# ASYMPTOTIC BEHAVIOR AND DENJOY-WOLFF THEOREMS FOR HILBERT METRIC NONEXPANSIVE MAPS

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## ABSTRACT OF THE DISSERTATION

# Asymptotic Behavior and Denjoy-Wolff Theorems for Hilbert Metric Nonexpansive Maps

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We study the asymptotic behavior of fixed point free Hilbert metric nonexpansive maps on bounded convex domains. For such maps, we prove that the omega limit sets are contained in a convex subset of the boundary when the domain is either polyhedral or two dimensional. Similar results are obtained for several classes of positive operators defined on closed cones, including linear maps, affine linear maps, max-min operators, and reproduction-decimation operators. We discuss the relationship between these results and other Denjoy-Wolff type theorems. In particular, we investigate the interaction of nonexpansive maps with the horofunction boundary in the Hilbert geometry and in finite dimensional normed spaces.

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I think that everyone who has spent time in the math department at Rutgers will understand why I want to thank Stephen Greenfield. It is hard to imagine how one person can do so much for so many and yet always make it seem like nothing at all.

Of course, I cannot forget my parents. They always taught me to seek the things which matter most in life, and by their example, my parents have shown me how to find them.

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# Chapter 1

# Introduction

In his 1957 paper [7], Garrett Birkhoff demonstrated the utility of the Hilbert metric for studying the eigenvectors of linear maps that preserve a cone. He observed that a linear map which takes a cone into itself is nonexpanding with respect to Hilbert's projective metric. He also gave conditions under which a linear map is a strict contraction with respect to the Hilbert metric. Around the same time, Samelson [51] also observed the connection between the Hilbert metric and linear maps. Bushell [13] notes that any homogeneous of degree one map which is order-preserving with respect to a closed cone is nonexpansive with respect to Hilbert's projective metric. In this thesis, we call such maps positive operators.

Positive operators arise in a wide variety of applications. In some applications, such as the $DAD$-problem in linear algebra ([41], [44], [10]) and the existence problem for diffusions on fractals ([36], [50]), it is crucial to know whether or not a given positive operator has an eigenvector in the interior of a closed cone. Such problems are related to the Perron-Frobenius theorem which states that every irreducible nonnegative matrix has a unique eigenvector in the interior of the cone of nonnegative vectors. There are theorems which ensure, under suitable compactness conditions, that a continuous, order-preserving, homogeneous of degree one map on a closed cone always has an eigenvector in the cone with eigenvalue equal to the cone spectral radius of the map (e.g., see section 9 in [32] and Theorem 3.4 in [35]). However, these theorems do not establish the existence of an eigenvector in the interior of the cone. In fact, ascertaining whether or not such an eigenvector exists can be a very difficult problem, even in finite dimensions.

With the Hilbert metric, questions about eigenvectors of positive operators become questions about fixed points of nonexpansive maps. Nussbaum ([40], [41]) has used this approach to give conditions for finding eigenvectors in the interior of a cone, as has Sabot ([50], see also [37]). Of course, nonexpansive maps do not always have fixed points. The main focus of this thesis is to better understand the asymptotic behavior of Hilbert metric nonexpansive maps in the absence of fixed points.

There is a classical result in complex analysis concerning the iterates of fixed point free holomorphic maps on the open unit disc $D$ in $\mathbb{C}$. In 1926, Wolff [56] proved that if $f$ is a holomorphic map from the unit disc into itself and $f$ has no fixed points in $D$, then there is a point $z \in \partial D$ such that $f^k(x) \to z$ as $k \to \infty$ for every $x \in D$. Originally, Wolff assumed that $f$ extended continuously to the boundary, $\partial D$, but a few weeks later Denjoy [17] and Wolff [57] independently showed that this assumption is unnecessary. This result has come to be known as the Denjoy-Wolff theorem.

Beardon [5] has observed that the Denjoy-Wolff theorem in complex analysis can really be thought of as a geometrical result which applies to nonexpansive maps in a wide variety of metric spaces. In particular, he proves a version of the Denjoy-Wolff theorem for Hilbert metric nonexpansive maps on strictly convex domains (Theorem 1a, [6]).

The Hilbert metric nonexpansive maps that appear in applications are typically defined on domains which are not strictly convex. In chapter 3 of this thesis, we establish a Denjoy-Wolff type theorem for Hilbert metric nonexpansive maps on polyhedral domains and also for arbitrary convex domains in two dimensions. In section 3.5, we make an observation which amounts to a fixed point theorem for nonexpansive maps on finite dimensional normed spaces.

In chapter 4, we focus on the asymptotic behavior of iterates of linear and affine linear maps which preserve a cone. For these maps we are able to prove stronger results than the general Denjoy-Wolff type results found in chapter 3. Chapter 5 establishes a Denjoy-Wolff type theorem for a class of nonlinear operators used to study diffusion on fractals. This class of "reproduction-decimation" operators is defined on a domain which is neither strictly convex nor polyhedral, so the results of chapter 3 must be specially adapted for them.

In the final chapter, we consider order-preserving homogeneous of degree one maps on the standard cone in $\mathbb{R}^n$. We note their connection to topical maps and also construct an example which shows that the main theorem of chapter 3 is the strongest possible result for general Hilbert metric nonexpansive maps on polyhedral domains.

# Chapter 2

# Preliminaries

## 2.1 Closed Cones and Convex Sets

Let $X$ be a Banach space with norm $||\cdot||$. A *closed cone* is a closed convex set $C \subset X$ such that $\lambda C \subseteq C$ for all $\lambda \geq 0$ and $C \cap (-C) = \{0\}$. If $C$ has nonempty interior in $X$, then we let $\mathrm{int\,C}$ denote the interior of $C$. A closed cone $C$ induces a partial ordering $\leq_C$ on $X$ as follows: for any $x, y \in X$ we say that $x \leq_C y$ if $y - x \in C$. If there are positive real constants $\alpha$ and $\beta$ such that $\alpha x \leq_C y$ and $y \leq_C \beta x$, then we say that $x$ and $y$ are *comparable* and we write $x \sim_C y$. The relationship $\sim_C$ is an equivalence relation and the equivalence classes of the cone $C$ under $\sim_C$ are called the *parts* of $C$. Observe that $x \sim_C y$ for any two points $x, y \in \mathrm{int\,C}$, thus the interior of $C$ is a part. When the cone $C$ is understood, we write $\leq$ and $\sim$ instead of $\leq_C$ and $\sim_C$.

A closed cone in $C$ in a Banach space $X$ is called *normal* if there is a constant $M > 0$ such that $x \leq y$ implies that $||x|| \leq M||y||$ for all $x, y \in C$. A cone $C$ is called *reproducing* if $C - C = X$, that is $X = \{x - y \mid x, y \in C\}$. We say that $C$ is *total* if $\mathrm{cl\,(C - C) = X}$. Any closed cone in a finite dimensional normed space is normal, and a cone in a finite dimensional vector space is reproducing if and only if it has nonempty interior.

For any Banach space $X$, we let $X^*$ denote the dual space. If $C \subset X$ is a closed cone, we let $C^* = \{\varphi \in X^* \mid \varphi(x) \geq 0 \text{ for all } x \in C\}$. If $C$ is total, then $C^*$ is a closed cone in $X^*$ which we call the *dual cone* of $C$. A set $S \subset C^*\backslash\{0\}$ is called a *sufficient set* for $C$ if $C = \{x \in X \mid \varphi(x) \geq 0 \; \forall \varphi \in S\}$.

A subset $U$ of a normed space $X$ is *affine* if $(1-\lambda)x + \lambda y \in U$ for all $x, y \in U$ and $\lambda \in \mathbb{R}$. For any subset $U \subset X$, the *affine hull* of $U$ is the smallest affine set containing $U$, and is denoted $\mathrm{aff\,U}$. The norm closure of the affine hull, $\mathrm{cl\,(aff\,U)}$ inherits the relative topology from $X$. For any convex set $U \subset X$, we define the *relative interior* of $U$, $\mathrm{ri\,U}$, to be the union of all subsets of $U$ which are relatively open in $\mathrm{cl\,(aff\,U)}$. We will refer to any convex set which is relatively open in the closure of its affine hull as a *convex domain.* For any convex set $U \subset X$, we define the *boundary* of $U$ to be $\partial U = \mathrm{cl\,U}\backslash\mathrm{ri\,U}$. At first glance, this is not the usual topological definition of the boundary. However, it is the boundary with respect to the relative topology on $\mathrm{cl\,(aff\,U)}$. A convex set $U$ is called *strictly convex* if $\lambda x + (1-\lambda)y \in \mathrm{ri\,U}$ whenever $0 < \lambda < 1$ and $x, y \in \partial U$ with $x \neq y$. In other words, $U$ is strictly convex if $\partial U$ does not contain any line segments.

If $X$ is finite dimensional and $U \subset X$ is convex, then we follow the terminology of [49] in defining a *face* of $U$ to be a convex subset $F \subset U$ such that, if $\lambda x + (1-\lambda)y \in F$ for some $x, y \in U$ and $0 < \lambda < 1$, then $x, y \in U$. Note that if $U$ is closed, then the faces of $U$ are closed.

**Lemma 2.1.1** *Let $C$ be a closed cone in a finite dimensional Banach space $X$. The parts of $C$ are precisely the relative interiors of the faces of $C$.*

**Proof** By Theorem 18.2 in [49], the relative interiors of the faces of $C$ form a partition of $C$. Suppose that $F$ is a face of $C$. Note that $F$ is a closed cone. For any $x, y \in \mathrm{ri\,F}$, there exist $\alpha, \beta > 0$ such that $y - \alpha x \in F$ and $\beta x - y \in F$. Since $F \subset C$ it follows that $\alpha x \leq_C y \leq_C \beta x$ and therefore $x \sim_C y$. To complete the proof it suffices to show that, for any $x, y \in C$, if $x \sim_C y$ and $x \in \mathrm{ri\,F}$, then $y \in \mathrm{ri\,F}$. Note that if $x \sim_C y$, then there exists $\epsilon > 0$ small enough so that $x_\epsilon = (1+\epsilon)x - \epsilon y \sim_C x$ and $y_\epsilon = -\epsilon x + (1+\epsilon)y \sim_C y$. Since $x \in F$ and $x = ax_\epsilon + by_\epsilon$ where $a = (\epsilon+1)/(2\epsilon+1)$ and $b = \epsilon/(2\epsilon+1)$, it follows that $x_\epsilon$ and $y_\epsilon$ are in $F$. Then, since $x \in \mathrm{ri\,F}$ and $y = \lambda y_\epsilon + (1-\lambda)x$ when $\lambda = 1/(1+\epsilon)$, we see that $y \in \mathrm{ri\,F}$. ■

A convex set $U$ in a finite dimensional normed space $X$ is called *polyhedral* if it is the intersection of finitely many half-spaces (which may be either open or closed). If $U$ is a relatively open polyhedral subset of $\mathrm{aff\,U}$, then we say that $U$ is a *polyhedral domain.* Note that a closed cone $C$ is a polyhedral cone if and only if it has a finite sufficient set. It turns out that every face $F$ of a polyhedral cone $C$ is an exposed face, that is, there is a linear functional $\varphi \in C^*$ such that $F = \{x \in C \mid \varphi(x) = 0\}$. For details, see chapter 19 of [49]. If $\{\theta_1, ..., \theta_N\}$ is a minimal sufficient set for a polyhedral cone $C$ with nonempty interior, then we call the faces $F_i = \{x \in C \mid \theta_i(x) = 0\}$ the *facets* of $C$.

## 2.2 The Hilbert Metric

The Hilbert geometry provides an example of a metric space where the shortest connection between any two points is given by a straight line. Let $D$ be a bounded convex domain in a Banach space $X$. The definition of the Hilbert metric preferred by geometers makes use of the *cross ratio*:

$$[a, x, y, b] = \frac{||y-a||\,||x-b||}{||x-a||\,||y-b||}.$$

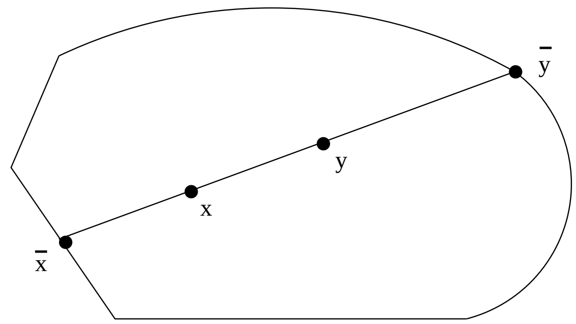


Figure 2.1: The definition of $d(x, y)$.

For any two points $x, y \in D$ the Hilbert metric $d$ is given by the logarithm of the cross ratio:

$$d(x, y) = \log\left([\bar{x}, x, y, \bar{y}]\right), \tag{2.1}$$

where $\bar{x}$ is the unique point in $\partial D$ which lies on the ray from $y$ passing through $x$ and $\bar{y} \in \partial D$ is the point on the ray from $x$ through $y$ (see figure 2.1).

An alternative formulation of the Hilbert metric has appeared in the analysis of positive operators. To introduce this construction, let us fix a closed cone $C$ in a Banach space $X$. For $x, y \in X$, note that $x \sim y$ if and only if there exist real numbers $\alpha, \beta > 0$ such that

$$\alpha x \leq y \leq \beta x. \tag{2.2}$$

Following the notation of [40] we define the *Hilbert projective metric* for points $x \sim y$ as

$$d(x, y) = \log\left(\frac{M(y/x)}{m(y/x)}\right), \tag{2.3}$$

where $M(y/x) = \inf\{\beta > 0 \mid y \leq \beta x\}$ and $m(y/x) = \sup\{\alpha > 0 \mid \alpha x \leq y\}$.

Hilbert's projective metric is not a true metric on the parts of $C$. However it does satisfy the following properties.

**Proposition 2.2.1** *Let $C$ be a closed cone in a Banach space $X$. Let $d$ denote Hilbert's projective metric on $C$. If $C_u$ is a part of $C$ and $x, y, z \in C_u$, then*

$$d(x, y) = 0 \text{ if and only if } y = \lambda x \text{ where } \lambda > 0, \tag{2.4}$$

$$d(x, y) = d(y, x), \tag{2.5}$$

$$d(x, z) \leq d(x, y) + d(y, z), \tag{2.6}$$

$$d(\alpha x, \beta y) = d(x, y) \text{ for any } \alpha, \beta > 0. \tag{2.7}$$

The proof of this proposition is elementary, and can be found in [13].

Suppose that $C$ is a closed cone with nonempty interior and that $S$ is a sufficient set for $C$. An alternative formula for Hilbert's projective metric on $\operatorname{int} \mathrm{C}$ is given below.

$$d(x, y) = \sup_{\chi, \psi \in S} \log \left( \frac{\chi(x)\psi(y)}{\chi(y)\psi(x)} \right) \quad \text{for } x, y \in \operatorname{int} \mathrm{C}. \tag{2.8}$$

To see that equation 2.3 and equation 2.8 are equivalent it suffices to prove the following characterization of $M(y/x)$ and $m(y/x)$.

**Lemma 2.2.1** *Let $C$ be closed cone with nonempty interior in a Banach space. Let $x, y \in \operatorname{int} \mathrm{C}$ and let $S \subset C^*$ be a sufficient set for $C$. Then*

$$M(y/x) = \sup_{\varphi \in S} \frac{\varphi(y)}{\varphi(x)}, \quad m(y/x) = \inf_{\varphi \in S} \frac{\varphi(y)}{\varphi(x)}.$$

**Proof** Let $b = \sup_{\varphi \in S} \varphi(y)/\varphi(x)$. Then for any $\varphi \in S$,

$$\varphi(bx - y) = b\varphi(x) - \varphi(y) \geq 0.$$

However, if $b' < b$, then there exists $\varphi \in S$ such that $\varphi(y)/\varphi(x) > b'$ and therefore $\varphi(b'x - y) < 0$. That means that $b'x - y \notin C$ and therefore $b = \inf\{\beta > 0 \mid y \leq \beta x\} = M(y/x)$.

Now, let $a = \inf_{\varphi \in S} \varphi(y)/\varphi(x)$. Then for any $\varphi \in S$,

$$\varphi(y - ax) = \varphi(y) - a\varphi(x) \geq 0.$$

This implies that $y - ax \in C$. If $a' > a$, then there is some $\varphi \in S$ such that $\varphi(y)/\varphi(x) < a'$. Therefore $\varphi(y - a'x) < 0$ and thus $a = \sup\{\alpha > 0 \mid \alpha x \leq y\} = m(y/x)$. ■

If $C$ is a polyhedral cone with nonempty interior, then it has a finite sufficient set $\{\theta_1, ..., \theta_N\}$ and equation 2.8 can be simplified as follows.

$$d(x, y) = \max_{1 \leq i,j \leq N} \log \left( \frac{\theta_i(x)\theta_j(y)}{\theta_i(y)\theta_j(x)} \right) \quad \text{for } x, y \in \operatorname{int} \mathrm{C}. \tag{2.9}$$

The projective metric defined by equation 2.3 is not a metric on the parts of a cone because it does not distinguish between two points on the same ray emanating from the origin. We can work around this difficulty by focusing on a projective subset of the cone. For a closed cone $C$ in a Banach space $X$, let $q : C \to \mathbb{R}$ be a homogeneous of degree one map such that $q(x) > 0$ for all $x \in C \backslash \{0\}$. For any part $C_u$ of $C$ we let $\Sigma_u = \{x \in C_u \mid q(x) = 1\}$ and we refer to $\Sigma_u$ as a *projective subset* of $C$. The projective metric $d$ is a metric when restricted to $\Sigma_u$. We call $d$ the *Hilbert metric* on $\Sigma_u$. Note that if $q$ is linear, then the projective subsets $\Sigma_u$ are convex.

We have now introduced two different Hilbert metrics. The metric defined by equation 2.1 applies to bounded convex domains in a Banach space, while the expression given in equation 2.3 is a metric on projective subsets of a cone. It turns out that the two definitions really are the same. This is established in equation 3.15 of [31], for example. We will give a proof here, for simplicity. Suppose that $C$ is a closed cone in a Banach space $X$, $C_u$ is a part of $C$, and $q : C \to [0, \infty)$ is a continuous homogeneous of degree one map such that $q(x) > 0$ for all $x \in C \backslash \{0\}$. Let $\Sigma_u = \{x \in C_u \mid q(x) = 1\}$ and suppose that $\Sigma_u$ is a bounded convex domain in $X$.

Let $\hat{d}$ denote Hilbert's projective metric on $C$, as given by equation 2.3, and let $d$ be the Hilbert metric on $\Sigma_u$ defined by equation 2.1. We will show that for any $x, y \in \Sigma_u$, $\hat{d}(x,y) = d(x,y)$. Note that if $x, y \in \Sigma_u$ and $x \neq y$, then $W = \operatorname{span}\{x, y\}$ is a two-dimensional subspace of $X$. Let $C_W = C \cap W$ and note that $C_W$ is a closed cone in $W$. Furthermore for any two elements $u, v \in C_W$, $u \leq_{C_W} v$ if and only if $u \leq_C v$. Therefore Hilbert's projective metric on $C_W$ agrees with Hilbert's projective metric on $C$ for any pair $u, v \in C_W$. Therefore it suffices to prove that $d(x,y) = \hat{d}(x,y)$ when $C$ is a closed cone with nonempty interior in a two-dimensional normed space. For such a cone, there will be a sufficient set $S = \{\chi, \psi\} \subset C^*$ containing exactly two elements. By equation 2.9,

$$\hat{d}(x,y) = \left| \log\left( \frac{\chi(x)\psi(y)}{\chi(y)\psi(x)} \right) \right|.$$

The line containing $x$ and $y$ intersects $\partial\Sigma_u$ at two points, $\bar{x}$ and $\bar{y}$. Furthermore the points on that line appear in the following order: $\bar{x}, x, y, \bar{y}$. Therefore, we may assume without loss of generality that $\chi(\bar{y}) = 0$ and that $\psi(\bar{x}) = 0$. We see that

$$\frac{||x - \bar{y}||}{||y - \bar{y}||} = \frac{\chi(x)}{\chi(y)} \quad \text{and} \quad \frac{||y - \bar{x}||}{||x - \bar{x}||} = \frac{\psi(y)}{\psi(x)}.$$

This implies that $d(x,y)$ and $\hat{d}(x,y)$ really are equivalent.

Note that if $D$ is a bounded convex domain in a Banach space $Y$, then we can think of $D$ as a projective subset of a cone in the Banach space $X = Y \times \mathbb{R}$. We let $C \subset X$ be the closed cone $C = \{(\lambda y, \lambda) \mid \lambda \geq 0 \text{ and } y \in \mathrm{cl}\, \mathrm{D}\}$ and $q((y,t)) = t$ for all $(y,t) \in Y \times \mathbb{R}$. Then $D$ can be identified with the projective subset $\Sigma_u = \{(y,1) \mid y \in D\}$, and the above argument shows that the Hilbert metric on $D$ can be thought of in terms of the definition given in equation 2.3.

If $D$ is a bounded convex domain in a finite dimensional normed space $X$, then each part of the cone $C = \{(\lambda x, \lambda) \mid \lambda \geq 0 \text{ and } x \in \operatorname{cl} \mathrm{D}\}$ will correspond to the relative interior of a face of $\operatorname{cl} \mathrm{D}$ and vice versa. Thus it makes sense to refer to the relative interiors of the faces of $\operatorname{cl} \mathrm{D}$ as the parts of $\operatorname{cl} \mathrm{D}$ and to write $x \sim_D y$ whenever $x$ and $y$ are contained in the same part of $\operatorname{cl} \mathrm{D}$.

We have seen that any bounded convex domain in a Banach space can be naturally identified with a bounded convex projective subset of a cone. Not every projective subset of a cone will be bounded or convex, however. If $q : X \to \mathbb{R}$ is a linear functional such that $q(x) > 0$ for all $x \in C\backslash\{0\}$, then $S = \{x \in C \mid q(x) = 1\}$ will be convex. Unfortunately, in infinite dimensions, it may not be possible to find a linear functional $q \in C^*$ such that $S$ is bounded. The following lemma shows that we can find such a functional in finite dimensions.

**Lemma 2.2.2** *If $X$ is a finite dimensional vector space and $C$ is a closed cone in $X$, then there is a linear functional $q \in C^*$ such that $q(x) > 0$ for all $x \in C\backslash\{0\}$. Furthermore, the set $S_R = \{x \in C \mid q(x) = R\}$ is bounded in $X$ for all $R \geq 0$.*

**Proof** Let $\Sigma = \{x \in C : ||x|| = 1\}$. Then $\operatorname{co} \Sigma$ is a closed convex subset of $C$. Note that $0 \notin \operatorname{co} \Sigma$. Otherwise, $0 = (1 - \lambda)x + \lambda y$ for some $x, y \in \operatorname{co} \Sigma\backslash\{0\}$ and $\lambda \in (0, 1)$, which would imply that $-y \in C$, a contradiction. Since $0 \notin \operatorname{co} \Sigma$, the Hahn-Banach theorem implies that there is a linear functional $q \in X^*$ such that $q(x) > 0$ for all $x \in \operatorname{co} \Sigma$. Therefore $q(x) > 0$ for all $x \in C\backslash\{0\}$ and $q \in C^*$.

The set $C \cap \{x \in X : ||x|| = 1\}$ is compact and therefore there exists $\epsilon > 0$ such that $q(x) > \epsilon$ for all $x \in C$ with $||x|| = 1$. This implies that $||x|| \leq R/\epsilon$ for any $x \in C$ with $q(x) \leq R$ and therefore $S_R = \{x \in C \mid q(x) = R\}$ is a bounded subset of $X$. ■

**Remark 2.1** Suppose that $X$ is the Banach space of continuous real valued functions

$C(M, \mathbb{R})$ on a compact Hausdorff space $M$, with norm $||f|| = \sup_{x \in M} |f(x)|$. Let $K$ be the cone of nonnegative functions on $M$ and assume that $M$ is not finite. We claim that there does not exist a $\varphi \in K^*\backslash\{0\}$ such that the set

$$\Sigma = \{f \in \mathrm{int\,K} \mid \varphi(\mathrm{f}) = 1\}$$

is bounded in norm. Fix an element $\varphi \in K^*\backslash\{0\}$. By the Riesz representation theorem there is a regular Borel measure $\mu$ on $M$ such that $\mu(M) < \infty$ and such that $\varphi(f) = \int f d\mu$ for all $f \in X$. The set $M$ is not finite and therefore, for every $\epsilon > 0$, there is a point $x_\epsilon \in M$ such that $\mu(\{x_\epsilon\}) < \epsilon$. Since $\mu$ is regular, it follows that there is an open set $U$ in $M$ such that $x_\epsilon \in U$ and $\mu(U) < \epsilon$. For any $R > 0$, we may use Urysohn's lemma to construct a function $f_R \in K$ such that $f_R(x_\epsilon) = R$, $f_R = 0$ on $M\backslash U$, and $||f_R|| = R$. Thus, $\int f_R\, d\mu \leq R\epsilon$. By choosing $\epsilon$ small enough, we may ensure that $\int f_R\, d\mu \leq 1$. From this, we see that for any $\varphi \in K^*$, $\varphi \neq 0$, the set $\Sigma = \{f \in \mathrm{int\,K} \mid \varphi(\mathrm{f}) = 1\}$ is unbounded in $X$.

We will now turn our attention to some of the important properties of the Hilbert geometry. In a metric space $(M, d_M)$, a *minimal geodesic* from $x \in M$ to $y \in M$ is a continuous path $\gamma : [0,1] \to M$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $d_M(x,y) = d_M(x,\gamma(t)) + d_M(\gamma(t), y)$ for all $0 < t < 1$. One of the characteristic properties of the Hilbert metric on a bounded convex domain is that straight lines are minimal geodesics.

**Proposition 2.2.2** *Suppose that $D$ is a bounded convex domain in a Banach space $X$. Let $x, y, z$ be elements in $D$ such that $z$ lies on the line segment $[x, y]$. Then $d(x,y) = d(x,z) + d(z,y)$ and $[x,y]$ is a minimal geodesic connecting $x$ to $y$.*

This propostion is well known and a proof can be found in Proposition 1.9 of [40], for example. Note that the minimal geodesics in the Hilbert geometry may not be unique.

However, if $D$ is a bounded and strictly convex domain in a Banach space $X$, then the line segment $[x, y]$ is the unique minimal geodesic from $x$ to $y$ (see propostion 1.11, [40]).

Suppose that $C$ is a closed normal cone with nonempty interior in a Banach space $X$. If $\Sigma = \{x \in \mathrm{int\,C} \mid ||\mathrm{x}|| = 1\}$ then Nussbaum points out in remark 1.4 of [40] that the norm topology on $\Sigma$ is identical to the topology induced by the Hilbert metric $d$. In particular, $(\Sigma, d)$ is a complete metric space. Because any bounded convex domain in $X$ can be represented as a projective subset of a cone in $X \times \mathbb{R}$, remark 1.4 of [40] establishes the following lemma.

**Lemma 2.2.3** *Let $D$ be a bounded convex domain with Hilbert metric $d$ in a finite dimensional normed space $X$. Then $(D, d)$ is a complete metric space and the topology induced by $d$ is equivalent to the norm topology on $D$.*

Note that Zabreiko, Kransnoselskii, and Pokornyi give general conditions in [59] which imply that a projective subset of a cone is a complete metric space. Birkhoff gives somewhat weaker conditions in Theorem 5 of [8].

Another useful property of the Hilbert geometry is that Hilbert metric balls are convex. This is proved as Lemma 4.1 in [40]. We restate the lemma here for convenience.

**Lemma 2.2.4** *If $D$ is a bounded convex domain with the Hilbert metric $d$, then the ball $B_R(x) = \{y \in D \mid d(x, y) \leq R\}$ is convex for any $x \in D$ and $R > 0$.*

The Hilbert geometry on bounded polyhedral domains has additional structure which we will make use of later. A bounded polyhedral domain equipped with the Hilbert metric can be isometrically embedded into a subset of a finite dimensional normed space. This fact will be the key to proving Theorem 3.2.2 in the following chapter. Recall that the supremum norm $||\cdot||_\infty$ on $\mathbb{R}^n$ is defined to be $||x||_\infty = \max_{1\leq i\leq n}|x_i|$ where $x_i$ is the $i^{th}$ entry of $x$.

**Lemma 2.2.5** *If $D$ is a bounded polyhedral domain in a finite dimensional normed space $X$, then there is an isometric embedding of $(D, d)$ into a subset of $(\mathbb{R}^{N\times N}, ||\cdot||_\infty)$ where $N$ is an integer depending on $D$.*

**Proof** Let $C$ be the cone $C = \{(\lambda x, \lambda) \mid \lambda \geq 0 \text{ and } x \in \operatorname{cl} \mathrm{D}\}$. Since $D$ is a polyhedral domain, $C$ is a polyhedral cone. We may assume that $C$ has nonempty interior, otherwise we restrict ourselves to the subspace of $X \times \mathbb{R}$ spanned by $C$. Since $C$ is polyhedral, it has a finite sufficient set $\{\theta_1, ..., \theta_N\} \subset C^*$, where $N$ is the number of facets of $C$. Let $\hat{x} = (x, 1)$ for each $x \in D$. By equation 2.9 we have

$$d(x, y) = \sup_{1\leq i,j\leq N} \log\left(\frac{\theta_i(\hat{x})\theta_j(\hat{y})}{\theta_j(\hat{x})\theta_i(\hat{y})}\right),$$

for any $x, y \in D$. If $\Phi : D \to \mathbb{R}^{N\times N}$ is given by $\Phi_{ij}(x) = \log(\theta_i(\hat{x})/\theta_j(\hat{x}))$, we see immediately that $\Phi$ is one-to-one and $||\Phi(x) - \Phi(y)||_\infty := \max_{ij}|\Phi_{ij}(x) - \Phi_{ij}(y)| = d(x, y)$. Therefore $\Phi$ is an isometric embedding from $D$ with the Hilbert metric $d$ into a subset of $\mathbb{R}^{N\times N}$ with the sup-norm $||\cdot||_\infty$. ■

**Remark 2.2** The embedding described in Lemma 2.2.5 maps $D$ into a subset of a finite dimensional normed space. Foertsch and Karlsson prove in [19] that $D$ is isometric to a normed linear space if and only if $D$ is a simplex.

Thompson's metric is another metric which arises in applications to positive operators and it is closely related to the Hilbert metric. Unlike the Hilbert metric, which is only a projective metric on the parts of a closed cone, the Thompson metric will be a true metric on each part of a cone. It is often referred to as the part metric. Let $C$ be a closed cone in a Banach space $X$ and let $C_u$ be a part of $C$. For any $x, y \in C_u$ we define Thompson's metric $\bar{d}$ to be

$$\bar{d}(x, y) = \max(\log M(y/x), \log M(x/y)). \tag{2.10}$$

Like the Hilbert metric, balls in the Thompson metric are convex (see Lemma 4.2, [40]). If $C$ is a normal cone with nonempty interior, then the topology induced by $\bar{d}$ is the same as the norm topology on $\mathrm{int\,C}$ (see remark 1.4, [40]). The following lemma shows that we can isometrically embed any part of a polyhedral cone with the Thompson metric into a subset of a finite dimensional normed space.

**Lemma 2.2.6** *Let $C$ be a closed polyhedral cone in a finite dimensional normed space and suppose that $C_u$ is a part of $C$. Then there is an isometric embedding of $(C_u, \bar{d})$ into a subset of $(\mathbb{R}^N, ||\cdot||_\infty)$ where $N$ is an integer depending on $C_u$.*

**Proof** We may assume that $C_u = \mathrm{int\,C}$ since $C_u$ is the interior of $\mathrm{cl\,C_u}$ in $\mathrm{aff\,C_u}$. Since $C$ is polyhedral there is a finite sufficient set $\{\theta_1, ..., \theta_N\} \subset C^*$, where $N$ is the number of facets of $C$. Let $\Phi : \mathrm{int\,C} \to \mathrm{\mathbb{R}^N}$ be the map given by $\Phi_i(x) = \log(\theta_i(x))$. By Lemma 2.2.1,

$$\log M(x/y) = \sup_{1 \le i \le N} (\Phi_i(x) - \Phi_i(y)) \text{ and}$$

$$\log M(y/x) = \sup_{1 \le i \le N} (\Phi_i(y) - \Phi_i(x)) \text{ for any } x, y \in \mathrm{int\,C}.$$

Therefore $\bar{d}(x, y) = ||\Phi_i(x) - \Phi_i(y)||_\infty$, and $\Phi$ is an isometric embedding of $(\mathrm{int\,C}, \bar{\mathrm{d}})$ into a subset of $(\mathbb{R}^N, ||\cdot||_\infty)$. ■

## 2.3 Nonexpansive Maps and Omega Limit Sets

Suppose that $M$ is a metric space with metric $d_M$. We say that a map $f : M \to M$ is *nonexpansive* with respect to $d_M$ if $d_M(f(x), f(y)) \leq d_M(x, y)$ for all $x, y \in M$. We say that $f$ is a *contraction* if $d_M(f(x), f(y)) < d_M(x, y)$ for all $x, y \in M$. If there is a constant $c < 1$ such that $d_M(f(x), f(y)) \leq c d_M(x, y)$ for all $x, y \in M$, then we say that $f$ is a *strict contraction*. The contraction mapping principle tells us that if $(M, d_M)$ is a complete metric space and $f : M \to M$ is a strict contraction, then $f$ has a unique fixed point in $M$. Moreover, if $x$ is the fixed point of $f$ and $y$ is any other point in $M$, then the iterates of $y$ under repeated application of $f$ will converge to $x$. That is, $\lim_{k\to\infty} f^k(y) = x$ for all $y \in M$.

If $f$ is nonexpansive and $f$ has a fixed point $x \in M$, then the orbit $\mathcal{O}(y; f) = \{f^k(y) \mid k \geq 0\}$ of any other point $y \in M$ will remain within a bounded distance of $x$. The behavior of orbits is quite different when $f$ does not have a fixed point in $M$. The following theorem appears in [40] as Theorems 4.2 and 4.4 where it is stated specifically for the Hilbert and Thompson metrics. The proof is a consequence of a theorem of Całka (Theorem 5.6, [14]) which states that, if an orbit of a nonexpansive map in a finitely totally bounded metric space contains a bounded subsequence, then the whole orbit is bounded. A metric space $M$ is *finitely totally bounded* if any bounded subset of $M$ can be covered by finitely many balls of radius $\epsilon$, for every $\epsilon > 0$. In particular, a proper metric space will satisfy the conditions of Całka's theorem.

**Theorem 2.3.1** *Let $M$ be a convex domain in a finite dimensional normed space $X$. Suppose that $d_M$ is a metric on $M$ such that every open ball in $(M, d_M)$ is convex. Furthermore, suppose that the topology on $M$ induced by $d_M$ is equivalent to the norm* *topology. If $f : M \to M$ is nonexpansive with respect to $d_M$ and $f$ has no fixed point in $M$, then for any compact subset $K \subset M$ and any $x \in M$ there exists $N \geq 0$ such that $f^k(x) \in M \backslash K$ for all $k \geq N$. In particular, if $\mathcal{O}(x; f)$ is bounded in norm, then* $\lim_{k\to\infty} \inf\{||f^k(x) - y|| : y \in \partial M\} = 0$.

**Proof** If $K \subset M$ is compact, then $K$ is bounded with respect to $d_M$. Suppose that $f^k(x) \subset K$ for infinitely many $k \geq 0$. Then by Theorem 5.6 in [14] it follows that $\{f^k(x)\}_{k\geq 0}$ is bounded with respect to $d_M$. Choose $R > \text{diam}(\mathcal{O}(x; f))$ where the diameter is measured in the $d_M$-metric. Let $B_R(y) = \{x \in M \mid d_M(x, y) \leq R\}$ for each $y \in M$. The set $U = \bigcap_{k\geq 0} B_R(f^k(x))$ is a bounded, nonempty, closed (in the norm topology), convex subset of $M$. Note that $f(U) \subset U$. Therefore $U$ contains a fixed point of $f$ by the Brouwer fixed point theorem. This contradicts the hypothesis, so we concluded that only finitely many $f^k(x)$ are contained in $K$. If $f^k(x)$ is bounded in norm, we conclude that $\lim_{k\to\infty} \inf\{||f^k(x) - y|| : y \in \partial M\} = 0$. ■

If $(M, d_M)$ is a metric space and $f : M \to M$ is a map, then for any $x \in M$ the *omega limit set* of $x$ is defined to be

$$\omega(x; f) = \bigcap_{N\geq 0} (\mathrm{cl} \bigcup_{\mathrm{k=N}}^{\infty} \mathrm{f^k(x)}). \tag{2.11}$$

The omega limit set can be thought of as the set of accumulation points of the orbit $\mathcal{O}(x; f)$. In fact, if $y \in \omega(x; f)$, then there is a sequence of integers $k_i$ such that $f^{k_i}(x) \to y$ as $i \to \infty$. Dafermos and Slemrod have shown (Theorem 1, [16]) that if $(M, d_M)$ is a complete metric space and $f : M \to M$ is a $d_M$-nonexpansive mapping such that $\omega(x; f)$ is a nonempty subset of $M$ for some $x \in M$, then $f$ restricted to $\omega(x; f)$ is an invertible isometry onto $\omega(x; f)$ and if $y \in \omega(x; f)$, then $\omega(y; f) = \omega(x; f)$.

Actually, Dafermos and Slemrod restrict their attention to Banach spaces, but their argument also applies to complete metric spaces.

When $M$ is contained in a Banach space $X$ and the topology induced by $d_M$ is equivalent to the topology on $M$ induced by the norm, the definition of omega limit set given by equation 2.11 is ambiguous. It is not clear whether we are to take the closure in the norm topology or the metric topology. Because we wish to allow $\omega(x; f)$ to contain points on the boundary of $M$, we will take the closure in the norm topology when applying equation 2.11. This gives us an alternative formula for the omega limit set,

$$\omega(x; f) = \{z \in \mathrm{cl\,M} \mid \exists\, \mathrm{a\ sequence\ k_i\ such\ that} \lim_{\mathrm{i}\to\infty} ||\mathrm{f}^{\mathrm{k_i}}(\mathrm{x}) - \mathrm{z}|| = 0\}. \tag{2.12}$$

Note that if the hypotheses of Theorem 2.3.1 are satisfied, then $\omega(x; f) \subset \partial M$ for all $x \in M$. If $\omega(x; f) \subset \partial M$, then the results of Dafermos and Slemrod can fail. However, we will show that this will not be the case for several important classes of Hilbert metric nonexpansive maps. See Theorem 4.2.1 and Theorem 6.2.2.

We are primarily interested in the dynamics of Hilbert metric nonexpansive maps and for the remainder of this section we will collect facts which are particular to the omega limit sets of such maps. The following result is Lemma 5.1 in [45]. An earlier version, which only applies to finite dimensional spaces, can be found in Lemma 1 of [34].

**Lemma 2.3.1** *Let $C$ be a closed cone in a Banach space $(X, ||\cdot||)$. Suppose that $\{x^k\}_{k\in\mathbb{N}}$ and $\{y^k\}_{k\in\mathbb{N}}$ are sequences in $C$ such that $x^k \sim y^k$ and $d(x^k, y^k) \le R < \infty$ for all $k \ge 1$. If $\lim_{k\to\infty} ||x^k - \zeta|| = 0$ and $\lim_{k\to\infty} ||y^k - \eta|| = 0$, where $\zeta \ne 0$ and $\eta \ne 0$, then $\zeta$ and $\eta$ are comparable and $d(\zeta, \eta) \le R$.*

**Proof** Since $\zeta, \eta \in C\backslash\{0\}$, there exist linear functionals $\varphi_1, \varphi_2 \in C^*$ with $\varphi_1(\zeta) > 0$ and $\varphi_2(\eta) > 0$. We define $\varphi = \varphi_1 + \varphi_2$, so that $\varphi(\zeta) > 0$ and $\varphi(\eta) > 0$. Let $\zeta' = \zeta/\varphi(\zeta)$ and $\eta' = \eta/\varphi(\eta)$. Let $x_*^k = x^k/\varphi(x^k)$ and $y_*^k = y^k/\varphi(y^k)$ for $k$ large. Then, $\lim_{k\to\infty} ||x_*^k - \zeta'|| = 0$, $\lim_{k\to\infty} ||y_*^k - \eta'|| = 0$, $d(x_*^k, y_*^k) = d(x^k, y^k) \leq R$, and $d(\zeta', \eta') = d(\zeta, \eta)$. Thus, we may as well assume from the beginning that $\varphi(x^k) = \varphi(y^k) = 1$ and $\varphi(\zeta) = \varphi(\eta) = 1$.

For each $k \in \mathbb{N}$, there exist $\alpha_k > 0$ and $\beta_k > 0$ with $\alpha_k x^k \leq y^k \leq \beta_k x^k$ and $\log(\beta_k/\alpha_k) \leq R$. Since $\varphi \in C^*$ it follows that $\varphi(x) \leq \varphi(y)$ whenever $x \leq y$. Therefore

$$\alpha_k = \alpha_k\varphi(x^k) \leq \varphi(y^k) = 1 \quad \text{and} \quad 1 = \varphi(y^k) \leq \varphi(\beta_k x^k) = \beta_k.$$

Since $\beta_k/\alpha_k \leq \exp(R)$ we deduce that

$$\beta_k \leq \alpha_k \exp(R) \leq \exp(R) \quad \text{and} \quad \alpha_k \geq \beta_k \exp(-R) \geq \exp(-R).$$

By taking a subsequence we can assume that $\alpha_k \to \alpha > 0$ and $\beta_k \to \beta < \infty$, and we deduce that $\alpha\zeta \leq \eta \leq \beta\zeta$, with $\log(\beta/\alpha) \leq R$. ■

An immediate corollary of Lemma 2.3.1 is the following result.

**Lemma 2.3.2** *Let $D$ be a bounded convex domain in a finite dimensional normed space $X$ and let $d$ denote the Hilbert metric on $D$. Suppose that $f : D \to D$ is nonexpansive with respect to $d$. For any $x, y \in D$, if $\zeta \in \omega(x; f)$, then there is an $\eta \in \omega(y; f)$ such that $\eta \sim_D \zeta$ and $d(\zeta, \eta) \leq d(x, y)$.*

Another result which can be derived from Lemma 2.3.1 is the following lemma.

**Lemma 2.3.3** *Let $D$ be a bounded convex domain in a finite dimensional normed space $X$. Suppose that $f : D \to D$ is nonexpansive with respect to the Hilbert metric $d$ on $D$*

*and that* $z \in \omega(x; f)$ *for some* $x \in D$*. If* $f$ *extends continuously to* $z$*, then* $f(z) \sim_D z$ *and* $d(f(z), z) \le \inf_{k\ge 0} d(f^{k+1}(x), f^k(x))$*.*

**Proof** Suppose that $f^{k_i}(x) \to z$ in norm as $i \to \infty$. Then $f^{k_i+1}(x) \to f(z)$. Since $f$ is nonexpansive $d(f^{k_i+1}(x), f^{k_i}(x))$ is a decreasing sequence. By Lemma 2.3.1 $d(f(z), z) \le d(f^{k_i+1}(x), f^{k_i}(x))$ for all $i \ge 0$. Therefore $d(f(z), z) \le \lim_{i\to\infty} d(f^{k_i+1}(x), f^{k_i}(x)) = \inf_{k\ge 0} d(f^{k+1}(x), f^k(x))$. ■

We are particularly interested in omega limit sets which are contained in the boundary of $D$. The following lemma shows that if one omega limit set is contained in a convex subset of the boundary, then all of the omega limit sets will be contained in a convex subset of the boundary.

**Lemma 2.3.4** *Let* $D$ *be a bounded convex domain in a finite dimensional normed space* $X$ *and suppose that* $f : D \to D$ *is nonexpansive with respect to the Hilbert metric* $d$ *on* $D$*. If for some* $x \in D$*,* $\mathrm{co}\,(\omega(\mathrm{x};\mathrm{f})) \subset \partial\mathrm{D}$*, then* $\mathrm{co}\,(\bigcup_{\mathrm{y}\in\mathrm{D}} \omega(\mathrm{y};\mathrm{f})) \subset \partial\mathrm{D}$*.*

**Proof** If $f$ contains a fixed point, then $\omega(x; f) \subset D$ for all $x \in D$, and therefore there is nothing to prove. If $f$ does not contain a fixed point in $D$, then $\omega(y; f) \subset \partial D$ for all $y \in D$ by Theorem 2.3.1. Each element $z \in \omega(x; f)$ is contained in a part $D_z$ of $\mathrm{cl}\,\mathrm{D}$ and I claim that $\mathrm{co}\,(\bigcup_{\mathrm{z}\in\omega(\mathrm{x};\mathrm{f})} \mathrm{D_z})$ is contained in $\partial D$. Suppose that $y \in \mathrm{co}\,(\bigcup_{\mathrm{z}\in\omega(\mathrm{x};\mathrm{f})} \mathrm{D_z})$. Then $y = \sum_{i=1}^n \lambda_i \zeta_i$ where each $\lambda_i \ge 0$, $\sum_{i=1}^n \lambda_i = 1$, and $\zeta_i \sim_D z_i$ where each $z_i \in \omega(x; f)$. Therefore $y$ is comparable to $\sum_{i=1}^n \lambda_i z_i$. Since $\sum_{i=1}^n \lambda_i z_i \in \mathrm{co}\,(\omega(\mathrm{x};\mathrm{f})) \subset \partial\mathrm{D}$, it follows that $y \in \partial D$. Therefore $\mathrm{co}\,(\bigcup_{\mathrm{z}\in\omega(\mathrm{x};\mathrm{f})} \mathrm{D_z}) \subset \partial\mathrm{D}$. By Corollary 2.3.2, $\omega(y; f) \subset \bigcup_{z\in\omega(x;f)} D_z$ for all $y \in D$. Therefore $\mathrm{co}\,(\bigcup_{\mathrm{y}\in\mathrm{D}} \omega(\mathrm{y};\mathrm{f})) \subset \mathrm{co}\,(\bigcup_{\mathrm{z}\in\omega(\mathrm{x};\mathrm{f})} \mathrm{D_z}) \subset \partial\mathrm{D}$. ■

Lemmas 2.3.2 through 2.3.4 are given as Theorem 1 in [34]. Infinite dimensional versions of these lemmas are proved by Nussbaum in Theorem 5.3 of [45].

## 2.4 Positive Operators

Suppose that $C$ is a closed cone in a Banach space $X$. For any subset $U \subset X$ and any Banach space $Y$, we say that a map $f : U \to Y$ is a *homogeneous of degree one* if $f(\lambda x) = \lambda f(x)$ for all $x \in U$ and $\lambda > 0$. A map $f : U \to X$ is *order-preserving* with respect to the partial ordering on $C$ if $f(x) \leq_C f(y)$ whenever $x \leq_C y$. Order-preserving and homogeneous of degree one maps on a cone are nonexpansive with respect to Hilbert's projective metric. They are also nonexpansive with respect to Thompson's metric.

**Theorem 2.4.1** *Let $C$ be a closed cone in a Banach space $X$. Let $d$ denote Hilbert's projective metric on $C$ and let $\bar{d}$ denote Thompson's metric on $C$. Suppose that $C_u$ is a part of $C$ and $f : C_u \to C$ is homogeneous of degree one and order-preserving. Then for any $x, y \in C_u$, $d(f(x), f(y)) \leq d(x, y)$ and $\bar{d}(f(x), f(y)) \leq \bar{d}(x, y)$.*

**Proof** Since $x, y \in C_u$ there are constants $\alpha, \beta > 0$ such that

$$\alpha x \leq y \leq \beta x.$$

Since $f$ is homogeneous of degree one and order-preserving,

$$\alpha f(x) \leq f(y) \leq \beta f(x).$$

It follows immediately from equations 2.3 and 2.10 that $d(f(x), f(y)) \leq d(x, y)$ and $\bar{d}(f(x), f(y)) \leq \bar{d}(x, y)$. ■

Let $q : C \to \mathbb{R}$ be a continuous homogeneous of degree one map with $q(x) > 0$ for all $x \in C \backslash \{0\}$. Let $C_u$ be a part of $C$ and suppose that $f : C_u \to C_u$ is order-preserving and homogeneous of degree one. We define $\Sigma_u = \{x \in C_u \mid q(x) = 1\}$. For any $x \in \Sigma_u$, let $\hat{f}(x) = f(x)/q(f(x))$, so that $\hat{f} : \Sigma_u \to \Sigma_u$. Theorem 2.4.1 and equation 2.4 imply that the map $\hat{f} : \Sigma_u \to \Sigma_u$ is *nonexpansive* with respect to $d$, that is, for any $x, y \in \Sigma_u$, $d(\hat{f}(x), \hat{f}(y)) \leq d(x, y)$.

Note that $x \in C$ with $q(x) = 1$ is an eigenvector of $f$ if and only if $\hat{f}(x) = x$. It is often useful to study the iterates of the normalized map $\hat{f}$ instead of the iterates of $f$ itself. The iterates of $f$ may diverge while the iterates of $\hat{f}$ remain bounded. In chapter 2 of [40], Nussbaum looks at conditions where the iterates $\hat{f}^k(x)$ approach an eigenvector of $f$. One of the main motivations for this thesis is to study the orbit $\mathcal{O}(x; \hat{f})$ when there is no eigenvector of $f$ in the part of the cone containing $x$.

Let $X$ be a Banach space and let $\mathcal{B}(X)$ denote the set of bounded linear operators from $X$ into $X$. For any $A \in \mathcal{B}(X)$ the *spectral radius* of $A$ is

$$r(A) = \lim_{k \to \infty} ||A^k||^{1/k}. \tag{2.13}$$

For order-preserving homogeneous of degree one maps on a cone there is a notion of spectral radius which is similar to the spectral radius defined above. Let $C$ be a closed cone in $X$ and let $f : C \to C$ be an order-preserving homogeneous of degree one map. We let

$$||f||_C = \sup\{||f(x)|| : x \in C \text{ and } ||x|| \leq 1\}. \tag{2.14}$$

We then define the *cone spectral radius* $r_c(f)$ to be

$$r_C(f) = \lim_{k \to \infty} ||f^k||_C^{1/k}. \tag{2.15}$$

Suppose that $C$ is a closed cone with nonempty interior in a finite dimensional normed space $X$ and $f : C \to C$ is order-preserving and homogeneous of degree one. Then general versions of the Krein-Rutman theorem (see section 9 in [32] and also Theorem 3.4 in [35]) ensure that there is an eigenvector $u \in C$ such that $f(u) = r_C(f)u$. If

$x \in \operatorname{int} \mathrm{C}$, there is some constant $\alpha > 0$ such that $\alpha u \leq x$ and therefore $\alpha ||f^k(u)||_C \leq M||f^k(x)||_C$ for some fixed $M > 0$ and all $k > 0$. Thus $r_C(f) \leq \lim_{k\to\infty} ||f^k(x)||_C^{1/k}$. Since the opposite inequality is obvious, we obtain an alternative formula for the cone spectral radius when $C$ has nonempty interior in a finite dimensional normed space:

$$r_C(f) = \lim_{k\to\infty} ||f^k(x)||_C^{1/k} \quad \text{for any } x \in \operatorname{int}\mathrm{C}. \tag{2.16}$$

Note that if $A : X \to X$ is a bounded linear operator such that $A(C) \subset C$, then $r_C(A) \leq r(A)$. The following lemma gives a condition which can be used to show that the cone spectral radius of a map is greater than zero.

**Lemma 2.4.1** *Let $C$ be a closed cone in a Banach space $X$. Let $f : C \to C$ be an order-preserving homogeneous of degree one map. If $f(x) \geq \delta x$ for some $x \in C\backslash\{0\}$, then $r_C(f) \geq \delta$.*

**Proof** By definition, $r_C(f) = \lim_{k\to\infty} ||f^k||_C^{1/k}$. Suppose that $r_C(f) < \delta$. Then there is a $\delta' > 0$ such that $r_C(f) < \delta' < \delta$. Note that $f^k(x) \geq \delta^k x$ for all $k > 0$. Therefore $\delta^{-k} f^k(x) \geq x$ for all $k > 0$. This is equivalent to $\delta^{-k} f^k(x) - x \in C$ for all $k > 0$. Note that $||\delta^{-k} f^k(x)||^{1/k} \leq \delta^{-1} ||f^k||_C^{1/k} ||x||^{1/k} \leq \delta'/\delta ||x||^{1/k}$ for all $k$ sufficiently large. Therefore $\lim_{k\to\infty} ||\delta^{-k} f^k(x)|| = 0$ and so $\lim_{k\to\infty} \delta^{-k} f^k(x) - x = -x \in C$, a contradiction. ■

# Chapter 3

# Denjoy-Wolff Type Theorems

## 3.1 Classical Results and Modern Generalizations

The classical Denjoy-Wolff theorem concerns holomorphic maps on the unit disc.

**Theorem 3.1.1 (Denjoy-Wolff)** *Let $D$ denote the open unit disc in $\mathbb{C}$. If $f : D \to D$ is a holomorphic map with no fixed point in $D$, then there is a point $z \in \partial D$ such that $\lim_{k\to\infty} f^k(x) = z$ for all $x \in D$.*

Proofs of the Denjoy-Wolff theorem make use of the Schwarz-Pick lemma which asserts that a holomorphic self-map of the open unit disc is nonexpansive with respect to the Poincare metric. Generalizations of the Denjoy-Wolff theorem have been studied by many authors working in several complex variables, see [48] for a survey of these results. Beardon has argued in [5] that the Denjoy-Wolff theorem is best understood as a geometric result. In Theorem 1a of [6], Beardon proves the following Denjoy-Wolff type theorem for Hilbert metric nonexpansive maps. Actually, Beardon proves this theorem for Hilbert metric contractions, but the proof applies to nonexpansive maps as well.

**Theorem 3.1.2** *Let $D$ be a bounded strictly convex domain in a finite dimensional normed space $X$. Let $f : D \to D$ be nonexpansive with respect to the Hilbert metric $d$ on*

*$D$. If $f$ has no fixed point in $D$, then there is a point $z \in \partial D$ such that $\lim_{k\to\infty} f^k(x) = z$ for all $x \in D$.*

For convex domains which are not strictly convex Karlsson and Noskov have shown that the omega limit sets of a fixed point free Hilbert metric nonexpansive map are contained in a star-shaped subset of the boundary. Recall that a set $S$ is *star-shaped* if there is a point $z \in S$ such that for any other $y \in S$ and any $0 < \lambda < 1$ we have $\lambda z + (1-\lambda)y \in S$. This result is Theorem 5.5 in [25] (see also [24]).

**Theorem 3.1.3** *Let $D$ be a bounded convex domain in a finite dimensional normed space $X$. If $f : D \to D$ is nonexpansive with respect to the Hilbert metric $d$ on $D$ and $f$ has no fixed point in $D$, then there is a $z \in \omega(x; f)$ such that for any $\zeta \in \bigcup_{y\in D} \omega(y; f)$, the line segment $[z, \zeta]$ is contained in $\partial D$. In particular, there is a star-shaped subset of $\partial D$ which contains $\omega(x; f)$ for all $x \in D$.*

Nussbaum has extended Theorem 3.1.3 to Hilbert metric nonexpansive maps in infinite dimensions which satisfy suitable compactness conditions (Theorem 4.17, [45]). Both Karlsson and Nussbaum have proposed the following conjecture.

**Conjecture 1** *Let $D$ be a bounded convex domain in a finite dimensional normed space $X$. Let $f : D \to D$ be nonexpansive with respect to the Hilbert metric $d$ on $D$. If $f$ has no fixed point in $D$, then there is a convex subset of $\partial D$ which contains $\omega(x; f)$ for all $x \in D$.*

Of course, Theorem 3.1.2 proves the conjecture in the case where $D$ is strictly convex. In the following sections we give a proof of the conjecture when $D$ is polyhedral (Theorem 3.2.2) and when $D$ is two-dimensional (Theorem 3.3.2). A general proof for any convex domain remains undiscovered. We will discuss some special cases where the conjecture can be proved even when the domain is neither strictly convex nor polyhedral. We will even prove that a version of this conjecture is true for linear maps in infinite dimensions, satisfying certain compactness conditions.

## 3.2 Polyhedral Domains

Before proving the main theorem of this section, we will need to review some terminology from metric geometry. Recall that a metric space $(M, d_M)$ is *proper* if every closed bounded subset of $M$ is compact. We choose a fixed reference point $z \in M$ and define a map $\Phi : x \mapsto f_x$ where

$$f_x(y) = d_M(y, x) - d_M(z, x).$$

If $(M, d_M)$ is a proper metric space, then $\Phi$ is a continuous embedding of $M$ into $C(M)$, the set of continuous real-valued maps on $M$ endowed with the topology of uniform convergence on compacta. If $M$ is not compact, then the image $\Phi(M)$ of $M$ under this embedding will not be closed. The closure of $\Phi(M)$ is called the *Busemann compactification* of $M$. The boundary of $M$ under this compactification is $M(\infty) = \mathrm{cl}\,\Phi(\mathrm{M}) \backslash \Phi(\mathrm{M})$ and is referred to as the *Busemann boundary* by some authors and the *horofunction boundary* by others. The elements of $M(\infty)$ are called *horofunctions* (or *Busemann functions*). Note that every horofunction $h \in M(\infty)$ can be written

$$h(y) = \lim_{k \to \infty} d_M(y, x^k) - d_M(z, x^k) \tag{3.1}$$

where $x^k$ is a sequence of points in $M$ and $z \in M$ is a fixed reference point. For a given horofunction $h$ and a constant $R \in \mathbb{R}$, the sublevel set $H_R = \{x \in M \mid h(x) \leq R\}$ is called a *horoball*. For more details of this compactification see [11]. For recent work concerning the horofunction boundary of Hilbert geometries see [26] and [55].

Suppose that $X$ is a finite dimensional normed space with norm $||\cdot||$. Let $B^* = \{\varphi \in X^* \mid \varphi(x) \leq 1 \ \forall x \in X$ with $||\mathrm{x}|| \leq 1\}$. In this framework, we prove the following lemma.

**Lemma 3.2.1** *Let $y \in X$ be an element with $||y|| = 1$. Let $0 < \lambda < 1$. For any $R > r > 0$ and any $z \in X$ with $||z|| \leq R$, if $||z - Ry|| \leq \lambda R$, then $||z - ry|| \leq R - (1-\lambda)r$.*

**Proof** Suppose that $||z-ry|| > R-(1-\lambda)r$. By the Hahn-Banach theorem there is some $\varphi \in B^*$ such that $||z-ry|| = \varphi(z-ry) > R-(1-\lambda)r$. Then, $\varphi(z)-\varphi(ry) > R-(1-\lambda)r$ so $\varphi(ry) < \varphi(z) - R + (1-\lambda)r$. Since $\varphi(z) \leq ||z|| \leq R$ it follows that $\varphi(ry) < (1-\lambda)r$ and hence $\varphi(y) < (1-\lambda)$. By scaling, $(R-r)\varphi(y) = \varphi(Ry - ry) < (1-\lambda)(R-r)$. So

$$\varphi(z - Ry) = \varphi(z - ry) - \varphi(Ry - ry) > R - (1-\lambda)r - (1-\lambda)(R-r) = \lambda R.$$

Since $||z - Ry|| \geq \varphi(z - Ry) > \lambda R$, we have a contradiction. ■

Lemma 3.2.1 allows us to prove the following result about the interaction of non-expansive mappings and the horofunction boundary in certain proper metric spaces, namely those which admit an almost isometric embedding into a subset of a finite dimensional normed space. In particular this theorem is true for finite dimensional normed spaces, thus it generalizes Theorem 2.1 in [33].

**Theorem 3.2.1** *Let $(M, d_M)$ be a complete proper metric space and let $(X, ||\cdot||)$ be a finite dimensional normed space. Let $U \subset M$ and suppose that there is a one-to-one map $\Phi : U \to X$ and a constant $K > 0$ with $-K \leq d_M(x, y) - ||\Phi(x) - \Phi(y)|| < K$ for all $x, y \in U$. Let $x \in U$ and suppose that $f : M \to M$ is a $d_M$-nonexpansive map* *such that $f^k(x) \in U$ for all $k \geq 0$ and $\lim_{k\to\infty} d_M(x, f^k(x)) = \infty$. Then there is a horofunction $h$ defined on $M$ such that $\lim_{k\to\infty} h(f^k(y)) = -\infty$ for all $y \in M$.*

**Proof** Let $x^k = f^k(x)$. For each $x^k$ let $\hat{x}^k = \Phi(x^k)$. We may assume without loss of generality that $\Phi(x) = 0$. Observe that $||\hat{x}^k|| \geq d_M(x^k, x) - K$. Therefore $\lim_{k\to\infty} ||\hat{x}^k|| = \infty$. A simple observation about unbounded sequences implies that we may choose a subsequence $\{x^{k_i}\}$ such that

$$||\hat{x}^{k_i}|| > ||\hat{x}^m|| \quad \text{for all } m < k_i.$$

We shall say that a subsequence $\{x^{k_i}\}$ satisfying this inequality has property (A).

Since the unit ball in $X$ is compact, there is a point $\bar{y} \in X$ with $||\bar{y}|| = 1$ which is an accumulation point of the sequence $\hat{x}^{k_i}/||\hat{x}^{k_i}||$ $(i \geq 1)$. By taking a further refinement we may assume that:

$$\left|\left|\frac{\hat{x}^{k_i}}{||\hat{x}^{k_i}||} - \bar{y}\right|\right| \leq 2^{-i} \quad \text{for all } i \geq 1.$$

Thus,

$$||\hat{x}^{k_i} - (||\hat{x}^{k_i}||\bar{y})|| \leq 2^{-i}||\hat{x}^{k_i}||, \quad \forall\, i \geq 1.$$

If we denote $||\hat{x}^{k_i}||\bar{y}$ by $y^i$ we get:

$$||\hat{x}^{k_i} - y^i|| \leq 2^{-i}||\hat{x}^{k_i}||, \quad \forall\, i \geq 1. \tag{3.2}$$

Fix some $i \geq 1$. Note that $||\hat{x}^{k_j - m}|| < ||\hat{x}^{k_j}||$ by property (A). Also,

$$||\hat{x}^{k_j - m} - y^j|| \leq ||\hat{x}^{k_j - m} - \hat{x}^{k_j}|| + ||\hat{x}^{k_j} - y^j||$$

$$\leq d_M(x^{k_j - m}, x^{k_j}) + K + 2^{-j}||\hat{x}^{k_j}|| \leq m d_M(x, f(x)) + K + 2^{-j}||\hat{x}^{k_j}||$$

by the nonexpansiveness of $f$ with respect to the metric $d_M$. Note that

$$K_m = m d_M(x, f(x)) + K$$

is a constant which depends only on $m$. Thus

$$||\hat{x}^{k_j - m} - y^j|| \leq K_m + 2^{-j}||\hat{x}^{k_j}||. \tag{3.3}$$

Equation 3.3 implies that for $j$ large enough $||\hat{x}^{k_j-m} - y^j|| \leq \frac{1}{4}||\hat{x}^{k_j}|| = \frac{1}{4}||y^j||$. We will now use Lemma 3.2.1 with $\lambda = \frac{1}{4}$, $R = ||\hat{x}^{k_j}||$, and $r = ||\hat{x}^{k_i}||$ so that $\hat{x}^{k_j-m}$ takes the role of $z$, $y^i$ takes the role of $ry$, and $y^j$ takes the role of $Ry$. We obtain

$$||\hat{x}^{k_j - m} - y^i|| \leq ||\hat{x}^{k_j}|| - \frac{3}{4}||\hat{x}^{k_i}||. \tag{3.4}$$

Combining equations 3.2 and 3.4

$$||\hat{x}^{k_i} - \hat{x}^{k_j-m}|| \leq ||\hat{x}^{k_i} - y^i|| + ||y^i - \hat{x}^{k_j-m}||$$

$$\leq 2^{-i}||\hat{x}^{k_i}|| + ||\hat{x}^{k_j}|| - \frac{3}{4}||\hat{x}^{k_i}||.$$

Thus,

$$||\hat{x}^{k_i} - \hat{x}^{k_j-m}|| \leq ||\hat{x}^{k_j}|| - \frac{1}{4}||\hat{x}^{k_i}||,$$

or translating back to $M$,

$$d_M(x^{k_i}, x^{k_j-m}) \leq d_M(x, x^{k_j}) - \frac{1}{4}d_M(x, x^{k_i}) + 2K \tag{3.5}$$

whenever $k_i$ and $m$ are fixed and $k_j$ is large enough.

Since $M$ is proper, the Ascoli-Arzela theorem implies that by taking a further subsequence of $x^{k_i}$ we may assume that the horofunction

$$h(y) = \lim_{j\to\infty} d_M(y, x^{k_j}) - d_M(x, x^{k_j})$$

exists for all $y \in M$. Observe that

$$h(x^{k_i+m}) = \lim_{j\to\infty} d_M(x^{k_i+m}, x^{k_j}) - d_M(x, x^{k_j}) \leq d_M(x^{k_i}, x^{k_j-m}) - d_M(x^0, x^{k_j})$$

$$\leq -\frac{1}{4} d_M(x, x^{k_i}) + 2K$$

by equation 3.5 and the fact that $f$ is nonexpansive. Since $d_M(x, x^{k_i}) \to \infty$ as $k_i \to \infty$ it follows from the inequality above that $\lim_{m\to\infty} h(x^m) = -\infty$.

To complete the proof, observe that $d_M(f^m(x), f^m(y)) \leq d_M(x, y)$ for all $y \in M$ and $m > 0$. Therefore,

$$h(f^m(y)) = \lim_{j\to\infty} d_M(f^m(y), x^{k_j}) - d_M(x, x^{k_j})$$

$$\leq \lim_{j\to\infty} d_M(f^m(x), f^m(y)) + d_M(f^m(x), x^{k_j}) - d_M(x, x^{k_j})$$

$$\leq d_M(x, y) + h(f^m(x)).$$

It follows that $\lim_{m\to\infty} h(f^m(y)) = -\infty$ for all $y \in M$. ∎

We can now prove the main goal of this section.

**Theorem 3.2.2** *Let $D$ be a bounded polyhedral domain in a finite dimensional normed space $X$. Let $f : D \to D$ be nonexpansive with respect to the Hilbert metric $d$ on $D$. If $f$ has no fixed point in $D$, then there is a convex subset of $\partial D$ which contains $\omega(x; f)$ for all $x \in D$.*

**Proof** By Lemma 2.2.5 there is an isometric embedding $\Phi : D \to \mathbb{R}^{N\times N}$ of $D$ with the Hilbert metric into a subset of $\mathbb{R}^{N\times N}$ with the sup-norm $||\cdot||_\infty$. Since $f$ has no fixed point in $D$, we must have $d(x, f^k(x)) \to \infty$ as $k \to \infty$ by Theorem 2.3.1. Thus Theorem 3.2.1 implies that there is a horofunction $h$ on $D$ such that $\lim_{k\to\infty} h(f^k(x)) = -\infty$ for all $x \in D$. By Lemma 2.2.4 the Hilbert metric balls in $D$ are convex and so it follows from equation 3.1 that the horoballs $H_R = \{x \in D \mid h(x) \leq R\}$ are convex for every $R \in \mathbb{R}$. Let $\mathrm{cl\,H_R}$ denote the norm closure of $H_R$. Because $h(f^k(x)) \to -\infty$ it follows that $\omega(x; f) \subset \mathrm{cl}\,\mathrm{H_R}$ for every $x \in D$ and $R < 0$. Therefore $\omega(x; f) \subset \bigcap_{R<0} \mathrm{cl}\,\mathrm{H_R}$ which is a convex subset of $\partial D$. ■

**Remark 3.1** In section 6.3 we show that, for a simplex in $\mathbb{R}^n$, any convex subset of the boundary is contained in the omega limit set of a Hilbert metric nonexpansive map (Theorem 6.3.1). This partial converse to Theorem 3.2.2 shows that Theorem 3.2.2 is the strongest possible restriction on the omega limit sets of general fixed point free Hilbert metric nonexpansive maps on polyhedral domains.

Theorem 3.2.1 can be applied to any metric space which is isometric to a subset of a finite dimensional Banach space. In addition to the Hilbert metric on a polyhedral domain, this is also true for any part of a polyhedral cone equipped with Thompson's metric by Lemma 2.2.6. Repeating the argument given in the proof of Theorem 3.2.2 will also prove the following.

**Theorem 3.2.3** *Suppose that $C \subset \mathbb{R}^n$ is a polyhedral cone and $C_u \subset C$ is a part of $C$. Let $f : C_u \to C_u$ be a Thompson metric nonexpansive map with no fixed point in $C_u$. If $\omega(x; f) \neq \varnothing$ for some $x \in C_u$, then there is a convex subset of $\partial C_u$ which contains $\omega(y; f)$ for all $y \in C_u$.*

**Remark 3.2** The proof of Theorem 3.2.2 will not work for general convex domains. In fact, if $D$ is the open unit disc in $\mathbb{R}^2$ with the Hilbert metric $d$, then there is a fixed point free $d$-nonexpansive map $f : D \to D$ such that for every horofunction $h$ on $D$ and every $x \in D$, $h(f^k(x))$ is bounded from below. We will sketch the proof of this fact. The open unit disc in $\mathbb{R}^2$ with the Hilbert metric is precisely the Klein model of the hyperbolic plane. It is well known that there is an isometry from the Klein model of the hyperbolic plane onto the open unit disc in $\mathbb{C}$ with the Poincaré metric (see section

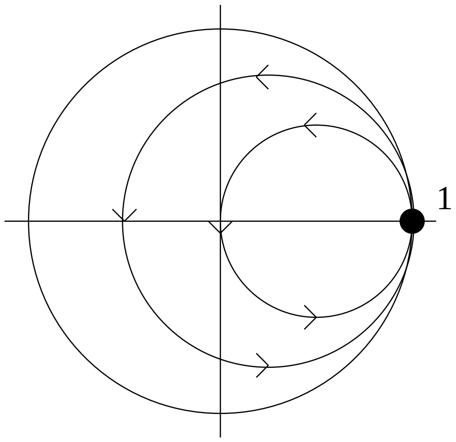


Figure 3.1: The map $f(z) = \frac{(1-2i)z-1}{z-(1+2i)}$ on the open unit disc in $\mathbb{C}$.

6.1 in [47], for example). The open unit disc in $\mathbb{C}$ with the Poincaré metric is sometimes referred to as the Poincaré model of the hyperbolic plane. The balls in the Poincaré model are discs. Since a horoball is a limit of balls, it follows that the horoballs in the Poincaré model are discs that are internally tangent to a single point on the boundary of $D$ (see section 4.5 of [47]). Let $f$ be the Möbius transform

$$f(z) = \frac{(1-2i)z-1}{z-(1+2i)}.$$

The map $f$ sends the open unit disc into itself and is nonexpansive with respect to the Poincaré metric by the Schwarz-Pick lemma. Note that $f$ is actually the composition $f = g \circ T \circ g^{-1}$ where

$$g(z) = \frac{z-i}{z+1} \quad \text{and} \quad T(z) = z+1,$$

that is, $g$ is the Cayley transform from the upper half-plane into the unit disc and $T$ is a horizontal translation. From this, we see that $\lim_{k\to\infty} f^k(z) = 1$ for any $z \in \mathbb{C}$ with $|z| < 1$, and every circle inside the open unit disc which is internally tangent to 1 is invariant under $f$ (see figure 3.1). Furthermore, these circles are the level sets of any horofunction based at 1. It follows that $h(f^k(z))$ is bounded below for every horofunction $h$ and $z \in \mathbb{C}$ with $|z| < 1$.

## 3.3 Two Dimensional Domains

In the two dimensional case there is an elegant argument which shows that $\mathrm{co}\,(\omega(\mathrm{x};\mathrm{f})) \subset \partial\mathrm{D}$ even when $D$ is neither polyhedral nor strictly convex. Before giving the proof, we need to review some facts about horofunctions in the Hilbert geometry. The following lemma is Theorem 5.2 in [25]. Nussbaum points out that this result is true in infinite dimensions in Theorem 4.13 of [45].

**Lemma 3.3.1** *Let $D$ be a bounded convex domain with Hilbert metric $d$ in a finite dimensional normed space. Let $w \in D$ be fixed. Let $\{x^k\}$ and $\{y^k\}$ be two sequences in $D$ such that $x^k$ converges in norm to $\zeta \in \partial D$ and $y^k$ converges in norm to $\eta \in \partial D$. If the line segment $[\zeta, \eta]$ is not contained in $\partial D$, then*

$$\limsup_{k\to\infty} \; [d(x^k, w) + d(y^k, w) - d(x^k, y^k)] < \infty.$$

**Proof** Since $[\zeta, \eta] \cap D$ is non-empty, an elementary convexity argument implies that $u = \frac{1}{2}(\zeta + \eta) \in D$. For each $k$, let $u^k = \frac{1}{2}(x^k + y^k)$. Since $u^k \to u$ and $u \in D$, it follows that $\lim_{k\to\infty} d(w, u^k) = d(w, u) < \infty$. Observe that

$$d(x^k, y^k) = d(x^k, u^k) + d(y^k, u^k) \geq d(x^k, w) + d(y^k, w) - 2d(w, u^k)$$

by the triangle inequality. Therefore

$$\limsup_{k\to\infty} \; [d(x^k, w) + d(y^k, w) - d(x^k, y^k)] \leq \lim_{k\to\infty} 2d(w, u^k) < \infty.$$

■

In order to prove the main result of this section, we need to construct a horofunction with somewhat different properties than the ones discussed in the proof of Theorem

3.2.1. The following theorem is a generalization of a result of Beardon (see Proposition 4.5 of [6]). Note that it is closely related to Theorem 3.1.3. Nussbaum has proved an infinite dimensional version of this result (Theorem 4.14, [45]).

**Theorem 3.3.1** *Let $D$ be a bounded convex domain with Hilbert metric $d$ in a finite dimensional normed space $X$. Suppose that $f : D \to D$ is $d$-nonexpansive and $f$ has no fixed point in $D$. Then there is a point $b \in D$ and a sequence $\{b^i\}$ in $D$ such that the horofunction*

$$h(x) = \lim_{i\to\infty} d(x, b^i) - d(b, b^i)$$

*exists and has the property that $h(f(x)) \le h(x)$ for all $x \in D$. Furthermore, the sequence $\{b^i\}$ converges in norm to a point $z \in \partial D$ such that, for any $x \in D$ and $\zeta \in \omega(x; f)$, the line segment $[z, \zeta]$ is contained in $\partial D$.*

**Proof** Choose an arbitrary point $b \in D$. For each $i \ge 1$, let $f_i(x) = (1 - 1/i)f(x) + (1/i)b$. Each $f_i$ is a contraction with respect to $d$. Since $f_i(D)$ is contained in the compact, convex set $K_i = \{(1 - 1/i)x + (1/i)b \mid x \in \mathrm{cl\,D}\}$, it follows from the Brouwer fixed point theorem that each $f_i$ has a unique fixed point $b^i \in K_i \subset D$. Observe that $f_i$ converges uniformly on $D$ to $f$ in norm, and therefore $f_i$ converges pointwise on $D$ to $f$ with respect to the metric $d$. Since $f$ has no fixed point in $D$, it follows that $b^i \to \partial D$. By taking a subsequence, we may assume that $b^i \to z \in \partial D$ and the horofunction $h(x) = \lim_{i\to\infty} d(x, b^i) - d(b, b^i)$ exists. Note that,

$$h(f(x)) = \lim_{i\to\infty} d(f(x), b^i) - d(b, b^i)$$

$$\le \liminf_{i\to\infty} \; [d(f(x), f_i(x)) + d(f_i(x), b^i) - d(b, b^i)]$$

$$\le \liminf_{i\to\infty} \; [d(f_i(x), f_i(b^i)) - d(b, b^i)] \le \lim_{i\to\infty} d(x, b^i) - d(b, b^i) = h(x).$$

It remains to show that if $f^{k_i}(x) \to \zeta \in \partial D$ as $i \to \infty$, then $[z, \zeta] \subset \partial D$. Suppose by way of contradiction that $[z, \zeta]$ is not contained in $\partial D$. For each $i \geq 1$ and $R \in \mathbb{R}$, let $V_R^i = \{y \in D \mid d(y, b^i) \leq d(b, b^i) + R\}$. Suppose that $h(x) = R$. Then there is a sequence of points $v^i \in V_R^i$ such that $\lim_{i\to\infty} v^i = x$. Since $h(f^k(x)) \leq R$ for all $k \geq 0$, it follows that we may choose a sequence $y^i \in V_R^i$ so that $||y^i - f^{k_i}(x)|| \leq 1/i$ for all integers $i$ large enough. Since $\lim_{i\to\infty} f^{k_i}(x) = \zeta$, we have $\lim_{i\to\infty} y^i = \zeta$ as well. Using the fact that $b^i \to z$ and $[z, \zeta]$ is not contained in $\partial D$, we may use Lemma 3.3.1 to conclude that

$$\limsup_{i\to\infty} \; [d(y^i, b) + d(b^i, b) - d(y^i, b^i)] < \infty.$$

Therefore,

$$\liminf_{i\to\infty} \; d(y^i, b^i) - d(b^i, b) = \infty,$$

because $\lim_{i\to\infty} d(y^i, b) = \infty$. This is a contradiction, since $y^i \in V_R^i$ for every $i$ large enough. ■

With Theorem 3.3.1, we can now extend Theorem 3.2.2 to any two dimensional convex domain.

**Theorem 3.3.2** *Let $D$ be a bounded convex domain in a two dimensional normed space. Let $f : D \to D$ be nonexpansive with respect to the Hilbert metric on $D$. If $f$ has no fixed point in $D$, then there is a convex subset of $\partial D$ which contains $\omega(x; f)$ for all $x \in D$.*

**Proof** Theorem 3.3.2 is trivial when aff D is one dimensional, so assume that aff D = X. By Theorem 3.3.1, there is a horofunction $h$ such that $h(f(y)) \leq h(y)$ for all $y \in D$. In particular, sequence $h(f^k(x))$ is non-increasing. Let $H_R = \{y \in D \mid h(y) < R\}$. Note that each horoball $H_R$ is convex. This is an immediate consequence of equation 3.1 and

Lemma 2.2.4. Since $H_R$ is convex, the norm closure of $H_R$, $\mathrm{cl\,H_R}$, is also convex. If $h(f^k(x)) \to -\infty$, then $\omega(x;f) \subset \mathrm{cl\,H_R}$ for all $R \in \mathbb{R}$. Thus, $\omega(x;f) \subset \bigcap_{R\in\mathbb{R}} \mathrm{cl\,H_R}$. But $\bigcap_{R\in\mathbb{R}} \mathrm{cl\,H_R}$ is a convex subset of $\partial D$, so Lemma 2.3.4 implies that $\omega(y;f)$ is contained in a convex subset of the boundary for all $y \in D$.

Let us assume, therefore, that $h(f^k(x)) \to R > -\infty$. Suppose by way of contradiction that there exists $\zeta, \eta \in \omega(x;f)$ such that the line segment $[\zeta, \eta]$ has non-empty intersection with $D$. By Theorem 3.1.3 there is a point $z \in \omega(x;f)$ such that $[z, \zeta] \subset \partial D$ and $[z, \eta] \subset \partial D$. Because $D$ is two-dimensional, $z$ is the only point in $\partial D$ with this property.

Since we used Theorem 3.3.1 to construct $h$, there is a point $b \in D$ and a sequence $\{b^i\}$ in $D$ such that $h(y) = \lim_{i\to\infty} d(y, b^i) - d(b, b^i)$ for all $y \in D$. Furthermore, $\lim_{i\to\infty} b^i = z$ since $z$ is the only point in $\partial D$ such that $[z, \zeta]$ and $[z, \eta]$ are contained in $\partial D$. Choose an $\bar{x} \in D$ on the line segment $[z, x]$ so that $h(x) - d(x, \bar{x}) < R' < R$. Let $r = ||\bar{x} - z||$. For all $i$ large enough, $||b^i - z|| < r$. Therefore, for each $i$ large enough there is unique point $y^i$ on the line segment $[x, b^i]$ such that $||y^i - z|| = r$. By construction, $\lim_{i\to\infty} y^i = \bar{x}$. Note that

$$h(x) = \lim_{i\to\infty} d(x, b^i) - d(b, b^i)$$

$$= \lim_{i\to\infty} d(x, y^i) + d(y^i, b^i) - d(b, b^i)$$

$$\geq \limsup_{i\to\infty} \; [d(x, y^i) + d(\bar{x}, b^i) - d(\bar{x}, y^i) - d(b, b^i)]$$

$$= \limsup_{i\to\infty} \; [d(x, \bar{x}) + d(\bar{x}, b^i) - d(b, b^i)] = \alpha + h(\bar{x}).$$

Therefore, $h(\bar{x}) \leq h(x) - d(x, \bar{x}) < R' < R$.

By Lemma 2.3.2 there are points $\bar{z}, \bar{\zeta}, \bar{\eta} \in \omega(\bar{x}, f)$ such that $\bar{z} \sim_D z$, $\bar{\zeta} \sim_D \zeta$, and $\bar{\eta} \sim_D \eta$. Furthermore, since $h(\bar{x}) < R'$ it follows that $\bar{z}, \bar{\zeta}, \bar{\eta} \in \mathrm{cl\,H_{R'}}$. Note that $\bar{z}$ must

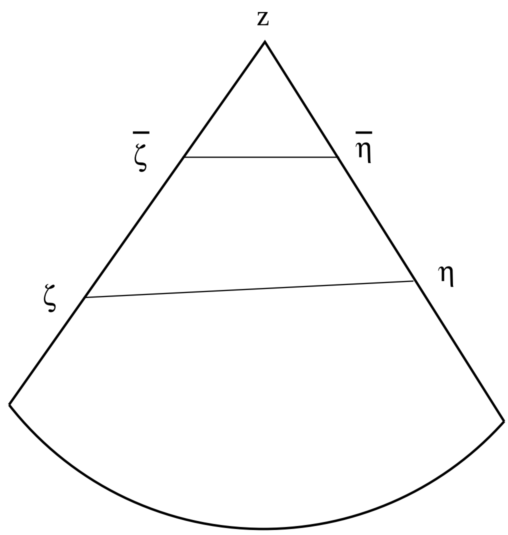


Figure 3.2: An illustration of the domain $D$ in the proof of Theorem 3.3.2.

equal $z$ because $D$ is two-dimensional (see figure 3.2). Since $z \in \omega(x; f)$ there is an increasing sequence of integers $k_i$ such that $f^{k_i}(x) \to z$. This means that there is some $N > 0$ such that $f^{k_i}(x) \in D \cap \operatorname{co}\{\mathrm{z}, \bar{\zeta}, \bar{\eta}\}$ for all $k_i \geq N$. Since $z, \bar{\zeta}, \bar{\eta}$ are each limit points of elements of $H_{R'}$, and $H_{R'}$ is a convex set, it follows that $h(f^{k_i}(x)) \in H_{R'}$ and $h(f^{k_i}(x)) < R'$ for all $k_i \geq N$. This contradicts the assumption that $h(f^k(x)) \to R$, proving that $\omega(x; f)$ is contained in a convex subset of the boundary. Lemma 2.3.4 then shows that $\omega(y; f)$ is contained in a convex subset of $\partial D$ for all $y \in D$. ■

## 3.4 A Special Case

Beyond two dimensions, we are unaware of a proof of Conjecture 1 for general convex domains. In applications however, a Hilbert metric nonexpansive map may have enough extra structure to ensure that the result of Conjecture 1 is true even when the domain is neither strictly convex nor polyhedral.

Suppose that $C_1 \subset C_2$ are closed cones in a finite dimensional normed space $X$. Assume that $C_1$ and $C_2$ have nonempty interiors. Let $C_1^*$ and $C_2^*$ denote the dual cones of $C_1$ and $C_2$ respectively. Note that $C_2^* \subset C_1^*$. Let $d_1(\cdot,\cdot)$ denote Hilbert's projective metric for the cone $C_1$ and $d_2(\cdot,\cdot)$ denote Hilbert's projective metric for the cone $C_2$. For any $x, y \in C_1 \cap \mathrm{int\,C_2}$, $d_2(x,y) \leq d_1(x,y)$. The following lemma gives conditions under which $d_2$ restricted to $C_1 \cap \mathrm{int\,C_2}$ is almost equivalent to the projective metric of a polyhedral cone.

**Lemma 3.4.1** *Let $C_1 \subset C_2$ be closed cones with nonempty interiors in a finite dimensional normed space $X$. Let $d_2(\cdot,\cdot)$ denote Hilbert's projective metric induced by $C_2$. If there is a polyhedral cone $C_p$ such that $C_1 \subset C_p \subset C_2$ and such that every element of $C_p^*$ is comparable to an element of $C_2^*$, in the partial ordering induced by $C_1^*$, then there is a constant $K \geq 0$ such that Hilbert's projective metric with respect to $C_p$, denoted $d_p(\cdot,\cdot)$, satisfies*

$$d_2(x,y) \leq d_p(x,y) \leq d_2(x,y) + K \tag{3.6}$$

*for all $x, y \in C_1 \cap \mathrm{int\,C_2}$.*

**Proof** Since $C_1 \subset C_p \subset C_2$ it follows immediately that $d_2(x,y) \leq d_p(x,y)$ for all $x, y \in C_1 \cap \mathrm{int\,C_2}$. Since $C_p$ is polyhedral, there is a finite collection $\{\theta_i\}_{i\in I} \subset C_p^*$ such that

$$d_p(x,y) = \max_{i,j\in I} \log\left(\frac{\theta_i(x)\theta_j(y)}{\theta_i(y)\theta_j(x)}\right)$$

whenever $x$ and $y$ are comparable in the partial ordering induced by $C_p$. For each $i \in I$ there is a $\varphi_i \in C_2^*$ such that $\theta_i$ is comparable to $\varphi_i$ in the partial ordering induced by $C_1^*$. This means that there is an $\epsilon_i > 0$ such that $\epsilon_i\varphi_i(x) \leq \theta_i(x) \leq \epsilon_i^{-1}\varphi_i(x)$ for all $x \in C_1$. Letting $\epsilon = \min_{i\in I}\epsilon_i$ we see that for each $i, j \in I$ and all $x, y \in C_1 \cap \mathrm{int\,C_2}$,

$$\log\left(\frac{\theta_i(x)\theta_j(y)}{\theta_i(y)\theta_j(x)}\right) \leq \log\left(\frac{\epsilon^{-2}\varphi_i(x)\varphi_j(y)}{\epsilon^2\varphi_i(y)\varphi_j(x)}\right)$$

$$= \log\left(\frac{\varphi_i(x)\varphi_j(y)}{\varphi_i(y)\varphi_j(x)}\right) + \log\left(\frac{1}{\epsilon^4}\right) \le d_2(x,y) + \log\left(\frac{1}{\epsilon^4}\right)$$

since

$$d_2(x,y) = \sup_{\chi,\psi \in C_2*} \log\left(\frac{\chi(x)\psi(y)}{\chi(y)\psi(x)}\right)$$

and $\varphi_i, \varphi_j \in C_2^*$. Therefore $d_p(x,y) \le d_2(x,y) + \log(1/\epsilon^4)$ which completes the proof.

■

Using Lemma 3.4.1 and Theorem 3.2.1 we obtain the following corollary of Theorem 3.2.2. We will use this theorem in chapter 5 when we study reproduction-decimation operators.

**Theorem 3.4.1** *Let $C_1 \subset C_2$ be closed cones with nonempty interiors in a finite dimensional normed space $X$. Let $d_2(\cdot,\cdot)$ denote Hilbert's projective metric induced by $C_2$. Suppose that there is a polyhedral cone $C_p$ such that $C_1 \subset C_p \subset C_2$ and such that every element of $C_p^*$ is comparable to an element of $C_2^*$ with respect to the partial ordering induced by $C_1^*$. Let $f : \mathrm{int\,C_2} \to \mathrm{int\,C_2}$ be order-preserving with respect to $C_2$ and homogeneous of degree one. Let $q \in X^*$ be a linear functional such that $q(x) > 0$ for all $x \in C_2 \backslash \{0\}$. Let $\Sigma = \{x \in \mathrm{int\,C_2} \mid \mathrm{q(x) = 1}\}$ and $\hat{f}(x) = f(x)/q(f(x))$ for $x \in \Sigma$. If for some $x^0 \in C_1 \cap \mathrm{int\,C_2}$, $f^k(x^0) \in C_1$ for all $k \in \mathbb{N}$ and $\hat{f}$ has no fixed point in $\Sigma$, then there is a convex subset of $\partial\Sigma$ which contains $\omega(x;\hat{f})$ for all $x \in \Sigma$.*

**Proof** By Lemma 3.4.1 and Lemma 2.2.5, there is a one-to-one map $\Phi : \Sigma \cap C_1 \to \mathbb{R}^{N\times N}$ such that $-K \le d_2(x,y) - ||\Phi(x) - \Phi(y)||_\infty \le K$ where $N$ and $K$ are constants depending on $C_1$, $C_p$, and $C_2$. Since $\hat{f}$ has no fixed point in $\Sigma$, we must have $d_2(x, \hat{f}^k(x)) \to \infty$ as $k \to \infty$ by Theorem 2.3.1. Thus, Theorem 3.2.1 implies that there is a horofunction $h$ on $\Sigma$ such that $\lim_{k\to\infty} h(\hat{f}^k(x)) = -\infty$ for all $x \in \Sigma$. By Lemma 2.2.4, the $d_2$-balls in $\Sigma$ are convex and so it follows from equation 3.1 that the horoballs $H_R = \{x \in \Sigma \mid h(x) \leq R\}$ are convex for every $R \in \mathbb{R}$. Let $\mathrm{cl\,H_R}$ denote the norm closure of $H_R$. Because $h(\hat{f}^k(x)) \to -\infty$, it follows that $\omega(x; \hat{f}) \subset \mathrm{cl\,H_R}$ for every $x \in \Sigma$ and $R < 0$. Therefore $\omega(x; \hat{f}) \subset \bigcap_{R<0} \mathrm{cl\,H_R}$ which is a convex subset of $\partial\Sigma$. ∎

## 3.5 Nonexpansive Maps in Finite Dimensional Normed Spaces

Another interesting application of Theorem 3.2.1 is the following result.

**Theorem 3.5.1** *Suppose that $U$ is a closed convex set in a finite dimensional normed space $X$ and $f : U \to U$ is a norm nonexpansive map. If $f$ does not have a fixed point in $U$, then there is a linear functional $\varphi \in X^*$ such that $\lim_{k\to\infty} \varphi(f^k(x)) = \infty$ for all $x \in U$.*

Theorem 3.5.1 is similar to the following theorem of Kohlberg and Neyman (Theorem 1.1 in [30]).

**Theorem 3.5.2** *Let $U$ be a convex subset of a normed space $X$ and let $f : U \to U$ be nonexpansive. Then there exists a linear functional $\varphi \in X^*$ with $||\varphi|| = 1$ such that for every $x \in U$,*

$$\lim_{k\to\infty} \varphi\left(\frac{f^k(x)}{k}\right) = \lim_{k\to\infty} \left|\left|\frac{f^k(x)}{k}\right|\right| = \inf_{y\in U} ||f(y) - y||.$$

This result of Kohlberg and Neyman will imply Theorem 3.5.1 if $\inf_{y\in X} ||f(y) - y|| > 0$. Note that Theorem 3.5.2 applies to infinite dimensional normed spaces unlike Theorem 3.5.1. In section 6.3 we give an example of a nonexpansive map in $(\mathbb{R}^n, || \cdot ||_\infty||)$ which does not have a fixed point even though $\inf_{y\in\mathbb{R}^n} ||f(y) - y||_\infty = 0$. This shows that Theorem 3.5.1 is independent of Theorem 3.5.2.

In order to prove Theorem 3.5.1, we will make some quick observations about convex sets in finite dimensional vector spaces. Recall that if $U$ is convex, then the *polar set* of $U$ is $U^\circ = \{\varphi \in X^* \mid \varphi(x) \leq 1 \; \forall \, x \in U\}$.

**Lemma 3.5.1** *Let $U$ be a convex set in a finite dimensional normed space $X$. If $\operatorname{span} U^\circ = X^*$, then there is a closed cone $C \subset X$ and a point $z \in X$ such that $U \subset C + \{z\}$.*

**Proof** Choose a basis $\{\varphi_i\}_{i=1}^n \subset U^\circ$ for $X^*$. Then each $\varphi_i$ has $\varphi_i(x) \leq 1$ for all $x \in U$. Note that there is a unique point $z \in X$ such that $\varphi_i(z) = 1$ for all $i \in \{1, ..., n\}$. Let $C = \{x \in X \mid \varphi_i(x) \leq 0 \; \forall \, 1 \leq i \leq n\}$. Then $C$ is a closed cone and every element of $U$ is contained in $C + \{z\}$. ■

**Lemma 3.5.2** *Let $U$ be a closed convex subset of a finite dimensional normed space $X$. Let $V = \{x \in X \mid \varphi(x) = 0 \; \forall \, \varphi \in U^\circ\}$. Then $U + V = U$.*

**Proof** Suppose that $u \in U$ and $v \in V$. If $u + v \notin U$, then by the Hahn-Banach theorem there is a linear functional $\varphi \in X^*$ and a constant $a \in \mathbb{R}$ such that $\varphi(u + v) > a$ and $\varphi(x) \leq a$ for all $x \in U$. In particular $\varphi(u) \leq a$ so we must have $\varphi(v) > 0$. We may assume $a \leq 1$ by scaling $\varphi$ if $a$ is positive. Then $\varphi(x) \leq a \leq 1$ for all $x \in U$ and so $\varphi \in U^\circ$. This is a contraction, since $v \in V$ and $V = \{x \in X \mid \varphi(x) = 0 \; \forall \, \varphi \in U^\circ\}$. Therefore $U + V \subset U$ and since it is obvious that $U \subset U + V$ we are done. ■

**Theorem 3.5.3** *Let $X$ be a finite dimensional normed space. Suppose that $\{U_k\}_{k \in \mathbb{N}}$ is a collection of nonempty closed convex sets in $X$ such that $U_{k+1} \subset U_k$ for all $k \geq 1$. If $\bigcap_{k \geq 1} U_k = \varnothing$, then there is a linear functional $\varphi \in X^*$ such that*

$$\lim_{k \to \infty} \left( \inf_{x \in U_k} \varphi(x) \right) = \infty.$$

**Proof** For each $U_k$ let $V_k = \{x \in X \mid \varphi(x) = 0 \; \forall \varphi \in U_k^\circ\}$. Note that each $V_k$ is a subspace of $X$. Since $U_{k+1}^\circ \supset U_k^\circ$ for all $k \geq 1$, $V_k$ is a decreasing sequence of subspaces. This implies that there is some $K > 0$ and a subspace $V \subset X$ such that $V_k = V$ for all $k \geq K$. Choose a subspace $W \subset X$ such that $X = V \oplus W$. By Lemma 3.5.2, $U_k = U_k + V$ for all $k \geq K$. For any $u \in U_k$, we have $u = w + v$ with $w \in W$, $v \in V$. Then $w = u - v \in U_k$ since $U_k + V = U_k$. This implies that $U_k = (W \cap U_k) + V$ for all $k \geq K$.

Every $\varphi \in W^*$ can be extended to a continuous linear functional on $X$ by letting $\varphi(v) = 0$ for all $v \in V$. Therefore, we may say that $W^* = \{\varphi \in X^* \mid \varphi(v) = 0 \; \forall v \in V\}$. Note that $(W \cap U_K)^\circ = W^\circ \cup U_K^\circ$. Since $W$ is a subspace, if $x \in W$ and $\varphi \in W^\circ$, then $\varphi(x) = 0$. Thus $(W \cap U_K)^\circ = U_K^\circ$. For every $x \in W \backslash \{0\}$ there is some $\varphi \in U_K^\circ$ such that $\varphi(x) \neq 0$, otherwise $x$ would be in $V$. This means that $(W \cap U_K)^\circ$ spans $W^*$. Lemma 3.5.1 implies that there is a closed cone $C \subset W$ and a point $z \in W$ such that $W \cap U_K \subset C + \{z\}$. By Lemma 2.2.2 there is a nonzero linear functional $\varphi \in W^*$ such that $\varphi(x) > 0$ for all $x \in C \backslash \{0\}$ and the set $S_R = \{x \in C \mid \varphi(x) = R\}$ is bounded for all $R \geq 0$. Since $S_R$ is closed for every $R \geq 0$, it is also compact. Furthermore, the set $\{x \in C \mid \varphi(x) \leq R\} = \mathrm{co}\,(\mathrm{S_R} \cup \{0\})$ is also compact for every $R \geq 0$.

Let $A_R = \{x \in C + \{z\} \mid \varphi(x) \leq R\}$. Then $x \in A_R$ if and only if $x = y + z$ where $y \in C$ and $\varphi(y) \leq R - \varphi(z)$. Thus $A_R$ is compact. We know that $W \cap U_k \subset W \cap U_K \subset C + \{z\}$ for all $k \geq K$. Suppose that there is some $R > 0$ such that $W \cap U_k \cap A_R$ is nonempty for every $k \geq K$. Since $W \cap U_k \cap A_R$ is compact for every $k \geq K$ we would have $\bigcap_{k \geq K} (W \cap U_k \cap A_R) \neq \varnothing$ which is a contradiction. Therefore, for every $R > 0$ there is some $k$ large enough so that $\varphi(W \cap U_k) > R$. Because $U_k = (W \cap U_k) + V$ for all $k \geq K$ and $\varphi(x) = 0$ for all $x \in V$ we see that $\inf_{x \in U_k} \varphi(x) \to \infty$ as $k \to \infty$. ■

**Proof of Theorem 3.5.1** By Theorem 2.3.1 $\lim_{k\to\infty} ||f^k(x)|| = \infty$ for all $x \in U$. Therefore, Theorem 3.2.1 implies that there is a horofunction $h$ defined on $X$ such that $\lim_{k\to\infty} h(f^k(x)) = -\infty$. Let $H_R = \{x \in X \mid h(x) \leq R\}$ for every integer $R < 0$. By equation 3.1 we can see that $H_R$ is convex for every $R < 0$. Since $H_{R-1} \subset H_R$ for every $R < 0$ and $\bigcap_{R<0} H_R = \varnothing$, we can use Theorem 3.5.3 to find a linear functional $\varphi \in X^*$ such that $\inf_{x\in H_R} \varphi(x) \to \infty$ as $R \to -\infty$. Then $\lim_{k\to\infty} \varphi(f^k(x)) = \infty$ for all $x \in U$.

■

**Remark 3.3** Theorem 3.5.1 is not true in infinite dimensions. Consider the Banach space $X = \ell^1(\mathbb{N})$. Let $U = \{x \in X \mid x_i \geq 0 \ \forall i \in \mathbb{N} \text{ and } \sum_{i\in\mathbb{N}} x_i = 1\}$. Note that $U$ is closed, bounded, and convex. However, if $f$ is the right-shift operator, $f(x) = (0, x_1, x_2, ...)$, then $f(U) \subset U$ and $f$ is nonexpansive. The only fixed point of $f$ is 0 which is not in $U$. Since $U$ is bounded, there cannot be a linear functional $\varphi \in X^*$ such that $\lim_{k\to\infty} \varphi(f^k(x)) = \infty$ for $x \in U$.

# Chapter 4

# Positive Linear Operators

## 4.1 Spectral Projections and the Essential Spectral Radius

Let $X$ be a Banach space and let $\mathcal{B}(X)$ be the set of bounded linear maps from $X$ into $X$. For now, assume that $X$ is a complex Banach space, although in the applications that we have in mind, $X$ will be real. For any $A \in \mathcal{B}(X)$ we let $\sigma(A)$ denote the spectrum of $A$. Recall that the spectral radius, $r(A)$, is given by equation 2.13. It is well known that $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$. We define the *peripheral spectrum* of $A$ to be the set $\{\lambda \in \sigma(A) : |\lambda| = r(A)\}$.

Let $q(A)$ be the seminorm:

$$q(A) = \inf\{||A + B|| : B \text{ is a compact linear map}\}.$$

The *essential spectral radius* of $A$, denoted $\rho(A)$, is defined to be:

$$\rho(A) = \lim_{k\to\infty} q(A^k)^{1/k}. \tag{4.1}$$

Clearly, $\rho(A) \leq r(A)$. If $A$ is compact, then $\rho(A) = 0$. It is proved in [38] that if $\lambda \in \sigma(A)$ and $|\lambda| > \rho(A)$, then $\lambda$ is an eigenvalue of $A$ with finite algebraic multiplicity and $\lambda$ is an isolated point of $\sigma(A)$.

Suppose that $A \in \mathcal{B}(X)$ has $\rho(A) < r(A)$. Since each $\lambda \in \sigma(A)$ with $|\lambda| = r(A)$ is an isolated point, there are only finitely many eigenvalues in the peripheral spectrum of

$A$. Because every such eigenvalue is isolated, we may define the spectral projection $P$ corresponding to the peripheral spectrum of $A$. This projection is given by the integral

$$P = \frac{1}{2\pi i} \oint_{\gamma} (\lambda I - A)^{-1} d\lambda$$

where $\gamma$ is a cycle winding once around each eigenvalue in the peripheral spectrum of $A$ and zero times around every other $\lambda \in \sigma(A)$. The spectral projection $P$ is a linear projection, that is $P^2 = P$, and $A$ commutes with $P$. Note that $P$ does not depend on the choice of $\gamma$. See section VIII.8 of [58] for more details about this construction.

We let $Y$ and $Z$ denote the ranges of $P$ and $I - P$ respectively. Since $A$ commutes with $P$ and $I - P$, the subspaces $Y$ and $Z$ are invariant under $A$ and $X = Y \oplus Z$. Moreover, the spectrum of $A$ restricted to $Y$ is the peripheral spectrum of $A$, while the spectrum of $A$ restricted to $Z$ is the interior spectrum of $A$. Since there are only finitely many eigenvalues in the peripheral spectrum, and each one has finite algebraic multiplicity, the subspace $Y$ is finite dimensional. We collect these facts in the following proposition.

**Proposition 4.1.1** *Let $X$ be a Banach space and let $A \in \mathcal{B}(X)$. Suppose that $\rho(A) < r(A)$. Then there is a spectral projection $P$ such that $P$ commutes with $A$, the range of $P$ is finite dimensional, and $r((I-P)A) < r(A)$.*

The remainder of this section is devoted to some minor lemmas about iterates of operators in $\mathcal{B}(X)$. We will need these results in order to prove the main goal of this chapter, Theorem 4.2.1. Let $M_n(\mathbb{C})$ denote the set of $n \times n$ complex matrices and let $J_n(\lambda) \in M_n(\mathbb{C})$ denote the $n \times n$ Jordan block corresponding to an eigenvalue $\lambda \in \mathbb{C}$. We can write

$$J_n(\lambda) = \lambda I_n + N_n \tag{4.2}$$

where $I_n$ is the $n \times n$ identity matrix, and $N_n$ is the nilpotent matrix with entries $\nu_{ij} = 1$ if $j - i = 1$ and $\nu_{ij} = 0$ otherwise.

**Lemma 4.1.1** *Suppose that* $J = J_n(\lambda)$ *and* $|\lambda| = 1$. *If* $\lim_{i\to\infty} \lambda^{k_i} = \alpha$ *for some increasing sequence of integers* $k_i$, *then*

$$\lim_{i\to\infty} k_i^{-n+1} J^{k_i} = \frac{\alpha\lambda^{-n+1}}{(n-1)!} N_n^{n-1}.$$

**Proof** Applying the binomial theorem to equation 4.2 gives:

$$J^k = J_n^k(\lambda) = \lambda^k I_n + \sum_{t=1}^{k} \begin{pmatrix} k \\ t \end{pmatrix} \lambda^{k-t} N_n^t.$$

In particular, if $t > n-1$, $N_n^t = 0$. Consider the limit:

$$\lim_{k\to\infty} \frac{J^k}{\lambda^k k^{n-1}} = \lim_{k\to\infty} \left[ \frac{I_n}{k^{n-1}} + \sum_{t=1}^{n-1} \frac{\lambda^{-t}}{k^{n-1}} \begin{pmatrix} k \\ t \end{pmatrix} N_n^t \right].$$

We can see that

$$\lim_{k\to\infty} \frac{1}{k^{n-1}} \begin{pmatrix} k \\ t \end{pmatrix} = 0$$

for all $1 \leq t < n-1$ while

$$\lim_{k\to\infty} \frac{1}{k^{n-1}} \begin{pmatrix} k \\ n-1 \end{pmatrix} = \frac{1}{(n-1)!}.$$

From this and the fact that $N_n^t = 0$ for $t > n-1$, we conclude that

$$\lim_{k\to\infty} \frac{J_n^k(\lambda)}{\lambda^k k^{n-1}} = \lim_{k\to\infty} \left[ \frac{I_n}{k^{n-1}} + \sum_{t=1}^{n-1} \frac{\lambda^{-t}}{k^{n-1}} \begin{pmatrix} k \\ t \end{pmatrix} N_n^t \right] =$$

$$= \lim_{k\to\infty} \left[ \frac{\lambda^{-n+1}}{k^{n-1}} \begin{pmatrix} k \\ n-1 \end{pmatrix} N_n^{n-1} \right] = \frac{\lambda^{-n+1} N_n^{n-1}}{(n-1)!}.$$

Since $\lim_{i\to\infty}\lambda^{k_i}=\alpha$ we have:

$$\lim_{i\to\infty} k_i^{-n+1}J^{k_i}=\lim_{i\to\infty}\lambda^{k_i}\left(\frac{J^{k_i}}{\lambda^{k_i}{k_i}^{n-1}}\right)=\alpha\left(\frac{\lambda^{-n+1}N_n^{n-1}}{(n-1)!}\right).$$

■

**Lemma 4.1.2** *Let $J=J_n(\lambda)$ and $|\lambda|=1$. For any $x\in\mathbb{C}^n\backslash\{0\}$ there is a $y\in\mathbb{C}^n\backslash\{0\}$ with $Jy=\lambda y$ and an integer $q$, $0\le q<n$, such that for any increasing sequence of integers $k_i$,*

$$\lim_{i\to\infty} k_i^{-q}J^{k_i}x=\left(\lim_{i\to\infty}\lambda^{k_i}\right)y.$$

*In particular, $\lim_{i\to\infty}k_i^{-q}J^{k_i}x$ exists if and only if $\lim_{i\to\infty}\lambda^{k_i}$ exists.*

**Proof** Let $\alpha=\lim_{i\to\infty}\lambda^{k_i}$. Note that $x=(x_1,x_2,...,x_n)$. For now, assume that $x_n\neq 0$. If $q=n-1$, then Lemma 4.1.1 implies that,

$$\lim_{i\to\infty} k_i^{-q}J^{k_i}x=\alpha\lambda^{-n+1}N_n^{n-1}x.$$

Thus $y=\lambda^{-n+1}N_n^{n-1}x$. Note that $y\neq 0$ since $(N_n^{n-1}x)_1=x_n\neq 0$. Because $N_n^n=0$, we see that:

$$Jy=J\lambda^{-n+1}N_n^{n-1}x=(\lambda I_n+N_n)\lambda^{-n+1}N_n^{n-1}x=\lambda y.$$

If $x_n=0$, then suppose that $x_p\neq 0$ and $x_{p+1},...,x_n=0$ for some $1\le p<n$. Let $V=\{y\in\mathbb{C}^n \mid y_{p+1},...,y_n=0\}$. Note that $J(V)\subseteq V$, and furthermore $J|_V$ is represented by the $p\times p$ Jordan block $J_p(\lambda)$. We may simply repeat the argument above, with $p$ replacing $n$. ■

**Lemma 4.1.3** *Suppose that $A\in\mathcal{B}(X)$ has $r(A)=1$ and $\rho(A)<1$. Let $P$ be the spectral projection operator corresponding to the peripheral spectrum of $A$. If $x\in X$*

*and $Px \neq 0$, then there is an $\epsilon > 0$ such that $||A^k x|| \geq \epsilon$ for all $k \geq 0$. Furthermore, the sequence $\{A^k x/||A^k x||\}_{k\geq 0}$ has a convergent subsequence.*

**Proof** Let $Y$ be the image of $X$ under $P$ and let $Z$ be the image of $X$ under $I - P$. By Proposition 4.1.1, $Y$ is finite dimensional. Therefore, we can decompose $Y$ into $Y_1 \oplus Y_2 \oplus ... \oplus Y_p$ where each $Y_j$ is an $A$-invariant subspace on which there is a basis such that $A|_{Y_j}$ can be represented by an $n_j \times n_j$ Jordan block matrix $J_{n_j}(\lambda_j)$. Since $Y$ corresponds to the peripheral spectrum of $A$, each $\lambda_j$ will have $|\lambda_j| = 1$. For each $Y_j$ there is a projection $P_j$ from $X$ onto $Y_j$ which commutes with $A$. Since $Px \neq 0$, there is a $j \in \{1, ..., p\}$ such that $P_j x \neq 0$. By Lemma 4.1.2, it is clear that there is an $\epsilon > 0$ such that $||A^k P_j x|| \geq \epsilon$ for all $k \geq 0$. Since $||P_j|| = 1$ and $P_j$ commutes with $A$, it follows that $\epsilon \leq ||P_j A^k x|| \leq ||A^k x||$.

Every element in the set $\{A^k x/||A^k x|| \mid k \geq 0\}$ has norm one. Note that $r(A|_Z) < 1$ so $\lim_{k\to\infty} A^k|_Z = 0$. Since $||A^k x|| \geq \epsilon$ for all $k \geq 0$, it follows that $(I-P)A^k x/||A^k x|| \to 0$ as $k \to \infty$. Therefore $A^k x/||A^k x||$ approaches the unit ball in $Y$ as $k \to \infty$. Since $Y$ is finite dimensional, the unit ball in $Y$ is compact and therefore $A^k x/||A^k x||$ has a convergent subsequence. ∎

**Lemma 4.1.4** *Suppose that $A \in \mathcal{B}(X)$ has $r(A) = 1$ and $\rho(A) < 1$. Let $P$ be the spectral projection operator corresponding to the peripheral spectrum of $A$. Let $x \in X$. If $Px \neq 0$ and*

$$\lim_{i\to\infty} \frac{A^{k_i} x}{||A^{k_i} x||} = \zeta,$$

*then there is an integer $q \geq 0$ such that $\lim_{i\to\infty} {k_i}^{-q} A^{k_i} x = c\zeta$ where $c > 0$. Furthermore, $\zeta \in W = \operatorname{span}\{x \in X \mid Ax = \lambda x \text{ where } |\lambda| = 1\}$.*

**Proof** Let the notation be as in the proof of Lemma 4.1.3. Since $r(A|_Z) < 1$, it immediately follows that $\lim_{i\to\infty} A^{k_i}|_Z = 0$. Let $Px = x_1 + x_2 + ... + x_p$ where each $x_j \in Y_j$, for $1 \leq j \leq p$. We may choose a refinement $\{m_i\}$ of the sequence $\{k_i\}$ such that $\lim_{i\to\infty} \lambda_j^{m_i}$ exists for every $j \in \{1, ..., p\}$. If $x_j \neq 0$, then Lemma 4.1.2 implies that there is an integer $q_j \geq 0$ such that

$$\lim_{i\to\infty} \frac{A^{m_i} x_j}{m_i^{q_j}} = y_j$$

where $y_j \neq 0$ and $Ay_j = \lambda_j y_j$. If we let $q = \max_{1\leq j\leq p} q_j$, then by the linearity of $A$, $\lim_{i\to\infty} m_i^{-q} A^{m_i} x = y$ where $y \neq 0$ and $y \in W$. Since

$$\zeta = \lim_{i\to\infty} \frac{A^{m_i} x}{||A^{m_i} x||} = \lim_{i\to\infty} \frac{m_i^{-q} A^{m_i} x}{||m_i^{-q} A^{m_i} x||} = \frac{y}{||y||}$$

it follows that $\lim_{i\to\infty} m_i^{-q} A^{m_i} x = c\zeta$ where $c = ||y|| = \lim_{i\to\infty} ||m_i^{-q} A^{m_i} x||$.

Note that $P_j \zeta = P_j y/c$ for all $j \in \{1, ..., p\}$. Therefore,

$$||P_j \zeta|| = \lim_{i\to\infty} \left|\left| P_j \left( \frac{A^{k_i} x}{||A^{k_i} x||} \right) \right|\right| = \lim_{i\to\infty} \frac{||k_i^{-q} A^{k_i} x_j||}{||k_i^{-q} A^{k_i} x||} = \frac{||y_j||}{c}.$$

By Lemma 4.1.2, $\lim_{i\to\infty} ||k_i^{-q} A^{k_i} x_j|| = ||y_j||$. Therefore, $\lim_{i\to\infty} ||k_i^{-q} A^{k_i} x|| = c$. Since

$$P_j \zeta = \lim_{i\to\infty} \frac{k_i^{-q} A^{k_i} x_j}{||k_i^{-q} A^{k_i} x||},$$

we conclude that $\lim_{i\to\infty} k_i^{-q} A^{k_i} x_j$ exists, and must equal $\lim_{i\to\infty} m_i^{-q} A^{m_i} x_j$ for each $j \in \{1, ..., p\}$. Therefore, $\lim_{i\to\infty} k_i^{-q} A^{k_i} x = c\zeta$. ∎

**Lemma 4.1.5** *Suppose that* $A \in \mathcal{B}(X)$ *has* $r(A) = 1$ *and* $\rho(A) < 1$. *Suppose that there is an integer* $q \geq 0$ *and sequences* $\{k_i\}$ *and* $\{m_i\}$ *such that*

$$\lim_{i\to\infty} \frac{A^{k_i} x}{k_i{}^q} = \zeta \quad \text{and} \quad \lim_{i\to\infty} \frac{A^{k_i + m_i} x}{(k_i + m_i)^q} = \eta$$

*where* $\zeta$ *and* $\eta$ *are each nonzero. Then* $\eta = \lim_{i\to\infty} A^{m_i} \zeta$.

**Proof** Let the notation be as in the proof of Lemma 4.1.3. For each $j \in \{1, ..., p\}$, $A|_{Y_j}$ corresponds to a Jordan block $J_{n_j}(\lambda_j)$ where $|\lambda_j| = 1$. By Lemma 4.1.2, if $\zeta_j = \lim_{i\to\infty} k_i^{-q} A^{k_i} x_j$ is a nonzero vector in $Y_j$, then there is a $y_j \in Y_j \backslash \{0\}$ such that $\zeta_j = \alpha_j y_j$ where $\alpha_j = \lim_{i\to\infty} \lambda_j^{k_i}$. Furthermore, $y_j$ is an eigenvector of $A$ with $Ay_j = \lambda_j y_j$. For this same $j$, Lemma 4.1.2 also implies that

$$\eta_j = \lim_{i\to\infty} \frac{A^{k_i+m_i} x_j}{(k_i + m_i)^q} = \alpha_j \beta_j y_j$$

where $\beta_j = \lim_{i\to\infty} \lambda_j^{m_i}$. Since $\zeta_j = \alpha_j y_j$ is an eigenvector of $A$ with eigenvalue $\lambda_j$, it follows that $\lim_{i\to\infty} A^{m_i} \zeta_j = \eta_j$.

Since $\lim_{k\to\infty} A^k|_Z = 0$ and since $\zeta \neq 0$, it follows that there is some $j \in \{1, ..., p\}$ such that $\zeta_j$ is nonzero. For each such $j$, $\lim_{i\to\infty} A^{m_i} \zeta_j = \eta_j$. For the rest, $\zeta_j = \eta_j = 0$. Thus, $\lim_{i\to\infty} A^{m_i} \zeta = \eta$. ■

## 4.2 Positive Linear Operators

In what follows we will assume that $X$ is a real Banach space. In order to do spectral theory on $X$, we need the *complexification* of $X$, that is, the complex linear space $\tilde{X} = X \oplus X$ where $\alpha(x, y) = (a_1 x - a_2 y, a_2 x + a_1 y)$ for any $\alpha = a_1 + ia_2 \in \mathbb{C}$ and $x, y \in X$. The complexification $\tilde{X}$ can be given a norm,

$$|||(x, y)||| = \sup_{0\leq t\leq 2\pi} ||(\cos t)x + (\sin t)y||,$$

and $\tilde{X}$ is a complex Banach space with this norm. We may identify $X$ with the subset $\{(x, 0) \mid x \in X\} \subset \tilde{X}$. Any $A \in \mathcal{B}(X)$ extends to a linear map $\tilde{A} \in \mathcal{B}(\tilde{X})$ as follows, $\tilde{A}(x, y) = (Ax, Ay)$. Note that for $A \in \mathcal{B}(X)$, the spectrum is defined to be $\sigma(\tilde{A})$. This implies that $r(A) = r(\tilde{A})$ and $\rho(A) = \rho(\tilde{A})$.

Suppose that $C$ is a closed cone in $X$. If $A \in \mathcal{B}(X)$ is a linear map such that $A(C) \subset C$, then we know that $A$ is nonexpansive with respect to Hilbert's projective metric on $C$. If $C$ has nonempty interior and $A(\mathrm{int\,C}) \subset \mathrm{int\,C}$, then we can ask whether the normalized iterates of $A$ satisfy a Denjoy-Wolff type theorem. Theorem 4.2.1 answers this question even when the cone $C$ is neither strictly convex nor polyhedral. Moreover, Theorem 4.2.1 gives us a Denjoy-Wolff type result when $C$ is infinite dimensional. Note that the results of this theorem are much stronger than merely demonstrating that the omega limit sets of the normalized map are contained in a convex subset of the boundary. In fact, Theorem 4.2.1 reproduces the results of Dafermos and Slemrod (Theorem 1 of [16]) for omega limit sets of nonexpansive maps even though the omega limit sets described below may be contained in the boundary of the domain rather than in the domain itself.

We say that $x \in C$ is a *quasi-interior* point of $C$ if the closed linear span of the set $[0, y] = \{y \in X \mid 0 \leq y \leq x\}$ is all of $X$. Note that if $C$ has nonempty interior, then $x$ is a quasi-interior point if and only if $x \in \mathrm{int\,C}$.

**Theorem 4.2.1** *Let $C$ be a closed total cone in a real Banach space $X$. Let $d$ denote Hilbert's projective metric on $C$. Let $A \in \mathcal{B}(X)$ be a linear map such that $A(C) \subset C$ and $r(A) > \rho(A)$. Let $T(x) = Ax/||Ax||$ for all $x \in C$ such that $Ax \neq 0$. Then for any quasi-interior point $x \in C$ such that $Ax \sim x$, there is an eigenvector $z \in C\backslash\{0\}$ with $Az = r(A)z$ such that $\omega(x;T) \subset C_z$ where $C_z$ is the part of $C$ containing $z$. Furthermore, $T$ is an invertible $d$-isometry on $\omega(x;T)$ and if $\zeta \in \omega(x;T)$, then $\omega(\zeta;T) = \omega(x;T)$.*

One of the conclusions of Theorem 4.2.1 is that there is an eigenvector $z \in C$ with eigenvalue $r(A)$. This is not a new result. For compact linear maps, this is the Krein-Rutman theorem, see [32] and the appendix of [53]. Bonsall and Schaefer give generalizations of the Krein-Rutman theorem in [9] and [52], section 10. Nussbaum establishes the existence of an eigenvector $z \in C$ with $Az = r(A)z$ for maps $A \in \mathcal{B}(X)$ with $A(C) \subset C$ and $\rho(A) < r(A)$ in [39] (see also [43]).

In order to prove Theorem 4.2.1, we need to know that the omega limit sets of the map $T$ are nonempty. The following lemma, together with Lemma 4.1.3 will prove this.

**Lemma 4.2.1** *Let $C$ be a closed total cone in a Banach space $X$. Let $\tilde{X}$ be the complexification of $X$. Suppose that $A \in \mathcal{B}(X)$ satisfies $A(C) \subset C$ and $r(A) > \rho(A)$. Let $P$ be the spectral projection in $\tilde{X}$ corresponding to the peripheral spectrum of $A$. If $x$ is a quasi-interior point of $C$, then $Px \neq 0$.*

**Proof** If $x$ is a quasi-interior point of $C$, then the closed linear span of $[0, x] = \{y \in X \mid 0 \leq y \leq x\}$ is all of $X$. Therefore, if $Py = 0$ for every $y \in [0, x]$, then $Pv = 0$ for all $v \in X$. Since every $z \in \tilde{X}$ has the form $z = (v, w) = v + iw$, with $v, w \in X$, it follows that $Pz = 0$ for all $z \in \tilde{X}$. This cannot be the case since $\rho(A) < r(A)$. Thus, there is some $y \in [0, x]$ such that $Py \neq 0$. Now, suppose by way of contradiction that $Px = 0$. Since $A$ is order-preserving and $y \leq x$, $A^k y \leq A^k x$ for all $k \geq 0$. By Lemma 4.1.3, we can find an increasing sequence of integers $k_i$ such that $\lim_{i \to \infty} A^{k_i} y / ||A^{k_i} y|| = u$ where $u \in C$ and $||u|| = 1$. Then Lemma 4.1.4 implies that there is a $q \geq 0$ such that $\lim_{i \to \infty} {k_i}^{-q} A^{k_i} y = cu$ where $c > 0$. At the same time, since $Px = 0$, we must have $\lim_{i \to \infty} {k_i}^{-q} A^{k_i} x = 0$ by Proposition 4.1.1. However $k_i^{-q} A^{k_i} x \geq k_i^{-q} A^{k_i} y$ for all $i > 0$. In other words $k_i^{-q} A^{k_i} x - k_i^{-q} A^{k_i} y \in C$ and since $C$ is closed, we would have $-cu \in C$

which is a contradiction. Therefore $Px \neq 0$ for every quasi-interior point $x \in C$. ■

**Proof of Theorem 4.2.1** We can assume without loss of generality that $r(A) = 1$ by replacing $A$ with $r(A)^{-1}A$. Let $\tilde{X}$ be the complexification of $X$ and let $\tilde{A}$ be the natural extension of $A$ to $\tilde{X}$. Let $P$ be the spectral projection on $\tilde{X}$ corresponding to the peripheral spectrum of $A$. By Lemma 4.2.1, $Px \neq 0$. Therefore Lemma 4.1.3 implies that $\omega(x;T)$ exists and is nonempty.

If $\zeta \in \omega(x;T)$ we may assume that

$$\zeta = \lim_{i\to\infty} T^{k_i}x = \lim_{i\to\infty} \frac{A^{k_i}x}{||A^{k_i}x||}$$

for some sequence of integers $k_i$. Lemma 4.1.4 implies that there is a $q \geq 0$ such that $\lim_{i\to\infty} k_i^{-q}A^{k_i}x = c_1\zeta$ where $c_1 > 0$. Furthermore, $\zeta \in W \cap X$ where $W = \text{span}\{z \in \tilde{X} \mid \tilde{A}z = \lambda z \text{ where } |\lambda| = 1\}$.

Let $C_\zeta$ be the part of $C$ containing $\zeta$. We know that $A$ is nonexpansive with respect to Hilbert's projective metric $d$ by Theorem 2.4.1. Since $Ax \sim x$, Lemma 2.3.1 implies that $A\zeta \sim \zeta$ and therefore $A(C_\zeta) \subset C_\zeta$ since $A$ is order-preserving and homogeneous of degree one. It follows that $T$ is defined on all of $C_\zeta$ and $T : C_\zeta \to C_\zeta$ is nonexpansive with respect to $d$ by equation 2.4. Since $\tilde{A}|_W$ is invertible and $\omega(x;T) \subset W \cap X$, it follows that $T$ is an invertible map on $\omega(x;T)$.

Now, suppose that $\eta \in \omega(x;T)$. There is a sequence of integers $m_i$ such that

$$\eta = \lim_{i\to\infty} \frac{A^{k_i+m_i}x}{||A^{k_i+m_i}x||}.$$

By Lemma 4.1.4, there is an integer $q \geq 0$ and a constant $c_2 > 0$ such that

$$c_2\eta = \lim_{i\to\infty} \frac{A^{k_i+m_i}x}{(k_i+m_i)^q}.$$

Lemma 4.1.5 implies that

$$\lim_{i\to\infty} \frac{A^{k_i+m_i}x}{(k_i+m_i)^q} = \lim_{i\to\infty} A^{m_i}(c_1\zeta).$$

Therefore

$$\eta = \lim_{i\to\infty} \frac{A^{m_i}\zeta}{||A^{m_i}\zeta||}.$$

This tells us that if $\eta \in \omega(x;T)$, then $\eta$ is also in $\omega(\zeta;T)$.

Since $\tilde{A}|_W$ can be represented by a diagonal matrix with each diagonal entry having modulus one, it follows that there is a sequence of integers $k_i$ such that $\lim_{i\to\infty} \tilde{A}^{k_i}|_W = I_W$ where $I_W$ is the identity map on $W$. Therefore $\lim_{i\to\infty} A^{k_i}\zeta = \zeta$. Note that by Theorem 2.4.1, $A$ is nonexpansive with respect to Thompson's metric on $C_\zeta$. Since $A^{k_i}\zeta \to \zeta$ as $i \to \infty$, it follows from Theorem 2.3.1 that $A$ has a fixed point in $C_\zeta$. Any fixed point of $A$ in $C_\zeta$ will be an eigenvector with eigenvalue one. Let $z \in C_\zeta$ be one such eigenvector, normalized so that $||z|| = 1$.

Let $k_i$ be a sequence of integers such that $\lim_{i\to\infty} \tilde{A}^{k_i}|_W = I_W$ where $I_W$ is the identity map on $W$. Then $\lim_{i\to\infty} T^{k_i}(\zeta) = \zeta$ for all $\zeta \in \omega(x;T)$. Suppose that $T$ were not a $d$-isometry on $\omega(x;T)$. This would imply that there is a pair $\zeta, \eta \in \omega(x;T)$ such that $d(T(\zeta), T(\eta)) < d(\zeta, \eta)$. Then $d(T^{k_i}(\zeta), T^{k_i}(\eta)) < d(\zeta, \eta)$ for all $i > 0$ by the nonexpansiveness of $T$. Since $T^{k_i}(\zeta) \to \zeta$ and $T^{k_i}(\eta) \to \eta$, we get a contradiction. Therefore $T$ is an isometry on $\omega(x;T)$. ■

**Remark 4.1** If $C$ has nonempty interior in $X$, $A(\operatorname{int} \mathrm{C}) \subset \operatorname{int} \mathrm{C}$, and $x \in \operatorname{int} \mathrm{C}$, then Theorem 4.2.1 tells us that $\omega(x;T)$ is contained in a single part, $C_z$, of $C$. If $A$ has no eigenvector in $\operatorname{int} \mathrm{C}$, then $C_z$ will be a convex subset of the boundary of $C$. Thus, Theorem 4.2.1 implies that conjecture 1 is true for such maps, even in infinite dimensions. Note that, if $C$ has nonempty interior in $X$ and $A$ is a compact linear map such that $A(\mathrm{int\,C}) \subset \mathrm{int\,C}$, then $\rho(A) < r(A)$ automatically. After all, $\rho(A) = 0$ since $A$ is compact, and $r(A) \geq r_C(A) > 0$ by Lemma 2.4.1.

**Remark 4.2** There are important examples of closed cones which do not have a nonempty interior. For example, if $X = L^p[0,1]$ with $1 \leq p < \infty$ and $C$ is the closed cone consisting of functions in $X$ that are nonnegative almost everywhere, then $C$ does not have an interior. Note that any function $f \in L^p[0,1]$ that is positive almost everywhere is a quasi-interior point of $C$.

## 4.3 Linear Maps on Polyhedral Cones

Suppose that $X$ is a finite dimensional normed space and $C \subset X$ is a closed polyhedral cone with nonempty interior in $X$. If $A : X \to X$ is a linear map such that $A(C) \subset C$, then we have the following lemma about the eigenvalues in the peripheral spectrum of $A$.

**Lemma 4.3.1** *Suppose that $C$ is a closed polyhedral cone with nonempty interior in a finite dimensional normed space and $A$ is a linear map such that $A(C) \subset C$. If $C$ has $N$ facets, then each eigenvalue in the peripheral spectrum of $A$ is equal to $r(A)$ times a $k^{th}$-root of unity where $1 \leq k \leq N$.*

Lemma 4.3.1 originally appeared in [4] and a corrected proof can be found in Theorem 7.6 of [54]. A special case of Lemma 4.3.1 appears in the paper of Krein and Rutman [32]. Using Lemma 4.3.1, we can strengthen the results of Theorem 4.2.1 when $C$ is a polyhedral cone in a finite dimensional normed space. The following theorem appeared as Theorem 2 in [34], although we give a different proof here.

**Theorem 4.3.1** *Let $C$ be a closed polyhedral cone with nonempty interior in a finite dimensional normed space $X$. Suppose that $A : X \to X$ is a linear map such that $A(\mathrm{int\,C}) \subset \mathrm{int\,C}$. Let $T(x) = Ax/||Ax||$ for all $x \in \mathrm{int\,C}$. There is an integer $p > 0$ such that for each $x \in \mathrm{int\,C}$, $\lim_{k\to\infty} T^{kp}(x) = \zeta$ where $\zeta$ is a point which depends on $x$ and $p$ can be chosen to be the least common multiple of $\{1, ..., N\}$ where $N$ is the number of facets of $C$. Furthermore, the omega limit sets, $\omega(x;T)$, are finite periodic orbits of $T$.*

**Proof** Since $A(\mathrm{int\,C}) \subset \mathrm{int\,C}$, it follows that $r(A) > 0$ (by Lemma 2.4.1). Then by replacing $A$ with $r(A)^{-1}A$ we can assume that $r(A) = 1$. By Lemma 4.3.1 every element of the peripheral spectrum of $A$ is a $k^{th}$-root of unity where $1 \leq k \leq N$. By letting $p = \mathrm{lcm}\{1, ..., N\}$, we ensure that the peripheral spectrum of $A^p$ contains only 1. If $\zeta \in \omega(x;T^p)$, then we know from the proof of Theorem 4.2.1 that $\zeta \in W = \{y \in X \mid A^p y = y\}$. Therefore $A^p(\zeta) = \zeta$ and so $T^p(\zeta) = \zeta$. Theorem 4.2.1 tells us that $\omega(\zeta;T^p) = \omega(x;T^p)$. We conclude that $\omega(x;T^p) = \{\zeta\}$ and $\zeta = \lim_{k\to\infty} T^{kp}(x)$. It follows that $\omega(x;T)$ is a finite periodic orbit of $T$ with a period that is a divisor of $p$. ∎

## 4.4 Affine Linear Maps

Suppose that $X$ is a finite dimensional normed space and $C$ is a closed cone with nonempty interior in $X$. Suppose that $q \in X^*$ is a linear functional such that $q(x) > 0$ for all $x \in C\backslash 0$. Let $\Sigma = \{x \in \mathrm{int\,C} \mid \mathrm{q(x)} = 1\}$. Suppose that $A : C \to C$ is a linear map and $b \in C$. If the affine linear map $f(x) = Ax + b$ has $f(\mathrm{int\,C}) \subset \mathrm{int\,C}$, then we define $\hat{f} = f(x)/q(f(x))$ for all $x \in \Sigma$. Even though $f$ is not homogeneous of degree one, $\hat{f}$ will still be nonexpansive with respect to the Hilbert metric on $\Sigma$. After all, if $x, y \in \Sigma$, then there exist constants $\alpha, \beta > 0$ such that $\alpha x \leq y \leq \beta x$. Furthermore, the proof of Lemma 2.3.1 shows that $\alpha < 1$ and $\beta > 1$. Thus $\alpha(Ax+b) \leq Ay+b \leq \beta(Ax+b)$ and therefore equation 2.3 gives $d(\hat{f}(x), \hat{f}(y)) \leq d(x,y)$ where $d$ is the Hilbert metric on $\Sigma$.

As with other Hilbert metric nonexpansive maps we can ask whether the omega limit sets of affine linear maps satisfy a Denjoy-Wolff type theorem. Certainly if $\Sigma$ is strictly convex or polyhedral we have such results. However, in the case of affine linear maps in finite dimensions we can prove that $\mathrm{co}\,(\omega(\mathrm{x}; \hat{\mathrm{f}})) \subset \partial\Sigma$ when $\hat{f}$ has no fixed point in $\Sigma$, even when $\Sigma$ is neither strictly convex nor polyhedral. In order to prove this, we need the following lemma.

**Lemma 4.4.1** *Suppose that $A : X \to X$ is linear and $b \in X$. Let $f(x) = Ax + b$ and let $\hat{f}(x) = f(x)/q(f(x))$ where $q \in X^*$. Let $S = \{x \in X \mid q(f(x)) \neq 0\}$. Then $\hat{f}$ is a convexity-preserving map on $S$. That is, if $x \in S$ is a convex combination of $z_1, z_2, ..., z_k$ in $S$, then $\hat{f}(x)$ will be a convex combination of $\hat{f}(z_1), \hat{f}(z_2), ..., \hat{f}(z_k)$.*

**Proof** For $x = \lambda_1 z_1 + \lambda_2 z_2 + ... + \lambda_k z_k$, with $\lambda_i > 0$ and $\sum_i \lambda_i = 1$

$$\hat{f}(x) = \frac{f(x)}{q(f(x))} = \frac{Ax+b}{q(Ax+b)} =$$

$$= \frac{\lambda_1 Az_1 + ... + \lambda_k Az_k + b}{\lambda_1 q(Az_1) + ... + \lambda_k q(Az_k) + q(b)}$$

$$= \frac{\lambda_1(Az_1 + b) + ... + \lambda_k(Az_k + b)}{\lambda_1 q(Az_1 + b) + ... + \lambda_k q(Az_k + b)}$$

$$= \frac{\lambda_1 q(Az_1 + b)\hat{f}(z_1) + ... + \lambda_k q(Az_k + b)\hat{f}(z_k)}{\lambda_1 q(Az_1 + b) + ... + \lambda_k q(Az_k + b)}$$

$$= \mu_1 \hat{f}(z_1) + \mu_2 \hat{f}(z_2) + ... + \mu_k \hat{f}(z_k),$$

with:

$$\mu_i = \frac{\lambda_i q(Az_i + b)}{\lambda_1 q(Az_1 + b) + ... + \lambda_k q(Az_k + b)}.$$

■

**Remark 4.3** For a characterization of convexity-preserving maps on a subset of a vector space see [3].

**Theorem 4.4.1** *Let $C$ be a closed cone with nonempty interior in a finite dimensional normed space $X$. Let $A : C \to C$ be a linear map and $b \in C$ such that the affine map $f(x) = Ax + b$ satisfies $f(\mathrm{int\,C}) \subset \mathrm{int\,C}$. Let $q \in C^*$ be a linear functional such that $q(x) > 0$ for all $x \in C\backslash\{0\}$ and define $\Sigma = \{x \in \mathrm{int\,C} \mid \mathrm{q(x)} = 1\}$. Let $\hat{f} = f(x)/q(f(x))$. If $\hat{f}$ has no fixed point in $\Sigma$, then for any $x \in \Sigma$, $\mathrm{co}\,(\omega(\mathrm{x};\hat{\mathrm{f}})) \subset \partial\Sigma$.*

**Proof** Let $X$ be a normed space with $\dim(X) = n$ and let $C$ and $f$ be as in the theorem. Note that if $b = 0$, then $f$ is linear and we can use Theorem 4.2.1. Therefore we assume that $b \neq 0$. This will imply that $\hat{f}$ is defined continuously on $\mathrm{cl}\,\Sigma$. Let us suppose by way of contradiction that there is a point $x \in \Sigma$ such that $\mathrm{co}\,(\omega(\mathrm{x};\hat{\mathrm{f}})) \cap \Sigma$ is non-empty. Let $y \in \mathrm{co}\,(\omega(\mathrm{x};\hat{\mathrm{f}})) \cap \Sigma$. Thus $y$ is a convex combination of points in $\omega(x;\hat{f})$. By Carathéodory's theorem (see [49]), we may assume that $y$ is a convex combination of at most $n$ points $z_1, z_2, ..., z_n \in \omega(x;\hat{f})$.

By Lemma 2.3.2, we know that for each $z_i \in \omega(x;\hat{f})$ there exists $\zeta_i \in \omega(y;\hat{f})$ such that $\zeta_i \sim z_i$. Furthermore, $\mathrm{co}\,\{\zeta_1, ..., \zeta_\mathrm{n}\} \cap \Sigma \neq \varnothing$. After all, if $y = \lambda_1 z_1 + ... + \lambda_n z_n \in \Sigma$ where each $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$, then it is easy to check that $y' = \lambda_1\zeta_1 + \lambda_2\zeta_2 + ... + \lambda_n\zeta_n \in \Sigma$.

At the same time, we claim that $\omega(y;\hat{f}) \subset \mathrm{co}\,(\mathrm{U})$ where $U = \omega(z_1;\hat{f}) \cup \omega(z_2;\hat{f}) \cup ... \cup \omega(z_n;\hat{f})$. To prove the claim, note that for each $k \geq 0$, $\hat{f}^k(y) = \lambda_1^{(k)}\hat{f}^k(z_1) + ... + \lambda_n^{(k)}\hat{f}^k(z_n)$, with each $\lambda_i^{(k)} \geq 0$ and $\sum_i \lambda_i^{(k)} = 1$, by Lemma 4.4.1. Taking a subsequence, we can arrange that $\hat{f}^{k_j}(z_i) \to z_i' \in \omega(z_i, \hat{f})$ as $j \to \infty$ for $1 \leq i \leq n$ and simultaneously $\lambda_i^{(k_j)} \to \lambda_i'$ with each $\lambda_i' \geq 0$ and $\sum_i \lambda_i' = 1$. Thus, each point $z' \in \omega(y;\hat{f})$ is a convex combination $\sum_{i=1}^{n} \lambda_i' z_i'$ with each $z_i' \in \omega(z_i; \hat{f})$, proving the claim.

Let $y^1$ be a point in $\mathrm{co}\,(\omega(\mathrm{y}; \hat{\mathrm{f}})) \cap \Sigma$, and choose $z_1^1, z_2^1, ..., z_n^1 \in U$ such that $y^1 \in \mathrm{co}\,(\{\mathrm{z}_1^1, \mathrm{z}_2^1, ..., \mathrm{z}_n^1\})$ Note that each point $z_j^1 \in \omega(z_i; \hat{f})$ for some $i$. Furthermore, since $z_i \in \partial\Sigma$, we know that $z_i$ must lie in a part $C_{z_i}$ of $C$ which has dimension at most $n-1$. Let $\Sigma_{z_i} = \{x \in C_{z_i} \mid q(x) = 1\}$. Lemma 2.3.3 implies that $\hat{f}(\Sigma_{z_i}) \subset \Sigma_{z_i}$. If $\Sigma_{z_i}$ contains a fixed point of $\hat{f}$, then since $\hat{f}$ is nonexpansive in the Hilbert metric on $\Sigma_{z_i}$, every $\hat{f}$-orbit in $\Sigma_{z_i}$ must remain within a bounded Hilbert metric distance of that fixed point. On the other hand, if $\Sigma_{z_i}$ does not contain a fixed point, then $z_j^1$ is contained in a part of $C$ on the boundary of $C_{z_i}$ by Theorem 2.3.1. Such a part would have to have dimension strictly less than $n-1$.

Repeat this process to obtain a sequence of points $y^1, y^2, ..., y^{n-2} \in \Sigma$ with the property that each $y^i \in \mathrm{co}\,(\omega(\mathrm{y}^{\mathrm{i}-1}; \hat{\mathrm{f}}))$, and more importantly $y^i \in \mathrm{co}\,(\{\mathrm{z}_1^{\mathrm{i}}, \mathrm{z}_2^{\mathrm{i}}, ...\mathrm{z}_n^{\mathrm{i}}\})$ where each $z_j^i$ is contained in a part of $C$ with dimension less than $n-i$ or is contained in a part of $C$ on which $\hat{f}$ has a fixed point. This means that $y^{n-2}$ is a point in $\Sigma$ which is a convex combination of points $z_1^{n-2}, z_2^{n-2}, ..., z_n^{n-2}$ which all lie in parts of $C$ containing fixed points of $\Sigma$. For each $1 \leq i \leq n$, let $p_i$ be a fixed point of $\hat{f}$ in the part which contains $z_i^{n-2}$. Suppose that $y^{n-2} = \lambda_1 z_1^{n-2} + \lambda_2 z_2^{n-2} + ... + \lambda_n z_n^{n-2}$ and let $\zeta = \lambda_1 p_1 + \lambda_2 p_2 + ... + \lambda_n p_n$. Observe that $\zeta$ is comparable to $y^{n-2}$ since $z_i^{n-2} \sim p_i$ for all $1 \leq i \leq n$. Thus $\zeta \in \Sigma$. Now, since $\hat{f}(p_i) = p_i$ if and only if $f(p_i) = r_i p_i$ for some constant $r_i > 0$, we have:

$$\hat{f}^k(\zeta) = \frac{f^k(\zeta)}{q(f^k(\zeta))} = \frac{\sum \lambda_i r_i^k p_i}{\sum \lambda_i r_i^k q(p_i)}$$

If $r = \max r_i$ and $J = \{i \mid r_i = r\}$, then the reader can verify that as $k \to \infty$

$$\hat{f}^k(\zeta) \to \frac{\sum_{i \in J} \lambda_i p_i}{\sum_{i \in J} \lambda_i q(p_i)}$$

which is a single point in $\operatorname{cl}\Sigma$. Since there are no bounded orbits in $\Sigma$, this limit point must be on the boundary $\partial\Sigma$. However, if $\omega(\zeta; \hat{f})$ is a single point, then $\operatorname{co}(\omega(\zeta; \hat{\mathrm{f}})) \subset \partial\Sigma$ which gives us the contradiction we need, by Lemma 2.3.4. ■

# Chapter 5

# Reproduction-Decimation Operators

## 5.1 Positive Semi-Definite Forms and Discrete Dirichlet Forms

A class of nonlinear order-preserving homogeneous of degree one maps appears in the study of diffusion on fractals. These "reproduction-decimation operators" are defined on the interior of the cone of positive semi-definite forms. In this section we will introduce the cone of positive semi-definite forms as well as the cone of discrete Dirichlet forms. In the following section we will define a general class of reproduction-decimation operators and show how the results of the chapter 3 allow us to establish a Denjoy-Wolff type result for these operators even though the cone of positive semi-definite forms is neither polyhedral nor strictly convex.

Let $S$ be a finite set. If we think of $S$ as a measure space with the counting measure, then $L^2(S)$ is a finite dimensional Hilbert space consisting of the functions $x : S \to \mathbb{R}$. The inner product on $L^2(S)$ is $\langle x, y \rangle = \sum_{i \in S} x(i)y(i)$. On $L^2(S)$ we have a *standard basis* consisting of the functions $e_i$, $i \in S$ where $e_i(j) = \delta_{ij}$, the Kronecker delta. We let $\mathbb{1}_S$ be the function $\mathbb{1}_S(i) = 1$ for all $i \in S$.

We say that a bounded self-adjoint linear operator $A$ on a Hilbert space $H$ is *positive semi-definite* if $\langle Ax, x \rangle > 0$ and $A$ is *positive definite* if there is a constant $c > 0$ such that $\langle Ax, x \rangle \geq c\langle x, x \rangle$ for all $x \in H$. Since $S$ is finite, any bounded linear operator $A : L^2(S) \to L^2(S)$ can be represented by a matrix $(a_{ij})_{i,j \in S}$ where $(a_{ij}) = \langle Ae_j, e_i \rangle$.

We shall denote by $X_S$ the set of all bounded self-adjoint linear operators $A : L^2(S) \to L^2(S)$ such that $A(\mathbb{1}_S) = 0$. When we refer to elements $A \in X_S$, we will not always make a sharp distinction between the operator $A$ and the quadratic form $\langle Ax, x\rangle$ defined by $A$. Thus, we may refer to $A$ as a quadratic form when it is convenient. In the space $X_S$ we let $K_S$ denote the cone of positive semi-definite operators, that is

$$K_S = \{A \in X_S \mid \langle Ax, x\rangle \geq 0 \;\forall\, x \in L^2(S)\}.$$

The cone of discrete Dirichlet forms, $D_S$, is defined

$$D_S = \{A \in X_S \mid (a_{ij}) \leq 0 \text{ for all } i, j \in S \text{ with } i \neq j\}.$$

Both $K_S$ and $D_S$ have nonempty interior in $X_S$. In fact it is not hard to show that

$$\mathrm{int\,K_S = \{A \in X_S \mid \exists\, c > 0 \text{ with } \langle Ax, x\rangle \geq c\langle x, x\rangle \;\forall\, x \perp \mathbb{1}_S\}},$$

$$\mathrm{int\,D_S = \{A \in X_S \mid (a_{ij}) < 0 \text{ for all } i, j \in S \text{ with } i \neq j\}}.$$

For any operator $A$ in $X_S$, there is a nice formula for the quadratic form $\langle Ax, x\rangle$ which we state here as a lemma.

**Lemma 5.1.1** *If $A \in X_S$ and $x \in L^2(S)$, then*

$$\langle Ax, x\rangle = -\frac{1}{2} \sum_{i\neq j\in S} a_{ij}(x(i) - x(j))^2. \tag{5.1}$$

**Proof** Since $A \in X_S$, we have $A(\mathbb{1}_S) = 0$ and therefore $\sum_{j\in S} a_{ij} = 0$ for each $i \in S$. Alternatively, we may write $a_{ii} = -\sum_{j\neq i} a_{ij}$ for each $i \in S$. Now,

$$\langle Ax, x\rangle = \sum_{i\in S}\sum_{j\in S} a_{ij}x(i)x(j).$$

Each $x(i)x(j)$ term with $i \neq j$ will appear twice in this sum. For each $i \in S$, the term $x(i)^2$ will appear only once in the sum, but the coefficient on $x(i)^2$ will be $-\sum_{j\neq i} a_{ij}$.

Since $A$ is self-adjoint, $a_{ij} = a_{ji}$ for all $i, j \in S$. From these facts it is clear that

$$\sum_{i \in S} \sum_{j \in S} a_{ij} x(i) x(j) = -\frac{1}{2} \sum_{i \neq j \in S} a_{ij} (x(i) - x(j))^2.$$

■

Using equation 5.1 it follows immediately that $D_S \subset K_S$. We say that a Dirichlet form $A \in D_S$ is *irreducible* if $A \in D_S \cap \operatorname{int} \mathrm{K_S}$. The reproduction-decimation operators will be defined on these irreducible Dirichlet forms. We would like to use Theorem 3.4.1 to establish a Denjoy-Wolff type theorem for this class of maps. In order to do this, we must first prove the following proposition. Note that Theorem 3.2.2 does not apply to the cone $K_S$ because $K_S$ is neither polyhedral nor strictly convex when $\operatorname{card} \mathrm{S} > 3$.

**Proposition 5.1.1** *If $S$ is a finite set with $\operatorname{card} \mathrm{S} \geq 3$ and $D_S$ and $K_S$ are defined as above, then there is a closed polyhedral cone $C_p \subset X_S$ such that $D_S \subset C_p \subset K_S$ and every element in $C_p^*$ is comparable to an element of $K_S^*$ in the partial ordering induced by $D_S^*$.*

In order to prove this proposition, we need to consider the dual cones of $D_S$ and $K_S$. One can easily show that

$$D_S^* = \{ \sum_{i \neq j \in S} b_{ij} \psi_{ij} \mid b_{ij} \geq 0 \text{ and } b_{ij} = b_{ji} \text{ for all } i \neq j \}$$

where $\psi_{ij}(A) = -\langle A e_j, e_i \rangle = -a_{ij}$ for all $A \in X$. Finding a nice characterization of $K_S^*$ takes a little more work. In what follows, for any $x \in L^2(S)$, let $|x|$ denote the "variation norm" of $x$, that is,

$$|x| = \max_{i,j \in S} |x(i) - x(j)|.$$

Although $| \cdot |$ is not norm on $L^2(S)$, it is a norm on the subspace $\{\mathbb{1}_S\}^\perp = \{x \in L^2(S) \mid \langle x, \mathbb{1}_S \rangle = 0\}$.

**Lemma 5.1.2** *Let* $n = \mathrm{card\,S}$*. The dual cone of* $K_S$ *is*

$$K_S^* = \{ \sum_{k=1}^{n(n-1)/2+1} c_k \chi_{x_k} \mid c_k \geq 0,\ x_k \in L^2(S) \text{ with } |x_k| = 1\}$$

*where, for any* $x \in L^2(S)$*,* $\chi_x \in X_S^*$ *is the linear functional such that* $\chi_x(A) = \langle Ax, x\rangle$ *for* $A \in X_S$*.*

**Proof** $A \in X_S$ is positive semi-definite if and only if $\langle Ax, x\rangle \geq 0$ for all $x \in L^2(S)$ with $|x| = 1$. Thus the set of linear functionals $\{\chi_x : |x| = 1\}$ is a sufficient set for $K_S$. Therefore,

$$K_S^* = \mathrm{cl}\,\{\sum_{\mathrm{k=1}}^{\mathrm{N}} \mathrm{c_k}\chi_{\mathrm{x_k}} \mid \mathrm{N} \in \mathbb{N},\ \mathrm{c_k} \geq 0,\ \mathrm{x_k} \in \mathrm{L^2(S)} \text{ with } |\mathrm{x_k}| = 1\}.$$

We will now show that the set $\{\sum_{k=1}^{N} c_k\chi_{x_k} \mid N \in \mathbb{N}, c_k \geq 0,\ |x_k| = 1\}$ is closed. Since the set $\{\chi_x : |x| = 1\}$ is closed and bounded in $X_S^*$, an application of Carathéodory's theorem proves that $\mathrm{co}\,\{\chi_\mathrm{x} : |\mathrm{x}| = 1\}$ is compact (see [49], Theorem 17.2). Observe that if $A \in \mathrm{int\,K_S}$ and $|x| = 1$, then $\chi_x(A) > 0$. This implies that $0 \notin \mathrm{co}\,\{\chi_\mathrm{x} \mid |\mathrm{x}| = 1\}$.

Since $\mathrm{co}\,\{\chi_\mathrm{x} : |\mathrm{x}| = 1\}$ is compact and does not contain zero, the set $\bigcup_{\lambda\geq 0} \lambda(\mathrm{co}\,\{\chi_\mathrm{x} : |\mathrm{x}| = 1\})$ is closed. To see this, suppose that $\upsilon_k$ is a sequence in $\mathrm{co}\,\{\chi_\mathrm{x} : |\mathrm{x}| = 1\}$ and $b_k \geq 0$ is a sequence of real numbers such that $b_k\upsilon_k \to \varphi$. Then since $\mathrm{co}\,\{\chi_\mathrm{x} : |\mathrm{x}| = 1\}$ is compact, a subsequence $\upsilon_{k_i}$ converges to some $\upsilon_\infty \in \mathrm{co}\,\{\chi_\mathrm{x} : |\mathrm{x}| = 1\}$. Since $\upsilon_\infty \neq 0$, the corresponding subsequence $b_{k_i}$ must also converge to some $b_\infty \geq 0$ as $i \to \infty$. Then $\varphi = b_\infty\upsilon_\infty$, so $\varphi \in \bigcup_{\lambda\geq 0} \lambda(\mathrm{co}\,\{\chi_\mathrm{x} : |\mathrm{x}| = 1\})$ and therefore, $\bigcup_{\lambda\geq 0} \lambda(\mathrm{co}\,\{\chi_\mathrm{x} : |\mathrm{x}| = 1\})$ is closed. Now observe that

$$\bigcup_{\lambda\geq 0} \lambda(\mathrm{co}\,\{\chi_\mathrm{x} : |\mathrm{x}| = 1\}) = \{\sum_{\mathrm{k=1}}^{\mathrm{N}} \mathrm{c_k}\chi_{\mathrm{x_k}} \mid \mathrm{N} \in \mathbb{N},\ \mathrm{c_k} \geq 0,\ \mathrm{x_k} \in \mathrm{L^2(S)} \text{ with } |\mathrm{x_k}| = 1\},$$

and by Carathéodory's theorem for convex sets (see [49], Theorem 17.1) we may assume that $N = \dim X_S^* + 1 = n(n-1)/2 + 1$. ∎

Since $D_S \subset K_S$, it follows that $K_S^* \subset D_S^*$. Also note that $D_S^*$ is a polyhedral cone. Every face of $D_S^*$ has the form $F_I = \{\sum_{i \neq j \in S} b_{ij}\psi_{ij} \in D_S^* \mid b_{ij} = 0 \text{ if } (i,j) \in I\}$ where $I \subset S \times S$ is a symmetric collection of pairs, that is $(i,j) \in I$ if and only if $(j,i) \in I$.

Since each face $F_I$ of $D_S^*$ corresponds to a collection of pairs $I$, we may also associate to $F_I$ the graph $\Gamma(I)$ on $n$ vertices obtained by connecting the $i$ and $j$ vertices with an edge whenever $(i,j) \in I$.

By Lemma 2.1.1, the parts of the cone $D_S^*$ are the relative interiors of the faces $F_I$. That is, any two elements $\theta, \varphi \in D_S^*$ are comparable in the partial order induced by $D_S^*$ if and only if there is some $F_I$ such that $\theta, \varphi \in \mathrm{ri}\,\mathrm{F_I}$.

**Lemma 5.1.3** *If $F_I$ is a face of $D_S^*$ corresponding to the graph $\Gamma(I)$ and $F_I \cap K_S^* \neq \varnothing$, then $F_I \cap K_S^* \subset F_J$ where $F_J$ is the closed subface of $F_I$ corresponding to the graph $\Gamma(J)$ which is the minimal graph containing $\Gamma(I)$ such that every connected component of $\Gamma(J)$ is complete. In particular, the relative interior $\mathrm{ri}\,\mathrm{F_I}$ contains an element of $K_S^*$ if and only if the connected components of $\Gamma(I)$ are all complete.*

**Proof** Observe that $\mathrm{ri}\,\mathrm{F_I} = \{\sum_{\mathrm{i \neq j \in S}} \mathrm{b_{ij}}\psi_{\mathrm{ij}} \in \mathrm{F_I} \mid \mathrm{b_{ij}} = 0 \text{ if and only if } (\mathrm{i,j}) \in \mathrm{I}\}$. For any $\chi_x$, equation 5.1 implies that

$$\chi_x(A) = \langle Ax, x\rangle = -\frac{1}{2}\sum_{i \neq j \in S} a_{ij}(x(i) - x(j))^2,$$

thus,

$$\chi_x = \sum_{i \neq j \in S} \frac{1}{2}(x(i) - x(j))^2 \psi_{ij},$$

so $\chi_x$ is a sum $\sum_{i \neq j \in S} b_{ij}\psi_{ij}$ with $b_{ij} = \frac{1}{2}(x(i) - x(j))^2$. Thus, $\chi_x \in \mathrm{ri}\,\mathrm{F_I}$ if and only if $x(i) = x(j)$ exactly when $(i,j) \in I$.

If the graph $\Gamma(I)$ has a connected component which is not complete, then there is a pair $(i,j) \notin I$ such that there is a path $i_k$, $k = 1, ..., N$ with $i_1 = i$, and $i_N = j$ and

$(i_k, i_{k+1}) \in I$ for all $k \in \{1, ..., N-1\}$. If $\chi_x \in F_I$, then $x(i_k) = x(i_{k+1})$ for each $k$ and therefore, $x(i) = x(j)$. Thus the constant $b_{ij} = 0$ for that particular $\chi_x$ even though $(i,j) \notin I$. Therefore $\chi_x \notin \mathrm{ri}\,\mathrm{F_I}$.

Observe that, if $\varphi = \sum_{k=1}^{N} c_k \chi_{x_k} \in F_I$ with $c_k > 0$ for each $k$, then $x_k(i) = x_k(j)$ for all $k \in \{1, ..., N\}$ and $(i,j) \in I$. If $\Gamma(J)$ is the minimal graph containing $\Gamma(I)$ such that every connected component of $\Gamma(J)$ is complete, then for all $k$ and $(i,j) \in J$, $x_k(i) = x_k(j)$. Thus $\chi_{x_k} \in F_J$ and therefore, $\varphi \in F_J$. Thus $K_S^* \cap F_I \subset F_J$.

Conversely, if every connected component of $\Gamma(I)$ is complete, then we may choose an $x \in L^2(S)$ such that $x(i) = x(j)$ if and only if $i$ and $j$ correspond to vertices in the same connected component of $\Gamma(I)$. In other words, $x(i) = x(j)$ if and only if $(i,j) \in I$. It is then clear that the functional $\chi_x = \sum_{i \neq j \in S} \frac{1}{2}(x(i) - x(j))^2 \psi_{ij}$ is in $\mathrm{ri}\,\mathrm{F_I} \cap \mathrm{K_S^*}$. ∎

**Lemma 5.1.4** *If $F_I$ is a face of $D_S^*$ such that $F_I \cap K_S^* \neq \varnothing$, but $\mathrm{ri}\,\mathrm{F_I} \cap \mathrm{K_S^*} = \varnothing$, then there is an $A \in X_S$ such that $\varphi(A) \geq 0$ for all $\varphi \in K_S^*$ but $\theta(A) < 0$ for all $\theta \in \mathrm{ri}\,\mathrm{F_I}$.*

**Proof** By the lemma above, the fact that $K_S^* \cap \mathrm{ri}\,\mathrm{F_I} = \varnothing$ implies that there exists $J \supset I$ such that all of the connected components of $\Gamma(J)$ are complete and $K_S^* \cap F_I \subset F_J$. To construct the operator $A$, let $a_{ij} = 0$ if $(i,j) \notin J$, let $a_{ij} = 1$ if $(i,j) \in J \backslash I$, and for $(i,j) \in I$ let $a_{ij} = -K$ where $K > 0$ is some large constant which we will specify later. Then for any $\chi_x$,

$$\chi_x(A) = \sum_{i \neq j \in S} -\frac{1}{2} a_{ij}(x(i) - x(j))^2 = \frac{1}{2} \sum_{(i,j) \in I} K(x(i) - x(j))^2 - \frac{1}{2} \sum_{(i,j) \in J \backslash I} (x(i) - x(j))^2.$$

Suppose that $(p,q) \in I$ is the pair which attains the maximum value over $(i,j) \in I$ of the expression $(x(i) - x(j))^2$. Note that for any pair $(i,j) \in J$,

$$(x(i) - x(j))^2 = |x(i) - x(j)|^2 \leq (|x(i_1) - x(i_2)| + |x(i_2) - x(i_3)| + ... + |x(i_{N-1}) - x(i_N)|)^2,$$

where each pair $(i_k, i_{k+1}) \in I$ (or possibly, $i_k = i_{k+1}$), $i_1 = i$, $i_N = j$, and $N$ is the largest distance on the graph $\Gamma(I)$ between two vertices in any connected component. Thus,

$$(x(i) - x(j))^2 \leq N^2(x(p) - x(q))^2 \quad \text{for all } (i, j) \in J.$$

By letting $K > \mathrm{card}\,(\mathrm{J}\backslash\mathrm{I})\mathrm{N}^2$ we can see that

$$\begin{aligned}
\chi_x(A) &= \frac{1}{2} \sum_{(i,j)\in I} K(x(i) - x(j))^2 - \frac{1}{2} \sum_{(i,j)\in J\backslash I} (x(i) - x(j))^2 \\
&\geq \frac{1}{2}\mathrm{card}\,(\mathrm{J}\backslash\mathrm{I})\mathrm{N}^2(\mathrm{x(p)} - \mathrm{x(q)})^2 - \frac{1}{2} \sum_{(\mathrm{i,j})\in \mathrm{J}\backslash\mathrm{I}} (\mathrm{x(i)} - \mathrm{x(j)})^2 \\
&\geq \frac{1}{2}\mathrm{card}\,(\mathrm{J}\backslash\mathrm{I})\mathrm{N}^2(\mathrm{x(p)} - \mathrm{x(q)})^2 - \frac{1}{2}\mathrm{card}\,(\mathrm{J}\backslash\mathrm{I}) \max_{(\mathrm{i,j})\in\mathrm{J}} (\mathrm{x(i)} - \mathrm{x(j)})^2 \geq 0.
\end{aligned}$$

It remains to show that $\theta(A) < 0$ for all $\theta \in \mathrm{ri}\,(\mathrm{F_I})$. However, if $\theta \in \mathrm{ri}\,\mathrm{F_I}$, then $\theta = \sum_{i\neq j\in S} b_{ij}\psi_{ij}$ with $b_{ij} \geq 0$ for all pairs $(i, j) \in S \times S$, and $b_{ij} = 0$ if and only if $(i, j) \in I$. Therefore

$$\theta(A) = \sum_{(i,j)\in J} b_{ij}\psi_{ij}(A) = \sum_{(i,j)\in J\backslash I} -b_{ij} < 0.$$

∎

**Proof of Proposition 5.1.1** We will construct a polyhedral cone $C_p^*$ such that $K_S^* \subset C_p^* \subset D_S^*$ and such that every element $\theta \in C_p^*$ is comparable to an element $\varphi \in K_S^*$ under the partial ordering induced by $D_S^*$.

To construct the polyhedral cone, we will intersect the cone $D_S^*$ with finitely many closed half-spaces of the form $H_A = \{\theta \in X_S^* \mid \theta(A) \geq 0\}$ where $A \in X_S$. Suppose that $F_I$ is a face of $D_S^*$. If $F_I$ is disjoint from $K_S^*$, then the Hahn-Banach theorem implies that we may find an $A \in X_S$ such that $\varphi(A) \geq 0$ for all $\varphi \in K_S^*$ but $\theta(A) < 0$ for all $\theta \in F_I$.

If $F_I$ is not disjoint from $K_S^*$ but $\mathrm{ri}\,\mathrm{F_I} \cap \mathrm{K_S^*} = \varnothing$, then Lemma 5.1.4 implies that there is an $A \in X_S$ such that $\theta(A) < 0$ for all $\theta \in \mathrm{ri}\,\mathrm{F_I}$ while $\varphi(A) \geq 0$ for all $\varphi \in K_S^*$. Therefore, for any face $F_I$ such that $\mathrm{ri}\,\mathrm{F_I} \cap \mathrm{K_S^*} = \varnothing$ there is a half-space $H_A$ such that $H_A$ contains $K_S^*$ but is disjoint from $\mathrm{ri}\,\mathrm{F_I}$. Since there are only finitely many faces of $D_S^*$, it follows that by intersecting the cone $D_S^*$ with finitely many half-spaces we may obtain a polyhedral cone $C_p^*$ such that $K_S^* \subset C_p^* \subset D_S^*$ and every element of $C_p^*$ lies in the relative interior of a face $F_I$ of $D_S^*$ such that $\mathrm{ri}\,\mathrm{F_I} \cap \mathrm{K_S^*} \neq \varnothing$. This implies that every element of $C_p^*$ is comparable to an element of $K_S^*$ in the partial ordering of $D_S^*$ since the parts of $D_S^*$ are precisely the relative interiors of its faces (see Lemma 2.1.1).

■

## 5.2 Reproduction-Decimation Operators

A fractal is a set defined by a finite family of functions $\{\psi_1, ...\psi_N\}$ where each $\psi_k : \mathbb{R}^n \to \mathbb{R}^n$ has the property that $||\psi_k(x) - \psi_k(y)|| = \kappa_k ||x - y||$ with $0 < \kappa_k < 1$. For any set $U \subset \mathbb{R}^n$ we let $\Psi(U) = \bigcup_{k=1}^n \psi_k(U)$. The fractal is then the unique compact set $\mathcal{F} \subset \mathbb{R}^n$ such that $\Psi(\mathcal{F}) = \mathcal{F}$. We let $I = \{x \in \mathcal{F} \mid x \in \psi_i(\mathcal{F}) \cap \psi_j(\mathcal{F}) \text{ for some } i \neq j\}$ and $V = \{x \in \mathcal{F} \mid \Psi(x) \cap I \neq \varnothing\}$. If $I$ is finite, then we say that $\mathcal{F}$ is *finitely ramified*. From now on, we shall always assume that $\mathcal{F}$ is post critically finite which implies that it is finitely ramified (see section 1.3 of [28] for the definition of post critically finite). We let $W = \Psi(V)$. Note that $V$ and $W$ are finite sets when $\mathcal{F}$ is finitely ramified, so we may let $L^2(V)$ and $L^2(W)$ be the corresponding $L^2$ spaces when $V$ and $W$ are equipped with the counting measure.

We are now ready to define the reproduction operators, which are a class of linear maps from $L^2(V)$ to $L^2(W)$ determined by the fractal $\mathcal{F}$. For each $k \in \{1, ..., N\}$ define

$\Phi_k : L^2(W) \to L_2(V)$ by $(\Phi_k x)(i) = x(\psi_k(i))$. Note that each $\Phi_k$ is a bounded linear map, so it has a Hilbert space adjoint $\Phi_k^* : L^2(V) \to L^2(W)$. Also observe that

$$\Phi_k(\mathbb{1}_W) = \mathbb{1}_V \text{ and } \Phi_k(x \wedge \mathbb{1}_W) = \Phi_k(x) \wedge \mathbb{1}_V \ \forall x \in L^2(W). \tag{5.2}$$

We define a *reproduction operator* to be a map $R : X_V \to X_W$ given by

$$R(A) = \sum_{k=1}^{N} \eta_k \Phi_k^* A \Phi_k \tag{5.3}$$

where each $\eta_k$ is a positive real constant. It is immediately clear that $R(K_V) \subset K_W$. If $A \in D_S$, then recall that $a_{ij} \leq 0$ for $i, j \in S$ with $i \neq j$. Thus for any $x \in L^2(W)$

$$\langle R(A)x, x\rangle = \langle \sum_{k=1}^{N} \eta_k \Phi_k^* A \Phi_k x, x\rangle = \sum_{k=1}^{N} \eta_k \langle A\Phi_k x, \Phi_k x\rangle =$$

$$= -\frac{1}{2}\sum_{k=1}^{N} \eta_k \sum_{i,j\in V} a_{ij}(\Phi_k x(i) - \Phi_k x(j))^2 = -\frac{1}{2}\sum_{k=1}^{N} \eta_k \sum_{i,j\in V} a_{ij}[x(\psi_k(i)) - x(\psi_k(j))]^2.$$

Therefore $R(D_V) \subset D_W$. Because $R$ is linear, it follows that $R$ is homogeneous of degree one, and if $A \leq_{K_V} B$, then $R(A) \leq_{K_W} R(B)$.

Corollary 1.6.5 in [28] states that a post critically finite fractal $\mathcal{F}$ determined by a family of contractions $\{\psi_k \mid 1 \leq k \leq N\}$ is *connected* if and only if, for all $j, j' \in W$, there exists $k_0, k_1, ..., k_p$ with $j \in \psi_{k_0}(V)$, $j' \in \psi_{k_p}(V)$, and $\psi_{k_s}(V) \cap \psi_{k_{s+1}}(V) \neq \varnothing$ for $0 \leq s < p$. If the fractal $\mathcal{F}$ is connected, then $R$ will map irreducible Dirichlet forms on $L^2(V)$ to irreducible Dirichlet forms on $L^2(W)$.

**Lemma 5.2.1** *If $\mathcal{F}$ is a post critically finite connected fractal and $R$ is a reproduction operator defined by equation 5.3, then*

$$R(D_V \cap \operatorname{int} \mathrm{K_V}) \subset \mathrm{D_W} \cap \operatorname{int} \mathrm{K_W}. \tag{5.4}$$

**Proof** We have already seen that $R(D_V \cap \operatorname{int} \mathrm{K_V}) \subset \mathrm{D_W}$. By equation 5.1, it suffices to prove that if $x \in L^2(W)$, $A \in D_V \cap \operatorname{int} \mathrm{K_V}$, and $\langle R(A)x, x\rangle = 0$, then $x = \lambda \mathbb{1}_W$ for some $\lambda \in \mathbb{R}$. However,

$$\langle R(A)x, x\rangle = \sum_{k=1}^{N} \eta_k \langle A\Phi_k x, \Phi_k x\rangle,$$

so, if $\langle R(A)x, x\rangle = 0$, then $\langle A\Phi_k x, \Phi_k x\rangle = 0$ for $1 \leq k \leq N$. Because $A \in D_V \cap \operatorname{int} \mathrm{K_V}$, it follows that there exists $\lambda_k \in \mathbb{R}$ with

$$\Phi_k(x) = x \circ \varphi_k = \lambda_k \mathbb{1}_V \quad \text{for } 1 \leq k \leq N.$$

Therefore $x(w) = \lambda_k$ for all $w \in \varphi_k(V), 1 \leq k \leq N$. If $j, j' \in W$, select $k_0, k_1, ..., k_p$ as in the characterization of connectivity. It follows that $\lambda_{k_s} = \lambda_{k_{s+1}}$ for $0 \leq s < p$, so $\lambda_{k_0} = x(j) = \lambda_{k_p} = x(j')$, and $x$ is a scalar multiple of $\mathbb{1}_W$. ■

We let $H = L^2(W)$ and define an orthogonal projection $P : H \to H$ by $(Px)(w) = x(w)$ for $w \in V$ and $(Px)(w) = 0$ for $w \in W \backslash V$. We let $Q = I - P$ and $H_1 = P(H)$ and $H_2 = Q(H)$. Then $H = H_1 \oplus H_2$. It is easy to see that $H_1$ is isomorphic to $L^2(V)$. We will identify $L^2(V)$ with the subspace $H_1$, and thus any $x \in L^2(V)$ can be treated as an element of $L^2(W)$ with $x(w) = 0$ for all $w \in W \backslash V$.

We define the *decimation* operator $\Psi : X_W \to X_V$ by letting $\Psi(A)$ be the unique element in $X_V$ such that

$$\langle \Psi(A)x, x\rangle = \inf\{\langle A(x + y), x + y\rangle \mid y \in H_2\}. \tag{5.5}$$

The operator $\Psi$ is not linear. The following lemma shows that $\Psi$ has several nice properties, however.

**Lemma 5.2.2** *If $\Psi : X_W \to X_V$ is a decimation operator, then*

(*a*) *$\Psi$ is homogeneous of degree one.*

(*b*) *If $A, B \in X_W$ satisfy $A \leq_{K_W} B$, then $\Psi(A) \leq_{K_V} \Psi(B)$.*

(c) $\Psi(\mathrm{int\,K_W}) \subset \mathrm{int\,K_V}$.

(d) $\Psi(D_W) \subset D_V$.

**Proof** (a) It is clear from the definition that $\Psi$ is homogeneous of degree one.

(b) To prove that $\Psi(A) \leq_{K_V} \Psi(B)$ when $A \leq_{K_W} B$, it suffices to prove that $\Psi(B) - \Psi(A) \in K_V$. Note that

$$\langle(\Psi(B) - \Psi(A))x, x\rangle = \inf\{\langle B(x+y), x+y\rangle \mid y \in H_2\} - \inf\{\langle A(x+y), x+y\rangle \mid y \in H_2\}$$

$$\geq \inf\{\langle B(x+y) - A(x+y), x+y\rangle \mid y \in H_2\} \geq 0$$

since $B - A \in K_W$.

(c) If $A \in \mathrm{int\,K_W}$, then it has a unique square root $A^{1/2} \in \mathrm{int\,K_W}$ and

$$\inf\{\langle A(x+y), x+y\rangle \mid y \in H_2\} = \inf ||A^{1/2}(x+y)||^2.$$

Since $H$ is a Hilbert space, there is some $y_0 \in H_2$ such that

$$\inf ||A^{1/2}(x+y)||^2 = ||A^{1/2}(x+y_0)||^2$$

and therefore

$$\langle\Psi(A)x, x\rangle = \langle A(x+y_0), x+y_0\rangle.$$

If $\langle\Psi(A)x, x\rangle = 0$, then the above equation implies that $x = \lambda \mathbb{1}_V$ for some $\lambda \in \mathbb{R}$. Therefore $\Psi(A) \in \mathrm{int\,K_V}$.

(d) If $y \in L^2(V)$, $A \in D_W$, and we identify $L^2(V)$ with $H_1$ as above, equation 5.5 gives $\langle\Psi(A)(y \wedge \mathbb{1}_V), y \wedge \mathbb{1}_V\rangle = \inf\{\langle A(y \wedge \mathbb{1}_W + z), y \wedge \mathbb{1}_W + z\rangle \mid z \in H_2\}$. Because $z \wedge \mathbb{1}_W \in H_2$ for $z \in H_2$ and because $(y+z) \wedge \mathbb{1}_W = y \wedge \mathbb{1}_W + z \wedge \mathbb{1}_W$, we see that

$$\inf\{\langle A(y \wedge \mathbb{1}_W + z), y \wedge \mathbb{1}_W + z\rangle \mid z \in H_2\}$$

$$\leq \inf\{\langle A(y \wedge \mathbb{1}_W + z \wedge \mathbb{1}_W), y \wedge \mathbb{1}_W + z \wedge \mathbb{1}_W\rangle \mid z \in H_2\}$$

$$= \inf\{\langle A((y+z)\wedge \mathbb{1}_W), (y+z)\wedge \mathbb{1}_W\rangle \mid z \in H_2\}$$

$$\leq \inf\{\langle A(y+z), y+z\rangle \mid z \in H_2\} = \langle \Psi(A)y, y\rangle.$$

This proves that $\Psi(D_W) \subset D_V$. ■

We define a *reproduction-decimation* operator to be a function $f = \Psi \circ R$ where $\Psi$ is a decimation operator and $R$ is a reproduction operator.

**Theorem 5.2.1** *Let notation be as above and let $R$ and $\Psi$ be as defined in equations 5.3 and 5.5. Assume that $\mathcal{F}$ is connected. Let $q$ be a linear functional which is positive on $K_V\backslash\{0\}$ and define $\Sigma = \{A \in \mathrm{int}\,\mathrm{K_V} \mid \mathrm{q(A)} = 1\}$. For $A \in \mathrm{int}\,\mathrm{K_V}$, define $f = \Psi \circ R$ and $\hat{f}(A) = f(A)/q(f(A))$. Then $f(\mathrm{int}\,\mathrm{K_V}) \subset \mathrm{int}\,\mathrm{K_V}$, $f(D_V \cap \mathrm{int}\,\mathrm{K_V}) \subset \mathrm{D_V} \cap \mathrm{int}\,\mathrm{K_V}$, $f$ is homogeneous of degree one and $f$ is order-preserving in the partial ordering from $K_V$. If $A \in \Sigma$, let $\omega(A;\hat{f})$ denote the omega limit set of $A$ under the map $\hat{f}$. If $\hat{f}$ has no fixed points in $\Sigma$ (or equivalently, $f$ has no eigenvectors in $\mathrm{int}\,\mathrm{K_V}$), then we have*

$$\mathrm{co}\,(\bigcup_{\mathrm{A}\in\Sigma} \omega(\mathrm{A};\hat{\mathrm{f}})) \subset \partial\Sigma.$$

*Furthermore, for $A, B \in \Sigma$, every element of $\omega(A;\hat{f})$ is comparable to an element of $\omega(B;\hat{f})$ in the partial ordering from $K_V$.*

**Proof** Under the given assumptions we have proved that $R(\mathrm{int}\,\mathrm{K_V}) \subset \mathrm{int}\,\mathrm{K_V}$ and Lemma 5.2.1 showed that $R(D_V \cap \mathrm{int}\,\mathrm{K_V}) \subset \mathrm{D_W} \cap \mathrm{int}\,\mathrm{K_W}$. Because $R$ is linear it follows that $R$ is order-preserving as a map from $K_V$ to $K_W$ and also homogeneous of degree one. Lemma 5.2.2 shows that $\Psi : \mathrm{int}\,\mathrm{K_W} \to \mathrm{int}\,\mathrm{K_V}$ is order-preserving and homogeneous of degree one and that $\Psi(D_W \cap \mathrm{int}\,\mathrm{K_W}) \subset \mathrm{D_V} \cap \mathrm{int}\,\mathrm{K_V}$. These facts, combined with Proposition 5.1.1 allow us to use Theorem 3.4.1. To prove the last claim of the theorem, we use Lemma 2.3.1 and the fact that $\hat{f}$ is nonexpansive with respect to Hilbert's projective metric on $K_V$. ■

We can say more about the omega limit sets $\omega(A;\hat{f})$ of normalized reproduction-decimation operators in the special case when $\mathrm{card}\,(\mathrm{V}) = 3$. In this case, the set $\Sigma = \{A \in \mathrm{int}\,\mathrm{K_V} \mid \mathrm{q(A)} = 1\}$ is strictly convex. After all, when $\mathrm{card}\,(\mathrm{V}) = 3$, $X_V$ is naturally isomorphic to the set $\tilde{X}_V$ of real symmetric $3 \times 3$ matrices with row sums equal to zero. Under this isomorphism, $\mathrm{int}\,\mathrm{K_V}$ corresponds to the rank 2 positive semi-definite elements of $\tilde{X}_V$. The boundary $\partial K_V$ corresponds to those positive semi-definite matrices in $\tilde{X}_V$ with rank less than 2. For any $A, B \in \partial K_V$ such that $A$ is not a scalar multiple of $B$, $\mathrm{rank}\,(\lambda A + (1-\lambda)B) = 2$ for all $0 < \lambda < 1$. Therefore $\Sigma$ is strictly convex. We can now apply the Denjoy-Wolff type theorem established by Beardon for Hilbert metric nonexpansive maps on strictly convex domains (theorem 3.1.2) to conclude that there exists $B \in \partial K_V$ such that $\hat{f}^k(A) \to B$ as $k \to \infty$, for all $A \in \Sigma$. This is a stronger result than we are able to prove when $n > 3$. Moreover, since $f(\mathrm{int}\,\mathrm{K_V} \cap \mathrm{D_V}) \subset \mathrm{D_V}$, it follows that $B \in D_V \cap \partial K_V$ and we also know that $q(B) = 1$. An easy argument then implies that

$$(a)\quad B = \beta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{or } (b)\quad B = \beta \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{or } (c)\quad B = \beta \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

where $\beta$ is determined by the condition that $q(B) = 1$.

# Chapter 6

# Topical Maps and Positive Operators on $\mathbb{R}^n_+$

## 6.1 Topical Maps

In $\mathbb{R}^n$, we let $\mathbb{R}^n_+$ denote the set of vectors with all nonnegative entries, that is $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i \geq 0 \; \forall \, 1 \leq i \leq n\}$. Note that $\mathbb{R}^n_+$ is a closed cone and we refer to it as the *standard cone* in $\mathbb{R}^n$. The standard cone induces a partial ordering $\leq$ on $\mathbb{R}^n$ given by:

$$x \leq y \text{ if and only if } x_i \leq y_i \text{ for each } i \in \{1, ..., n\}.$$

Let $\mathbb{1} = (1, 1, ..., 1) \in \mathbb{R}^n$ be the vector with every entry equal to one. We say that a map $g : \mathbb{R}^n \to \mathbb{R}^n$ is *additively homogeneous* if $g(x + \lambda\mathbb{1}) = g(x) + \lambda\mathbb{1}$ for all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. If $g : \mathbb{R}^n \to \mathbb{R}^n$ is both additively homogeneous and order-preserving with respect to $\leq$, then $g$ is said to be *topical*. The following proposition due to Crandall and Tartar [15] shows that topical maps are nonexpansive with respect to $||\cdot||_\infty$.

**Proposition 6.1.1** *A map $g : \mathbb{R}^n \to \mathbb{R}^n$ is topical if and only if $g$ is additively homogeneous and nonexpansive with respect to $||\cdot||_\infty$.*

We define a map $L : \operatorname{int} \mathbb{R}^n_+ \to \mathbb{R}^n$ by $L_i(x) = \log(x_i)$. The inverse map $E : \mathbb{R}^n \to \operatorname{int} \mathbb{R}^n_+$ is given by $E_i(y) = \exp(y_i)$. Note that if $g$ is topical, then $f = E \circ g \circ L$ is an order-preserving homogeneous of degree one map which takes $\operatorname{int} \mathbb{R}^n_+$ into itself. By Corollary 4.8 in [12], every such $f$ extends continuously to the whole cone $\mathbb{R}^n_+$.

If $g : \mathbb{R}^n \to \mathbb{R}^n$ is topical and $\chi(g) = \lim_{k\to\infty} g^k(x)/k$ exists for some $x \in \mathbb{R}^n$, then $\chi = \chi(g)$ is called the *cycle time* vector of $g$. It is not hard to verify that $\chi$ does not depend on the choice of $x \in \mathbb{R}^n$. The cycle time vector $\chi$ is closely related to the cone spectral radius of $f = E \circ g \circ L$. In fact, if $r_C(f)$ is the cone spectral radius of $f$ with respect to $C = \mathbb{R}^n_+$, then $r_C(f) = \exp(\max_{1\leq i\leq n}(\chi_i))$ by equation 2.16.

Unlike the cone spectral radius, the cycle time vector is not guaranteed to exist for arbitrary topical maps. In fact, Gunawardena and Keane construct an example of a topical map with no cycle time vector in [22]. For certain classes of topical maps, the cycle time vector is known to exist. Katirtzoglou proves this for $DAD$-maps in [27]. We discuss another such class of maps in the next section.

## 6.2 Max-Min Operators

The linear operators in the max-plus algebra have received a great deal of attention because of their applications to transportation and communication networks (see [23]). These max-plus operators are part of a larger hierarchy of topical maps which has been studied by Gaubert and Gunawardena (see [21] and the references in that paper). If $a, b \in \mathbb{R}$, then we let $a \vee b = \min(a, b)$ and $a \wedge b = \max(a, b)$. For $x, y \in \mathbb{R}^n$ we let $(x \vee y)_i = \min(x_i, y_i)$ and $(x \wedge y)_i = \max(x_i, y_i)$ for $1 \leq i \leq n$. The following proposition is easy to verify.

**Proposition 6.2.1** *Suppose that $g, g' : \mathbb{R}^n \to \mathbb{R}^n$ are topical. For any $u \in \mathbb{R}^n$ and $0 < \lambda < 1$ the following maps are also topical: $g + u$, $g \vee g'$, $g \wedge g'$, $\lambda g + (1-\lambda)g'$.*

Following the notation of [21] we let $\mathrm{Sim(n, n)}$ denote the set of functions $g : \mathbb{R}^n \to \mathbb{R}^n$ such that each component $g_i(x) = x_j$ for some $1 \leq j \leq n$. We then let $\mathcal{A}^*$ denote the closure of $\mathrm{Sim(n, n)}$ under the operations in Proposition 6.2.1. Note that the max-plus linear operators are just the closure of $\mathrm{Sim(n, n)}$ under the operations of max and adding a fixed vector. Every element in $\mathcal{A}^*$ is piecewise affine and nonexpansive with respect to $||\cdot||_\infty$. Therefore a theorem of Kohlberg (Theorem 2.1 in [29]) implies that every $g \in \mathcal{A}^*$ has an invariant half-line. Gaubert and Gunawardena originally made this observation in [20].

**Theorem 6.2.1** *If $g \in \mathcal{A}^*$, then $g$ has an invariant half-line. That is, there exists a vector $u \in \mathbb{R}^n$ and a unique vector $v \in \mathbb{R}^n$ such that $g(u + tv) = u + (t + 1)v$ for all $t \geq 0$.*

Note that $\lim_{k\to\infty} g^k(u)/k = v$ and therefore $v$ is the cycle time vector of $g$. Thus, one consequence of Theorem 6.2.1 is that the cycle time vector exists for all $g \in \mathcal{A}^*$. If $g : \mathbb{R}^n \to \mathbb{R}^n$ is topical and $u \in \mathbb{R}^n$ satisfies $g(u+kv) = u+(k+1)v$ for some $v \in \mathbb{R}^n$ and all $k \geq 0$, then $u$ is called a *generalized additive eigenvector* of $g$. We see that every element $g \in \mathcal{A}^*$ has a generalized additive eigenvector $u$ such that $g(u + k\chi) = u + (k + 1)\chi$ where $\chi = \chi(g)$ is the cycle time vector of $g$.

The maps in $\mathcal{A}^*$ correspond to order-preserving homogeneous of degree one maps on the standard cone $\mathbb{R}^n_+$ via the transformation $\Phi : g \mapsto E \circ g \circ L$. We let $\Phi(\mathcal{A}^*)$ denote the set of all order-preserving homogeneous of degree one maps $f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ such that $f(x) = E \circ g \circ L(x)$ for all $x \in \operatorname{int} \mathbb{R}^n_+$. Normalized operators in the class $\Phi(\mathcal{A}^*)$ have simple omega limit sets, as the following theorem shows.

**Theorem 6.2.2** *Suppose that $f \in \Phi(\mathcal{A}^*)$ and let $r_C(f)$ be the cone spectral radius of $f$ with respect to the standard cone $C = \mathbb{R}^n_+$. Let $\hat{f} = r_C(f)^{-1} f$. Then $\omega(x; \hat{f})$ is a finite periodic orbit of $\hat{f}$ contained in a single part of $\mathbb{R}^n_+$.*

**Proof** Since $f \in \Phi(\mathcal{A}^*)$ there is a $g \in \mathcal{A}^*$ such that $f(x) = E \circ g \circ L(x)$ for all $x \in \operatorname{int} \mathbb{R}^n_+$. By Theorem 6.2.1 there is a generalized additive eigenvector $u \in \mathbb{R}^n$ such that $g(u + t\chi) = u + (t+1)\chi$ for all $t \geq 0$ where $\chi = \chi(g)$. Let $y = E(u)$. Note that $f^k(y) = E(u + k\chi)$, so $r_C(f) = \lim_{k\to\infty} ||f^k(y)||_\infty^{1/k} = \exp(\max_{1\leq i\leq n} \chi_i)$. Let $\bar{\chi}$ be the vector with each entry equal to $\max_{1\leq i\leq n} \chi_i$. Then $\hat{f}^k(y) = r_C(f)^{-k} f^k(y) = E(u + k(\chi - \bar{\chi}))$. Since $\chi - \bar{\chi} \leq 0$ it is clear that the orbit $\mathcal{O}(y; \hat{f})$ is bounded. For any other $x \in \operatorname{int} \mathbb{R}^n_+$, there is a constant $\lambda > 0$ such that $x \leq \lambda y$, and since $\hat{f}$ is order-preserving it follows that the orbit $\mathcal{O}(x; \hat{f})$ is bounded. By Theorem 6.8 and Lemma 6.7 of [1], this implies that $\omega(x; \hat{f})$ is a finite periodic orbit of $\hat{f}$ and is contained in a single part of $\operatorname{int} \mathbb{R}^n_+$. ■

### 6.3 A Pathological Example

We have proved that the omega limit sets of normalized linear maps on a polyhedral cone are finite (Theorem 4.3.1) as are the omega limit sets of normalized max-min type operators on the standard cone (Theorem 6.2.2). There are, however, examples of order-preserving homogeneous of degree one maps on the standard cone whose omega limit sets contain infinitely many points on the boundary and even contain points from more than one part of the cone. In this section we will introduce one such example. Let $V = \{x \in \mathbb{R}^n \mid v_1 = 0\}$.

**Lemma 6.3.1** *For any sequence $\{a^i\}_{i\geq 1} \subset V$ such that $a^i \geq 0$ and $a^{i+1} \leq a^i$ for all $i \geq 1$, there is a topical map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $g^k(0) = \sum_{i=1}^k a^i$ for each $k \geq 0$.*

**Proof** Let $v^k = \sum_{i=1}^k a^i$. For each $2 \leq j \leq n$ there is an order-preserving Lipschitz function $\gamma_j : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant $\operatorname{Lip}(\gamma_j) \leq 1$ such that $\gamma_j(v_j^k) = v_j^{k+1}$.

Indeed, by constructing each $\gamma_j$ piecewise linear we see immediately that this is the case. Let $G : V \to V$ be the map $G_j(x) = \gamma_j(x_j)$ for each $2 \leq j \leq n$. For any $x \in \mathbb{R}^n$, let $g(x) = G(x - x_1\mathbb{1}) + x_1\mathbb{1}$. It is easy to see that $g$ is additively homogeneous since

$$g(x + \lambda\mathbb{1}) = G(x + \lambda\mathbb{1} - (x_1 + \lambda)\mathbb{1}) + (x_1 + \lambda)\mathbb{1} =$$

$$= G(x - x_1\mathbb{1}) + (x_1 + \lambda)\mathbb{1} = g(x) + \lambda\mathbb{1}.$$

Suppose that $x, y \in \mathbb{R}^n$ and $x \leq y$. If $x_j - x_1 \leq y_j - y_1$, then because $\gamma_j$ is order-preserving,

$$g_j(x) = \gamma_j(x_j - x_1) + x_1 \leq \gamma_j(y_j - y_1) + y_1 = g_j(y).$$

If $x_j - x_1 > y_j - y_1$, then because $\mathrm{Lip}\,(\gamma_j) \leq 1$,

$$0 \leq \gamma_j(x_j - x_1) - \gamma_j(y_j - y_1) \leq (x_j - x_1) - (y_j - y_1) \leq y_1 - x_1.$$

Therefore

$$g_j(x) = \gamma_j(x_j - x_1) + x_1 \leq \gamma_j(y_j - y_1) + y_1 = g_j(y).$$

Thus $g : \mathbb{R}^n \to \mathbb{R}^n$ is a topical map such that $g^k(0) = G^k(0) = v^k$. ■

Every topical map $g$ corresponds to an order-preserving homogeneous of degree one map defined on the interior of the cone $\mathbb{R}^n_+$ by $f = E \circ g \circ L$. We will use this correspondence to state an alternative version of Lemma 6.3.1. In the following lemma, $\prod_{i=1}^k b^i$ is understood to be the entry-wise product of the vectors $b^i$.

**Lemma 6.3.2** *Suppose that* $\{b^i\}_{i\geq 1}$ *is a sequence of vectors in* $\mathbb{R}^n$ *such that* $b^i \geq \mathbb{1}$, $b^{i+1} \leq b^i$ *and* $b^i_1 = 1$ *for all* $i \geq 1$. *Then there is an order-preserving homogeneous of degree one map* $f : \mathrm{int}\,\mathbb{R}^n_+ \to \mathbb{R}^n_+$ *such that*

$$f^k(\mathbb{1}) = \prod_{i=1}^{k} b^i \quad \text{for all } k \geq 1. \tag{6.1}$$

**Proof** If $a^i = L(b^i)$ for each $i \geq 1$, then $\{a^i\}_{i\geq 1}$ is a sequence in $V$ which satisfies the hypotheses of Lemma 6.3.1. Therefore there is a topical map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$g^k(0) = \sum_{i=1}^{k} a^i.$$

The corresponding map $f = E \circ g \circ L$ will have

$$f^k(\mathbb{1}) = E(g^k(0)) = E\left(\sum_{i=1}^{k} a^i\right) = \prod_{i=1}^{k} b^i.$$

■

Let $\Sigma = \{x \in \operatorname{int} \mathbb{R}^n_+ \mid ||x||_1 = 1\}$, where $|| \cdot ||_1$ is the norm $||x||_1 = \sum_{i=1}^n |x_i|$. Let $d$ denote the Hilbert metric on $\Sigma$. If $f : \operatorname{int} \mathbb{R}^n_+ \to \operatorname{int} \mathbb{R}^n_+$ is order-preserving and homogeneous of degree one, then $\hat{f}(x) = f(x)/||f(x)||_1$ is a $d$-nonexpansive map from $\Sigma$ into $\Sigma$ by theorem 2.4.1 and property (i) of Hilbert's projective metric. Furthermore, if $f$ satisfies equation 6.1 and $\xi = \frac{1}{n}\mathbb{1}$, then

$$\hat{f}^k(\xi) = \frac{\prod_{i=1}^k b^i}{|| \prod_{i=1}^k b^i||_1}. \tag{6.2}$$

Note that $\hat{f}^k(\xi)$ will have $\hat{f}^k(\xi)_1 \to 0$ as $k \to \infty$ if and only if $|| \prod_{i=1}^k b^i||_1 \to \infty$.

**Theorem 6.3.1** *For any convex subset $U \subset \partial\Sigma$ there is a Hilbert metric nonexpansive map $T : \Sigma \to \Sigma$ and a point $\xi \in \Sigma$ such that the omega limit set $\omega(\xi; T)$ contains $U$.*

**Proof** Since $U$ is convex, and $U \subset \partial\Sigma$, there is some coordinate $j$ such that $x_j = 0$ for all $x \in U$. Assume without loss of generality that $j = 1$ and let $F_1$ be the closed face $F_1 = \{x \in \operatorname{cl}\Sigma \mid x_1 = 0\}$. Then $U \subseteq F_1$. We will choose a sequence $b^i$ with $b^i_1 = 1$, $b^i \geq \mathbb{1}$ and $b^{i+1} \leq b^i$ for all $i \geq 1$. We will let $\xi = \frac{1}{n}\mathbb{1}$. The above comments and Lemma 6.3.2 will then imply that there is a Hilbert metric nonexpansive map $T : \Sigma \to \Sigma$ satisfying equation 6.2. Let $\xi^k = T^k(\xi)$ for each $k > 0$. We wish to choose the $b^i$ in such a way that subsequences of $\xi^k$ converge to a countable dense collection of points in $F_1$. Suppose that the vectors $b^1, ..., b^N$ are fixed for some $N > 0$. Observe that for any two vectors $x, y \in \mathbb{R}^n_+$ with $x \leq y$ and $x_1 = y_1$, we may choose a finite sequence of vectors $b^{N+i}$, $1 \leq i \leq m$, such that $b_1^{N+i} = 1$, $b^{N+i} \leq b^{N+i-1}$, and $b^{N+i} \geq \mathbb{1}$ for all $1 \leq i \leq m$, and such that the entry-wise product of $x$ with $\prod_{i=1}^m b^{N+i}$ equals $y$. For example, by choosing $m$ large enough, let each $b_j^{N+i} = (y_j/x_j)^{1/m}$. Now, suppose that $z \in F_1$ and $x \in \Sigma$ is arbitrarily close (in norm) to $z$. Suppose also that $z'$ is any other point in $F_1$. We may choose a $y \in \mathbb{R}^n_+$ with $x \leq y$ and $x_1 = y_1$ such that $y/||y||_1$ is arbitrarily close to $z'$. This implies that if $\xi^N$ is arbitrarily close to some $z \in F_1$, and $z'$ is any other point in $F_1$, we may find an $m > 0$ such that $\xi^{N+m}$ is the entry-wise product of $\xi^N$ with $\prod_{i=1}^m b^{N+i}$ scaled to have norm one, and $\xi^{N+m}$ is arbitrarily close to $z'$. Repeating this process, the sequence $\xi^k$ can accumulate at any countable collection of points $\{z^i \mid i \geq 1\}$ contained in $F_1$. In particular, by choosing a countable dense subset of $F_1$ and using the fact that $\omega(\xi; T)$ is closed, we may ensure that the omega limit set $\omega(\xi; T)$ contains all of $F_1$. Thus $U \subseteq \omega(\xi; T)$. ■

The example above shows that the restrictions on the omega limit sets we found in Theorems 4.3.1 and 6.2.2 are stronger than can be expected in general. In fact, this example shows that Theorem 3.2.2 is the best general result we can hope for on polyhedral domains.

Note that if $\{a^i\}$ is any sequence in $\mathbb{R}^n$ such that $a^i \geq 0$ and $a^{i+1} \leq a^i$ for all $i \geq 1$, then $\lim_{i\to\infty} a^i$ exists. If $g$ is a topical map such that $g^k(0) = \sum_{i=1}^k a^i$, as in Lemma 6.3.1, then the cycle time vector $\chi = \chi(g)$ exists and $\chi = \lim_{k\to\infty} g^k(0)/k = \lim_{i\to\infty} a^i$. It is a simple matter to construct a sequence $a^i \geq 0$ with $a^{i+1} \leq a^i$ for all $i \geq 1$ such that $\lim_{i\to\infty} a^i = 0$ and such that the partial sums $\sum_{i=1}^k a^i$ are unbounded in $\mathbb{R}^n$. Then the cycle time vector of the corresponding topical map $g$ is $\chi(g) = 0$ even though $g$ cannot have any fixed points. For such a map, we can use Theorem 3.5.1 to at least prove that there is a linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\varphi(g^k(x)) \to \infty$ as $k \to \infty$. However, the asymptotic behavior of the map $g$ can be quite complicated, despite the existence of a cycle time vector. This suggest that the cycle time vector may be less useful for understanding general topical maps than one might hope from studying special classes such as $\mathcal{A}^*$.

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