MULTIPLE TESTING METHODS IN DEPENDENT CASES

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ABSTRACT OF THE DISSERTATION

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The most popular multiple testing procedures are stepwise procedures based on P-values for individual test statistics. Included among these are the false discovery rate (FDR) controlling procedures of Benjamini-Hochberg(1995) and their offsprings. For many models including the case where model variables are multivariate normal, dependent and alternatives are two sided, these stepwise procedures lack an intuitive convexity property which is also needed for admissibility. Here we present two new stepwise methods that do in fact have the convexity property. Furthermore unlike the method using P-values based on marginal distributions, the new methods take dependency into account in all stages. Still further the new methodology is computationally feasible. Applications are detailed for models such as testing for change points of variances and testing treatments against control of variances.

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Chapter 1

Introduction

The need for multiple testing procedures (MTPs) has been given great impetus by diverse fields of application such as microarrays, astronomy, mutual fund evaluations, proteomics, disclosure risk, cytometry, imaging and others. Traditional methods to deal with multiple testing when the number of tests is large are deemed too conservative (do not detect significant effects often enough). New approaches to multiple testing have arisen. Many of the new approaches are classified as stepwise procedures such as step-up and step-down in contrast to single step procedures. See Hochberg and Tamhane (1987) and also Dudoit, Shaffer and Boldrick (DSB) (2003) where 18 procedures are listed as single step, step-up or step-down. Among the more popular procedures is the Benjamini-Hochberg (1995) false discovery rate (FDR) controlling procedure. Many offspring have followed. See for example, Efron, Tibshirani, Storey and Tusher (2001), Storey and Tusher (2001), Storey and Tibshirani (2003), Sarkar (2002), Benjamini and Yekutieli (2001), Lehmann and Romano (2005), Cai and Sarkar (2006) and Dudoit and van der Laan (2008). Typically, the stepwise procedures deal with P-values determined from marginal distributions. Even when the model entails random vectors with correlated variates, P-values from marginal distributions, ignoring correlations, are the basis of the procedures.

In a series of papers Cohen and Sackrowitz (CS) (2005), (2007), (2008), and Cohen, Kolassa, and Sackrowitz (2007) demonstrated that given a typical step-up or step-down procedure, there exist other procedures whose expected numbers of type I and type II errors are smaller. In fact in CS(2007b) for multivariate normal models when correlation is nonzero, for two-sided alternatives of means, there exist procedures whose individual tests have smaller expected type I and type II errors.

The goal of this thesis is to develop good MTPs in the case of correlated variables. To begin with we realize that every MTP induces individual tests, ϕ_i, for the individual hypothesis testing problems H_i vs K_i. The behavior of these tests should be of fundamental concern. However, the stepwise construction of most MTPs often makes it difficult to describe and study the individual tests.

In particular, suppose an individual test induced by an MTP is inadmissible for the standard hypothesis testing loss. That is, for that individual hypothesis testing problem, a test exists whose size is no greater than the stepwise procedure test and whose power is no less with some strict inequality. It would then follow that the overall procedure would be inadmissible whenever the risk function is a monotone function of the expected numbers of type I and type II errors.

We use a convexity property (A.Cohen, H.Sackrowitz and M.Xu (2007)) that is necessary and sufficient for admissibility of the individual tests. In CS (2008) it has been shown that most popular stepwise procedures do not possess the convexity property when there is correlation in the two-sided alternative case. In this thesis we have constructed two step-down type MTPs whose individual tests do have the required convexity property for the problems we studied in the thesis. As is typical in problems where no single optimal procedure exists, the selection of a procedure is somewhat subjective. In evaluating procedures we focus mainly on the expected number of Type I and Type II errors that the procedures make.

One of the new stepwise testing methods proposed is based on the maximum of adaptively formed residuals. The method is called maximum residual down (MRD) procedure. The other one is called "maximum-likelihood ratio down (M-LRD)" procedure, as the name says, it is based on the maximum of a collection of likelihood ratios. Both of them are step-down type MTPs. These two methods have several advantages over the stepwise methods that are currently recommended in the literature.

(1) They can't be improved upon in terms of both type I and type II errors at the same time. That is, they are admissible for a vector risk function, each of whose components is the risk for the individual testing problems. The loss function for the individual tests is the typical zero-one loss function entailing type I and type II errors.

(2) They take into account the correlation among the variates, thus utilizing information oftentimes not used by the current P-value methods.

(3) For the change point model of variances in this thesis, we found if the variances have only one spot of consecutive changes, then MRD is quite efficient in detecting them. For the treatment vs control model of variances, simulations demonstrate that the MRD and M-LRD make substantially fewer mistakes that the popular FDR controlling procedures.

For the testing of means case, we assume $\boldsymbol{X}$ is an $M \times 1$ vector which is multivariate normal with mean vector $\boldsymbol{\mu}$ and known intraclass covariance matrix Σ. Applications of the intraclass model include the model of testing several treatments vs control. We test two sided alternatives, i.e. $H_i : \mu_i = 0 \;\; vs \;\; K_i : \mu_i \neq 0$, $i = 1, ..., M$. We also test one sided alternatives, i.e. $H_i : \mu_i = 0 \;\; vs \;\; K_i^* : \mu_i > 0$, $i = 1, ..., M$.

A seemingly logical step-down method that would take correlations into account is to successively perform likelihood ratio tests (LRT) of global hypotheses, that is, it continues in a step-down fashion in determining the LRT-based MTP. Call this procedure LRSD. At step one, LRSD employs the closure method (see Marcus, Peritz, and Gabriel (1976)) using a LRT for $\boldsymbol{\mu} = \boldsymbol{0} \;\; vs \;\; \boldsymbol{\mu} \neq \boldsymbol{0}$. If the global test rejects, then eliminate the variate corresponding to $\max\limits_{1 \leq i \leq M} |X_i|$. One continues in a step-down fashion. Similar for one sided alternatives.

With this intraclass covariance matrix , for one-sided alternatives, LRSD is admissible. For two sided alternatives LRSD is admissible for any monotone collection of critical constants only when M=2 or M=3. For $M \geq 4$, counterexamples abound. That is, there are many critical constants for which LRSD is inadmissible. Furthermore critical constants are found for $M \geq 5$ which relate to constants that are likely to be used.

The inadmissibility of LRSD is what prompted and led to MRD and M-LRD.

We have already applied our MRD method to the mean case of two special problems in the paper (A.Cohen, H.Sackrowitz and M.Xu (2007)). One problem is to detect the change points in mean. The other problem is testing for means of several treatments against a control. Advantages and limitations of MRD method to these two projects have been discussed in detail in this paper. And later we found that the test statistics for M-LRD and MRD are linear functions of each other for the two sided mean case. Thus a similar proof of admissibility works for M-LRD.

For the testing of variance case, we assume $\mathbf{z_j} = (z_{j1}, z_{j2}, ..., z_{j(M+1)})'$ is a sequence of independent normal variables with parameters$(\mu_1, \sigma_1^2), (\mu_2, \sigma_2^2), ..., (\mu_{M+1}, \sigma_{M+1}^2)$. $j = 1, 2, ..., n$. i.e., for each distribution with parameters (μ_j, σ_j^2), there are n independent sample points. Let $s_i^2 = \frac{\sum_{j=1}^{n}(z_{ji}-\bar{z}_i)^2}{n-1}$ be the sample variance, where $\bar{z}_i = \frac{\sum_{j=1}^{n} z_{ji}}{n}$, $i = 1, ..., (M+1)$. For this variance case, similarly, we mainly studied two problems. That is, one is to detect the change points in variance for a sequence of data. Another one is to test for variance of each of several treatments against a control.

The first problem is simplified into testing two sided alternatives, i.e. $H_i : \sigma_i^2 = \sigma_{i+1}^2 \;\; vs \;\; K_i : \sigma_i^2 \neq \sigma_{i+1}^2, i = 1, ..., M$. or test one sided alternatives, i.e. $H_i : \sigma_i^2 = \sigma_{i+1}^2 \;\; vs \;\; K_i : \sigma_i^2 > \sigma_{i+1}^2, i = 1, ..., M$. In either case the step-up and step-down methods are inadmissible. The LRSD step-down method is mostly inadmissible while the MRD method is admissible for both cases. The statistics for M-LRD and MRD are not linear functions of each other for this testing of variance case. M-LRD is studied only for two sided alternatives and M-LRD is admissible for such cases.

For the second problem, we test two sided alternatives, i.e. $H_i : \sigma_i^2 = \sigma_{M+1}^2 \;\; vs \;\; K_i : \sigma_i^2 \neq \sigma_{M+1}^2, i = 1, ..., M$. We also test one sided alternatives, i.e. $H_i : \sigma_i^2 = \sigma_{M+1}^2 \;\; vs \;\; K_i : \sigma_i^2 > \sigma_{M+1}^2, i = 1, ..., M$. For one-sided alternatives, step-up , step-down and LRSD methods are all admissible. For two sided alternatives, step-up and step-down methods are inadmissible while LRSD is admissible for any monotone collection of critical constants only when M=2. For $M \geq 3$, counterexamples abound. That is, there are many critical constants for which LRSD is inadmissible, while the MRD method is admissible for both cases and M-LRD is admissible for the two sided case.

One issue of concern in any MTP is computational feasibility. It is an issue because in some instances the number of tests to be performed is very large. The only obstacle to computational feasibility would be the possible need to invert high dimensional matrices numerically. Oftentimes covariance matrices are such that the inversion process can be simplified algebraically so that the computations present no problem. This is the case for the practical applications we consider here. The general case however involves inverting higher order matrices which may not be feasible if M is extremely large.

In the next Chapter we describe the LRT based step-down procedure (LRSD) for the mean case. Here there are both admissibility and inadmissibility results of interest. Chapter 3 and Chapter 4 are focused on change point problems and treatment vs control problems of variance individually, MRD, M-LRD, LRSD, step-up and step-down procedures are studied here. Admissibility and inadmissibility of each procedure is assessed. Chapter 5 provides a set of C's controlling strong FWER for the MRD procedure. Simulations and analysis are given in Chapter 6.

Chapter 2

Testing of means with intraclass covariance matrix

Assume $\mathbf{X}$ is an $M \times 1$ vector which is distributed as multivariate normal with unknown mean vector $\boldsymbol{\mu}$ and known covariance matrix $\Gamma = \sigma^2\Sigma$. The Σ matrix is an intraclass matrix here. Without loss of generality we take the diagonal elements of Σ to be 1 and the off diagonal elements to be ρ, that is

$$\Sigma = \begin{pmatrix} 1 & \rho & \rho & \cdots & \rho & \rho \\ \rho & 1 & \rho & \cdots & \rho & \rho \\ & & \ldots\ldots & & & \\ \rho & \rho & \rho & \cdots & 1 & \rho \\ \rho & \rho & \rho & \cdots & \rho & 1 \end{pmatrix}, \text{ which is a } M \times M \text{ matrix.}$$

We are interested in testing two sided alternatives, i.e

$$H_i : \mu_i = 0 \quad vs \quad K_i : \mu_i \neq 0, \quad i = 1, ..., M \tag{2.1}$$

We also interested in testing one sided alternatives, i.e

$$H_i : \mu_i = 0 \quad vs \quad K_i^* : \mu_i > 0, \quad i = 1, ..., M \tag{2.2}$$

One of the applications to the intraclass model is the model of testing several means of treatments vs control. For example, we have $(M+1)$ independent random samples from $(M+1)$ normal populations, i.e. $Z_i \sim N(\nu_i, \sigma^2)$, $i = 1, 2, ..., (M+1)$. Without loss of generality we assume $\sigma^2 = 1$. The treatments correspond to $i = 1, 2, ..., M$ while the control population corresponds to the $(M+1)^{\text{st}}$ population. And we are interested in testing

$$H_i : \nu_i - \nu_{M+1} = 0 \quad vs \quad K_i : \nu_i - \nu_{M+1} \neq 0, \quad i = 1, ..., M \tag{2.3}$$

or one sided alternatives:

$$H_i : \nu_i - \nu_{M+1} = 0 \quad vs \quad K_i^* : \nu_i - \nu_{M+1} > 0, \quad i = 1, ..., M \tag{2.4}$$

Let $X_i = Z_i - Z_{M+1}$, $i = 1, 2, ..., M$ so that $\boldsymbol{X}$ is distributed as multivariate normal with mean vector $\boldsymbol{\mu}$, $\mu_i = \nu_i - \nu_{M+1}$ and covariance matrix Γ . That is

$$\Gamma = 2 \times \begin{pmatrix} 1 & 0.5 & 0.5 & \cdots & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 & \cdots & 0.5 & 0.5 \\ & & \cdots\cdots & & & \\ 0.5 & 0.5 & 0.5 & 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 1 \end{pmatrix}, \text{ which is a } M \times M \text{ intraclass matrix.}$$

To solve these problems, we only studied Likelihood Ratio Step-Down Method(LRSD) method here which naturally takes correlation into account. It continues in a step-down fashion in determining the LRT-based MTP. We have already applied our new method MRD to these cases in the paper (A.Cohen, H.Sackrowitz and M.Xu (2007)). Advantages and limitations of MRD method have been discussed in detail in this paper. And we found that the test statistics for M-LRD and MRD are linear functions of each other for the two sided mean case. Thus a similar proof of admissibility works for M-LRD.

By way of notation, let $\mathbf{X}^{(j_1,j_2,...,j_{m-1})}$ be the (M-(m-1)) vector consisting of the components of $\mathbf{X}$ with $X_{j_1}, ..., X_{j_{m-1}}$ left out. $\boldsymbol{\mu}^{(j_1,j_2,...,j_{m-1})}$ is the (M-(m-1)) vector consisting of the components of $\boldsymbol{\mu}$ with $\mu_{j_1}, ..., \mu_{j_{m-1}}$ left out. $\Sigma_{(j_1,j_2,...,j_{m-1})}$ is the $(M - (m - 1)) \times (M - (m - 1))$ covariance matrix of $\mathbf{X}^{(j_1,j_2,...,j_{m-1})}$.

2.1 Likelihood Ratio Step-Down Method(LRSD)

LRSD Procedure for two sided alternatives:

Let $c_1 > c_2 > \cdots > c_M > 0$ be a given set of constants.

Stage 1: Let $I_1 = \{1, 2, ..., M\}$ be the indices of the hypotheses of (2.1) . We test $H_{1G} : \boldsymbol{\mu} = \mathbf{0}$ vs $K_{1G} : \boldsymbol{\mu} \neq \mathbf{0}$. The likelihood ratio for this test is $L_1 = \frac{\sup_{\boldsymbol{\mu}} \frac{1}{(2\pi)^{M/2}|\Sigma|^{1/2}} \exp\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\}}{\frac{1}{(2\pi)^{M/2}|\Sigma|^{1/2}} \exp\{-\frac{1}{2}\boldsymbol{x}'\Sigma^{-1}\boldsymbol{x}\}} = \exp\{\frac{1}{2}\boldsymbol{x}'\Sigma^{-1}\boldsymbol{x}\}$. If $L_1 < c_1$, then accept H_{1G} and stop; Otherwise, reject H_{j_1} where j_1 is the index for which $|X_{j_1}| = \max\{|X_j| : j \in I_1\}$, then continue.

Stage 2: Let I_2 be the indices of the hypotheses not previously rejected. Now we test $H_{2G} : \boldsymbol{\mu}^{(j_1)} = \mathbf{0}$ vs $K_{2G} : \boldsymbol{\mu}^{(j_1)} \neq \mathbf{0}$. Let L_2 be the likelihood ratio for this test.

If $L_2 < c_2$, then accept H_{2G} and stop; otherwise, reject H_{j_2} where j_2 is the index for which $|X_{j_2}| = \max\{|X_j| : j \in I_2\}$ and continue.

$\vdots$

In general at stage m: $1 \leq m \leq M$, let I_m be the indices of the hypotheses not previously rejected. Now we test $H_{mG} : \boldsymbol{\mu}^{(j_1,\ldots,j_{m-1})} = \mathbf{0}$ vs $K_{mG} : \boldsymbol{\mu}^{(j_1,\ldots,j_{m-1})} \neq \mathbf{0}$. Let L_m be the likelihood ratio for this test. If $L_m < c_m$, then accept H_0^m and stop; otherwise, reject H_{j_m} where j_m is the index for which $|X_{j_m}| = \max\{|X_j| : j \in I_m\}$ and continue.

We will demonstrate that the LRSD is admissible for M=2 and M=3. For $M \geq 4$ there exist counterexamples for certain collections of critical values and certain values of ρ. We offer a counterexample when $M = 4$ and when $M = 5$ we demonstrate inadmissibility for a large class of practical critical values for logical values of ρ. In fact for large M, using χ^2 critical values it turns out that for most ρ values ($\rho \neq 0$) counterexamples demonstrate that LRSD is inadmissible.

On the other hand should the alternatives for the individual hypotheses be the one-sided alternatives given in (2.2), then the LRSD is admissible.

Now we express the density of $\boldsymbol{X}$ as

$$f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\mu}) = \frac{1}{(2\pi)^{M/2}|\Sigma|^{1/2}} \exp\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\} \tag{2.5}$$

which in exponential family form is

$$f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\mu}) = h(\boldsymbol{x})\beta(\boldsymbol{\mu}) \exp\{\boldsymbol{x}'\Sigma^{-1}\boldsymbol{\mu}\} \tag{2.6}$$

Next let $\boldsymbol{Y} = \Sigma^{-1}\boldsymbol{X}$ so that

$$f_{\boldsymbol{Y}}(\boldsymbol{y}|\boldsymbol{\mu}) = h^*(\boldsymbol{y})\beta(\boldsymbol{\mu}) \exp\{\sum_{i=1}^{M} y_i\mu_i\} \tag{2.7}$$

Lemma 2.1.1. *A necessary and sufficient condition for a test, $\psi(\mathbf{y})$, of $H_1 : \mu_1 = 0$ vs $K_1 : \mu_1 \neq 0$ to be admissible, is that for almost every fixed $y_2, \ldots, y_{(M+1)}$, the acceptance region of the test is an interval in y_1.*

Proof. See Matthes and Truax (1967). $\square$

Note, to study the test function $\psi(\mathbf{y}) = \phi(\boldsymbol{x})$ as y_1 varies and $(y_2, ..., y_{(M+1)})$ remain fixed we can consider sample points $\boldsymbol{x} + r\boldsymbol{g}$ where $\boldsymbol{g}$ is the first row of Σ and r varies. This is true since $\boldsymbol{y}$ is a function of $\boldsymbol{x}$ and so $\boldsymbol{y}$ evaluated at $(\boldsymbol{x}+r\boldsymbol{g})$ is $(\Sigma)^{-1}(\boldsymbol{x}+r\boldsymbol{g}) = \boldsymbol{y} + (r, 0, ..., 0)' = (y_1 + r, y_2, ..., y_{(M+1)})$.

Focusing firstly on the two-sided alternative case we note that the LRT for H_{1G} vs K_{1G} is to reject if

$$\boldsymbol{x}'\Sigma^{-1}\boldsymbol{x} \geq C_1 \tag{2.8}$$

where $\Sigma^{-1} = \begin{pmatrix} 1+(M-2)\rho & -\rho & -\rho & \cdots & -\rho \\ -\rho & 1+(M-2)\rho & -\rho & \cdots & -\rho \\ & & \cdots & \cdots & \\ -\rho & -\rho & -\rho & \cdots & 1+(M-2)\rho \end{pmatrix}$

Theorem 2.1.1. *For the two-sided alternative case LRSD is admissible for M=2 and M=3.*

Proof of Theorem 2.1.1. We prove the theorem for M=3. For M=2 the method is the same and the proof is simpler. Note when $x_1^* = 0$, H_1 is accepted. In light of Lemma 2.1.1 we need to show that the LRSD test for H_1 vs K_1, say $\phi_1(\boldsymbol{x})$, as a function of $\boldsymbol{x}+r\boldsymbol{g}$ goes from reject to accept to reject as r varies from $(-\infty, \infty)$, where $\boldsymbol{g} = (1, \rho, \rho)'$. Another way of stating this requirement is suppose $\phi_1(\boldsymbol{x}^*) = 1$ when $x_1^* > 0$. Then we must have $\phi_1(\boldsymbol{x}^* + r\boldsymbol{g}) = 1$ for all $r > 0$ while if $\phi_1(\boldsymbol{x}^*) = 1$ when $x_1^* < 0$, we must have $\phi_1(\boldsymbol{x}^* - r\boldsymbol{g}) = 1$ for all $r > 0$.

H_1 can be rejected at three different stages:

(1) If H_1 is rejected at stage 1 for $\boldsymbol{x} = \boldsymbol{x}^* \Longrightarrow |x_1^*| > |x_2^*|$, $|x_1^*| > |x_3^*|$ and ${\boldsymbol{x}^*}'\Sigma^{-1}\boldsymbol{x}^* \geq C_1$,

when $x_1^* > 0$, this implies

$$(\boldsymbol{x}^* + r\boldsymbol{g})'\Sigma^{-1}(\boldsymbol{x}^* + r\boldsymbol{g}) = {\boldsymbol{x}^*}'\Sigma^{-1}\boldsymbol{x}^* + 2rx_1^* + r^2 > C_1 \tag{2.9}$$

Also $x_1^* + r > |x_2^* + r\rho|$ and $x_1^* + r > |x_3^* + r\rho|$, so $\phi_1(\boldsymbol{x}^* + r\boldsymbol{g}) = 1$ too, for all $r > 0$.

When $x_1^* < 0$, a similar argument works for $(\boldsymbol{x}^* - r\boldsymbol{g})'\Sigma^{-1}(\boldsymbol{x}^* - r\boldsymbol{g})$

(2) If H_1 is rejected at stage 2 for $\boldsymbol{x} = \boldsymbol{x}^*$, suppose x_3^* is out first $\Longrightarrow |x_3^*| > |x_1^*| > |x_2^*|$, $\boldsymbol{x}^{*'}\Sigma^{-1}\boldsymbol{x}^* \geq C_1$ and $\boldsymbol{x}^{*(3)'}\Sigma_{(3)}^{-1}\boldsymbol{x}^{*(3)} = x_1^{*2} + x_2^{*2} - 2\rho x_1^* x_2^* \geq C_2$.

When $x_1^* > 0 \Longrightarrow$ (2.9) $> C_1$ and

$$\begin{aligned}
&(\boldsymbol{x}^{*(3)} + r\boldsymbol{g}^{(3)})'\Sigma_{(3)}^{-1}(\boldsymbol{x}^{*(3)} + r\boldsymbol{g}^{(3)}) \\
&= (x_1^* + r)^2 + (x_2^* + r\rho)^2 - 2\rho(x_1^* + r)(x_2^* + r\rho) \qquad (2.10) \\
&= x_1^{*2} + x_2^{*2} - 2\rho x_1^* x_2^* + 2r x_1^* + r^2 + \rho^2 r^2 - 2\rho^2 x_1^* r - 2\rho^2 r^2
\end{aligned}$$

But since $r^2 + \rho^2 r^2 > 2\rho^2 r^2$ and $2r x_1^* \geq 2\rho^2 r x_1^*$ it follows that (2.10)$> C_2$ for all $r > 0$. Hence $\phi_1(\boldsymbol{x}^* + r\boldsymbol{g}) = 1$ for all $r > 0$.

When $x_1^* < 0$, a similar argument works for $(\boldsymbol{x}^{*(3)} - r\boldsymbol{g}^{(3)})'\Sigma_{(3)}^{-1}(\boldsymbol{x}^{*(3)} - r\boldsymbol{g}^{(3)})$.

(3) If H_1 is rejected at stage 3 for $\boldsymbol{x} = \boldsymbol{x}^*$, suppose x_3^* is out first and x_2^* is out second $\Longrightarrow |x_3^*| > |x_2^*| > |x_1^*|$.

When $x_1^* > 0$, subcases where the ordering of the components of $\boldsymbol{x}^*$ is maintained with $(\boldsymbol{x}^* + r\boldsymbol{g})$, it is easy to prove the required monotonicity property. The most challenging subcases is if $|x_3^*| > x_2^* > x_1^* > 0$ with $x_3^* < 0$ but

$$|x_3^* + r\rho| < x_2^* + r\rho \qquad (2.11)$$

In this case when $\rho > 0$ we use the fact that $x_3^{*2} > x_2^{*2}$ and use the inequalities as in the previous case to prove the result. When $\rho < 0$ we observe that if $|x_3^*| > x_2^* > x_1^* > 0$ and $x_3^* < 0$ then $|x_3^* + r\rho| > x_2^* + r\rho$ and so (2.11) can't happen. It's easy to verify that if $\phi_1(\boldsymbol{x}^*) = 1$ then $\phi_1(\boldsymbol{x}^* + r\boldsymbol{g}) = 1$ for all $r > 0$.

Similar argument works for $x_1^* < 0$. $\square$

For M=4 we exhibit a set of critical values for which LRSD is inadmissible. To do so we find a sample point $\boldsymbol{x}^*(x_1^* > 0)$ at which H_1 is rejected and for which H_1 is accepted at $\boldsymbol{x}^* + r\boldsymbol{g}$. In fact let $\boldsymbol{x}^* = (a, -a - \Delta, b, -b - \varepsilon)'$ for $b > a + \Delta > a > 0$, $\varepsilon > 0$ and $b + \varepsilon > a + \varepsilon/\rho$. Thus using (2.8) at stage 1 choose C_1 so that $\boldsymbol{x}^{*'}\Sigma^{-1}\boldsymbol{x}^* \geq C_1$ and H_4 is rejected and x_4^* is eliminated at stage 1. At stage 2 we calculate

$$\boldsymbol{x}^{*(4)'}\Sigma_{(4)}^{-1}\boldsymbol{x}^{*(4)} = \frac{1}{1 + \rho - 2\rho^2}\{(1 + \rho)b^2 + 2a^2(1 + 2\rho) + 2\Delta[a + 2a\rho + \rho b + (1 + \rho)\Delta/2]\} \qquad (2.12)$$

We set $\boldsymbol{x}^{*(4)'}\Sigma_{(4)}^{-1}\boldsymbol{x}^{*(4)} = C_2$, so H_3 is rejected. At stage 3, H_2 is rejected and at stage 4, H_1 is rejected. Now if $\rho > 0$ let $r = \varepsilon/\rho$ and note that $(\boldsymbol{x}^* + r\boldsymbol{g})'\Sigma^{-1}(\boldsymbol{x}^* + r\boldsymbol{g}) \geq C_1$. This time however, H_3 is rejected at stage 1. At stage 2 we calculate for $r = \varepsilon/\rho$,

$$\begin{aligned}
&\boldsymbol{x}^{*(3)'}\Sigma_{(3)}^{-1}\boldsymbol{x}^{*(3)} \\
&= \frac{1}{1+\rho-2\rho^2}\{(1+\rho)b^2 + 2a^2(1+2\rho) + 2\Delta(a + 2a\rho - b\rho + (1+\rho)\Delta/2) \\
&+ (\rho - 1 + \frac{1}{\rho} + \frac{1}{\rho^2})\varepsilon^2 + \varepsilon(\frac{2a}{\rho} + 2a - 4a\rho - 2\rho\Delta + 2(1+\rho)b)\}
\end{aligned} \tag{2.13}$$

We note that (2.12) minus (2.13) is

$$\frac{1}{1+\rho-2\rho^2}\{4\Delta b\rho - (\rho - 1 + \frac{1}{\rho} + \frac{1}{\rho^2})\varepsilon^2 - \varepsilon(\frac{2a}{\rho} + 2a - 4a\rho - 2\rho\Delta + 2(1+\rho)b)\} \tag{2.14}$$

There are many choices of $a, b, \Delta, \varepsilon, \rho$ for which (2.14) is positive (e.g., $a = 2, b = 4, \Delta = 1, \varepsilon = .1, \rho = .5, r = .2$). The fact that (2.14) > 0 implies that we can choose C_2 such that $\boldsymbol{x}^{*(3)'}\Sigma_{(3)}^{-1}\boldsymbol{x}^{*(3)} < C_2$ so that at $\boldsymbol{x}^* + r\boldsymbol{g}$ the overall procedure rejects H_3 and accepts H_1, H_2 and H_4. Note since $x_1^* > 0$, $\boldsymbol{x}^* - a\boldsymbol{g}$ is an accept point. Now if H_1 is rejected for $\boldsymbol{x} = \boldsymbol{x}^*$ but accepted for $\boldsymbol{x}^* + r\boldsymbol{g}$, that implies the test for H_1 is inadmissible.

For $M = 5$ it can be shown that if the critical values correspond to critical values of chi-square with m degrees of freedom, m=1,2,3,4,5, at level, say .05, then for most values of ρ, LRSD is also inadmissible. The same is true for any $M > 5$.

Next for the intraclass model we consider testing one-sided alternatives, i.e. we test $H_i : \mu_i = \mu_{i+1}$ vs $K_i^* : \mu_i > \mu_{i+1}$. The LRSD method in this case is the same as in the two-sided alternative case except that $|X_{j_1}|$ is replaced by $X_{j_1} = \max(X_1, ..., X_M)$, the likelihood ratio $L_1 = \exp\{\frac{1}{2}\boldsymbol{x}'\Sigma^{-1}\boldsymbol{x}\}$ is replaced by $L_1 = \sup_{\mathbf{u}\geq 0} \exp\{\mathbf{x}'\Sigma^{-1}\mathbf{u} - \frac{1}{2}\mathbf{u}'\Sigma^{-1}\mathbf{u}\}$, and similar changes for subsequent stages. For this setup we have

Theorem 2.1.2. *For the one-sided alternative case LRSD is admissible.*

Proof of Theorem 2.1.2. Once again we focus on H_1 vs K_1^* and demonstrate that if $\phi(\boldsymbol{x}^*) = 1$ then $\phi(\boldsymbol{x}^* + r\boldsymbol{g}) = 1$ for all $r > 0$. Suppose H_1 is rejected at stage m for $\boldsymbol{x} = \boldsymbol{x}^*$. Then $x_1^* > 0$, $x_{j_1}^* > x_{j_2}^* > \cdots > x_{j_{m-1}}^* > x_1^* > x_{j_{m+1}}^* > \cdots > x_{j_M}^*$ and

$L_1 > c_1, L_2 > c_2, ..., L_m > c_m$. Note at $\boldsymbol{x}^{**} = \boldsymbol{x}^* + r\boldsymbol{g}$ the orders of all coordinates are preserved except perhaps the first coordinate which now can be anywhere among the m largest coordinates. The k stage global hypothesis is considered if $H_{j_1}, ..., H_{j_{k-1}}$ have been rejected. This global testing problem is $H_{kG} : \boldsymbol{\mu}^{(j_1,\dots,j_{k-1})} = \mathbf{0}$ vs $K_{kG} : \boldsymbol{\mu}^{(j_1,\dots,j_{m-1})} \geq \mathbf{0}$ but at least one $\mu_i > 0, i \in K_k$. The likelihood ratio test rejects H_{kG} if $L_k > c_k$, i.e

$$\begin{aligned}&\sup_{\{\mu_i \geq 0, i \in K_k\}} \exp\{\boldsymbol{x}^{*(j_1,\dots,j_{k-1})'}\Sigma^{-1}_{(j_1,\dots,j_{k-1})}\boldsymbol{\mu}^{(j_1,\dots,j_{k-1})} - \frac{1}{2}\boldsymbol{\mu}^{(j_1,\dots,j_{k-1})'}\Sigma^{-1}_{(j_1,\dots,j_{k-1})}\boldsymbol{\mu}^{(j_1,\dots,j_{k-1})}\}\\ &= \exp\{\boldsymbol{x}^{*(j_1,\dots,j_{k-1})'}\Sigma^{-1}_{(j_1,\dots,j_{k-1})}\hat{\boldsymbol{\mu}}^{*(j_1,\dots,j_{k-1})} - \frac{1}{2}\hat{\boldsymbol{\mu}}^{(j_1,\dots,j_{k-1})*'}\Sigma^{-1}_{(j_1,\dots,j_{k-1})}\hat{\boldsymbol{\mu}}^{*(j_1,\dots,j_{k-1})}\}\end{aligned} \tag{2.15}$$

$> c_k$, where $\hat{\boldsymbol{\mu}}^{*(j_1,\dots,j_{k-1})}$ is the maximum likelihood estimator on $[0, +\infty)$ of $\boldsymbol{\mu}^{(j_1,\dots,j_{k-1})}$ when $\boldsymbol{x} = \boldsymbol{x}^*$.

Next consider the likelihood ratio test statistic L_k at $\boldsymbol{x}^{**}$. It is

$$\begin{aligned}&\sup_{\{\mu_i \geq 0, i \in K_k\}} \exp\{(\boldsymbol{x}^{*(j_1,\dots,j_{k-1})'} + r\boldsymbol{g}^{(j_1,\dots,j_{k-1})})\Sigma^{-1}_{(j_1,\dots,j_{k-1})}\boldsymbol{\mu}^{(j_1,\dots,j_{k-1})}\\ &\qquad - \frac{1}{2}\boldsymbol{\mu}^{(j_1,\dots,j_{k-1})'}\Sigma^{-1}_{(j_1,\dots,j_{k-1})}\boldsymbol{\mu}^{(j_1,\dots,j_{k-1})}\}\\ &\geq \exp\{(\boldsymbol{x}^{*(j_1,\dots,j_{k-1})'} + r\boldsymbol{g}^{(j_1,\dots,j_{k-1})})\Sigma^{-1}_{(j_1,\dots,j_{k-1})}\hat{\boldsymbol{\mu}}^{*(j_1,\dots,j_{k-1})}\\ &\qquad - \frac{1}{2}\hat{\boldsymbol{\mu}}^{*(j_1,\dots,j_{k-1})'}\Sigma^{-1}_{(j_1,\dots,j_{k-1})}\hat{\boldsymbol{\mu}}^{*(j_1,\dots,j_{k-1})}\}\\ &= \exp\{\boldsymbol{x}^{*(j_1,\dots,j_{k-1})'}\Sigma^{-1}_{(j_1,\dots,j_{k-1})}\hat{\boldsymbol{\mu}}^{*(j_1,\dots,j_{k-1})} - \frac{1}{2}\hat{\boldsymbol{\mu}}^{*(j_1,\dots,j_{k-1})'}\Sigma^{-1}_{(j_1,\dots,j_{k-1})}\hat{\boldsymbol{\mu}}^{*(j_1,\dots,j_{k-1})}\\ &\qquad + r\hat{\mu}_1^{*(j_1,\dots,j_{k-1})}\}\end{aligned} \tag{2.16}$$

Recognize that the right-hand side of (2.16) is the maximized likelihood in (2.15) times $\exp\{r\hat{\mu}_1^{(j_1,\dots,j_{k-1})*}\}$. Since $\hat{\mu}_1^{(j_1,\dots,j_{k-1})*} \geq 0$, it follows from (2.15) and (2.16) that (2.16) $\geq c_k$, which means there is a rejection at stage k at $\boldsymbol{x}^{**}$ if there was a rejection at stage k at $\boldsymbol{x}^*, k = 1, ..., M$. Since the order of the coordinates of $x^{**}_{j_1}, x^{**}_{j_2}, \cdots, x^{**}_{j_{m-1}}$ remains unchanged and x^{**}_1 is among the m largest coordinates of $\boldsymbol{x}^{**}$ it follows that H_1 is rejected at stage m or sooner. $\square$

Chapter 3

Variance Change

Let $\mathbf{z_j} = (z_{j1}, z_{j2}, ..., z_{j(M+1)})'$ be a sequence of independent normal variables with parameters$(\mu_1, \sigma_1^2), (\mu_2, \sigma_2^2), ..., (\mu_{M+1}, \sigma_{M+1}^2)$, $j = 1, 2, ..., n$, i.e., for each distribution with parameters (μ_i, σ_i^2), there are n independent sample points. Let $s_i^2 = \frac{\sum_{j=1}^n (z_{ji} - \bar{z}_i)^2}{n-1}$ be the sample variance for the i^{th} population, where $\bar{z}_i = \frac{\sum_{j=1}^n z_{ji}}{n}$, $i = 1, ..., (M+1)$.

The interest here is to test the hypothesis testing:

$$H_i : \sigma_i^2 = \sigma_{i+1}^2 \quad vs \quad K_i : \sigma_i^2 \neq \sigma_{i+1}^2, \quad i = 1, ..., M \tag{3.1}$$

So rejecting any H_i indicates a change point in variance occurs at position i.

We will also consider one-sided alternative problems

$$H_i : \sigma_i^2 = \sigma_{i+1}^2 \quad vs \quad K_i^* : \sigma_i^2 > \sigma_{i+1}^2, \quad i = 1, ..., M \tag{3.2}$$

We know that $\frac{(n-1)s_i^2}{\sigma_i^2} \sim \chi_{n-1}^2$, so the density of $\mathbf{s^2} = (s_1^2, s_2^2, ..., s_{M+1}^2)'$ is:

$$f_{\mathbf{s^2}}(\mathbf{s}^2 | \sigma^{\mathbf{2}}) = \prod_{i=1}^{M+1} \frac{(n-1)}{\Gamma(\frac{n-1}{2}) 2^{(n-1)/2}} \frac{((n-1)s_i^2)^{(\frac{n-1}{2} - 1)}}{(\sigma_i^2)^{\frac{n-1}{2}}} e^{-\frac{(n-1)s_i^2}{2\sigma_i^2}} \tag{3.3}$$

Now let $\tilde{z}_i = (n-1)s_i^2$, $u_i = -\frac{1}{2\sigma_i^2}$, so that (3.3) becomes:

$$f_{\tilde{\mathbf{z}}}(\tilde{\mathbf{z}} | \mathbf{u}) = h(\tilde{\mathbf{z}}) \beta(\mathbf{u}) \exp\{\tilde{\mathbf{z}}' \mathbf{u}\} \tag{3.4}$$

where $\tilde{\mathbf{z}} = (\tilde{z}_1, \tilde{z}_2, ..., \tilde{z}_{M+1})'$ and $\mathbf{u} = (u_1, u_2, ..., u_{M+1})'$.

Let A= $\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ & & \cdots\cdots & & & \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$, which is a $(M+1) \times (M+1)$ matrix, Then

$$f_{\tilde{\mathbf{z}}}(\tilde{\mathbf{z}}|\mathbf{u}) = h(\tilde{\mathbf{z}})\beta(\mathbf{u})\exp\{\tilde{\mathbf{z}}'A^{-1}A\mathbf{u}\} \tag{3.5}$$

Denote $\boldsymbol{\nu} = A\mathbf{u}$, so

$$f_{\tilde{\mathbf{z}}}(\tilde{\mathbf{z}}|\mathbf{u}) = h(\tilde{\mathbf{z}})\beta^*(\boldsymbol{\nu})\exp\{\tilde{\mathbf{z}}'A^{-1}\boldsymbol{\nu}\} \tag{3.6}$$

And testing (3.1) and (3.2) are equivalent to test

$$H_i : \nu_i = 0 \quad vs \quad K_i : \nu_i \neq 0, \quad i = 1, ..., M \tag{3.7}$$

$$H_i : \nu_i = 0 \quad vs \quad K_i^* : \nu_i > 0, \quad i = 1, ..., M \tag{3.8}$$

3.1 MRD

The maximum residual down (MRD) method is based on the maximum of adaptively formed residuals. It is step-down type MTP. For each stage, we calculate the residuals for the hypotheses not previously rejected, and compare the biggest one with some constant c, then make decision of rejecting or accepting.

Let $\mathbf{X} = A\tilde{\boldsymbol{z}}$, $\Sigma = AA'$,then from (3.6) we can get

$$f_{\mathbf{X}}(\mathbf{x}|\boldsymbol{\nu}) = h^*(\mathbf{x})\beta^*(\boldsymbol{\nu})\exp\{\mathbf{x}'\Sigma^{-1}\boldsymbol{\nu}\} \tag{3.9}$$

Note that

$$\Sigma = AA' = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ & & \cdots\cdots\cdots & & & & \\ 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & (M+1) \end{pmatrix}$$

which is a $(M+1) \times (M+1)$ matrix.

Using the similar notation as in Chapter 2, let $\mathbf{X}^{(j_1,j_2,...,j_r,i)}$ be the (M-r) vector consisting of the components of $\mathbf{X}$ with $X_{j_1}, ..., X_{j_r}, X_i$ left out; $\Sigma_{(j_1,j_2,...,j_r,i)}$ is the $(M-r) \times (M-r)$ covariance matrix of $\mathbf{X}^{(j_1,j_2,...,j_r,i)}$; $\boldsymbol{\sigma}_{(i)}^{(j_1,j_2,...,j_r)}$ is the $(M-r) \times 1$ vector of covariances between X_i and all variables except $X_{j_1}, ..., X_{j_r}$ and X_i.

So for stage m after rejecting $H_{j_1}, H_{j_2}, ...H_{j_{m-1}}$, we define Residual$_{m,i}$ as follows:

$$\text{Residual}_{m,i} \quad = \quad X_i - \boldsymbol{\sigma}_{(i)}^{(j_1,j_2,...,j_{m-1})'}\Sigma^{-1}_{(j_1,j_2,...,j_{m-1},i)}\boldsymbol{X}^{(j_1,j_2,...,j_{m-1},i)} \tag{3.10}$$

for any i, $i \in \{1, 2, ..., M\} \setminus \{j_1, ..., j_{m-1}\}$,

Let $(j_{(1)}, ...j_{(m-1)})$ be the ordered sequence of $(j_1, ..., j_{m-1})$.

If i is in the range of $(j_{(k)}, \, j_{(k+1)})$, where $k = 0, 1, ..., m-1$. Here denote $j_{(0)} = 0$, $j_{(m)} = M + 1$. After calculating (3.10), we get

$$\text{Residual}_{m,i} = \frac{\tilde{z}_{j_{(k)}+1} + \cdots + \tilde{z}_i}{i - j_{(k)}} - \frac{\tilde{z}_{i+1} + \cdots + \tilde{z}_{j_{(k+1)}}}{j_{(k+1)} - i} \tag{3.11}$$

which only involves $(\tilde{z}_{j_{(k)}+1}, \tilde{z}_{j_{(k)}+2}, ..., \tilde{z}_{j_{(k+1)}})'$.

To make Residual$_{m,i}$ invariant in scale, let $W_{m,i}$ be defined as Residual$_{m,i}$ divided by $\tilde{z}_{j_{(k)}+1} + \cdots + \tilde{z}_{j_{(k+1)}}$. That is

$$W_{m,i} = \frac{Residual_{m,i}(\tilde{\boldsymbol{z}})}{\tilde{z}_{j_{(k)}+1} + \cdots + \tilde{z}_{j_{(k+1)}}} = \frac{\frac{\tilde{z}_{j_{(k)}+1}+\cdots+\tilde{z}_i}{i-j_{(k)}} - \frac{\tilde{z}_{i+1}+\cdots+\tilde{z}_{j_{(k+1)}}}{j_{(k+1)}-i}}{\tilde{z}_{j_{(k)}+1} + \cdots + \tilde{z}_{j_{(k+1)}}} \tag{3.12}$$

Then our test statistics $U_{m,i}$ is defined as:

$$U_{m,i} = (W_{m,i})^2 \tag{3.13}$$

for the two sided (3.1) case, $m = 1, ..., M$.

And

$$U_{m,i} = W_{m,i} \tag{3.14}$$

for the one sided (3.2) case, $m = 1, ..., M$

3.1.1 MRD Procedure

MRD Procedure:

Let $c_1 > c_2 > \cdots > c_M > 0$ be a given set of constants.

Stage 1: Let $I_1 = \{1, 2, ..., M\}$. If $U_{1,j_1} = \max\{U_{1,i} : i \in I_1\} < c_1$, then accept all hypotheses and stop; otherwise, reject H_{j_1} and continue.

Stage 2: Let I_2 be the indices of the hypotheses not previously rejected. If $U_{2,j_2} = \max\{U_{2,i} : i \in I_2\} < c_2$, then accept all hypotheses in I_2 and stop; otherwise, reject H_{j_2} and continue.

$\vdots$

In general at stage m: $1 \leq m \leq M$, let I_m be the indices of the hypotheses not previously rejected. If $U_{m,j_m} = \max\{U_{m,i} : i \in I_m\} < c_m$, then accept all hypotheses in I_m and stop; otherwise, reject H_{j_m} and continue.

3.1.2 Admissibility of MRD

We will demonstrate that for each individual testing problem that the MTP based on MRD method is admissible. Without loss of generality we focus on H_1 vs K_1. Again our plan is to use a result of Matthes and Truax (1967) stated as Lemma 2.1.1 which offers a necessary and sufficient condition for admissibility of a test of H_1 vs K_1 when the joint distribution of $\tilde{\boldsymbol{z}}$ is an exponential family. We next demonstrate in Lemma 3.1.1 that $W_{m,i}(\tilde{\mathbf{z}})$ function given in (3.12) has certain monotonicity properties. These monotonicity properties will enable us to prove in Lemma 3.1.2 that the individual test function for H_i vs K_i have the convexity property that is necessary and sufficient for admissibility. Theorem 3.1.1 summarizes and states the admissibility of the MRD procedure.

The density of $\tilde{\boldsymbol{z}}$ is expressed in (3.6), now let $\boldsymbol{Y} = (A')^{-1}\tilde{\boldsymbol{z}}$ so that

$$f_{\mathbf{Y}}(\mathbf{y}|\boldsymbol{\nu}) = h^{**}(\mathbf{y})\beta^*(\boldsymbol{\nu})\exp\{\sum_{i=1}^{M+1} y_i\nu_i\} \tag{3.15}$$

Similar to the proofs in Chapter 2, to study the test function $\psi(\mathbf{y}) = \phi_U(\tilde{\boldsymbol{z}})$ as y_1 varies and $(y_2, ..., y_{(M+1)})$ remain fixed we can consider sample points $\tilde{\boldsymbol{z}} + r\boldsymbol{g}$ where $\boldsymbol{g}$ is the first row of A and r varies. This is true since $\boldsymbol{y}$ is a function of $\tilde{\boldsymbol{z}}$ and so $\boldsymbol{y}$ evaluated at $(\tilde{\boldsymbol{z}} + r\boldsymbol{g})$ is $(A')^{-1}(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = \boldsymbol{y} + (r, 0, ..., 0)' = (y_1 + r, y_2, ..., y_{(M+1)})$

Lemma 3.1.1. *The function $W_{m,j}(\tilde{\boldsymbol{z}})$ given in (3.12) have the following properties:*

At any stage m, as far as H_1 has not been rejected, for any $i \neq 1$, i.e., $i \in \{2, ..., M\} \setminus \{j_1, ..., j_{m-1}\}$, $j_1 \neq 1, ..., j_{m-1} \neq 1$,

$$W_{m,i}(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = W_{m,i}(\tilde{\boldsymbol{z}}) \tag{3.16}$$

and

$$W_{m,1}(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = W_{m,1}(\tilde{\boldsymbol{z}}) + ar \tag{3.17}$$

where a is some constant and $a > 0$;

Proof of Lemma 3.1.1. For $i = 1$, use (3.12) and recall $\boldsymbol{g} = (1, -1, 0, ..., 0)'$ is the first row of A to see that

$$W_{m,1}(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = \frac{(\tilde{z}_1 + r) - \frac{(\tilde{z}_2 - r) + \tilde{z}_3 + \cdots + \tilde{z}_{j_{(1)}}}{j_{(1)} - 1}}{(\tilde{z}_1 + r) + (\tilde{z}_2 - r) + \cdots + \tilde{z}_{j_{(1)}}}$$
$$= W_{m,i}(\tilde{\boldsymbol{z}}) + ar$$

where $a = \frac{1 + \frac{1}{j_{(1)} - 1}}{\sum_{k=1}^{j_1} \tilde{z}_k}$, $j_{(1)} \geq 2$, so $a > 0$. This establishes (3.17).

Now for $i \neq 1$, if $j_{(k)} < i < j_{(k+1)}$, $j_{(k)} \neq 1, k = 0, 1, ..., m$, where $j_0 = 0, j_m = M+1$,

$$W_{m,i}(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = \frac{\frac{\tilde{z}_{j_{(k)}+1} + \cdots + \tilde{z}_i}{i - j_{(k)}} - \frac{\tilde{z}_{(i+1)} + \cdots + \tilde{z}_{j_{(k+1)}}}{j_{(k+1)} - i}}{\tilde{z}_{j_{(k)}+1} + \cdots + \tilde{z}_{j_{(k+1)}}}$$
$$= W_{m,i}(\tilde{\boldsymbol{z}})$$

since $i \geq 2$ and $j_{(k)} = 0$ or $j_{(k)} \geq 2$. This establishes (3.16).

□

Lemma 3.1.2. *Suppose that for some $\tilde{\boldsymbol{z}}^*$ and $r_0 > 0, \phi_U(\tilde{\boldsymbol{z}}^*) = 0$ and $\phi_U(\tilde{\boldsymbol{z}}^* + r_0\boldsymbol{g}) = 1$. Then $\phi_U(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$ for all $r > r_0$. This is true both for the one sided alternatives (3.2) and two sided alternatives (3.1) of the variance change problem in this Chapter.*

Proof of Lemma 3.1.2. If $\phi_U(\tilde{\boldsymbol{z}}^*) = 0$ when $\tilde{\boldsymbol{z}}^*$ is observed, the process must stop before H_1 is rejected. Suppose it stops at stage m without having rejected H_1. That means that $U_{m,j_m} < c_m$ which is equivalent to $U_{m,i} < c_m$ for all $i \in \{1, 2, ..., M\} \setminus \{j_1, ..., j_{m-1}\}, j_i \neq 1$. Also $U_{i,j_i} > c_i, i = 1, ..., m-1, j_i \neq 1$. Next consider $\tilde{\boldsymbol{z}}^* + r_0\boldsymbol{g}$ which is a rejecting H_1 point. By Lemma 3.1.1, (3.16) and (3.17) imply that only the function $U_{h,1}$ can change from $\tilde{\boldsymbol{z}}^*$ to $\tilde{\boldsymbol{z}}^* + r_0\boldsymbol{g}$ at each stage $h \leq m$. For some stage s, $s \leq m$, $W_{s,1}$ must have increased to become positive and $U_{s,1}$ become the maximum function at that stage and also be $\geq c_s$. By (3.17) $U_{s,1}(U_{s,1} = W_{s,1}$ for one sided alternatives and $U_{s,1} = (W_{s,1})^2$ for two sided alternatives) will be at least this large for all $r \geq r_0$. Thus H_1 will also be rejected for all $\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}$, $r > r_0$. □

Not that Lemma 3.1.2 implies that the acceptance region in y_1, for fixed $y_2, ..., y_{M+1}$ is an interval.

Theorem 3.1.1. *Both for the one sided alternatives (3.2)and two sided alternatives (3.1), the MRD procedure based on $U_{m,i}$ is admissible.*

Proof. Admissible means that each individual test for each hypothesis testing problem is admissible. Without loss of generality we show admissibility of $\phi_U(\tilde{\boldsymbol{z}})$ for H_1 vs K_1. Proof that the other tests are admissible for the other hypotheses would be done in the same way. That $\phi_U(\tilde{\boldsymbol{z}}^{)}$ is admissible for H_1 vs K_1 follows readily from Lemma 2.1.1 and Lemma 3.1.2. □

3.2 M-LRD

The Maximum-Likelihood Ratio down (M-LRD) method is step-down type MTP too. It's also based on the maximum of a collection of likelihood ratios. Only two-sided alternatives will be addressed here.

3.2.1 M-LRD Procedure

M-LRD procedure calculates M likelihood ratios for the first stage and calculate $(M-1)$ likelihood ratios for the second stage and so on.

M-LRD Procedure:

Let $c_1 > c_2 > \cdots > c_M > 0$ be a given set of constants.

Stage 1: Let $I_1 = \{1, 2, ..., M\}$ be the indices of the hypotheses of (3.7). We test $H_{1G} : \nu_1 = \nu_2 = ... = \nu_M = 0$ vs $K_i^1 : H_{1G}$ but $\nu_i \neq 0$. Let $L_{1,i}$ be the likelihood ratio for H_{1G} vs K_i^1. If $L_{1,j_1} = \max\{L_{1,i} : i \in I_1\} < c_1$, then accept H_{1G} and stop, i.e., there is no change point; Otherwise, reject H_{j_1}, say these is a change point at position j_1, and continue.

Stage 2: Let I_2 be the indices of the hypotheses not previously rejected. Now we test $H_{2G} : \nu_1 = ... = \nu_{j_1-1} = \nu_{j_1+1} = ... = \nu_M = 0$ vs $K_i^2 : H_{2G}$ but $\nu_i \neq 0$, $i \in I_2$. Let $L_{2,i}$ be the likelihood ratio for H_{2G} vs K_i^2. If $L_{2,j_2} = \max\{L_{2,i} : i \in I_2\} < c_2$, then accept H_{2G} and stop; otherwise, reject H_{j_2} and continue.

$\vdots$

In general at stage m: $1 \leq m \leq M$, let I_m be the indices of the hypotheses not previously rejected. Now we test H_{mG} : all the $\nu_i = 0, i \in I_m$ vs $K_i^m : H_{mG}$ but $\nu_i \neq 0$, $i \in I_m$. Let $L_{m,i}$ be the likelihood ratio for H_{mG} vs K_i^m. If $L_{m,j_m} = \max\{L_{m,i} : i \in I_m\} < c_m$, then accept H_{mG} and stop; otherwise, reject H_{j_m} and continue.

3.2.2 Admissibility of M-LRD

For stage m after rejecting $H_{j_1}, H_{j_2}, ...H_{j_{m-1}}$, $I_m = \{1, 2, ..., M\} \setminus \{j_{(1)}, ...j_{(m-1)}\}$, let $(j_{(1)}, ...j_{(m-1)})$ be the ordered sequence of $(j_1, ..., j_{m-1})$.

Then if i is in the range of $(j_{(k)},\ j_{(k+1)})$, where $k = 0, 1, ..., m-1$, with $j_{(0)} = 0, j_{(m)} = M + 1$, testing

$$H_{mG}: \text{ all the } \nu_i = 0, i \in I_m \quad vs \quad K_i^m : H_{mG} \text{ but } \nu_i \neq 0, i \in I_m \tag{3.18}$$

is equivalent to:

$$\begin{aligned} H'_{mG}: \quad & \sigma_1^2 = ... = \sigma_{j_{(1)}}^2 = \sigma_1'^2 \\ & \sigma_{j_{(1)}+1}^2 = ... = \sigma_{j_{(2)}}^2 = \sigma_2'^2 \\ & \cdots \\ & \sigma_{j_{(k)}+1}^2 = ... = \sigma_{j_{(k+1)}}^2 = \sigma_{k+1}'^2 \\ & \cdots \\ & \sigma_{j_{(m-1)}+1}^2 = ... = \sigma_{M+1}^2 = \sigma_m'^2 \end{aligned}$$

vs

$$\begin{aligned} K_i'^m: \quad & H'_{mG} \quad \text{except} \\ & \sigma_{j_{(k)}+1}^2 = ... = \sigma_{j_{(k+1)}}^2 = \sigma_{k+1}'^2 \quad \text{changes to} \\ & \sigma_{j_{(k)+1}}^2 = ... = \sigma_i^2 = \sigma_{k_1}'^2 \quad \text{and} \quad \sigma_{i+1}^2 = ... = \sigma_{j_{(k+1)}}^2 = \sigma_{k_2}'^2 \end{aligned}$$

So under H'_{mG}, the likelihood function of $\mathbf{s^2} = (s_1^2, s_2^2, ..., s_{M+1}^2)'$ is

$$
\begin{aligned}
&L_0(\sigma_1'^2, ..., \sigma_k'^2, ..., \sigma_m'^2) \\
&= \prod_{h=1}^{m} \prod_{t=(j_{(h-1)}+1)}^{j_{(h)}} \left(\frac{(n-1)}{\Gamma(\frac{n-1}{2})2^{(n-1)/2}} \frac{((n-1)s_t^2)^{(\frac{n-1}{2}-1)}}{(\sigma_h'^2)^{\frac{n-1}{2}}} e^{-\frac{(n-1)s_t^2}{2\sigma_h'^2}} \right) \\
&= \left(\prod_{\substack{1\le h\le m \\ \text{and } h\neq(k+1)}} \prod_{t=j_{(h-1)}+1}^{j_h} \left(\frac{(n-1)}{\Gamma(\frac{n-1}{2})2^{(n-1)/2}} \frac{((n-1)s_t^2)^{(\frac{n-1}{2}-1)}}{(\sigma_h'^2)^{\frac{n-1}{2}}} e^{-\frac{(n-1)s_t^2}{2\sigma_h'^2}} \right) \right) \\
&\quad \times \prod_{t=j_{(k)}+1}^{j_{(k+1)}} \left(\frac{(n-1)}{\Gamma(\frac{n-1}{2})2^{(n-1)/2}} \frac{((n-1)s_t^2)^{(\frac{n-1}{2}-1)}}{(\sigma_{k+1}'^2)^{\frac{n-1}{2}}} e^{-\frac{(n-1)s_t^2}{2\sigma_{k+1}'^2}} \right)
\end{aligned}
$$

And under $K_i'^m$, the likelihood function is

$$
\begin{aligned}
&L_1(\sigma_1'^2, ..., \sigma_{k_1}'^2, \sigma_{k_2}'^2, ..., \sigma_m'^2) \\
&= \left(\prod_{\substack{1\le h\le m \\ \text{and } h\neq(k+1)}} \prod_{t=(j_{(h-1)}+1)}^{j_{(h)}} \left(\frac{(n-1)}{\Gamma(\frac{n-1}{2})2^{(n-1)/2}} \frac{((n-1)s_t^2)^{(\frac{n-1}{2}-1)}}{(\sigma_h'^2)^{\frac{n-1}{2}}} e^{-\frac{(n-1)s_t^2}{2\sigma_h'^2}} \right) \right) \\
&\quad \times \prod_{t=j_{(k)}+1}^{i} \left(\frac{(n-1)}{\Gamma(\frac{n-1}{2})2^{(n-1)/2}} \frac{((n-1)s_t^2)^{(\frac{n-1}{2}-1)}}{(\sigma_{k_1}'^2)^{\frac{n-1}{2}}} e^{-\frac{(n-1)s_t^2}{2\sigma_{k_1}'^2}} \right) \\
&\quad \times \prod_{t=i+1}^{j_{(k+1)}} \left(\frac{(n-1)}{\Gamma(\frac{n-1}{2})2^{(n-1)/2}} \frac{((n-1)s_t^2)^{(\frac{n-1}{2}-1)}}{(\sigma_{k_2}'^2)^{\frac{n-1}{2}}} e^{-\frac{(n-1)s_t^2}{2\sigma_{k_2}'^2}} \right)
\end{aligned}
$$

So the likelihood ratio is

$$
\begin{aligned}
L_{m,i} &= \frac{\sup_{\{\sigma_1'^2, ..., \sigma_{k_1}'^2, \sigma_{k_2}'^2, ..., \sigma_m'^2\}} L_1}{\sup_{\{\sigma_1'^2, ..., \sigma_k'^2, ..., \sigma_m'^2\}} L_0} \\
&= \frac{\sup_{\sigma_{k_1}'^2, \sigma_{k_2}'^2} \left(\prod_{t=j_{(k)}+1}^{i} \left(\frac{((n-1)s_t^2)^{(\frac{n-1}{2}-1)}}{(\sigma_{k_1}'^2)^{\frac{n-1}{2}}} e^{-\frac{(n-1)s_t^2}{2\sigma_{k_1}'^2}} \right) \times \prod_{t=i+1}^{j_{(k+1)}} \left(\frac{((n-1)s_t^2)^{(\frac{n-1}{2}-1)}}{(\sigma_{k_2}'^2)^{\frac{n-1}{2}}} e^{-\frac{(n-1)s_t^2}{2\sigma_{k_2}'^2}} \right) \right)}{\sup_{\sigma_{k+1}'^2} \prod_{t=j_{(k)}+1}^{j_{(k+1)}} \left(\frac{((n-1)s_t^2)^{(\frac{n-1}{2}-1)}}{(\sigma_{k+1}'^2)^{\frac{n-1}{2}}} e^{-\frac{(n-1)s_t^2}{2\sigma_{k+1}'^2}} \right)}
\end{aligned}
$$

and the maximum likelihood estimator(mle) of $\sigma_{k+1}'^2, \sigma_{k_1}'^2, \sigma_{k_2}'^2$ are

$$
\hat{\sigma}_{k+1}'^2 = \frac{\sum_{t=j_{(k)}+1}^{j_{(k+1)}} s_t^2}{j_{(k+1)} - j_{(k)}}, \hat{\sigma}_{k_1}'^2 = \frac{\sum_{t=j_{(k)}+1}^{i} s_t^2}{i - j_{(k)}}, \hat{\sigma}_{k_2}'^2 = \frac{\sum_{t=i+1}^{j_{(k+1)}} s_t^2}{j_{(k+1)} - i}
$$

So

$$L_{m,i} = \left(\left(\frac{i - j_{(k)}}{\sum\limits_{t=j_{(k)}+1}^{i} s_t^2} \right)^{i-j_{(k)}} \left(\frac{j_{(k+1)} - i}{\sum\limits_{t=i+1}^{j_{(k+1)}} s_t^2} \right)^{j_{(k+1)}-i} \left(\frac{\sum\limits_{t=j_{(k)}+1}^{j_{(k+1)}} s_t^2}{j_{(k+1)} - j_{(k)}} \right)^{j_{(k+1)}-j_{(k)}} \right)^{\frac{(n-1)}{2}} \tag{3.19}$$

Since $\tilde{z}_i$ is defined as $\tilde{z}_i = (n-1)s_i^2$, then

$$L_{m,i} = \left(\left(\frac{i - j_{(k)}}{\sum\limits_{t=j_{(k)}+1}^{i} \tilde{z}_t} \right)^{i-j_{(k)}} \left(\frac{j_{(k+1)} - i}{\sum\limits_{t=i+1}^{j_{(k+1)}} \tilde{z}_t} \right)^{j_{(k+1)}-i} \left(\frac{\sum\limits_{t=j_{(k)}+1}^{j_{(k+1)}} \tilde{z}_t}{j_{(k+1)} - j_{(k)}} \right)^{j_{(k+1)}-j_{(k)}} \right)^{\frac{(n-1)}{2}} \tag{3.20}$$

Lemma 3.2.1. *The function $L_{m,j}(\tilde{\boldsymbol{z}})$ given in (3.20) has the following properties:*

(1) At any stage m, as far as H_1 has not been rejected, then for any $i \neq 1$, i.e., $i \in \{2, ..., M\} \setminus \{j_1, ..., j_{m-1}\}$, $j_1 \neq 1, ..., j_{m-1} \neq 1$,

$$L_{m,i}(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = L_{m,i}(\tilde{\boldsymbol{z}}) \tag{3.21}$$

for any $r > 0$.

(2) For $i = 1$, regard $L_{m,1}(\tilde{\boldsymbol{z}} + r\boldsymbol{g})$ as a function of r, then:

If for any $0 < r_1 < r_2$, $L_{m,1}(\tilde{\boldsymbol{z}} + r_2\boldsymbol{g}) > L_{m,1}(\tilde{\boldsymbol{z}} + r_1\boldsymbol{g})$, then for any $r > r_2$, $L_{m,1}(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) > L_{m,1}(\tilde{\boldsymbol{z}} + r_2\boldsymbol{g})$.

Proof of Lemma 3.2.1. Use (3.20) and recall $\boldsymbol{g} = (1, -1, 0, ..., 0)'$ is the first row of A.

For $i \neq 1$: if i falls into the range $0 < i < j_{(1)}$, then

$$
\begin{aligned}
&L_{m,i}(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) \\
&= \left(\left(\frac{i}{(\tilde{z}_1 + r) + (\tilde{z}_2 - r) + \sum\limits_{t=3}^{i} \tilde{z}_t} \right)^{i} \left(\frac{j_{(1)} - i}{\sum\limits_{t=i+1}^{j_{(1)}} \tilde{z}_t} \right)^{j_{(1)}-i} \left(\frac{(\tilde{z}_1 + r) + (\tilde{z}_2 - r) + \sum\limits_{t=3}^{j_{(1)}} \tilde{z}_t}{j_{(1)}} \right)^{j_{(1)}} \right)^{\frac{(n-1)}{2}} \\
&= \left(\left(\frac{i}{\sum\limits_{t=1}^{i} \tilde{z}_t} \right)^{i} \left(\frac{j_{(1)} - i}{\sum\limits_{t=i+1}^{j_{(1)}} \tilde{z}_t} \right)^{j_{(1)}-i} \left(\frac{\sum\limits_{t=1}^{j_{(1)}} \tilde{z}_t}{j_{(1)}} \right)^{j_{(1)}} \right)^{\frac{(n-1)}{2}} \\
&= L_{m,i}(\tilde{\boldsymbol{z}})
\end{aligned}
$$

if i falls into the range $j_{(k)} < i < j_{(k+1)}$ and $j_{(k)} \neq 0$, since $j_{(k)} \geq 2 \Longrightarrow i \geq 3 \Longrightarrow \tilde{z}_i + rg_i = \tilde{z}_i$, then it's obvious that

$$L_{m,i}(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = L_{m,i}(\tilde{\boldsymbol{z}})$$

This establishes (3.21).

For $i = 1$,

$$L_{m,1}(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = \left(\left(\frac{1}{\tilde{z}_1 + r} \right) \left(\frac{j_{(1)} - 1}{\sum\limits_{t=2}^{j_{(1)}} \tilde{z}_t - r} \right)^{j_{(1)}-1} \left(\frac{\sum\limits_{t=1}^{j_{(1)}} \tilde{z}_t}{j_{(1)}} \right)^{j_{(1)}} \right)^{\frac{(n-1)}{2}}$$

Let

$$
\begin{aligned}
l_{m,1}(r) &= \log\{L_{m,1}(\tilde{\boldsymbol{z}} + r\boldsymbol{g})\} \\
&= \frac{(n-1)}{2} \left(-\log(\tilde{z}_1 + r) + (j_{(1)} - 1) \log \left(\frac{j_{(1)} - 1}{\sum\limits_{t=2}^{j_{(1)}} \tilde{z}_t - r} \right) + j_{(1)} \log \left(\frac{\sum\limits_{t=1}^{j_{(1)}} \tilde{z}_t}{j_{(1)}} \right) \right)
\end{aligned}
$$

Now take derivative of $l_{m,1}(r)$ with respect to r

$$\frac{\mathrm{d}l_{m,1}(r)}{\mathrm{d}r} = \frac{(n-1)}{2} \left(-\frac{1}{\tilde{z}_1 + r} + (j_{(1)} - 1) \frac{1}{\sum\limits_{t=2}^{j_{(1)}} \tilde{z}_t - r} \right)$$

So as r increases, $r > 0$, $\frac{\mathrm{d}l_{m,1}(r)}{\mathrm{d}r}$ increases $\Longrightarrow$ once $\frac{\mathrm{d}l_{m,1}(r)}{\mathrm{d}r}$ becomes positive, it will stay positive $\Longrightarrow$ Once $L_{m,1}(\tilde{\boldsymbol{z}} + r\boldsymbol{g})$ increases, it will keep increasing.

□

Lemma 3.2.2. *Suppose that for some* $\tilde{\boldsymbol{z}}^*$ *and* $r_0 > 0, \phi_L(\tilde{\boldsymbol{z}}^*) = 0$ *and* $\phi_L(\tilde{\boldsymbol{z}}^* + r_0\boldsymbol{g}) = 1$. *Then* $\phi_L(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$ *for all* $r > r_0$.

Proof. Same as proof of Lemma 3.1.2. □

Theorem 3.2.1. *For the two sided case the M-LRD procedure based on* $L_{m,i}$ *is admissible.*

Proof. Same as proof of Theorem 3.1.1. □

3.3 Likelihood Ratio Step-Down Method(LRSD)

Similar to the mean case in Chapter 2, for one-sided variance change case, the LRSD method is as following:

Let $c_1 > c_2 > \cdots > c_M > 1$ be a given set of constants. At Stage 1: Let $I_1 = \{1, 2, ..., M\}$ be the indices of the hypotheses of (3.8). We test $H_{1G} : \boldsymbol{\nu} = \mathbf{0}$ vs $K_{1G} : \boldsymbol{\nu} \geq \mathbf{0}$ and at least one $\nu_i > 0$, $i \in I_1$. The likelihood ratio for this test is L_1. If $L_1 < c_1$, then accept H_{1G} and stop; Otherwise, reject H_{j_1} where j_1 is the index for which $F_{j_1} = \max\{F_j : j \in I_1\}$,where

$$F_j = \frac{s_j^2}{s_{j+1}^2} = \frac{\tilde{z}_j}{\tilde{z}_{j+1}} \tag{3.22}$$

and continue similarly for the hypotheses not rejected.

In general, the Stage m global hypothesis is considered if $H_{j_1}, ..., H_{j_{m-1}}$ have been rejected. This global testing problem is $H_{mG} : \boldsymbol{\nu}^{(j_1,...,j_{m-1})} = \mathbf{0}$ vs $K_{mG} : \boldsymbol{\nu}^{(j_1,...,j_{m-1})} \geq \mathbf{0}$ but at least one $\nu_i > 0$, $i \in I_m$, where I_m is the indices of the hypotheses not previously rejected. The likelihood ratio test rejects H_{mG} if $L_m \geq c_m$, i.e

$$L_m = \frac{\sup\limits_{\{\sigma_i^2 \geq \sigma_{i+1}^2, i \in I_m\}} \prod_{i=1}^{M+1} (\frac{1}{\sigma_i^2})^{\frac{n-1}{2}} e^{-\frac{(n-1)s_i^2}{2\sigma_i^2}}}{\sup\limits_{\{\sigma_i^2 = \sigma_{i+1}^2, i \in I_m\}} \prod_{i=1}^{M+1} (\frac{1}{\sigma_i^2})^{\frac{n-1}{2}} e^{-\frac{(n-1)s_i^2}{2\sigma_i^2}}}$$

$$= \frac{\sup\limits_{\{\sigma_i^2 \geq \sigma_{i+1}^2, i \in I_m\}} \prod_{i=1}^{M+1} (\frac{1}{\sigma_i^2})^{\frac{n-1}{2}} e^{-\frac{(n-1)s_i^2}{2\sigma_i^2}}}{\sup\limits_{\{\sigma_1'^2, \sigma_2'^2, ..., \sigma_m'^2\}} \left(\prod_{i=1}^{j_{(1)}} (\frac{1}{\sigma_1'^2})^{\frac{n-1}{2}} e^{-\frac{(n-1)s_i^2}{2\sigma_1'^2}} \right) \left(\prod_{i=j_{(1)}+1}^{j_{(2)}} (\frac{1}{\sigma_2'^2})^{\frac{n-1}{2}} e^{-\frac{(n-1)s_i^2}{2\sigma_2'^2}} \right) \cdots \left(\prod_{i=j_{(m-1)}+1}^{M+1} (\frac{1}{\sigma_m'^2})^{\frac{n-1}{2}} e^{-\frac{(n-1)s_i^2}{2\sigma_m'^2}} \right)} \tag{3.23}$$

$\geq c_m$

For the denominator the maximum likelihood estimators are: $\hat{\sigma}_1'^2 = \frac{\sum_1^{j_{(1)}} s_j^2}{j_{(1)}}, \hat{\sigma}_2'^2 = \frac{\sum_{j_{(1)}+1}^{j_{(2)}} s_j^2}{j_{(2)} - j_{(1)}}, ..., \hat{\sigma}_m'^2 = \frac{\sum_{j_{(m-1)}+1}^{M+1} s_j^2}{M - j_{(m-1)}+1}$, replace $\sigma_1'^2, \sigma_2'^2, ..., \sigma_m'^2$ with them in (3.23), we get:

$$\begin{aligned}
&(3.23)\\
&= e^{\frac{(M+1)(n-1)}{2}}\\
&\times \left(\left(\frac{\sum_1^{j_{(1)}} s_j^2}{j_{(1)}} \right)^{j_1} \left(\frac{\sum_{j_{(1)}+1}^{j_{(2)}} s_j^2}{j_{(2)} - j_{(1)}} \right)^{j_{(2)} - j_{(1)}} \cdots \left(\frac{\sum_{j_{(m-1)}+1}^{M+1} s_j^2}{M - j_{(m-1)} + 1} \right)^{M - j_{(m-1)}+1} \right)^{(n-1)/2}\\
&\times \left(\sup_{\{\sigma_i^2 \geq \sigma_{i+1}^2, i \in I_m\}} \prod_{i=1}^{M+1} (\frac{1}{\sigma_i^2}) e^{-\frac{s_i^2}{\sigma_i^2}} \right)^{(n-1)/2}
\end{aligned} \tag{3.24}$$

Define

L'_m

$$= \left(\frac{\sum_{1}^{j_{(1)}} s_j^2}{j_{(1)}}\right)^{j_{(1)}} \left(\frac{\sum_{j_{(1)}+1}^{j_{(2)}} s_j^2}{j_{(2)}-j_{(1)}}\right)^{j_{(2)}-j_{(1)}} \cdots \left(\frac{\sum_{j_{(m-1)}+1}^{M+1} s_j^2}{M-j_{(m-1)}+1}\right)^{M-j_{(m-1)}+1}$$

$$\times \sup_{\{\sigma_i^2 \geq \sigma_{i+1}^2, i \in I_m\}} \prod_{i=1}^{M+1} (\frac{1}{\sigma_i^2}) e^{-\frac{s_i^2}{\sigma_i^2}}$$

$$= \left(\frac{\sum_{1}^{j_{(1)}} \tilde{z}_j}{(n-1)j_{(1)}}\right)^{j_{(1)}} \left(\frac{\sum_{j_{(1)}+1}^{j_{(2)}} \tilde{z}_j}{(n-1)(j_{(2)}-j_{(1)})}\right)^{j_{(2)}-j_{(1)}} \cdots \left(\frac{\sum_{j_{(m-1)}+1}^{M+1} \tilde{z}_j}{(n-1)(M-j_{(m-1)}+1)}\right)^{M-j_{(m-1)}+1}$$

$$\times \sup_{\{\sigma_i^2 \geq \sigma_{i+1}^2, i \in I_m\}} \prod_{i=1}^{M+1} (\frac{1}{\sigma_i^2}) e^{-\frac{\tilde{z}_i}{(n-1)\sigma_i^2}}$$

$$= \left(\frac{\sum_{1}^{j_{(1)}} \tilde{z}_j}{(n-1)j_{(1)}}\right)^{j_{(1)}} \left(\frac{\sum_{j_{(1)}+1}^{j_{(2)}} \tilde{z}_j}{(n-1)(j_{(2)}-j_{(1)})}\right)^{j_{(2)}-j_{(1)}} \cdots \left(\frac{\sum_{j_{(m-1)}+1}^{M+1} \tilde{z}_j}{(n-1)(M-j_{(m-1)}+1)}\right)^{M-j_{(m-1)}+1}$$

$$\times \prod_{i=1}^{M+1} (\frac{1}{\hat{\sigma}_i^2}) e^{-\frac{\tilde{z}_i}{(n-1)\hat{\sigma}_i^2}} \tag{3.25}$$

where $\hat{\sigma}_i^2$ is the maximum likelihood estimator of σ_i^2 when $\tilde{\boldsymbol{z}} = \tilde{\boldsymbol{z}}$. Thus $L_m \geq c_m \Longleftrightarrow L'_m \geq C_m$, where $c_m = e^{\frac{(M+1)(n-1)}{2}} \times C_m^{(n-1)/2}$.

Lemma 3.3.1. *When* $\tilde{\boldsymbol{z}}^* = \tilde{\boldsymbol{z}} + r\boldsymbol{g} = \begin{pmatrix} \tilde{z}_1+r \\ \tilde{z}_2-r \\ \tilde{z}_3 \\ \cdots \\ \tilde{z}_{M+1} \end{pmatrix}$, *if* $j_{(1)} > 1$, *i.e.*H_1 *has not been rejected,* $L_m^{*'} \geq L'_m$.

Proof of Lemma 3.3.1. From (3.25),

$L_m^{*'}$

$$= \left(\frac{\sum_{1}^{j_{(1)}} \tilde{z}_j^*}{(n-1)j_{(1)}}\right)^{j_{(1)}} \left(\frac{\sum_{j_{(1)}+1}^{j_{(2)}} \tilde{z}_j^*}{(n-1)(j_{(2)}-j_{(1)})}\right)^{j_{(2)}-j_{(1)}} \cdots \left(\frac{\sum_{j_{(m-1)}+1}^{M+1} \tilde{z}_j^*}{(n-1)(M-j_{(m-1)}+1)}\right)^{M-j_{(m-1)}+1}$$

$$\times \sup_{\{\sigma_i^2 \geq \sigma_{i+1}^2, i \in I_m\}} \prod_{i=1}^{M+1} (\frac{1}{\sigma_i^2}) e^{-\frac{\tilde{z}_i^*}{(n-1)\sigma_i^2}}$$

since $j_{(1)} > 1 \Longrightarrow$

$L_m^{*'}$

$$= \left(\frac{\sum_{1}^{j_{(1)}} \tilde{z}_j}{(n-1)j_{(1)}} \right)^{j_{(1)}} \left(\frac{\sum_{j_{(1)}+1}^{j_{(2)}} \tilde{z}_j}{(n-1)(j_{(2)} - j_{(1)})} \right)^{j_{(2)} - j_{(1)}} \cdots \left(\frac{\sum_{j_{(m-1)}+1}^{M+1} \tilde{z}_j}{(n-1)(M - j_{(m-1)} + 1)} \right)^{M - j_{(m-1)} + 1}$$

$$\times \sup_{\{\sigma_i^2 \geq \sigma_{i+1}^2, i \in I_m\}} \prod_{i=1}^{M+1} (\frac{1}{\sigma_i^2}) e^{-\frac{\tilde{z}_i^*}{(n-1)\sigma_i^2}}$$

$$\geq \left(\frac{\sum_{1}^{j_{(1)}} \tilde{z}_j}{(n-1)j_{(1)}} \right)^{j_{(1)}} \left(\frac{\sum_{j_{(1)}+1}^{j_{(2)}} \tilde{z}_j}{(n-1)(j_{(2)} - j_{(1)})} \right)^{j_{(2)} - j_{(1)}} \cdots \left(\frac{\sum_{j_{(m-1)}+1}^{M+1} \tilde{z}_j}{(n-1)(M - j_{(m-1)} + 1)} \right)^{M - j_{(m-1)} + 1}$$

$$\times \prod_{i=1}^{M+1} (\frac{1}{\hat{\sigma}_i^2}) e^{-\frac{\tilde{z}_i^*}{(n-1)\hat{\sigma}_i^2}}$$

$$= \left(\frac{\sum_{1}^{j_{(1)}} \tilde{z}_j}{(n-1)j_{(1)}} \right)^{j_{(1)}} \left(\frac{\sum_{j_{(1)}+1}^{j_{(2)}} \tilde{z}_j}{(n-1)(j_{(2)} - j_{(1)})} \right)^{j_{(2)} - j_{(1)}} \cdots \left(\frac{\sum_{j_{(m-1)}+1}^{M+1} \tilde{z}_j}{(n-1)(M - j_{(m-1)} + 1)} \right)^{M - j_{(m-1)} + 1}$$

$$\times (\frac{1}{\hat{\sigma}_1^2}) e^{-\frac{(\tilde{z}_1 + r)}{(n-1)\hat{\sigma}_1^2}} (\frac{1}{\hat{\sigma}_2^2}) e^{-\frac{(\tilde{z}_2 - r)}{(n-1)\hat{\sigma}_2^2}} \prod_{i=3}^{M+1} (\frac{1}{\hat{\sigma}_i^2}) e^{-\frac{\tilde{z}_i}{(n-1)\hat{\sigma}_i^2}}$$

$$= e^{\frac{r}{n-1}(\frac{1}{\hat{\sigma}_2^2} - \frac{1}{\hat{\sigma}_1^2})} \times L'_m$$

where $\hat{\sigma}_i^2$ is the maximum likelihood estimator of σ_i^2 when $\tilde{\boldsymbol{z}} = \tilde{\boldsymbol{z}}$.

Thus $L_m^{*'} \geq L'_m$. Since $\hat{\sigma}_1^2 \geq \hat{\sigma}_2^2$. □

Theorem 3.3.1. *For the one-sided alternatives (3.2) LRSD is admissible for M=2 and M=3.*

Proof of Theorem 3.3.1. We proof the theorem for M=3. For M=2 the method is the same and the proof is simpler. Once again we focus on H_1 *vs* K_1^* and demonstrate that if $\phi(\tilde{\boldsymbol{z}}^*) = 1$ then $\phi(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$ for all $r > 0$. H_1 can be rejected at three different stages:

(1) If H_1 is rejected at stage 1 for $\tilde{\boldsymbol{z}} = \tilde{\boldsymbol{z}}^* \Longrightarrow F_1^* > F_2^*, F_1^* > F_3^*$ and $L_1^{*'} \geq C_1$. When at $\tilde{\boldsymbol{z}}^{**} = \tilde{\boldsymbol{z}}^* + r\boldsymbol{g}$, $F_1^{**} = \frac{\tilde{z}_1^*+r}{\tilde{z}_2^*-r} > F_1^* = \frac{\tilde{z}_1^*}{\tilde{z}_2^*}$, $F_2^{**} = \frac{\tilde{z}_2^*-r}{\tilde{z}_3^*} < F_2^* = \frac{\tilde{z}_2^*}{\tilde{z}_3^*}$, $F_3^{**} = \frac{\tilde{z}_3^*}{\tilde{z}_4^*} = F_3^*$ and from Lemma 3.3.1, we know that $L_1^{**'} \geq L_1^{*'} \geq C_1$, so $\phi_1(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$ too, for all $r > 0$.

(2) If H_1 is rejected at stage 2 for $\tilde{\boldsymbol{z}} = \tilde{\boldsymbol{z}}^*$,

when H_3 is rejected first $\Longrightarrow F_3^* > F_1^* > F_2^*$ and $L_1^{*'} \geq C_1, L_2^{*'} \geq C_2$. When at $\tilde{\boldsymbol{z}}^{**} = \tilde{\boldsymbol{z}}^* + r\boldsymbol{g}$, we know that $F_1^{**} > F_1^*$, $F_2^{**} < F_2^*$, $F_3^{**} = F_3^*$, if the ordering of $\boldsymbol{F}^{**}$ changes to $F_1^{**} > F_3^{**} > F_2^{**}$, by Lemma 3.3.1 that $L_1^{**'} \geq L_1^{*'} \geq C_1$, so $\phi_1(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$ too, for all $r > 0$; if the ordering of $\boldsymbol{F}^{**}$ keeps unchanged,i.e., $F_3^{**} > F_1^{**} > F_2^{**}$, also by Lemma 3.3.1 that $L_1^{**'} \geq L_1^{*'} \geq C_1, L_2^{**'} \geq L_2^{*'} \geq C_2$, so $\phi_1(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$ too, for all $r > 0$;

when H_2 is rejected first, a similar argument works too.

(3)If H_1 is rejected at stage 3 at $\tilde{\boldsymbol{z}} = \tilde{\boldsymbol{z}}^*$,

when H_3 is rejected first $\Longrightarrow F_3^* > F_2^* > F_1^* > 1$ (since if $F_1^* < 1$, it can be proved that $L_3^* < 1 < c_3$, thus H_1 can't be rejected on at stage 3) and $L_1^{*'} \geq C_1$, $L_2^{*'} \geq C_2$, $L_3^{*'} \geq C_3$. If the ordering of $\boldsymbol{F}^{**}$ keeps unchanged, similar argument like above assures that $\phi_1(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$, for all $r > 0$; If the ordering of $\boldsymbol{F}^{**}$ changes to: $F_3^{**} > F_1^{**} > F_2^{**}$ or $F_1^{**} > F_3^{**} > F_2^{**}$; no matter for which case, Lemma 3.3.1 assures that $L_1^{**'} \geq L_1^{*'} \geq C_1$ for both cases and $L_2^{**'} \geq L_2^{*'} > C_2$ for the first case, thus $\phi_1(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$ too, for all $r > 0$.

when H_2 is rejected first $\Longrightarrow F_2^* > F_3^* > F_1^* > 1$ and $L_1^{*'} \geq C_1$, $L_2^{*'} \geq C_2$, $L_3^{*'} \geq C_3$. If the ordering of $\boldsymbol{F}^{**}$ keeps unchanged, it's not difficult to verify that $\phi_1(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$, for all $r > 0$ by using the Lemma 3.3.1; If the ordering of $\boldsymbol{F}^{**}$ changes to: $F_2^{**} > F_1^{**} > F_3^{**}$ or $F_1^{**} > F_2^{**} > F_3^{**}$ or $F_1^{**} > F_3^{**} > F_2^{**}$, using the similar argument we can verify that $\phi_1(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$ too, for all $r > 0$; The most difficult subcases are: if the ordering of $\boldsymbol{F}^{**}$ changes to: $F_3^{**} > F_2^{**} > F_1^{**}$ and $F_3^{**} > F_1^{**} > F_2^{**}$. For these two cases, Lemma 3.3.1 assures that $L_1^{**'} \geq C_1$. So H_3 is rejected first.

For $\tilde{\boldsymbol{z}}^*$ at stage 2: by (3.25),

$$L_2^{*'} = \left(\frac{\tilde{z}_1^* + \tilde{z}_2^*}{2(n-1)}\right)^2 \left(\frac{\tilde{z}_3^* + \tilde{z}_4^*}{2(n-1)}\right)^2 \times \sup_{\substack{\sigma_1^2 \geq \sigma_2^2 \\ \sigma_3^2 \geq \sigma_4^2}} \prod_{i=1}^4 (\frac{1}{\sigma_i^2}) e^{-\frac{\tilde{z}_i^*}{(n-1)\sigma_i^2}} \tag{3.26}$$

Since $F_2^* > F_3^* > F_1^* > 1 \Longrightarrow \tilde{z}_1^* > \tilde{z}_2^* > \tilde{z}_3^* > \tilde{z}_4^* \Longrightarrow \hat{\sigma}_i^{*2} = \frac{\tilde{z}_i^*}{n-1} \Longrightarrow$

$$L_2^{*'} = \left(\frac{\tilde{z}_1^* + \tilde{z}_2^*}{2}\right)^2 \left(\frac{\tilde{z}_3^* + \tilde{z}_4^*}{2}\right)^2 \frac{1}{\tilde{z}_1^*}\frac{1}{\tilde{z}_2^*}\frac{1}{\tilde{z}_3^*}\frac{1}{\tilde{z}_4^*} e^{-4} \tag{3.27}$$

For $\tilde{\boldsymbol{z}}^{**}$ at stage 2: by (3.25),

$$L_2^{*'} = \left(\frac{\tilde{z}_1^{**} + \tilde{z}_2^{**} + \tilde{z}_3^{**}}{3(n-1)}\right)^3 \frac{\tilde{z}_4^{**}}{(n-1)} \times \sup_{\substack{\sigma_1^2 \geq \sigma_2^2 \geq \sigma_3^2 \\ \sigma_4^2}} \prod_{i=1}^4 (\frac{1}{\sigma_i^2}) e^{-\frac{\tilde{z}_i^{**}}{(n-1)\sigma_i^2}} \tag{3.28}$$

Since $F_1^{**} = \frac{\tilde{z}_1^*+r}{\tilde{z}_2^*-r} < F_3^{**} = \frac{\tilde{z}_3^*}{\tilde{z}_4^*} \Longrightarrow r < \frac{\tilde{z}_2^*\tilde{z}_3^* - \tilde{z}_1^*\tilde{z}_4^*}{\tilde{z}_3^*+\tilde{z}_4^*} \Longrightarrow F_2^{**} = \frac{\tilde{z}_2^*-r}{\tilde{z}_3^*} > \frac{\tilde{z}_2^* - \frac{\tilde{z}_2^*\tilde{z}_3^* - \tilde{z}_1^*\tilde{z}_4^*}{\tilde{z}_3^*+\tilde{z}_4^*}}{\tilde{z}_3^*} = \frac{\tilde{z}_2^*\tilde{z}_4^* + \tilde{z}_1^*\tilde{z}_4^*}{\tilde{z}_3^*(\tilde{z}_3^*+\tilde{z}_4^*)}$ since $\tilde{z}_1^* > \tilde{z}_2^* > \tilde{z}_3^* > \tilde{z}_4^* \Longrightarrow F_2^{**} > \frac{2\tilde{z}_2^*\tilde{z}_4^*}{2\tilde{z}_3^*\tilde{z}_3^*}$ since $F_2^* > F_3^*$ i.e., $\frac{\tilde{z}_2^*}{\tilde{z}_3^*} > \frac{\tilde{z}_3^*}{\tilde{z}_4^*} \Longrightarrow$ $F_2^{**} > 1$ and $F_3^{**} > F_1^{**} > F_1^* > 1 \Longrightarrow \tilde{z}_1^{**} > \tilde{z}_2^{**} > \tilde{z}_3^{**} > \tilde{z}_4^{**} \Longrightarrow \hat{\sigma}_i^{**2} = \frac{\tilde{z}_i^{**}}{n-1} \Longrightarrow$

$$\begin{aligned} L_2^{**'} &= \left(\frac{\tilde{z}_1^{**} + \tilde{z}_2^{**} + \tilde{z}_3^{**}}{3}\right)^3 \tilde{z}_4^{**} \frac{1}{\tilde{z}_1^{**}}\frac{1}{\tilde{z}_2^{**}}\frac{1}{\tilde{z}_3^{**}}\frac{1}{\tilde{z}_4^{**}} e^{-4} \\ &= \left(\frac{\tilde{z}_1^* + \tilde{z}_2^* + \tilde{z}_3^*}{3}\right)^3 \tilde{z}_4^* \frac{1}{(\tilde{z}_1^*+r)}\frac{1}{(\tilde{z}_2^*-r)}\frac{1}{\tilde{z}_3^*}\frac{1}{\tilde{z}_4^*} e^{-4} \end{aligned} \tag{3.29}$$

If we can prove $L_2^{**'} \geq L_2^{*'}$, then if $F_1^{**} > F_2^{**}$ we reject H_1 at second stage; if $F_2^{**} > F_1^{**}$, by Lemma 3.3.1 we know that $L_3^{**'} > L_3^{*'} \geq C_3$, thus we reject H_1 at the third stage. Thus $\phi_1(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$, for all $r > 0$.

So in order to prove (3.29) $\geq$ (3.27), we want to prove

$$\frac{1}{(\tilde{z}_1^*+r)}\frac{1}{(\tilde{z}_2^*-r)} \geq \frac{1}{\tilde{z}_1^*}\frac{1}{\tilde{z}_2^*} \tag{3.30}$$

and

$$\left(\frac{\tilde{z}_1^* + \tilde{z}_2^* + \tilde{z}_3^*}{3}\right)^3 \tilde{z}_4^* \geq \left(\frac{\tilde{z}_1^* + \tilde{z}_2^*}{2}\right)^2 \left(\frac{\tilde{z}_3^* + \tilde{z}_4^*}{2}\right)^2 \tag{3.31}$$

For (3.30), $(\tilde{z}_1^*+r)(\tilde{z}_2^*-r) = -(r - \frac{(\tilde{z}_2^*-\tilde{z}_1^*)}{2})^2 + (\frac{(\tilde{z}_2^*-\tilde{z}_1^*)}{2})^2 + \tilde{z}_1^*\tilde{z}_2^* \leq \tilde{z}_1^*\tilde{z}_2^*$ since $\tilde{z}_1^* > \tilde{z}_2^*$ and $r \geq 0$.

For (3.31), let

$$\begin{aligned} f &= \left(\left(\frac{\tilde{z}_1^* + \tilde{z}_2^* + \tilde{z}_3^*}{3}\right)^3 \tilde{z}_4^* - \left(\frac{\tilde{z}_1^* + \tilde{z}_2^*}{2}\right)^2 \left(\frac{\tilde{z}_3^* + \tilde{z}_4^*}{2}\right)^2\right) / \tilde{z}_4^{*4} \\ &= \left(\frac{D^* + F_3^*}{3}\right)^3 - \left(\frac{D^*}{2}\right)^2 \left(\frac{F_3^* + 1}{2}\right)^2 \end{aligned} \tag{3.32}$$

where $D^* = \frac{\tilde{z}_1^*+\tilde{z}_2^*}{\tilde{z}_4^*}$. Since $F_2^* > F_3^* > F_1^* > 1$, i.e., $\frac{\tilde{z}_2^*}{\tilde{z}_3^*} > \frac{\tilde{z}_3^*}{\tilde{z}_4^*} > \frac{\tilde{z}_1^*}{\tilde{z}_2^*}$ and $D > 2\frac{\tilde{z}_2^*}{\tilde{z}_4^*} \Longrightarrow D^* > 2F_3^{*2}$.

Now we think f as a function of D^* only. In order to prove $f > 0$, we need to prove that (1) $f(D^* = 2F_3^{*2}) > 0$ and (2)$\frac{df}{dD^*} > 0$, for $D^* > 2F_3^{*2}$.

For (1), $f(D^* = 2F_3^{*2}) = \left(\frac{2F_3^{*2}+F_3^*}{3}\right)^3 - \left(\frac{2F_3^{*2}}{2}\right)^2\left(\frac{F_3^*+1}{2}\right)^2 = \frac{F_3^{*3}}{108}(F_3^*-1)^2(5F_3^*+4) > 0$, since $F_3^* > 1$.

For (2), $\frac{df}{dD^*} = \left(\frac{D^*+F_3^*}{3}\right)^2 - \frac{D^*}{2}\left(\frac{F_3^*+1}{2}\right)^2 = \frac{1}{9}\{D^{*2} - \frac{(9F_3^{*2}+2F_3^*+9)}{8}D^* + F_3^{*2}\}$ which is a function of D^*, whose graph is open upward and symmetric with $\frac{(9F_3^{*2}+2F_3^*+9)}{16}$. It's not difficult to verify that $2F_3^{*2} > \frac{(9F_3^{*2}+2F_3^*+9)}{16}$ by using $F_3^* > 1$. And at $D^* = 2F_3^{*2}$, $\frac{df}{dD^*} = \frac{F_3^{*2}}{36}(F_3^*-1)(7F_3^*+5) > 0$, since $F_3^* > 1$. Thus $\frac{df}{dD^*} > 0$, for $D^* > 2F_3^{*2}$.

Combine (1) and (2), we know that $f > 0$. Thus (3.31) holds.

□

For M=4 we exhibit a set of critical values for which LRSD is inadmissible. To do so we find a sample point $\tilde{\boldsymbol{z}}$ at which H_1 is rejected and for which H_1 is accepted at $\tilde{\boldsymbol{z}} + r\boldsymbol{g}$. In fact let $\tilde{\boldsymbol{z}} = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4, \tilde{z}_5)'$ for $\frac{\tilde{z}_2}{\tilde{z}_3} > \frac{\tilde{z}_3}{\tilde{z}_4} > \frac{\tilde{z}_4}{\tilde{z}_5} > \frac{\tilde{z}_1}{\tilde{z}_2} > 1$, i.e., $F_2 > F_3 > F_4 > F_1 > 1$. Thus using (3.25) at stage 1 choose C_1 so that $L_1' = \left(\frac{\sum_{j=1}^5 \tilde{z}_j}{5(n-1)}\right)^5 \times \sup_{\{\sigma_1^2\ge\sigma_2^2\ge\sigma_3^2\ge\sigma_4^2\ge\sigma_5^2\}} \prod_{i=1}^5 (\frac{1}{\sigma_i^2})e^{-\frac{\tilde{z}_i}{(n-1)\sigma_i^2}} \ge C_1$ so that H_2 is rejected. At stage 2 we calculate

$$L_2' = \left(\frac{\sum_{j=1}^2 \tilde{z}_j}{2(n-1)}\right)^2 \left(\frac{\sum_{j=3}^5 \tilde{z}_j}{3(n-1)}\right)^3 \times \sup_{\{\sigma_1^2\ge\sigma_2^2,\sigma_3^2\ge\sigma_4^2\ge\sigma_5^2\}} \prod_{i=1}^5 (\frac{1}{\sigma_i^2})e^{-\frac{\tilde{z}_i}{(n-1)\sigma_i^2}} \tag{3.33}$$

Since $\tilde{z}_1 > \tilde{z}_2 > \tilde{z}_3 > \tilde{z}_4 > \tilde{z}_5 \Longrightarrow \hat{\sigma}_i^2 = \frac{\tilde{z}_i}{n-1} \Longrightarrow$

$$L_2' = \left(\frac{\sum_{j=1}^2 \tilde{z}_j}{2}\right)^2 \left(\frac{\sum_{j=3}^5 \tilde{z}_j}{3}\right)^3 \frac{1}{\tilde{z}_1}\frac{1}{\tilde{z}_2}\frac{1}{\tilde{z}_3}\frac{1}{\tilde{z}_4}\frac{1}{\tilde{z}_5} \times e^{-5} \tag{3.34}$$

We set $\left(\frac{\sum_{j=1}^2 \tilde{z}_j}{2}\right)^2 \left(\frac{\sum_{j=3}^5 \tilde{z}_j}{3}\right)^3 \frac{1}{\tilde{z}_1}\frac{1}{\tilde{z}_2}\frac{1}{\tilde{z}_3}\frac{1}{\tilde{z}_4}\frac{1}{\tilde{z}_5} \times e^{-5} = C_2$. At stage 3, H_4 is rejected and at stage 4, H_1 is rejected.

Now at $\tilde{\boldsymbol{z}}^{\boldsymbol{*}} = \tilde{\boldsymbol{z}} + r\boldsymbol{g}$, let r such that $\frac{\tilde{z}_3}{\tilde{z}_4} > \frac{\tilde{z}_2 - r}{\tilde{z}_3} > 1$ and $\frac{\tilde{z}_3}{\tilde{z}_4} > \frac{\tilde{z}_1 + r}{\tilde{z}_2 - r}$, i.e., $F_3^* > F_2^*$ and $F_3^* > F_1^*$. Note that by Lemma 3.3.1, we know $L_1^{*'} \geq L_1' \geq C_1$. This time however, H_3 is rejected at stage 1. At stage 2 we calculate at $\tilde{\boldsymbol{z}}^{\boldsymbol{*}}$,

$$\begin{aligned} L_2^{*'} &= \left(\frac{\sum_{j=1}^{3} \tilde{z}_j^*}{3(n-1)} \right)^3 \left(\frac{\sum_{j=4}^{5} \tilde{z}_j^*}{2(n-1)} \right)^2 \times \sup_{\{\sigma_1^2 \geq \sigma_2^2 \geq \sigma_3^2, \sigma_4^2 \geq \sigma_5^2\}} \prod_{i=1}^{5} (\frac{1}{\sigma_i^2}) e^{-\frac{\tilde{z}_i^*}{(n-1)\sigma_i^2}} \\ &= \left(\frac{\sum_{j=1}^{3} \tilde{z}_j}{3} \right)^3 \left(\frac{\sum_{j=4}^{5} \tilde{z}_j}{2} \right)^2 \frac{1}{(\tilde{z}_1 + r)} \frac{1}{(\tilde{z}_2 - r)} \frac{1}{\tilde{z}_3} \frac{1}{\tilde{z}_4} \frac{1}{\tilde{z}_5} \times e^{-5} \end{aligned} \tag{3.35}$$

since $\tilde{z}_1^* = \tilde{z}_1 + r > \tilde{z}_2^* = \tilde{z}_2 - r > \tilde{z}_3^* = \tilde{z}_3 > \tilde{z}_4^* = \tilde{z}_4 > \tilde{z}_5^* = \tilde{z}_5$.

We note that (3.34) divided by (3.35) is

$$\frac{(\sum_{j=1}^{2} \tilde{z}_j)^2 (\sum_{j=3}^{5} \tilde{z}_j)^3 (\tilde{z}_1 + r)(\tilde{z}_2 - r)}{(\sum_{j=1}^{3} \tilde{z}_j)^3 (\sum_{j=4}^{5} \tilde{z}_j)^2 \tilde{z}_1 \tilde{z}_2} \tag{3.36}$$

There are many choices of $\tilde{z}_1$, $\tilde{z}_2$, $\tilde{z}_3$, $\tilde{z}_4$, $\tilde{z}_5$, r for which (3.36) is greater than 1 (e.g., $\tilde{z}_1 = 99, \tilde{z}_2 = 96, \tilde{z}_3 = 70, \tilde{z}_4 = 52, \tilde{z}_5 = 43.2, r = 10$). The fact that (3.36) > 1 implies that we can choose C_2 such that $L_2^{*'} < C_2$ so that at $\boldsymbol{x}^* + r\boldsymbol{g}$ the overall procedure rejects H_3 and accepts H_1, H_2 and H_4. Note when $r > \frac{\tilde{z}_1 - \tilde{z}_2}{2}$, $\tilde{\boldsymbol{z}} - r\boldsymbol{g}$ is an accept point, since then $F_2 = \frac{(\tilde{z}_2 + r)}{\tilde{z}_3} > F_3 = \frac{\tilde{z}_3}{\tilde{z}_4} > F_4 = \frac{\tilde{z}_4}{\tilde{z}_5} > 1 > F_1 = \frac{(\tilde{z}_1 - r)}{(\tilde{z}_2 + r)}$, then $L_4 < 1 < c_4$, then H_1 is accepted at stage 4. Now if H_1 is rejected for $\tilde{\boldsymbol{z}}$ but accepted for $\tilde{\boldsymbol{z}} + r\boldsymbol{g}$, that implies the test for H_1 is inadmissible.

The same is true for $M \geq 5$.

Next for the variance change model we consider testing two-sided alternatives, i.e. we test $H_i : \nu_i = 0 \ vs \ K_i : \nu_i \neq 0$. The LRSD method in this case is the same as in the one-sided alternative case except that F_j is replaced by

$$F_j = \frac{\max\{s_j^2, s_{j+1}^2\}}{\min\{s_j^2, s_{j+1}^2\}} = \frac{\max\{\tilde{z}_j, \tilde{z}_{j+1}\}}{\min\{\tilde{z}_j, \tilde{z}_{j+1}\}} \tag{3.37}$$

.

In general, the Stage m global hypothesis is considered if $H_{j_1}, ..., H_{j_{m-1}}$ have been rejected. This global testing problem is $H_{mG} : \boldsymbol{\nu}^{(j_1,...,j_{m-1})} = \mathbf{0}$ vs $K_{mG} : \boldsymbol{\nu}^{(j_1,...,j_{m-1})} \neq$

0. The likelihood ratio test rejects H_{mG} if $L_m \geq c_m$, i.e

$$\frac{\sup\limits_{\{\sigma_i^2\}} \prod\limits_{i=1}^{M+1} (\frac{1}{\sigma_i^2})^{\frac{n-1}{2}} e^{-\frac{(n-1)s_i^2}{2\sigma_i^2}}}{\sup\limits_{\{\sigma_1'^2,\sigma_2'^2,...,\sigma_m'^2\}} \left(\prod\limits_{i=1}^{j_{(1)}} (\frac{1}{\sigma_1'^2})^{\frac{n-1}{2}} e^{-\frac{(n-1)s_i^2}{2\sigma_1'^2}}\right) \left(\prod\limits_{i=j_{(1)}+1}^{j_{(2)}} (\frac{1}{\sigma_2'^2})^{\frac{n-1}{2}} e^{-\frac{(n-1)s_i^2}{2\sigma_2'^2}}\right) \cdots \left(\prod\limits_{i=j_{(m-1)}+1}^{M+1} (\frac{1}{\sigma_m'^2})^{\frac{n-1}{2}} e^{-\frac{(n-1)s_i^2}{2\sigma_m'^2}}\right)} \tag{3.38}$$

$\geq c_m$

For the numerator the maximum likelihood estimators are: $\hat{\sigma}_i^2 = s_i^2$. For the denominator the maximum likelihood estimators are: $\hat{\sigma}_1'^2 = \frac{\sum_1^{j_{(1)}} s_j^2}{j_{(1)}}$, $\hat{\sigma}_2'^2 = \frac{\sum_{j_{(1)}+1}^{j_{(2)}} s_j^2}{j_{(2)}-j_{(1)}}$, ..., $\hat{\sigma}_m'^2 = \frac{\sum_{j_{(m-1)}+1}^{M+1} s_j^2}{M-j_{(m-1)}+1}$, replace $\sigma_1'^2, \sigma_2'^2, ..., \sigma_m'^2$ with them in (3.38), we get:

$$\begin{aligned} L_m &= \left(\prod_{i=1}^{j_{(1)}} \left(\frac{\sum_1^{j_{(1)}} s_j^2}{j_{(1)} s_i^2}\right) \prod_{i=j_{(1)}+1}^{j_{(2)}} \left(\frac{\sum_{j_{(1)}+1}^{j_{(2)}} s_j^2}{(j_{(2)} - j_{(1)}) s_i^2}\right) \cdots \prod_{i=j_{(m-1)}+1}^{M+1} \left(\frac{\sum_{j_{(m-1)}+1}^{M+1} s_j^2}{(M - j_{(m-1)} + 1) s_i^2}\right)\right)^{(n-1)/2} \\ &= \left(\prod_{i=1}^{j_{(1)}} \left(\frac{\sum_1^{j_{(1)}} \tilde{z}_j}{j_{(1)} \tilde{z}_i}\right) \prod_{i=j_{(1)}+1}^{j_{(2)}} \left(\frac{\sum_{j_{(1)}+1}^{j_{(2)}} \tilde{z}_j}{(j_{(2)} - j_{(1)}) \tilde{z}_i}\right) \cdots \prod_{i=j_{(m-1)}+1}^{M+1} \left(\frac{\sum_{j_{(m-1)}+1}^{M+1} \tilde{z}_j}{(M - j_{(m-1)} + 1) \tilde{z}_i}\right)\right)^{(n-1)/2} \end{aligned} \tag{3.39}$$

Define

$$L_m' = \prod_{i=1}^{j_{(1)}} \left(\frac{\sum_1^{j_{(1)}} \tilde{z}_j}{j_{(1)} \tilde{z}_i}\right) \prod_{i=j_{(1)}+1}^{j_{(2)}} \left(\frac{\sum_{j_{(1)}+1}^{j_{(2)}} \tilde{z}_j}{(j_{(2)} - j_{(1)}) \tilde{z}_i}\right) \cdots \prod_{i=j_{(m-1)}+1}^{M+1} \left(\frac{\sum_{j_{(m-1)}+1}^{M+1} \tilde{z}_j}{(M - j_{(m-1)} + 1) \tilde{z}_i}\right) \tag{3.40}$$

Thus $L_m \geq c_m \Longleftrightarrow L_m' \geq C_m$, where $C_m^{\frac{n-1}{2}} = c_m$. For this set up we have

Theorem 3.3.2. *For the two-sided alternative case (3.1) LRSD is admissible for M=2.*

Proof of Theorem 3.3.2. For M=2, once again we focus on H_1 *vs* K_1:

(1) If $\tilde{z}_1 > \tilde{z}_2$, we will demonstrate that if $\phi(\tilde{\boldsymbol{z}}) = 1$ then $\phi(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = 1$ for all $r > 0$.

When H_1 is rejected first $\Longrightarrow F_1 = \frac{\tilde{z}_1}{\tilde{z}_2} > F_2 = \frac{\max\{\tilde{z}_2,\tilde{z}_3\}}{\min\{\tilde{z}_2,\tilde{z}_3\}}$ and $L_1' \geq C_1$. At $\tilde{\boldsymbol{z}}^{\boldsymbol{*}} = \tilde{\boldsymbol{z}} + r\boldsymbol{g}$, $r > 0$, $F_1^* = \frac{\tilde{z}_1+r}{\tilde{z}_2-r} > F_2^* = \frac{\max\{\tilde{z}_2-r,\tilde{z}_3\}}{\min\{\tilde{z}_2-r,\tilde{z}_3\}}$, $L_1^{*'} = (\frac{\tilde{z}_1+\tilde{z}_2+\tilde{z}_3}{3(\tilde{z}_1+r)})(\frac{\tilde{z}_1+\tilde{z}_2+\tilde{z}_3}{3(\tilde{z}_2-r)})(\frac{\tilde{z}_1+\tilde{z}_2+\tilde{z}_3}{3\tilde{z}_3}) = \frac{\tilde{z}_1\tilde{z}_2}{(\tilde{z}_1+r)(\tilde{z}_2-r)}L_1' > L_1'$ by (3.30), so $\phi_1(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = 1$ too, for all $r > 0$.

When H_1 is rejected secondly $\Longrightarrow F_1 = \frac{\tilde{z}_1}{\tilde{z}_2} < F_2 = \frac{\max\{\tilde{z}_2,\tilde{z}_3\}}{\min\{\tilde{z}_2,\tilde{z}_3\}}, L_1' \geq C_1$ and $L_2' \geq C_2$. At $\tilde{\boldsymbol{z}}^{\boldsymbol{*}} = \tilde{\boldsymbol{z}} + r\boldsymbol{g}$, $r > 0$, $F_1^* = \frac{\tilde{z}_1+r}{\tilde{z}_2-r}$, $F_2^* = \frac{\max\{\tilde{z}_2-r,\tilde{z}_3\}}{\min\{\tilde{z}_2-r,\tilde{z}_3\}}$, $L_1^{*'} = \frac{\tilde{z}_1\tilde{z}_2}{(\tilde{z}_1+r)(\tilde{z}_2-r)}L_1' \geq L_1'$. If $F_1^* > F_2^*$, we reject H_1 firstly at $\tilde{\boldsymbol{z}}^{\boldsymbol{*}}$; If $F_2^* > F_1^*$, we reject H_2 firstly, since $L_2^{*'} = (\frac{\tilde{z}_1+\tilde{z}_2}{2(\tilde{z}_1+r)})(\frac{\tilde{z}_1+\tilde{z}_2}{2(\tilde{z}_2-r)}) = \frac{\tilde{z}_1\tilde{z}_2}{(\tilde{z}_1+r)(\tilde{z}_2-r)}L_2' > L_2'$, we reject H_1 at second stage. Thus $\phi(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = 1$ for all $r > 0$.

(2) If $\tilde{z}_2 > \tilde{z}_1$, we will demonstrate that if $\phi(\tilde{\boldsymbol{z}}) = 0$, and if $\phi(\tilde{\boldsymbol{z}}^{\boldsymbol{*}}) = \phi(\tilde{\boldsymbol{z}} + r_1\boldsymbol{g}) = 1$ for certain $r_1 > 0$, then $\phi(\tilde{\boldsymbol{z}}^{\boldsymbol{*}} + r\boldsymbol{g}) = 1$ for all $r > r_1$.

When both H_1, H_2 are not rejected at $\tilde{\boldsymbol{z}} \Longrightarrow L_1' < C_1$. In order to reject H_1, r_1 must $> (\tilde{z}_2 - \tilde{z}_1)$, then at $\tilde{\boldsymbol{z}}^{\boldsymbol{*}} = \tilde{\boldsymbol{z}} + r_1\boldsymbol{g}$, $L_1^{*'} = \frac{\tilde{z}_1\tilde{z}_2}{(\tilde{z}_1+r_1)(\tilde{z}_2-r_1)}L_1' = \frac{\tilde{z}_1\tilde{z}_2}{-(r_1-\frac{1}{2}(\tilde{z}_2-\tilde{z}_1))^2+\frac{1}{4}(\tilde{z}_2-\tilde{z}_1)^2+\tilde{z}_1\tilde{z}_2}L_1'$ $> L_1'$, and $\tilde{z}_1^* = \tilde{z}_1 + r_1 > \tilde{z}_2$, $\tilde{z}_2^* = \tilde{z}_2 - r_1 < \tilde{z}_1$, so $\tilde{z}_1^* > \tilde{z}_2^*$, by the above part (1) we know that $\phi_1(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$, for all $r > 0$.

When H_2 is rejected and H_1 is accepted at $\tilde{\boldsymbol{z}} \Longrightarrow L_1' \geq C_1, F_1 = \frac{\tilde{z}_2}{\tilde{z}_1} < F_2 = \frac{\max\{\tilde{z}_2,\tilde{z}_3\}}{\min\{\tilde{z}_2,\tilde{z}_3\}}$ and $L_2' < C_2$. To reject H_1 at $\tilde{\boldsymbol{z}}^{\boldsymbol{*}} = \tilde{\boldsymbol{z}} + r_1\boldsymbol{g}$, there are two cases. One is that at $\tilde{\boldsymbol{z}}^{\boldsymbol{*}}$, $L_1^{*'} \geq C_1, F_1^* < F_2^*$ and $L_2^{*'} \geq C_2$; the other one is that $L_1^{*'} \geq C_1, F_1^* > F_2^*$.

For the first case, $L_2^{*'} \geq C_2 \Longrightarrow L_2^{*'} > L_2'$, i.e.

$$L_2^{*'} = \frac{\tilde{z}_1\tilde{z}_2}{(\tilde{z}_1 + r_1)(\tilde{z}_2 - r_1)}L_2' = \frac{\tilde{z}_1\tilde{z}_2}{-(r_1 - \frac{1}{2}(\tilde{z}_2 - \tilde{z}_1))^2 + \frac{1}{4}(\tilde{z}_2 - \tilde{z}_1)^2 + \tilde{z}_1\tilde{z}_2}L_2' > L_2'$$

$\Longrightarrow r_1 > (\tilde{z}_2 - \tilde{z}_1) \Longrightarrow \tilde{z}_1^* = \tilde{z}_1 + r_1 > \tilde{z}_2$, $\tilde{z}_2^* = \tilde{z}_2 - r_1 < \tilde{z}_1 \Longrightarrow \tilde{z}_1^* > \tilde{z}_2^*$, then by part (1) we know that $\phi_1(\tilde{\boldsymbol{z}}^{\boldsymbol{*}} + \boldsymbol{g}) = 1$, for all $r > r_1$.

For the second case, $F_1^* > F_2^* \Longrightarrow \tilde{z}_1^* > \tilde{z}_2^*$. Since if $\tilde{z}_1^* < \tilde{z}_2^*$, $F_1^* = \frac{\tilde{z}_2^*}{\tilde{z}_1^*} = \frac{\tilde{z}_2-r}{\tilde{z}_1+r} < F_1$,if $F_2 = \frac{\tilde{z}_3}{\tilde{z}_2} \Longrightarrow F_2^* = \frac{\tilde{z}_3}{\tilde{z}_2-r} > F_2 \Longrightarrow F_2^* > F_1^*$ contradicted with $F_1^* > F_2^*$; if $F_2 = \frac{\tilde{z}_2}{\tilde{z}_3}$ and if $F_2^* = \frac{\tilde{z}_2-r}{\tilde{z}_3}$, since $F_1 = \frac{\tilde{z}_2}{\tilde{z}_1} < F_2 = \frac{\tilde{z}_2}{\tilde{z}_3} \Longrightarrow F_1^* = \frac{\tilde{z}_2-r}{\tilde{z}_1+r} < F_2^* = \frac{\tilde{z}_2-r}{\tilde{z}_3}$ contradicted with $F_1^* > F_2^*$; if $F_2 = \frac{\tilde{z}_2}{\tilde{z}_3}$ and if $F_2^* = \frac{\tilde{z}_3}{\tilde{z}_2-r}$, since $F_1 = \frac{\tilde{z}_2}{\tilde{z}_1} < F_2 = \frac{\tilde{z}_2}{\tilde{z}_3} \Longrightarrow F_1^* = \frac{\tilde{z}_2-r}{\tilde{z}_1+r} < \frac{\tilde{z}_2-r}{\tilde{z}_3} < \frac{\tilde{z}_3}{\tilde{z}_2-r} = F_2^*$ contradicted with $F_1^* > F_2^*$. Thus for this case, $\tilde{z}_1^* > \tilde{z}_2^*$ and H_1 is rejected firstly at $\tilde{\boldsymbol{z}}^{\boldsymbol{*}}$, by part (1), we know that $\phi_1(\tilde{\boldsymbol{z}}^{\boldsymbol{*}} + r\boldsymbol{g}) = 1$, for all $r > r_1$.

□

For M=3 we exhibit a set of critical values for which LRSD is inadmissible. To do so we find a sample point $\tilde{\boldsymbol{z}}^*$ at which H_1 is rejected and for which H_1 is accepted at $\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}$. In fact let $\tilde{\boldsymbol{z}}^* = (\tilde{z}_1^*, \tilde{z}_2^*, \tilde{z}_3^*, \tilde{z}_4^*)'$ for $\tilde{z}_3^* > \tilde{z}_1^* > \tilde{z}_2^* > \tilde{z}_4^*$ and $\frac{\tilde{z}_3^*}{\tilde{z}_4^*} > \frac{\tilde{z}_3^*}{\tilde{z}_2^*} > \frac{\tilde{z}_1^*}{\tilde{z}_2^*}$, i.e. $F_3^* > F_2^* > F_1^*$. Thus using (3.40) at stage 1 choose C_1 so that $L_1^{*'} = (\frac{\tilde{z}_1^*+\tilde{z}_2^*+\tilde{z}_3^*+\tilde{z}_4^*}{4\tilde{z}_1^*})(\frac{\tilde{z}_1^*+\tilde{z}_2^*+\tilde{z}_3^*+\tilde{z}_4^*}{4\tilde{z}_2^*})(\frac{\tilde{z}_1^*+\tilde{z}_2^*+\tilde{z}_3^*+\tilde{z}_4^*}{4\tilde{z}_3^*})(\frac{\tilde{z}_1^*+\tilde{z}_2^*+\tilde{z}_3^*+\tilde{z}_4^*}{4\tilde{z}_4^*}) = C_1$ so that H_3 is rejected. At stage 2 we calculate

$$L_2^{*'} = (\frac{\tilde{z}_1^* + \tilde{z}_2^* + \tilde{z}_3^*}{3\tilde{z}_1^*})(\frac{\tilde{z}_1^* + \tilde{z}_2^* + \tilde{z}_3^*}{3\tilde{z}_2^*})(\frac{\tilde{z}_1^* + \tilde{z}_2^* + \tilde{z}_3^*}{3\tilde{z}_3^*}) \tag{3.41}$$

We set $(\frac{\tilde{z}_1^*+\tilde{z}_2^*+\tilde{z}_3^*}{3\tilde{z}_1^*})(\frac{\tilde{z}_1^*+\tilde{z}_2^*+\tilde{z}_3^*}{3\tilde{z}_2^*})(\frac{\tilde{z}_1^*+\tilde{z}_2^*+\tilde{z}_3^*}{3\tilde{z}_3^*}) = C_2$, so H_2 is rejected. At stage 3, H_1 is rejected.

Now let r such that $\tilde{z}_2^* - \tilde{z}_4^* < r < \tilde{z}_3^* - \tilde{z}_1^*$. Thus at $\tilde{\boldsymbol{z}}^{**} = \tilde{\boldsymbol{z}}^* + r\boldsymbol{g}$, $F_2^{**} = \frac{\tilde{z}_3^*}{\tilde{z}_2^* - r} > F_3^{**} = \frac{\tilde{z}_3^*}{\tilde{z}_4^*}$, $F_2^{**} = \frac{\tilde{z}_3^*}{\tilde{z}_2^* - r} > F_1^{**} = \frac{\tilde{z}_1^* + r}{\tilde{z}_2^* - r}$, and

$$\begin{aligned} L_1^{**'} &= (\frac{\tilde{z}_1^* + \tilde{z}_2^* + \tilde{z}_3^* + \tilde{z}_4^*}{4(\tilde{z}_1^* + r)})(\frac{\tilde{z}_1^* + \tilde{z}_2^* + \tilde{z}_3^* + \tilde{z}_4^*}{4(\tilde{z}_2^* - r)})(\frac{\tilde{z}_1^* + \tilde{z}_2^* + \tilde{z}_3^* + \tilde{z}_4^*}{4\tilde{z}_3^*})(\frac{\tilde{z}_1^* + \tilde{z}_2^* + \tilde{z}_3^* + \tilde{z}_4^*}{4\tilde{z}_4^*}) \\ &= \frac{\tilde{z}_1^*\tilde{z}_2^*}{(\tilde{z}_1^* + r)(\tilde{z}_2^* - r)} L_1^{*'} \\ &> L_1^{*'} \end{aligned}$$

This time however, H_2 is rejected at stage 1. At stage 2 we calculate for $\tilde{\boldsymbol{z}}^{**}$,

$$L_2^{**'} = (\frac{\tilde{z}_1^* + \tilde{z}_2^*}{2(\tilde{z}_1^* + r)})(\frac{\tilde{z}_1^* + \tilde{z}_2^*}{2(\tilde{z}_2^* - r)})(\frac{\tilde{z}_3^* + \tilde{z}_4^*}{2\tilde{z}_3^*})(\frac{\tilde{z}_3^* + \tilde{z}_4^*}{2\tilde{z}_4^*}) \tag{3.42}$$

We note that (3.41) divided by (3.42) is

$$\frac{16}{27}\frac{(\tilde{z}_1^* + \tilde{z}_2^* + \tilde{z}_3^*)^3\tilde{z}_4^*(\tilde{z}_1^* + r)(\tilde{z}_2^* - r)}{(\tilde{z}_1^* + \tilde{z}_2^*)^2(\tilde{z}_3^* + \tilde{z}_4^*)^2\tilde{z}_1^*\tilde{z}_2^*} \tag{3.43}$$

There are many choices of $\tilde{z}_1^*$, $\tilde{z}_2^*$, $\tilde{z}_3^*$, $\tilde{z}_4^*$, r for which (3.43) is greater than 1 (e.g., $\tilde{z}_1^* = 1.2, \tilde{z}_2^* = 1.1, \tilde{z}_3^* = 3, \tilde{z}_4^* = 1, r = 0.001$). The fact that (3.43) > 1 implies that we can choose C_2 such that $L_2^{**'} < C_2$ so that at $\boldsymbol{x}^* + r\boldsymbol{g}$ the overall procedure rejects H_2 and accepts H_1, H_3. Note since $\tilde{\boldsymbol{z}} = \tilde{\boldsymbol{z}}^{\boldsymbol{*}} - r\boldsymbol{g}, r < \frac{\tilde{z}_1^* - \tilde{z}_2^*}{2}$ is an accept point (because $L_1' < L_1^{*'} = C_1$). Now if H_1 is rejected for $\tilde{\boldsymbol{z}}^{\boldsymbol{*}}$ but accepted for $\tilde{\boldsymbol{z}}^{\boldsymbol{*}} + r\boldsymbol{g}$, that implies the test for H_1 is inadmissible.

The same is true for $M \geq 5$.

3.4 Step-Up

Now we study two of the most popular stepwise procedures. We demonstrate that the individual tests they induce are inadmissible for these one-sided and two-sided testing hypotheses of variance change.

For step-up, let $1 \leq C_1 < C_2 < \cdots < C_M$ be a sequence of increasing of critical values and let $F_{(1)} \leq F_{(2)} \leq \cdots \leq F_{(M)}$ be the ordered statistics of F_1, F_2, ..., F_M, where for one side testing hypotheses of (3.8), F_j is defined in (3.22); and for two sided testing hypotheses of (3.7), F_j is defined in (3.37).

Stage 1: If $F_{(1)} \leq C_1$, accept $H_{(1)}$ where $H_{(1)}$ is the hypothesis corresponding to $F_{(1)}$. Otherwise reject all H_i.

Stage 2: If $H_{(1)}$ is accepted, accept $H_{(2)}$ if $F_{(2)} \leq C_2$. Otherwise reject $H_{(2)}, ..., H_{(M)}$.

$\cdots\cdots$

In general, at stage m, if $F_{(m)} \leq C_m$ accept $H_{(m)}$. Otherwise reject $H_{(m)}, ..., H_{(M)}$.

Theorem 3.4.1. *Consider the variance change problem of this chapter, the step-up procedure is inadmissible for the one sided testing problem (3.8).*

Proof of Theorem 3.4.1. Again we focus on H_1 vs K_1^*. To show $\phi_1(\tilde{\boldsymbol{z}})$ is inadmissible we will find three points $\tilde{\boldsymbol{z}}^*, \tilde{\boldsymbol{z}}^{**}, \tilde{\boldsymbol{z}}^{***}$ with $\tilde{\boldsymbol{z}}^{**} = \tilde{\boldsymbol{z}}^* - r_1\boldsymbol{g}$, $\tilde{\boldsymbol{z}}^{***} = \tilde{\boldsymbol{z}}^* - r_2\boldsymbol{g}$, $r_2 > r_1 > 0$ such that $\phi_1(\tilde{\boldsymbol{z}}^*) = 0$, $\phi_1(\tilde{\boldsymbol{z}}^{**}) = 1$, $\phi_1(\tilde{\boldsymbol{z}}^{***}) = 0$. This will prove the theorem by Lemma 2.1.1.

At $\tilde{\boldsymbol{z}}^*$, let $\tilde{z}_1^* = C_1 + C_2$, $\tilde{z}_2^* = 2$, $\tilde{z}_3^* = \frac{2}{C_1}$, $\tilde{z}_j^* = \frac{1}{C_j+1}\tilde{z}_{j-1}^*$, $j = 4, ..., M+1$, so $F_1^* = \frac{C_1+C_2}{2}$, $F_2^* = C_1$, $F_j^* = C_j + 1$, $j = 3, ..., M$. Since for stage 1, $F_{(1)}^* = \min\{F_j^*,\ j = 1, 2, ..., M\} = F_2^* \leq C_1 \Longrightarrow \phi_2(\tilde{\boldsymbol{z}}^*) = 0$; for stage 2, $F_{(2)}^* = \min\{F_j^*,\ j = 1, 3, 4, ..., M\} = F_1^* \leq C_2 \Longrightarrow \phi_1(\tilde{\boldsymbol{z}}^*) = 0$ at $\tilde{\boldsymbol{z}}^*$.

Let $r_1 = \frac{(C_2-C_1)}{2(1+C_1)}$, so at $\tilde{\boldsymbol{z}}^{**} = \tilde{\boldsymbol{z}}^* - r_1\boldsymbol{g}$, $\tilde{z}_1^{**} = C_1 + C_2 - \frac{(C_2-C_1)}{2(1+C_1)} = \frac{2C_1^2+2C_1C_2+3C_1+C_2}{2(1+C_1)}$, $\tilde{z}_2^{**} = 2 + \frac{(C_2-C_1)}{2(1+C_1)} = \frac{4+3C_1+C_2}{2(1+C_1)}$, $\tilde{z}_3^{**} = \tilde{z}_3^* = \frac{2}{C_1}$, $\tilde{z}_j^{**} = \tilde{z}_j^*$, $j = 4, ..., M+1$. So $F_1^{**} = \frac{2C_1^2+2C_1C_2+3C_1+C_2}{4+3C_1+C_2} > C_1$, $F_2^{**} = \frac{(4+3C_1+C_2)C_1}{4(1+C_1)} > C_1$, $F_j^{**} = C_j + 1 > C_1$, $j = 3, ..., M$, so we reject all $\Longrightarrow \phi_1(\tilde{\boldsymbol{z}}^{**}) = 1$ at $\tilde{\boldsymbol{z}}^{**}$.

Let $r_2 = \frac{C_2+C_1-2}{2} > r_1$, so at $\tilde{\boldsymbol{z}}^{***} = \tilde{\boldsymbol{z}}^* - r_2\boldsymbol{g}$, $\tilde{z}_1^{***} = C_1 + C_2 - \frac{C_2+C_1-2}{2} = \frac{C_1+C_2+2}{2}$,

$\tilde{z}_2^{***} = 2 + \frac{C_2+C_1-2}{2} = \frac{C_1+C_2+2}{2}$, $\tilde{z}_3^{***} = \tilde{z}_3^* = \frac{2}{C_1}$, $\tilde{z}_j^{***} = \tilde{z}_j^*$, $j = 4, ..., M+1$. So $F_1^{***} = 1 \le C_1$, $F_2^{***} = \frac{(C_1+C_2+2)C_1}{4} > C_1$, $F_j^{***} = C_j + 1 > C_1$, $j = 3, ..., M$, so at stage 1, we accept H_1, i.e., $\phi_1(\tilde{\boldsymbol{z}}^{***}) = 0$ at $\tilde{\boldsymbol{z}}^{***}$. □

Theorem 3.4.2. *Consider the variance change problem of this chapter, the step-up procedure is inadmissible for the two sided testing problem (3.7).*

Proof of Theorem 3.4.2. Again we focus on H_1 vs K_1. For this two sided case problem, we use F_j is defined in (3.37). To show $\phi_1(\tilde{\boldsymbol{z}})$ is inadmissible, the three points $\tilde{\boldsymbol{z}}^*, \tilde{\boldsymbol{z}}^{**}, \tilde{\boldsymbol{z}}^{***}$ defined in the above proof for the one sided case with $\tilde{\boldsymbol{z}}^{**} = \tilde{\boldsymbol{z}}^* - r_1\boldsymbol{g}$, $\tilde{\boldsymbol{z}}^{***} = \tilde{\boldsymbol{z}}^* - r_2\boldsymbol{g}$, $r_2 > r_1 > 0$ can also be used here satisfying $\phi_1(\tilde{\boldsymbol{z}}^*) = 0$, $\phi_1(\tilde{\boldsymbol{z}}^{**}) = 1$, $\phi_1(\tilde{\boldsymbol{z}}^{***}) = 0$. This prove the theorem by Lemma 2.1.1.

□

3.5 Step-Down

For step-down, let $1 \le C_1 < C_2 < \cdots < C_M$ be a sequence of increasing of critical values and let $F_{(1)} \le F_{(2)} \le \cdots \le F_{(M)}$ be the order statistics of F_1, F_2, ..., F_M, where for one side testing hypotheses of (3.8), F_j is defined in (3.22); and for two sided testing hypotheses of (3.7), F_j is defined in (3.37).

Stage 1: If $F_{(M)} > C_M$, reject $H_{(M)}$ where $H_{(M)}$ is the hypothesis corresponding to $F_{(M)}$. Otherwise accept all H_i.

Stage 2: If $H_{(M)}$ is rejected, reject $H_{(M-1)}$ if $F_{(M-1)} > C_{M-1}$. Otherwise accept $H_{(1)}, ..., H_{(M-1)}$.

......

In general, at stage m, if $F_{(M-m+1)} > C_{M-m+1}$ reject $H_{(m)}$. Otherwise accept $H_{(1)}, ..., H_{(M-m+1)}$.

Theorem 3.5.1. *Consider the variance change problem of this section, the step-down procedure is inadmissible for the one sided testing problem (3.8).*

Proof of Theorem 3.5.1. Similar to the proof of Theorem 3.4.1, we focus on H_1 vs K_1^*. To show $\phi_1(\tilde{\boldsymbol{z}})$ is inadmissible we will find three points $\tilde{\boldsymbol{z}}^*$, $\tilde{\boldsymbol{z}}^{**}$, $\tilde{\boldsymbol{z}}^{***}$ with $\tilde{\boldsymbol{z}}^{**} = \tilde{\boldsymbol{z}}^* - r_1\boldsymbol{g}$,

$\tilde{\boldsymbol{z}}^{***} = \tilde{\boldsymbol{z}}^{*} - r_2\boldsymbol{g}$, $r_2 > r_1 > 0$ such that $\phi_1(\tilde{\boldsymbol{z}}^{*}) = 0$, $\phi_1(\tilde{\boldsymbol{z}}^{**}) = 1$, $\phi_1(\tilde{\boldsymbol{z}}^{***}) = 0$. This will prove the theorem by Lemma 2.1.1.

At $\tilde{\boldsymbol{z}}^{*}$, use the same $\tilde{\boldsymbol{z}}^{*}$ for the proof of Theorem 3.4.1, except change $\tilde{z}_3^*$ to $\tilde{z}_3^* = \frac{2}{C_2}$. i.e., $\tilde{z}_1^* = C_1 + C_2$, $\tilde{z}_2^* = 2$, $\tilde{z}_3^* = \frac{2}{C_2}$, $\tilde{z}_j^* = \frac{1}{C_j+1}\tilde{z}_{j-1}^*$, $j = 4, ..., M+1$, so use the definition of F_j in (3.22), $F_1^* = \frac{C_1+C_2}{2} < C_2$, $F_2^* = C_2$, $F_j^* = C_j + 1 > C_j$, $j = 3, ..., M$. From the above step-down procedure, we accept H_1 and H_2, i.e., $\phi_1(\tilde{\boldsymbol{z}}^{*}) = 0$ at $\tilde{\boldsymbol{z}}^{*}$.

Use the same $r_1 = \frac{(C_2-C_1)}{2(1+C_1)}$, so at $\tilde{\boldsymbol{z}}^{**} = \tilde{\boldsymbol{z}}^{*} - r_1\boldsymbol{g}$, $\tilde{z}_1^{**} = C_1 + C_2 - \frac{(C_2-C_1)}{2(1+C_1)} = \frac{2C_1^2+2C_1C_2+3C_1+C_2}{2(1+C_1)}$, $\tilde{z}_2^{**} = 2 + \frac{(C_2-C_1)}{2(1+C_1)} = \frac{4+3C_1+C_2}{2(1+C_1)}$, $\tilde{z}_3^{**} = \tilde{z}_3^{*} = \frac{2}{C_2}$, $\tilde{z}_j^{**} = \tilde{z}_j^{*}$, $j = 4, ..., M+1$. So $F_1^{**} = \frac{2C_1^2+2C_1C_2+3C_1+C_2}{4+3C_1+C_2} > C_1$, $F_2^{**} = \frac{(4+3C_1+C_2)C_2}{4(1+C_1)} > C_2$, $F_j^{**} = C_j + 1 > C_j$, $j = 3, ..., M$, so we reject all $\Longrightarrow \phi_1(\tilde{\boldsymbol{z}}^{**}) = 1$ at $\tilde{\boldsymbol{z}}^{**}$.

Use the same $r_2 = \frac{C_2+C_1-2}{2} > r_1$, so at $\tilde{\boldsymbol{z}}^{**} = \tilde{\boldsymbol{z}}^{*} - r_2\boldsymbol{g}$, $\tilde{z}_1^{***} = C_1 + C_2 - \frac{C_2+C_1-2}{2} = \frac{C_1+C_2+2}{2}$, $\tilde{z}_2^{***} = 2 + \frac{C_2+C_1-2}{2} = \frac{C_1+C_2+2}{2}$, $\tilde{z}_3^{***} = \tilde{z}_3^{*} = \frac{2}{C_2}$, $\tilde{z}_j^{***} = \tilde{z}_j^{*}$, $j = 4, ..., M+1$. so $F_1^{***} = 1 \leq C_1$, $F_2^{***} = \frac{(C_1+C_2+2)C_2}{4} > C_2$, $F_j^{***} = C_j + 1 > C_j$, $j = 3, ..., M$, so we accept H_1, i.e., $\phi_1(\tilde{\boldsymbol{z}}^{***}) = 0$ at $\tilde{\boldsymbol{z}}^{***}$. □

Theorem 3.5.2. *Consider the variance change problem of this section, the step-down procedure is inadmissible for the two sided testing problem (3.7).*

Proof of Theorem 3.5.2. Again we focus on H_1 vs K_1. For this two sided case problem, we use F_j is defined in (3.37). To show $\phi_1(\tilde{\boldsymbol{z}})$ is inadmissible, the three points $\tilde{\boldsymbol{z}}^{*}$, $\tilde{\boldsymbol{z}}^{**}$, $\tilde{\boldsymbol{z}}^{***}$ defined in the above proof of Theorem 3.5.1 for the one sided case with $\tilde{\boldsymbol{z}}^{**} = \tilde{\boldsymbol{z}}^{*} - r_1\boldsymbol{g}$, $\tilde{\boldsymbol{z}}^{***} = \tilde{\boldsymbol{z}}^{*} - r_2\boldsymbol{g}$, $r_2 > r_1 > 0$ can also be used here satisfying $\phi_1(\tilde{\boldsymbol{z}}^{*}) = 0$, $\phi_1(\tilde{\boldsymbol{z}}^{**}) = 1$, $\phi_1(\tilde{\boldsymbol{z}}^{***}) = 0$. This prove the theorem by Lemma 2.1.1. □

Chapter 4

Testing of variances of treatments against a control

The setting for testing of variances of treatments against a control is same to the variance change problem in Chapter 3, i.e., we have $(M+1)$ independent random samples $\mathbf{z_j} = (z_{j1}, z_{j2}, ..., z_{j(M+1)})'$ from $(M+1)$ normal populations with parameters(μ_1, σ_1^2), (μ_2, σ_2^2), ..., $(\mu_{M+1}, \sigma_{M+1}^2)$. And there are n such independent sequences. The treatments correspond to $j = 1, 2, ...M$ while the control population corresponds to the $(M+1)^{\text{th}}$ population. The testing problem we are interested in this chapter is:

$$H_i : \sigma_i^2 = \sigma_{M+1}^2 \quad vs \quad K_i : \sigma_i^2 \neq \sigma_{M+1}^2, \quad i = 1, ..., M \tag{4.1}$$

So rejecting any H_i indicates the variance for ith population is different from the control.

We will also consider one-sided alternative problems

$$H_i : \sigma_i^2 = \sigma_{M+1}^2 \quad vs \quad K_i^* : \sigma_i^2 > \sigma_{M+1}^2, \quad i = 1, ..., M \tag{4.2}$$

Same as in Chapter 3, let $s_i^2 = \frac{\sum_{j=1}^n (z_{ji} - \bar{z}_i)^2}{n-1}$ be the sample variance, where $\bar{z}_i = \frac{\sum_{j=1}^n z_{ji}}{n}$, $i = 1, ..., (M+1)$. So $\boldsymbol{s^2} = (s_1^2, s_2^2, ..., s_{M+1}^2)$ follows a distribution in (3.3), i.e.,

$$f_{\mathbf{s^2}}(\mathbf{s}^2|\sigma^{\mathbf{2}}) = \prod_{i=1}^{M+1} \frac{(n-1)}{\Gamma(\frac{n-1}{2})2^{(n-1)/2}} \frac{((n-1)s_i^2)^{(\frac{n-1}{2}-1)}}{(\sigma_i^2)^{\frac{n-1}{2}}} e^{-\frac{(n-1)s_i^2}{2\sigma_i^2}}$$

Now let $\tilde{z}_i = (n-1)s_i^2$, $u_i = -\frac{1}{2\sigma_i^2}$, so

$$f_{\tilde{\mathbf{z}}}(\tilde{\mathbf{z}}|\mathbf{u}) = h(\tilde{\mathbf{z}})\beta(\mathbf{u})\exp\{\tilde{\mathbf{z}}'\mathbf{u}\}$$

Let A= $\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ & & \cdots\cdots & & & \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$, which is a $(M+1) \times (M+1)$ matrix.

Then

$$f_{\tilde{\mathbf{z}}}(\tilde{\mathbf{z}}|\mathbf{u}) = h(\tilde{\mathbf{z}})\beta(\mathbf{u})\exp\{\tilde{\mathbf{z}}'A^{-1}A\mathbf{u}\} \tag{4.3}$$

Define $\boldsymbol{\nu} = A\mathbf{u}$ then

$$f_{\tilde{\mathbf{z}}}(\tilde{\mathbf{z}}|\mathbf{u}) = h(\tilde{\mathbf{z}})\beta^*(\boldsymbol{\nu})\exp\{\tilde{\mathbf{z}}'A^{-1}\boldsymbol{\nu}\} \tag{4.4}$$

So Testing (4.1) and (4.2) are equivalent to test

$$H_i : \nu_i = 0 \quad vs \quad K_i : \nu_i \neq 0, \quad i = 1, ..., M \tag{4.5}$$

$$H_i : \nu_i = 0 \quad vs \quad K_i^* : \nu_i > 0, \quad i = 1, ..., M \tag{4.6}$$

4.1 MRD procedure

Similar to the variance change problem, the maximum residual down (MRD) method is based on the maximum of adaptively formed residuals for treatment vs control problems. It is step-down type MTPs. For each stage, we calculate the residuals for the hypotheses not previously rejected, and compare the biggest one with some constant c, then make decision of rejecting or accepting.

Let $\mathbf{X} = A\tilde{\boldsymbol{z}}$, $\Sigma = AA'$,then from (4.4) we can get

$$f_{\mathbf{X}}(\mathbf{x}|\boldsymbol{\nu}) = h^*(\mathbf{x})\beta^*(\boldsymbol{\nu})\exp\{\mathbf{x}'\Sigma^{-1}\boldsymbol{\nu}\} \tag{4.7}$$

Note that $\Sigma = AA' = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 2 & 1 & \cdots & 1 & 1 & 0 \\ & & \cdots\cdots\cdots & & & & \\ 1 & 1 & 1 & \cdots & 1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & (M+1) \end{pmatrix}$ which is a $(M+1) \times (M+1)$ matrix.

Use the same notation as in the previous chapter, let $\mathbf{X}^{(j_1,j_2,...,j_r,i)}$ be the (M-r) vector consisting of the components of $\mathbf{X}$ with X_{j_1}, ..., X_{j_r}, X_i left out. $\Sigma_{(j_1,j_2,...,j_r,i)}$ is the $(M-r)\times(M-r)$ covariance matrix of $\mathbf{X}^{(j_1,j_2,...,j_r,i)}$. $\boldsymbol{\sigma}_{(i)}^{(j_1,j_2,...,j_r)}$ is the $(M-r)\times 1$ vector of covariances between X_i and all variables except $X_{j_1}, ..., X_{j_r}$ and X_i.

So for Stage m after rejecting $H_{j_1}, H_{j_2}, ...H_{j_{m-1}}$, let $(j_{(1)}, ...j_{(m-1)})$ be the ordered sequence of $(j_1, ..., j_{m-1})$, then for $j_{(k)} < i < j_{(k+1)}$, where $k = 0, 1, ..., m-1$, with $j_{(0)} = 0$, $j_{(m)} = M+1$, we define $\text{Residual}_{m,i}$ like this:

$$\begin{aligned} \text{Residual}_{m,i} &= X_i - \boldsymbol{\sigma}_{(i)}^{(j_1,j_2,...,j_{m-1})'} \Sigma^{-1}_{(j_1,j_2,...,j_{m-1},i)} \boldsymbol{X}^{(j_1,j_2,...,j_{m-1},i)} && (4.8) \\ &= X_i - \frac{1}{M-m+1} \sum_{\substack{1\le j\le M \\ j\ne j_1,j_2,...,j_{m-1},i}} X_j && (4.9) \\ &= \tilde{z}_i - \frac{1}{M-m+1} \sum_{\substack{1\le j\le (M+1) \\ j\ne j_1,j_2,...,j_{m-1},i}} \tilde{z}_j && (4.10) \end{aligned}$$

and let $W_{m,i}$ be defined as $\text{Residual}_{m,i}$ divide by $\sum_{\substack{1\le j\le (M+1) \\ j\ne j_1,j_2,...,j_{m-1}}} \tilde{z}_j$ to make it invariant in scale. That is

$$W_{m,i} = \frac{\text{Residual}_{m,i}(\tilde{\boldsymbol{z}})}{\sum_{\substack{1\le j\le (M+1) \\ j\ne j_1,j_2,...,j_{m-1}}} \tilde{z}_j} = \frac{\tilde{z}_i - \frac{1}{M-m+1} \sum_{\substack{1\le j\le (M+1) \\ j\ne j_1,j_2,...,j_{m-1},i}} \tilde{z}_j}{\sum_{\substack{1\le j\le (M+1) \\ j\ne j_1,j_2,...,j_{m-1}}} \tilde{z}_j} \tag{4.11}$$

Then our test statistics $U_{m,i}$ is defined as:

$$U_{m,i} = (W_{m,i})^2 \tag{4.12}$$

for the two sided (4.5) case, $m = 1, ..., M$.

And

$$U_{m,i} = W_{m,i} \tag{4.13}$$

for the one sided (4.6) case, $m = 1, ..., M$.

4.1.1 MRD Procedure

MRD Procedure:

Let $c_1 > c_2 > \cdots > c_M > 0$ be a given set of constants.

Stage 1: Let $I_1 = \{1, 2, ..., M\}$. If $U_{1,j_1} = \max\{U_{1,i} : i \in I_1\} < c_1$, then accept all hypotheses and stop; otherwise, reject H_{j_1} and continue.

Stage 2: Let I_2 be the indices of the hypotheses not previously rejected. If $U_{2,j_2} = \max\{U_{2,i} : i \in I_2\} < c_2$, then accept all hypotheses in I_2 and stop; otherwise, reject H_{j_2} and continue.

$\vdots$

In general at stage m: $1 \leq m \leq M$, let I_m be the indices of the hypotheses not previously rejected. If $U_{m,j_m} = \max\{U_{m,i} : i \in I_m\} < c_m$, then accept all hypotheses in I_m and stop; otherwise, reject H_{j_m} and continue.

4.1.2 Admissibility of MRD

Similarly we will demonstrate that for each individual testing problem that the MTP based on MRD method is admissible. Without loss of generality we focus on H_1 vs K_1. Again we will use the result of Matthes and Truax (1967) and demonstrate in Lemma 4.1.1 that $W_{m,i}(\tilde{\mathbf{z}})$ function given in (4.11) has the monotonicity properties which enable us to prove in Lemma 4.1.2 that the individual test functions for H_i vs K_i have the convexity property that is necessary and sufficient for admissibility. Theorem 4.1.1 summarizes and states the admissibility of the MRD procedure.

The density of $\tilde{\boldsymbol{z}}$ is expressed in (4.4), now let $\boldsymbol{Y} = (A')^{-1}\tilde{\boldsymbol{z}}$ so that

$$f_{\mathbf{Y}}(\mathbf{y}|\boldsymbol{\nu}) = h^{**}(\mathbf{y})\beta^*(\boldsymbol{\nu})\exp\{\sum_{i=1}^{M+1} y_i\nu_i\} \tag{4.14}$$

Note, to study the test function $\psi(\mathbf{y}) = \phi_U(\tilde{\boldsymbol{z}})$ as y_1 varies and $(y_2, ..., y_{(M+1)})$ remain fixed, we can consider sample points $\tilde{\boldsymbol{z}} + r\boldsymbol{g}$ where $\boldsymbol{g}$ is the first row of A and r varies. This is true since $\boldsymbol{y}$ is a function of $\tilde{\boldsymbol{z}}$ and so $\boldsymbol{y}$ evaluated at $(\tilde{\boldsymbol{z}} + r\boldsymbol{g})$ is $(A')^{-1}(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = \boldsymbol{y} + (r, 0, ..., 0)' = (y_1 + r, y_2, ..., y_{(M+1)})$.

Lemma 4.1.1. *The function $W_{m,j}(\tilde{\boldsymbol{z}})$ given in (4.11) have the following properties:*

At any stage m, as far as H_1 has not been rejected, for any $i \neq 1$, i.e., $i \in \{2, ..., M\} \setminus \{j_1, ..., j_{m-1}\}$, $j_1 \neq 1, ..., j_{m-1} \neq 1$,

$$W_{m,i}(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = W_{m,i}(\tilde{\boldsymbol{z}}) \tag{4.15}$$

and

$$W_{m,1}(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = W_{m,1}(\tilde{\boldsymbol{z}}) + ar \tag{4.16}$$

where a is some constant and $a > 0$;

Proof of Lemma 4.1.1. For $i = 1$, use (4.11) and recall $\boldsymbol{g} = (1, 0, 0, ..., -1)'$ is the first row of A to see that

$$W_{m,1}(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = \frac{(\tilde{z}_1 + r) - \frac{1}{M-m+1}\sum_{\substack{1\le j\le(M+1)\\ j\ne 1, j_1, j_2, \ldots, j_{m-1}}} \tilde{z}_j + \frac{1}{M-m+1}r}{\sum_{\substack{1\le j\le(M+1)\\ j\ne j_1, j_2, \ldots, j_{m-1}}} \tilde{z}_j}$$

$$= W_{m,i}(\tilde{\boldsymbol{z}}) + ar$$

where $a = \frac{1+\frac{1}{M-m+1}}{\sum_{\substack{1\le j\le(M+1)\\ j\ne j_1, j_2, \ldots, j_{m-1}}} \tilde{z}_j}$, so $a > 0$. This establishes (4.16).

Now for $i \neq 1$, $j_k \neq 1, k = 1, ..., (m-1)$,

$$W_{m,i}(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = \frac{\tilde{z}_i - \frac{1}{M-m+1}\sum_{\substack{1\le j\le(M+1)\\ j\ne j_1, j_2, \ldots, j_{m-1}, i}} \tilde{z}_j}{\sum_{\substack{1\le j\le(M+1)\\ j\ne j_1, j_2, \ldots, j_{m-1}}} \tilde{z}_j}$$

$$= W_{m,i}(\tilde{\boldsymbol{z}})$$

This establishes (4.15).

□

Lemma 4.1.2. *Suppose that for some* $\tilde{\boldsymbol{z}}^*$ *and* $r_0 > 0, \phi_U(\tilde{\boldsymbol{z}}^*) = 0$ *and* $\phi_U(\tilde{\boldsymbol{z}}^* + r_0\boldsymbol{g}) = 1$. *Then* $\phi_U(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$ *for all* $r > r_0$. *This is true both for the one sided alternatives (4.6)and two sided alternatives (4.5) of the treatment vs control problem of variance in this Chapter.*

Proof. Same as proof of Lemma 3.1.2. □

Not that Lemma 4.1.2 again implies that the acceptance region in y_1, for fixed y_2, ..., y_{M+1} is an interval.

Theorem 4.1.1. *Both for the one sided alternatives (4.6) and two sided alternatives (4.5), the MRD procedure based on* $U_{m,i}$ *is admissible*

Proof. Same as proof of Theorem 3.1.1. □

4.2 M-LRD

The Maximum-Likelihood Ratio down (M-LRD) method depends on Likelihood Ratios for each stage. Again, only the two-sided test is addressed here.

4.2.1 M-LRD Procedure

M-LRD Procedure:

Let $c_1 > c_2 > \cdots > c_M > 0$ be a given set of constants.

Stage 1: Let $I_1 = \{1, 2, ..., M\}$ be the indices of the hypotheses of (4.5). We test $H_{1G} : \nu_1 = \nu_2 = ... = \nu_M = 0$ vs $K_i^1 : H_{1G}$ but $\nu_i \neq 0$. Let $L_{1,i}$ be the likelihood ratio for H_{1G} vs K_i^1. If $L_{1,j_1} = \max\{L_{1,i} : i \in I_1\} < c_1$, then accept H_{1G} and stop, i.e., the variances of treatments and control are equal; Otherwise, reject H_{j_1} and continue, then the variance of the j_1^{th} treatment is different from the variance of control.

Stage 2: Let I_2 be the indices of the hypotheses not previously rejected. Now we test $H_{2G} : \nu_1 = ... = \nu_{j_1-1} = \nu_{j_1+1} = ... = \nu_M = 0$ vs $K_i^2 : H_{2G}$ but $\nu_i \neq 0$, $i \in I_2$. Let $L_{2,i}$ be the likelihood ratio for H_{2G} vs K_i^2. If $L_{2,j_2} = \max\{L_{2,i} : i \in I_2\} < c_2$, then accept H_{2G} and stop; otherwise, reject H_{j_2} and continue.

$\vdots$

In general at stage m: $1 \leq m \leq M$, let I_m be the indices of the hypotheses not previously rejected. Now we test H_{mG} : all the $\nu_i = 0, i \in I_m$ vs $K_i^m : H_{mG}$ but $\nu_i \neq 0$, $i \in I_m$. Let $L_{m,i}$ be the likelihood ratio for H_{mG} vs K_i^m. If $L_{m,j_m} = \max\{L_{m,i} : i \in I_m\} < c_m$, then accept H_{mG} and stop; otherwise, reject H_{j_m} and continue.

4.2.2 Admissibility of M-LRD

For stage m after rejecting $H_{j_1}, H_{j_2}, ...H_{j_{m-1}}$, test

$$H_{mG} : all\ the\ \nu_i = 0, i \in K_m \quad vs \quad K_i^m : H_{mG}\ but\ \nu_i \neq 0, i \in K_m \tag{4.17}$$

is equivalent to test:

$$H'_{mG} : \quad \sigma_k^2 = \sigma_{M+1}^2, k \neq j_1, ...j_{m-1}$$

vs

$$K_i'^m : \quad H'_{mG} \quad \text{but} \quad \sigma_i^2 \neq \sigma_{M+1}^2$$

So under H'_{mG}, the likelihood function of $\boldsymbol{s^{2(j_1,\ldots j_{m-1})}}$ which is the (M-m+2) vector consisting of the components of $\boldsymbol{s}^2$ with $s^2_{j_1},\ldots,s^2_{j_{m-1}}$ left out is

$$L_0(\sigma^2_{M+1}) = \prod_{\substack{1\le k\le(M+1)\\ k\ne j_1,\cdots j_{m-1}}} \left(\frac{(n-1)}{\Gamma(\frac{n-1}{2})2^{(n-1)/2}} \frac{((n-1)s_k^2)^{(\frac{n-1}{2}-1)}}{(\sigma^2_{M+1})^{\frac{n-1}{2}}} e^{-\frac{(n-1)s_k^2}{2\sigma^2_{M+1}}} \right)$$

And under $K_i'^m$, the likelihood function is

$$L_1(\sigma_i^2,\sigma^2_{M+1}) = \left(\prod_{\substack{1\le k\le(M+1)\\ k\ne j_1,\cdots j_{m-1},i}} \left(\frac{(n-1)}{\Gamma(\frac{n-1}{2})2^{(n-1)/2}} \frac{((n-1)s_k^2)^{(\frac{n-1}{2}-1)}}{(\sigma^2_{M+1})^{\frac{n-1}{2}}} e^{-\frac{(n-1)s_k^2}{2\sigma^2_{M+1}}} \right) \right)$$
$$\times \left(\frac{(n-1)}{\Gamma(\frac{n-1}{2})2^{(n-1)/2}} \frac{((n-1)s_i^2)^{(\frac{n-1}{2}-1)}}{(\sigma_i^2)^{\frac{n-1}{2}}} e^{-\frac{(n-1)s_k^2}{2\sigma_i^2}} \right)$$

So the likelihood ratio is

$$L_{m,i} = \frac{\sup\limits_{\{\sigma_i^2,\sigma^2_{M+1}\}} L_1}{\sup\limits_{\{\sigma^2_{M+1}\}} L_0}$$
$$= \frac{\sup\limits_{\{\sigma_i^2,\sigma^2_{M+1}\}} \prod_{\substack{1\le k\le(M+1)\\ k\ne j_1,\cdots j_{m-1},i}} \left(\frac{((n-1)s_k^2)^{(\frac{n-1}{2}-1)}}{(\sigma^2_{M+1})^{\frac{n-1}{2}}} e^{-\frac{(n-1)s_k^2}{2\sigma^2_{M+1}}} \right) \times \left(\frac{((n-1)s_i^2)^{(\frac{n-1}{2}-1)}}{(\sigma_i^2)^{\frac{n-1}{2}}} e^{-\frac{(n-1)s_k^2}{2\sigma_i^2}} \right)}{\sup\limits_{\{\sigma^2_{M+1}\}} \prod_{\substack{1\le k\le(M+1)\\ k\ne j_1,\cdots j_{m-1}}} \left(\frac{((n-1)s_k^2)^{(\frac{n-1}{2}-1)}}{(\sigma^2_{M+1})^{\frac{n-1}{2}}} e^{-\frac{(n-1)s_k^2}{2\sigma^2_{M+1}}} \right)}.$$

For the numerator the maximum likelihood estimator(mle) of $\sigma^2_{M+1},\sigma_i^2$ are

$$\hat\sigma^2_{M+1} = \frac{\sum\limits_{\substack{1\le k\le(M+1)\\ k\ne j_1,\cdots j_{m-1},i}} s_k^2}{M-m+1},\ \hat\sigma_i^2 = s_i^2.$$

For the denominator the maximum likelihood estimator(mle) of σ^2_{M+1} is

$$\hat\sigma^2_{M+1} = \frac{\sum\limits_{\substack{1\le k\le(M+1)\\ k\ne j_1,\cdots j_{m-1}}} s_k^2}{M-m+2}$$

.

So

$$L_{m,i} = \left(\left(\frac{\sum\limits_{\substack{1\le k\le(M+1)\\ k\ne j_1,\cdots j_{m-1}}} s_k^2}{M-m+2} \right)^{M-m+2} \left(\frac{1}{s_i^2}\right) \left(\frac{M-m+1}{\sum\limits_{\substack{1\le k\le(M+1)\\ k\ne j_1,\cdots j_{m-1},i}} s_k^2} \right)^{M-m+1} \right)^{\frac{(n-1)}{2}} \tag{4.18}$$

Since $\tilde{z}_i = (n-1)s_i^2$, so

$$L_{m,i} = \left(\left(\frac{\sum\limits_{\substack{1\le k\le (M+1) \\ k\ne j_1,\cdots j_{m-1}}} \tilde{z}_k}{M-m+2} \right)^{M-m+2} \left(\frac{1}{\tilde{z}_i}\right) \left(\frac{M-m+1}{\sum\limits_{\substack{1\le k\le (M+1) \\ k\ne j_1,\cdots j_{m-1},i}} \tilde{z}_k} \right)^{M-m+1} \right)^{\frac{(n-1)}{2}} \tag{4.19}$$

Lemma 4.2.1. *The function $L_{m,j}(\tilde{\boldsymbol{z}})$ given in (4.19) have the following properties:*

(1) At any stage m, as far as H_1 has not been rejected, then for any $i \neq 1$, i.e., $i \in \{2, ..., M\} \setminus \{j_1, ..., j_{m-1}\}$, $j_1 \neq 1, ..., j_{m-1} \neq 1$,

$$L_{m,i}(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = L_{m,i}(\tilde{\boldsymbol{z}}) \tag{4.20}$$

for any $r > 0$.

(2) For $i = 1$, regard $L_{m,1}(\tilde{\boldsymbol{z}} + r\boldsymbol{g})$ as a function of r, then:

If for any $0 < r_1 < r_2$, $L_{m,1}(\tilde{\boldsymbol{z}} + r_2\boldsymbol{g}) > L_{m,1}(\tilde{\boldsymbol{z}} + r_1\boldsymbol{g})$, then for any $r > r_2$, $L_{m,1}(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) > L_{m,1}(\tilde{\boldsymbol{z}} + r_2\boldsymbol{g})$.

Proof of Lemma 4.2.1. For $i = 1$, use (4.19) and recall $\boldsymbol{g} = (1, 0, 0, ..., -1)'$ is the first row of A to see that

$$L_{m,1}(\tilde{\boldsymbol{z}}+r\boldsymbol{g}) = \left(\left(\frac{\sum\limits_{\substack{1\le k\le (M+1) \\ k\ne j_1,\cdots j_{m-1}}} \tilde{z}_k}{M-m+2} \right)^{M-m+2} \left(\frac{1}{\tilde{z}_1 + r}\right) \left(\frac{M-m+1}{\sum\limits_{\substack{1\le k\le (M+1) \\ k\ne j_1,\cdots j_{m-1},1}} \tilde{z}_k - r} \right)^{M-m+1} \right)^{\frac{(n-1)}{2}}$$

Let

$$\begin{aligned} &l_{m,1}(r) \\ &= \log\{L_{m,1}(\tilde{\boldsymbol{z}} + r\boldsymbol{g})\} \\ &= \frac{(n-1)}{2} \\ &\times \left((M-m+2)\log\left(\frac{\sum\limits_{\substack{1\le k\le (M+1) \\ k\ne j_1,\cdots j_{m-1}}} \tilde{z}_k}{M-m+2} \right) - \log(\tilde{z}_1 + r) - (M-m+1)\log\left(\frac{\sum\limits_{\substack{1\le k\le (M+1) \\ k\ne j_1,\cdots j_{m-1},1}} \tilde{z}_k - r}{M-m+1} \right) \right) \end{aligned}$$

Now take derivative of $l_{m,1}(r)$ with respect to r

$$\frac{\mathrm{d}l_{m,1}(r)}{\mathrm{d}r} = \frac{(n-1)}{2} \left(-\frac{1}{\tilde{z}_1 + r} + (M-m+1)\frac{1}{\sum\limits_{\substack{1\le k\le (M+1) \\ k\ne j_1,\cdots j_{m-1},1}} \tilde{z}_k - r} \right)$$

So as r increases, $\frac{\mathrm{d}l_{m,1}(r)}{\mathrm{d}r}$ increases $\Longrightarrow$ once $\frac{\mathrm{d}l_{m,1}(r)}{\mathrm{d}r}$ becomes positive, it will stay positive $\Longrightarrow$ once $L_{m,1}(\tilde{\boldsymbol{z}}+r)$ increases, it will keep increasing.

For $m = 1, ..., M$; $i \in \{2, ..., M\} \setminus \{j_1, ..., j_{m-1}\}$, $j_1 \neq 1, ..., j_{m-1} \neq 1$, it's obvious that

$$L_{m,i}(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = L_{m,i}(\tilde{\boldsymbol{z}})$$

. □

Lemma 4.2.2. *Suppose that for some* $\tilde{\boldsymbol{z}}^*$ *and* $r_0 > 0, \phi_L(\tilde{\boldsymbol{z}}^*) = 0$ *and* $\phi_L(\tilde{\boldsymbol{z}}^* + r_0\boldsymbol{g}) = 1$. *Then* $\phi_L(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$ *for all* $r > r_0$.

Proof. Same as proof of Lemma 3.1.2. □

Theorem 4.2.1. *For the two sided case the M-LRD procedure based on* $L_{m,i}$ *is admissible.*

Proof. Same as proof of Theorem 3.1.1. □

4.3 Likelihood Ratio Step-Down Method(LRSD)

Similar to the variance change cases in Chapter 3, the LRSD method for one-sided alternatives in this case is as follows:

Let $c_1 > c_2 > \cdots > c_M > 1$ be a given set of constants. At Stage 1: Let $I_1 = \{1, 2, ..., M\}$ be the indices of the hypotheses of (4.6). We test $H_{1G} : \boldsymbol{\nu} = \boldsymbol{0}$ vs $K_{1G} : \boldsymbol{\nu} \geq \boldsymbol{0}$ and at least one $\nu_i > 0$, $i \in I_1$. The likelihood ratio for this test is L_1. If $L_1 < c_1$, accept H_{1G} and stop; Otherwise, reject H_{j_1} where j_1 is the index for which $F_{j_1} = \max\{F_j : j \in I_1\}$, where F_j is defined as:

$$F_j = \frac{s_j^2}{s_{M+1}^2} = \frac{\tilde{z}_j}{\tilde{z}_{M+1}}. \tag{4.21}$$

Continue similarly for the hypotheses not rejected.

In general, the Stage m global hypothesis is considered if H_{j_1}, ..., $H_{j_{m-1}}$ have been rejected. This global testing problem is $H_{mG} : \boldsymbol{\nu}^{(j_1,...,j_{m-1})} = \boldsymbol{0}$ vs $K_{mG} : \boldsymbol{\nu}^{(j_1,...,j_{m-1})} \geq$

$\mathbf{0}$ but at least one $\nu_i > 0$, $i \in I_m$, where I_m is the indices of the hypotheses not previously rejected. The likelihood ratio test rejects H_{mG} if $L_m \geq c_m$, i.e

$$\begin{aligned} &L_m \\ &= \frac{\sup\limits_{\{\sigma_i^2 \geq \sigma_{M+1}^2, i \in I_m\}} \prod\limits_{\substack{1 \leq i \leq (M+1) \\ i \neq j_1, \cdots j_{m-1}}} (\frac{1}{\sigma_i^2})^{\frac{n-1}{2}} e^{-\frac{(n-1)s_i^2}{2\sigma_i^2}}}{\sup\limits_{\sigma_{M+1}^2} \left(\prod\limits_{\substack{1 \leq i \leq (M+1) \\ i \neq j_1, \cdots j_{m-1}}} (\frac{1}{\sigma_{M+1}^2})^{\frac{n-1}{2}} e^{-\frac{(n-1)s_i^2}{2\sigma_{M+1}^2}} \right)} \\ &\geq c_m \end{aligned} \tag{4.22}$$

For the denominator the maximum likelihood estimator is: $\hat{\sigma}_{M+1}^2 = \frac{\sum\limits_{\substack{1 \leq k \leq (M+1) \\ k \neq j_1, \cdots j_{m-1}}} s_k^2}{M-m+2}$, replace σ_{M+1}^2 with it in (4.22), we get:

$$\begin{aligned} &L_m \\ &= e^{\frac{(M-m+2)(n-1)}{2}} \times \\ &\left(\left(\frac{\sum\limits_{\substack{1 \leq k \leq (M+1) \\ k \neq j_1, \cdots j_{m-1}}} s_k^2}{M-m+2} \right)^{M-m+2} \sup_{\{\sigma_i^2 \geq \sigma_{M+1}^2, i \in I_m\}} \prod_{\substack{1 \leq i \leq (M+1) \\ k \neq j_1, \cdots j_{m-1}}} (\frac{1}{\sigma_i^2}) e^{-\frac{s_i^2}{\sigma_i^2}} \right)^{(n-1)/2} \end{aligned} \tag{4.23}$$

Define

$$\begin{aligned} &L'_m \\ &= \left(\frac{\sum\limits_{\substack{1 \leq k \leq (M+1) \\ k \neq j_1, \cdots j_{m-1}}} s_k^2}{M-m+2} \right)^{M-m+2} \sup_{\{\sigma_i^2 \geq \sigma_{M+1}^2, i \in I_m\}} \prod_{\substack{1 \leq i \leq (M+1) \\ k \neq j_1, \cdots j_{m-1}}} (\frac{1}{\sigma_i^2}) e^{-\frac{s_i^2}{\sigma_i^2}} \\ &= \left(\frac{\sum\limits_{\substack{1 \leq k \leq (M+1) \\ k \neq j_1, \cdots j_{m-1}}} \tilde{z}_k}{(n-1)(M-m+2)} \right)^{M-m+2} \sup_{\{\sigma_i^2 \geq \sigma_{M+1}^2, i \in I_m\}} \prod_{\substack{1 \leq i \leq (M+1) \\ k \neq j_1, \cdots j_{m-1}}} (\frac{1}{\sigma_i^2}) e^{-\frac{\tilde{z}_i}{(n-1)\sigma_i^2}} \\ &= \left(\frac{\sum\limits_{\substack{1 \leq k \leq (M+1) \\ k \neq j_1, \cdots j_{m-1}}} \tilde{z}_k}{(n-1)(M-m+2)} \right)^{M-m+2} \prod_{\substack{1 \leq i \leq (M+1) \\ k \neq j_1, \cdots j_{m-1}}} (\frac{1}{\hat{\sigma}_i^2}) e^{-\frac{\tilde{z}_i}{(n-1)\hat{\sigma}_i^2}} \end{aligned} \tag{4.24}$$

where $\hat{\sigma}_i^2$ is the maximum likelihood estimator of σ_i^2 when $\tilde{\boldsymbol{z}} = \tilde{\boldsymbol{z}}$. And $L_m > c_m \iff L'_m > C_m$, where $c_m = e^{\frac{(M-m+2)(n-1)}{2}} \times C_m^{(n-1)/2}$.

Lemma 4.3.1. *When* $\tilde{\boldsymbol{z}}^* = \tilde{\boldsymbol{z}} + r\boldsymbol{g} = \begin{pmatrix} \tilde{z}_1+r \\ \tilde{z}_2 \\ \tilde{z}_3 \\ \cdots \\ \tilde{z}_{M+1}-r \end{pmatrix}$, *if* $j_{(1)} > 1$, *i.e.* H_1 *has not been rejected,* $L_m^{*'} \geq L_m'$.

Proof of Lemma 4.3.1. From (4.24),

$$
\begin{aligned}
&L_m^{*'} \\
&= \left(\frac{\sum\limits_{\substack{1\leq k\leq (M+1)\\ k\neq j_1,\cdots j_{m-1}}} \tilde{z}_k^*}{(n-1)(M-m+2)} \right)^{M-m+2} \sup_{\{\sigma_i^2 \geq \sigma_{M+1}^2, i\in K_m\}} \prod_{\substack{1\leq i\leq (M+1)\\ k\neq j_1,\cdots j_{m-1}}} (\frac{1}{\sigma_i^2}) e^{-\frac{\tilde{z}_i^*}{(n-1)\sigma_i^2}} \\
&= \left(\frac{\sum\limits_{\substack{1\leq k\leq (M+1)\\ k\neq j_1,\cdots j_{m-1}}} \tilde{z}_k}{(n-1)(M-m+2)} \right)^{M-m+2} \sup_{\{\sigma_i^2 \geq \sigma_{M+1}^2, i\in K_m\}} \prod_{\substack{1\leq i\leq (M+1)\\ k\neq j_1,\cdots j_{m-1}}} (\frac{1}{\sigma_i^2}) e^{-\frac{\tilde{z}_i^*}{(n-1)\sigma_i^2}} \\
&\geq \left(\frac{\sum\limits_{\substack{1\leq k\leq (M+1)\\ k\neq j_1,\cdots j_{m-1}}} \tilde{z}_k}{(n-1)(M-m+2)} \right)^{M-m+2} \prod_{\substack{1\leq i\leq (M+1)\\ k\neq j_1,\cdots j_{m-1}}} (\frac{1}{\hat{\sigma}_i^2}) e^{-\frac{\tilde{z}_i^*}{(n-1)\hat{\sigma}_i^2}} \\
&= \left(\frac{\sum\limits_{\substack{1\leq k\leq (M+1)\\ k\neq j_1,\cdots j_{m-1}}} \tilde{z}_k}{(n-1)(M-m+2)} \right)^{M-m+2} (\frac{1}{\hat{\sigma}_1^2}) e^{-\frac{(\tilde{z}_1+r)}{(n-1)\hat{\sigma}_1^2}} (\frac{1}{\hat{\sigma}_{M+1}^2}) e^{-\frac{(\tilde{z}_{M+1}-r)}{(n-1)\hat{\sigma}_{M+1}^2}} \prod_{\substack{2\leq i\leq M\\ k\neq j_1,\cdots j_{m-1}}} (\frac{1}{\hat{\sigma}_i^2}) e^{-\frac{\tilde{z}_i}{(n-1)\hat{\sigma}_i^2}} \\
&= e^{\frac{r}{n-1}(\frac{1}{\hat{\sigma}_{M+1}^2} - \frac{1}{\hat{\sigma}_1^2})} \times L_m' \\
&\geq L_m'
\end{aligned}
$$

since $\hat{\sigma}_1^2 \geq \hat{\sigma}_{M+1}^2$, where $\hat{\sigma}_i^2$ is the maximum likelihood estimator of σ_i^2 when $\tilde{\boldsymbol{z}} = \tilde{\boldsymbol{z}}$. □

Theorem 4.3.1. *For the one-sided alternative case (4.6) LRSD is admissible.*

Proof of Theorem 4.4.1. Once again we focus on H_1 *vs* K_1^* and demonstrate that if $\phi(\tilde{\boldsymbol{z}}^*) = 1$ then $\phi(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$ for all $r > 0$. Suppose H_1 is rejected at stage m for $\tilde{\boldsymbol{z}} = \tilde{\boldsymbol{z}}^*$. Then $F_{j_1}^* > F_{j_2}^* > \cdots > F_{j_{m-1}}^* > F_1^* > F_{j_{m+1}}^* > \cdots > F_{j_M}^*$ and $L_1' \geq C_1$, $L_2' \geq C_2$,..., $L_m' \geq C_m$. Note at $\tilde{\boldsymbol{z}}^{**} = \tilde{\boldsymbol{z}}^* + r\boldsymbol{g}$, $F_i^{**} = \frac{\tilde{z}_i}{\tilde{z}_{M+1}-r}$ for $i \neq 1$ and $F_1^{**} = \frac{\tilde{z}_1+r}{\tilde{z}_{M+1}-r}$, so the orders of all coordinates are preserved except perhaps the first coordinate which now can be anywhere among the m largest coordinates. It follows form Lemma 4.3.1 that $L_k^{**'} \geq L_k^{*'} \geq C_k$, which means there is a rejection at stage k at $\tilde{\boldsymbol{z}}^{**}$ if there was a rejection at stage k at $\tilde{\boldsymbol{z}}^*, k = 1, ..., M$. Since the order of the coordinates of $F_{j_1}^{**}$, $F_{j_2}^{**}, \cdots, F_{j_{m-1}}^{**}$ remains unchanged and F_1^{**} is among the m largest coordinates of $\tilde{\boldsymbol{z}}^{**}$ it follows that H_1 is rejected at stage m or sooner. □

Next we consider testing two-sided alternatives for this treatment vs control model of variance, i.e. we test $H_i : \nu_i = 0 \ vs \ K_i : \nu_i \neq 0$. The LRSD method in this case is the same as in the one-sided alternative case except that F_j is replaced by

$$F_j = \frac{\max\{s_j^2, s_{M+1}^2\}}{\min\{s_j^2, s_{M+1}^2\}} = \frac{\max\{\tilde{z}_j, \tilde{z}_{M+1}\}}{\min\{\tilde{z}_j, \tilde{z}_{M+1}\}} \tag{4.25}$$

.

In general, the Stage m global hypothesis is considered if $H_{j_1}, ..., H_{j_{m-1}}$ have been rejected. This global testing problem is $H_{mG} : \boldsymbol{\nu}^{(j_1,...,j_{m-1})} = \mathbf{0}$ vs $K_{mG} : \boldsymbol{\nu}^{(j_1,...,j_{m-1})} \neq \mathbf{0}$. The likelihood ratio test rejects H_{mG} if $L_m \geq c_m$, i.e

$$\begin{aligned} &L_m \\ &= \frac{\sup\limits_{\{\sigma_i^2, i \in I_m\}} \prod\limits_{\substack{1 \leq i \leq (M+1) \\ i \neq j_1, \cdots j_{m-1}}} (\frac{1}{\sigma_i^2})^{\frac{n-1}{2}} e^{-\frac{(n-1)s_i^2}{2\sigma_i^2}}}{\sup\limits_{\sigma_{M+1}^2} \left(\prod\limits_{\substack{1 \leq i \leq (M+1) \\ k \neq j_1, \cdots j_{m-1}}} (\frac{1}{\sigma_{M+1}^2})^{\frac{n-1}{2}} e^{-\frac{(n-1)s_i^2}{2\sigma_{M+1}^2}} \right)} \\ &\geq c_m \end{aligned} \tag{4.26}$$

For the numerator the maximum likelihood estimators are:

$$\hat{\sigma}_i^2 = s_i^2.$$

For the denominator the maximum likelihood estimator is:

$$\hat{\sigma}_{M+1}^2 = \frac{\sum\limits_{\substack{1 \leq k \leq (M+1) \\ k \neq j_1, \cdots j_{m-1}}} s_k^2}{M - m + 2}.$$

Put them into (4.26), we get:

$$L_m = \prod_{\substack{1 \leq i \leq (M+1) \\ i \neq j_1, \cdots j_{m-1}}} \left(\frac{\sum\limits_{\substack{1 \leq k \leq (M+1) \\ k \neq j_1, \cdots j_{m-1}}} s_k^2}{(M - m + 2) s_i^2} \right)^{(n-1)/2} \tag{4.27}$$

Define

$$L'_m = \prod_{\substack{1\le i\le (M+1)\\ i\ne j_1,\cdots j_{m-1}}} \left(\frac{\sum\limits_{\substack{1\le k\le (M+1)\\ k\ne j_1,\cdots j_{m-1}}} s_k^2}{(M-m+2)s_i^2} \right) = \prod_{\substack{1\le i\le (M+1)\\ i\ne j_1,\cdots j_{m-1}}} \left(\frac{\sum\limits_{\substack{1\le k\le (M+1)\\ k\ne j_1,\cdots j_{m-1}}} \tilde{z}_k}{(M-m+2)\tilde{z}_i} \right) \tag{4.28}$$

Then $L_m \ge c_m \Longleftrightarrow L'_m \ge C_m$, where $((\frac{1}{M-m+2})^{M-m+2} C_m)^{\frac{n-1}{2}} = c_m$. For this set up we have

Theorem 4.3.2. *For the two-sided alternative case (4.5) LRSD is admissible for M=2.*

Proof of Theorem 4.3.2. For M=2, once again we focus on H_1 vs K_1:

(1) If $\tilde{z}_1 > \tilde{z}_3$, we will demonstrate that if $\phi(\tilde{\boldsymbol{z}}) = 1$ then $\phi(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = 1$ for all $r > 0$.

When H_1 is rejected first $\Longrightarrow F_1 = \frac{\tilde{z}_1}{\tilde{z}_3} > F_2 = \frac{\max\{\tilde{z}_2,\tilde{z}_3\}}{\min\{\tilde{z}_2,\tilde{z}_3\}}$ and $L'_1 \ge C_1$. At $\tilde{\boldsymbol{z}}^* = \tilde{\boldsymbol{z}} + r\boldsymbol{g}$, $r > 0$, $F_1^* = \frac{\tilde{z}_1+r}{\tilde{z}_3-r} > F_2^* = \frac{\max\{\tilde{z}_2,\tilde{z}_3-r\}}{\min\{\tilde{z}_2,\tilde{z}_3-r\}}$, $L_1^{*'} = (\frac{\tilde{z}_1+\tilde{z}_2+\tilde{z}_3}{3(\tilde{z}_1+r)})(\frac{\tilde{z}_1+\tilde{z}_2+\tilde{z}_3}{3\tilde{z}_2})(\frac{\tilde{z}_1+\tilde{z}_2+\tilde{z}_3}{3(\tilde{z}_3-r)})) = \frac{\tilde{z}_1\tilde{z}_3}{(\tilde{z}_1+r)(\tilde{z}_3-r)} L'_1 = \frac{\tilde{z}_1\tilde{z}_3}{-(r-\frac{1}{2}(\tilde{z}_3-\tilde{z}_1))^2+\frac{1}{4}(\tilde{z}_3-\tilde{z}_1)^2+\tilde{z}_1\tilde{z}_3} L'_1 > L'_1$, so $\phi_1(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = 1$ too, for all $r > 0$.

When H_1 is rejected secondly $\Longrightarrow F_1 = \frac{\tilde{z}_1}{\tilde{z}_3} < F_2 = \frac{\max\{\tilde{z}_2,\tilde{z}_3\}}{\min\{\tilde{z}_2,\tilde{z}_3\}}$, $L'_1 \ge C_1$ and $L'_2 \ge C_2$. At $\tilde{\boldsymbol{z}}^* = \tilde{\boldsymbol{z}} + r\boldsymbol{g}$, $r > 0$, $F_1^* = \frac{\tilde{z}_1+r}{\tilde{z}_3-r}$, $F_2^* = \frac{\max\{\tilde{z}_2,\tilde{z}_3-r\}}{\min\{\tilde{z}_2,\tilde{z}_3-r\}}$, $L_1^{*'} = \frac{\tilde{z}_1\tilde{z}_3}{(\tilde{z}_1+r)(\tilde{z}_3-r)} L'_1 > L'_1$. If $F_1^* > F_2^*$, we reject H_1 firstly for $\tilde{\boldsymbol{z}}^*$; If $F_2^* > F_1^*$, we reject H_2 firstly, since $L_2^{*'} = (\frac{\tilde{z}_1+\tilde{z}_3}{2(\tilde{z}_1+r)})(\frac{\tilde{z}_1+\tilde{z}_3}{2(\tilde{z}_3-r)}) = \frac{\tilde{z}_1\tilde{z}_3}{(\tilde{z}_1+r)(\tilde{z}_3-r)} L'_2 > L'_2$, we reject H_1 at second stage. Thus $\phi(\tilde{\boldsymbol{z}} + r\boldsymbol{g}) = 1$ for all $r > 0$.

(2) If $\tilde{z}_3 > \tilde{z}_1$, we will demonstrate that if $\phi(\tilde{\boldsymbol{z}}) = 0$, and if $\phi(\tilde{\boldsymbol{z}}^*) = \phi(\tilde{\boldsymbol{z}} + r_1\boldsymbol{g}) = 1$ for certain $r_1 > 0$, then $\phi(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$ for all $r > r_1$.

When both H_1, H_2 are not rejected at $\tilde{\boldsymbol{z}} \Longrightarrow L'_1 < C_1$. In order to reject H_1, r_1 must $> (\tilde{z}_3 - \tilde{z}_1)$, then at $\tilde{\boldsymbol{z}}^* = \tilde{\boldsymbol{z}} + r_1\boldsymbol{g}$, $L_1^{*'} = \frac{\tilde{z}_1\tilde{z}_3}{(\tilde{z}_1+r_1)(\tilde{z}_3-r_1)} L'_1 = \frac{\tilde{z}_1\tilde{z}_3}{-(r_1-\frac{1}{2}(\tilde{z}_3-\tilde{z}_1))^2+\frac{1}{4}(\tilde{z}_3-\tilde{z}_1)^2+\tilde{z}_1\tilde{z}_3} L'_1$ $> L'_1$, then $\tilde{z}_1^* = \tilde{z}_1 + r_1 > \tilde{z}_3$, $\tilde{z}_3^* = \tilde{z}_3 - r_1 < \tilde{z}_1 \Longrightarrow \tilde{z}_1^* > \tilde{z}_3^*$, by the above part (1) we know that $\phi_1(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$, for all $r > 0$.

When H_2 is rejected and H_1 is accepted at $\tilde{\boldsymbol{z}} \Longrightarrow L'_1 \ge C_1$, $F_1 = \frac{\tilde{z}_3}{\tilde{z}_1} < F_2 = \frac{\max\{\tilde{z}_2,\tilde{z}_3\}}{\min\{\tilde{z}_2,\tilde{z}_3\}}$ and $L'_2 < C_2$. To reject H_1 at $\tilde{\boldsymbol{z}}^* = \tilde{\boldsymbol{z}} + r_1\boldsymbol{g}$, there are two cases. One is that at $\tilde{\boldsymbol{z}}^*$,

$L_1^{*'} \geq C_1, F_1^* < F_2^*$ and $L_2^{*'} \geq C_2$; the other one is that $L_1^{*'} \geq C_1, F_1^* > F_2^*$.

For the first case, $L_2^{*'} \geq C_2 \Longrightarrow L_2^{*'} > L_2'$, i.e.,

$$L_2^{*'} = \frac{\tilde{z}_1\tilde{z}_3}{(\tilde{z}_1+r_1)(\tilde{z}_3-r_1)}L_2' = \frac{\tilde{z}_1\tilde{z}_3}{-(r_1-\frac{1}{2}(\tilde{z}_3-\tilde{z}_1))^2+\frac{1}{4}(\tilde{z}_3-\tilde{z}_1)^2+\tilde{z}_1\tilde{z}_3}L_2' > L_2'$$

$\Longrightarrow r_1 > (\tilde{z}_3 - \tilde{z}_1) \Longrightarrow \tilde{z}_1^* = \tilde{z}_1 + r_1 > \tilde{z}_3$, $\tilde{z}_3^* = \tilde{z}_3 - r_1 < \tilde{z}_1 \Longrightarrow \tilde{z}_1^* > \tilde{z}_3^*$, then by part (1) we know that $\phi_1(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$, for all $r > r_1$.

For the second case, $F_1^* > F_2^* \Longrightarrow \tilde{z}_1^* > \tilde{z}_3^*$. Since if $\tilde{z}_1^* < \tilde{z}_3^*$, $F_1^* = \frac{\tilde{z}_3^*}{\tilde{z}_1^*} = \frac{\tilde{z}_3-r}{\tilde{z}_1+r} < F_1$, if $F_2 = \frac{\tilde{z}_2}{\tilde{z}_3} \Longrightarrow F_2^* = \frac{\tilde{z}_2}{\tilde{z}_3-r} > F_2 \Longrightarrow F_2^* > F_1^*$ contradicted with $F_1^* > F_2^*$; if $F_2 = \frac{\tilde{z}_3}{\tilde{z}_2}$ and if $F_2^* = \frac{\tilde{z}_3-r}{\tilde{z}_2}$, since $F_1 = \frac{\tilde{z}_3}{\tilde{z}_1} < F_2 = \frac{\tilde{z}_3}{\tilde{z}_2} \Longrightarrow F_1^* = \frac{\tilde{z}_3-r}{\tilde{z}_1+r} < F_2^* = \frac{\tilde{z}_3-r}{\tilde{z}_2}$ contradicted with $F_1^* > F_2^*$; if $F_2 = \frac{\tilde{z}_3}{\tilde{z}_2}$ and if $F_2^* = \frac{\tilde{z}_2}{\tilde{z}_3-r}$, since $F_1 = \frac{\tilde{z}_3}{\tilde{z}_1} < F_2 = \frac{\tilde{z}_3}{\tilde{z}_2} \Longrightarrow F_1^* = \frac{\tilde{z}_3-r}{\tilde{z}_1+r} < \frac{\tilde{z}_3-r}{\tilde{z}_2} < \frac{\tilde{z}_2}{\tilde{z}_3-r} = F_2^*$ contradicted with $F_1^* > F_2^*$. Thus for this case, $\tilde{z}_1^* > \tilde{z}_3^*$ and H_1 is rejected firstly at $\tilde{\boldsymbol{z}}^*$, by part (1), we know that $\phi_1(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$, for all $r > r_1$.

$\square$

For M=3 we exhibit a set of critical values for which LRSD is inadmissible. To do so we find a sample point $\tilde{\boldsymbol{z}}^*$ at which H_1 is rejected and for which H_1 is accepted at $\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}$. In fact let $\tilde{\boldsymbol{z}}^* = (\tilde{z}_1^*, \tilde{z}_2^*, \tilde{z}_3^*, \tilde{z}_4^*)'$ for $\tilde{z}_2^* > \tilde{z}_1^* > \tilde{z}_4^* > \tilde{z}_3^*$ and $\frac{\tilde{z}_4^*}{\tilde{z}_3^*} > \frac{\tilde{z}_2^*}{\tilde{z}_4^*} > \frac{\tilde{z}_1^*}{\tilde{z}_4^*}$, i.e. $F_3^* > F_2^* > F_1^*$. Thus using (4.28) at stage 1 choose C_1 so that $L_1^{*'} = (\frac{\tilde{z}_1^*+\tilde{z}_2^*+\tilde{z}_3^*+\tilde{z}_4^*}{4\tilde{z}_1^*})(\frac{\tilde{z}_1^*+\tilde{z}_2^*+\tilde{z}_3^*+\tilde{z}_4^*}{4\tilde{z}_2^*})(\frac{\tilde{z}_1^*+\tilde{z}_2^*+\tilde{z}_3^*+\tilde{z}_4^*}{4\tilde{z}_3^*})(\frac{\tilde{z}_1^*+\tilde{z}_2^*+\tilde{z}_3^*+\tilde{z}_4^*}{4\tilde{z}_4^*}) = C_1$, so that H_3 is rejected. At stage 2 we calculate

$$L_2^{*'} = (\frac{\tilde{z}_1^*+\tilde{z}_2^*+\tilde{z}_4^*}{3\tilde{z}_1^*})(\frac{\tilde{z}_1^*+\tilde{z}_2^*+\tilde{z}_4^*}{3\tilde{z}_2^*})(\frac{\tilde{z}_1^*+\tilde{z}_2^*+\tilde{z}_4^*}{3\tilde{z}_4^*}) \tag{4.29}$$

We set $(\frac{\tilde{z}_1^*+\tilde{z}_2^*+\tilde{z}_4^*}{3\tilde{z}_1^*})(\frac{\tilde{z}_1^*+\tilde{z}_2^*+\tilde{z}_4^*}{3\tilde{z}_2^*})(\frac{\tilde{z}_1^*+\tilde{z}_2^*+\tilde{z}_4^*}{3\tilde{z}_4^*}) = C_2$, so H_2 is rejected. At stage 3, H_1 is rejected.

Now let r such that $r < \tilde{z}_2^* - \tilde{z}_1^*$, $r < \tilde{z}_4^* - \tilde{z}_3^*$ and $(\tilde{z}_4^* - r)^2 < \tilde{z}_2^*\tilde{z}_3^*$. Thus at

$\tilde{\boldsymbol{z}}^{**} = \tilde{\boldsymbol{z}}^{*} + r\boldsymbol{g}$, $F_2^{**} = \frac{\tilde{z}_2^*}{\tilde{z}_4^*-r} > F_3^{**} = \frac{\tilde{z}_4^*-r}{\tilde{z}_3^*}$, $F_2^{**} = \frac{\tilde{z}_2^*}{\tilde{z}_4^*-r} > F_1^{**} = \frac{\tilde{z}_1^*+r}{\tilde{z}_4^*-r}$ and

$$
\begin{aligned}
&L_1^{**'} \\
&= \left(\frac{\tilde{z}_1^* + \tilde{z}_2^* + \tilde{z}_3^* + \tilde{z}_4^*}{4(\tilde{z}_1^* + r)}\right)\left(\frac{\tilde{z}_1^* + \tilde{z}_2^* + \tilde{z}_3^* + \tilde{z}_4^*}{4\tilde{z}_2^*}\right)\left(\frac{\tilde{z}_1^* + \tilde{z}_2^* + \tilde{z}_3^* + \tilde{z}_4^*}{4\tilde{z}_3^*}\right)\left(\frac{\tilde{z}_1^* + \tilde{z}_2^* + \tilde{z}_3^* + \tilde{z}_4^*}{4(\tilde{z}_4^* - r)}\right) \\
&= \frac{\tilde{z}_1^*\tilde{z}_4^*}{(\tilde{z}_1^* + r)(\tilde{z}_4^* - r)} L_1^{*'} \\
&> L_1^{*'}
\end{aligned}
$$

This time however, H_2 is rejected at stage 1. At stage 2 we calculate,

$$L_2^{**'} = \left(\frac{\tilde{z}_1^* + \tilde{z}_3^* + \tilde{z}_4^*}{3(\tilde{z}_1^* + r)}\right)\left(\frac{\tilde{z}_1^* + \tilde{z}_3^* + \tilde{z}_4^*}{3\tilde{z}_3^*}\right)\left(\frac{\tilde{z}_1^* + \tilde{z}_3^* + \tilde{z}_4^*}{3(\tilde{z}_4^* - r)}\right) \tag{4.30}$$

We note that (4.29) divided by (4.30) is

$$\frac{(\tilde{z}_1^* + \tilde{z}_2^* + \tilde{z}_4^*)^3(\tilde{z}_1^* + r)\tilde{z}_3^*(\tilde{z}_4^* - r)}{(\tilde{z}_1^* + \tilde{z}_3^* + \tilde{z}_4^*)^3\tilde{z}_1^*\tilde{z}_2^*\tilde{z}_4^*} \tag{4.31}$$

There are many choices of $\tilde{z}_1^*$, $\tilde{z}_2^*$, $\tilde{z}_3^*$, $\tilde{z}_4^*$, r for which (4.31) is greater than 1 (e.g., $\tilde{z}_1^* = 1.6568, \tilde{z}_2^* = 2.7, \tilde{z}_3^* = 1, \tilde{z}_4^* = 1.6432, r = 0.0002$). The fact that (4.31) > 1 implies that we can choose C_2 such that $L_2^{**'} < C_2$ so that at $\boldsymbol{x}^* + r\boldsymbol{g}$ the overall procedure rejects H_2 and accepts H_1, H_3. Note since $\tilde{\boldsymbol{z}} = \tilde{\boldsymbol{z}}^{\boldsymbol{*}} - r\boldsymbol{g}, r < \frac{\tilde{z}_1^* - \tilde{z}_4^*}{2}$ is an accept point ($L_1' < L_1^{*'} = C_1$). Now if H_1 is rejected for $\tilde{\boldsymbol{z}}^{\boldsymbol{*}}$ but accepted for $\tilde{\boldsymbol{z}}^{\boldsymbol{*}} + r\boldsymbol{g}$, that implies the test for H_1 is inadmissible.

The same is true for $M \geq 5$.

4.4 Step-Up

Now again we study two of the most popular stepwise procedures. We demonstrate that the individual tests they induce are inadmissible for these two-sided testing hypotheses of treatment vs control of variances, but admissible for these one-sided testing hypotheses.

For step-up, let $1 \leq C_1 < C_2 < \cdots < C_M$ be a sequence of increasing of critical values and let $F_{(1)} \leq F_{(2)} \leq \cdots \leq F_{(M)}$ be the order statistics of F_1, F_2, ..., F_M, where for one side testing hypotheses of (4.6), F_j is defined in (4.21); and for two sided testing hypotheses of (4.5), F_j is defined in (4.25).

Stage 1: If $F_{(1)} \leq C_1$, accept $H_{(1)}$ where $H_{(1)}$ is the hypothesis corresponding to $F_{(1)}$. Otherwise reject all H_i.

Stage 2: If $H_{(1)}$ is accepted, accept $H_{(2)}$ if $F_{(2)} \leq C_2$. Otherwise reject $H_{(2)}, ..., H_{(M)}$.

......

In general, at stage m, if $F_{(m)} \leq C_m$ accept $H_{(m)}$. Otherwise reject $H_{(m)}, ..., H_{(M)}$.

Theorem 4.4.1. *Consider the treatment vs control problem of this chapter, the step-up procedure is admissible for the one sided testing problem (4.6).*

Proof of Theorem 4.4.1. Once again we focus on H_1 *vs* K_1^* and demonstrate that if $\phi(\tilde{\boldsymbol{z}}^*) = 1$ then $\phi(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$ for all $r > 0$. At $\tilde{\boldsymbol{z}}^*$, $F_j^* = \frac{\tilde{z}_j^*}{\tilde{z}_{M+1}^*}$, for $j = 1, 2, ..., M$. Suppose H_1 is rejected at stage m, then $F_1^* = \frac{\tilde{z}_1^*}{\tilde{z}_{M+1}^*}$ it the m^{th} smallest among the $\boldsymbol{F^*}$. And $F_{(1)}^* \leq C_1, F_{(2)}^* \leq C_2, ..., F_{(m-1)}^* \leq C_{m-1}, F_{(m)}^* = F_1^* > C_m$ and $F_{(j)}^* > C_m$ for $j > m$. Note at $\tilde{\boldsymbol{z}}^{**} = \tilde{\boldsymbol{z}}^* + r\boldsymbol{g}$, $F_1^{**} = \frac{(\tilde{z}_1^* + r)}{(\tilde{z}_{M+1}^* - r)}$ and $F_j^* = \frac{\tilde{z}_j^*}{(\tilde{z}_{M+1}^* - r)}$ for $j \neq 1$, so the value of $\boldsymbol{F^{**}}$ increased, and the order of the coordinates of $\boldsymbol{F^{**}}$ remains unchanged, except the order of F_1^{**} increases, it follows that H_1 is rejected at stage m or sooner.

□

Theorem 4.4.2. *Consider the treatment vs control problem of this chapter, the step-up procedure is inadmissible for the two sided testing problem (4.5).*

Proof of Theorem 4.4.2. Again we focus on H_1 vs K_1. To show $\phi_1(\tilde{\boldsymbol{z}})$ is inadmissible we will find three points $\tilde{\boldsymbol{z}}^{\boldsymbol{*}}, \tilde{\boldsymbol{z}}^{\boldsymbol{**}}, \tilde{\boldsymbol{z}}^{\boldsymbol{***}}$ with $\tilde{\boldsymbol{z}}^{\boldsymbol{**}} = \tilde{\boldsymbol{z}}^{\boldsymbol{*}} - r_1\boldsymbol{g}$, $\tilde{\boldsymbol{z}}^{\boldsymbol{***}} = \tilde{\boldsymbol{z}}^{\boldsymbol{*}} - r_2\boldsymbol{g}$, $r_2 > r_1 > 0$ such that $\phi_1(\tilde{\boldsymbol{z}}^{\boldsymbol{*}}) = 0$, $\phi_1(\tilde{\boldsymbol{z}}^{\boldsymbol{**}}) = 1$, $\phi_1(\tilde{\boldsymbol{z}}^{\boldsymbol{***}}) = 0$. This will prove the theorem by Lemma 2.1.1.

At $\tilde{\boldsymbol{z}}^{\boldsymbol{*}}$, let $\tilde{z}_1^* = C_1 + C_2$, $\tilde{z}_2^* = \frac{2}{C_1}$, $\tilde{z}_j^* = \frac{2}{C_j+1}$, $j = 3, ..., M$, and $\tilde{z}_{M+1}^* = 2$, so $F_1^* = \frac{C_1+C_2}{2}$, $F_2^* = C_1$, $F_j^* = C_j + 1$, $j = 3, ..., M$. Then at stage 1, $F_{(1)}^* = \min\{F_j^*$, $j = 1, 2, ..., M\} = F_2^* \leq C_1 \Longrightarrow \phi_2(\tilde{\boldsymbol{z}}^{\boldsymbol{*}}) = 0$; at stage 2, $F_{(2)}^* = F_1^* \leq C_2 \Longrightarrow \phi_1(\tilde{\boldsymbol{z}}^{\boldsymbol{*}}) = 0$ at $\tilde{\boldsymbol{z}}^{\boldsymbol{*}}$.

Let $r_1 = \frac{(C_2-C_1)}{2(1+C_1)}$, so at $\tilde{\boldsymbol{z}}^{\boldsymbol{**}} = \tilde{\boldsymbol{z}}^{\boldsymbol{*}} - r_1\boldsymbol{g}$, $\tilde{z}_1^{**} = C_1 + C_2 - \frac{(C_2-C_1)}{2(1+C_1)} = \frac{2C_1^2+2C_1C_2+3C_1+C_2}{2(1+C_1)}$, $\tilde{z}_{M+1}^{**} = 2 + \frac{(C_2-C_1)}{2(1+C_1)} = \frac{4+3C_1+C_2}{2(1+C_1)}$, $\tilde{z}_j^{**} = \tilde{z}_j^*$, $j = 2, ..., M$. So $F_1^{**} = \frac{2C_1^2+2C_1C_2+3C_1+C_2}{4+3C_1+C_2} >$

C_1, $F_2^{**} = \frac{(4+3C_1+C_2)C_1}{4(1+C_1)} > C_1$, $F_j^{**} = (2 + \frac{(C_2-C_1)}{2(1+C_1)})\frac{(C_j+1)}{2} > C_j + 1 > C_1$, $j = 3, ..., M$, so we reject all $\Longrightarrow \phi_1(\tilde{\boldsymbol{z}}^{**}) = 1$ at $\tilde{\boldsymbol{z}}^{**}$.

Let $r_2 = \frac{C_2+C_1-2}{2} > r_1$, so at $\tilde{\boldsymbol{z}}^{**} = \tilde{\boldsymbol{z}}^{*} - r_2\boldsymbol{g}$, $\tilde{z}_1^{***} = C_1 + C_2 - \frac{C_2+C_1-2}{2} = \frac{C_1+C_2+2}{2}$, $\tilde{z}_{M+1}^{***} = 2 + \frac{C_2+C_1-2}{2} = \frac{C_1+C_2+2}{2}$, $\tilde{z}_j^{***} = \tilde{z}_j^{*}$, $j = 2, ..., M$, so $F_1^{***} = 1 \leq C_1$, $F_2^{***} = \frac{(C_1+C_2+2)C_1}{4} > C_1$, $F_j^{***} = (2 + \frac{C_2+C_1-2}{2})\frac{(C_j+1)}{2} > C_j + 1 > C_1$, $j = 3, ..., M$, so at stage 1, we accept H_1, i.e., $\phi_1(\tilde{\boldsymbol{z}}^{***}) = 0$ at $\tilde{\boldsymbol{z}}^{***}$. □

4.5 Step-Down

For step-down, let $1 \leq C_1 < C_2 < \cdots < C_M$ be a sequence of increasing of critical values and let $F_{(1)} \leq F_{(2)} \leq \cdots \leq F_{(M)}$ be the order statistics of F_1, F_2, ..., F_M, where for one side testing hypotheses of (4.6), F_j is defined in (4.21); and for two sided testing hypotheses of (4.5), F_j is defined in (4.25).

Stage 1: If $F_{(M)} > C_M$, reject $H_{(M)}$ where $H_{(M)}$ is the hypothesis corresponding to $F_{(M)}$. Otherwise accept all H_i.

Stage 2: If $H_{(M)}$ is rejected, reject $H_{(M-1)}$ if $F_{(M-1)} > C_{M-1}$. Otherwise accept $H_{(1)}, ..., H_{(M-1)}$.

$\cdots\cdots$

In general, at stage m, if $F_{(M-m+1)} > C_{M-m+1}$ reject $H_{(M-m+1)}$. Otherwise accept $H_{(1)}, ..., H_{(M-m+1)}$.

Theorem 4.5.1. *Consider the variance change problem of this chapter, the step-down procedure is admissible for the one sided testing problem (4.6).*

Proof of Theorem 4.5.1. Similar to the proof of Theorem 4.4.1, we focus on H_1 *vs* K_1^* and demonstrate that if $\phi(\tilde{\boldsymbol{z}}^*) = 1$ then $\phi(\tilde{\boldsymbol{z}}^* + r\boldsymbol{g}) = 1$ for all $r > 0$. At $\tilde{\boldsymbol{z}} = \tilde{\boldsymbol{z}}^*$, $F_j^* = \frac{\tilde{z}_j^*}{\tilde{z}_{M+1}^*}$, for $j = 1, 2, ..., M$. Suppose H_1 is rejected at stage m, then $F_1^* = \frac{\tilde{z}_1^*}{\tilde{z}_{M+1}^*}$ it the m^{th} largest among the $\boldsymbol{F}^*$. And $F_{(M)}^* > C_M$, $F_{(M-1)}^* > C_{M-1}$, ..., $F_{(M-m+2)}^* > C_{M-m+2}$, $F_{(M-m+1)}^* = F_1^* > C_{M-m+1}$. Note at $\tilde{\boldsymbol{z}}^{**} = \tilde{\boldsymbol{z}}^* + r\boldsymbol{g}$, $F_1^{**} = \frac{(\tilde{z}_1^*+r)}{(\tilde{z}_{M+1}^*-r)}$ and $F_j^* = \frac{\tilde{z}_j^*}{(\tilde{z}_{M+1}^*-r)}$ for $j \neq 1$, so the value of $\boldsymbol{F}^{**}$ increased without changing the order of $(F_2^{**}, ..., F_M^{**})$, and F_1^{**} is among the m largest coordinates of $\boldsymbol{F}^{**}$, it follows that H_1 is rejected at stage m or sooner. □

Theorem 4.5.2. *Consider the treatment vs control problem of this chapter, the step-down procedure is inadmissible for the two sided testing problem (4.5).*

Proof of Theorem 4.5.2. Similar to the proof of Theorem 4.4.2, we focus on H_1 vs K_1. To show $\phi_1(\tilde{z})$ is inadmissible we will find three points $\tilde{\boldsymbol{z}}^{\boldsymbol{*}}$, $\tilde{\boldsymbol{z}}^{\boldsymbol{**}}$, $\tilde{\boldsymbol{z}}^{\boldsymbol{***}}$ with $\tilde{\boldsymbol{z}}^{\boldsymbol{**}} = \tilde{\boldsymbol{z}}^{\boldsymbol{*}} - r_1\boldsymbol{g}$, $\tilde{\boldsymbol{z}}^{\boldsymbol{***}} = \tilde{\boldsymbol{z}}^{\boldsymbol{*}} - r_2\boldsymbol{g}$, $r_2 > r_1 > 0$ such that $\phi_1(\tilde{\boldsymbol{z}}^{\boldsymbol{*}}) = 0$, $\phi_1(\tilde{\boldsymbol{z}}^{\boldsymbol{**}}) = 1$, $\phi_1(\tilde{\boldsymbol{z}}^{\boldsymbol{***}}) = 0$. This will prove the theorem by Lemma 2.1.1.

At $\tilde{\boldsymbol{z}}^{\boldsymbol{*}}$, use the same $\tilde{\boldsymbol{z}}^{\boldsymbol{*}}$ for the proof of Theorem 4.4.2, except change $\tilde{z}_2^*$ to $\tilde{z}_2^* = \frac{2}{C_2}$. i.e., $\tilde{z}_1^* = C_1 + C_2$, $\tilde{z}_{M+1}^* = 2$, $\tilde{z}_j^* = \frac{2}{C_j+1}$, $j = 3, ..., M$, so use the definition of F_j in (4.25), $F_1^* = \frac{C_1+C_2}{2} < C_2$, $F_2^* = C_2$, $F_j^* = C_j + 1 > C_j$, $j = 3, ..., M$. From the above step-down procedure, we accept H_1 and H_2, i.e., $\phi_1(\tilde{\boldsymbol{z}}^{\boldsymbol{*}}) = 0$ at $\tilde{\boldsymbol{z}}^{\boldsymbol{*}}$.

Use the same $r_1 = \frac{(C_2-C_1)}{2(1+C_1)}$, so at $\tilde{\boldsymbol{z}}^{\boldsymbol{**}} = \tilde{\boldsymbol{z}}^{\boldsymbol{*}} - r_1\boldsymbol{g}$, $\tilde{z}_1^{**} = C_1 + C_2 - \frac{(C_2-C_1)}{2(1+C_1)} = \frac{2C_1^2+2C_1C_2+3C_1+C_2}{2(1+C_1)}$, $\tilde{z}_{M+1}^{**} = 2 + \frac{(C_2-C_1)}{2(1+C_1)} = \frac{4+3C_1+C_2}{2(1+C_1)}$, $\tilde{z}_j^{**} = \tilde{z}_j^*$, $j = 3, ..., M$. So $F_1^{**} = \frac{2C_1^2+2C_1C_2+3C_1+C_2}{4+3C_1+C_2} > C_1$, $F_2^{**} = \frac{(4+3C_1+C_2)C_2}{4(1+C_1)} > C_2$, $F_j^{**} = (2+\frac{(C_2-C_1)}{2(1+C_1)})(\frac{C_j+1}{2}) > C_j+1$, $j = 3, ..., M$, so we reject all $\Longrightarrow \phi_1(\tilde{\boldsymbol{z}}^{\boldsymbol{**}}) = 1$ at $\tilde{\boldsymbol{z}}^{\boldsymbol{**}}$.

Use the same $r_2 = \frac{C_2+C_1-2}{2} > r_1$, so at $\tilde{\boldsymbol{z}}^{\boldsymbol{**}} = \tilde{\boldsymbol{z}}^{\boldsymbol{*}} - r_2\boldsymbol{g}$, $\tilde{z}_1^{***} = C_1 + C_2 - \frac{C_2+C_1-2}{2} = \frac{C_1+C_2+2}{2}$, $\tilde{z}_{M+1}^{***} = 2 + \frac{C_2+C_1-2}{2} = \frac{C_1+C_2+2}{2}$, $\tilde{z}_j^{***} = \tilde{z}_j^*$, $j = 3, ..., M$. So $F_1^{***} = 1 \le C_1$, $F_2^{***} = \frac{(C_1+C_2+2)C_2}{4} > C_2$, $F_j^{***} = (2 + \frac{C_2+C_1-2}{2})(\frac{C_j+1}{2}) > C_j + 1$, $j = 3, ..., M$, so we accept H_1, i.e., $\phi_1(\tilde{\boldsymbol{z}}^{\boldsymbol{***}}) = 0$ at $\tilde{\boldsymbol{z}}^{\boldsymbol{***}}$. $\square$

Chapter 5

Choosing critical values to control strong FWER for MRD procedure

The MRD procedure can be viewed as a family of admissible procedures parameterized by a set of constants $c_1, ..., c_M$. It is shown that using an inequality due to Sidák (1968) that $c_1, ..., c_M$ can be chosen so that the MRD procedure controls the strong FWER at level α, thus controls FDR at level α (see Lehamann and Romano).

Assume P is the true probability distribution generating the data, let $I = I(P) \subset \{1, 2, ..., M\}$ denote the indices of the set of true hypotheses. For $K \subset \{1, 2, ..., M\}$, let H_K denote the intersection hypothesis that all H_i with $i \in K$ are true.

Let the critical value be $\hat{c}_{j,K}(1 - \frac{\alpha}{M})$, which is designed for testing the intersection hypothesis H_K, at nominal level $\frac{\alpha}{M}$, at stage j, when assuming that U's for that stage are independent. I.e.,

$$P_{\{\text{independent}\}}\{\max\{U_{j,i}, i \in K\} \geq \hat{c}_{j,K}(1 - \frac{\alpha}{M})\} = \frac{\alpha}{M}$$

Then we will prove that this set of critical values control strong FWER for MRD procedure at level α.

Consider the event that MRD procedure commits a false rejection, so that for some $i \in I(P)$, hypothesis H_i is rejected. Let j be the earliest stage in the method where this occurs, which means

$$\begin{aligned}
&\max\{U_{1,i}, i \in I_1\} = \max\{U_{1,i}, i \in I_1 \setminus I(P)\} \geq \hat{c}_{1,I_1}(1 - \frac{\alpha}{M}) \\
&\max\{U_{2,i}, i \in I_2\} = \max\{U_{2,i}, i \in I_2 \setminus I(P)\} \geq \hat{c}_{2,I_2}(1 - \frac{\alpha}{M}) \\
&\qquad\qquad\vdots \\
&\max\{U_{j-1,i}, i \in I_{j-1}\} = \max\{U_{j-1,i}, i \in I_{j-1} \setminus I(P)\} \geq \hat{c}_{j-1,I_{j-1}}(1 - \frac{\alpha}{M}) \\
&\max\{U_{j,i}, i \in I_j\} = \max\{U_{j,i}, i \in I(P)\} \geq \hat{c}_{j,I_j}(1 - \frac{\alpha}{M})
\end{aligned} \tag{5.1}$$

Note that this can only happens before or at the $(M-|I|+1)^{\text{th}}$ stage, so,

$$
\begin{aligned}
& FWER \\
& = P\{\bigcup_{j=1}^{M-|I|+1}(5.1)\text{happens}\} \\
& \leq P\{\bigcup_{j=1}^{M-|I|+1}(\max\{U_{j,i}, i \in I_j\} = \max\{U_{j,i}, i \in I(P)\} \geq \hat{c}_{j,I_j}(1-\frac{\alpha}{M}))\} \\
& \leq \sum_{j=1}^{M-|I|+1} P\{\max\{U_{j,i}, i \in I_j\} = \max\{U_{j,i}, i \in I(P)\} \geq \hat{c}_{j,I_j}(1-\frac{\alpha}{M})\}
\end{aligned}
\tag{5.2}
$$

When U's are independent, according to Sidák (1968),

$$
\leq \sum_{j=1}^{M-|I|+1} P_{\{\text{independent}\}}\{\max\{U_{j,i}, i \in I_j\} = \max\{U_{j,i}, i \in I(P)\} \geq \hat{c}_{j,I_j}(1-\frac{\alpha}{M})\}
\tag{5.3}
$$

Since $I_j \supset I(P) \Longrightarrow \hat{c}_{j,I_j}(1-\frac{\alpha}{M}) \geq \hat{c}_{j,I(P)}(1-\frac{\alpha}{M}) \Longrightarrow$

$$
FWER \leq \sum_{j=1}^{M-|I|+1} P_{\{\text{independent}\}}\{\max\{U_{j,i}, i \in I(P)\} \geq \hat{c}_{j,I(P)}(1-\frac{\alpha}{M})\}. \tag{5.4}
$$

So by the definition of $\hat{c}_{j,I(P)}(1-\frac{\alpha}{M})$

$$
\begin{aligned}
FWER \leq & \sum_{j=1}^{M-|I|+1} \frac{\alpha}{M} \\
\leq & \ \alpha
\end{aligned}
\tag{5.5}
$$

Chapter 6

Simulations

The multiple hypothesis testing procedures in this thesis can be viewed as families of procedures parameterized by a set of constants $c_1, ..., c_M$. It is shown in the above chapter that using an inequality due to Sidák (1968) that $C_1, ..., C_M$ can be chosen so that the MRD procedure controls the strong FWER, this implies it also controls FDR. However such a choice of C's would be extremely conservative and would sacrifice the gains achieved by MRD which takes advantage of the correlation among the variables. It may also be possible to choose C's to control FWER and FDR for the M-LRD or LRSD procedures. However this too is likely to lead to an overly conservative procedure. To determine a reasonable set of constants one must study the risks (errors and error rates) for various choices of constants. As is the case in a typical decision theory problem where no optimal procedure exists one must choose from a number of admissible procedures. Of course this process needs to be done prior to looking at the data. To make this choice in practice one must consider the particular application. In the examples we present, a large variety of sets of constants were evaluated through simulation. Those presented gave a good balance of performance in terms of expected numbers of Type I and Type II errors committed.

We have seen in Chapter 2, Chapter 3 and Chapter 4 that the LRSD procedures for the one-sided alternatives of mean cases, the MRD procedures for the change points of variances cases and the M-LRD procedures for the two sided variances of treatment versus control cases possess the intuitive convexity property needed for admissibility. These stepwise procedures make extensive use of the covariance structure at every stage. To see the types of improvements that can be made over usual stepwise methods we now present some simulation studies. We present a comparison of these three methods with either the step-up or step-down method (whichever did best in the given situation). The step-up and step-down methods used in the comparison are those based on P-values determined from marginal distributions. We report the expected number of Type I errors, the expected number of Type II errors and the FDR. To obtain the probabilities of Type I and Type II errors we can divide the expected number of errors in the tables below by the number of true nulls and alternatives respectively. For all simulations we used 1000 iterations.

Table 6.1 gives the results for the one sided treatment versus control model of means. So $\rho = 0.5$ for the intraclass covariance matrix. The difficulty of using one sided LRSD is calculating the likelihood ratio in each stage which involves finding the solution to a quadratic optimization problem. Here we use the package "quadprog" in R which implements the dual method of Goldfarb and Idnani (1982, 1983). This method was found to be very satisfactory compared to other quadratic programming methods. This quadratic programming procedure involves calculating inverse matrices which can take a considerable amount of time. Hence we only present results for M=100. The step-up procedure in the table is based on the difference of two normal variables, each with variance 1. This procedure is the Benjamini-Hochberg(1995) FDR controlling procedure where $FDR = .05$. The critical values for LRSD are somewhat related to the FWER controlling step-down procedure where the control is at level .05. Specifically these critical values for LRSD are as follows: For $\alpha = .05$, $M = 100$, $C_1 = 1.25\Phi^{-1}(1 - .05/M)$, $C_i = 1.2\Phi^{-1}(1 - .05/(M - i + 1))$, $1 < i \leq M$. These critical values were selected by trial and error using simulations with 1000 iterations. They were chosen so that a desirable procedure would ensue and and also to suggest a way to get critical values in other cases. Here M=100 and the results are dramatic in almost all the cases presented here. There is improvement (usually substantial) in the expected number of Type II errors, while the Type I errors remain comparable, though step-up and step-down procedures can be proved admissible for this positive ρ case.

Table 6.2 gives results for the treatment versus control model of variances. The variance of control equals 1. MRD and M-LRD procedures are both presented here. Step-up works better than step-down in this case. So only step-up is presented here.

Here $M = 1000$, $n = 10$, $\alpha = .05$. For MRD, $C_1 = 0.00012F^{-1}_{(n-1),(n-1)}(1 - .05/2M)$, $C_i = 0.00007F^{-1}_{(n-1),(n-1)}(1 - .05/2(M - i + 1))$, $1 < i \leq M$; For M-LRD, $C_1 = 0.11F^{-1}_{(n-1),(n-1)}(1 - .05/2M)$, $C_i = 0.07F^{-1}_{(n-1),(n-1)}(1 - .05/2(M - i + 1))$, $1 < i \leq M$. The step-up procedure in the table is based on P-values of the marginal distributions of $F_{(n-1),(n-1)}$-statistics. The step-up procedure controls FDR at $\alpha = .05$. From it we can see that for a small proportion of true alternatives ($\leq 20\%$) MRD and M-LRD have fewer numbers of mistakes compared to step-up procedure. For the proportion of alternative $> 20\%$, M-LRD performs much better than the other two procedures and M-LRD has smaller number of Type I errors and Type II errors than step-up procedures in almost all the cases here.

Table 6.3 to Table 6.5 deal with the change point model for variances. Unlike the previous two models, the variables in this problem are not exchangeable. Thus the pattern of true variance values as well as the choice of true variance values impacts the operating characteristics of the procedures. It would be difficult to select a particular portion of the parameter space to study without knowing the specific application. We have tried three types of patterns. For all the cases M=1000, $\alpha = .05$.

Pattern 1: The sequence of differences in consecutive variances are of the form: $1, .., 1, 11, 8, 2, 1, ..., 1, 11, 8, 2, 1, ..., 1$ where the triple sets of (11,8,2) are equally spaced. So there are 4 changes (the present variance comparing to the previous variance) accompanied with this tripe set, they are $(-10, -3, -6, -1)$. The results are shown in Table 6.3. For MRD, $C_1 = 0.00005F^{-1}_{(n-1),(n-1)}(1 - .05/2M)$, $C_i = 0.00003F^{-1}_{(n-1),(n-1)}(1 - .05/2(M - i + 1))$, $1 < i \leq M$; For M-LRD, $C_i = 0.55F^{-1}_{(n-1),(n-1)}(1 - .05/2(M - i + 1))$, $1 \leq i \leq M$. The step-up procedure in the table is based on P-values of the marginal distributions of $F_{(n-1),(n-1)}$-statistics. The step-up procedure controls FDR at $\alpha = .05$. The message in Table 6.3 is that MRD has least number of errors for small number of consecutive changes; while M-LRD performs best for larger number of consecutive changes by a slight elevation in the number of Type I errors in exchange for a substantial improvement in Type II errors.

Pattern 2: There is only one spot of consecutive variances changes. The results are shown in Table 6.4. The step-down (Holm (1979)) procedures deals with p-values determined form marginal distributions of $F_{(n-1),(n-1)}$-statistics. It controls FWER at $\alpha = 0.05$. For MRD, $C_i = 0.00005F^{-1}_{(n-1),(n-1)}(1 - .05/2(M - i + 1))$, $1 \leq i \leq M$; For M-LRD, $C_i = 0.7F^{-1}_{(n-1),(n-1)}(1 - .05/2(M - i + 1))$, $1 \leq i \leq M$. Table 6.4 shows that MRD performs best for these one spot of consecutive variances changes situations, most time it almost detects all the changes, while step-down seldom detects the changes.

Pattern 3: The sequence of differences in consecutive variances are of the form: $1, .., 1, 5, 5, 5, 1, ..., 1, 5, 5, 5, 1, ..., 1$ where the triple sets of (5,5,5) are equally spaced. So there are two changes (the present variance comparing to the previous variance) accompanied with this tripe set. The results are shown in Table 6.5. We used the same Cs as for Table 6.4 for MRD and M-LRD. The step-up procedure in the table is based on P-values of the marginal distributions of $F_{(n-1),(n-1)}$-statistics controlling FDR at $\alpha = .05$. From the table we can see that the three methods's performance are comparable. They are quite weak in detecting the change points of variances for this kind of situation.

Tables from simulations

Number of means equal to				Expected # of Type I errors		Expected # of Type II errors		FDR		Number of rejects		Total errors	
0	0.5	2	4	LRSD	SU	LRSD	SU	LRSD	SU	LRSD	SU	LRSD	SU
100	0	0	0	0.145	0.508	0	0	0.03	0.027	0.145	0.508	0.145	0.508
70	0	0	30	0.203	1.282	4.326	9.854	0.007	0.031	25.877	21.428	4.529	11.136
70	0	30	0	0.917	0.869	18.41	26.821	0.066	0.025	12.507	4.048	19.327	27.69
70	30	0	0	0.206	0.565	29.753	29.587	0.038	0.022	0.453	0.978	29.959	30.152
70	0	10	20	0.384	1.303	9.373	15.421	0.016	0.032	21.011	15.882	9.757	16.724
70	10	20	0	0.85	0.826	22.575	28.007	0.085	0.027	8.275	2.819	23.425	28.833
70	10	0	20	0.227	0.986	12.923	17.389	0.012	0.029	17.304	13.597	13.15	18.375
70	10	10	10	0.514	1.015	17.829	23.084	0.035	0.028	12.685	7.931	18.343	24.099
60	40	0	0	0.135	0.468	39.716	39.537	0.026	0.022	0.419	0.931	39.851	40.005
60	0	0	40	0.145	1.419	5.613	11.501	0.004	0.027	34.532	29.918	5.758	12.92
60	0	40	0	0.81	1.113	24.363	34.555	0.045	0.026	16.447	6.558	25.173	35.668
60	20	20	0	0.654	0.962	32.492	37.513	0.065	0.026	8.162	3.449	33.146	38.475
60	10	10	20	0.339	0.857	19.253	25.306	0.015	0.022	21.086	15.551	19.592	26.163
50	10	20	20	0.346	0.975	25.579	33.024	0.013	0.021	24.767	17.951	25.925	33.999
40	20	20	20	0.238	0.807	35.879	42.417	0.009	0.017	24.359	18.39	36.117	43.224
30	30	20	20	0.158	0.672	46.459	51.984	0.006	0.013	23.699	18.688	46.617	52.656

Table 7.1: Comparison of LRSD and step-up procedures for treatment vs. control of means

Number of variances equal to					Expected # of Type I errors			Expected # of Type II errors			FDR			Total errors		
1	0.1	0.5	2.5	5	MRD	M-LRD	SU	MRD	M-LRD	SU	MRD	M-LRD	SU	MRD	M-LRD	SU
1000	0	0	0	0	0.19	0.12	1.64	0	0	0	0.05	0.04	0.02	0.19	0.12	1.64
950	50	0	0	0	0.45	1.95	1.66	50	9.35	29.77	0.07	0.045	0.029	50.45	11.3	31.43
950	0	50	0	0	0.09	0.04	4.49	50	49.96	49.92	0.03	0.007	0.01	50.09	50	54.41
950	0	0	50	0	2.81	1.38	16.84	32.43	41.5	48.26	0.13	0.121	0.052	35.24	42.88	65.1
950	0	0	0	50	4.17	1.87	8.36	7.04	13.15	45.77	0.086	0.047	0.038	11.21	15.02	54.13
900	50	0	50	0	4.64	1.87	9.07	79.69	48.01	77.5	0.182	0.035	0.035	84.33	49.88	86.57
900	50	0	0	50	6.6	1.91	1.44	56.41	21.56	75.15	0.129	0.024	0.027	63.01	23.47	76.59
900	0	50	50	0	3.28	1.89	0.07	81.26	88.69	99.91	0.144	0.141	0.025	84.54	90.58	99.98
900	0	50	0	50	5.13	1.7	3.22	57.16	62.15	95.31	0.104	0.042	0.027	62.29	63.85	98.53
850	100	0	0	50	9.78	1.89	6.27	105.41	27.83	94.2	0.176	0.015	0.023	115.19	29.72	100.47
850	0	100	50	0	4.36	1.51	4.44	129.35	137.33	149.41	0.171	0.097	0.044	133.71	138.84	153.85
850	50	0	100	0	3.7	1.9	2.14	111.48	90.02	128.76	0.085	0.03	0.031	115.18	91.92	130.9
800	100	0	0	100	14.2	2.05	4.73	109.35	38.49	120.18	0.133	0.012	0.025	123.55	40.54	124.91
800	0	100	100	0	3.65	1.66	4.27	160.73	178.65	198.14	0.084	0.072	0.034	164.38	180.31	202.41
800	50	0	0	150	11.45	2.33	4.81	64.68	46.74	150.29	0.077	0.015	0.026	76.13	49.07	155.1
800	0	50	150	0	2.03	1.82	3.87	148.03	177.71	195.99	0.036	0.076	0.032	150.06	179.53	199.86
800	150	0	50	0	11.7	2.09	10.53	174.64	58.39	106.73	0.31	0.014	0.037	186.34	60.48	117.26
800	50	50	50	50	7.22	1.98	12.03	133.19	109.24	170.29	0.095	0.021	0.044	140.41	111.22	182.32
700	0	150	0	150	13.75	1.62	0.03	162.76	181.34	286.03	0.09	0.013	0.001	176.51	182.96	286.06
650	150	50	150	0	16.44	2.05	4.87	264.13	186.26	254.95	0.156	0.012	0.022	280.57	188.31	259.82
600	200	100	100	0	43.05	1.48	6.71	330.21	197.34	248.77	0.37	0.007	0.025	373.26	198.82	255.48
600	100	100	100	100	28.9	1.98	9.35	240.82	217.14	308.1	0.15	0.011	0.035	269.72	219.12	317.45
600	100	200	100	0	19.29	1.51	3.51	339.22	288.81	335.48	0.237	0.013	0.022	358.51	290.32	338.99
500	50	200	150	100	39.14	1.32	14.59	298.26	350.39	440.33	0.156	0.009	0.037	337.4	351.71	454.92

Table 7.2: Comparison of MRD, M-LRD and step-up procedures for treatment vs. control of variances

Number of			Expected # of Type I errors			Expected # of Type II errors			FDR			Total errors		
nulls	triples	changes	MRD	M-LRD	SU	MRD	M-LRD	SU	MRD	M-LRD	SU	MRD	M-LRD	SU
1000	0	0	0.05	0.023	0.072	0	0	0	0.039	0.023	0.047	0.05	0.023	0.072
996	1	4	0.24	0.084	0.07	0.022	2.398	3.876	0.043	0.054	0.043	0.262	2.482	3.946
988	3	12	0.235	0.347	0.088	8.02	9.974	11.52	0.042	0.084	0.037	8.255	10.321	11.608
980	5	20	0.211	0.783	0.101	16.018	17.143	19.116	0.038	0.139	0.04	16.229	17.926	19.217
960	10	40	0.147	2.541	0.221	36.038	32.74	37.65	0.028	0.216	0.047	36.185	35.281	37.871
900	25	100	0.111	7.832	0.554	96.209	68.82	89.785	0.021	0.2	0.043	96.32	76.652	90.339
860	35	140	0.183	9.844	0.897	136.424	89.511	122.182	0.031	0.163	0.042	136.607	99.355	123.079
800	50	200	0.289	9.984	1.327	196.864	129.04	169.53	0.047	0.127	0.039	197.153	139.024	170.857
760	60	240	0.337	8.558	1.566	237.567	164.085	200.201	0.051	0.104	0.036	237.904	172.643	201.767

Table 7.3: Comparison of MRD, M-LRD and step-up procedures for the chance point model of variances, with variances of the pattern: 1,..,1,11,8,2,1,...,1,11,8,2,1,...,1,…

Different Situations		Expected # of Type I errors			Expected # of Type II errors			FDR			Total errors		
Variances	Number of changes	MRD	M-LRD	SD	MRD	M-LRD	SD	MRD	M-LRD	SD	MRD	M-LRD	SD
1,…,1,2,…,2	1	0.01	0.04	0.05	0.69	0.96	1	0.005	0.04	0.04	0.7	1	1.05
1,…,1,6,…,,6	1	0.02	0.1	0.05	0.08	0.36	0.99	0.01	0.095	0.04	0.1	0.46	1.04
1,…,1,2,3,…,3	2	0.02	0.14	0.03	0.94	1.66	2	0.008	0.135	0.03	0.96	1.8	2.03
1,…,1,6,11,…,11,16,21,…,21	2	0.07	0.08	0.03	0.04	1.07	1.97	0.023	0.075	0.03	0.11	1.15	2
1,…,1,2,3,4,…,4	3	0.03	0.09	0.02	1.18	2.25	2.99	0.018	0.08	0.02	1.21	2.34	3.01
1,…,1,6,11,16,...,16	3	0.04	0.05	0.06	0.06	1.99	3	0.01	0.03	0.06	0.1	2.04	3.06
1,…,1,0.5,1.5,3,…,3	3	0.04	0.07	0.04	1.93	2.48	2.99	0.017	0.05	0.03	1.97	2.55	3.03
1,…,1,2,3,4,5,…,5	4	0.03	0.1	0.05	1.21	3.12	4	0.006	0.09	0.04	1.24	3.22	4.05
1,…,1,3,4,5,1,…,1	4	0.05	0.11	0.03	0.53	3.03	3.98	0.018	0.095	0.03	0.58	3.14	4.01
1,…,1,6,11,16,21,…,21	4	0.03	0.03	0.03	0.09	2.96	3.97	0.006	0.015	0.03	0.12	2.99	4
1,…,1,11,8,12,11,…,11	4	0.01	0.04	0.03	0.03	3	3.88	0.002	0.02	0.02	0.04	3.04	3.91
1,…,1,11,21,36,26,…,26	4	0.04	0.02	0.04	0.03	3	3.89	0.008	0.01	0.03	0.07	3.02	3.93
1,…,1,2,3,4,5,6,…,6	5	0.05	0.08	0.06	0.98	4.05	5	0.009	0.07	0.06	1.03	4.13	5.06
1,…,1,6,11,16,21,26,…,26	5	0.03	0.03	0.03	0.07	3.92	5	0.005	0.015	0.03	0.1	3.95	5.03

Table 7.4: Comparison of MRD, M-LRD and step-down procedures for the chance point model of variances, with only one spot of consecutive changes in variances.

Number of			Expected # of Type I errors			Expected # of Type II errors			FDR			Total errors		
nulls	triples	changes	MRD	M-LRD	SU	MRD	M-LRD	SU	MRD	M-LRD	SU	MRD	M-LRD	SU
1000	0	0	0.05	0.02	0.08	0	0	0	0.05	0.02	0.07	0.05	0.02	0.08
998	1	2	1.89	0.18	0.05	0.15	1.16	1.99	0.481	0.165	0.04	2.04	1.34	2.04
996	2	4	2	0.14	0.13	2.05	2.93	3.94	0.501	0.102	0.078	4.05	3.07	4.07
994	3	6	1.93	0.32	0.1	4.17	4.97	5.88	0.489	0.178	0.042	6.1	5.29	5.98
992	4	8	1.89	0.34	0.1	6.18	6.76	7.87	0.496	0.149	0.075	8.07	7.1	7.97
990	5	10	1.88	0.45	0.08	8.21	8.8	9.84	0.478	0.191	0.041	10.09	9.25	9.92
980	10	20	1.89	0.74	0.14	18.22	18.05	19.62	0.493	0.216	0.065	20.11	18.79	19.76
970	15	30	1.79	1.07	0.14	28.29	27.6	29.34	0.457	0.272	0.05	30.08	28.67	29.48
960	20	40	1.85	1.23	0.11	38.27	36.64	39.27	0.485	0.214	0.05	40.12	37.87	39.38
940	30	60	1.62	2.24	0.13	58.53	54.35	57.86	0.44	0.288	0.037	60.15	56.59	57.99
920	40	80	1.37	2.6	0.2	78.71	73.26	76.85	0.358	0.275	0.037	80.08	75.86	77.05
900	50	100	1.52	2.86	0.47	98.61	91.12	94.77	0.412	0.243	0.058	100.13	93.98	95.24
880	60	120	1.16	2.53	0.46	118.98	110.9	112.79	0.318	0.239	0.042	120.14	113.43	113.25
860	70	140	1.05	1.96	0.72	139.05	132.83	128.97	0.292	0.234	0.051	140.1	134.79	129.69

Table 7.5: Comparison of MRD, M-LRD and step-up procedures for the chance point model of variances, with variances of the pattern: 1,..,1,5,5,5,1,...,1,5,5,5,1,...,1,…

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