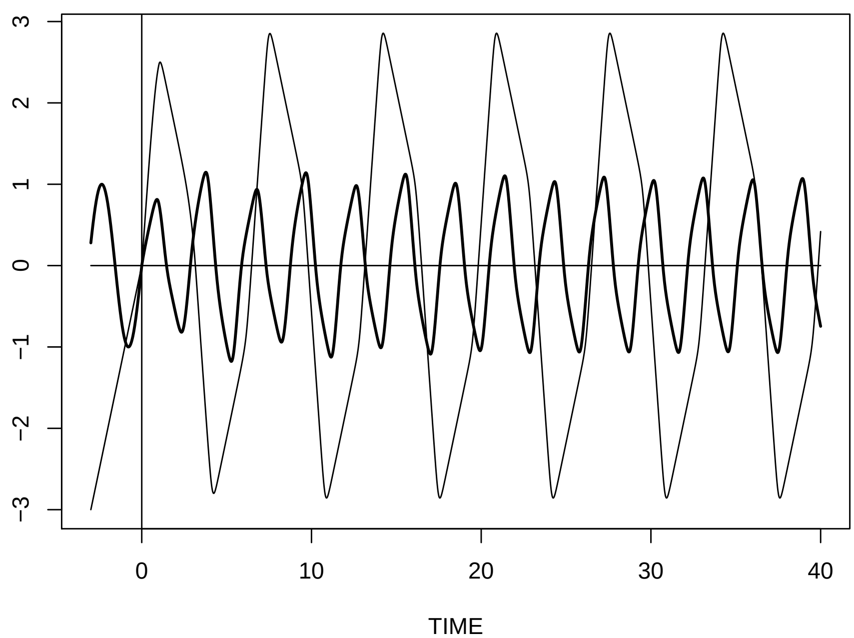
A theorem of Nussbaum's (Theorem 2.3 in [27]) implies that, for sufficiently large r, (E_r) has two periodic solutions, both with zeros separated by at least $3/2$. We amplify this result here.

The corresponding step problem is

$$y'(t) = -2\text{sgn}(y(t-1)) - \text{sgn}(y(t-3)). \tag{SE}$$

It is easy to verify that (SE) has (among many others) three periodic solutions: one with zeros separated by $10/3$, another with zeros separated by 2, and another with zeros separated by $10/7$. (With the apparatus we'll develop later — and a little bit of practice — finding these solutions in the first place is not too difficult either.) Moreover, the first and third solutions are asymptotically stable in a sense that we shall make precise in sections 2.3 and 3.1 (see especially remark 3.1.15). Our results imply that, for r sufficiently large, (E_r) has periodic solutions similar to these three, respectively, and that the first and third are asymptotically stable. Indeed, simulation seems to detect the two asymptotically stable solutions; we show such simulated solutions (with $r = 5$) in Figure 2.

Figure 2



The usefulness of our results rests, of course, on the fact that particular periodic solutions of the step-feedback problems $(SDDE)$ are often quite easy to find. It also turns out that the global dynamics of $(SDDE)$ are compelling: tractable enough to invite thorough investigation, but not trivial. Accordingly a study of $(SDDE)$ develops interest in its own right — while hopefully also giving additional (heuristic) insight into (1.1.1).

In the next section we present some general background on (1.1.1). In the following three sections we describe some known results on, respectively, (1.1.1) with a single delay, (1.1.1) with multiple delays, and a generalization of ($SDDE$) with a single delay. We conclude this introductory chapter with a more specific description of the problems we shall be considering and the introduction of some notation.

The simulations and figures

We have generated all of our simulated solutions and created all of our figures with the software package R. We also used R for the matrix calculations in the analysis of equation (4.4.10) and in example 5.2.3. R is developed by the R Foundation for Statistical Computing, Vienna, Austria; see `www.r-project.org`.

We use the following forward Euler method for simulating solutions of

$$y'(t) = \sum_{i=1}^{D} F_i(y(t-d_i))$$

(for F_i continuous or step). Select a step size $h > 0$ such that d_i is a positive integer multiple of h for all i (note that this requires that the d_i be rationally related). We approximate y with a function Y defined at points kh, where k belongs to the set of integers greater than or equal to $-d_M/h$. We first specify initial values for Y at the points

$$-d_M, -d_M + h, -d_M + 2h, \ldots, -h, 0.$$

We then put

$$Y(h) = Y(0) + h\left[\sum_{i=1}^{D} F_i(Y(-d_i))\right].$$

We similarly generate values $y(kh)$ for $k > 1$, as many as desired. Detailed information on the code used to generate the simulated solutions is available from the author.

Most of the simulations and figures in this thesis use a step size of $h = 10^{-2}$; occasionally we have used a step size of 10^{-3}.

1.2 Background on equation (1.1.1)

In this section we recall some well-known results on equation (1.1.1):

$$x'(t) = \sum_{i=1}^{D} F_i(x(t-d_i)),\ t > 0; \quad x(t) = x_0(t),\ t \in [-d_D, 0]$$
$$0 < d_1 < \cdots < d_D = \gamma; \qquad\qquad F_i \text{ continuous.}$$

Our discussion is in the spirit of the now-standard [11] or its later incarnation [13]. The presentation of the return map is essentially that of Xie (see [41] and [39]).

Given $d_D = \gamma > 0$, write $C = C[-\gamma, 0]$ for the space of continuous real-valued functions on $[-\gamma, 0]$, equipped with the sup norm. Given any real-valued function $y(\cdot)$ defined on $[t-\gamma, t]$ for some t, in the standard way we write y_t for the element of C given by

$$y_t(s) = y(t+s), \quad s \in [-\gamma, 0].$$

To every initial condition $x_0 \in C$ corresponds a unique solution $x(\cdot)$ of (1.1.1) that is defined on $[-\gamma, \infty)$ and satisfies $x(t) = x_0(t)$ for $t \in [-\gamma, 0]$. We call $x(\cdot)$ the *continuation of* x_0 *as a solution of* (1.1.1); we call any x_t with $t \geq 0$ a *section* of $x(\cdot)$.

We write $T : \mathbf{R}_+ \times C \to C$ for the *solution operator*

$$T(t, x_0) = x_t,$$

where $x(\cdot)$ is the continuation of x_0 as a solution of (1.1.1). The map T is continuous, and the map $T(\tau, \cdot)$ is completely continuous for any given $\tau \geq \gamma$. If the feedback functions F_i are C^1, the map $T(\tau, \cdot)$ is C^1 for any fixed $\tau > 0$ and the map T is C^1 on $(\gamma, \infty) \times C$.

Given a solution $x(\cdot)$ continuing $x_0 \in C$, there might or might not be a way to extend $x(\cdot)$ so that equation (1.1.1) is satisfied for all $t \in \mathbf{R}$. We can certainly do this, for example, if $x(t)$ is periodic; more generally, we can do this if x_0 lies in the attractor – if one exists – for the solution semiflow on C. (For a general discussion of attractors for so-called dissapative systems, and how semiflows extend to flows on attractors, see Hale's book [12]). Therefore, while solutions are in general defined on $[-\gamma, \infty)$, we will not hesitate to view periodic solutions as defined for all time.

Suppose that the F_i are C^1. Given some particular solution $x(t)$ of (1.1.1), the *linearization about* $x(t)$ is the differential delay equation

$$y'(t) = \sum_{i=1}^{D} F_i'(x(t-d_i)) \cdot y(t-d_i), \quad y_0 \in C.$$

Unique solutions of this equation also exist for all positive time; write $L_x : \mathbf{R}_+ \times C \to C$ for the solution operator. The operator L_x and the partial derivative of T with respect to its functional coordinate are related as follows: for any $\tau > 0$ and $y_0 \in C$,

$$D_2T[\tau, x_0]y_0 = L_x(\tau, y_0).$$

(Throughout we shall use notation like the above for Fréchet derivatives: $D_iG[a,b]$ is the derivative of G with respect to the ith coordinate at the point (a,b).)

Write C_0 for the subspace $\{x_0 \in C \ : \ x_0(0) = 0\}$. Take $x_0 \in C_0$ with continuation $x(t)$ as a solution of (1.1.1) and suppose that there is some time $\tau(x_0) > \gamma$ for which

$$T(\tau(x_0), x_0) \in C_0 \text{ and } x'(\tau(x_0)) \neq 0.$$

Then, by the implicit function theorem, $\tau(x_0)$ extends to a unique C^1 real-valued map on a neighborhood $U \subset C_0$ about x_0 with the feature that, for all $y_0 \in U$ with continuation $y(t)$ as a solution of (1.1.1),

$$\tau(y_0) > \gamma; \quad y(\tau(y_0)) = 0; \quad y'(\tau(y_0)) \neq 0.$$

$\tau(y_0)$ is called the *return time* of y_0. The map

$$R : U \to C_0 \ \text{ given by } \ R(y_0) = T(\tau(y_0), y_0)$$

is called a *return map* (R is a Poincaré map on the hyperplane C_0). Return maps can, of course, be defined on subsets of C other than C_0; but the return map we have described is the only one we shall need.

Any fixed point y_0 of R is a section of a periodic solution $y(\cdot)$ of (1.1.1), with minimal period dividing $\tau(y_0)$. Finding fixed points of maps like R is one of the most prominent techniques for establishing the existence of periodic solutions of differential delay equations, and is what we do in this thesis.

Proposition 1.2.1. Facts about R **(see [41], [39])** *As described, the map* $R : U \to C_0$ *is completely continuous. If the functions* F_i *are* C^1*,* R *is* C^1 *also.*

Suppose that the F_i *are* C^1*, and that* $R(x_0)$ *is defined. Then* $DR[x_0]$*, the derivative of* R *at* x_0*, is a compact linear operator. Writing* $\tau = \tau(x_0)$*, where* τ *is as above, we have the following formula for the derivative of* $DR[x_0]$*: given* $y_0 \in C_0$*,*

$$DR[x_0]y_0 = L_x(\tau, y_0) - \frac{L_x(\tau, y_0)(0)}{R(x_0)'(0)} \cdot R(x_0)',$$

where $R(x_0)'$ *is the pointwise derivative of the function* $R(x_0) \in C_0$*.*

Suppose in particular that x_0 *is a fixed point of* R*. Then*

$$DR[x_0]y_0 = L_x(\tau, y_0) - \frac{L_x(\tau, y_0)(0)}{x_0'(0)} \cdot x_0'.$$

If the spectrum of $DR[x_0]$ *lies strictly inside the unit circle, then the periodic solution* $x(t)$ *is asymptotically stable: that is, given* $y_0 \in C_0$ *close enough to* x_0*,* $R^k(y_0) \to x_0$ *as* $k \to \infty$*.*

It is sometimes useful to cast (1.1.1) more abstractly. Rendered in the general form for retarded functional differential equations with bounded delay used, for example, in [13], (1.1.1) reads

$$x'(t) = f(x_t),$$
$$f(x) = \sum_{i=1}^{D} F_i(x(-d_i)).$$

If the F_i are C^1, the map $f : C \to \mathbf{R}$ is C^1 also, with derivative given by the formula

$$Df[x]h = \sum_{i=1}^{D} F_i'(x(-d_i))h(-d_i).$$

Let Λ be an open set about zero in some Banach space, write $f(x) = f(x, 0)$, and suppose that f extends to a C^1 map

$$f : C \times \Lambda \to \mathbf{R}.$$

Assume also that solutions of the retarded functional differential equation

$$x'(t) = f(x_t, \lambda),\ x_0 \in C \qquad (RFDE_\lambda)$$

exist and are unique for all positive time, for all $\lambda \in \Lambda$. Suppose in this case that $x(\cdot)$ is a periodic solution of the equation $(RFDE_0)$ (which is just equation (1.1.1)), that $x(0) = 0$, and that a return map R as described above is defined in a neighborhood of $x_0 \in C_0$. Finally, suppose that $DR[x_0]$ does not have eigenvalue 1; in this case the periodic solution $x(\cdot)$ is said to be *nondegenerate*. Then the implicit function theorem tells us that there is a neighborhood about 0 in Λ where there is defined a unique C^1 map $\lambda \mapsto x_0^\lambda \in C_0$ such that $x_0^0 = x_0$ and x_0^λ is a section of a periodic solution of $(RDDE_\lambda)$. Conversationally, a nondegenerate periodic solution $x(\cdot)$ persists under perturbations in the equation.

We now describe an estimate that we will use repeatedly with minimal comment. Let x and y be two functions continuous on $[-\gamma, \infty)$ and differentiable on $(0, \infty)$. Let $b > \gamma$ and suppose that we know that $|x(t) - y(t)| \leq M$ on $[0, b]$ and that $|y'(t)| \leq \mu$ on $(0, b)$. Then, for any $t_1, t_2 \in (0, b)$,

$$|x(t_1) - y(t_2)| \leq |x(t_1) - y(t_1)| + |y(t_1) - y(t_2)| \leq M + \mu|t_1 - t_2|.$$

Therefore if τ_1 and τ_2 lie in (γ, b) we have

$$\|x_{\tau_1} - y_{\tau_2}\| \leq M + \mu|\tau_1 - \tau_2|. \tag{1.2.2}$$

We will use this estimate in the following way. Suppose now that x and y are solutions of (1.1.1) and that $R(x_0)$ and $R(y_0)$ are both defined. Suppose also that there are numbers a and b such that we know that the following are true:

- $\gamma < a < b$;
- $\tau(x_0)$ and $\tau(y_0)$ both lie in (a, b);
- On (a, b), $x'(t)$ and $y'(t)$ are of the same fixed sign and $|y'(t)|$ is greater than some $\sigma > 0$;
- $|x(t) - y(t)| \leq M$ for $t \in [0, b]$;
- $|y'(t)| \leq \mu$ for $t \in (0, b)$.

In this case, since $|y(t)| \geq M$ for

$$t \in (a,b) \setminus (\tau(y_0) - M/\sigma, \tau(y_0) + M/\sigma),$$

we have that $|\tau(x_0) - \tau(y_0)| \leq M/\sigma$; the estimate (1.2.2) then reads

$$\|R(x_0) - R(y_0)\| \leq M\left(1 + \frac{\mu}{\sigma}\right). \tag{1.2.3}$$

Two definitions

We close this section by introducing two definitions that, while not really needed until later, will make some parts of our intervening discussion easier.

Definition 1.2.4. Proper zero. Suppose that y is a continuous function defined in a neighborhood of z, and that $y(z) = 0$. We call z a *proper zero* of y if $y(z-\epsilon)$ and $y(z+\epsilon)$ are of strictly opposite signs for all sufficiently small positive ϵ. If $y(z+\epsilon) < 0$ for such z and ϵ, we call z a *downward proper zero.* If $y(z+\epsilon) > 0$ for such z and ϵ, we call z an *upward proper zero.*

Definition 1.2.5. k-cyclic periodic function. We say that a periodic function $x : \mathbf{R} \to \mathbf{R}$ with period P is *k-cyclic* if x has exactly $2k$ zeros on any interval of the form $(t, t+P]$.

The periodic solutions pictured in Figure 1, for example, are 2-cyclic while those in Figure 2 are 1-cyclic.

1.3 One delay and negative feedback

Probably the best-studied nonlinear delay equation (particularly if one counts work on certain of its generalizations) is

$$x'(t) = F(x(t-1)), \tag{DDE}$$

where F is a smooth function that is bounded above and satisfies the so-called negative feedback condition that $xF(x) < 0$ for nonzero x. In this section we review some of the known theory for equation (DDE). We will assume throughout that F satisfies the above hypotheses.

To simplify and unify our presentation, we will describe many results as weaker than they actually are; in particular, many of the results we review here apply to various equations of the form $x'(t) = g(x(t), x(t-1))$. Not all such equations are reasonably described as mere extensions of (DDE): for example, the "singular perturbation" equation

$$\frac{1}{\mu}x'(t) = -x(t) + F(x(t-1)), \tag{1.3.1}$$

where F is as above and $\mu > 0$, admits distinctive treatment (see, for example, [21]).

Of the techniques used to study periodic solutions of (DDE) many fall into two rough categories. The first are what we might call "phase-plane" techniques: studying the properties of the trajectories traced out by solutions x in the $(x(t), x(t-1))$- or the $(x(t), x'(t))$-plane. This kind of approach does not seem as fruitful when there is more than one delay (but see section 1.4). The second are fixed-point techniques: studying periodic solutions cast as fixed points of return maps like the map R introduced in section 1.2. As already mentioned, this is the sort of approach we will take to our several-delay problems.

Observe that the constant function $x(t) \equiv 0$ is a solution of (DDE). This zero solution is called *hyperbolic* if the characteristic equation

$$\lambda = F'(0)e^{-\lambda},$$

obtained by looking for solutions of the form $e^{\lambda t}$ of the linearization of (DDE) about the zero solution, has no solutions on the real axis. The zero solution is hyperbolic except when $-F'(0) = 2n\pi + \pi/2$, $n \in \mathbf{Z}_+$.

Given a solution x of (DDE), let us define the *oscillation speed* of x at t to be the number of proper zeros of x in $(\tau - 1, \tau)$, where

$$\tau = \inf\{\ s \geq t\ :\ s \text{ is a proper zero of } x\ \}$$

(if τ does not exist, we set the oscillation speed of x at t equal to 0). It has long been recognized that the negative feedback condition in (DDE) implies, roughly speaking, that the oscillation speed of a solution is nonincreasing in time (for a precise statement, see [22]; our definition of oscillation speed is similar but not identical to the definition given there). Accordingly, oscillation speed provides a convenient way to organize the global solution semiflow. This organization has been exploited perhaps most fully by Mallet-Paret, who in 1988 [22] proved (among much else) the following:

Theorem 1.3.2. Collected and simplified results from [22]. *If* $|F'(0)| > 2\pi N + \pi/2$, $N \in \mathbf{Z}_+$, *then* (DDE) *has a periodic solution of oscillation speed* M *for every nonnegative even* $M \leq 2N$.

Suppose that $x(\cdot)$ *solves* (DDE). *If the zero solution is hyperbolic, then there is some finite integer* K *such that either* $x(\cdot)$ *eventually has oscillation speed less than* K, *or* $x(t) \to 0$ *as* $t \to \infty$.

Remark 1.3.3. For problems (1.1.1) with several delays, the notion of oscillation speed has neither an obvious definition nor an obvious utility. We will find in chapter 5, though, that the concept is useful for the study of certain step problems $(SDDE)$.

We call *slowly oscillating* any solution x of (DDE) that has oscillation speed 0 at (and after) some time t; we will call a periodic solution with oscillation speed 0 a *slowly oscillating periodic solution.* (Any periodic solution that has oscillation speed 0 at some time necessarily has oscillation speed 0 at all times.) This terminology is widespread, but not quite standard: many papers use a narrower definition of "slowly oscillating periodic solution," and for general solutions the phrase "eventually slowly oscillating" is sometimes used instead of "slowly oscillating."

[22] shows that, for $F'(0) < -\pi/2$, (DDE) has a slowly oscillating periodic solution. (This particular existence result was obtained by Nussbaum in 1974 [26] by fixed point arguments). Numerical studies have long suggested that certain equations (DDE) have slowly oscillating periodic solutions that are strongly attractive; accordingly there has been intensive study of the global dynamics of (DDE). By far the most complete results have been obtained in the case that F is strictly decreasing. In early work focusing

on this case, Kaplan and Yorke [17] showed that, if $F'(0) < -\pi/2$, there is an annulus A about 0 in $\mathbf{R}^2$ that attracts all trajectories of the form $(x(t), -x(t-1))$ traced by solutions x of (DDE) whose initial conditions have at most one zero. The boundaries of A are such trajectories traced by 1-cyclic slowly oscillating periodic solutions of (DDE). In particular, if F is strictly decreasing and (DDE) has only one slowly oscillating periodic solution, other slowly oscillating solutions are attracted to it. The question of the uniqueness of slowly oscillating periodic solutions is therefore of special interest in the case that F is strictly decreasing. In 1979 Nussbaum [29] showed that if, in addition to our other hypotheses, F is odd and strictly decreasing, $F'(0) < -\pi/2$, and $F'(x)$ and $F(x)/x$ are strictly increasing for positive x, then (DDE) has a unique slowly oscillating periodic solution; in 1996 Cao [4] showed that the same holds without the hypothesis of oddness but with the added requirement that $xF'(x)/F(x)$ be less than 1, strictly decreasing for positive x, and strictly increasing for negative x (we have given simplified statements of both results). Both of these papers study the $(x(t), x'(t))$-trajectory in the plane.

In a series of papers in the early 1990s ([40], [41], [42] — see also [39]) Xie used a different approach to prove uniqueness of periodic solutions of various types — all varieties of 1-cyclic slowly oscillating periodic solutions — for several different classes of equations (DDE), without the assumption that F is strictly decreasing. The idea, put roughly, was to find conditions under which, first, any solution of interest must be asymptotically stable and, second, every asymptotically stable solution of interest contributes 1 to a total fixed point index of 1. At the expense of asymptotic results, Xie required relatively little of the shape of F. A simplified example result is the following [40]: if, in addition to our standing hypotheses, $F(x)$ has finite nonzero limits as $x \to \infty$ and $x \to -\infty$, $F'(x)$ is integrable, and $xF'(x) \to 0$ as $|x| \to \infty$, then the equation

$$x'(t) = \alpha F(x(t-1))$$

has a unique 1-cyclic slowly oscillating periodic solution for α sufficiently large.

In 1975 Kaplan and Yorke [17] conjectured that a dense set of initial conditions have slowly oscillating continuations if F is strictly decreasing with $F'(0) < -\pi/2$.

This conjecture was proven by Walther in 1981 [38] for F satisfying the additional condition

$$|F(x)| \leq c|x|, \; c < \sqrt{2} + \frac{1}{2};$$

the full conjecture has been proven more recently by Mallet-Paret and Walther [24]. Mallet-Paret and Sell [23] have proven a Poincaré-Bendixon type theorem for (DDE) when F is strictly decreasing: if the zero solution is hyperbolic, every solution of (DDE) approaches a periodic solution (perhaps the zero solution). In the same paper the authors prove that, if F is strictly decreasing, every periodic solution of (DDE) is 1-cyclic (recall definition 1.2.5).

The situation is very different when F is not strictly decreasing. In 1977 Mackey and Glass [20] published numerical studies — of a single-delay equation not quite of the form (DDE) — that suggested that solutions of (DDE) can exhibit chaotic behavior, in particular if $F'(0) < 0$ is not too small and $F(x)$ is nondecreasing far from the origin (so-called "hump-shaped" F). This has indeed proven to be the case. In a pair of papers that together constitute an approach similar to the one we take in this thesis, Peters [33] and Siegberg [36] exhibited a (DDE), with a hump-shaped F, that has chaotic slowly oscillating solutions. Peters first exhibited chaotic behavior for a nonmonotonic step F, and then Siegberg studied a similar F with the discontinuities smoothed out.

Equations (DDE) with odd and hump-shaped feedback functions F have proven fruitful ground for exhibiting non-uniqueness of slowly oscillating periodic solutions; see, for example, [29] and [30].

1.4 Several delays

The theory of differential delay equations with several fixed delays is less developed.

Some authors have successfully used Lyapunov functions to show that equilibrium solutions of certain equations are globally attractive; see for example [18] (where a several-delay equation different from (1.1.1) is studied) or Gopalsamy's book [9].

In 1974 Kaplan and Yorke [16] studied the equation

$$x'(t) = f(x(t-1)) + f(x(t-2)),$$

where f is odd and satisfies the negative feedback condition. (In the same paper the authors study the analogous single-delay equation with a similar approach.) Kaplan and Yorke observed that periodic solutions with period 6 of the three-dimensional ODE

$$\begin{aligned}\dot{x} &= f(y) + f(z)\\ \dot{y} &= -f(x) + f(z)\\ \dot{z} &= -f(x) - f(y)\end{aligned}$$

correspond to periodic solutions $x(\cdot)$ of the delay equation (where $y(t) = x(t-1)$ and $z(t) = x(t-2)$). Using the fact that the ODE has Hamiltonian

$$H(x, y, z) = -\int_0^x f(s)\ ds - \int_0^y f(s)\ ds - \int_0^z f(s)\ ds,$$

Kaplan and Yorke proved that there is a solution of period six provided that f is not integrable and that the limits

$$\alpha = \lim_{x\to 0} f(x)/x, \quad \beta = \lim_{x\to\infty} f(x)/x$$

exist and strictly straddle $\pi/(3\sqrt{3})$. Kaplan and Yorke conjectured that a similar existence theorem should hold for equations of the form

$$x'(t) = \sum_{i=1}^{n} f(x(t-i)).$$

This conjecture was proven in 1978 by Nussbaum [28] using fixed point techniques. The results in [28] are actually considerably more general and concern the existence of periodic solutions of equations of the form (this is in fact still a simplification)

$$x'(t) = \sum_{i=1}^{n} [f_i(x(t-\gamma_i)) + f_i(x(t-q+\gamma_i))]$$

where $\gamma_i \in [0, q]$. The hypotheses on the f_i, though, are similar to those in [16]: the f_i are continuous, odd, have negative feedback, and satisfy certain conditions on the limits

$$\alpha_i = \lim_{x\to 0} f_i(x)/x, \quad \beta_i = \lim_{x\to\infty} f_i(x)/x.$$

In the years since, other authors have pursued Kaplan and Yorke's Hamiltonian approach, both for equations with single and with several delays: see, for example, [10], [19], and [31].

In 1978 Nussbaum [27], in the main inspiration for this thesis, studied a class of (1.1.1) with two delays. As an example of the kind of results obtained we quote here a simplified version of Theorem 2.3 in [27] (which we already invoked in example 1.1.4).

Theorem 1.4.1. Theorem 2.3 in [27]. *Suppose that* $f : \mathbf{R} \to \mathbf{R}$ *is continuous and strictly decreasing with* $f(0) = 0$*, and that* $lim_{x\to-\infty} f(x) = -\lim_{x\to\infty} f(x) = b > 0$*. Consider the equation*

$$x'(t) = r\alpha f(x(t-1)) + r\beta f(x(t-3)), \tag{E}$$

where $r > 0$ *and* $\alpha \geq 0$ *and* $\beta \geq 0$ *satisfy*

$$\alpha^2 + \beta^2 = 1; \quad \beta \in \left(\frac{\alpha}{3}, \alpha\right).$$

Then, for r *sufficiently large,* (E) *has two distinct* 1*-cyclic periodic solutions of period greater than* 3*.*

[27] demonstrated that varying the feedback at each delay, or the relative sizes of the delays, exerts an important influence on the appearance of whatever periodic solutions might be present. [27] also showed that there can exist multiple periodic solutions with what we might regard as “comparable oscillation speed” (as in the above-quoted theorem). Nussbaum’s approach in [27] is to make careful estimates on solutions with initial conditions in certain subsets, and then to apply fixed point arguments to return maps; to explore a wide variety of problems and solutions thus requires many careful estimates. The labor involved in these estimates motivates the search for a class of equations with several delays for which we can generate, with somewhat less effort, examples of many substantially different periodic solutions. This is a central objective of this thesis.

1.5 Step feedback

Many authors have used delay equations with step feedback to shed light on more classical delay equations. We have already mentioned [33] and [36]. We also mention [37], where Stoffer has found stable rapidly oscillating periodic solutions for the singular perturbation equation (1.3.1) with a non-monotonic F. Stoffer’s approach bears an affinity to ours in that he uses an F that is similar to a step function — in particular, that is locally constant except on a few small intervals. In [27] Nussbaum used solutions of equations with step feedback to help formulate estimates for solutions of continuous two-delay problems. In [40] Xie used slowly oscillating periodic solutions of the problem

$$y'(t) = -ky(t) + h(y(t-1)),$$

where $k \geq 0$ and h is step with negative feedback, to describe the asymptotic appearance of slowly oscillating periodic solutions (known to exist) of

$$\epsilon x'(t) = -\sigma x(t) + F(x(t-1)); \quad \epsilon \to 0,\ \sigma/\epsilon \to k$$

where F satisfies certain conditions.

In this section we describe some results for the well-studied equation

$$x'(t) = f(x(t)) - \operatorname{sgn}(x(t-1)),\ t > 0; \quad x_0 \in C = C[-1,0] \tag{1.5.1}$$

where f is continuous. While this equation certainly sheds light on continuous delay equations with one delay, it also possesses considerable inherent interest. (This is a small part of the topic of "functional differential equations with discontinuous right hand side," which ranges far beyond our current focus and uses substantial additional hardware.) By a solution to (1.5.1) we mean a solution to the corresponding integral equation

$$x(t) = x(0) + \int_0^t f(x(s)) - \operatorname{sgn}(x(s-1))\ ds,\ t \geq 0; \quad x_0 \in C.$$

In 1993, Fridman, Fridman and Shustin [8] studied equation (1.5.1) where f is C^1 with $|f(x)| \leq p < 1$ for all x. They focused in particular on the forward-invariant set $C' \subset C[-1,0]$ of continuous functions with finitely many zeros. For $x_0 \in C'$ with continuation x, the positive proper zeros of x form a countable sequence $z_1 < z_2 < \ldots$ with infinite limit and no accumulation point. The oscillation speed of x at z_n (that is, the number of proper zeros of x in $(z_n - 1, z_n)$) is even and nonincreasing in n and so has a limit $N(x_0)$ that is attained in finite time. In [8] it is proven that to each nonnegative even N there corresponds a unique periodic solution with oscillation speed N, and that the set of initial conditions with slowly oscillating continuations is dense in C. The dynamics of (1.5.1) thus evoke the dynamics of the one-delay problem (DDE) with strictly decreasing F.

Remark 1.5.2. The version of (1.5.1) studied by Fridman, Fridman and Shustin essentially subsumes $(SDDE)$ with one delay and negative feedback, at least on the forward-invariant subspace C': just take $f(x)$ to be a constant of absolute value less than one.

The equation $x'(t) = -\text{sgn}(x(t-1))$ is described, in similar completeness, in section XVI.2 of [7]. In 2002 Martin [25] gave results almost as complete for the equation

$$x'(t) = \mu x - a\text{sgn}(x(t-1)),$$

where $\mu \geq 0$ and $a > 0$.

To complete the analysis of the $|f(x)| \leq p < 1$ case, we need to consider solutions that do not lie in C' — otherwise put, solutions with infinite oscillation speed. In 1995, Shustin [35] showed that, for a restricted (but nonautonomous) f, such solutions enter C' in finite time unless $x_0 \equiv 0$. The rough idea was to introduce a functional on the set of zeros of x_t that grows in such a way that, if x_t has infinitely many zeros for all $t \geq 0$, we must have $x_0 = 0$. This result was strengthened (in the autonomous case) by two articles appearing in 2000 and 2001. Akian and Bliman proved Shustin's result for $f \in L^\infty(\mathbf{R})$ with $\|f\|_\infty < 1$ [1], and Nussbaum and Shustin proved the result for locally Lipschitz f with $|f(x)| < 1$ [32]. (Dix [6] has shown that infinite oscillations can persist under nonautonomous feedback or nonconstant delays.)

In section 2.2 we will present a similar but weaker result for solutions of $(SDDE)$ with infinitely many zeros.

1.6 The problems under consideration

We take $\gamma > 0$ and write $C = C[-\gamma, 0]$ for the set of real-valued continuous functions on $[-\gamma, 0]$, equipped with the sup norm. The problems we are studying are of the form

$$\begin{aligned} & x'(t) = \sum_{i=1}^{D} F_i(x(t-d_i)), \ t > 0; \quad x_0 \in C = C[-\gamma, 0]; \\ & F_i \text{ continuous}; \quad 0 < d_1 < \cdots < d_D = \gamma; \\ & \lim_{x \to -\infty} F_i(x) = b_i \neq 0; \quad \lim_{x \to \infty} F_i(x) = -a_i \neq 0 \end{aligned} \tag{E}$$

(this is just (1.1.1) with added requirements on the F_i at $\pm\infty$). We point out some distinguishing features of (E).

- $x'(t)$ is a nonlinear function of x_t. The theory of linear retarded functional differential equations — of forms much more general than (E) — is highly developed; see, for example, the early chapters of [7].
- The delays are discrete and solutions are real-valued.
- The equation is autonomous. Non-autonomous equations, especially those where the relationship between $x'(t)$ and x_t varies periodically, have also been studied; see, for example, [34].
- The delays are constant. For a recent survey on differential delay equations with variable delays, see [14].
- The shortest delay d_1 is strictly positive — that is, the value $x(t)$ does not directly affect $x'(t)$. Since so many results for $x'(t) = F(x(t-1))$ extend to equations of the form $x'(t) = g(x(t), x(t-1))$, we can reasonably regard this as a serious restriction.

Throughout, we will adhere to our notation above — in particular γ, C, d_i, D, F_i, b_i and $-a_i$ will always be used as in (E). As described in section 1.1, our approach is to link the problem (E) to the related "step differential delay equation"

$$y'(t) = \sum_{i=1}^{D} h_i(y(t-d_i)), \;\; t > 0; \quad y_0 \in C; \qquad (SDDE)$$

$$h_i(y) = \begin{cases} b_i, & y < 0; \\ 0, & y = 0; \\ -a_i, & y > 0. \end{cases}$$

We will always write h_i for these step functions.

There is of course no intimate connection between (E) and $(SDDE)$ in general, but if the feedback functions F_i are close to the h_i in some reasonable sense, we expect $(SDDE)$ to be rigorously informative for (E). This thesis is devoted to this idea.

Chapter 2

Basic theory for $(SDDE)$

2.1 Existence and uniqueness

By a solution of the initial value problem

$$y'(t) = \sum_{i=1}^{D} h_i(y(t-d_i)), \; t > 0; \quad y_0 \in C \tag{$SDDE$}$$

we mean a continuous function $y : [-\gamma, \infty) \to \mathbf{R}$ that satisfies the corresponding integral initial value problem

$$y(t) = y_0(t), \; t \in [-\gamma, 0]; \quad y(t) = y(0) + \int_0^t \left[\sum_{i=1}^{D} h_i(y(s-d_i)) \; ds \right], \; t > 0. \tag{2.1.1}$$

Given $y_0 \in C$, the function

$$f_{y_0} : [0, d_1] \to \mathbf{R} \; : \; f_{y_0}(t) = \sum_{i=1}^{D} h_i(y_0(t-d_i)),$$

being the sum of compositions of Borel measurable functions, is Borel measurable and hence Lebesgue measurable. The indefinite integral $y(t)$ of f from 0 to d_1 is therefore (uniquely) defined and is absolutely continuous. $y(t)$ is a solution of (2.1.1) on $[0, d_1]$, and we can continue this solution to $2d_1$, to $3d_1$, and so on. Furthermore, by Lebesgue's differentiation theorem this indefinite integral is differentiable for almost every $t \in [0, d_1]$, with derivative $f_{y_0}(t)$. At such points, $y(t)$ is a solution of $(SDDE)$ as usually written.

We have established

Proposition 2.1.2. Existence and uniqueness for $(SDDE)$**.** *The initial value problem (2.1.1) has, for any initial condition $y_0 \in C$, a unique solution y defined on $[-\gamma, \infty)$.* $\square$

In the situation described in the above result, we call y the *continuation of* y_0 *as a solution of* $(SDDE)$. For such a y and any $t \geq 0$, we call the function $y_t \in C$ given by

$$y_t(s) = y(t+s), \quad s \in [-\gamma, 0]$$

a *section* of y. Just as for continuous differential delay equations, we think of solutions as evolving in C; the solution semiflow sends $(t, y_0) \in \mathbf{R}_+ \times C$ to $y_t \in C$.

Continuous dependence of solutions on initial conditions does not hold.

Example 2.1.3. Continuous dependence on initial conditions does not hold for $(SDDE)$**.** Consider the equation with one delay

$$y'(t) = -\mathrm{sgn}(y(t-1)),$$

where $\mathrm{sgn}(y)$ is either -1, 0, or 1. The zero initial condition continues as 0, but arbitrarily small positive initial conditions have continuations $y(t)$ with $y(1)$ arbitrarily close to -1. □

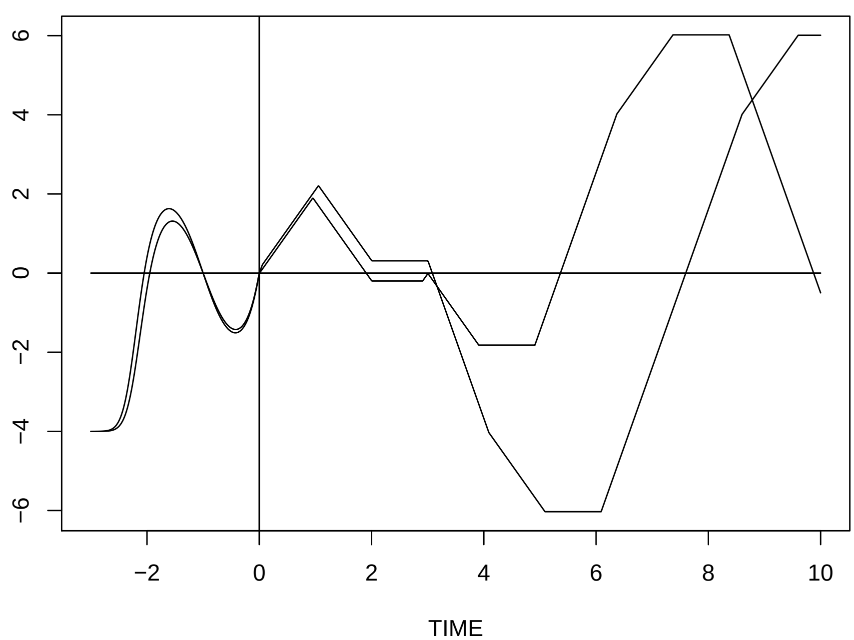
Write w for the continuation of $w_0 \in C$ as a solution of $(SDDE)$. If $y_0 \in C$ has only isolated zeros, then the map $w_0 \mapsto w_t$ is continuous at y_0 for small t (see proposition 2.1.7 below). It is possible, though, for a solution whose initial condition has only isolated zeros to eventually have an interval of zeros, at which point continuity fails for the reason illustrated in the above example.

Example 2.1.4. Consider, for instance, the equation

$$y'(t) = -\mathrm{sgn}(y(t-1)) - \mathrm{sgn}(y(t-2)) - 2\mathrm{sgn}(y(t-3)).$$

Any initial condition y_0 that is negative on $(-3,-2) \cup (-1,0)$, positive on $(-2,-1)$, and 0 at 0 will have continuation y with $y'(t) = 2$ for $t \in (0,1)$, $y'(t) = -2$ for $t \in (1,2)$, and $y'(t) = 0$ for $t \in (2,3)$. Figure 3 shows Euler method approximations for two solutions whose initial conditions are both close to such a y_0. The two solutions are close until time $t = 3$ and then diverge. (Notice that no alternative definition of our feedback function at 0 would prevent this divergence.) □

Figure 3



Remark 2.1.5. The above example prompts one to ask whether nonzero initial conditions can have continuations that are eventually identically zero; under our hypotheses on $(SDDE)$ (namely, that $h_i(y) = 0$ if and only if $y = 0$), the answer is no. For imagine that $y_0 \neq 0$ but that $y_\tau = 0$ for some minimum positive τ. Then there is an open interval I in $(\tau - (d_D - d_{D-1}), \tau)$ where

$$y(t) = y(t - d_1) = \cdots = y(t - d_{D-1}) = 0$$

but where $y(t - d_D)$ is nonzero and of constant sign. This means that, on this same interval, $y'(t) = h_D(y(t - d_D))$ is nonzero and of constant sign, a contradiction. $\square$

Definition 2.1.6. Write $C' \subset C$ for the space of continuous real-valued functions on $[-\gamma, 0]$ with isolated zeros. Since $[-\gamma, 0]$ is compact, we can define C' equivalently as the subset of C whose members have finitely many zeros. We equip C' with the subspace topology.

Loosely speaking, continuity with respect to initial data holds, for short time, at points of C'. For initial conditions in C', we can also give a concrete description of how differentiability of solutions fails.

Proposition 2.1.7. *Given $y_0 \in C'$ there is a unique continuation $y : [-\gamma, d_1] \to \mathbf{R}$ of y_0 such that y solves $(SDDE)$ in the following sense:*

- *$y(t)$ is differentiable for every $t \in (0, d_1)$ except perhaps at the finitely many points for which $y(t - d_i) = 0$ for some delay d_i;*
- *Everywhere that $y'(t)$ exists, it satisfies $(SDDE)$.*

Write w for the continuation of $w_0 \in C$ as a solution of $(SDDE)$. For all $t \in [0, d_1]$, the map from C to C given by

$$w_0 \mapsto w_t$$

is continuous at y_0.

PROOF. Existence and uniqueness are clear: the slope of y is prescribed all at but finitely many points.

About every zero of $y_0 \in C'$ put an open interval of radius so small that the sum of the lengths of all such intervals is less than ϵ. Off of these intervals, $|y_0(s)|$ has a minimum value δ. Take $w_0 \in C$ with $\|w_0 - y_0\| < \min\{\delta/2, \epsilon\}$, and write w and y for the continuations of w_0 and y_0, respectively. Then, as t runs from 0 to d_1, $h_i(w(t-d_i))$ and $h_i(y(t - d_i))$ disagree for at most ϵ units; and if $h_i(w(t - d_i)) \neq h_i(y(t - d_i))$ we have

$$|h_i(w(t - d_i)) - h_i(y(t - d_i))| \leq |a_i| + |b_i|.$$

Thus we estimate, for $t \in [0, d_1]$,

$$\begin{aligned} &|w(t) - y(t)| \leq \epsilon + \int_0^t |w'(s) - y'(s)| \, ds \\ &\leq \epsilon + \sum_i \int_0^t |h_i(w(s - d_i)) - h_i(y(s - d_i))| \, ds \leq \epsilon + \epsilon \sum_i (|a_i| + |b_i|) \, . \quad \square \end{aligned}$$

We restrict our study of $(SDDE)$ almost exclusively to those solutions (or parts or solutions) whose sections lie in C'. Proposition 2.1.7 lends some appeal to this restriction, but it is not the real reason for it. The continuation of $y_0 \in C'$ as a solution of $(SDDE)$ is completely determined by the signs and locations of the proper zeros of y_0, and the value $y_0(0)$. Since y_0 has only finitely many zeros, then, we can view the solution semiflow as a function of finite-dimensional data, at least until time d_1. This observation is the key to the tractability of $(SDDE)$.

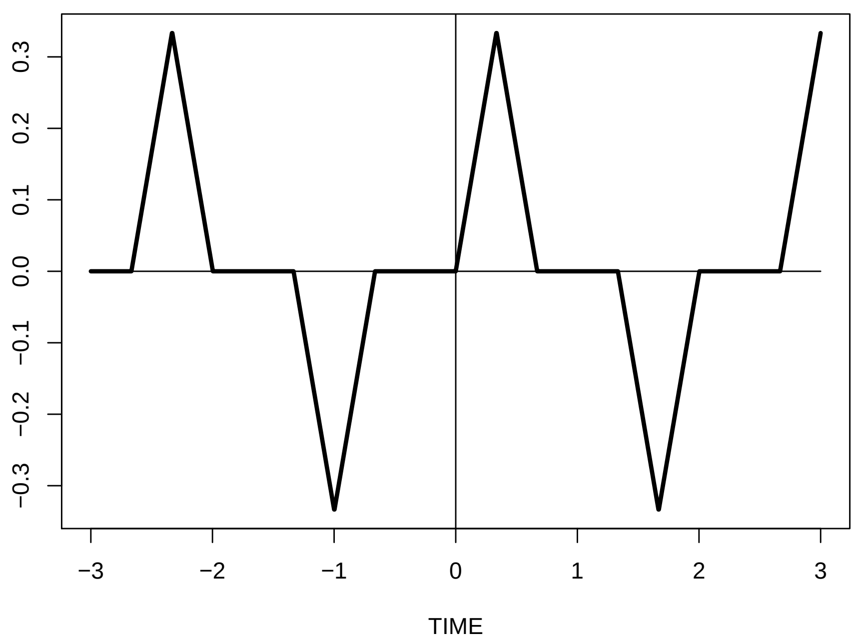
Example 2.1.4 shows that the space C' is not, in general, forward-invariant under the solution semiflow. Moreover, solutions of $(SDDE)$ need never enter C', as the following example shows.

Example 2.1.8. Consider the $(SDDE)$

$$y'(t) = -\text{sgn}(y(t-1)) - \text{sgn}(y(t-2)) - \text{sgn}(y(t-3)).$$

This equation has a periodic solution none of whose sections lies in C'. Figure 4 shows a picture of this periodic solution (the slopes change at points of the form $n/3$, $n \in \mathbf{Z}$, and the intervals of zeros are each $2/3$ units long). $\square$

Figure 4



The restriction of our attention to C' thus does entail some loss, and we would like to articulate conditions under which this loss is minimal. We conclude this section with a fairly mild condition under which the subspace C' is forward-invariant. In section 2.2 we shall describe conditions that ensure that solutions enter C'.

Proposition 2.1.9. A forward-invariance condition for C'. *Suppose that the* *equation*

$$y'(t) = \sum_{i=1}^{D} h_i(y(t-d_i)) \tag{$SDDE$}$$

satisfies the following condition:

$$y_i \neq 0 \ \forall i \implies \sum_{i=1}^{D} h_i(y_i) \neq 0. \tag{2.1.10}$$

Then C' is forward-invariant under the solution semiflow for $(SDDE)$: that is, if $y_0 \in C'$ with continuation y as a solution of $(SDDE)$, then $y_t \in C'$ for all $t \geq 0$.

Observe that condition (2.1.10) is generic in the sense that, if $(SDDE)$ does not satisfy (2.1.10), we can make (2.1.10) hold with an arbitrarily small adjustment to the $\{a_i, b_i\}$.

PROOF. Take $(SDDE)$ satisfying (2.1.10) and $y_0 \in C'$ with continuation y. Write E for the set of nonisolated zeros of y. Since y is continuous, its zeros form a closed set; E is therefore also a closed set and so, if we imagine E to be nonempty, E must have a minimum $\zeta \geq 0$ (remember that $y(t)$ is defined for $t \geq -\gamma$, but that y has only finitely many zeros on $[-\gamma, 0]$ by hypothesis). Since all the zeros of y less than ζ are isolated, there is some $\epsilon > 0$ such that $y(\zeta - d_i + t)$ is nonzero and of constant sign for all $t \in (0, \epsilon)$. $y'(t)$ therefore exists and is constant on $(\zeta, \zeta + \epsilon)$, and condition (2.1.10) guarantees that $y'(t)$ is nonzero on this interval. Thus $y(t)$ has no zeros on $(\zeta, \zeta + \epsilon)$. If $\zeta > 0$, a similar argument shows that there is some $\epsilon > 0$ such that $y(t)$ has no zeros on $(\zeta - \epsilon, \zeta)$ (if $\zeta = 0$, there is such an ϵ by the hypothesis that y_0 has only finitely many zeros). We therefore see that ζ is not a nonisolated zero after all, and conclude that E is empty. $\Box$

2.2 Zeroing behavior

Proposition 2.1.9 tells us that C' is forward-invariant under the solution semiflow given condition (2.1.10). The equation in example 2.1.8 satisfies this condition, but (as the example shows) has solutions that never enter C'. In this section we examine when solutions of $(SDDE)$ enter and remain in C'. We have found it necessary to restrict the set of initial conditions we consider.

Definition 2.2.1. The space C''. We define the space C'' as follows:

$$C'' = \{\ y_0 \in C\ :\ \partial y_0^{-1}(0) \text{ is finite } \}.$$

Equivalently, $y_0 \in C''$ if and only if the inverse image of $\mathbf{R} \setminus \{0\}$ under y_0 has finitely many connected components.

To insist that an initial condition lie in C'' is a substantial restriction. The following observation makes the situation more palatable by showing that C'' is a natural, if narrow, setting for $(SDDE)$.

Proposition 2.2.2. *The space C'' is forward-invariant under the solution semiflow for $(SDDE)$. Furthermore, given $M > 0$, the differentiability of y fails only at finitely many points on $[0, M]$.*

PROOF. If $y_0 \in C$ with continuation $y(t)$ as a solution of $(SDDE)$, the differentiability of y on $(0, d_1)$ can fail only at those points $t \in (0, d_1)$ such that

$$t - d_i \in \partial y_0^{-1}(0) \text{ for some } d_i.$$

Thus, if in fact $y_0 \in C''$, then y is differentiable at all but finitely many points $t \in (0, d_1)$. Away from these points, $y(t)$ is of constant slope; and any open interval where $y(t)$ is of constant slope intersects $\partial y^{-1}(0)$ in at most one point. Therefore $[0, d_1] \cap \partial y^{-1}(0)$ is a finite set, and $y_t \in C''$ for all $t \in [0, d_1]$. Stepping forward, we see that $y_t \in C''$ for all $t \geq 0$.

The above paragraph shows that $y'(t)$ is defined of all but finitely many points of the intervals $(0, d_1/2]$, $(d_1/2, d_1]$, and so on. The last part of the proposition follows. $\square$

Here is a feature of C'' that will be useful below.

Lemma 2.2.3. *If $y_0 \in C''$ with continuation $y(t)$, the nonisolated zeros of $y(t)$ occur in closed intervals of positive length.*

PROOF. Write E for the set of nonisolated zeros of $y(t)$; E is a closed set. Suppose that $p \in E$. This means that, for any $\epsilon > 0$, the interval $I_\epsilon(p)$ about p of radius ϵ

contains zeros of $y(t)$ distinct from p. In particular, there are zeros of $y(t)$ in either all of the sets

$$(p, p+\epsilon) \text{ or } (p-\epsilon, p),\ \epsilon > 0.$$

Suppose that we are in the first case (the other is similar). We claim that, for ϵ sufficiently small, $y(t)$ is identically zero on $[p, p+\epsilon]$. For suppose not; then there is a sequence q_n approaching p from above with $y(q_n) \neq 0$, and accordingly $\partial y^{-1}(0)$ is infinite on $[p, p+\gamma]$. This is impossible by the forward-invariance of C''. □

We now define a class of $(SDDE)$ for which solutions in C'' flow into C'.

Definition 2.2.4. $(SDDE)$ **of type I.** We say that $(SDDE)$ is of *type I* if the functions h_i have the feature that

$$\sum_{i=1}^{D} h_i(y_i) = 0 \implies y_i = 0 \text{ for all } i.$$

Remark 2.2.5. There are 3^D combinations of the b_i and $-a_i$ that represent possible slopes of solutions of $(SDDE)$; to check the type I condition one just computes these 3^D combinations and checks that only the trivial combination sums to zero.

The point of the condition is that if $(SDDE)$ is type I and $y(t)$ is any solution of $(SDDE)$, then $y'(t) = 0$ only if $y(t-d_i) = 0$ for all i. Observe that any $(SDDE)$ with only one delay is type I, by the definition of $(SDDE)$.

The type I condition implies condition (2.1.10), but not conversely (observe that the equation in example 2.1.8 is not type I). Even so, the type I condition is "generic" in the sense that if $(SDDE)$ is not type I an arbitrarily small perturbation in the $\{a_i, b_i\}$ will make it type I.

Here is the main result for this section.

Theorem 2.2.6. *If $(SDDE)$ is of type I, then for any nonzero $y_0 \in C''$ with continuation y there is some finite $\tau \geq 0$ such that $y_t \in C'$ for all $t \geq \tau$.*

Since any type I $(SDDE)$ satisfies condition (2.1.10), to prove this theorem we need only show that solutions starting in C'' eventually reach C'.

Remark 2.2.7. We conjecture that any nonzero solution of a type I ($SDDE$) eventually reaches C' (recall analogous results for equation (1.5.1) in section 1.5), but we have been unable to prove this.

Proof of 2.2.6

We shall lean heavily on the particular values of the delays d_i, and need the following observation. Suppose that $y(t)$ is a solution of

$$y'(t) = \sum_{i=1}^{D} h_i(y(t-d_i)). \tag{$SDDE$}$$

Then, given $\kappa > 0$, the function $\hat{y}(t) = y(\kappa t)/\kappa$ satisfies the equation

$$\hat{y}'(t) = y'(\kappa t) = \sum_{i=1}^{D} h_i(y(\kappa t - d_i)) = \sum_{i=1}^{D} h_i(\hat{y}(t - d_i/\kappa)) \tag{$SDDE_\kappa$}$$

— that is, $\hat{y}(t)$ is a solution of a version of ($SDDE$) where all the delays have been divided by κ. We make three obvious points: first, ($SDDE_\kappa$) is type I if and only if ($SDDE$) is; second, for any $t \geq 0$,

$$\left(\hat{y}^{-1}(0) \cap [t/\kappa - \gamma/\kappa, t/\kappa]\right) \text{ is finite} \quad \iff \quad \left(y^{-1}(0) \cap [t-\gamma, t]\right) \text{ is finite;}$$

third, for any $t \geq 0$,

$$\partial\left(\hat{y}^{-1}(0) \cap [t/\kappa - \gamma/\kappa, t/\kappa]\right) \text{ is finite} \quad \iff \quad \partial\left(y^{-1}(0) \cap [t-\gamma, t]\right) \text{ is finite.}$$

Therefore, to prove 2.2.6, we are free to replace ($SDDE$) with ($SDDE_\kappa$) at the outset; otherwise put, we can divide all the delays in ($SDDE$) by a positive scalar of our choosing.

For the rest of the section, $y(\cdot)$ will always denote the continuation of some $y_0 \in C''$ as a solution of ($SDDE$). Given such a y_0 and some $t \geq -\gamma$, let us write

$$E_t = \{\ s \in [0,1]\ :\ t+s \text{ is a nonisolated zero of } y(\cdot)\ \}.$$

(Please observe that, in the definition of E_t, s lies in $[0,1]$, *not* in $[-\gamma, 0]$.) Each E_t is closed; write S_t for the complement of this set in $[0,1]$. S_t is open (relative to $[0,1]$), and to say that $y_0 \neq 0$ is just to say that S_t is nonempty for some t (more particularly, for some $t < 0$).

Remark 2.2.8. The members of E_t are translates of nonisolated zeros of y, not translates of zeros of y that are nonisolated in $[t, t+1]$. Thus, for example, if $\epsilon \in (0, \gamma)$, $y(t) = 0$ for $t \in [-\epsilon, 0]$, and $y(t) > 0$ for $t \in (0, \epsilon)$, then 0 does belong to E_0.

$t + E_t$ is just the intersection of $[t, t+1]$ with the set E of all nonisolated zeros of y; accordingly, by proposition 2.2.2 and lemma 2.2.3, $t + E_t$ is the intersection of $[t, t+1]$ with a finite set of closed intervals of positive length. Therefore E_t itself consists of a finite set of closed intervals of positive length, along with perhaps one or both of the points $\{0\}$ and $\{1\}$.

Let $(SDDE)$ be of type I and $y_0 \in C''$. Suppose that there is some point $t > 0$ and some i such that $t - d_i$ is not in the set of nonisolated zeros of y. In this case there is an interval around t where y' is nonzero everywhere it exists (since $(SDDE)$ is type I) and exists at all but finitely many points (since C'' is forward-invariant). It follows that any zeros of y in this interval are isolated. We see, then, what the hypotheses of theorem 2.2.6 do for us:

Lemma 2.2.9. *If $(SDDE)$ is of type I and $y_0 \in C''$, then*

$$S_t \subset S_{t+d_i}$$

whenever $t + d_i > 0$. $\square$

Our objective is to show, given $(SDDE)$ of type I and nonzero $y_0 \in C''$, that $S_t = [0, 1]$ for all sufficiently large t. It is enough to show that $S_n = [0, 1]$ for all sufficiently large $n \in \mathbf{N}$: for suppose that $S_n = [0, 1]$ for all $n \geq N$ but that there is some noninteger $t > N$ such that $S_t \neq [0, 1]$. This means that $[t, t+1]$ contains a nonisolated zero p of $y(\cdot)$, and accordingly that p lies in some closed interval of positive length of zeros of $y(\cdot)$. This interval will intersect some $[n, n+1]$ with $n \geq N$ and so E_n will not be empty — a contradiction.

Our proof of theorem 2.2.6 divides into two cases: that where all the delays are rationally related (i.e. the ratio of any two delays is a rational), and otherwise. In both cases we show that the hypotheses of 2.2.6 imply that $S_n = [0,1]$ for all sufficiently large natural n.

We consider the first case. If all the delays are rationally related, by rescaling the delays as demonstrated above we can take all the delays to be integers. Furthermore, by scaling out the greatest common divisor of these integers, we may assume that $gcd(d_1, \dots, d_D) = 1$.

We will need the following number-theoretic lemma. The proof here is drawn from [15], chapter 1.

Lemma 2.2.10. *Suppose that $d_1 < \cdots < d_D$ are positive integers such that $gcd(d_1, \dots, d_D) = 1$. Then there is an integer k such that any integer $n \geq k$ can be expressed as a positive integer combination*

$$n = \sum_{i=1}^{D} c_i d_i, \quad c_i \in \mathbf{N}.$$

PROOF. Write

$$\Sigma = \{ \sum_{i=1}^{D} c_i d_i, \quad c_i \in \mathbf{Z} \}.$$

Σ contains positive integers and so contains a least positive integer c. Given $x \in \Sigma$, write $x = mc + r$, where $m \in \mathbf{Z}$ and $0 \leq r < c$. Since Σ is closed under addition and multiplication by integers, we see that $r \in \Sigma$ and conclude that $r = 0$. Therefore c divides every $x \in \Sigma$, and so $c = 1$.

Write $1 = \sum_{i=1}^{D} u_i d_i$, and put

$$K = d_1 \sum_{i=1}^{D} |u_i| d_i.$$

Given $n \geq K$, write $n = K + md_1 + r$, where $m \in \mathbf{Z}_+$ and $0 \leq r < d_1$. Then n is expressible as the non-negative integer combination

$$n = md_1 + \sum_{i=1}^{D} (d_1|u_i| + ru_i) d_i.$$

That we can express any sufficiently large integer as a *non-negative* integer combination of the $\{d_1, \dots, d_D\}$ is a standard fact. We want some k such that any $n \geq k$ is expressible as a *positive* integer combination of the $\{d_1, \ldots, d_D\}$; to accomplish this, we just put

$$k = K + \sum_{i=1}^{D} d_i,$$

where K is as above. $\square$

The following is inspired by a similar idea in [32].

Proposition 2.2.11. *Suppose that* $(SDDE)$ *is of type I, that* $y_0 \neq 0$ *lies in* C''*, and that the delays* $d_1 < \cdots < d_D$ *are integers with* $gcd(d_1, \ldots, d_D) = 1$.

Imagine that $y_t \notin C'$ *for all* $t \geq 0$. *Then the intersection*

$$\cap_{n \in \mathbf{Z}_+} E_n$$

is nonempty.

PROOF. We show that the collection $\{E_n \ : n \in \mathbf{Z}_+\}$ satisfies the finite intersection property; it will follow that $\cap_n E_n$ is nonempty.

2.2.9 and 2.2.10 together imply that there is some positive integer k such that $S_n \subset S_{n+m}$ for any $n \in \mathbf{Z}_+$ and any integer $m \geq k$; otherwise put, $E_n \supset E_{n+m}$. If we imagine that some $E_n = \emptyset$, we have that $E_{n+m} = \emptyset$ for all $m \geq k$ — contrary to hypothesis. Therefore every E_n is nonempty. Now choose any finite subcollection $\{E_{n_j}\}$, where $n_1 < \ldots < n_p$. These sets all contain the nonempty set

$$E_{n_p+k},$$

and so have nonempty intersection. $\square$

Suppose now that the hypotheses and conclusion of proposition 2.2.11 hold. By shifting our solution less than one unit to the right or left, we may assume that $0 \in E_n$ for all nonnegative integers n. In this case, of course, it is also true that $1 \in E_n$ for all nonnegative integers n. We now show that this is impossible.

Proposition 2.2.12. *Suppose that* $(SDDE)$ *is type I and that the delays in* $(SDDE)$ *are integers with greatest common divisor* 1. *Then if* $y_0 \in C''$ *is nonzero it is impossible that* $0, 1 \in E_n$ *for all nonnegative integers* n.

Corollary 2.2.13. *Theorem 2.2.6 holds when all the delays in* $(SDDE)$ *are rationally related.* $\square$

PROOF OF PROPOSITION. We imagine that $0, 1 \in E_n$ for all nonnegative integers n and derive a contradiction.

Our assumption that there is some nonempty S_n implies that S_n is nonempty for all n large enough; by shifting we may assume that S_n is nonempty for all nonnegative n. Doing this, for every nonnegative integer n we put

$$\alpha_n = \inf S_n.$$

On the interval $[n, n+\alpha_n]$, $y(t)$ is identically zero. α_n is the left endpoint of a connected component I_n of S_n; since all sections of y_t lie in C'', $y(t)$ has only finitely many zeros on $I_n + n$ (by $I_n + n$ we mean the interval I_n translated to the right by n units). Therefore, $y'(t)$ exists and is nonzero at all but finitely many points on $I_n + n + d_i$ for any delay d_i. On the other hand, $y'(t) = 0$ if, for all delays d_i, $t - d_i$ lies in some $(n, n + \alpha_n)$. It follows that, for all $n \geq \gamma$, we have

$$\alpha_n = \min_{1 \leq i \leq D} \alpha_{n-d_i}.$$

Thus α_n is eventually some constant α. By again shifting our solution, we may assume that $\alpha_n = \alpha$ for all nonnegative integers n; by shifting once more, we may assume that $\alpha = 0$. Since the nonisolated zeros of y occur in closed intervals of positive length, this assumption just says that $y(0) = 0$ and that every $n \in \mathbf{N}$ is the right endpoint of a nontrivial interval of zeros of y.

Since the connected components of S_n contain only finitely many zeros, for every n there is a $\beta_n \in (0, 1)$ such that $n + \beta_n$ is the minimum zero of $y(t)$ on $(n, n+1)$ (our assumption that every natural n is the right endpoint of a nontrivial interval of zeros of y guarantees that $\beta_n \in (0, 1)$). Choose some $n \geq \gamma$ and write

$$\nu = \min_{1 \leq i \leq D} \beta_{n-d_i};$$

we have that $y(n) = 0$ and that $y'(t)$ is a nonzero constant on the interval $(n, n + \nu)$. Write μ for the largest possible value of $|y'(t)|$ and σ for the smallest nonzero value of $|y'(t)|$. Then $|y(n + \nu)| \geq \sigma\nu$, and so it will take at least $(\sigma\nu/\mu)$ additional time units after time $n + \nu$ for $y(t)$ to recover to zero. Thus we have that

$$\beta_n \geq (1 + \sigma/\mu)\nu = (1 + \sigma/\mu) \min_{1 \leq i \leq D} \beta_{n-d_i}.$$

In particular, if

$$\beta = \min_{0 \le m \le \gamma} \beta_m,$$

then we have

$$\beta_n \ge (1 + \sigma/\mu)\beta, \quad \gamma + 1 \le n \le 2\gamma;$$

$$\beta_n \ge (1 + \sigma/\mu)^2\beta, \quad 2\gamma + 1 \le n \le 3\gamma;$$

and more generally

$$\beta_n \ge (1 + \sigma/\mu)^k\beta, \quad k\gamma + 1 \le n \le (k+1)\gamma$$

for all $k \in \mathbf{N}$. It follows that $\beta_n \to \infty$ as $n \to \infty$, a contradiction. □

We now turn to the case that two of the delays in $(SDDE)$ are not rationally related. In this case, by rescaling time we may take these two delays to be 1 and $N + \alpha$, where N is a natural number and $\alpha \in (0, 1)$ is irrational. (Note that we are not assuming that these are the only two delays, or that 1 is the shortest delay.) We continue, of course, to assume that $(SDDE)$ is type I and that y_0 is a nonzero member of C''.

In this case we have the set inclusions $S_t \subset S_{t+1}$ and $S_t \subset S_{t+N+\alpha}$ for all t. Let us examine how these inclusions behave on the sets S_n, $n \in \mathbf{Z}_+$. Choose $p \in S_n$. This means, remember, that $n + p$ does not lie in the set E of nonisolated zeros of $y(t)$. $p + n + N + 1$ is not a nonisolated zero either, so we certainly have that $p \in S_{n+N+1}$. Suppose now that $p \ge 1 - \alpha$. Then, since

$$n + p + N + \alpha = n + p + N + \alpha + 1 - 1 = n + p + N + 1 - (1 - \alpha)$$

is not in E, we see that

$$p - (1 - \alpha) \in S_{n+N+1}.$$

On the other hand, if $p < 1 - \alpha$ then

$$n + p + N + \alpha + 1 \notin E,$$

and so $p + \alpha \in S_{n+N+1}$.

Thus we see that S_{n+N+1} contains both S_n and the image of $S_n \cap [0,1)$ under the following map:

$$F_\alpha : [0,1) \to [0,1) \quad \text{given by} \quad F_\alpha(x) = \begin{cases} x+\alpha, & x < 1-\alpha \\ x-(1-\alpha), & x \geq 1-\alpha. \end{cases}$$

If $S_n \neq \emptyset$, that $S_m = [0,1]$ for large enough m now follows from

Proposition 2.2.14. *The orbit of single point $x_0 \in [0,1)$ under F_α is dense in $[0,1)$. Furthermore, given any open set $U \subset [0,1)$, there is a positive integer k such that*

$$\cup_{i=1}^{k} F^i(U) = [0,1).$$

Corollary 2.2.15. *Theorem 2.2.6 holds when the delays in $(SDDE)$ are not all rationally related.* $\square$

The map F_α is equivalent to the rotation ρ_α of the unit circle S^1 through the angle $2\pi\alpha$ in the sense that

$$\rho_\alpha \circ h = h \circ F_\alpha,$$

where $h(t) = e^{2\pi i t}$. Especially as stated in terms of ρ_α, result 2.2.14 is well known (see, for example, chapter 8 of [3]); we sketch a proof here.

SKETCH OF PROOF OF 2.2.14. While not homeomorphisms, the bijections h and h^{-1} both take dense sets to dense sets, and if $U \subset [0,1)$ is open, $h(U)$ contains an open set of S^1. Therefore it is sufficient to show that ρ_α-orbits are dense in S^1, and that given an open set $W \subset S^1$ there is some k such that

$$\cup_{j=1}^{k} \rho_\alpha^j(W) = S^1.$$

Choose $p \in S^1$. Imagine that $\rho_\alpha^j(p) = p$ for some $j \geq 1$; this implies that $j2\pi\alpha$ is an integer multiple of 2π, contrary to the irrationality of α. It follows that the points of any ρ_α-orbit are all distinct and so contain a Cauchy subsequence. Given $\epsilon > 0$, then, there are two powers $\rho_\alpha^m(p)$ and $\rho_\alpha^n(p)$, $n > m$, that are separated by an arc of circle of angle less than ϵ; the power ρ_α^{n-m} therefore rotates the circle through an angle less than ϵ. If $W \subset S^1$ is an open set containing an open arc of angle greater than ϵ, then, we see that $\{\rho_\alpha^j(W) \ : \ j \in \mathbf{Z}_+\}$ covers S^1; the desired conclusion follows from the compactness of S^1. $\square$

2.3 Finite-dimensional apparatus

In this section we develop tools for studying $(SDDE)$ when solutions, or sections of solutions, lie in C'.

Definition 2.3.1. The subset C_0'. We define C_0' to be the set of $y_0 \in C'$ with $y_0(0) = 0$; that is,

$$C_0' = C' \cap C_0.$$

If $y_t \in C'$ for all $t \in [0, \tau]$, the proper zeros of y on $(-\gamma, \tau)$ strictly alternate in sign. (By contrast, if y has an interval of zeros it is possible, say, for two successive proper zeros to both be upward.) Accordingly, if y_0 lies in C_0', the following finite-dimensional information is all that we need to determine its continuation y:

- The locations of the proper zeros of y_0 on $(-\gamma, 0)$;
- The sign of $y_0(-\epsilon)$ for sufficiently small positive ϵ.

This observation lies at the heart of the tractability of $(SDDE)$; we now build an apparatus to exploit it.

Definition 2.3.2. The sets $\mathcal{S}_\pm^n$ and the function V. For each $n \in \mathbf{N}$ write

$$\mathcal{S}^n = \{ \ x \in \mathbf{R}^n \ : \ 0 < x_1 < \cdots < x_n < \gamma \ \}.$$

We will always equip $\mathcal{S}^n$ with the sup metric. We take $\mathcal{S}^0$ to be a one-point set.

For every $n \in \mathbf{Z}_+$, we take two copies of $\mathcal{S}^n$, labeled $\mathcal{S}_-^n$ and $\mathcal{S}_+^n$.

Suppose that $y_0 \in C_0'$. Take $i \in \{-, +\}$ to be the opposite of the sign of $y(-\epsilon)$ for small positive ϵ. If y_0 has proper zeros $-x_1 > -x_2 > \cdots > -x_n$ in $(-\gamma, 0)$, write

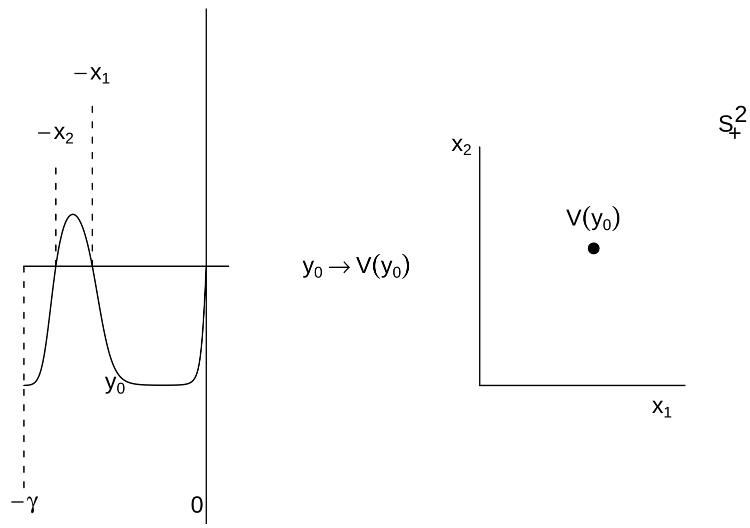
$$V(y_0) = (x_1, x_2, \ldots, x_n) \in \mathcal{S}_i^n;$$

if y_0 has no proper zeros in $(\gamma, 0)$, write

$$V(y_0) \in \mathcal{S}_i^0.$$

Note the signs: the zeros of y_0 are negative, but the entries of $V(y_0)$ are positive to allow us to work in the more customary positive orthant. See Figure 5.

Figure 5



Definition 2.3.3. The functions z and ϕ. Given $y_0 \in C_0'$, write $x = V(y_0)$ and suppose that $x \in \mathcal{S}_-^n$ (or $\mathcal{S}_+^n$). Suppose that the continuation $y(t)$ of y_0 as a solution of $(SDDE)$ has the following features:

- $y(t)$ has a downward (upward) proper zero at 0;
- $y(t)$ has a first positive upward (downward) proper zero ζ;
- All zeros of $y(t)$ on $[0, \zeta]$ are isolated.

In this case, we define

$$z(x) = \zeta; \ \phi(x) = V(y_\zeta).$$

Figure 6 shows the continuation of an initial condition $y_0 \in C_0'$ as a solution of $y'(t) = h(y(t-1))$, where

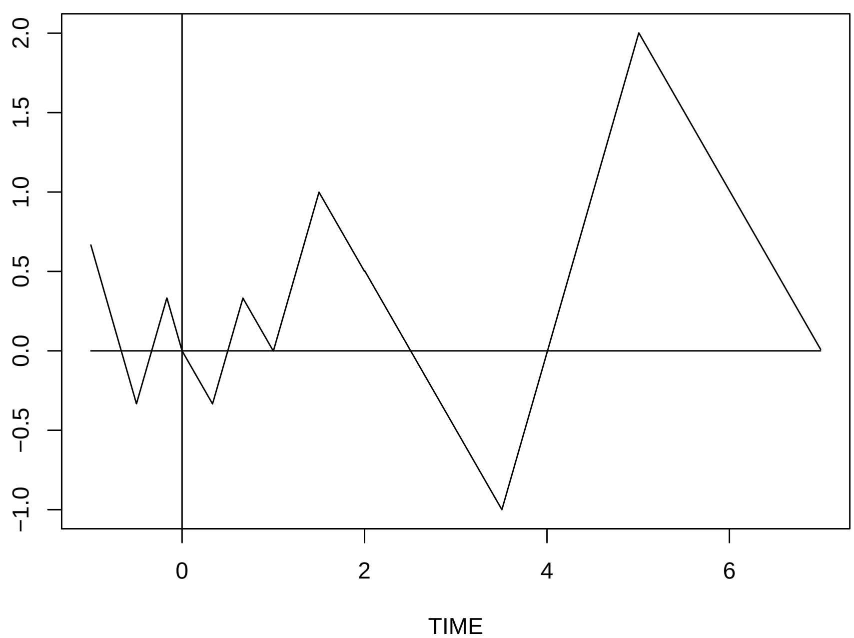
$$h(y) = 2,\ y < 0; \quad h(y) = -1,\ y > 0.$$

Writing $x = V(y_0)$, for this example we have

$$x = \left(\frac{1}{3}, \frac{2}{3}\right) \in \mathcal{S}_-^2;\ \phi(x) = \left(\frac{1}{2}, \frac{5}{6}\right) \in \mathcal{S}_+^2;\ \phi^2(x) \in \mathcal{S}_-^0;\ \phi^3(x) \in \mathcal{S}_+^0 \ldots.$$

Observe that the second positive zero of this solution (at time $t = 1$) is improper, and so is "ignored" by the sequence $\{\phi^k(x)\}$.

Figure 6



Example 2.3.4. z **and** ϕ **need not be defined.** Consider the equation $y'(t) = -\text{sgn}(y(t-1))$ and the initial condition given by

$$y_0(s) = -s(s + 1/2),\ s \in [-1, 0].$$

$y_0 \in C_0'$ and $V(y_0) \in \mathcal{S}_-^1$, but $z(V(y_0))$ and $\phi(V(y_0))$ are not defined (0 is not a proper zero of the continuation $y(t)$ of y_0). If we consider instead the same initial condition y_0 with the equation $y'(t) = \text{sgn}(y(t-1))$, then $z(V(y_0))$ and $\phi(V(y_0))$ are not defined either because, while the continuation of y_0 has 0 as a downward proper zero, it has no positive proper zeros. $\square$

Informally speaking, we think of the map ϕ as the distillation of a return map on C'_0. Let us suppose that C' is forward-invariant and that $y_0 \in C'_0$ has continuation y. If $\phi^m(y_0)$ is defined for all $m \in \mathbf{N}$, then the sequence $\{\phi^m(y_0)\}$ captures essentially all information about the evolution of $y_t \in C'$: for if ζ is the mth positive proper zero of y, then $\phi^m(y_0) = V(y_\zeta)$ suffices to compute the continuation of y_ζ. If $\phi^m(y_0)$ is defined only for m up to some natural M, then y has a maximum positive proper zero ζ and $y(t)$ has constant slope for $t > \zeta + \gamma$. If $\phi(y_0)$ is not defined, then y either has at most one positive proper zero — in which case $y(t)$ eventually has constant slope — or there is some minimum $\tau > 0$ where $\phi(y_\tau)$ is defined; in this case, the sequence $\{\phi^m(y_\tau)\}$ describes the subsequent evolution of y (for as many terms as it is defined). In any case we see that, except for solutions that eventually have constant slope, the map ϕ comprehends the solution semiflow for $(SDDE)$ when C' is forward-invariant. Somewhat more generally, if $y_0 \in C'_0$ and $y_s \in C'$ for $s \in [0,t]$, then $\{\phi^m(y_0)\}$ describes the evolution of y_s up to time τ, where

$$\tau = \sup\{\ s \leq t\ :\ s \text{ is a proper zero of } y\ \}.$$

In particular, periodic points of ϕ^2 correspond to periodic solutions of $(SDDE)$.

Definition 2.3.5. We write $D_-(n)$ for the subset of $\mathcal{S}^n_-$ where z and ϕ are defined, and similarly for $D_+(n)$. We extend this notation as follows:

$$D_-(n,j) = D_-(n) \cap \phi^{-1}\left(\mathcal{S}^j_+\right); \quad D_+(n,j) = D_+(n) \cap \phi^{-1}\left(\mathcal{S}^j_-\right).$$

Given $x = (x_1, \ldots, x_n) \in D_-(n)$, if $z(x) \geq \gamma$ we of course have $x \in D_-(n,0)$. If $z(x) < \gamma$, then $x \in D_-(n,j)$ exactly if

$$z(x) + x_{j-1} < \gamma \leq z(x) + x_j$$

(with the notational convention $x_0 = 0$) and in this case we have the following formula:

$$\phi(x) = (z(x), z(x) + x_1, \ldots, z(x) + x_{j-1}). \tag{2.3.6}$$

Observe that $D_-(n,j)$ is always empty for $j > n+1$.

All of the analogous statements of course hold with the opposite signs.

Example 2.3.7. We describe all of the sets $D_-(2,j) \subset \mathbf{R}^2$ associated with the equation

$$y'(t) = -2\text{sgn}(y(t-1)) - \text{sgn}(y(t-2)).$$

Observe that

$$\begin{aligned} y'(t) = 3 &\text{ when } y(t-2) < 0,\ y(t-1) < 0; \\ y'(t) = 1 &\text{ when } y(t-2) > 0,\ y(t-1) < 0; \\ y'(t) = -1 &\text{ when } y(t-2) < 0,\ y(t-1) > 0; \\ y'(t) = -3 &\text{ when } y(t-2) > 0,\ y(t-1) > 0. \end{aligned}$$

We start by describing $D_-(2)$. Suppose that $y_0 \in C_0'$ has two proper zeros in $(-2,0)$, and write y for the continuation of y_0. All positive zeros of y will be isolated (condition (2.1.10) is satisfied, so C' is forward-invariant) and $y'(t)$ will exist for $t > 0$ except at isolated points. If we imagine that y has a downward proper zero at 0 but no positive upward proper zero, then we have that $y(t) < 0$ and $y'(t) = 3$ for all $t > 2$ except a set of isolated points; this is impossible. For this particular equation, then, y having a downward proper zero at 0 guarantees that $z(V(y_0))$ and $\phi(V(y_0))$ are defined.

If $y'(t)$ exists, it is negative if and only if $y(t-1) > 0$. Thus, if $y_0 \in C_0'$ has two proper zeros on $(-2,0)$, the continuation y of y_0 will have a downward proper zero at 0 exactly when $y(-\epsilon) > 0$ for small $\epsilon > 0$ and both proper zeros of y_0 occur either in $(-2,-1]$ or in $(-1,0)$. Therefore, for this equation,

$$D_-(2) = \{\ (x_1,x_2) \in \mathcal{S}_-^2\ :\ x_1 \geq 1 \text{ or } x_2 < 1\ \}.$$

Suppose that $x = (x_1,x_2) \in D_-(2)$. Direct computation shows that if $x_1 \geq 1$, then $x \in D_-(2,0)$. If $x_2 < 1$, though, x might lie in $D_-(2,0)$, $D_-(2,1)$, $D_-(2,2)$, or $D_-(2,3)$. To see this, choose $(x_1,x_2) \in D_-(2)$, with $0 < x_1 < x_2 < 1$. The corresponding solution y will behave as follows.

- Drop with slope -3 for $(1 - x_2)$ units. The value of y at the end of this interval is $-3(1 - x_2)$. Next step:

- Rise with slope 1 for $(x_2 - x_1)$ units. The value of y at the end of this interval is $-3 + 4x_2 - x_1$. If this quantity is greater than zero, $x \in D_-(2, 3)$. Otherwise, next step:

- Drop with slope -3 for x_1 units. The value of y at the end of this interval is $-3 + 4x_2 - 4x_1$. We have now gone forward one time unit. Next step:

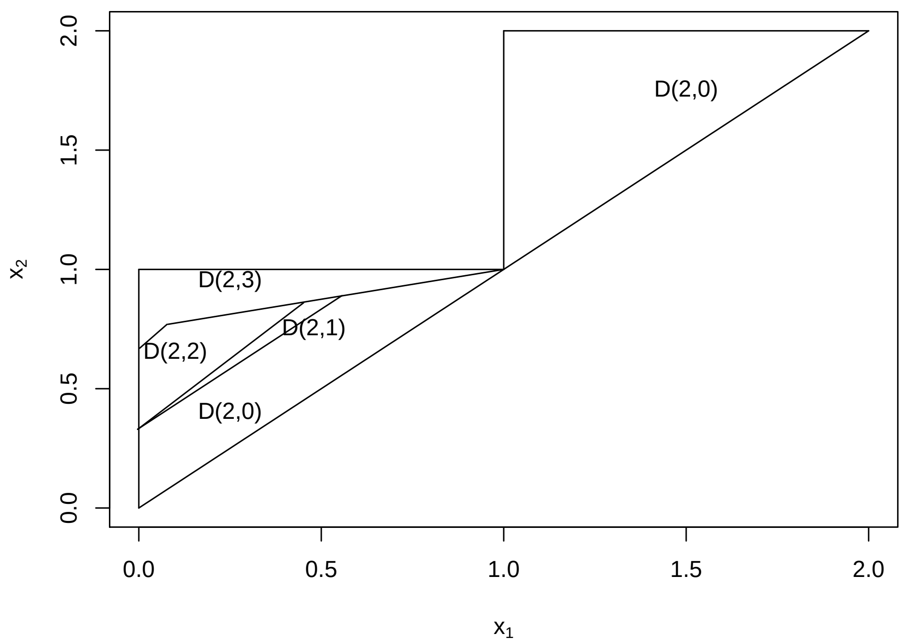
- Rise with slope 1 for $1 - x_2$ units. The value of y at the end of this interval is $-2 + 3x_2 - 4x_1$. If this quantity is greater than 0, then $x \in D_-(2, 3)$. Otherwise, y_t has "lost a zero" — x does not lie in $D_-(2, 3)$. Next step:

- Rise with slope 3 for $x_2 - x_1$ units. The value of y at the end of this interval is $-2 + 6x_2 - 7x_1$. If this quantity is greater than zero, $x \in D_-(2, 2)$. Otherwise, y_t has "lost a zero" — x does not lie in $D_-(2, 2)$. Next step:

- Rise with slope 1 for x_1 units. The value of y at the end of this interval is $-2 + 6x_2 - 6x_1$. If this quantity is greater than zero, $x \in D_-(2, 1)$. Otherwise, $x \in D_-(2, 0)$. We have now gone forward two time units. $\square$

Figure 7 displays a decomposition of $D_-(2)$ into the four regions $D_-(2, 0)$, $D_-(2, 1)$, $D_-(2, 2)$, and $D_-(2, 3)$. We see from this example that the sets $D_\pm(n, j)$ need not be especially nice: in general they are neither connected nor convex, are neither open nor closed in $\mathcal{S}^n_\pm$, and can contain points of discontinuity of z.

Figure 7



To forge a connection between $(SDDE)$ and equations with continuous feedback, we will need to work at points where z and ϕ are differentiable. ((2.3.6) shows that the differentiability of z and ϕ are equivalent.) The following two examples illustrate how differentiability can fail.

Example 2.3.8. A point of discontinuity of z interior to $D_-(n,j)$. Even if x is interior to $D_-(n,j)$, z might be discontinuous there. We revisit the equation from example 2.3.7:

$$y'(t) = -2\text{sgn}(y(t-1)) - \text{sgn}(y(t-2)).$$

Consider the point

$$x = (0.1, 0.5, 0.6, 0.9) \in D_-(4).$$

$z(x) = 0.8$, and there is a neighborhood about x in $\mathcal{S}_-^4$ where z is strictly less than 1 — this point is therefore interior to $D_-(4,5)$. Nevertheless, z is discontinuous at x in the third coordinate: for any $\epsilon \in (0, 0.1)$ we have

$$z\left((0.1, 0.5, 0.6-\epsilon, 0.9)\right) = 0.4.$$

The trouble comes from the fact that the continuation $y(t)$ of $y_0 \in V^{-1}(x)$ has an improper zero at $t = 0.4$. $\square$

Example 2.3.9. A point where z is continuous but not differentiable. Consider the equation

$$y'(t) = -\text{sgn}(y(t-1)) - 2\text{sgn}(y(t-2)).$$

Observe that

$$\begin{aligned}
y'(t) &= 3 \quad \text{when} \quad y(t-2) < 0,\ y(t-1) < 0; \\
y'(t) &= 1 \quad \text{when} \quad y(t-2) < 0,\ y(t-1) > 0; \\
y'(t) &= -1 \quad \text{when} \quad y(t-2) > 0,\ y(t-1) < 0; \\
y'(t) &= -3 \quad \text{when} \quad y(t-2) > 0,\ y(t-1) > 0.
\end{aligned}$$

Consider the point $x = (x_1, x_2) = (1, 9/5) \in \mathcal{S}^2_-$. First, let us calculate $z(x)$. Suppose that $y_0 \in C'_0$ with $V(y_0) = x$, and write y for the continuation of y_0. $y'(t) = -3$ for $t \in (0, 1/5)$ and then $y'(t) = 1$ for $t \in (1/5, 4/5)$; we therefore have that $z(x) = 4/5$ and that $x \in D_-(2, 2)$. In fact, there is an open set about x in $\mathcal{S}^2_-$ that is contained in $D_-(2, 2)$, and z is continuous at x.

Nevertheless, z is not differentiable at x, as we now show. If x_1 is increased slightly to $1 + \epsilon$, $z(x)$ is unchanged; thus the right-hand derivative of z at x with respect to x_1 is zero.

On the other hand, suppose that x_1 is decreased slightly to $1 - \epsilon$. In this case, $y'(t) = -1$ for $t \in (0, \epsilon)$ and $y'(t) = -3$ for $t \in (\epsilon, 1/5)$; we therefore have

$$y(1/5) = -3(1/5 - \epsilon) - \epsilon = -3/5 + 2\epsilon.$$

Then $y'(t) = 1$ for the next $3/5 - 2\epsilon$ units, and $z(x) = 4/5 - 2\epsilon$. Therefore the left-hand derivative of z at x with respect to x_1 is 2, and z is not differentiable at $(1, 9/5)$. As we see, the problem stems from the fact that $y(0 - d_1) = y(-1) = 0$. $\square$

We now turn to the articulation of a condition on x that circumvents the sources of nondifferentiability illustrated in examples 2.3.8 and 2.3.9. The condition is practical to check in particular cases and implies that the behavior of z at x is particularly nice: x is interior to $D_\pm(n, j)$, and the derivative of z is locally constant around x.

Definition 2.3.10. Change points and change index pairs. Let y be a solution of $(SDDE)$. Any point c such that $(c - d_i)$ is a proper zero of y for some delay d_i is called a *change point of* y.

Suppose that $y_0 \in C_0'$ with $V(y_0) = x = (x_1, \ldots, x_n) \in D_-(n)$, and write $x_0 = 0$. Given $0 \le k \le n$ and $1 \le i \le D$ (where D is the number of delays in $(SDDE)$), we say that (k, i) is a *change index pair of* x if there is a change point $c = c(k, i)$ of $y(t)$ satisfying

- $c \in [0, z(x)]$;
- $c - d_i = -x_k$.

Similarly if $x \in D_+(n)$.

Remark 2.3.11. The reason for this terminology is obvious — the change points are, by and large, the places where it is possible for $y'(t)$ to change — but we amplify a few points. First, if $c - d_i$ is (for example) the endpoint of an interval of zeros of y, $y'(t)$ might change at c, but we do not regard c as a change point (in the definition, $c - d_i$ has to be a *proper* zero). Second, at a change point $c > 0$, $y'(t)$ does not have to change (for instance, the sign of y might change at $c - d_i$ for multiple delays d_i, with the changes in the $h_i(y(c - d_i))$ canceling one another). Third, if y_0 is an initial condition and $c < 0$ happens to be a change point, this of course raises no expectation on the behavior of $y'(t)$ around c because $y(t)$ does not solve $(SDDE)$ for $t < 0$.

Suppose that y is a solution of $(SDDE)$. If $y_t \in C'$ for t in some positive interval $(\tau - \delta, \tau + \delta)$, then $y'(\tau)$ exists if τ is not a change point. For even if $y(\tau - d_i) = 0$ for some delay d_i, since τ is not a change point the zero $\tau - d_i$ is isolated and improper; there is thus a small interval about τ where $y(t - d_i)$ is of constant sign, except perhaps at finitely many points, for all delays d_i. The solution y therefore has constant slope on some interval about τ.

From a bookkeeping standpoint, change index pairs are more useful than change points, since the change index pairs corresponding to a change point c (there might be more than one) encode how $y'(t)$ changes at c.

Definition 2.3.12. Simple points in $D_-(n)$ and $D_+(n)$. Suppose that $y_0 \in C_0'$ with continuation y. We say that $x = V(y_0)$ is *simple* if $z(x)$ is defined and

- y has no zeros (at all, not just no proper zeros) between 0 and $z(x)$;
- neither 0 nor $z(x)$ is a change point of y.

Notice that the two parts of this definition remove precisely the sources of discontinuity and nondifferentiability, respectively, in examples 2.3.8 and 2.3.9. We describe the properties of simple points in the next theorem.

Theorem 2.3.13. Near simple points, the functions z and ϕ are affine and change index pairs are invariant. *Suppose that $n \in \mathbf{N}$ and that $x \in \mathcal{S}_-^n$ is simple. Then there is an open neighborhood U about x in $\mathcal{S}_-^n$ such that the following hold. Every point $\xi \in U$*

- *is contained in the domain of z;*
- *is simple;*
- *has image $\phi(\xi)$ of the same dimension as $\phi(x)$;*
- *has the same set of change index pairs as does x.*

Furthermore, z is differentiable on U and the derivative of z is constant on U — that is, z is affine on U. Therefore ϕ is affine on U also.

Similarly for $x \in \mathcal{S}_+^n$.

PROOF. Write

$$\mu = \sum_{i=1}^{D} |a_i| + |b_i|;$$

μ is an upper bound on $|u'(t) - v'(t)|$ for any two solutions u, v of $(SDDE)$. Write $\sigma > 0$ for the smallest nonzero value attainable by $|u'(t)|$, where u is any solution of $(SDDE)$.

Let $x = (x_1, \ldots, x_n) \in \mathcal{S}_-^n$ be simple (the proof is the same for $x \in \mathcal{S}_+^n$). Choose $y_0 \in V^{-1}(x)$ with continuation y, and write $z(x) = z$. Finally, choose $\rho > 0$ such that

- $[-2\rho, 2\rho] \cup [z - 2\rho, z + 2\rho]$ contains no change points of y.

  Such a ρ does exist, since $y(t)$ will have finitely many change points on $[-\gamma, z+d_1]$. Notice in particular that $y(t)$ has constant negative slope on $(0, 2\rho)$ and constant positive slope on $(z - 2\rho, z + 2\rho)$. Observe also that we must have $2\rho < d_1$, since d_1 and $z + d_1$ are change points. Finally, we point out that the condition that $[-2\rho, 2\rho]$ contain no change points of y is equivalent to the condition that

  $$| - d_i - (-x_k)| > 2\rho \;\forall\; 0 \leq k \leq n \text{ and } 1 \leq i \leq D,$$

  where we are writing $x_0 = 0$.

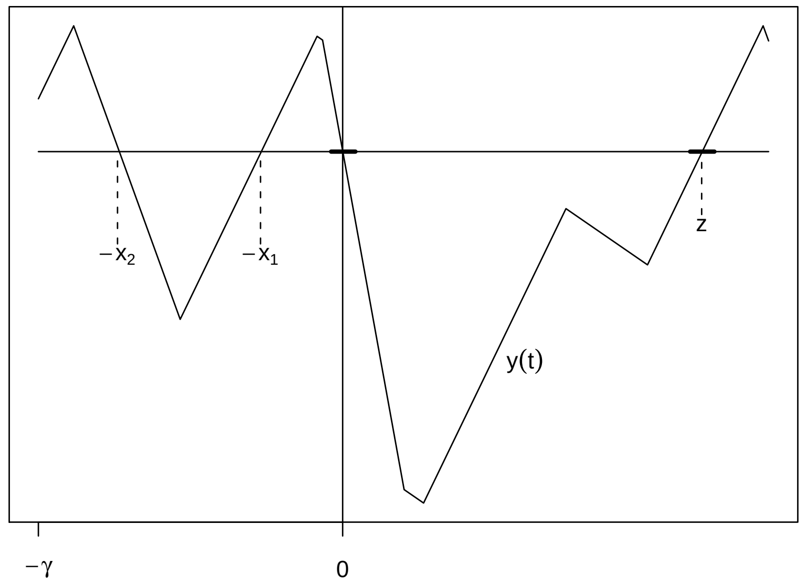
- $y(t) < -\rho\sigma \;\forall\; t \in (\rho, z - \rho)$.

  This second condition is not automatic given the first, since $y(t)$ might get quite close to 0 for t between ρ and $z - \rho$; but since $y(t)$ is strictly negative for $t \in (0, z)$, this condition does hold for ρ small enough.

This notation is illustrated in Figure 8. The dark bands on the time axis around $t = 0$ and $t = z$ are of radius 2ρ.

Remark 2.3.14. Making this choice of ρ is exactly where and how we use the hypothesis that x is simple.

Figure 8



Now choose $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{S}^n_-$ with $|\xi - x| < \rho$ (recall that we have endowed $\mathcal{S}^n_-$ with the sup metric). Choose $w_0 \in V^{-1}(\xi)$ with continuation w as a solution of $(SDDE)$.

For notational simplicity, we shall write $x_0 = \xi_0 = 0$ throughout the proof.

We claim that $y'(t) = w'(t)$ for $t \in (0, \rho)$. For given any $1 \leq i \leq D$, the definition of ρ implies that there is some $1 \leq k \leq n$ such that

$$t - d_i \in (-x_k, -x_{k-1}) \ \forall \ t \in [-2\rho, 2\rho].$$

Since $|\xi - x| < \rho$ it follows that

$$t - d_i \in (-\xi_k, -\xi_{k-1}) \ \forall \ t \in [-\rho, \rho].$$

Therefore $w(t - d_i)$ and $y(t - d_i)$ are of the same sign for all $t \in [0, \rho]$ (except perhaps at finitely many points where one or the other initial has an improper zero). The claim follows, and implies that 0 is a downward proper zero of w. We write

$$\zeta = \inf\{ \ t > 0 \ : \ w(t) = 0 \ \}$$

and put $\tau = \min\{ \ z, \ \zeta \ \}$.

Suppose that $c \in (0, \tau + d_1)$ is a change point of y. Then, since $y(t) < 0$ for $t \in (0, \tau)$, we must have that $c - d_i = -x_k$ for some $0 \leq k \leq n$ and some $1 \leq i \leq D$. For every such pair (k, i) (there might be more than one) we write $c = c(k, i)$. $c(k, i) > 2\rho$ by the definition of ρ. Since $|-\xi_k - (-x_k)| < \rho$, there is a positive time point

$$\kappa(k, i) \in (c(k, i) - \rho, c(k, i) + \rho)$$

such that

$$\kappa(k, i) - d_i = -\xi_k.$$

Conversely, suppose that $\kappa \in (0, \tau + d_1)$ is a change point of w. Then, since $w(t) < 0$ for all $t \in (0, \tau)$, we must have $\kappa - d_i = -\xi_k$ for some $0 \leq k \leq n$ and some $1 \leq i \leq D$. For every such pair (k, i) (there might be more than one) we write $\kappa = \kappa(k, i)$. We know from the above claim that $\kappa(k, i) > \rho$; therefore there is a positive time point

$$c(k, i) \in (\kappa(k, i) - \rho, \kappa(k, i) + \rho)$$

such that

$$c(k, i) - d_i = -x_k.$$

Let us now suppose that $t \in (0, \tau + d_1)$ is more than ρ units from any change point of y. Then the last paragraph tells us that t is not a change point of w either, and so $w'(t)$ and $y'(t)$ are both defined (see remark 2.3.11). The fact that $y(s - d_i)$ is of constant sign (except at finitely many points) for all s in the interval of radius ρ around t implies that $w(s - d_i)$ is of the same constant sign (except at finitely many points) for all s in the interval of radius

$$\rho - |\xi - x| > 0$$

about t (imitate the proof of the above claim). These observations establish that, if $t \in (0, \tau + d_1)$ is more than ρ units from any change point of y,

- t is not a change point of w;
- $w'(t) = y'(t)$;

- the change points of y occurring on $[0,t]$ and the change points of w occurring on $[0,t]$ correspond to identical sets of change index pairs.

The total time over the interval $(0, \tau + d_1)$ that $h_i(w(t-d_i))$ and $h_i(y(t-d_i))$ can disagree is $n|\xi - x|$, and therefore we have

$$\sup_{t\in[0,\tau+d_1]} |w(t) - y(t)| \leq \mu n|\xi - x|.$$

If we therefore choose

$$U = \{\ \xi\ :\ |\xi - x| < \frac{\rho\sigma}{\mu n}\ \},$$

for any $\xi \in U$ we have that

$$\sup_{t\in[0,\tau+d_1]} |w(t) - y(t)| < \rho\sigma.$$

Since $y(t)$ has constant slope at least as large as σ on $(z-2\rho, z+2\rho)$ and is strictly less than $-\rho\sigma$ on $(\rho, z-\rho)$, we see that $\xi \in U$ implies that $\zeta \in (z-\rho, z+\rho)$. Thus $z + \rho < \tau + 2\rho < \tau + d_1$, and since no point of $(z-\rho, z+\rho)$ is within ρ of a change point of y we have that $w'(t) = y'(t) = y'(z)$ on $(z-\rho, z+\rho)$. It follows that $z(\xi)$ is defined, that ξ is simple, and that ξ and x have the same set of change index pairs. This last implies that $\phi(x)$ and $\phi(\xi)$ are of the same dimension: for to say that $\phi(x)$ has dimension j is just to say that x has the change index pair (j, D) but not the change index pair $(j-1, D)$.

It remains to show that z is differentiable on U. Suppose that in fact $\xi_k = x_k + \epsilon$, where $|\epsilon| < \rho\sigma/\mu n$, and that $\xi_p = x_p$ for $p \neq k$. Then, for $t \in (0, \tau + d_1)$, $w'(t)$ and $y'(t)$ are equal unless $t - d_i$ lies between $-\xi_k$ and $-x_k$ for some i, in which case

$$h_i(w(t-d_i)) - h_i(y(t-d_i)) = (-1)^k \mathrm{sgn}(\xi_k - x_k)(b_i + a_i). \tag{2.3.15}$$

For $t \in (z-\rho, z+\rho)$, then, we actually have

$$w(t) - y(t) = \epsilon \left[\sum_{i:(k,i) \text{ is a change index pair of } x} (-1)^k (b_i + a_i) \right].$$

Therefore

$$\zeta - z = \frac{-\epsilon}{y'(z)} \left[\sum_{i:(k,i) \text{ is a change index pair of } x} (-1)^k (b_i + a_i) \right].$$

Thus we see that $\partial z(x)/\partial x_k$ exists, and depends only on the change index pairs of x and on $y'(z)$. But we have established that these things are locally invariant, and so conclude that $\partial z/\partial x_k$ is locally constant.

That ϕ is affine around x follows from the fact that z is, and the formula (2.3.6). □

The following formula will be useful later.

Proposition 2.3.16. Determinant of the Jacobian of ϕ. *Suppose that $x \in D_-(n,n)$ is simple. Then the determinant of the Jacobian of ϕ at x is given by*

$$\frac{-1^{n+1} \partial z(x)}{\partial x_n}.$$

PROOF (Thanks to Roger Nussbaum). The Jacobian of ϕ at x is given by

$$D\phi[x] = \begin{pmatrix} \frac{\partial z(x)}{\partial x_1} & \frac{\partial z(x)}{\partial x_2} & \cdots & \frac{\partial z(x)}{\partial x_{n-1}} & \frac{\partial z(x)}{\partial x_n} \\ 1 + \frac{\partial z(x)}{\partial x_1} & \frac{\partial z(x)}{\partial x_2} & \cdots & \frac{\partial z(x)}{\partial x_{n-1}} & \frac{\partial z(x)}{\partial x_n} \\ \frac{\partial z(x)}{\partial x_1} & 1 + \frac{\partial z(x)}{\partial x_2} & \cdots & \frac{\partial z(x)}{\partial x_{n-1}} & \frac{\partial z(x)}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial z(x)}{\partial x_1} & \frac{\partial z(x)}{\partial x_2} & \cdots & 1 + \frac{\partial z(x)}{\partial x_{n-1}} & \frac{\partial z(x)}{\partial x_n} \end{pmatrix}.$$

If we subtract the first row from each of the subsequent rows, the determinant is unchanged. Cofactor expansion along the first row then gives the desired result. □

Example 2.3.17. The nonsimple points of $D_-(n)$ need not have empty interior. The set of nonsimple points of $D_-(n)$ seems frequently to have empty interior in $\mathcal{S}^n_-$. We have not determined conditions that guarantee that this is the case; it is not always.

The single element of $D_-(0)$ might not be simple. For example, if our $(SDDE)$ is

$$y'(t) = -2\text{sgn}(y(t-1)) - \text{sgn}(y(t-4)),$$

the member of $D_-(0)$ is not simple: for if $x \in D_-(0)$ then $z(x) = 4$.

We can use almost the same example to exhibit an interval of non-simple points in $D_{-}(1)$. If our $(SDDE)$ is

$$y'(t) = -\operatorname{sgn}(y(t-1)) - \operatorname{sgn}(y(t-2)) - \operatorname{sgn}(y(t-10)),$$

then there is an interval about $x = 5$ that is contained in $D_{-}(1)$ and on which z has the constant value 2. None of the points in this interval is simple. □

Definition 2.3.18. Suppose that p is a periodic solution of $(SDDE)$. We say that p is *simple* if $V(p_z)$ is simple (recall definition 2.3.12) for every zero z of p.

In particular, a periodic solution p of $(SDDE)$ is simple if and only if it has the following properties:

- The only zeros of p are proper zeros;
- No zero of p is a change point.

Definition 2.3.19. Suppose that p is a simple periodic solution with a downward proper zero at 0. Write $0 < s_1 < s_2 < \cdots$ for the positive zeros of p. We say that p is *nondegenerate* if there is some even positive integer m such that

- $s_m > \gamma$;
- s_m is an integer multiple of the period of p (so $V(p_0)$ is a fixed point of ϕ^m);
- The Jacobian $D\phi^m[V(p_0)]$ of ϕ^m at $V(p_0)$ does not have eigenvalue 1.

Intuitively, we regard simple, nondegenerate periodic solutions as "robust." Our main theorems relating $(SDDE)$ to equations with continuous feedback apply to simple, nondegenerate periodic solutions of $(SDDE)$ — roughly speaking, if p is a simple nondegenerate periodic solution then an equation whose feedback functions are close enough to the step feedback functions in $(SDDE)$ will have a periodic solution close to p.

The simplicity and nondegeneracy conditions seem "usually" to hold, but we have been unable to articulate any precise statement of this sort. Such a statement is desirable, because it can be tedious to verify that a periodic solution of $(SDDE)$ is simple and nondegenerate.

Chapter 3

Steplike problems

3.1 Existence and uniqueness of periodic solutions

In this chapter we prove our main existence, uniqueness, and stability theorems for a special (and admittedly artificial) subclass of equations with continuous feedback. In the next chapter we will extend these results to more natural problems.

We choose and fix, for the duration of the chapter, some particular $(SDDE)$:

$$y'(t) = \sum_{i=1}^{D} h_i(y(t-d_i)),\ t > 0; \quad y_0 \in C; \tag{$SDDE$}$$

$$h_i(y) = \begin{cases} b_i, & y < 0; \\ 0, & y = 0; \\ -a_i, & y > 0. \end{cases}$$

Definition 3.1.1. Steplike problems. Given $(SDDE)$ as above and numbers $\eta > 0$, $\mu \geq 0$ and $B \geq 0$ (with $B = \infty$ allowed), we write

$$\mathcal{P}_{(SDDE)}(\eta, \mu, B)$$

for the set of differential delay equations of the form

$$y'(t) = \sum_{i=1}^{D} H_i(y(t-d_i)),\ t > 0; \quad y_0 \in C \tag{DDE}$$

where the following hold:

- Each H_i is continuous;
- $2\sum_{i=1}^{D} \sup |H_i| \leq \mu$;

- Each H_i has total variation less than or equal to B;
- $H_i(y) = b_i$ for all $y \le -\eta$ and $H_i(y) = -a_i$ for all $y \ge \eta$.

We call (DDE) an η-*steplike* problem corresponding to $(SDDE)$, μ, and B. Throughout this chapter, the label (DDE) will always refer to a member of a class $\mathcal{P}_{(SDDE)}(\eta, \mu, B)$.

If the functions H_i are C^k, we say that the problem (DDE) is C^k.

For the duration of this chapter, we will fix μ and B with

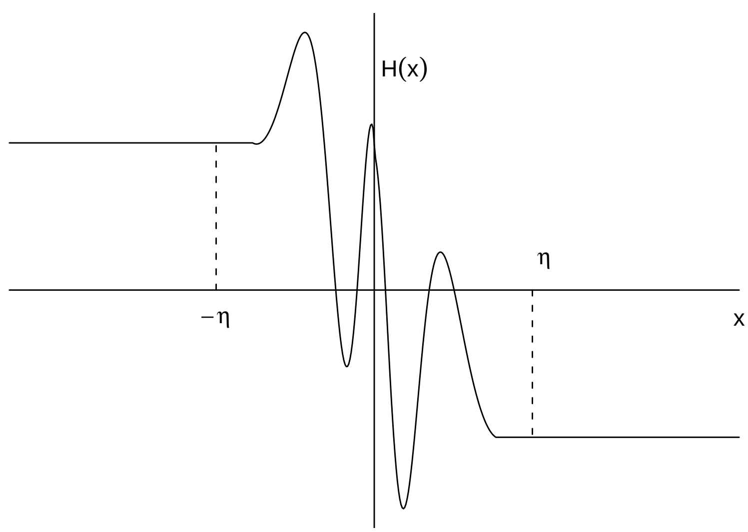
$$2\sum_{i=1}^{D} \sup |h_i| \le \mu; \quad \mu \le B.$$

Since we regard $(SDDE)$, μ and B as fixed, we will use the shortened notation

$$\mathcal{P}_{(SDDE)}(\eta, \mu, B) = \mathcal{P}(\eta).$$

Figure 9 shows a typical "steplike" feedback function H.

Figure 9



Our main results refer to particular periodic solutions of $(SDDE)$. We name such a periodic solution and its various features in the following definition; we will adhere to this notation throughout chapters 3 and 4.

Definition 3.1.2. Standing notation — the solution p. p is a simple nondegenerate periodic solution of $(SDDE)$, and p has a downward proper zero at 0. We write

$$V(p_0) = \pi,$$

where the function V is as in 2.3.2. We write n for the dimension of π:

$$V(p_0) = \pi = (\pi_1, \ldots, \pi_n).$$

We write $s_1 < s_2 < \cdots$ for the positive zeros of p. We write m for an even number such that s_m is both greater than γ and a multiple of the period of p, and also such that the $n \times n$ Jacobian matrix

$$A = D\phi^m[\pi] = D\phi^m[V(p_0)],$$

where ϕ is as in 2.3.3, does not have eigenvalue 1. (That we can do this is just the definition of nondegeneracy; recall definition 2.3.19.)

We write σ for the minimum nonzero value of $|p'(t)|$.

We choose $\rho > 0$ such that

- no closed interval of radius 6ρ centered at a zero of p contains a change point of p. Note that 6ρ is less than the shortest delay d_1, that $s_m > \gamma + 6\rho$, that $-\pi_n > -\gamma + 6\rho$, and that any two zeros of p are more than 12ρ units apart.

- $|p(t)| > 3\rho\sigma$ if t is not within 3ρ of a zero of p (that is, if $|t - z| > 3\rho$ for any zero z of p).

We write $t^* = s_m + 3\rho$.

Remark 3.1.3. Figure 10 illustrates this notation. The dark intervals about the zeros of p have radius 6ρ.

The particular solution p pictured in Figure 10, and throughout this section, is a periodic solution of

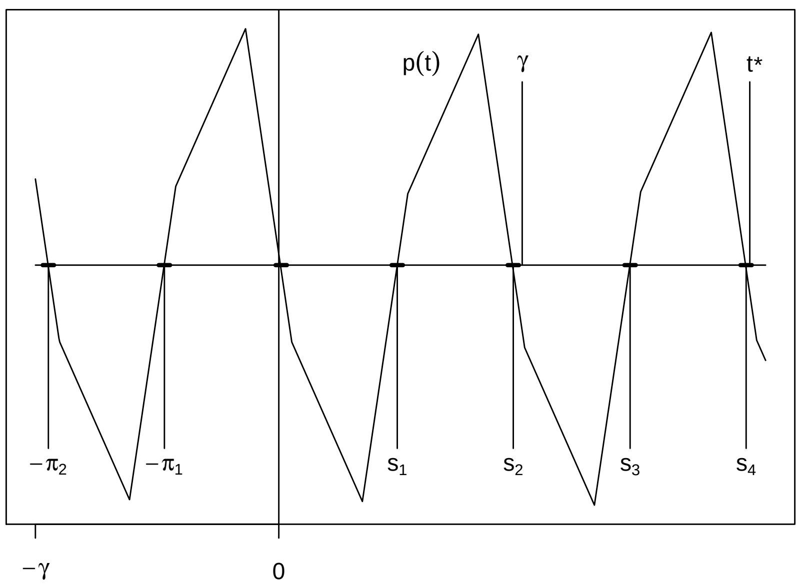
$$y'(t) = -2\text{sgn}(y(t-1)) - \text{sgn}(y(t-3)).$$

This solution has zeros separated by 10/7, and is simple and nondegenerate. For this solution, $n = 2$ and we can take $m = 4$. We will discuss this solution in more detail in section 4.4.

For the sake of clarity and readability, the quantities used to generate the figures in this section are not accurate: the figures should be considered schematic.

Much of our work in this section and in chapter 4 will consist of comparing solutions of various equations on the interval $[0, t^*]$.

Figure 10



As in the proof of 2.3.13, the choice of ρ in 3.1.2 is exactly where and how we use the hypothesis that p is simple (recall definition 2.3.18). A basic consequence of this choice is

Lemma 3.1.4. *If t is within 3ρ of a zero z of p (that is, if $|t - z| \leq 3\rho$), then*

$$|p(t - d_i)| > 3\rho\sigma$$

for all delays d_i.

PROOF. If $|p(t - d_i)| \leq 3\rho\sigma$, then $t - d_i$ is within 3ρ of a zero of p, contradicting that p has no change points within 6ρ of z. □

Since this section is technical, we first outline the general ideas. Choose $y_0 \in C_0$ with continuation y as a solution of (DDE). When η is small, we expect y to behave much as if it were a solution of $(SDDE)$ — at least if $y(t)$ is "usually" not close to zero. In particular, if y_0 is close to p_0 we expect that

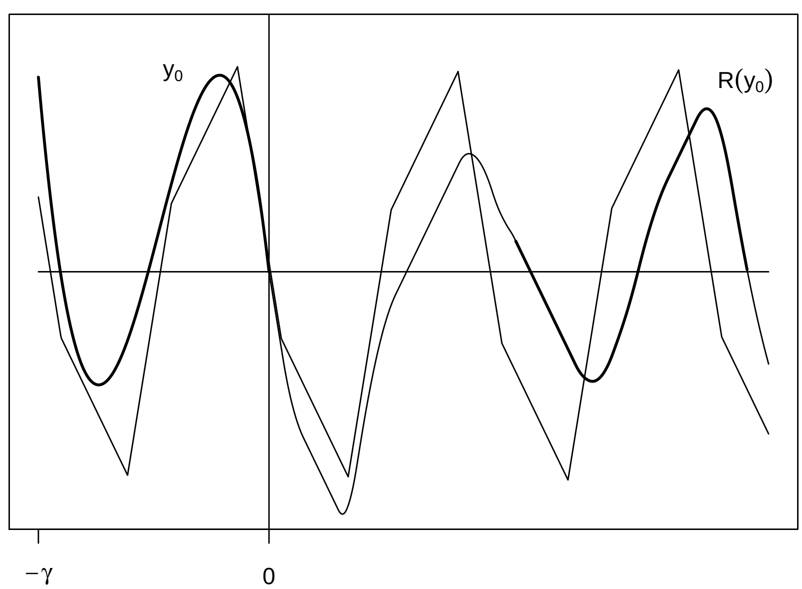
- $y(t)$ should be close to $p(t)$ for $t > 0$ not too large;
- the mth positive proper zero z of y should be defined;
- y_z should be close to p_0.

Definition 3.1.5. Notation — the return map R. Let $y_0 \in C_0$ with continuation y as a solution of (DDE). If the mth positive proper zero z of y is defined, we write

$$R(y_0) = y_z \in C_0.$$

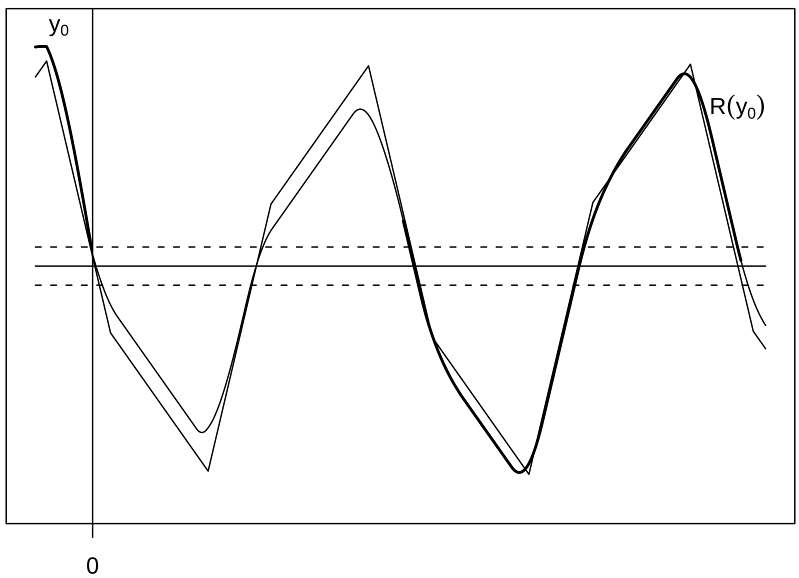
Figure 11 shows a y_0 close to p_0 and its continuation y as a solution of (DDE), along with p. y_0 and (a translate of) $R(y_0)$ are marked with bold lines.

Figure 11



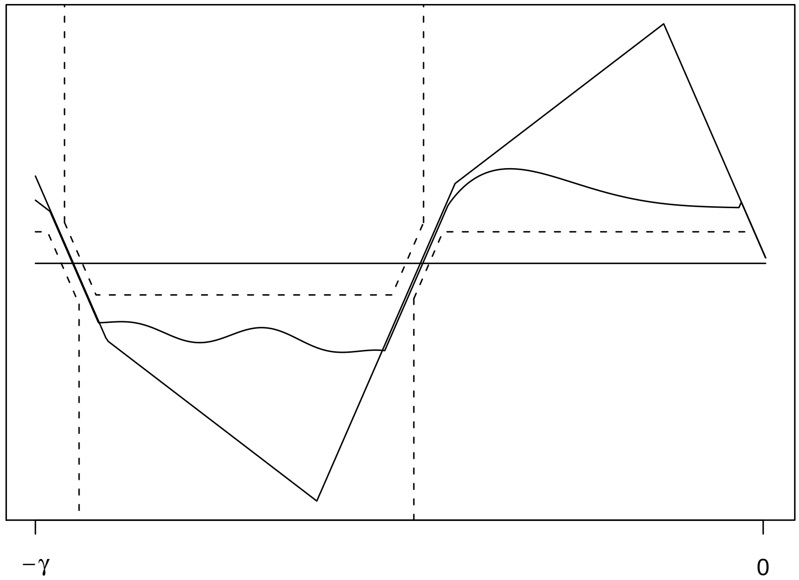
The simplicity of p implies that, when $p(t)$ is close to zero, the delayed values $p(t-d_i)$ are far from zero (this is the substance of lemma 3.1.4). If y_0 is close to p_0 and η is small, then, since we expect y and p to track each other closely we in turn expect delayed values $y(t-d_i)$ to be far from zero when $y(t)$ is close to zero. Since $H_i(y(t-d_i)) = h_i(p(t-d_i))$ when $y(t-d_i)$ and $p(t-d_i)$ are of the same sign and far from zero, we expect y to have constant slope, equal to that of p, near its zeros. Figure 12 illustrates the idea.

Figure 12



We therefore expect, if y_0 is close to p_0 and η is small, that $R(y_0)$ will not only be close to p_0 but will also lie in a special subset of C_0 — namely, a set of initial conditions whose zeros are close to the zeros $-\pi_k$ of p_0, and whose slopes are constant and equal to the slopes of p_0 around those zeros. Such a subset of C_0 is illustrated in Figure 13. The dashed lines bound the possible graphs of initial conditions in the subset, and the slopes of these initial conditions are equal to the slopes of p_0 near its zeros.

Figure 13



If y_0 lies in a set like that pictured in Figure 13 and η is small enough, then the zeros of y_0 completely determine its continuation y as a solution of (DDE) (for the behavior of y_0, where $|y_0(t)|$ is small, is determined by the location of its zeros; and $H_i(y_0(t))$, when $|y_0(t)|$ is large, is locally constant). Therefore we expect that, for η small enough, there is a neighborhood $U \subset C_0$ about p_0 such that R^2 is defined on U and such that R is finite-dimensional on $R(U)$. We will be able to give a formula for $V \circ R$ on $R(U)$ that is intimately related to the return map ϕ^m for $(SDDE)$. This formula will allow us to assert that R has a fixed point.

We introduce some more necessary machinery, and then follow the program outlined above. Given $y_0, w_0 \in C_0$, we define the semimetric

$$\langle y_0, w_0 \rangle = \|\bar{y}_0 - \bar{w}_0\|,$$

where

$$\bar{y}_0(s) = \begin{cases} y_0(s), & |y_0(s)| \leq \rho\sigma; \\ \rho\sigma \ \mathrm{sgn}(y_0(s)), & |y_0(s)| \geq \rho\sigma \end{cases} \tag{3.1.6}$$

and similarly for $\bar{w}_0(s)$.

Figure 13 shows two initial conditions that are not close in norm, but are close in the $\langle\cdot,\cdot\rangle$ semimetric.

Lemma 3.1.7. Facts about $\langle\cdot,\cdot\rangle$. *Suppose that $y_0, w_0, v_0 \in C_0$. Then the following hold:*

- *Symmetry:* $\langle w_0, y_0\rangle = \langle y_0, w_0\rangle$.
- *The triangle inequalities:*

$$\langle w_0, y_0\rangle \leq \langle w_0, v_0\rangle + \langle v_0, y_0\rangle, \quad \langle w_0, y_0\rangle \geq \langle w_0, v_0\rangle - \langle v_0, y_0\rangle.$$

- $\langle w_0, y_0\rangle \leq \|w_0 - y_0\|$.
- *Given $\delta > 0$, the set $\{\ v_0 \in C_0\ :\ \langle v_0, y_0\rangle < \delta\ \}$ is an open set about y_0, and the set $\{\ v_0 \in C_0\ :\ \langle v_0, y_0\rangle \leq \delta\ \}$ is a closed set about y_0.*

PROOF. Symmetry and the triangle inequality follow immediately from the corresponding properties of the norm; $\langle w_0, y_0\rangle \leq \|w_0 - y_0\|$ simply because

$$|\bar{w}_0(s) - \bar{y}_0(s)| \leq |w_0(s) - y_0(s)|$$

for all $s \in [-\gamma, 0]$, where $\bar{w}_0$ and $\bar{y}_0$ are as in (3.1.6).

Suppose that $\langle v_0, y_0\rangle < \delta$, and suppose that w_0 is in the open ball of radius

$$\delta' = \delta - \langle v_0, y_0\rangle$$

about v_0. Then

$$\langle w_0, y_0\rangle \leq \langle w_0, v_0\rangle + \langle v_0, y_0\rangle \leq \|w_0 - v_0\| + \langle v_0, y_0\rangle < \delta' + \langle v_0, y_0\rangle = \delta.$$

Thus $\{\ v_0 \in C_0\ :\ \langle v_0, y_0\rangle < \delta\ \}$ is open.

Suppose now that $\langle v_0, y_0\rangle > \delta$, and suppose that w_0 is in the open ball of radius $\delta' = \langle v_0, y_0\rangle - \delta$ about v_0. Then

$$\langle w_0, y_0\rangle \geq \langle y_0, v_0\rangle - \langle v_0, w_0\rangle \geq \langle y_0, v_0\rangle - \|v_0 - w_0\| > \langle y_0, v_0\rangle - \delta' = \delta.$$

Thus $\{\ v_0 \in C_0\ :\ \langle v_0, y_0\rangle \leq \delta\ \}$, having open complement, is closed. $\square$

Remark 3.1.8. Write

$$S = \{ \ v_0 \in C_0 \ : \ \langle v_0, y_0 \rangle < \delta \ \};$$

$$S' = \{ \ v_0 \in C_0 \ : \ \langle v_0, y_0 \rangle = \delta \ \}.$$

$S \cup S'$ is *not* equal to the closure of S. S' certainly contains ∂S, but might also contain much more — for S' can have non-empty interior. For example, if $y_0 \in C_0$ is any initial condition that attains an absolute value of greater than $\rho\sigma + \epsilon$, then for any v_0 with $\|v_0 - y_0\| < \epsilon$ we have

$$\langle v_0, 0 \rangle = \rho\sigma.$$

The continuation of $y_0 \in C_0$ as a solution of (DDE) is not affected by perturbations in y_0 at points where $|y_0(s)|$ is large. Indeed, if $\eta < \rho\sigma$ and $y_0, w_0 \in C_0$ with $\langle y_0, w_0 \rangle = 0$, the continuations of y_0 and w_0 as solutions of $(DDE) \in \mathcal{P}(\eta)$ are identical for all positive time. We have introduced the $\langle \cdot, \cdot \rangle$ semimetric to detect differences between points in the space C_0 that are relevant for $(DDE) \in \mathcal{P}$.

Our goal for the current section is to find an open set U in C_0 about p_0 such that, for any $(DDE) \in \mathcal{P}(\eta)$ with η small enough, the return map R described in 3.1.5 is defined on U and has a fixed point u_0 whose continuation u, as a solution of (DDE), is close to p in a sense we'll make precise. In section 3.2 we will develop some properties of the derivative $DR[u_0]$ that will later aid us in extending to non-steplike problems. If we regard (and we do) $(SDDE)$, p, μ, and B as fixed, then the results of these two sections (with a handful of exceptions that we will identify explicitly) depend only on the size of η — that is, the particular shapes of the feedback functions H_i in (DDE) don't matter. This uniformity will be pivotal when we extend to non-steplike problems.

Definition 3.1.9. The subspace W and the subset $\tilde{W}$. We write $W \subset C_0$ for the closed linear subspace defined as follows: W consists of all initial conditions h_0 that are equal to zero on $[-2\rho, 0]$ and are constant on each of the n disjoint intervals $[-\pi_k - 2\rho, -\pi_k + 2\rho]$.

We define affine subset

$$\tilde{W} = \{ \ p_0 + h_0 \ : \ h_0 \in W, \ \|h_0\| \leq \rho\sigma \ \}.$$

Lemma 3.1.10. *If $y_0 \in \tilde{W}$, then y_0 has the following properties:*

- *y_0 has exactly n zeros $-x_n < \ldots < -x_1$ on $(-\gamma, 0)$, and each of these $-x_k$ is within ρ units of $-\pi_k$;*
- *The connected components of $y_0^{-1}[-\rho\sigma, \rho\sigma]$ are precisely intervals about $-x_k$, where y_0 has constant slope $p'(-\pi_k)$, and an interval with right endpoint 0, where y_0 has constant slope $p'(0)$.*

PROOF. Suppose that $w_0 \in \tilde{W}$. Since w_0 is everywhere within $\rho\sigma$ of p_0 and $|p_0(s)| \leq 2\rho\sigma$ only on the intervals $[-\pi_k - 2\rho, -\pi_k + 2\rho]$, $|w_0(s)| \leq \rho\sigma$ only on these same intervals (we are writing $\pi_0 = 0$). On these intervals, w_0 has constant slope $p'(-\pi_k)$ by definition, and has a unique zero within ρ units of $-\pi_k$. □

Remark 3.1.11. In the current section, it is the properties of $\tilde{W}$ described in 3.1.10 that will really concern us. Figure 13 suggests the possible graphs of functions having these features. The more rigid definition of $\tilde{W}$ that we have given will prove convenient in section 3.2.

We will use the following observations repeatedly. The proof is immediate from the definitions of $\langle \cdot, \cdot \rangle$ and $\tilde{W}$ and the fact that μ is an upper bound for $|p'(t)|$. Recall that, if $w_0, y_0 \in \tilde{W}$, the distance $|V(w_0) - V(y_0)|$ is measured in the n-dimensional sup norm.

Lemma 3.1.12. *Suppose that $w_0, y_0 \in \tilde{W}$. Then*

$$\sigma |V(w_0) - V(y_0)| \leq \langle w_0, y_0 \rangle \leq \mu |V(w_0) - V(y_0)|,$$

where V is as in section 2.3. If furthermore $\eta < \rho\sigma$ and $V(w_0) = V(y_0)$, the continuations of w_0 and y_0 as solutions of $(DDE) \in \mathcal{P}(\eta)$ are identical for all positive time. □

We now turn to the heart of the chapter — the comparison of solutions of (DDE) to p. (The reader will find it convenient to review the notation in definition 3.1.2.)

Lemma 3.1.13. *Write*

$$I = (0, 3\rho) \cup (s_1 - 3\rho, s_1 + 3\rho) \cup \cdots \cup (s_m - 3\rho, s_m + 3\rho).$$

Given $\epsilon_0 > 0$, there are $\eta > 0$ and $\epsilon > 0$ such that, given $y_0 \in C_0$ with continuation y as a solution of $(DDE) \in \mathcal{P}(\eta)$ and $\langle y_0, p_0 \rangle < \epsilon$, the following hold.

1. $|y(t) - p(t)| < \epsilon_0$ *for all* $t \in [0, s_m + 3\rho] = [0, t^*]$.

2. $[0, t^*] \cap y^{-1}[-2\rho\sigma, 2\rho\sigma] \subset I$.

3. $t \in I$ *and* $(t - d_i) > 0$ *implies that* $|y(t - d_i)| > 2\rho\sigma$.

4. $y'(t) = p'(t)$ *for all* $t \in I$.

5. *The first* m *positive zeros* $z_1, \ldots, z_m$ *of* y *are defined and proper, and for all* $1 \le j \le m$ *we have* $|z_j - s_j| < \epsilon_0$*; also,*

$$\|R(y_0) - p_0\| = \|y_{z_m} - p_0\| < \epsilon_0.$$

6. *There are in fact positive constants* k_1 *and* k_2 *such that*

$$\|R(y_0) - p_0\| < k_1\eta + k_2\epsilon.$$

7. $R(y_0) \in \tilde{W}$.

PROOF. Choose $y_0 \in C_0$ with $\langle y_0, p_0 \rangle < \epsilon$. Write y for the continuation of y_0 as a solution of $(DDE) \in \mathcal{P}(\eta)$.

Let us write

$$S = \{\ t \in [-\gamma, t^*]\ :\ |p(t)| \le \eta \text{ or } |y(t)| \le \eta \text{ or } p(t)y(t) \le 0\ \}.$$

S is a collection of intervals. The two basic observations are the following. First, given $t \in (0, t^*)$, $y'(t) = p'(t)$ unless $t - d_i \in S$ for some i. Second, for $t \in [0, t^*]$,

$$\begin{aligned} |y(t) - p(t)| &\le \int_0^t |y'(s) - p'(s)|\ ds \\ &\le \sum_i \int_0^t |H_i(y(s - d_i)) - h_i(p(s - d_i))|\ ds \\ &\le \sum_i \text{measure}\,(S \cap [-\gamma, t - d_i])\,[\sup|H_i| + \sup|h_i|] \\ &\le \mu \cdot \text{measure}\,(S \cap [-\gamma, t - d_1])\,. \end{aligned} \tag{3.1.14}$$

For each $0 \le j \le m$, write $N(j)$ for the measure of $S \cap [-\gamma, s_j + 3\rho]$ (where we are writing $s_0 = 0$). For each $0 \le j \le m$, also write

$$M(j) = \sup_{t \in [0, s_j + 3\rho]} |y(t) - p(t)|.$$

Suppose that $\eta + \epsilon < \rho\sigma$. Since $|p(t - d_i)| > 3\rho\sigma$ for all $t \in [0, 3\rho]$ and all d_i, $|\bar{p}_0(t - d_i)| = \rho\sigma$ for all such t and d_i; therefore $\bar{y}_0(t - d_i)\bar{p}_0(t - d_i) > 0$ and $|\bar{y}_0(t - d_i)| > \eta$ for all such t and d_i (here $\bar{p}_0$ and $\bar{y}_0$ are as in (3.1.6)). Thus we see that $t - d_i \notin S$ for all $t \in [0, 3\rho]$ and all d_i, and hence that $y'(t) = p'(t)$ and $y(t) = p(t)$ for all $t \in (0, 3\rho)$. Similar reasoning shows that $t - d_i \notin S$ whenever $t \in I$ and $t - d_i \le 0$.

The set $S \cap [-\gamma, 3\rho]$ is contained in the subset of $[-\gamma, 3\rho]$ where $|p(t)| \le \eta + \epsilon < \rho\sigma$, and we accordingly have the bound

$$N(0) \le \frac{2(n+1)}{\sigma} (\eta + \epsilon) .$$

Now, for $1 \le j \le m$ and $t \in (3\rho, s_j + 3\rho)$, we have the implication

$$p(t) > \eta + M(j) \implies y(t)p(t) > 0 \text{ and } |y(t)| > \eta \implies t \notin S.$$

In particular, as long as $M(j) < \rho\sigma$ the connected components of S on $(3\rho, s_j + 3\rho)$ will be contained in intervals where $p(t)$ is of absolute value no more than $\eta + M(j) < 2\rho\sigma$ — these are intervals where p is of constant slope. Therefore, as long as $M(j) < \rho\sigma$, the connected components of S on $(3\rho, s_j + 3\rho)$ will be contained in intervals about $s_1, \ldots, s_j$ each of length no more than

$$\frac{2}{\sigma} (\eta + M(j)) .$$

When $M(j) < \rho\sigma$, then, the $N(j)$ satisfy

$$N(0) \le \frac{2(n+1)}{\sigma} (\eta + \epsilon) ;$$
$$N(j) \le N(j-1) + \frac{2}{\sigma} (\eta + M(j)) , \quad 1 \le j \le m.$$

Again as long as $M(j) < \rho\sigma$, since the positive connected components of S are contained in intervals of radius 2ρ about $s_1, \ldots, s_j$ and $d_1 > 6\rho$, we have

$$\text{measure} (S \cap [-\gamma, s_j + 3\rho - d_1]) = N(j-1).$$

Thus if $M(j) < \rho\sigma$ the estimate (3.1.14) yields

$$M(j) \leq \mu N(j-1).$$

Combining the inequalities for $N(j)$ and $M(j)$ we get

$$N(j) \leq \left(1 + \frac{2\mu}{\sigma}\right) N(j-1) + \frac{2\eta}{\sigma}.$$

Therefore we see that, by choosing η and ϵ small, we can make $N(m-1)$, and hence $M(m)$, as small as we like. This yields the first part of the lemma.

Let us suppose in particular that we have chosen η and ϵ so that $M(m) < \rho\sigma$. In this case, since $|p(t)| \leq 3\rho\sigma$ only for $t \in I$ and $|p(t-d_i)| > 3\rho\sigma$ for all d_i and $t \in I$, we conclude that $|y(t)| \leq 2\rho\sigma$ only for $t \in I$ and that $|y(t-d_i)| > 2\rho\sigma$ whenever $t \in I$ and $t - d_i \geq 0$. These are the second and third parts of the lemma.

We have already seen that $t - d_i \notin S$ whenever $t \in I$ and $t - d_i \leq 0$; if $M(m) < \rho\sigma$ the last paragraph shows that $t - d_i \notin S$ for all $t \in I$. This yields the fourth part of the lemma.

If $M(m) < \rho\sigma$ then, since $y'(t) = p'(t)$ for $t \in I$ and $|y(t)| \leq 2\rho\sigma$ only for $t \in I$, we see that the first m positive zeros of $y(t)$ are defined and proper. Furthermore, we have the estimate

$$|z_j - s_j| < \frac{M(m)}{\sigma} < \rho$$

for every $1 \leq j \leq m$. Recalling (1.2.3), we also have

$$\|R(y_0) - p_0\| < M(m)\left(1 + \frac{\mu}{\sigma}\right).$$

This yields the fifth part of the lemma.

The above estimate expresses $\|R(y_0) - p_0\|$ as a constant times $M(m)$, which we bounded earlier by a constant times $N(m-1)$. For the sixth part of the lemma, then, we just need to see that $N(m-1)$ can be bounded by a constant times ϵ plus a constant times η. An examination of our estimates on the $N(j)$ confirms this:

$$N(0) \leq \frac{2(n+1)}{\sigma}(\eta + \epsilon) = c_0\eta + c_0'\epsilon;$$

and writing $N(j-1) \leq c_{j-1}\eta + c'_{j-1}\epsilon$ gives us

$$\begin{aligned} N(j) &\leq \left(1+\frac{2\mu}{\sigma}\right) N(j-1) + \frac{2\eta}{\sigma} \\ &\leq \left[\left(1+\frac{2\mu}{\sigma}\right) c_{j-1} + \frac{2}{\sigma}\right]\eta + \left(1+\frac{2\mu}{\sigma}\right) c'_{j-1}\epsilon. \end{aligned}$$

We turn to the last part of the lemma. Suppose that we have chosen η and ϵ so small that in fact $M(m) + \mu M(m)/\sigma \leq \rho\sigma$ and $|z_j - s_j| < \rho$ for all $0 \leq j \leq m$. Define

$$\tilde{y}(t) = y(t - (s_m - z_m)).$$

The graph of $\tilde{y}$ is obtained by shifting the graph of y to the right by $s_m - z_m$ units. Since $p_{s_m} = p_0$, to prove that $y_{z_m} = R(y_0) \in \tilde{W}$ it is sufficient to prove that

- $\tilde{y}$ is of constant slope, equal to that of p, on each of the intervals

$$(s_{m-n} - 2\rho, s_{m-n} + 2\rho), \cdots, (s_{m-1} - 2\rho, s_{m-1} + 2\rho), (s_m - 2\rho, s_m + 2\rho);$$

- $|\tilde{y}(t) - p(t)| \leq \rho\sigma$ for all $t \in [s_m - \gamma, s_m]$.

The first point follows from the facts that $y'(t) = p'(t)$ for all $t \in I$ and that $|z_j - p_j| < \rho$ for all $0 \leq j \leq m$. For $t \in [s_m - \gamma, s_m]$, the same idea we used for the estimate (1.2.3) yields

$$|\tilde{y}(t) - p(t)| \leq M(m) + \mu|z_m - s_m| \leq M(m) + \mu M(m)/\sigma \leq \rho\sigma.$$

This completes the proof. □

Remark 3.1.15. Asymptotic stability of simple periodic solutions of $(SDDE)$**.** The above lemma holds just as well if y_0 is continued as a member of $(SDDE)$ — just take $\eta = 0$. It follows that, if π is asymptotically stable as a fixed point of ϕ^m (that is, if all of the eigenvalues of $A = D\phi^m[\pi]$ lie strictly inside the unit circle), then p is asymptotically stable as a solution of $(SDDE)$ in the usual sense. For given $y_0 \in C_0$ with continuation y as a solution of $(SDDE)$, lemma 3.1.13 tells us that, if y_0 is close enough to p_0, then y has a positive zero z such that y_z is close to p_0 and $V(y_z)$ is close to π. Since $\phi^k(V(y_z)) \to \pi$, we have as well that $R^k(y_0) \to \pi$ and $y(t) \to p(t)$.

We have established that, for η and $\langle y_0, p_0 \rangle$ small, $R(y_0)$ is close to p_0 and lies in $\tilde{W}$. This is helpful because we can say a great deal about how initial conditions near p_0 that are also in $\tilde{W}$ continue as solutions of (DDE).

Lemma 3.1.16. *Given $\epsilon > 0$ sufficiently small and η sufficiently small, for every $(DDE) \in \mathcal{P}(\eta)$ there is a constant κ such that the following holds.*

Given $y_0 \in \tilde{W}$ with $\langle y_0, p_0 \rangle < \epsilon$, write y for the continuation of y_0 as a solution of (DDE) and w for the continuation of y_0 as a solution of $(SDDE)$. Then the minimum positive zeros z and ζ of y and w, respectively, both exist and

$$z - \zeta = \kappa.$$

κ depends on (DDE), but is $O(\eta)$; in particular,

$$\kappa \leq \frac{2(n+1)\mu\eta}{\sigma^2}.$$

PROOF. Write

$$V(y_0) = (x_1, \ldots, x_n)$$

and $x_0 = 0$.

Applying lemma 3.1.13 and remark 3.1.15, we choose ϵ and η such that

- z and ζ both exist and lie in $(s_1 - \rho, s_1 + \rho)$;
- $|y(t - d_i)| \geq \rho\sigma$ and $|w(t - d_i)| \geq \rho\sigma$ for all $1 \leq i \leq D$ and all $t \in (0, 3\rho) \cup (s_1 - 3\rho, s_1 + 3\rho)$;
- $y(t)$ and $w(t)$ are both strictly less than $-\rho\sigma$ on $(3\rho, s_1 - 3\rho)$.

We assume that $\eta < \rho\sigma$. Then the second point above implies the following:

- $y'(t) = w'(t) = p'(0)$ and $y(t) = w(t)$ for all $t \in (0, 3\rho)$;
- $y'(t) = w'(t) = p'(s_1)$ for all $t \in (s_1 - 3\rho, s_1 + 3\rho)$.

Still assuming that $\eta < \rho\sigma$, let us compute

$$\int_0^{s_1 + 3\rho} H_i(y(t - d_i)) - h_i(w(t - d_i)) \, dt. \tag{3.1.17}$$

The integrand is nonzero only if

$$|y(t-d_i)| < \eta \text{ or } |w(t-d_i)| < \eta \text{ or } y(t-d_i)w(t-d_i) \leq 0.$$

Since the integrand is zero for $t \in (0, 3\rho) \cup (s_1 - 3\rho, s_1 + 3\rho)$, any subinterval where the integrand is nonzero is contained in the interval $t \in (3\rho, s_1 - 3\rho)$.

Since $y(t)$ and $w(t)$ are strictly less than $-\rho\sigma$ on the interval

$$(3\rho, s_1 - 3\rho)$$

and $t - d_i \leq t - d_1 < t - 6\rho < s_1 - 3\rho$ for all $t \leq s_1 + 3\rho$, the integrand is nonzero only if $t - d_i < 3\rho$. When $t - d_i < 3\rho$, though, $y(t-d_i) = w(t-d_i)$ and — since $\eta < \rho\sigma$, $y_0 \in \tilde{W}$, and $y'(t) = p'(0)$ for $t \in (0, 3\rho)$ — we know precisely how $y(t-d_i) = w(t-d_i)$ behaves. In particular, writing $s = t - d_i$ we have that

- $|y(s)| < \eta$ exactly on intervals of the form
$$\left(-x_k - \frac{\eta}{|p'(-\pi_k)|}, -x_k + \frac{\eta}{|p'(-\pi_k)|}\right)$$
and on these intervals $y'(s) = p'(-\pi_k)$;
- If s enters any such interval, it moves through the entire interval;
- s enters such an interval about $-x_k$ if and only if (k, i) is a change index pair of $V(y_0)$ (recall definition 2.3.10).

Therefore we see that, if and only if (k, i) is a change index pair of $V(y_0)$, there is a contribution to the integral (3.1.17) of the form

$$\begin{aligned} &\int_{-\eta/|p'(-\pi_k)|}^{0} H_i(|p'(-\pi_k)|s) - b_i \; ds + \int_0^{\eta/|p'(-\pi_k)|} H_i(|p'(-\pi_k)|s) + a_i \; ds \\ &\qquad = \frac{1}{|p'(-\pi_k)|}\left[\int_{-\eta}^{0} H_i(s) - b_i \; ds + \int_0^{\eta} H_i(s) + a_i \; ds\right]. \end{aligned} \tag{3.1.18}$$

Observe that this quantity does not depend on the particular y_0, and that it is bounded by $2\eta(\sup|H_i| + \sup|h_i|)/\sigma$. Given i, $V(y_0)$ has at most $(n+1)$ change index pairs of the form (k, i). Summing across i, we see that, for $t \in (s_1 - 3\rho, s_1 + 3\rho)$,

$$|w(t) - y(t)| \leq \frac{2(n+1)\mu\eta}{\sigma};$$

it follows that

$$|\zeta - z| \leq \frac{2(n+1)\mu\eta}{\sigma^2}.$$

Now, shrinking ϵ further if necessary, we can take ϵ to be small enough that theorem 2.3.13 applies to y_0 and p_0 (since $y_0 \in \tilde{W}$, by 3.1.12 we can make $|V(y_0) - \pi|$ small by taking ϵ small). In particular, by choosing ϵ small enough we make the set of change index pairs of $V(y_0)$ invariant — and this means that the set of summands of the form (3.1.18) contributing to the difference $y(t) - w(t)$ for $t \in (s_1 - 3\rho, s_1 + 3\rho)$ is invariant. We conclude that there is some $\bar{\kappa}$ such that $y(t) - w(t) = \bar{\kappa}$ on $(s_1 - 3\rho, s_1 + 3\rho)$ whenever $\langle y_0, p_0 \rangle$ is small enough; it follows that

$$z - \zeta = \frac{-\bar{\kappa}}{p'(s_1)} := \kappa. \quad \square$$

Let y_0, y, z, and κ be as above. In this case, observe that the map

$$V(y_0) \mapsto V(y_z)$$

is affine: for writing $K_0 = (\kappa, \kappa, \ldots, \kappa)^t$, we have

$$V(y_z) = V(w_\zeta) + K_0 = D\phi[\pi](V(w_0) - \pi) + V(p_{s_1}) + K_0 = D\phi[\pi](V(y_0) - \pi) + V(p_{s_1}) + K_0.$$

What we really needed in the above proof was that y_0 have the features of $\tilde{W}$ articulated in 3.1.10 and that $\langle y_0, p_0 \rangle$ be small. If $y_0 \in \tilde{W}$ and $\langle y_0, p_0 \rangle$ and η are small enough, the same sort of argument as in lemma 3.1.13 tells us that $\langle y_z, p_{s_1} \rangle$ will be small and that y_z will have features directly analogous to those in 3.1.10 (with p_{s_1} playing the role of p_0). In this case we can adapt lemma 3.1.16 to show that the map

$$V(y_z) \mapsto V(y_{z_2}),$$

where z_2 is the second positive zero of y, is also affine. In particular, writing $\pi' = V(p_{s_1})$ and $\pi'' = V(p_{s_2})$ there is some Euclidean constant K_1, with $|K_1|$ of order $O(\eta)$, such that

$$V(y_{z_2}) = D\phi[\pi'](V(y_{s_1}) - \pi') + \pi'' + K_1.$$

Composing, we find that the map $V(y_0) \mapsto V(y_{z_2})$ is affine as well:

$$
\begin{aligned}
V(y_{z_2}) &= D\phi[\pi'](V(y_{s_1}) - \pi') + \pi'' + K_1 \\
&= D\phi[\pi']\left(D\phi[\pi](V(y_0) - \pi) + K_0\right) + \pi'' + K_1 \\
&= D\phi^2[\pi](V(y_0) - \pi) + D\phi[\pi']K_0 + \pi'' + K_1 \\
&= D\phi^2[\pi](V(y_0) - \pi) + \pi'' + \tilde{K},
\end{aligned}
$$

where $|\tilde{K}|$ is $O(\eta)$.

Continuing this way for m steps, and using the fact that the linear part of a composition of affine maps is the composition of its linear parts, we obtain the following proposition.

Proposition 3.1.19. *Let $\epsilon_0 > 0$ be given. Then there is an $\epsilon^* > 0$ such that the following hold. Write*

$$U = \{\ y_0 \in C_0\ :\ \langle y_0, p_0\rangle < \epsilon^* \text{ and } \|y_0\| < Z\ \},$$

where $Z > \|p_0\|$.

Then for any $\eta > 0$ sufficiently small, given $y_0 \in \overline{U}$ with continuation y as a solution of $(DDE) \in \mathcal{P}(\eta)$, y satisfies all of the conclusions of lemma 3.1.13. In particular,

- *the first m positive zeros $z_1 < \ldots < z_m$ of y are well-defined and proper — in particular, $R : \overline{U} \to C_0$ is defined;*
- *$|s_j - z_j| < \epsilon_0$ for all $1 \leq j \leq m$;*
- *$|y(t) - p(t)| < \epsilon_0$ for all $t \in [0, t^*]$;*
- *$\|R(y_0) - p_0\| < \epsilon_0$;*
- *$y'(t) = p'(t)$ on $(0, 3\rho) \cup (s_1 - 3\rho, s_1 + 3\rho) \cup \cdots \cup (s_m - 3\rho, s_m + 3\rho)$;*
- *$R(y_0) \in \tilde{W}$.*

Furthermore, if y_0 also lies in $\tilde{W}$, we have the formula

$$V(R(y_0)) = A(V(y_0) - \pi) + \pi + K,$$

where

$$A = D\phi^m[\pi]$$

and $K \in \mathbf{R}^n$ *depends on* (DDE) *but is* $O(\eta)$. □

Remark 3.1.20. We include the second part of the definition of the set U to make U into a bounded open set. This maneuver allows consideration of the Leray-Schauder degree of $I - R$ on U with minimal technical complication.

Now, the map on $\mathbf{R}^n$ given by

$$v \mapsto A(v - \pi) + \pi + K,$$

where A, π, and K are as above, has a unique fixed point: for such a fixed point is a solution of the equation

$$(I - A)(v - \pi) = K,$$

and this equation has a unique solution by the hypothesis that p is nondegenerate. Furthermore, the distance of this solution from π is $O(|K|)$ and hence $O(\eta)$. In particular, for η small enough this solution v must lie within ϵ^*/σ of π. In this case (recall 3.1.12) there is some $\bar{u}_0 \in \overline{U} \cap \tilde{W}$ such that

$$V(\bar{u}_0) = v,$$

and hence such that

$$V(R(\bar{u}_0)) = V(\bar{u}_0).$$

3.1.12 now tells us that $\bar{u}_0$ and $R(\bar{u}_0)$ must have identical continuations as solutions of (DDE), and so we see that $u_0 = R(\bar{u}_0)$ is the unique fixed point of R with initial condition in $\overline{U}$.

Note that $\langle u_0, p_0 \rangle < \epsilon^*$; indeed, we can make $\langle u_0, p_0 \rangle$ as small as we wish by taking η small enough.

We want to bound $\|I - R\|$ below on ∂U. We can do this by choosing η and Z in 3.1.19 properly. This is the substance of the next two lemmas.

Lemma 3.1.21. *Let* U *be as above. Suppose that* $y_0 \in \overline{U}$ *and that* $w_0 \in \tilde{W} \cap \overline{U}$, *and write* $y(t)$ *and* $w(t)$ *for the continuations, respectively, of* y_0 *and* w_0 *as solutions of*

$(DDE) \in \mathcal{P}(\eta)$. Then there are constants k_1 and k_2 such that, if η is sufficiently small,

$$\|R(y_0) - R(w_0)\| \leq k_1\eta + k_2\langle y_0, w_0\rangle.$$

PROOF. Given $c \leq \rho\sigma$, we know that any connected component of the set $\{t \in [0, t^*] \ : \ |y(t)| \leq c\}$ is no more than $2c/\sigma$ units long. The point is that, given the extra information that $w_0 \in \tilde{W}$, we know that any connected component of the set $\{t \in [-\gamma, t^*] \ : \ |w(t)| \leq c\}$ is no more than $2c/\sigma$ units long. This allows us to make just the same sort of comparison between $y(t)$ and $w(t)$ as we made between $y(t)$ and $p(t)$ in the proof of lemma 3.1.13. In particular, a direct analog of the sixth part of 3.1.13 can be proven with the same estimates on (analogs of) the numbers $N(j)$ and $M(j)$. □

Lemma 3.1.22. *Let U and ϵ^* be as above. Then there is a positive constant ν such that, for $\eta > 0$ sufficiently small,*

$$y_0 \in \overline{U} \ \textit{and} \ \langle y_0, p_0\rangle > \frac{3\epsilon^*}{4} \implies \|R(y_0) - y_0\| > \nu.$$

If Z in 3.1.19 is chosen large enough, this implies that $\|R(y_0) - y_0\| > \nu$ for $y_0 \in \partial U$.

PROOF. We prove the last statement first. We have

$$\partial U \subset \{\ y_0 \in C_0 \ : \ \langle y_0, p_0\rangle = \epsilon^* \ \text{ or } \|y_0\| = Z\}.$$

If $\langle y_0, p_0\rangle = \epsilon^*$, the desired bound on $\|R(y_0) - y_0\|$ follows immediately from the first statement of the lemma.

If $\|y_0\| = Z$, then since $R(y_0)$ is within ϵ_0 of p_0 we have

$$\|R(y_0) - y_0\| \geq Z - \|p_0\| - \epsilon_0;$$

just choose Z so that the right-hand side is bigger than ν.

We now turn to the first statement of the lemma. We first choose η small enough that R has a unique fixed point $u_0 \in U$, with

$$\langle u_0, p_0\rangle < \epsilon^*/4.$$

Suppose that $w_0 \in \overline{U} \cap \tilde{W}$ with $\langle w_0, p_0 \rangle > \epsilon^*/2$. Then $\langle w_0, u_0 \rangle > \epsilon^*/4$, and

$$|V(w_0) - V(u_0)| \geq \frac{\epsilon^*}{4\mu}.$$

Since $V(w_0)$ is not the unique fixed point $V(u_0)$ of

$$v \mapsto \pi + K + A(v - \pi),$$

we have that

$$|V(w_0) - VR(w_0)| \geq \frac{\epsilon^* \alpha}{4\mu},$$

where $\alpha = \|(I - A)^{-1}\|^{-1}$. Thus (again by 3.1.12)

$$\|w_0 - R(w_0)\| \geq \langle w_0, R(w_0) \rangle \geq \frac{\sigma \alpha \epsilon^*}{4\mu} := k.$$

We take η and δ small enough so that lemma 3.1.21 applies: if $y_0 \in \overline{U}$ and $w_0 \in \tilde{W} \cap \overline{U}$ with $\langle y_0, w_0 \rangle < \delta$, then

$$\|R(y_0) - R(w_0)\| \leq k_1 \eta + k_2 \langle y_0, w_0 \rangle \leq k_1 \eta + k_2 \delta.$$

We also choose $\nu > 0$ and $\eta > 0$ small enough so that

$$\nu < \delta \text{ and } \nu < \frac{\epsilon^*}{4} \text{ and } k_1 \eta + k_2 \nu + 2\nu < k.$$

Now, suppose that $y_0 \in \overline{U}$ with $\langle y_0, p_0 \rangle > 3\epsilon^*/4$. Let w_0 be a member of $\tilde{W} \cap \overline{U}$ of minimal distance from y_0 (there is such a point, because $\tilde{W} \cap \overline{U}$ is a closed set). If $\|y_0 - w_0\| > \nu$, then $\|R(y_0) - y_0\| > \nu$ because R maps into $\tilde{W}$. If $\|y_0 - w_0\| \leq \nu$, then we see that we must have

$$\langle w_0, u_0 \rangle > \epsilon^*/2$$

(because $\nu < \epsilon^*/4$ and $\langle y_0, p_0 \rangle > 3\epsilon^*/4$) and so we have

$$\begin{aligned} \|R(y_0) - y_0\| &\geq \|R(w_0) - w_0\| - \|R(y_0) - R(w_0)\| - \|w_0 - y_0\| \\ &\geq k - k_1 \eta - k_2 \delta - \nu > \nu. \quad \square \end{aligned}$$

We summarize.

Theorem 3.1.23. Existence and uniqueness of periodic solutions for steplike problems. *Let ϵ_0, ϵ^*, $\overline{U}$, and R be as in 3.1.19 (with Z large enough that 3.1.22 applies). Then there is an $\eta^* \in (0, \rho\sigma)$ such that, if $(DDE) \in \mathcal{P}(\eta^*)$,*

- *all the conclusions of 3.1.19 hold.*
- *R has a unique fixed point u_0 on $\overline{U}$, and $\langle u_0, p_0 \rangle < \epsilon^*/4$.*
- *There is a $\nu > 0$ such that*

$$\|R(y_0) - y_0\| > \nu \text{ whenever } y_0 \in \partial U.$$

Remark 3.1.24. For the rest of the chapter the reader should regard $\overline{U}$, ϵ^*, ν, and η^* as fixed. We will henceforth use the fact that $\eta^* \in (0, \rho\sigma)$ without comment.

Below, whenever some problem $(DDE) \in \mathcal{P}(\eta^*)$ is under discussion, we will always write R for its above-described return map, and we will always write u_0, with continuation u as a solution of (DDE), for the unique fixed point of R on $\overline{U}$.

Here and throughout this thesis, we shall always consider Leray-Schauder degree with respect to 0; for simplicity we will write

$$\deg(F, \Omega) := \deg(F, \Omega, 0)$$

for the Leray-Schauder degree of F on Ω (with respect to 0). We close this section with the following proposition on the degree of $I - R$.

Proposition 3.1.25. *The Leray-Schauder degree of $I - R$ on $\overline{U}$ is defined for all return maps R for $(DDE) \in \mathcal{P}(\eta^*)$, and is constant across all such maps.*

PROOF. Let us choose two problems in $\mathcal{P}(\eta^*)$ as follows:

$$y'(t) = \sum_{i=1}^{D} H_i(y(t - d_i)),$$
$$y'(t) = \sum_{i=1}^{D} G_i(y(t - d_i)).$$

For every $s \in [0, 1]$, write

$$y'(t) = \sum_{i=1}^{D} (1 - s) H_i(y(t - d_i)) + s G_i(y(t - d_i)). \qquad (DDE_s)$$

Each (DDE_s) is a member of $\mathcal{P}(\eta^*)$; write $R_s : \overline{U} \to C_0$ for the corresponding return map. Each R_s is compact and fixed-point free on ∂U, and so $\deg(I - R_s, U)$ is defined. Moreover, the map

$$\overline{U} \times [0, 1] \to C_0 \ : \ (y_0, s) \mapsto R_s(y_0)$$

is a homotopy from R_0 to R_1. Therefore $\deg(I - R_0, U) = \deg(I - R_1, U)$. □

3.2 The derivative $DR[u_0]$

We retain all of the notation of the last section. With the exception of corollary 3.2.5, throughout this section we will assume that the problem (DDE) is C^1 (that is, that the feedback functions in (DDE) are C^1). General theory (recall section 1.2) tells us that R is C^1 also and that $DR[u_0]$, the derivative of R at its fixed point u_0, is a compact linear operator. In this section we develop some further properties of $DR[u_0]$. These features will be uniform across C^1 problems $(DDE) \in \mathcal{P}(\eta^*)$, and will reflect the connection between $DR[u_0]$ and the matrix

$$A = D\phi^m[\pi].$$

R maps a neighborhood of u_0 into the affine space $\tilde{W}$: that is, $R(u_0 + h_0) \in \tilde{W}$ for sufficiently small $h_0 \in C_0$. Since $u_0 = R(u_0) \in \tilde{W}$, we see that

$$R(u_0 + h_0) - R(u_0) \in W$$

for all sufficiently small $h_0 \in C_0$. Therefore we see that $DR[u_0]$ is a continuous linear map from C_0 into W.

On $\tilde{W}$, the map V is affine (here V is as defined in 2.3.2 and used in section 3.1). Let us write $L : W \to \mathbf{R}^n$ for the linear part of V about u_0: so if $w \in W$ and $u_0 + w \in \tilde{W}$, we have

$$V(u_0 + w) = V(u_0) + Lw.$$

If $w \in W$ and $w(s) = c$ for $s \in [-\pi_k - 2\rho, \pi_k + 2\rho]$, then the kth coordinate of Lw is $-c/p'(-\pi_k)$. We see therefore that L is a surjective linear map onto $\mathbf{R}^n$. Observe also that the kernel of L is exactly the linear subspace of $w \in W$ with $\langle u_0, u_0 + w\rangle = 0$. This set is contained in the kernel of $DR[u_0]$: for if $\langle u_0, u_0 + w\rangle = 0$ we have $R(u_0 + w) = R(u_0) = u_0$. Finally, note that

$$\sigma|Lw| \le \langle u_0, u_0 + w\rangle \le \mu|Lw|.$$

For $w \in W$ sufficiently small we have the formula

$$
\begin{aligned}
&VR(u_0 + w) = \pi + K + A(V(u_0 + w) - \pi) \\
= \quad & \pi + K + A(V(u_0) + Lw - \pi) = \pi + K + A(V(u_0) - \pi) + ALw \\
= \quad & V(u_0) + ALw.
\end{aligned}
$$

This is an affine map from $\tilde{W}$ into $\mathbf{R}^n$. By taking the derivative on the left (with the chain rule) and the linear part on the right, we get the formula

$$LDR[u_0]w = ALw, \ w \in W.$$

Similarly, for $w \in W$ sufficiently small,

$$VR^2(u_0 + w) = \pi + K + A(V(u_0) + ALw - \pi) = V(u_0) + A^2Lw$$

and

$$LDR^2[u_0]w = A^2Lw;$$

indeed, for any $k \in \mathbf{N}$ and $w \in W$ we have

$$LDR^k[u_0]w = A^kLw. \tag{3.2.1}$$

Proposition 3.2.2. Comparison of spectra. *$DR[u_0]$ and A have the same nonzero spectrum:*

$$\mathit{spectrum}(DR[u_0]) \setminus \{0\} = \mathit{spectrum}(A) \setminus \{0\}.$$

PROOF. By the spectra of $DR[u_0]$ and A we mean the spectra of their complexifications. Since $DR[u_0]$, L and A are all real operators, though, we obtain these complexifications by applying the operators separately to the real and imaginary parts of vectors. In particular, $DR[u_0]$ maps the complexification of C_0 into the complexification of W, and on the latter space we still have the formula

$$LDR[u_0] = AL.$$

Since $DR[u_0]$ is a compact linear operator, its nonzero spectrum consists of eigenvalues.

Suppose that w is an eigenvector of $DR[u_0]$ with nonzero eigenvalue λ. Then w must lie in the complexification of W, and since w is not in the kernel of $DR[u_0]$ it is not in the kernel of L either. Therefore we have

$$ALw = LDR[u_0]w = L\lambda w = \lambda Lw,$$

and $Lw \neq 0$ is an eigenvector of A with eigenvalue λ.

Conversely, suppose that λ is a nonzero eigenvalue of A. Since L is surjective, there is some w in the complexification of W such that $ALw = \lambda Lw$; but in this case we have

$$LDR[u_0]w = ALw = \lambda Lw = L\lambda w.$$

Therefore we can write $DR[u_0]w = \lambda w + v$, where v is in the kernel of L (and hence in the kernel of $DR[u_0]$). We now put $\tilde{w} = w + v/\lambda$ and compute

$$DR[u_0]\tilde{w} = DR[u_0]w = \lambda w + v = \lambda\tilde{w},$$

whence λ is an eigenvalue of $DR[u_0]$. □

Recalling section 1.2, we obtain

Corollary 3.2.3. Asymptotic stability of u. *If $(DDE) \in \mathcal{P}(\eta^*)$ is C^1 and the spectrum of A lies strictly inside the unit circle, then the periodic solution u of (DDE) is asymptotically stable.* □

The hypothesis that A is nondegenerate now implies

Corollary 3.2.4. *If (DDE) is C^1, $DR[u_0]$ does not have eigenvalue* 1. □

Since u_0 is the unique fixed point of R in $\overline{U}$, the excision and linearization properties of Leray-Schauder degree (see, for example, [2]), together with propostion 3.1.25, now yield

Corollary 3.2.5. *For all $(DDE) \in \mathcal{P}(\eta^*)$ (whether C^1 or not),*

$$deg(I - R, U) \neq 0. \quad \square$$

Proposition 3.2.4 tells us that the periodic solution u_0 will persist if $(DDE) \in \mathcal{P}(\eta^*)$ is appropriately perturbed (recall section 1.2). In this way we get existence results for non-steplike problems sufficiently close to some fixed (DDE). The reason that results like those stated in examples 1.1.3 and 1.1.4 don't follow immediately is that the parameterized problems in those examples do not approach a particular, fixed member of $\mathcal{P}(\eta^*)$ in any suitable sense. We devote chapter four to closing this gap.

For the rest of this section we will need to assume that $B < \infty$. (Recall that we are writing B for the common bound on the total variation of the feedback functions H_i in the problems $(DDE) \in \mathcal{P}(\eta^*)$.)

Lemma 3.2.6. *If (DDE) is C^1 and $B < \infty$ then, for any $1 \leq i \leq D$,*

$$\int_{-\gamma}^{t^*} |H_i'(u(t))| \; dt \leq (n+1+m)\frac{B}{\sigma}.$$

PROOF. On $[-\gamma, t^*]$, there are precisely $n+1+m$ subintervals where $|u(t)| \leq \eta$; these are exactly the subintervals where $H_i'(u(t))$ is nonzero. On each such subinterval I, $u(t)$ runs from $-\eta$ to η (or vice-versa) at constant slope s, where $|s| \geq \sigma$. We therefore have

$$\int_I |H_i'(u(t))| \; dt = \frac{1}{|s|} \int_{-\eta}^{\eta} |H_i'(t)| \; dt \leq \frac{B}{\sigma}.$$

The lemma follows. □

Proposition 3.2.7. Uniformly bounded derivative of R. *Suppose that $B < \infty$. Then there is a number k_1 (depending on B) such that, uniformly across C^1 problems $(DDE) \in \mathcal{P}(\eta^*)$,*

$$\|DR[u_0]\| \leq k_1.$$

PROOF. The linearization about u is the equation

$$y'(t) = \sum_{i=1}^{D} H_i'(u(t-d_i))y(t-d_i);$$

write $T_u : \mathbf{R}_+ \times C \to C$ for its solution operator. The derivative $DR[u_0]$ is given by the formula (recall 1.2.1)

$$DR[u_0]y_0 = T_u(\tau(u), y_0) - \frac{T_u(\tau(u), y_0)(0)}{u_0'(0)} \cdot u_0', \quad y_0 \in C_0,$$

where $\tau(u)$ is the mth positive zero of $u(t)$.

Since the solution $u(t)$ has derivative uniformly bounded (by μ) across $\mathcal{P}(\eta^*)$ and $u_0'(0) = p'(0)$ is constant across $\mathcal{P}(\eta^*)$, it suffices to show that the linear operator $T_u(\tau, \cdot)$ has norm uniformly bounded across $\tau \in [0, t^*]$ and C^1 problems $(DDE) \in \mathcal{P}(\eta^*)$.

Given any $\tau \in [0, t^*]$, we can write

$$\tau = N\bar{\tau},$$

where $\tau \in [0, d_1]$ and N is some positive integer no more than the integer ceiling of t^*/d_1.

Choose some $y_0 \in C_0$. Lemma 3.2.6 yields that

$$\|T_u(\bar{\tau}, y_0)\| \le D(n+1+m)\frac{B}{\sigma}\|y_0\|;$$

another application of 3.2.6 yields that

$$\|T_u(2\bar{\tau}, y_0)\| \le \left[D(n+1+m)\frac{B}{\sigma}\right]^2 \|y_0\|,$$

and so on. Applying the lemma N times proves the proposition. $\square$.

Out of the above proof we extract a corollary that we will need later.

Corollary 3.2.8. *Suppose that $B < \infty$. Then there is a number k_1 (depending on B) such that, uniformly across C^1 problems $(DDE) \in \mathcal{P}(\eta^*)$, for all $\tau \in [0, t^*]$ we have*

$$\|T_u(\tau, \cdot)\| \le k_1,$$

where u_0 is the unique fixed point of R in $\overline{U}$, u is the continuation of u_0 as a solution of (DDE), and T_u is the solution operator for the linearization of (DDE) about u. $\square$

Our next goal is to show that not only is $(I - DR[u_0])$ invertible but that, when $B < \infty$, its inverse has uniformly bounded norm across C^1 problems in $\mathcal{P}(\eta^*)$. As a first step we study the restriction of $I - DR[u_0]$ to W. (The reader might find it helpful to review definitions 3.1.2 and 3.1.9.)

Given $w \in W$, we will write

$$c(w) = \sup\{\ |w(s)|\ :\ s \in \bigcup_{k=1}^{n} [-\pi_k - 2\rho, -\pi_k + 2\rho]\ \}.$$

$c(w)$ is the maximal size of $w \in W$ on the intervals where w is required to be constant by the definition of W; if $c(w) \leq \rho\sigma$ and $u_0 + w \in \tilde{W}$, we in fact have

$$c(w) = \langle u_0, u_0 + w \rangle.$$

(This identity does not necessarily hold if $u_0 + w \notin \tilde{W}$; this, roughly speaking, is why we have found it convenient to introduce the quantity $c(w)$ for the next few pages.) It is intuitively clear that $DR[u_0]w$ depends only on the restriction of the function w to the set

$$\bigcup_{k=1}^{n} [-\pi_k - 2\rho, -\pi_k + 2\rho];$$

the following lemma makes this precise. We continue to write L for the linear part of the restriction of V to $\tilde{W}$ (see the beginning of the current section 3.2).

Lemma 3.2.9. *Suppose that $c(w) \leq \rho\sigma$ and that $u_0 + w \in \tilde{W}$. Then $c(w) \leq \|w\|$, and*

$$\sigma |Lw| \leq c(w) \leq \mu |Lw|.$$

Moreover, there is a vector $\hat{w} \in W$ such that

$$\|\hat{w}\| = c(\hat{w}) = c(w)$$

and such that, for all $\delta > 0$ sufficiently small,

$$R(u_0 + \delta w) = R(u_0 + \delta \hat{w}).$$

Consequently $DR[u_0]w = DR[u_0]\hat{w}$.

PROOF. The first statement is immediate; the second follows from 3.1.12 and the fact that $|Lw| = |V(u_0 + w) - V(u_0)|$.

To construct $\hat{w}$, just take $\hat{w}$ to agree with w on the intervals $[-\pi_k - 2\rho, -\pi_k + 2\rho]$ and extend $\hat{w}$ polygonally off of those intervals. $\square$

Lemma 3.2.10. *Suppose that $B < \infty$. Then for any C^1 problem $(DDE) \in \mathcal{P}(\eta^*)$, if $w \in W$ we have*

$$\|DR[u_0]w\| \leq k_1 c(w) \leq k_1 \mu |Lw|$$

where k_1 is as in 3.2.7.

PROOF. For any $s > 0$ we have

$$\|DR[u_0]sw\| = s\|DR[u_0]w\|; \; c(sw) = sc(w); \; |Lsw| = s|Lw|.$$

Therefore we can, without loss of generality, scale w so that the hypotheses of lemma 3.2.9 hold.

That $k_1 c(w) \leq k_1 \mu |Lw|$ is just the second point of 3.2.9. For the first inequality, just choose $\hat{w}$ as in 3.2.9: we have

$$\|DR[u_0]w\| = \|DR[u_0]\hat{w}\| \leq k_1 \|\hat{w}\| = k_1 c(w). \quad \square$$

Lemma 3.2.11. *Suppose that $B < \infty$. Then there is a constant β such that, for any C^1 problem $(DDE) \in \mathcal{P}(\eta^*)$ and $w \in W$,*

$$\|w - DR[u_0]w\| \geq \beta \|w\|.$$

PROOF. Claim: there is some k_2 such that

$$\|DR[u_0]w - w\| \geq k_2 c(w).$$

For we have

$$\|DR[u_0]w - w\| \geq \sigma |LDR[u_0]w - Lw| = \sigma |ALw - Lw| \geq \alpha\sigma |Lw| \geq \frac{\alpha\sigma}{\mu} c(w),$$

where $\alpha = \|(I - A)^{-1}\|^{-1}$.

Now put $\beta = k_2/(k_1 + k_2)$, where k_2 is as above and k_1 is as in 3.2.7 and 3.2.10. There are two cases to consider.

Case 1: $c(w) \leq \|w\|/(k_1 + k_2)$. In this case lemma 3.2.10 yields

$$\|DR[u_0]w - w\| \geq \|w\| - \|DR[u_0]w\| \geq \beta \|w\|.$$

Case 2: $c(w) \geq \|w\|/(k_1 + k_2)$. In this case our above claim yields

$$\|DR[u_0]w - w\| \geq \beta\|w\|. \quad \Box$$

Proposition 3.2.12. *Suppose that $B < \infty$. There is a constant M such that, for any C^1 problem $(DDE) \in \mathcal{P}(\eta^*)$,*

$$\|(I - DR[u_0])^{-1}\| \leq M.$$

PROOF. The result follows immediately from the following general lemma, with $DR[u_0]$ in the role of T and the verifications of the hypotheses on T coming from our work so far this section. $\Box$

Lemma 3.2.13. *Let X be a Banach space with $W \subset X$ a closed linear subspace. Suppose that T is a continuous linear operator on X with the following features:*

- *$T(X) \subset W$;*
- *$\|w - T(w)\| \geq \beta\|w\|$ for all $w \in W$, where β is some positive constant;*
- *$\|T\| \leq k_1$ where k_1 is some constant.*
- *$(I - T)^{-1}$ exists.*

Then we have

$$\|(I - T)^{-1}\| \leq \frac{\beta + k_1 + 2}{\beta}.$$

PROOF. The restriction of $I - T$ to W certainly maps into W; we claim that this map is surjective. For every $w \in W$ is of the form $(I - T)x$ for some $x \in X$ (since $I - T$ is invertible) and so we have

$$w = x - T(x) \implies w + T(x) = x \in W$$

(since $T(x) \in W$ by hypothesis). Accordingly, given $w \in W$ write $w = (I - T)h$ where $h \in W$. We have

$$\|w\| = \|(I - T)h\| \geq \beta\|h\| = \beta\|(I - T)^{-1}w\| \implies \|(I - T)^{-1}w\| \leq \frac{1}{\beta}\|w\|,$$

and so we see that the restriction of $(I-T)^{-1}$ to W has norm no more than $1/\beta$.

We now need to consider the action of $(I-T)^{-1}$ on vectors outside of W. Write

$$c = \frac{\beta}{\beta + k_1 + 2};$$

observe that

$$c = \beta(1-c) - (1+k_1)c.$$

There are now two possibilities to consider.

Case 1: $h \in X$ has the feature that distance$(h, W) \geq c\|h\|$. In this case, it is certainly the case that $\|h - T(h)\| \geq c\|h\|$, because T maps into W.

Case 2: distance$(h, W) < c\|h\|$. In this case, we can write $h = h_0 + h_1$, where $h_0 \in W$ has norm at least $(1-c)\|h\|$ and h_1 has norm at most $c\|h\|$. In this case, we have

$$\begin{aligned}
& \|(I-T)h\| \\
= \ & \|(I-T)(h_0+h_1)\| \\
= \ & \|(I-T)(h_0) + (I-T)(h_1)\| \\
\geq \ & \|(I-T)h_0\| - \|(I-T)h_1\| \\
\geq \ & \beta(1-c)\|h\| - (1+k_1)c\|h\| \\
= \ & \left(\beta(1-c) - (1+k_1)c\right)\|h\| = c\|h\|.
\end{aligned}$$

Therefore we see that $\|(I-T)h\| \geq c\|h\|$ for all $h \in X$. The lemma follows. □

Corollary 3.2.14. *Suppose that $B < \infty$. Then there is a number ϵ such that, for all C^1 problems (DDE) $\in \mathcal{P}(\eta^*)$, if S is a continuous linear operator with $\|S - DR[u_0]\| < \epsilon$, then $I - S$ is invertible.*

PROOF. If $\|S - DR[u_0]\| < \epsilon$, then $\|(I-S) - (I - DR[u_0])\| < \epsilon$ as well. The corollary now follows immediately from the following general linear functional analysis lemma, with $I - DR[u_0]$ in the role of T and $I - S$ in the role of Q. □

Lemma 3.2.15. *Suppose that X is a Banach space, and that $T : X \to X$ is an invertible linear operator with*

$$\|T^{-1}\| < M.$$

Then, if $\|Q - T\| < M_1 < \frac{1}{M}$, *$Q$ is invertible with*

$$\|Q^{-1}\| < \frac{M}{1 - MM_1}.$$

PROOF. We have

$$\begin{aligned} &\|T - Q\| < M_1 \implies \\ &\|I - T^{-1}Q\| \le \|T^{-1}\|\|T - Q\| < MM_1 < 1. \end{aligned}$$

Thus $(T^{-1}Q)^{-1} = Q^{-1}T$ has a series expansion

$$\begin{aligned} Q^{-1}T &= \sum_{k=0}^{\infty}(I - T^{-1}Q)^k \implies \\ \|Q^{-1}T\| &< \sum_{k=0}^{\infty}(MM_1)^k \implies \\ \|Q^{-1}\| &= \|Q^{-1}TT^{-1}\| \le \|Q^{-1}T\|\|T^{-1}\| < \frac{M}{1 - MM_1}. \quad \square \end{aligned}$$

We close this section by considering the spectral radius of perturbations of $DR[u_0]$.

Lemma 3.2.16. *Suppose that $B < \infty$. Write r for the spectral radius of A, and let $\epsilon > 0$ be given. Then there is some $N \in \mathbf{N}$ such that, for all C^1 problems $(DDE) \in \mathcal{P}(\eta^*)$, $j \ge N$ implies that*

$$\|(DR[u_0])^j\|^{1/j} \le r + \epsilon.$$

The above lemma, together with the spectral radius formula, doubles as an alternative proof that the spectral radius of $DR[u_0]$ is no more than that of A (though we already know, from 3.2.2, that these radii are equal).

PROOF. Given $h_0 \in C_0$, write

$$h^j = DR[u_0]^j h_0.$$

$h^1 \in W$, and 3.2.7 tells us that $\|h^1\| \le k_1\|h_0\|$; we therefore have

$$|Lh^1| \le \frac{k_1}{\sigma}\|h_0\|.$$

Since $h^j \in W$ for all $j \in \mathbf{N}$, by 3.2.10 we have that

$$\|h^{j+1}\| \leq k_1\mu|Lh^j|.$$

Choose any $\epsilon' \in (0,\epsilon)$ and then choose $j \in \mathbf{N}$ so large that

$$\frac{k_1^2\mu/\sigma}{(r+\epsilon)^2} \leq \frac{(r+\epsilon)^j}{(r+\epsilon')^j}$$

and

$$\|A^j\| \leq (r+\epsilon')^j.$$

We have

$$\begin{aligned} & \|h^{j+2}\| \leq k_1\mu|Lh^{j+1}| = k_1\mu|A^jLh^1| \leq k_1\mu(r+\epsilon')^j|Lh^1| \\ \leq \; & \frac{k_1^2\mu(r+\epsilon')^j}{\sigma}\|h_0\| \leq (r+\epsilon)^{j+2}\|h_0\|. \quad \square \end{aligned}$$

Proposition 3.2.17. *Suppose that $B < \infty$ and that $(DDE) \in \mathcal{P}(\eta^*)$ is C^1. Let T be a linear operator on C_0. Suppose that A has spectral radius $r < 1$. Then there is an $\epsilon > 0$ such that, if $\|T\| < \epsilon$, then $DR[u_0] + T$ has spectral radius less than 1. ϵ does not depend on the particular problem $(DDE) \in \mathcal{P}(\eta^*)$.*

PROOF (Thanks to Roger Nussbaum). Choose $r_1 \in (r,1)$. By lemma 3.2.16 there is an $N \in \mathbf{N}$ such that, for all C^1 problems $(DDE) \in \mathcal{P}(\eta^*)$, $j \geq N$ implies that

$$\|DR[u_0]^j\|^{1/j} \leq r_1.$$

Choose any $\lambda \in \mathbf{C}$ with $|\lambda| \geq 1$. Since $r < 1$, the operator $(\lambda I - DR[u_0])$ is invertible, and the inverse has series expansion

$$(\lambda I - DR[u_0])^{-1} = \sum_{j=0}^{\infty} \frac{1}{\lambda^{j+1}} DR[u_0]^j.$$

We therefore have the estimate, uniformly across C^1 problems $(DDE) \in \mathcal{P}(\eta^*)$ and $|\lambda| \geq 1$,

$$\begin{aligned} \|(\lambda I - DR[u_0])^{-1}\| &\leq \sum_{j=0}^{N-1} \|DR[u_0]\|^j + \sum_{j=N}^{\infty} \|DR[u_0]^j\| \\ &\leq \sum_{j=0}^{N-1} k_1^j + \sum_{j=N}^{\infty} r_1^j := M^*. \end{aligned}$$

If we now choose $\epsilon < 1/M^*$, lemma 3.2.15 implies that

$$\lambda I - DR[u_0] - T = \lambda I - (DR[u_0] + T)$$

is invertible for all $|\lambda| \geq 1$. It follows that the spectral radius of $DR[u_0] + T$ is less than 1. $\square$

Chapter 4

A particular parameterized family

4.1 Introduction

In [27] Nussbaum studied families of differential delay equations with two delays of the type

$$x'(t) = r \sum_{i=1}^{D} F_i(x(t - d_i)),$$

where we regard $r \geq 1$ as a parameter — recall section 1.4. (This kind of parameterization has also played an important role in the study of delay equations with a single delay.) In example 1.1.3 we showed how, with a change of variables, to rewrite the above problem as

$$x'(t) = \sum_{i=1}^{D} F_i(rx(t - d_i)).$$

In this chapter we apply the work we have done so far to problems of this type. We define the following class and notation.

Definition 4.1.1. The problems (E_r), and notation. Suppose that the functions F_i are continuous, with

$$\lim_{x \to -\infty} F_i(x) = b_i \neq 0, \quad \lim_{x \to \infty} F_i(x) = -a_i \neq 0.$$

Write $F_i^r(x) = F_i(rx)$ for all $r \geq 1$ (please take careful note of this nonstandard notation, which we will use throughout this chapter).

We consider the parameterized family of equations

$$x'(t) = \sum_{i=1}^{D} F_i^r(x(t - d_i)), \; t > 0; \quad x_0 \in C. \tag{E_r}$$

We will write $\kappa \geq 0$ for an exponent and $\ell(\cdot)$ for a function satisfying the following. $\ell(\cdot) > 1$; and given any $\epsilon > 0$, $|x| > \ell(\epsilon)$ implies, for all i, that $F_i(x)$ is within

$$\frac{\epsilon}{2D|x|^\kappa}$$

of the limit of F_i at $\operatorname{sgn}(x) \cdot \infty$ (D is the number of delays in (E_r)).

If each F_i is in fact differentiable with

$$\lim_{|x|\to\infty} |F_i'(x)| = 0,$$

we will similarly write $\kappa' \geq 0$ for an exponent such that, given any $\epsilon > 0$, $|x| > \ell(\epsilon)$ also implies that

$$|F_i'(x)| \leq \frac{\epsilon}{2D|x|^{\kappa'}}.$$

To say that $\kappa \geq 0$ is just to say that each F_i has limits at $\pm\infty$; the larger κ is, the faster the functions F_i approach their limits.

As $r \to \infty$, the feedback functions F_i^r approach, pointwise except perhaps at 0, the step functions h_i in the problem

$$y'(t) = \sum_{i=1}^{D} h_i(y(t - d_i)),\ t > 0; \quad y_0 \in C; \tag{$SDDE$}$$

$$h_i(y) = \begin{cases} b_i, & y < 0; \\ 0, & y = 0; \\ -a_i, & y > 0. \end{cases}$$

We accordingly regard $(SDDE)$ as the "limit" of the problems (E_r) as $r \to \infty$.

We will retain all of our notation from the previous chapter. In particular, we suppose that p is a simple, nondegenerate periodic solution of $(SDDE)$, and we name various features of p as in 3.1.2. (In particular, throughout this chapter we will use s_i, m, n, σ, and ρ as in 3.1.2.)

Our main theorems for this chapter are the following.

Theorem 4.1.2. Existence. *Let $\epsilon > 0$ be given. Then, for r sufficiently large, (E_r) has a periodic solution q that is close to p in the following sense:*

- *q has a downward proper zero at* 0 *and all zeros of q are proper;*
- *If $\zeta_1 < \zeta_2 < \cdots < \zeta_m$ are the first m positive zeros of q, then $q_{\zeta_m} = q_0$, $|\zeta_k - s_k| < \epsilon$ for all $1 \le k \le m$, and $|q(t) - p(t)| < \epsilon$ for all $t \in [0, t^*]$.*

Theorem 4.1.3. Uniqueness and stability. *Assume further that the F_i are C^2, with bounded second derivative, and of bounded variation. Assume too that*

$$\lim_{|x| \to \infty} |F_i'(x)| = 0,$$

and that $\kappa > 2$ and $\kappa' > 1$ (where κ and κ' are as in 4.1.1).

Then there is an open set U in C_0 about p_0 such that, for all r sufficiently large, there is exactly one periodic solution q as described in 4.1.2 with $q_0 \in U$.

Furthermore, if all of the eigenvalues of

$$A = D\phi^m[V(p_0)]$$

lie strictly inside the unit circle and r is large enough, then this periodic solution q is asymptotically stable.

4.2 Linking (E_r) to steplike problems

We choose $\mu \ge 1$ such that the functions F_i defining (E_r) satisfy

$$2 \sum_i \sup |F_i| \le \mu.$$

Let $B \le \infty$ be some fixed constant (we will specify B shortly). Choose $\epsilon_0 \in (0, \rho\sigma)$. Given μ, B, ϵ_0, and the simple nondegenerate solution p of $(SDDE)$, we define the open set U in C_0 about p_0 as in 3.1.19 and 3.1.23 (ϵ_0 is as in 3.1.19, with the additional requirement that $\epsilon_0 \in (0, \rho\sigma)$). We define the class of steplike problems (recall definition 3.1.1)

$$\mathcal{P} := \mathcal{P}_{(SDDE)}(\eta^*, \mu, B),$$

where η^* is as in 3.1.23.

This class $\mathcal{P}$ will be fixed throughout the chapter. All that we learned in chapter 3 applies: every $(DDE) \in \mathcal{P}$ has a periodic solution u close to p in the sense of 3.1.19

and 3.1.23 (in particular, $|u(t) - p(t)| < \epsilon_0 < \rho\sigma$ for all $t \in [0, t^*]$); u_0 is the unique fixed point of the return map $R : \overline{U} \to C_0$ that "advances solutions by m zeros"; $\|y_0 - R(y_0)\| > \nu$ for all $(DDE) \in \mathcal{P}$ and $y_0 \in \partial U$; and $\deg(I - R, U)$ is a nonzero constant across $(DDE) \in \mathcal{P}$. Moreover, if $(DDE) \in \mathcal{P}$ is C^1 then the nonzero spectrum of $DR[u_0]$ is equal to the nonzero spectrum of $A = D\phi^m[V(p_0)]$. If furthermore $B < 0$ then there are constants k_1 and M such that

$$\|DR[u_0]\| \leq k_1 \text{ and } \|(I - DR[u_0])^{-1}\| \leq M$$

for all C^1 problems $(DDE) \in \mathcal{P}$.

Throughout this chapter, whenever a particular member of $\mathcal{P}$ is understood, we will write u for this periodic solution and R for this return map.

Our strategy for proving 4.1.2 and 4.1.3 is to link (E_r) with a "nearby" member of $\mathcal{P}$ for each $r \geq 1$. In this section we describe this link and prove theorem 4.1.2. We prove theorem 4.1.3 in section 4.3.

Recall that, given $\epsilon > 0$, we write $\ell(\epsilon) > 1$ for a number large enough so that $|x| > \ell(\epsilon)$ implies that $F_i(x)$ is within $\epsilon/2D|x|^\kappa$ of the limit of F_i at $\operatorname{sgn}(x) \cdot \infty$. If $|F_i'(x)| \to 0$ as $|x| \to \infty$, then $|x| > \ell(\epsilon)$ also implies that $|F_i'(x)| < \epsilon/2D|x|^{\kappa'}$.

We will use these bounds in the following way. Given $\epsilon > 0$ and $\alpha \in [0, 1]$, if $|x| > \ell(\epsilon)/r^\alpha$ then $|rx| > \ell(\epsilon)$, and $F_i^r(x) = F_i(rx)$ differs from its asymptotic values by no more than

$$\frac{\epsilon}{2D|rx|^\kappa} \leq \frac{\epsilon}{2Dr^{\kappa(1-\alpha)}\ell(\epsilon)^\kappa} \leq \frac{\epsilon}{2Dr^{\kappa(1-\alpha)}}. \tag{4.2.1}$$

Similarly for κ': suppose that $|x| > \ell(\epsilon)/r^\alpha$. Then we have

$$|(F_i^r)'(x)| = |rF_i'(rx)| \leq \frac{r\epsilon}{2D|rx|^{\kappa'}} \leq \frac{r\epsilon}{2Dr^{\kappa'(1-\alpha)}}. \tag{4.2.2}$$

Definition 4.2.3. The problems (DDE_r). Choose and fix $\eta \in (0, \eta^*)$. Write $\psi : (-\infty, 0] \to [0, 1]$ for a smooth monotonic function satisfying $\psi(x) = 0$ for $x \in [-\eta, 0]$

and $\psi(x) = 1$ for $x \le -\eta^*$. Then, given $r \ge 1$ and $1 \le i \le D$, define the functions

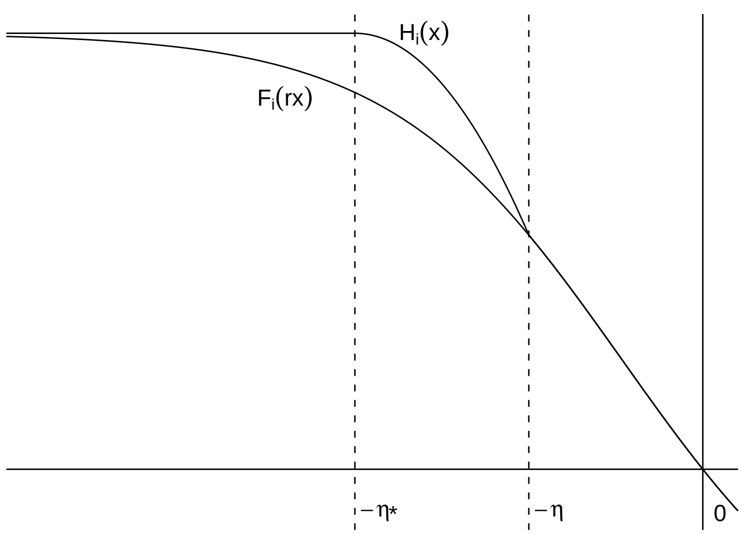
$$H_i(x) = \begin{cases} \psi(x)h_i(x) + (1-\psi(x))F_i^r(x), & x \le 0; \\ \psi(-x)h_i(x) + (1-\psi(-x))F_i^r(x), & x \ge 0. \end{cases}$$

For each $r \ge 1$ we define the steplike problem

$$y'(t) = \sum_{i=1}^{D} H_i(y(t-d_i)),\ t > 0;\ y_0 \in C. \qquad (DDE_r)$$

We think of (DDE_r) as a steplike problem "similar" to (E_r). The feedback functions H_i in (DDE_r) of course depend on r; for simplicity we do not express this dependence in our notation. Figure 14 illustrates example feedback functions $F_i^r(x) = F_i(rx)$ and $H_i(x)$, for negative values of x.

Figure 14



The next proposition records some features of the problems (DDE_r), and establishes the crucial point that all of the problems (DDE_r), $r \ge 1$, fall into a single class $\mathcal{P}$.

Proposition 4.2.4. The member of $\mathcal{P}$ linked to (E_r). *We can choose the constant B in the definition of $\mathcal{P}$ such that $(DDE_r) \in \mathcal{P}$ for all $r \ge 1$. Furthermore, if all of the*

F_i are C^1 and of bounded variation, then we can take $B < \infty$ and B larger than the total variation on $(-\infty, \infty)$ of any F_i.

In addition, for any $r \geq 1$, the functions H_i have the following features.

1. *$H_i(x) = F_i^r(x)$ for $|x| \leq \eta$.*

2. *For any $\alpha \in [0,1]$ and $\epsilon > 0$,*

$$|y|, |x| > \frac{\ell(\epsilon)}{r^\alpha},\ yx > 0 \implies |F_i^r(x) - H_i(y)| < \frac{\epsilon}{Dr^{\kappa(1-\alpha)}}.$$

   In particular, if r is large enough that $\ell(\epsilon)/r^\alpha < \eta$, then

$$|F_i^r(x) - H_i(x)| < \frac{\epsilon}{Dr^{\kappa(1-\alpha)}} \qquad \forall\ x.$$

3. *If the F_i are C^2 with bounded second derivative, and $|F_i'(x)| \to 0$ as $|x| \to \infty$, there is some $\triangle > 0$, independent of r, such that for all $r \geq 1$,*

$$\sup_i |(F_i^r)'|,\ \sup_i |H_i'| < r\triangle; \qquad \sup_i |(F_i^r)''|,\ \sup_i |H_i''| < r^2\triangle.$$

4. *$\triangle$ can also be chosen so that, for any $\alpha \in [0,1]$ and $\epsilon > 0$,*

$$|y| > \frac{\ell(\epsilon)}{r^\alpha} \Longrightarrow |H_i'(y)| < \frac{\triangle\epsilon}{2Dr^{\kappa(1-\alpha)}} + \frac{r\epsilon}{2Dr^{\kappa'(1-\alpha)}}.$$

Remark 4.2.5. Notation. For the rest of the chapter, we will of course assume that B is large enough for 4.2.4 to hold (and finite but larger than $TV(F_i)$, the total variation of F_i on $(-\infty, \infty)$, if all of the F_i are C^1 and of bounded variation). We will always write (DDE) for a general member of $\mathcal{P}$ and $(DDE)_r$ for the particular member of $\mathcal{P}$ described in 4.2.3 and 4.2.4.

In addition to fixing μ, B, ϵ_0 and η^*, we will also fix the numbers η and $\triangle$ as described in 4.2.4.

Throughout this chapter, if f is a real-valued function, we shall write $\|f\|$ for the supremum of f (if it exists) over its domain.

PROOF OF 4.2.4. We need to do two things. First, we need to find $B \leq \infty$ — finite if the F_i are all C^1 and of bounded variation — such that $(DDE_r) \in \mathcal{P}$ for every $r \geq 1$. Second, we need to verify that the H_i satisfy the stated properties.

$H_i(x)$ is certainly equal to its asymptotic values outside of $[-\eta^*, \eta^*]$ (and so is steplike); we also certainly have

$$\sum_i 2 \sup |H_i| \leq \mu.$$

If we can exhibit a common bound B on the total variation of the H_i, then, we will have found B such that all of the problems (DDE_r) lie in $\mathcal{P}$.

If any of the F_i are not C^1 or have infinite variation, we take $B = \infty$ and we are done.

Now suppose that the F_i are C^1 and of bounded variation. The H_i are as smooth as the F_i are. The total variation of F_i^r is the same as that of F_i:

$$\int_{-\infty}^{0} |(F_i^r)'(x)| \; dx = \int_{-\infty}^{0} r|F_i'(rx)| \; dx = \int_{-\infty}^{0} |F_i'(u)| \; du,$$

where $u = rx$, and similarly on the positive half-line. Therefore we have, very crudely,

$$\begin{aligned}
\int_{-\infty}^{0} |H_i'(x)| \; dx \quad &\leq \quad |b_i| \int_{-\infty}^{0} |\psi'(x)| \; dx + \int_{-\infty}^{0} |(F_i^r)'(x)| \; dx \\
&\quad + \int_{-\infty}^{0} |\psi(x)||(F_i^r)'(x)| \; dx + \int_{-\infty}^{0} |\psi'(x)||(F_i^r)(x)| \; dx \\
&\leq \quad b_i + 2TV(F_i) + \mu \leq 2\mu + 2TV(F_i),
\end{aligned}$$

and similarly on the positive half-line. Thus we see that there is a finite bound, independent of r, on the total variation of the H_i. We take this bound as our B.

Feature (1) of the H_i in proposition 4.2.4 clearly holds. For the second feature, just note that if x and y are of the same sign and both greater than $\ell(\epsilon)/r^\alpha$ in absolute value, then $F_i^r(x)$ and $H_i(y)$ are both within $\epsilon/2Dr^{\kappa(1-\alpha)}$ of the asymptotic value of F_i at $\operatorname{sgn}(x) \cdot \infty$ (this is true of F_i^r by the definition of $\ell(\epsilon)$ and κ; this is true of H_i because $H_i(x)$ lies between $F_i^r(x)$ and its asymptotic value).

Suppose now that the F_i are C^2 with bounded second derivative, and that $|F_i'(x)| \to 0$ as $|x| \to \infty$ for all i. We compute

$$\left|(F_i^r)'(x)\right| = r \left|F_i'(rx)\right| \leq r\|F_i'\|;$$

$$\left|(F_i^r)''(x)\right| = r^2 \left|F_i''(rx)\right| \leq r^2\|F_i''\|.$$

For $x \leq 0$,

$$\begin{aligned}
\left|H_i'(x)\right| &= \left|\ (b_i - F_i(rx))\psi'(x) + (1-\psi(x))rF_i'(rx))\ \right| \\
&\leq r(2\mu\|\psi'\| + \|F_i'\|); \\
\left|H_i''(x)\right| &= \left|\ (b_i - F_i(rx))\psi''(x) - 2rF_i'(rx)\psi'(x) + (1-\psi(x))r^2F_i''(rx)\ \right| \\
&\leq r^2(2\mu\|\psi''\| + 2\|F_i'\|\|\psi'\| + \|F_i''\|);
\end{aligned}$$

similarly for $x \geq 0$. Therefore we see that the constant

$$2\mu\|\psi'\| + 2\mu\|\psi''\| + 2\max_i \|F_i'\|\|\psi'\| + \max_i \|F_i'\| + \max_i \|F_i''\|$$

is a possible value for $\triangle$ in part 3 of proposition 4.2.4.

Taking $\triangle$ as above we of course have $\triangle \geq \|\psi'\|$; if $x \leq -\ell(\epsilon)/r^\alpha$ we have

$$\begin{aligned}
|H_i'(x)| &\leq |b_i - F_i(rx)||\psi'(x)| + (1-\psi(x))|(F_i^r)'(x)| \\
&\leq \frac{\triangle\epsilon}{r^{\kappa(1-\alpha)}} + \frac{r\epsilon}{2Dr^{\kappa'(1-\alpha)}},
\end{aligned}$$

and similarly if $x \geq \ell(\epsilon)/r^\alpha$. This completes the proof. □

We now compare solutions of (E_r) and (DDE_r) on $[0, t^*]$.

Lemma 4.2.6. *Given $\delta > 0$, for any r sufficiently large the following holds. Given $y_0 \in \overline{U}$ with continuation w as a solution of (DDE_r) and continuation v as a solution of (E_r),*

- $|v(t) - w(t)| < \delta$ *for all* $t \in [0, t^*]$.
- *The first m positive zeros $\zeta_1, \ldots, \zeta_m$ of v are defined and proper, and for all $1 \leq k \leq m$*

  $$|\zeta_k - z_k| < \delta,$$

  where the z_k are the first m positive zeros of w.
- $$\|P(y_0) - R(y_0)\| < \delta,$$

  where $P(y_0) = v_{\zeta_m}$.

Throughout this chapter, whenever a particular problem (E_r) is understood, we will write P for the above-described return map. $P : \overline{U} \to C_0$ "advances solutions of (E_r) by m positive zeros." The substance of the above lemma is that P is indeed defined, and close to R, for r sufficiently large.

PROOF. The approach is like that used in section 3.1. Take $y_0 \in \overline{U}$ and write w for its continuation as a solution of (DDE_r) and v for its continuation as a solution of (E_r). We begin with a reminder of how w behaves (recall 3.1.13 and 3.1.19).

- w has exactly m zeros on $(0, t^*)$ and $w(t^*) \leq -2\rho\sigma$.
- w has constant slope, of absolute value at least σ, on every connected component of the sets

$$[(0, 3\rho) \cup (s_1 - 3\rho, s_1 + 3\rho) \cup \cdots \cup (s_m - 3\rho, s_m + 3\rho)]$$
$$\supset w^{-1}[-2\rho\sigma, 2\rho\sigma] \supset w^{-1}[-2\eta^*, 2\eta^*] \supset w^{-1}[-2\eta, 2\eta].$$

- If $t \in [0, t^*]$, $|w(t)| \leq 2\rho\sigma$, and $t - d_i \geq 0$, then $|w(t - d_i)| > 2\rho\sigma$.

Choose $\epsilon > 0$, and suppose that r is large enough that $\ell(\epsilon)/r < \eta$. Write

$$S = \{\ t \in [0, t^*]\ :\ |w(t)| < \ell(\epsilon)/r \text{ or } |v(t)| < \ell(\epsilon)/r \text{ or } w(t)v(t) < 0\ \}.$$

For all $t \in (0, t^*)$ we of course have the bound $|v'(t) - w'(t)| < \mu$. The basic observation is that, for $t \in (0, t^*)$, we in fact have the bound

$$|v'(t) - w'(t)| < \epsilon \text{ unless } t - d_i \in S \text{ for some } d_i.$$

(This is an immediate consequence of 4.2.4 — the second part of part 2, with $\kappa = 0$ — and the fact that $v(t)$ and $w(t)$ have a common initial condition). Please note that all points in S are nonnegative.

For each $0 \leq j \leq m$, write

$$M(j) = \sup_{t \in [0, s_j + 3\rho]} |v(t) - w(t)|$$

(where we are writing $s_0 = 0$). Similarly, for $0 \leq j \leq m$ write

$$N(j) = \text{measure}\,(S \cap [0, s_j + 3\rho])\,.$$

Since $t - d_i < 0$ for $t \in (0, 3\rho)$, we have $M(0) \leq 3\rho\epsilon$.

If $t \in S \cap [0, s_j + 3\rho]$, then $|w(t)|$ must be less than $M(j) + \ell(\epsilon)/r$. Accordingly, for all j such that $M(j) + \ell(\epsilon)/r < \rho\sigma$, the connected components of $S \cap [0, s_j + 3\rho]$ are contained in connected components where $|w(t)| < \rho\sigma$, and where w has constant slope of absolute value at least σ. Hence the length of such a connected component is no more than

$$\frac{2M(j) + 2\ell(\epsilon)/r}{\sigma}$$

units. In other words, as long as $M(j) + \ell(\epsilon)/r < \rho\sigma$, we have

$$N(j) \leq \frac{2(j+1)}{\sigma}\left[M(j) + \frac{\ell(\epsilon)}{r}\right].$$

Suppose now that $t \in [0, s_j + 3\rho]$, and that $t - d_i \in S$ for some i. Since $d_i > 6\rho$, we see that $t - d_i < s_j - 3\rho$. If $M(j) + \ell(\epsilon)/r < \rho\sigma$, we must have $|w(t - d_i)| < \rho\sigma$. Since $|w(s)| > 2\rho\sigma$ for $s \in (s_{j-1} + 3\rho, s_j - 3\rho)$, $w(t - d_i) < \rho\sigma$ implies that $t - d_i \leq s_{j-1} + 3\rho$. Thus $t - d_i$ is in that part of S measured by $N(j-1)$. This tells us that, as long as $M(j) + \ell(\epsilon)/r < \rho\sigma$ (compare to the proof of 3.1.13),

$$M(j) \leq (s_j + 3\rho)\epsilon + \mu N(j-1).$$

Combining the estimates for $N(j)$ and $M(j)$ yields the following (crude) difference inequality for $M(j)$, for $0 < j \leq m$ and as long as $M(j) + \ell(\epsilon)/r < \rho\sigma$:

$$M(j) \leq t^*\epsilon + \frac{2(m+1)\mu}{\sigma}\left(M(j-1) + \ell(\epsilon)/r\right).$$

We have established that, by choosing ϵ small and then r large, we can make $M(m)$ as small as we like. This is the first point of 4.2.6.

Assume that we have chosen $\epsilon < \sigma/2$ and then r so large that $M(m) + \ell(\epsilon)/r < \rho\sigma$. In this case, $|w(t)| \leq 2\rho\sigma$, where $t \in (0, t^*)$, implies that $(t - d_i) \notin S$ for all d_i (since either $|w(t - d_i)| > 2\rho\sigma$ or $t - d_i < 0$). By our choice of ϵ, it follows that $|w(t)| < 2\rho\sigma$ implies that $|v'(t) - w'(t)| < \sigma/2$. For such an ϵ and r, then, we see that $v(t)$ has exactly m zeros on $(0, t^*)$, all of them proper. Corresponding zeros of $w(t)$ and $v(t)$ are no more than $M(m)/\sigma$ units apart, and the sort of estimate described in (1.2.3) tells us that

$$\|P(y_0) - R(y_0)\| \leq M(m)\left(1 + \frac{\mu}{\sigma}\right).$$

This completes the proof of 4.2.6. □

PROOF OF THEOREM 4.1.2. We can choose $\overline{U}$ so that the continuation w of $y_0 \in \overline{U}$ as a solution of (DDE) is as close as we like to p in the sense of 4.1.2 and 3.1.19. Lemma 4.2.6 now tells us that we can arrange for the continuation of an initial condition in $\overline{U}$ as a solution of (E_r) to also be close to p in the sense of 4.1.2 and 3.1.19. To complete the proof, then, we need to show that the return map P has a fixed point on $\overline{U}$.

Let $\nu > 0$ be as in 3.1.23: for any $(DDE) \in \mathcal{P}$, we have

$$y_0 \in \partial U \implies \|R(y_0) - y_0\| > \nu.$$

Applying 4.2.6, we choose r large enough that P is defined and $\|P(y_0) - R(y_0)\| < \nu/2$. Then given $y_0 \in \partial U$ and $s \in [0, 1]$ we have

$$\begin{aligned} \|y_0 - (sP(y_0) + (1-s)R(y_0))\| &\geq \|y_0 - R(y_0)\| - s\|R(y_0) - P(y_0)\| \\ > \ & \nu - s\nu/2 \geq \nu/2. \end{aligned}$$

$sP + (1 - s)R$ is a compact map and (as we have just shown) is fixed-point-free on ∂U for all $s \in [0, 1]$. Thus, by the homotopy property of Leray-Schauder degree, we find that $\deg(I - P, U)$ is defined and is equal to $\deg(I - R, U)$. Since $\deg(I - R, U)$ is nonzero (recall 3.2.5), $\deg(I - P, U)$ is nonzero too and we conclude that P has a fixed point on $\overline{U}$. □

4.3 Uniqueness and stability

We now assume that (E_r) is as in 4.1.1 and 4.1.3 — in particular, we assume that the F_i are C^2, with bounded second derivative, and of bounded variation with $\lim_{|x|\to\infty} F_i'(x) = 0$. We also assume that

$$\kappa > 2, \ \kappa' > 1,$$

where κ and κ' are as in 4.1.1.

We will take $\mathcal{P}$ and $(DDE_r) \in \mathcal{P}$ as described in 4.2.3 and 4.2.4. In particular, we are assuming that all of the F_i and all of the H_i have total variation less than or equal to some $B < \infty$.

The proof of 4.1.3 relies on the following lemma.

Lemma 4.3.1. Main lemma. *With all the above assumptions, there are numbers $a > 2$, $b > 0$, and $N > 0$ such that the following holds.*

Let $\delta > 0$ be given. For any $r \geq 1$, let R be the return map (with fixed point u_0) for (DDE_r), and let P be the return map for (E_r). Then, for r sufficiently large,

1. *Everywhere on $\overline{U}$,*
$$\|P - R\| < \frac{\delta}{r^a};$$
2. *any fixed point $\bar{v}_0 \in \overline{U}$ of P satisfies*
$$\langle \bar{v}_0, u_0 \rangle < \frac{\delta}{r^a};$$
3. *Given $s \in [0,1]$, if $v_0 \in \overline{U}$ lies in the image of $sR + (1-s)P$ and $\langle v_0, u_0 \rangle < 1/r^a$, then*
$$\|DP[v_0] - DR[u_0]\| < \frac{N}{r^b}$$
and
$$\|DR[v_0] - DR[u_0]\| < \frac{N}{r^b}.$$

PROOF OF THEOREM 4.1.3. Theorem 4.1.2 tells us that P has a fixed point in $\overline{U}$ for r large enough. The second part of lemma 4.3.1 tells us that, for r large enough, any fixed point $\bar{v}_0$ of P satisfies
$$\langle \bar{v}_0, u_0 \rangle < \frac{\delta}{r^a}.$$
If we take $\delta < 1$, the third part of lemma 4.3.1 then tells us that
$$\|DP[\bar{v}_0] - DR[u_0]\| < \frac{N}{r^b}.$$
3.2.17 now implies that, if r is large enough and the spectral radius of $DR[u_0]$ is less than 1, then the spectral radius of $DP[\bar{v}_0]$ is less than one also. 3.2.2 tells us that if the matrix $A = D\phi^m[V(p_0)]$ has spectral radius less than 1 then $DR[u_0]$ does too. This proves the part of theorem 4.1.3 concerning stability.

Recall (see 3.2.12) that, since $B < \infty$, there is a fixed M such that
$$\|(I - DR[u_0])^{-1}\| < M$$

for all (DDE_r), $r \geq 1$. 3.2.15 now implies that if T is a linear map with

$$\|T - DR[u_0]\| < M_1 < M^{-1},$$

then $(I-T)^{-1}$ exists and

$$\|(I-T)^{-1}\| < \frac{M}{1 - M_1 M}.$$

Now fix $\delta < 1$ and choose r large enough for lemma 4.3.1 to apply, and also so large that $N/r^b < M^{-1}$, where M is as in last paragraph. This choice guarantees, in particular, that

$$I - DP[\bar{v}_0]$$

is invertible for any fixed point $\bar{v}_0$ of P.

We now define the straight-line homotopy

$$Q : \overline{U} \times [0,1] \to C_0 \quad : \quad Q(y_0, s) = sR + (1-s)P.$$

Given $y_0 \in Q(\overline{U}, s)$ with

$$\langle y_0, u_0 \rangle < \frac{1}{r^a},$$

we have

$$D_1 Q[y_0, s] = sDR[y_0] + (1-s)DP[y_0],$$

and so, by the third part of 4.3.1,

$$\|D_1 Q[y_0, s] - DR[u_0]\| < \frac{N}{r^b} < M^{-1}.$$

(In particular, $I - D_1Q[y_0, s]$ is invertible for such y_0 and all $s \in [0,1]$). Also, for any $0 \leq s_1 \leq s_2 \leq 1$ we have

$$\|Q(y_0, s_1) - Q(y_0, s_2)\| = (s_2 - s_1)\|R(y_0) - P(y_0)\| < \frac{\delta(s_2 - s_1)}{r^a},$$

whence

$$\|D_2 Q[y_0, s]\| < \frac{\delta}{r^a}.$$

Now, assume that $\bar{v}_0$ is a fixed point of $P = Q(\cdot, 0)$. By the implicit function theorem, for small $s \geq 0$ there is a unique smooth path $s \mapsto v^s$, where $v^0 = \bar{v}_0$ and each

v^s is a fixed point of $Q(\cdot, s)$. Since $\langle \bar{v}_0, u_0 \rangle < \delta / r^a$, if $\delta < 1$ and for s^* sufficiently small we have that

$$\langle v^s, u_0 \rangle < \frac{1}{r^a} \ \forall \ s \in [0, s^*].$$

Suppose that $s^* < 1$. By the implicit function theorem and the third part of 4.3.1 — v^s of course lies in the image of $sR + (1-s)P$ — the derivative of v^s in s is bounded by

$$\left| \frac{dv^s}{ds} \right| \leq \|D_2 Q[v^s, s]\| \|(I - D_1 Q[v^s, s])^{-1}\| \leq \frac{\delta}{r^a} \frac{1}{1 - NM/r^b}.$$

By choosing δ small enough and r large enough, then, we can arrange that

$$\left| \frac{dv^s}{ds} \right| \leq \frac{1-\delta}{r^a},$$

whence $\|v^{s^*} - \bar{v}_0\| < s^*(1-\delta)/r^a < (1-\delta)/r^a$ and $\langle v^{s^*}, u_0 \rangle < 1/r^a$. If r is large enough, we of course have that v^{s^*} lies in $\overline{U}$. In this case, through, we can extend the path $s \mapsto v^s$ uniquely beyond s^*. This reasoning shows that, for r large enough, the path $s \mapsto v^s$ is defined (uniquely) on the whole unit interval and runs from $\bar{v}_0$ to u_0. This path can be reversed, and we see that as s goes from 1 to 0 we follow a unique path of fixed points of $Q(\cdot, s)$ from u_0 to $\bar{v}_0$.

If we now imagine that there is another fixed point v_0 of P in U, exactly the same argument shows that the above mentioned unique "backward" path actually goes from u_0 to v_0. Thus we see that, for r large enough, $\bar{v}_0$ is the unique fixed point of P in $\overline{U}$.

□

Proof of 4.3.1

We devote the rest of this section to the lengthy and technical proof of lemma 4.3.1. Recall our assumptions: the F_i are C^2, with bounded second derivative, and of bounded variation with $\lim_{|x|\to\infty} F_i'(x) = 0$. The exponents κ and κ' satisfy

$$\kappa > 2, \ \kappa' > 1. \tag{4.3.2}$$

We choose and fix $\alpha \in (0, 1)$ such that

$$\kappa(1-\alpha) > 2. \tag{4.3.3}$$

Let us write $\ell = \ell(1)$, where the function $\ell(\cdot)$ is as in definition 4.1.1. We will always assume $r \geq 1$ large enough that $\ell/r^\alpha < \eta$, where η is as in 4.2.3. Recalling 4.2.4, we have the following for all such r and all $a \in [0, \kappa(1-\alpha)]$:

- For all x,
$$|F_i^r(x) - H_i(x)| < \frac{1}{Dr^a};$$
- If x and y are both greater than ℓ/r^α in absolute value and are of the same sign, then
$$|F_i^r(x) - H_i(y)| < \frac{1}{Dr^a}.$$

Let us choose $v_0 \in \overline{U}$ with continuations v and w as solutions of (E_r) and (DDE_r), respectively. Recall from lemma 4.2.6 that we can ensure that $v(t)$ and $w(t)$ are uniformly close for all $t \in [0, t^*]$ if r is large enough. In particular, since $|w(t)| \leq 2\rho\sigma$ implies either that $t - d_i < 0$ or that $|w(t - d_i)| > 2\rho\sigma$ for all i, if we insist that $|v(t) - w(t)| < \rho\sigma/2$ for all $t \in [0, t^*]$ we have the following.

Lemma 4.3.4. *If r is sufficiently large, then for all $t \in [0, t^*]$ we have, for all i,*

$$|v(t)| \leq \frac{3\rho\sigma}{2} \text{ or } |w(t)| \leq \frac{3\rho\sigma}{2} \implies t - d_i < 0 \text{ or } |w(t-d_i)|, |v(t-d_i)| > \rho\sigma. \quad \square$$

Since we are assuming that $\ell/r^\alpha < \eta < \rho\sigma$, applying 4.2.4 immediately yields that, for any $a \in [0, \kappa(1-\alpha)]$,

Corollary 4.3.5. *For r sufficiently large, for all $t \in [0, t^*]$ we have that*

$$|v(t)| \leq \frac{3\rho\sigma}{2} \text{ or } |w(t)| \leq \frac{3\rho\sigma}{2} \implies |v'(t) - w'(t)| \leq \frac{1}{r^a}. \quad \square$$

If we further insist that $1/r^{\kappa(1-\alpha)} < \sigma/2$, we get the following.

Corollary 4.3.6. Summary of behavior of solutions of (E_r). *For r sufficiently large, given any $v_0 \in \overline{U}$ with continuation v as a solution of (E_r) the following hold:*

- *if w is the continuation of v_0 as a solution of (DDE_r), $|v(t) - w(t)| < \rho\sigma/2$ for all $t \in [0, t^*]$;*

- *$v(t)$ has exactly m positive zeros, all proper, on $[0, t^*]$;*

- *The set*
$$S = \{\ t \in [0, t^*]\ :\ |v(t)| \leq 3\rho\sigma/2\ \}$$
has exactly $m+1$ components $S_0, S_1, \ldots, S_m$, with
$$S_0 \subset [0, 3\rho],\ \ S_1 \subset [s_1 - 3\rho, s_1 + 3\rho], \ldots, S_m \subset [s_m - 3\rho, s_m + 3\rho]$$
(where the s_k are the positive zeros of p);

- *Each S_k contains one zero of v, and on S_k we have*
$$|v'(t) - p'(s_k)| < \frac{1}{r^a}$$
for all $a \in [0, \kappa(1-\alpha)]$. □

We will henceforth assume r large enough that corollary 4.3.6 holds.

We now define a subset $\hat{W}$ of C_0 into which P maps. $\hat{W}$ is very similar to the space $\tilde{W}$, and plays an analogous role, but we can afford to make its definition somewhat simpler.

Definition 4.3.7. The space $\hat{W}$. Given $r \geq 1$, write $\hat{W}$ for the set of initial conditions $v_0 \in C_0$ that satisfy the following:

- v_0 has exactly n zeros $-\xi_n < \cdots < -\xi_1$ on $(-\gamma, 0)$.

- On the connected component of $\{\ s \in (-\gamma, 0)\ :\ |v_0(s)| \leq \rho\sigma\ \}$ whose right endpoint is 0,
$$|v_0'(s) - p'(0)| < \frac{1}{r^{\kappa(1-\alpha)}}.$$

- On the connected component of $\{\ s \in (-\gamma, 0)\ :\ |v_0(s)| \leq \rho\sigma\ \}$ containing $-\xi_k$,
$$|v_0'(s) - p'(-\pi_k)| < \frac{1}{r^{\kappa(1-\alpha)}}$$
(recall that $-\pi_k$ is the kth negative zero of p_0).

$\hat{W}$ of course depends on r, but for simplicity we do not express this dependence in our notation. Observe that $\tilde{W} \subset \hat{W}$ for all $r \geq 1$.

Given $v_0 \in \overline{U}$ with continuation v as a solution of (E_r), write ζ for the mth positive zero of v; we know from lemma 4.3.6 that $\zeta \in [s_m - 3\rho, s_m + 3\rho]$. Since

$$s_m - \gamma \in [s_{m-n-1} + 6\rho, s_{m-n} - 6\rho]$$

(recall 3.1.2), we have

$$\zeta - \gamma \in [s_{m-n-1} + 3\rho, s_{m-n} - 3\rho].$$

4.3.6 now implies that $P(v_0) = v_\zeta$ has exactly n proper zeros on $(-\gamma, 0)$, and having established this it is easy to see (again by 4.3.6) that $P(v_0) \in \hat{W}$. We in fact need a bit more, namely

Lemma 4.3.8. *If r is sufficiently large, for any $y_0 \in \overline{U}$ and any $\tau \in [0,1]$ we have*

$$\tau R(y_0) + (1-\tau) P(y_0) \in \hat{W}.$$

PROOF. Write v and w for the continuations of y_0 as solutions of (E_r) and (DDE_r), respectively. Assume that r is large enough that 4.3.6 holds, and also so that $\|P(y_0) - R(y_0)\| < \rho\sigma/2$ for all $y_0 \in \overline{U}$.

Write

$$\bar{y}_0 = \tau R(y_0) - (1-\tau) P(y_0).$$

The set of $s \in [-\gamma, 0]$ for which $|\bar{y}_0(s)| \leq \rho\sigma$ contains the set where both $|R(y_0)(s)| \leq \rho\sigma$ and $|P(y_0)(s)| \leq \rho\sigma$, and is contained in the set where both $|R(y_0)(s)| \leq 3\rho\sigma/2$ and $|P(y_0)(s)| \leq 3\rho\sigma/2$. We see therefore that

$$\{\ s \in [-\gamma, 0]\ :\ |\bar{y}_0(s)| \leq \rho\sigma\ \}$$

is nonempty and is contained in a collection of subintervals. On each such subinterval, both $P(y_0)$ and $R(y_0)$ assume all values on $[-\rho\sigma, \rho\sigma]$ and have slopes of the same constant sign. It follows that $\bar{y}_0(s)$ has exactly $n+1$ zeros $-\xi_n < \cdots < -\xi_1 < -\xi_0 = 0$

on $[-\gamma, 0]$. On the connected component of $\{ s \in [-\gamma, 0] \ : \ |\bar{y}_0(s)| \leq \rho\sigma \}$ containing $-\xi_k$, we have

$$R(y_0)'(s) = p'(-\pi_k)$$

while 4.3.6 implies that

$$|P(y_0)'(s) - p'(-\pi_k)| < \frac{1}{r^{\kappa(1-\alpha)}}.$$

It follows that

$$|\bar{y}_0'(s) - p'(-\pi_k)| < \frac{1}{r^{\kappa(1-\alpha)}},$$

and we conclude that $\bar{y}_0 \in \hat{W}$. □

Our objective is to make a careful comparison of solutions of (E_r) with initial conditions in $\hat{W}$ to solutions of (DDE_r) with initial conditions in $\tilde{W}$. We begin with the following calculus lemma.

Lemma 4.3.9. *Suppose that f and g are two continuous functions satisfying*

$$\|f - g\| \leq c_1; \quad \|f\|, \|g\| \leq M.$$

Suppose too that $[a, b]$ and $[c, d]$ are nontrivial intervals each of length no more than L and that

$$|a - c| \leq c_2 \text{ and } |b - d| \leq c_2.$$

Then

$$\left| \int_a^b f(s)\ ds - \int_c^d g(s)\ ds \right| \leq 2c_2 M + c_1 L.$$

PROOF. WOLOG suppose that $a \leq c$. There are then three subcases to consider.

Case 1: $b \leq c$. Then $[a, b]$ and $[c, d]$ are each of length less than c_2 and the quantity in question is less than

$$2c_2 M.$$

Case 2: $c < b \leq d$. Then we have

$$\begin{aligned} & \left| \int_a^b f(s)\ ds - \int_c^d g(s)\ ds \right| \\ = & \left| \int_a^c f(s)\ ds + \int_c^b f(s) - g(s)\ ds - \int_b^d g(s)\ ds \right| \\ \leq & \ 2c_2 M + c_1 L. \end{aligned}$$

Case 3: $c < d < b$. Then we have

$$\begin{aligned}
& \left| \int_a^b f(s)\ ds - \int_c^d g(s)\ ds \right| \\
= & \left| \int_a^c f(s)\ ds + \int_c^d f(s) - g(s)\ ds + \int_d^b f(s)\ ds \right| \\
\leq & \ 2c_2 M + c_1 L.
\end{aligned}$$

This completes the proof. □

Lemma 4.3.10. *Let $v_0 \in \hat{W} \cap \overline{U}$ with continuation v as a solution of (E_r), and let $w_0 \in \tilde{W} \cap \overline{U}$ with continuation w as a solution of (DDE_r). Then there is a constant K_1 such that, given any $a \in (0, \kappa(1-\alpha)]$, if r is sufficiently large we have*

$$\langle v_0, w_0 \rangle < \frac{1}{r^a} \implies |v(t) - w(t)| < \frac{K_1}{r^a} \ \textit{for all} \ t \in [0, t^*].$$

PROOF. The idea is this. Suppose that there is some $\tau \in [0, t^*]$ and some $c \geq 1$ such that

$$\langle v_0, w_0 \rangle < \frac{c}{r^a} \text{ and } |v(t) - w(t)| < \frac{c}{r^a} \text{ for all } t \in [0, \tau].$$

Suppose also that r is so large that 4.3.5 and 4.3.6 hold, and also so large that $c/r^a < \rho\sigma - \eta$. In this case, we have the following two implications for all $t \in [-\gamma, \tau]$: first,

$$|v(t)| \leq \eta \text{ or } |w(t)| \leq \eta \text{ or } v(t)w(t) < 0 \implies |v(t)| \leq \rho\sigma \text{ and } |w(t)| \leq \rho\sigma;$$

second,

$$|v(t)| \leq \rho\sigma \text{ and } |w(t)| \leq \rho\sigma \implies |v'(t) - w'(t)| \leq \frac{1}{r^{\kappa(1-\alpha)}} \leq \frac{c}{r^a}.$$

We claim that, in this case, there is a number $K(c) \geq c$, that depends on c but not on a or r (assuming r is large enough to meet the above-stated conditions), such that, for all $1 \leq i \leq D$ and all $t \in [0, \tau]$,

$$\left| \int_{-d_i}^t F_i^r(v(s))\ ds - \int_{-d_i}^t H_i(w(s))\ ds \right| \leq \frac{K(c)}{r^a}.$$

This claim implies that

$$|v(t) - w(t)| \leq \frac{DK(c)}{r^a} \text{ for all } t \in [0, \tau + d_1].$$

By increasing r further if necessary to get $DK(c)/r^a < \rho\sigma - \eta$, we can then apply our claim again to assert that

$$|v(t) - w(t)| \leq \frac{DK(DK(c))}{r^a} \text{ for all } t \in [0, \tau + 2d_1],$$

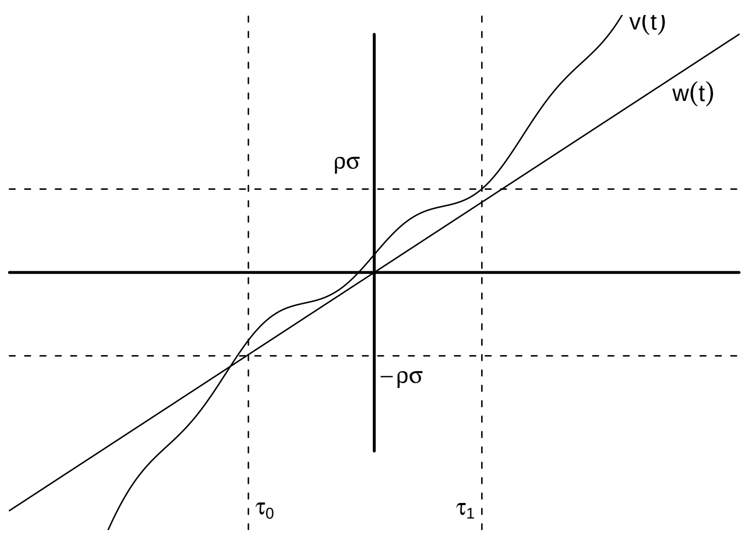
and so on. Finitely many such steps, starting with $\tau = 0$ and $c = 1$, prove the lemma.

We now prove our claim. As s ranges from $-d_i$ to $t \in [-d_i, \tau]$, $|F_i^r(v(s)) - H_i(w(s))| \leq 1/Dr^a$ unless

$$|v(s)| < \eta \text{ or } |w(s)| < \eta \text{ or } v(s)w(s) < 0.$$

The connected components where this happens are contained in the connected components where both $|v(s)|$ and $|w(s)|$ are less than or equal to $\rho\sigma$; there are at most $m + n + 1$ such components in $[-d_i, \tau]$. Let $[\tau_0, \tau_1]$ be such a component (observe that the length of this component is less than 2ρ). We illustrate this component in Figure 15.

Figure 15



To prove our claim it suffices to exhibit a constant c' such that, for any $t \in [\tau_0, \tau_1]$,

$$\left| \int_{\tau_0}^{t} F_i^r(v(s))\, ds - \int_{\tau_0}^{t} H_i(w(s))\, ds \right| \leq \frac{c'}{r^a}.$$

On $[\tau_0, \tau_1]$, $w'(s)$ is equal to some constant $\bar{\sigma}$ and $v'(s)$ is uniformly within $1/r^a$ of $\bar{\sigma}$. Given any s on this interval, write

$$v'(s) = \bar{\sigma} + z;$$

for r sufficiently large we have

$$\begin{aligned} & \left| \frac{v'(s)}{\bar{\sigma}} - \frac{v'(s)}{v'(s)} \right| \\ = & \left| \frac{v'(s)\bar{\sigma} + zv'(s) - v'(s)\bar{\sigma}}{\bar{\sigma}^2 + \bar{\sigma}z} \right| \\ \leq & \left| \frac{zv'(s)}{\bar{\sigma}^2/2} \right| \\ \leq & \frac{c_1}{r^a}. \end{aligned}$$

The point is that $v'(s)/v'(s)$ is uniformly within c_1/r^a of $v'(s)/\bar{\sigma}$ for all $s \in [\tau_0, \tau_1]$. Since $|F_i^r| \leq \mu$, it follows that

$$\left| \frac{v'(s)}{v'(s)} F_i^r(v(s)) - \frac{v'(s)}{\bar{\sigma}} F_i^r(v(s)) \right| \leq \frac{c_1 \mu}{r^a}$$

for all such s. We therefore have the estimate

$$\left| \int_{\tau_0}^t F_i^r(v(s)) \, ds - \int_{\tau_0}^t H_i(w(s)) \, ds \right| \leq \left| \int_{\tau_0}^t \frac{v'(s)}{\bar{\sigma}} F_i^r(v(s)) \, ds - \int_{\tau_0}^t H_i(w(s)) \, ds \right| + \frac{2\rho c_1 \mu}{r^a}.$$

Changing variables, we rewrite the above difference of integrals as

$$\frac{1}{\bar{\sigma}} \left| \int_{v(\tau_0)}^{v(t)} F_i^r(u) \, du - \int_{w(\tau_0)}^{w(t)} H_i(u) \, du \right|.$$

Since $|F_i^r|$ and $|H_i|$ are both bounded by μ, and $|F_i^r - H_i|$, $|v(\tau_0) - w(\tau_0)|$, and $|v(t) - w(t)|$ are all bounded by some constant times $1/r^a$, our desired result follows from lemma 4.3.9. $\square$

Similar to 4.3.10 is

Lemma 4.3.11. *Suppose that $v_0 \in \overline{U}$. Write v and w for the continuations of v_0 as solutions of (E_r) and (DDE_r), respectively. Then there is a constant K_1 such that, given any $a \in (0, \kappa(1-\alpha)]$, if r is sufficiently large we have*

$$|v(t) - w(t)| \leq \frac{K_1}{r^a}$$

for all $t \in [0, t^*]$.

PROOF. The argument is the same the same as for 4.3.10, except that

$$|F_i^r(v(t)) - H_i(w(t))| \leq \frac{1}{Dr^a}$$

whenever $t < 0$. $\square$

The above calculations give us our basic control over how solutions of (E_r) and (DDE_r) compare. The next lemma translates this control into a comparison of the maps P and R.

Lemma 4.3.12. *There is a constant K such that the following hold. Choose any $a \in (1, \kappa(1-\alpha)]$. Take $v_0 \in \overline{U}$ and $w_0 \in \overline{U}$ with continuations $v(t)$ and $w(t)$ as solutions of (E_r) and (DDE_r), respectively. Assume either that $v_0 = w_0$ or that $v_0 \in \hat{W}$ and $w_0 \in \tilde{W}$ with $\langle v_0, w_0 \rangle < 1/r^a$. Then, for r sufficiently large,*

1. $$\sup_{t \in [0,t^*]} |v(t) - w(t)| \leq \frac{K}{r^a}.$$

2. $$\|P(v_0) - R(w_0)\| \leq \frac{K}{r^a}.$$

3. $$\sup_{t \in (0,t^*)} |v'(t) - w'(t)| \leq \frac{K}{r^{a-1}}.$$

4. $$\|P(v_0)' - R(w_0)'\| \leq \frac{K}{r^{a-1}}.$$

5. $$\left\| \frac{P(v_0)'}{P(v_0)'(0)} - \frac{R(w_0)'}{R(w_0)'(0)} \right\| \leq \frac{K}{r^{a-1}}.$$

PROOF. We know that, by choosing r large enough, we have

$$|v(t) - w(t)| \leq \frac{K_1}{r^a}$$

for all $t \in [0, t^*]$. Take r large enough that this difference is less than $\rho\sigma$. In this case the mth positive zero of $w(t)$ and the mth positive zero of $v(t)$ are less than

$$\frac{K_1}{\sigma r^a}$$

units apart. Accordingly, (1.2.3) tells us that

$$\|P(v_0) - R(w_0)\| \leq \frac{K_1}{r^a}\left(1 + \frac{\mu}{\sigma}\right).$$

This the second part of the lemma.

Using the estimate

$$\|F(v) - H(w)\| \leq \|F(w) - H(w)\| + \|F(v) - F(w)\| \leq \|F - H\| + \|F'\||u - w|,$$

we see that that $v'(t)$ and $w'(t)$ differ on $(0, t^*)$ by no more than

$$D\left[\frac{1}{r^{\kappa(1-\alpha)}} + \frac{r\triangle K}{r^\alpha}\right] \leq \frac{D + D\triangle K}{r^{a-1}} := \frac{K_2}{r^{a-1}}.$$

This is the third point of the lemma.

The second derivative of $v(t)$ is bounded above by $D\mu r\triangle$, and by again using the idea of (1.2.3) we get

$$\|P(v_0)' - R(w_0)'\| \leq \frac{K_2}{r^{a-1}} + \frac{D\mu r\triangle K_1}{\sigma r^a} := \frac{K_3}{r^{a-1}}$$

This yields the fourth part of the lemma.

Recall that $R(w_0)'(0) = p'(0)$, that $\|R(w_0)'\| \leq \mu$, and that

$$\|P(v_0)'(0) - R(w_0)'(0)\| \leq \frac{K_3}{r^{a-1}}.$$

Therefore we have, writing $z = P(v_0)'(0) - p'(0)$,

$$\begin{aligned} & \left\|\frac{P(v_0)'}{P(v_0)'(0)} - \frac{R(w_0)'}{R(w_0)'(0)}\right\| \\ = & \left\|\frac{P(v_0)'}{p'(0) + z} - \frac{R(w_0)'}{p'(0)}\right\| \\ = & \left\|\frac{(P(v_0)' - R(w_0)')p'(0) - zR(w_0)'}{p'(0)^2 + zp'(0)}\right\| \\ \leq & \frac{\|P(v_0)' - R(w_0)'\|}{|p'(0) + z|} + \frac{|z|\|R(w_0)'\|}{|p'(0)^2 + zp'(0)|}. \end{aligned}$$

Since $\|R(w_0)'\| \leq \mu$ and $|z| \leq K_2/r^{a-1}$, the fifth part of the lemma follows. □

The following corollary immediately implies the first part of our main lemma 4.3.1:

Corollary 4.3.13. *Let $0 < a < \kappa(1-\alpha)$. Then, given any $\delta > 0$, for r sufficiently large we have*

$$\|P(y_0) - R(y_0)\| < \frac{\delta}{r^a}.$$

PROOF. By 4.3.12,

$$\|P(y_0) - R(y_0)\| \le \frac{K}{r^{\kappa(1-\alpha)}}.$$

Given $\delta > 0$ and $0 < a < \kappa(1-\alpha)$, for r large enough we have

$$\frac{K}{r^{\kappa(1-\alpha)}} < \frac{\delta}{r^a}. \quad \square$$

Suppose now that $\bar{v}_0$ is a fixed point of P. Then by lemma 4.3.12 we have

$$\|\bar{v}_0 - R(\bar{v}_0)\| \le \frac{K}{r^{\kappa(1-\alpha)}}.$$

For r sufficiently large, the quantity on the right can be made less than both $\epsilon^*/4$ and ν, where ϵ^* and ν are as in lemma 3.1.22 and theorem 3.1.23. This tells us that we must have $\langle \bar{v}_0, p_0 \rangle < 3\epsilon^*/4$, and that $\langle R(\bar{v}_0), p_0 \rangle < \epsilon^*$. This last just says that $R(\bar{v}_0) \in \overline{U}$, and so $R^2(\bar{v}_0)$ is defined. Indeed, since $R(\bar{v}_0)$ actually lies in $\tilde{W} \cap \overline{U}$, we can apply lemma 4.3.12 to the pair $\bar{v}_0$ and $R(\bar{v}_0)$.

Choosing $0 < a < \kappa(1-\alpha)$, if r is large enough we have that

$$\|\bar{v}_0 - R(\bar{v}_0)\| = \|P(\bar{v}_0) - R(\bar{v}_0)\| \le \frac{K}{r^{\kappa(1-\alpha)}} < 1/r^a,$$

and so we have $\langle \bar{v}_0, R(\bar{v}_0) \rangle < 1/r^a$ and

$$\|\bar{v}_0 - R^2(\bar{v}_0)\| < \frac{K}{r^a}.$$

Thus, for $a' < a$ and r sufficiently large, we have both

$$\|\bar{v}_0 - R(\bar{v}_0)\| < \frac{1}{r^{a'}} \text{ and } \|\bar{v}_0 - R^2(\bar{v}_0)\| < \frac{1}{r^{a'}}.$$

Now we know, since $R(\bar{v}_0) \in \tilde{W}$, that there is a constant β such that

$$\|R(\bar{v}_0) - R^2(\bar{v}_0)\| \ge \frac{\sigma\beta}{\mu} \langle R(\bar{v}_0), u_0 \rangle$$

(see the proof of 3.1.22). On the other hand,

$$\|R(\bar{v}_0) - R^2(\bar{v}_0)\| \le \|R(\bar{v}_0) - \bar{v}_0\| + \|R^2(\bar{v}_0) - \bar{v}_0\| \le \frac{2}{r^{a'}}.$$

Therefore we see that

$$\langle R(\bar{v}_0), u_0 \rangle \le \frac{2\mu}{\sigma\beta r^{a'}},$$

whence

$$\langle \bar{v}_0, u_0 \rangle \leq \frac{2\mu/\sigma\beta}{r^{a'}} + \frac{1}{r^{a'}}.$$

Choosing $b \in (1, a')$ establishes the following corollary, which immediately yields the second point of our main lemma 4.3.1.

Lemma 4.3.14. *Given any $\delta > 0$ and any $0 < b < \kappa(1-\alpha)$, for r sufficiently large any fixed point of P must be within δ/r^b of u_0 in the $\langle \cdot, \cdot \rangle$ semimetric.* □

Before proving the last point of 4.3.1, we need two more technical estimates. It is here that we will make use of the hypothesis that $\kappa' > 1$.

Lemma 4.3.15. *There is a constant $\bar{K}$ such that the following holds. Let $v_0 \in \overline{U}$ with continuation v as a solution of (E_r). Then, for r sufficiently large,*

$$\int_0^{t^*} |(F_i^r)'(v(s))| \; ds < \bar{K}.$$

PROOF. By 4.3.6 we know that, if r is sufficiently large, we can arrange that $|v(t)| \leq \rho\sigma$ on exactly $m+1$ connected components of $[0, t^*]$, and that $|v'(t)| \geq \sigma/2$ on each of those components.

Assume that r is large enough so that

$$|x| \geq \rho\sigma \implies |(F_i^r)'(x)| < \frac{1}{t^*}$$

(we can do this because $\kappa' > 1$). Now, the integral

$$\int_0^{t^*} |(F_i^r)'(v(s))| \; ds$$

can be broken up into components where $|v(s)| \leq \rho\sigma$ and components where $|v(s)| > \rho\sigma$. The total integral over all components of the latter type is less than 1. To complete the proof, then, we show that

$$\int_{t_1}^{t_2} |(F_i^r)'(v(s))| \; ds < \frac{2B}{\sigma}$$

where B is as in the definition of $\mathcal{P}$ (recall 4.2.4)and (t_1, t_2) is any connected component where $|v(s)| \leq \rho\sigma$. For simplicity, let us assume that $v'(t) \geq \sigma/2$ on this component; the proof if the derivative of v is negative on this component is similar.

We have

$$\int_{t_1}^{t_2} |(F_i^r)'(v(s))| \, ds = \int_{t_1}^{t_2} \frac{v'(s)}{v'(s)} |(F_i^r)'(v(s))| \, ds$$
$$\leq \frac{2}{\sigma} \int_{t_1}^{t_2} v'(s) |(F_i^r)'(v(s))| \, ds = \frac{2}{\sigma} \int_{v(t_1)}^{v(t_2)} |(F_i^r)'(s)| \, ds.$$

Since B is larger than the total variation of F_i^r, this last integral is less than $2B/\sigma$. □

Lemma 4.3.16. *Choose $a \in (2, \kappa(1-\alpha)]$.*

Choose $\beta \in (0,1)$ such that $\kappa'(1-\beta) > 1$, and choose $b > 0$ such that

$$b < \min\{ \ \kappa'(1-\beta) - 1 \ , \kappa(1-\beta), \ a - 2 \ \}.$$

There is a constant $N_1 > 0$ such that, if $v_0 \in \hat{W}$ and

$$\langle v_0, u_0 \rangle < \frac{1}{r^a},$$

then for r sufficiently large we have

$$\sum_i \int_{-\gamma}^{t^*} \left| (F_i^r)'(v(s)) - H_i'(u(s)) \right| \ ds \leq \frac{N_1}{r^b}.$$

PROOF. We show that, for all i and for r sufficiently large, we have

$$\left| (F_i^r)'(v(t)) - H_i'(u(t)) \right| \leq \frac{1}{Dr^b}$$

for all $t \in [-\gamma, t^*]$.

We choose r large enough so that the following hold (here, the constant K is as in 4.3.12):

- $|v(t) - u(t)| < K/r^a$ for all $t \in [0, t^*]$;
- $\ell/r^\beta < \eta$;
- $K/r^a + \ell/r^\beta < \rho\sigma$.

We also insist that $K > 1$. Then the condition $\langle v_0, u_0 \rangle < 1/r^a$ guarantees that, for $t \in [-\gamma, 0]$, $|v(t) - w(t)| < K/r^a$ whenever both $|v(t)|$ and $|w(t)|$ are less than $\rho\sigma$.

There are two cases to consider. Case 1: $t < 0$ and either $v(t)$ or $u(t)$ is greater than $\rho\sigma$. Then, since $\langle v_0, u_0\rangle < K/r^a$ and $K/r^a + \ell/r^\beta < \rho\sigma$, $v(t)$ and $u(t)$ must be of the same sign and both greater than ℓ/r^β. In this case, though, we have (by (4.2.2) and 4.2.4)

$$|(F_i^r)'(v(t))| \leq \frac{r}{2Dr^{\kappa'(1-\beta)}} \quad \text{and} \quad |H_i'(u(t))| \leq \frac{\triangle}{2Dr^{\kappa(1-\beta)}} + \frac{r}{2Dr^{\kappa'(1-\beta)}}.$$

For r sufficiently large, the sum of these two estimates is less than $1/Dr^b$.

Case 2: either $t < 0$ and $v(t)$ and $u(t)$ are both less than or equal to $\rho\sigma$, or $t \geq 0$. In this case, $|v(t) - u(t)| \leq K/r^a$. The same sort of estimate as in the proof of 4.3.12 gives

$$|(F_i^r)'(v(t)) - H_i'(u(t))| \quad \leq \quad |(F_i^r)'(u(t)) - H_i'(u(t))| + \|(F_i^r)''\||v(t) - u(t)|.$$

The second summand on the right is smaller than

$$\frac{K\triangle r^2}{r^a}$$

(recall from 4.2.4 that the second derivative of F_i^r is smaller than $\triangle r^2$ in absolute value), which is smaller than $1/(2Dr^b)$ for r large enough. For the first summand, we estimate in three cases. If $|u(t)| < \eta$ then $|(F_i^r)'(u(t)) - H_i'(u(t))| = 0$. If $u(t) \leq -\eta$, then $|u(t)| > \ell/r^\beta$ and we have

$$\begin{aligned}
\left|(F_i^r)'(u(t)) - H_i'(u(t))\right| &= \left|\ (F_i^r)'(u(t)) - (b_i - F_i^r(u(t))\psi'(u(t))) - (F_i^r)'(u(t))(1 - \psi(u(t)))\ \right| \\
&\leq 2|(F_i^r)'(u(t))| + \frac{\|\psi'\|}{2Dr^{\kappa(1-\beta)}} \\
&\leq \frac{2r}{2Dr^{\kappa'(1-\beta)}} + \frac{\|\psi'\|}{2Dr^{\kappa(1-\beta)}}.
\end{aligned}$$

This quantity too is smaller than $1/(2Dr^b)$ for r large enough. Similarly for $u(t) > \eta$. $\square$

Let us now choose $a \in (2, \kappa(1-\alpha))$. This is the a of lemma 4.3.1. We choose $\beta > 0$ such that $\kappa'(1-\beta) > 1$, and choose $b > 0$ in lemma 4.3.1 to satisfy

$$b < \min\{\ \kappa'(1-\beta) - 1, \kappa(1-\beta), a - 2\ \}.$$

Let us recap what our work so far tells us. Let $v_0 \in \overline{U}$ with continuation v as a solution of (E_r). Given $\delta > 0$, for r sufficiently large we have

- $$\|P(v_0) - R(v_0)\| < \frac{\delta}{r^a}.$$

- Any fixed point $\bar{v}_0$ of P satisfies

$$\langle \bar{v}_0, u_0 \rangle < \frac{\delta}{r^a}.$$

- If $v_0 \in \hat{W}$ and $\langle v_0, u_0 \rangle < 1/r^a$, for all $t \in [0, t^*]$

$$|v(t) - u(t)| < \frac{K}{r^a} < \rho\sigma.$$

- If $v_0 \in \hat{W}$ and $\langle v_0, u_0 \rangle < 1/r^a$,

$$\|P(v_0) - u_0)\| \leq \frac{K}{r^a}.$$

- If $v_0 \in \hat{W}$ and $\langle v_0, u_0 \rangle < 1/r^a$,

$$\|P(v_0)' - u_0'\| \leq \frac{K}{r^{a-1}}.$$

- If $v_0 \in \hat{W}$ and $\langle v_0, u_0 \rangle < 1/r^a$,

$$\left\| \frac{P(v_0)'}{P(v_0)'(0)} - \frac{u_0'}{u_0'(0)} \right\| \leq \frac{K}{r^{a-1}}.$$

- $$\int_0^{t^*} |(F_i^r)'(v(s))| \; ds < \bar{K}.$$

- If $v_0 \in \hat{W}$ and $\langle v_0, u_0 \rangle < 1/r^a$,

$$\sum_i \int_{-\gamma}^{t^*} \left| (F_i^r)'(v(s)) - H_i'(u(s)) \right| \; ds \leq \frac{N_1}{r^b}.$$

We will maintain the notation in the above list for the short remainder of the section. To complete the proof of lemma 4.3.1 (and hence of theorem 4.1.3) it remains to prove the following.

Lemma 4.3.17. Last part of lemma 4.3.1. *Suppose that $v_0 \in \overline{U}$ is in the image of $\tau R + (1-\tau)P$ for $\tau \in [0, 1]$, and suppose that*

$$\langle v_0, u_0 \rangle < \frac{1}{r^a}.$$

Then there is an $N > 0$ *such that, for* r *sufficiently large,*

$$\|DP[v_0] - DR[u_0]\| < \frac{N}{r^b}$$

and

$$\|DR[v_0] - DR[u_0]\| < \frac{N}{r^b}.$$

We will only prove the first estimate. The second estimate is a special case of the first: just assume that (E_r) is steplike and take $(DDE_r) = (E_r)$ for all large r.

PROOF OF 4.3.17. Our hypothesis, together with lemma 4.3.8, tells us that we can assume that $v_0 \in \hat{W}$. As usual we write v and u for the continuations of v_0 and u_0 as solutions of (E_r) and (DDE_r), respectively.

Let us write $T_u : \mathbf{R}_+ \times C_0 \to C_0$ for the solution operator for the linearization of (DDE_r) about u and $S_v : \mathbf{R}_+ \times C_0 \to C_0$ for the solution operator for the linearization of (E_r) about v. Let us write $\tau(u)$ for the mth positive zero of $u(t)$ and $\tau(v)$ for the mth positive zero of $v(t)$. We have the formula (recall 1.2.1)

$$\begin{aligned}
DR[u_0]y_0 &= T_u(\tau(u), y_0) - \frac{T_u(\tau(u), y_0)(0)}{u_0'(0)} \cdot u_0'; \\
DP[v_0]y_0 &= S_v(\tau(v), y_0) - \frac{S_v(\tau(v), y_0)(0)}{P(v_0)'(0)} \cdot P(v_0)'.
\end{aligned}$$

We write out

$$\begin{aligned}
& DP[v_0]y_0 - DR[u_0]y_0 \\
= \;& S_v(\tau(v), y_0) - S_v(\tau(v), y_0)(0) \cdot \frac{P(v_0)'}{P(v_0)'(0)} - T_u(\tau(u), y_0) \\
& + T_u(\tau(u), y_0)(0) \cdot \frac{u_0'}{u_0'(0)} \\
= \;& (S_v(\tau(v), y_0) - T_u(\tau(u), y_0)) \\
- \;& (S_v(\tau(v), y_0)(0) - T_u(\tau(u), y_0)(0)) \left(\frac{P(v_0)'}{P(v_0)'(0)} - \frac{u_0'}{u_0'(0)} \right) \\
- \;& (S_v(\tau(v), y_0)(0) - T_u(\tau(u), y_0)(0)) \left(\frac{u_0'}{u_0'(0)} \right) \\
- \;& (T_u(\tau(u), y_0)(0)) \left(\frac{P(v_0)'}{P(v_0)'(0)} - \frac{u_0'}{u_0'(0)} \right).
\end{aligned}$$

Therefore we have that $\|DP[v_0] - DR[u_0]\|$ is less than or equal to

$$\begin{aligned} & \|S_v(\tau(v), \cdot) - T_u(\tau(u), \cdot)\| \\ + \quad & \|S_v(\tau(v), \cdot) - T_u(\tau(u), \cdot)\| \left\| \frac{P(v_0)'}{P(v_0)'(0)} - \frac{u_0'}{u_0'(0)} \right\| \\ + \quad & \|S_v(\tau(v), \cdot) - T_u(\tau(u), \cdot)\| \left\| \frac{u_0'}{u_0'(0)} \right\| \\ + \quad & \|T_u(\tau(u), \cdot)\| \left\| \frac{P(v_0)'}{P(v_0)'(0)} - \frac{u_0'}{u_0'(0)} \right\|. \end{aligned}$$

Now, we know from corollary 3.2.8 that there is some k_1 such that

$$\|T_u(t, \cdot)\| \leq k_1$$

for all $t \in [0, t^*]$. We also know that $\|u_0'\|/\|u_0'(0)\| \leq \mu/\sigma$. Combining the above estimate with these facts and 4.3.12 yields

$$\|DP[v_0] - DR[u_0]\| \leq \|S_v(\tau(v), \cdot) - T_u(\tau(u), \cdot)\| \left[1 + \frac{K}{r^{a-1}} + \frac{\mu}{\sigma}\right] + \frac{k_1 K}{r^{a-1}}.$$

Since $a - 1 > b$, to complete our proof we need only show that

$$\|S_v(\tau(v), \cdot) - T_u(\tau(u), \cdot)\| \leq \frac{N}{r^b}$$

for some $N > 0$.

Given $y_0 \in C_0$, let us write y for the continuation of y_0 under $T_u(t, \cdot)$, and let us write z for the continuation of y_0 under $S_v(t, \cdot)$. (Of course $z_0 = y_0$.) We have the formulas (recall section 1.2)

$$z(t) = S_v(t, y_0) = \sum_i \int_0^t (F_i^r)'(v(s - d_i)) z(s - d_i)\; ds;$$

$$y(t) = T_u(t, y_0) = \sum_i \int_0^t H_i'(u(s - d_i)) y(s - d_i)\; ds.$$

We know that

$$|\tau(u) - \tau(v)| < \frac{K}{\sigma r^a}.$$

Since $|y(t)| \leq k_1\|y_0\|$, the above formulas show that $|y'(t)|$ is bounded above by

$Dr\triangle k_1\|y_0\|$. Making the kind of estimate described in (1.2.3) yields

$$\begin{aligned}
& \|S_v(\tau(v),y_0)-T_u(\tau(u),y_0)\| \\
\leq\ & \|S_v(\tau(u),y_0)-T_u(\tau(u),y_0)\| + Dr\triangle k_1\|y_0\|\frac{K}{\sigma r^a} \\
\leq\ & \|S_v(\tau(u),y_0)-T_u(\tau(u),y_0)\| + D\triangle k_1\|y_0\|\frac{K}{\sigma r^{a-1}}.
\end{aligned}$$

Therefore to complete the proof it is enough to show that

$$\|S_v(t,\cdot)-T_u(t,\cdot)\| \leq \frac{N}{r^b}$$

for some $N > 0$ and all $t \in [0,t^*]$. Otherwise put, it is enough to show that, for all $t \in [0,t^*]$,

$$|z(t)-y(t)| \leq \frac{N\|y_0\|}{r^b}.$$

Now, let $[t_0,t_1] \subset [0,t^*]$ be any interval of length at most d_1. We have

$$\begin{aligned}
\|z_{t_1}-y_{t_1}\| \leq\ & \|(z_{t_0}-y_{t_0})\| + \sup_{t\in[t_0,t_1]}\left|\sum_i \int_{t_0}^{t} (F_i^r)'(v(s-d_i))z(s-d_i) - H_i'(u(s-d_i))y(s-d_i)\,ds\right| \\
\leq\ & \|z_{t_0}-y_{t_0}\| + \sum_i \left[\int_{t_0-d_i}^{t_1-d_i} |(F_i^r)'(v(s))(z(s)-y(s))|\,ds\right] \\
& + \sum_i \left[\int_{t_0-d_i}^{t_1-d_i} |(F_i^r)'(v(s)) - H_i'(u(s))||y(s)|\,ds\right] \\
\leq\ & \|z_{t_0}-y_{t_0}\| + \sum_i \left[\|z_{t_0}-y_{t_0}\| \int_{\max(0,t_0-d_i)}^{\max(0,t_1-d_i)} |(F_i^r)'(v(s))|\,ds\right] \\
& + \|y_{t_0}\| \sum_i \left[\int_{t_0-d_i}^{t_1-d_i} |(F_i^r)'(v(s)) - H_i'(u(s))|\,ds\right].
\end{aligned}$$

Appealing to lemmas 4.3.15 and 4.3.16, we obtain

$$\|z_{t_1}-y_{t_1}\| \leq \|z_{t_0}-y_{t_0}\|\left[1+D\bar{K}\right] + \|y_{t_0}\|\frac{N_1}{r^b}.$$

Now choose a finite number of points

$$0 = t_0 < t_1 < \cdots < t_J = t^*,$$

with $t_i - t_{i-1} < d_1$ for all $1 \leq i \leq J$. Since $\|z_0 - y_0\| = 0$, the above estimates give

$$\begin{aligned}
\|z_{t_1} - y_{t_1}\| &\leq \|y_0\| \frac{N_1}{r^b}; \\
\|z_{t_2} - y_{t_2}\| &\leq \left[1 + D\bar{K}\right] \|z_{t_1} - y_{t_1}\| + \frac{N_1}{r^b} \|y_{t_1}\| \\
&\leq \left[1 + D\bar{K}\right] \frac{N_1}{r^b} \|y_0\| + \frac{k_1 N_1}{r^b} \|y_0\| \\
&\leq \left[1 + D\bar{K} + k_1\right] \frac{N_1}{r^b} \|y_0\|; \\
\|z_{t_3} - y_{t_3}\| &\leq \left[1 + D\bar{K}\right] \|z_{t_2} - y_{t_2}\| + \frac{N_1}{r^b} \|y_{t_2}\| \\
&\leq \left[1 + D\bar{K} + 2k_1\right]^2 \frac{N_1}{r^b} \|y_0\|;
\end{aligned}$$

and so on.

Finitely many such steps prove the lemma, and we in fact see that we can take

$$N = N_1 \left[1 + D\bar{K} + Jk_1\right]^J. \quad \square$$

4.4 Some examples

In this section we give some examples motivated by our work in this chapter and the last.

Very slowly oscillating periodic solutions

We revisit example 1.1.3. Consider the parameterized family of equations

$$\begin{aligned}
&x'(t) = F^r(x(t-1)) + F^r(x(t-5)), \ t > 0; \quad x_0 \in C \qquad (4.4.1) \\
&r \geq 1, \ F^r(x) = F(rx)
\end{aligned}$$

where

$$F(x) = \frac{1 - e^x}{e^x + \frac{1}{2}}.$$

The corresponding $(SDDE)$ is

$$y'(t) = h(y(t-1)) + h(y(t-5)), \qquad (4.4.2)$$

where

$$h(y) = \begin{cases} 2, & y < 0; \\ 0, & y = 0; \\ -1, & y > 0. \end{cases}$$

Direct computation shows that, if we take $y_0 \in C_0'$ with $y_0 \geq 0$, its continuation y as a solution of (4.4.2) will have zeros (all proper) at the following points:

$$t = 3;\ t = 4.5;\ t = 5.625;\ t = 12.375.$$

Since $12.375 - 5.625 > 5 = \gamma$, $y_{12.375}$ is nonnegative and lies in C_0' — and so has the same continuation as y_0. Thus y agrees with a periodic solution p for $t \geq 0$. We readily see that p is simple (no zero is a change point, and all the zeros are proper). Since any initial condition for $(SDDE)$ that is entirely of one sign can only continue in one way, we see that the derivative of ϕ^4 at $V(p_z)$, where z is any zero of $p(z)$, is zero. Therefore p is, trivially, both nondegenerate and asymptotically stable (recall remark 3.1.15). Theorems 4.1.2 and 4.1.3 apply; we conclude that, for r sufficiently large, (4.4.1) has an asymptotically stable periodic solution q that is similar to p.

Theorem 4.1.2 tells us that (a translate of) q satisfies $q(0) = 0$ and $q_z = q_0$, where z is the fourth positive proper zero of q. Therefore q is either 1- or 2-cyclic. Since, for r large, the zeros of q must be spaced similarly as the zeros of p, q must in fact be 2-cyclic (recall definition 1.2.5). Note the contrast with the one-delay case, where equations with strictly monotonic negative feedback have only one-cyclic periodic solutions (recall section 1.3).

Definition 4.4.3. We call a periodic solution of $(SDDE)$ that ever has two consecutive proper zeros separated by γ or more units *very slowly oscillating.*

We have adopted this label so as to reserve the term "slowly oscillating" for a concept, which we will explore next chapter, that is both more useful and bears stronger affinity with the "slowly oscillating" solutions studied in the one-delay case.

For the reasons described above, any simple (nonconstant) very slowly oscillating periodic solution of an equation $(SDDE)$ is both nondegenerate and asymptotically stable.

Given a particular problem ($SDDE$), finding a very slowly oscillating periodic solution (or ruling one out) is often quite easy; here is an example. Consider the equation

$$y'(t) = \operatorname{sgn}(y(t-1)) - 1.5 \operatorname{sgn}(y(t-3)) - 0.6 \operatorname{sgn}(y(t-5)). \tag{4.4.4}$$

Take $y_0 \in C_0'$ with $y_0 \leq 0$; write y for the continuation of y_0 as a solution of (4.4.4) and z for the first positive zero of y. If we can show that $z > 5$, then z is clearly a proper zero and not a change point. Since the feedback functions are odd, if $z > 5$ then $y(t)$ will satisfy the symmetry $y(z+t) = -y(t)$ for all $t \geq 0$, and y will coincide with a simple 1-cyclic very slowly oscillating periodic solution for $t \geq 0$.

For $t \in (0,1)$, $y'(t) = 1.5 + .6 - 1 = 1.1$. For $t \in (1,3)$, $y'(t) = 1 + 1.5 + .6 = 3.1$. For $t \in (3,5)$, $y'(t) = 1 - 1.5 + .6 = 0.1$. Therefore we see that y is increasing on $(0,5)$, and so $z > 5$. Thus (4.4.4) has a very slowly oscillating periodic solution. This solution is (up to time translation) the only very slowly oscillating periodic solution of (4.4.4), and it is simple and 1-cyclic.

It is of course a simple matter to concoct continuous problems for which (4.4.4) is informative.

Consider now the equation

$$y'(t) = \operatorname{sgn}(y(t-1)) - \operatorname{sgn}(y(t-3)). \tag{4.4.5}$$

The idea here, loosely speaking, is that there is a positive feedback effect with time delay 1 that is undone two periods later. This equation is an imitation of equation (1.1.2). This equation has no very slowly oscillating periodic solutions: for if $y_0 \in C_0'$ is nonpositive, we see that $y'(t) = 0$ for $t \in (0,1)$, that $y'(t) = 1$ for $t \in (1,2)$, that $y'(t) = 2$ for $t \in (2,3)$, that $y'(t) = 1$ for $t \in (3,4)$, and that $y'(t) = 0$ for $t > 4$.

Suppose that we change the equation so that the feedback with time delay 1 is negative instead of positive:

$$y'(t) = -\operatorname{sgn}(y(t-1)) + \operatorname{sgn}(y(t-3)). \tag{4.4.6}$$

then the continuation of $y_0 \in C_0'$ with $y_0 \leq 0$ does coincide with a (non-simple) very slowly oscillating periodic solution for positive time: for $y'(t) = 0$ on $(0,1)$, $y'(t) = -1$ on $(1,2)$, $y'(t) = 0$ on $(2,3)$, and $y'(t) = 1$ on $(3,4)$.

Degeneracy, and a continuum of periodic solutions

The above calculations show that (4.4.5) and (4.4.6) are certainly not similar. Here is another observation in the same vein. We claim that (4.4.5) has no simple periodic solutions p with the property that

$$V(p_z) \in \mathcal{S}_-^1 \cup \mathcal{S}_+^1$$

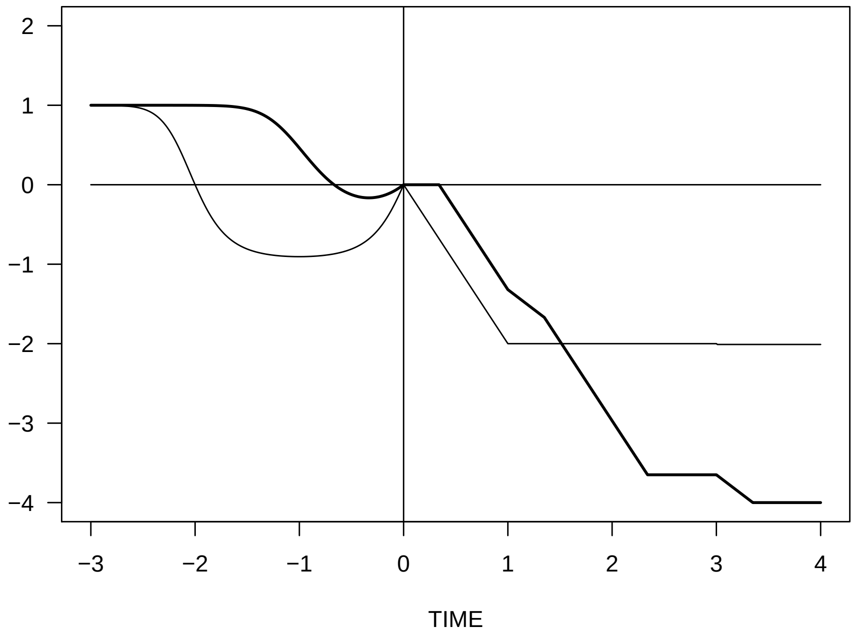
for some zero z of p. For choose $y_0 \in C_0'$ with $V(y_0) = x \in \mathcal{S}_+^1$ and write y for the continuation of y_0 as a solution of (4.4.5). There are two cases to consider.

Case 1: $x < 1$. Then $y'(t) = 0$ for $t \in (0, 1-x)$, $y'(t) = -2$ for $t \in (1-x, 1)$, $y'(t) = -1$ for $t \in (1, 2-x)$, $y'(t) = -2$ for $t \in (2-x, 3-x)$, $y'(t) = 0$ for $x \in (3-x, 3)$, $y'(t) = -1$ for $t \in (3, 4-x)$, and $y'(t) = 0$ for $t > 4-x$.

Case 2: $x \geq 1$. In this case, $y'(t) = -2$ for $t \in (0, 3-x)$ and $y'(t) = 0$ for $t > 3-x$.

See Figure 16. The thicker line is a plot of a solution as described in case 1; the thinner line is a plot of a solution as described in case 2.

Figure 16



On the other hand, (4.4.6) has a continuum of simple periodic solutions passing through $\mathcal{S}^1_{\pm}$. For choose y_0 as just above, but now let y be the continuation of y_0 as a solution of (4.4.6). We consider three cases.

Case 1: $x < 1$. Then $y'(t) = 0$ for $t \in (0, 1-x)$, $y'(t) = 2$ for $t \in (1-x, 1)$, $y'(t) = 1$ for $t \in (1, 2-x)$, $y'(t) = 0$ for $t \in (2-x, 3-x)$, $y'(t) = -2$ for $t \in (3-x, 3)$, and $y'(t) = -1$ for $t \in (3, 4-x)$. We have

$$y(4-x) = 2x + (1-x) - 2x - (1-x) = 0;$$

and for $t > 4-x$, y will coincide with the negative reflection of the very slowly oscillating periodic solution of (4.4.6) described earlier.

Case 2a: $x \in [1, 2)$. In this case $y'(t) = 2$ for $t \in (0, 1)$, $y'(t) = 0$ for $t \in (1, 3-x)$, and $y'(t) = -2$ for $t \in (3-x, 4-x)$. $y(4-x) = 0$, and

$$\phi(x) = 4 - x \in \mathcal{S}^1_{-}.$$

Case 2b: $x \in [2, 3)$. In this case $y'(t) = 2$ for $t \in (0, 3-x)$, then $y'(t) = 0$ for $t \in (3-x, 1)$, and $y'(t) = -2$ for $t \in (1, 4-x)$. $y(4-x) = 0$, and

$$\phi(x) = 4 - x \in \mathcal{S}^1_{-}.$$

Appealing to the oddness of the feedback functions, we see the the orbit of any $x \in [1,3) \subset \mathcal{S}^1_+$ under ϕ is

$$4 - x \in \mathcal{S}^1_-,\ x \in \mathcal{S}^1_+,\ 4 - x \in \mathcal{S}^1_-,\ x \in \mathcal{S}^1_+, \ldots.$$

Figure 17

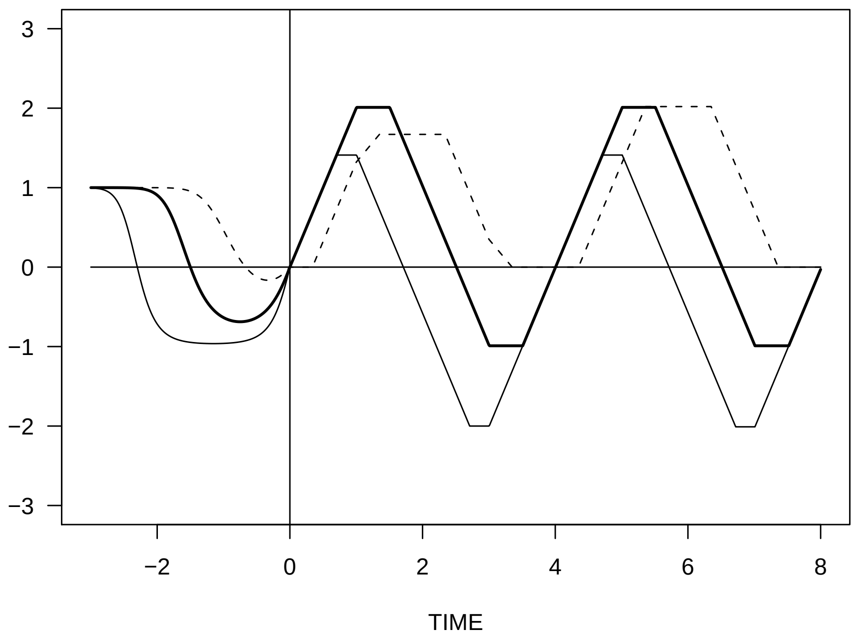


Figure 17 shows example solutions. The dashed line is a plot of a solution as described in case 1, the thick line as in case 2a, the thin line as in case 2b. These latter two periodic solutions are simple, but are degenerate: for ϕ^2 is the identity map on $[1,3) \subset \mathcal{S}^1_+$.

The point we want to make is that the nondegeneracy hypothesis is indeed crucial to our main existence theorems. For if we now consider a continuous parameterized problem related to (4.4.6), depending on how we choose our feedback functions we can exhibit either nonexistence of periodic solutions, or existence of a continuum of periodic solutions.

Let us first consider the steplike problem

$$x'(t) = f(rx(t-1)) - f(rx(t-3)) \tag{4.4.7}$$

where $f(x) = -\text{sgn}(x)$ when $|x| \geq 1$. Suppose that $p_0 \in C_0$ is a section of a simple periodic solution of (4.4.6) with

$$V(p_0) \in (1, 3) \subset \mathcal{S}^1_+.$$

Take $y_0 \in C_0$ close to p_0 with continuation y as a solution of (4.4.7). Let us write R for the map that advances solutions of (4.4.7) by two proper zeros. Since p is simple, the work we did in section 3.1 shows that, for r large enough, $R(y_0)$ lies in an analog of the space $\tilde{W}$. Given y_0 in such a space, though, direct computation (using the formula discovered in the proof of lemma 3.1.16) shows that, near the first zero of p, $y(t)$ and $p(t)$ differ by

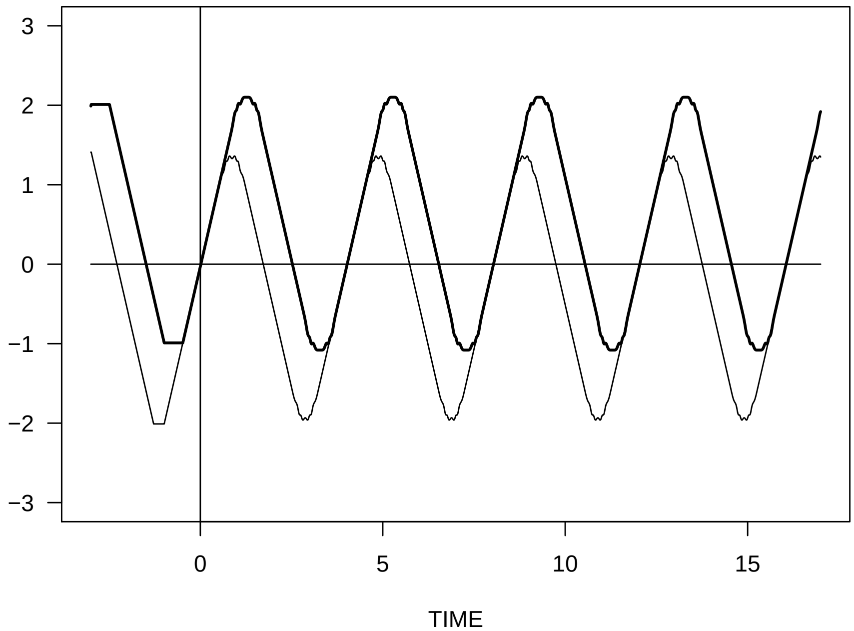
$$\int_{-1/r}^{0} f(x) - 1 \; dx + \int_{0}^{1/r} f(x) + 1 \; dx - \int_{-1/r}^{0} f(x) - 1 \; dx - \int_{0}^{1/r} f(x) + 1 \; dx = 0.$$

A similar calculation for the next zero shows that

$$V(R(y_0)) = V(y_0).$$

Therefore we find that (4.4.7) has, for large r, a continuum of periodic solutions close to the solutions of (4.4.6) described above. We illustrate two simulated periodic solutions of (4.4.7) in Figure 18 (using the f pictured in Figure 9 and $r = 2.5$). The initial conditions are segments of the simple periodic solutions of (4.4.6) illustrated in Figure 17.

Figure 18



On the other hand, if we consider an equation of the form

$$x'(t) = f(rx(t-1)) - g(rx(t-3)) \tag{4.4.8}$$

where $f(x) = g(x) = -\mathrm{sgn}(x)$ for $|x| \geq 1$ but f and g are not equal, we no longer expect this to be true. Rather, if y_0 lies in the analog of the space $\tilde{W}$ we expect to have

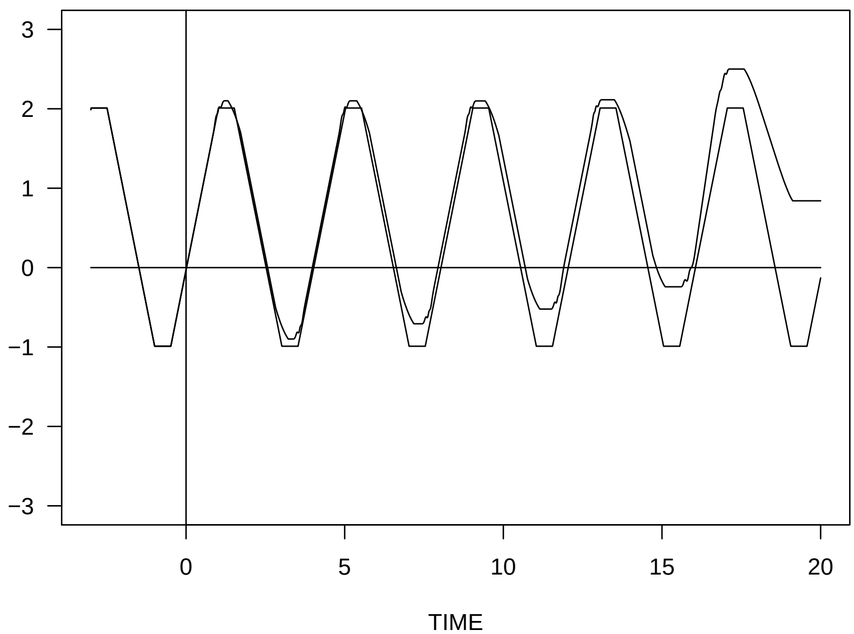
$$V(R(y_0)) = V(y_0) + K,$$

where K is a nonzero constant. This formula will hold as long as $R^k(y_0)$ is close to p_0, and so eventually $R^k(y_0)$ must not be close to p_0. Therefore we see that, in this case, (4.4.8) has *no* periodic solutions close to the solutions p of (4.4.6) that we described above. Figure 19 illustrates this. We have kept the same f that we used above, taken

$$g(x) = \begin{cases} 1, & x \leq -1 \\ -x, & x \in [-1,1] \\ -1, & x \geq 1, \end{cases}$$

and $r = 2.5$. Figure 19 takes a segment of one of the periodic solutions of (4.4.6) as an initial condition and continues it both as a solution of (4.4.6) and as a solution of (4.4.8).

Figure 19



Multiple stable periodic solutions

We close this section by fleshing out example 1.1.4. Consider the function

$$G(x) = \frac{1-e^x}{1+e^x}$$

and the equation

$$x'(t) = 2G^r(x(t-1)) + G^r(x(t-3)); \quad r \geq 1, \quad G^r(x) = G(rx). \tag{4.4.9}$$

The corresponding $(SDDE)$ is

$$y'(t) = -2\text{sgn}(y(t-1)) - \text{sgn}(y(t-3)). \tag{4.4.10}$$

(4.4.10) has a periodic solution p^1 with a downward proper zero at 0 whose zeros are separated by $10/3$. This periodic solution is simple and very slowly oscillating, so it is trivially nondegenerate and asymptotically stable.

There is also a periodic solution p^2 with a downward proper zero at 0 whose zeros are separated by 2. This periodic solution is simple. To check nondegeneracy, we need to compute $D\phi^2[V(p_0^2)] = D\phi^2[2 \in \mathcal{S}_-^1]$. In this particular case, we find

$$\frac{\partial z(2 \in \mathcal{S}_-^1)}{\partial x} = \frac{\partial \phi(2 \in \mathcal{S}_+^1)}{\partial x} = 2.$$

Therefore the derivative of ϕ^2 at $2 \in \mathcal{S}_-^1$ is 4. We see that $p^2(t)$ is nondegenerate and unstable. $p^1(t)$ and $p^2(t)$ are the solutions of (4.4.10) that correspond to the two solutions of (4.4.9) shown to exist in Theorem 2.3 of [27].

There is also a periodic solution p^3 with a downward zero at 0 whose zeros are separated by 10/7. (This is the solution pictured in the figures in section 3.1.) Direct computation shows that, at

$$V(p_0^3) = (10/7, 20/7) \in \mathcal{S}_-^2,$$

we have

$$\frac{\partial z}{\partial x_1} = 0 \text{ and } \frac{\partial z}{\partial x_2} = -\frac{2}{3}.$$

Thus applying (2.3.6) we find that

$$D\phi[(10/7, 20/7) \in \mathcal{S}_-^2] = \begin{pmatrix} 0 & -2/3 \\ 1 & -2/3 \end{pmatrix} := M.$$

Since the zeros of p^3 are evenly spaced we see that

$$V(p_z) = (10/7, 20/7) \in \mathcal{S}_-^2 \cup \mathcal{S}_+^2$$

for any zero z of p^3; since the feedback functions in (4.4.10) are odd we have

$$D\phi[V(p_z^3)] = M$$

for any zero z of p^3.

Since the minimal even m such that $10m/7 > 3$ is $m = 4$, to verify that p^3 is nondegenerate we first examine

$$D\phi^4[V(p_0^3)] = M^4 = \frac{1}{81}\begin{pmatrix} 12 & -32 \\ 48 & -20 \end{pmatrix}.$$

Numerical calculation by computer shows that the spectral radius of this matrix is approximately 0.44. Thus we see that p^3 is nondegenerate and asymptotically stable (recall remark 3.1.15).

Theorem 4.1.2 therefore tells us that (4.4.9) has, for large r, three periodic solutions close to p^1, p^2, and p^3 respectively; theorem 4.1.3 tells us that the first and third are asymptotically stable. Again we see a contrast with the one-delay case: the equation

$$x'(t) = G(rx(t-1))$$

where G is as in the current example has exactly one stable nontrivial periodic solution for r large (this is a consequence of [24], which excludes such solutions whose zeros are separated by one unit or less, and [40], which gives uniqueness and stability of such solutions whose zeros are separated by more than one unit). See Figure 2 in section 1.1, which illustrates simulated solutions of (4.4.9) close to p^1 and p^3 with $r = 5$.

Chapter 5

$(SDDE)$ with negative feedback

5.1 Introduction

Motivated by chapters 3 and 4, we return to the study of $(SDDE)$. The reader might find it convenient to review some of the machinery introduced in section 2.3. We have been able to build a relatively good understanding of the following class of problems, which will be our focus for the current chapter.

Definition 5.1.1. $(SDDE)$ **with negative feedback.** We say that an $(SDDE)$

$$y'(t) = \sum_{i=1}^{D} h_i(y(t-d_i)) \tag{$SDDE$}$$

$$h_i(y) = \begin{cases} b_i \neq 0, & y < 0; \\ 0, & y = 0; \\ -a_i \neq 0, & y > 0 \end{cases}$$

has *negative feedback* if each of the feedback functions h_i satisfies the so-called *negative feedback condition*

$$b_i, a_i > 0 \iff yh_i(y) < 0 \text{ when } y \neq 0.$$

Numerical studies suggest that solutions of $(SDDE)$ with negative feedback usually converge to periodic solutions; we have not been able to prove any such far-reaching statement. In this brief section we describe some general features of the solution semi-flow for $(SDDE)$ with negative feedback. In later sections we look at some subclasses of $(SDDE)$ with negative feedback for which we have been able to develop a somewhat fuller picture of the global dynamics. We will use the functions V, z, and ϕ described in section 2.3 (see especially definitions 2.3.2 and 2.3.3). We will also make free and frequent use of the fact that, if $y_0 \in C' \subset C''$ has continuation $y(t)$, $y'(t)$ is defined except at finitely many points on $[0, M]$, for any $M > 0$ (proposition 2.2.2).

Lemma 5.1.2. Oscillation about zero. *If $(SDDE)$ has negative feedback, any solution y has a sequence of zeros approaching ∞.*

PROOF. Imagine not and suppose that z is the largest zero of y. Then $y(t)$ is all of one sign for $t \in (z, \infty)$ and so $y'(t)$ will be a nonzero constant, of sign opposite that of $y(t)$, for all $t \in (z + \gamma, \infty)$. This is impossible. $\square$

In the following proposition, and throughout this chapter, by "increasing" and "decreasing" we mean non-decreasing and non-increasing, respectively.

Proposition 5.1.3. Monotonicity in coordinates. *On each set $D_\pm(n)$, $n \geq 1$, the function z is increasing in odd coordinates and decreasing in even coordinates.*

PROOF. Take a point $x = (x_1, x_2, \ldots, x_n) \in D_-(n)$. Choose $y_0 \in V^{-1}(x)$ and write y for the continuation of y_0.

Suppose that k is odd and that we increase x_k to $x_k + \delta$; write $\tilde{x}$ for x so altered and suppose that $\tilde{x}$ is also in $D_-(n)$. Choose $\tilde{y}_0 \in V^{-1}(\tilde{x})$ and write $\tilde{y}$ for the continuation of $\tilde{y}_0$.

The effect of our alteration is to lengthen an interval where y_0 is positive at the expense of an interval where y_0 is negative, with no compensating change in y_0 elsewhere. $\tilde{y}'(t)$ and $y'(t)$ are both defined for all but finitely many $t \in (0, \min\{z(x), z(\tilde{x})\})$. For all such t, $\tilde{y}'(t)$ will be equal to $y'(t)$ except if some $t - d_i$ is between $-(x_k + \delta)$ and $-x_k$; in this case, $\tilde{y}'(t) \leq y'(t)$. This means that $\tilde{y}(t) \leq y(t)$ for all $t \in (0, \min\{z(x), z(\tilde{x})\})$; it follows that $z(\tilde{x}) \geq z(x)$. The argument for $z(x)$ decreasing in the even coordinates of x is similar; the parallel argument for $D_+(n)$ is again similar. $\square$

Here is a fact in the same spirit:

Proposition 5.1.4. *$D_-(0)$ is always nonempty, and the value of z on $D_-(0)$ is greater than or equal to*

$$\sup_{n \in \mathbf{N}} \sup\{\ z(x)\ :\ x \in D_-(n)\ \}.$$

Similarly for $D_+(0)$.

PROOF. To see that $D_-(0)$ always coincides with $\mathcal{S}^0_-$, choose $y_0 \in C'_0$ with $V(y_0) \in \mathcal{S}^0_-$. Since $y_0 \geq 0$ and y_0 has only isolated zeros, $y'(t)$ is a negative constant for all $t \in (0, d_1)$. y therefore has a downward proper zero at 0 and a minimum positive zero z. Moreover, y' will be nondecreasing (or undefined) on $(0, z + d_1)$; but since y' must assume a positive value on $(0, z)$ we see that z is an upward proper zero of y. Thus $V(y_0)$ satisfies all the conditions for $z(V(y_0))$ to be defined — that is, $V(y_0) \in D_-(0)$.

For the second assertion, take $x \in D_-(0)$ and $x' \in D_-(n)$ for any n, and write $y_0 \in V^{-1}(x)$ and $w_0 \in V^{-1}(x')$ with continuations $y(t)$ and $w(t)$ respectively. Then $y'(t) \leq w'(t)$ for all $t \in (0, \min\{z(x), z(x')\}$ where both derivatives are defined; we conclude that $z(x) \geq z(x')$. □

Proposition 5.1.5. Injectivity of ϕ. *Suppose that $x, x' \in D_-(n,j)$ and that $\phi(x) = \phi(x')$. If $j = n+1$, then $x = x'$. If $j = n$ and z is strictly increasing or strictly decreasing in x_n at x, then $x = x'$. Similarly for $D_+(n,j)$.*

PROOF. To say that $\phi(x) = \phi(x')$ is to say that

$$(z(x), x_1{+}z(x), x_2{+}z(x), \ldots, x_{j-1}{+}z(x)) = (z(x'), x'_1{+}z(x'), x'_2{+}z(x'), \ldots, x'_{j-1}{+}z(x')).$$

The first coordinate yields $z(x) = z(x')$, and the later coordinates in turn yield $x_1 = x'_1, \;\; x_2 = x'_2, \;\ldots,\; x_{j-1} = x'_{j-1}$. If $j = n+1$, we're done. Suppose now that $j = n$. If z is strictly increasing or strictly decreasing in x_n at x, then to have $z(x) = z(x')$ we must have (since x and x' agree in all but perhaps the last coordinate) that $x = x'$. □

Definition 5.1.6. Dominant delay. Suppose that $(SDDE)$ has negative feedback. We say that $(SDDE)$ has *dominant Mth delay* if

$$b_M > \sum_{i \neq M} a_i \;\text{ and }\; a_M > \sum_{i \neq M} b_i.$$

This means that the sign of $y'(t)$ is strictly opposite the sign of $y(t - d_M)$ whenever $y'(t)$ is defined and $y(t{-}d_M) \neq 0$. If $(SDDE)$ has negative feedback and dominant Mth delay, then condition (2.1.10) is satisfied, and so the space C' is forward-invariant under the solution semiflow. An $(SDDE)$ with negative feedback need not have a dominant delay, but if it does the dominant delay is obviously unique.

When $(SDDE)$ has negative feedback and dominant delay, we have a reasonable and useful notion of "oscillation speed" of solutions.

Definition 5.1.7. Oscillation speed. Suppose that $(SDDE)$ has negative feedback and dominant Mth delay. Given some solution y of $(SDDE)$ with $y_t \in C'$, we define the *oscillation speed* $S(y_t)$ of y_t, which we will also call the oscillation speed of y at t, to be the number of proper zeros of y_τ occurring in the interval $(-d_M, 0)$, where

$$\tau = \inf\{\ s \geq t\ :\ s \text{ is a proper zero of } y(\cdot)\ \}.$$

Please note that, in the above definition, we count zeros in the interval $(-d_M, 0)$, *not* in $(-\gamma, 0)$.

Proposition 5.1.8. *Suppose $(SDDE)$ has negative feedback and dominant Mth delay, and let $y_0 \in C'$ with continuation y. The oscillation speed $S(y_t)$ is even and decreasing in t.*

PROOF. Suppose that z is a positive proper zero of y. Since C' is forward-invariant, y has a least proper zero z' greater than z. It suffices to show that $S(y_z)$ is even and that $S(y_{z'}) \leq S(y_z)$. We assume that z is a downward proper zero; the other case is similar.

Since $y'(t)$ is defined at all but finitely many points on any bounded interval about z, there is some $\delta > 0$ such that $y'(t)$ is constant on $(z-\delta, z)$ and on $(z, z+\delta)$. Since z is a downward proper zero of y, we see that $y'(t)$ must in fact be negative on both of these intervals. It follows that $y(t-d_M) \geq 0$ for $t \in (z-\delta, z+\delta)$, and since C' is forward-invariant we actually have that $y(t-d_M) > 0$ for all but finitely many $t \in (z-\delta, z+\delta)$. The number of proper zeros of y on $(z-d_M, z)$ is therefore even.

As t ranges from z to z', $t - d_M$ must cross a proper zero ζ of y, because $y'(t)$ changes from negative to positive. If we write Z for the collection of proper zeros of y, then, we see that

$$Z \cap (z' - d_M, z') \subset [(Z \cap (z - d_M, z)) \cup \{z\}] \setminus \zeta.$$

The proposition follows. $\square$

Since oscillation speed is even, nonnegative, and nonincreasing, every solution y of $(SDDE)$ has a limiting oscillation speed

$$\overline{S}(y_0) = \lim_{t\to\infty} S(y_t) \leq S(y_0)$$

that is attained in finite time.

We now make the definition that accounts for the awkward term "very slowly oscillating" introduced in section 4.4.

Definition 5.1.9. Slowly oscillating solutions. Suppose that $(SDDE)$ has negative feedback and dominant Mth delay. We call a solution y *slowly oscillating* if $\overline{S}(y_0) = 0$.

In section 1.3 we discussed some of the ways in which oscillation speed has proven an important organizing principle in the study of differential delay equations with continuous negative feedback and one delay. The rest of this chapter is devoted to showing that many properties of solutions of $(SDDE)$ that hold in the single-delay negative feedback case (and have, at least, analogs when the feedback is continuous) hold in the case of $(SDDE)$ with negative feedback and a dominant delay. Not surprisingly, we can make the strongest statements when the dominant delay is the longest delay; we turn to this case first.

5.2 $(SDDE)$ with negative feedback and dominant longest delay

Suppose that $(SDDE)$ has negative feedback and dominant longest delay. In this case the oscillation speed $S(y_t)$ of a solution y at t is given by the dimension of $V(y_\tau)$, where

$$\tau = \inf\{\ s \geq t\ :\ s \text{ is a proper zero of } y\ \}.$$

Existence of a unique slowly oscillating periodic solution is clear: for choose $y_0 \in C_0'$ with $y_0 \geq 0$ and continuation y. If z is the first positive zero of y, then z is proper and greater than γ; we therefore have $V(y_z) \in \mathcal{S}_+^0$. Similarly,

$$\phi(V(y_z)) = V(y_0) \in \mathcal{S}_-^0.$$

Thus there is a particular periodic solution p such that $y(t) = p(t)$ for all $t \geq 0$. This solution is 1-cyclic and attracts (indeed, eventually coincides with) every solution y with $\overline{S}(y_0) = 0$.

Given some nonnegative integer K, we now ask what we can say about the set of initial conditions $y_0 \in D_-(2K)$ for which $\overline{S}(y_0) = 2K$ — that is, the set of initial conditions whose continuations have oscillation speed $2K$ for all positive time. The main results for the current section are theorems 5.2.1 and 5.2.2, and example 5.2.3.

Theorem 5.2.1. Oscillation speed usually decreases. *Suppose that* $(SDDE)$ *has negative feedback and dominant longest delay, and that* $K \in \mathbf{N}$*. The set*

$$\{\ x \in D_-(2K)\ :\ \overline{S}(y_0) = 2K,\ y_0 \in V^{-1}(x)\ \}$$

has measure zero in $\mathcal{S}_-^{2K}$*.*

Theorem 5.2.2. Unique 1-cyclic periodic solutions at each oscillation speed. *Suppose that* $(SDDE)$ *has negative feedback and dominant longest delay, and that* K *is a nonnegative integer.* ϕ^2 *has exactly one fixed point on* $D_-(2K)$*.*

Example 5.2.3. Other rapidly oscillating solutions are possible. Suppose that $(SDDE)$ has negative feedback and dominant longest delay, and that $K \in \mathbf{N}$. It is possible that there is a point $x \in D_-(2K)$ such that $\phi^{2n}(x) \in D_-(2K)$ for all $n \in \mathbf{N}$ but $\phi^{2n}(x)$ is never the unique fixed point of ϕ^2.

Consider, for example, the parameterized family of problems

$$y'(t) = -a\mathrm{sgn}(y(t-5/9)) - \mathrm{sgn}(y(t-1)), \quad a \in [0,1). \tag{$SDDE_a$}$$

The 1-cyclic periodic solution with oscillation speed four has zeros spaced 2/9 units apart. To see this, let us take

$$x = (2/9, 4/9, 6/9, 8/9) \in \mathcal{S}_-^4$$

and compute $z(x)$. Given $y_0 \in V^{-1}(x)$ with continuation y as a solution of $(SDDE_a)$, we find that $y'(t) = -1 - a$ for $t \in (0, 1/9)$ and that $y'(t) = 1 + a$ for $t \in (1/9, 3/9)$.

Thus $z(x) = 2/9$ and

$$\phi(x) = (2/9, 4/9, 6/9, 8/9) \in \mathcal{S}^4_+.$$

Since the feedback functions in $(SDDE_a)$ are odd, we see that $z(\phi(x)) = 2/9$ also, and hence that $\phi^2(x) = x$.

Observe that x is simple; therefore ϕ is affine near x. Making the computation described in the proof of 2.3.13 we find that

$$\frac{\partial z(x)}{\partial x_1} = 0; \quad \frac{\partial z(x)}{\partial x_2} = \frac{-2a}{1+a}; \quad \frac{\partial z(x)}{\partial x_3} = 0; \quad \frac{\partial z(x)}{\partial x_4} = \frac{-2}{1+a}.$$

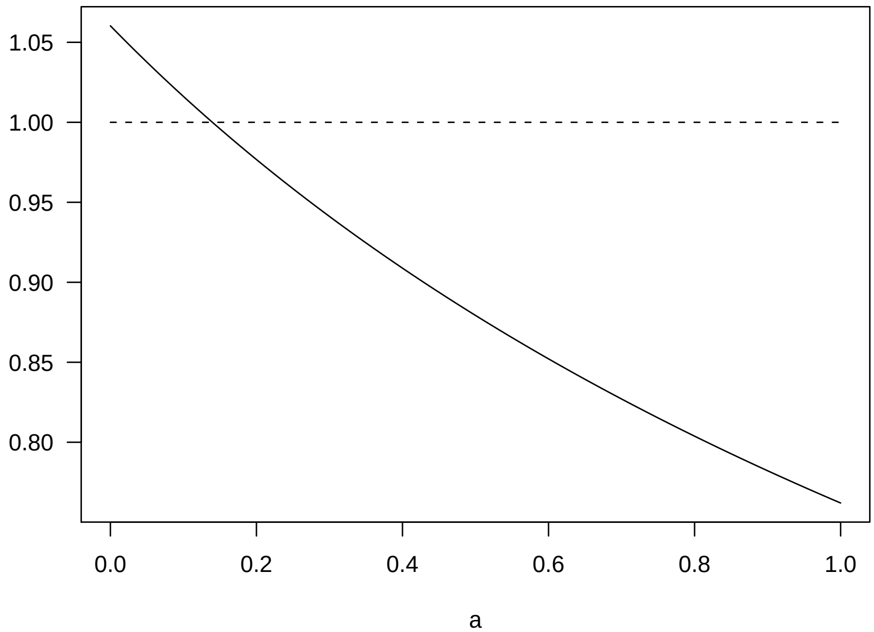
Thus the Jacobian of ϕ at x is

$$M_a = \begin{pmatrix} 0 & \frac{-2a}{1+a} & 0 & \frac{-2}{1+a} \\ 1 & \frac{-2a}{1+a} & 0 & \frac{-2}{1+a} \\ 0 & \frac{-2a}{1+a}+1 & 0 & \frac{-2}{1+a} \\ 0 & \frac{-2a}{1+a} & 1 & \frac{-2}{1+a} \end{pmatrix},$$

and the Jacobian of ϕ^2 at x is M_a^2.

Figure 20 shows a plot of the approximate smallest absolute value of the eigenvalues of M_a as a ranges from 0 to 1. The figure shows that there is some value of a such that M_a has spectrum on the unit circle. Suppose that v is a vector in the corresponding generalized eigenspace. Then, if v is small enough in norm, $\phi^{2n}(x+v) \in \mathcal{S}^4_-$ for all nonnegative integers n. $\phi^{2n}(x+v)$ will never be equal to x, though, and so by theorem 5.2.2 will never equal the unique fixed point of ϕ^2. $\square$

Figure 20



Remark 5.2.4. Theorems 5.2.1 and 5.2.2 both hold in the one-delay case, of course; but a stronger version of theorem 5.2.1 holds in the one-delay case because there can be no solutions like those described in example 5.2.3. In the one-delay case, if $V(y_0) \in D_-(2K, 2K)$ is not equal to the unique periodic point of ϕ^2 in $D_-(2K, 2K)$, then $\overline{S}(y_0)$ is less than $2K$ (see [8]).

When we move to the case where the dominant delay is not the longest delay we will see that, while an analog of theorem 5.2.2 "mostly" holds, multiple slowly oscillating periodic solutions can occur.

For the rest of this section we will assume that $(SDDE)$ has negative feedback and dominant longest delay. We first describe some general features of the sets $D_\pm(2K, 2K)$, and of the maps z and ϕ restricted to those sets. We will then prove theorems 5.2.1 and 5.2.2 in turn.

Write

$$\mu = \sum_{i=1}^{D} a_i + b_i,$$

and write σ for the smallest nonzero value obtainable by $|y'(t)|$ for $t > 0$, where y is any solution of $(SDDE)$. We will adhere to this notation for the rest of the section.

Choose $x \in \mathcal{S}_-^{2K}$ and $y_0 \in V^{-1}(x)$ with continuation y. y will strictly decrease on $(0, \gamma - x_{2K})$ and so has a proper downard zero at 0. We claim that y has a first positive proper zero, and that it is upward. For since C' is forward-invariant, all positive zeros of y are isolated and so the first positive proper zero of y will be upward. If there is no such zero on the interval $[0, \gamma]$, then for $t > \gamma$ the derivative $y'(t) \geq \sigma$ will be a positive constant until d_1 time units after an upward proper zero occurs. Thus we see that $D_-(2K)$ and $\mathcal{S}_-^{2K}$ coincide.

To say that $x = (x_1, \ldots, x_{2K}) \in D_-(2K, 2K)$ is just to say that

$$z(x) \in [\gamma - x_{2K}, \gamma - x_{2K-1}),$$

and in fact we know more: if $x \in D_-(2K, 2K)$, y strictly decreases on $(0, \gamma - x_{2K})$ and strictly increases on $(\gamma - x_{2K}, \gamma - x_{2K-1})$; and this latter interval contains $z(x)$. When y is decreasing, its derivative is between $-\mu$ and $-\sigma$ everywhere it is defined; when y is increasing, its derivative is between σ and μ everywhere it is defined. Therefore, for $x \in D_-(2K, 2K)$, we have the bounds

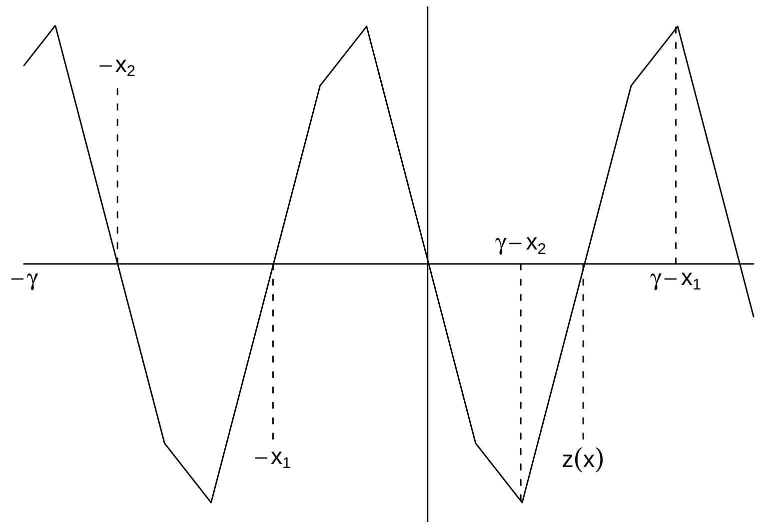
$$z(x) \in \left[\left(1 + \frac{\sigma}{\mu}\right)(\gamma - x_{2K}), \left(1 + \frac{\mu}{\sigma}\right)(\gamma - x_{2K}) \right]. \tag{5.2.5}$$

Similarly, if $x \in D_-(2K)$ with

$$\gamma - x_{2K-1} > \left(1 + \frac{\mu}{\sigma}\right)(\gamma - x_{2K}), \tag{5.2.6}$$

then x in fact lies in $D_-(2K, 2K)$. See Figure 21, which illustrates the $K = 1$ case.

Figure 21



Proposition 5.2.7. Properties of $D_{\pm}(2K, 2K)$, z, and ϕ. *Suppose that (SDDE) has negative feedback and dominant longest delay. Given $K \in \mathbf{N}$, the following hold.*

- *$D_{-}(2K, 2K)$ is an open and path-connected subset of $\mathcal{S}_{-}^{2K}$;*
- *z and ϕ are continuous on $D_{-}(2K, 2K)$;*
- *ϕ is injective on $D_{-}(2K, 2K)$.*

Similarly for $D_{+}(2K, 2K)$.

PROOF. We give the proof for $D_{-}(2K, 2K)$; the proof for $D_{+}(2K, 2K)$ is essentially the same.

Take $u = (u_1, \ldots, u_{2K}) \in D_{-}(2K, 2K)$. Let $\epsilon > 0$ be given, and choose $\rho > 0$ such that

$$\rho < \min\{\ \gamma - u_{2K},\ z(u) - (\gamma - u_{2K}),\ (\gamma - u_{2K-1}) - z(u), d_1, \epsilon\ \}.$$

Choose $y_0 \in V^{-1}(u)$ with continuation y. $y'(t) \leq -\sigma$ for all but finitely many $t \in (0, \rho)$, $y'(t) \geq \sigma$ for all but finitely many $t \in (z(u) - \rho, z(u) + \rho)$, and $y(t) \leq -\rho\sigma$ for $t \in [\rho, z(u) - \rho]$.

Suppose now that $\delta > 0$ and that $v = (v_1, \ldots, v_{2K}) \in \mathcal{S}_-^{2K}$ with $|v - u| < \delta$. We will show that, for δ small enough, $v \in D_-(2K, 2K)$ and $|z(v) - z(u)| < \epsilon$. Choose $w_0 \in V^{-1}(v)$ with continuation w. For $t \in (0, \min(z(u), z(v)) + d_1)$ we have

$$\begin{aligned} |y(t) - w(t)| &\leq \sum_i \int_0^{\min(z(u),z(v))+d_1} |h_i(y(s - d_i)) - h_i(w(s - d_i))|\, ds \\ &\leq \sum_i (2K)\delta|a_i + b_i| \leq (2K)\mu\delta. \end{aligned}$$

Now choose δ small enough that $2K\mu\delta < \rho\sigma$. Then $z(v)$ lies in the interval $(z(u) - \rho, z(u) + \rho)$, and we see that z is continuous at u. We can also choose δ small enough that $z(u) + \rho + \delta < \gamma - u_{2K-1}$, which implies that $z(v) < \gamma - v_{2K-1}$ and hence that $v \in D_-(2K, 2K)$. Thus $D_-(2K, 2K)$ is open.

Now take $v, u \in D_-(2K)$ with $v_k = u_k$ for $1 \leq k \leq 2K - 1$ and $v_{2K} > u_{2K}$. We know that $z(v) \leq z(u)$ by proposition 5.1.3; we claim that this inequality is strict. Again choose $y_0 \in V^{-1}(u)$ and $w_0 \in V^{-1}(v)$ with continuations y and w, respectively. For $t \in (0, \min(z(u), z(v)) + d_1)$ such that $w'(t)$ and $y'(t)$ are defined, we have that $w'(t) = y'(t)$ unless $t - d_i \in (-v_{2K}, -u_{2K})$ for some i, in which case $w'(t) > y'(t)$ by the negative feedback hypothesis. Since $z(v) > \gamma - v_{2K}$, we know that $w'(t) > y'(t)$ for at least some time interval in $(0, \min(z(u), z(v))$; the claim follows. By proposition 5.1.5, $\phi(x)$ is injective on $D_-(2K, 2K)$.

Finally, we prove that $D_-(2K, 2K)$ is path-connected. We need two preliminary observations. The first is that, given any $x = (x_1, x_2, \ldots, x_{2K}) \in D_-(2K, 2K)$, the line segment

$$\{(x_1, \ldots, x_{2K-1}, s) \ : \ s \in [x_{2K}, \gamma)\}$$

is contained in $D_-(2K, 2K)$, and

$$z(x_1, \ldots, x_{2K-1}, s) \to 0 \ \text{ as } \ s \to \gamma.$$

The second is that, given any

$$x = (x_1, \ldots, x_{2K-1}) \in \mathcal{S}_-^{2K-1},$$

(5.2.6) implies that for s sufficiently close to γ we have

$$x_s = (x_1, \ldots, x_{2K-1}, s) \in D_-(2K, 2K).$$

Now, choose any two points $u, v \in D_-(2K, 2K)$. Write

$$\pi(u) = (u_1, \ldots, u_{2K-1}) \in \mathcal{S}_-^{2K-1}; \quad \pi(v) = (v_1, \ldots, v_{2K-1}) \in \mathcal{S}_-^{2K-1}.$$

Let $\Gamma \subset \mathcal{S}^{2K-1}$ be the straight line segment between $\pi(u)$ and $\pi(v)$ ($\mathcal{S}^{2K-1}$ is convex, so Γ is contained in $\mathcal{S}_-^{2K-1}$). For every $n \in \mathbf{N}$, put

$$U_n = \left\{ \begin{array}{l} (x_1, \ldots, x_{2K-1}) \in \Gamma \text{ such that} \\ (x_1, \ldots, x_{2K-1}, \gamma - 1/n) \in D_-(2K, 2K) \end{array} \right\}.$$

Since z is continuous on $D_-(2K, 2K)$, each U_n is open in Γ, and the sets U_n are nested and cover Γ. Since Γ is compact there is therefore some $N \in \mathbf{N}$ with $\Gamma = U_N$. We can also take $\gamma - 1/N > \max(u_{2K}, v_{2K})$. Then the union of the three line segments

$$\{(u_1, \ldots, u_{2K-1}, s) \ : \ s \in [u_{2K}, \gamma - 1/N] \ \}$$
$$\{(v_1, \ldots, v_{2K-1}, s) \ : \ s \in [v_{2K}, \gamma - 1/N] \ \}$$
$$\{(x_1, \ldots, x_{2K-1}, \gamma - 1/N) \ : \ (x_1, \ldots, x_{2K-1}) \in \Gamma \ \}$$

lies in $D_-(2K, 2K)$ and constitutes a path in $D_-(2K, 2K)$ from u to v. $\square$

In the proof, we discovered

Corollary 5.2.8. *Suppose that (SDDE) has negative feedback and dominant longest delay. Then, on $D_-(2K)$, z is strictly decreasing in its last coordinate.* $\square$

The following lemma is essentially about the partial derivatives of z on $D_-(2K)$, but since z is not differentiable everywhere we need to formulate the statement carefully. We shall use the first statement immediately, and the second statement later. We shall be considering the change index pairs of points in $D_-(2K)$; the reader might find it useful to review definition 2.3.10.

Lemma 5.2.9. *Suppose that (SDDE) has negative feedback and dominant longest delay. There is some $\beta > 1$ such that the following holds. Given $K \in \mathbf{N}$, if $x \in D_-(2K)$ is simple then*

$$\frac{\partial z(x)}{\partial x_{2K}} \leq -\beta.$$

If x and x' are points in $D_-(2K, 2K)$ (simple or not) with

$$x = (x_1, \ldots, x_{2K}) \text{ and } x' = (x_1 + \delta, \ldots, x_{2K} + \delta),$$

then if $\delta > 0$ *is sufficiently small we have*

$$z(x') < z(x).$$

Similarly for $D_+(2K)$ *and* $D_+(2K, 2K)$.

PROOF. We will prove the lemma for $D_-(2K)$ and $D_-(2K, 2K)$; the proof of the other case is the same.

We consider the first statement. Suppose that x is simple and take $y_0 \in V^{-1}(x)$ with continuation y. Choose $\delta > 0$ and write

$$x' = (x_1, \ldots, x_{2K} + \delta).$$

Observe that $(2K, D)$ is a change index pair for x (for $t - \gamma = -x_{2K}$ at time $\gamma - x_{2K} < z(x)$). The computation in the proof of theorem 2.3.13 shows that, for $\delta > 0$ sufficiently small,

$$z(x') - z(x) = \sum_{i\,:\,(2K,i) \text{ is a change index pair of } x} \frac{-\delta(a_i + b_i)}{y'(z(x))} \le \frac{-\delta(a_D + b_D)}{y'(z(x))}.$$

The slope $y'(z(x))$, being positive, is of the form

$$y'(z(x)) = b_D + \sum_{i=1}^{D-1} s_i,$$

where each s_i is either $-a_i$ or b_i. All the positive terms in the latter sum total less than a_D, and all the negative terms total more than $-b_D$ (this is the definition of dominant longest delay). Therefore we have

$$\frac{a_D + b_D}{y'(z(x))} = \frac{a_D + b_D}{b_D + \sum_{i=1}^{D-1} s_i} \ge \frac{a_D + b_D}{b_D + \sum_{i=1}^{D-1} b_i} \ge \beta,$$

where $\beta > 1$ is some constant. It follows that $z(x') \le z(x) - \beta\delta$, and the first part of the lemma is proved.

We now turn to the second part of the lemma. Choose $x, x' \in D_-(2K, 2K)$ with

$$x = (x_1, \ldots, x_{2K}) \text{ and } x' = (x_1 + \delta, \ldots, x_{2K} + \delta), \quad \delta > 0.$$

Take $y_0 \in V^{-1}(x)$ and $w_0 \in V^{-1}(x')$ with continuations y and w respectively.

Imagine that $z(x') \geq z(x)$. We will show that $w(z(x)) > y(z(x)) = 0$, obtaining a contradiction.

Write $\{(k,i)\}$ for the change index pairs for x. Recall that to say that (k,i) is a change index pair of x is to say that $-d_i \leq -x_k$ and that $z(x) - d_i \geq -x_k$. (As in chapter 2, we shall write $x_0 = 0$.)

Suppose that $0 \leq j \leq 2K$, $1 \leq i \leq D$, and that (j,i) is *not* a change index pair of x. Then there is some minimum $\nu(j,i) > 0$ such that

$$|(t - d_i) - (-x_j)| \geq \nu(j,i) \ \forall\ t \in [0, z(x)].$$

We now choose $\delta > 0$ small enough so that

$$\delta < \min\{\nu(j,i) \ : \ (j,i) \text{ is not a change index pair of } x\}$$

and also such that the intervals

$$[x_k, x_k + \delta], \ 0 \leq k \leq 2K$$

are pairwise disjoint.

Given such a δ, and still assuming that $z(x') \geq z(x)$, every change point of w on $[0, z(x)]$ corresponds to a change point y with the same change index pair (or set of change index pairs), though not conversely. For suppose that $t \in [0, z(x)]$ is such that $t - d_i = -x_k - \delta$ — and so (k,i) is a change index pair of x' — and imagine that (k,i) is not a change index pair of x. This implies that $z(x) - d_i < -x_k$; by the definition of δ, though, we in fact have

$$t - d_i < -x_k - \delta \quad \forall \quad t \in [0, z(x)],$$

a contradiction. On the other hand, it is perfectly possible for (k,i) to be a change index pair of x but not of x': this will happen if $-d_i$ lies in the interval

$$(-x_k - \delta, -x_k].$$

For each change index pair (k,i) of x, put

$$\delta(k,i) = \begin{cases} \delta, & -x_k - (-d_i) \geq \delta; \\ -x_k - (-d_i), & \text{otherwise.} \end{cases}$$

The point is that, as t runs from 0 to $z(x)$, $t-d_i$ will be between $-x_k-\delta$ and $-x_k$ for $\delta(k,i)$ time units.

We make three observations. First, $(2K,D)$ is a change index pair of x and $\delta(2K,D)=\delta$ (since $x_{2K}+\delta<\gamma$ by our assumption that $x'\in D_-(2K)$). Second, (k,D) is not a change index pair of x for any $k<2K$ (since $x\in D_-(2K,2K)$ by assumption). Finally, if we fix some particular i, the sum

$$\sum_{k\,:\,(k,i)\text{ is a change index pair of }x}(-1)^k\delta(k,i)$$

consists of alternating terms, and each except perhaps the first is of absolute value δ (since the intervals $[x_k,x_k+\delta]$ are pairwise disjoint). Therefore this whole sum is no more than δ in absolute value.

The assumption that $z(x')\geq z(x)$ implies that $w(t-d_i)<0$ for all $t\in(0,z(x))$ such that $t-d_i>0$. If $z(x')\geq z(x)$, then, we have the formua

$$\int_0^{z(x)} h_i(w(t-d_i))-h_i(y(t-d_i))\,dt=\sum_{k\,:\,(k,i)\text{ is a change index pair of }x}(-1)^k\delta(k,i)(a_i+b_i).$$

Summing across $1\leq i\leq D$ yields

$$w(z(x))-y(z(x))=\sum_{i=1}^{D}\sum_{k\,:\,(k,i)\text{ is a change index pair of }x}(-1)^k\delta(k,i)(a_i+b_i).$$

Since $\delta(2K,D)=\delta$ and (k,D) is a change index pair only for $k=2K$, the double sum on the right rewrites

$$\delta(a_D+b_D)+\sum_{i=1}^{D-1}\sum_{k\,:\,(k,i)\text{ is a change index pair of }x}(-1)^k\delta(k,i)(a_i+b_i).$$

The absolute value of the latter sum is no more than

$$\delta\sum_{i=1}^{D-1}(a_i+b_i),$$

which is less that $\delta(a_D+b_D)$ by the definition of dominant longest delay. It follows that $w(z(x))-y(z(x))$ is strictly positive; we have a contradiction and conclude that in fact $z(x')<z(x)$. $\square$

Lemma 5.2.9 and proposition 2.3.16 yield the following corollary.

Proposition 5.2.10. The Jacobian of ϕ at simple points in $D_-(2K, 2K)$. *Let $\beta > 1$ be as in lemma 5.2.9. If $x \in D_-(2K, 2K)$ is simple, then*

$$\frac{\partial z(x)}{\partial x_{2K}} \leq -\beta < -1$$

and

$$\det D\phi[x] \geq \beta.$$

Similarly for $D_+(2K, 2K)$. $\square$

We need to develop a more detailed picture of the set of non-simple points in $D_-(2K, 2K)$. Any point $x \in D_-(2K, 2K)$ certainly satisfies the first part of the definition of a simple point (recall definition 2.3.12): if $y_0 \in V^{-1}(x)$ with continuation y, y can have no zeros between 0 and $z(x)$. It might happen, though, that either 0 or $z(x)$ is a change point of y. To say that 0 is a change point of y is just to say that $d_i = x_k$ for some $1 \leq i < D$ and some $1 \leq k \leq 2K$. We will, just for the current discussion, call points x where this happens *points of the first kind.* Points of the first kind all lie on the finite collection $\mathcal{H}_1 \subset \mathcal{S}_-^{2K}$ of hyperplanes

$$\mathcal{H}_1 = \bigcup_{\substack{1 \leq i < D \\ 1 \leq k \leq 2K}} \{x \ : \ x_k = d_i \ \}.$$

On the other hand, to say that $z(x)$ is a change point of y is to say that

$$z(x) - d_i = -x_k \iff z(x) + x_k = d_i$$

for some $0 \leq k < 2K$ and some $1 \leq i < D$, where we are writing $x_0 = 0$ (note that, since $z(x)$ lies strictly between $\gamma - 2K$ and $\gamma - x_{2K-1}$, we cannot have $k = 2K$ or $i = D$). We will call such an x a *point of the second kind.*

Where do these simple points lie? Fix some pair (k, i), with $0 \leq k < 2K$ and $1 \leq i < D$, and suppose that $u = (u_1, \ldots, u_{2K-1}) \in \mathcal{S}_-^{2K-1}$. Since z is strictly decreasing with respect to its last coordinate, there is at most one point $s_{k,i}(u) \in (u_{2K-1}, \gamma)$ such that

$$(u, s_{k,i}(u)) \in D_-(2K, 2K) \text{ and } z(u, s_{k,i}(u)) + u_k = d_i.$$

By the continuity of z on $D_-(2K,2K)$, the set of $x \in \mathcal{S}_-^{2K-1}$ for which $s_{k,i}$ is defined is open, and the function $s_{k,i}$ is continuous on this set. Let us write $\Gamma_{k,i}$ for the graph of $s_{k,i}$ over its domain in $\mathcal{S}_-^{2K-1}$. The collection of all such graphs $\Gamma_{k,i}$ contains all points of the second kind.

If ever $(u, s_{k,i}) = (u, s_{k',i'})$ for two distinct pairs (k,i) and (k',i') (we must have $k \neq k'$ and $i \neq i'$), then u lies on the hyperplane

$$\{\ u_k - u_{k'} = d_i - d_{i'}\ \}.$$

Away from this hyperplane, the graphs of $s_{k,i}$ and $s_{k',i'}$ do not intersect.

Let us write Γ for the union of the graphs $\Gamma_{k,i}$, and let us write $\mathcal{H}_2 \subset \mathcal{S}_-^{2K}$ for the collection of hyperplanes

$$\mathcal{H}_2 = \bigcup_{\substack{1 \leq i \neq i' < D \\ 0 \leq k \neq k' < 2K}} \{\ x\ :\ x_k - x_{k'} = d_i - d_{i'}\ \}.$$

The set

$$S = (\mathcal{H}_1 \cup \mathcal{H}_2 \cup \Gamma) \cap D_-(2K,2K)$$

contains all the non-simple points in $D_-(2K,2K)$. Its complement consists of finitely many open connected components $W_1, \ldots W_M$. The restriction of ϕ to any W_m is affine: for if we choose $x, x' \in W_m$ and draw a path in W_m between them, this path has a finite cover by neighborhoods where ϕ is affine (recall theorem 2.3.13). We conclude that ϕ is described by the same affine map near x and x'. Write $\overline{W}_m$ for the closure of W_m in $D_-(2K,2K)$. By the continuity of ϕ on $D_-(2K,2K)$, the restriction of ϕ to each $\overline{W}_m$ is affine. Since S has no interior, and there are only finitely many sets W_m, every point of S lies in the closure of some W_m, and

$$D_-(2K,2K) = \cup_m \overline{W}_m.$$

We have shown the following.

Proposition 5.2.11. Structure of ϕ on $D_-(2K,2K)$. *The set $D_-(2K,2K)$ is expressible as the finite union of closed sets*

$$D_-(2K,2K) = \cup_m \overline{W}_m,$$

where the restriction of ϕ to each $\overline{W}_m$ is affine. Similarly for $D_+(2K, 2K)$. $\square$

Proof of theorem 5.2.1

Continuing to assume that $(SDDE)$ has negative feedback and dominant longest delay, we now prove theorem 5.2.1.

ϕ is an injective continuous map from $D_-(2K, 2K)$ to $D_+(2K)$. By invariance of domain, ϕ is actually a homeomorphism onto its image. We claim that this image is in fact all of $D_+(2K)$. For choose $x = (x_1, \ldots, x_{2K}) \in D_+(2K)$, and consider the following set of points in $D_-(2K)$:

$$B = \{x_s = (x_2 - x_1, x_3 - x_1, \ldots, x_{2K} - x_1, s)\},$$

where $s \in (x_{2K} - x_1, \gamma)$. As s runs from γ to $x_{2K} - x_1$, $z(x_s)$ increases strictly from 0 to some quantity strictly larger than

$$\gamma - (x_{2K} - x_1) > x_1;$$

as long as $z(x_s) < \gamma - (x_{2K} - x_1)$ the increase is continuous in s. This means that there is a unique x_s such that $z(x) = x_1$ and hence such that $\phi(x_s) = x$. Thus we conclude that the restriction of ϕ to $D_-(2K, 2K)$ is a homeomorphism onto $D_+(2K)$.

The restriction of ϕ to the open set $D_+(2K, 2K)$ is likewise a homeomorphism onto $D_-(2K)$. Write

$$U = D_-(2K, 2K) \cap \phi^{-1}(D_+(2K, 2K)).$$

The restriction of ϕ^2 to U is a homeomorphism onto $D_-(2K)$.

Now, let W_m be a finite collection of open sets in $D_-(2K, 2K)$ whose closures $\overline{W}_m$ (relative to $D_-(2K, 2K)$) cover $D_-(2K, 2K)$, and such that the restriction of ϕ to each $\overline{W}_m$ is affine. Similarly, let V_j be a finite collection of open sets in $D_+(2K, 2K)$ whose closures $\overline{V}_j$ (relative to $D_+(2K, 2K)$) cover $D_+(2K, 2K)$ and such that the restriction of ϕ to each $\overline{V}_j$ is affine. Then the sets

$$\overline{W}_m \cap \phi^{-1}\left(\overline{V}_j\right) \cap U$$

form a finite cover of U, and the restriction of ϕ to each such set is affine.

Let us write F for the inverse of $\phi^2|_U$ on $D_-(2K)$. Since ϕ^2 is a homeomorphism on U, $D_-(2K)$ can be written as the finite union of relatively closed sets

$$D_-(2K) = Y_1 \cup \cdots \cup Y_J$$

such that the restriction of F to each Y_j is affine. Moreover, the linear part of each such restriction (being the inverse of the linear part of an affine portion of ϕ^2) has determinant in the range $(0, \beta^{-2}]$, where $\beta > 1$ is as in proposition 5.2.10. The Jacobi transformation formula now implies that

$$\text{measure}(F^k(D_-(2K)) \leq \beta^{-2k} \sum_j \text{measure}(Y_j).$$

We conclude that the set

$$\cap_{k \in \mathbf{N}} F^k(D_-(2K))$$

has measure 0. But this is precisely the set

$$\{x \in D_-(2K, 2K) \ : \ \overline{S}(y_0) = 2K, \ y_0 \in V^{-1}(x) \ \}.$$

This proves theorem 5.2.1. □

Proof of theorem 5.2.2

We continue to assume that $(SDDE)$ has negative feedback and dominant longest delay. In this subsection we prove that, given $K \in \mathbf{N}$, $(SDDE)$ has a unique 1-cyclic periodic solution of oscillation speed $2K$. This will complete the proof of theorem 5.2.2; the slowly oscillating periodic solution (i.e., the $K = 0$ case) has already been discussed.

Fix some $K \in \mathbf{N}$. Choose $\ell < \gamma/K$ and $\alpha < \ell$, and let us write $p^-(\ell, \alpha)$ for the point in $D_-(2K)$ of the following form:

$$p^-(\ell, \alpha) = (\alpha, \ell, \ell + \alpha, 2\ell, \ldots, K\ell) \in D_-(2K).$$

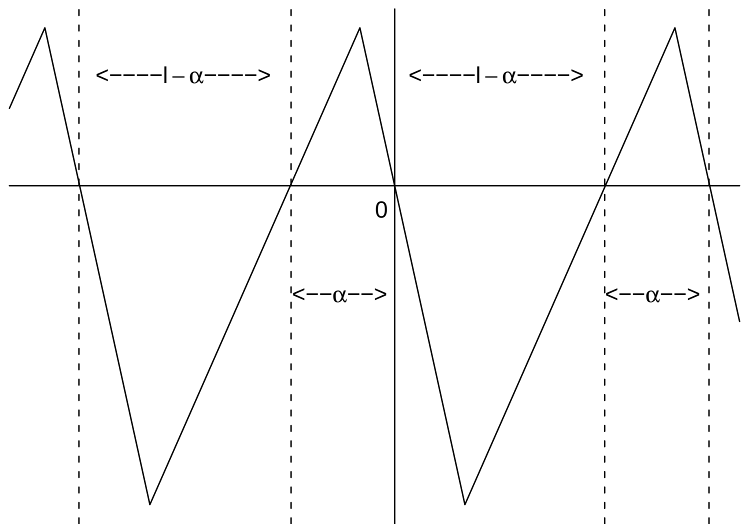
We define $p^+(\ell, \alpha) \in D_+(2K)$ similarly.

If x is a fixed point of ϕ^2 lying in $D_-(2K, 2K)$, then we see that, in fact, x must be of the form just introduced:

$$x = p^-(\ell, \alpha) = (\alpha, \ell, \ell + \alpha, 2\ell, \ldots, K\ell) \in D_-(2K, 2K).$$

ℓ is the period of the corresponding 1-cyclic periodic solution, and this solution is positive for α units at a time and negative for $\ell - \alpha$ units at a time. See Figure 22.

Figure 22



If $p^-(\ell, \alpha)$ is in fact a fixed point for ϕ^2, we have

$$\phi(p^-(\ell, \alpha)) = p^+(\ell, \ell - \alpha) \in D_+(2K, 2K) \text{ and } \phi^2(p^-(\ell, \alpha)) = p^-(\ell, \alpha).$$

Focusing on the function z, we get the identities

$$z(p^-(\ell, \alpha)) = \ell - \alpha; \quad z(p^+(\ell, \ell - \alpha)) = \alpha. \tag{5.2.12}$$

Conversely, if there are numbers $0 < \ell < \gamma/K$ and $0 < \alpha < \ell$ such that conditions (5.2.12) are satisfied, then the points $p^\pm(\ell, \alpha)$ lie in $D_\pm(2K, 2K)$ and are fixed points of ϕ^2. In this subsection we prove theorem 5.2.2 by showing that, for each $K \in \mathbf{N}$, there is exactly one pair (ℓ, α) such that identities (5.2.12) are satisfied.

We make our search easier by remarking restrictions on where ℓ and α lie. By construction $K\ell < \gamma$ and $\alpha < \ell$. Suppose that q is a 1-cyclic periodic solution of $(SDDE)$ with $V(q_0) = p^-(\ell, \alpha)$. Then $q(t)$ is positive for $t \in (-\gamma, -K\ell)$ and, since q is positive for α units at a time, we have that $\alpha \geq \gamma - K\ell$. This forces $(K+1)\ell > \gamma$. Thus to have such a periodic solution we actually have the following pair of size conditions:

$$\ell \in \left(\frac{\gamma}{K+1}, \frac{\gamma}{K}\right); \quad \alpha \in [\gamma - K\ell, \ell). \tag{5.2.13}$$

Now, if identities (5.2.12) are satisfied, then not only the pair (ℓ, α) but also the pair $(\ell, \ell - \alpha)$ must satisfy (5.2.13). In particular, we must have

$$\ell - \alpha \geq \gamma - K\ell \iff \alpha \leq (K+1)\ell - \gamma.$$

If there is to be any range that α can occupy, then, we must also have

$$\gamma - K\ell \leq (K+1)\ell - \gamma \iff \ell \geq \frac{\gamma}{K + \frac{1}{2}}.$$

Therefore we uncover the following more stringent size conditions: the pair (ℓ, α) must satisfy

$$\ell \in \left[\frac{\gamma}{K+\frac{1}{2}}, \frac{\gamma}{K}\right); \quad \alpha \in [\gamma - K\ell, (K+1)\ell - \gamma]. \tag{5.2.14}$$

These conditions are symmetric in α and $\ell - \alpha$.

Lemma 5.2.15. *The pair (ℓ, α) satisfies (5.2.14) if and only if the pair $(\ell, \ell - \alpha)$ does.*

PROOF. By symmetry, we only need to prove one implication. Suppose that (ℓ, α) satisfies (5.2.14). Then, as we have already shown,

$$\alpha \leq (K+1)\ell - \gamma \implies \ell - \alpha \geq \gamma - K\ell.$$

On the other hand,

$$\begin{aligned} \alpha \geq \gamma - K\ell &\implies \\ \alpha \geq \gamma - (K+1)\ell + \ell &\implies \\ (K+1)\ell - \gamma \geq \ell - \alpha. &\quad \square \end{aligned}$$

We summarize our approach in the following lemma:

Lemma 5.2.16. The approach to proving 5.2.2. 1-*cyclic periodic solutions of* ($SDDE$) *with oscillation speed* $2K$ *are in bijective correspondence with pairs* (ℓ, α) *satisfying (5.2.12) and (5.2.14).* □

We now show that there is exactly one such pair (ℓ, α). We will need two easy preliminary lemmas and one more difficult lemma. The first lemma is an immediate consequence of propositions 5.1.3 and 5.2.7, corollary 5.2.8, and lemma 5.2.9, together with the fact that $p^-(\ell, \alpha) \in D_-(2K, 2K)$ if and only if $z(p^-(\ell, \alpha)) < \gamma - K\ell + (\ell - \alpha)$.

Lemma 5.2.17. *Suppose that the pair* (ℓ, α) *satisfies (5.2.14). Then* $z(p^-(\ell, \alpha))$ *is defined, is strictly decreasing in* ℓ*, and is increasing in* α*. Moreover, at any* (ℓ, α) *such that*

$$z(p^-(\ell, \alpha)) < \gamma - K\ell + (\ell - \alpha)$$

— that is, such that $p^-(\ell, \alpha) \in D_-(2K, 2K)$ *—* $z(p^-(\ell, \alpha))$ *is continuous and for* $\epsilon > 0$ *sufficiently small we have*

$$z(p^-(\ell + \epsilon, \alpha + \epsilon)) < z(p^-(\ell, \alpha)).$$

Similarly for $z(p^+(\ell, \alpha))$. □

Lemma 5.2.18. *Suppose that* ℓ *is minimal and that* α *is minimal given* ℓ*: that is,*

$$\ell = \frac{\gamma}{K + \frac{1}{2}}, \quad \alpha = \gamma - K\ell.$$

Then

$$z(p^\pm(\ell, \alpha)) > \ell - \alpha.$$

For any ℓ *and* α *maximal given* ℓ*, that is,*

$$\ell \in \left[\frac{\gamma}{K + \frac{1}{2}}, \frac{\gamma}{K}\right); \quad \alpha = (K + 1)\ell - \gamma,$$

then

$$z(p^\pm(\ell, \alpha)) > \ell - \alpha.$$

PROOF. We give the proof for $p^-(\ell,\alpha)$; the other case is similar. Suppose that ℓ is minimal and α is minimal given ℓ. Then

$$\alpha = \gamma - \frac{K\gamma}{K+\frac{1}{2}} = \frac{\gamma}{2K+1}.$$

Also,

$$\ell - \alpha = \frac{\gamma}{K+\frac{1}{2}} - \gamma + \frac{K\gamma}{K+\frac{1}{2}} = \frac{\gamma}{2K+1}.$$

We have the estimate

$$z(p^-(\ell,\alpha)) \geq \left(1+\frac{\sigma}{\mu}\right)(\gamma - K\ell) > \gamma - K\ell = \alpha = \ell - \alpha.$$

The second statement is similar: if α is maximal given ℓ, then $\ell - \alpha$ is equal to $\gamma - K\ell$, which is strictly less than $z(p^-(\ell,\alpha))$. $\square$

Here is the main lemma for the proof of theorem 5.2.2.

Lemma 5.2.19. *There is some least*

$$\ell_- \in \left(\frac{\gamma}{K+\frac{1}{2}}, \frac{\gamma}{K}\right)$$

such that, for every

$$\ell \in \left[\ell_-, \frac{\gamma}{K}\right),$$

there is a unique $\alpha_-(\ell) \in [\gamma - K\ell, (K+1)\ell - \gamma]$ *with the property that*

$$z(p^-(\ell, \alpha_-(\ell))) = \ell - \alpha_-(\ell).$$

This function $\alpha_-(\ell)$ *is continuous and strictly increasing in* ℓ. *We have the limits*

$$\alpha_-(\ell_-) = \gamma - K\ell_-; \quad \alpha_-(\ell) \to \frac{\gamma}{K} \text{ as } \ell \to \frac{\gamma}{K}.$$

For $\ell \in [\ell_-, \gamma/K)$, *the function* $\ell - \alpha_-(\ell)$ *is strictly decreasing in* ℓ.

There is similarly some least

$$\ell_+ \in \left(\frac{\gamma}{K+\frac{1}{2}}, \frac{\gamma}{K}\right)$$

such that, for every

$$\ell \in \left[\ell_+, \frac{\gamma}{K}\right),$$

there is a unique $\alpha_+(\ell) \in [\gamma - K\ell, (K+1)\ell - \gamma]$ *with the feature that*

$$z(p^+(\ell, \alpha_+(\ell))) = \ell - \alpha_+(\ell).$$

This function $\alpha_+(\ell)$ *is continuous and strictly increasing in* ℓ*. We have the limits*

$$\alpha_+(\ell_+) = \gamma - K\ell_+; \quad \alpha_+(\ell) \to \frac{\gamma}{K} \text{ as } \ell \to \frac{\gamma}{K}.$$

For $\ell \in [\ell_+, \gamma/K)$*, the function* $\ell - \alpha_+(\ell)$ *is strictly decreasing in* ℓ*.*

PROOF. We give the proof for ℓ_- and $\alpha_-(\cdot)$; the other case is the same.

Given any fixed $\ell \in [\gamma/(K + \frac{1}{2}), \gamma/K)$, the function $z(p^-(\ell, \alpha))$ is increasing in α while $\ell - \alpha$ is strictly decreasing in α. Therefore, given ℓ, there is at most one $\alpha_-(\ell)$ satisfying

$$z(p^-(\ell, \alpha_-(\ell))) = \ell - \alpha_-(\ell).$$

Furthermore, lemma 5.2.18 implies that there is no such $\alpha_-(\ell)$ for $\ell = \gamma/(K + \frac{1}{2})$.

For $\ell \in [\gamma/(K + \frac{1}{2}), \gamma/K)$, the function

$$\ell \mapsto z(p^-(\ell, \gamma - K\ell))$$

is a strictly decreasing function that tends to 0 as ℓ approaches γ/K. Moreover, lemma 5.2.17 tells us that this function will be continuous as long as

$$z(p^-(\ell, \gamma - K\ell)) < \gamma - K\ell + (\ell - (\gamma - K\ell)) = \ell.$$

On the other hand, the function

$$\ell \mapsto \ell - (\gamma - K\ell) < \ell$$

is continuous and strictly increasing, and approaches γ/K as ℓ approaches γ/K. Since lemma 5.2.18 tells us that

$$z(p^-(\ell, \gamma - K\ell)) > \ell - (\gamma - K\ell)$$

for the minimal value $\gamma/(K + \frac{1}{2})$ of ℓ, we see that there must be a unique ℓ_- such that

$$z(p^-(\ell_-, \gamma - K\ell_-)) = \ell_- - (\gamma - K\ell_-) < \ell_-.$$

Write $\alpha_-(\ell_-) = \gamma - K\ell_-$.

For any $\ell < \ell_-$, there is no $\alpha_-(\ell) \in [\gamma - K\ell, (K+1)\ell - \gamma]$ such that

$$z(p^-(\ell, \alpha_-(\ell))) = \ell - \alpha_-(\ell).$$

For suppose there were: then since $z(p^-(\ell, \alpha))$ is strictly decreasing in ℓ,

$$z(p^-(\ell_-, \alpha_-(\ell))) < z(p^-(\ell, \alpha_-(\ell))) = \ell - \alpha_-(\ell) < \ell_- - \alpha_-(\ell). \tag{5.2.20}$$

Since we must have

$$\alpha_-(\ell) \geq \gamma - K\ell > \gamma - K\ell_- = \alpha_-(\ell_-),$$

inequality (5.2.20) yields

$$z(p^-(\ell_-, \alpha_-(\ell_-))) \leq z(p^-(\ell_-, \alpha_-(\ell))) < \ell_- - \alpha_-(\ell) < \ell_- - \alpha_-(\ell_-),$$

a contradiction. Therefore the point ℓ_- really is the minimum value where $\alpha_-(\cdot)$ is defined, and $\alpha_-(\ell^-) = \gamma - K\ell_-$. To complete the proof, we need to show that $\alpha_-(\ell) \in [\gamma - K\ell, (K+1)\ell - \gamma]$ is defined for all $\ell \in [\ell_-, \gamma/K)$, that it is continuous and strictly increasing, that $\alpha_-(\ell) \to \gamma/K$ as $\ell \to \gamma/K$, and that $\ell - \alpha_-(\ell)$ is strictly decreasing.

Choose any $\ell \in [\ell_-, \gamma/K)$ and suppose that $\alpha_-(\ell)$ is defined: that is, there is $\alpha_-(\ell) \in [\gamma - K\ell, (K+1)\ell - \gamma]$ such that

$$z(p^-(\ell, \alpha_-(\ell))) = \ell - \alpha_-(\ell).$$

Note that this implies that $p^-(\ell, \alpha_-(\ell)) \in D_-(2K, 2K)$: for $p^-(\ell, \alpha_-(\ell)) \in D_-(2K, 2K)$ exactly when

$$z(p^-(\ell, \alpha_-(\ell))) < \gamma - K\ell + (\ell - \alpha_-(\ell)).$$

Let $\epsilon > 0$ be a small enough that

$$p^-(\ell + \epsilon, \alpha_-(\ell) + \epsilon)$$

lies in $D_-(2K, 2K)$ and also small enough (invoking lemma 5.2.17) that

$$z(p^-(\ell + \epsilon, \alpha_-(\ell) + \epsilon)) < z(p^-(\ell, \alpha_-(\ell)) = \ell - \alpha = (\ell + \epsilon) - (\alpha_-(\ell) + \epsilon).$$

Now suppose that we increase some variable δ from 0. Then

$$z(p^-(\ell+\epsilon, \alpha_-(\ell)+\epsilon+\delta))$$

will be increasing in δ. By lemma 5.2.17 this increase is in fact continuous as long as

$$z(p^-(\ell+\epsilon, \alpha_-(\ell)+\epsilon+\delta)) < \gamma - K(\ell+\epsilon) + (\ell - \alpha_-(\ell) - \delta);$$

in particular, this increase is continuous as long as

$$z(p^-(\ell+\epsilon, \alpha_-(\ell)+\epsilon+\delta)) \leq \ell - \alpha_-(\ell) - \delta.$$

On the other hand, the quantity

$$(\ell+\epsilon) - (\alpha_-(\ell)+\epsilon+\delta) = \ell - \alpha_-(\ell) - \delta$$

is strictly decreasing in δ. The second part of lemma 5.2.18 tells us that, as δ increases to the point that $\alpha_-(\ell)+\epsilon+\delta$ is maximal given $\ell+\epsilon$ — that is, as

$$\alpha_-(\ell)+\epsilon+\delta \to (K+1)(\ell+\epsilon) - \gamma$$

— we will eventually get

$$z(p^-(\ell+\epsilon, \alpha_-(\ell)+\epsilon+\delta)) > \ell+\epsilon-\alpha_-(\ell)-\epsilon-\delta = \ell - \alpha_-(\ell) - \delta.$$

Thus there is some unique $\delta > 0$ such that, first,

$$(\ell+\epsilon, \alpha_-(\ell)+\epsilon+\delta)$$

does in fact satisfy (5.2.14) and, second,

$$z(p^-(\ell+\epsilon, \alpha_-(\ell)+\epsilon+\delta)) = (\ell+\epsilon) - (\alpha_-(\ell)+\epsilon+\delta).$$

Continuity follows from the continuity of z: if ϵ is small enough, δ will be small.

We of course put

$$\alpha_-(\ell+\epsilon) = \alpha_-(\ell)+\epsilon+\delta;$$

observe that

$$p^-(\ell+\epsilon, \alpha_-(\ell+\epsilon)) \in D_-(2K, 2K).$$

Thus we see that $\alpha_-(\cdot)$ is a continuous and strictly increasing function on $[\ell_-, \nu)$, where $\nu \leq \gamma/K$. Since $\alpha_-(\cdot)$ increases with respect to ℓ at rate greater than 1, we see that $\ell - \alpha_-(\ell)$ is strictly decreasing. If we imagine that $\nu < \gamma/K$, we can extend α_- by continuity to ν, and we see that, since

$$z(p^-(\nu, \alpha_-(\nu))) = \nu - \alpha_-(\nu) < \gamma - K\nu + \nu - \alpha_-(\nu),$$

the point $p^-(\nu, \alpha_-(\nu))$ lies in $D_-(2K, 2K)$ and we can in fact, using the argument used just above, extend $\alpha_-(\cdot)$ to values greater than ν. We conclude that $\alpha_-(\cdot)$ is defined on all of $[\ell_-, \gamma/K)$.

It remains to show that $\alpha_-(\ell) \to \gamma/K$ as $\ell \to \gamma/K$. As $\ell \to \gamma/K$, $z(p^-(\ell, \alpha)) \to 0$ uniformly across all allowable values of α, so we must have $\alpha_-(\ell) \to \gamma/K$ as $\ell \to \gamma/K$. This completes the proof. □

PROOF OF THEOREM 5.2.2 ($K \in \mathbf{N}$ case). As explained above, we show that there is exactly one pair (ℓ, α) satisfying both (5.2.12) and (5.2.14).

Suppose that $\ell_+ \geq \ell_-$ (the other case is similar). Then of course

$$\ell_+ - \alpha_+(\ell_+) = \ell_+ - (\gamma - K\ell_+) = (K+1)\ell_+ - \gamma \geq \alpha_-(\ell_+).$$

As ℓ grows from ℓ_+ to γ/K, $\ell - \alpha_+(\ell)$ decreases strictly to zero, while $\alpha_-(\ell)$ increases strictly to γ/K. Therefore there is a unique ℓ^* where

$$\alpha_-(\ell^*) = \ell^* - \alpha_+(\ell^*).$$

Now consider the point

$$p = p^-(\ell^*, \alpha_-(\ell^*)).$$

By the definition of α_- and α_+, we have

$$z(p) = \ell^* - \alpha_-(\ell^*) = \alpha_+(\ell^*).$$

The first of conditions (5.2.12) is satisfied, and computing we have

$$\begin{aligned}\phi(p) &= (\ell^* - \alpha_-(\ell^*), \ell^*, \ell^* + (\ell^* - \alpha_-(\ell^*)), 2\ell^*, \ldots, K\ell^*) \\ &= (\alpha_+(\ell^*), \ell^*, \ell^* + \alpha_+(\ell^*), 2\ell^*, \ldots, K\ell^*) \\ &= p^+(\ell^*, \alpha_+(\ell^*)).\end{aligned}$$

Since $\ell^* - \alpha_+(\ell^*) = \alpha_-(\ell^*)$, we of course have

$$\alpha_+(\ell^*) = \ell^* - \alpha_-(\ell^*),$$

and an argument symmetric to the one we just gave now shows that the second of conditions (5.2.12) is satisfied, and that we in fact have

$$\phi^2(p) = p.$$

This completes the proof. $\square$

If all of the feedback functions h_i in $(SDDE)$ are odd, then α_- and α_+ agree, and we see that the zeros of any 1-cyclic periodic solution are evenly spaced. In particular, we have the following corollary.

Corollary 5.2.21. *Write q for the unique 1-cyclic periodic solution of $(SDDE)$ with oscillation speed $2K$, translated so that $q(0) = 0$. If all of the feedback functions h_i are odd, then there is some z such that the zeros of q occur at points nz, $n \in \mathbf{Z}$, and q has the symmetry*

$$q(t) = -q(-t)\ \forall\ t.\quad \square$$

5.3 $(SDDE)$ with negative feedback and other dominant delay

Consider the equation

$$y'(t) = -2\mathrm{sgn}(y(t-1)) - \mathrm{sgn}(y(t-3)).$$

This equation has negative feedback and dominant first delay. In section 4.4 we described three distinct slowly oscillating 1-cyclic periodic solutions for this equation (recall definition 5.1.9). Therefore we see that the closest analog to theorem 5.2.2 does not hold when some delay other than the longest is dominant: slowly oscillating 1-cyclic periodic solutions need not be unique. Nevertheless, a version of 5.2.2 that is not too different does hold: in particular, 1-cyclic periodic solutions exist at every oscillation speed, and are unique at every positive oscillation speed. Here are the main results of this section.

Theorem 5.3.1. Existence and uniqueness of rapidly oscillating 1-cyclic periodic solutions. *Suppose that (SDDE) has negative feedback and dominant Mth delay. For each $K \in \mathbf{N}$, (SDDE) has a unique 1-cyclic periodic solution of oscillation speed 2K.*

Theorem 5.3.2. Existence of slowly oscillating 1-cyclic periodic solutions. *Suppose that (SDDE) has negative feedback and dominant Mth delay. Then (SDDE) has a slowly oscillating 1-cyclic periodic solution.*

Throughout this section, we will assume that $(SDDE)$ has negative feedback and that the Mth delay, $M < D$, is dominant.

As in the last section, we shall write

$$\mu = \sum_{i=1}^{D} a_i + b_i,$$

and write σ for the smallest nonzero value attainable by $|y'(t)|$ for $t > 0$, where y is any solution of $(SDDE)$.

Proof of theorem 5.3.1

Our approach is very similar to the long-delay dominant case. Accordingly, we will only sketch the proofs in this section.

Choose and fix some natural number K. Let ℓ be any number in the range

$$\ell \in \left[\frac{d_M}{K+\frac{1}{2}}, \frac{d_M}{K}\right),$$

and given this ℓ let α be any number in the range

$$\alpha \in [d_M - K\ell, (K+1)\ell - d_M].$$

We now define $p^-(\ell, \alpha)$ to be the vector in $\cup_n \mathcal{S}^n_-$ of the form

$$(\alpha, \ell, \ell + \alpha, 2\ell, \ldots, K\ell < d_M, K\ell + \alpha \geq d_M, \ldots, c),$$

where c is the greatest number in the sequence $\alpha, \ell, \ell+\alpha, 2\ell, 2\ell+\alpha, 3\ell, \ldots$ that is less than γ. We define $p^+(\ell, \alpha)$ similarly. We do not know the dimension of $p^\pm(\ell, \alpha)$, but it will turn out that these points are special enough that we do not need to. (The loss of control over the dimension of zero vectors of solutions is what makes the Mth-delay-dominant case cumbersome.)

Whatever the dimension of $p^-(\ell,\alpha)$, if $y_0 \in V^{-1}(p^-(\ell,\alpha))$ with continuation y, then y has a proper downward zero at 0; indeed, y will be strictly decreasing for $t \in (0, d_M - K\ell)$, and then strictly increasing for $t \in (d_M - K\ell, d_M - K\ell + \ell - \alpha)$. If the point $z(p^-(\ell,\alpha))$ is in this latter interval, the oscillation speed of y has not dropped from time 0 to time $z(p^-(\ell,\alpha))$. In any event, y will have a first positive proper zero, and so $z(p^-(\ell,\alpha))$ is defined. Similarly as in the last section, we have the bound

$$z(p^-(\ell,\alpha)) \geq \left(1 + \frac{\sigma}{\mu}\right)(d_M - K\ell);$$

if $z(p^-(\ell,\alpha)) < d_M - K\ell + \ell - \alpha$, we also have the bound

$$z(p^-(\ell,\alpha)) \leq \left(1 + \frac{\mu}{\sigma}\right)(d_M - K\ell).$$

All of the above holds similarly for $p^+(\ell,\alpha)$.

We define the following set:

$$L^- = \left\{ (\ell,\alpha) \ : \ \begin{array}{c} \ell \in \left[\frac{d_M}{K+\frac{1}{2}}, \frac{d_M}{K}\right); \\ \alpha \in [d_M - K\ell, (K+1)\ell - d_M]; \\ z(p^-(\ell,\alpha)) \in (d_M - K\ell, d_M - K\ell + \ell - \alpha)\,. \end{array} \right\}$$

In words, L^- is the set of pairs (ℓ,α) such that analogs of the size conditions (5.2.14) hold and, given $(\ell,\alpha) \in L^-$ and $y_0 \in V^{-1}(p^-(\ell,\alpha))$ with continuation y, the oscillation speed of y does not drop between 0 and its first positive zero. We define L^+ similarly.

The same sort of reasoning as in section 5.2 yields the following

Lemma 5.3.3. Approach to proving 5.3.1. *The* 1*-cyclic periodic solutions of* (*SDDE*) *with oscillation speed* $2K$ *are in bijective correspondence with pairs* $(\ell,\alpha) \in L^-$ *that satisfy*

$$z(p^-(\ell,\alpha)) = \ell - \alpha \quad \textit{and} \quad z(p^+(\ell,\ell-\alpha)) = \alpha. \quad \Box$$

Observe that the condition in the lemma implies that $(\ell, \ell-\alpha) \in L^+$.

As in the last section, our strategy is to prove that there is exactly one pair $(\ell, \alpha) \in L^-$ as described in lemma 5.3.3. The following two lemmas are analogs of lemmas 5.2.17 and 5.2.18.

Lemma 5.3.4. *Write L for the subset in $\mathbf{R}^2$ of pairs (ℓ, α) satisfying*

$$L = \left\{ (\ell, \alpha) \ : \ \begin{array}{c} \ell \in \left[\frac{d_M}{K+\frac{1}{2}}, \frac{d_M}{K}\right); \\ \alpha \in [d_M - K\ell, (K+1)\ell - d_M]. \end{array} \right\}.$$

Then L^- is open in L. The function

$$z(p^-(\ell, \alpha))$$

is continuous on L^-, is strictly decreasing in ℓ, and is increasing in α. If $(\ell, \alpha) \in L^-$, for $\epsilon > 0$ small enough we have

$$z(p^-(\ell + \epsilon, \alpha + \epsilon)) < z(p^-(\ell, \alpha)).$$

Similarly for L^+ and $z(p^+(\ell, \alpha))$.

PROOF. We give a proof for L^-; the other case is the same.

Observe that the dimension of $p^-(\ell, \alpha)$ is no more than $N = (2K+1) \times \gamma/d_M$. A basic difficulty, though, is that $p^-(\ell, \alpha)$ is not necessarily of constant dimension as ℓ and α range over their allowed values. We are going to adopt an ad hoc way of de-emphasizing the dimension of $p^-(\ell, \alpha)$. Namely, for each $(\ell, \alpha) \in L$ we write $Y = F_-(\ell, \alpha)$ for a 1-cyclic periodic function defined on $(-\infty, 0]$ satisfying $V(Y_0) = p^-(\ell, \alpha)$. We define $F_+(\ell, \alpha)$ similarly.

Take $(\ell, \alpha) \in L^-$ and choose $Y = F_-(\ell, \alpha)$ with continuation y as a solution of $(SDDE)$. We know that y is strictly decreasing on $[0, d_M - K\ell]$ with derivative less than $-\sigma$ almost everywhere, and is strictly increasing on $[d_M - K\ell, d_M - K\ell + \ell - \alpha]$ with derivative greater than σ almost everywhere, and has its first positive zero $z = z(p^-(\ell, \alpha))$ on the interval

$$(d_M - K\ell, d_M - K\ell + \ell - \alpha).$$

Write $\rho > 0$ for a number satisfying

$$\rho < \min\{\ d_M - K\ell,\ z - (d_M - K\ell),\ (d_M - K\ell + \ell - \alpha) - z\ \}.$$

Then y is strictly decreasing at rate less than or equal to $-\sigma$ on $(0, \rho)$, is strictly increasing at rate greater than or equal to σ on $(z - \rho, z + \rho)$, and is less than $-\rho\sigma$ on $(\rho, z - \rho)$.

Given $\delta > 0$, choose $(\ell', \alpha') \in L$ with

$$|\ell - \ell'| < \delta, \ |\alpha - \alpha'| < \delta.$$

Take $W = F_-(\ell', \alpha')$ with continuation w. The same kind of argument as in the proof of proposition 5.2.7 now shows that, if δ is small enough, for

$$t \in [0, \min\{z(p^-(\ell, \alpha)), z(p^-(\ell', \alpha'))\} + d_1$$

we have the bound

$$|w(t) - y(t)| \leq N\mu\delta.$$

(This bound holds even if $p^-(\ell, \alpha)$ and $p^-(\ell', \alpha')$ are not of the same dimension; picture the initial conditions W_0 and Y_0 to see this.) If we choose δ small enough, then, we can guarantee that

$$|z(p^-(\ell', \alpha')) - z(p^-(\ell, \alpha))| < \rho \text{ and } z(p^-(\ell', \alpha')) < d_M - K\ell' + \ell' - \alpha'.$$

This establishes the openness of L^- and the continuity of $z(p^-(\ell, \alpha))$ on L^-.

Consider $Y = F_-(\ell, \alpha)$ with continuation y and $W = F_-(\ell, \alpha + \delta)$ with continuation w, where (ℓ, α) and $(\ell, \alpha + \delta)$ are both in L. For simplicity write $V(Y_0) = x = (x_1, \ldots, x_n)$. Then, for almost all

$$t \in (0, \min\{z(p^-(\ell, \alpha)), z(p^-(\ell, \alpha + \delta))\} + d_1),$$

$w'(t) = y'(t)$ unless $t - d_i \in (-x_k - \delta, -x_k)$ for some odd k. If this ever happens, $w'(t) < y'(t)$. We conclude that z is increasing in α.

Similarly, consider $Y = F_-(\ell, \alpha)$ with continuation y and $W = F_-(\ell + \delta, \alpha)$ with continuation w, where (ℓ, α) and $(\ell + \delta, \alpha)$ are both in L^-. Again write $V(Y_0) = x = (x_1, \ldots, x_n)$. Then, for almost all

$$t \in (0, \min\{z(p^-(\ell, \alpha)), z(p^-(\ell + \delta, \alpha))\} + d_1),$$

$w'(t) = y'(t)$ unless $t - d_i \in (-x_k - \delta, -x_k)$ for some positive even k. If this ever happens, $w'(t) > y'(t)$. But this does happen, for $t - d_M$ must lie between $-x_{2K} - \delta$ and $-x_{2K}$ for some subinterval of t. We conclude that z is strictly decreasing in ℓ.

Finally, consider $Y = F_{-}(\ell, \alpha)$ with continuation y and $W = F_{-}(\ell + \delta, \alpha + \delta)$ with continuation w, where (ℓ, α) and $(\ell + \delta, \alpha + \delta)$ are both in L^{-}. Let us write

$$x = p^{-}(\ell, \alpha);\ x' = p^{-}(\ell + \delta, \alpha + \delta).$$

We choose δ small enough that x and x' have the same dimension N.

Imagine that

$$z(x') \geq z(x).$$

We will show that, for sufficiently small δ, we get

$$w(z(x)) > y(z(x)),$$

yielding a contradiction.

We will write (k, i) for the change index pairs for x. Recall that to say that (k, i) is a change index pair of x is to say that $-d_i \leq -x_k$ and that $z(x) - d_i \geq -x_k$.

Suppose that $0 \leq j \leq N$, $1 \leq i \leq D$, and that (j, i) is *not* a change index pair of x. Then there is some $\nu(j, i) > 0$ such that

$$|(t - d_i) - (-x_j)| \geq \nu(j, i)\ \forall\ t \in [0, z(x)].$$

We now choose $\delta > 0$ small enough so that

$$\delta < \min\{\ \nu(j, i)\ :\ (j, i) \text{ is not a change index pair of } x\ \}$$

and also such that the intervals

$$[x_k, x_k + \delta],\ 0 \leq k \leq N$$

are pairwise disjoint. Given such a δ, and assuming still that $z(x') \geq z(x)$, every change point of w on $[0, z(x)]$ corresponds to a change point of y with the same change index pair (though not conversely).

For each change index pair (k,i) of x, put

$$\delta(k,i) = \begin{cases} \delta, & -x_k + d_i \geq \delta; \\ -x_k + d_i, & \text{otherwise.} \end{cases}$$

As t runs from 0 to $z(x) \leq z(x')$, $t - d_i$ will be between $-x_k - \delta$ and $-x_k$ for exactly $\delta(k,i)$ time units. Observe that $(2K, M)$ is a change index pair and that $\delta(2K, M) = \delta$ (since $x_{2K} + \delta < d_M$ by our assumption that $x' \in L^-$). Observe also that (k, M) is a change index pair only for $k = 2K$ by our assumption that $x \in L^-$. Finally note that, if we fix some particular i, the sum

$$\sum_{\substack{k \;:\; (k,i) \text{ is a change index} \\ \text{pair of } x}} (-1)^k \delta(k,i)$$

consists of alternating terms, and each except perhaps the first is of absolute value δ. Therefore this whole sum is no more than δ in absolute value.

Our assumption that $z(x') \geq z(x)$ implies that $w(t - d_i) < 0$ for all $t \in (0, z(x))$ such that $t - d_i > 0$. We therefore have that $w(z(x)) - y(z(x))$ is equal to

$$\sum_{i=1}^{D} \sum_{\substack{k \;:\; (k,i) \text{ is a change index} \\ \text{pair of } x}} (-1)^k \delta(k,i)(a_i + b_i)$$

$$= \delta(a_M + b_M) + \sum_{i \neq M} \sum_{\substack{k \;:\; (k,i) \text{ is a change index} \\ \text{pair of } x}} (-1)^k \delta(k,i)(a_i + b_i)$$

(recall the proof of lemma 5.2.9). The definition of dominant delay ensures that this sum is strictly positive . We have a contradiction and conclude that in fact $z(x') < z(x)$. $\square$

Lemma 5.3.5. *Let $(\ell, \alpha) \in L$. Suppose that ℓ is minimal and that α is minimal given ℓ: that is,*

$$\ell = \frac{d_M}{K + \frac{1}{2}}, \quad \alpha = d_M - K\ell.$$

Then

$$z(p^{\pm}(\ell,\alpha)) > \ell - \alpha.$$

For any ℓ and α maximal given ℓ, that is,

$$\ell \in \left[\frac{d_M}{K+\frac{1}{2}}, \frac{d_M}{K}\right); \quad \alpha = (K+1)\ell - d_M,$$

then

$$z(p^{\pm}(\ell,\alpha)) > \ell - \alpha.$$

PROOF. The same as the proof of 5.2.18. □

With the above two lemmas in hand, the analogs of lemma 5.2.19 and the proof of theorem 5.2.2 go through similarly as in section 5.2. Making the necessary changes to these arguments (mostly substituting d_M for γ) proves theorem 5.3.1. □

Proof of theorem 5.3.2

.

Write

$$\beta = d_M/\gamma < 1$$

and choose $N \in \mathbf{N}$ so large that

$$(1-\beta)^N \gamma < d_M.$$

We make three easy observations about these choices.

Lemma 5.3.6. *Let β and N be as above. Suppose that x_1 and x_2 are numbers with $0 \leq x_1 \leq x_2 \leq \gamma$. Then*

$$x_2 \geq x_1 + d_M \implies x_2 \geq x_1 + \beta(\gamma - x_1).$$

Suppose that $0 \leq x_1 \leq x_2 \leq \gamma$, and let $\triangle \geq d_M$. Write

$$\tilde{x}_1 = \min(x_1 + \triangle, \gamma); \; \tilde{x}_2 = \min(x_2 + \triangle, \gamma).$$

Then

$$x_2 \geq x_1 + \beta(\gamma - x_1) \implies \tilde{x}_2 \geq \tilde{x}_1 + \beta(\gamma - \tilde{x}_1).$$

Finally, suppose that $0 \leq x_1 \leq x_2 \leq \ldots \leq x_N \leq \gamma$ *is a sequence such that*

$$x_i \geq x_{i-1} + \beta(\gamma - x_{i-1}) \ \forall \ 2 \leq i \leq N.$$

Then $x_N + d_M > \gamma$.

PROOF. We verify the three statements in turn. Given $0 \leq x_1 \leq x_2 \leq \gamma$, suppose that in fact $x_2 \geq x_1 + d_M$. Then, since

$$\beta(\gamma - x_1) \leq \beta\gamma = d_M,$$

the first implication is proven.

We turn to the second implication. If $\tilde{x}_2 = \gamma$, the implication holds automatically. Suppose that both $\tilde{x}_1$ and $\tilde{x}_2$ are less than γ. Then

$$\tilde{x}_1 + \beta(\gamma - \tilde{x}_1) = x_1 + \triangle + \beta(\gamma - x_1) - \beta\triangle \leq x_2 + \triangle - \beta\triangle \leq \tilde{x}_2.$$

The hypothesis of the last part of the lemma rewrites as

$$\gamma - x_i \leq (1 - \beta)(\gamma - x_{i-1}).$$

Since $(\gamma - x_1) \leq \gamma$, we must have $\gamma - x_N < d_M$ by our choice of N. $\square$

N serves as a bound on the dimension of the zero vectors of slowly oscillating solutions of $(SDDE)$. We first devise a way to store these zero vectors — of varying dimension — in $\mathbf{R}^N$.

Let us take two copies of $\mathbf{R}^N$, labeled $\mathbf{R}^N_+$ and $\mathbf{R}^N_-$. We define a map

$$\pi : \left(\cup_{n=0}^N \mathcal{S}^n_-\right) \bigcup \left(\cup_{n=0}^N \mathcal{S}^n_+\right) \to \mathbf{R}^N_- \cup \mathbf{R}^N_+$$

as follows. If, for example,

$$x = (x_1, \ldots, x_n) \in \mathcal{S}^n_-, \ n \leq N,$$

put

$$\pi(x) = (x_1, \ldots, x_n, \gamma, \gamma, \ldots, \gamma) \in \mathbf{R}^N_-;$$

similarly for $x \in \mathcal{S}_+^n$. The map π is continuous and injective.

We define the following subsets of $\mathbf{R}_-^N$ and $\mathbf{R}_+^N$:

$$D_- \subset \mathbf{R}_-^N = \left\{ \begin{array}{l} v = (v_1, \ldots, v_N) \text{ such that} \\ d_M \leq v_i \leq \gamma \;\forall\; 1 \leq i \leq N; \quad v_i \geq v_{i-1} + \beta(\gamma - v_{i-1}) \;\forall\; 2 \leq i \leq N \end{array} \right\},$$

and similarly for $D_+ \subset \mathbf{R}_+^N$.

Observe that $D_- \cup D_+$ lies in the image of π, and so $\pi^{-1}(v)$ is unambiguously defined for each $v \in D_- \cup D_+$. The following observation is immediate, but central to why we have given $D_\pm$ its somewhat elaborate definition:

Lemma 5.3.7. *D_- and D_+ are closed and convex sets of $\mathbf{R}_-^N$ and $\mathbf{R}_+^N$, respectively.* □

We define the following map ψ from $D_- \cup D_+$ to $\mathbf{R}_-^N \cup \mathbf{R}_+^N$:

$$\psi(v) = \pi \circ \phi \circ \pi^{-1}(v).$$

Our main result for the subsection is the following:

Proposition 5.3.8. *ψ is defined and continuous everywhere on $D_- \cup D_+$. ψ maps D_- into D_+ and maps D_+ into D_-.*

PROOF OF THEOREM 5.3.2. Proposition 5.3.8 implies that ψ^2 (restricted to D_-) is a continuous mapping of D_- into itself; this mapping has a fixed point v by Brouwer's theorem. Since π is injective and π^{-1} is injective on D_-, $\pi^{-1}(v)$ must be a fixed point of ϕ^2. This fixed point is the zero vector of a 1-cyclic slowly oscillating periodic solution. □

Remark 5.3.9. It might seem to the reader that the definition of $D_\pm$ is needlessly complicated. An obvious thing to try would be to take the vectors $(x_1, \ldots, x_N)$ in $D_\pm$ to satisy $d_M \leq x_i \leq \gamma$ and, for all $2 \leq i \leq N$, either $v_i \geq v_{i-1} + d_M$ or $v_i = \gamma$. If the $D_\pm$ were defined this way they would have the desired self-mapping properties under ψ, but would not be convex (consider the convex combination of two points where the number of coordinates equal to γ differs by more than one).

PROOF OF PROPOSITION 5.3.8. We will prove the proposition for the restriction of ψ to D_-; the other case is similar. Choose $v \in D_-$ and write $x = (x_1, \dots, x_n) = \pi^{-1}(v) \in \mathcal{S}_-^n$. We first show that $\phi(x)$ is defined and that $\pi(\phi(x))$ lies in D_+.

Choose $y_0 \in V^{-1}(x)$ with continuation y. Then, since $x_1 \geq d_M$, we know that y will be strictly decreasing (at rate less than or equal to $-\sigma$) for $t \in (0, d_M)$ and then strictly increasing (at rate greater than or equal to σ) until d_M time units after the first positive zero of y. Therefore $z(x) > d_M$ is defined, and so $\phi(x)$ is defined too. $\phi(x)$ is of the the form

$$\phi(x) = (z(x), z(x) + x_1, \dots, z(x) + x_m),$$

where m is the maximal k such that $z(x) + x_k < \gamma$. We don't know the dimension of $\phi(x)$ in general, but the last part of lemma 5.3.6 assures us that it is no more than N. In fact, lemma 5.3.6 assures us that $\pi(\phi(x))$ lies in D_+. Thus ψ is defined and maps D_- into D_+.

It remains to show that ψ is continuous on D_-. Intuitively, the idea is this: if $x = (x_1, \dots, x_n) \in \mathcal{S}_-^n$ and $y_0 \in V^{-1}(x)$, we can regard y_0 as having its "$n+1$st through Nth" negative zeros be equal to $-\gamma$ without altering anything about the continuation of y_0.

Choose $v \in D_-$. v has some maximum coordinate $v_n < \gamma$. Choose δ such that $\gamma - 2\delta > v_n$, and then choose $v' \in D_-$ such that $|v - v'| < \delta$ (sup metric). Write $x = \pi^{-1}(v)$ and $x' = \pi^{-1}(v')$; the dimension of x' is greater than or equal to the dimension of x (which is n). Take $y_0 \in V^{-1}(x)$ and $w_0 \in V^{-1}(x')$ with continuations y and w, respectively. Write $\rho > 0$ for a number such that y is strictly increasing on an interval of radius ρ about $z(x)$ (in particular, $z(x) - \rho > d_M$).

We now seek to bound the following quantity:

$$\int_0^{\min(z(x), z(x')) + d_1} |h_i(w(t - d_i)) - h_i(y(t - d_i))| \, dt.$$

$h_i(w(t - d_i))$ and $h(y(t - d_i))$ will disagree for at most an interval of length δ around every point $t = d_i - x_k$, $1 \leq k \leq n$, as well as perhaps an interval of length δ about 0 (every x'_k, $k > n$, is greater than $\gamma - \delta$). Therefore we have

$$\int_0^{\min(z(x), z(x')) + d_1} |h_i(w(t - d_i)) - h_i(y(t - d_i))| \, dt \leq (a_i + b_i)\delta(N + 1).$$

Summing across i we see that, if δ satisfies our size conditions, for

$$t \in [0, \min\{z(x), z(x')\} + d_1$$

we have the bound

$$|w(t) - y(t)| \leq (N+1)\mu\delta.$$

If δ is small enough, this quantity is less than $\rho\sigma$ and we see that

$$|z(x') - z(x)| \leq \frac{(N+1)\mu\delta}{\sigma}.$$

If k is such that both $\phi(x)$ and $\phi(x')$ are of dimension greater than or equal to k, then, we have

$$|\phi(x')_k - \phi(x)_k| \leq \delta + \frac{(N+1)\mu\delta}{\sigma}.$$

By choosing δ small enough we can guarantee that $z(x') > \delta$; in this case $\phi(x)$ will have dimension greater than or equal to the dimension of $\phi(x')$. Suppose that $k \geq 1$ is such that $\phi(x)_k$ exists but $\phi(x')_k$ does not. Then, since $\phi(x)_k = x_{k-1} + z(x)$ and we have both

$$|z(x) + x_{k-1} - (z(x') - x'_{k-1})| \leq \delta + \frac{(N+1)\mu\delta}{\sigma} \quad \text{and} \quad z(x') + x'_{k-1} \geq \gamma,$$

we have that $\phi(x)_k$ is within

$$\delta + \frac{(N+1)\mu\delta}{\sigma}$$

of γ. It follows that

$$|\phi(x') - \phi(x)| \leq \delta + \frac{(N+1)\mu\delta}{\sigma}.$$

This shows that ψ is continuous. The proof of 5.3.8, and hence of 5.3.2, is complete. □

5.4 Some additional remarks on $(SDDE)$ with negative feedback

The situation when $(SDDE)$ has negative feedback but no dominant delay is much less clear. The lack of any clear nonincreasing "oscillation speed" of solutions deprives us both of a salutary organizational principle and of some strong — and very helpful — strictures on the dynamics, and we are able to say relatively little about $(SDDE)$ with negative feedback when no delay is dominant. On the other hand, it is apparent that in this case some interesting phenonema occur: namely, this case is an easy source of periodic solutions of cyclicity higher than 1 (recall definition 1.2.5 and example 1.1.3).

Example 5.4.1. Let us look at a parameterized version of the equation presented in example 1.1.3.

$$y'(t) = h(y(t-1)) + h(y(t-\gamma)), \tag{$SDDE_\gamma$}$$

where $\gamma > 1$ and

$$h(y) = \begin{cases} 2, & y < 0; \\ 0, & y = 0; \\ -1, & y > 0. \end{cases}$$

Observe that C' is forward-invariant under the solution semiflow for $(SDDE_\gamma)$ (recall condition (2.2.10)).

If y is a solution then $y'(t) < 0$ if and only if both $y(t-1)$ and $y(t-\gamma)$ are positive. It follows that, if $V(y_0) \in \mathcal{S}_+^0$, then $\phi(V(y_0)) \in \mathcal{S}_-^0$. It is not necessarily true, though, that ϕ maps $\mathcal{S}_-^0$ into $\mathcal{S}_+^0$. What is true is that $x \in \mathcal{S}_-^0$ is always a periodic point for ϕ, with the period (i.e. the cyclicity of the corresponding periodic solution of $(SDDE_\gamma)$) increasing in γ. We show this now.

Let us write $x \in \mathcal{S}_-^0$. Choose $y_0 \in V^{-1}(x)$ with continuation y as a solution of $(SDDE_\gamma)$.

Let us first assume that $\gamma \leq 3$. In this case, we have that $y'(t) = -2$ for $t \in (0,1)$ and that $y'(t) = 1$ for $t \in (1,\gamma)$. Therefore, if $\gamma \leq 3$, $z(x) \geq \gamma$, $\phi(x) \in \mathcal{S}_+^0$, and x is a 2-periodic point of ϕ.

Let us now assume that $\gamma > 3$. For $t \in (0,\gamma)$, y will behave like the slowly oscillating periodic solution of

$$y'(t) = -h(-y(t-1))$$

with a downward zero at 0 — in particular, y will be negative for 3 units at a time and positive for 3/2 units at a time, with local minima equal to -2 and local maxima equal to 1.

Suppose now that k is the largest nonnegative integer such that $9k/2 \leq \gamma$. From time 0 to time γ, y completes k oscillations about 0 of the type described above. Over the time period $t \in (\gamma, \gamma+3)$, $y(t-\gamma)$ will be negative and so y will be strictly increasing at rate at least 1. We claim that $y(\gamma+3) \geq 3$. If $y(\gamma) \geq 0$, this is clear. If $y(\gamma) < 0$, then $y(t)$ is negative for at least $|y(\gamma)|/2$ units of $(\gamma-1, \gamma)$ (since y cannot decrease at rate more negative than -2). Therefore, for $|y(\gamma)|/2$ units of $(\gamma, \gamma+3)$, we have $y(t-1) < 0$ and $y'(t) = 4$. Thus if $y(\gamma) < 0$ we have

$$y(\gamma+3) \geq y(\gamma) + 2|y(\gamma)| + 3 - |y(\gamma)|/2 \geq 3 + |y(\gamma)|/2.$$

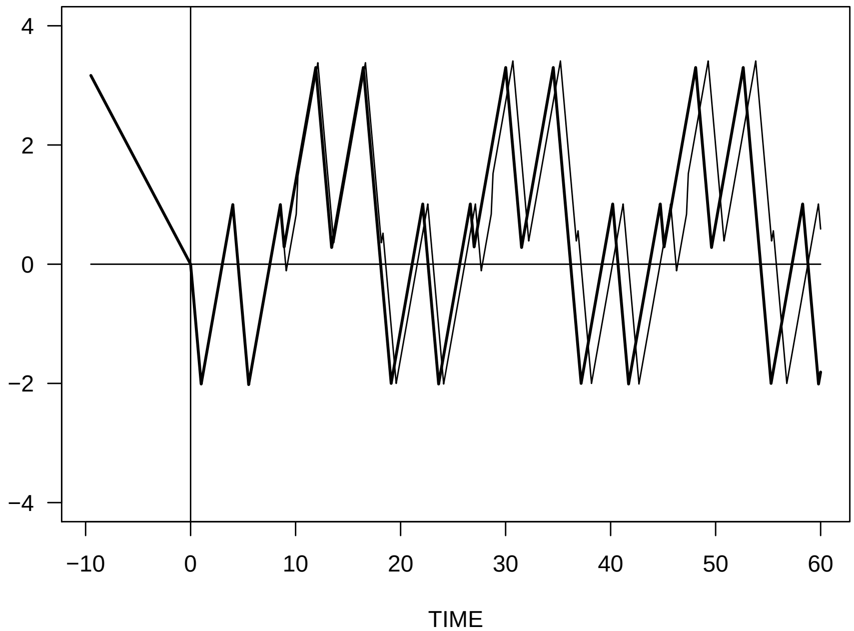
Write $\gamma + \tau$ for the first positive zero of y greater than or equal to γ. From time $\tau + 1$ to time 2γ, $y(t-1)$ is positive while $y(t-\gamma)$ is oscillating; in particular, y acts like a solution of

$$y'(t) = -h(-y(t-\gamma)),$$

shifted upward by $y(\gamma+3)$ units. Since $y(\gamma+3) \geq 3$, it follows that $y(t)$ is nonnegative for $t \in [\gamma+\tau, 2\gamma]$. Since $y(t-\gamma)$ is negative for $t \in (2\gamma, 2\gamma+\tau)$ (if $\tau > 0$), we see in fact that $y(t)$ is nonnegative for $t \in [\gamma+\tau, 2\gamma+\tau]$. We conclude that y coincides, for $t \geq 0$, with a very slowly oscillating $(k+1)$-cyclic periodic solution. The bifurcation points occur at positive integer multiples of $9/2$. Figure 23 shows bifurcation from a 2-cyclic to a 3-cyclic periodic solution. Both solutions have the nonnegative initial condition $y_0(t) = -t$. The bold line is a simulated solution for $\gamma = 8.9$; the other line is a simulated solution for $\gamma = 9.1$.

This example shows that periodic solutions of $(SDDE)$ of any cyclicity are possible. Moreover, since the zeros of y are separated either by 3, $3/2$, or more than γ units, y is simple; since y is very slowly oscillating, it is trivially nondegenerate and asymptotically stable. Therefore our main theorems 4.1.2 and 4.1.3 imply that, for differential delay equations with two delays, stable periodic solutions of any cyclicity are possible (including when there is strictly monotonic negative feedback in each delay).

Figure 23



Here are two basic questions for $(SDDE)$ with negative feedback that we have not been able to answer:

- Given $y_0 \in C_0'$, is the dimension of $V(y_t)$ for $t \geq 0$ bounded by some function (depending on $(SDDE)$) of the dimension of $V(y_0)$?
- Is there any sense in which "most" solutions converge to periodic solutions?

We close by presenting, for intrigue and amusement, some simulated solutions of $(SDDE)$ with negative feedback. Simulated solutions seem most often to have the general appearance pictured in Figure 24. Figure 25 is meant to illustrate the basic mechanism by which oscillations often slow down: successive critical points with no intervening sign changes. Figure 26 shows a solution that apparently converges to a 2-cyclic periodic solution. Figure 27 shows a simulated solution with no apparent convergence in short time (here $\gamma = 11.5$ and the simulated continuation is shown to time $t = 100$). Continuing the simulation further suggests that the solution might converge — slowly — to a periodic solution, but for this particular solution our simulations are not definitive.

These figures were found by generating, within a certain class, random problems $(SDDE)$ and initial conditions. Detailed information about the generation of the figures is available from the author.

Figure 24

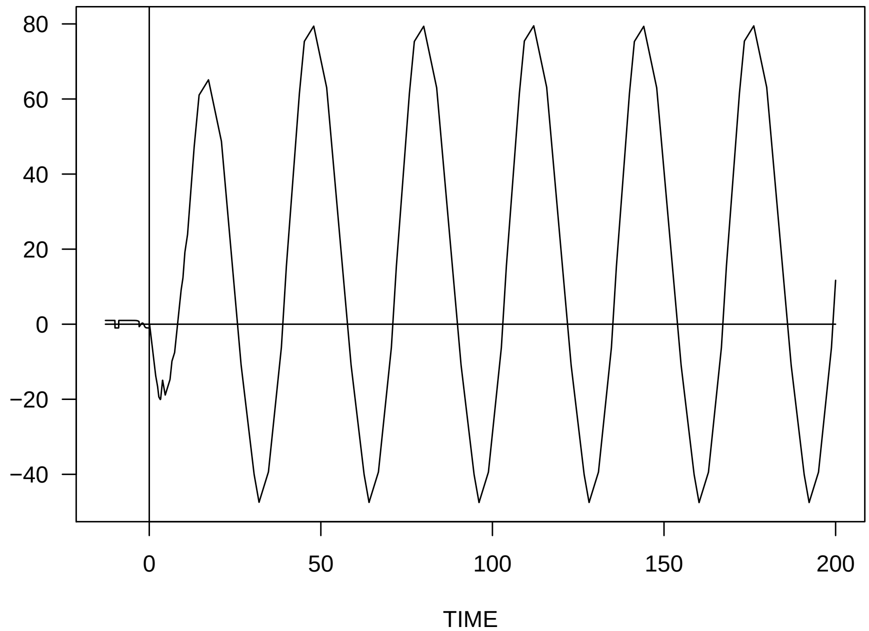


Figure 25

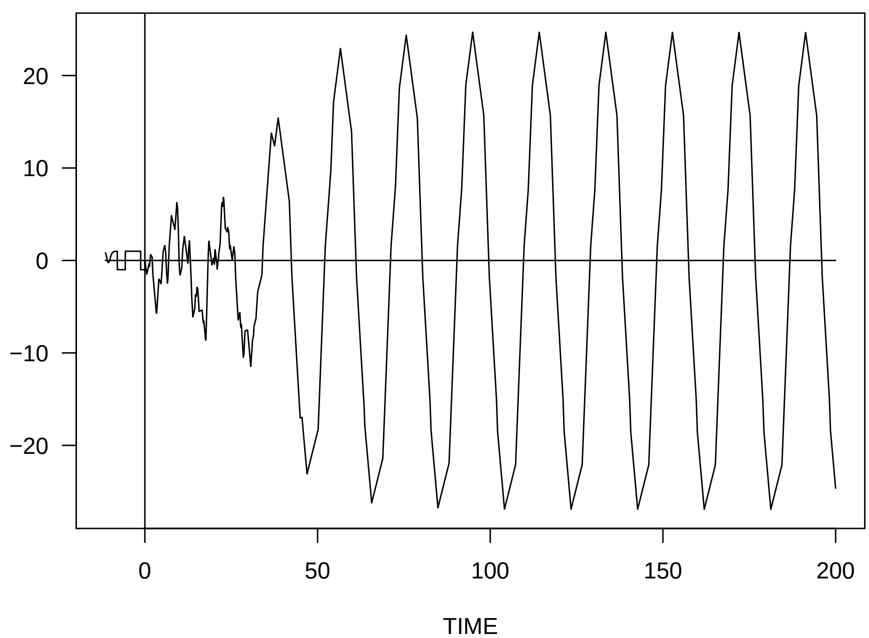


Figure 26

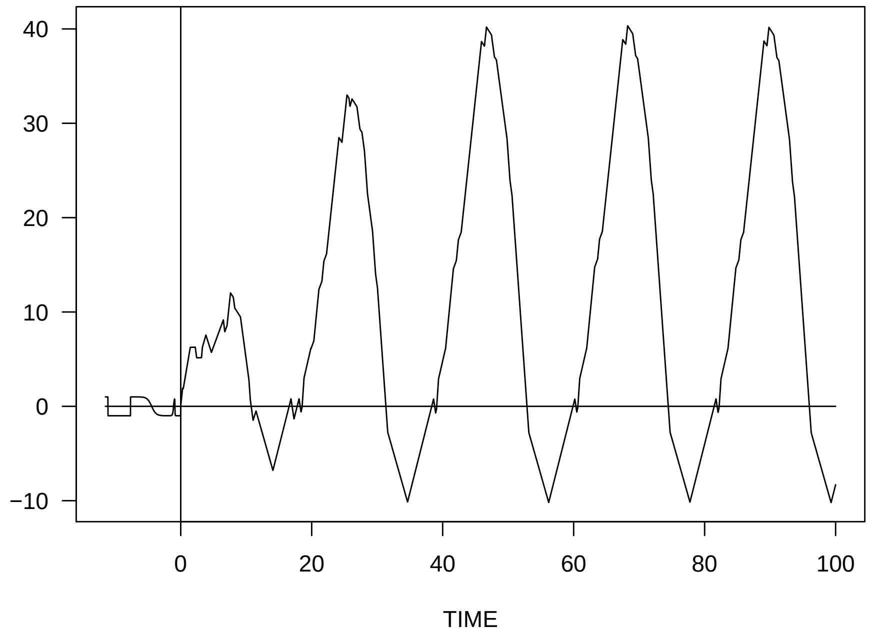
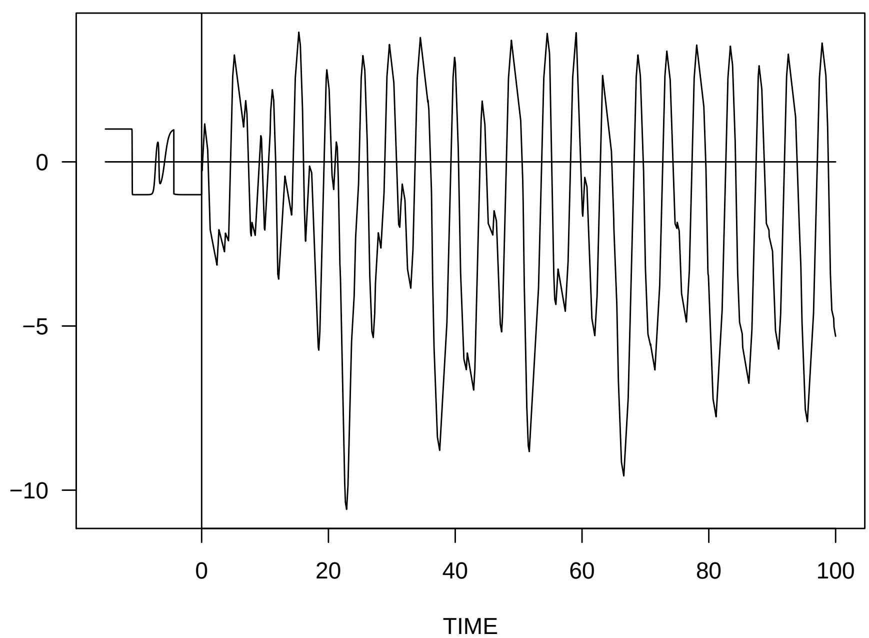


Figure 27



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