ASYMPTOTIC PERTURBATION FORMULAS FOR THE EFFECT OF SCATTERING BY SMALL OBJECTS: AN ANALYSIS OVER A BROAD BAND OF FREQUENCIES

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ABSTRACT OF THE DISSERTATION

Asymptotic Perturbation Formulas for the Effect of Scattering by Small Objects: an Analysis over a Broad Band of Frequencies

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This thesis is a study of the asymptotic perturbation formulas that result from electromagnetic (or acoustic) wave scattering by small, penetrable objects. The ultimate purpose of these formulas is to aid in solving the inverse problem of reconstructing small inhomogeneities embedded within an otherwise known background medium. For simplicity, we consider the time-harmonic, transverse magnetic setting, in which case the scalar electric field satisfies a two-dimensional Helmholtz equation.

We first derive, in the case of fixed frequency, a rigorous asymptotic formula for the boundary field perturbation caused by small inhomogeneities of arbitrary shape within a bounded domain. We then derive formal asymptotic formulas in the case where frequency is allowed to grow as the size of a single, smooth inhomogeneity tends to zero. For high frequencies, we use the technique of geometric optics to derive an integral formula for the scattered field, which we then simplify by a stationary phase analysis. The resulting asymptotic formula is ripe with geometric information to aid in solving the inverse problem. In a step toward a rigorous proof of this high frequency asymptotic formula, we prove an estimate of a Sobolev norm of the scattered field in the case of a penetrable, though conducting, circular scatterer.

Acknowledgements

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While writing Chapter 2, I received help from Frédéric de Gournay in the form of several fruitful discussions. A significant portion of Chapter 3 is based on joint work with Michael Vogelius that appears in [HV]. Most of Chapter 4 is based on joint work with Clair Poignard and Michael Vogelius that appears in [HPV07].

Table of Contents

Chapter 1

Introduction

1.1 Synopsis

The ultimate purpose of this thesis is to provide tools for solving a particular inverse problem—that of detecting and reconstructing electromagnetic inhomogeneities of low volume-fraction within a bounded medium. For simplicity, we consider the time harmonic, transverse magnetic setting, in which case the scalar electric field satisfies a two-dimensional Helmholtz equation. To solve the inverse problem of determining the location, size and electromagnetic profile of the inhomogeneities, one may prescribe Neumann boundary data—that is, apply a tangential magnetic field to the boundary—and then measure the resulting Dirichlet data. This resulting boundary data is a perturbation of what it would be were the medium flawless. The key to reconstructing the inhomogeneities is to study this perturbation.

Recent years have seen the development of noniterative reconstruction algorithms based on asymptotic formulas for these boundary perturbations (for a survey, see [AK04b]). An example of such an asymptotic formula follows: Suppose a finite number of diametrically small inhomogeneities $z_j + \rho D_j$ lie within a domain Ω. Here D_j is fixed, smooth domain, and the parameter ρ is small. Suppose that, at a given time harmonic frequency ω, the permeability and complex permittivity[1] are, respectively, the constants μ_0 and $\boldsymbol{\epsilon}_0$ in the background medium and the constants μ_j and $\boldsymbol{\epsilon}_j$ in the j^{th} inhomogeneity. Then, with u_0 representing the background field and u_ρ the

[1] The complex permittivity $\boldsymbol{\epsilon} = \varepsilon + i\frac{\sigma}{\omega}$, where ε is the permittivity and σ is the conductivity.

perturbed field,

$$\frac{1}{2\mu_0}(u_\rho - u_0)(y) + \int_{\partial\Omega} \frac{1}{\mu_0} \partial_{\nu_x} \Phi^\omega(x,y)(u_\rho - u_0)(x)\, \mathrm{d}\sigma_x$$
$$= \sum_j \rho^2 |D_j| \Big\{ - \Big(\frac{1}{\mu_j} - \frac{1}{\mu_0}\Big) \nabla_x \Phi^\omega(z_j, y) \cdot \big(M(z_j)\nabla u_0(z_j)\big) \tag{1.1}$$
$$+ \omega^2 (\boldsymbol{\epsilon}_j - \boldsymbol{\epsilon}_0) \Phi^\omega(z_j, y) u_0(z_j) \Big\} + o(\rho^2)$$

as $\rho \to 0$, uniformly in $y \in \partial\Omega$. Here Φ^ω is the free-space Green's function for the background Helmholtz operator and the polarization tensors[2] $M(z_j)$ are independent of the prescribed boundary data [VV00]. A similar asymptotic expansion holds for the related scattering problem, wherein $\Omega = \mathbb{R}^2$ and a prescribed wave u^i is incident upon the small inhomogeneities $z_j + \rho D_j$. In this case,

$$u^s_\rho(y) := (u_\rho - u^i)(y)$$
$$= \sum_j \rho^2 |D_j| \Big\{ - \Big(\frac{1}{\mu_j} - \frac{1}{\mu_0}\Big) \nabla_x \Phi^\omega(z_j, y) \cdot \big(M(z_j)\nabla u^i(z_j)\big) \tag{1.2}$$
$$+ \omega^2 (\boldsymbol{\epsilon}_j - \boldsymbol{\epsilon}_0) \Phi^\omega(z_j, y) u^i(z_j) \Big\} + o(\rho^2)$$

as $\rho \to 0$ for y bounded away from the inhomogeneities. This scattering problem distills the essence of the conditions that give rise to formula (1.1) and is generally more amenable to study.

The aforementioned reconstruction algorithms are based on a model where frequency ω remains fixed as the inhomogeneity shrinks, and thus are meant to be applied in situations where $\rho \ll \omega^{-1}$. But at the higher frequencies, say in the regimes where $\rho \approx \omega^{-1}$ or $\rho \gg \omega^{-1}$, the interaction between the incident field and the inhomogeneity is greater. This stronger interaction transmits a stronger, and therefore more detectable, signal to the boundary. The plots in Figure 1.1 demonstrate that for higher frequencies the leading term of the asymptotic expansion in ρ of the field perturbation would be on the order of $\sqrt{\rho}$, which is much larger than the order ρ^2 that results when frequency is fixed. This suggests that a new analysis of high frequency asymptotics should lead

[2]The Pólya-Szëgo polarization tensor was first defined in [SS49]. See also [PS51].

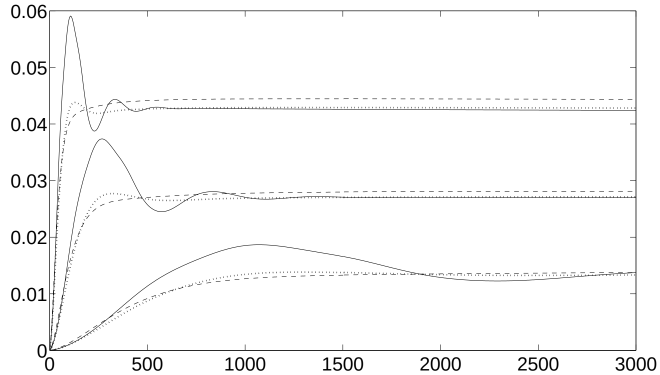

Figure 1.1: Plots of $\|u^s\|_{L^2(\partial B(0,2),\mathrm{d}\theta)}$ as a function of ω, where u^s is the scattered field when a plane wave of frequency ω is incident upon a penetrable disk of radius ρ centered at the origin. Each group of three plots corresponds to a radius of the disk. From top to bottom: $\rho = 0.01$, $\rho = 0.004$, $\rho = 0.001$. For each group, the solid graph corresponds to the value $\boldsymbol{\epsilon}_1 = 3+i$ within the disk, the dotted graph to $\boldsymbol{\epsilon}_1 = 2+2i$ and the dashed graph to $\boldsymbol{\epsilon}_1 = 1+3i$. In all cases, $\mu_1 = 2$ and $\boldsymbol{\epsilon}_0 = \mu_0 = 1$.

to new detection algorithms that are more robust.

In Chapter 3 we derive the following high frequency approximation formula in the case of a plane wave propagating in the direction η and incident upon a finite collection of smooth, convex scatterers $z_j + \rho D_j$ that are well separated:

$$u^s(y) \approx \sum_j \left(\frac{\sqrt{\rho}}{\sqrt{2}} \sqrt{\sin(\theta_j/2)}\, \frac{\frac{\mu_j}{\mu_0}\sin(\theta_j/2) - \sqrt{\frac{\boldsymbol{\epsilon}_j}{\boldsymbol{\epsilon}_0}\frac{\mu_j}{\mu_0} - 1 + \sin^2(\theta_j/2)}}{\frac{\mu_j}{\mu_0}\sin(\theta_j/2) + \sqrt{\frac{\boldsymbol{\epsilon}_j}{\boldsymbol{\epsilon}_0}\frac{\mu_j}{\mu_0} - 1 + \sin^2(\theta_j/2)}} \right.$$
$$\left. \times \frac{e^{i\rho\omega\sqrt{\boldsymbol{\epsilon}_0\mu_0}\left(\eta - (b_j^{y,\eta} - z_j)/|b_j^{y,\eta} - z_j|\right)\cdot(y - z_j)}}{\sqrt{K(b_j^{y,\eta})}} \frac{e^{i\omega\sqrt{\boldsymbol{\epsilon}_0\mu_0}\,|y - z_j|}}{\sqrt{|y - z_j|}} \right), \tag{1.3}$$

for y bounded away from the scatterers, where $0 < \theta_j < 2\pi$ is the counterclockwise angle of rotation between η and $y - z_j$, $K(\cdot) > 0$ is the curvature, and $b_j^{y,\eta}$ is the unique point on the boundary ∂D_j with outward normal pointing in the opposite direction of $\eta - (y - z_j)/|y - z_j|$. While not a rigorous asymptotic formula, numerical computations show this formula to be a close approximation in the backscattered region, which, in the case of a single convex scatterer, is the semi-infinite region bounded by the illuminated portion of the boundary of the scatterer and by the semi-infinite rays that follow the normal vectors at the two points on the boundary that are grazed by the incidence wave (in other words, the points on the boundary where the normal is perpendicular to the direction of propagation of the incident wave). We derive this approximation in the case of one convex scatterer by way of the technique of geometric optics [FK55]: formally expand the amplitude and phase parts of the appropriate ansatz ($Ae^{i\omega\phi}$) in powers of $(\omega\rho)^{-1}$ so that the highest order terms of the incident, transmitted and scattered fields satisfy a certain transmission problem. The solution of this transmission problem is inserted into the Green's representation formula for the scattered field, and we arrive at the above approximation formula after performing a stationary phase analysis.

In the case of fixed frequency, the known asymptotic formula for the scattered field would be just as in (1.1), except with the left-hand side replaced by $u^s(y)$ and u_0 replaced by the incident plane wave u^i. This formula lacks detailed geometric information about the inhomogeneities—each polarization tensor holds information about only the *average* curvature of the boundary of the corresponding inhomogeneity—and therefore has limited use for shape reconstruction. This is not surprising given that it arises in cases where the diameter of each scatterer is small relative to the wavelength of the incident wave, and given that the polarization tensor is completely determined by its action upon just two (any two) non-parallel incident waves. More detailed shape information does appears in higher order terms of this fixed frequency expansion (described in [AK04b]), but reconstruction algorithms that depend on these terms risk being overwhelmed by noise.

The high frequency formula (1.3), on the other hand, contains local information about the curvature of the boundary. With multiple testing from several incident directions, this formula may serve as the basis for new methods of shape reconstruction. If one only seeks the location and size of the inhomogeneities, the formula (1.3) may also prove to be, in many cases, more useful than (1.1) since it is a stronger signal by a factor of $\rho^{-3/2}$ and is thus less prone to corruption by noise.

In Chapter 4 we rigorously estimate the size of the scattered field at high frequencies in the case of a shrinking, penetrable disk. The L^2-based bounds we find are consistent with the formula (1.3) in that they are of the order $\sqrt{\rho}$. Though we prove such estimates only in the case of circular scatterers, we expect our method, based on wave equation factoring within the scatterer, could be modified to apply to general convex domains.

Such bounds will likely serve as a step toward a proof that formula (1.3), or a slight modification of it, is indeed the highest order term of a true asymptotic expansion.

But before we get to this high frequency analysis, we will first prove in Chapter 2 a general asymptotic perturbation formula in the case of fixed frequency and a bounded domain—general in that it applies to any sequence of inhomogeneities $\mathcal{I}_\rho$ with Lebesgue measure tending to zero. This formula is analogous to a similar formula for the conductivity problem (cf. [CV03a, CV03b, CV04]), and, like that formula, has applications to size estimation of the inhomogeneous set in cases where this set is small in volume but highly irregular in shape or not small in diameter.

1.2 Background

Let ε, μ and σ denote respectively the electric permittivity, magnetic permeability and electric conductivity with a given medium, represented as a region in $\mathbb{R}^3$. Maxwell's equations take the form

$$\begin{aligned} \nabla_x \times \mathcal{E} &= -\mu \partial_t \mathcal{H} \\ \nabla_x \times \mathcal{H} &= \varepsilon \partial_t \mathcal{E} + \mathcal{J}, \end{aligned}$$

where $\mathcal{E}$ is the electric field, $\mathcal{H}$ is the magnetic field and $\mathcal{J}$ is the electric current, which is the sum of the free current, $\mathcal{J}_f = \sigma \mathcal{E}$, and any prescribed current source [Gri98, Jac99]. The time-harmonic form of the Maxwell system is

$$\begin{aligned} \nabla \times \mathbf{E} &= i\omega\mu \mathbf{H} \\ \nabla \times \mathbf{H} &= (-i\omega\varepsilon + \sigma)\mathbf{E} + \mathbf{J}. \end{aligned} \tag{1.4}$$

If $\mathbf{E}$ and $\mathbf{H}$ solve this system, then

$$\mathcal{E}(x,t) = \operatorname{Re}\left\{\mathbf{E}(x) e^{-i\omega t}\right\} \quad \text{and} \quad \mathcal{H}(x,t) = \operatorname{Re}\left\{\mathbf{H}(x) e^{-i\omega t}\right\}$$

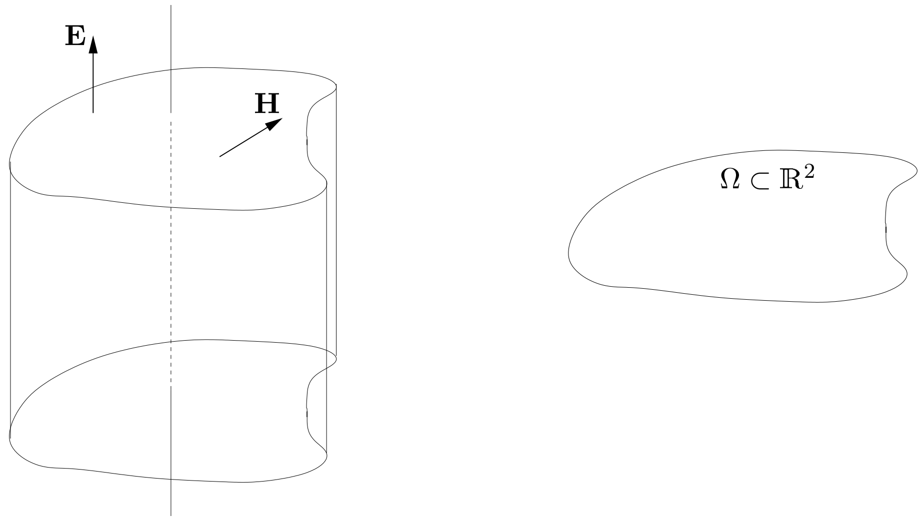

Figure 1.2: TM symmetry

are solutions to the time-dependent Maxwell's equations, with

$$\mathcal{J} = \mathcal{J}_f + \mathrm{Re}\left\{\mathbf{J}(x)e^{-i\omega t}\right\}.$$

Now suppose the region is a cylinder of the form $\Omega \times \mathbb{R}$, where $\Omega \subset \mathbb{R}^2$ is a simply connected domain (possibly all of $\mathbb{R}^2$). If μ, ε, σ, $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{J}$ are independent of the variable x_3 corresponding to the axis of the cylinder, the time-harmonic Maxwell system may be decomposed into independent systems: one satisfied by $\mathbf{E}^* = (0, 0, E_3)$ and $\mathbf{H}^* = (H_1, H_2, 0)$ with the current source term $\mathbf{J}^* = (0, 0, J_3)$, and the other satisfied by $\mathbf{E}^{**} = (E_1, E_2, 0)$ and $\mathbf{H}^{**} = (0, 0, H_3)$ with the current source term $\mathbf{J}^{**} = (J_1, J_2, 0)$. A system of the first type is called *transverse magnetic*, as the magnetic field is always transverse to the axis of the cylinder (see Figure 1.2). A system of the second type is *transverse electric*. We will restrict our attention to transverse magnetic systems, and so we necessarily assume the prescribed current travels only in the direction parallel to the axis of the cylinder: $\mathbf{J} = (0, 0, J)$.[3] By straightforward calculations one can show

[3] In many models the current source is taken to be a wire, or a collection of wires, carrying electrical current (alternating at frequency ω) parallel to the axis of the cylinder.

the scalar electric field $E := E_3$ satisfies the Helmholtz equation

$$\nabla \cdot \left(\frac{1}{\mu} \nabla E \right) + \omega^2 \boldsymbol{\epsilon} E = -i\omega J, \tag{1.5}$$

where $\boldsymbol{\epsilon} = \varepsilon + i\frac{\sigma}{\omega}$ is the complex permittivity. If Ω is a bounded domain, we may specify a unique solution to (1.5) by imposing boundary conditions as long as the homogeneous form of (1.5) admits only the trivial solution to the corresponding problem with zero boundary data. Prescribing the Dirichlet data $E|_{\partial\Omega}$ is equivalent to prescribing the data $\mathbf{E}^* \times \nu$ to the boundary of the cylinder, which is typically how Dirichlet data are assigned to a bounded domain in $\mathbb{R}^3$ in the case of the full Maxwell's equations. Prescribing the Neumann data $\partial_\nu E$ is equivalent to prescribing the tangential magnetic field to the boundary of the cylinder, as

$$i\mu\omega\, \mathbf{H}^* \cdot \tau = (\nabla \times \mathbf{E}^*) \cdot \tau = \nabla E \cdot \nu.$$

The scattering problem. Consider now the case where $\Omega = \mathbb{R}^2$, $J = 0$ and ε, μ and σ are constant except within a bounded scatterer. For simplicity, we assume the background conductivity, i.e., the conductivity outside the scatterer, is zero. Let ε_0 and μ_0 denote the constant background coefficients. The prescribed boundary condition is replaced with a prescribed incident wave $\mathcal{E}^i(x,t) = \operatorname{Re}\{\mathbf{E}^i(x)e^{-i\omega t}\}$, which is usually taken to be a plane wave $\mathbf{E}^i(x) = \xi E^i(x) = \xi e^{i\omega\sqrt{\varepsilon_0\mu_0}\; x\cdot\eta}$, where the polarization vector $\xi \in \mathbb{S}^2$ is parallel to the axis of the cylindrical inhomogeneity and the propagation direction $\eta \in \mathbb{S}^2$ is perpendicular to ξ. To limit the number of solutions to one, we require that the solution field satisfy Sommerfeld's outgoing radiation condition

$$(\partial_r - i\omega\sqrt{\varepsilon_0\mu_0})(E - E^i) = o\big(r^{-1/2}\big) \quad \text{as } r = |x| \to \infty.$$

1.2.1 The inverse problem

Suppose there is a (cylindrical) inhomogeneity within the medium, which manifests as a discontinuity in at least one of the three parameters ε, μ and σ. We are interested in the inverse problem of determining information about this inhomogeneity—for instance, its size, shape, location, or EM parameters—based on measurements at the boundary. It has been shown in the case of a bounded domain in $\mathbb{R}^3$ with $\mathbf{J} = 0$ that, at any fixed frequency ω that is not a resonant frequency, full knowledge of the mapping $\mathbf{\Lambda}_\omega : \mathbf{E} \times \nu \mapsto \mathbf{H} \times \nu$ uniquely determines ε, μ and σ within the domain, assuming these functions are sufficiently smooth [OPS93]. In the transverse magnetic setting, $\mathbf{\Lambda}_\omega$ is equivalent to the Dirichlet-to-Neumann map $\Lambda_\omega : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$, which maps $E|_{\partial\Omega} \mapsto \sigma\partial_\nu E|_{\partial\Omega}$. For this two dimensional problem, it can be shown that full knowledge of Λ_ω at two distinct frequencies is sufficient to determine the coefficients μ, ε and ω, provided they are sufficiently smooth [GC96, VV00]. Such a result extends what has previously been shown in the context of the related conductivity problem, i.e.,

$$\begin{cases} \nabla \cdot (\sigma \nabla U) = 0 & \text{in } \Omega, \\ U = f & \text{on } \partial\Omega \\ \big(\text{or } \sigma\partial_\nu U = g & \text{on } \partial\Omega\big), \end{cases} \tag{1.6}$$

where U represents the voltage potential within the bounded, simply connected domain Ω of conductivity profile σ, and $f \in H^{1/2}(\partial\Omega)$ (or $g \in H_\diamond^{-1/2}(\partial\Omega) = \{u \in H^{-1/2}(\partial\Omega) : \int_{\partial\Omega} u \, d\sigma = 0\}$) is prescribed. The inverse problem of determining the interior conductivity profile σ from boundary measurements has attracted significant attention since it was posed by Calderón in 1980 [Cal80]. Presently, it is known in two dimensions that full knowledge of the Dirichlet-to-Neumann map $\Lambda : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$, which maps the boundary voltage $U|_{\partial\Omega}$ to the boundary current $\sigma\partial_\nu U|_{\partial\Omega}$, uniquely determines the isotropic conductivity $\sigma \in L^\infty(\Omega)$ within the domain, so long as $c^{-1} \leq \sigma \leq c$ for some $c > 0$ [AP06, Nac96, BU97]. In higher dimensions, such a result is known to hold if σ is sufficiently regular, for instance, if σ is assumed piecewise analytic [KV84, KV85], or if σ is assumed to belong to $\bigcup_{\alpha>1/2} C^{1,\alpha}(\overline{\Omega})$ [Bro96, SU87] (see also [Isa88] for such a uniqueness result in certain cases when σ is assumed piecewise C^2).

In the case of the scattering problem, the inverse problem is to determine the coefficients ε, μ and σ from the measurements of E away from the inhomogeneity that result from a number of prescribed incident waves. Typically, the measured data is modeled as the scattering amplitude (also called the far field pattern) $E^s_\infty : \mathbb{S}^1 \to \mathbb{C}$, which is the unique function satisfying

$$(E - E^i)(x) = \frac{e^{i\omega\sqrt{\varepsilon_0\mu_0}\,|x|}}{\sqrt{|x|}} \left\{ E^s_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\} \quad \text{as } |x| \to \infty,$$

uniformly in all directions $\hat{x} = x/|x|$. In the way of general results on the solvability of this inverse problem, it is known that, at a fixed frequency ω, the discontinuities in the coefficient μ, ε and σ are completely determined by knowledge of the functions E^s_∞ that result from every incident plane wave $e^{i\omega\sqrt{\varepsilon_0\mu_0}\,x\cdot\eta}$, that is, by the mapping $\eta \mapsto E^s_\infty$ [SU93].[4] There are other such results (see [CK98, Ch. 10] for a discussion), but we will not dwell on these, as our objective is to find tools that will aid in the practical problem of reconstructing the inhomogeneity.

Many algorithms for solving inverse problems for the Helmholtz equation have been developed; see, for example, [CK98] and the references therein. But these algorithms typically are designed to reconstruct *all* of the unseen interior of the object, and are thus not well suited for the distinct problem of detecting small inhomogeneities within an *otherwise known* body.

1.2.2 Asymptotic perturbation formulas: diametrically small inclusions

A class of methods for solving this special inverse problem, based on asymptotic formulas akin to (1.1), began with the work of Friedman and Vogelius in 1989 [FV89]. In [FV89], the authors derived an asymptotic formula of the boundary voltage perturbation due to the presence of diametrically small inhomogeneities of extreme conductivity (perfectly conducting or perfectly insulating) within a finite body for which the positive conductivity profile, in the absence of the inhomogeneities, is known. Later, Cedio-Fengya, Moskow and Vogelius [CFMV98] extended this result to diametrically small

[4] Stronger results are known for the three dimensional acoustic scattering problem. See [CK98, Ch. 10].

inhomogeneities of *finite* conductivity, and they proposed an algorithm for detecting the locations and sizes of the inhomogeneities based on the method of least squares. A version of the expansion in [CFMV98] is as follows: suppose the bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, has background conductivity $\sigma_0 > 0$ and conductivity $\sigma_j > 0$ within the inhomogeneity $z_j + \rho D_j$, $j = 1, 2, \ldots, m$. suppose u_0 is the background voltage potential and u_ρ is the perturbed potential that has the same Neumann data as u_0 and is normalized so that $\int_{\partial\Omega} u_\rho = \int_{\partial\Omega} u_0$. Then for $y \in \partial\Omega$,

$$
\begin{aligned}
&(u_\rho - u_0)(y) \\
&\qquad = \sum_j \rho^n |D_j| \big(\sigma_0(z_j) - \sigma_j(z_j)\big) \nabla_x N(z_j, y) \cdot \big(M_j(z_j) \nabla u_0(z_j)\big) + o\big(\rho^n\big),
\end{aligned} \tag{1.7}
$$

where N is a Neumann function satisfying

$$
\begin{aligned}
\nabla_x \cdot \big(\sigma_0 \nabla_x N(x, y)\big) &= -\delta_y(x) \quad \text{for } x \in \Omega \\
\sigma_0 \partial_{\nu_x} N(x, y) &= -\frac{1}{|\partial\Omega|} \quad \text{for } x \in \partial\Omega
\end{aligned}
$$

for all $y \in \Omega$, and where the polarization tensors M_j are independent of the Neumann data.

Analogous expansions for the Helmholtz problem, such as the formula (1.1), were then obtained [VV00] and subsequently generalized to the full, three dimensional, time harmonic Maxwell's system [AVV01].[5] In the case of a single diametrically small inhomogeneity, the full asymptotic perturbation expansion (all higher integral orders of ρ) for the conductivity problem [AK03] and the expansion for the related elasticity problem [AKNT02] were later obtained using layer potential techniques based on those developed in [KS96]. These techniques are easily adapted to obtain a full asymptotic expansion for the boundary perturbation in the context of the Helmholtz problem [AK04a]. It should be noted, however, that these techniques require the parameters—ε and μ for the Helmholtz problem, σ for the conductivity problem—to be piecewise constant, and

[5] In [VV00] and [AVV01] the EM parameters were assumed to be constant in the background medium and within each inhomogeneity. But this assumption was made for simplicity—the results therein could be generalized to smooth parameters.

require that $\sigma = 0$ for the Helmholtz problem.

1.2.3 Reconstruction of diametrically small inclusions using asymptotic formulas

We will now illustrate the utility of asymptotic perturbation formulas such as (1.1) in solving the inverse problem of detecting and reconstructing diametrically small electromagnetic inhomogeneities from boundary measurements. Methods similar to those discussed below for fixed frequency testing, but based instead on formula (1.3), will undoubtedly prove to be useful for high frequency testing.

In addition to (1.1), we also have the expansion

$$\begin{aligned}\int_{\partial\Omega} \frac{1}{\mu_0}\big\{E_\rho(\partial_\nu v) - (\partial_\nu E_\rho)v\big\}\,\mathrm{d}\sigma \\ = \sum_j \rho^2|D_j|\bigg\{ -\Big(\frac{1}{\mu_j} - \frac{1}{\mu_0}\Big)\nabla v(z_j)\cdot\big(M(z_j)\nabla E_0(z_j)\big) \\ + \omega^2(\boldsymbol{\epsilon}_j - \boldsymbol{\epsilon}_0)v(z_j)E_0(z_j)\bigg\} + o(\rho^2),\end{aligned} \tag{1.8}$$

which holds for any $v \in H^1(\Omega)$ that solves the background Helmholtz equation,

$$\nabla\cdot\left(\frac{1}{\mu_0}\nabla v\right) + \omega^2\boldsymbol{\epsilon}_0 v = 0$$

[AMV03] (cf. Corollary 2.22). Suppose the background parameters μ_0, ε_0 and σ_0 are constant and also that the parameters μ_j, ε_j and σ_j within the j^{th} inhomogeneity $z_j + \rho D_j$, $j = 1, 2, \ldots, m$, are constant. To determine the unknown values μ_j, ε_j, σ_j and the entries of the $M(z_j)$ (which hold some information about the shape of the inhomogeneities), one may choose E_0 and v of the forms

$$E_0(x) = e^{ik_0 x\cdot\alpha} \quad \text{and} \quad v(x) = e^{ik_0 x\cdot\beta}, \quad \text{where} \quad k_0 = \omega\sqrt{\varepsilon_0\mu_0}.$$

Evaluating the boundary integral from (1.8) for several appropriately chosen values of $\alpha, \beta \in \mathbb{C}^2$, and disregarding the $o(\rho^2)$ term, results in equations that may be solved simultaneously for the desired unknown values. For this process to be ultimately successful, one must have additional *a priori* knowledge about the inhomogeneities, such as their sizes $\rho^2|D_j|$, or the knowledge that they all have the same rescaled polarization tensor, or the values of some of the EM parameters. This method is described in detail for the conductivity problem and for the full time harmonic Maxwell's equations in [AMV03].

THE FOURIER METHOD. We now describe a method (based on ideas in [Cal80] for the conductivity problem) for locating the positions of the small inhomogeneities, i.e., the points z_j. For any $\xi \in \mathbb{R}^2$, one may choose $E_0(x) = e^{i(\xi+\gamma\hat{\eta})\cdot x}$ and $v(x) = e^{i(\xi-\gamma\hat{\eta})\cdot x}$, where $\hat{\eta}$ is a unit vector perpendicular to ξ and $\gamma \in \mathbb{C}$ is such that $\omega^2\mu_0\epsilon_0 = |\xi|^2 + \gamma^2$. The last condition ensures that E_0 and v both solve the background Helmholtz equation. Disregarding the $o(\rho^2)$ term in (1.8) then, the function $\xi \mapsto \int_{\partial\Omega} \left(\partial_\nu E_\rho v - E_\rho \partial_\nu v\right) \mathrm{d}x$ becomes a linear combination of the Fourier transforms $\mathcal{F}(\partial^\alpha \delta_{z_j})(\xi)$, $|\alpha| \leq 2$. Thus, to locate the positions z_j, one may perform a numerical Fourier inversion of the measured data as a function ξ and see where the graph of this inversion "spikes" (cf. [AMV03], [Vol01], [Vol03] and [AK04b, §13.2.1]).

One disadvantage of this approach is that the imaginary part of γ will cause the test field E_0 to grow (or decay) exponentially—$E_0(x) = e^{\alpha\cdot x}p(x)$, where $\alpha \in \mathbb{R}^2$ and p is a plane wave—and therefore may yield measurements that are overwhelmed with noise. A second disadvantage is the large number of measurements required to reach adequate resolution when performing the numerical Fourier inversion. Fortunately, the following algorithm does not suffer from these disadvantages.

THE MUSIC ALGORITHM. The MUSIC (MUltiple SIgnal Classification) algorithm, which was originally developed for signal processing [Sch86], is based on the fact that the range of a self-adjoint operator is orthogonal to the kernel. If the operator is slightly perturbed, the original kernel is slightly perturbed and becomes the noise subspace corresponding to negligibly small eigenvalues. The orthogonal complement of this noise subspace is the so-called essential range of the perturbed operator. To test whether a given vector g is in the essential range, one may calculate $\|Pg\|^{-1}$, where P is the projection onto the noise subspace, and see whether the value is large.

The essence of this method has been adapted to the inverse problem of determining the locations of point scatterers within an infinite, homogeneous medium [Dev99], and the problem of determining the support of a scatterer within an otherwise homogeneous medium [Kir02] (see [Che01] for a lucid description of the MUSIC algorithm and its relevance to imaging). In the context of the conductivity problem, Brühl, Hanke, and Vogelius [BHV03] developed a MUSIC-like algorithm for determining the location of small inhomogeneities based on the asymptotic expansion (1.7). Using similar techniques, Ammari, Iakovleva and Lesselier [AIL05] developed and successfully numerically tested a MUSIC algorithm based on the asymptotic expansion (1.8) to determine the locations of small electric inhomogeneities. We will outline an algorithm essentially the same as that presented in [AIL05] and [Iak04].

Assume the background medium is nonconducting ($\sigma_0 = 0$) with ε_0 and μ_0 constant. Assume also that each ε_j, μ_j and σ_j are constant, $j = 1, 2, \ldots, m$, with each

$$\mu_j \neq \mu_0 \quad \text{and} \quad \boldsymbol{\epsilon}_j \neq \varepsilon_0.$$

In formula (1.8), if we take $E_0(x) = e^{ik_0 x\cdot\eta}$ and $v(x) = e^{-ik_0 x\cdot\xi}$, with η and ξ unit vectors in $\mathbb{R}^2$, we get

$$\begin{aligned}\mathcal{P}_\rho(\xi,\eta,\omega) := &\int_{\partial\Omega} \left\{ E_\rho\big(\partial_\nu e^{-ik_0 y\cdot\xi}\big) - (\partial_\nu E_\rho) e^{-ik_0 y\cdot\xi} \right\} \mathrm{d}\sigma_y \\ &\text{(= a measurement of the boundary perturbation)} \\ &= \rho^2 k_0^2 \sum_{l=1}^m \big[- a_l(\xi^T M_l\, \eta) + b_l \big] e^{ik_0(\eta-\xi)\cdot z_l} + O(\rho^3)\end{aligned}$$

where $a_l = -|D_l|\big(\frac{1}{\mu_l} - \frac{1}{\mu_0}\big)$ and $b_l = |D_l|(\boldsymbol{\epsilon}_l - \varepsilon_0)/\varepsilon_0\mu_0$. (Note that E_0 is a pure plane wave that does not grow exponentially, unlike in the Fourier method.) Given n distinct directions of incidence, $\eta^1, \eta^2, \ldots, \eta^n$, we define the matrix $A \in \mathbb{C}^{n\times n}$ by

$$A_{ij} = \sum_{l=1}^{m} \left[a_l\, \eta^i \cdot (M_l\, \eta^j) + b_l\right] e^{ik_0(\eta^j+\eta^i)\cdot z_l},$$

so that

$$\mathcal{P}_\rho(-\eta^i, \eta^j, \omega) = \rho^2 k_0^2 A_{ij} + O(\rho^3).$$

Because each M_l is symmetric, A is also symmetric, and therefore the matrix $\mathcal{A} := A\overline{A}$ is positive semidefinite. The maximum possible rank of $\mathcal{A}$ is $3m$, which can be seen by representing A as

$$A = VDV^T,$$

where V^T is the $3m \times n$ matrix

$$V^T = \begin{bmatrix} \begin{bmatrix} \eta^1 \\ 1 \end{bmatrix} e^{ik_0\eta^1\cdot z_1} & \begin{bmatrix} \eta^2 \\ 1 \end{bmatrix} e^{ik_0\eta^2\cdot z_1} & \cdots & \begin{bmatrix} \eta^n \\ 1 \end{bmatrix} e^{ik_0\eta^n\cdot z_1} \\ \begin{bmatrix} \eta^1 \\ 1 \end{bmatrix} e^{ik_0\eta^1\cdot z_2} & \begin{bmatrix} \eta^2 \\ 1 \end{bmatrix} e^{ik_0\eta^2\cdot z_2} & \cdots & \begin{bmatrix} \eta^n \\ 1 \end{bmatrix} e^{ik_0\eta^n\cdot z_2} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{bmatrix} \eta^1 \\ 1 \end{bmatrix} e^{ik_0\eta^1\cdot z_m} & \begin{bmatrix} \eta^2 \\ 1 \end{bmatrix} e^{ik_0\eta^2\cdot z_m} & \cdots & \begin{bmatrix} \eta^n \\ 1 \end{bmatrix} e^{ik_0\eta^n\cdot z_m} \end{bmatrix}$$

and D is the $3m \times 3m$ block diagonal matrix

$$D = \operatorname{diag}\left(\begin{bmatrix} a_1 M_1 & \mathbf{0} \\ \mathbf{0} & b_1 \end{bmatrix}, \begin{bmatrix} a_2 M_2 & \mathbf{0} \\ \mathbf{0} & b_2 \end{bmatrix}, \ldots, \begin{bmatrix} a_m M_m & \mathbf{0} \\ \mathbf{0} & b_m \end{bmatrix} \right).$$

For $z \in \Omega$ and $w \in \mathbb{R}^3$, let $g_z^w \in \mathbb{C}^n$ denote the vector

$$g_z^w = \left(w \cdot \begin{bmatrix} \eta^1 \\ 1 \end{bmatrix} e^{ik_0 \eta^1 \cdot z} \; , \; w \cdot \begin{bmatrix} \eta^2 \\ 1 \end{bmatrix} e^{ik_0 \eta^2 \cdot z} \; , \; \dots \; , \; w \cdot \begin{bmatrix} \eta^n \\ 1 \end{bmatrix} e^{ik_0 \eta^n \cdot z} \right). \tag{1.9}$$

As each of the m matrices $\operatorname{diag}\big(a_j M_j, b_j\big)$ is invertible, it follows that D is invertible. Therefore, if $n \geq 3m$ and V achieves the maximal rank $3m$ then $\operatorname{Ran}(\mathcal{A}) = \operatorname{Ran}(V)$, which would imply that $g_{z_j}^w \in \operatorname{Ran}(\mathcal{A})$ for every $w \in \mathbb{C}^3$, $j = 1, 2, \dots, m$. Moreover, it can be shown [AIL05, Kir02] that given a sequence of directions $\{\eta^j\}_{j=1}^\infty$ dense in $\mathbb{S}^1$, there exists an $N \geq 3m$ such that if $n \geq N$ then for any $z \in \Omega$ and nonzero $w \in \mathbb{R}^3$, g_z^w defined by (1.9) satisfies

$$g_z^w \in \operatorname{Ran}(\mathcal{A}) \quad \Longleftrightarrow \quad z \in \{z_1, z_2, \dots, z_m\}. \tag{1.10}$$

Assume we have a sufficient number of directions $\{\eta^1, \eta^2, \dots, \eta^n\}$ so that (1.10) holds. Since $\mathcal{A}$ is Hermitian, $\operatorname{Ran}(\mathcal{A}) \perp \operatorname{Ker}(\mathcal{A})$, and therefore

$$g_z^w \in \operatorname{Ran}(\mathcal{A}) \quad \Longleftrightarrow \quad \operatorname{Proj}_{\operatorname{Ker}(\mathcal{A})} g_z^w = \big(I - \operatorname{Proj}_{\operatorname{Ran}(\mathcal{A})}\big) g_z^w = 0. \tag{1.11}$$

Because $\mathcal{A}$ is positive semidefinite, it is diagonalizable with nonnegative eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and corresponding orthonormal eigenvectors $v^1, v^2, \dots, v^n$. If we let k denote the smallest index such that $\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_n = 0$, then for any vector $g \in \mathbb{C}^n$,

$$\operatorname{Proj}_{\operatorname{Ran}(\mathcal{A})}(g) = \sum_{j=1}^{k} ([v^j]^* \, g) \, v^j.$$

Consider now the matrix

$$B_{ij} = \mathcal{P}_\rho(-\eta^i, \eta^j, \omega),$$

which is determined by measured data. B is a perturbation of $\rho^2 k_0^2 A$ satisfying $B = \rho^2 k_0^2 A + O(\rho^3)$. Likewise, $\mathcal{B} := BB^* = \rho^4 k_0^4 \mathcal{A} + O(\rho^5)$. We may numerically calculate the eigenvalues of $\mathcal{B}$—which are necessarily real since $\mathcal{B}$ is Hermitian—and arrange them in decreasing order: $\lambda_{\rho,1} \geq \lambda_{\rho,2} \geq \cdots \geq \lambda_{\rho,n}$, with corresponding orthonormal eigenvectors $v_\rho^1, v_\rho^2, \ldots, v_\rho^n$. By standard perturbation theory [Kat76],

$$\lambda_{\rho,j} = \rho^4 k_0^4 \lambda_j + O(\rho^5).$$

We may therefore determine the index k by seeing where the decreasing sequence of eigenvalues $\lambda_{\rho,j}$ drops sharply to a tail of negligible values. $v_\rho^{k+1}, v_\rho^{k+2}, \ldots, v_\rho^n$ span what is known as the noise subspace. We define P_R to be the projection onto the orthogonal complement of the noise subspace,

$$P_R(g) = \sum_{j=1}^{k} ([v_\rho^j]^* \, g) \, v_\rho^j \quad \text{for } g \in \mathbb{C}^n,$$

and we define

$$P_{\text{noise}} := I - P_R.$$

By using standard results of perturbation theory [Kat76], it can be shown that

$$P_R = \rho^4 k_0^4 \operatorname{Proj}_{\operatorname{Ran}(\mathcal{A})} + O(\rho^5). \tag{1.12}$$

Choose any nonzero $w \in \mathbb{R}^2$ and plot $z \mapsto \|(I - P_R) g_z^w\|^{-1}$. Assuming the number n of test directions is sufficiently large, by a combination of (1.11) and (1.12), one should expect sharp spikes in the graph at the locations of the inhomogeneities.[6]

Using similar methods, it is possible to also determine the EM parameters and certain geometric features of the inhomogeneities, in addition to their locations [AILP07]. (In [BHV03], the authors outline a procedure to recover geometric information about the inhomogeneities from the polarization tensor in formula (1.7).)

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[6]We should note that the N^2 boundary measurements required for this algorithm to succeed is far less than the number of measurements that would be needed to achieve a satisfactory resolution when performing the numerical Fourier inversion of the Fourier method.

1.2.4 General asymptotic formulas

In the context of the conductivity problem, formula (1.7) applies to diametrically small inhomogeneities that are well separated and away from the boundary.[7] If an inhomogeneity of low volume fraction within a two dimensional conducting body has a high aspect ratio such that its width is small but is length is not particularly small relative to the larger body—a crack, for example—then formula (1.7) ceases to be a useful. If the j^{th} inhomogeneity is represented by a tubular neighborhood of width ρ about the curve γ_j, then the appropriate asymptotic expansion of the boundary voltage perturbation is

$$(u_\rho - u_0)(y) = \sum_j \rho \int_{\gamma_j} (\sigma_j - \sigma_0)(z) \nabla_x N(z, y) \cdot \big(M_j(z) \nabla u_0(z)\big) \, \mathrm{d}\sigma_z + o(\rho) \qquad (1.13)$$

for all $y \in \partial\Omega$, where the tensor M_j is a symmetric and positive definite dσ-a.e. on γ_j [BMV01, BFV03]. The similarities between formulas (1.7) and (1.13) suggest that they are special cases of a general asymptotic expansion. In fact, such a generalization was achieved in [CV03a], where the following was shown: suppose the inhomogeneous set is represented as any sequence of measurable sets $\mathcal{I}_\rho$ that are well contained in Ω (i.e., they do not approach the boundary) and satisfy $|\mathcal{I}_\rho| \to 0$ as $\rho \to 0$. Let σ_0 denote the positive background conductivity and let σ_1 denote that of the inhomogeneous set. Then there exists a subsequence $\mathcal{I}_{\rho_n}$, a probability measure α supported on $\overline{\bigcup \mathcal{I}_{\rho_n}}$, and a polarization tensor $M \in L^2(\Omega, \mathrm{d}\alpha)$ that is symmetric and positive definite dα-a.e., such that if the voltage functions u_ρ and u_0 result from the same prescribed Neumann data and are normalized so that $\int_{\partial\Omega} u_\rho = \int_{\partial\Omega} u_0$, then

$$(u_\rho - u_0)(y) = |\mathcal{I}_{\rho_n}| \int_\Omega (\sigma_1 - \sigma_0)(z) \nabla_x N(z, y) \cdot \big(M(z) \nabla u_0(z)\big) \, \mathrm{d}\alpha_z + o(|\mathcal{I}_{\rho_n}|) \qquad (1.14)$$

for all $y \in \partial\Omega$. This formula, along with Hashin-Shtrikman bounds of the polarization tensor, can be used to estimate from boundary measurements alone the volume of a very

[7] The $o(\rho^2)$ term from formula (1.7) depends on the distance between pairs of inhomogeneities and the distance from the inhomogeneities to the boundary. There are known expansions that address the case of small inclusions that are closely spaced [AAK05] and the case of inclusions that are close to the boundary [AKKL05].

general class of small inhomogeneities [CV03a, CV03b, CV04, CV06]. In Chapter 2 we will derive an analogous asymptotic formula for the Helmholtz problem, in other words, in the case of nonzero frequency. We will achieve this by following the same idea of proof for the zero frequency case in [CV03a]. This proof requires certain L^2 perturbation estimates, which we will derive for the Helmholtz problem by borrowing techniques from [VV00]. We will also show the polarization tensor arising from discontinuities in the permeability satisfies bounds similar to those in [CV03b]. Thus it should be possible to effectively estimate the volume of small electromagnetic inhomogeneities within a body from boundary measurements, even if the background material is nonconducting.

1.2.5 Acoustic waves

It should be noted that the Helmholtz equation is more commonly used to model acoustic waves. If

$$\nabla \cdot \left(\varrho \nabla u\right) + \frac{\omega^2}{\varrho c^2} u = 0$$

in $\mathbb{R}^2$ or $\mathbb{R}^3$, then $p(x,t) := \mathrm{Re}\{u(x)e^{-i\omega t}\}$ represents a pressure wave[8] of frequency ω in a medium of equilibrium density ϱ and sound speed c [Che90, KFCS82]. As in the case of time harmonic electromagnetic waves, absorbtion, due to, say, viscosity or thermal conduction, can be modeled by ascribing a positive imaginary part to c^{-2} [Jon86, §6.5], [KFCS82, Ch. 7]. In the electromagnetic setting, this imaginary part has a simple inverse relationship with frequency ($\mathrm{Im}(\boldsymbol{\epsilon}) = \sigma/\omega$) that holds for all frequencies. In the acoustic setting, however, though the rate of attenuation of acoustic waves does eventually decreases as frequency increases, the relationship of this absorption parameter to frequency cannot simply be expected to obey the same simple inverse relationship. Moreover, different acoustic absorption mechanisms give rise to different behaviors with changing frequency (see [KFCS82, Ch. 7] for a discussion of this). Therefore, much of our high frequency analysis of scattering by penetrable but absorbing objects may not

[8]In fact, p is a perturbation. If p_0 denotes the equilibrium pressure, then the pressure function in the presence of the wave is $p_0 + p$.

apply to an acoustic setting without appropriate modifications. That being said, the case of a perfectly conducting scatterer embedded within a nonconducting background medium does coincide with that of a perfectly sound-soft scatterer embedded within a nonabsorbent medium: in both cases, the total field must vanish on the surface of the scatterer, thus giving rise to exterior Dirichlet problems for the scattered fields that are identical in form.

1.2.6 Nondimensionalization

Throughout this thesis, all quantities are assumed to be dimensionless. In other words, we assume that the following procedure has been performed. Suppose $\bigcup_\rho \mathcal{I}_\rho \subset \mathcal{I}$, where $\mathcal{I} \subset\subset \Omega$. Translate the coordinates, if necessary, so that $0 \in \mathcal{I}$. We first normalize E_0 and E_ρ, which may be achieved, for example, by taking

$$\widehat{E}_0(x) = E_0(x) \Big/ \tfrac{1}{|\mathcal{I}|} \textstyle\int_{\mathcal{I}} |E_0(y)| \, \mathrm{d}y,$$
$$\widehat{E}_\rho(x) = E_\rho(x) \Big/ \tfrac{1}{|\mathcal{I}|} \textstyle\int_{\mathcal{I}} |E_0(y)| \, \mathrm{d}y.$$

We then perform the rescaling

$$u_\rho(x) = \widehat{E}_\rho(dx),$$

where d is the "size" of the set $\mathcal{I}$ in whichever units of length have been chosen. For instance, we could take

$$d := \sup\{|x| : x \in \mathcal{I}\} \quad \text{or} \quad d := \operatorname{diam}(\mathcal{I}).$$

u_ρ then satisfies

$$\nabla \cdot \Big(\frac{1}{\tilde{\mu}_\rho} \nabla u_\rho \Big) + \Big(\tilde{\varepsilon}_\rho + i \frac{\tilde{\sigma}_\rho}{\tilde{\omega}} \Big) \tilde{\omega}^2 u_\rho = 0 \quad \text{in } \Omega',$$

where

$$\tilde{\mu}_\rho(x) = \begin{cases} \dfrac{\mu_1(dx)}{\mu_0(0)} & \text{in } \mathcal{I}'_\rho \\ \dfrac{\mu_0(dx)}{\mu_0(0)} & \text{in } \Omega' \setminus \overline{\mathcal{I}'_\rho}, \end{cases}$$

$$\tilde{\varepsilon}_\rho(x) = \begin{cases} \dfrac{\varepsilon_1(dx)}{\varepsilon_0(0)} & \text{in } \mathcal{I}'_\rho \\ \dfrac{\varepsilon_0(dx)}{\varepsilon_0(0)} & \text{in } \Omega' \setminus \overline{\mathcal{I}'_\rho}, \end{cases}$$

$$\tilde{\sigma}_\rho(x) = \begin{cases} \sigma_1(dx)d\left[\dfrac{\mu_0(0)}{\varepsilon_0(0)}\right]^{1/2} & \text{in } \mathcal{I}'_\rho \\ \sigma_0(dx)d\left[\dfrac{\mu_0(0)}{\varepsilon_0(0)}\right]^{1/2} & \text{in } \Omega' \setminus \overline{\mathcal{I}'_\rho}, \end{cases}$$

$$\tilde{\omega} = \omega d\sqrt{\mu_0(0)\varepsilon_0(0)},$$

and

$$\Omega' = \tfrac{1}{d}\Omega,$$

$$\mathcal{I}'_\rho = \tfrac{1}{d}\mathcal{I}_\rho.$$

The quantities $\tilde{\mu}_\rho$ $\tilde{\varepsilon}_\rho$ $\tilde{\sigma}_\rho$, $\tilde{\omega}$, u_ρ and the argument of u_ρ are dimensionless. $\mathcal{I}' = \frac{1}{d}\mathcal{I}$ represents, in a sense, a set of unit size. If Ω is a bounded domain, $\mathcal{I}$ should be taken to be a set sufficiently large so that the rescaled set Ω' has dimensionless size on the order of 1.

Chapter 2

General asymptotic formulas in the case of fixed frequency

2.1 The problem

We consider a cylindrical object with transverse magnetic symmetry, the cross-section of which is represented by the bounded, open and connected domain $\Omega \subset \mathbb{R}^2$. This domain contains a small inhomogeneous set denoted $\mathcal{I}_\rho$. This set is nearly arbitrary in shape, the only restrictions being that it must be measurable and that it must be compactly contained in the interior of Ω. $0 < \rho \leq 1$ is a parameter we introduce for our asymptotic analysis: we assume $\{\mathcal{I}_\rho\}$ is a family of measurable sets such that $|\mathcal{I}_\rho| \to 0$ as $\rho \to 0$. $\bigcup_\rho \mathcal{I}_\rho$ is assumed to be bounded away from the boundary $\partial\Omega$, and therefore one may construct a smooth domain $\mathcal{I}$ depending only on $\bigcup_\rho \mathcal{I}_\rho$ and $\operatorname{dist}\left(\bigcup_\rho \mathcal{I}_\rho, \partial\Omega\right)$ such that

$$\textstyle\bigcup_\rho \mathcal{I}_\rho \subset\subset \mathcal{I} \subset\subset \Omega.$$

The boundary $\partial\Omega$ is assumed to be smooth—or at least sufficiently regular for our purposes[1] ($C^{1,1}$ regularity will suffice). The electromagnetic profile of the object is given by

$$\mu_\rho = \begin{cases} \mu_1 & \text{in } \mathcal{I}_\rho \\ \mu_0 & \text{in } \Omega \setminus \mathcal{I}_\rho \end{cases}, \quad \varepsilon_\rho = \begin{cases} \varepsilon_1 & \text{in } \mathcal{I}_\rho \\ \varepsilon_0 & \text{in } \Omega \setminus \mathcal{I}_\rho \end{cases}, \quad \sigma_\rho = \begin{cases} \sigma_1 & \text{in } \mathcal{I}_\rho \\ \sigma_0 & \text{in } \Omega \setminus \mathcal{I}_\rho \end{cases},$$

[1] We will require regularity of the boundary $\partial\Omega$ sufficient for certain elliptic estimates and sufficient to properly pose, and prove the unique existence of solutions to, the problems (2.1a) and (2.1b).

where $\mu > 0$ is the magnetic permeability, $\varepsilon > 0$ the electric permittivity and $\sigma \geq 0$ the conductivity. The background EM parameters ε_0 and σ_0 belong to $C^0(\overline{\Omega})$,[2] while μ_0 belongs to $C^{0,1}(\overline{\Omega})$. The EM parameters of the inhomogeneity, μ_1, ε_1 and σ_1, all belong to $C^0(\mathcal{I})$. Let $\mathfrak{C}$ denote the maximum of the suprema of μ_0, ε_0, σ_0, μ_1, ε_1 and σ_1. The ellipticity constant

$$\vartheta = \min\left\{\inf_{\Omega} \frac{1}{\mu_0}, \inf_{\mathcal{I}} \frac{1}{\mu_1}\right\}$$

is assumed to be strictly positive. We consider the time harmonic situation and denote the frequency by $\omega > 0$. $\boldsymbol{\epsilon} = \varepsilon + i\frac{\sigma}{\omega}$ will denote the complex permittivity, and we let $\kappa = \omega\sqrt{\boldsymbol{\epsilon}}$ and $k = \omega\sqrt{\mu\boldsymbol{\epsilon}}$, where the square root is the principle square root, so that the real and imaginary parts of κ and k are nonnegative.

Let $\Gamma_D \subset \partial\Omega$ be a finite union of open, connected subsets of $\partial\Omega$ and let $\Gamma_N = \partial\Omega \backslash \overline{\Gamma_D}$. Given a source current $J = \frac{1}{i\omega}F$ in Ω, an electric field f on Γ_D and a (tangentially directed) magnetic field $\frac{1}{i\omega\mu_0}g$ on Γ_N, the resulting electric fields u_0 and u_ρ—in the absence and presence of the inhomogeneity, respectively—satisfy

$$\begin{cases} \nabla \cdot \left(\dfrac{1}{\mu_0}\nabla u_0\right) + \kappa_0^2 u_0 = F & \text{in } \Omega \\ u_0 = f & \text{on } \Gamma_D \\ \dfrac{1}{\mu_0}\partial_\nu u_0 = g & \text{on } \Gamma_N \end{cases} \tag{2.1a}$$

and

$$\begin{cases} \nabla \cdot \left(\dfrac{1}{\mu_\rho}\nabla u_\rho\right) + \kappa_\rho^2 u_\rho = F & \text{in } \Omega \\ u_\rho = f & \text{on } \Gamma_D \\ \dfrac{1}{\mu_\rho}\partial_\nu u_\rho = g & \text{on } \Gamma_N. \end{cases} \tag{2.1b}$$

Remark 2.1. *The subsequent analysis of these problems does allow $\omega = 0$, in which case we take $\kappa_0 = 0$ and $\kappa_1 = 0$. This situation can be physically (re)interpreted as a model for the conductivity problem, where $\frac{1}{\mu_\rho}$ would represent conductivity, u_ρ the voltage potential, f a boundary potential, g an applied current and F the divergence*

[2] In fact, we need only assume that ε_0 and σ_0 in $L^\infty(\Omega)$ are continuous on $\mathcal{I}$.

of a source current.

2.1.1 Spaces for $\boldsymbol{F}$, $\boldsymbol{f}$ and $\boldsymbol{g}$

To properly pose these problems, we restrict the given functions to appropriate spaces. With

$$H^1_D(\Omega) = \big\{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\big\},$$

the restrictions are:

$$F \in \big(H^1_D(\Omega)\big)', \qquad f \in H^{1/2}(\Gamma_D) \qquad \text{and} \qquad g \in \big(H^{1/2}_{00}(\Gamma_N)\big)', \tag{2.2}$$

where V' denotes the antidual of the Hilbert space V, and

$$\begin{aligned} H^{1/2}_{00}(\Gamma_N) =& \big\{v \in L^2(\Gamma_N) : \exists\, w \in H^1_D(\Omega) \text{ such that } w|_{\Gamma_N} = v\big\} \\ =& \big\{v \in L^2(\Gamma_N) : \text{ the trivial extension to } \partial\Omega \text{ belongs to } H^{1/2}(\partial\Omega)\big\}, \end{aligned}$$

with

$$\|f\|_{H^{1/2}_{00}(\Gamma_N)} := \inf\big\{\|w\|_{H^1(\Omega)} :\ w \in H^1_D(\Omega) \text{ and } w|_{\Gamma_N} = f\big\}$$

(see, for instance, [BC84], [DL88] or [LM72]). Observe that $H^1_D(\Omega) = H^1_0(\Omega)$ when $\Gamma_D = \partial\Omega$, $H^1_D(\Omega) = H^1(\Omega)$ when $\Gamma_D = \varnothing$, and

$$\big(H^{1/2}_{00}(\partial\Omega)\big)' = \big(H^{1/2}(\partial\Omega)\big)' = H^{-1/2}(\partial\Omega).$$

2.1.2 Variational formulation

To state problems (2.1a) and (2.1b) variationally, we define the following sesquilinear form:

Definition 2.2. *For $u, v \in H^1_D(\Omega)$ and $0 \le \rho \le 1$ let*

$$\mathcal{H}_\rho(u, v) = \int_\Omega \Big(\frac{1}{\mu_\rho}\nabla u \cdot \nabla\overline{v} - \kappa_\rho^2 u\overline{v}\Big)\mathrm{d}x.$$

We will also use the notation $\mathcal{H}_\rho(u, v)$ in cases where the integral is defined but u and v are not necessarily in $H^1_D(\Omega)$.

Suppose F, f and g lie within the appropriate spaces (2.2). Let $\tilde{f} \in H^{1/2}(\partial\Omega)$ be an extension of f with

$$\|\tilde{f}\|_{H^{1/2}(\partial\Omega)} \le C_{\partial\Omega,\Gamma_D}\|f\|_{H^{1/2}(\Gamma_D)}.$$

Let $h \in H^1(\Omega)$ satisfy

$$\begin{cases} h|_{\partial\Omega} = \tilde{f}, \\ h|_{\mathcal{I}} = 0, \\ \|h\|_{H^1(\Omega)} \le C\|\tilde{f}\|_{H^{1/2}(\partial\Omega)}. \end{cases}$$

(Naturally, we take $h = 0$ if $\Gamma_D = \varnothing$.) Problem (2.1b), for $0 \le \rho \le 1$, reduces to the following

Problem 2.3. Given $F \in (H^1_D(\Omega))'$, $f \in H^{1/2}(\Gamma_D)$ and $g \in \big(H^{1/2}_{00}(\Gamma_N)\big)'$, find $\tilde{u}_\rho \in H^1_D(\Omega)$ such that

$$\begin{aligned} -\mathcal{H}_\rho(\tilde{u}_\rho, v) = {}& {}_{(H^1_D(\Omega))'}\langle F, v\rangle_{H^1_D(\Omega)} + \mathcal{H}_0(h, v) - {}_{(H^{1/2}_{00}(\Gamma_N))'}\langle g, v|_{\Gamma_N}\rangle_{H^{1/2}_{00}(\Gamma_N)} \\ =:{}& {}_{(H^1_D(\Omega))'}\langle \mathcal{F}, v\rangle_{H^1_D(\Omega)} \quad \text{for all } v \in H^1_D(\Omega). \end{aligned} \tag{2.3}$$

We then interpret $u_\rho = \tilde{u}_\rho + h$ as a solution to (2.1b). To see why this is a proper formulation of (2.1b), suppose we have $F \in L^2(\Omega)$ and $f \in H^{3/2}(\Gamma_D)$. We may then choose $h \in H^1_{\mu_0}(\Omega)$, where

$$H^1_{\mu_\rho}(\Omega) = \Big\{u \in H^1(\Omega) : \nabla \cdot \big(\tfrac{1}{\mu_\rho}\nabla u\big) \in L^2(\Omega)\Big\}$$

with

$$\|u\|^2_{H^1_{\mu_\rho}(\Omega)} := \|u\|^2_{H^1(\Omega)} + \|\nabla\cdot(\tfrac{1}{\mu_\rho}\nabla u)\|^2_{L^2(\Omega)}$$

(see [BC84], [LM72] or [Lio61]). Consequently, any solution $\tilde{u}_\rho$ of Problem 2.3 will belong to the space

$$H^1_{D,\mu_\rho}(\Omega) = H^1_{\mu_\rho}(\Omega) \cap H^1_D(\Omega).$$

The bounded trace operator

$$\frac{1}{\mu_\rho}\partial_\nu : H^1_{\mu_\rho}(\Omega) \longrightarrow H^{-1/2}(\partial\Omega)$$

is well defined and may be used in a generalized form of Green's formula: for $u \in H^1_{\mu_\rho}(\Omega)$ and $v \in H^1_D(\Omega)$,

$$\int_\Omega \nabla\cdot\Big(\frac{1}{\mu_\rho}\nabla u\Big)\overline{v}\,\mathrm{d}x = -\int_\Omega \frac{1}{\mu_\rho}\nabla u\cdot\nabla\overline{v}\,\mathrm{d}x \;+\; {}_{(H^{1/2}_{00}(\Gamma_N))'}\Big\langle \tfrac{1}{\mu_0}\partial_\nu u, v\Big\rangle_{H^{1/2}_{00}(\Gamma_N)}$$

[BC84]. It follows that

$$\begin{aligned}\int_\Omega \Big[\nabla\cdot\Big(\frac{1}{\mu_\rho}\nabla(\tilde{u}_\rho+h)\Big) + \kappa_\rho^2(\tilde{u}_\rho+h)\Big]\,\overline{v}\,\mathrm{d}x - \int_\Omega F\overline{v}\,\mathrm{d}x& \\ = {}_{(H^{1/2}_{00}(\Gamma_N))'}\Big\langle \tfrac{1}{\mu_\rho}\partial_\nu\tilde{u}_\rho - g, v|_{\Gamma_N}\Big\rangle_{H^{1/2}_{00}(\Gamma_N)} &+ {}_{H^{-1/2}(\partial\Omega)}\Big\langle \tfrac{1}{\mu_0}\partial_\nu h, v|_{\partial\Omega}\Big\rangle_{H^{1/2}(\partial\Omega)}\end{aligned}$$

for all $v \in H^1_D(\Omega)$. If we also assume $g \in L^2(\Gamma_N)$, $u_\rho = \tilde{u}_\rho + h$ would satisfy

$$\begin{aligned}\nabla\cdot\Big(\frac{1}{\mu_\rho}\nabla u_\rho\Big) + u_\rho = F &\quad \text{in } L^2(\Omega),\\ u_\rho = f &\quad \text{in } H^{1/2}(\Gamma_D),\\ \frac{1}{\mu_\rho}\partial_\nu u_\rho = g &\quad \text{in } L^2(\Gamma_N).\end{aligned}$$

Remark 2.4. *Note that* $\mathcal{F} \in (H^1_D(\Omega))'$ *as defined in* (2.3) *satisfies*

$$\|\mathcal{F}\|_{(H^1_D(\Omega))'} \leq C \left(\|F\|_{(H^1_D(\Omega))'} + \|f\|_{H^{1/2}(\Gamma_D)} + \|g\|_{(H^{1/2}_{00}(\Gamma_N))'} \right),$$

where C *depends on* $\mathfrak{C}$, ω *and* $\operatorname{dist}(\partial\Omega, \mathcal{I})$.[3]

Throughout this chapter, we assume ω and the EM parameters μ_0, ε_0, σ_0, μ_1, ε_1 and σ_1 remain fixed as $\rho \to 0$. The resulting asymptotic formulas will thus model physical situations where $\omega|\mathcal{I}_\rho| \ll 1$.

2.2 Well-posedness of the problem

Consider the case where $\Gamma_D = \partial\Omega$, the EM parameters are all constant with μ_0, μ_1, ε_0 and ε_1 strictly positive, and the inhomogeneity is represented by a finite collection of smooth, diametrically shrinking domains, $\mathcal{I}_\rho = \bigcup_j (z_j + \rho D_j)$. We know from [VV00] that, so long as k_0^2 is not an eigenvalue for the operator $-\Delta$ with Dirichlet boundary conditions in Ω, the problem (2.1b) is well-posed for ρ sufficiently small. We will follow essentially the same argument to prove as much in the case of nonconstant coefficients, mixed boundary conditions and inhomogeneous sets of arbitrary shape.

Definition 2.5. *For* $0 \leq \rho \leq 1$ *we define the operator*

$$\mathcal{L}_\rho : H^1_D(\Omega) \longrightarrow \left(H^1_D(\Omega)\right)'$$

by

$${}_{(H^1_D(\Omega))'}\langle \mathcal{L}_\rho(u), \cdot \rangle_{H^1_D(\Omega)} = -\mathcal{H}_\rho(u, \cdot).$$

Definition 2.6. *Given* μ_ρ, ε_ρ, σ_ρ *and* Γ_D, *we call* ω *an* eigenfrequency *if there are nontrivial solutions to the homogeneous form of* (2.1b) *(i.e., when* F, f *and* g *are zero).*

As one would expect, we have the following

Lemma 2.7. *Let* μ_ρ, ε_ρ, σ_ρ *and* Γ_D *be given. The following are equivalent:*

[3] Recall that $\mathfrak{C}$ is the maximum of the suprema of the functions μ_0, ε_0, σ_0, μ_1, ε_1 and σ_1.

(a) *ω is not an eigenfrequency.*

(b) *$\mathcal{L}_\rho$ is invertible.*

(c) *Given any $F \in (H^1_D(\Omega))'$, $f \in H^{1/2}(\Gamma_D)$ and $g \in \big(H^{1/2}_{00}(\Gamma_N)\big)'$,* (2.1b) *is uniquely solvable.*

Proof. We will show $(c) \Rightarrow (a) \Rightarrow (b) \Rightarrow (c)$. $(c) \Rightarrow (a)$ is immediate. Assume (a). Then the only solution of $\mathcal{L}_\rho u = 0$ is $u = 0$. From this, the injectivity of $\mathcal{L}_\rho$, a standard argument using the Lax-Milgram theorem and the Fredholm alternative can be used to show $\mathcal{L}_\rho$ is invertible: Let

$$\mathcal{H}^\gamma_\rho(u,v) = \mathcal{H}_\rho(u,v) + \gamma \int_\Omega u\bar{v}\,\mathrm{d}x, \quad \rho \geq 0.$$

For $\gamma > 0$ satisfying

$$\gamma > \|\operatorname{Re}(\kappa_\rho^2)\|_{L^\infty(\Omega)},$$

$\mathcal{H}^\gamma_\rho$ is bounded and coercive on $H^1_D(\Omega)$. Consequently, by the Lax-Milgram theorem, the operator

$$\mathcal{L}^\gamma_\rho : H^1_D(\Omega) \longrightarrow (H^1_D(\Omega))'$$

defined by

$$_{(H^1_D(\Omega))'}\big\langle \mathcal{L}^\gamma_\rho(u), v\big\rangle_{H^1_D(\Omega)} = -\mathcal{H}^\gamma_\rho(u,v) \quad \text{for all } v \in H^1_D(\Omega) \tag{2.4}$$

is a continuous isomorphism of $H^1_D(\Omega)$ onto $(H^1_D(\Omega))'$. Let

$$I : H^1_D(\Omega) \longrightarrow (H^1_D(\Omega))' \tag{2.5}$$

be the natural injection

$$ {}_{(H^1_D(\Omega))'}\langle I(u), v\rangle_{H^1_D(\Omega)} = \int_\Omega u\bar{v}\,\mathrm{d}x. $$

Since

$$ I = \imath_1\imath_2 : H^1_D(\Omega) \xrightarrow{\subset\subset} L^2(\Omega) \longrightarrow (H^1_D(\Omega))', $$

I is compact, and so

$$ (\mathcal{L}^\gamma_\rho)^{-1} I : H^1_D(\Omega) \longrightarrow H^1_D(\Omega) $$

is compact as well. Since $\mathcal{L}_\rho$ is injective,

$$ (\mathcal{L}^\gamma_\rho)^{-1}\mathcal{L}_\rho = 1 + \gamma(\mathcal{L}^\gamma_\rho)^{-1} I $$

is also injective (here 1 is the identity operator on $H^1_D(\Omega)$). Therefore, by the Fredholm alternative, $(\mathcal{L}^\gamma_\rho)^{-1}\mathcal{L}_\rho$ must be a continuous automorphism of $H^1_D(\Omega)$. Consequently, $\mathcal{L}_\rho$ is invertible. Hence $(a) \Rightarrow (b)$.

Now assume (b) and suppose that, for some given F, f and g, $u_\rho = \tilde{u}_\rho + h_1$ and $v_\rho = \tilde{v}_\rho + h_2$ both solve problem 2.3. Then

$$ -\mathcal{H}_\rho(\tilde{u}_\rho - \tilde{v}_\rho, v) = \mathcal{H}_0(h_1 - h_2, v) = \mathcal{H}_\rho(h_1 - h_2, v) \quad \text{for all } v \in H^1_D(\Omega), $$

where the last equality follows from the fact that $h_1|_\mathcal{I} = h_2|_\mathcal{I} = 0$. Therefore, by the assumption that $\mathcal{L}_\rho$ is invertible and the fact that $h_1 - h_2 \in H^1_D(\Omega)$, $\tilde{u}_\rho - \tilde{v}_\rho = h_1 - h_2$. Thus $u_\rho = v_\rho$, which concludes the proof that $(b) \Rightarrow (c)$. □

We are now ready to state

Theorem 1. *If $\omega \geq 0$ is not an eigenfrequency relative to μ_0, ε_0, σ_0 and Γ_D (in other words, if $\mathcal{L}_0$ is invertible) then there exists a $\rho_0 > 0$ such that, given any $F \in (H^1_D(\Omega))'$, $f \in H^{1/2}(\Gamma_D)$ and $g \in \left(H^{1/2}_{00}(\Gamma_N)\right)'$, problem* (2.1b) *has a unique solution in $H^1(\Omega)$ for*

$0 \le \rho \le \rho_0$. *Furthermore,*

$$\|u_\rho\|_{H^1(\Omega)} \le C \left(\|F\|_{(H^1_D(\Omega))'} + \|f\|_{H^{1/2}(\Gamma_D)} + \|g\|_{(H^{1/2}_{00}(\Gamma_N))'} \right), \tag{2.6}$$

where C depends on ω, the ellipticity constant ϑ, the suprema of the EM parameter functions ($\mathfrak{C}$), $\operatorname{dist}(\mathcal{I}, \partial\Omega)$, Ω, Γ_D, *and* $\|\mathcal{L}_0^{-1}\|_{\mathscr{L}((H^1_D(\Omega))', H^1_D(\Omega))}$, *but is independent of ρ, F, f and g.*

Remark 2.8. *The assumption that the EM parameter functions ε_0, σ_0, μ_1, ε_1 and σ_1 are continuous, and the assumption that μ_0 is Lipschitz continuous, are not necessary for Theorem 1 to hold. All that is required of these functions is that they belong to L^∞. The continuity of these parameters will be needed later for the asymptotic expansion that is the main result of this chapter (Theorem 3).*

Before we prove this theorem, we must establish some preliminary results. With $\mathcal{L}_\rho^\gamma$ as defined at (2.4), we have the following

Property 2.9. *Let*

$$\gamma = \max\left\{\|\operatorname{Re}(\kappa_0^2)\|_{L^\infty(\Omega)}, \|\operatorname{Re}(\kappa_1^2)\|_{L^\infty(\mathcal{I})}\right\} + 1. \tag{2.7}$$

Then

$$\|\mathcal{L}_\rho^\gamma\|_{\mathscr{L}(H^1_D(\Omega),(H^1_D(\Omega))')} \le C_1, \tag{2.8}$$

$$\|(\mathcal{L}_\rho^\gamma)^{-1}\|_{\mathscr{L}((H^1_D(\Omega))', H^1_D(\Omega))} \le C_2 := \max\left\{\frac{1}{\vartheta}, 1\right\}. \tag{2.9}$$

where C_1 depends only on ω and the L^∞ norms of the EM parameter functions.

Proof. (2.8) is obvious, and (2.9) immediately follows from the fact that

$$\min\{\vartheta, 1\}\|u\|^2_{H^1_D(\Omega)} \le \operatorname{Re}\{\mathcal{H}_\rho^\gamma(u,u)\}. \qquad \square$$

Recall that

$$(\mathcal{L}_\rho^\gamma)^{-1} I : H_D^1(\Omega) \longrightarrow H_D^1(\Omega)$$

is compact, where $I : H_D^1(\Omega) \to (H_D^1(\Omega))'$ is the natural injection (2.5). This compactness of the $(\mathcal{L}_\rho^\gamma)^{-1} I$ is uniform in ρ in a sense we shall now make precise.

Definition 2.10. *A sequence of compact operators T_n on a Banach space V is* collectively compact *if*

$$\bigcup_n T_n\big(\{v \in V : \|v\| \le 1\}\big)$$

is precompact in V.

Lemma 2.11. *Given a sequence of parameters ρ_n with $\rho_n \to 0$, the operators $(\mathcal{L}_{\rho_n}^\gamma)^{-1} I$ are collectively compact and converge pointwise to $(\mathcal{L}_0^\gamma)^{-1} I$.*

Proof. Our proof is essentially the same as that found in [VV00]. We first prove $(\mathcal{L}_{\rho_n}^\gamma)^{-1}$ converges pointwise to $(\mathcal{L}_0^\gamma)^{-1}$, which implies the pointwise convergence of $(\mathcal{L}_{\rho_n}^\gamma)^{-1} I$ to $(\mathcal{L}_0^\gamma)^{-1} I$. To do this, we will prove $\mathcal{L}_{\rho_n} \to \mathcal{L}_0$ pointwise. From this the pointwise convergence of $(\mathcal{L}_{\rho_n}^\gamma)^{-1}$ to $(\mathcal{L}_0^\gamma)^{-1}$ will follow: for given any $G \in (H_D^1(\Omega))'$,

$$\mathcal{L}_{\rho_n}^\gamma (\mathcal{L}_0^\gamma)^{-1} G \longrightarrow \mathcal{L}_0^\gamma (\mathcal{L}_0^\gamma)^{-1} G = G,$$

and therefore

$$\begin{aligned} \big\|(\mathcal{L}_{\rho_n}^\gamma)^{-1} G - (\mathcal{L}_0^\gamma)^{-1} G\big\|_{H_D^1(\Omega)} &\le \big\|(\mathcal{L}_\rho^\gamma)^{-1}\big\| \big\|G - \mathcal{L}_{\rho_n}^\gamma (\mathcal{L}_0^\gamma)^{-1} G\big\|_{(H_D^1(\Omega))'} \\ &\le C \big\|G - \mathcal{L}_{\rho_n}^\gamma (\mathcal{L}_0^\gamma)^{-1} G\big\|_{(H_D^1(\Omega))'} \\ &\longrightarrow 0, \end{aligned}$$

where $C = C_2$ from Property 2.9. To prove $\mathcal{L}_{\rho_n} \to \mathcal{L}_0$ pointwise, observe that for any

$u, v \in H^1_D(\Omega)$,

$$\Big|\, {}_{(H^1_D(\Omega))'}\langle (\mathcal{L}_{\rho_n} - \mathcal{L}_0)u, v\rangle_{H^1_D(\Omega)} \Big| = \left| \int_{\mathcal{I}_{\rho_n}} \left\{ \Big(\frac{1}{\mu_0} - \frac{1}{\mu_1}\Big) \nabla u \cdot \nabla \overline{v} + (\kappa_1^2 - \kappa_0^2) u \overline{v} \right\} \mathrm{d}x \right|$$
$$\leq C \|u\|_{H^1(\mathcal{I}_{\rho_n})} \|v\|_{H^1_D(\Omega)}.$$

This means

$$\|(\mathcal{L}_{\rho_n} - \mathcal{L}_0)u\|_{(H^1_D(\Omega))'} \leq C\|u\|_{H^1(\mathcal{I}_{\rho_n})} \longrightarrow 0,$$

which is the desired result.

As for the collective compactness of the $(\mathcal{L}^\gamma_{\rho_n})^{-1} I$: Suppose $\{u_j\}$ is a sequence in $H^1_D(\Omega)$ satisfying $\|u_j\|_{H^1_D(\Omega)} \leq 1$. We will show that any sequence of the form $(\mathcal{L}^\gamma_{\rho_{n_j}})^{-1} I u_j$ (here n_j is an arbitrary sequence of indices, not necessarily a subsequence of n) has a strongly convergent subsequence.

First note that in the possible case where the sequence n_j is bounded, and thus ranges over a finite set, the result is an immediate consequence of the fact that a finite union of compact sets is itself compact. So we assume $n_j \to \infty$.

By passing to a subsequence if necessary, we may assume by the compactness of I that Iu_j converges strongly to some $G \in (H^1_D(\Omega))'$. We therefore have

$$\|(\mathcal{L}^\gamma_{\rho_{n_j}})^{-1} I u_j - (\mathcal{L}^\gamma_0)^{-1} G\|_{H^1_D(\Omega)}$$
$$\leq \|(\mathcal{L}^\gamma_{\rho_{n_j}})^{-1} I u_j - (\mathcal{L}^\gamma_{\rho_{n_j}})^{-1} G\|_{H^1_D(\Omega)} + \|(\mathcal{L}^\gamma_{\rho_{n_j}})^{-1} G - (\mathcal{L}^\gamma_0)^{-1} G\|_{H^1_D(\Omega)}$$
$$\leq C\|Iu_j - G\|_{(H^1_D(\Omega))'} + \|(\mathcal{L}^\gamma_{\rho_{n_j}})^{-1} G - (\mathcal{L}^\gamma_0)^{-1} G\|_{H^1_D(\Omega)},$$

with C independent of j. The proof is finished since $(\mathcal{L}^\gamma_{\rho_{n_j}})^{-1}$ converges pointwise to $(\mathcal{L}^\gamma_0)^{-1}$. □

To prove Theorem 1, we will need the following lemma concerning collectively compact operators. For a proof, see Theorem 1.6, Corollary 1.8 and Theorem 1.11 in [Ans71].

Lemma 2.12. *Assume the collectively compact sequence of operators $\{T_n\}$ on the Banach space V converges pointwise to the (necessarily compact) operator T. Then*

$$\|(T_n - T)T_n\|_{\mathscr{L}(V)} \longrightarrow 0. \tag{2.10}$$

Furthermore,

(a) $1 - T$ *is invertible*

if and only if

(b) there exists an N such that for $n \geq N$, $1 - T_n$ is invertible and the norms $\|(1 - T_n)^{-1}\|_{\mathscr{L}(V)}$ are bounded uniformly.

If (a) *and* (b) *hold,*

$$(1 - T_n)^{-1} \xrightarrow{\text{pointwise}} (1 - T)^{-1} \tag{2.11}$$

and

$$\|(1 - T_n)^{-1}\|_{\mathscr{L}(V)} \leq \frac{1 + \|(1 - T)^{-1}\|_{\mathscr{L}(V)}\|T_n\|_{\mathscr{L}(V)}}{1 - \|(1 - T)^{-1}\|_{\mathscr{L}(V)}\|(T_n - T)T_n\|_{\mathscr{L}(V)}}. \tag{2.12}$$

Proof of Theorem 1. Let $F \in (H^1_D(\Omega))'$, $f \in H^{1/2}(\Gamma_D)$ and $g \in \left(H^{1/2}_{00}(\Gamma_N)\right)'$. Assume $\mathcal{L}_0$ is invertible. With $\gamma > 0$ as in (2.7), $\mathcal{L}^\gamma_0$ and the $\mathcal{L}^\gamma_\rho$ for $\rho > 0$ are isomorphisms of $H^1_D(\Omega)$ onto $(H^1_D(\Omega))'$ satisfying the uniform bounds of Property 2.9. Since $\mathcal{L}_0 = \mathcal{L}^\gamma_0 + \gamma I$ and $\mathcal{L}^\gamma_0$ are both invertible,

$$1 + \gamma(\mathcal{L}^\gamma_0)^{-1} I$$

is a continuous automorphism of $H^1_D(\Omega)$.

Let $\rho_n \to 0$. With $T = -\gamma(\mathcal{L}^\gamma_0)^{-1} I$ and $T_n = -\gamma(\mathcal{L}^\gamma_{\rho_n})^{-1} I$, we apply Lemmas 2.11 and 2.12 to conclude there exists an N such that

$$1 + \gamma(\mathcal{L}^\gamma_{\rho_n})^{-1} I$$

is invertible for all $n \geq N$.

Now, if the there did not exist a ρ_0 as asserted in the statement of the theorem, there would necessarily exist a sequence $\rho_n \to 0$ and a sequence $\mathcal{F}_n \in (H^1_D(\Omega))'$ such that for each n the equation

$$(\mathcal{L}^{\gamma}_{\rho_n} + \gamma I)u = \mathcal{F}_n$$

either would have no solution or would have non-unique solutions in $H^1_D(\Omega)$. Either way, the operators $1 + \gamma(\mathcal{L}^{\gamma}_{\rho_n})^{-1}I$ would each be non-invertible, which would contradict the conclusion of the previous paragraph.

As for the bound (2.6), first note that, by (2.10), ρ_0 can be chosen so that

$$\|(T_\rho - T)T_\rho\| \leq \frac{1}{2\|(1-T)^{-1}\|} \quad \text{for all} \quad 0 < \rho \leq \rho_0,$$

where $T = -\gamma(\mathcal{L}^{\gamma}_0)^{-1}I$ and $T_\rho = -\gamma(\mathcal{L}^{\gamma}_\rho)^{-1}I$. For these ρ,

$$\begin{aligned}
\|\mathcal{L}_\rho^{-1}\| &= \|(1 + \gamma(\mathcal{L}^{\gamma}_\rho)^{-1}I)^{-1}(\mathcal{L}^{\gamma}_\rho)^{-1}\| \\
&\leq \|(1 + \gamma(\mathcal{L}^{\gamma}_\rho)^{-1}I)^{-1}\| \, \|(\mathcal{L}^{\gamma}_\rho)^{-1}\| \\
&\leq C_2\|(1 + \gamma(\mathcal{L}^{\gamma}_\rho)^{-1}I)^{-1}\| && \text{(by Property 2.9)} \\
&\leq C_2\left(\frac{1 + \|(1-T)^{-1}\| \, \|T_\rho\|}{1 - \|(1-T)^{-1}\| \, \|(T_\rho - T)T_\rho\|}\right) && \text{(by Lemma 2.12)} \\
&\leq 2C_2(1 + C_2\|(1-T)^{-1}\|) && \text{(by Property 2.9)} \\
&= 2C_2(1 + C_2\|\mathcal{L}_0^{-1}\mathcal{L}^{\gamma}_0\|) \\
&\leq 2C_2(1 + C_1C_2\|\mathcal{L}_0^{-1}\|) && \text{(by Property 2.9).}
\end{aligned}$$

Now, if u_ρ is a solution to (2.1b) then $u_\rho = \tilde{u}_\rho + h$ for some $\tilde{u}_\rho \in H^1_D(\Omega)$ and $h \in H^1(\Omega)$ as in Problem 2.3. Since $\tilde{u}_\rho = (\mathcal{L}_\rho)^{-1}\mathcal{F}$, where $\mathcal{F} \in (H^1_D(\Omega))'$ is as defined in Problem 2.3,

$$\|\tilde{u}_\rho\|_{H^1_D(\Omega)} \leq \|\mathcal{L}_\rho^{-1}\|\|\mathcal{F}\|_{(H^1_D(\Omega))'}.$$

The bound (2.6) now follows from Remark 2.4 concerning the bound on $\mathcal{F}$, and also from the fact that h must satisfy $\|h\|_{H^1(\Omega)} \le C\|f\|_{\Gamma_D}$, with C depending only on Γ_D, $\partial\Omega$ and $\operatorname{dist}(\mathcal{I}, \partial\Omega)$. $\square$

In the case where μ_0, ε_0 and σ_0 are constant, the bound on $\|\mathcal{L}_\rho^{-1}\|$ in terms of $\|\mathcal{L}_0^{-1}\|$ translates into a bound in terms of the distance between k_0^2 and the spectrum of $-\Delta$ on Ω with homogeneous mixed boundary data. We define the operator $\mathcal{L} : H^1_D(\Omega) \to (H^1_D(\Omega))'$ by

$$_{(H^1_D(\Omega))'}\langle \mathcal{L}(u), v\rangle_{H^1_D(\Omega)} = \int_\Omega \nabla u \nabla \overline{v}\, \mathrm{d}x \quad \text{for all } v \in H^1_D(\Omega),$$

and note that the problem of finding $u \in H^1_D(\Omega)$ solving

$$\mathcal{L}u = (F, \cdot)_{L^2(\Omega)} \quad \text{given } F \in L^2(\Omega)$$

is properly interpreted as the problem of finding u such that

$$\begin{cases} -\Delta u = F & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \partial_\nu u = 0 & \text{on } \Gamma_N. \end{cases}$$

Assume for the moment $\Gamma_D \neq \varnothing$. Using a standard Poincaré-type inequality, we may take $(u, v) \mapsto \int_\Omega \nabla u \nabla \overline{v}\, \mathrm{d}x$ to be the inner-product for $H^1_D(\Omega)$. Consequently, the Riesz theorem implies that $\mathcal{L}$ is invertible. Let $1/\lambda_n \searrow 0$, $n = 1, 2 \ldots$, denote the decreasing sequence of eigenvalues of $\mathcal{L}^{-1}I$ with corresponding eigenfunctions ϕ_n, normalized so that $\|\phi_n\|_{L^2(\Omega)} = 1$. For any $\mathcal{G} \in (H^1_D(\Omega))'$,

$$\begin{aligned} & \mathcal{L}_0 u = \mathcal{G} \\ \iff \quad & (-\mathcal{L} + k_0^2 I)u = \mu_0 \mathcal{G} \\ \iff \quad & \sum (k_0^2 - \lambda_n)\widehat{u}_n \overline{\widehat{v}_n} = \mu_0 \langle \mathcal{G}, v\rangle \quad \text{for all } v \in H^1_D(\Omega), \end{aligned}$$

where

$$\widehat{u}_n = \int_\Omega u\overline{\phi_n}\,\mathrm{d}x.$$

For the appropriate choice of v (that is, for the appropriate choice of θ_n, where we take $\widehat{v}_n = e^{i\theta_n}\widehat{u}_n$), $\|v\|_{H^1_D(\Omega)} = \|u\|_{H^1_D(\Omega)}$ and

$$\begin{aligned}
\mu_0\|\mathcal{G}\|_{(H^1_D(\Omega))'}\|u\|_{H^1_D(\Omega)} &\geq \mu_0|\langle\mathcal{G}, v\rangle| \\
&= \sum |k_0^2 - \lambda_n||\widehat{u}_n|^2 \\
&\geq \left(\min_n \left|\frac{k_0^2}{\lambda_n} - 1\right|\right) \sum |\lambda_n||\widehat{u}_n|^2 \\
&= \left(\min_n \left|\frac{k_0^2}{\lambda_n} - 1\right|\right) \|u\|^2_{H^1_D(\Omega)}.
\end{aligned}$$

Thus,

$$\|\mathcal{L}_0^{-1}\|_{\mathscr{L}((H^1_D(\Omega))', H^1_D(\Omega))} \leq \frac{\mu_0}{\min\limits_n \left|\frac{k_0^2}{\lambda_n} - 1\right|}.$$

In the case where $\Gamma_D = \varnothing$, $\ker\mathcal{L} = \{\text{constants}\}$, and the quotient map

$$\widetilde{\mathcal{L}} : H^1_\diamond(\Omega) \longrightarrow (H^1_\diamond(\Omega))'$$

is invertible. Let $1/\lambda_n \searrow 0$, $n = 1, 2, \ldots$, be the sequence of eigenvalues of $\widetilde{\mathcal{L}}^{-1}I$, with corresponding eigenfunctions ϕ_n, normalized by $\|\phi_n\|_{L^2(\Omega)} = 1$. With $\phi_0 := 1/|\Omega|$, $\{\phi_n\}_{n=0}^\infty$ is then an orthonormal basis of $H^1(\Omega)$. The above argument (for the case $\Gamma_D \neq \varnothing$) can be easily modified to yield

$$\begin{aligned}
\mu_0\|\mathcal{G}\|_{(H^1(\Omega))'}\|u\|_{H^1(\Omega)} &\geq \mu_0|\langle\mathcal{G}, v\rangle| \\
&= |k_0^2||\widehat{u}_0|^2 + \sum_{n\geq 1} |k_0^2 - \lambda_n||\widehat{u}_n|^2 \\
&\geq \min\left\{|k_0^2|\ ,\ \min_{n\geq 1}\left|\frac{k_0^2}{\lambda_n} - 1\right|\right\}\left(|\widehat{u}_0|^2 + \sum_{n\geq 1}|\lambda_n||\widehat{u}_n|^2\right) \\
&= \min\left\{|k_0^2|\ ,\ \min_{n\geq 1}\left|\frac{k_0^2}{\lambda_n} - 1\right|\right\}\|u\|^2_{H^1(\Omega)}.
\end{aligned}$$

Thus we have

Corollary 2.13. *Suppose the functions μ_0, ε_0 and σ_0 are constants, and that $k_0^2 = \omega^2\mu_0(\varepsilon_0 + i\sigma_0/\omega) \notin \{\lambda_n\}$ = the spectrum of $-\Delta$ on Ω with homogeneous mixed boundary data. With ρ_0 as in the statement of Theorem 1, for $0 \le \rho \le \rho_0$, given any $F \in (H^1_D(\Omega))'$, $f \in H^{1/2}(\Gamma_D)$ and $g \in \big(H^{1/2}_{00}(\Gamma_N)\big)'$, the solution u_ρ to* (2.1b) *satisfies*

$$\|u_\rho\|_{H^1_D(\Omega)} \le C\left(\|F\|_{(H^1_D(\Omega))'} + \|f\|_{H^{1/2}(\Gamma_D)} + \|g\|_{(H^{1/2}_{00}(\Gamma_N))'}\right), \tag{2.13}$$

where C depends on ω, ϑ, $\mathfrak{C}$, $\operatorname{dist}(\mathcal{I}, \partial\Omega)$, Ω, Γ_D, and on

$$\min_n \left|\frac{k_0^2}{\lambda_n} - 1\right| \quad \textit{when } \Gamma_D \neq \varnothing$$

and

$$\min\left\{|k_0^2|, \min_{\lambda_n \neq 0}\left|\frac{k_0^2}{\lambda_n} - 1\right|\right\} \quad \textit{when } \Gamma_D = \varnothing.$$

Remark 2.14. *The goal of this chapter, the asymptotic expansion of Theorem 3, is similar to, and motivated by, a prior result for the conductivity problem with a boundary condition of strictly Neumann type [CV03a]. That problem corresponds to the eigenfrequency $\omega = 0$ for problems* (2.1a) *and* (2.1b) *when $\Gamma_D = \varnothing$, with $\frac{1}{\mu}$ reinterpreted as the conductivity profile and u as the voltage potential (cf. Remark 2.1). In this case, problems* (2.1a) *and* (2.1b) *each require an additional normalization condition in order to be well-posed:*

$$\begin{cases} \nabla \cdot \left(\dfrac{1}{\mu_0}\nabla u_0\right) = F & \text{in } \Omega \\ \dfrac{1}{\mu_0}\partial_\nu u_0 = g & \text{on } \partial\Omega \\ \displaystyle\int_{\partial\Omega} u_0 \, \mathrm{d}\sigma = 0 & \end{cases} \tag{2.14a}$$

and

$$\begin{cases} \nabla \cdot \Big(\dfrac{1}{\mu_\rho} \nabla u_\rho\Big) = F & \text{in } \Omega \\ \qquad \dfrac{1}{\mu_\rho} \partial_\nu u_\rho = g & \text{on } \partial\Omega \\ \displaystyle\int_{\partial\Omega} u_\rho \,\mathrm{d}\sigma = 0. \end{cases} \tag{2.14b}$$

We must also require that $F \in (H^1(\Omega))'$ *and* $g \in H^{-1/2}(\partial\Omega)$ *satisfy*

$$\int_\Omega F \,\mathrm{d}\sigma = \int_{\partial\Omega} g \,\mathrm{d}\sigma \tag{2.15}$$

(i.e., $\langle F, 1|_\Omega \rangle = \langle g, 1|_{\partial\Omega} \rangle$*). To see that* (2.14b) *is well-posed for all* ρ*, consider the operator*

$$\widetilde{\mathcal{L}}_\rho := \mathcal{L}_\rho|_{H^1_\diamond(\Omega)} : H^1_\diamond(\Omega) \longrightarrow \mathfrak{U}, \tag{2.16}$$

where

$$H^1_\diamond(\Omega) = \big\{ u \in H^1(\Omega) : \textstyle\int_{\partial\Omega} u \,\mathrm{d}\sigma = 0 \big\}$$

and

$$\mathfrak{U} = \big\{ \mathcal{F} \in (H^1(\Omega))' : \langle \mathcal{F}, 1 \rangle = 0 \big\}.$$

(Note: $F \in (H^1_D(\Omega))'$ *and* $g \in H^{-1/2}(\partial\Omega)$ *satisfy condition* (2.15) *if and only if* $\mathcal{F}$ *as defined in* (2.3) *lies in* $\mathfrak{U}$*.) Taking*

$$((u, v)) := \int_\Omega \nabla u \cdot \nabla \overline{v} \,\mathrm{d}x + \Big(\int_{\partial\Omega} u \,\mathrm{d}\sigma \Big) \Big(\int_{\partial\Omega} \overline{v} \,\mathrm{d}\sigma \Big)$$

as the inner-product for $H^1(\Omega)$*, which is possible thanks to a simple Poincaré-type inequality,* $H^1_\diamond(\Omega)$ *is the orthogonal complement of the space of constants in* $H^1(\Omega)$*, and the mapping* $\mathcal{F} \mapsto \mathcal{F}|_{H^1_\diamond(\Omega)}$ *is a linear isometry of* $\mathfrak{U}$ *onto* $(H^1_\diamond(\Omega))'$*. We identify* $\mathfrak{U}$ *and* $(H^1_\diamond(\Omega))'$*, and then, noting that* $\mathcal{H}_\rho$ *is bounded and coercive on* $H^1_\diamond(\Omega)$*, we apply the Lax-Milgram theorem to conclude* $\widetilde{\mathcal{L}}_\rho$ *is invertible.*

2.3 Convergence of the perturbed field to the background field

The following lemma was proved in [CV03a]:

Lemma 2.15. *Suppose $F \in (H^1(\Omega))'$ and $g \in H^{-1/2}(\partial\Omega)$ with $\int_\Omega F \,\mathrm{d}x = \int_{\partial\Omega} g \,\mathrm{d}\sigma$. Let V_0 denote the solution to*

$$\begin{cases} \nabla \cdot \Big(\dfrac{1}{\mu_0}\nabla V_0\Big) = F & \text{in } \Omega \\ \quad \dfrac{1}{\mu_0}\partial_\nu V_0 = g & \text{on } \partial\Omega \\ \quad \displaystyle\int_{\partial\Omega} V_0 \,\mathrm{d}\sigma = 0, & \end{cases} \tag{2.17}$$

and let V_ρ denote the solution to the perturbed problem

$$\begin{cases} \nabla \cdot \Big(\dfrac{1}{\mu_\rho}\nabla V_\rho\Big) = F & \text{in } \Omega \\ \quad \dfrac{1}{\mu_\rho}\partial_\nu V_\rho = g & \text{on } \partial\Omega \\ \quad \displaystyle\int_{\partial\Omega} V_\rho \,\mathrm{d}\sigma = 0. & \end{cases} \tag{2.18}$$

Then

$$\|V_\rho - V_0\|_{H^1_D(\Omega)} \le C|\mathcal{I}_\rho|^{1/2}\|V_0\|_{C^{0,1}(\mathcal{I}_\rho)} \tag{2.19}$$

for some C independent of ρ. If we also assume $\mu_0 \in C^{0,1}(\overline{\Omega})$ then for any $\delta > 0$,

$$\|V_\rho - V_0\|_{L^2(\Omega)} \le C_\delta|\mathcal{I}_\rho|^{1-\delta}\|V_0\|_{C^{0,1}(\mathcal{I}_\rho)}. \tag{2.20}$$

We will prove a similar result for the Helmholtz problem. Our proof of the analogue of (2.19), namely (2.21), is essentially the same as that used to prove an energy estimate in [VV00]. Our proof of the analogue of (2.20), namely (2.22), is an adaptation of the proof of (2.20) found in [CV03a].

Before stating and proving these estimates of the strength of convergence of u_ρ to u_0, we present the following lemma, which is of central importance. The proof is a simple exercise.

Lemma 2.16. *Let* $F \in \left(H^1_D(\Omega)\right)'$, $f \in H^{1/2}(\Gamma_D)$ *and* $g \in \left(H^{1/2}_{00}(\Gamma_N)\right)'$. *If* U_0 *satisfies*

$$\begin{cases} \nabla \cdot \Big(\dfrac{1}{\mu_0}\nabla U_0\Big) + \kappa_0^2 U_0 = F & \text{in } \Omega \\ U_0 = f & \text{on } \Gamma_D \\ \dfrac{1}{\mu_0}\partial_\nu U_0 = g & \text{on } \Gamma_N \end{cases}$$

and U_ρ *satisfies the perturbed problem (that is, the problem in the presence of the inhomogeneity, with* μ_0 *replaced by* μ_ρ*) then for any* $\phi \in H^1_D(\Omega)$,

$$\mathcal{H}_0(U_\rho - U_0, \phi) = \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla U_\rho \cdot \nabla\overline{\phi}\,\mathrm{d}x + \int_{\mathcal{I}_\rho}(\kappa_1^2 - \kappa_0^2)U_\rho\overline{\phi}\,\mathrm{d}x$$

and

$$\mathcal{H}_\rho(U_\rho - U_0, \phi) = \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla U_0 \cdot \nabla\overline{\phi}\,\mathrm{d}x + \int_{\mathcal{I}_\rho}(\kappa_1^2 - \kappa_0^2)U_0\overline{\phi}\,\mathrm{d}x.$$

We now state the main result of this section.

Theorem 2. *Suppose* ω *is not an eigenfrequency relative to* μ_0, ε_0, σ_0 *and* Γ_D *(in other words, suppose* $\mathcal{L}_0$ *is invertible). Let* $\rho_0 > 0$ *be as in Theorem 1 and suppose* $F \in (H^1_D(\Omega))'$, $f \in H^{1/2}(\Gamma_D)$ *and* $g \in \left(H^{1/2}_{00}(\Gamma_N)\right)'$. *For* $\rho \le \rho_0$, *if* u_0 *and* u_ρ *are the solutions to problems* (2.1a) *and* (2.1b) *respectively, then*

$$\|u_\rho - u_0\|_{H^1_D(\Omega)} \le C|\mathcal{I}_\rho|^{1/2}\|u_0\|_{C^{0,1}(\mathcal{I}_\rho)}, \tag{2.21}$$

where C *depends on* ω, *the ellipticity constant* ϑ, *the suprema of the EM profile functions* ($\mathfrak{C}$), $\mathrm{dist}(\mathcal{I}, \partial\Omega)$, Ω, Γ_D *and* $\|\mathcal{L}_0^{-1}\|_{\mathscr{L}((H^1_D(\Omega))', H^1_D(\Omega))}$ *but is independent of* ρ, F, f *and* g. *Moreover, for any* $\delta > 0$ *and* $\rho \le \rho_0$,

$$\|u_\rho - u_0\|_{L^2(\Omega)} \le C|\mathcal{I}_\rho|^{1-\delta}\|u_0\|_{C^{0,1}(\mathcal{I}_\rho)}, \tag{2.22}$$

where C *depends on* ω, ϑ, $\mathfrak{C}$, $\|\mu_0\|_{C^{0,1}(\overline{\Omega})}$, $\mathrm{dist}(\mathcal{I}, \partial\Omega)$, Ω, Γ_D, $\|\mathcal{L}_0^{-1}\|$ *and* δ *but is independent of* ρ, F, f *and* g.

Proof. By Lemma 2.16, if $v \in H^1_D(\Omega)$ then

$$\begin{aligned}\mathcal{H}_\rho(u_\rho - u_0, v) &= \int_{\mathcal{I}_\rho} \left[-\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big) \nabla u_0 \cdot \nabla v + (\kappa_1^2 - \kappa_0^2) u_0 v \right] \mathrm{d}x \\ &\leq C|\mathcal{I}_\rho|^{1/2} \|u_0\|_{C^{0,1}(\mathcal{I}_\rho)} \|v\|_{H^1(\Omega)}.\end{aligned}$$

In other words,

$$\mathcal{L}_\rho(u_\rho - u_0) = G$$

with

$$\|G\|_{(H^1_D(\Omega))'} \leq C|\mathcal{I}_\rho|^{1/2} \|u_0\|_{C^{0,1}(\mathcal{I}_\rho)}.$$

By Theorem 1, for $\rho \leq \rho_0$,

$$\|u_\rho - u_0\|_{H^1_D(\Omega)} = \|(\mathcal{L}_\rho)^{-1} G\|_{H^1_D(\Omega)} \leq C \|G\|_{(H^1_D(\Omega))'}$$

with C as in the statement of that theorem. This completes the proof of (2.21). To prove (2.22), let $w \in H^2_{\text{loc}}(\Omega) \cap H^1_D(\Omega)$ solve

$$\begin{cases} \nabla \cdot \Big(\dfrac{1}{\mu_0} \nabla w \Big) + \kappa_0^2 w = \overline{u_0 - u_\rho} & \text{in } \Omega \\ w = 0 & \text{on } \Gamma_D \\ \dfrac{1}{\mu_0} \partial_\nu w = 0 & \text{on } \Gamma_N. \end{cases}$$

Since $\mathcal{I} \subset\subset \Omega$, we may choose a smooth domain Ω' depending only on Ω and $\text{dist}(\mathcal{I}, \partial\Omega)$ such that $\mathcal{I} \subset\subset \Omega' \subset\subset \Omega$. Using Lemma 2.16, we find that for any $1 < p, q < \infty$

satisfying $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}\int_\Omega |u_\rho-u_0|^2\,\mathrm{d}x &= \int_\Omega \frac{1}{\mu_0}\nabla w\cdot\nabla(u_\rho-u_0)\,\mathrm{d}x-\int_\Omega \kappa_0^2 w(u_\rho-u_0)\,\mathrm{d}x\\ &= \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1}-\frac{1}{\mu_0}\Big)\nabla u_\rho\cdot\nabla w\,\mathrm{d}x+\int_{\mathcal{I}_\rho}(\kappa_1^2-\kappa_0^2)u_\rho w\,\mathrm{d}x\\ &\le C\big(\|\nabla u_\rho\|_{L^q(\mathcal{I}_\rho)}\|\nabla w\|_{L^p(\Omega')}+\omega^2\|u_\rho\|_{L^1(\mathcal{I}_\rho)}\|w\|_{L^\infty(\Omega')}\big)\\ &\le C\big(\|\nabla u_\rho\|_{L^q(\mathcal{I}_\rho)}+\omega^2\|u_\rho\|_{L^1(\mathcal{I}_\rho)}\big)\|w\|_{H^2(\Omega')},\end{aligned}\tag{2.23}$$

where C depends on Ω' and $\mathfrak{C}$. (We note that the above use of Sobolev's embedding theorem requires that the ambient space be two dimensional, as we would like to choose p arbitrarily large.) By elliptic estimates,

$$\begin{aligned}\|w\|_{H^2(\Omega')} &\le C\big(\|w\|_{L^2(\Omega)}+\|u_\rho-u_0\|_{L^2(\Omega)}\big)\\ &\le C\Big(\|\mathcal{L}_0^{-1}\|\,\|I(u_\rho-u_0)\|_{(H^1_D(\Omega))'}+\|u_\rho-u_0\|_{L^2(\Omega)}\Big)\\ &= C\big(\|\mathcal{L}_0^{-1}\|+1\big)\|u_\rho-u_0\|_{L^2(\Omega)}\end{aligned}\tag{2.24}$$

for some C depending on $\operatorname{dist}(\Omega',\partial\Omega)$, ϑ, $\mathfrak{C}$, $\|\mu_0\|_{C^{0,1}(\overline{\Omega})}$, ω and p. With the freedom to choose $p<\infty$ arbitrarily large, we make the appropriate choice so that $1-\delta=1/q$, hence

$$\begin{aligned}\|\nabla u_\rho\|_{L^q(\mathcal{I}_\rho)} &\le \|\nabla(u_\rho-u_0)\|_{L^q(\mathcal{I}_\rho)}+\|\nabla u_0\|_{L^q(\mathcal{I}_\rho)}\\ &\le |\mathcal{I}_\rho|^{\frac{1}{2}-\delta}\|\nabla(u_\rho-u_0)\|_{L^2(\mathcal{I}_\rho)}+|\mathcal{I}_\rho|^{1-\delta}\|\nabla u_0\|_{L^\infty(\mathcal{I}_\rho)}.\end{aligned}\tag{2.25}$$

Similarly, but more easily, we have

$$\begin{aligned}\|u_\rho\|_{L^1(\mathcal{I}_\rho)} &\le \|u_\rho-u_0\|_{L^1(\mathcal{I}_\rho)}+\|u_0\|_{L^1(\mathcal{I}_\rho)}\\ &\le |\mathcal{I}_\rho|^{1/2}\|u_\rho-u_0\|_{L^2(\mathcal{I}_\rho)}+|\mathcal{I}_\rho|\|u_0\|_{L^\infty(\mathcal{I}_\rho)}.\end{aligned}\tag{2.26}$$

Combining (2.23), (2.24), (2.25) and (2.26), and then using (2.21), yields

$$\begin{aligned}\|u_\rho - u_0\|_{L^2(\Omega)} &\leq C\Big(|\mathcal{I}_\rho|^{\frac{1}{2}-\delta}\|u_\rho - u_0\|_{H^1(\mathcal{I}_\rho)} + |\mathcal{I}_\rho|^{1-\delta}\|u_0\|_{C^{0,1}(\mathcal{I}_\rho)}\Big)\\ &\leq C|\mathcal{I}_\rho|^{1-\delta}\|u_0\|_{C^{0,1}(\mathcal{I}_\rho)}.\end{aligned}$$

□

Corollary 2.17. *In addition to the hypotheses of Theorem 2, assume $F|_{\Omega'} \in L^p(\Omega')$ for some $p > 2$ and some open set Ω' with $\mathcal{I} \subset\subset \Omega' \subset\subset \Omega$. Let*

$$\|(F, f, g)\| = \|F\|_{L^p(\Omega')} + \|F\|_{(H^1_D(\Omega))'} + \|f\|_{H^{1/2}(\Gamma_D)} + \|g\|_{(H^{1/2}_{00}(\Gamma_N))'}.$$

Then for $\rho \leq \rho_0$, if u_0 and u_ρ are the solutions to problems (2.1a) *and* (2.1b) *respectively,*

$$\|u_\rho - u_0\|_{H^1_D(\Omega)} \leq C|\mathcal{I}_\rho|^{1/2}\|(F, f, g)\|, \tag{2.27}$$

where C depends on ω, ϑ, $\mathfrak{C}$, $\mathrm{dist}(\mathcal{I}, \partial\Omega')$, $\mathrm{dist}(\Omega', \partial\Omega)$ Ω, Γ_D and $\|\mathcal{L}_0^{-1}\|$ but is independent of ρ, F, f and g. Moreover, given any $\delta > 0$,

$$\|u_\rho - u_0\|_{L^2(\Omega)} \leq C|\mathcal{I}_\rho|^{1-\delta}\|(F, f, g)\|, \tag{2.28}$$

with C depending on ω, ϑ, $\mathfrak{C}$, $\|\mu_0\|_{C^{0,1}(\overline{\Omega})}$, $\mathrm{dist}(\mathcal{I}, \partial\Omega')$, $\mathrm{dist}(\Omega', \partial\Omega)$ Ω, Γ_D and $\|\mathcal{L}_0^{-1}\|$ and δ but is independent of ρ, F, f and g.

Proof. By interior elliptic estimates [GT01, Theorem 9.11],

$$\|u_0\|_{W^{2,p}(\mathcal{I}')} \leq C\big(\|u_0\|_{L^p(\Omega')} + \|F\|_{L^p(\Omega')}\big)$$

for some open set $\mathcal{I}'$ chosen to satisfy $\mathcal{I} \subset\subset \mathcal{I}' \subset\subset \Omega'$. C depends on $\mathfrak{C}$, $\|\mu_0\|_{C^{0,1}(\overline{\Omega})}$, ω and $\mathrm{dist}(\mathcal{I}, \partial\Omega')$. Since $p > 2 =$ the dimension of the ambient space,

$$\|u_0\|_{C^{0,1}(\mathcal{I}_\rho)} \leq C_{\mathrm{dist}(\mathcal{I},\mathcal{I}')}\|u_0\|_{W^{2,p}(\mathcal{I}')}.$$

□

Remark 2.18. *Theorem 2 and Corollary 2.17 continue to hold when the dimension n*

of the ambient space is greater than two. However, the factor $|\mathcal{I}_\rho|^{1-\delta}$ *must be replaced with* $|\mathcal{I}_\rho|^{\frac{1}{2}+\frac{1}{n}-\delta}$*, and* p *must be greater than* n*.*

2.4 Derivation of the asymptotic formula

Before proceeding, we state a stronger form of Lemma 2.16.

Lemma 2.19. *Let* $F \in \left(H^1_D(\Omega)\right)'$*,* $f \in H^{1/2}(\Gamma_D)$ *and* $g \in \left(H^{1/2}_{00}(\Gamma_N)\right)'$*.*

(i) Suppose ω *is not an eigenfrequency relative to* μ_0*,* ε_0*,* σ_0 *and* Γ_D*. If* U_0 *is the unique solution to*

$$\begin{cases} \nabla \cdot \left(\dfrac{1}{\mu_0}\nabla U_0\right) + \kappa_0^2 U_0 = F & \text{in } \Omega \\ U_0 = f & \text{on } \Gamma_D \\ \dfrac{1}{\mu_0}\partial_\nu U_0 = g & \text{on } \Gamma_N \end{cases}$$

and U_ρ*,* $0 < \rho < \rho_0$*, is the unique solution to the perturbed problem (that is, the problem with* μ_0 *and* κ_0^2 *replaced by* μ_ρ *and* κ_ρ^2*) then for any* $\phi \in H^1(\Omega)$*,*

$$\begin{aligned} \mathcal{H}_0(U_\rho - U_0, \phi) - &\int_{\Gamma_D} \frac{1}{\mu_0}\partial_\nu (U_\rho - U_0)\overline{\phi}\,\mathrm{d}\sigma \\ &= \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla U_\rho \cdot \nabla\overline{\phi}\,\mathrm{d}x + \int_{\mathcal{I}_\rho} (\kappa_1^2 - \kappa_0^2) U_\rho \overline{\phi}\,\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}_\rho(U_\rho - U_0, \phi) - &\int_{\Gamma_D} \frac{1}{\mu_0}\partial_\nu (U_\rho - U_0)\overline{\phi}\,\mathrm{d}\sigma \\ &= \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla U_0 \cdot \nabla\overline{\phi}\,\mathrm{d}x + \int_{\mathcal{I}_\rho} (\kappa_1^2 - \kappa_0^2) U_0 \overline{\phi}\,\mathrm{d}x, \end{aligned}$$

where

$$\text{``}\int_{\Gamma_D} \frac{1}{\mu_0}\partial_\nu (U_\rho - U_0)\overline{\phi}\,\mathrm{d}\sigma\text{''} = {}_{H^{-1/2}(\partial\Omega)}\Big\langle \tfrac{1}{\mu_0}\partial_\nu (U_\rho - U_0), \phi \Big\rangle_{H^{1/2}(\partial\Omega)}.$$

(ii) If W_0 satisfies

$$\begin{cases} \nabla \cdot \Big(\dfrac{1}{\mu_0}\nabla W_0\Big) = F & \text{in } \Omega \\ W_0 = f & \text{on } \Gamma_D \\ \dfrac{1}{\mu_0}\partial_\nu W_0 = g & \text{on } \Gamma_N \end{cases}$$

and W_ρ, $\rho > 0$, satisfies the perturbed problem then for any $\phi \in H^1(\Omega)$,

$$\int_\Omega \frac{1}{\mu_0}\nabla(W_\rho - W_0)\cdot\nabla\overline{\phi}\,\mathrm{d}x - \int_{\Gamma_D}\frac{1}{\mu_0}\partial_\nu(W_\rho - W_0)\overline{\phi}\,\mathrm{d}\sigma = \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla W_\rho\cdot\nabla\overline{\phi}\,\mathrm{d}x$$

and

$$\int_\Omega \frac{1}{\mu_\rho}\nabla(W_\rho - W_0)\cdot\nabla\overline{\phi}\,\mathrm{d}x - \int_{\Gamma_D}\frac{1}{\mu_0}\partial_\nu(W_\rho - W_0)\overline{\phi}\,\mathrm{d}\sigma = \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla W_0\cdot\nabla\overline{\phi}\,\mathrm{d}x.$$

Of course, (ii) is just a special case of (i).

Proof. These formulas clearly hold in the purely Neumann and purely Dirichlet cases, hence we assume the boundary data is of strictly mixed type. We prove only the second formula asserted in (i), as all other cases are similar. Observe that for $v \in H^1_D(\Omega)$,

$$\begin{aligned}\mathcal{H}_\rho(U_\rho - U_0, v) &= \int_{\mathcal{I}_\rho}\Big[-\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla U_0\cdot\nabla v + (\kappa_1^2 - \kappa_0^2)U_0 v\Big]\,\mathrm{d}x \\ &=: {}_{(H^1_D(\Omega))'}\langle G, v\rangle_{H^1_D(\Omega)}\,,\end{aligned}$$

as in the proof of Theorem 2. This means $U_\rho - U_0$ is the solution to

$$\begin{cases} \nabla\cdot\Big(\dfrac{1}{\mu_\rho}\nabla(U_\rho - U_0)\Big) + \kappa_\rho^2(U_\rho - U_0) = G & \text{in } \Omega \\ U_\rho - U_0 = 0 & \text{on } \Gamma_D \\ \dfrac{1}{\mu_\rho}\partial_\nu(U_\rho - U_0) = 0 & \text{on } \Gamma_N, \end{cases}$$

with G supported in $\mathcal{I}$. As a result, $U_\rho - U_0 \in H^1_{\mu_0}(\Omega\setminus\overline{\mathcal{I}})$, so that the trace $\frac{1}{\mu_0}\partial_\nu(U_\rho -$

$U_0) \in H^{-1/2}(\partial\Omega)$ is well-defined.

Choose open sets $\mathcal{I}'$ and Ω' with $\mathcal{I} \subset\subset \mathcal{I}' \subset\subset \Omega' \subset\subset \Omega$. Any $\phi \in H^1(\Omega)$ can be decomposed as $\phi = \phi_1 + \phi_2$ with $\phi_1|_{\mathcal{I}'} \equiv 0$, $\phi_2|_{\Omega\setminus\Omega'} \equiv 0$ and both $\phi_j \in H^1(\Omega)$. Therefore,

$$\begin{aligned}
\mathcal{H}_\rho(U_\rho - U_0, \phi) &= {}_{H^{-1/2}(\partial\Omega)}\big\langle \tfrac{1}{\mu_0}\partial_\nu(U_\rho - U_0), \phi|_{\partial\Omega}\big\rangle_{H^{1/2}(\partial\Omega)} + \mathcal{H}_\rho(U_\rho - U_0, \phi_2) \\
&= \int_{\Gamma_D} \frac{1}{\mu_0}\partial_\nu(U_\rho - U_0)\overline{\phi}\,\mathrm{d}\sigma + {}_{(H^1_D(\Omega))'}\langle G, \phi_2\rangle_{H^1_D(\Omega)} \\
&= \int_{\Gamma_D} \frac{1}{\mu_0}\partial_\nu(U_\rho - U_0)\overline{\phi}\,\mathrm{d}\sigma + \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla U_0 \cdot \nabla\overline{\phi}\,\mathrm{d}x \\
&\qquad + \int_{\mathcal{I}_\rho} (\kappa_1^2 - \kappa_0^2)U_0\overline{\phi}\,\mathrm{d}x. \qquad \square
\end{aligned}$$

Part of the significance of Lemma 2.19 is that is provides a way to equate boundary measurements with integrals concentrated solely on the inhomogeneities. For instance, if $\phi \in H^1(\Omega)$ solves the background Helmholtz equation then, after integrating by parts, the first identity of Lemma 2.19(i) becomes

$$\begin{aligned}
&\int_{\partial\Omega} \Big\{ U_\rho\Big(\frac{1}{\mu_0}\partial_\nu\overline{\phi}\Big) - \Big(\frac{1}{\mu_0}\partial_\nu U_\rho\Big)\overline{\phi}\Big\}\,\mathrm{d}\sigma \\
&\qquad = \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla U_\rho \cdot \nabla\overline{\phi}\,\mathrm{d}x + \int_{\mathcal{I}_\rho} (\kappa_1^2 - \kappa_0^2)U_\rho\overline{\phi}\,\mathrm{d}x. \qquad (2.29)
\end{aligned}$$

The left side of this equality represents a sort of measurement of the disturbance caused by the presence of the inhomogeneity. Hidden within this measurement is information about the inhomogeneity—its size, shape, etc. To extract this information, we manipulate the right-hand side to bring it to a simplified, asymptotic form. But before proceeding with this manipulation, we will discuss another useful class of choices for the test function ϕ, namely Green's functions. Suppose G is a Green's function for the background operator:

$$\nabla_x \cdot \Big(\frac{1}{\mu_0}\nabla_x G(\cdot, y)\Big) + \kappa_0^2 G(\cdot, y) = -\delta_y \quad \text{in } \Omega.$$

Let u_0 and u_ρ solve (2.1). Because $u_\rho - u_0$ satisfies

$$\begin{cases} \nabla \cdot \Big(\dfrac{1}{\mu_0}\nabla(u_\rho - u_0)\Big) + \kappa_0^2(u_\rho - u_0) = 0 & \text{in } \Omega \setminus \overline{\mathcal{I}} \\ u_\rho - u_0 = 0 & \text{on } \Gamma_D \\ \dfrac{1}{\mu_0}\partial_\nu(u_\rho - u_0) = 0 & \text{on } \Gamma_N, \end{cases}$$

$u_\rho - u_0$ is continuous on $(\Omega \setminus \overline{\mathcal{I}}) \cup \Gamma_D \cup \Gamma_N$, but not necessarily at the points where the boundary condition changes type. Consequently, for $y \in (\Omega \setminus \overline{\mathcal{I}})$,

$$(u_\rho - u_0)(y) + \int_{\Gamma_N} \frac{1}{\mu_0}\partial_{\nu_x} G(\cdot, y)(u_\rho - u_0)\,\mathrm{d}\sigma = \mathcal{H}_0\big(u_\rho - u_0, \overline{G(\cdot, y)}\big). \tag{2.30}$$

For many choices of G, this formula may be extended to $y \in \partial\Omega$, in which case the left-hand side represents a measurable perturbation in the boundary field caused by the presence of the inhomogeneity. By an application[4] of Lemma 2.19(i), for $y \in (\Omega \setminus \overline{\mathcal{I}})$,

$$\begin{aligned} &(u_\rho - u_0)(y) - \int_{\Gamma_D} \frac{1}{\mu_0}\partial_\nu(u_\rho - u_0)G(\cdot, y)\,\mathrm{d}\sigma + \int_{\Gamma_N} \frac{1}{\mu_0}\partial_{\nu_x} G(\cdot, y)(u_\rho - u_0)\,\mathrm{d}\sigma \\ &= \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla u_\rho \cdot \nabla G(\cdot, y)\,\mathrm{d}x + \int_{\mathcal{I}_\rho} (\kappa_1^2 - \kappa_0^2) u_\rho G(\cdot, y)\,\mathrm{d}x. \end{aligned} \tag{2.31}$$

2.4.1 Examples of Green's functions

I. Suppose ω is not an eigenfrequency with respect to μ_0, ε_0, σ_0 and a boundary condition of strictly Neumann type. For $y \in \Omega$ we define the Neumann function $\mathcal{N}(\cdot, y)$ to be the solution to

$$\begin{cases} \nabla \cdot \Big(\dfrac{1}{\mu_0}\nabla \mathcal{N}(\cdot, y)\Big) + \kappa_0^2 \mathcal{N}(\cdot, y) = -\delta_y & \text{in } \Omega \\ \dfrac{1}{\mu_0}\partial_\nu \mathcal{N}(\cdot, y) = 0 & \text{on } \partial\Omega, \end{cases}$$

[4] Since $G(\cdot, y) \notin H^1(\Omega)$, Lemma 2.19(i) cannot be applied directly. However, the result follows from a straightforward limiting approximation argument.

which extends to $y \in \partial\Omega$ as

$$\begin{cases} \nabla \cdot \Big(\dfrac{1}{\mu_0}\nabla\mathcal{N}(\cdot,y)\Big) + \kappa_0^2\mathcal{N}(\cdot,y) = 0 & \text{in } \Omega \\ \dfrac{1}{\mu_0}\partial_\nu\mathcal{N}(\cdot,y) = \delta_y & \text{on } \partial\Omega. \end{cases}$$

Suppose ω is also not an eigenfrequency with respect to μ_0, ε_0, σ_0 and Γ_D, so that (2.1a) and (2.1b) have unique solutions u_0 and u_ρ for $\rho > 0$ sufficiently small. With $G = \mathcal{N}$, (2.30) holds for all $y \in (\Omega \setminus \overline{\mathcal{I}}) \cup \Gamma_D \cup \Gamma_N$, and so we have

$$\begin{aligned}(u_\rho - u_0)(y) &- \int_{\Gamma_D} \frac{1}{\mu_0}\partial_\nu(u_\rho - u_0)\mathcal{N}(\cdot,y)\,\mathrm{d}\sigma \\ &= \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla u_\rho \cdot \nabla\mathcal{N}(\cdot,y)\,\mathrm{d}x + \int_{\mathcal{I}_\rho} (\kappa_1^2 - \kappa_0^2)u_\rho\mathcal{N}(\cdot,y)\,\mathrm{d}x. \quad (2.32)\end{aligned}$$

II. Suppose ω is not an eigenfrequency with respect to μ_0, ε_0, σ_0 and a boundary condition of strictly Dirichlet type. For $y \in \Omega$, we define the Dirichlet function $\mathcal{D}$ to be the solution to

$$\begin{cases} \nabla \cdot \Big(\dfrac{1}{\mu_0}\nabla\mathcal{D}(\cdot,y)\Big) + \kappa_0^2\mathcal{D}(\cdot,y) = -\delta_y & \text{in } \Omega \\ \mathcal{D}(\cdot,y) = 0 & \text{on } \partial\Omega. \end{cases}$$

In the case of a Dirichlet problem ($\Gamma_D = \partial\Omega$), for all $y \in (\Omega \setminus \overline{\mathcal{I}})$,

$$(u_\rho - u_0)(y) = \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla u_\rho \cdot \nabla\mathcal{D}(\cdot,y)\,\mathrm{d}x + \int_{\mathcal{I}_\rho} (\kappa_1^2 - \kappa_0^2)u_\rho\mathcal{D}(\cdot,y)\,\mathrm{d}x.$$

If $\partial\Omega$ is sufficiently smooth then for $y \in \partial\Omega$,

$$\begin{aligned}&\partial_\nu(u_\rho - u_0)(y) \\ &\quad= \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla u_\rho \cdot \nabla_x\partial_{\nu_y}\mathcal{D}(\cdot,y)\,\mathrm{d}x + \int_{\mathcal{I}_\rho} (\kappa_1^2 - \kappa_0^2)u_\rho\partial_{\nu_y}\mathcal{D}(\cdot,y)\,\mathrm{d}x. \\ &\hfill (2.33)\end{aligned}$$

III. If the background EM parameters ε_0, μ_0 and σ_0 are constant, let Φ^{k_0} denote the free-space Green's function for the Helmholtz operator $\Delta + k_0^2$ satisfying Sommerfeld's outgoing radiation condition.[5] Then

$$\Phi^{k_0}(x,y) = \frac{i}{4} H_0^{(1)}(k_0|x-y|),$$

where $H_0^{(1)}$ is the 0^{th} order Hankel function of the first kind [CK98, Néd01]. For $y \in \Omega$ and $\phi \in C^\infty(\overline{\Omega})$,

$$\int_\Omega \nabla\phi \cdot \nabla\Phi^{k_0}(\cdot, y)\,\mathrm{d}x - \int_\Omega k_0^2 \phi\, \Phi^{k_0}(\cdot,y)\,\mathrm{d}x \\ = \phi(y) + \int_{\partial\Omega} \partial_{\nu_x}\Phi^{k_0}(x,y)\phi(x)\,\mathrm{d}\sigma_x.$$

Assuming $\partial\Omega$ is sufficiently smooth ($C^{1,\alpha}$ for example), the well-known limiting formula for double-layer potentials [CK83] implies that for $y \in \partial\Omega$,

$$\int_\Omega \nabla\phi \cdot \nabla\Phi^{k_0}(\cdot, y)\,\mathrm{d}x - \int_\Omega k_0^2 \phi\, \Phi^{k_0}(\cdot,y)\,\mathrm{d}x \\ = \frac{1}{2}\phi(y) + \int_{\partial\Omega} \partial_{\nu_x}\Phi^{k_0}(x,y)\phi(x)\,\mathrm{d}\sigma_x.$$

Suppose k_0^2 is not an eigenvalue of $-\Delta$ on Ω with mixed boundary conditions. Fix $y \in \partial\Omega$ such that y is a point of continuity of $u_\rho - u_0$. Choose a sequence $\phi_n \in C^\infty(\overline{\Omega})$ with $\phi_n \to (u_\rho - u_0)$ in $H^1(\Omega)$ and $\phi_n \to (u_\rho - u_0)$ uniformly in some neighborhood $B(y,\epsilon) \cap \partial\Omega$ of y.[6] If $\partial\Omega$ is at least $C^{1,\alpha}$-regular, the above formula

[5] Recall Sommerfeld's outgoing radiation condition: $\partial_r u - ik_0 u = o(r^{-1/2})$ as $r = |x| \to \infty$.

[6] This can be done since $(u_\rho - u_0) \in C^\infty(B \cap (\overline{\Omega} \setminus \overline{\mathcal{I}}))$ for every open set B away from $\partial\Gamma_D$.

continues to hold in the limit.[7] Then by Lemma 2.19(i)[8], for $y \in \Gamma_D \cup \Gamma_N$,

$$\begin{aligned}\frac{1}{2}(u_\rho - u_0)(y) - \int_{\Gamma_D} \partial_\nu (u_\rho - u_0)\Phi^{k_0}(\cdot, y)\,\mathrm{d}\sigma + \int_{\Gamma_N} \partial_{\nu_x}\Phi^{k_0}(\cdot, y)(u_\rho - u_0)\,\mathrm{d}\sigma \\ = \mu_0\Big(\int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla u_\rho \cdot \nabla\Phi^{k_0}(\cdot, y)\,\mathrm{d}x + \int_{\mathcal{I}_\rho}(\kappa_1^2 - \kappa_0^2)u_\rho \Phi^{k_0}(\cdot, y)\,\mathrm{d}x\Big).\end{aligned} \tag{2.34}$$

2.4.2 The asymptotic formula

Assume u_0 and u_ρ solve (2.1a) and (2.1b), and let

$$\mathcal{R}(u_\rho, s_0) := \underbrace{\int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla u_\rho \cdot \nabla\overline{s_0}\,\mathrm{d}x}_{I_1} + \underbrace{\int_{\mathcal{I}_\rho}(\kappa_1^2 - \kappa_0^2)u_\rho \overline{s_0}\,\mathrm{d}x}_{I_2}. \tag{2.35}$$

We have seen that for certain functions s_0, namely the Green's functions of the previous section, $\mathcal{R}(u_\rho, s_0)$ is equal to a boundary measurement. We now state our main

Goal: *To find the asymptotic expansion of the right-hand side of* (2.35) *for any* $s_0 \in H^1(\Omega)$, *or for any* s_0 *sufficiently regular, such as* $s_0 = G(\cdot, y)$, $y \in \overline{\Omega} \setminus \overline{\mathcal{I}}$, *where* G *is a Green's function.*

Tackling I_2 is easy:

$$I_2 = \int_{\mathcal{I}_\rho}(\kappa_1^2 - \kappa_0^2)u_0\overline{s_0}\,\mathrm{d}x + \underbrace{\int_{\mathcal{I}_\rho}(\kappa_1^2 - \kappa_0^2)(u_\rho - u_0)\overline{s_0}\,\mathrm{d}x}_{R_0}. \tag{2.36}$$

---

[7] This is because $H^{1/2}(\partial\Omega) \hookrightarrow L^p(\partial\Omega)$ continuously for all $p < \infty$ when $|\partial_{\nu_x}\Phi^{k_0}(x, y)| = O(|x-y|^{\alpha-1})$ as $x \to y$.

[8] Again, this application of Lemma 2.19(i) requires a limiting approximation argument because $\Phi^{k_0}(\cdot, y) \notin H^1(\Omega)$.

As for I_1:

$$\begin{aligned}
&\int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla\overline{s_0}\cdot\nabla u_\rho \,\mathrm{d}x \\
&\qquad\qquad = \int_{\mathcal{I}_\rho} \left(-\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla\overline{s_0}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \nabla u_\rho \right) \mathrm{d}x \\
&\qquad\qquad = \int_{\mathcal{I}_\rho} \left(-\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla\overline{s_0}^T \begin{bmatrix} \big(\nabla v_0^{(1)}\big)^T \\ \big(\nabla v_0^{(2)}\big)^T \end{bmatrix} \nabla u_\rho \right) \mathrm{d}x \\
&\qquad\qquad = \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)(\nabla v_0^{(i)}\cdot\nabla u_\rho)\partial_i\overline{s_0}\,\mathrm{d}x,
\end{aligned} \tag{2.37}$$

where

$$v_0^{(i)} = x_i + C_i$$

with C_i any constant. We choose

$$C_i = -\frac{1}{|\partial\Omega|}\int_{\partial\Omega} x_i \,\mathrm{d}\sigma,$$

so that

$$\int_{\partial\Omega} v_0^{(i)}\mathrm{d}\sigma = 0.$$

Now consider the problem of finding the perturbation $v_\rho^{(i)}$ satisfying

$$\begin{cases}
\nabla\cdot\Big(\dfrac{1}{\mu_\rho}\nabla v_\rho^{(i)}\Big) = \partial_i\Big(\dfrac{1}{\mu_0}\Big) & \text{in } \Omega \\
v_\rho^{(i)} = x_i + C_i & \text{on } \Gamma_D \\
\dfrac{1}{\mu_\rho}\partial_\nu v_\rho^{(i)} = \dfrac{1}{\mu_0}\nu^i & \text{on } \Gamma_N \\
\Big(\text{and } \int_{\partial\Omega} v_\rho^{(i)}\mathrm{d}\sigma = 0 \ \text{ if } \ \Gamma_N = \partial\Omega\Big).
\end{cases} \tag{2.38}$$

Such a $v_\rho^{(i)}$ exists uniquely for all $0 \le \rho \le 1$. (When $\Gamma_D = \varnothing$, this follows from Remark 2.14. Otherwise, it follows from the standard argument using the Lax-Milgram theorem.) Moreover, by Theorem 2 (Lemma 2.15 when $\Gamma_N = \partial\Omega$),

$$\big\|v_\rho^{(i)} - v_0^{(i)}\big\|_{H^1_D(\Omega)} \leq C|\mathcal{I}_\rho|^{1/2} \tag{2.39}$$

and

$$\big\|v_\rho^{(i)} - v_0^{(i)}\big\|_{L^2(\Omega)} \leq C_\delta|\mathcal{I}_\rho|^{1-\delta} \tag{2.40}$$

for all $\delta > 0$, where C and C_δ are independent of ρ. The next step toward our goal will be to show that

$$\begin{aligned}
&\int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)(\nabla v_0^{(i)} \cdot \nabla u_\rho)\partial_i\overline{s_0}\,\mathrm{d}x \\
&= \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)(\nabla v_\rho^{(i)} \cdot \nabla u_0)\partial_i\overline{s_0}\,\mathrm{d}x + o(|\mathcal{I}_\rho|).
\end{aligned}$$

To achieve this, we will use the following

Lemma 2.20. *Suppose ω is not an eigenfrequency with respect to μ_0, ε_0, σ_0 and Γ_D. Let $F \in \big(H^1_D(\Omega)\big)'$, $f \in H^{1/2}(\Gamma_D)$ and $g \in \big(H^{1/2}_{00}(\Gamma_N)\big)'$ be given. Let u_0 be the solution to* (2.1a) *and u_ρ the solution to* (2.1b) *guaranteed for $\rho > 0$ sufficiently small by Theorem 1. Let $F_1 \in \big(H^1_D(\Omega)\big)'$, $f_1 \in H^{1/2}(\Gamma_D)$ and $g_1 \in \big(H^{1/2}_{00}(\Gamma_N)\big)'$ be given (with $\int_\Omega F_1\,\mathrm{d}\sigma = \int_{\partial\Omega} g_1\,\mathrm{d}\sigma$ if $\Gamma_N = \partial\Omega$). Let $v_0 \in H^1_D(\Omega)$ denote the solution to*

$$\begin{cases}
\nabla \cdot \Big(\dfrac{1}{\mu_0}\nabla v_0\Big) = F_1 & \text{in } \Omega \\
v_0 = f_1 & \text{on } \Gamma_D \\
\dfrac{1}{\mu_0}\partial_\nu v_0 = g_1 & \text{on } \Gamma_N \\
\Big(\text{and } \int_{\partial\Omega} v_0\,\mathrm{d}\sigma = 0 & \text{if } \Gamma_N = \partial\Omega\Big),
\end{cases}$$

and let v_ρ denote the solution of the perturbed problem (that is, the problem with μ_0

replaced by μ_ρ). Then for any $\phi \in C^{0,1}(\overline{\Omega})$,

$$\int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)(\nabla v_0 \cdot \nabla u_\rho)\phi = \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)(\nabla v_\rho \cdot \nabla u_0)\phi$$
$$+ R_1 + R_2 + R_3 + R_4,$$

where

$$\begin{aligned}
R_1 &= \int_\Omega \frac{1}{\mu_\rho}\big(\nabla(v_\rho - v_0) \cdot \nabla\phi\big)(u_\rho - u_0),\\
R_2 &= \int_{\mathcal{I}_\rho} \Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\big(\nabla v_0 \cdot \nabla\phi\big)(u_\rho - u_0),\\
R_3 &= -\int_\Omega \frac{1}{\mu_\rho}\big(\nabla(u_\rho - u_0) \cdot \nabla\phi\big)(v_\rho - v_0),\\
R_4 &= -\int_{\mathcal{I}_\rho} \Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\big(\nabla u_0 \cdot \nabla\phi\big)(v_\rho - v_0),\\
R_5 &= \int_{\mathcal{I}_\rho} (\kappa_1^2 - \kappa_0^2)u_0(v_\rho - v_0)\phi,\\
R_6 &= \int_\Omega \kappa_\rho^2(u_\rho - u_0)(v_\rho - v_0)\phi.
\end{aligned}$$

Proof.

$$\begin{aligned}
\int_\Omega \frac{1}{\mu_\rho}\big(\nabla(v_\rho - v_0) \cdot \nabla(u_\rho - u_0)\big)\phi &= \int_\Omega \frac{1}{\mu_\rho}\nabla(v_\rho - v_0) \cdot \nabla\big((u_\rho - u_0)\phi\big) - R_1\\
&= \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla v_0 \cdot \nabla\big((u_\rho - u_0)\phi\big) - R_1\\
&= \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\big(\nabla v_0 \cdot \nabla(u_\rho - u_0)\big)\phi\\
&\qquad - R_1 - R_2
\end{aligned}$$

by Lemma 2.19(ii) with $W_\rho = v_\rho$ and $W_0 = v_0$. Similarly, but with the roles of $v_\rho - v_0$

and $u_\rho - u_0$ switched, we apply Lemma 2.19(i) with $U_\rho = u_\rho$ and $U_0 = u_0$ to find

$$\begin{aligned}\int_\Omega \frac{1}{\mu_\rho}\big(\nabla(v_\rho - v_0)\cdot\nabla(u_\rho - u_0)\big)\phi \\ &= \int_\Omega \frac{1}{\mu_\rho}\nabla(u_\rho - u_0)\cdot\nabla\big((v_\rho - v_0)\phi\big) + R_3 \\ &= \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\big(\nabla u_0\cdot\nabla(v_\rho - v_0)\big)\phi + R_3 + R_4 + R_5 + R_6. \qquad \square\end{aligned}$$

We apply this lemma to (2.37) with $v_\rho = v_\rho^{(i)}$, $v_0 = v_0^{(i)}$ and $\phi \in C^{0,1}(\overline{\Omega})$ satisfying $\phi = \partial_i \overline{s_0}$ in $\mathcal{I}$. Doing this, of course, requires that we assume

$$s_0|_{\overline{\mathcal{I}}} \in C^{1,1}(\overline{\mathcal{I}}), \tag{2.41}$$

with $\partial\mathcal{I}$ sufficiently regular for $W^{1,\infty}$-extension. Then, using (2.36) and (2.35), it follows that with $\mathcal{R}$ as defined by (2.35),

$$\begin{aligned}\mathcal{R}(u_\rho, s_0) &= \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)(\nabla v_\rho^{(i)}\cdot\nabla u_0)\partial_i\overline{s_0}\,\mathrm{d}x + \int_{\mathcal{I}_\rho}(\kappa_1^2 - \kappa_0^2)u_0\overline{s_0}\,\mathrm{d}x \\ &\qquad + R_0 + R_1^{(i)} + R_2^{(i)} + R_3^{(i)} + R_4^{(i)} + R_5^{(i)} + R_6^{(i)} \\ &= \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla\overline{s_0}\cdot(M_\rho\nabla u_0)\,\mathrm{d}x + \int_{\mathcal{I}_\rho}(\kappa_1^2 - \kappa_0^2)u_0\overline{s_0}\,\mathrm{d}x \\ &\qquad + R_0 + R_1^{(i)} + R_2^{(i)} + R_3^{(i)} + R_4^{(i)} + R_5^{(i)} + R_6^{(i)},\end{aligned}$$

where the $\mathbb{R}^{2\times 2}$-valued function M_ρ is defined by

$$[M_\rho]_{ij} = \partial_j v_\rho^{(i)}.$$

Observe that $|\mathcal{I}_\rho|^{-1}\mathbb{1}_{\mathcal{I}_\rho}$ is bounded in $L^1(\Omega)$, and therefore bounded in $(C^0(\overline{\Omega}))'$ as $\rho \to 0$. By the Banach-Alaoğlu theorem, the $|\mathcal{I}_\rho|^{-1}\mathbb{1}_{\mathcal{I}_\rho}$ lie within a weak-$*$ compact subset of $(C^0(\overline{\Omega}))'$. Since $C^0(\overline{\Omega})$ is a separable space, weak-$*$ compact subsets of its dual are sequentially weak-$*$ compact. Thus, by the Riesz representation theorem, there exists a positive regular Borel measure α and a subsequence $\rho_n \to 0$ such that

$$|\mathcal{I}_{\rho_n}|^{-1}\mathbb{1}_{\mathcal{I}_{\rho_n}}\mathrm{d}x \xrightarrow{\text{weak-}*} \mathrm{d}\alpha \quad \text{in } \big(C^0(\overline{\Omega})\big)'.$$

As a consequence of this weak convergence, $\int_\Omega \mathrm{d}\alpha = 1$, and α is therefore a probability measure. A simple argument (as in [CV03a]) using (2.19) shows

$$\big\| |\mathcal{I}_\rho|^{-1} \mathbb{1}_{\mathcal{I}_\rho} \partial_j v_\rho^{(i)} \big\|_{L^1(\Omega)}$$

is bounded as $\rho \to 0$, hence there exists a regular Borel measure $\mathcal{M}_{ij}$ such that, after passing to a further subsequence,

$$|\mathcal{I}_{\rho_n}|^{-1} \mathbb{1}_{\mathcal{I}_{\rho_n}} \partial_j v_{\rho_n}^{(i)} \,\mathrm{d}x \xrightarrow{\textit{weak-*}} \mathrm{d}\mathcal{M}_{ij} \quad \text{in } \big(C^0(\overline{\Omega})\big)'.$$

Another simple argument (as in [CV03a]), again using (2.19), shows the linear functional on $C^0(\overline{\Omega})$ defined by

$$\phi \mapsto \int_\Omega \phi \,\mathrm{d}\mathcal{M}_{ij}$$

can be extended to a bounded linear functional on $L^2(\Omega, \mathrm{d}\alpha)$. Consequently, there exists a matrix M with entries $M_{ij} \in L^2(\Omega, \mathrm{d}\alpha)$ such that

$$\mathrm{d}\mathcal{M}_{ij} = M_{ij} \,\mathrm{d}\alpha.$$

By taking any $\psi \in C^{1,1}(\overline{\Omega})$ satisfying $\psi|_{\overline{\mathcal{I}}} = s_0|_{\overline{\mathcal{I}}}$[9], we have

$$\begin{aligned}
\mathcal{H}_0(u_{\rho_n} - u_0, s_0) &- \int_{\Gamma_D} \left(\frac{1}{\mu_0} \partial_\nu u_{\rho_n} - \frac{1}{\mu_0} \partial_\nu u_0 \right) \overline{s_0} \,\mathrm{d}\sigma \\
&= \mathcal{R}(u_\rho, s_0) := \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0} \Big) \nabla u_\rho \cdot \nabla \overline{s_0} \,\mathrm{d}x + \int_{\mathcal{I}_\rho} (\kappa_1^2 - \kappa_0^2) u_\rho \overline{s_0} \,\mathrm{d}x \\
&= |\mathcal{I}_{\rho_n}| \left(\int_\Omega -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0} \Big) \nabla \overline{s_0} \cdot (M \nabla u_0) \,\mathrm{d}\alpha + \int_\Omega (\kappa_1^2 - \kappa_0^2) u_0 \overline{s_0} \,\mathrm{d}\alpha \right) \\
&\qquad\qquad + R_0 + R_1^{(i)} + R_2^{(i)} + R_3^{(i)} + R_4^{(i)} + R_5 + R_6,
\end{aligned}$$

[9] We do this because we do not want to restrict our choices for s_0 to only functions that are $C^{1,1}(\overline{\Omega})$. Such a restriction would disqualify $s_0 = G(\cdot, y)$, where G is any Green's function for the background Helmoltz equation and $y \in \overline{\Omega} \setminus \overline{\mathcal{I}}$.

with

$$
\begin{aligned}
R_0 &= \int_{\mathcal{I}_{\rho n}} (\kappa_1^2 - \kappa_0^2)(u_{\rho_n} - u_0)\overline{s_0}\,\mathrm{d}x,\\
R_1^{(i)} &= \int_\Omega \frac{1}{\mu_0}\big(\nabla(v_{\rho_n}^{(i)} - v_0^{(i)}) \cdot \nabla(\partial_i\overline{\psi})\big)(u_{\rho_n} - u_0)\,\mathrm{d}x,\\
R_2^{(i)} &= \int_{\mathcal{I}_{\rho n}} \Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\big(\nabla v_0^{(i)} \cdot \nabla(\partial_i\overline{s_0})\big)(u_{\rho_n} - u_0)\,\mathrm{d}x,\\
R_3^{(i)} &= -\int_\Omega \frac{1}{\mu_0}\big(\nabla(u_{\rho_n} - u_0) \cdot \nabla(\partial_i\overline{\psi})\big)\big(v_{\rho_n}^{(i)} - v_0^{(i)}\big)\,\mathrm{d}x,\\
R_4^{(i)} &= -\int_{\mathcal{I}_{\rho n}} \Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\big(\nabla u_0 \cdot \nabla(\partial_i\overline{s_0})\big)\big(v_{\rho_n}^{(i)} - v_0^{(i)}\big)\,\mathrm{d}x,\\
R_5 &= \int_{\mathcal{I}_\rho} (\kappa_1^2 - \kappa_0^2)u_0\big(v_{\rho_n}^{(i)} - v_0^{(i)}\big)\overline{s_0}\,\mathrm{d}x,\\
R_6 &= \int_\Omega \kappa_\rho^2(u_\rho - u_0)\big(v_{\rho_n}^{(i)} - v_0^{(i)}\big)\overline{\psi}\,\mathrm{d}x,\\
R_7 &= \int_{\mathcal{I}_{\rho n}} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla\overline{s_0} \cdot \big(M_{\rho_n}\nabla u_0\big)\,\mathrm{d}x\\
&\qquad\qquad - |\mathcal{I}_{\rho_n}| \int_\Omega -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla\overline{s_0} \cdot (M\nabla u_0)\,\mathrm{d}\alpha,\\
R_8 &= \int_{\mathcal{I}_{\rho n}} (\kappa_1^2 - \kappa_0^2)u_0\overline{s_0}\,\mathrm{d}x - |\mathcal{I}_{\rho_n}| \int_\Omega (\kappa_1^2 - \kappa_0^2)u_0\overline{s_0}\,\mathrm{d}\alpha.
\end{aligned}
$$

Both R_7 and R_8 are clearly $o(|\mathcal{I}_{\rho_n}|)$ as $\rho_n \to 0$. Using Hölder's inequality followed by the H^1 and L^2 estimates (2.39) and (2.40) of the error $v_{\rho_n}^{(i)} - v_0^{(i)}$ and the H^1 and L^2 estimates of $u_\rho - u_0$ from Theorem 2, we see that the remainder terms R_0 through R_6 are each bounded in absolute value by

$$
C|\mathcal{I}_{\rho_n}|^{\frac{3}{2}-\delta}\|\psi\|_{C^{1,1}(\overline{\Omega})}\|u_0\|_{C^{0,1}(\overline{\mathcal{I}})}.
$$

If $F|_{\Omega'} \in L^p(\Omega')$ for some $\mathcal{I} \subset\subset \Omega' \subset\subset \Omega$ and some $p > 2$, we may appeal to Corollary 2.17 to replace the above bound with

$$
C|\mathcal{I}_{\rho_n}|^{\frac{3}{2}-\delta}\|\psi\|_{C^{1,1}(\overline{\Omega})}\|(F, f, g)\|.
$$

If $s_0|_{\overline{\mathcal{I}}} \in C^{1,1}(\overline{\mathcal{I}})$, ψ may be chosen so that

$$
\|\psi\|_{C^{1,1}(\overline{\Omega})} \le C_{\mathrm{dist}(\Omega',\partial\Omega)}\|s_0|_{\mathcal{I}}\|_{C^{1,1}(\overline{\mathcal{I}})}.
$$

This brings us to the main theorem of this chapter.

Theorem 3. *Assume $\omega \geq 0$ is not an eigenfrequency with respect to μ_0, ε_0, σ_0 and Γ_D, (that is, suppose there are no nontrivial solutions to the homogeneous form of* (2.1a)*) and let $\rho_0 > 0$ be as in Theorem 1. Given any sequence $\rho_n \leq \rho_0$ satisfying $\rho_n \to 0$, there exists a subsequence—which we continue to denote by ρ_n—a regular Borel probability measure α on Ω supported in $\mathcal{I}$, and a polarization tensor M with real-valued entries $M_{ij} \in L^2(\Omega, \mathrm{d}\alpha)$ such that the following holds: for any given $f \in H^{1/2}(\Gamma_D)$, $g \in \big(H^{1/2}_{00}(\Gamma_N)\big)'$ and $F \in (H^1_D(\Omega))'$ satisfying $F|_{\Omega'} \in L^p(\Omega')$ for some $\mathcal{I} \subset\subset \Omega' \subset\subset \Omega$ and some $p > 2$, and for any given function s_0 defined on Ω with $s_0|_{\overline{\mathcal{I}}} \in C^{1,1}(\overline{\mathcal{I}})$, we have*

$$
\begin{aligned}
\mathcal{R}(u_\rho, s_0) :=& \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla u_\rho \cdot \nabla \overline{s_0}\,\mathrm{d}x + \int_{\mathcal{I}_\rho} (\kappa_1^2 - \kappa_0^2) u_\rho \overline{s_0}\,\mathrm{d}x \\
=& |\mathcal{I}_{\rho_n}| \left(\int_\Omega -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla \overline{s_0} \cdot (M \nabla u_0)\,\mathrm{d}\alpha + \int_\Omega (\kappa_1^2 - \kappa_0^2) u_0 \overline{s_0}\,\mathrm{d}\alpha \right) \\
& + o(|\mathcal{I}_{\rho_n}|), \quad (2.42)
\end{aligned}
$$

where u_0 and u_ρ are the solutions to (2.1a) *and* (2.1b) *respectively. The remainder term satisfies*

$$
\big|o(|\mathcal{I}_{\rho_n}|)\big| \leq |R_7| + |R_8| + C|\mathcal{I}_{\rho_n}|^{\frac{3}{2}-\delta} \|s_0|_{\overline{\mathcal{I}}}\|_{C^{1,1}(\overline{\mathcal{I}})} \|(F, f, g)\|,
$$

where

$$
\|(F, f, g)\| = \|F\|_{L^p(\Omega')} + \|F\|_{(H^1_D(\Omega))'} + \|f\|_{H^{1/2}(\Gamma_D)} + \|g\|_{(H^{1/2}_{00}(\Gamma_N))'}
$$

and C depends on ω, the ellipticity constant ϑ, the suprema of the EM parameters (μ_0, μ_1, ε_0, ε_1, σ_0 and σ_1), $\mathrm{dist}(\mathcal{I}, \partial\Omega')$, $\mathrm{dist}(\Omega', \partial\Omega)$, Ω, Γ_D and $\|\mathcal{L}_0^{-1}\|$. The subsequence $\mathcal{I}_{\rho_n}$ and the measure α are independent of F, f and g (as well as ω, ε_0, ε_1, σ_0 and σ_1). M depends on the subsequence $\mathcal{I}_{\rho_n}$ and on μ_0 and μ_1, but is independent of F, f and g (as well as ω, ε_0, ε_1, σ_0 and σ_1). M is symmetric and positive definite $\mathrm{d}\alpha$*-a.e. in the* *set* $\{\mu_0 \neq \mu_1\}$*, where it satisfies the bounds*

$$\min\Big\{1, \frac{\mu_1}{\mu_0}\Big\}|\xi^2| \le \xi^T M \xi \le \max\Big\{1, \frac{\mu_1}{\mu_0}\Big\}|\xi^2| \quad \textit{for all } \xi \in \mathbb{R}^2. \tag{2.43}$$

Proof. All that remains to be shown is the symmetry and positive definiteness of M with the stated bounds. The proofs in [CV03a] for the conductivity problem carry over to the present context with essentially no change. We include these proofs for completeness only.

To see that M is symmetric, observe that by Lemma 2.20 with $v_0 = v_0^{(i)}$, $v_\rho = v_\rho^{(i)}$, $u_0 = v_0^{(j)}$ and $u_\rho = v_\rho^{(j)}$,

$$\int_{\mathcal{I}_\rho} \Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\big(\nabla v_0^{(i)} \cdot \nabla v_\rho^{(j)}\big)\phi \,\mathrm{d}x = \int_{\mathcal{I}_\rho} \Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\big(\nabla v_\rho^{(i)} \cdot \nabla v_0^{(j)}\big)\phi \,\mathrm{d}x + o(|\mathcal{I}_\rho|)$$

for all $\phi \in C^{0,1}(\overline{\Omega})$.[10] Thus

$$\begin{aligned}
\int_\Omega \Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big) M_{ji}\phi \,\mathrm{d}\sigma &= \lim \frac{1}{|\mathcal{I}_{\rho_n}|}\int_{\mathcal{I}_{\rho_n}} \Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\big(\nabla v_0^{(i)} \cdot \nabla v_{\rho_n}^{(j)}\big)\phi \,\mathrm{d}x \\
&= \lim \frac{1}{|\mathcal{I}_{\rho_n}|}\int_{\mathcal{I}_{\rho_n}} \Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\big(\nabla v_{\rho_n}^{(i)} \cdot \nabla v_0^{(j)}\big)\phi \,\mathrm{d}x \\
&= \int_\Omega \Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big) M_{ij}\phi \,\mathrm{d}\sigma.
\end{aligned}$$

We now show that M is positive definite with the bounds (2.43). Recall from the proof of Lemma 2.20 that, with u_0, u_ρ, v_0 and v_ρ as in the statement of that lemma, for any $\phi \in C^{0,1}(\overline{\Omega})$,

$$\begin{aligned}
\int_\Omega &\frac{1}{\mu_\rho}\big(\nabla(v_\rho - v_0)\cdot\nabla(u_\rho - u_0)\big)\phi \,\mathrm{d}x \\
&= \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\big(\nabla v_0 \cdot \nabla(u_\rho - u_0)\big)\phi \,\mathrm{d}x - R_1 - R_2,
\end{aligned} \tag{2.44}$$

[10] In the event that $\Gamma_N = \partial\Omega$, this choice of u_0 solves (2.1a) (with the appropriate F and g) at the eigenfrequency $\omega = 0$, and therefore doesn't meet the stated requirement for Lemma 2.20 to apply. However, the proof of Lemma 2.20 is still valid in this case. To see that the remainder terms continue to be $o(|\mathcal{I}_\rho|)$, simply appeal to Lemma 2.15 instead of Theorem 2 to estimate the error $u_\rho - u_0$.

where

$$R_1 = \int_\Omega \frac{1}{\mu_\rho}\big(\nabla(v_\rho - v_0)\cdot\nabla\phi\big)(u_\rho - u_0),$$
$$R_2 = \int_{\mathcal{I}_\rho}\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\big(\nabla v_0\cdot\nabla\phi\big)(u_\rho - u_0).$$

As we have already noted, R_1 and R_2 are both $o(\mathcal{I}_\rho)$. With

$$v_0^{(i)}(x) = x_i - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} x_i \,\mathrm{d}\sigma$$

and $v_\rho^{(i)}$ as defined in (2.38), we take $v_0 = v_0^{(j)}$, $v_\rho = v_\rho^{(j)}$, $u_0 = v_0^{(i)}$, $u_\rho = v_\rho^{(i)}$ in (2.44) and get

$$\begin{aligned}\int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla v_0^{(j)}\cdot\nabla v_\rho^{(i)}\phi \,\mathrm{d}x = &\int_\Omega \frac{1}{\mu_\rho}\Big[\nabla\big(v_\rho^{(j)} - v_0^{(j)}\big)\cdot\nabla\big(v_\rho^{(i)} - v_0^{(i)}\big)\Big]\phi \,\mathrm{d}x \\ &- \int_{\mathcal{I}_\rho}\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla v_0^{(i)}\cdot\nabla v_0^{(j)}\phi \,\mathrm{d}x + o(\mathcal{I}_\rho)\end{aligned} \tag{2.45}$$

for any fixed $i, j \in \{1, 2\}$. Fix any $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, multiply both sides of this equation by $\xi_1\xi_2$ and then sum over the indices. We may write the resulting equation as

$$\begin{aligned}\int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)[M_\rho]_{ij}\xi_i\xi_j\phi \,\mathrm{d}x = &\int_\Omega \frac{1}{\mu_\rho}|\nabla(V_\rho - V_0)|^2\phi \,\mathrm{d}x \\ &- \int_{\mathcal{I}_\rho}\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)|\nabla V_0|^2\phi \,\mathrm{d}x + o(\mathcal{I}_\rho),\end{aligned} \tag{2.46}$$

where

$$V_\rho = \xi_1 v_\rho^{(1)} + \xi_2 v_\rho^{(2)} \quad \text{and} \quad V_0 = \xi_1 v_0^{(1)} + \xi_2 v_0^{(2)}$$

(recall that $[M_\rho]_{ij} := \partial_j v_\rho^{(i)} = \nabla v_0^{(j)}\cdot\nabla v_\rho^{(i)}$.) Note that $|\nabla V_0| = \xi$. Along the subsequence $\mathcal{I}_{\rho_n}$, we divide both sides of (2.46) by $|\mathcal{I}_{\rho_n}|$ and get, in the limit,

$$\int_\Omega -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\xi^T M\xi\phi \,\mathrm{d}\alpha \geq \int_\Omega -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)|\xi|^2\phi \,\mathrm{d}\alpha \tag{2.47}$$

for all $\phi \in C^{0,1}(\overline{\Omega})$ such that $\phi \geq 0$.

Claim: For any nonnegative $\phi \in C^{0,1}(\overline{\Omega})$,

$$\int_\Omega \frac{1}{\mu_\rho}|\nabla(V_\rho - V_0)|^2\phi \,\mathrm{d}x \leq \int_{\mathcal{I}_\rho} \mu_1\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)^2|\nabla V_0|^2\phi \,\mathrm{d}x + o(\mathcal{I}_\rho). \tag{2.48}$$

Assuming this were true, from (2.46) it would then follow that for any nonnegative $\phi \in C^{0,1}(\overline{\Omega})$,

$$\int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)[M_\rho]_{ij}\xi_i\xi_j\phi \,\mathrm{d}x \leq \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\frac{\mu_1}{\mu_0}|\xi|^2\phi \,\mathrm{d}x + o(\mathcal{I}_\rho).$$

This in turn implies that for any nonnegative $\phi \in C^{0,1}(\overline{\Omega})$,

$$\int_\Omega -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\xi^T M\xi\phi \,\mathrm{d}\alpha \leq \int_\Omega -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\frac{\mu_1}{\mu_0}|\xi|^2\phi \,\mathrm{d}\alpha.$$

The above inequality combined with (2.47) would yield that

$$-\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)|\xi|^2 \leq -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\xi^T M\xi \leq -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\frac{\mu_1}{\mu_0}|\xi|^2$$

$\mathrm{d}\alpha$-a.e. in the set $\{\mu_0 \neq \mu_1\}$, which would imply Property 2.43. To complete the proof, then, we must verify (2.48). To this end, observe that by (2.45),

$$\begin{aligned}\int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla V_\rho \cdot \nabla V_0\phi \,\mathrm{d}x = &\int_\Omega \frac{1}{\mu_\rho}|\nabla(V_\rho - V_0)|^2\phi \,\mathrm{d}x \\ &- \int_{\mathcal{I}_\rho}\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)|\nabla V_0|^2\phi \,\mathrm{d}x + o(\mathcal{I}_\rho),\end{aligned}$$

and therefore

$$\begin{aligned}\int_\Omega \frac{1}{\mu_\rho}|\nabla(V_\rho - V_0)|^2\phi \,\mathrm{d}x &= \int_{\mathcal{I}_\rho} -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla V_0 \cdot \nabla(V_\rho - V_0)\phi \,\mathrm{d}x + o(\mathcal{I}_\rho) \\ &\leq \Big[\int_{\mathcal{I}_\rho} \mu_1\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)^2|\nabla V_0|^2\phi \,\mathrm{d}x\Big]^{\frac{1}{2}} \\ &\qquad \times \Big[\int_{\mathcal{I}_\rho} \frac{1}{\mu_1}|\nabla(V_\rho - V_0)|^2\phi \,\mathrm{d}x\Big]^{\frac{1}{2}} + o(\mathcal{I}_\rho).\end{aligned}$$

(2.48) now follows from an application of the inequality $ab \le (a^2+b^2)/2$ for $a, b \ge 0$. $\square$

Remark 2.21. *The polarization tensor M is local; that is, it can be shown to be independent of the type of boundary conditions and to depend only on $\mu_0|_{\Omega'}$, $\mu_1|_{\Omega'}$ and the subsequence $\mathcal{I}_{\rho_n}$, where Ω' is any smooth set satisfying $\bigcup \mathcal{I}_\rho \subset\subset \Omega' \subset\subset \Omega$. This was done in [CV03b] for the conductivity problem. The idea of the proof is as follows: recall that $v_\rho^{(i)}$ is defined on Ω to be the perturbation of $v_0^{(i)} := x_i + C_i$ satisfying (2.38). Define $w_\rho^{(i)}$ on Ω' as a perturbation of $v_0^{(i)}|_{\Omega'}$ with the same Neumann data. Using elliptic estimates and the L^2 estimate from Theorem 2, one can show that $\left\|\nabla(v_\rho^{(i)} - w_\rho^{(i)})\right\|_{L^2(\Omega')} \le C|\mathcal{I}_\rho|^{1-\delta}$. Consequently, for any sequence $\rho_n \to 0$, $\left(\partial_j v_{\rho_n}^{(i)} - \partial_j w_{\rho_n}^{(i)}\right) \xrightarrow{\text{weak-}*} 0$ in $(C^0(\overline{\Omega'}))'$.*

Since M does not depend on the type of boundary conditions of the problem (2.38) defining $v_\rho^{(i)}$, we now know that it is identical to the polarization tensor derived in [CV03a] for the conductivity problem with a Neumann condition, except with the role of σ_ρ replaced by $1/\mu_\rho$.

A consequence of this local dependence of M is the following: suppose $\mathcal{I}_\rho = \bigcup_{j=1}^m \mathcal{I}_\rho^j$ such that there exist mutually disjoint sets $\Omega'_j \subset\subset \Omega$ with each $\bigcup_\rho \mathcal{I}_\rho^j \subset\subset \Omega'_j$. Let $\rho_n \to 0$ be a subsequence for which $|\mathcal{I}_\rho|^{-1}\mathrm{d}x \xrightarrow{\text{weak-}*} \mathrm{d}\alpha$ and, for some probability measures α_j supported in Ω'_j, $|\mathcal{I}_\rho^j|^{-1}\mathrm{d}x \xrightarrow{\text{weak-}*} \mathrm{d}\alpha^j$. (At least one such subsequence must exist.) Suppose also that

$$\frac{|\mathcal{I}_\rho^j|}{|\mathcal{I}_\rho|} = \frac{|\mathcal{I}_\rho^j|}{\sum_i |\mathcal{I}_\rho^i|} \longrightarrow c_j \quad \text{as } \rho \to 0$$

for some $c_j > 0$, so that

$$\sum_{j=1}^m \left(\frac{|\mathcal{I}_\rho^j|}{\sum_i |\mathcal{I}_\rho^i|} \right) \frac{1}{|\mathcal{I}_\rho^j|}\mathrm{d}x \xrightarrow{\text{weak-}*} \begin{cases} \mathrm{d}\alpha, \\ \sum_j c_j \, \mathrm{d}\alpha^j, \end{cases}$$

and therefore

$$\mathrm{d}\alpha = \sum_{j=1}^m c_j \, \mathrm{d}\alpha^j.$$

Then if $M^j \in L^2(\Omega, \mathrm{d}\alpha^j)$ is the polarization tensor determined by $\mu_\rho|_{\mathcal{I}_\rho^j}$ and supported in Ω_j',

$$M \,\mathrm{d}\alpha = \sum_{j=1}^{m} M^j \,\mathrm{d}\alpha^j.$$

From (2.29), (2.32), (2.33) and (2.34) we get the following corollaries.

Corollary 2.22. *If $s_0 \in H^1(\Omega)$ solves the background Helmholtz equation,*

$$\nabla \cdot \left(\frac{1}{\mu_0} \nabla s_0 \right) + \kappa_0^2 s_0 = 0,$$

and $s_0|_{\overline{\mathcal{I}}} \in C^{1,1}(\overline{\mathcal{I}})$, then

$$\begin{aligned}\int_{\partial\Omega} \left\{ u_{\rho_n} \left(\frac{1}{\mu_0} \partial_\nu \overline{s_0} \right) - \left(\frac{1}{\mu_0} \partial_\nu u_{\rho_n} \right) \overline{s_0} \right\} \mathrm{d}\sigma \\ = |\mathcal{I}_{\rho_n}| \left[\int_\Omega -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0} \Big) \nabla \overline{s_0} \cdot (M \nabla u_0) \,\mathrm{d}\alpha + \int_\Omega \big(\kappa_1^2 - \kappa_0^2 \big) u_0 \overline{s_0} \,\mathrm{d}\alpha \right] + o(|\mathcal{I}_{\rho_n}|).\end{aligned}$$

Corollary 2.23. *For all $y \in (\Omega \setminus \overline{\mathcal{I}}) \cup \Gamma_D \cup \Gamma_N$ (but not necessarily at the points on $\partial\Omega$ where the boundary condition changes type),*

$$\begin{aligned}(u_\rho - u_0)(y) - \int_{\Gamma_D} \frac{1}{\mu_0} \partial_\nu (u_\rho - u_0) \mathcal{N}(\cdot, y) \,\mathrm{d}\sigma \\ = |\mathcal{I}_{\rho_n}| \bigg(\int_\Omega -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0} \Big) \nabla_x \mathcal{N}(\cdot, y) \cdot (M \nabla u_0) \,\mathrm{d}\alpha \\ + \int_\Omega (\kappa_1^2 - \kappa_0^2) u_0 \mathcal{N}(\cdot, y) \,\mathrm{d}\alpha \bigg) + o(|\mathcal{I}_{\rho_n}|).\end{aligned}$$

Corollary 2.24. *In the case of a purely Dirichlet problem, if $\partial\Omega$ is sufficiently smooth then for all $y \in \partial\Omega$,*

$$\begin{aligned}\partial_\nu (u_\rho - u_0)(y) = |\mathcal{I}_{\rho_n}| \bigg(\int_\Omega -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0} \Big) \big(\nabla_x \partial_{\nu_y} \mathcal{D}(\cdot, y) \big) \cdot (M \nabla u_0) \,\mathrm{d}\alpha \\ + \int_\Omega (\kappa_1^2 - \kappa_0^2) u_0 \big(\partial_{\nu_y} \mathcal{D}(\cdot, y) \big) \,\mathrm{d}\alpha \bigg) + o(|\mathcal{I}_{\rho_n}|).\end{aligned}$$

Corollary 2.25. *When the background coefficients are constant, we have for all $y \in$*

$\Gamma_D \cup \Gamma_N$,

$$\begin{aligned}
\frac{1}{2\mu_0}(u_\rho - u_0)(y) - \int_{\Gamma_D} \frac{1}{\mu_0}\partial_\nu(u_\rho - u_0)\Phi^{k_0}(\cdot, y)\,\mathrm{d}\sigma& \\
+ \int_{\Gamma_N} \frac{1}{\mu_0}\partial_{\nu_x}\Phi^{k_0}(\cdot, y)(u_\rho - u_0)\,\mathrm{d}\sigma& \\
= |\mathcal{I}_{\rho_n}|\biggl(\int_\Omega -\Bigl(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Bigr)\nabla_x\Phi^{k_0}(\cdot, y)\cdot(M\nabla u_0)\,\mathrm{d}\alpha& \\
+ \int_\Omega (\kappa_1^2 - \kappa_0^2)u_0\Phi^{k_0}(\cdot, y)\,\mathrm{d}\alpha\biggr) + o(|\mathcal{I}_{\rho_n}|).&
\end{aligned}$$

Remark 2.26. *In Corollaries 2.23, 2.24 and 2.25, by continuity we see that for any set* $\Gamma \subset\subset (\Omega \setminus \overline{\mathcal{I}}) \cup \Gamma_D \cup \Gamma_N$,

$$\|o(|\mathcal{I}_{\rho_n}|)\|_{L^\infty(\Gamma)}/|\mathcal{I}_{\rho_n}| \to 0$$

for any fixed $f \in H^{1/2}(\Gamma_D)$, $g \in \bigl(H_{00}^{1/2}(\Gamma_N)\bigr)'$ *and* $F \in (H_D^1(\Omega))'$ *satisfying* $F|_{\mathcal{I}} \in L^p(\mathcal{I})$. *In particular, if* $\Gamma_D = \partial\Omega$ *or* $\Gamma_N = \partial\Omega$, *we may take* $\Gamma = \partial\Omega$.

Remark 2.27. *Recalling Remark 2.18, it is clear that Theorem 1 and its corollaries continue to hold when the dimension* n *of the ambient space is greater than two. However,* p *must be greater than* n, *and the factor* $|\mathcal{I}_\rho|^{\frac{3}{2}-\delta}$ *(from the bound on the remainder term) must be replaced with* $|\mathcal{I}_\rho|^{1+\frac{1}{n}-\delta}$.

2.4.3 Particular cases

If $\mathcal{I}_\rho = \bigcup_{l=1}^m (z_l + \rho D_l)$, where each D_l is a smooth, simply connected domain, the measure $\alpha = \frac{1}{\sum_i |D_i|}\sum_{l=1}^m |D_l|\delta_{z_l}$ (cf. Remark 2.21). In this case, we may calculate M by the formula

$$M_{ij}(z_l) = |D_l|^{-1}\int_{\partial D_l} \nu_i w_j\,\mathrm{d}\sigma = |D_l|^{-1}\int_{D_l} \partial_{y_i} w_j\,\mathrm{d}y,$$

where w_j is the solution to

$$\begin{cases} \Delta w_j = 0 & \text{in } D_l \text{ and } \mathbb{R}^2 \setminus \overline{D_l}, \\ {w_j}^- = {w_j}^+ & \text{on } \partial D_l, \\ \dfrac{1}{\mu}\partial_\nu^- w_j = \partial_\nu^+ w_j & \text{on } \partial D_l, \\ |w_j(x) - x_j| \to 0 & \text{as } |x| \to \infty. \end{cases}$$

(This was proved in [CFMV98]. See also Section 3.2.3 of Chapter 3 for a justification.) If Each $D_j = B(0,1) =$ the unit disk centered at the origin, the above formula can be used to show $M = 2\left(1 + \frac{\mu_0}{\mu_1}\right)^{-1}\mathbf{I}^{2\times 2}$ (cf. Remarks 3.1 and 3.4 of Chapter 3).

Another notable case is that when $\mathcal{I}_\rho$ is a thin inhomogeneity, or a collection of thin inhomogeneities. Given a smooth nonintersecting curve γ of finite length compactly contained in Ω, let $\mathcal{I}_\rho = [-\rho/2, \rho/2] \times [0, \text{length}\,\gamma]$ in the local coordinate system determined by the normal ν and tangent τ directions of γ. In this case it can be shown that $\mathrm{d}\alpha = \frac{1}{\text{length}\,\gamma}\mathrm{d}\sigma_\gamma$, where $\mathrm{d}\sigma_\gamma$ is the arc-length measure on γ [BFV03, CV03a]. Furthermore, it can be shown that for $x \in \gamma$,

$$M(x) \text{ in the basis determined by } \tau(x) \text{ and } \nu(x) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\mu_1(x)}{\mu_0(x)} \end{bmatrix}.$$

Note that the bounds (2.43) are achieved with $\xi = \tau(x)$ and $\xi = \nu(x)$.

2.4.4 Estimating the size of the inclusion

Techniques for using boundary voltage measurements to estimate the size of a small conducting inhomogeneity within a bounded, conducting background medium were provided in [CV03b] and [CV04]. These techniques, based on the analogue of Theorem 3 for the conductivity problem, do not directly apply when the background material is nonconducting. In such a situation, we may apply Theorem 3 to see how to use electromagnetic boundary measurements to estimate the size of the inhomogeneity.

Suppose μ_0, μ_1, ε_0, ε_1, σ_1 and σ_0 are all constant. Take any $\xi \in \mathbb{R}^2$ and apply appropriate boundary data to produce the background field $u_0 = e^{ik_0 x\cdot\xi}$. Let $s_0 =$

$e^{ik_0 x\cdot\xi}$. By Corollary 2.22,

$$
\begin{aligned}
\mathcal{P}_\xi &:= \int_{\partial\Omega} \left\{ u_{\rho_n} \left(\frac{1}{\mu_0} \partial_\nu \overline{s_0} \right) - \left(\frac{1}{\mu_0} \partial_\nu u_{\rho_n} \right) \overline{s_0} \right\} \mathrm{d}\sigma \\
&= |\mathcal{I}_{\rho_n}| \left[\int_\Omega -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0} \Big) \nabla \overline{s_0} \cdot (M \nabla u_0) \, \mathrm{d}\alpha + \int_\Omega \big(\kappa_1^2 - \kappa_0^2 \big) u_0 \overline{s_0} \, \mathrm{d}\alpha + o(1) \right] \\
&= |\mathcal{I}_{\rho_n}| \bigg[- \Big(\frac{1}{\mu_1} - \frac{1}{\mu_0} \Big) |k_0|^2 \int_\Omega \xi^T M \xi e^{-2\,\mathrm{Im}(k_0) x\cdot\xi} \, \mathrm{d}\alpha \\
&\qquad\qquad + \big(\kappa_1^2 - \kappa_0^2 \big) \int_\Omega e^{-2\,\mathrm{Im}(k_0) x\cdot\xi} \, \mathrm{d}\alpha + o(1) \bigg].
\end{aligned}
$$

If $\mathrm{Im}(k_0) = 0$ (i.e., if $\sigma_0 = 0$ or $\omega = 0$), the above simplifies as

$$
\mathcal{P}_\xi = |\mathcal{I}_{\rho_n}| \bigg[- \Big(\frac{1}{\mu_1} - \frac{1}{\mu_0} \Big) k_0^2 \int_\Omega \xi^T M \xi \, \mathrm{d}\alpha + \big(\kappa_1^2 - \kappa_0^2 \big) + o(1) \bigg]. \tag{2.49}
$$

Consider the case where $\mathrm{Im}(k_0) = 0$ and $\mu_0 = \mu_1$. (2.49) then becomes

$$
\begin{aligned}
\mathcal{P}_\xi &= |\mathcal{I}_{\rho_n}| \big[\big(\kappa_1^2 - \kappa_0^2 \big) + o(1) \big] \\
&= |\mathcal{I}_{\rho_n}| \big[\omega^2 (\varepsilon_1 - \varepsilon_0) + i\omega\sigma_1 + o(1) \big].
\end{aligned}
$$

Taking real and imaginary parts gives

$$
\begin{aligned}
\mathrm{Re}\{\mathcal{P}_\xi\} &= |\mathcal{I}_{\rho_n}| \big[\omega^2 (\varepsilon_1 - \varepsilon_0) + o(1) \big], \\
\mathrm{Im}\{\mathcal{P}_\xi\} &= |\mathcal{I}_{\rho_n}| [\omega\sigma_1 + o(1)].
\end{aligned}
$$

In the event that at least one of ε_1 and σ_1 are known, the equation corresponding to the known quantity provides a simple and efficient tool for estimating the size of the arbitrarily shaped small inhomogeneity from the boundary measurement $\mathcal{P}_\xi$. This approximation of $|\mathcal{I}_{\rho_n}|$ can then be used to solve for the unknown parameter ε_1 or σ_1, if one of these is unknown.

If $\sigma_0 = 0$ and $\sigma_1 > 0$,

$$
\mathrm{Im}\{\mathcal{P}_\xi\} = |\mathcal{I}_{\rho_n}| \big[\omega\sigma_1 + o(1) \big]
$$

holds even if $\mu_1 \neq \mu_0$. Again, assuming σ_1 is known, this equation can be used to estimate the size of the inhomogeneity from $\mathcal{P}_\xi$.

The situation is more complicated when $\sigma_0 = \sigma_1 = 0$ and $\mu_0 \neq \mu_1$. Fortunately, the ideas of [CV03b] carry over with minor changes. The significance of the positive definiteness bounds (2.43) will now become clear. By applying that inequality to (2.49), we find

$$-\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)k_0^2|\xi|^2 \leq \frac{\mathrm{Re}\{\mathcal{P}_\xi\}}{|\mathcal{I}_{\rho_n}|} - \omega^2(\varepsilon_1 - \varepsilon_0) + o(1) \leq -\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\frac{\mu_1}{\mu_0}k_0^2|\xi|^2. \quad (2.50)$$

Assume $|\xi| = 1$. If $\mathrm{Re}\{\mathcal{P}_\xi\} > 0$, the above inequality yields the lower bound

$$\frac{\mathrm{Re}\{\mathcal{P}_\xi\}}{-\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\frac{\mu_1}{\mu_0}k_0^2 + \omega^2(\varepsilon_1 - \varepsilon_0) + o(1)} \leq |\mathcal{I}_{\rho_n}|. \quad (2.51a)$$

If $\mathrm{Re}\{\mathcal{P}_\xi\} < 0$, we instead get the lower bound

$$\frac{\mathrm{Re}\{\mathcal{P}_\xi\}}{-\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)k_0^2 + \omega^2(\varepsilon_1 - \varepsilon_0) + o(1)} \leq |\mathcal{I}_{\rho_n}|. \quad (2.51b)$$

As for upper bounds, we have: if

$$-\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)k_0^2 + \omega^2(\varepsilon_1 - \varepsilon_0) > 0$$

(and therefore, by (2.50), $\mathrm{Re}\{\mathcal{P}_\xi\} > 0$, assuming the $o(1)$ term is sufficiently small) then

$$|\mathcal{I}_{\rho_n}| \leq \frac{\mathrm{Re}\{\mathcal{P}_\xi\}}{-\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)k_0^2 + \omega^2(\varepsilon_1 - \varepsilon_0) + o(1)}. \quad (2.51c)$$

If

$$-\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\frac{\mu_1}{\mu_0}k_0^2 + \omega^2(\varepsilon_1 - \varepsilon_0) < 0$$

(and therefore, by (2.50), $\mathrm{Re}\{\mathcal{P}_\xi\} < 0$, assuming the $o(1)$ term is sufficiently small) then

$$|\mathcal{I}_{\rho_n}| \leq \frac{\mathrm{Re}\{\mathcal{P}_\xi\}}{-\left(\frac{1}{\mu_1} - \frac{1}{\mu_0}\right)\frac{\mu_1}{\mu_0}k_0^2 + \omega^2(\varepsilon_1 - \varepsilon_0) + o(1)}. \tag{2.51d}$$

The bounds (2.51a-d) can be used to estimate the size of the inhomogeneity in the case where $\sigma_0 = \sigma_1 = 0$ and $\mu_0 \neq \mu_1$. These are analogous to similar single measurement estimates in [CV03b]. In [CV03b] and [CV04] it was shown that, in the context of the conductivity problem, better estimates can be made from multiple measurements using bounds on the trace of M.

Property 2.28. *Suppose μ_0 and μ_1 are C^∞. With M as in Theorem 3,*

$$\begin{aligned} \frac{4}{1+\frac{\mu_0}{\mu_1}} &\leq \ \mathrm{trace}(M) \quad \leq 1 + \frac{\mu_1}{\mu_0}, \\ \frac{4}{1+\frac{\mu_1}{\mu_0}} &\leq \mathrm{trace}(M^{-1}) \leq 1 + \frac{\mu_0}{\mu_1} \end{aligned}$$

$\mathrm{d}\alpha$*-a.e. in the set* $\{\mu_0 \neq \mu_1\}$.

The proof of this property found in [CV06] (cf. [CV03b, CV04]) is based on a variational approach in the spirit of the work of Hashin and Shtrikman [HS63]. To illustrate the utility of the these bounds, we observe that by (2.49),

$$\mathcal{P}_{\mathbf{e}_1} + \mathcal{P}_{\mathbf{e}_2} = |\mathcal{I}_{\rho_n}|\left[-\left(\frac{1}{\mu_1} - \frac{1}{\mu_0}\right)k_0^2\int_\Omega \mathrm{trace}(M)\ \mathrm{d}\alpha + 2\left(\kappa_1^2 - \kappa_0^2\right) + o(1)\right]$$

($\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$), which can be used in conjunction with Property (2.28) to estimate $|\mathcal{I}_{\rho_n}|$ in the same manner as in the single measurement case.

Chapter 3

The Scattering Problem

3.1 Introduction

In this chapter we consider the two-dimensional scattering problem, arising from a three-dimensional problem with transverse-magnetic symmetry, wherein a given time-harmonic wave traveling in a background medium is incident upon a diametrically small, penetrable obstacle. The background medium is isotropic and nonconducting, and it is assumed to occupy all of the space exterior to the inhomogeneity. The inhomogeneity is also isotropic, though possibly conducting. We normalize the background permeability and permittivity to be 1 and denote the (dimensionless) permeability, permittivity and conductivity within the inhomogeneity by the constants μ, ε and σ, respectively. To simplify our notation, we let $q = \mu\boldsymbol{\epsilon} = \mu(\varepsilon + i\frac{\sigma}{\omega})$. Also, we let μ_ρ denote the piecewise constant function that equals the permeability throughout $\mathbb{R}^2$ when the inhomogeneity is present, and we do likewise for ε_ρ, σ_ρ and q_ρ.

The inhomogeneity (or, more precisely, the two-dimensional cross section of the inhomogeneity) is represented by $\mathcal{I}_\rho = \rho D$, where $D \subset \mathbb{R}^2$ is a smooth, simply connected domain with $0 \in D$. The prescribed incident wave u^i satisfies

$$(\Delta + \omega^2)u^i = 0 \quad \text{in } \mathbb{R}^2.$$

We will usually take u^i to be a plane wave propagating in the direction η:

$$u^i(x) = e^{i\omega x\cdot\eta},$$

where $\eta \in \mathbb{R}^2$, $|\eta| = 1$. The resulting scalar electric field u_ρ is the unique solution to the problem consisting of the equation

$$\nabla \cdot \Big(\frac{1}{\mu_\rho} \nabla u_\rho \Big) + \omega^2 \frac{q_\rho}{\mu_\rho} u_\rho = 0 \tag{3.1a}$$

and Sommerfeld's outgoing radiation condition,

$$(\partial_r - i\omega)(u_\rho - u^i) = O(r^{-3/2}) \quad \text{as } r \to \infty. \tag{3.1b}$$

We define the transmitted and scattered fields u_ρ^{tr} and u_ρ^s by

$$u_\rho = \begin{cases} u^i + u_\rho^s & \text{in } \mathbb{R}^2 \setminus \rho\overline{D}, \\ u_\rho^{tr} & \text{in } \rho D. \end{cases}$$

Since D is smooth, we may rewrite (3.1) as the problem of finding (u_ρ^{tr}, u_ρ^s) satisfying

$$\begin{cases} \Delta u_\rho^s + \omega^2 u_\rho^s = 0 & \text{in } \mathbb{R}^2 \setminus \rho\overline{D}, \\ \Delta u_\rho^{tr} + q\omega^2 u_\rho^{tr} = 0 & \text{in } \rho D, \end{cases} \tag{3.2a}$$

with the transmission conditions

$$\begin{cases} u^{tr} = u_\rho^s + u^i & \text{on } \rho\partial D, \\ \frac{1}{\mu}\partial_\nu u_\rho^{tr} = \partial_\nu (u_\rho^s + u^i) & \text{on } \rho\partial D, \end{cases} \tag{3.2b}$$

along with the radiation condition

$$(\partial_r - i\omega)u_\rho^s = O(r^{-3/2}) \quad \text{as } r \to \infty. \tag{3.2c}$$

We remark that the radiation condition can be replaced with the weaker condition

$$(\partial_r - i\omega)u_\rho^s = o(r^{-1/2}),$$

or even

$$\int_{\partial B(0,r)} \left|(\partial_r - i\omega)u_\rho^s\right|^2 \mathrm{d}\sigma \to 0,$$

but any solution will automatically satisfy the stronger form, thanks to the Green's representation formula (3.27) (see [Wil56] or [CK98]). The unique existence of a solution u_ρ to this problem can be proved by replacing the radiation condition with the boundary condition

$$\partial_r u_\rho^s = \Lambda_R\big(u_\rho^s\big|_{\partial B_R}\big) \quad \text{on } \partial B_R, \tag{3.3}$$

for some R sufficiently large so that $\rho D \subset\subset B_R$, where $\Lambda_R : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ is the Dirichlet-to-Neumann operator, which assigns to a given function $f \in H^{1/2}(\partial B_R)$ the normal derivative of the solution of the exterior problem

$$\begin{cases} (\Delta + \omega^2)u = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{B_R}, \\ \qquad\qquad u = f \quad \text{on } \partial B_R, \\ (\partial_r - i\omega)u = O(r^{-3/2}) \quad \text{as } r \to \infty, \end{cases}$$

which may easily be solved using the method of separation of variables. This equivalent formulation of the problem, where the radiation condition (3.1b) is replaced with (3.3), may then be solved variationally. This method applies just as well when the inhomogeneity $\mathcal{I}_\rho$ is assumed only to be bounded and measurable. The scattered field u_ρ^s will belong to the space

$$\mathcal{H} := \Big\{ u : \ \frac{u}{\sqrt{1+r^2}} \in L^2(\mathbb{R}^2 \setminus \overline{\mathcal{I}_\rho}),\ \frac{\nabla u}{\sqrt{1+r^2}} \in L^2(\mathbb{R}^2 \setminus \overline{\mathcal{I}_\rho}), \\ (\partial_r - i\omega)u \in L^2(\mathbb{R}^2 \setminus \overline{\mathcal{I}_\rho}) \Big\},$$

and the transmitted field u^{tr} will belong to $H^1(\mathcal{I}_\rho)$. Details can be found in [Néd01] for the exterior Dirichlet and Neumann problems—that is, for the case when $\mathcal{I}_\rho$ is perfectly conducting (sound-soft) and for the case when $\mathcal{I}_\rho$ is sound-hard. The ideas found there can be easily modified to accommodate penetrable scatterers.

Alternatively, in the case of a smooth scatterer and constant permeability, existence and uniqueness can be proved using layer potential techniques, with the solution being a convergent Born series (also called a Neumann series) [Roa92]. As such, this method of proof spells out an iterative procedure for numerically computing the solution.[1]

Our goal in the chapter will be to derive approximation formulas for the scattered field as $\rho \to 0$. In the case of a nonconducting inhomogeneity and fixed frequency, such an approximation has been derived rigorously as the leading order term of an asymptotic expansion [AIM03]. But here we will allow the frequency to vary as a function of ρ, and we will derive formal asymptotic approximations for three separate regimes: $\rho\omega \to 0$, $\rho\omega \to \lambda_0$ (finite and nonzero), and $\rho\omega \to \infty$.

To illustrate the effect of changes in frequency, we consider the case of a plane wave incident upon a disk of radius ρ centered at the origin. Let $\left\| u_\rho^s|_{\partial B(0,2)} \right\|_{L^2(\mathbb{T})}^2 = \frac{1}{2\pi} \int_0^{2\pi} \left| u_\rho^s|_{\partial B(0,2)} \right|^2 \mathrm{d}\theta$, where $\partial B(0,2)$ is the circle of radius 2 centered at the origin. In Figures 3.1 and 3.2 we show plots of $\left\| u_\rho^s|_{\partial B(0,2)} \right\|_{L^2(\mathbb{T})}$ as a function of ω for various values of ρ, μ, ε and σ/ω (σ is assumed to grow at a rate proportional to ω, which means the scatterer is well absorbing). The asymptotic behavior as $\rho \to 0$ for a fixed small ω

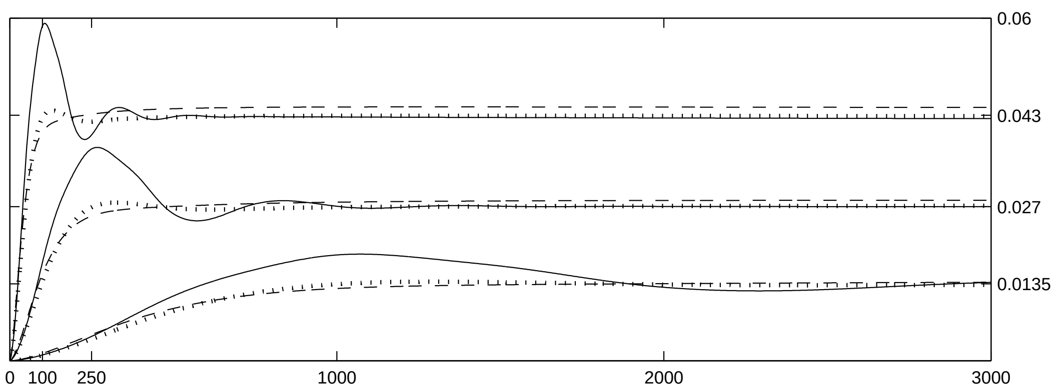

Figure 3.1: Plots of the $\left\| u_\rho^s|_{\partial B(0,2)} \right\|_{L^2(\mathbb{T})}$ as a function of ω. Of these nine plots, the cluster of three at the top correspond to the scattering disk of radius $\rho = 0.01$, the middle cluster to $\rho = 0.004$ and the bottom cluster to $\rho = 0.001$. The dotted graphs correspond to the values $\varepsilon = 2$ and $\sigma/\omega = 2$, the solid graphs to $\varepsilon = 3$ and $\sigma/\omega = 1$ and the dashed graphs to $\varepsilon = 1$ and $\sigma/\omega = 3$. In all cases $\mu = 2$. The three values of ρ^{-1} are labeled on the ω-axis.

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[1] See also [CK98, Ch. 8] for a proof based on the principle of unique continuation.

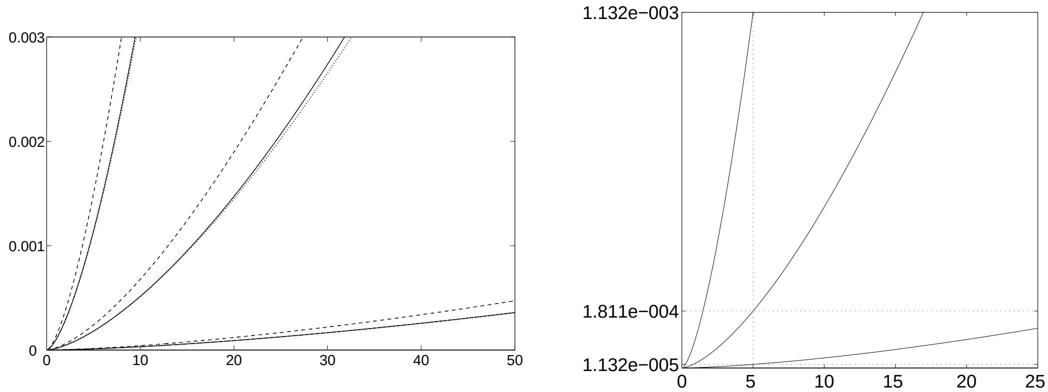


Figure 3.2: The left frame is a close-up view of the plots from Figure 3.1 for ω small. The right frame shows only those plots corresponding to the values $\varepsilon = 2$ and $\sigma/\omega = 2$, which were dotted in the left frame. On the vertical axis are the values of $\left\|u^s_\rho|_{\partial B(0,2)}\right\|_{L^2(\mathbb{T})}$ when $\omega = 5$ for $\rho = 0.01$, 0.004 and 0.001 (from top to bottom).

can be seen in Figure 3.2: at $\omega = 5$, $\rho^{-2}\left\|u^s_\rho|_{\partial B(0,2)}\right\|_{L^2(\mathbb{T})}$ is nearly constant (≈ 11.32) for the three values $\rho = 0.01$, $\rho = 0.004$ and $\rho = 0.001$. That $\left\|u^s_\rho|_{\partial B(0,2)}\right\|_{L^2(\mathbb{T})} \sim \rho^2$ is consistent with the asymptotic expansion we derived in Chapter 1 and with the rigorous expansion in [AIM03]. If ω is allowed to vary as a function of ρ, Figure 3.1 suggests that this asymptotic behavior as $\rho \to 0$ is valid so long as $\omega \ll \rho^{-1}$, but there is a change when $\omega \sim \rho^{-1}$. The plots suggest that if ω grows as $\rho \to 0$ in such a way that $\omega \sim \rho^{-1}$ or $\omega \gg \rho^{-1}$ then $\left\|u^s_\rho|_{\partial B(0,2)}\right\|_{L^2(\mathbb{T})} \sim \sqrt{\rho}$. For instance, note that

$$\frac{0.0135}{\sqrt{0.001}} \approx \frac{0.027}{\sqrt{0.004}} \approx \frac{0.043}{\sqrt{0.01}} = 0.43.$$

Figure 3.3 shows what happens when the scatterer has zero conductivity. The oscillations in the plots are due to resonance effects caused by the transmitted field, which, in the absence of conductivity, does not rapidly attenuate within the scatterer. It is not surprising then that the task of finding an asymptotic expression for the scattered field when $\omega \gg \rho^{-1}$ in the case of a nonconducting scatterer is more difficult than in the case where $\sigma/\omega \geq c > 0$. We will avoid this complication by only considering well absorbing scatterers. However, we note that the regularity of the oscillations in Figure 3.3 suggests that testing over a broad band of high frequencies followed by an appropriate averaging may yield stable data that will aid in solving the inverse problem

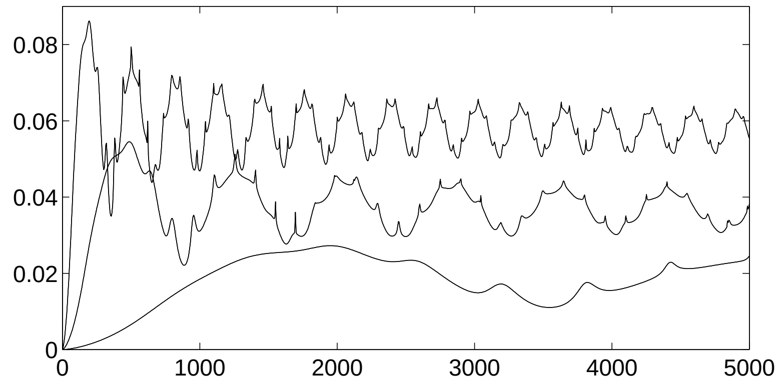


Figure 3.3: Plots of the $\left\| u^s_\rho|_{\partial B(0,2)} \right\|_{L^2(\mathbb{T})}$ as a function of ω in the case of a nonconducting scatterer. The highest graph corresponds to the disk of radius $\rho = 0.01$, the middle to $\rho = 0.004$ and the bottom graph to $\rho = 0.001$. In each case, $\mu = 2$, $\varepsilon = 2$ and $\sigma = 0$.

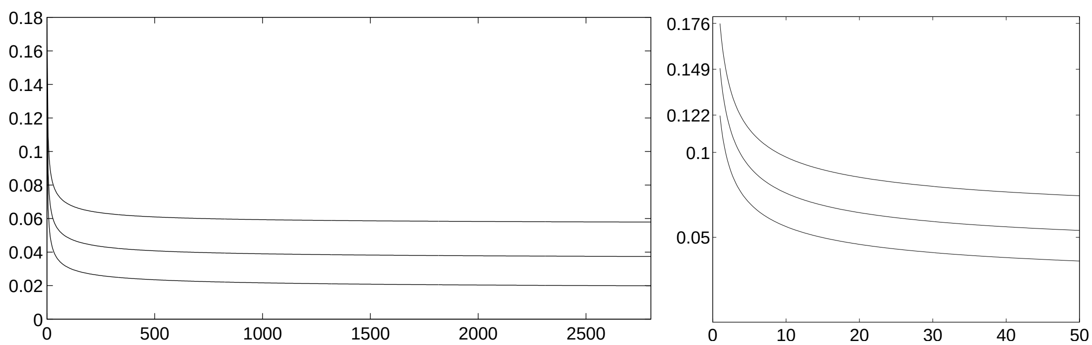


Figure 3.4: Plots of the $\left\| u^s_\rho|_{\partial B(0,2)} \right\|_{L^2(\mathbb{T})}$ as a function of ω in the case of a perfectly conducting scatterer. The highest graph corresponds to the disk of radius $\rho = 0.01$, the middle to $\rho = 0.004$ and the bottom graph to $\rho = 0.001$. The right frame is a close-up view of for $1 \le \omega \le 50$.

even when the inhomogeneity has low, or zero, conductivity.

Figure 3.4 is just as in Figures 3.1 and 3.2 except now the scatterer is perfectly conducting. That is, u^s_ρ solves the problem[2]

$$\begin{cases} \Delta u^s_\rho + q_0 \omega^2 u^s_\rho = 0 & \text{in } \mathbb{R}^2 \setminus \rho\overline{D}, \\ u^s_\rho = -u^i & \text{on } \rho\partial D, \\ (\partial_r - i\sqrt{q_0}\omega) u^s_\rho = o(r^{-1/2}) & \text{as } r \to \infty. \end{cases} \tag{3.4}$$

The asymptotic behavior as $\rho \to 0$ for high frequencies $\omega \gg \rho^{-1}$ is the same as in the case of a moderately well absorbing scatterer (Figure 3.1): $\left\| u^s_\rho|_{\partial B(0,2)} \right\|_{L^2(\mathbb{T})} \sim \sqrt{\rho}$. But

[2]The total electric field vanishes within the perfect conductor, hence the boundary condition $u_\rho = u^s_\rho + u^i = 0$ on ∂D.

for low frequencies the magnitude shrinks on the order of $|\log\rho|^{-1}$:

$$0.176 \times |\log 0.01| \approx 0.149 \times |\log 0.004| \approx 0.122 \times |\log 0.001|.$$

3.2 Low to moderate frequency

By *low frequency* we of course mean low relative frequency: ω may tend to ∞, but it must do so at a slow pace relative to ρ^{-1}. *Moderate frequency* refers to the situation where ω grows in such a way that $1/C \leq \omega\rho \leq C$ for some $C > 0$.

3.2.1 The case of a disk

Suppose the scatterer is the unit disk $D = B(0,1)$. In this case, the method of separation of variables gives us an exact solution to problem (3.2). From the expression of this solution it is not difficult to prove a rigorous expansion in $\omega\rho$ of the scattered field.

For now, we may assume u^i is of a more general form than simply that of a plane wave. Since $(\Delta + \omega^2)u^i = 0$ in all of $\mathbb{R}^2$, u^i must have the form

$$u^i(x) = \sum_{n=-\infty}^{\infty} a_n J_n(\omega r) e^{in\theta}, \tag{3.5}$$

where J_n the Bessel function of the first kind of order n. We assume that $\{a_n\}_{n=-\infty}^{\infty}$ satisfies

$$\|\{a_n\}\|_{h^\alpha}^2 := \sum_{n=-\infty}^{\infty} |a_n|^2 (1+|n|)^{2\alpha} < \infty \tag{3.6}$$

for some $\alpha \in \mathbb{R}$. Since

$$|J_n(t)| \leq \frac{\left|\frac{z}{2}\right|^{|n|}}{|n|!} e^{\frac{1}{4}|z|^2} \quad \text{for all } n \in \mathbb{Z},\, z \in \mathbb{C}$$

(cf. [Wat44, §2.1]), the condition (3.6) guarantees that the sum (3.5) converges to a C^∞ function.[3] We should note that this condition easily allows for u^i to be a plane

[3] Here we use the identity $zJ_n'(z) = nJ_n(z) - zJ_{n+1}(z)$ to show that when the terms of the sum (3.5) are all differentiated to any order of r and θ (all terms to the same order), the resulting sum converges

wave: if $\eta = (\cos\theta_0, \sin\theta_0)$ then

$$e^{i\omega x\cdot\eta} = \sum_{n=-\infty}^{\infty} J_n(\omega r) e^{in(\theta-\theta_0+\pi/2)},$$

so that $|a_n| = 1$ for each n. This follows from the fact that

$$\begin{aligned}(\widehat{e^{i\omega x\cdot\eta}})_n(r) &= \frac{1}{2\pi}\int_{\mathbb{T}} e^{i\omega r\cos(\theta-\theta_0)} e^{-in\theta}\,\mathrm{d}\theta \\ &= e^{in(\pi/2-\theta_0)} \frac{1}{2\pi}\int_{\mathbb{T}} e^{i\omega(r\sin\theta - n\theta)}\,\mathrm{d}\theta \\ &= e^{in(\pi/2-\theta_0)} J_n(\omega r),\end{aligned}$$

the last equality being a well known integral representation of J_n.

Theorem 4. *Let u_ρ^s be the scattered field that results when a given wave u^i is incident on the scatterer ρD, as in* (3.2)*, where $D = B(0,1)$. Assume u^i is such that* (3.5) *and* (3.6) *hold. As $\rho \to 0$, assume $\lambda := \omega\rho \to 0$ and assume ω stays bounded away from zero. We allow the possibility that σ varies as a function of ρ as $\rho \to 0$, with the restriction that σ/ω be bounded. Let $r_0 > 0$. Then for $x \in \mathbb{R}^2 \setminus B(0, r_0)$,*

$$\begin{aligned}u_\rho^s(x) = \rho^2\omega^2|B(0,1)|\big\{(\varepsilon + i\sigma/\omega) - 1\big\}\Phi^\omega(x,0)u^i(0) \\ + \rho^2|B(0,1)|\Big\{1 - \frac{1}{\mu}\Big\}\nabla\Phi^\omega(x,0)\cdot\big(M\nabla u^i(0)\big) + O(\lambda^3),\end{aligned} \quad (3.7)$$

where the polarization tensor $M = 2\Big(1 + \frac{1}{\mu}\Big)^{-1}\mathbf{I}^{2\times 2}$ and the remainder $O(\lambda^3)$ is uniform in x.[4]

If the scatterer is perfectly conducting, that is, if u_ρ^s solves problem (3.4)*,*

$$u_\rho^s(x) = \frac{-u^i(0)}{1 + \frac{2i}{\pi}\big[\log(\lambda/2) + \gamma\big]} + O(\lambda^2), \quad (3.8)$$

uniformly in (r,θ) restricted to compact sets. Of course, one need only verify this convergence of derivatives to order two—smoothness will then follow from elliptic regularity.

[4] $\Phi^\omega(x,y) = \frac{i}{4}H_0^{(1)}(\omega|x-y|)$, where $H_0^{(1)}$ is the 0^{th} order Hankel function of the first kind, is the free-space Green's function for the background Helmholtz operator $\Delta + \omega^2$ satisfying Sommerfeld's outgoing radiation condition.

with the remainder $O(\lambda^2)$ uniform in x for $|x| = r \geq r_0$.

Remark 3.1. *Formula (3.7) for the scattering problem is consistent with the remarks of section 2.4.3 of Chapter 2 concerning the problem in a bounded domain. To see this, suppose u_ρ solves the problem (3.1), except with the background constants μ_0 and ε_0 not necessarily equal to 1. With notation as in Chapter 2, observe that for any domain Ω such that $B(0, r_0) \subset\subset \Omega$,*

$$\mathcal{H}_0\big(u_\rho - u_0, \overline{\Phi^{k_0}(x,\cdot)}\big) = \frac{1}{\mu_0}(u_\rho - u_0)(x) + \int_{\partial\Omega} \frac{1}{\mu_0}(u_\rho - u_0)(y)\partial_{\nu_y}\Phi^{k_0}(x,y)\,\mathrm{d}\sigma_y$$

for any $x \in \Omega \setminus \overline{B(0,r_0)}$ (here $u^i = u_0$ and $u^s_\rho = u_\rho - u_0$). But at the same time,

$$\begin{aligned}
&\mathcal{H}_0\big(u_\rho - u_0, \overline{\Phi^{k_0}(x,\cdot)}\big)\\
&\quad= \int_{B(0,\rho)} \Big\{-\Big(\frac{1}{\mu_1} - \frac{1}{\mu_0}\Big)\nabla u_\rho(y)\cdot\nabla\Phi^{k_0}(x,y) + \big(\kappa_1^2 - \kappa_0^2\big)u_\rho(y)\Phi^{k_0}(x,y)\Big\}\,\mathrm{d}y\\
&\qquad\qquad + \int_{\partial\Omega} \frac{1}{\mu_0}\partial_\nu(u_\rho - u_0)(y)\Phi^{k_0}(x,y)\,\mathrm{d}\sigma_y,
\end{aligned}$$

and consequently,

$$\begin{aligned}
\frac{1}{\mu_0}(u_\rho - u_0)(x) &= \int_{B(0,\rho)} \Big\{\text{as above}\Big\}\mathrm{d}y\\
&\quad+ \int_{\partial\Omega} \frac{1}{\mu_0}\Big\{\partial_\nu(u_\rho - u_0)(y)\Phi^{k_0}(x,y) - (u_\rho - u_0)(y)\partial_{\nu_y}\Phi^{k_0}(x,y)\Big\}\,\mathrm{d}\sigma_y.
\end{aligned}$$

Suppose $\Omega = B(0,R)$. Since $u^s_\rho = u_\rho - u_0$ and $\Phi^{k_0}(x,\cdot)$ both satisfy Sommerfeld's outgoing radiation condition, the above integral over $\partial\Omega$ is on the order of $|\partial\Omega|R^{-3/2}\log(R)$, or $R^{-1/2}\log R$, and therefore vanishes as $R \to \infty$. In section 2.4.3 we remarked that when when $\mathcal{I}_\rho = B(0,\rho)$, the measure $\mathrm{d}\alpha$ guaranteed by Theorem 3 is simply the Dirac measure δ_0, and the polarization tensor $M = 2\Big(1 + \frac{\mu_0}{\mu_1}\Big)^{-1}\mathbf{I}^{2\times 2}$. As a result,

$$\begin{aligned}
u^s_\rho(x) &= \rho^2\omega^2|B(0,1)|\big\{(\varepsilon_1 + i\sigma_1/\omega) - \varepsilon_0\big\}\Phi^{k_0}(x,0)u^i(0)\\
&\qquad + \rho^2|B(0,1)|\Big\{\frac{1}{\mu_0} - \frac{1}{\mu_1}\Big\}\nabla\Phi^{k_0}(x,0)\cdot\big(M\nabla u^i(0)\big) + o(\rho^2)
\end{aligned}$$

as $\rho \to 0$ with ω fixed. The significance of Theorem 4 is that the same asymptotic *formula is shown to hold even when ω is allowed to grow at any rate slower than ρ^{-1}.*

Proof of Theorem 4. We first consider the case of a penetrable scatterer. To begin, we write[5]

$$u^s_\rho = \sum_{n=-\infty}^{\infty} \alpha_n H_n(\omega r) e^{in\theta}, \quad u^{tr}_\rho = \sum_{n=-\infty}^{\infty} \beta_n J_n(\sqrt{q}\omega r) e^{in\theta}, \tag{3.9}$$

where H_n denotes the Hankel function $H_n^{(1)} := J_n + iY_n$. Using the transmission conditions, we find a system of two equations for each pair (α_n, β_n), which we solve to get

$$\alpha_n = a_n \frac{\sqrt{q}\frac{1}{\mu} J_n'(\sqrt{q}\lambda) J_n(\lambda) - J_n'(\lambda) J_n(\sqrt{q}\lambda)}{H_n'(\lambda) J_n(\sqrt{q}\lambda) - \sqrt{q}\frac{1}{\mu} J_n'(\sqrt{q}\lambda) H_n(\lambda)} \tag{3.10}$$

$$= a_n \left[\frac{\sqrt{q}\lambda \frac{1}{\mu} \frac{J_n'(\sqrt{q}\lambda)}{J_n(\sqrt{q}\lambda)} - \lambda \frac{J_n'(\lambda)}{J_n(\lambda)}}{\lambda \frac{H_n'(\lambda)}{H_n(\lambda)} - \sqrt{q}\lambda \frac{1}{\mu} \frac{J_n'(\sqrt{q}\lambda)}{J_n(\sqrt{q}\lambda)}} \right] \frac{J_n(\lambda)}{H_n(\lambda)}, \tag{3.11}$$

$$\beta_n = a_n \frac{H_n'(\lambda) J_n(\lambda) - J_n'(\lambda) H_n(\lambda)}{H_n'(\lambda) J_n(\sqrt{q}\lambda) - \sqrt{q}\frac{1}{\mu} J_n'(\sqrt{q}\lambda) H_n(\lambda)},$$

$$= a_n \frac{2i}{\pi\lambda} \left(\frac{1}{H_n'(\lambda) J_n(\sqrt{q}\lambda) - \sqrt{q}\frac{1}{\mu} J_n'(\sqrt{q}\lambda) H_n(\lambda)} \right), \tag{3.12}$$

the last equality because of the well known identity: Wronsk$(J_n(z), Y_n(z)) = \frac{2}{\pi z}$. To write the expression (3.11) for α_n, we must assume that $J_n(\lambda)$ and $J_n(\sqrt{q}\lambda)$ are both nonzero for all n. We therefore assume that λ, and $\sqrt{q}\lambda$, if q is real[6], are less than the smallest positive zero of J_0. This will suffice since $z_n < z_{n+1}$ for all $n \geq 0$, where z_n denotes the smallest positive zero of J_n.

Note: *That the denominator in* (3.10) *and* (3.12) *remains nonzero for all λ and q is an immediate consequence of the fact that the transmission problem* (3.2) *is well posed. However, it also has a simple, direct proof: Suppose the expression in the denominator of* (3.10) *is zero for some n, λ and q. Since the zeros of J_n away from the origin are all*

[5] In other words, $\widehat{(u^s_\rho)}_n = \alpha_n H_n(\omega r)$ and $\widehat{(u^{tr}_\rho)}_n = \beta_n J_n(\sqrt{q}\omega r)$.

[6] We need not worry if q is not real since J_n has only real zeros [Wat44, 15.25].

simple, either $J_n(\sqrt{q}\lambda) \neq 0$ *or* $J_n'(\sqrt{q}\lambda) \neq 0$. *If* $J_n(\sqrt{q}\lambda) \neq 0$, *let*

$$v(r,\theta) = \begin{cases} \frac{H_n(\lambda)}{J_n(\sqrt{q}\lambda)} J_n(\sqrt{q}r)e^{in\theta} & \text{for } r < \lambda, \\ H_n(r)e^{in\theta} & \text{for } r > \lambda. \end{cases}$$

If $J_n'(\sqrt{q}\lambda) \neq 0$, *let*

$$v(r,\theta) = \begin{cases} \frac{H_n'(\lambda)}{\frac{1}{\mu}J_n'(\sqrt{q}\lambda)} J_n(\sqrt{q}r)e^{in\theta} & \text{for } r < \lambda, \\ H_n(r)e^{in\theta} & \text{for } r > \lambda. \end{cases}$$

In either case, v *is the solution to*

$$\nabla \cdot \left(\frac{1}{\mu^*}\nabla v\right) + \frac{q^*}{\mu^*}v = 0 \tag{3.13}$$

with the radiation condition

$$(\partial_r - i)v = O(r^{-3/2}).$$

Here $q^* = q$ *and* $\mu^* = \mu$ *for* $r < \lambda$, *and* $q^* = \mu^* = 1$ *for* $r > \lambda$. *Multiply* (3.13) *by* $\overline{v}$, *then integrate by parts, and then use the radiation condition to get*

$$\lim_{R\to\infty}\left[\int_{B_R}\left\{-\frac{1}{\mu^*}|\nabla v|^2 + q^*|v|^2\right\}\mathrm{d}x + i\int_{\partial B_R}|v|^2\,\mathrm{d}\sigma\right] = 0,$$

which implies that $\lim_{R\to\infty}\int_{\partial B_R}|v|^2\,\mathrm{d}\sigma = 0$ *since* $\operatorname{Im} q^* \geq 0$. *However, using the well known asymptotic formula for the Hankel function,*

$$H_n(R) = \sqrt{\frac{2}{\pi R}}e^{i(R-n\pi/2-\pi/4)} + O(R^{-3/2}),$$

we calculate

$$\lim_{R\to\infty}\int_{\partial B_R}|v|^2\,\mathrm{d}\sigma = 4,$$

a contradiction. Therefore, it must be the case that for all $n \in \mathbb{Z}$ and $\lambda \geq 0$,

$$H_n'(\lambda) J_n(\sqrt{q}\lambda) - \sqrt{q}\frac{1}{\mu} J_n'(\sqrt{q}\lambda) H_n(\lambda) \neq 0.$$

Since any n^{th} order Bessel function, or the derivative of such a function, satisfies $F_{-n} = (-1)^n F_n$, it follows that $\alpha_{-n}/a_{-n} = \alpha_n/a_n$ (and similarly for β_n). Using the identity $zF_n'(z) = nF_n(z) - zF_{n+1}(z)$, we find

$$\alpha_n = -a_n \left[\frac{\left(\frac{1}{\mu} - 1\right)n + \lambda \frac{J_{n+1}(\lambda)}{J_n(\lambda)} - \lambda \frac{\sqrt{q}}{\mu} \frac{J_{n+1}(\sqrt{q}\lambda)}{J_n(\sqrt{q}\lambda)}}{\left(\frac{1}{\mu} - 1\right)n + \lambda \frac{H_{n+1}(\lambda)}{H_n(\lambda)} - \lambda \frac{\sqrt{q}}{\mu} \frac{J_{n+1}(\sqrt{q}\lambda)}{J_n(\sqrt{q}\lambda)}} \right] \frac{J_n(\lambda)}{H_n(\lambda)}. \tag{3.14}$$

In order to prove the asymptotic expansion (3.7) of u_ρ^s, we must first study the asymptotics of the ratios of Bessel functions that appear in (3.14).

First we consider the case of $n \geq 1$. In the following steps we assume $t \in \mathbb{C}$ with $0 < |t| \leq T$ for some finite T. The formula

$$J_n(t) = \left(\frac{t}{2}\right)^n \sum_{j=0}^{\infty} \frac{(-1)^j (\frac{t^2}{4})^j}{j!(n+j)!} \tag{3.15}$$

implies that, given any $\delta > 0$, there exists some $0 < t_0 < 1$, independent of n, such that for $|t| < t_0$, $J_n(t) = \frac{1}{n!}\left(\frac{t}{2}\right)^n \left(1 + t\mathcal{R}_n^J\right)$, with $|\mathcal{R}_n^J| \leq \frac{|t|}{4} e^{|t|^2/4} \leq \delta$. The formula

$$\begin{aligned} Y_n(t) = \frac{2}{\pi}\Big(\log\frac{t}{2} + \gamma\Big) J_n(t) - \frac{1}{\pi}\left(\frac{2}{t}\right)^n \sum_{j=0}^{n-1} \frac{(n-1-j)!}{j!}\left(\frac{t}{2}\right)^{2j} \\ - \frac{1}{\pi}\left(\frac{t}{2}\right)^n \sum_{j=0}^{\infty}\left(\sum_{m=1}^{n+j}\frac{1}{m} + \sum_{m=1}^{j}\frac{1}{m}\right)\frac{(-1)^j}{j!(n+j)!}\left(\frac{t}{2}\right)^{2j} \end{aligned} \tag{3.16}$$

implies that, given any $\delta > 0$, there exists some $0 < t_0 < 1$, independent of n, such that for $0 < |t| < t_0$, $H_n(t) = \frac{-i(n-1)!}{\pi}\left(\frac{2}{t}\right)^n \left(1 + \mathcal{R}_n^H\right)$, with $|\mathcal{R}_n^H| \leq 8\left|\frac{t}{2}\right|^\zeta \leq \delta$. Here $\zeta = 1$ when $n = 1$ and $\zeta = 2$ when $n > 1$. These two asymptotic formulas yield asymptotic expressions for the ratios of Bessel functions in (3.14):

$$\frac{J_{n+1}(t)}{J_n(t)} = \frac{\frac{1}{(n+1)!}\left(\frac{t}{2}\right)^{n+1}\left(1+t\mathcal{R}^J_{n+1}\right)}{\frac{1}{n!}\left(\frac{t}{2}\right)^{n}\left(1+t\mathcal{R}^J_{n}\right)}$$
$$= \frac{t}{2(n+1)}\big\{1+t\mathcal{A}_n\big\}, \tag{3.17}$$

where, given any $\delta > 0$, $|\mathcal{A}_n| \leq \delta$ for all $0 < |t| \leq t_0$, some $0 < t_0 < 1$ independent of n.

$$\frac{H_{n+1}(t)}{H_n(t)} = \frac{\frac{-in!}{\pi}\left(\frac{2}{t}\right)^{n+1}\left(1+\mathcal{R}^H_{n+1}\right)}{\frac{-i(n-1)!}{\pi}\left(\frac{2}{t}\right)^{n}\left(1+\mathcal{R}^H_{n}\right)}$$
$$= \frac{2n}{t}\big\{1+t^{\zeta}\mathcal{B}_n\big\}, \tag{3.18}$$

where, given any $\delta > 0$, $|\mathcal{B}_n| \leq \delta$ for all $0 < |t| \leq t_0$, some $0 < t_0 < 1$ independent of n. Here $\zeta = 1$ if $n = 1$ and $\zeta = 2$ if $n > 1$.

$$\frac{J_n(t)}{H_n(t)} = \frac{\frac{1}{n!}\left(\frac{t}{2}\right)^{n}\left(1+t\mathcal{R}^J_{n}\right)}{\frac{-i(n-1)!}{\pi}\left(\frac{2}{t}\right)^{n}\left(1+\mathcal{R}^H_{n}\right)}$$
$$= \frac{i\pi}{n!(n-1)!}\left(\frac{t}{2}\right)^{2n}\big\{1+t^{\zeta}\mathcal{C}_n\big\}, \tag{3.19}$$

where, given any $\delta > 0$, $|\mathcal{C}_n| \leq \delta$ for all $0 < |t| \leq t_0$, some $0 < t_0 < 1$ independent of n. Using (3.17), (3.18) and (3.19) we get, for $n \geq 1$,

$$\alpha_n = -a_n\left[\frac{\left(\frac{1}{\mu}-1\right)n + \frac{\lambda^2}{2(n+1)}\Big\{\left(1-\frac{q}{\mu}\right)+\lambda\left[\mathcal{A}_n(\lambda)-\sqrt{q}\mathcal{A}_n(\sqrt{q}\lambda)\right]\Big\}}{\left(\frac{1}{\mu}-1\right)n+2n+\lambda^{\zeta}\mathcal{B}_n(\lambda)-\frac{\lambda^2}{2(n+1)}\frac{q}{\mu}\left[1+\lambda\sqrt{q}\mathcal{A}_n(\sqrt{q}\lambda)\right]}\right]\frac{J_n(\lambda)}{H_n(\lambda)}$$
$$= a_n\left(\frac{1-\frac{1}{\mu}}{1+\frac{1}{\mu}}\right)\frac{1}{n!}\left(\frac{\lambda}{2}\right)^{n}\big\{1+\lambda^{\zeta}\mathcal{R}^1_n\big\}\frac{1}{H_n(\lambda)} \tag{3.20a}$$
$$= a_n\left(\frac{1-\frac{1}{\mu}}{1+\frac{1}{\mu}}\right)\frac{i\pi}{n!(n-1)!}\left(\frac{\lambda}{2}\right)^{2n}\big\{1+\lambda^{\zeta}\mathcal{R}^2_n\big\}, \tag{3.20b}$$

where, given any $\delta > 0$, $|\mathcal{R}^1_n|, |\mathcal{R}^2_n| \leq \delta$ for all $\lambda \leq \lambda_0$, some $0 < \lambda_0 < 1$ depending on $q_{\text{sup}} := \mu(\varepsilon + i\sup\{\sigma/\omega\})$ and μ but independent of n. We use the formula (3.20b) when $n = 1$ to see that

$$\begin{aligned}
&\alpha_1 H_1(\omega r)e^{i\theta} + \alpha_{-1} H_{-1}(\omega r)e^{-i\theta} \\
&\qquad = H_1(\omega r)(\alpha_1 e^{i\theta} - \alpha_{-1} e^{-i\theta}) \\
&\qquad = H_1(\omega r)\left(\frac{1-\frac{1}{\mu}}{1+\frac{1}{\mu}}\right) i\pi \left(\frac{\lambda}{2}\right)^2 (a_1 e^{i\theta} - a_{-1}e^{-i\theta}) + O(\lambda^3). \qquad (3.21)
\end{aligned}$$

Here we used the assumption that ω is bounded away from zero to ensure that $H_1(\omega r)$ can be absorbed into the $O(\lambda^3)$ term. The assumption that $r \geq r_0 > 0$ ensures $O(\lambda^3)$ remains uniform in x. Now, observe that

$$\begin{aligned}
u^i(x) &= (a_1 e^{i\theta} - a_{-1}e^{-i\theta})J_1(\omega r) + \mathcal{R}_1 \\
&= \frac{\omega}{2}\big[a_1(x_1 + ix_2) - a_{-1}(x_1 - ix_2)\big] \\
&\qquad + \underbrace{(x_1 + ix_2)\mathcal{R}_2(r^2) + (x_1 - ix_2)\mathcal{R}_3(r^2) + \mathcal{R}_1}_{\text{gradient vanishes at } x=0}.
\end{aligned}$$

As a result,

$$\nabla u^i(0) = \frac{\omega}{2}\big[a_1(1, i) - a_{-1}(1, -i)\big],$$

and so

$$\nabla u^i(0)\cdot \hat{x} = \frac{\omega}{2}\big[a_1 e^{i\theta} - a_{-1}e^{-i\theta}\big].$$

Noting also that

$$\nabla_y \Phi^\omega(x, 0) = \omega\frac{i}{4}H_1(\omega r)\hat{x},$$

we may rewrite (3.21) as

$$\begin{aligned}
&\alpha_1 H_1(\omega r)e^{i\theta} + \alpha_{-1} H_{-1}(\omega r)e^{-i\theta} \\
&\qquad = \rho^2 |B(0,1)| \Big\{1 - \frac{1}{\mu}\Big\} \nabla\Phi^\omega(x, 0)\cdot \big(M\nabla u^i(0)\big) + O(\lambda^3), \qquad (3.22)
\end{aligned}$$

where the polarization tensor $M = 2\Big(1 + \frac{1}{\mu}\Big)^{-1}\mathbf{I}^{2\times 2}$.

For the $n = 0$ term, we use the above asymptotic expressions for $J_n(t)$ and $H_n(t)$

when $n=1$, as well as those corresponding to the $n=0$ case,

$$J_0(t) = 1 + O(t^2)$$

and

$$H_0(t) = \frac{2i}{\pi}\log(t/2) + O(1),$$

to get

$$\begin{aligned}
\alpha_0 H_0(\omega r) &= -a_0 \left[\frac{\frac{J_1(\lambda)}{J_0(\lambda)} - \frac{\sqrt{q}}{\mu}\frac{J_1(\sqrt{q}\lambda)}{J_0(\sqrt{q}\lambda)}}{H_1(\lambda) - \frac{\sqrt{q}}{\mu}\frac{J_1(\sqrt{q}\lambda)}{J_0(\sqrt{q}\lambda)} H_0(\lambda)} \right] J_0(\lambda) H_0(\omega r) \\
&= -u^i(0) \left[\frac{\left(1 - \frac{q}{\mu}\right)\frac{\lambda}{2} + O(\lambda^2)}{-\frac{2i}{\pi\lambda} + O(\lambda \log \lambda)} \right] (1 + O(\lambda^2)) H_0(\omega r) \\
&= \lambda^2 |B(0,1)| \big\{(\varepsilon + i\sigma/\omega) - 1\big\} \Phi^\omega(x,0) u^i(0) + O(\lambda^3), \qquad (3.23)
\end{aligned}$$

where $O(\lambda^3)$ is uniform in x for $|x| \geq r_0$. From (3.22) and (3.23) we get

$$\begin{aligned}
u^s_\rho = \rho^2\omega^2 |B(0,1)| &\big\{(\varepsilon + i\sigma/\omega) - 1\big\} \Phi^\omega(x,0) u^i(0) \\
&+ \rho^2 |B(0,1)| \Big\{1 - \frac{1}{\mu}\Big\} \nabla\Phi^\omega(x,0) \cdot \big(M \nabla u^i(0)\big) \\
&\qquad + \sum_{|n|\geq 2} \alpha_n H_n(\omega r) e^{in\theta} + O(\lambda^3).
\end{aligned}$$

To complete the proof, it remains to be shown that $\sum_{|n|\geq 2} \alpha_n H_n(\omega r) e^{in\theta} = O(\lambda^3)$ uniformly in x as $t \to 0$. To this end, observe that

$$\begin{aligned}
\left| \sum_{|n|\geq 2} \alpha_n H_n(\omega r) e^{in\theta} \right| &= \left| \sum_{|n|\geq 2} a_n \left[\left(\frac{1 - \frac{1}{\mu}}{1 + \frac{1}{\mu}} \right) \frac{i\pi}{n!(n-1)!} \left(\frac{\lambda}{2}\right)^{2n} \right.\right. \\
&\qquad \left.\left. + \frac{1}{2^{2n} n!(n-1)!} O(\lambda^{2n+2}) \right] H_n(\omega r) e^{in\theta} \right| \\
&\leq \sqrt{\frac{\delta}{\omega r}} \left(\frac{\lambda}{2}\right)^4 \sum_{|n|\geq 2} \frac{C|a_n|}{n!(n-1)!} \left(\frac{\lambda}{2}\right)^{2n-4} |H_n(\delta)|,
\end{aligned}$$

where $\delta > 0$ is a sufficiently small so that $\delta \leq \omega r_0$ for all ω (such a δ exists since ω is assumed bounded away from zero). In forming the inequality, we used the fact that, for $n \geq 1$, $t \mapsto \sqrt{t}|H_n(t)|$ is a decreasing function of positive t [Wat44, 13.74]. Finally, since $\|\{a_n\}\|_{h^\alpha} < \infty$ (for some real α) and

$$|H_n(\delta)| \leq C(n-1)!\left[\left(\frac{2}{\delta}\right)^n + \left(\frac{2}{\delta}\right)^{n-2}\right],^7$$

it follows that $\sum_{|n|\geq 2} \alpha_n H_n(\omega r)e^{in\theta} = O(\lambda^4)$, and the proof of (3.7) is complete.

In the case of a perfectly conducting disk, since $u^s_\rho\big|_{r=\rho} = -u^i\big|_{r=\rho}$,

$$\alpha_n = -a_n \frac{J_n(\lambda)}{H_n(\lambda)}.$$

Note that, as would be expected, this is the limit of (3.10) as $\operatorname{Im} q = \mu\sigma/\omega \to \infty$. Using (3.19) we can show

$$\sum_{|n|\geq 1} \alpha_n H_n(\omega r)e^{in\theta} = O(\lambda^2),$$

uniformly in x for $|x| \geq r_0$. Then since $J_0(\lambda) = 1 + O(\lambda^2)$ and

$$H_0(\lambda) = 1 + \frac{2i}{\pi}\big[\log(\lambda/2) + \gamma\big] + O(\lambda^2 \log \lambda),$$

we conclude

$$u^s_\rho(x) = \frac{-a_0}{1 + \frac{2i}{\pi}\big[\log(\lambda/2)+\gamma\big]} + O(\lambda^2),$$

where $O(\lambda^2)$ is bounded uniformly in x. $\square$

Remark 3.2. *The above proof implies the following bound in the case of a penetrable scatterer: there exists a* $\lambda_0 > 0$ *depending only on* $q_{\text{sup}} = \mu(\varepsilon + i \sup\{\sigma/\omega\})$ *such that for any* $\alpha \in \mathbb{R}$, $v \in \mathbb{R}$, $\rho > 0$, $r_0 \geq \rho$ *and* $\omega \geq c > 0$ *with* $\omega\rho \leq \lambda_0$, *we have*

$$\big\|u^s_\rho\big|_{r=r_0}\big\|_{H^v(\mathbb{T})} \leq C\big\|\{a_n\}\big\|_{h^\alpha}(\omega\rho)^2|H_0(\omega r_0)| \tag{3.24}$$

for some $C > 0$, *depending on* μ, ε, $\sup\{\sigma/\omega\}$, α, v *and* c, *but independent of* $\{a_n\}$,

[7] This follows from the fact that $H_n = J_n + iY_n$ and from the expansions (3.15) and (3.16).

ρ, r_0 and ω. If the permeability is constant in all of $\mathbb{R}^2$, i.e., if $\mu = 1$, then the above bound holds uniformly in $\omega > 0$. That is, we need not require that ω be bounded away from zero.[8] *This is because (3.20a) becomes: for $n \geq 1$,*

$$\alpha_n = a_n \frac{1}{n(n+1)} \left(\frac{\lambda}{2}\right)^{n+2} \frac{1}{n!} \{1 + \lambda^\zeta \mathcal{R}_n^1\} \frac{1}{H_n(\lambda)}.$$

Therefore, for all $\lambda > 0$ sufficiently small,

$$\begin{aligned}
\left\| u_\rho^s \big|_{r=r_0} \right\|_{H^v(\mathbb{T})}^2 &= 2\pi \sum_{n=-\infty}^{\infty} (1+|n|)^{2v} |\alpha_n|^2 |H_n(\omega r_0)|^2 \\
&= 2\pi |\alpha_0|^2 |H_0(\omega r_0)|^2 + 2\pi \sum_{|n|\geq 1} \left((1+|n|)^{2v} |\alpha_n H_n(\lambda)|^2 \left| \frac{H_n(\omega r_0)}{H_n(\omega\rho)} \right|^2 \right) \\
&\leq C\lambda^4 |a_0|^2 |H_0(\omega r_0)|^2 \\
&\quad + C\lambda^4 \sum_{|n|\geq 1} \left(\frac{|a_n|^2 (1+|n|)^{2v}}{n^2((n+1)!)^2} \left(\frac{\lambda}{2}\right)^{2(n-1)} |1 + \lambda^\zeta \mathcal{R}_n^1|^2 \left| \frac{H_n(\omega r_0)}{H_n(\omega\rho)} \right|^2 \right) \\
&\leq C\lambda^4 \left\| \{a_n\} \right\|_{h^\alpha}^2 \left(|H_0(\omega r_0)|^2 + \lambda^2 \frac{\rho}{r_0} \right), \\
&\leq C\lambda^4 \left\| \{a_n\} \right\|_{h^\alpha}^2 |H_0(\omega r_0)|^2,
\end{aligned}$$

where we have used the fact that $\left|1 + \lambda^\zeta \mathcal{R}_n^1\right|$ is bounded uniformly in n and λ (for λ sufficiently small), and the fact that, when $|n| \geq 1$, $|H_n(\omega r_0)|^2 / |H_n(\omega\rho)|^2 \leq \rho / r_0$ since $t \mapsto t|H_n(t)|^2$ is a decreasing function of positive t. We have also used the fact that

$$\begin{aligned}
\lambda^2 \frac{\rho}{r_0} &\leq \min\left\{ (\omega r_0)^2, \frac{1}{\omega r_0} \right\} \\
&\leq C|H_0(\omega r_0)|^2, \quad \text{for } 0 < \lambda < 1,\ \rho \leq r_0.
\end{aligned}$$

The requirement that λ_0 be sufficiently small is likely an artifact of the method that is not actually necessary for the bound to hold. We conjecture that, given a $\lambda_0 > 0$, there exists a C depending on μ, $q_{\sup}$ and λ_0 but independent of ρ, ω and $\{a_n\}$ such

[8] Though we still must require σ/ω be bounded, which means σ must approach zero as $\omega \to 0$. Without such a condition, the frequency dependent conductivity $\frac{\sigma}{\varepsilon\omega}$ could approach infinity. But the asymptotic size of the scattered field in the case of a perfect conductor is of a larger order (cf. Remark 3.3).

that for any $r_0 \geq \rho$,

$$\left\| u_\rho^s \big|_{r=r_0} \right\|_{L^2(\mathbb{T})} \leq C \frac{\sqrt{\rho}}{\sqrt{r_0}} \|\{a_n\}\|_{l^\infty}$$

for all $\lambda = \omega\rho \geq \lambda_0$ *(in the next chapter we will prove a weaker form of this). If this were the case, we would then know that the bound (3.24), when* $v \leq 0$ *and* $\alpha < -1/2$, *holds for any* $\lambda_0 > 0$, *with* C *depending on* λ_0. *The proof of this is immediate when we assume* ω *is bounded away from zero. When* $\mu = 1$ *and and* ω *is allowed to approach zero, the straightforward proof relies on the fact that* $t \mapsto \sqrt{t}|H_0(t)|$ *is a decreasing function of positive* t.

Remark 3.3. *For a perfectly conducting scatterer, we have the following bound: there exists a* $\lambda_0 > 0$ *such that for any* $\alpha \in \mathbb{R}$, $v \in \mathbb{R}$, $\rho > 0$, $r_0 \geq \rho$ *and* $\omega > 0$ *with* $\omega\rho \leq \lambda_0$, *we have*

$$\left\| u_\rho^s \big|_{r=r_0} \right\|_{H^v(\mathbb{T})} \leq C \left\| \{a_n\} \right\|_{h^\alpha} \frac{|H_0(\omega r_0)|}{1 + |\log(\omega\rho)|} \tag{3.25}$$

for some $C > 0$, *depending on* α *and* v *but independent of* $\{a_n\}$, ρ, r_0 *and* ω. *This is because*

$$\begin{aligned}
\left\| u_\rho^s \big|_{r=r_0} \right\|_{H^v(\mathbb{T})}^2 &= 2\pi \sum_{n=-\infty}^{\infty} (1+|n|)^{2v} |\alpha_n|^2 |H_n(\omega r_0)|^2 \\
&= 2\pi \sum_{n=-\infty}^{\infty} (1+|n|)^{2v} |a_n|^2 |J_n(\omega\rho)|^2 \frac{|H_n(\omega r_0)|^2}{|H_n(\omega\rho)|^2} \\
&\leq C|a_0|^2 |J_0(\omega\rho)|^2 \frac{|H_0(\omega r_0)|^2}{|H_0(\omega\rho)|^2} + C\frac{\rho}{r_0} \sum_{|n|\geq 1} (1+|n|)^{2v} |a_n|^2 |J_n(\omega\rho)|^2.
\end{aligned}$$

Here we have used the fact that $0 < t \mapsto t|H_n(t)|^2$ *is decreasing for integers* $n \neq 0$. *To estimate the* $n = 0$ *term, observe that given any* $\lambda_0 > 0$, *there exists a constant* C_{λ_0} *such that for* $0 < \lambda \leq \lambda_0$,

$$\frac{|J_0(\lambda)|^2}{|H_0(\lambda)|^2} \leq C_{\lambda_0} \frac{1}{(1+|\log\lambda|)^2}.$$

For the $n \neq 0$ *terms, observe that there exists a* $0 < t_0 < 1$ *independent of* n *such that* *for all* $|t| \le t_0$, $|J_n(t)| \le 2\frac{1}{n!}\left(\frac{1}{2}\right)^n$. *Therefore, there exists a* λ_{q0} *and a* $C_{\alpha,v}$ *such that for* $0 < \lambda \le \lambda_{q0}$,

$$\sum_{|n|\ge 1} (1+|n|)^{2v} |a_n|^2 |J_n(\omega\rho)|^2 \le C_{\alpha,v} \|\{a_n\}\|^2_{h^\alpha}.$$

To finish the proof, observe that

$$\frac{\rho}{r_0} = \frac{\omega\rho}{\omega r_0} \le \frac{|H_0(\omega r_0)|^2}{|H_0(\omega\rho)|^2}$$

since $0 < t \mapsto t|H_0(t)|^2$ *is increasing, and observe that there exists a constant* C *such that for* $0 < \omega\rho \le 1$,

$$\frac{|H_0(\omega r_0)|^2}{|H_0(\omega\rho)|^2} \le C\frac{|H_0(\omega r_0)|^2}{(1+|\log(\omega\rho)|)^2}.$$

This establishes (3.25). *In the next chapter we will prove that, given any* $\lambda_0 > 0$, *there exists a constant* $C = C(\lambda_0)$, *independent of* $\omega > 0$ *and* $\rho > 0$, *such that for any* $r_0 \ge \rho$,

$$\left\| u^s_\rho \big|_{r=r_0} \right\|_{L^2(\mathbb{T})} \le C\frac{\sqrt{\rho}}{\sqrt{r_0}} \|\{a_n\}\|_{l^\infty} \quad \text{for } \omega\rho \ge \lambda_0$$

(Theorem 7 of Chapter 4). From this it is not hard to see that when $v \le 0$ *and* $\alpha < -1/2$, (3.25) *holds for* any $\lambda_0 > 0$, *with* C *depending on* λ_0.

3.2.2 The case of a general simply connected domain

Assume D is a bounded, smooth domain. Let $\lambda = \rho\omega$ and define $\tilde{u}_\lambda(x) := u_\rho(\rho x)$. The subscript λ is warranted because $\tilde{u}_\lambda$ is the solution to the transmission problem

$$\begin{cases} \Delta\tilde{u}_\lambda + \lambda^2\tilde{u}_\lambda = 0 & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ \Delta\tilde{u}_\lambda + q\lambda^2\tilde{u}_\lambda = 0 & \text{in } D, \\ u^-_\lambda = u^+_\lambda & \text{on } \partial D, \\ \frac{1}{\mu}\partial^-_\nu \tilde{u}_\lambda = \partial^+_\nu \tilde{u}_\lambda & \text{on } \partial D, \end{cases} \tag{3.26a}$$

along with the radiation condition

$$|(\partial_r - i\lambda)\tilde{u}^s_\lambda| = O(r^{-3/2}) \quad \text{as } r \to \infty, \tag{3.26b}$$

where

$$\tilde{u}_\lambda = \begin{cases} \tilde{u}^i_\lambda + \tilde{u}^s_\lambda & \text{in } \mathbb{R}^2 \setminus D, \\ \tilde{u}^{tr}_\lambda & \text{in } D, \end{cases}$$

with

$$\tilde{u}^i_\lambda(x) = u^i(\rho x) = \sum_{n=-\infty}^{\infty} a_n J_n(\lambda r) e^{in\theta}.$$

For simplicity, we assume the incident wave is a plane wave in the direction η, so that

$$\tilde{u}^i_\lambda(x) = e^{i\lambda\eta\cdot x}.$$

Let $\Phi^\lambda(x,y)$ denote the free-space Green's function for the Helmholtz operator $\Delta + \lambda^2$ satisfying the outgoing radiation condition:

$$\begin{cases} (\Delta_x + \lambda^2)\Phi^\lambda(x,y) = -\delta_y(x), \\ \left(\frac{\partial}{\partial|x|} - i\lambda\right)\Phi^\lambda(x,y) = O(|x|^{-3/2}) \quad \text{as } |x| \to \infty \text{ for fixed } y. \end{cases}$$

One can easily verify that

$$\Phi^\lambda(x,y) = \frac{i}{4} H^{(1)}(|x-y|),$$

where $H^{(1)}$ is the Hankel function of the first kind. Owing to the radiation condition, we have the representation formula

$$\tilde{u}^s_\lambda(x) = \int_{\partial D} \left\{ \tilde{u}^s_\lambda(y)\partial_{\nu_y}\Phi^\lambda(x,y) - \partial_\nu \tilde{u}^s_\lambda(y)\Phi^\lambda(x,y) \right\} \mathrm{d}\sigma_y, \quad x \in \mathbb{R}^2 \setminus \overline{D}, \tag{3.27}$$

or, equivalently,

$$\tilde{u}_\lambda(x) = \tilde{u}^i_\lambda(x) + \int_{\partial D} \Big\{ \tilde{u}_\lambda(y) \partial_{\nu_y} \Phi^\lambda(x,y) - \partial^+_\nu \tilde{u}_\lambda(y) \Phi^\lambda(x,y) \Big\} \, \mathrm{d}\sigma_y, \quad x \in \mathbb{R}^2 \setminus \overline{D}.$$

We use the transmission condition for the normal derivative and then perform an integration by parts to get

$$\begin{aligned} \tilde{u}_\lambda(x) = \tilde{u}^i_\lambda(x) + \int_D \Big\{ \tilde{u}_\lambda(y) \Delta \Phi^\lambda(x,y) - \Delta \tilde{u}_\lambda(y) \Phi^\lambda(x,y) \Big\} \, \mathrm{d}y \\ + \Big(1 - \frac{1}{\mu}\Big) \int_{\partial D} \partial^-_\nu \tilde{u}_\lambda(y) \Phi^\lambda(x,y) \, \mathrm{d}\sigma_y, \end{aligned}$$

which simplifies as

$$\begin{aligned} \tilde{u}_\lambda(x) = \tilde{u}^i_\lambda(x) + \lambda^2 (q-1) \int_D \tilde{u}_\lambda(y) \Phi^\lambda(x,y) \, \mathrm{d}y \\ + \Big(1 - \frac{1}{\mu}\Big) \int_{\partial D} \partial^-_\nu \tilde{u}_\lambda(y) \Phi^\lambda(x,y) \, \mathrm{d}\sigma_y. \end{aligned} \tag{3.28}$$

A simple calculation verifies that this formula is in fact valid in the entire plane. Let

$$\begin{aligned} \mathcal{U}(x) = \tilde{u}^i_\lambda(x) + \lambda^2 (q-1) \frac{1}{\mu} \int_D \tilde{u}_\lambda(y) \Phi^\lambda(x,y) \, \mathrm{d}y \\ + \Big(1 - \frac{1}{\mu}\Big) \int_{\partial D} \tilde{u}_\lambda(y) \partial_{\nu_y} \Phi^\lambda(x,y) \, \mathrm{d}\sigma_y. \end{aligned}$$

Using (3.28) and the jump property of the double layer potential, we find

$$\mathcal{U} = \begin{cases} \frac{1}{\mu} \tilde{u}_\lambda & \text{in } D, \\ \tilde{u}_\lambda & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ \frac{1}{2}\big(1 + \frac{1}{\mu}\big) \tilde{u}_\lambda & \text{on } \partial D. \end{cases} \tag{3.29}$$

3.2.3 Low frequency: $\lambda \to 0$

The main result of this section will be a formal asymptotic approximation of the scattered field: for ρ and $\lambda = \omega\rho$ small[9],

$$
\begin{aligned}
u^s_\rho(x) \approx \rho^2 |D| \Big(1 - \frac{1}{\mu}\Big) \nabla_y \Phi^\omega(x,0) \cdot \big[M \nabla u^i(0)\big] \\
+ \rho^2 \omega^2 \Big(\varepsilon + i\frac{\sigma}{\omega} - 1\Big) |D| \Phi^\omega(x,0) u^i(0). \quad (3.30)
\end{aligned}
$$

Such a formula could be rigorously justified using the technique of [AIM03]. We present a formal derivation here to unify the technique for the low frequency regime with those of the moderate and high frequency regimes, which will be discussed in subsequent sections. We begin by expressing

$$
\tilde{u}^i_\lambda = e^{i\lambda\eta\cdot x} = 1 + i\lambda\eta\cdot x - \lambda^2(\eta\cdot x)^2 + \cdots, \quad (3.31)
$$

$$
\begin{aligned}
\Phi^\lambda(x,y) &= \frac{i}{4} H^{(1)}_0(\lambda|x-y|) \\
&= -\frac{1}{2\pi}\log\lambda + \Big[\Big(\frac{i}{4} - \frac{\gamma}{2\pi} + \frac{1}{2\pi}\log 2\Big) + \Phi_0(x,y)\Big] \quad (3.32) \\
&\qquad + \frac{1}{2\pi}|x-y|^2 \lambda^2 \log\lambda + O(\lambda^2)
\end{aligned}
$$

and

$$
\partial_{\nu_y}\Phi^\lambda(x,y) = \partial_{\nu_y}\Phi_0(x,y) + \frac{1}{4\pi}\big\{(y-x)\cdot\nu_y\big\}\lambda^2\log\lambda + O(\lambda^2). \quad (3.33)
$$

The remainder is $O(\lambda^2)$ uniformly in x and y when restricted to bounded sets. γ denotes Euler's constant and

$$
\Phi_0(x,y) = -\frac{1}{2\pi}\log|x-y|
$$

[9] Note that is includes the possibility that $\omega \to \infty$ as $\rho \to 0$, so long as $\rho \ll \omega^{-1}$.

is the (radial) fundamental solution of Laplace's equation:

$$\Delta_x \Phi_0(x,y) = -\delta_y(x).$$

Formally expand

$$\tilde{u}_\lambda^{tr} = \tilde{u}_0^{tr} + i\lambda\tilde{u}_1^{tr} - \lambda^2\tilde{u}_2^{tr} + \cdots, \tag{3.34}$$

$$\tilde{u}_\lambda^{s} = \tilde{u}_0^{s} + i\lambda\tilde{u}_1^{s} - \lambda^2\tilde{u}_2^{s} + \cdots, \tag{3.35}$$

and define the coefficients $\tilde{u}_j$ for the total field in the obvious way, $\tilde{u}_j := \tilde{u}_j^i + \tilde{u}_j^s$ in $\mathbb{R}^2 \setminus \overline{D}$ and $\tilde{u}_j := \tilde{u}_j^{tr}$ in D. Then insert these expansions into (3.26a) and collect coefficients. Assuming

$$\frac{\sigma}{\omega} \ll \frac{1}{\lambda^2} \quad \Big(\text{i.e., } \sigma \ll \frac{1}{\omega\rho^2}\Big), \tag{3.36}$$

$\tilde{u}_0$ solves the transmission problem

$$\begin{cases} \Delta\tilde{u}_0 = 0 & \text{in } D \text{ and } \mathbb{R}^2 \setminus \overline{D}, \\ \tilde{u}_0^- = \tilde{u}_0^+ & \text{on } \partial D, \\ \dfrac{1}{\mu}\partial_\nu^- \tilde{u}_0 = \partial_\nu^+ \tilde{u}_0 & \text{on } \partial D, \\ |\tilde{u}_0(x) - 1| \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{3.37}$$

The asymptotic condition follows by substituting the expansions (3.31), (3.32), (3.33) and the formal expansion of $\tilde{u}_\lambda$ into (3.29) and then collecting coefficients. In fact, the resulting integral equation[10] for $\tilde{u}_0$ implies all of (3.37)—the transmission condition for the flux follows from the fact that the normal derivative of the double layer potential does not "jump" [CK83]. $\tilde{u}_0 \equiv 1$ uniquely solves (3.37) since the zero function is the

[10]The integral equation is $\tilde{u}_0(x) = 1 + \left(1 - \frac{1}{\mu}\right)\int_{\partial D} \partial_\nu^- \tilde{u}_0(y)\Phi_0(x,y)\,\mathrm{d}\sigma_y$.

unique solution to

$$\begin{cases} \Delta w = 0 & \text{in } D \text{ and } \mathbb{R}^2 \setminus \overline{D}, \\ w^- = w^+ & \text{on } \partial D, \\ \dfrac{1}{\mu}\partial_\nu^- w = \partial_\nu^+ w & \text{on } \partial D, \\ |w(x)| \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{3.38}$$

Uniqueness follows from the fact that w, being harmonic at ∞, satisfies $|\partial_r w(x)| = O\big(|x|^{-2}\big)$ as $|x| \to \infty$ [Fol95, Prop. 2.75]: for then

$$\int_{\partial B(0,R)} \big\{\partial_\nu w(y)\Phi_0(x,y) - w(y)\partial_{\nu_y}\Phi_0(x,y)\big\}\, \mathrm{d}\sigma_y \longrightarrow 0 \quad \text{as } R \to \infty$$

for any $x \in \mathbb{R}^2 \setminus \overline{D}$, and consequently,

$$\begin{aligned} w(x) &= -\int_{\partial D} \Big\{\partial_\nu^+ w(y)\Phi_0(x,y) - w(y)\partial_{\nu_y}\Phi_0(x,y)\Big\}\, \mathrm{d}\sigma_y \\ &= -w(x) - \int_{\partial D} \Big\{\partial_\nu^- w(y)\Phi_0(x,y) - w(y)\partial_{\nu_y}\Phi_0(x,y)\Big\}\, \mathrm{d}\sigma_y \\ &= -w(x). \end{aligned}$$

$\tilde{u}_1$ is the unique solution to the transmission problem

$$\begin{cases} \Delta w = 0 & \text{in } D \text{ and } \mathbb{R}^2 \setminus \overline{D}, \\ w^- = w^+ & \text{on } \partial D, \\ \dfrac{1}{\mu}\partial_\nu^- w = \partial_\nu^+ w & \text{on } \partial D, \\ |w(x) - x \cdot \eta| \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{3.39}$$

Uniqueness is a consequence of the the fact that (3.38) has only the trivial solution. The existence of a solution can be easily demonstrated:

$$w(x) = (\mathcal{S}_D\phi)(x) + \eta \cdot x,$$

with $\phi \in L^2_\diamond(\partial D) := \left\{u \in L^2(\partial D) : \int_{\partial D} u \,\mathrm{d}\sigma = 0\right\}$ satisfying the integral equation

$$\frac{1}{2}\Big(1+\frac{1}{\mu}\Big)\phi(y) + \Big(1-\frac{1}{\mu}\Big)\int_{\partial D} \phi(z)\partial_{\nu_z}\Phi_0(y,z)\,\mathrm{d}\sigma_z = \Big(\frac{1}{\mu}-1\Big)\eta\cdot\nu_y. \tag{3.40}$$

(Here $\mathcal{S}_D\phi$ is the single layer potential: $(\mathcal{S}_D\phi)(x) = \int_{\partial D}\phi(z)\Phi_0(y,z)\,\mathrm{d}\sigma_z$.) The Fredholm alternative guarantees the unique existence of such a ϕ since a nonzero solution to the homogeneous form of (3.40) would provide a nonzero[11] solution to (3.38), namely $w = \mathcal{S}_D\phi$.

Equivalently, $\tilde{u}_1 = \mathbf{w}\cdot\eta$, where $\mathbf{w} := (w_1, w_2)$ with w_j denoting the unique solution to the transmission problem

$$\begin{cases} \Delta w_j = 0 & \text{in } D \text{ and } \mathbb{R}^2\setminus\overline{D}, \\ {w_j}^- = {w_j}^+ & \text{on } \partial D, \\ \frac{1}{\mu}\partial_\nu^- w_j = \partial_\nu^+ w_j & \text{on } \partial D, \\ |w_j(x) - x_j| \to 0 & \text{as } |x|\to\infty. \end{cases} \tag{3.41}$$

Assuming $x \in \mathbb{R}^2\setminus\overline{D}$ (therefore bounded away from zero) and $y \in \partial D$,

$$\begin{aligned} \Phi^\lambda(x/\rho, y) &= \frac{i}{4}H_0^{(1)}\Big(\lambda\Big|\frac{x}{\rho}-y\Big|\Big) \\ &= \frac{i}{4}H_0^{(1)}(\omega|x|+\delta) \\ &= \Phi^\omega(x,0) + O(\lambda), \end{aligned} \tag{3.42}$$

where $\delta = -\lambda\hat{x}\cdot y + O(\rho\lambda/|x|)$, and

$$\begin{aligned} \partial_{\nu_y}\Phi^\lambda(x/\rho, y) &= \frac{i\lambda}{4}\frac{x-\rho y}{|x-\rho y|}\cdot\nu_y\; H_1^{(1)}(\omega|x-\rho y|) \\ &= \frac{i\lambda}{4|x|}\Big\{1+\rho\frac{\hat{x}\cdot y}{|x|} + O(\rho^2/|x|^2)\Big\}(x-\rho y)\cdot\nu_y\; H_1^{(1)}(\omega|x|+\delta). \end{aligned}$$

[11]Nonzero since $\phi = \partial_\nu^-(\mathcal{S}_D\phi) - \partial_\nu^+(\mathcal{S}_D\phi)$.

With the aid of the identity

$$\big(H_1^{(1)}\big)'(z) = H_0^{(1)}(z) - \frac{1}{z}H_1^{(1)}(z),$$

we see that

$$\begin{aligned}
H_1^{(1)}(\omega|x|+\delta) &= H_1^{(1)}(\omega|x|) + \delta\big(H_1^{(1)}\big)'(\omega|x|) + \delta^2\big(H_1^{(1)}\big)''(\omega|x|+\delta') \\
&= \Big(1 - \frac{\delta}{\omega|x|}\Big)H_1^{(1)}(\omega|x|) + \delta H_0^{(1)}(\omega|x|) + \delta^2\big(H_1^{(1)}\big)''(\omega|x|+\delta')
\end{aligned}$$

for some δ' between zero and δ. Since

$$\big(H_1^{(1)}\big)''(z) = \Big(\frac{2}{z^2} - 1\Big)H_1^{(1)}(z) - \frac{1}{z}H_0^{(1)}(z),$$

$\big(H_1^{(1)}\big)''(\omega|x|+\delta') = O(1)$ as $\lambda \to 0$. Thus

$$H_1^{(1)}(\omega|x|+\delta) = \Big(1 + \frac{\lambda}{\omega|x|}\hat{x}\cdot y\Big)H_1^{(1)}(\omega|x|) - \lambda\hat{x}\cdot y H_0^{(1)}(\omega|x|) + O(\lambda^2),$$

and so

$$\begin{aligned}
\partial_{\nu_y}\Phi^\lambda(x/\rho, y) &= \frac{i\lambda}{4}\Big\{H_1^{(1)}(\omega|x|)\hat{x}\cdot\nu_y - \frac{\rho}{|x|}H_1^{(1)}(\omega|x|)y\cdot\nu_y \\
&\quad + 2\frac{\rho}{|x|^2}H_1^{(1)}(\omega|x|)\hat{x}\cdot y - \lambda H_0^{(1)}(\omega|x|)(\hat{x}\cdot y)(\hat{x}\cdot\nu_y)\Big\} + O(\lambda^3) \\
&= \rho\nabla_y\Phi^\omega(x,0)\cdot\nu_y - \lambda^2\Phi^\omega(x,0)(\hat{x}\cdot y)(\hat{x}\cdot\nu_y) \\
&\quad + \rho\lambda\Big\{-\frac{1}{|x|}H_1^{(1)}(\omega|x|)y\cdot\nu_y + 2\frac{1}{|x|^2}H_1^{(1)}(\omega|x|)\hat{x}\cdot y\Big\} + O(\lambda^3). \quad (3.43)
\end{aligned}$$

We now insert (3.42), (3.43) and

$$\tilde{u}_\lambda(y) \asymp 1 + i\lambda\mathbf{w}(y)\cdot\eta + O(\lambda^3)$$ [12]

into the integral representation formula (3.29) for u_ρ^s, which we restate here:

$$u_\rho^s(x) = \tilde{u}_\lambda^s(x/\rho) = \lambda^2(q-1)\frac{1}{\mu}\int_D \tilde{u}_\lambda(y)\Phi^\lambda(x/\rho,y)\,\mathrm{d}y + \Big(1-\frac{1}{\mu}\Big)\int_{\partial D}\tilde{u}_\lambda(y)\partial_{\nu_y}\Phi^\lambda(x/\rho,y)\,\mathrm{d}\sigma_y.$$

Noting that the integral over ∂D of the expression from (3.43) within the curly braces is zero, we find

$$\begin{aligned} u_\rho^s(x) &\asymp \lambda^2(q-1)\frac{1}{\mu}|D|\Phi^\omega(x,0) - \lambda^2\Big(1-\frac{1}{\mu}\Big)|D|\Phi^\omega(x,0) \\ &\quad + \rho^2\Big(1-\frac{1}{\mu}\Big)\int_{\partial D}\big(\nabla u^i(0)\cdot\mathbf{w}(y)\big)\big(\nabla_y\Phi^\omega(x,0)\cdot\nu_y\big)\,\mathrm{d}\sigma_y\Big\} + O(\lambda^3) \\ &= \rho^2\omega^2\Big(\frac{q}{\mu}-1\Big)|D|\Phi^\omega(x,0) \\ &\quad + \rho^2\Big(1-\frac{1}{\mu}\Big)\nabla_y\Phi^\omega(x,0)\cdot\bigg(\Big[\int_D D\mathbf{w}\,\mathrm{d}y\Big]^T\nabla u^i(0)\bigg) + O(\lambda^3), \end{aligned}$$

or

$$u_\rho^s(x) \asymp |\rho D|\Big\{\omega^2\big\{(\varepsilon + i\sigma/\omega)-1\big\}\Phi^\omega(x,0)u^i(0) + \Big(1-\frac{1}{\mu}\Big)\nabla_y\Phi^\omega(x,0)\cdot\big(M\nabla u^i(0)\big)\Big\} + O(\lambda^3), \quad (3.44a)$$

where the polarization tensor M has entries

$$M_{ij} = |D|^{-1}\int_{\partial D}\nu_i w_j\,\mathrm{d}\sigma = |D|^{-1}\int_D \partial_{y_i} w_j\,\mathrm{d}y. \quad (3.44b)$$

In the case of fixed frequency and zero conductivity, this is the same asymptotic formula as that found in [AIM03] (see also [VV00] for essentially the same formula in the case of a bounded domain and fixed frequency). Formula (3.44) serves as a good approximation only as long as the scatterer is not too highly conducting, as noted in (3.36). For very highly absorbing scatterers, we expect the intensity of the scattered field to decrease on the order of $(\log\rho)^{-1}$ since this is the case when the scatterer is perfectly conducting.

[12] We use the symbol $\asymp$ instead of $\sim$ to indicate that the asymptotic expansion is merely formal.

Remark 3.4. *When $D = B(0,1)$, formula (3.44) does reduce to the form we had rigorously verified in Theorem 4. For in this case, a simple calculation shows that*[13]

$$\mathbf{w} = \begin{cases} \frac{2}{1+\frac{1}{\mu}} x & \textit{in } B(0,1), \\ \frac{2}{1+\frac{1}{\mu}} \frac{x}{|x|^2} + x & \textit{in } \mathbb{R}^2 \setminus \overline{B(0,1)}. \end{cases}$$

Consequently, $M = 2\left(1 + \frac{1}{\mu}\right)^{-1} \mathbf{I}^{2\times 2}$.

3.2.4 Comparison with an approximation formula of Jones

The method used by Jones [Jon79, Jon86] to derive a similar approximation formula in the case of three dimensional acoustic scattering from a penetrable obstacle is as follows: assume $\tilde{u}_\lambda$ solves

$$\begin{cases} \Delta \tilde{u}_\lambda + \lambda^2 \tilde{u}_\lambda = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ \Delta \tilde{u}_\lambda + q\lambda^2 \tilde{u}_\lambda = 0 & \text{in } D, \\ u_\lambda^- = u_\lambda^+ & \text{on } \partial D, \\ \frac{1}{\mu} \partial_\nu^- \tilde{u}_\lambda = \partial_\nu^+ \tilde{u}_\lambda & \text{on } \partial D, \end{cases} \tag{3.45a}$$

along with the radiation condition

$$|(\partial_r - i\lambda)\tilde{u}_\lambda^s| = O(r^{-2}) \quad \text{as } r \to \infty. \tag{3.45b}$$

Substitute

$$\Phi^\lambda(x,y) = \frac{e^{i\lambda|x-y|}}{4\pi|x-y|} = \frac{e^{i\lambda(r - \hat{x}\cdot y)}}{4\pi r} + O(\lambda r^{-2}) \quad \text{as } r \to \infty$$

into the approximation formula (3.28) to get

$$\tilde{u}_\lambda^s(x) \approx \frac{e^{i\lambda r}}{4\pi r} \left[\lambda^2 (q-1) \int_D \tilde{u}_\lambda(y) e^{-i\lambda \hat{x}\cdot y} \, \mathrm{d}y + \left(1 - \frac{1}{\mu}\right) \int_{\partial D} \partial_\nu^- \tilde{u}_\lambda(y) e^{-i\lambda\hat{x}\cdot y} \, \mathrm{d}\sigma_y \right],$$

---

[13] This was observed in [CFMV98] in the context of the conductivity problem.

which should be valid for large r. An application of Green's theorem yields the far field pattern

$$\tilde{u}^s_\lambda(x) \approx \frac{e^{i\lambda r}}{4\pi r}\Bigg[\lambda^2\Big(\frac{q}{\mu}-1\Big)\int_D \tilde{u}_\lambda(y)e^{-i\lambda\hat{x}\cdot y}\,\mathrm{d}y \\ - i\lambda\Big(1-\frac{1}{\mu}\Big)\hat{x}\cdot\int_D \nabla\tilde{u}_\lambda(y)e^{-i\lambda\hat{x}\cdot y}\,\mathrm{d}y\Bigg]. \quad (3.46)$$

Formally expand

$$\tilde{u}_\lambda = \tilde{u}_0 + i\lambda\tilde{u}_1 - \lambda^2\tilde{u}_2 + \cdots ,$$

and note also that

$$\tilde{u}^i_\lambda = e^{i\lambda\eta\cdot x} = 1 + i\lambda\eta\cdot x - \lambda^2(\eta\cdot x)^2 + \cdots$$

and

$$\Phi^\lambda(x,y) = \Phi_0(x,y) + \frac{i\lambda}{4\pi} + O(\lambda^2) \quad (3.47)$$

as $\lambda \to 0$, where $\Phi_0(x,y) = \frac{1}{4\pi}|x-y|^{-1}$ is the free-space Green's function for Laplace's equation:

$$\Delta_x\Phi_0(x,y) = -\delta_y(x).$$

Inserting these expansions into (3.28) and equating coefficients gives

$$\begin{aligned}
\tilde{u}_0(x) &= 1 + \Big(1-\frac{1}{\mu}\Big)\int_{\partial D}\partial_\nu^-\tilde{u}_0(y)\Phi_0(x,y)\,\mathrm{d}\sigma_y,\\
\tilde{u}_1(x) &= \eta\cdot x + \Big(1-\frac{1}{\mu}\Big)\int_{\partial D}\Big\{\partial_\nu^-\tilde{u}_1(y)\Phi_0(x,y) + \frac{1}{4\pi}\partial_\nu^-\tilde{u}_0(y)\Big\}\,\mathrm{d}\sigma_y.
\end{aligned}$$

Jones notes that these formulas necessitate that $\tilde{u}_0 \equiv 1$ and $\tilde{u}_1 = \mathbf{w}\cdot\eta$, where $\mathbf{w} = (w_1,w_2,w_3)$ is defined just as it was in the two dimensional case. Inserting $\tilde{u}_\lambda \approx$

$1 + i\lambda\mathbf{w} \cdot \eta$ into the far field pattern (3.46) yields the far field approximation formula

$$\tilde{u}^s_\lambda(x) \approx \lambda^2 \frac{e^{i\lambda r}}{4\pi r} \Big\{ \Big(\frac{q}{\mu} - 1\Big)|D| + \Big(1 - \frac{1}{\mu}\Big)\, \hat{x} \cdot \Big(\Big[\int_D D\mathbf{w}\, \mathrm{d}y \Big]^T \eta \Big) \Big\}.$$

This is essentially the formula derived in [Jon79, Jon86], though Jones expresses it in terms of the content matrix $\mathbf{C} := \int_D D\mathbf{t}\, \mathrm{d}y$, where $\mathbf{w} = x + \big(1 - \frac{1}{\mu}\big)\mathbf{t}$, instead of the polarization tensor M. There it is asserted that this formula is applicable in the regime of Rayleigh scattering, where $\lambda\, \mathrm{diam}(D) \ll 1$. To make this precise, we assume that $\tilde{u}^s_\lambda$ is a rescaling just as before, i.e. $\lambda = \rho\omega$ and $\tilde{u}_\lambda(x) := u_\rho(\rho x)$, so that the formula becomes

$$\begin{aligned} u^s_\rho(x) = \tilde{u}^s_\lambda(x/\rho) \approx \rho^3\omega^2 \Big(\frac{q}{\mu} - 1\Big)|D|\Phi^\omega(x,0) \\ + \rho^3\Big(1 - \frac{1}{\mu}\Big)\nabla_y \Phi^\omega(x,0) \cdot \Big(\Big[\int_D D\mathbf{w}\, \mathrm{d}y \Big]^T \nabla u^i(0) \Big), \end{aligned}$$

or

$$u^s_\rho(x) \approx |\rho D| \Big\{ \omega^2 \Big(\frac{q}{\mu} - 1\Big)\Phi^\omega(x,0) + \Big(1 - \frac{1}{\mu}\Big)\nabla_y \Phi^\omega(x,0) \cdot \big(M \nabla u^i(0)\big) \Big\},$$

with

$$M_{ij} = |D|^{-1} \int_D \partial_{y_i} w_j\, \mathrm{d}y.$$

The method used to derive this approximation suggests it is valid as long as $\lambda \ll 1$ and $r/\rho \gg 1$. However, the requirement that $r/\rho \gg 1$ is unnecessary, as it was unnecessary for the analogue in two dimensions, namely (3.44). In other words, the method used to derive (3.44) applies just as well in three dimension (or better, since $\Phi^\lambda(x,y) := e^{i\lambda|x-y|}/|x-y|$ is then analytic in λ).

3.2.5 Moderate frequency: $\lambda \to \lambda_0$

Now we examine the case where, as $\rho \to 0$, ω grows so that $\lambda = \rho\omega \to \lambda_0$ for some $0 < \lambda_0 < \infty$. Assuming $x \in \mathbb{R}^2 \setminus \overline{D}$ (and therefore bounded away from zero) and

$y \in \partial D$,

$$\begin{aligned}\Phi^\lambda(x/\rho, y) &= \frac{i}{4} H_0^{(1)}\left(\lambda\left|\frac{x}{\rho} - y\right|\right) \\ &= \frac{i}{4} H_0^{(1)}(\omega|x| - \lambda\hat{x}\cdot y + O(\rho)) \\ &\approx \sqrt{\rho}\sqrt{\frac{1}{8\pi\lambda_0|x|}} e^{i(\omega|x| - \lambda_0\hat{x}\cdot y + \pi/4)},\end{aligned}$$

and

$$\begin{aligned}\partial_{\nu_y}\Phi^\lambda(x/\rho, y) &= \frac{i\lambda}{4}\frac{x-\rho y}{|x-\rho y|}\cdot \nu_y\, H_1^{(1)}(\omega|x-\rho y|) \\ &\approx \sqrt{\rho}\,\hat{x}\cdot\nu_y\sqrt{\frac{\lambda_0}{8\pi|x|}} e^{i(\omega|x| - \lambda_0\hat{x}\cdot y - \pi/4)},\end{aligned}$$

where we have used the well known asymptotic formula for the Hankel function of large argument:

$$H_n^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{i(z - n\pi/2 - \pi/4)}.$$

Noting that $\tilde{u}_\lambda$ is continuous in λ, we arrive at the approximation

$$\begin{aligned}u_\rho^s(x) &= \int_{\partial D}\left\{\tilde{u}_\lambda^s(y)\partial_{\nu_y}\Phi^\lambda(x/\rho, y) - \partial_\nu \tilde{u}_\lambda^s(y)\Phi^\lambda(x/\rho, y)\right\} \mathrm{d}\sigma_y \\ &\approx \sqrt{\rho}\sqrt{\frac{\lambda_0}{8\pi|x|}} e^{i(\omega|x|-\pi/4)} \int_{\partial D}\left\{\hat{x}\cdot\nu_y\,\tilde{u}_{\lambda_0}^s(y) - \frac{i}{\lambda_0}\partial_\nu\tilde{u}_{\lambda_0}^s(y)\right\} e^{-i\lambda_0\hat{x}\cdot y}\,\mathrm{d}\sigma_y. \quad (3.48)\end{aligned}$$

This asymptotic behavior, where the size of the scattered field is on the order of $\sqrt{\rho}$ when ω is on the order of ρ^{-1} as $\rho \to 0$, is consistent with Figure 3.1, as the height of the peaks of the plots decreases at roughly the rate $\sqrt{\rho}$, for $\rho = 0.01$, 0.004 and 0.001. Recall that the formula (3.44) for the scattered field has a magnitude on the order of ρ^2 as $\rho \to 0$ for fixed ω. If we take into account ω when estimating the size of formula (3.44), we find that it is an estimate of order $\omega^{3/2}\rho^2$, since $\Phi^\omega(x, 0)$ is of order $1/\sqrt{\omega}$ and $\nabla_y\Phi^\omega(x, 0)$ is of order $\sqrt{\omega}$. If we let $\omega = \rho^{-\alpha}$, $0 < \alpha < 1$, (3.44) is therefore of order $\rho^{2-\frac{3}{2}\alpha}$, which transitions smoothly between ρ^2 and $\sqrt{\rho}$ as α goes from

0 to 1. Thus we have consistency between our low frequency and moderate frequency approximations—at least in terms of their magnitudes.

3.3 High frequency: $\lambda \to \infty$

Here we must distinguish among different possible rates of growth of ω relative to ρ^{-1} as $\rho \to 0$. Listed in increasing orders of frequency, these distinct regimes are

$$\begin{aligned} \rho\omega \to \infty \quad &\text{but} \quad \rho^2\omega \to 0, \\ \rho^2\omega \to \infty \quad &\text{but} \quad \rho^3\omega \to 0, \\ \rho^3\omega \to \infty \quad &\text{but} \quad \rho^4\omega \to 0, \end{aligned}$$

and so on. We will derive an approximation that is applicable in lowest regime, but our method can be easily modified for any higher order. One could derive a high frequency approximation formula by letting λ_0 tend to ∞ in (3.48) and performing a stationary phase analysis on the boundary integral. Such an approach does in fact yield a good approximation in the regime where $\rho\omega \to \infty$ but $\rho^2\omega \to 0$. However, to better understand this limitation and to better understand how to derive accurate approximations for higher frequencies, we take a different approach. Our approach will be to first find approximations of $\tilde{u}_\lambda$ and $\partial_\nu \tilde{u}_\lambda$ on ∂D using the technique of geometric optics [Lun64, BW02, FK55], and to then approximate the limit as $\lambda \to \infty$ of

$$u^s_\rho(x) = \int_{\partial D} \left\{ \tilde{u}^s_\lambda(y) \partial_{\nu_y} \Phi^\lambda(x/\rho, y) - \partial_\nu \tilde{u}^s_\lambda(y) \Phi^\lambda(x/\rho, y) \right\} \mathrm{d}\sigma_y$$

by using the method of stationary phase. Our analysis will not account for internal reflections of the transmitted field, and will thus only apply to well absorbing scatterers. Furthermore, to avoid trapping effects we will consider only convex scatterers. A similar approach to ours appears in [Bru03] (see also [BGMR04, BGR05]), but there the goal is a fast reconstruction algorithm, not a representation formula. Future work may integrate our approach with the scheme for dealing with multiple scattering effects presented in [BGR05] to construct a high frequency representation formula that is valid for scatterers that are neither convex nor well conducting.

3.3.1 The case of a half-space

Before we begin our formal asymptotic analysis, we will examine the special case of a plane wave incident upon a half-space composed of a homogenous, isotropic material distinct from the background medium. This example will serve as a guide in the coming analysis.

Let $D = \mathbb{R}^2_+$ and suppose $\eta_2 > 0$, so that u^i is incident upon the interface $x_2 = 0$ from the lower half plane. We assume u^s is a reflected plane wave of the form

$$u^s(x) = A^s e^{i\omega x\cdot\eta^s}$$

with $\eta^s = (\eta_1, -\eta_2)$, and also that for some possibly complex "direction" vector ξ satisfying $\xi \cdot \xi = 1$,

$$u^{tr}(x) = A^{tr} e^{i\omega\sqrt{q}x\cdot\xi}.$$

At the interface $x_2 = 0$ we have the transmission conditions

$$\begin{aligned} A^{tr} e^{i\omega\sqrt{q}x_1\xi_1} &= (1 + A^s)e^{i\omega x_1\eta_1}, \\ \frac{1}{\mu} i\omega\sqrt{q}\xi_2 A^{tr} e^{i\omega\sqrt{q}x_1\xi_1} &= i\omega\eta_2(1 - A^s)e^{i\omega x_1\eta_1}. \end{aligned}$$

In order for these equations to hold, it is necessary that

$$\begin{aligned} \operatorname{Re}\sqrt{q}\operatorname{Re}\xi_1 - \operatorname{Im}\sqrt{q}\operatorname{Im}\xi_1 &= \eta_1, \\ \operatorname{Re}\sqrt{q}\operatorname{Im}\xi_1 + \operatorname{Im}\sqrt{q}\operatorname{Re}\xi_1 &= 0, \end{aligned}$$

which in turn imply

$$\begin{aligned} A^{tr} &= 1 + A^s, \\ \frac{1}{\mu}\sqrt{q}\xi_2 A^{tr} &= \eta_2(1 - A^s). \end{aligned} \tag{3.49}$$

We therefore find $\xi_1 = \eta_1\overline{\sqrt{q}}/|q|$. For $\xi_2 = \pm\sqrt{1-\xi_1^2} = \pm\frac{1}{\sqrt{q}}\sqrt{q-\eta_1^2}$, we choose the sign so that the intensity of the transmitted field,

$$|u^{tr}(x)| = |A^{tr}|e^{-\omega\,\mathrm{Im}(\sqrt{q}\xi_2)x_2} = |A^{tr}|e^{\mp\omega x_2\,\mathrm{Im}\sqrt{q-\eta_1^2}},$$

will decay, rather than grow exponentially, as it proceeds farther into the medium when $\sigma > 0$; that is, we choose $\xi_2 = \frac{1}{\sqrt{q}}\sqrt{q-\eta_1^2}$. Solving (3.49) leads to the solution [Jon86, §6.5]

$$u^s(x) = A^s e^{i\omega x\cdot\eta^s}, \tag{3.50a}$$

$$u^{tr}(x) = A^{tr}e^{-\omega x_2\,\mathrm{Im}\sqrt{q-\eta_1^2}}e^{i\omega(x_1\eta_1+x_2\,\mathrm{Re}\sqrt{q-\eta_1^2})}, \tag{3.50b}$$

where

$$A^s = \frac{\mu\eta_2 - \sqrt{q-\eta_1^2}}{\mu\eta_2 + \sqrt{q-\eta_1^2}}, \tag{3.50c}$$

$$A^{tr} = \frac{2\mu\eta_2}{\mu\eta_2 + \sqrt{q-\eta_1^2}}. \tag{3.50d}$$

The exponential attenuation of the intensity of the electric field within D when $\sigma > 0$ is the well known *skin effect*. The rate of this attenuation is

$$\omega\,\mathrm{Im}\sqrt{q-\eta_1^2} = \omega\sqrt{\frac{\mu\varepsilon}{2}}\sqrt{\sqrt{\left(1-\frac{\eta_1^2}{\mu\varepsilon}\right)^2 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - \left(1-\frac{\eta_1^2}{\mu\varepsilon}\right)}. \tag{3.51}$$

Now suppose that $\mu > 0$ and $\varepsilon > 0$ are fixed with with respect to changing ω, but that the nonnegative value of σ varies as a function of ω. As $\omega \to \infty$, we consider the following three possibilities: $\sigma \ll \omega$, $\sigma = \Theta(\omega)$ and $\sigma \gg \omega$.[14] If $\sigma \ll \omega$ then the rate of

[14] The Θ notation denotes asymptotic equivalence of order: $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$.

this exponential attenuation of $|\tilde{u}_\rho|$ depends on the angle of incidence:

$$\omega \operatorname{Im} \sqrt{q-\eta_1^2} \approx \begin{cases} \dfrac{\sigma}{4} \dfrac{\sqrt{\mu/\varepsilon}}{\sqrt{1-\frac{\eta_1^2}{\mu\varepsilon}}} & \text{if } \sigma \ll \omega \text{ and } \frac{\eta_1^2}{\mu\varepsilon} < 1 \text{ with } \frac{\sigma}{\omega\varepsilon}\big/\big(1-\frac{\eta_1^2}{\mu\varepsilon}\big) \ll 1, \\ \omega\sqrt{\dfrac{\sigma}{\omega}}\sqrt{\dfrac{\mu}{2}} & \text{if } \sigma \ll \omega \text{ and } \frac{\eta_1^2}{\mu\varepsilon} \approx 1, \\ \omega\sqrt{\mu\varepsilon}\sqrt{\frac{\eta_1^2}{\mu\varepsilon}-1} & \text{if } \sigma \ll \omega \text{ and } \frac{\eta_1^2}{\mu\varepsilon} > 1 \text{ with } \frac{\sigma}{\omega\varepsilon}\big/\big(\frac{\eta_1^2}{\mu\varepsilon}-1\big) \ll 1. \end{cases}$$

Thus, for incident directions sufficiently close to normal (and, if $\mu\varepsilon > 1$, for all possible incident directions $\eta \in \mathbb{S}^1$ with $\eta_2 > 0$), if $\sigma = o(\omega)$ as $\omega \to \infty$ then

$$\omega \operatorname{Im} \sqrt{q-\eta_1^2} = \Theta(\sigma) \quad \text{as } \omega \to \infty. \tag{3.52a}$$

If $\sigma = \Theta(\omega)$ as $\omega \to \infty$ then from (3.51) we see that for all incident directions,

$$\omega \operatorname{Im} \sqrt{q-\eta_1^2} = \Theta(\omega) \quad \text{as } \omega \to \infty. \tag{3.52b}$$

Likewise, if $\sigma \gg \omega$ as $\omega \to \infty$ then for all incident directions,

$$\omega \operatorname{Im} \sqrt{q-\eta_1^2} = \Theta\big(\omega\sqrt{\sigma/\omega}\big) \quad \text{as } \omega \to \infty. \tag{3.52c}$$

3.3.2 Geometric optics approximation

We assume that as $\rho \to 0$, $\varepsilon > 0$ and $\mu > 0$ are fixed, ω grows so that $\lambda = \omega\rho \to \infty$, and $\sigma > 0$ is allowed to vary as a function of ρ. If we represent the rescaled scattered fields as

$$\tilde{u}_\lambda^s = |\tilde{u}_\lambda^s| e^{i\vartheta^s},$$

it would be reasonable to expect that $|\tilde{u}_\lambda^s(x)| = O(1)$ and $\vartheta^s(x) = O(\lambda)$ as $\lambda \to \infty$ for each fixed x. We therefore make the simplifying assumption that $\vartheta^s = \lambda\phi^s + \tilde{\vartheta}$, with ϕ^s independent of λ and $\tilde{\vartheta} = O(1)$. Let $A^s = |\tilde{u}_\lambda| e^{i\tilde{\vartheta}}$ to get the high frequency ansatz that is the basis of the geometric optics technique:

$$\tilde{u}^s_\lambda(x) = A^s(x)e^{i\lambda\phi^s(x)}, \tag{3.53}$$

where ϕ^s is independent of λ and where A^s may be approximated by a formal expansion in powers of $(i\lambda)^{-1}$:

$$A^s = A^s_0 + \frac{A^s_1}{i\lambda} - \frac{A^s_2}{\lambda^2} + \cdots .$$

This formal series is the well-known Luneburg-Kline expansion [BSU87, Lun64, Kli51]. Within the conducting inhomogeneity, the field is subject to the skin effect. Hence, in light of (3.52), it is reasonable to suppose the transmitted field will decay in such a way that

$$|u^{tr}_\rho(x)| \approx e^{-\alpha\,\mathrm{dist}(x,\rho\Gamma_I)},$$

where Γ_I is the illuminated portion of ∂D and

$$\alpha = \begin{cases} \Theta(\sigma) & \text{if } \sigma \ll \omega, \\ \Theta(\omega) & \text{if } \sigma = \Theta(\omega), \\ \Theta\big(\omega\sqrt{\sigma/\omega}\big) & \text{if } \sigma \gg \omega \end{cases}$$

as $\lambda \to \infty$. Rescaled, this becomes

$$|\tilde{u}^{tr}_\lambda(x)| = e^{-\tilde{\alpha}\,\mathrm{dist}(x,\Gamma_I)},$$

where

$$\tilde{\alpha}(\cdot) = \alpha(\cdot/\rho) = \begin{cases} \Theta(\rho\sigma) & \text{if } \sigma \ll \omega, \\ \Theta(\lambda) & \text{if } \sigma = \Theta(\omega), \\ \Theta\big(\lambda\sqrt{\sigma/\omega}\big) & \text{if } \sigma \gg \omega. \end{cases}$$

In order, then, to ensure $|\tilde{u}^{tr}_\lambda(x)|$ decays with sufficiently rapidity to avoid the complications strong backscattering would bring, we should (minimally) assume that σ grows sufficiently fast so that, as $\rho \to 0$ and $\lambda = \omega\rho \to \infty$,

$$\rho\sigma \to \infty. \tag{3.54}$$

To simplify matters, we assume a stronger condition is met:

$$\text{as } \rho \to 0 \text{ and } \lambda = \omega\rho \to \infty,\ \sigma \text{ grows so that } \sigma/\omega \geq C > 0. \tag{3.55}$$

Then, if we represent the rescaled transmitted field as

$$\tilde{u}_\lambda^{tr} = |\tilde{u}_\lambda^{tr}| e^{i\vartheta^{tr}},$$

we suppose, in light of (3.50), that $\vartheta^{tr} = \lambda[\beta_1 + \mathrm{dist}(x, \partial D)\beta_2]$, where $\beta_1 = O(1)$, $\beta_2 = O\big(\max\big\{\sqrt{\sigma/\omega}, 1\big\}\big)$. A reasonable ansatz for $\tilde{u}_\lambda^{tr}$ is then

$$\tilde{u}_\lambda^{tr}(x) = A^{tr}(x) e^{i\lambda\phi^{tr}(x)}, \tag{3.56}$$

where $\phi^{tr} = O\big(\max\big\{\sqrt{\sigma/\omega}, 1\big\}\big)$, $\mathrm{Im}\,\phi^{tr} \geq 0$, and A^{tr} depends on λ in such a way that it can reasonably be approximated by a formal expansion in powers of $(i\lambda)^{-1}$:

$$A^{tr} = A_0^{tr} + \frac{A_1^{tr}}{i\lambda} - \frac{A_2^{tr}}{\lambda^2} + \cdots .$$

We recall that the field $\tilde{u}_\lambda$ satisfies

$$\begin{aligned} \Delta\tilde{u}_\lambda^{tr} + q\lambda^2\tilde{u}_\lambda^{tr} &= 0 \quad \text{in } D, \\ \Delta\tilde{u}_\lambda^{s} + \lambda^2\tilde{u}_\lambda^{s} &= 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}, \end{aligned} \tag{3.57a}$$

along with the transmission conditions

$$\tilde{u}_\lambda^{tr} = \tilde{u}_\lambda^{i} + \tilde{u}_\lambda^{s} \qquad \text{on } \partial D, \tag{3.57b}$$

$$\frac{1}{\mu}\partial_\nu\tilde{u}_\lambda^{tr} = \partial_\nu(\tilde{u}_\lambda^{i} + \tilde{u}_\lambda^{s}) \quad \text{on } \partial D, \tag{3.57c}$$

and the radiation condition

$$|(\partial_r - i\lambda)\tilde{u}^s_\lambda| = O\big(r^{-3/2}\big) \quad \text{as } r \to \infty. \tag{3.57d}$$

Noting that

$$\Delta(Ae^{i\lambda\phi}) = \big\{-\lambda^2 A\nabla\phi\cdot\nabla\phi + i\lambda(2\nabla A\cdot\nabla\phi + A\Delta\phi) + \Delta A\big\}e^{i\lambda\phi},$$

the expressions (3.53) and (3.56) are inserted into (3.57a), whereupon A^s and A^{tr} are formally expanded in powers of $(i\lambda)^{-1}$ and ϕ^{tr} is expressed as

$$\phi^{tr} = \begin{cases} \sqrt{\sigma/\omega}\,\phi_0^{tr} + O(1) & \text{if } \sigma \gg \omega, \\ \phi_0^{tr} + o(1) & \text{if } \sigma = \Theta(\omega). \end{cases}$$

Collecting the coefficients of leading order, i.e. coefficients of λ^2 for the scattered field and of $\lambda^2\sigma/\omega$ (or just λ^2) for the transmitted field, yields the Eikonal equations

$$\nabla\phi^s\cdot\nabla\phi^s = 1 \quad \text{in } \mathbb{R}^2\setminus\overline{D}, \tag{3.58a}$$

$$\nabla\phi_0^{tr}\cdot\nabla\phi_0^{tr} = q \quad \text{in } D. \tag{3.58b}$$

As a consequence of (3.57b), it is not hard to see that

$$A_0^{tr} = 1 + A_0^s \quad \text{on } \partial D \tag{3.59}$$

and

$$\phi_0^{tr}(x) = \phi^s(x) = x\cdot\eta \quad \text{on } \partial D. \tag{3.60}$$

From (3.57c) we get

$$\frac{1}{\mu}A_0^{tr}\partial_\nu\phi_0^{tr} = \eta\cdot\nu + A_0^s\partial_\nu\phi^s \quad \text{on } \partial D, \tag{3.61}$$

which combined with (3.59) gives

$$A_0^s = \frac{\eta \cdot \nu - \frac{1}{\mu}\partial_\nu \phi_0^{tr}}{\frac{1}{\mu}\partial_\nu \phi_0^{tr} - \partial_\nu \phi^s} \quad \text{on } \partial D. \tag{3.62}$$

The Eikonal equations (3.58a) and (3.58b), together with the boundary conditions (3.60), imply

$$\partial_\nu \phi_0^{tr} = \pm\sqrt{q - (\eta \cdot \tau)^2} \quad \text{on } \partial D$$

and

$$\partial_\nu \phi^s = \pm\sqrt{1 - (\eta \cdot \tau)^2} = \pm\eta \cdot \nu \quad \text{on } \partial D.$$

These equations each translate into separate possibilities, though only one is the proper choice that best approximates the actual solution. We will first consider $\partial_\nu \phi^s$, for which their are four possible continuous expressions:

$$\partial_\nu \phi^s = \eta \cdot \nu, \quad \partial_\nu \phi^s = -\eta \cdot \nu, \quad \partial_\nu \phi^s = |\eta \cdot \nu| \quad \text{or} \quad \partial_\nu \phi^s = -|\eta \cdot \nu|.$$

One way to identify the proper choice is by a principle of limiting absorption. Suppose the background medium is a conductor such that $\sigma_0/\omega = \delta > 0$. Then there are two possible choices,

$$\partial_\nu \phi^s = \sqrt{i\delta + (\eta \cdot \nu)^2} \tag{3.63}$$

or

$$\partial_\nu \phi^s = -\sqrt{i\delta + (\eta \cdot \nu)^2}. \tag{3.64}$$

Of these, only (3.63) has a positive imaginary part. Since $\operatorname{Im} \phi^s|_{\partial D} = 0$, this choice corresponds to a situation where $\operatorname{Im} \phi^s(x) > 0$ for $x \in \mathbb{R}^2 \setminus \overline{D}$ near ∂D. Consequently, the ansatz (3.53) would decay exponentially as $\lambda \to \infty$ near ∂D in accordance with the skin effect. A negative exponent of $\partial_\nu \phi^s$ would imply exponential growth, which is physically unreasonable. We therefore choose (3.63) and then let the absorption parameter δ tend to zero, leaving

$$\partial_\nu \phi^s = |\eta \cdot \nu|. \tag{3.65}$$

This choice is further justified by the law of reflection. For $x_0 \in \partial D$, $\lambda \gg 1$ and $|x - x_0| \ll \lambda^{-1}$, the ansatz for $\tilde{u}^s_\lambda$ satisfies

$$A^s(x)e^{i\lambda\phi^s(x)} \approx A^s_0(x_0)e^{i\lambda[\phi^s(x_0)+\nabla\phi^s(x_0)\cdot(x-x_0)]},$$

which is to say that near any point x_0 on ∂D, $\tilde{u}^s_\lambda$ behaves like a plane wave propagating in the direction $\nabla\phi^s(x_0)$. The choice (3.65) ensures that this direction of propagation obeys the law of reflection in the illuminated region (Figure 3.5). The same is true of

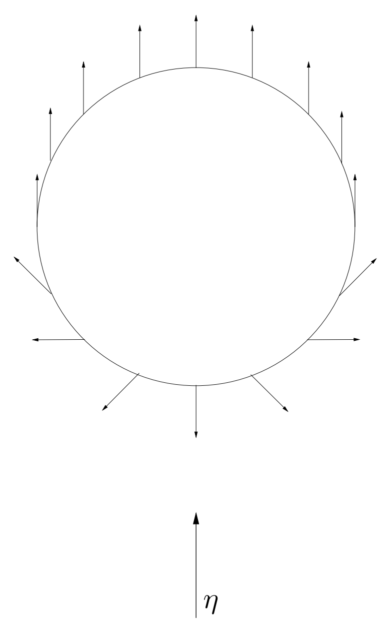


Figure 3.5: The characteristic rays when D is a disk and η is directed upward, so that the bottom half of the circle is illuminated. The characteristic ray originating at the point $x_0 \in \partial D$ points in the direction $\nabla\phi^s(x_0) = (\eta \cdot \tau)\tau + |\eta \cdot \nu|\nu$.

the choice $\partial_\nu \phi^s = -\eta \cdot \nu$, but this choice is unsatisfactory in the shadow. To see this, note that in the case of a well absorbing (conducting) scatterer, the intensity of the total field $\tilde{u}_\lambda$ should be small in the shadow. In fact, it would be zero were it not for the small contribution from creeping waves—a contribution that only decreases as λ grows. Therefore, for λ large and x near ∂D in the shadow, $\tilde{u}^s_\lambda(x) \approx -\tilde{u}^i_\lambda(x)$, which means $\tilde{u}^s_\lambda$ behaves like a plane wave propagating in the direction η.

To make the proper choice for $\partial_\nu \phi_0^{tr}$, we again make the only choice consistent with the skin effect. If we suppose $\operatorname{Im} q = \delta > 0$, then only the choice

$$\partial_\nu \phi_0^{tr} = -\sqrt{q - (\eta \cdot \tau)^2} \quad \text{on } \partial D \tag{3.66}$$

has a negative imaginary part. Since $\operatorname{Im} \phi|_{\partial D} = 0$, this choice corresponds to a situation where $\operatorname{Im} \phi_0^{tr}(x) > 0$ for $x \in D$ near ∂D. The ansatz (3.56) for $\tilde{u}^{tr}_\lambda$ therefore obeys the expected exponential decay as $\lambda \to \infty$. Since this argument applies for δ arbitrarily small, we continue to choose (3.66) when the scatterer is nonconducting. Alternatively, this choice can be justified by noting that it is the only choice consistent with Snell's law of refraction.

Combining (3.65), (3.66) and (3.62) results in the following

Approximation Assertion: *Suppose D is bounded and convex and that as $\rho \to 0$ ε and μ are fixed and $\omega \to \infty$ such that $\lambda := \rho\omega \to \infty$. For $\lambda \gg 1$ we have the approximations*

$$\tilde{u}^s_\lambda|_{\partial D} \approx [\tilde{u}^s_\lambda]_{\text{geo}} := A_0^s e^{i\lambda\eta\cdot x} \tag{3.67a}$$

and

$$\partial_\nu \tilde{u}^s_\lambda|_{\partial D} \approx [\partial_\nu \tilde{u}^s_\lambda]_{\text{geo}} := i\lambda|\eta\cdot\nu| A_0^s e^{i\lambda\eta\cdot x}, \tag{3.67b}$$

with

$$A_0^s = -\frac{\mu\eta\cdot\nu + \sqrt{q - (\eta\cdot\tau)^2}}{\mu|\eta\cdot\nu| + \sqrt{q - (\eta\cdot\tau)^2}}. \tag{3.67c}$$

Figure 3.7 shows numerical examples in the case of a plane wave incident upon a disk (Figure 3.6).

Remark 3.5. *We expect these to be good approximations only in the regime of moderate to high (frequency dependent) conductivity, i.e. for σ/ω bounded away from zero.*

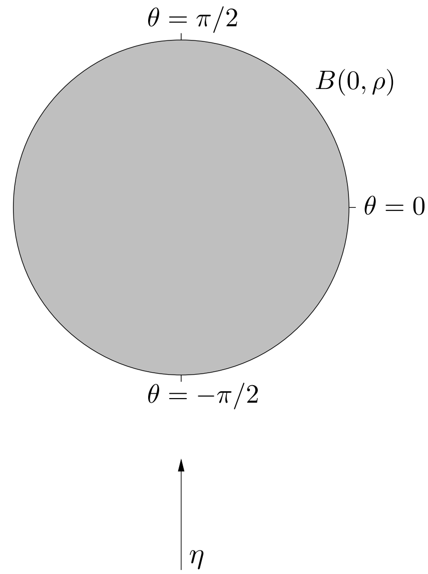
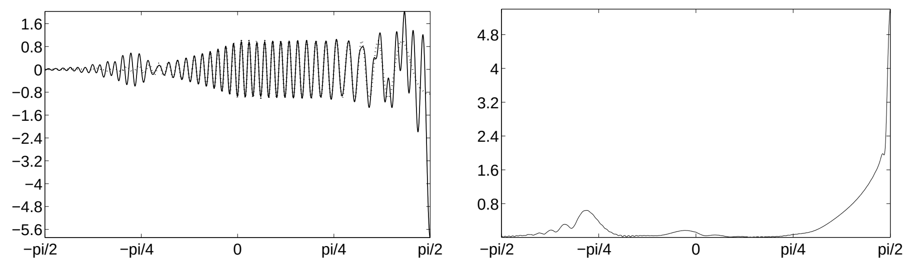


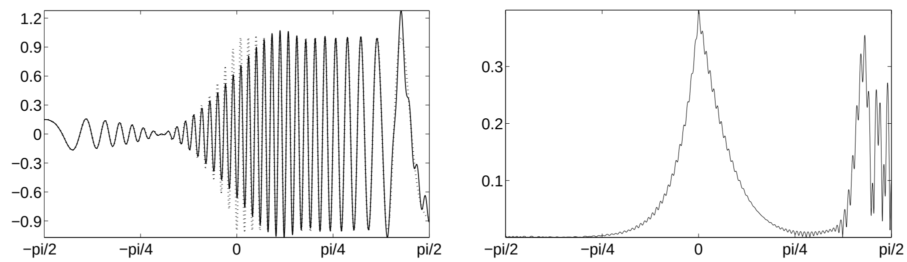
Figure 3.6: The orientation of the problem in our numerical computations. The plane wave $e^{i\omega x \cdot \eta}$ is incident upon the scatterer ρD, where $D = B(0, 1)$ and $\eta = \binom{0}{1}$.

Since the boundary of the scatterer is not flat, a transmitted wave that does not attenuate rapidly will transmit a sufficiently strong signal through the scatterer for which the contributions following rays from different points of incidence will significantly interfere with one another. Furthermore, there will be multiple internal reflections of the transmitted waves, as well as refractions that will reemerge to augment the scattered field. Hence the transmitted field cannot be expected to behave as it would in the case of a half-plane (3.50)*. The ansatz* (3.56) *was motivated by* (3.50) *and therefore cannot be trusted when* $\sigma/\omega \ll 1$*. This deficiency of our approach for poorly conducting scatters is evident in Figure 3.7.*

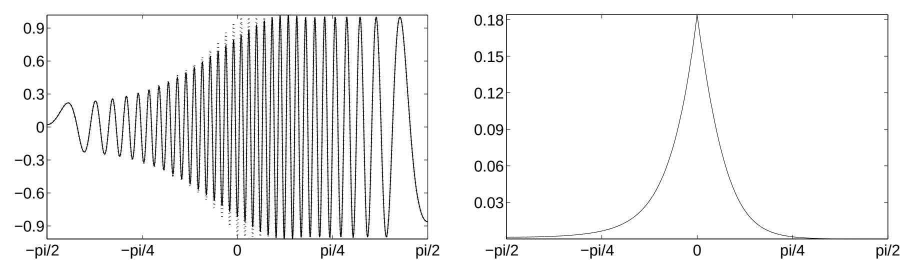
Also note that the error is greatest near the two points where the incident rays graze the scatterer (only one is shown in the figure). This is not surprising as the diffraction effects missed by the geometric optics method are greatest near these points.



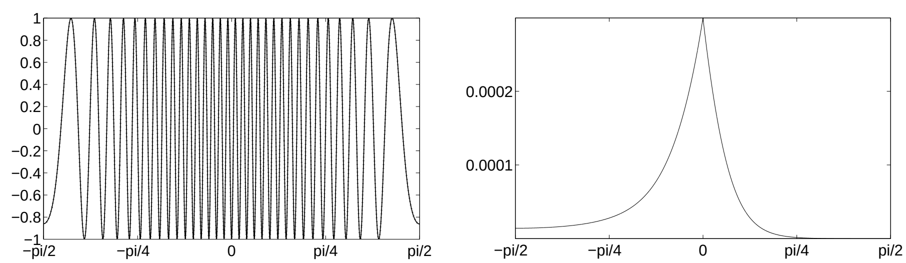
(a) $\mu = 2$, $\varepsilon = 2$ and $\sigma = 0$.



(b) $\mu = 2$, $\varepsilon = 1$ and $\sigma/\omega = 1/50$, so that $q = 2 + i/50$.



(c) Here $\mu = 2$, $\varepsilon = 2$ and $\sigma/\omega = 2$, so that $q = 4 + 4i$.



(d) $\mu = 2$, $\varepsilon = 2$ and $\sigma/\omega = 10^6$, so that $q = 4 + 2 \times 10^6\, i$.

Figure 3.7: Plots on the right half of the boundary of the scatterer $\rho D = B(0, \rho)$ running counterclockwise from $-\pi/2$ to the highest point at $\pi/2$. The orientation is as in Figure 3.6. The left half of the boundary has been omitted because of the obvious symmetry. The left frames show the graphs of $\operatorname{Re} \tilde{u}^s_\lambda$ (solid) and $\operatorname{Re}[\tilde{u}^s_\lambda]_{\text{geo}}$ (dotted). The right frames show the error $|\tilde{u}^s_\lambda - [\tilde{u}^s_\lambda]_{\text{geo}}|$. In all cases, $\rho = 10^{-4}$ and $\omega = 10^6$, so that $\lambda = \rho\omega = 100$ and $\rho^2\omega = 1/100$.

3.3.3 The case of a perfectly conducting scatterer

In the case of a perfectly conducting scatterer, the above analysis simplifies. The scattered field satisfies

$$\Delta \tilde{u}^s_\lambda + \lambda^2 \tilde{u}^s_\lambda = 0 \qquad \text{in } \mathbb{R}^2 \setminus \overline{D}, \tag{3.68a}$$

$$\tilde{u}^s_\lambda = -\tilde{u}^i_\lambda \quad \text{on } \partial D, \tag{3.68b}$$

along with the radiation condition

$$|(\partial_r - i\lambda)\tilde{u}^s_\lambda| = O(r^{-3/2}) \quad \text{as } r \to \infty. \tag{3.68c}$$

Here we find easily that $A^s_0 = -1$ on ∂D, and therefore (3.67) becomes

$$\tilde{u}^s_\lambda|_{\partial D} = [\tilde{u}^s_\lambda]_{\text{geo}} := -e^{i\lambda\eta\cdot x} \tag{3.69a}$$

and

$$\partial_\nu \tilde{u}^s_\lambda|_{\partial D} \approx [\partial_\nu \tilde{u}^s_\lambda]_{\text{geo}} := -i\lambda|\eta \cdot \nu| e^{i\lambda\eta\cdot x}. \tag{3.69b}$$

This is the well known physical optics approximation [Jon86, Néd01]. In Figure 3.8 we compare the physical optics approximation of the total field, $[\partial_\nu \tilde{u}_\lambda]_{\text{geo}} = [\partial_\nu \tilde{u}^s_\lambda]_{\text{geo}} + \partial_\nu \tilde{u}^i_\lambda$, with the actual total field, $\partial_\nu \tilde{u}_\lambda|_{\partial D}$.

3.3.4 Green's formula and stationary phase

Once we have (3.67), the most obvious way to approximate the scattered field away from the boundary is to construct ϕ^s and A^s_0. The approximation $\tilde{u}^s_\lambda \approx A^s e^{i\lambda\phi^s}$ is known as the geometric optics field. Finding ϕ^s is simple since it satisfies the Eikonal equation (3.58a). Using the method of characteristics we find the solution: given $x \in \mathbb{R}^2 \setminus \overline{D}$ uniquely represented as $x = x_0 + t\nabla\phi^s(x_0)$ for some $t > 0$ and $x_0 \in \partial D$,

$$\phi^s(x) = t + \phi^s(x_0).$$

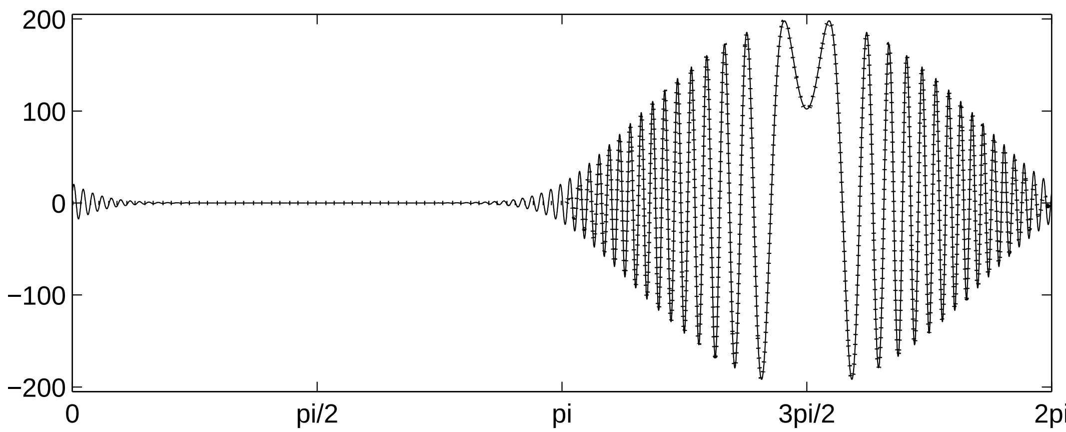


Figure 3.8: Plots on the boundary of the perfectly conducting scatterer $\rho D = B(0,\rho)$, oriented as in Figure 3.6 (with the convention that θ is defined modulo 2π). The graph of $\mathrm{Re}(\partial_\nu \tilde{u}_\lambda|_{\partial D})$ is solid and the graph of the physical optics approximation, $\mathrm{Re}[\partial_\nu \tilde{u}_\lambda]_{\mathrm{geo}}$, is dotted. Here $\rho = 10^{-4}$ and $\omega = 10^6$, so that $\lambda = \rho\omega = 100$ and $\rho^2\omega = 1/100$.

Once ϕ^s is known we find A_0^s by solving the transport equation

$$2\nabla A_0^s \cdot \nabla \phi^s + A_0^s \Delta \phi^s = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D},$$

which follows by collecting the coefficients of $i\lambda$ in the Hemholtz equation satisfied by $\left(\sum A_n^s (i\lambda)^{-n}\right) e^{i\lambda\phi^s}$. This transport equation can be solved by integrating along characteristic paths. The characteristic ODE is

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \nabla \phi^s(\mathbf{x}(t)), \\ \dot{z}(t) &= -\frac{1}{2}\Delta\phi^s(\mathbf{x}(t))z(t), \end{aligned}$$

where $z(t) = A_0^s(\mathbf{x}(t))$. The solution is

$$A_0^s(x) = A_0^s(x_0) e^{-\frac{1}{2}\int_0^t \Delta\phi^s(x_0 + s\nabla\phi^s(x_0))\, ds},$$

where $A_0^s(x_0)$ is given by (3.62). We therefore have the approximation

$$\tilde{u}_\lambda^s(x) \approx A_0^s(x_0) e^{-\frac{1}{2}\int_0^t \Delta\phi^s(x_0 + s\nabla\phi^s(x_0))\, ds} e^{i\lambda\phi^s}.$$

This approach can be useful when approximating the solution to the direct scattering problem. However, since our ultimate goal is a tool for solving the *inverse* problem, we will instead employ a method that will lead to a simpler representation formula for the scattered field. In a sentence, our approach will be to substitute the approximations (3.67) of $\tilde{u}^s_\lambda|_{\partial D}$ and $\partial_\nu \tilde{u}^s_\lambda|_{\partial D}$ into the Green's representation

$$u^s_\rho(x) = \int_{\partial D} \left\{ \tilde{u}^s_\lambda(y) \partial_{\nu_y} \Phi^\lambda(x/\rho, y) - \partial_\nu \tilde{u}^s_\lambda(y) \Phi^\lambda(x/\rho, y) \right\} \mathrm{d}\sigma_y \tag{3.70}$$

and then perform a stationary phase analysis of this integral. The technique of substituting approximations of $\tilde{u}^s_\lambda|_{\partial D}$ and $\partial_\nu \tilde{u}^s_\lambda|_{\partial D}$ into the Green's representation is often used in the context of the physical optics approximation and in aperture calculations.

The physical optics approximation [Jon86, Néd01] applies in the case of perfectly conducting (or sound-soft) scatterers. By treating the scattering at each point of incidence on the illuminated portion of the boundary as though the scatterer were a half space with boundary determined by the tangent line to the actual scatterer at that point, and by assuming the total field in the shadow portion of the boundary vanishes, we take $\partial_\nu \tilde{u}_\lambda|_{\partial D} = 2\partial_\nu u^i$ (that is, $\partial_\nu \tilde{u}^s_\lambda|_{\partial D} = \partial_\nu u^i$) on the illuminated side of the boundary and $\partial_\nu \tilde{u}_\lambda|_{\partial D} = 0$ (or $\partial_\nu \tilde{u}^s_\lambda|_{\partial D} = -\partial_\nu u^i$) in the shadow.[15] The physical optics approximation then yields

$$u^s_\rho(x) \approx -2 \int_{\Gamma_I} \partial_\nu \tilde{u}^i_\lambda(y) \Phi^\lambda(x/\rho, y) \, \mathrm{d}\sigma_y$$

where Γ_I is the illuminated portion of the boundary.[16]

When applied in the context of a light source on one side of an opaque screen with an aperture, this method is often referred to as Kirchoff's approximation [Jon86, BW02]. In this case, ∂D is replaced by $\mathcal{S}^- \cup \mathcal{A}$, where $\mathcal{S}^-$ is the dark side of the screen and $\mathcal{A}$

[15] This is the same approximation as (3.69b).

[16] Here we used the fact that $\int_{\partial D} \tilde{u}^i_\lambda(y) \partial_{\nu_y} \Phi^\lambda(x/\rho, y) \, \mathrm{d}\sigma_y = \int_{\partial D} \partial_\nu \tilde{u}^i_\lambda(y) \Phi^\lambda(x/\rho, y) \, \mathrm{d}\sigma_y$ for x exterior to $\overline{D}$.

is the aperture, so that the total field u satisfies

$$\int_{\mathcal{S}^- \cup \mathcal{A}} \Big\{ \partial_\nu^- u(y) \Phi^\omega(x,y) - u(y) \partial_{\nu_y} \Phi^\omega(x,y) \Big\} \, \mathrm{d}\sigma_y$$
$$= \begin{cases} u(x) & \text{in the non-illuminated side of screen,} \\ 0 & \text{in the illuminated side,} \end{cases}$$

where the normal vector ν is directed into the non-illuminated side of the screen (Figure 3.9). By taking $u|_{\mathcal{A}} = u^i|_{\mathcal{A}}$, $\partial_\nu u|_{\mathcal{A}} = \partial_\nu u^i|_{\mathcal{A}}$ and $u|_{\mathcal{S}}^- = \partial_\nu^- u|_{\mathcal{S}} = 0$, we arrive at

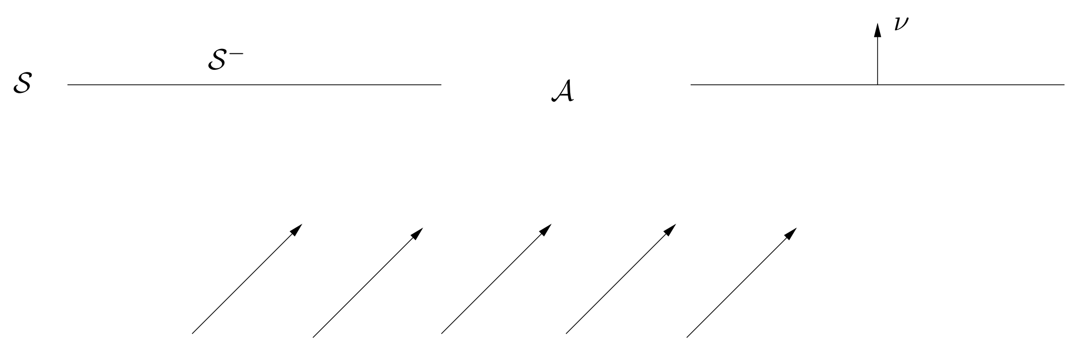


Figure 3.9: The infinite screen $\mathcal{S}$ with aperture $\mathcal{A}$. $\mathcal{S}^-$ is the dark (non-illuminated) side of screen.

Kirchoff's approximation

$$u(x) \approx \int_{\mathcal{A}} \Big\{ \partial_\nu^- u^i(y) \Phi^\omega(x,y) - u^i(y) \partial_{\nu_y} \Phi^\omega(x,y) \Big\} \, \mathrm{d}\sigma_y \tag{3.71}$$

for x in the non-illuminated side of the screen. For high frequencies, this integral can be approximated using the technique of stationary phase [BW02]. We will apply this technique in the similar but distinct context of scattering by a finite penetrable object.

As $\rho \to 0$, the functions $\Phi^\lambda(x/\rho, y)$ and $\partial_{\nu_y}\Phi^\lambda(x/\rho, y)$ satisfy [Wat44]

$$\begin{aligned}
\Phi^\lambda(x/\rho, y) &= \frac{i}{4} H_0^{(1)}\Big(\lambda\Big|\frac{x}{\rho} - y\Big|\Big) \\
&= \Big(8\pi\lambda\Big|\frac{x}{\rho} - y\Big|\Big)^{-1/2} e^{i(\lambda|\frac{x}{\rho} - y| + \pi/4)}\Big\{1 + O\Big(\Big[\lambda\Big|\frac{x}{\rho} - y\Big|\Big]^{-1}\Big)\Big\}, \\
\partial_{\nu_y}\Phi^\lambda(x/\rho, y) &= \frac{i\lambda}{4}\frac{\frac{x}{\rho} - y}{|\frac{x}{\rho} - y|} \cdot \nu_y \, H_1^1\Big(\lambda\Big|\frac{x}{\rho} - y\Big|\Big) \\
&= \lambda\frac{\frac{x}{\rho} - y}{|\frac{x}{\rho} - y|} \cdot \nu_y \Big(8\pi\lambda\Big|\frac{x}{\rho} - y\Big|\Big)^{-1/2} e^{i(\lambda|\frac{x}{\rho} - y| - \pi/4)} \\
&\qquad\qquad \times \Big\{1 + O\Big(\Big[\lambda\Big|\frac{x}{\rho} - y\Big|\Big]^{-1}\Big)\Big\},
\end{aligned}$$

uniformly in $y \in \partial D$ and x bounded away from ρD.[17] Before inserting these approximations of $\Phi^\lambda(x/\rho, y)$ and $\partial_{\nu_y}\Phi^\lambda(x/\rho, y)$, along with the geometric optics approximations (3.67), into (3.70), we first refine the asymptotics to derive forms that are more amenable to the coming stationary phase analysis. We expand

$$\begin{aligned}
\lambda\Big|\frac{x}{\rho} - y\Big| &= \omega|x|\left\{1 - \rho\frac{\hat{x}\cdot y}{|x|} + \rho^2\left[\frac{|y|^2 - (\hat{x}\cdot y)^2}{2|x|^2}\right] + \rho^3\left[\frac{\hat{x}\cdot y|y|^2 - (\hat{x}\cdot y)^3}{2|x|^3}\right] + O(\rho^4)\right\} \\
&= \omega|x| - \omega\rho\,\hat{x}\cdot y + \omega\rho^2\left[\frac{|y|^2 - (\hat{x}\cdot y)^2}{2|x|}\right] + \omega\rho^3\left[\frac{\hat{x}\cdot y|y|^2 - (\hat{x}\cdot y)^3}{2|x|^2}\right] \\
&\qquad + O(\omega\rho^4),
\end{aligned}$$

and note that at this point it is necessary to distinguish among the many possible high frequency regimes. The above will suffice in the regime where $\omega\rho^3 \to \infty$ but $\omega\rho^4 \to 0$, but in higher order regimes, more terms must be included. We will restrict our attention to the lowest regime, where $\omega\rho \to \infty$ but $\omega\rho^2 \to 0$, with the understanding that our method could be easily modified to accommodate the higher order regimes. Hence we will only need that

$$\lambda\Big|\frac{x}{\rho} - y\Big| = \omega|x| - \omega\rho\,\hat{x}\cdot y + O(\omega\rho^2),$$

[17] Specifically, we must assume x is in the complement of a domain $\mathcal{I}$ independent of ρ satisfying $\rho D \subset\subset \mathcal{I}$.

which yields the asymptotic formulas

$$\begin{aligned}
\Phi^\lambda(x/\rho, y) &= \frac{1}{\sqrt{8\pi\omega|x|}} e^{i(\lambda|\frac{x}{\rho}-y|+\pi/4)}(1+O(\rho+\omega^{-1})) \\
&= \frac{1}{\sqrt{\omega}} \frac{e^{i\pi/4}}{\sqrt{8\pi|x|}} e^{i(\omega|x|-\lambda\hat{x}\cdot y)}(1+O(\rho+\omega^{-1}+\omega\rho^2)) \qquad (3.72)\\
\partial_{\nu_y}\Phi^\lambda(x/\rho, y) &= \lambda\,\hat{x}\cdot\nu_y \frac{1}{\sqrt{8\pi\omega|x|}} e^{i(\lambda|\frac{x}{\rho}-y|-\pi/4)}(1+O(\rho+\omega^{-1})) \\
&= \rho\sqrt{\omega}\,\hat{x}\cdot\nu_y \frac{e^{-i\pi/4}}{\sqrt{8\pi|x|}} e^{i(\omega|x|-\lambda\hat{x}\cdot y)}(1+O(\rho+\omega^{-1}+\omega\rho^2)). \qquad (3.73)
\end{aligned}$$

The amplitude of the approximation (3.72) will have small absolute error in all high frequency regimes. The amplitude of (3.73) will have small absolute error so long as $\omega\rho^4 \to 0$, but the relative error will be small regardless of the relative growth of ω and ρ^{-1}. The phase, however, is sensitive to this relative rate of growth.

On substituting (3.72) and (3.73) along with the geometric optics approximations (3.67) into (3.70) we obtain

$$\begin{aligned}
u^s_\rho(x) &\approx \rho\sqrt{\omega}e^{-i\pi/4}\frac{e^{i\omega|x|}}{\sqrt{8\pi|x|}}\,\hat{x}\cdot\int_{\partial D}\nu_y A^s_0(y)e^{i\lambda(\eta-\hat{x})\cdot y}\,\mathrm{d}\sigma_y \\
&\qquad - \frac{i\lambda}{\sqrt{\omega}}e^{i\pi/4}\frac{e^{i\omega|x|}}{\sqrt{8\pi|x|}}\int_{\partial D}|\eta\cdot\nu_y|A^s_0(y)e^{i\lambda(\eta-\hat{x})\cdot y}\,\mathrm{d}\sigma_y \\
&= \rho\sqrt{\omega}e^{-i\pi/4}\frac{e^{i\omega|x|}}{\sqrt{8\pi|x|}}\int_{\partial D}\big(\hat{x}\cdot\nu_y+|\eta\cdot\nu_y|\big)A^s_0(y)e^{i\lambda(\eta-\hat{x})\cdot y}\,\mathrm{d}\sigma_y, \qquad (3.74)
\end{aligned}$$

with A^s_0 given by (3.67c). If $\omega\rho^2$ is not small, say, for instance, $\omega\rho^3 \to \infty$ but $\omega\rho^4 \to 0$, we would replace the phase $\lambda(\eta-\hat{x})\cdot y$ with

$$\lambda(\eta-\hat{x})\cdot y + \omega\rho^2\left[\frac{|y|^2-(\hat{x}\cdot y)^2}{2|x|}\right] + \omega\rho^3\left[\frac{\hat{x}\cdot y|y|^2-(\hat{x}\cdot y)^3}{2|x|^2}\right].$$

We will now perform a stationary phase analysis on the integral in the formula (3.74). We begin with

Lemma 3.6. *Let D be a bounded, smooth, strictly convex domain. Let x be a nonzero* *vector and let η be a unit vector such that $\hat{x} \neq \eta$. The function*

$$y \mapsto \psi_\eta(x,y) := (\eta - \hat{x}) \cdot y, \quad y \in \partial D,$$

has two stationary points: y_1 and y_2 satisfying

$$\psi_\eta(x,y_1) = \min_{y\in\partial D} \psi_\eta(x,y) \quad \textit{and} \quad \psi_\eta(x,y_2) = \max_{y\in\partial D} \psi_\eta(x,y). \tag{3.75}$$

y_1 and y_2 are also the unique points on ∂D satisfying

$$\nu_{y_1} = -\frac{\eta - \hat{x}}{|\eta - \hat{x}|} \quad \textit{and} \quad \nu_{y_2} = \frac{\eta - \hat{x}}{|\eta - \hat{x}|}.$$

Proof. The stationary points of $y \mapsto \psi_\eta(x,y)$ are the points where $\partial_{\tau_y}\psi_\eta(x,y) = (\eta - \hat{x}) \cdot \tau_y = 0$. These are precisely the points y where $\nu_y = \pm(\eta - \hat{x})/|\eta - \hat{x}|$. That there are exactly two such points follows from the strict convexity of D. Since there are only two stationary points, they must be the stationary points characterized by (3.75). $\square$

We now evaluate the density of the oscillatory integral in (3.74) at the stationary points.

Lemma 3.7. *Let D, η, x, ψ_η, y_1 and y_2 be as in Lemma 3.6 and let A_0^s be given by* (3.62)*. Let ϑ represent the angle of counterclockwise rotation from η to $\hat{x}$, $0 < \vartheta < 2\pi$. Then*

$$(\hat{x} \cdot \nu_{y_1} + |\eta \cdot \nu_{y_1}|)A_0^s(y_1) = 2\sin(\vartheta/2)\frac{\mu\sin(\vartheta/2) - \sqrt{q - 1 + \sin^2(\vartheta/2)}}{\mu\sin(\vartheta/2) + \sqrt{q - 1 + \sin^2(\vartheta/2)}}$$

and

$$(\hat{x} \cdot \nu_{y_2} + |\eta \cdot \nu_{y_2}|)A_0^s(y_2) = 0.$$

Proof. By Lemma 3.6,

$$\begin{aligned}
\hat{x} \cdot \nu_{y_1} + |\eta \cdot \nu_{y_1}| &= -\hat{x} \cdot \frac{\eta - \hat{x}}{|\eta - \hat{x}|} + \eta \cdot \frac{\eta - \hat{x}}{|\eta - \hat{x}|} \\
&= |\eta - \hat{x}| \\
&= \sqrt{2 - 2\eta \cdot \hat{x}} \\
&= \sqrt{2(1 - \cos\vartheta)} \\
&= 2\sin(\vartheta/2)
\end{aligned}$$

and

$$\begin{aligned}
\hat{x} \cdot \nu_{y_2} + |\eta \cdot \nu_{y_2}| &= \hat{x} \cdot \frac{\eta - \hat{x}}{|\eta - \hat{x}|} + \eta \cdot \frac{\eta - \hat{x}}{|\eta - \hat{x}|} \\
&= 0.
\end{aligned}$$

Since $|\eta \cdot \nu_{y_1}| = \eta \cdot \nu_{y_1} = \sin(\vartheta/2)$ and $|\eta \cdot \nu_{y_2}| = \eta \cdot \nu_{y_2}$, we easily calculate

$$\begin{aligned}
A_0^s(y_1) &= -\frac{\mu\eta \cdot \nu_{y_1} + \sqrt{q - (\eta \cdot \tau_{y_1})^2}}{\mu|\eta \cdot \nu_{y_1}| + \sqrt{q - (\eta \cdot \tau_{y_1})^2}} \\
&= \frac{\mu \sin(\vartheta/2) - \sqrt{q - 1 + \sin^2(\vartheta/2)}}{\mu \sin(\vartheta/2) + \sqrt{q - 1 + \sin^2(\vartheta/2)}}
\end{aligned}$$

and

$$\begin{aligned}
A_0^s(y_2) &= -\frac{\mu\eta \cdot \nu_{y_2} + \sqrt{q - (\eta \cdot \tau_{y_2})^2}}{\mu|\eta \cdot \nu_{y_2}| + \sqrt{q - (\eta \cdot \tau_{y_2})^2}} \\
&= -1.
\end{aligned}$$

□

The core of our stationary phase analysis relies on the following

Lemma 3.8. *Let D, η, x, ψ_η, y_1 and y_2 be as in Lemma 3.6 and let ϑ be as in Lemma 3.7. Let $K(x)$ denote the nonnegative curvature of ∂D at $x \in \partial D$, and suppose that*

$K(y_1)$ and $K(y_2)$ are both nonzero. For any piecewise C^1 function $a(\cdot)$ on ∂D,

$$\int_{\partial D} a(y)e^{i\lambda\psi_\eta(x,y)}\,\mathrm{d}\sigma_y = \frac{\sqrt{\pi}}{\sqrt{\lambda}\sqrt{\sin(\vartheta/2)}}\left(\frac{e^{i\pi/4}}{\sqrt{K(y_1)}}e^{i\lambda\psi_\eta(x,y_1)}a(y_1) + \frac{e^{-i\pi/4}}{\sqrt{K(y_2)}}e^{i\lambda\psi_\eta(x,y_2)}a(y_2)\right) + o(\lambda^{-1/2})$$

as $\lambda \to \infty$.

Proof. If Γ is connected segment of ∂D along which $|\partial_{\tau_y}\psi_\eta(x,y)| > c > 0$ then a straightforward argument using an integration by parts yields

$$\int_{\Gamma} a(y)e^{i\lambda\psi_\eta(x,y)}\,\mathrm{d}\sigma_y = O(\lambda^{-1}).$$

It follows that

$$\int_{\partial D} a(y)e^{i\lambda\psi_\eta(x,y)}\,\mathrm{d}\sigma_y = \left(\int_{\Gamma_{1,\delta}} + \int_{\Gamma_{2,\delta}}\right)a(y)e^{i\lambda\psi_\eta(x,y)}\,\mathrm{d}\sigma_y + O(\lambda^{-1}),$$

where $\Gamma_{1,\delta}$ and $\Gamma_{2,\delta}$ are arbitrarily small ∂D-neighborhoods of y_1 and y_2, respectively. We take

$$\Gamma_{1,\delta} = \partial D \cap \{y : \psi_\eta(x,y) < \psi_\eta(x,y_1) + \delta\}$$

for some $\delta > 0$. Let s denote the signed arclength of $\Gamma_{1,\delta}$ beginning at y_1 and proceeding counterclockwise for $s > 0$ and clockwise for $s < 0$. Then

$$\int_{\Gamma_{1,\delta}} a(y)e^{i\lambda\psi_\eta(x,y)}\,\mathrm{d}\sigma_y = \underbrace{\int_{\Gamma_{1,\delta}^+} a(y)e^{i\lambda\psi_\eta(x,y)}\,\mathrm{d}\sigma_y}_{I_1} + \underbrace{\int_{\Gamma_{1,\delta}^-} a(y)e^{i\lambda\psi_\eta(x,y)}\,\mathrm{d}\sigma_y}_{I_2}$$

where

$$\Gamma_{1,\delta}^+ = \Gamma_{1,\delta} \cap \{y(s) \in \Gamma_{1,\delta} : s > 0\},$$
$$\Gamma_{1,\delta}^- = \Gamma_{1,\delta} \cap \{y(s) \in \Gamma_{1,\delta} : s < 0\}.$$

Because

$$y(s) = y(0) + sy'(0) + \frac{1}{2}s^2 y''(0) + O(s^3)$$
$$= y_1 + s\tau_{y_1} - \frac{1}{2}s^2 K(y_1)\nu_{y_1} + O(s^3),$$

it follows from Lemma 3.6 that

$$\psi_\eta(x, y(s)) = (\eta - \hat{x}) \cdot y(s) = \psi_\eta(x, y_1) + |\eta - \hat{x}| K(y_1)\frac{1}{2}s^2 + O(s^3)$$
$$= \psi_\eta(x, y_1) + \sin(\vartheta/2)K(y_1)s^2 + O(s^3).$$

In the integral I_1 we change to the variable $t = \psi_\eta(x, y(s)) - \psi_\eta(x, y_1)$ to get

$$\int_{\Gamma_{1,\delta}^+} a(y)e^{i\lambda\psi_\eta(x,y)}\,\mathrm{d}\sigma_y = e^{i\lambda\psi_\eta(x,y_1)}\int_0^\delta \tilde{a}(t)e^{i\lambda t}\left|\frac{dt}{ds}\right|^{-1}\mathrm{d}t$$
$$= e^{i\lambda\psi_\eta(x,y_1)}\int_0^\delta \tilde{a}(t)e^{i\lambda t}\frac{1}{2\sin(\vartheta/2)K(y_1)s}(1 + O(s))\,\mathrm{d}t$$

where $\tilde{a}(t) = a(y(s(t)))$. Since

$$s = \frac{\sqrt{t}}{\sqrt{\sin(\vartheta/2)K(y_1)}} + O(t),$$

the above integral becomes

$$\int_{\Gamma_{1,\delta}^+} a(y)e^{i\lambda\psi_\eta(x,y)}\,\mathrm{d}\sigma_y = \frac{e^{i\lambda\psi_\eta(x,y_1)}}{2\sqrt{\sin(\vartheta/2)K(y_1)}}\int_0^\delta \tilde{a}(t)e^{i\lambda t}\big(1 + O(\sqrt{t})\big)\frac{\mathrm{d}t}{\sqrt{t}}$$
$$= \frac{e^{i\lambda\psi_\eta(x,y_1)}}{2\sqrt{\sin(\vartheta/2)K(y_1)}}\int_0^\delta \tilde{a}(t)e^{i\lambda t}\frac{\mathrm{d}t}{\sqrt{t}} + O(\lambda^{-1}).$$

A standard asymptotic result [Olv97, Chapter 3, Theorem 13.1] for integrals of this form is

$$\int_0^\delta \tilde{a}(t)e^{i\lambda t}\frac{\mathrm{d}t}{\sqrt{t}} = \sqrt{\frac{\pi}{\lambda}}e^{i\pi/4}\tilde{a}(0) + o(\lambda^{-1/2}) \quad \text{as } \lambda \to \infty.$$

Therefore

$$\int_{\Gamma_{1,\delta}^{+}} a(y)e^{i\lambda\psi_\eta(x,y)}\,\mathrm{d}\sigma_y = \frac{\sqrt{\pi}}{2\sqrt{\lambda}}\frac{e^{i(\lambda\psi_\eta(x,y_1)+\pi/4)}}{\sqrt{\sin(\vartheta/2)K(y_1)}}a(y_1) + o(\lambda^{-1/2}).$$

By symmetry, I_2 has this same asymptotic formula, and thus

$$\int_{\Gamma_{1,\delta}} a(y)e^{i\lambda\psi_\eta(x,y)}\,\mathrm{d}\sigma_y = \frac{\sqrt{\pi}}{\sqrt{\lambda}}\frac{e^{i(\lambda\psi_\eta(x,y_1)+\pi/4)}}{\sqrt{\sin(\vartheta/2)K(y_1)}}a(y_1) + o(\lambda^{-1/2}).$$

Finally, for $\Gamma_{2,\delta}$ we take

$$\Gamma_{2,\delta} = \partial D \cap \{y : \psi_\eta(x,y) > \psi_\eta(x,y_2) - \delta\}$$

and proceed as we did for y_1 to find

$$\int_{\Gamma_{2,\delta}} a(y)e^{i\lambda\psi_\eta(x,y)}\,\mathrm{d}\sigma_y = \frac{\sqrt{\pi}}{\sqrt{\lambda}}\frac{e^{i(\lambda\psi_\eta(x,y_1)-\pi/4)}}{\sqrt{\sin(\vartheta/2)K(y_2)}}a(y_2) + o(\lambda^{-1/2}). \qquad \square$$

Using (3.74), Lemma 3.6 and Lemma 3.8, we find the approximation

$$\begin{aligned} u_\rho^s(x) \approx \rho\sqrt{\omega}e^{-i\pi/4}\frac{e^{i\omega|x|}}{\sqrt{8\pi|x|}}\frac{\sqrt{\pi}}{\sqrt{\lambda}\sqrt{\sin(\vartheta/2)}}\frac{e^{i\pi/4}}{\sqrt{K(y_1)}}e^{i\lambda\psi_\eta(x,y_1)} \\ \times 2\sin(\vartheta/2)\frac{\mu\sin(\vartheta/2) - \sqrt{q-1+\sin^2(\vartheta/2)}}{\mu\sin(\vartheta/2) + \sqrt{q-1+\sin^2(\vartheta/2)}} \end{aligned} \tag{3.76}$$

for $x \in \mathbb{R}^2 \setminus \overline{D}$, which is the main result of this section.

Approximation Assertion: *In problem* (3.2)*, assume D is a bounded, smooth, strictly convex domain such that the nonnegative curvature function K on ∂D is strictly positive. Also assume that as $\rho \to 0$, ε and μ are fixed, $\omega \to \infty$ in such a way that $\lambda := \rho\omega \to \infty$, and $\sigma \to \infty$ in such a way that σ/ω is bounded away from zero. Given any $x \in \mathbb{R}^2 \setminus \overline{D}$, let ϑ denote the angle measured counterclockwise from η to x, with $0 \le \vartheta < 2\pi$, and let y_1 be the arg-min*[18] *of $\psi_\eta(x,\cdot)$, where $\psi_\eta(x,y) = (\eta - \hat{x})\cdot y$. An*

[18]the arg-min of $\psi_\eta(x,\cdot)$ the solution y_1 to $\psi_\eta(x,y_1) = \min_{y\in\partial D}\psi_\eta(x,y)$.

approximation to the scattered field at x bounded away from ρD satisfying $\hat{x} \neq \eta$ is

$$\begin{aligned} u^s_\rho(x) &\approx [u^s_\rho]_{\text{geo}}(x) \\ &:= \sqrt{\rho}\frac{e^{i\omega[|x|+\rho(\eta-\hat{x})\cdot y_1]}}{\sqrt{2|x|K(y_1)}}\sqrt{\sin(\vartheta/2)}\frac{\mu\sin(\vartheta/2) - \sqrt{q-1+\sin^2(\vartheta/2)}}{\mu\sin(\vartheta/2) + \sqrt{q-1+\sin^2(\vartheta/2)}}. \end{aligned} \tag{3.77}$$

If D is perfectly conducting,

$$u^s_\rho(x) \approx [u^s_\rho]_{\text{geo}}(x) := \sqrt{\rho}\frac{e^{i\omega[|x|+\rho(\eta-\hat{x})\cdot y_1]}}{\sqrt{2|x|K(y_1)}}\sqrt{\sin(\vartheta/2)}. \tag{3.78}$$

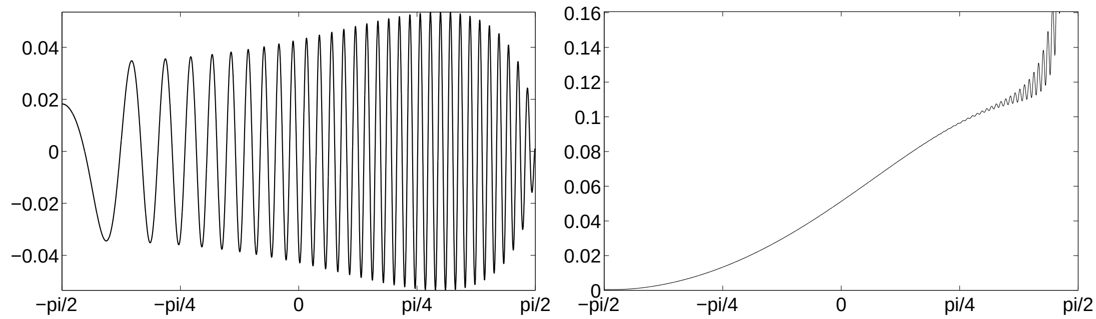
The formula (3.78) follows either as a limit of (3.77) as $\operatorname{Im} q \to \infty$ or by recalling (3.69). Figure 3.10 provide numerical examples of this approximation. For comparison, we include a graph (Figure 3.11) of the actual scattered field in the case corresponding to Figure 3.10(b). The spike in the intensity of the scattered field within the narrow shade region is missed by our geometric optics approach. We also provide an example in the case of a perfectly conducting disk (Figure 3.12).

Figure 3.14 shows the graphs of the real part of the approximation (3.77) of the scattered field on the circle $r = 2$ in the case when the scatterer is an ellipse of aspect ratio 1:2 oriented as in Figure 3.13.

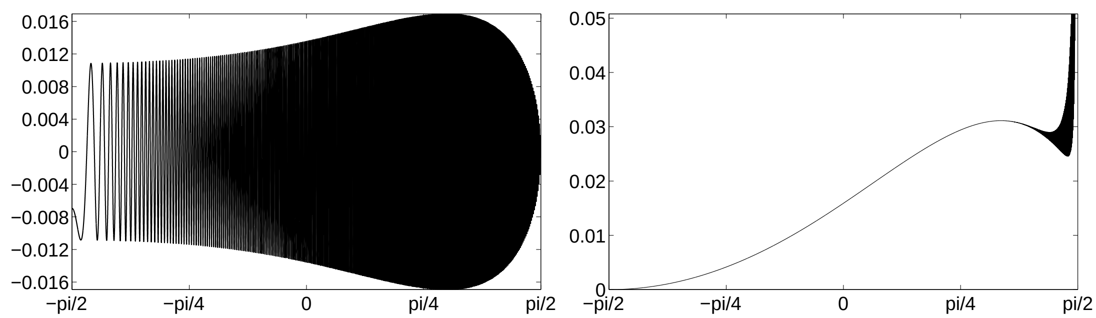
Remark 3.9. *The formula* (3.77) *may also be derived from the formula* (3.48) *for moderate frequencies by letting λ_0 tend to ∞ and appealing to the method of stationary phase. The disadvantage of such an approach, as compared to our approach based on geometric optics, is that it sheds no light on why it fails for frequencies ω of order greater than or equal to ρ^{-2}.*

3.3.5 The three dimensional problem

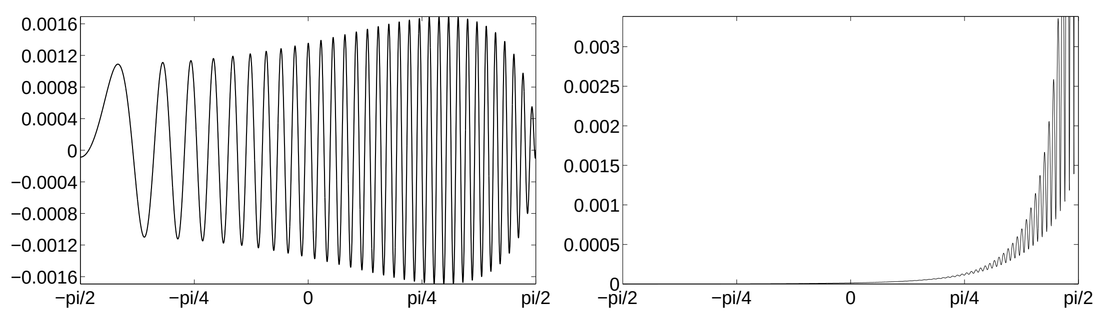
The procedure we followed to arrive at the approximation formula (3.77) as $\lambda = \rho\omega \to \infty$ but $\rho^2\omega \to 0$ applies just as well to the scattering problem in three dimensions. In this case, the radiating free space Green's function $\Phi^\lambda(x, y) = \frac{1}{4\pi}e^{i\lambda|x-y|}/|x-y|$. We



(a) $\rho = 10^{-1}$ and $\omega = 10^3$ so that $\omega\rho = 100$ and $\omega\rho^2 = 10$.



(b) $\rho = 10^{-2}$ and $\omega = 10^5$ so that $\omega\rho = 1000$ and $\omega\rho^2 = 10$.



(c) $\rho = 10^{-4}$ and $\omega = 10^6$ so that $\omega\rho = 100$ and $\omega\rho^2 = 10^{-2}$.

Figure 3.10: Plots on the right half of the circle $r = 2$. Here a plane wave is incident upon a disk as in Figure 3.6. The plots run counterclockwise from $\theta = -\pi/2$ to the highest point at $\theta\pi/2$, where θ is as in Figure 3.6 (which differs from the angle ϑ defined in the approximation formula (3.77); the angle ϑ from (3.77) runs on the horizontal axis from π to 2π). The left half of the circle has been omitted because of the obvious symmetry. The left frames show the graph of $\mathrm{Re}[u_\rho^s]_{\mathrm{geo}}$. The right frames show the error $|u_\rho^s - [u_\rho^s]_{\mathrm{geo}}|$. In each case, $\varepsilon = 2$, $\mu = 2$ and $\sigma/\omega = 2$. Only in the third example is $\omega\rho^2$ small, and so it is not surprising that the error is larger in the first two.

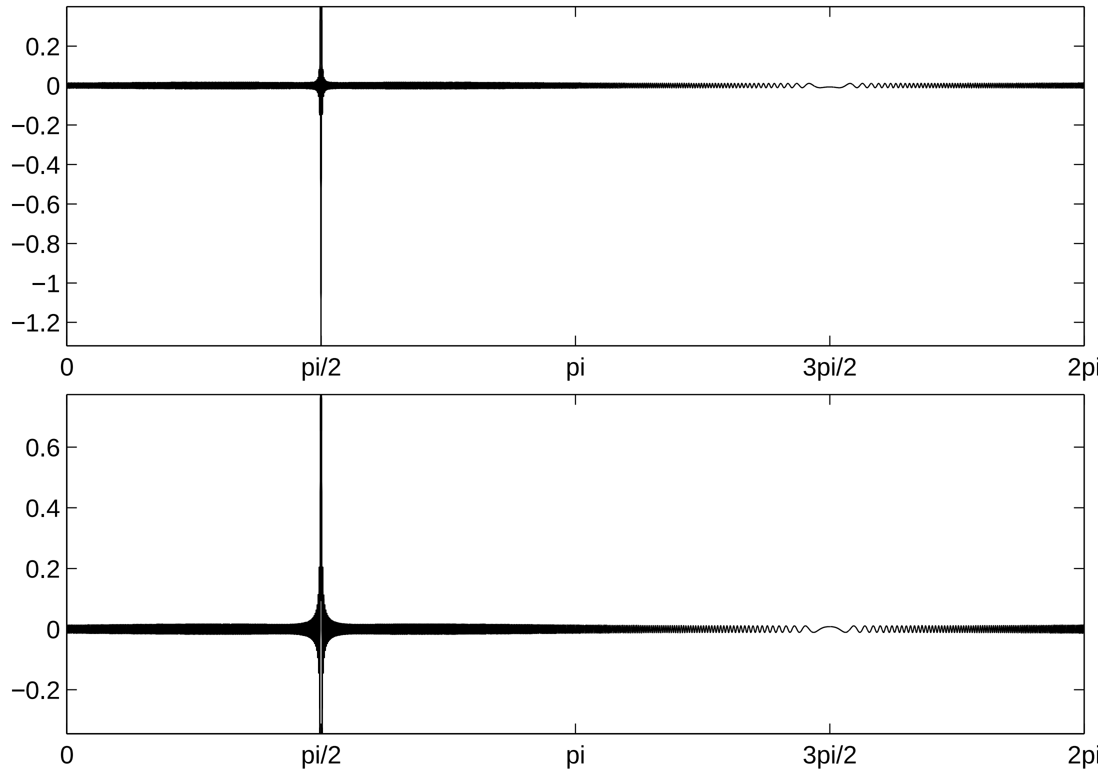


Figure 3.11: $\mathrm{Re}\, u^s_\rho$ (top) and $\mathrm{Im}\, u^s_\rho$ (bottom) at $r = 2$ when $\varepsilon = 2$, $\mu = 2$, $\rho = 10^{-2}$ and $\omega = 10^5$, so that $\rho\omega = 1000$. The lowest point corresponds to the angle $\theta = 3\pi/2$. The shadow portion of $r = 2$ is in a small neighborhood of $\theta = \pi/2$. θ is as in Figure 3.6 (modulo 2π), which differs from the angle ϑ in the approximation formula (3.78).

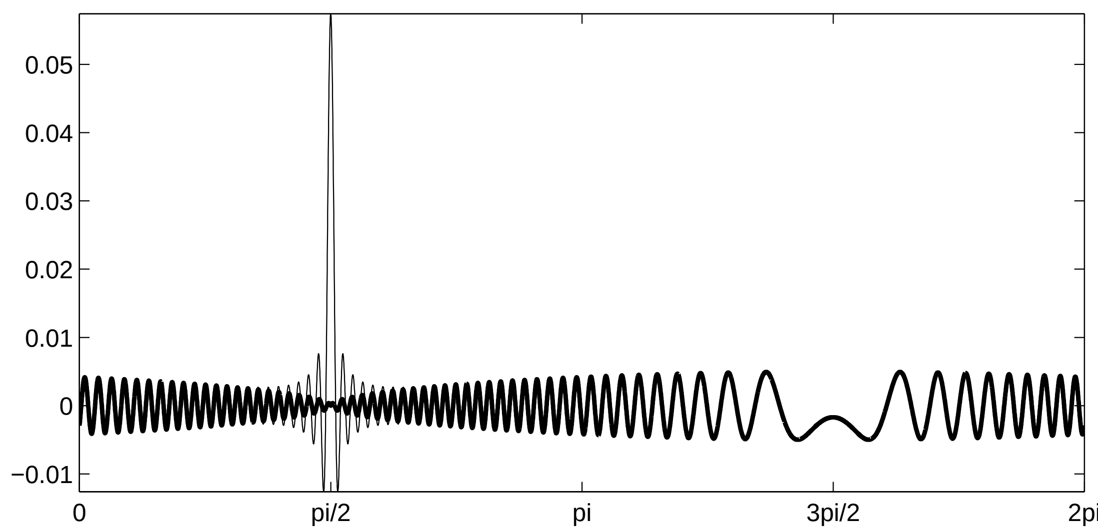


Figure 3.12: The graphs of $\mathrm{Im}\, u^s_\rho$ and $\mathrm{Im}[u^s_\rho]_{\mathrm{geo}}$ (the latter in boldface) on the circle $r = 2$ in the case of a plane wave incident upon a perfectly conducting disk $D = B(0, \rho)$, as in Figure 3.6. On the horizontal axis is the angle θ from Figure 3.6 (which differs from the angle ϑ defined in the approximation formula (3.77)). Here $\rho = 10^{-4}$ and $\omega = 10^6$, so that $\rho\omega = 100$ and $\rho^2\omega = 10^{-2}$. $[u^s_\rho]_{\mathrm{geo}}$ is a good approximation of u^s_ρ on the circle $r = 2$ except in a narrow arc around the shadow cast by the scatterer (centered at $\theta = \pi/2$).

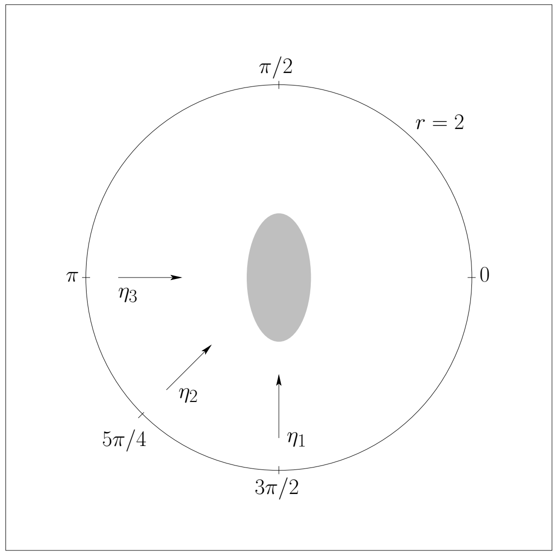


Figure 3.13: Diagram of ellipse orientation and incident directions.

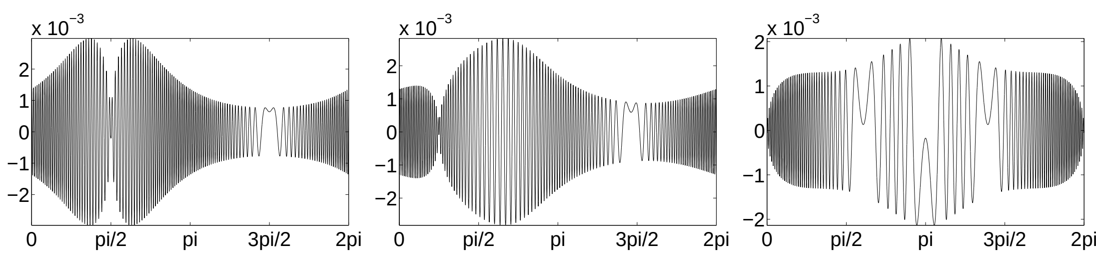


Figure 3.14: The real part of $[u^s_\rho]_{\text{geo}}$ at $r = 2$ in the case of an elliptical scatterer as in Figure 3.13. The left frame corresponds to the incident direction η_1, the middle frame to η_2 and the right frame to η_3. In each case, $\varepsilon = 2$, $\mu = 2$, $\sigma/\omega = 2$, $\omega = 10^6$ and $\rho = 10^{-4}$ so that $\omega\rho = 100$ and $\omega\rho^2 = 10^{-2}$. One can see that the intensity of the approximate field shrinks to zero at the point x in the shadow satisfying $\hat{x} = \eta$, which is a defect of this method.

therefore find the following analogues to (3.72) and (3.73):

$$\begin{aligned}
\Phi^\lambda(x/\rho, y) &= \frac{1}{4\pi}\frac{e^{i\lambda|\frac{x}{\rho}-y|}}{|\frac{x}{\rho}-y|} \\
&\approx \frac{1}{4\pi}\frac{\rho}{|x|}e^{i(\omega|x|-\lambda\hat{x}\cdot y)}, \\
\partial_{\nu_y}\Phi^\lambda(x/\rho, y) &\approx \frac{-i}{4\pi}\rho^2\omega\frac{\hat{x}\cdot\nu_y}{|x|}e^{i(\omega|x|-\lambda\hat{x}\cdot y)}.
\end{aligned}$$

Then, inserting these approximations, along with the approximations $[\tilde{u}^s_\lambda]_{\text{geo}}$ and $\partial_\nu\tilde{u}^s_\lambda|_{\partial D}$ from (3.67) (which remain unchanged in three dimensions) into the Green's representation formula (3.70) yields the approximation

$$u^s_\rho(x) \approx \frac{-i}{4\pi}\rho^2\omega\frac{e^{i\omega|x|}}{|x|}\int_{\partial D}\big(\hat{x}\cdot\nu_y + |\eta\cdot\nu_y|\big)A^s_0(y)e^{i\lambda(\eta-\hat{x})\cdot y}\,\mathrm{d}\sigma_y.$$

As $\lambda\to\infty$, the integral becomes concentrated near the arg-min y_1 of $\psi(y) = (\eta-\hat{x})\cdot y$. (The other stationary point, i.e., the arg-max of ψ, does not contribute to the highest order term of the asymptotic expansion of the integral since the integrand at that point is zero, just as in the two dimensional case). Write the integral as

$$\int_{\partial D} f(y)e^{i\lambda\psi(y)}\,\mathrm{d}\sigma_y,$$

$f(y) = \big(\hat{x}\cdot\nu_y + |\eta\cdot\nu_y|\big)A^s_0(y)$. Rotate the coordinates so that $\nu_{y_1} = (0,0,-1)$, and let ∂D near y_1 be the graph of the function g, with $g(\xi_1) = y_1$. Then, since $\eta-\hat{x}$ points in the same direction as ν_{y_1}, $\psi(y) = (\eta-\hat{x})g(\xi)$, and therefore

$$\int_{\partial D} f(y)e^{i\lambda\psi(y)}\,\mathrm{d}\sigma_y \overset{\text{(to highest order)}}{=} \int_{N_{\xi_1}} f(\xi,\phi(\xi))e^{i\lambda(\eta-\hat{x})g(\xi)}\sqrt{|\nabla g(\xi)|^2+1}\,\mathrm{d}\xi,$$

where N_{ξ_1} is a neighborhood of ξ_1. We now use the method of stationary phase for functions of two variables: if ξ_0 is an isolated stationary point of the function $h:\mathbb{R}^2\to\mathbb{R}$, then

$$\int_{N_{\xi_0}} F(\xi)e^{i\lambda h(\xi)}\,\mathrm{d}\xi = \frac{2\pi}{\lambda}e^{i\pi(\delta+1)\sigma/4}e^{i\lambda h(\xi_0)}F(\xi_0)\big|\det(D^2h)(\xi_0)\big|^{-1/2} + o(\lambda^{-1})$$

as $\lambda \to \infty$, where $\delta = \operatorname{sgn}\big(\det(D^2h)(\xi_0)\big)$ and $\sigma = \operatorname{sgn}\big(\partial_1^2 h(\xi_0)\big)$ (second partial derivative in a coordinate direction) [Coo82]. We therefore find

$$\int_{N_{\xi_1}} f(\xi,\phi(\xi)) e^{i\lambda(\eta-\hat{x})g(\xi)} \sqrt{|\nabla g(\xi)|^2+1}\,\mathrm{d}\xi \\ = \frac{2\pi i}{\lambda} e^{i\lambda(\eta-\hat{x})\cdot y_1} f(y_1) \frac{1}{|\eta-\hat{x}|\sqrt{K(y_1)}} + o(\lambda^{-1}),$$

where $K(\cdot)$ is the Gaussian curvature function on $\partial\Omega$. This leads us to the approximation formula[19]

$$u_\rho^s(x) \approx -\rho\frac{1}{2}\frac{e^{i\omega|x|}}{|x|}\frac{e^{i\lambda(\eta-\hat{x})\cdot y_1}}{\sqrt{K(y_1)}}\frac{\mu\,\eta\cdot\nu_{y_1} + \sqrt{q(y_1)-1+|\eta\cdot\nu_{y_1}|^2}}{\mu|\eta\cdot\nu_{y_1}| + \sqrt{q(y_1)-1+|\eta\cdot\nu_{y_1}|^2}}.$$

Noting that $\eta\cdot\nu_{y_1} = -\hat{x}\cdot\nu_{y_1}$, we may also write this formula as

$$u_\rho^s(x) \approx \rho\frac{1}{2}\frac{e^{i\omega|x|}}{|x|}\frac{e^{i\lambda(\eta-\hat{x})\cdot y_1}}{\sqrt{K(y_1)}}\frac{\mu\,\hat{x}\cdot\nu_{y_1} - \sqrt{q(y_1)-1+|\hat{x}\cdot\nu_{y_1}|^2}}{\mu\,\hat{x}\cdot\nu_{y_1} + \sqrt{q(y_1)-1+|\hat{x}\cdot\nu_{y_1}|^2}}. \tag{3.79}$$

As in the two dimension case, this formula applies only when there is ample absorption within the strictly convex scatterer.

Remark 3.10. *Majda and Taylor [MT77] derived a similar approximation formula for the scattering amplitude*[20] *(or, more precisely, a convolution of the scattering amplitude by a smoothing kernel, which they call the filtered scattering amplitude). Their result applies in the context of the following three dimensional scattering problem: given* $u^i(x) = e^{i\omega\eta\cdot x}$, *with* $\eta\in\mathbb{R}^3$ *satisfying* $|\eta|=1$, *determine* (u^{tr},u^s) *such that*

$$\begin{cases} \Delta u^{tr} + \omega^2 q(x) u^{tr} = 0 \quad \text{in } \Omega, \\ \Delta u^s + \omega^2 u^s = 0 \quad \text{in } \mathbb{R}^3\setminus\Omega, \end{cases}$$

[19] Here we have used the fact that $\hat{x}\cdot\nu_{y_1} + |\eta\cdot\nu_{y_1}| = |\eta-\hat{x}|$.

[20] For an exterior scattered field u^s satisfying $(\Delta+\omega^2)u^s = 0$, the scattering amplitude u_∞, also called the far field pattern, is defined on $\mathbb{T}$ for two dimensional problems as the unique function satisfying $u^s(x) = \frac{e^{i\omega r}}{\sqrt{r}}\big[u_\infty(\hat{x}) + O(r^{-1})\big]$, and on $\mathbb{S}^2$ for three dimensional problems as the unique function such that $u^s(x) = \frac{e^{i\omega r}}{r}\big[u_\infty(\hat{x}) + O(r^{-1})\big]$.

with the transmission conditions

$$\begin{cases} u^{tr} = u^s + u^i & \text{on } \partial\Omega, \\ \partial_\nu u^{tr} = \partial_\nu u^s + \partial_\nu u^i & \text{on } \partial\Omega, \end{cases}$$

and the radiation condition

$$(\partial_r - i\omega)u^s = o(r^{-1}) \quad \text{as } \omega \to \infty,$$

where Ω *is a smooth, bounded, convex domain in* $\mathbb{R}^3$ *and the refractive index* $q > 1$ *is a smooth function on* $\overline{\Omega}$. *They proved that the filtered scattering amplitude* $a(\hat{x}, \eta, \omega)$ *admits, for* $\hat{x} \neq \eta$, *the asymptotic expansion*

$$a(\hat{x}, \eta, \omega) = -\frac{1}{4\pi} \frac{e^{i\omega(\eta - \hat{x})\cdot y_1}}{\sqrt{K(y_1)}} \frac{\nu_{y_1} \cdot \hat{x} - \sqrt{q(y_1) - 1 + |\nu_{y_1} \cdot \hat{x}|^2}}{\nu_{y_1} \cdot \hat{x} + \sqrt{q(y_1) - 1 + |\nu_{y_1} \cdot \hat{x}|^2}} + O(\omega^{-1}) \tag{3.80}$$

as $\omega \to \infty$, *where* y_1 *is the arg-min of* $y \mapsto y \cdot (\eta - \hat{x})$ *on* $\partial\Omega$ *and* $K(\cdot)$ *is the Gaussian curvature function on* $\partial\Omega$.[21] *This result is achieved by performing a microlocal analysis of a Fourier integral operator associated with the scattering amplitude, which includes a stationary phase analysis similar to ours. Such a method may perhaps be used to prove that the approximation* (3.77) *is in fact the leading order term of an asymptotic expansion of the scattered field as* $\omega\rho \to \infty$, $\omega\rho^2 \to 0$.

3.3.6 An alternate approach for high frequencies

In the previous sections we: 1) found the geometric optics approximations (3.67) of $\tilde{u}^s_\lambda|_{\partial D}$ and $\partial_\nu \tilde{u}^s_\lambda|_{\partial D}$, 2) substituted these approximations into Green's formula for the scattered field, and then 3) performed a stationary phase analysis on this integral. It would be natural to ask what would happen if we, in a sense, were to reverse the order of 2) and 3). That is, if we were to perform a stationary phase analysis to calculate the

[21] This formula was presented as

$$a(\hat{x}, \eta, \omega) = -\frac{1}{2\pi} \frac{\nu_{y_*} \cdot \hat{x}}{|\eta - \hat{x}|} \frac{e^{i\omega(\eta - \hat{x})\cdot y_*}}{\sqrt{K(y_*)}} \frac{\nu_{y_*} \cdot \hat{x} - \sqrt{q(y_*) - 1 + |\nu_{y_*} \cdot \hat{x}|^2}}{\nu_{y_*} \cdot \hat{x} + \sqrt{q(y_*) - 1 + |\nu_{y_*} \cdot \hat{x}|^2}} + O(\omega^{-1})$$

in [MT77], but note that $2(\nu_{y_*} \cdot \hat{x}) = |\eta - \hat{x}|$ since $\nu_{y_*} = -(\eta - \hat{x})/|\eta - \hat{x}|$.

distributional limit of (3.67a) and (3.67b) and then insert the resulting sum of delta functions into the Green's representation formula for the scattered field.

By choosing $\hat{x} = -\eta$ and then replacing λ with $\lambda/2$, we get

$$\int_{\partial D} a(y) e^{i\lambda y\cdot\eta}\,\mathrm{d}\sigma_y = \frac{\sqrt{2\pi}}{\sqrt{\lambda}}\left(\frac{e^{i\pi/4}}{\sqrt{K(y_1^*)}} e^{i\lambda y_1^*\cdot\eta} a(y_1^*) \right.$$
$$\left. + \frac{e^{-i\pi/4}}{\sqrt{K(y_2^*)}} e^{i\lambda y_2^*\cdot\eta} a(y_2^*)\right) + o(\lambda^{-1/2})$$

as $\lambda \to \infty$, where y_1^* is the arg-min of $y\cdot\eta$ on ∂D and y_2^* is the arg-max ($\nu_{y_1^*} = -\eta$ and $\nu_{y_2^*} = \eta$). At the points y_1^* and y_2^*,

$$A_0^s(y_1^*) = \frac{\mu - \sqrt{q}}{\mu + \sqrt{q}} \quad \text{and} \quad A_0^s(y_2^*) = -1.$$

Therefore, the distributional limits as $\lambda \to \infty$ of (3.67a) and (3.67b) are

$$[\tilde{u}_\lambda^s]_{\mathrm{geo}} \to \sqrt{\frac{2\pi}{\lambda}}\left(\frac{\mu-\sqrt{q}}{\mu+\sqrt{q}}\frac{e^{i\pi/4}}{\sqrt{K(y_1^*)}} e^{i\lambda y_1^*\cdot\eta}\delta_{y_1^*} - \frac{e^{-i\pi/4}}{\sqrt{K(y_2^*)}} e^{i\lambda y_2^*\cdot\eta}\delta_{y_2^*}\right), \tag{3.81}$$

$$[\partial_\nu \tilde{u}_\lambda^s]_{\mathrm{geo}} \to i\sqrt{2\pi\lambda}\left(\frac{\mu-\sqrt{q}}{\mu+\sqrt{q}}\frac{e^{i\pi/4}}{\sqrt{K(y_1^*)}} e^{i\lambda y_1^*\cdot\eta}\delta_{y_1^*} - \frac{e^{-i\pi/4}}{\sqrt{K(y_2^*)}} e^{i\lambda y_2^*\cdot\eta}\delta_{y_2^*}\right). \tag{3.82}$$

We then insert (3.81) and (3.82) into Green's formula (3.70) and find

$$u_\rho^s(x) \approx \sqrt{\frac{2\pi}{\lambda}}\left[\frac{\mu-\sqrt{q}}{\mu+\sqrt{q}}\frac{e^{i\pi/4}}{\sqrt{K(y_1^*)}} e^{i\lambda y_1^*\cdot\eta}\Big(\partial_{\nu_y}\Phi^\lambda(x/\rho, y_1^*) - i\lambda\Phi^\lambda(x/\rho, y_1^*)\Big)\right.$$
$$\left. + \frac{e^{-i\pi/4}}{\sqrt{K(y_2^*)}} e^{i\lambda y_2^*\cdot\eta}\Big(\partial_{\nu_y}\Phi^\lambda(x/\rho, y_2^*) - i\lambda\Phi^\lambda(x/\rho, y_2^*)\Big)\right].$$

By (3.72) and (3.73),

$$\partial_{\nu_y}\Phi^\lambda(x/\rho, y) - i\lambda\Phi^\lambda(x/\rho, y)$$
$$= \rho\sqrt{\omega}\frac{e^{-i\pi/4}}{\sqrt{8\pi|x|}} e^{i(\omega|x| - \lambda\hat{x}\cdot y)}\left[\hat{x}\cdot\nu_y + 1\right]\big(1 + O(\rho + \omega^{-1} + \omega\rho^2)\big),$$

and so we have the approximation

$$u^s_\rho(x) \approx \frac{\sqrt{\rho}}{2}\frac{e^{i\omega|x|}}{\sqrt{|x|}}\left(\frac{\mu-\sqrt{q}}{\mu+\sqrt{q}}(1-\cos\vartheta)\frac{e^{i\lambda(\eta-\hat{x})\cdot y_1^*}}{\sqrt{K(y_1^*)}}+i(1+\cos\vartheta)\frac{e^{i\lambda(\eta-\hat{x})\cdot y_1^*}}{\sqrt{K(y_2^*)}}\right). \quad (3.83)$$

This approximation is far worse than (3.77), as demonstrated in Figure 3.15. The inad-

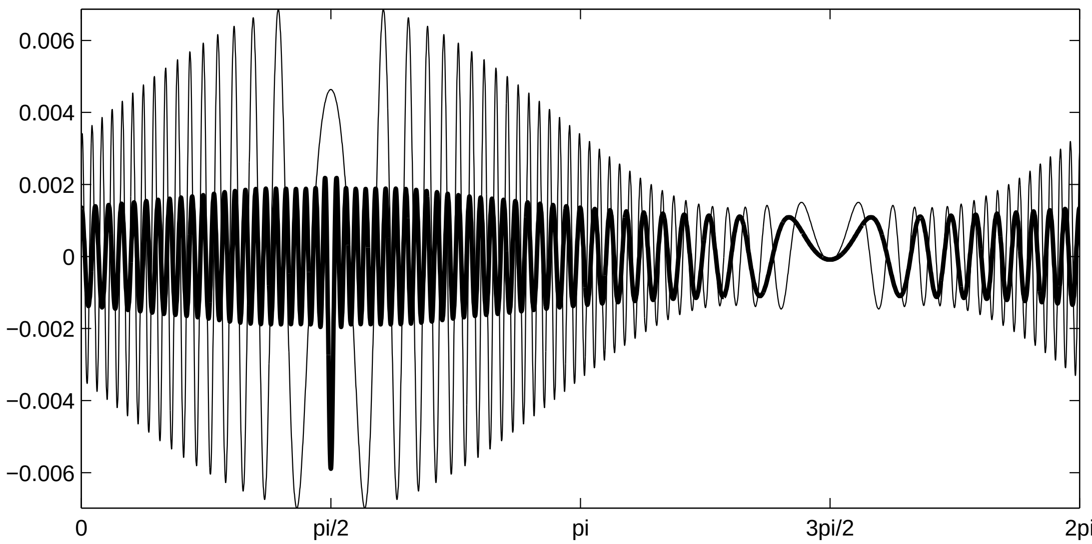


Figure 3.15: The real parts of $u^s_\rho|_{r=2}$ (in boldface) and the alternate approximation formula (3.83) in the case of a plane wave incident upon a disk as in Figure 3.6. In this example, $\mu = 2$, $\varepsilon = 2$, $\sigma/\omega = 2$, $\rho = 10^{-4}$ and $\omega = 10^6$, so that $\omega\rho = 100$ and $\omega\rho^2 = 10^{-2}$.

equacy of this alternate method stems from the fact that, unlike the prior method, the oscillations in the Green's function and the oscillations in the Cauchy data ($[u^s_\rho]_{\text{geo}}\big|_{\partial D}$ and $[\partial_\nu u^s_\rho]_{\text{geo}}\big|_{\partial D}$) are not treated simultaneously. We have presented this alternative method to illustrate the importance of this simultaneous treatment of the oscillations.

3.3.7 Multiple scatterers

If there are multiple scatterers $z_j + \rho D_j$, $j = 1, 2, \ldots, m$, then (3.77) can be replaced with the sum

$$u^s(x) \approx \sum_j \left(\frac{\sqrt{\rho}}{\sqrt{2}} \sqrt{\sin(\vartheta_j/2)} \frac{\mu_j \sin(\vartheta_j/2) - \sqrt{q_j - 1 + \sin^2(\vartheta_j/2)}}{\mu_j \sin(\vartheta_j/2) + \sqrt{q_j - 1 + \sin^2(\vartheta_j/2)}} \right. \\ \left. \times \frac{e^{i\rho\omega\left(\eta - (b_j^{x,\eta} - z_j)/|b_j^{x,\eta} - z_j|\right)\cdot(x - z_j)}}{\sqrt{K(b_j^{x,\eta})}} \frac{e^{i\omega|x - z_j|}}{\sqrt{|x - z_j|}} \right), \tag{3.84}$$

for x bounded away from the scatterers, where $0 < \vartheta_j < 2\pi$ is the counterclockwise angle of rotation between η and $x - z_j$, $K(\cdot) > 0$ is the curvature, and $b_j^{x,\eta}$ is the unique point on the boundary ∂D_j with outward normal pointing in the direction of $(x - z_j) - \eta$. This formula should be valid as long as the scatterers are well separated (i.e., the minimum distance between pairs of scatterers is on the order of 1) and the observation point x is sufficiently far outside a convex set containing all the scatterers. This is because the disturbance transmitted from any one scatterer to another will be of the order $\sqrt{\rho}/d$, where d is the distance between the scatterers. The disturbance will be transmitted, from the second scatter, back to the observation point will an amplitude on the order of ρ. Adding together all other higher order multiply scattered signals should amount to, at most, a total signal of order $\rho + \rho^{3/2} + \rho^2 + \rho^{5/2} + \cdots = O(\rho)$. This by no means rigorous argument. After all, the number of n^{th} order signals grows exponentially in n. Nonetheless, we are confident in the validity of (3.84).

We note that formula (3.84) may not be accurate if one scatterer lies in the shadow of another. Testing from multiple directions will remedy this.

3.3.8 Future directions: the inverse problem

The (approximate) representation formula (3.77) for the scattered field is ripe with information that should be useful in solving the inverse problem of determining characteristics of the inhomogeneity (size, shape, etc.); for instance, the factors $\sqrt{\rho}$ and $1/K(y_1)$.

In the context of the three dimensional scattering problem of Remark 3.10, Majda and Taylor [MT77] showed, using the asymptotic formula (3.80), that knowledge of the high frequency asymptotics of the filtered differential scattering cross section, $|a(\hat{x},\eta,\omega)|^2$, over a sufficient sample of values[22] of $\hat{x}$ and η, determines the unique convex shape of the scatterer Ω, and the index of refraction $q(\cdot)\big|_{\Omega}$, assuming, for the latter, that $q > 1$. (See [Maj76a], [Maj77] and, for a brief overview, [Maj76b]). At the heart of this proof is the observation that, given a function of the form

$$g(t,\alpha,\beta) = \alpha\left(\frac{t-\sqrt{\beta+t^2}}{t+\sqrt{\beta+t^2}}\right), \quad \text{for } t,\alpha,\beta > 0,$$

if t_1 and t_2 are distinct positive numbers, the values $g(t_1,\alpha,\beta)$ and $g(t_2,\alpha,\beta)$ uniquely determine α and β. If one can first show that (3.77) is indeed a genuine asymptotic expansion, then this sort of approach would surely be useful in proving an analogous theoretical result in the context of the problem of a two dimensional conducting scatterer.

[22]Specifically, the high frequency asymptotics must be known for all $(\hat{x},\eta) \in V_1 \cap V_2$, where V_1 and V_2 are complementary *subsets of determinacy.* A subset of determinacy V is a finite union of open subsets of $\mathbb{S}^2 \times \mathbb{S}^2 \setminus \{\hat{x} = \eta\}$ such that the function $(\hat{x},\eta) \mapsto \frac{\eta-\hat{x}}{|\eta-\hat{x}|}$ maps V onto $\mathbb{S}^2$.

Chapter 4

Rigorous norm estimates of the scattered field

In the previous chapter we derived a formal approximation of the scattered field when $\rho \to 0$ and $\omega \to \infty$ in such a way that $\lambda = \omega\rho \to \infty$ based on a geometric optics approach. This first step toward proving a *rigorous* asymptotic estimate is to show that the scattered field is bounded in some Sobolev space as $\rho \to 0$, for then we would know that a subsequence converges weakly to what would surely be the asymptotic limit (cf. Chapter 2). This task turns out to surprisingly difficult. In this chapter we will prove such bounds in the case where the scatterer is a disk. First we will use a method based on separation of variables to establish a bound on

$$\|u_\rho^s|_{r=r_0}\|_{H^\sigma(\mathbb{T})} = \left[\sum_{n=-\infty}^{\infty} \left|\widehat{(u_\rho^s|_{r=r_0})}_n\right|^2 (1+n^2)^{2\sigma}\right]^{1/2}$$

of order $\sqrt{\rho}$ that depends on the incident field u^i. (The order σ here has no relation to conductivity.) The circular geometry of the scatterer is inherent to such a method, so there is little hope that it will generalize to arbitrary convex scatterers. Fortunately, there is another method, one based on wave equation factorization and pseudodifferential operators, that is very likely to generalize. We present this alternate method in the latter part of this chapter.

4.1 A bound via separation of variables

Let $\mu_0 > 0$, $\varepsilon_0 > 0$, and $\sigma_0 \geq 0$ denote, respectively, the magnetic permeability, the electric permittivity and the conductivity of the background medium, and let $\mu > 0$, $\varepsilon > 0$ and $\sigma > 0$ denote those of the inhomogeneity ρD, $D = B(0,1)$. Let $q = \mu\boldsymbol{\epsilon} =$

$\mu(\varepsilon + i\sigma/\omega)$ and $q_0 = \mu_0 \boldsymbol{\epsilon}_0 = \mu_0(\varepsilon_0 + i\sigma_0/\omega_0)$. For simplicity, we write

$$q = a + ib,$$

where $a, b > 0$. Recall the transmission problem: Given an incident wave u^i satisfying $(\Delta + q_0\omega^2)u^i = 0$ in $\mathbb{R}^2$, the transmitted and scattered fields, u_ρ^{tr} and u_ρ^s, are the unique solutions to

$$\begin{cases} \Delta u_\rho^{tr} + q\omega^2 u_\rho^{tr} = 0 \quad (r < \rho), \\ \Delta u_\rho^s + q_0\omega^2 u_\rho^s = 0 \quad (r > \rho), \end{cases} \tag{4.1a}$$

satisfying the transmission conditions

$$\begin{cases} \dfrac{1}{\mu}\partial_r u_\rho^{tr}\big|_{r=\rho} = \dfrac{1}{\mu_0}\left(\partial_r u_\rho^s\big|_{r=\rho} + \partial_r u^i\big|_{r=\rho}\right), \\ \qquad u_\rho^{tr}\big|_{r=\rho} = u_\rho^s\big|_{r=\rho} + u^i\big|_{r=\rho}, \end{cases} \tag{4.1b}$$

as well as Sommerfeld's outgoing radiation condition,

$$\partial_r u_\rho^s - i\sqrt{q_0}\,\omega u_\rho^s = O(r^{-3/2}) \quad \text{as } r \to \infty \tag{4.1c}$$

(here we have not normalized q_0 to be 1, as we did in Chapter 3).

Note: *If we let $\gamma = \frac{\mu_0}{\mu}$, the transmission condition for the normal derivative becomes*

$$\gamma \partial_r u_\rho^{tr}\big|_{r=\rho} = \left(\partial_r u_\rho^s\big|_{r=\rho} + \partial_r u^i\big|_{r=\rho}\right).$$

For the problem (4.1) to be properly posed, we need only require a given q, q_0, γ, ω, ρ, and a given incident wave u^i satisfying $(\Delta + q_0\omega^2)u^i = 0$.

We suppose the incident wave has the form

$$u^i(x) = \sum_{n=-\infty}^{\infty} a_n J_n(\sqrt{q_0}\,\omega r) e^{in\theta}, \tag{4.2}$$

where $\{a_n\}_{n=-\infty}^{\infty}$ satisfies

$$\|\{a_n\}\|_{h^s}^2 := \sum_{n=-\infty}^{\infty} |a_n|^2(1+|n|)^{2s} < \infty \tag{4.3}$$

for some $s \in \mathbb{R}$ (cf. (3.5) and (3.6) from Chapter 3). Recall from the proof of Theorem 4 of Chapter 3 that

$$u_\rho^s = \sum_{n=-\infty}^{\infty} \alpha_n H_n(\omega r)e^{in\theta}, \quad u_\rho^{tr} = \sum_{n=-\infty}^{\infty} \beta_n J_n(\sqrt{q}\omega r)e^{in\theta}, \tag{4.4}$$

where

$$\alpha_n = a_n \frac{\gamma\sqrt{q}J_n'(\sqrt{q}\lambda)J_n(\sqrt{q_0}\lambda) - \sqrt{q_0}J_n'(\sqrt{q_0}\lambda)J_n(\sqrt{q}\lambda)}{\sqrt{q_0}H_n'(\sqrt{q_0}\lambda)J_n(\sqrt{q}\lambda) - \gamma\sqrt{q}J_n'(\sqrt{q}\lambda)H_n(\sqrt{q_0}\lambda)}, \tag{4.5}$$

$$\beta_n = a_n \frac{2i}{\pi\lambda}\left(\frac{1}{\sqrt{q_0}H_n'(\sqrt{q_0}\lambda)J_n(\sqrt{q}\lambda) - \gamma\sqrt{q}J_n'(\sqrt{q}\lambda)H_n(\sqrt{q_0}\lambda)}\right), \tag{4.6}$$

with $\lambda = \omega\rho$.

4.1.1 Incident waves with finitely many Fourier coefficients

We will first study the case of where the incident wave u^i has only finitely many coefficients. To estimate the size of the scattered field u_ρ^s, we must estimate the coefficients α_n and β_n. To do this, we appeal to the asymptotic properties of Bessel functions.

Lemma 4.1. *Let $c = \alpha + i\beta$, with $\beta > 0$. Then for any integer $n \geq 0$, as $0 < t \to \infty$,*

$$H_n'(t)J_n(ct) = i\frac{e^{\beta t}e^{i(1-\alpha)t}}{\pi t\sqrt{c}}\big(1 + O(t^{-1})\big),$$
$$J_n'(ct)H_n(t) = -i\frac{e^{\beta t}e^{i(1-\alpha)t}}{\pi t\sqrt{c}}\big(1 + O(t^{-1})\big).$$

Furthermore, for each $n \in \mathbb{Z}$ there exists a constant $C_{n,c} > 0$ such that for all $t > 0$,

$$|J_n'(t)J_n(ct)| \leq C_{n,c}\frac{e^{\beta t}}{t},$$
$$|J_n'(ct)J_n(t)| \leq C_{n,c}\frac{e^{\beta t}}{t}.$$

Proof. We will use the Identity

$$F_n'(z) = \frac{1}{2}\big[F_{n-1}(z) - F_{n+1}(z)\big],$$

which holds for both J_n and H_n, and the asymptotic formula

$$H_n(z) = \sqrt{\frac{2}{\pi z}} e^{i(z - n\pi/2 - \pi/4)} \Big(1 + O(z^{-1})\Big) \quad \text{as } |z| \to \infty$$

[Wat44, §7.21]. We will also use the formula

$$\begin{aligned} J_n(z) = c_1 \sqrt{\frac{2}{\pi z}} e^{i(z - n\pi/2 - \pi/4)} &\Big(1 + O(z^{-1})\Big) \\ &+ c_2 \sqrt{\frac{2}{\pi z}} e^{-i(z - n\pi/2 - \pi/4)} \Big(1 + O(z^{-1})\Big) \quad \text{as } |z| \to \infty, \end{aligned}$$

which is valid in the annular sector $\{|\arg z| < \pi,\ |z| \geq c > 0\}$ (any $c > 0$) with $c_1 = c_2 = 1/2$. In the annular sector $\{0 < \arg z < 2\pi,\ |z| \geq c > 0\}$, the formula, with the branch of $\sqrt{\cdot}$ changed appropriately, is valid with $c_1 = -1/2$, $c_2 = 1/2$ [Wat44, §7.22].[1] Because of this, and because of the continuity of J_n at zero, we obtain the inequality

$$|J_n(z)| \leq C_n \sqrt{\frac{2}{\pi |z|}} e^{|\operatorname{Im} z|} \tag{4.7}$$

for all $z \in \mathbb{C}$.

We have

$$\begin{aligned} H_n'(t) J_n(ct) &= \frac{1}{2}\big[H_{n-1}(t) - H_{n+1}(t)\big] J_n(ct) \\ &= \frac{1}{2}\sqrt{\frac{2}{\pi t}} e^{i(t - n\pi/2 - \pi/4)} \Big[e^{i\pi/2}\Big(1 + O(t^{-1})\Big) \\ &\qquad\qquad - e^{-i\pi/2}\Big(1 + O(t^{-1})\Big)\Big] J_n(ct) \\ &= i\sqrt{\frac{2}{\pi t}} e^{i(t - n\pi/2 - \pi/4)} \Big(1 + O(t^{-1})\Big) J_n(ct). \end{aligned}$$

[1] This discontinuity of the constants is known as Stokes' phenomenon.

Since $\beta = \operatorname{Im} t > 0$, the asymptotic formula for J_n implies

$$J_n(ct) = \frac{1}{2}\sqrt{\frac{2}{\pi ct}}e^{-i(ct-n\pi/2-\pi/4)}\Big(1 + O(t^{-1})\Big) \quad \text{as } |t| \to \infty.$$

Therefore,

$$H_n'(t)J_n(ct) = \frac{i}{\pi t\sqrt{c}}e^{i(1-c)t}\Big(1 + O(t^{-1})\Big),$$

which is the first asymptotic identity. The proof of the second is similar. To prove the first inequality, we use (4.7):

$$\begin{aligned}
|J_n'(t)J_n(ct)| &= \Big|\frac{1}{2}\big[J_{n-1}(t) - J_{n+1}(t)\big]J_n(ct)\Big| \\
&\leq C_n\sqrt{\frac{2}{\pi t}}\sqrt{\frac{2}{\pi ct}}e^{|\operatorname{Im} ct|} \\
&= C_{n,c}\frac{e^{bt}}{t}.
\end{aligned}$$

The proof of the second inequality is similar. □

We may now estimate the size of α_n and β_n in terms of $\lambda = \omega\rho$.

Lemma 4.2. *Suppose $\gamma > 0$, $q_0 > 0$ and $q = a + ib$ with $a, b > 0$, and let α_n and β_n be defined as (4.5) and (4.6). Given any fixed $n \in \mathbb{Z}$, there exists a constant $D_n = D_n(q, q_0, \gamma)$, independent of $\lambda > 0$ and a_n, such that*

$$|\alpha_n| \leq D_n|a_n|.$$

Furthermore, given any fixed $n \in \mathbb{Z}$ and $\lambda_0 > 0$, there exists a constant E_n depending on q, q_0, γ and λ_0 but independent of λ and a_n, such that for all $\lambda \geq \lambda_0$,

$$|\beta_n| \leq E_n|a_n|e^{-\lambda \operatorname{Im}\sqrt{q}}.$$

Proof. Using Lemma 4.1 with $c = \sqrt{q}/\sqrt{q_0}$ and $t = \sqrt{q_0}\lambda$, we find that the denominator from (4.5) and (4.6),

$$\begin{aligned}\sqrt{q_0}H_n'(\sqrt{q_0}\lambda)J_n(\sqrt{q}\lambda)-&\gamma\sqrt{q}J_n'(\sqrt{q}\lambda)H_n(\sqrt{q_0}\lambda)\\ &= \frac{i}{\pi\lambda}\frac{\sqrt{q_0}+\gamma\sqrt{q}}{(q_0q)^{1/4}}e^{\lambda(\operatorname{Im}\sqrt{q})}e^{i(\sqrt{q_0}-\operatorname{Re}\sqrt{q})\lambda}\big(1+O(\lambda^{-1})\big)\end{aligned}$$

as $\lambda \to \infty$. We also find, for the numerator from (4.5),

$$\left|\gamma\sqrt{q}J_n'(\sqrt{q}\lambda)J_n(\sqrt{q_0}\lambda)-\sqrt{q_0}J_n'(\sqrt{q_0}\lambda)J_n(\sqrt{q}\lambda)\right| \le C_{n,q,q_0,\gamma}\frac{e^{\lambda(\operatorname{Im}\sqrt{q})}}{\lambda}$$

for all $\lambda > 0$. Combining these, and appealing to the continuity of α_n with respect to λ, we get, for all $\lambda \ge c$ (any fixed $c > 0$),

$$\begin{aligned}|\alpha_n| &= \left|a_n\frac{\gamma\sqrt{q}J_n'(\sqrt{q}\lambda)J_n(\sqrt{q_0}\lambda)-\sqrt{q_0}J_n'(\sqrt{q_0}\lambda)J_n(\sqrt{q}\lambda)}{\sqrt{q_0}H_n'(\sqrt{q_0}\lambda)J_n(\sqrt{q}\lambda)-\gamma\sqrt{q}J_n'(\sqrt{q}\lambda)H_n(\sqrt{q_0}\lambda)}\right|\\ &\le D|a_n|,\end{aligned}$$

with $D = D(n, q, \gamma, c)$. But from the proof of Theorem 4 of Chapter 3 (adapted slightly to accommodate $q_0 \neq 1$), we see easily that

$$|\alpha_n| \le C_{q,q_0,\gamma}\lambda^2|a_n|$$

for all $n \in \mathbb{Z}$ and all $\lambda > 0$ sufficiently small. It is therefore clear that D may be chosen independently of c, and the desired estimate of $|\alpha_n|$ follows. The desired estimate of $|\beta_n|$ follows by a similar argument. □

Remark 4.3. *The requirement that* $\operatorname{Im} q$ *be positive turns out to be unnecessary. We will defer the proof of this until Remark 4.6.*

We are now ready to present a bound on the scattered field in the special case of an incident wave with finitely many Fourier coefficients.

Theorem 5. *Suppose* $\gamma > 0$, $q_0 > 0$ *and* $q = a + ib$ *with* $a > 0$, $b \ge 0$. *Let the incident* *wave be of the form*

$$u^i(x) = \sum_{|n|\le N} a_n J_n(\sqrt{q_0}\omega r)e^{in\theta} \tag{4.8}$$

for some fixed $N > 0$ *and any coefficients* $a_n \in \mathbb{C}$, $-N \le n \le N$. *Let* (u^{tr}_ρ, u^s_ρ) *solve the transmission problem* (4.1). *Given any* $\lambda_0 > 0$, *there exists a constant* $C_N = C(N, \gamma, q, q_0, \lambda_0)$, *independent of* $\rho > 0$, $\omega > 0$ *and* $\{a_n\}$, *such that for any* $r_0 \ge \rho$,

$$\left\| u^s_\rho\big|_{r=r_0} \right\|_{L^2(\mathbb{T})} \le C_N \Big(\max_{|n|\le N} |a_n| \Big) \frac{1}{\sqrt{\omega r_0}} \quad \text{for } \omega\rho \ge \lambda_0.$$

(cf. Remark 3.2 of Chapter 3 for the L^2 *estimate when* $\omega\rho$ *is near zero.)*

Proof. Using the bound on α_n from Lemma 4.2 (which holds even when $b = 0$; cf. Remark 4.6), we get

$$\begin{aligned} \left\| u^s_\rho\big|_{r=r_0} \right\|^2_{L^2(\mathbb{T})} &= 2\pi \sum_{|n|\le N} |\alpha_n|^2 |H_n(\sqrt{q_0}\omega r_0)|^2 \\ &\le C_N \max_{|n|\le N} \{|a_n|^2\} \max_{|n|\le N} |H_n(\sqrt{q_0}\omega r_0)|^2. \end{aligned}$$

The estimate follows from the fact that $0 < t \mapsto t|H_n(t)|^2$ is decreasing for integers $n \ne 0$, and the fact that $t|H_0(t)|^2 \le 2/\pi$ for $t > 0$ [Wat44, 13.74]. □

The optimality of the estimate is illustrated by Figure 4.1.

Remark 4.4. *This estimate will not hold in general for incident waves with infinite Fourier series, as seen in Figure 3.1 of Chapter 3. However, this estimate does imply that*

$$\left\| u^s_\rho\big|_{r=r_0} \right\|_{L^2(\mathbb{T})} \le C_N \left\| \{a_n\} \right\|_{l^\infty} \frac{\sqrt{\rho}}{\sqrt{r_0}} \quad \text{for } \omega\rho \ge \lambda_0 > 1,$$

and this sort of bound is consistent with Figure 3.1 of Chapter 3, and with the formal asymptotics performed later in that chapter. We conjecture that such a bound holds for general $\{a_n\}_{n\in\mathbb{Z}} \in l^\infty$, *with* C_N *replaced by a constant independent of* n.

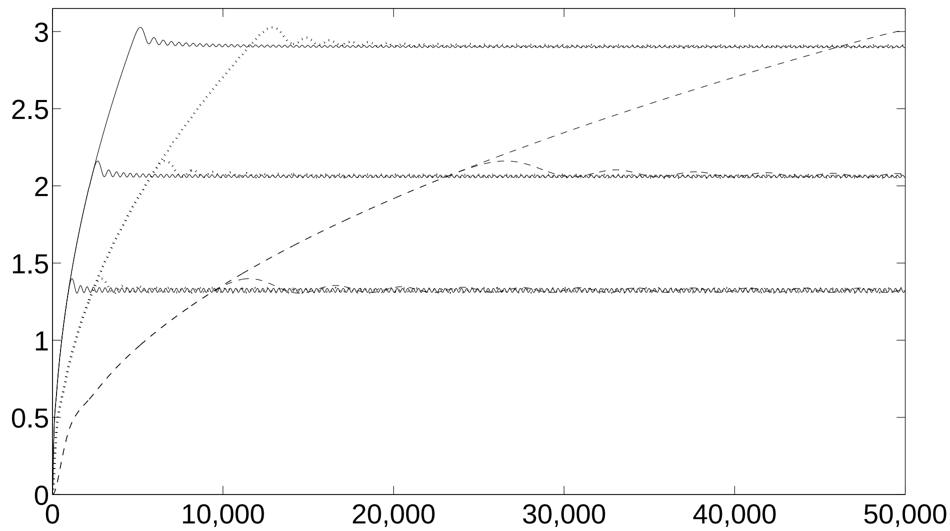


Figure 4.1: Plots of $\sqrt{\omega}\left\|u_\rho^s|_{\partial B(0,\rho)}\right\|_{L^2(\mathbb{T})}$ as a function of ω when $u^i(r,\theta) = \sum_{|n|\le N} J_n(\omega r)e^{in\theta}$ (the truncated series of the plane wave $e^{i\omega x\cdot\eta}$ when $\eta = \binom{0}{1}$), for $N = 10$, 25 and 50. In all cases, $q = 2+2i$, $q_0 = 1$ and $\gamma := \mu_0/\mu = 1/2$. Of these nine plots, the three that are solid correspond to the scattering disk of radius $\rho = 0.01$, the three plots that are dotted to $\rho = 0.004$ and the three that are dashed to $\rho = 0.001$. Each plot first reaches its asymptotic limit when ω is on the order of N/ρ; that is, when $\lambda \sim N$.

4.1.2 General incident waves

In order to estimate the size of u_ρ^s when u^i has infinitely many Fourier coefficients, we must find bounds on $|\alpha_n|$ that are uniform in n. To this end, we have

Lemma 4.5. *Suppose $\gamma > 0$, $q_0 > 0$ and $q = a + ib$ with $a > 0$, $b \ge 0$. There exists a constant $C > 0$, depending on q but independent of γ, q_0, n, ω and ρ, such that*

$$\begin{aligned}\left|\sqrt{q_0}H_n'(\sqrt{q_0}\lambda)J_n(\sqrt{q}\lambda) - \gamma\sqrt{q}J_n'(\sqrt{q}\lambda)H_n(\sqrt{q_0}\lambda)\right|& \\ \ge C\sqrt{b}\sqrt{\gamma}\frac{\sqrt{1+|n|}}{1+\lambda+|n|}|J_n(\sqrt{q}\lambda)|& \qquad (4.9)\end{aligned}$$

for all $n \in \mathbb{Z}$ and $\lambda > 0$. Furthermore, if we assume $b > 0$ then given any $\lambda_0 > 0$, there exists a constant C, depending on q, γ and λ_0 but independent of q_0, n, ω and ρ, such that

$$|\alpha_n| \le C|a_n|\left(\sqrt{1+|n|} + |J_n(\sqrt{q_0}\lambda)|\right)|H_n(\sqrt{q_0}\lambda)|^{-1}, \qquad (4.10)$$

$$|\beta_n| \le C|a_n|\sqrt{1+|n|}\,|J_n(\sqrt{q}\lambda)|^{-1}, \qquad (4.11)$$

for all $n \in \mathbb{Z}$ and $\lambda = \omega\rho \geq \lambda_0$.

Proof. Let $\mathcal{W} = \mathcal{W}(\gamma, q_0, q, n, \lambda)$ denote the (Wronskian-like) expression we wish to bound in modulus from below:

$$\mathcal{W} := \sqrt{q_0} H_n'(\sqrt{q_0}\lambda) J_n(\sqrt{q}\lambda) - \gamma\sqrt{q} J_n'(\sqrt{q}\lambda) H_n(\sqrt{q_0}\lambda).$$

By a well known identity for the Wronskian of J_n and Y_n [Olv97, Ch. 7, §5.2],

$$\begin{aligned}\operatorname{Im}\left\{\sqrt{q_0} H_n'(\sqrt{q_0}\lambda)\overline{H_n(\sqrt{q_0}\lambda)}\right\} &= \sqrt{q_0}\,\mathrm{Wronsk}(J_n(\sqrt{q_0}\lambda), Y_n(\sqrt{q_0}\lambda)) \\ &= \frac{2}{\pi\lambda}.\end{aligned}$$

Using Green's formula, and the fact that $(\Delta + q\lambda^2)v_n = 0$, where

$$v_n(r, \theta) := J_n(\sqrt{q}\lambda r)e^{in\theta},$$

we get

$$\begin{aligned}\sqrt{q} J_n'(\sqrt{q}\lambda)\overline{J_n(\sqrt{q}\lambda)} &= \frac{1}{\lambda}(\partial_r v_n)\overline{v_n}\big|_{r=1} \\ &= \frac{1}{2\pi\lambda}\int_{\partial B}(\partial_r v_n)\overline{v_n}\,\mathrm{d}\sigma \\ &= \frac{1}{2\pi\lambda}\|\nabla v_n\|^2_{L^2(B)} + \frac{1}{2\pi\lambda}\int_B \Delta v_n \overline{v_n}\,\mathrm{d}x \\ &= \frac{1}{2\pi\lambda}\|\nabla v_n\|^2_{L^2(B)} - \frac{q\lambda}{2\pi}\|v_n\|^2_{L^2(B)}.\end{aligned}$$

Then, since

$$\begin{aligned}&\mathcal{W}\overline{J_n(\sqrt{q}\lambda)H_n(\sqrt{q_0}\lambda)} \\ &\quad = \sqrt{q_0}H_n'(\sqrt{q_0}\lambda)\overline{H_n(\sqrt{q_0}\lambda)}\big|J_n(\sqrt{q}\lambda)\big|^2 - \gamma\sqrt{q}J_n'(\sqrt{q}\lambda)\overline{J_n(\sqrt{q}\lambda)}\big|H_n(\sqrt{q_0}\lambda)\big|^2,\end{aligned}$$

we obtain

$$\operatorname{Im}\left\{\mathcal{W}\overline{J_n(\sqrt{q}\lambda)H_n(\sqrt{q_0}\lambda)}\right\} = \frac{2}{\pi\lambda}\big|J_n(\sqrt{q}\lambda)\big|^2 + \frac{b\gamma\lambda}{2\pi}\|v_n\|^2_{L^2(B)}\big|H_n(\sqrt{q_0}\lambda)\big|^2. \tag{4.12}$$

But since we may estimate

$$\begin{aligned}\operatorname{Im}\left\{\mathcal{W}\overline{J_n(\sqrt{q}\lambda)H_n(\sqrt{q_0}\lambda)}\right\} &\le \left|\mathcal{W}\overline{H_n(\sqrt{q_0}\lambda)J_n(\sqrt{q}\lambda)}\right| \\ &\le \frac{\pi\lambda}{8}|\mathcal{W}|^2\left|H_n(\sqrt{q_0}\lambda)\right|^2 + \frac{2}{\pi\lambda}\left|J_n(\sqrt{q}\lambda)\right|^2,\end{aligned}$$

it follows that

$$|\mathcal{W}|^2 \ge \frac{4b\gamma}{\pi^2}\|v_n\|^2_{L^2(B)}.$$

Note: *We could just as well have estimated*

$$\operatorname{Im}\left\{\mathcal{W}\overline{J_n(\sqrt{q}\lambda)H_n(\sqrt{q_0}\lambda)}\right\} \le \frac{\pi\lambda}{4}|\mathcal{W}|^2\left|H_n(\sqrt{q_0}\lambda)\right|^2 + \frac{1}{\pi\lambda}\left|J_n(\sqrt{q}\lambda)\right|^2.$$

By (4.12), *this implies the bound*

$$|\mathcal{W}| \ge \frac{2}{\pi\lambda}\frac{\left|J_n(\sqrt{q}\lambda)\right|}{\left|H_n(\sqrt{q_0}\lambda)\right|}, \tag{4.13}$$

which holds whether b is zero or positive. We will remark on the significance of this after the proof of Lemma 4.5 is complete (Remark 4.6).

Our task now is to show that

$$\|v_n\|^2_{L^2(B)} \ge c\frac{1+|n|}{(1+\lambda+|n|)^2}|J_n(\sqrt{q}\lambda)|^2, \tag{4.14}$$

for some positive c independent of γ, q_0, n, ω and ρ. Since

$$\|\Delta v_n\|_{L^2(B)} = |q|\lambda^2\|v_n\|_{L^2(B)},$$

by elliptic regularity estimates we get

$$\begin{aligned}\|v_n\|_{H^2(B)} &\le C\left(\lambda^2|q|\|v_n\|_{L^2(B)} + \|v_n\|_{H^{3/2}(\partial B)}\right) \\ &\le C\left(\lambda^2\|v_n\|_{L^2(B)} + (1+|n|)^{3/2}|J_n(\sqrt{q}\lambda)|\right).\end{aligned}$$

The logarithmic convexity of Sobolev norms[2] therefore gives us

$$\begin{aligned}\|v_n\|^2_{H^1(B)} &\le C\|v_n\|_{H^2(B)}\|v_n\|_{L^2(B)}\\ &\le C\left(\lambda^2\|v_n\|_{L^2(B)} + (1+|n|)^{3/2}|J_n(\sqrt{q}\lambda)|\right)\|v_n\|_{L^2(B)}\end{aligned}$$

By the continuity of the trace operator from $H^1(B)$ to $H^{1/2}(\partial B)$,

$$\begin{aligned}(1+|n|)|J_n(\sqrt{q}\lambda)|^2 &\le C\|v_n\|^2_{H^{1/2}(\partial B)}\\ &\le C\|v_n\|^2_{H^1(B)}\\ &\le C\left(\lambda^2\|v_n\|_{L^2(B)} + (1+|n|)^{3/2}|J_n(\sqrt{q}\lambda)|\right)\|v_n\|_{L^2(B)}\\ &\le C\left(\lambda^2 + (1+|n|)^2\right)\|v_n\|^2_{L^2(B)} + \frac{1}{2}(1+|n|)|J_n(\sqrt{q}\lambda)|^2.\end{aligned}$$

From this easily follows (4.14), and thus (4.9). Now assume $b > 0$. It follows from the formula (4.6) for β_n and the bound (4.9) that

$$\begin{aligned}|\beta_n| &\le |a_n|\frac{2}{\pi\lambda}\frac{1+\lambda+|n|}{\sqrt{1+|n|}}\frac{1}{|J_n(\sqrt{q}\lambda)|}\\ &\le C|a_n|\sqrt{1+|n|}\,|J_n(\sqrt{q}\lambda)|^{-1},\end{aligned}$$

for all $n \in \mathbb{Z}$ and all $\lambda = \omega\rho \ge \lambda_0 > 0$, where C depends on q, γ and λ_0 but is independent of q_0, n, ω and ρ. Hence (4.11) holds, and (4.10) now follows from the formula

$$\alpha_n = \left(\beta_n J_n(\sqrt{q}\lambda) - a_n J_n(\sqrt{q_0}\lambda)\right)\frac{1}{H_n(\sqrt{q_0}\lambda)}. \tag{4.15}$$

□

Remark 4.6. *Using* (4.13) *and the formula* (4.6) *for* β_n*, we get the estimate*

$$|\beta_n| \le |a_n|\frac{|H_n(\sqrt{q_0}\lambda)|}{|J_n(\sqrt{q}\lambda)|}.$$

---

[2]If $\phi \in H^{s_0}(\Omega)$ is fixed, the function $s \mapsto \log\left(\|\phi\|_{H^s(\Omega)}\right)$ is convex for $s \in (-\infty, s_0]$.

This combined with (4.15) *yields the estimate*

$$|\alpha_n| \leq 2|a_n|,$$

which holds for all $n \in \mathbb{Z}$, $\omega > 0$, $\rho > 0$ *and* $q = a + ib$ *with* $a > 0$ *and* $b \geq 0$. *This improves Lemma 4.2 and completes the proof of Theorem 5. However, these estimates are not useful when the incident wave has infinitely many Fourier coefficients.*

Though the estimate of α_n in Lemma 4.5 is, for fixed n, weaker than that of Lemma 4.2 (when $\lambda \gg 1$), it is better suited as a tool for estimating the size of u_ρ^s when the incident wave has infinitely many Fourier coefficients.

Theorem 6. *Suppose* $\gamma > 0$, $q_0 > 0$ *and* $q = a + ib$ *with* $a > 0$, $b > 0$ *(note the requirement that* $\operatorname{Im} q$ *be strictly positive). Let the incident wave be of the form*

$$u^i(x) = \sum_{n=-\infty}^{\infty} a_n J_n(\sqrt{q_0}\omega r)e^{in\theta} \tag{4.16}$$

for some coefficients $a_n \in \mathbb{C}$. *Let* (u_ρ^{tr}, u_ρ^s) *solve the transmission problem* (4.1)*. Given any* $\lambda_0 > 0$, *there exists a constant* $C = C(\gamma, q, q_0, \lambda_0)$, *independent of* $\rho > 0$, $\omega > 0$ *and* $\{a_n\}$, *such that for any* $r_0 \geq \rho$ *and any* $s \in \mathbb{R}$,

$$\left\| u_\rho^s\big|_{r=r_0} \right\|_{H^s(\mathbb{T})} \leq C\frac{\sqrt{\rho}}{\sqrt{r_0}} \|\{a_n\}\|_{h^{s+1/2}} \quad \textit{for } \omega\rho \geq \lambda_0.$$ [3]

(cf. Remark 3.2 of Chapter 3 for the corresponding estimate when $\lambda = \omega\rho$ *is near zero.)*

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[3] Here the $s \in \mathbb{R}$ has, of course, no relation to the superscript ‘s’ in u_ρ^s, which stands for “scattered.”

Proof. Using Lemma the bound (4.10) from Lemma 4.5, we obtain

$$\begin{aligned}
\left\| u_\rho^s \big|_{r=r_0} \right\|_{H^s(\mathbb{T})}^2 &= 2\pi \sum_{n=-\infty}^{\infty} |\alpha_n|^2 (1+|n|)^{2s} |H_n(\sqrt{q_0}\omega r_0)|^2 \\
&\leq C \frac{1}{\omega r_0} |a_0|^2 + C \sum_{|n|\geq 1} \left(|\alpha_n|^2 (1+|n|)^{2s+1} \frac{|H_n(\sqrt{q_0}\omega r_0)|^2}{|H_n(\sqrt{q_0}\omega \rho)|^2} \right) \\
&\quad + C \sum_{|n|\geq 1} \left(|\alpha_n|^2 (1+|n|)^{2s} |J_n(\sqrt{q_0}\omega\rho)|^2 \frac{|H_n(\sqrt{q_0}\omega r_0)|^2}{|H_n(\sqrt{q_0}\omega \rho)|^2} \right) \\
&\leq C \frac{1}{\omega r_0} |a_0|^2 + C \frac{\rho}{r_0} \|\{a_n\}\|_{h^{s+1/2}}^2 + C \frac{\rho}{r_0} \|\{a_n\}\|_{h^s}^2 \\
&\leq C \frac{\rho}{r_0} \|\{a_n\}\|_{h^{s+1/2}}^2
\end{aligned}$$

Here, as in the proof of Theorem 5, we have used the fact that $0 < t \mapsto t|H_0(t)|^2$ increases to the limit $2/\pi$ and the fact that $0 < t \mapsto t|H_n(t)|^2$ is decreasing for integers $n \neq 0$. We also used the bound $|J(\sqrt{q_0}\lambda)|^2 \leq 1$, which is a simple consequence of the well known fact[4] that $\sum |J_n(t)|^2 = 1$ for all real t. ☐

Corollary 4.7. *With the same assumptions as in Theorem 6, we have that, given any $\lambda_0 > 0$, there exists a constant $C = C(\gamma, q, q_0, \lambda_0)$, independent of $\rho > 0$, $\omega > 0$ and $\{a_n\}$, such that for any $r_0 \geq \rho$,*

$$\left\| u_\rho^s \big|_{r=r_0} \right\|_{L^2(\mathbb{T})} \leq C \frac{\sqrt{\rho}}{\sqrt{r_0}} \|\{a_n\}\|_{h^{1/2}} \quad \text{for } \omega\rho \geq \lambda_0.$$

Furthermore, given any $s < -1$ and any $\lambda_0 > 0$, there exists a constant C_s, depending on γ, q, q_0, λ_0 and s, but independent of $\rho > 0$, $\omega > 0$ and $\{a_n\}$, such that for any $r_0 \geq \rho$,

$$\left\| u_\rho^s \big|_{r=r_0} \right\|_{H^s(\mathbb{T})} \leq C_s \frac{\sqrt{\rho}}{\sqrt{r_0}} \|\{a_n\}\|_{l^\infty} \quad \text{for } \omega\rho \geq \lambda_0.$$

Proof. The first estimate is simply Theorem 6 with $s = 0$. The second estimate follows from the fact that $\|\{a_n\}\|_{h^{-1/2-\delta}} \leq C_\delta \|\{a_n\}\|_{l^\infty}$ for all $\delta > 0$. ☐

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[4]This is simply Parseval's equality applied to a plane wave on the unit circle.

As stated earlier, we expect the stronger estimate

$$\left\| u^s_\rho\big|_{r=r_0} \right\|_{L^2(\mathbb{T})} \le C \frac{\sqrt{\rho}}{\sqrt{r_0}} \|\{a_n\}\|_{l^\infty} \tag{4.17}$$

to hold. Though we do not have a proof of this, we do have a stronger estimate than that of Theorem 6, which we will prove in the next section using methods likely to generalize to arbitrary smooth, convex scatterers (Theorem 8).

In the case of a perfectly conducting (sound-soft) circular scatterer, we can easily prove the estimate (4.17).

Theorem 7. *Let the incident wave be of the form*

$$u^i(x) = \sum_{n=-\infty}^{\infty} a_n J_n(\sqrt{q_0}\omega r) e^{in\theta}$$

for some coefficients $a_n \in \mathbb{C}$. Let u^s_ρ solve the problem

$$\begin{cases} \Delta u^s_\rho + q_0 \omega^2 u^s_\rho = 0 & (r > \rho), \\ u^s_\rho = -u^i & (r = \rho), \\ (\partial_r - i\sqrt{q_0}\omega) u^s_\rho = o(r^{-1/2}) & \text{as } r \to \infty. \end{cases}$$

Given any $\lambda_0 > 0$, there exists a constant $C = C(q_0, \lambda_0)$, independent of $\omega > 0$ and $\rho > 0$, such that for any $r_0 \ge \rho$,

$$\left\| u^s_\rho\big|_{r=r_0} \right\|_{L^2(\mathbb{T})} \le C \frac{\sqrt{\rho}}{\sqrt{r_0}} \|\{a_n\}\|_{l^\infty} \quad \text{for } \omega\rho \ge \lambda_0.$$

(cf. Remark 3.3 of Chapter 3 for the corresponding estimate when $\omega\rho$ is near zero.)

Proof. Since

$$u^s_\rho(r,\theta) = -\sum_{n=-\infty}^{\infty} a_n J_n(\sqrt{q_0}\omega\rho) \frac{H_n(\sqrt{q_0}\omega r_0)}{H_n(\sqrt{q_0}\omega\rho)} e^{in\theta},$$

it follows that

$$\left\|u_\rho^s\big|_{r=r_0}\right\|_{L^2(\mathbb{T})}^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 |J_n(\sqrt{q_0}\omega\rho)|^2 \frac{|H_n(\sqrt{q_0}\omega r_0)|^2}{|H_n(\sqrt{q_0}\omega\rho)|^2}$$
$$\leq C_{q_0,\lambda_0}\frac{\rho}{r_0}|a_0|^2 + C_{q_0,\lambda_0}\frac{\rho}{r_0}\sum_{|n|\geq 1}|a_n|^2|J_n(\sqrt{q_0}\omega\rho)|^2. \tag{4.18}$$

For the $n \neq 0$ terms, we used the fact that $0 < t \mapsto t|H_n(t)|^2$ is decreasing for integers $n \neq 0$. For the $n = 0$ term, we used the fact that $0 < t \mapsto t|H_0(t)|^2$ increases to the limit $2/\pi$, and therefore

$$|J_0(\sqrt{q_0}\omega\rho)|^2 \frac{|H_0(\sqrt{q_0}\omega r_0)|^2}{|H_0(\sqrt{q_0}\omega\rho)|^2} \leq \frac{2}{\pi}\frac{1}{\sqrt{q_0}\omega r_0}\frac{|J_0(\sqrt{q_0}\omega\rho)|^2}{|H_0(\sqrt{q_0}\omega\rho)|^2}.$$

Since the continuous function $0 < t \mapsto |J_0(t)|^2/|H_0(t)|^2$ is bounded as $t \to 0$ and as $t \to \infty$, it is bounded on $(0, \infty)$. We therefore have

$$|J_0(\sqrt{q_0}\omega\rho)|^2 \frac{|H_0(\sqrt{q_0}\omega r_0)|^2}{|H_0(\sqrt{q_0}\omega\rho)|^2} \leq C\frac{1}{\sqrt{q_0}\omega r_0} \leq \frac{C}{\sqrt{q_0}\lambda_0}\frac{\rho}{r_0}.$$

The theorem now follows from (4.18) and the identity $\sum |J_n(t)|^2 = 1$, $t \in \mathbb{R}$. $\square$

4.2 A bound via pseudodifferential operators

In this section, we explore an alternative route to L^2-based bounds on the scattered field in the asymptotic regime wherein $\lambda = \omega\rho \gg 1$. A brief outline of this alternative route follows: We find a factorization $\mathscr{L}_0 = \big(\partial_r - \widetilde{D}\big)\big(\partial_r - D\big)$ of the rescaled Helmoltz operator $\mathscr{L} := \Delta + q\lambda^2$ that well approximates $\mathscr{L}$. The solution to the transmission problem that results when $\mathscr{L}$ is replaced by $\mathscr{L}_0$ inside $B = B(0,1)$ (or, more precisely, inside the annulus $\mathbf{C} = \{x : 1/2 < r < 1\}$) is then equivalent to an exterior problem with a sort of impedance boundary condition, involving a nonlocal operator on ∂B. This means the transmitted signal is, in a sense, replaced by an impedance condition. The approximate solution satisfying this exterior problem is then shown to satisfy the desired L^2-based bounds. Finally, we quantify the degree to which the solution to the exterior problem approximates the actual scattered field, and thus prove that the actual solution also satisfies the desired L^2-based estimates.

We will discuss how these methods may be extended to arbitrary smooth domains of the form ρD at the end of this section.

In order to work in a fixed domain, we introduce the rescaled variable $R = r/\rho$, and let $U^i(R) = u^i(\rho R)$, $U^{tr}(R) = u^{tr}_\rho(\rho R)$ and $U^s(R) = u^s_\rho(\rho R)$. U^{tr} and U^s are therefore the unique solutions to

$$\begin{cases} \Delta U^{tr} + q\lambda^2 U^{tr} = 0 \quad (R < 1), \\ \Delta U^s + q_0\lambda^2 U^s = 0 \quad (R > 1), \end{cases} \tag{4.19a}$$

satisfying the transmission conditions

$$\begin{cases} \dfrac{1}{\mu}\partial_R U^{tr}\big|_{R=1} = \dfrac{1}{\mu_0}\left(\partial_R U^s\big|_{R=1} + \partial_R U^i\big|_{R=1}\right), \\ \qquad U^{tr}\big|_{R=1} = U^s\big|_{R=1} + U^i\big|_{R=1}, \end{cases} \tag{4.19b}$$

as well as the radiation condition

$$\partial_R U^s - i\sqrt{q_0}\lambda U^s = O(R^{-3/2}) \quad \text{as } R \to \infty. \tag{4.19c}$$

4.2.1 Factorization of Hemholtz operator in the unit disk

Denote by $\mathscr{L}$ the Helmholtz operator written in polar coordinates:

$$\begin{aligned} \mathscr{L} &= \Delta + q\lambda^2 \\ &= \partial_R^2 + \frac{1}{R}\partial_R + \frac{1}{R^2}\partial_\theta^2 + q\lambda^2 \quad \text{on } (0,1)\times\mathbb{T}. \end{aligned} \tag{4.20}$$

In this section we find two pseudodifferential operators (cf. [Fol95], [Tay81] or [CP82]) on the Torus $\mathbb{T} = (0, 2\pi)_{per}$, denoted D_q and $\widetilde{D}_q$, such that the Helmholtz operator $\mathscr{L}$ can be factorized as

$$\mathscr{L} = \left(\partial_R + \widetilde{D}_q\right)\left(\partial_R - D_q\right) + R^0_q,$$

where the operator R_q^0 is of order 0. To be precise, D_q and $\widetilde{D}_q$ will each be functions of R that assign, to each R, a pseudodifferential operator on $\mathbb{T}$. To say that R_q^0 is of order 0 means the symbol[5] σ_0 of R_q^0 will satisfy

$$\|\partial_\theta^m \sigma_0(R, n, \lambda, \theta)\|_{L^\infty(\mathbb{T})} \leq C_m (1 + |n|)^{-m}$$

for all nonnegative integral m. We will also need the symbol of R_q^0 to be well controlled in its dependence on λ.[6] Our first step toward this goal will be to find an operator D_q^1 such that

$$\mathscr{L} = \left(\partial_R + D_q^1\right)\left(\partial_R - D_q^1\right) + R_q^1,$$

with R_q^1 an operator of order 1. Then using the expression of the symbol of R_q^1, we will find two operators D_q^0 and $\widetilde{D}_q^0$ such that

$$\mathscr{L} = \left(\partial_R + D_q^1 + \widetilde{D}_q^0\right)\left(\partial_R - D_q^1 - D_q^0\right) + R_q^0.$$

Denote by $\mathbf{C}$ the annulus

$$\mathbf{C} = \left\{(R, \theta) \ : \ 1/2 \leq R \leq 1, \ \theta \in \mathbb{T}\right\}.$$

In the two variables R and θ, we say that σ is the symbol of the operator $\mathscr{O}$ if

$$\mathscr{O}u(R, \theta) = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{\mathbb{T}} \sigma(R, \theta, \xi, n) e^{in(\theta - \vartheta)} e^{i\xi(R - S)} u(S, \vartheta) \, \mathrm{d}\vartheta \, \mathrm{d}S \, \mathrm{d}\xi \tag{4.21}$$

---

[5] σ is the symbol of the operator $\mathscr{O}$ if, and only if, $\mathscr{O}u(\theta) = \sum_{n=-\infty}^{\infty} \sigma(\theta, n) \widehat{u}_n e^{in\theta}$ for all u sufficiently smooth. Here, $\widehat{u}_n = \frac{1}{2\pi} \int_{\mathbb{T}} u(\theta) e^{-in\theta} \mathrm{d}\theta$.

[6] Since the $q\lambda^2$ term in the Helmholtz operator $\mathscr{L}$ comes from the second derivative with respect to time in the wave operator, the problem of finding a factorization of $\mathscr{L}$ with the (zeroth order) error well controlled in its dependence on λ could be thought of as the equivalent problem of finding a factorization of the wave operator with error of order zero.

for all smooth and rapidly decaying[7] u defined on $(-\infty,\infty)\times\mathbb{T}$. Now, any smooth u defined on $\mathbf{C}$ can be extended to a smooth and rapidly decaying function on $(-\infty,\infty)\times\mathbb{T}$. For such u, we have

$$\mathscr{L}u(R,\theta)=\frac{1}{(2\pi)^2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\mathbb{T}}\mathscr{L}_{R,\theta}\left(e^{in(\theta-\vartheta)}e^{i\xi(R-S)}\right)u(S,\vartheta)\,\mathrm{d}\vartheta\,\mathrm{d}S\,\mathrm{d}\xi.$$

Therefore, the symbol $\sigma_{\mathscr{L}}$ of $\mathscr{L}$ in $\mathbf{C}$ is: for $\lambda\geq 1$, $n\in\mathbb{Z}$, $R\in[1/2,1]$ and $\xi\in\mathbb{R}$,

$$\sigma_{\mathscr{L}}(n,R,\xi,\lambda)=-\xi^2-\frac{n^2}{R^2}+q\lambda^2+\frac{i\xi}{R}.$$

Define the symbol

$$d_q^1(n,R,\lambda)=\sqrt{\frac{n^2}{R^2}-q\lambda^2},$$

where $\sqrt{\cdot}$ is the principal square root:

$$\sqrt{z}=\sqrt{|z|}e^{(\operatorname{Arg} z)/2},\quad -\pi<\operatorname{Arg} z\leq\pi.$$

Accordingly, $\operatorname{Re} d_q^1>0$. We then have the following factorization of $\sigma_{\mathscr{L}}$:

$$\sigma_{\mathscr{L}}=\left(i\xi+d_q^1\right)\left(i\xi-d_q^1\right)+\frac{i\xi}{R}.$$

Denote by D_q^1 the operator whose symbol is d_q^1 and by R_q^1 the operator of order 1 defined by

$$\begin{aligned}R_q^1&=\mathscr{L}-\left(\partial_R+D_q^1\right)\left(\partial_R-D_q^1\right)\\&=\mathscr{L}-\left(\partial_R^2-D_q^1D_q^1-\left[\partial_R,D_q^1\right]\right).\end{aligned}$$

[7] $\partial_S^m u(S,\vartheta)$ must decay faster than $|S|^{-l}$ as $|S|\to\infty$ for all orders m and l.

The symbol of R_q^1 is then

$$\begin{aligned}\sigma_{R_q^1} &= \frac{i\xi}{R} + \frac{\mathrm{d}}{\mathrm{d}R} d_q^1 \\ &= \frac{i\xi}{R} - \frac{n^2}{R^3 d_q^1}.\end{aligned}$$

Now we seek two operators $D_q^0 = \mathrm{Op}(d_q^0)$ and $\widetilde{D_q^0} = \mathrm{Op}\left(\widetilde{d_q^0}\right)$ such that

$$R_q^0 = \mathscr{L} - \left(\partial_R + D_q^1 + \widetilde{D_q^0}\right)\left(\partial_R - D_q^1 - D_q^0\right)$$

is of order 0, with its symbol r_q^0 independent of ξ and well controlled in its dependence on n, R and $\lambda \geq 1$. A simple computation leads to

$$R_q^0 = R_q^1 - \left(-D_q^1\big(\widetilde{D_q^0} + D_q^0\big) + \big(\widetilde{D_q^0} - D_q^0\big)\partial_R - \big[\partial_R, D_q^0\big] - \widetilde{D_q^0} D_q^0\right),$$

and therefore

$$r_q^0 = \sigma_{R_q^1} - \Big(- d_q^1(\widetilde{d_q^0} + d_q^0) + i\xi(\widetilde{d_q^0} - d_q^0) - \frac{\mathrm{d}}{\mathrm{d}R} d_q^0 - \widetilde{d_q^0} d_q^0\Big).$$

Naturally, we define d_q^0 and $\widetilde{d_q^0}$ by the following system:

$$\begin{aligned}\widetilde{d_q^0} - d_q^0 &= 1/R, \\ \widetilde{d_q^0} + d_q^0 &= \frac{n^2}{R^3 (d_q^1)^2}.\end{aligned}$$

Solving, we find

$$\begin{aligned}d_q^0 &= \frac{q\lambda^2}{2R(d_q^1)^2}, \\ \widetilde{d_q^0} &= \frac{1}{2}\left(\frac{1}{R} + \frac{n^2}{R^3 (d_q^1)^2}\right).\end{aligned}$$

We now define the symbols d_q and $\widetilde{d}_q$ by

$$d_q = d_q^1 + d_q^0,$$
$$\widetilde{d}_q = d_q^1 + \widetilde{d}_q^0,$$

or, more precisely, for $n \in \mathbb{Z}$, $R \in [1/2, 1]$ and $\lambda \geq 1$,

$$d_q(n, R, \lambda) = \sqrt{\frac{n^2}{R^2} - q\lambda^2} + \frac{1}{2R}\frac{q}{\frac{n^2}{R^2\lambda^2} - q}, \tag{4.22a}$$

$$\widetilde{d}_q(n, R, \lambda) = \sqrt{\frac{n^2}{R^2} - q\lambda^2} + \frac{1}{2R}\left(1 + \frac{\frac{n^2}{R^2\lambda^2}}{\frac{n^2}{R^2\lambda^2} - q}\right). \tag{4.22b}$$

We set $D_q = \mathrm{Op}\,(d_q)$ and $\widetilde{D}_q = \mathrm{Op}(\widetilde{d}_q)$. Since d_q and $\widetilde{d}_q$ are independent of ξ, for each fixed R these operators act on functions defined on the torus $\mathbb{T}$. Let us define $\mathscr{L}_0$ by

$$\mathscr{L}_0 = \big(\partial_R + \widetilde{D}_q\big)\big(\partial_R - D_q\big), \tag{4.23}$$

so that

$$\begin{aligned}\sigma_{\mathscr{L}-\mathscr{L}_0} &= \frac{\mathrm{d}}{\mathrm{d}R} d_q^0 + \widetilde{d}_q^0 d_q^0 \\ &= \frac{q\lambda^2(n^2 + qR^2\lambda^2)}{2(n^2 - qR^2\lambda^2)^2} + \frac{q\lambda^2(n^2 - qR^2\lambda^2/2)}{2(n^2 - qR^2\lambda^2)^2} \\ &= \frac{q}{R^2}\frac{\left(\frac{n^2}{\lambda^2R^2} + \frac{q}{8}\right)}{\left(\frac{n^2}{\lambda^2R^2} - q\right)^2}.\end{aligned}$$

As a result, since $b \neq 0$, there exists[8] a C depending only on b/a such that for all $R \in [1/2, 1]$ and $\lambda \geq 1$,

$$|\sigma_{\mathscr{L}-\mathscr{L}_0}| \leq C. \tag{4.24}$$

Therefore, we have

[8]Take $C = \max_{t\geq 0}\left|4(1 + ib/a)\big(t + \frac{1+ib/a}{8}\big)/\big(t - (1 + ib/a)\big)^2\right|$.

Lemma 4.8. *There exists C depending only on b/a such that for all $v \in L^2(\mathbf{C})$,*

$$\|(\mathscr{L} - \mathscr{L}_0)v\|_{L^2(\mathbf{C})} \leq C\|v\|_{L^2(\mathbf{C})}.$$

Remark 4.9. *$\sigma_{\mathscr{L}}$ does completely factorize (without remainder) as*

$$\sigma_{\mathscr{L}} = \Big(i\xi + \frac{1}{2R}\Big(1 + \sqrt{1 + 4(n^2 - qR^2\lambda^2)}\Big)\Big) \\ \times \Big(i\xi - \frac{1}{2R}\Big(-1 + \sqrt{1 + 4(n^2 - qR^2\lambda^2)}\Big)\Big).$$

However, this factorization will not meet our needs. For if $\mathscr{L}_0$ is defined to be the operator

$$\mathscr{L}_0 = \big(\partial_R + \widetilde{D}\big)\big(\partial_R - D\big),$$

where $\widetilde{D}$ has symbol

$$\sigma_{\widetilde{D}} = \frac{1}{2R}\big(1 + \sqrt{1 + 4(n^2 - qR^2\lambda^2)}\big)$$

and D has symbol

$$\sigma_D = \frac{1}{2R}\big(-1 + \sqrt{1 + 4(n^2 - qR^2\lambda^2)}\big),$$

then (4.24), and therefore Lemma 4.8, will no longer hold.

4.2.2 Properties of d_q and $\widetilde{d}_q$

In this section we present some essential properties of d_q and $\widetilde{d}_q$. The first is a simple consequence of the definition (4.22a) of d_q and the fact that $\operatorname{Im} q \neq 0$.

Property 4.10. ***Upper bound for $|d_q|$.***

There exists a constant $C > 0$ depending on q $(= a + ib$ with $a, b > 0)$ such that for all $R \in [1/2, 1]$, $n \in \mathbb{Z}$ and $\lambda \geq 1$ we have

$$|d_q| \leq C\,(|n| + \lambda)\,. \tag{4.25}$$

(We also have that $\big|\widetilde{d}_q\big| \leq C\,(|n| + \lambda)$, but we will not need this bound.)

Property 4.11. ***Lower bounds for*** $\operatorname{Re} d_q$ ***and*** $\operatorname{Re} \widetilde{d}_q$***.***

There exist constants $C > 0$ *and* $\lambda_0 \geq 1$*, both depending only on* q *(*$= a + ib$ *with* $a, b > 0$*), such that for all* $R \in [1/2, 1]$*,* $n \in \mathbb{Z}$ *and* $\lambda \geq \lambda_0$ *we have*

$$\operatorname{Re}\{d_q\,(n, R, \lambda)\} \geq C\,(\lambda + |n|)\,, \tag{4.26}$$

$$\operatorname{Re}\left\{\widetilde{d}_q\,(n, R, \lambda)\,\right\} \geq C\,(\lambda + |n|)\,. \tag{4.27}$$

Proof. We will prove only (4.26), the proof of (4.27) being similar. Let $s = \frac{n^2}{R^2\lambda^2}$, so that (4.22a) becomes

$$d_q = \lambda\sqrt{s-q} + \frac{1}{2R}\frac{q}{s-q}.$$

Write this as

$$d_q = \underbrace{\lambda\Big(\frac{1}{2}\sqrt{s-q} + \frac{1}{2R\lambda}\frac{q}{s-q}\Big)}_{A} + \underbrace{\frac{\lambda}{2}\sqrt{s-q}}_{B}.$$

To complete the proof we will demonstrate that $\operatorname{Re} A > C\lambda$ and $\operatorname{Re} B > C|n|$. Observe that

$$\operatorname{Re}\sqrt{s-q} \geq \operatorname{Re}\sqrt{-q} > 0$$

for all $s \geq 0$. Observe also that, since $\operatorname{Im} q \neq 0$, by simply choosing λ_0 sufficiently large the term $\frac{1}{2R\lambda}\frac{q}{s-q}$ can be bounded uniformly in $s \geq 0$, $R \in [1/2, 1]$ and $\lambda \geq \lambda_0$ by an arbitrarily small bound. It follows that there exists a $C > 0$ and a $\lambda_0 \geq 1$, both depending only on q, such that

$$\operatorname{Re} A \geq C\lambda.$$

Now observe that

$$\inf_{s>0}\left\{\operatorname{Re}\sqrt{1-q/s}\right\} > 0,$$

which implies that

$$\begin{aligned}\operatorname{Re} B &= \frac{|n|}{2R} \operatorname{Re} \sqrt{1 - q/s} \qquad \text{(for } l \neq 0\text{)} \\ &\geq C|n|\end{aligned}$$

for some $C > 0$ depending only on q. □

Upper bound for $\operatorname{Im}(d_q)$

We denote by d and ν the following functions:

$$\begin{gathered}\forall t > 0, \quad \nu(t) = \sqrt{1 - qt^2}, \\ \forall n \in \mathbb{Z}^*, \forall t > 0, \quad d(n,t) = |n|\nu(t) - \left(1 - 1/\nu^2(t)\right)/2.\end{gathered}$$

Property 4.12. ***Upper bound for*** $\operatorname{Im} d_q$**.**
There exist constants $C > 0$ and $\lambda_0 \geq 1$ depending only on q ($= a + ib$ with $a, b > 0$) such that for all $n \in \mathbb{Z}$ and $\lambda \geq \lambda_0$,

$$\operatorname{Im}\{d_q(n, R, \lambda)\}\big|_{R=1} \leq -C\lambda \min(1, \lambda/|n|).$$

(When $n = 0$, we take $\min(1, \lambda/|n|) = 1$.)

Proof. The case $n = 0$ is immediate, so we assume $n \neq 0$. Define for $t > 0$ and $n \in \mathbb{Z}^*$ the functions

$$\begin{aligned}\nu(t) &= \sqrt{1 - qt^2}, \\ d(n,t) &= |n|\nu(t) - \frac{1}{2}\left(1 - \frac{1}{\nu^2(t)}\right),\end{aligned}$$

so that

$$d_q(n, R, \lambda)\big|_{R=1} = d(n, \lambda/n).$$

Since $\mathrm{Im}(1-qt^2)<0$ for all $t>0$, we may write

$$\nu(t)=\rho e^{i\theta}, \quad -\pi/2<\theta<0,$$

so that

$$\begin{aligned}\mathrm{Im}\{d(n,t)\} &= |n|\rho\sin(\theta)-\frac{\sin(2\theta)}{2\rho^2}\\ &= \frac{\sin(\theta)}{\rho^2}\left(|n|\rho^3-\cos(\theta)\right).\end{aligned}$$

To prove $\mathrm{Im}\{d_q(n,1,\lambda)\}=\mathrm{Im}\{d(n,\lambda/|n|)\}<0$, it suffices to prove there exists a $\lambda_0\geq 1$ such that for every integer $n\neq 0$ and all $\lambda\geq\lambda_0$,

$$|n|\rho^3\geq 2.$$

Since

$$|\nu(t)|^4=\left(1-at^2\right)^2+b^2t^4,$$

there exists a $t_0>0$ depending only on q such that for all $t\geq t_0$, $|\nu(t)|^3\geq 2$. Moreover, a simple calculation shows that

$$\forall t\geq 0, \quad |\nu(t)|\geq\frac{ab}{a^2+b^2}.$$

Therefore, if $|n|\geq 2\left[(a^2+b^2)/ab\right]^3$ then $\mathrm{Im}\{d(n,\lambda/|n|)\}<0$ for all λ. Otherwise, we simply choose λ_0 such that

$$\lambda_0\geq t_0 2\left[(a^2+b^2)/ab\right]^3.$$

We now have that $\mathrm{Im}\{d_q(n,1,\lambda)\}<0$ for $n\neq 0$, $\lambda\geq\lambda_0$. To finish the proof we let

$t = \lambda/|n|$ and observe that for $\lambda \geq \lambda_0$,

$$\begin{aligned}\frac{1}{\lambda}\operatorname{Im}\{d(n,t)\} &\leq \frac{1}{\lambda}\frac{\sin(\theta)}{\rho^2}\frac{|n|\rho^3}{2} \\ &= \frac{1}{2t}\operatorname{Im}\nu(t).\end{aligned}$$

Since

$$\operatorname{Im}\nu(t) = -\frac{1}{\sqrt{2}}\sqrt{\sqrt{(1-at^2)^2 + b^2t^4} - (1-at^2)},$$

the continuous and strictly negative function

$$f(t) := \max\{1, 1/t\}\frac{1}{2t}\operatorname{Im}(\nu(t)), \quad 0 < t < \infty,$$

satisfies

$$f(t) \to \text{const}(q) < 0 \text{ as } t \to 0,$$

$$f(t) \to \text{const}(q) < 0 \text{ as } t \to \infty,$$

and therefore has a strictly negative supremum on $[0, \infty)$. □

4.2.3 The factorized problem: approximation of U^{tr}

In this section we suppose we know U^{tr} (or an approximation of U^{tr}) on ∂B, which we denote by g, and we study the solution U of the following Cauchy problem:

$$\begin{cases} \partial_R U - D_q(R)U = 0 \quad \text{in } (1/2, 1] \times \mathbb{T}, \\ U(R,\theta)\big|_{R=1} = g. \end{cases} \tag{4.28}$$

In Fourier space this becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}R}\widehat{U}_n - d_q\widehat{U}_n = 0 \quad \text{in } (1/2, 1), \\ \widehat{U}_n\big|_{R=1} = \widehat{g}_n,\end{aligned}$$

for all $n \in \mathbb{Z}$, which implies

$$\widehat{U}_n = \widehat{g}_n e^{-\int_R^1 d_q(s)\,\mathrm{d}s}. \tag{4.29}$$

Lemma 4.13. *Let g be in $H^{1/2}(\mathbb{T})$. Then the solution U of (4.28) is in $H^1(\mathbf{C})$ and the following estimates hold for a constant $C > 0$ depending only on q:*

$$\frac{1}{C}\|g\|_{H^{1/2}(\mathbb{T})} \leq \|U\|_{H^1(\mathbf{C})} \leq C\sqrt{\lambda}\,\|g\|_{H^{1/2}(\mathbb{T})}, \tag{4.30}$$

and

$$\frac{1}{C}\|g\|_{H^{-1/2}(\mathbb{T})} \leq \sqrt{\lambda}\|U\|_{L^2(\mathbf{C})} \leq C\,\|g\|_{L^2(\mathbb{T})}. \tag{4.31}$$

Moreover, for all $s, \sigma \in \mathbb{R}$ there exists two constants $C_{s,\sigma} > 0$ and $c > 0$, which also depend on q, such that

$$\left\|U|_{R=1/2}\right\|_{H^\sigma(\mathbb{T})} \leq C_{s,\sigma} e^{-c\lambda}\|g\|_{H^s(\mathbb{T})}.\,[9] \tag{4.32}$$

Proof. After we prove these second inequality of (4.30), the first inequality will follow as a direct consequence of the continuity of the trace operator from $H^1(\mathbf{C})$ into $H^{1/2}(\partial B)$. To prove the second inequality of (4.30), first observe that by the expression (4.29),

$$\begin{aligned}
\int_{1/2}^1 \left|\widehat{U}_n\right|^2 R\,\mathrm{d}R \leq \int_{1/2}^1 \left|\widehat{U}_n\right|^2 \mathrm{d}R \quad &= \left|\widehat{g}_n\right|^2 \int_{1/2}^1 e^{-2\int_R^1 \mathrm{Re}\{d_q(s)\}\,\mathrm{d}s}\mathrm{d}R, \\
\int_{1/2}^1 \left|n\widehat{U}_n\right|^2 R\,\mathrm{d}R \leq \int_{1/2}^1 \left|n\widehat{U}_n\right|^2 \mathrm{d}R \quad &= \left|\widehat{g}_n\right|^2 \int_{1/2}^1 n^2 e^{-2\int_R^1 \mathrm{Re}\{d_q(s)\}\,\mathrm{d}s}\mathrm{d}R, \\
\int_{1/2}^1 \left|\partial_R\widehat{U}_n\right|^2 R\,\mathrm{d}R \leq \int_{1/2}^1 \left|\partial_R\widehat{U}_n\right|^2 \mathrm{d}R &= \left|\widehat{g}_n\right|^2 \int_{1/2}^1 |d_q(R)|^2 e^{-2\int_R^1 \mathrm{Re}\{d_q(s)\}\,\mathrm{d}s}\mathrm{d}R.
\end{aligned}$$

[9] Here $\sigma \in \mathbb{R}$ has no relation to conductivity.

By Properties 4.11 and 4.10, there exists a $C > 0$ such that

$$\int_{1/2}^{1} e^{-2\int_R^1 \mathrm{Re}\{d_q(s)\}\,\mathrm{d}s}\mathrm{d}R \le \frac{C}{|n| + \lambda},$$
$$\int_{1/2}^{1} n^2 e^{-\int_R^1 \mathrm{Re}\{d_q(s)\}\,\mathrm{d}s}\mathrm{d}R \le \frac{Cn^2}{|n| + \lambda},$$
$$\int_{1/2}^{1} |d_q(R)|^2 e^{-2\int_R^1 \mathrm{Re}\{d_q(s)\}\,\mathrm{d}s}\mathrm{d}R \le C\left(|n| + \lambda\right).$$

We have therefore proved

$$\int_{1/2}^{1} \big|\widehat{U}_n\big|^2 R\,\mathrm{d}R \le \frac{C}{|n| + \lambda}\big|\widehat{g}_n\big|^2,$$
$$\int_{1/2}^{1} \big|n\widehat{U}_n\big|^2 R\,\mathrm{d}R \le \frac{Cn^2}{|n| + \lambda}\big|\widehat{g}_n\big|^2,$$
$$\int_{1/2}^{1} \big|\partial_R\widehat{U}_n\big|^2 R\,\mathrm{d}R \le C\big(|n| + \lambda\big)\big|\widehat{g}_n\big|^2,$$

hence (4.30).

We now prove (4.31). By the expression (4.29) and Properties 4.10 and 4.11, there exist constants $C_1, C_2 > 0$, both depending only on q, such that

$$\big|\widehat{g}_n\big|e^{-C_1(|n|+\lambda)(1-R)} \le \big|\widehat{U}_n\big| \le \big|\widehat{g}_n\big|e^{-C_2(|n|+\lambda)(1-R)}.$$

Consequently, there exists a $C > 0$ depending only on q such that for all $\lambda \ge 1$ and $n \in \mathbb{Z}$,

$$\frac{\big|\widehat{g}_n\big|^2}{C\big(|n| + \lambda\big)} \le \int_{1/2}^{1} \big|\widehat{U}_n(R)\big|^2 R\,\mathrm{d}R \le \frac{C}{|n| + \lambda}\big|\widehat{g}_n\big|^2,$$

which implies

$$\frac{1}{C}\frac{\big|\widehat{g}_n\big|^2}{1 + |n|} \le \lambda\int_{1/2}^{1} \big|\widehat{U}_n(R)\big|^2 R\,\mathrm{d}R \le C\big|\widehat{g}_n\big|^2,$$

which in turn implies (4.31). As for (4.32), we observe that

$$\widehat{U}_n\Big|_{R=1/2} = \widehat{g}_n e^{-\int_{1/2}^1 d_q(s)\,\mathrm{d}s}.$$

By Property 4.11, there exists a $c > 0$ such that

$$\left| e^{-\int_{1/2}^{1} d_q(s)\,\mathrm{d}s} \right| \leq e^{-c(|n|+\lambda)}.$$

Since, given any $s, \sigma \in \mathbb{R}$, there exists a $C_{s,\sigma} > 0$ such that

$$|n|^{\sigma-s} e^{-c|n|} \leq C_{s,\sigma}$$

for all $n \in \mathbb{Z}^*$, estimate (4.32) follows. $\square$

A natural way to extend U into $B_{1/2} := B(0, 1/2)$ is by defining it to be the solution to the Dirichlet problem

$$\begin{cases} \Delta U + q\lambda^2 U = 0 & \text{in } \ B_{1/2}, \\ U^- = U^+ & \text{on } \partial B_{1/2}. \end{cases} \tag{4.33}$$

Lemma 4.14. *Let $g \in H^{1/2}(\mathbb{T})$ and let U solve* (4.28) *with the extension* (4.33)*. Then for every $s, \sigma \in \mathbb{R}$, there exist constants $C_\sigma > 0$ and $c > 0$, both depending on q but independent of s, such that*

$$\|U\|_{H^\sigma\left(B_{1/2}\right)} \leq C_\sigma e^{-c\lambda} \|g\|_{H^s(\mathbb{T})} \tag{4.34}$$

and

$$\left\| \partial_R^- U|_{R=1/2} \right\|_{H^\sigma(\mathbb{T})} + \left\| \partial_R^+ U|_{R=1/2} \right\|_{H^\sigma(\mathbb{T})} \leq C_\sigma e^{-c\lambda} \|g\|_{H^s(\mathbb{T})}. \tag{4.35}$$

Proof. (4.34) is a consequence of (4.32) and standard elliptic regularity estimates. As for (4.35): the estimate

$$\left\| \partial_R^- U|_{R=1/2} \right\|_{H^\sigma(\mathbb{T})} \leq C_\sigma e^{-c\lambda} \|g\|_{H^s(\mathbb{T})}$$

follows from the continuity of the trace operator from $H^{\sigma+1/2}(B_{1/2})$ to $H^\sigma(\partial B_{1/2})$ for

$\sigma \geq 1/2$, and the estimate

$$\left\| \partial_R^+ U|_{R=1/2} \right\|_{H^\sigma(\mathbb{T})} \leq C_\sigma e^{-c\lambda} \|g\|_{H^s(\mathbb{T})}$$

follows by first observing that $\partial_R^+ U = D_q U^+$ at $R = 1/2$, then applying (4.32) and the fact that $|d_q| \leq C_q(|n| + \lambda)$. □

4.2.4 Approximation of U^s

Let (V^{tr}, V^s) be the approximation of (U^{tr}, U^s) that satisfies the following analogue of problem (4.19):

$$\Delta V^s + q_0 \lambda^2 V^s = 0 \quad \text{for } (R, \theta) \in (1, +\infty) \times \mathbb{T}, \tag{4.36a}$$

$$\partial_R V^{tr} - D_q V^{tr} = 0 \quad \text{for } (R, \theta) \in (1/2, 1) \times \mathbb{T}, \tag{4.36b}$$

with the transmission conditions

$$\frac{1}{\mu} \partial_R V^{tr}\big|_{R=1} = \frac{1}{\mu_0} \left(\partial_R V^s\big|_{R=1} + \partial_R U^i\big|_{R=1} \right), \tag{4.36c}$$

$$V^{tr}\big|_{R=1} = V^s\big|_{R=1} + U^i\big|_{R=1}, \tag{4.36d}$$

and the radiation condition

$$(\partial_R - i\sqrt{q_0}\lambda) V^s = O(R^{-3/2}). \tag{4.36e}$$

The fact that problem (4.36) has a unique solution can be seen, for instance, by first expanding V^s and V^{tr} as Fourier series then noting that the transmission conditions uniquely determine $\widehat{V}_n^{tr}\big|_{R=1}$ for each $n \in \mathbb{Z}$. Combining the transmission conditions, along with the impedance relation

$$\partial_R V^{tr}\big|_{R=1} - D_q V^{tr}\big|_{R=1} = 0,$$

yields the following boundary condition for V^s:

$$\partial_R V^s\big|_{R=1} - \frac{\mu_0}{\mu} D_q V^s\big|_{R=1} = \frac{\mu_0}{\mu} D_q\, U^i\big|_{R=1} - \partial_R\, U^i\big|_{R=1}. \tag{4.37}$$

This brings us to

Remark 4.15. *The transmission problem* (4.36) *may be equivalently formulated as follows. Let V^s be the unique solution to the following exterior problem with impedance condition:*

$$\begin{cases} \Delta V^s + q_0\lambda^2 V^s = 0 & \text{for } R > 1, \\ \Big(\partial_R - \dfrac{\mu_0}{\mu} D_q\Big) V^s = -\Big(\partial_R - \dfrac{\mu_0}{\mu} D_q\Big) U^i & \text{for } R = 1, \\ (\partial_R - i\sqrt{q_0}\lambda) V^s = O(R^{-3/2}) & \text{as } R \to \infty. \end{cases} \tag{4.38}$$

Then, with this V^s, let V^{tr} solve

$$\begin{cases} \partial_R V^{tr} - D_q V^{tr} = 0 \quad \text{for } (R,\theta) \in (1/2, 1) \times \mathbb{T}, \\ V^{tr}\big|_{R=1} = V^s\big|_{R=1} + U^i\big|_{R=1}. \end{cases}$$

For convenience, we extend V^{tr} into $B_{1/2}$ by defining it to be the solution to the Dirichlet problem

$$\begin{cases} \Delta V^{tr} + q\lambda^2 V^{tr} = 0 & \text{in } B_{1/2}, \\ (V^{tr})^- = (V^{tr})^+ & \text{on } \partial B_{1/2}. \end{cases}$$

We will soon demonstrate that this approximation V^s of U^s satisfies the $H^{-1/2}$ bound we seek for U^s. But first we will need

Lemma 4.16. *Let g be a given function in $H^\sigma(\mathbb{T})$, $\sigma \geq -1/2$. Let V^s be the solution of the following problem:*

$$\begin{cases} \Delta V^s + q_0\lambda^2 V^s = 0 \quad \text{for } R > 1, \\ \partial_R V^s - (\mu_0/\mu) D_q V^s = g \quad \text{for } R = 1, \end{cases}$$

with the radiation condition

$$(\partial_R - i\sqrt{q_0}\lambda)V^s = O(R^{-3/2}).$$

Then there exists a constant $C = \mathrm{const}(q, \mu/\mu_0) > 0$ *and a* $\lambda_0 \geq 1$ *depending on* q *such that for all* $\lambda \geq \lambda_0$*,*

$$\|V^s|_{R=1}\|_{H^\sigma(\mathbb{T})} \leq C\left(\frac{1}{\lambda}\|g\|_{H^\sigma(\mathbb{T})} + \frac{1}{\lambda^2}\|g\|_{H^{\sigma+1}(\mathbb{T})}\right).$$

Proof. The l^{th} Fourier coefficient $\widehat{V}_n^s$ of V^s satisfies the following ordinary differential equation:

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\mathrm{d}}{\mathrm{d}R}\widehat{V}_n^s\right) + \lambda^2\left(q_0 - \frac{n^2}{(R\lambda)^2}\right)\widehat{V}_n^s = 0 \quad \text{for } 1 < R < \infty, \tag{4.39a}$$

$$\frac{\mathrm{d}}{\mathrm{d}R}\widehat{V}_n^s - \frac{\mu_0}{\mu}d_q\widehat{V}_n^s = \widehat{g}_n \text{ at } R = 1, \tag{4.39b}$$

with the radiation condition as written in Fourier mode,

$$\partial_R\widehat{V}_n^s - i\sqrt{q_0}\lambda\widehat{V}_n^s = O(R^{-3/2}). \tag{4.39c}$$

We multiply (4.39a) by the conjugate of $R\widehat{V}_n^s$ and then integrate over $R \in [1, +\infty)$. Then integrate by parts using the boundary condition (4.39b) and the radiation condition. After taking the imaginary part of the resulting equation, we obtain

$$\lambda\,\mathrm{Im}(\sqrt{q_0})\lim_{R\to+\infty}\left(R\left|\widehat{V}_n^s\right|^2\right) - \frac{\mu_0}{\mu}\,\mathrm{Im}(d_q|_{R=1})\left|\widehat{V}_n^s|_{R=1}\right|^2 = \mathrm{Im}\left(\widehat{g}_n\overline{\widehat{V}_n^s}|_{R=1}\right).$$

Therefore, since $\mathrm{Im}(\sqrt{q_0}) > 0$ and $\mathrm{Im}(d_q|_{R=1}) < 0$,

$$\left|\widehat{V}_n^s|_{R=1}\right| \leq -\frac{\mu/\mu_0}{\mathrm{Im}(d_q|_{R=1})}\left|\widehat{g}_n\right|.$$

By Property 4.12, there exists a constant C and a $\lambda_0 \geq 1$, both depending on q, such that for all $\lambda \geq \lambda_0$,

$$\frac{1}{[\mathrm{Im}(d_q)]^2} \leq \frac{C}{\lambda^2}\left(1+\frac{|n|^2}{\lambda^2}\right),$$

and therefore

$$|n|^{2\sigma}\left|\widehat{V}_n^s\big|_{R=1}\right|^2 \leq C_{q,\mu/\mu_0}\left(\frac{|n|^{2\sigma}}{\lambda^2}+\frac{|n|^{2\sigma+2}}{\lambda^4}\right)|\widehat{g}_n|^2. \qquad \square$$

In the context of problem (4.36),

$$g = \left(\frac{\mu_0}{\mu}D_q - \partial_R\right)U^i\big|_{R=1}.$$

Using the bound $d_q|_{R=1} \leq C\,(\lambda+|n|)$ (Lemma 4.10), we find

$$\|g\|_{H^\sigma(\mathbb{T})} \leq C\left(\lambda\|U^i|_{R=1}\|_{H^\sigma(\mathbb{T})} + \|U^i|_{R=1}\|_{H^{\sigma+1}(\mathbb{T})}\right) + \|\partial_R U^i|_{R=1}\|_{H^\sigma(\mathbb{T})}, \tag{4.40}$$

where C depends only on q and μ_0/μ. To express this bound in terms of $\{a_n\}$, where

$$U^i(x) = \sum_{l\in\mathbb{Z}} a_n J_n(\sqrt{q_0}\lambda R)e^{in\theta},$$

we use the following

Lemma 4.17. *For all $\sigma \geq 0$,*

$$\left\|U^i|_{R=1}\right\|_{H^\sigma(\mathbb{T})} \leq C_{q_0,\sigma}\lambda^\sigma\|\{a_n\}\|_{l^\infty}, \tag{4.41a}$$

$$\left\|\partial_R U^i|_{R=1}\right\|_{H^\sigma(\mathbb{T})} \leq C_{q_0,\sigma}\lambda^{\sigma+1}\|\{a_n\}\|_{l^\infty}. \tag{4.41b}$$

Proof. The second inequality is an easy consequence of the first inequality and the identity

$$zJ_n'(z) = nJ_n(z) - zJ_{n+1}(z).$$

The first inequality will be a consequence of the identity

$$nJ_n(z) = \frac{z}{2}\big(J_{n-1}(z) + J_{n+1}(z)\big). \tag{4.42}$$

By repeatedly applying (4.42), we find that for any positive integer m,

$$|n|^m |J_n(z)| \le \frac{|z|^m}{2^m} \sum_{j=0}^{m} \binom{m}{j} |J_{n-m+2j}(z)|,$$

which implies

$$|n|^{2m} |J_n(z)|^2 \le C_m |z|^{2m} \sum_{j=0}^{m} |J_{n-m+2j}(z)|^2,$$

which in turn implies[10] (4.41a) for $\sigma = m$.

The case of non-integral $\sigma > 0$ follows by choosing any $m > \sigma$ and using the interpolation inequality

$$\|U^i\|_{H^\sigma(\mathbb{T})} \le \|U^i\|_{L^2(\mathbb{T})}^{1-\sigma/m} \|U^i\|_{H^m(\mathbb{T})}^{\sigma/m}. \qquad \square$$

Remark 4.18. *Note that this proof of Lemma 4.17 requires that $\sigma \ge 0$. In section 4.2.7 we will find a bound in the case where $\sigma = -1/2$ and U^i is a plane wave.*

Consequently, for $\sigma \ge 0$,

$$\|g\|_{H^\sigma(\mathbb{T})} \le C_{(q,q_0,\sigma,\mu/\mu_0)} \lambda^{\sigma+1} \|\{a_n\}\|_{l^\infty},$$

and thus

$$\begin{aligned} \|V^s|_{R=1}\|_{H^\sigma(\mathbb{T})} &\le C_q \left(\frac{1}{\lambda} \|g\|_{H^\sigma(\mathbb{T})} + \frac{1}{\lambda^2} \|g\|_{H^{\sigma+1}(\mathbb{T})} \right) \qquad (4.43) \\ &\le C_{(q,q_0,\sigma,\mu/\mu_0)} \lambda^\sigma \|\{a_n\}\|_{l^\infty}. \end{aligned}$$

---

[10] Here we use the well known fact that $\sum |J_n(t)|^2 = 1$ for all real t, which is simply Parseval's equality applied to a plane wave on the unit circle.

We have therefore proved

Proposition 4.19. *Let* (V^{tr}, V^s) *solve problem* (4.36) *with the incoming wave* U^i *satisfying*

$$U^i(x) = \sum_{l \in \mathbb{Z}} a_n J_n(\sqrt{q_0}\lambda R) e^{in\theta},$$

for some sequence $\{a_n\} \in l^\infty$*. Then for all* $\sigma \geq 0$*, there exists a constant* C *depending only on* q*,* q_0*,* σ *and* μ/μ_0*, and a* λ_0 *depending only on* q*, such that for all* $\lambda \geq \lambda_0$*,*

$$\|V^s|_{R=1}\|_{H^\sigma(\mathbb{T})} \leq C\lambda^\sigma \|\{a_n\}\|_{l^\infty}.$$

In particular, we have

$$\|V^s|_{R=1}\|_{L^2(\mathbb{T})} \leq C\|\{a_n\}\|_{l^\infty}. \tag{4.44}$$

4.2.5 The main estimate

Before proving our main theorem, we need the following

Lemma 4.20. *Let* $q = a + ib$ *with* $a, b > 0$*. Suppose* $u \in H^1(B)$ *satisfies*

$$\Delta u + \lambda^2 q u = 0, \ \text{in } B.$$

Then there exists a constant $C > 0$ *and a* $\lambda_0 \geq 1$*, both depending only on* q*, such that for all* $\lambda \geq \lambda_0$*,*

$$\|u|_{R=1}\|_{H^{-1/2}(\mathbb{T})} \leq C\lambda^{1/2}\|u\|_{L^2(B)}.$$

Proof. Let us denote by g the trace of u on the boundary $R = 1$. Let v be the solution of the factorized problem:

$$\partial_R v - D_q(R)v = 0 \quad \text{in } \mathbf{C},$$
$$v = g \quad \text{at } R = 1,$$

and in $B_{1/2}$,

$$\begin{aligned} \Delta v + \lambda^2 q v = 0 &\quad \text{in } B_{1/2}, \\ v^- = v^+ &\quad \text{at } R = 1/2. \end{aligned}$$

By applying (4.31) from Lemma 4.13 with $U = v$, the proof is finished if we can show

$$\|v\|_{L^2(B)} \le C\|u\|_{L^2(B)} \tag{4.45}$$

for λ sufficiently large. Recall that $\mathscr{L}$ denotes Helmholtz operator while $\mathscr{L}_0$ denotes its factorization to order zero (4.23). Let $w = u - v$. w then satisfies

$$\begin{aligned} &\Delta w + \lambda^2 q w = -(\mathscr{L} - \mathscr{L}_0)\, v && \text{in } \mathbf{C}, \\ &\Delta w + \lambda^2 q w = 0 && \text{in } B_{1/2}, \end{aligned}$$

with transmission conditions

$$\begin{aligned} w^- &= w^+ && \text{at } R = 1/2, \\ \partial_R w^- &= \partial_R w^+ + \partial_R v^- - \partial_R v^+ && \text{at } R = 1/2, \end{aligned}$$

and boundary condition

$$w = 0 \quad \text{at } R = 1.$$

From this we calculate

$$\begin{aligned} \int_B \big(- |\nabla w|^2 + \lambda^2 q |w|^2 \big)\, \mathrm{d}x = \int_{\partial B_{1/2}} \big(\partial_R v^- - \partial_R v^+ \big) \overline{w}\, \mathrm{d}\sigma& \\ - \int_{\mathbf{C}} \big[(\mathscr{L} - \mathscr{L}_0) v \big] \overline{w}\, \mathrm{d}x.& \end{aligned} \tag{4.46}$$

Using Lemma 4.14 with $U = v$ and $s = \sigma = -1/2$, we get

$$\left| \int_{\partial B_{1/2}} (\partial_R v^- - \partial_R v^+)\overline{w}\,\mathrm{d}\sigma \right| \leq Ce^{-c\lambda} \|v\|_{H^{-1/2}(\partial B)} \|w\|_{H^{1/2}(\partial B)}$$
$$\leq C\sqrt{\lambda} e^{-c\lambda} \|v\|_{L^2(\mathbf{C})} \|w\|_{H^1(B)},$$

where the second inequality follows from estimate (4.31) from Lemma 4.13. Then since $\mathscr{L} - \mathscr{L}_0 : L^2(\mathbf{C}) \to L^2(\mathbf{C})$ is bounded independently of λ (Lemma 4.8), by taking the imaginary part of (4.46) we find

$$\|w\|^2_{L^2(B)} \leq \frac{C_q}{\lambda^2} \|v\|_{L^2(\mathbf{C})} \|w\|_{H^1(B)}.$$

Appealing to (4.46) a second time therefore yields

$$\|w\|_{H^1(B)} \leq C_q \|v\|_{L^2(\mathbf{C})}.$$

which, combined with the previous inequality, yields

$$\|w\|_{L^2(B)} \leq \frac{C_q}{\lambda} \|v\|_{L^2(\mathbf{C})},$$

Since $\|v\|_{L^2(B)} - \|u\|_{L^2(B)} \leq \|w\|_{L^2(B)}$, (4.45) now follows by choosing λ sufficiently large. $\square$

Corollary 4.21. *Let $q = a + ib$ with $a, b > 0$. Suppose $f \in L^2(B)$ and $u \in H^1(B)$ such that*

$$\Delta u + \lambda^2 q u = f \quad \text{in } B.$$

Then there exists a constant $C > 0$ and a $\lambda_0 \geq 1$, both depending only on q, such that for all $\lambda \geq \lambda_0$,

$$\|u|_{R=1}\|_{H^{-1/2}(\mathbb{T})} \leq C\lambda^{1/2} \left(\|u\|_{L^2(B)} + \frac{1}{b\lambda^2} \|f\|_{L^2(B)} \right).$$

Proof. Let $u^0 \in H_0^1(B)$ solve $\mathscr{L}u^0 = f$. We write this equation in the Fourier mode:

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\mathrm{d}}{\mathrm{d}R}\widehat{u^0}_n\right) + \left(\lambda^2 q - n^2/R^2\right)\widehat{u^0}_n = \widehat{f}_n \quad \text{for } 0 < R < 1,$$
$$\widehat{u^0}_n = 0 \qquad \text{at } R = 1.$$

Multiplying by $R\overline{u^0}$, integrating over $0 < R < 1$ and then taking imaginary parts, we find

$$b\lambda^2\left(\int_0^1 |\widehat{u^0}_n|^2 R\,\mathrm{d}R\right)^{1/2} \leq \left(\int_0^1 |\widehat{f}_n|^2 R\,\mathrm{d}R\right)^{1/2},$$

and therefore,

$$\|u^0\|_{L^2(B)} \leq \frac{1}{b\lambda^2}\|f\|_{L^2(B)}.$$

The corollary follows by applying Lemma 4.20 to $u - u^0$. □

It remains to be shown that V^s is sufficiently well approximated by U^s, so that the Sobolev bound of Proposition 4.19 on V^s applies to U^s as well. This will be accomplished in the following

Proposition 4.22. *Let U^i be an incident wave of the form*

$$U^i(R,\theta) = \sum_{n\in\mathbb{Z}} a_n J_n\left(\sqrt{q_0}\lambda R\right) e^{in\theta},$$

where $\{a_n\} \in l^\infty$. Let $q = a + ib$ with $a, b > 0$. Let (U^{tr}, U^s) be the solution of problem (4.19) *and let (V^{tr}, V^s) be that of problem* (4.36)*. Then there exists a constant $C > 0$ depending on q and a λ_0 depending only on q such that for all $\lambda \geq \lambda_0$,*

$$\left\|U^s|_{R=1}\right\|_{H^{-1/2}(\mathbb{T})} \leq C\left\|V^s|_{R=1}\right\|_{H^{-1/2}(\mathbb{T})}.$$

Proof. Denote by W^s and W^{tr} the following fields:

$$W^s = U^s - V^s, \quad W^{tr} = U^{tr} - V^{tr}.$$

They satisfy

$$
\begin{aligned}
&\Delta W^{tr} + q\lambda^2 W^{tr} = 0 && \text{for } (R,\theta) \in (0,1/2)\times\mathbb{T},\\
&\Delta W^{tr} + q\lambda^2 W^{tr} = -\left(\mathscr{L} - \mathscr{L}_0\right) V^{tr} && \text{for } (R,\theta) \in (1/2,1)\times\mathbb{T},\\
&\Delta W^{s} + q_0\lambda^2 W^{s} = 0 && \text{for } (R,\theta) \in (1,\infty)\times\mathbb{T},
\end{aligned}
$$

with the transmission conditions

$$
\begin{aligned}
\partial_R^{+} W^{tr} &= \partial_R^{-} W^{tr} + \partial_R^{+} V^{tr} - \partial_R^{-} V^{tr} && \text{at } R = 1/2,\\
\left(W^{tr}\right)^{+} &= \left(W^{tr}\right)^{-} && \text{at } R = 1/2,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{\mu}\partial_R W^{tr} &= \frac{1}{\mu_0}\partial_R W^{s} && \text{at } R = 1,\\
W^{tr} &= W^{s} && \text{at } R = 1,
\end{aligned}
$$

and the radiation condition

$$
(\partial_R - i\sqrt{q_0}\lambda)V^s = O(R^{-3/2}).
$$

Claim: *There exists $C > 0$ such that*

$$
\lambda^2 b \|W^{tr}\|_{L^2(B)} \le C \|V^{tr}\|_{H^1(\mathbf{C})}. \tag{4.47}
$$

To prove this, note that because W^{tr} and W^s solve the above transmission problem, we have

$$
\begin{aligned}
-\int_B |\nabla W^{tr}|^2 &+ q\lambda^2 \int_B |W^{tr}|^2 \\
&= -\int_{\mathbf{C}} [(\mathscr{L} - \mathscr{L}_0)V^{tr}]\overline{W^{tr}} - \int_{\partial B} \partial_\nu W^{tr}\overline{W^{tr}}\,\mathrm{d}\sigma \\
&\qquad + \int_{\mathbb{T}} \left(\partial_R^{+} V^{tr}\big|_{R=1/2} - \partial_R^{-} V^{tr}\big|_{R=1/2}\right)\overline{W^{tr}}|_{R=1/2}\,\mathrm{d}\theta,
\end{aligned} \tag{4.48}
$$

and for any $R > 1$,

$$-\int_A |\nabla W^s|^2 + q_0\lambda^2 \int_A |W^s|^2 = -\int_{\partial A} \partial_{\mathbf{n}} W^s \overline{W^s}\,\mathrm{d}\sigma$$
$$= -\int_{\partial B_R} \partial_\nu W^s \overline{W^s}\,\mathrm{d}\sigma + \int_{\partial B} \partial_\nu W^s \overline{W^s}\,\mathrm{d}\sigma$$
$$= -i\sqrt{q_0}\lambda \int_{\partial B_R} |W^s|^2\,\mathrm{d}\sigma + \int_{\partial B_R} \mathrm{O}(R^{-3/2}) W^s\,\mathrm{d}\sigma$$
$$+ \frac{\mu_0}{\mu} \int_{\partial B} \partial_\nu W^{tr} \overline{W^{tr}}\,\mathrm{d}\sigma, \tag{4.49}$$

where $A = B_R \setminus B$. Taking the imaginary part of (4.48) yields

$$b\lambda^2 \|W^{tr}\|^2_{L^2(B)} = -\operatorname{Im}\left\{\int_{\mathbf{C}} [(\mathscr{L} - \mathscr{L}_0)V^{tr}]\overline{W^{tr}}\right\} - \operatorname{Im}\left\{ \int_{\partial B} \partial_\nu W^{tr} \overline{W^{tr}}\,\mathrm{d}\sigma\right\}$$
$$+ \operatorname{Im}\left\{\int_0^{2\pi} \left(\partial_R^+ V^{tr}\big|_{R=1/2} - \partial_R^- V^{tr}\big|_{R=1/2}\right) \overline{W^{tr}}|_{R=1/2}\,\mathrm{d}\theta\right\},$$

and taking the imaginary part of (4.49) yields

$$\operatorname{Im}\{q_0\}\lambda^2 \int_A |W^s|^2 = -\operatorname{Im}\{i\sqrt{q_0}\}\lambda \|W^s\|^2_{L^2(\partial B_R)}$$
$$+ \frac{\mu_0}{\mu} \operatorname{Im}\left\{ \int_{\partial B} \partial_\nu W^{tr} \overline{W^{tr}}\,\mathrm{d}\sigma\right\} + \mathrm{o}(1)_{R\to\infty},$$

which implies

$$\operatorname{Im}\left\{ \int_{\partial B} \partial_\nu W^{tr} \overline{W^{tr}}\,\mathrm{d}\sigma\right\} \geq 0,$$

since $\operatorname{Im}\{q_0\}$ and $\operatorname{Im}\{i\sqrt{q_0}\}$ are positive. We therefore get

$$\lambda^2 b \|W^{tr}\|^2_{L^2(B)} \leq \left|\int_{\mathbf{C}} [(\mathscr{L} - \mathscr{L}_0)V^{tr}]\overline{W^{tr}}\right|$$
$$+ \left|\int_{\mathbb{T}} \left(\partial_R^+ V^{tr}\big|_{R=1/2} - \partial_R^- V^{tr}\big|_{R=1/2}\right) \overline{W^{tr}}|_{R=1/2}\,\mathrm{d}\theta\right|.$$

Thus, for any $s \geq 0$,

$$\lambda^2 b \|W^{tr}\|^2_{L^2(B)} \leq \|(\mathscr{L} - \mathscr{L}_0)V^{tr}\|_{L^2(\mathbf{C})} \|W^{tr}\|_{L^2(B)} \\ + \left\| \partial_R^+ V^{tr}|_{R=1/2} - \partial_R^- V^{tr}|_{R=1/2} \right\|_{H^s(\mathbb{T})} \left\|W^{tr}|_{R=1/2}\right\|_{H^{-s}(\mathbb{T})},$$

According to Lemma (4.8),

$$\|(\mathscr{L} - \mathscr{L}_0)V^{tr}\|_{L^2(\mathbf{C})} \leq C_q \left\|V^{tr}\right\|_{L^2(\mathbf{C})},$$

and by Lemmas 4.14 and 4.13, there exists a $C_{q,s} > 0$ such that

$$\left\| \partial_R^+ V^{tr}|_{R=1/2} - \partial_R^- V^{tr}|_{R=1/2} \right\|_{H^s(\mathbb{T})} \leq C_{q,s} e^{-C\lambda} \left\|V^{tr}\right\|_{L^2(\mathbf{C})}.$$

Moreover, according to Lemma 4.20,

$$\left\|W^{tr}|_{R=1/2}\right\|_{H^{-1/2}(\mathbb{T})} \leq \sqrt{\lambda} \left\|W^{tr}\right\|_{L^2(B_{1/2})}.$$

The ***Claim*** (4.47) follows by combining the last four inequalities.

Thus, for some constant C depending on q, we have

$$\|W^{tr}\|_{L^2(B)} \leq \frac{C}{\lambda^2} \|V^{tr}\|_{L^2(\mathbf{C})},$$

hence

$$\|U^{tr}\|_{L^2(B)} \leq C \|V^{tr}\|_{L^2(\mathbf{C})}.$$

Now, by Lemma 4.20,

$$\frac{C}{\sqrt{\lambda}} \|U^{tr}|_{R=1}\|_{H^{-1/2}(\mathbb{T})} \leq \left\|U^{tr}\right\|_{L^2(B)},$$

and by Lemma 4.13,

$$\|V^{tr}\|_{L^2(B)} \leq \frac{C}{\sqrt{\lambda}} \left\|V^{tr}|_{R=1}\right\|_{L^2(\mathbb{T})}.$$

Putting the previous three inequalities together completes the proof. □

Recall estimate (4.44) (that is, Proposition 4.19 with $\sigma = 0$):

$$\left\| V^s|_{R=1} \right\|_{L^2(\mathbb{T})} \leq C \|\{a_n\}\|_{l^\infty}.$$

From this and Proposition 4.22 it follows that

$$\|U^s|_{R=1}\|_{H^{-1/2}(\partial B)} \leq C \|\{a_n\}\|_{l^\infty}.$$

We are now in a position to state and prove the main theorem of this chapter. For convenience, we will assume the background medium is non-conducting, i.e., that q_0 is real and positive.

Theorem 8. *Let $q = a + ib$ with $a, b > 0$, and let $q_0 > 0$. Let (u_ρ^{tr}, u_ρ^s) solve problem* (4.1)*, with the incident wave u^i of the form*

$$u^i(r, \theta) = \sum_{n \in \mathbb{Z}} a_n J_n\left(\sqrt{q_0}\omega r\right) e^{in\theta}.$$

There exists a constant $C = \mathrm{const}(q, q_0, \mu/\mu_0) > 0$ and a $\lambda_0 \geq 1$ depending only on q such that, for any $\rho > 0$ and $\omega > 0$ such that $\omega\rho \geq \lambda_0$, and for any $r \geq \rho$,

$$\left\| u_\rho^s|_{\partial B_r} \right\|_{H^{-1/2}(\mathbb{T})} \leq C \frac{\sqrt{\rho}}{\sqrt{r}} \|\{a_n\}\|_{l^\infty}.$$

Proof.

$$\begin{aligned}
\left\| u_\rho^s|_{\partial B_r} \right\|^2_{H^{-1/2}(\mathbb{T})} &= \sum_n \left| \widehat{\left[u_\rho^s|_{|x|=\rho} \right]}_n \frac{H_n(\sqrt{q_0}\omega r)}{H_n(\sqrt{q_0}\omega\rho)} \right|^2 \Big/ (1 + |n|) \\
&\leq C_{q_0} \frac{\rho}{r} \left\| u_\rho^s \right\|^2_{H^{-1/2}(\partial B_\rho)} \\
&= C_{q_0} \frac{\rho}{r} \left\| U^s \right\|^2_{H^{-1/2}(\partial B)},
\end{aligned}$$

where the inequality follows from the fact that for $n \neq 0$,

$$\left|\frac{H_n(\sqrt{q_0}\omega r)}{H_n(\sqrt{q_0}\omega\rho)}\right|^2 \leq \frac{\rho}{r},$$

and for $n = 0$,

$$\begin{aligned}\left|\frac{H_0(\sqrt{q_0}\omega r)}{H_0(\sqrt{q_0}\omega\rho)}\right|^2 &= \frac{\rho}{r}\frac{\sqrt{q_0}\omega r\left|H_0(\sqrt{q_0}\omega r)\right|^2}{\sqrt{q_0}\omega\rho\left|H_0(\sqrt{q_0}\omega\rho)\right|^2} \\ &\leq \frac{\rho}{r}\frac{2/\pi}{\sqrt{q_0}\left|H_0(\sqrt{q_0})\right|^2},\end{aligned}$$

since the function $0 < t \mapsto t\left|H_n(t)\right|^2$ is decreasing for integers $n \neq 0$ and increasing for $n = 0$ (to the limit $2/\pi$) [Wat44, 13.74], and since we are assuming $\lambda = \omega\rho \geq 1$. The theorem follows from Proposition 4.22 and Proposition 4.19 (see the remarks just before the statement of this theorem). □

4.2.6 A refinement of the bound

Numerical evidence[11] suggests that, for a fixed $r_0 > \rho$,

$$\|u_\rho^s\|_{L^2(\partial B_{r_0})} \leq C\sqrt{\rho} \quad \text{as } \omega\rho \to \infty,$$

and this is indeed the bound we found for $\left\|u_\rho^s\big|_{r=r_0}\right\|_{H^{-1/2}(\mathbb{T})}$ in Theorem 8. The numerical evidence, however, suggests that the optimal bound on the $H^{-1/2}$ norm should decrease as omega grows. In testing, this bound appears to be of the order such that

$$\frac{C}{\sqrt{r_0}}\frac{\sqrt{\log\lambda}}{\sqrt{\lambda}}\sqrt{\rho} \ll \left\|u_\rho^s\big|_{r=r_0}\right\|_{H^{-1/2}(\mathbb{T})} \ll \frac{1}{C\sqrt{r_0}}\frac{\log\lambda}{\sqrt{\lambda}}\sqrt{\rho} \quad \text{as } \lambda = \omega\rho \to \infty$$

(see Figure 4.2), which is a stronger bound than that found in Theorem 8. While we have not yet found the optimal bound, we do have the following

Theorem 9. *Let $q = a + ib$ with $a, b > 0$. Let (u_ρ^{tr}, u_ρ^s) solve problem* (4.1)*, where the*

[11] See Figure 3.1 in Chapter 3.

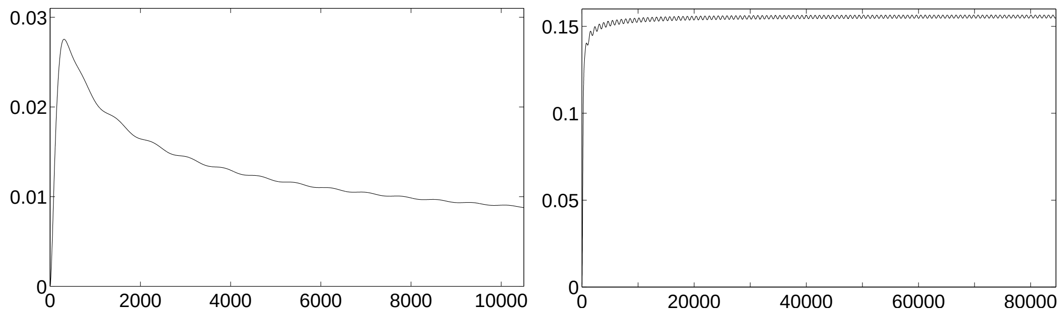


Figure 4.2: The left frame is a plot of $\|u^s_\rho|_{r=2}\|_{H^{-1/2}(\mathbb{T})}$ as a function of ω. The right frame is a plot of $s(\omega)\|u^s_\rho|_{r=2}\|_{H^{-1/2}(\mathbb{T})}$, where $s(\omega) = \sqrt{\omega}/(\log\omega)^{0.7975}$. Here we have taken $q_0 = 1$, $q_1 = 4 + 4i$, $\gamma := \mu_0/\mu = 2$ and $\rho = 0.004$. Numerical evidence suggests that $(\log\omega)^{0.8}/\sqrt{\omega} \ll \|u^s_\rho|_{r=2}\|_{H^{-1/2}(\mathbb{T})} \ll (\log\omega)^{0.795}/\sqrt{\omega}$ as $\omega \to \infty$.

incident wave u^i is a plane wave,

$$u^i(r,\theta) = e^{i\sqrt{q_0}\,\omega\, x\cdot\eta}, \quad \eta \in \mathbb{R}^2 \textit{ satisfying } |\eta| = 1,$$

and where $q_0 > 0$. Then there exists a constant $C = \mathrm{const}(q, q_0, \mu/\mu_0) > 0$ and a $\lambda_0 \geq 1$ depending only on q such that, for any $\rho > 0$ and $\omega > 0$ such that $\omega\rho \geq \lambda_0$, and for any $r \geq \rho$,

$$\|u^s_\rho|_{\partial B_r}\|_{H^{-1/2}(\mathbb{T})} \leq C\frac{\rho^{1/12}}{\omega^{5/12}} = C\frac{\sqrt{\rho}}{\lambda^{5/12}}.$$

To prove this, we simply improve the bound on $\|V^s|_{R=1}\|_{H^{-1/2}(\mathbb{T})}$ found in Proposition 4.19, and use the estimate

$$\|U^s|_{R=1}\|_{H^{-1/2}(\mathbb{T})} \leq C\|V^s|_{R=1}\|_{H^{-1/2}(\mathbb{T})}$$

(Proposition 4.22), just as before. To improve the bound on $\|V^s|_{R=1}\|_{H^{-1/2}(\mathbb{T})}$ found in Proposition 4.19, recall that we have shown, using Lemma 4.16, that for $\sigma \geq -1/2$,

$$\|V^s|_{R=1}\|_{H^\sigma(\mathbb{T})} \leq C_q\left(\frac{1}{\lambda}\|g\|_{H^\sigma(\mathbb{T})} + \frac{1}{\lambda^2}\|g\|_{H^{\sigma+1}(\mathbb{T})}\right)$$

(inequality (4.43)), where

$$g = \Big(\frac{\mu_0}{\mu} D_q - \partial_R\Big) U^i\big|_{R=1}.$$

Then since

$$\|g\|_{H^\sigma(\mathbb{T})} \le C\left(\lambda \|U^i|_{R=1}\|_{H^\sigma(\mathbb{T})} + \|U^i|_{R=1}\|_{H^{\sigma+1}(\mathbb{T})}\right) + \|\partial_R U^i|_{R=1}\|_{H^\sigma(\mathbb{T})},$$

(inequality (4.40)), we have

$$\|V^s|_{R=1}\|_{H^{-1/2}(\mathbb{T})} \le C_q \left(\|U^i|_{R=1}\|_{H^{-1/2}(\mathbb{T})} + \frac{1}{\sqrt{\lambda}}\right).$$

Here we used Lemma 4.17 and the fact that

$$\frac{1}{\lambda^2}\|\partial_R U^i|_{R=1}\|_{H^{-1/2}(\mathbb{T})} \le \frac{|\sqrt{q_0}|}{\lambda^2}\|U^i|_{R=1}\|_{H^{-1/2}(\mathbb{T})}.$$

We therefore must find a bound on $\|U^i|_{R=1}\|_{H^{-1/2}(\mathbb{T})}$.

4.2.7 Estimating the incident wave via stationary phase

By rotating the coordinates, we may assume $\eta = \binom{0}{1}$, so that

$$U^i(x) = e^{i\sqrt{q_0}\lambda R \sin\theta}.$$

Now we use the method of stationary phase: as $\lambda \to \infty$,

$$\begin{aligned}\|U^i|_{R=1}\|^2_{H^{-1}(\mathbb{T})} &\le C\int_0^{2\pi}\left|\int_0^\theta e^{i\sqrt{q_0}\lambda\sin t}\mathrm{d}t\right|^2\mathrm{d}\theta \\ &= \frac{2\pi C}{\sqrt{q_0}\lambda}\int_0^{2\pi}\left|\chi_{[\pi/2,2\pi]}e^{i(\lambda-\pi/4)} + \chi_{[3\pi/2,2\pi]}e^{-i(\lambda-\pi/4)}\right|^2\mathrm{d}\theta \\ &\qquad + O(\lambda^{-3/2}).\end{aligned}$$

By interpolating between H^{-1} and L^2 we therefore get

$$\|U^i|_{R=1}\|_{H^{-1/2}(\mathbb{T})} \leq C_{q_0}\lambda^{-1/4}.$$

This is a rather rough estimate, but it can be improved by appealing directly to an oscillatory integral expression of $\|U^i\|_{H^{-1/2}(\mathbb{T})}$ on which to perform a stationary phase analysis.

Proposition 4.23. *Let* (V^t, V^s) *solve problem* (4.36) *with the incoming wave* U^i *satisfying*

$$U^i(x) = e^{i\sqrt{q_0}\lambda\eta\cdot x}$$

with $|\eta| = 1$. *Then there exists a constant* C *depending only on* q, q_0, μ/μ_0 *and a* λ_0 *depending only on* q *such that for all* $\lambda \geq \lambda_0$,

$$\|V^s|_{R=1}\|_{H^{-1/2}(\mathbb{T})} \leq C\lambda^{-5/12}.$$

Proof. Define the function F on $\mathbb{T}$ by

$$F(\theta) = \int_0^\theta \left(U^i|_{R=1}(\tau) - \widehat{U}_0^i\right) \mathrm{d}\tau + \widehat{U}_0^i$$

so that

$$\begin{aligned}\|U^i|_{R=1}\|_{H^{-1/2}(\mathbb{T})} &= \|F\|_{H^{1/2}(\mathbb{T})}\\ &= \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{\left|\int_t^\theta e^{i\sqrt{q_0}\lambda\sin s}\,\mathrm{d}s\right|^2}{|t-\theta|^2}\,\mathrm{d}t\,\mathrm{d}\theta\end{aligned}$$

(see [Gri85] or [Ada75]). For simplicity, assume $q_0 = 1$ and let

$$I(t,\theta) = \int_t^\theta e^{i\lambda\sin s}\,\mathrm{d}s.$$

From the Figure 4.3 representing $\mathbb{T}^2$, let A denote the the unshaded region, D_1 the lightly shaded region, and D_2 the darkly shaded region. For $(t,\theta) \in A$, it follows from

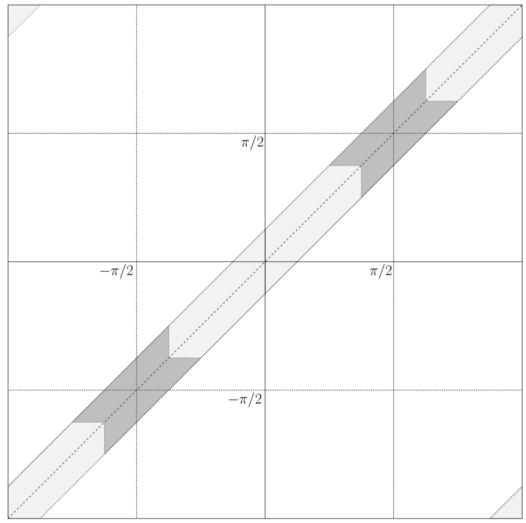


Figure 4.3: $\mathbb{T}^2$, where the shaded region around the diagonal is $\{(t,\theta) \in \mathbb{T}^2 : |t-\theta| < \delta\}$.

the method of stationary phase that

$$\begin{aligned}\iint_A \frac{|I(t,\theta)|^2}{|t-\theta|^2} &\leq \frac{C}{\lambda} \iint_A \frac{1}{|t-\theta|^2} \\ &\leq \frac{C}{\delta\lambda}\end{aligned}$$

(details omitted; cf. Lemma 3.8 of Chapter 3, or Chapter 3 of [Olv97]).

For $(t,\theta) \in D_1$ with $\theta < t < \theta + \delta$,

$$I(t,\theta) = \left.\frac{e^{i\lambda \sin s}}{i\lambda}\right|_{s=t}^{s=\theta} + \frac{1}{i\lambda} \int_t^\theta e^{i\lambda \sin s} \frac{\sin s}{\cos^2 s} \,\mathrm{d}s, \tag{4.50}$$

so that

$$|I(t,\theta)| \leq \frac{C}{\lambda}\left(1 + \left|\frac{1}{\cos\theta} - \frac{1}{\cos t}\right|\right).$$

Therefore,

$$\begin{aligned}\int_{\theta+1/(\delta\lambda)}^{\theta+\delta} \frac{|I(t,\theta)|^2}{|t-\theta|^2} \,\mathrm{d}t &\leq \frac{C}{\lambda^2} \max_{(t,\theta)\in D_1} \left(\frac{1}{\cos^2 t} + \frac{1}{\cos^2\theta}\right) \int_{\theta+1/(\delta\lambda)}^{\theta+\delta} \frac{1}{|t-\theta|^2} \,\mathrm{d}t \\ &\leq C \frac{1}{\lambda^2} \frac{1}{\delta^2} \delta\lambda \\ &\leq \frac{C}{\delta\lambda}.\end{aligned}$$

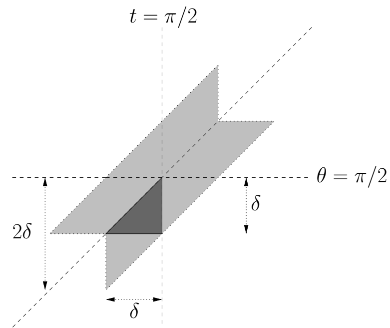


Figure 4.4: Close-up on one half of D_2. T denotes the shaded triangle.

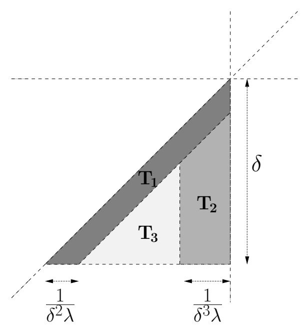


Figure 4.5: Close-up on T.

Since I also satisfies

$$|I(t,\theta)| \leq |t-\theta|,$$

it follows that

$$\iint_{D_1} \frac{|I(t,\theta)|^2}{|t-\theta|^2} \leq \frac{C}{\delta\lambda}.$$

For the region D_2, we need only consider the shaded triangle T from Figure 4.4. That is, for some constant C,

$$\iint_{D_2} \frac{|I(t,\theta)|^2}{|t-\theta|^2} \leq C \iint_T \frac{|I(t,\theta)|^2}{|t-\theta|^2}.$$

Let $T = T_1 \dot{\cup} T_2 \dot{\cup} T_3$ as in Figure 4.5. Clearly,

$$\iint_{T_1} \frac{|I(t,\theta)|^2}{|t-\theta|^2} \leq \frac{1}{\delta\lambda}.$$

To unclutter our notation, let $a = 1/(\delta^3\lambda)$ and $b = 1/(\delta^2\lambda)$. Using the method of stationary phase, we get

$$\begin{aligned}
\iint_{T_2} \frac{|I(t,\theta)|^2}{|t-\theta|^2} &\le \frac{C}{\lambda}\int_{\pi/2-a}^{\pi/2}\int_{\pi/2-\delta}^{t-b} \frac{1}{|t-\theta|^2}\,\mathrm{d}\theta\mathrm{d}t \\
&\le \frac{C}{\lambda}\int_{\pi/2-a}^{\pi/2}\left(\frac{1}{b} - \frac{1}{t-(\pi/2-\delta)}\right)\mathrm{d}t \\
&= \frac{C}{\lambda}\left(\frac{a}{b} + \log\left(1-\frac{a}{\delta}\right)\right) \\
&\le C\frac{a}{\lambda}\left(\frac{1}{b}-\frac{1}{\delta}\right) \\
&= C\left(\frac{1}{\delta\lambda} - \frac{1}{(\delta\lambda)^2}\right) \\
&\le \frac{C}{\delta\lambda}.
\end{aligned}$$

Finally, using (4.50),

$$\begin{aligned}
\iint_{T_3} \frac{|I(t,\theta)|^2}{|t-\theta|^2} &\le \frac{C}{\lambda^2}\int_{\pi/2-\delta+a}^{\pi/2-b}\int_{\pi/2-\delta}^{t-a} \frac{1}{\cos^2 t\,|t-\theta|^2}\,\mathrm{d}\theta\mathrm{d}t \\
&\le \frac{C}{\lambda^2}\int_{\pi/2-\delta+a}^{\pi/2-b}\int_{\pi/2-\delta}^{t-a} \frac{1}{|t-\pi/2|^2|t-\theta|^2}\,\mathrm{d}\theta\mathrm{d}t \\
&= \frac{C}{\lambda^2}\int_{\pi/2-\delta+a}^{\pi/2-b}\left(\left.\frac{1}{|t-\pi/2|^2(t-\theta)}\right|_{\theta=\pi/2-\delta}^{\theta=t-a}\right)\mathrm{d}t \\
&= \frac{C}{\lambda^2}\int_{\pi/2-\delta+a}^{\pi/2-b}\frac{1}{|t-\pi/2|^2}\left(\frac{1}{a} - \frac{1}{t-(\pi/2-\delta)}\right)\mathrm{d}t \\
&= \frac{C}{\lambda^2}\left[\frac{1}{a(\pi/2-t)} - \frac{\log(t-(\pi/2-\delta))}{\delta^2} + \frac{1}{\delta(t-\pi/2)}\right. \\
&\qquad\qquad \left. + \frac{\log(\pi/2-t)}{\delta^2}\right]_{\pi/2-\delta+a}^{\pi/2-b} \\
&= \frac{C}{\lambda^2}\left[\frac{1}{ab} - \frac{\log(\delta-b)}{\delta^2} - \frac{1}{\delta b} + \frac{\log b}{\delta^2}\right. \\
&\qquad\qquad \left. - \left(\frac{1}{a(\delta-a)} - \frac{\log a}{\delta^2} - \frac{1}{\delta(\delta-a)} + \frac{\log(\delta-a)}{\delta^2}\right)\right] \\
&\le \frac{C}{\lambda^2 ab} \qquad \text{(this holds if we assume } \delta^5 \gg \log\lambda/\lambda^2\text{)} \\
&\le C\delta^5.
\end{aligned}$$

Since,

$$\delta^5 \leq \frac{1}{\delta\lambda} \quad \Longleftrightarrow \quad \delta \leq \frac{1}{\lambda^{1/6}},$$

we let $\delta = \lambda^{-1/6}$ and conclude that

$$\|U^i|_{R=1}\|_{H^{-1/2}(\mathbb{T})} \leq C_{q_0}\lambda^{-5/12}.$$

$\square$

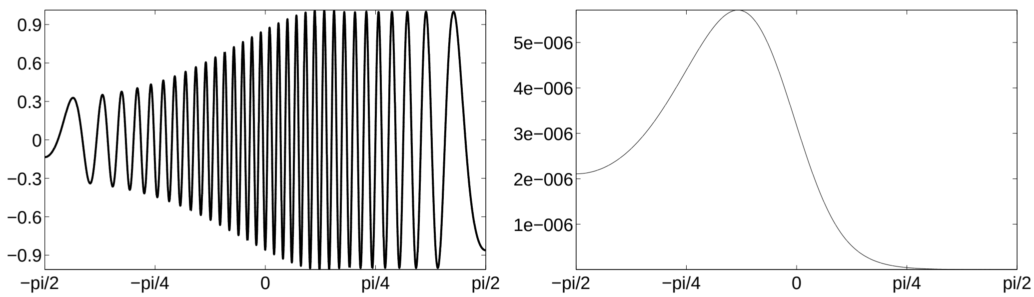
The proof of Theorem 9 is now complete.

4.2.8 Extensions and future work

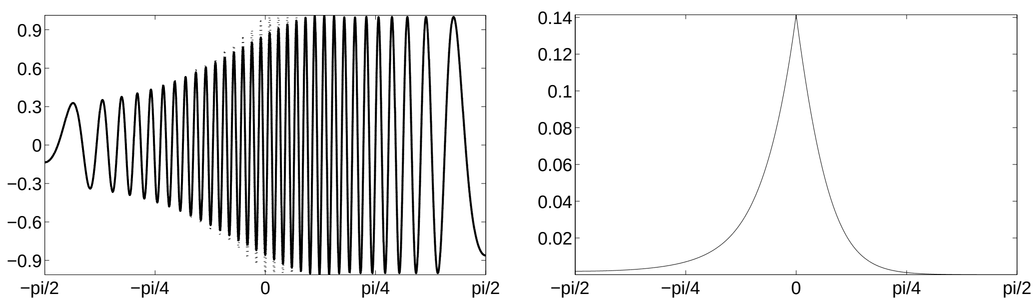
Even though the proof of Theorem 8 (and Theorem 9) relied somewhat on separation of variables, there is good reason to believe these methods can be modified to apply in the case of a general convex scatterer (cf. the microlocal analysis performed in [LL93], which relies on results found in [Laf92]). Once such bounds are established, they should prove useful in the construction of asymptotic formulas (as $\rho \to 0$) for the scattered field that are valid over a broad band of high frequencies (see Chapter 2 for such a construction when frequency is fixed.)

The impedance boundary condition satisfied by the approximate field V^s (see Remark 4.38) is also of independent interest, in that it may be a helpful tool in the search for approximations to the scattered field that are better at high frequencies than that found in Chapter 3. We compare that geometric optics-based approximation of Chapter 3 with the approximation $v^s_\rho := V^s(\cdot/\rho)$ in Figures 4.6 and 4.7. In Figure 4.6 it is clear that the v^s_ρ is the closer approximation. In Figure 4.7 we see this that this is still true, though less pronounced, when $\operatorname{Im} q$ is small. The approximation v^s_ρ is, however, not very useful by itself: attempts to numerically compute the solution to the exterior problem (4.38)[12] will be computationally expensive, owing to the fact that the operator

[12]Here we are referring to the exterior problem when the scatterer is an arbitrary convex domain D and the nonlocal operator D_q on ∂D is defined appropriately to account for the local geometry of the boundary.



(a) On the left: $\mathrm{Re}(u_\rho^s)$ (bold) and $\mathrm{Re}(v_\rho^s)$ (dotted). On the right: $|u_\rho^s - v_\rho^s|$. v_ρ^s is the approximation of the scattered field u_ρ^s that satisfies the impedance boundary condition.

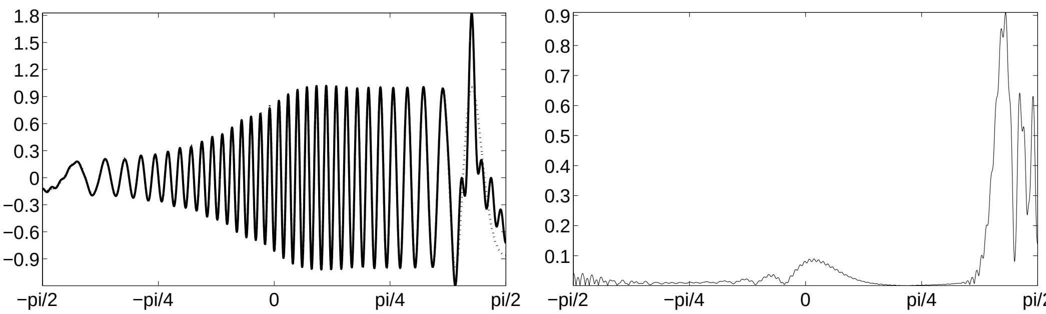


(b) On the left: $\mathrm{Re}(u_\rho^s)$ (bold) and $\mathrm{Re}\left(\{u_\rho^s\}_{\mathrm{geo}}\right)$ (dotted). On the right: $|u_\rho^s - \{u_\rho^s\}_{\mathrm{geo}}|$. $\{u_\rho^s\}_{\mathrm{geo}}$ is the approximation of the scattered field u_ρ^s based on geometric optics (see formula (3.67a) from Chapter 3).

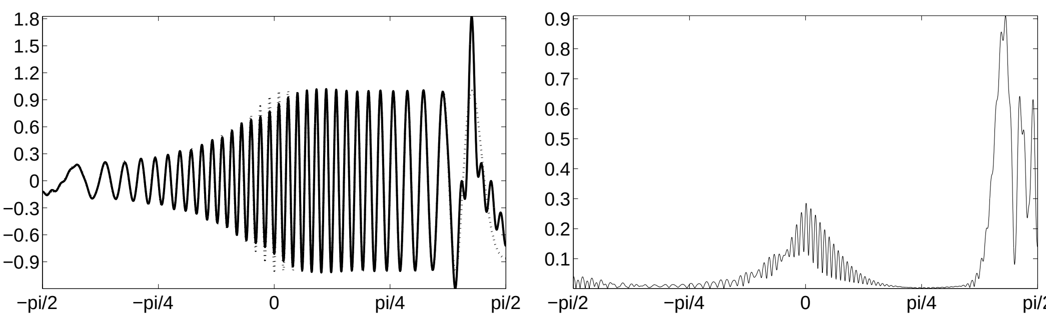
Figure 4.6: Plots on the right half of the scatterer $r = \rho$ when the plane wave is incident at the point corresponding to the angle $-\pi/2$. Here we have taken $q = 2 + 2i$, $q_0 = 1$, $\gamma = 1$, $\rho = 10^{-4}$ and $\omega = 10^6$.

D_q is nonlocal.[13] Nevertheless, this impedance condition may potentially be used to construct better approximation formulas. We cite relevant work developing *on surface radiation conditions* (OSRC), notably [ABV01], [ABB99] and [AB01], which improve upon the prior work in [KTU87] and [Jon92].

[13]A discretization of D_q by a finite element method results in a full matrix.



(a) On the left: $\mathrm{Re}(u_\rho^s)$ (bold) and $\mathrm{Re}(v_\rho^s)$ (dotted). On the right: $|u_\rho^s - v_\rho^s|$. v_ρ^s is the approximation of the scattered field u_ρ^s that satisfies the impedance boundary condition.



(b) On the left: $\mathrm{Re}(u_\rho^s)$ (bold) and $\mathrm{Re}\left(\{u_\rho^s\}_{\mathrm{geo}}\right)$ (dotted). On the right: $|u_\rho^s - \{u_\rho^s\}_{\mathrm{geo}}|$. $\{u_\rho^s\}_{\mathrm{geo}}$ is the approximation of the scattered field u_ρ^s based on geometric optics (see formula (3.67a) from Chapter 3).

Figure 4.7: As in Figure 4.6, but with $q = 2 + i/50$.

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