DESCRIPTIVE ASPECTS OF TORSION-FREE ABELIAN GROUPS

by

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ABSTRACT OF THE DISSERTATION

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In recent years, a major theme in descriptive set theory has been the study of the Borel complexity of naturally occurring classification problems. For example, Hjorth and Thomas have shown that the Borel complexity of the isomorphism problem for the torsion-free abelian groups of rank n increases strictly with the rank n.

In this thesis, we present some new applications of the theory of countable Borel equivalence relations to various classification problems for the p-local torsion-free abelian groups of finite rank. Our main result is that when $n \geq 3$, the isomorphism and quasi-isomorphism problems for the p-local torsion-free abelian groups of rank n have incomparable Borel complexities. (Here two abelian groups A and B are said to be quasi-isomorphic if A is abstractly commensurable with B.) We also introduce a new invariant, the divisible rank, for the class of p-local torsion-free abelian groups of finite rank; and we prove that if $n \geq 3$ and $1 \leq k \leq n-1$, then the isomorphism problems for the p-local torsion-free abelian groups of rank n and divisible rank k have incomparable Borel complexities as k varies.

Our proofs rely on the framework developed by Adams and Kechris, whereby cocycle superrigidity results from measurable group theory are applied in the purely Borel setting. In particular, we make use of the recent cocycle superrigidity theorem, due to Ioana, for free ergodic profinite actions of Kazhdan groups.

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Chapter 1

Introduction to superrigidity and Borel equivalence relations

1.1 Classification problems and equivalence relations

The study of Borel equivalence relations is motivated by the idea that a "classification problem" can often be viewed formally as an equivalence relation E on a suitable space X. (This idea was first introduced in Friedman–Stanley [FS] and Hjorth–Kechris [HK].) For this to be useful, we must require that X be a very concrete space; otherwise one could just take X to be the abstract set of objects to be classified and E the equality relation. Informally, let us say that the E has been solved if one can find invariants $i(x)$ for the elements of X such that x and y are E-equivalent exactly when $i(x) = i(y)$. Again, for this to be meaningful, the map i and its set of values must be very concrete or else one could simply take $i(x) =$ the equivalence class of x, considered as an element of X/E.

The first step in addressing these difficulties is to take X to be a *standard Borel space*, *i.e.*, a separable, complete metric space equipped only with its σ-algebra of Borel sets. Examples include the analytic spaces $\mathbb{R}^n$, $\mathbb{C}$, the p-adic numbers $\mathbb{Q}_p$, and also the descriptive set-theoretic spaces such as the Cantor space $2^{\mathbb{N}}$ (which is the same as $\mathcal{P}(\mathbb{N})$) and the Baire space $\mathbb{N}^{\mathbb{N}}$. Moreover, we have that a Borel subset of a standard Borel space is again standard Borel; so for instance the interval $(0,1)$ with its subspace σ-algebra is standard Borel, even though it isn't complete with the subspace topology. We will frequently use the remarkable fact due to Kuratowski (Theorem 15.6 of [Kec2]) that up to Borel isomorphism, there is a unique uncountable standard Borel space; it serves as the universe for all of the mathematical objects that we shall investigate.

As an example, we consider the classification problem for countable groups up to isomorphism. Let G be a countable group and suppose, without loss of generality, that the domain of G is $\mathbb{N}$. Then G is determined by the information contained in its multiplication function, $\times_G$. Hence, the space:

$$X = \{\times_G \subset \mathbb{N}^3 : \text{ the group axioms hold}\}$$

encompasses all countable groups. Moreover, X is easily seen to be a Borel subset of the standard Borel space $\mathcal{P}(\mathbb{N}^3)$, and so X is a standard Borel space in its own right. Since a permutation of the domain $\mathbb{N}$ takes any encoding of G to another one, it is clear that a single abstract group will occur many times in X. Understanding the classification problem for countable groups amounts to understanding the isomorphism equivalence relation $\cong$ on X.

We will soon see many more examples of classification problems from algebra as well as some important equivalence relations from descriptive set theory.

1.2 Borel reducibility

The equivalence relation E on X is called *completely classifiable*, or *smooth* for short, if there exists a standard Borel space Y and a Borel function $f : X \to Y$ such that:

$$x \; E \; x' \iff f(x) = f(x')$$

Here, f is called *Borel* if its graph is a Borel subset of $X \times Y$. For instance, the isomorphism relation on the space of countable divisible abelian groups is smooth. To see this, note that by Theorem 23.1 of [Fuc], any such group decomposes into a direct sum of copies of $\mathbb{Q}$ and the Prüfer groups $\mathbb{Z}(p^\infty) = \{z \in \mathbb{C}^* : (\exists n) z^{p^n} = 1\}$ and that they are classified up to isomorphism by the sequence consisting of the number of copies of each such factor.

It is necessary to explain the hypothesis that the function f is Borel. Recall that the Borel subsets of $X \times Y$ (or of any standard Borel space) are precisely those which can be constructed from the open sets by means of a countable process of unions, intersections, and complements. We take the view that the Borel subsets are precisely those which are *explicitly* described. Hence, E is smooth iff the elements of X can be classified up to E in an explicit manor using invariants from another standard Borel space Y.

Vastly generalizing the notion of smoothness, Friedman and Stanley defined in [FS] a preordering $\leq_B$ on equivalence relations, which should be thought of as a complexity comparison on classification problems. Suppose that E, F are equivalence relations on standard Borel spaces X, Y. Then E *Borel reduces* to F, written $E \leq_B F$, if there exists a Borel function $f : X \to Y$ such that for all $x, x' \in X$ we have:

$$x \; E \; x' \iff f(x) \; F \; f(x')$$

In this case, we call the associated map f a *Borel reduction*. With this notion in hand, the smooth relations are precisely those E which are Borel reducible to the equality relation on a standard Borel space Y. The existence of a Borel reduction from E to F means *all* of the following:

- The elements of X can be effectively classified up to E using complete invariants from the quotient space Y/F.
- The quotient X/E, itself considered as a space of potential invariants, is no more complicated than Y/F.
- The map f effectively reduces the problem of classifying elements of X up to E to that of classifying elements of Y up to F.

We remark that the last point does not capture the meaning $\leq_B$ on its own; f is unconvincing as a reduction in this sense if it has a much higher (descriptive set-theoretic) complexity than both E and F.

We say that E and F are *Borel equivalent*, written $E \equiv_B F$, if there exists a bijective function f that is a Borel reduction from E to F. We say that E is *Borel bireducible* with F, written $E \sim_B F$, if $E \leq_B F$ and $F \leq_B E$. The latter, weaker notion is more flexible and plays a much more central role in the theory. Finally, we say that E is *strictly less complex* than F, written $E <_B F$, if $E \leq_B F$ but $F \not\leq_B E$.

1.3 Borel equivalence relations

An equivalence relation E on the standard Borel space X is said to be *Borel* if it is a Borel subset of the product space $X \times X$. We shall see later that the example considered in the first section, namely the isomorphism relation on the space of countable groups, is *not* Borel. The following are examples of Borel equivalence relations on standard Borel spaces.

- The isomorphism relation on the subspace of $\mathcal{P}(\mathbb{N}^3)$ consisting of the finitely generated groups.
- The isomorphism relation on the subspace of $\mathcal{P}(\mathbb{N}^2)$ consisting of the connected locally finite graphs.
- The equivalence relation E_{set} on the space $\mathbb{R}^{\mathbb{N}}$ of functions $\mathbb{N} \to \mathbb{R}$ defined by $f \; E_{\text{set}} \; g$ iff f and g have the same range.

The structure of the partial order $<_B$ on the Borel equivalence relations is of great interest. The Borel equivalence relations with at most countably many classes are characterized up to Borel bireducibility by the number of classes, and so from now on we omit these from consideration. If the Borel equivalence relation E has uncountably many classes, then it necessarily has continuum many classes. Indeed, it is a consequence of Silver's theorem (Theorem 35.20 of [Kec2]) that the equality relation $\Delta(X)$ on an uncountable standard Borel space X reduces to any Borel equivalence relation with uncountably many classes. As a corollary, we have that up to Borel bireducibility, $\Delta(X)$ is the least complex Borel equivalence relation with uncountably many classes.

One might ask if similarly there is a *universal* Borel equivalence relation, that is, a Borel equivalence relation F such that $E \leq_B F$ for every Borel equivalence relation E. It follows from a result of Friedman and Stanley that no such Borel equivalence relation F exists. Indeed, in Section 1.2 of [FS], it is proved that there is a *jump* construction which for any Borel equivalence relation F yields a Borel equivalence relation F' such that $F <_B F'$.

However, [FS] does give examples of analytic equivalence relations F such that $E \leq_B F$ for every Borel equivalence relation E. Here, a subset A of the Polish space X is said to be *analytic* if there exists a standard Borel space Z and a Borel function $f : Z \to X$ such that $f(Z) = A$. For instance, the isomorphism relation on the space of countable groups is universal for the Borel equivalence relations, as is the isomorphism relation on the space of countable graphs. Indeed, the corresponding classification problems should be thought of as completely intractable.

1.4 Countable Borel equivalence relations

The equivalence relation E is called *countable* if each of its equivalence classes is countable. Few of the relations described so far have this property, but there are many natural examples.

- The isomorphism relation on the space $\mathcal{G}_2$ of 2-generator groups, where a group G is encoded as the set of words in x, y which represent the identity in G.
- The Turing equivalence relation on $\mathcal{P}(\mathbb{N})$ defined by $A \equiv_T B$ iff A and B lie in the same Turing degree.
- The orbit equivalence relation E_Γ induced by the Borel action of a countable group Γ on a standard Borel space, defined by $x \; E_\Gamma \; y$ iff x, y lie in the same Γ-orbit.

It is a remarkable fact that the last example includes all countable Borel equivalence relations. That is, a theorem of Feldman and Moore (Theorem 1 of [FM]) states that *any* countable Borel equivalence relation is the orbit equivalence relation arising from a Borel action of a suitably chosen countable group.

We shall now discuss the structure of the countable Borel equivalence relations under Borel reducibility. We have already observed that the smooth relations are the $<_B$-least among the Borel equivalence relations with uncountably many classes, and hence also among the countable Borel equivalence relations.

The simplest nonsmooth relation is the *eventual equality* relation, denoted by E_0 and defined on the Cantor space $2^{\mathbb{N}}$ of infinite binary sequences by:

$$\{a_n\} \; E_0 \; \{b_n\} \iff a_n = b_n \text{ for all but finitely many } n$$

Later we shall present a measure-theoretic argument that E_0 is indeed nonsmooth. It is easily observed that E_0 is *hyperfinite*, meaning that there exist Borel equivalence relations:

$$F_0 \subset F_1 \subset \cdots \subset F_n \subset \cdots$$

such that all classes of all F_i are finite and $E_0 = \cup F_i$. By Theorem 1 of [DJK], E_0 is a *universal* hyperfinite relation, in the sense that $F \leq_B E_0$ for every hyperfinite F. By Theorem 1.1 of [HKL], E is non-smooth iff $E_0 \leq_B E$. Thus, the class of nonsmooth hyperfinite Borel equivalence relations form the immediate $<_B$-successor to the class of smooth relations.

There is also a *universal countable* Borel equivalence relation E_∞, in the sense that $E \leq_B E_\infty$ for every countable Borel equivalence relation E. Clearly, all of the universal countable Borel equivalence relations lie in a single $\sim_B$-class. An example of such a relation is the orbit equivalence relation induced by the translation action of the free group F_2 on its power set $\mathcal{P}(F_2)$. For a proof that this is universal, see Section 1 of [DJK].

The results in this section are summarized in the following diagram (Figure 1.4.1) of the countable Borel equivalence relations.

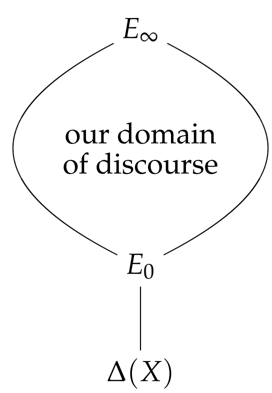

Figure 1.4.1. The countable Borel equivalence relations.

1.5 Essentially countable Borel equivalence relations

It is worth mentioning that there is a sense in which the countable Borel equivalence relations encompass a much larger collection of problems than would at first appear. This is because it is possible for a non-countable Borel equivalence relation to be Borel bireducible with a countable Borel equivalence relation. We call such relations *essentially countable*.

A large collection of essentially countable Borel equivalence relations can be obtained as follows. Let G be a locally compact second countable group, and suppose that G acts in a Borel fashion on the standard Borel space X. Then the orbit equivalence relation E_G induced by the action of G on X need not be countable, but by [Kec1] it is always essentially countable.

In [HK], Hjorth and Kechris provided many *natural* examples of essentially countable Borel equivalence relations. We first introduce a generalization of the construction of the space of countable groups. Let $\mathcal{L} = \{R_i\}$ be a countable (without loss of generality) relational language, and suppose that for all i, R_i is an n_i-ary relation. Then $X_{\mathcal{L}}$ denotes the space of $\mathcal{L}$-structures with domain $\mathbb{N}$, namely:

$$X_{\mathcal{L}} = \prod \mathcal{P}(\mathbb{N}^{n_i})$$

If σ is a sentence of the infinitary language $\mathcal{L}_{\omega_1,\omega}$, then the subset $X_\sigma \subset X_{\mathcal{L}}$ consisting of the models of σ will always be a standard Borel space. We let $\cong_\sigma$ denote the isomorphism relation on X_σ. As with the case of countable groups, $\cong_\sigma$ need not be Borel, but we have:

1.5.1. **Theorem** (Theorem 4.3 of [HK]). *Let σ be a sentence of $\mathcal{L}_{\omega_1,\omega}$. Then $\cong_\sigma$ is essentially countable iff every model M of σ is determined up to isomorphism by a tuple $\bar{a} \in M^n$ and the truth values in M of countably many $\mathcal{L}_{\omega_1,\omega}$ formulas over $\bar{a}$.*

With this in hand, it is easy to see that many of the examples discussed so far are essentially countable. For instance, the isomorphism relation on the space of finitely generated countable groups is essentially countable (for $\bar{a}$, use any finite generating set). Similarly, the isomorphism relation on the space of connected locally finite graphs is essentially countable. On the other hand, it can be shown that $E_\infty <_B E_{\text{set}}$, and so E_{set} is Borel but not essentially countable.

1.6 Torsion-free abelian groups

One of the first "real world" applications of the methods of countable Borel equivalence relations was to the classification problem for torsion-free abelian groups of finite rank. If A is an abelian group then A is *torsion-free* if $na \neq 0$ for all $n \in \mathbb{Z} \setminus \{0\}$ and $a \in A \setminus \{0\}$.

A *basis* for A is a maximal $\mathbb{Z}$-independent subset of A. The *rank* of A is the cardinality of a maximal basis; it is easy to check that this is well-defined.

In his 1937 paper [Bae], Baer classified the torsion-free abelian groups of rank 1 as follows. Let A be such a group and let $a \in A$ be any nonzero element. Then the *type* of A is the equivalence class of the set of pairs:

$$\langle (p,n) : \ p \text{ is prime and } p^n \text{ divides } a \text{ in } A \rangle$$

where two such sets are identified if their symmetric difference is finite. It is not hard to verify Baer's result: the type of A is independent of the choice of $a \in A \setminus \{0\}$, and A is determined up to isomorphism by its type.

Immediately after Baer's result, Kurosh and Malcev attempted to generalize it to higher ranks. Their efforts failed in the sense that the invariants they provided are no easier to distinguish than the groups themselves! Even the rank 2 torsion-free abelian groups resisted satisfactory classification for sixty years. In 1998, Hjorth used the theory of countable Borel equivalence relations to prove that the classification problem for the rank 2 groups is genuinely more complex than that for the rank 1 groups. We will now give a more detailed account of this result.

Following Hjorth, we wish to view the torsion-free abelian groups of finite rank as elements of a standard Borel space. As with all countable groups, we could again use a subspace of $\mathcal{P}(\mathbb{N}^3)$. However, the isomorphism relation on this subspace would be essentially countable and not countable, so in a sense it is not the simplest domain to use. Instead, observe that if A has rank n, then fixing a basis for A yields an embedding of A into $\mathbb{Q}^n$. Hence, studying the torsion-free abelian groups of rank n reduces to studying the subgroups of $\mathbb{Q}^n$. We shall denote by $R(n)$ the standard Borel space of subgroups of $\mathbb{Q}^n$ of rank exactly n, and the isomorphism relation on $R(n)$ by $\cong_n$. It is not hard to show that $\cong_n$ is Borel, and since for any group A there are only countably many choices of a basis for A, $\cong_n$ is a countable Borel equivalence relation.

With this set-up, it is easy to see that Baer's classification of torsion-free abelian groups of rank 1 by their type implies that $\cong_1 \ \leq_B \ E_0$. It is also easy to check that in fact $\cong_1 \sim_B E_0$. Hjorth's theorem can now be expressed as follows:

1.6.1. **Theorem** ([Hjo1]). $E_0 <_B \cong_2$.

Hence, the classification problem for torsion-free abelian groups of rank 2 is strictly more complex than that for rank 1.

This left open many questions about the torsion-free abelian groups of higher finite rank. Observe that we always have $\cong_n \ \leq_B \ \cong_{n+1}$, as it is easily checked that the map $A \mapsto A \oplus \mathbb{Q}$ is a Borel reduction. But after Hjorth's proof, it was still far from clear whether $\cong_3$ was strictly more complex than $\cong_2$, or whether $\cong_2$ was already universal for torsion-free abelian groups of all finite ranks. At the time, it was conceivable (and not out of the question) that $\cong_2$ is not universal but that $\cong_3$ was universal. This problem remained open for several years; we shall discuss the solution shortly.

1.7 Measurable dynamics

By the result of Feldman and Moore, every countable Borel equivalence relation is induced by the action of *some* countable group Γ. We remark that the group Γ and its action are not canonically determined by E_Γ. For instance, there is no known, "natural" group action that induces the Turing equivalence relation $\equiv_T$.

Fortunately for Hjorth, in the case of $\cong_n$, much more structure is available. Namely, for $A, B \in R(n)$, we have $A \cong_n B$ iff there exists $g \in GL_n(\mathbb{Q})$ such that $g(A) = B$. Hence, $\cong_n$ is precisely the orbit equivalence relation induced by a very natural action of the group $GL_n(\mathbb{Q})$.

In order to exploit this situation, it is necessary to study two key hypotheses on a group action: invariant measures and freeness. If Γ acts on X in a Borel fashion (which we shall frequently denote by $\Gamma \curvearrowright X$), then a Borel probability measure μ on X is said to be Γ-*invariant* if for every Borel $A \subset X$ and $\gamma \in \Gamma$, $\mu(\gamma A) = \mu(A)$. In this case we say that Γ preserves μ and we sometimes write $\Gamma \curvearrowright (X, \mu)$. We are exclusively interested in *nonatomic* measures, *i.e.*, measures with the property that every countable set is null.

For instance, the eventual equality relation E_0 is the orbit equivalence relation induced by a measure preserving action. To see this, identify the Cantor space $2^{\mathbb{N}}$ with the elements of the group $G = \prod_{i \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$. This space carries a measure that is invariant for the left multiplication action of G on itself. Namely, let μ be the product of the $\{\frac{1}{2}, \frac{1}{2}\}$ measures on the coordinates of G. Clearly, E_0 is precisely the orbit equivalence relation induced by the action of the countable subgroup $\bigoplus_{i \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$.

Next, we say that the action $\Gamma \curvearrowright X$ is *free* if for every $x \in X$ we have $1 \neq \gamma \in \Gamma$ implies that $\gamma x \neq x$. For instance, the action that we have just described, which induces E_0, is clearly free.

Returning to Theorem 1.6.1, recall that Hjorth wished to prove that $\cong_2$ does not reduce to E_0, *i.e.*, that $\cong_2$ is not hyperfinite. To do so, he made use of the result (Theorem 1.7 of [JKL]) that any free, measure preserving action of a nonamenable group is not hyperfinite. This is not directly applicable to $\cong_n$, since unfortunately $GL_n(\mathbb{Q})$ neither preserves a measure on $R(n)$ nor acts freely. However, for $n \geq 2$, Hjorth was able to find a measure on $R(n)$ that is preserved by the restricted action of $SL_n(\mathbb{Z})$. Moreover, this measure happens to concentrate on a set on which $SL_n(\mathbb{Z})$ acts freely. This information, together with a few other elementary results, was sufficient for Hjorth to complete the proof.

There is another extremely important property to consider when studying measure preserving actions. We say that $\Gamma \curvearrowright (X, \mu)$ is *ergodic* if whenever the Borel subset $A \subset X$ is Γ-invariant, A is null or conull. For instance, it is a standard exercise (which we shall explain in Chapter 3) that $\bigoplus_{i \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ acts ergodically on the Cantor space with the invariant measure μ described above.

The ergodicity hypothesis is extremely useful. For instance, if $\Gamma \curvearrowright (X, \mu)$ is ergodic then E_Γ already cannot be smooth. For, suppose that $f : X \to \mathbb{R}$ is a Borel map such that $x \; E_\Gamma \; x'$ implies $f(x) = f(x')$. Then f is Γ-invariant, and it is easy to show using ergodicity that f must be constant on a conull set. Since μ is nonatomic, such a map f clearly cannot be a Borel reduction. In particular, this verifies our earlier claim that E_0 is not smooth.

1.8 Initial applications of superrigidity

We have already mentioned that besides $\Delta(X)$, E_0, and E_∞, all countable Borel equivalence relations must lie in the interval (E_0, E_∞). Many open questions revolve around the structure of the partial order $<_B$ on this interval. We shall now discuss some of the principle results concerning this interval, beginning with the fact that it is nonempty.

It is a matter of folklore (due ultimately to Adams and explained in Section 3.5 of [JKL]) that the universal treeable equivalence relation $E_{T\infty}$ does indeed satisfy $E_0 <_B E_{T\infty} < E_\infty$. Here, an equivalence relation E on X is said to be *treeable* if there exists a Borel acyclic graph on X whose connected components are precisely the E-classes. For instance, any free action of a free group is treeable; the trees correspond to copies of the Cayley graph of the acting group. The universal treeable relation $E_{T\infty}$ can be realized as the orbit equivalence relation arising from the action of the free group F_2 not on its whole power set, but rather on the subset $X \subset \mathcal{P}(F_2)$ on which F_2 acts freely.

Adams's work can also be adapted to prove that the *product relation* $E_{T\infty}^2$ is non-treeable and non-universal. Here, $E_{T\infty}^2$ is the relation on $X \times X$ defined by $(x,y)\ E_{T\infty}^2\ (x',y')$ iff $x\ E_{T\infty}\ x'$ and $y\ E_{T\infty}\ y'$. Thus in 2000, there were five known $\sim_B$-classes of countable Borel equivalence relations, linearly ordered by $<_B$:

$$\Delta(X)\ <_B\ E_0\ <_B\ E_{T\infty}\ <_B\ E_{T\infty}^2\ <_B\ E_\infty$$

This set the scene for the biggest breakthrough to date in the study of countable Borel equivalence relations:

1.8.1. **Theorem** (Theorem 1 of [AK]). *Let $\mathcal{B}$ denote the set of Borel subsets of $\mathbb{R}$. Then the partial ordering $(\mathcal{B}, \subsetneq)$ embeds into the partial ordering $((E_0, E_\infty), <_B)$.*

In particular, there are uncountably many distinct countable Borel equivalence relations, and their structure is extremely rich. In proving this theorem, it was necessary for Adams and Kechris to find many pairs of equivalence relations E, F which are *Borel incomparable*, that is, $E \not\leq_B F$ and $F \not\leq_B E$. Their key insight was to mimic some methods from orbit equivalence theory. Here, two probability measure preserving actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are said to be *orbit equivalent* if there exists a measure preserving bijection $f : X \to Y$ satisfying:

$$\Gamma x = \Gamma x' \iff \Lambda f(x) = \Lambda f(x')$$

for almost all $x \in X$. We have the following well-known construction of orbit inequivalent ergodic actions.

1.8.2. **Theorem** (Theorem 5.2.2 of [Zim]). *Let G_0, G_1 be connected simple higher-rank Lie groups with finite center. Let $\Gamma_i < G_i$ be lattices and suppose that $\Gamma_i \curvearrowright X_i$ are free, ergodic probability measure preserving actions. If the two actions are orbit equivalent, then G_0 and G_1 are isogenous.*

This is called a "superrigidity" result because it gives conditions under which the orbit structure of a group action determines some information about the acting group. We will give further meaning to the term superrigidity in Chapter 4.

Theorem 1.8.2 implies that if Γ_α are lattices in sufficiently unrelated higher rank Lie groups G_α, and $\Gamma_\alpha \curvearrowright X_\alpha$ are free ergodic probability measure preserving actions, then their orbit equivalence relations E_{Γ_α} are pairwise orbit inequivalent. This result was not of direct use to Adams and Kechris, since of course a Borel reduction need not preserve any measure. Their proof required more care in the choice of Γ_α, as well as some finer results of Zimmer.

Adams next used superrigidity methods to provide in [Ada] the first example of countable Borel equivalence relations $E \subset F$ such that $E \not\leq_B F$. His examples are again orbit equivalence relations induced by the actions of lattices Γ in a higher rank Lie group G. Adams supposed additionally that Γ embeds densely into a compact simple Lie group K, and considered the orbit equivalence relation E_Γ induced by the left-translation action of Γ on K. If $\Lambda < \Gamma$ is a subgroup of finite index, then clearly $E_\Lambda \subset E_\Gamma$. However, superrigidity methods can be used to prove that E_Γ and E_Λ are in fact Borel incomparable.

Another question, similar to the one that Adams solved, asked whether there exists a Borel equivalence relation E such that $E <_B 2E$. Here, if E is an equivalence relation on X, then $2E$ is the equivalence relation on $X \times \{0,1\}$ defined by $(x,i)\ 2E\ (x',j)$ iff $x\ E\ x'$ and $i = j$. Thomas observed in [Tho2] that if $E = E_\Gamma$ is the equivalence relation described in the previous paragraph, then E satisfies $E <_B 2E$.

This example of Adams is important, at least in a motivating sense, for our results as well. Indeed, the relations that we shall consider in Theorem B (of Section 2.2) give new examples of relations $E \subset F$ such that E and F are incomparable. In our arguments, we shall work with $SL_n(\mathbb{Z})$, which is simultaneously a lattice in $SL_n(\mathbb{R})$ and a dense subgroup of the compact (but not Lie) group $SL_n(\mathbb{Z}_p)$.

This work of Adams and Thomas left open the very important problem of whether $E_\infty \subset F$ implies $E_\infty \leq_B F$. If this is the case, then it would follow that the Turing relation $\equiv_T$ is universal and it has been shown that this implies that the Martin conjecture is false!

Lastly, we remark that the analogous questions about the structure of the interval $(E_0, E_{T\infty})$ are almost all still open. The question of whether it is nonempty was solved by Hjorth, who proved in [Hjo2] that there indeed exists a non-universal, non-hyperfinite treeable equivalence relation. But so far, Hjorth's relation is the only one known to lie in this interval.

1.9 Applications to torsion-free abelian groups

In this section, we explore some extremely interesting applications of the methods of Adams and Kechris to the study of the complexity of the classification problem for torsion-free abelian groups of higher finite ranks.

We first discuss Thomas's answer in [Tho3] to Hjorth's question of whether $\cong_2$ is universal for all torsion-free abelian groups of finite rank. In fact, the $\cong_n$ form a chain of classification problems of strictly increasing complexity. This gave the first naturally occurring example of an infinite chain of countable Borel equivalence relations.

1.9.1. **Theorem** (Theorem 1.3 of [Tho3]). *For $n \geq 2$, we have $\cong_n <_B \cong_{n+1}$.*

Let us give a rough idea of how Theorem 1.9.1 is proved. Suppose that $f : R(n+1) \to R(n)$ is a Borel reduction from $\cong_{n+1}$ to $\cong_n$. Recall that $\cong_n$ is the orbit equivalence relation induced by the action of $GL_n(\mathbb{Q})$ on $R(n)$. Slightly simplifying matters, we instead consider the orbit equivalence relation E_n induced by the action of $SL_n(\mathbb{Z})$ on $R(n)$. In view of Zimmer's results, one might guess that if there is a Borel reduction from E_{n+1} to E_n then we should expect to obtain an embedding from $SL_{n+1}(\mathbb{R}) \to SL_n(\mathbb{R})$, a clear contradiction. This implication is far from true as stated; one key difficulty is that Hjorth's

$SL_n(\mathbb{Z})$-invariant measure on $R(n)$ fails to be ergodic. Yet Thomas does essentially proceed along these lines; we shall see more aspects of his proof later on.

Thomas next found a naturally occurring infinite $\leq_B$-antichain, again consisting of the isomorphism relations on various spaces of torsion-free abelian groups of finite rank. This was accomplished by considering the collection of local groups. Here, if A is a torsion-free abelian group and p is a prime, then A is said to be *p-local* if it is infinitely q-divisible for every prime $q \neq p$. We let $R(n,p)$ denote the space of p-local subgroups of $\mathbb{Q}^n$ of rank n, and $\cong_{n,p}$ denote the restriction of the isomorphism relation to $R(n,p)$.

The local groups are a natural class to consider; in their attempted classification of torsion-free abelian groups of finite rank, Kurosh and Malcev reduced the study of arbitrary $A \leq \mathbb{Q}^n$ to that of local groups by considering the sequence of p-localizations of A. Here, the *p-localization* $A_{(p)}$ of A is defined by:

$$A_{(p)} = A \otimes \mathbb{Z}_{(p)}$$

where $A \otimes \mathbb{Z}_{(p)}$ is set of $\mathbb{Z}_{(p)}$-linear combinations of elements of A, considered inside $\mathbb{Q}^n$. (Note that $\mathbb{Z}_{(p)}$ denotes the group $\{\frac{a}{b} \in \mathbb{Q} : p \nmid b\}$.) It is not hard to prove that A can be recovered as the intersection $\bigcap A_{(p)}$ of its p-localizations. One might wonder whether the isomorphism problem $\cong_{n,p}$ for p-local groups of rank n is just as hard as that for all torsion-free abelian groups of rank n. This is ruled out by the following result:

1.9.2. **Theorem** (Theorem 1.2 of [Tho1]). *Fix $n \geq 3$. Then for distinct primes p, q, we have that $\cong_{n,p}$ and $\cong_{n,q}$ are incomparable with respect to Borel reducibility.*

(Though only stated here for $n \geq 3$, we remark that the result was also proved for $n = 2$ by Hjorth and Thomas, see [HT].) The proof requires a finer version of Zimmer's theorem, which we are now ready to discuss. Let f be a Borel reduction from $\cong_{n,p}$ to $\cong_{n,q}$. Recall that both equivalence relations are induced by an action of $GL_n(\mathbb{Q})$. It is not difficult to imagine that matters would be much simpler if f had the additional property that it *preserved* the action of $GL_n(\mathbb{Q})$.

Here we have in mind the following. Suppose that $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are actions of countable groups on standard Borel spaces. A *permutation group homomorphism*

$\Gamma \curvearrowright X \longrightarrow \Lambda \curvearrowright Y$ is a pair (ϕ, f) where $\phi : \Gamma \to \Lambda$ is a group homomorphism and $f : X \to Y$ is a Borel map satisfying:

$$f(\gamma x) = \phi(\gamma) f(x)$$

for all $\gamma \in \Gamma$ and $x \in X$. A permutation group homomorphism need not be a Borel reduction from E_Γ to E_Λ, but it satisfies a weaker condition. A *Borel homomorphism* from E_Γ to E_Λ is a Borel map $f : X \to Y$ satisfying:

$$x \; E_\Gamma \; y \implies f(x) \; E_\Lambda \; f(y)$$

If (ϕ, f) is a permutation group homomorphism from $\Gamma \curvearrowright X$ to $\Lambda \curvearrowright Y$, then clearly f is a Borel homomorphism from E_Γ to E_Λ.

It is not so difficult to prove that there does not exist a "nontrivial" permutation group homomorphism between the actions $GL_n(\mathbb{Q}) \curvearrowright R(n,p)$ and $GL_n(\mathbb{Q}) \curvearrowright R(n,q)$. Hence, Theorem 1.9.2 would follow from the very implausible assertion that every Borel reduction from $\cong_{n,p}$ to $\cong_{n,q}$ comes from a permutation group homomorphism.

Again, there is no reason in general to expect something like this to hold, and this is where Zimmer's cocycle superrigidity theorem is needed. With strong hypotheses on the actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ (the hypotheses of Theorem 1.8.2 plus more), Zimmer's theorem implies that if f is a Borel homomorphism from E_Γ to E_Λ, then there exists a perturbation f' of f such that f' is a homomorphism of permutation groups. The perturbation itself has no adverse effect, as we will have that $f'(x) \; E_\Lambda \; f(x)$ for all $x \in X$.

Of course, $\cong_{n,p}$ is induced by the action of $GL_n(\mathbb{Q})$ and not by the action of any lattice. In order to use Zimmer's theorem to prove Theorem 1.9.2, it was necessary for Thomas to first reduce the question to the analogous question for $E_{n,p}$, where $E_{n,p}$ denotes the orbit equivalence relation induced by the action of the $SL_n(\mathbb{Z})$ on $R(n,p)$. Additionally, Thomas needed to find an ergodic, invariant measure for the action of $SL_n(\mathbb{Z})$. In this thesis, we shall investigate such measures in great detail. We shall also give an in-depth analysis of the equivalence relations $\cong_{n,p}$ and $E_{n,p}$.

Chapter 2

Precise statement of results

2.1 Another invariant: divisible rank

Let us revisit the theorem of Thomas which states that $\cong_{n+1} \not\leq_B \cong_n$ for all n. At the heart of Thomas's proof is a "dimension" argument, that is, the $\cong_{n+1}$-classes in some sense do not fit into $\cong_n$ classes. His next result states that the isomorphism equivalence relations on the spaces $R(n,p)$ of p-local torsion-free abelian groups of rank n are incomparable as p varies. Since the rank n is fixed, dimension does not play a large role in this argument. The incompatibility is due rather to a structural difference between the actions of $GL_n(\mathbb{Q})$ on $R(n,p)$ and on $R(n,q)$.

In this section, we will give a new result of the latter sort. For a fixed rank $n \geq 3$ and prime p, we consider the space $R(n,p)$ of p-local groups of rank n. Within this class, we will isolate a third, finer invariant for local groups which we call the divisible rank. We will show that as the divisible rank varies, the corresponding orbit equivalence relations are Borel incomparable.

Before defining the invariant, it is necessary to discuss the Kurosh-Malcev invariants, which were introduced by Kurosh and Malcev in their *unsatisfactory* classification of the torsion-free abelian groups of rank $n \geq 2$. First, recall that a torsion-free abelian group of rank n is determined by its sequence of p-localizations. Now, working with a fixed p-local subgroup A of $\mathbb{Q}^n$, Kurosh and Malcev considered its *p-adic completion*, *i.e.*, the $\mathbb{Z}_p$-submodule of $\mathbb{Q}_p^n$ defined by:

$$\Lambda_p(A) = A \otimes \mathbb{Z}_p$$

More precisely, $\Lambda_p(A)$ denotes the set of $\mathbb{Z}_p$-linear combinations of elements of A, considered as a subset of $\mathbb{Q}_p^n$. It is a basic consequence of the theory of $\mathbb{Z}_p$-modules that

$\Lambda_p(A)$ (and indeed any $\mathbb{Z}_p$-submodule of $\mathbb{Q}_p^n$) admits a decomposition:

$$\Lambda_p(A) = V_A \oplus L$$

where V_A is a vector subspace of $\mathbb{Q}_p^n$ and L is a free $\mathbb{Z}_p$-module of finite rank. The Kurosh-Malcev invariant for A essentially consists of a vector space basis for V_A together with a basis for L, modulo a very complex equivalence relation. Notice that V_A is the divisible part of $\Lambda_p(A)$ and hence it is uniquely determined by A.

2.1.1. **Definition**. The *divisible rank* of A is $k = \dim V_A$.

It is not difficult to see that if $g \in GL_n(\mathbb{Q})$, then we have $\dim V_A = \dim V_{gA}$. Hence, the divisible rank is an isomorphism invariant on the space $R(n,p)$. Refining our notation from the last chapter, let $R(n,p,k) \subset R(n,p)$ denote the subspace of p-local torsion-free abelian groups of rank n and divisible rank k, and let $\cong_{n,p}^k$ denote the restriction of the isomorphism relation to the space $R(n,p,k)$. Our first result is that these isomorphism equivalence relations are Borel incomparable.

Theorem A. *Fix $n \geq 3$ and p prime. Then for $1 \leq k \neq l \leq n-1$, we have that $\cong_{n,p}^k$ and $\cong_{n,p}^l$ are incomparable with respect Borel reducibility.*

This result is not surprising after Thomas's work, but may seem puzzling in view of the following abelian group-theoretic characterization of the divisible rank.

2.1.2. **Proposition** (essentially Exercise 93.1 of [Fuc]). *Let A be a p-local torsion-free abelian group of finite rank. Then the divisible rank k of A is precisely the maximum of the ranks of all divisible quotients of A.*

This result will be proved in Appendix 7.1. It says, in a sense, that if $k' < k$ then the elements of $R(n,p,k)$ are closer to being a divisible groups than the elements of $R(n,p,k')$. But as we have mentioned, the classification problem for countable divisible abelian groups is smooth, and so we might say that the groups of large divisible rank have a large factor that is easy to understand. Nevertheless, by Theorem A, the classification problems for groups of divisible rank k do not decrease in complexity as k increases.

2.2 Quasi-isomorphism

We next discuss an important weakening of the notion of isomorphism of torsion-free abelian groups called quasi-isomorphism. First, groups $A, B \leq \mathbb{Q}^n$ are said to be *quasi-equal*, written $A \approx B$, if $A \cap B$ has finite index in A and in B (in other words, iff A and B are commensurable). They are said to be *quasi-isomorphic*, written $A \sim B$, if A is quasi-equal to an isomorphic copy of B. It is straightforward to check that this is indeed symmetric and transitive. Write $\sim_n$ for the restriction of the quasi-isomorphism relation to the space $R(n)$ of torsion-free abelian groups of rank n.

In establishing Theorem 1.9.1, Thomas initially proved the corresponding result for the quasi-isomorphism relation, that is, he proved for all n that $\sim_n \ <_B \ \sim_{n+1}$. There are several reasons that this was a necessary stepping stone. First, he simply found the quasi-isomorphism relation easier to work with than the isomorphism relation. Second, we shall see that the quasi-isomorphism relation on $R(n, p)$ is Borel bireducible with the orbit equivalence relation induced by the action of $GL_n(\mathbb{Q})$ on the Grassmann space $Gr\mathbb{Q}_p^n$ of all vector subspaces of $\mathbb{Q}_p^n$, and it is much easier to apply Zimmer's theorem to such classical actions.

This situation led Thomas to ask in Questions 3.9, 3.10 of [Tho3] which of the two classification problems is more complex: the quasi-isomorphism problem or the isomorphism problem. Thomas suspected that the two equivalence relations were incomparable up to Borel reducibility; our main result precisely verifies this conjecture.

In analogy with our earlier notation, we let $\sim_{n,p}$ denote the quasi-isomorphism relation on the space $R(n, p)$ of p-local torsion-free abelian groups of rank n. Let $\sim_{n,p}^k$ denote the restriction of $\sim_{n,p}$ to the subspace of groups with divisible rank exactly k. Then we have:

Theorem B. *If $n \geq 3$ and $1 \leq k \leq n-2$, then $\cong_{n,p}^k$ and $\sim_{n,p}^k$ are incomparable with respect to Borel reducibility.*

The case when $k = n - 1$ must be omitted from the theorem, since it is not hard to show (see Theorem 4.4 of [Tho1]) that for the p-local groups of rank n and of divisible rank $n-1$, the quasi-isomorphism and isomorphism relations coincide.

One will immediately recognize that Theorem B provides an Adams-like example of equivalence relations $E \subset F$ such that $E \not\leq_B F$. Recall that in Adams' examples, $E \subset F$ and F is of *finite index* over E in the sense that each F-class is a union of finitely many E-classes. The situation in Theorem B is slightly different. If $E \subset F$ then let us say that F is a *smooth extension* of E if the following conditions are satisfied:

- F is the *join* of E with a smooth relation S, meaning that F is the smallest equivalence relation containing E and S, and
- E and S *commute*, meaning that whenever $x \ E \ y \ S \ z$ for some y, we also have $x \ S \ y' \ E \ z$ for some y'.

(For many results concerning these notions, see Sections 23 and 24 of [KM].) Now, the quasi-isomorphism relation $\sim^k_{n,p}$ is not of finite index over the isomorphism relation $\cong^k_{n,p}$. But by Lemma 4.4 of [Tho3], the quasi-equality relation on $R(n,p,k)$ is smooth, and it follows easily that $\sim^k_{n,p}$ is a smooth extension of $\cong^k_{n,p}$.

There is a second viewpoint from which the result of Theorem B appears unusual. This concerns the fact that the torsion-free abelian groups have a more reasonable structure theory with respect to quasi-isomorphism than they do with respect to isomorphism. For instance, there does not exist a "unique decomposition" theorem for torsion-free abelian groups of finite rank (in contrast with such classes of groups as the finitely generated abelian groups or the countable divisible groups).

To elaborate, we say that A is *indecomposable* if there does not exist a direct sum decomposition $A = A_1 \oplus A_2$ where $A_1, A_2 \neq 0$. Any torsion-free abelian group of finite rank may be decomposed into a direct sum of indecomposable groups, but in general this cannot be done in a unique way.

2.2.1. **Theorem** (Corner, Theorem 90.2 of [Fuc]). *Let $n \geq 2$ and fix $k \leq n$. There exists $A \leq \mathbb{Q}^n$ with the property that for any partition $n = r_1 + \cdots + r_k$ into $r_i \geq 1$, there is a decomposition:*

$$A = A_1 \oplus \cdots \oplus A_k$$

where A_i are indecomposable and $\operatorname{rank} A_i = r_i$.

The situation is much better in the category where quasi-isomorphism replaces isomorphism. Here, we say that $A \leq \mathbb{Q}^n$ is *strongly indecomposable* if A cannot be written as $A \approx A_1 \oplus A_2$, where $A_1, A_2 \neq 0$.

2.2.2. **Theorem** (Jónsson, Theorem 92.5 of [Fuc]). *Let A be a torsion-free abelian group of finite rank and suppose that A admits two quasi-decompositions:*

$$A \approx A_1 \oplus \cdots \oplus A_k \approx B_1 \oplus \cdots \oplus B_l$$

where A_i, B_j are strongly indecomposable. Then $k = l$ and there exists a permutation π of $\{1, \dots, k\}$ such that $A_i \sim B_{\pi(i)}$ for $1 \leq i \leq k$.

Despite this result, Theorem B implies that a solution to the isomorphism problem for p-local torsion-free abelian groups of rank n does not yield a solution for the quasi-isomorphism problem!

2.3 Strategy of the proofs

Theorems A and B are closely related, and their proofs will be intertwined. We now give an outline of the proof of one direction of Theorem B:

$$\cong^k_{n,p} \;\not\leq_B\; \sim^k_{n,p}$$

Although this is perhaps the less interesting direction of Theorem B, its proof is not too technical and it will serve as a template for the proofs of our other results. Now, if the proof was to be given all at once and in a purely mechanical order, then it might be broken into three steps.

Step 1. Transfer the domain of discourse from the rather complicated space of torsion-free abelian groups to the much simpler "Grassmann" space.

As a start, instead of working with the space $R(n, p)$ of p-local torsion-free abelian groups of rank n, we shall work with the space of completed groups. In Appendix 7.1, it will be shown that the completion map Λ_p is a $GL_n(\mathbb{Q})$-preserving bijection onto the space $\mathcal{M}(n,p)$ of $\mathbb{Z}_p$-submodules of $\mathbb{Q}_p^n$. Hence, $\cong_{n,p}$ is Borel equivalent to the orbit equivalence relation induced by the action of $GL_n(\mathbb{Q})$ on the space $\mathcal{M}(n,p)$.

Now, any $\mathbb{Z}_p$-submodule $M \leq \mathbb{Q}_p^n$ splits into $V_M \oplus L$, where V_M is a vector subspace and L is free. By definition of the divisible rank k, each subset $R(n,p,k)$ maps to the set $\mathcal{M}(n,p,k)$ of modules $M \leq \mathbb{Q}_p^n$ with $\dim V_M = k$. This shows that there is a close connection between $R(n,p,k)$ and the space $Gr_k\mathbb{Q}_p^n$ of k-dimensional vector subspaces of $\mathbb{Q}_p^n$. In Chapter 5, we shall work to establish the following results:

2.3.1. **Theorem** (Theorem 4.3 of [Tho1]). *Let $E^k_{GL_n\mathbb{Q}}$ denote the orbit equivalence relation induced by the action of $GL_n(\mathbb{Q})$ on $Gr_k\mathbb{Q}_p^n$. Then the quasi-isomorphism relation $\sim^k_{n,p}$ is Borel bireducible with $E^k_{GL_n\mathbb{Q}}$.*

2.3.2. **Theorem**. *Let $E^k_{SL_n\mathbb{Z}}$ denote the orbit equivalence relation induced by the action of $SL_n(\mathbb{Z})$ on $Gr_k\mathbb{Q}_p^n$. Then the isomorphism relation $\cong^k_{n,p}$ is Borel bireducible with an equivalence relation $E^k_{\cong}$ on $Gr_k\mathbb{Q}_p^n$ which satisfies $E^k_{SL_n\mathbb{Z}} \subset E^k_{\cong} \subset E^k_{GL_n\mathbb{Q}}$.*

We remark that $E^k_{\cong}$ will not be an orbit equivalence relation induced by a natural action of any group Γ such that $SL_n\mathbb{Z} \subset \Gamma \subset GL_n(\mathbb{Q})$. However, the containments provided by Theorem 2.3.2 are sufficient for our arguments, as well shall see.

Now, if there exists a Borel reduction from $\cong^k_{n,p}$ to $\sim^k_{n,p}$, then by Theorems 2.3.1 and 2.3.2, there exists a Borel reduction f from $E^k_{\cong}$ to $E^k_{GL_n\mathbb{Q}}$. Also by Theorem 2.3.2,

$$(SL_n\mathbb{Z})x = (SL_n\mathbb{Z})y \implies x \; E^k_{\cong} \; y$$

and so f is a Borel homomorphism from $E^k_{SL_n\mathbb{Z}}$ to $E^k_{GL_n\mathbb{Q}}$.

Step 2. Derive a superrigidity result which implies that any such Borel homomorphism is, after a slight perturbation, a permutation group homomorphism:

$$SL_n(\mathbb{Z}) \curvearrowright Gr_k\mathbb{Q}_p^n \longrightarrow GL_n(\mathbb{Q}) \curvearrowright Gr_k\mathbb{Q}_p^n$$

This part is more or less already done for us, thanks to a recent superrigidity theorem of Adrian Ioana (see Theorem 3.3.2 of [Ioa]); but we shall spend Chapters 3 and 4 developing the tools necessary to apply Ioana's theorem. Let $\Gamma \curvearrowright X$ be a measure preserving group action satisfying certain hypotheses (which are satisfied, for instance, by

$SL_n(\mathbb{Z}) \curvearrowright Gr_k\mathbb{Q}_p^n$, with respect to the p-adic probability measure). Let Λ be a countable group and $\Lambda \curvearrowright Y$ a free action, and suppose that f is a Borel homomorphism from E_Γ to E_Λ. Then, slightly oversimplifying matters, Ioana's theorem implies that there exists a homomorphism $\phi : \Gamma \to \Lambda$ and a slight perturbation f' of f such that $f'(\gamma x) = \phi(\gamma) f'(x)$ for all $\gamma \in \Gamma$ and almost every $x \in X$.

We must remark that $GL_n(\mathbb{Q}) \curvearrowright Gr_k\mathbb{Q}_p^n$ is not actually a free action. To get around this difficulty, we will rely on a highly nontrivial result of Thomas (essentially Lemma 5.1 of [Tho1]) that we may suppose that $GL_n(\mathbb{Q})$ acts freely on the range of f. We shall say more about the presence of this result in our argument shortly.

Step 3. Characterize the permutation group homomorphisms which arise in Step 2.

In Chapter 3, we shall establish a version of the following result:

2.3.3. **Theorem**. *Suppose that* $(\phi, f) : SL_n(\mathbb{Z}) \curvearrowright Gr_k\mathbb{Q}_p^n \longrightarrow GL_n(\mathbb{Q}) \curvearrowright Gr_k\mathbb{Q}_p^n$ *is a permutation group homomorphism, and that* ϕ *is injective. Then there exists* $h \in GL_n(\mathbb{Q})$ *such that* $f(x) = hx$*, for almost all* $x \in Gr_k\mathbb{Q}_p^n$ *(with respect to a certain probability measure).*

Finally, suppose that there exists a Borel reduction from $\cong_{n,p}^k$ to $\sim_{n,p}^k$. Then by Step 1 there exists a Borel reduction f from $E_\cong^k$ to $E_{GL_n\mathbb{Q}}^k$. By Steps 2 and 3, we may suppose that there exists $h \in GL_n(\mathbb{Q})$ such that $f(x) = hx$. Since $h \in GL_n(\mathbb{Q})$ and we are only interested in values of f *modulo* $GL_n(\mathbb{Q})$, we conclude that the *identity map* is a Borel reduction from $E_\cong^k$ to $E_{GL_n\mathbb{Q}}^k$. But this is impossible, since whenever E is a proper sub-equivalence relation of F, the identity map is not a Borel reduction from E to F! (In the actual proof, we will need to show that there is no positive measure subset of $Gr_k\mathbb{Q}_p^n$ on which the identity map is a Borel reduction; this will require a little more effort.)

This concludes the proof outline.

We should remark that it is almost certainly possible to apply Zimmer's theorem to reach our goals. So, one might ask why the use of Zimmer's theorem has been replaced by Ioana's more recent result; the answer is simply that Ioana's theorem is much more directly applicable. First, it applies directly to the $SL_n(\mathbb{Z})$-actions that we are considering. Second, since there are no restrictions whatsoever on the *target* group, we may greatly simplify the proof by considering the full action of $GL_n(\mathbb{Q})$ on the right-hand side (rather than attempting to restrict to just the action of the lattice $SL_n(\mathbb{Z})$).

It should be mentioned that Zimmer's theorem has not been totally eliminated from our arguments. We have already said that we will make use of an analog of Lemma 5.1 from [Tho1]; and this result relies on Zimmer's theorem. It is interesting to ask whether it is possible to completely eliminate the use of Zimmer from our arguments.

2.4 Future work

We conclude this chapter by presenting a couple of questions which naturally follow upon the work of this thesis.

Recall that Hjorth and Thomas proved that $\cong_{n+1} \not\leq_B \cong_n$. In other words, they showed that in some sense the rank n can be recovered from the $\sim_B$-class of the equivalence relation $\cong_n$. Their proof method can easily be adapted to show that similarly, $\cong_{n+1,p} \not\leq_B \cong_{n,q}$ for any primes p and q. In view of Thomas's result that $\cong_{n,p} \perp_B \cong_{n,q}$ for $p \neq q$, one might wonder if the following holds:

2.4.1. **Conjecture**. *If $m \geq n \geq 3$ and p, q are distinct primes, then $\cong_{n,p} \perp_B \cong_{m,q}$.*

Let us briefly consider this question. Simplifying matters somewhat, for $n \in \mathbb{N}$ and p prime, let $H_{n,p}$ denote the orbit equivalence relation arising from the action of $SL_n(\mathbb{Z})$ on $SL_n(\mathbb{Z}_p)$. In [Tho4], Thomas proved that if $n \geq 3$ and $p \neq q$, then $H_{n,p}$ and $H_{n,q}$ are incomparable. As a first step in answering the conjecture, one might attempt to show that if $m \geq n \geq 3$ and $p \neq q$, then $H_{n,p}$ and $H_{m,q}$ are Borel incomparable. Even this result would be interesting in its own right.

The solution to the latter problem, after an appeal to superrigidity, would involve a classification of the permutation group homomorphisms of the form:

$$(\phi, f) : SL_n(\mathbb{Z}) \curvearrowright SL_n(\mathbb{Z}_p) \longrightarrow SL_m(\mathbb{Z}) \curvearrowright SL_m(\mathbb{Z}_q)$$

For this, it is also necessary to consider the representations of $SL_n(\mathbb{Z})$ inside $SL_m(\mathbb{R})$. For instance, if $m = n + 1$ then there are only two distinct such representations up to conjugacy: the identity embedding, and the inverse-transpose embedding. Hence, the problem is likely to be much easier in this case.

Let us preface our next question by noting that it follows easily from Theorems A and B that the isomorphism and quasi-isomorphism relations for *all* p*-local* groups of rank n are Borel incomparable. Using this, together with Theorem 4.7 of [Tho1], it isn't hard to prove that the isomorphism and quasi-isomorphism relations on the space of all *local* (that is, p-local for any p) torsion-free abelian groups of rank n are also incomparable. It would be surprising if the following general result did not hold:

2.4.2. **Conjecture**. *For* $n \geq 3$*, the isomorphism and quasi-isomorphism relations on the space* $R(n)$ *of torsion-free abelian groups of rank n are Borel incomparable.*

In [Tho3], Thomas was able to study the quasi-isomorphism relation on torsion-free abelian groups of rank n using information about just the local groups. This was accomplished using Lady's theorem: A is quasi-equal to B iff the following hold:

(a) for all primes p, the localizations $A_{(p)}$ and $B_{(p)}$ are quasi-equal, and

(b) for all but finitely many primes p, $A_{(p)} = B_{(p)}$.

Despite Thomas's success, we remain unable to use this tool to extend our results on local groups to all torsion-free abelian groups of finite rank.

Chapter 3

Ergodic theory and Grassmann spaces

In this chapter, we remind the reader of the definition of ergodicity of a measure preserving action, which plays an essential role in the theory of countable Borel equivalence relations. We introduce the notion of a homogeneous space, *i.e.*, standard Borel spaces X on which a compact group K acts transitively, and we consider the action on X of a countable dense subgroup $\Gamma < K$. Such actions will be measure-preserving and ergodic with respect to a certain measure on X, and have many additional useful properties. We conclude with an analysis of the Grassmann space of k-dimensional subspaces of $\mathbb{Q}_p^n$; this is a homogeneous space for the compact group $K = SL_n(\mathbb{Z}_p)$.

Although some of the material of this chapter has been mentioned in the introduction, the present chapter shall be essentially self-contained. But recall that we are motivated by the fact, also explained in Section 2.3, that there is a close relationship between the k-Grassmann spaces and spaces of p-local torsion-free abelian groups.

3.1 Ergodicity

Let Γ be a countable group acting in a Borel fashion on the standard Borel space X, and suppose that X carries a Γ-invariant Borel probability measure μ. Recall that the action $\Gamma \curvearrowright (X, \mu)$ is *ergodic* if every Γ-invariant Borel subset of X is null or conull for μ. We shall use the following well-known characterization of ergodicity:

3.1.1. **Proposition**. *The action $\Gamma \curvearrowright (X, \mu)$ is ergodic iff for every Γ-invariant function $f : X \to Y$ into a standard Borel space Y, there exists a conull subset $A \subset X$ such that $f|_A$ is a constant function.*

The ergodic actions are in some sense the building blocks for all measure preserving actions. Indeed, there is a fundamental result of ergodic theory which says roughly that any Γ-invariant measure can be built up from ergodic measures as a "direct integral." Although we shall have no use for this fact, we shall often be motivated to work with orbit equivalence relations induced by ergodic actions. For example, in the introduction we have already proved the following fact:

3.1.2. **Proposition**. *Let μ be a nonatomic probability measure on X. Then for any ergodic action $\Gamma \curvearrowright (X, \mu)$, the induced orbit equivalence relation E_Γ is not smooth.*

3.2 F-ergodicity and weak Borel reducibility

We begin this section by recalling the definition of a Borel homomorphism. This notion is significantly more flexible than that of a Borel reduction, and it will frequently arise in our arguments. If E, F are equivalence relations on standard Borel spaces X, Y, then a *Borel homomorphism* from E to F is a Borel function $f : X \to Y$ satisfying:

$$x \mathrel{E} x' \implies f(x) \mathrel{F} f(x')$$

for all $x, x' \in X$. In other words, f is similar to a Borel reduction, but with some "collapsing" allowed.

Notice that a Γ-invariant function into the standard Borel space Y is exactly a Borel homomorphism from E_Γ to $\Delta(Y)$ (recall that $\Delta(Y)$ denotes the equality relation on Y). Hence, the characterization of ergodicity in Proposition 3.1.1 gives that the action of Γ on (X, μ) is ergodic iff every Borel homomorphism from E_Γ to $\Delta(Y)$ is constant on a μ-conull set.

Generalizing this property, if F is a Borel equivalence relation on the standard Borel space Y, then we say that the action $\Gamma \curvearrowright (X, \mu)$ is *F-ergodic* if for every Borel homomorphism f from E_Γ to F, there exists a μ-conull subset $A \subset X$ such that $f(A)$ is contained in a single F-class. This condition implies that the induced orbit equivalence relation E_Γ is very incompatible with F. When $\Gamma \curvearrowright (X, \mu)$ is F-ergodic and μ is clear from context, we often say that E_Γ is F-ergodic.

Although we are primarily interested in the $\leq_B$ partial ordering on equivalence relations, there is a second notion of reduction which will be useful in our arguments. If E, F are Borel equivalence relations on X, Y, then a *weak Borel reduction* from E to F is a countable-to-one Borel homomorphism from E to F. We write $E \leq_B^w F$ if there exists a weak Borel reduction from E to F. It is worth noting the following elementary facts:

3.2.1. **Proposition**. *Let E and F be countable Borel equivalence relations on X and Y respectively.*

(a) If $E \not\leq_B^w F$ then $E \not\leq_B F$.

(b) Suppose additionally that E is the orbit equivalence relation induced by the action $\Gamma \curvearrowright X$. If X carries a nonatomic Γ-invariant measure μ and E is F-ergodic, then $E \not\leq_B^w F$.

Proof. Part (a) is clear from the definitions. For part (b), if $E = E_\Gamma$ is F-ergodic and f is a weak Borel reduction from E to F, then there exists a conull subset $M \subset X$ such that $f(M)$ is contained in a single F-class. Since E and F are countable and f is countable-to-one, it follows that M is a countable conull set, contradicting the fact that μ is nonatomic. □

See Section 4 of [Tho5] for a discussion of further properties of weak Borel reductions.

3.3 Ergodic components

Suppose that Γ acts ergodically on the probability measure space (X, μ). If $\Lambda < \Gamma$ is an arbitrary subgroup, then of course Λ need not act ergodically on (X, μ). However, if Λ is a subgroup of finite index in Γ, then there exists a partition $X = Z_1 \sqcup \cdots \sqcup Z_N$ of X into Λ-invariant subsets such that for each $i \leq N$:

(a) $\mu(Z_i) > 0$, and

(b) Λ acts ergodically on (Z_i, μ_i), where μ_i denotes the (normalized) probability measure induced on Z_i by μ.

To see this, first observe that by ergodicity of $\Gamma \curvearrowright (X, \mu)$ we have that if $Z \subset X$ is a Λ-invariant subset of positive measure, then $\mu(Z) \geq 1/n$, where $n = [\Gamma : \Lambda]$. Now, if Λ does not act ergodically on X, then there exists a partition of X into two Λ-invariant subsets of positive measure. If Λ again fails to act ergodically on either subset, then it can be further subdivided. This splitting process must terminate, since each subdivision has measure at least $1/n$. The final partition consists of at most n cells, and Λ necessarily acts ergodically on each.

It is easily seen that the set of cells $\{Z_i\}$ in the partition satisfying (a), (b) is determined uniquely up to null sets by Λ and the action $\Gamma \curvearrowright (X, \mu)$. The cells Z_i (each together with its measure preserving action $\Lambda \curvearrowright (Z_i, \mu_i)$) are called the *ergodic components* for the action of Λ on X.

3.4 Homogeneous spaces

If the compact, second countable group K acts continuously and transitively on the standard Borel space X, then X is said to be a *homogeneous K-space*. Clearly, every homogeneous K-space X is isomorphic as a K-space with the quotient K/L, where L is the stabilizer in K of some point in X. It follows that any such X carries a unique K-invariant probability measure, called the *Haar measure*. Indeed, if we regard X as a quotient K/L, then this measure is the projection to K/L of the Haar probability measure on K.

We shall be interested in the action of a countable subgroup $\Gamma < K$ on a homogeneous K-space X. It is not difficult to prove that if Γ is a *dense* subgroup of K, then any Γ-invariant measure is necessarily K-invariant. It follows that in this case, the Haar measure is the unique Γ-invariant measure on X.

3.4.1. **Definition**. The action $\Gamma \curvearrowright Y$ is called *uniquely ergodic* if there exists a unique Γ-invariant measure on Y.

It is easy to see that if $\Gamma \curvearrowright Y$ is uniquely ergodic and μ is the Γ-invariant measure, then $\Gamma \curvearrowright (Y, \mu)$ is ergodic. Indeed, if $Y_0 \subset Y$ were a Γ-invariant subset such that $0 < \mu(Y_0) < 1$, then the normalized restriction of μ to Y_0 would be a Γ-invariant measure on Y that is distinct from μ. Using this together with our earlier remarks, we conclude that the action of a dense subgroup of K on a homogeneous K-space is always ergodic.

We now provide a characterization of the ergodic components for the action of a dense subgroup of K on a homogeneous K-space.

3.4.2. **Proposition** (Proposition 2.2 of [Tho4]). *Let K be compact and let K/L be a homogeneous K-space. Suppose that $\Gamma < K$ is a dense subgroup, and let $\Lambda \leq \Gamma$ be a subgroup of finite index.*

(a) The ergodic components for the action of Λ are precisely the orbits of $\bar{\Lambda}$ on K/L. Here, $\bar{\Lambda}$ denotes the closure in K of Λ.

(b) Each ergodic component is again a homogeneous space for the compact group $\bar{\Lambda}$.

(c) If $\Lambda \triangleleft \Gamma$ is a normal *subgroup of finite index, then Γ acts as a transitive permutation group on the $\bar{\Lambda}$-orbits,* i.e., *on the ergodic components for the action of Λ.*

Proof. First note that if $x \in K/L$ then the orbit $\bar{\Lambda}x$ has positive measure. Indeed, we have $[K : \bar{\Lambda}] < \infty$ and if $k_1, \ldots, k_m$ are coset representatives in K for $\bar{\Lambda}$ then $\bigcup k_i\bar{\Lambda}x = X$. Since $\bar{\Lambda}x$ is a homogeneous $\bar{\Lambda}$-space and Λ is dense in $\bar{\Lambda}$, by our earlier remarks Λ acts ergodically on $\bar{\Lambda}x$. This proves (a) and (b). For (c), note that if $\Lambda \triangleleft \Gamma$ then clearly Γ normalizes $\bar{\Lambda}$ and hence it acts on the $\bar{\Lambda}$-orbits. To see that this action is transitive, note that $\Gamma(\bar{\Lambda}x)$ is Γ-invariant and so by ergodicity of Γ it is conull. $\square$

3.5 Affine maps

In this section, we describe the natural morphisms between homogeneous spaces. For $i = 0, 1$ let K_i be compact and L_i a closed subgroup. Then a function $f : K_0/L_0 \to K_1/L_1$ is said to be *affine* if there exists an isomorphism $\phi : K_0 \to K_1$ and an element $t \in K_1$ with the following properties:

(a) $\phi(L_0) = tL_1t^{-1}$, and

(b) $f(kL_0) = \phi(k)tL_1$ for $k \in K_0$.

The terminology is motivated by the following special case. A map $f : K_0 \to K_0$ is affine iff it is obtained by composing an automorphism of K_0 with a translation by an element

K_0. We remark that condition (a) is redundant, as in fact f is affine iff there exists an isomorphism $\phi : K_0 \to K_1$ such that $f(kx) = \phi(k)f(x)$ for all $x \in K_0/L_0$ and $k \in K_0$.

If $f : K_0/L_0 \to K_1/L_1$ is an affine map with ϕ as in (a), (b), and $\Gamma_0 < K_0$ is a dense subgroup, then the pair (ϕ, f) is clearly a permutation group homomorphism from $\Gamma_0 \curvearrowright K_0/L_0$ to $\phi(\Gamma_0) \curvearrowright K_1/L_1$. We shall make use of the following result, which implies that many permutation group homomorphisms between homogeneous spaces actually come from affine maps.

3.5.1. **Lemma**. *For $i = 0, 1$ let K_i/L_i be a homogeneous space for the compact group K_i, let $\Gamma_i < K_i$ be a countable dense subgroup, and suppose that*

$$(\phi, f) : \Gamma_0 \curvearrowright K_0/L_0 \longrightarrow \Gamma_1 \curvearrowright K_1/L_1$$

is a permutation group homomorphism. If ϕ extends to an isomorphism $\Phi : K_0 \to K_1$, then after adjusting f on a set of measure zero, f is an affine map.

Proof. Following an argument of Gefter (see the proof of Theorem 3.3 of [Gef]), we define the map $\beta : K_0 \to K_1/L_1$ by:

$$\beta(k) = \Phi(k)^{-1}f(kL_0)$$

Notice first that β is Γ_0-invariant. Indeed, for $\gamma \in \Gamma_0$, we compute that:

$$\beta(\gamma k) = \Phi(\gamma k)^{-1}f(\gamma k L_0) = \Phi(k)^{-1}\Phi(\gamma)^{-1}\phi(\gamma)f(kL_0) = \Phi(k)^{-1}f(kL_0) = \beta(k)$$

Hence by ergodicity of $\Gamma_0 \curvearrowright K_0$, there exists $t \in K_1$ such that $\beta(k) = tL_1$ for (Haar) almost every $k \in K$. It follows that for almost every $kL_0 \in X$, we have that $f(kL_0) = \Phi(k)tL_1$, as desired. $\square$

We remark that there is a significantly more general version of this result, due to Furman. Specifically, Proposition 7.2 of [Fur] gives a similar conclusion without the hypothesis that ϕ extends to an isomorphism $K_0 \to K_1$. Furman used this result to compute the outer automorphism groups of some equivalence relations which are closely related to those considered in this thesis.

3.6 Grassmann spaces over the p-adics

We now introduce an important family of homogeneous spaces that we shall spend a great deal of time studying in future chapters.

If V is a vector space of dimension n and $k \leq n$, then the *k-Grassmann space* of V, denoted $Gr_k V$, is the set of k-dimensional subspaces of V. We will be interested in the spaces $Gr_k \mathbb{Q}_p^n$, where $\mathbb{Q}_p^n$ denotes the canonical n-dimensional vector space over the field $\mathbb{Q}_p$ of p-adic numbers. The next proposition shows that $Gr_k \mathbb{Q}_p^n$ is indeed a homogeneous space.

3.6.1. **Proposition** (Proposition 6.1 of [Tho4]). *The compact group $SL_n(\mathbb{Z}_p)$ acts transitively on $Gr_k \mathbb{Q}_p^n$, where $\mathbb{Z}_p$ denotes the ring of p-adic integers.*

Hence, we can view $Gr_k \mathbb{Q}_p^n$ as a homogeneous $SL_n(\mathbb{Z}_p)$-space by identifying $Gr_k \mathbb{Q}_p^n$ with $SL_n(\mathbb{Z}_p)/L$, where L is the stabilizer in $SL_n(\mathbb{Z}_p)$ of some point in $Gr_k \mathbb{Q}_p^n$. Accordingly, it carries a corresponding Haar probability measure and the dense subgroup $SL_n(\mathbb{Z}) < SL_n(\mathbb{Z}_p)$ acts (uniquely) ergodically on $Gr_k \mathbb{Q}_p^n$.

We now describe the "principle congruence components" of the k-Grassmann space. Recall that for any natural number m, the principal congruence subgroup $\Gamma_m \triangleleft SL_n \mathbb{Z}$ is defined by:

$$\Gamma_m = \ker[SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/m\mathbb{Z})]$$

where the map on the right-hand side is the canonical surjection. It is easily seen that the closure in $SL_n(\mathbb{Z}_p)$ of Γ_m is exactly K_m, where K_m is defined as:

$$K_m = \ker[SL_n(\mathbb{Z}_p) \to SL_n(\mathbb{Z}_p/m\mathbb{Z}_p)]$$

Hence by Proposition 3.4.2, the ergodic components of $Gr_k \mathbb{Q}_p^n$ corresponding to the action of Γ_m are precisely the K_m-orbits. We call these the m^{th} principle congruence components of the k-Grassmann space. Note that we always have $K_m = K_{p^t}$, where p^t is the highest power of p which divides m.

As an example, let us consider the ergodic component for Γ_{p^t} given by the K_{p^t}-orbit of $V_0 := \mathbb{Q}_p e_1 \oplus \cdots \oplus \mathbb{Q}_p e_k$. Clearly, any V in this orbit can be written as the column space of a matrix $\begin{bmatrix} a \\ v \end{bmatrix}$, where a is congruent to the $k \times k$ identity matrix I_k modulo p^t and v is congruent to 0 modulo p^t. Since any such a is invertible, one can use column operations to suppose without loss of generality that $a = I_k$. So we have:

$$(K_{p^t})V_0 = \left\{ \text{col} \begin{bmatrix} I_k \\ v \end{bmatrix} : p^t \mid v \right\} \tag{3.6.2}$$

where $p^t \mid v$ means that for each entry x of v, we have that x/p^t lies in $\mathbb{Z}_p$.

We shall use the fact (see [BLS]) that $SL_n(\mathbb{Z})$ has the *congruence subgroup property* for $n \geq 3$, which means that every subgroup of $SL_n(\mathbb{Z})$ of finite index contains a principle congruence subgroup. We immediately obtain the following:

3.6.3. **Proposition**. *If $n \geq 3$ and $\Gamma \leq SL_n(\mathbb{Z})$ is a subgroup of finite index, then any ergodic component for the action of Γ on $Gr_k\mathbb{Q}_p^n$ contains a principle congruence component.*

In Appendix 7.2, we shall find the exact number of principle congruence components of $Gr_k\mathbb{Q}_p^n$ for each subgroup Γ_m of $SL_n(\mathbb{Z})$.

3.7 Mappings between Grassmann spaces

Recall that superrigidity theorems can be used to show that Borel homomorphisms come from permutation group homomorphisms. We close this chapter with a characterization of the permutation group homomorphisms between ergodic components of the k-Grassmann spaces. This will be used in the next chapter to show that certain orbit equivalence relations on the k-Grassmann spaces are Borel incomparable.

3.7.1. **Theorem**. *Let $n \geq 3$ and suppose that $k, l \leq n$. Let Γ_0, Γ_1 be subgroups of $SL_n(\mathbb{Z})$ of finite index, X_0 an ergodic component for the action of Γ_0 on $Gr_k\mathbb{Q}_p^n$, and X_1 an ergodic component for the action of Γ_1 on $Gr_l\mathbb{Q}_p^n$. Suppose that:*

- *$\phi : \Gamma_0 \to \Gamma_1$ is an isomorphism,*
- *$f : X_0 \to X_1$ is a Borel function, and*
- *$(\phi, f) : \Gamma_0 \curvearrowright X_0 \longrightarrow \Gamma_1 \curvearrowright X_1$ is a permutation group homomorphism.*

Then $l = k$ or $l = n - k$, and:

(a) *In the case $l = k \neq n - k$, there exists $h \in GL_n(\mathbb{Q})$ such that f satisfies $f(x) = hx$ for almost every $x \in X_0$.*

(b) *In the case $l = n - k \neq k$, there exists $h \in GL_n(\mathbb{Q})$ such that f satisfies $f(x) = hx^\perp$ for almost every $x \in X_0$, where $x^\perp$ denotes the orthogonal complement of x with respect to the usual dot product.*

First, we give a characterization of the isomorphisms ϕ which can arise in Theorem 3.7.1. Although Lemma 3.7.2 applies more generally than necessary for this purpose, the full result will be useful later.

3.7.2. **Lemma**. *Let $n \geq 3$ and $\Gamma_0 \leq SL_n(\mathbb{Z})$ be a subgroup of finite index. Let $\phi : \Gamma_0 \to SL_n(\mathbb{Z})$ be an injective homomorphism. Then ϕ decomposes as $\phi = \epsilon \circ \chi_h \circ (-T)^i$ where:*

- *$\chi_h(g) = h^{-1}gh$ is conjugation by some element $h \in GL_n(\mathbb{Q})$,*
- *$-T$ is the inverse-transpose map and $i = 0$ or 1, and*
- *ϵ is an automorphism of $SL_n(\mathbb{Z})$ satisfying $\epsilon(\gamma) = \pm\gamma$.*

Proof. We first consider the image $\bar{\Gamma}_0$ of Γ_0 in $PSL_n(\mathbb{Z})$, so that $\bar{\Gamma}_0$ is a lattice in the simple Lie group $PSL_n(\mathbb{R})$ (of course, this is unnecessary if n is odd). Letting I denote the identity element of $SL_n(\mathbb{Z})$, if $-I \in \Gamma_0$ then $-I$ is in the center of Γ_0. It follows that $\phi(-I)$ is in the center of $SL_n(\mathbb{Z})$ and so $\phi(-I) = -I$. Hence, there exists a map $\bar{\phi}$ such that the following diagram commutes:

(3.7.3)
$$\begin{tikzcd} \Gamma_0 \arrow[r, "\phi"] \arrow[d, "\pi"'] & SL_n(\mathbb{Z}) \arrow[d, "\pi"] \\ \bar{\Gamma}_0 \arrow[r, "\bar{\phi}"'] & PSL_n(\mathbb{Z}) \end{tikzcd}$$

where π is the canonical projection.

We now argue that $\bar{\phi}$ extends to an automorphism Φ of $PSL_n(\mathbb{R})$. By Theorem VIII.3.10 of [Mar], the Zariski closure in H of $\bar{\phi}(\bar{\Gamma}_0)$ in $PSL_n(\mathbb{R})$ is semisimple. Let $\psi_i : H \to H_i$ denote the projections of H onto its simple factors, and let $\bar{\phi}_i := \psi_i \circ \bar{\phi}$. Then $\bar{\phi}_i(\bar{\Gamma}_0)$ is

Zariski dense in H_i, and so by the Mostow-Margulis superrigidity theorem (see Theorem 5.1.2 of [Zim]), $\bar{\phi}_i$ extends to a homomorphism $\Phi_i : PSL_n(\mathbb{R}) \to H_i$ for each i. Since $PSL_n(\mathbb{R})$ is simple, there is exactly one Φ_i with infinite image, and it follows that the corresponding factor H_i is actually $PSL_n(\mathbb{R})$ itself. Hence, we have that $\bar{\phi}$ extends to an automorphism Φ of $PSL_n(\mathbb{R})$.

Now, it is well known that any such Φ can be written as $\chi_s \circ \chi_r \circ (-T)^i$, where χ_s is conjugation by an element $s \in SL_n(\mathbb{R})$, χ_r is conjugation by a permutation matrix r, and $i = 0$ or 1. (When n is odd, then Φ can be written more simply as $\chi_s \circ (-T)^i$. However, when n is even, it is sometimes necessary to also include conjugation χ_r by a permutation matrix r of determinant -1.) Next, since $\Phi(\Gamma_0)$ is again a lattice of $PSL_n(\mathbb{R})$, we have that Φ commensurates $PSL_n(\mathbb{Z})$ and hence so does χ_s. By the proof of Proposition 6.2.2 of [Zim] (the statement found there is slightly inaccurate), there exists $a \in \mathbb{R}^*$ such that $as \in GL_n(\mathbb{Q})$. Taking $h = asr$, we have that $\bar{\phi} = \chi_h \circ (-T)^i$. Finally, $\bar{\phi}$ lifts to the map $\phi' : \Gamma_0 \to SL_n(\mathbb{Z})$ defined by the same formula, and it is clear from equation (3.7.3) that $\phi = \epsilon \circ \phi'$ where ϵ is as required. $\square$

In the next proposition we shall use the following notation. For $V \in Gr_k\mathbb{Q}_p^n$, let $\operatorname{stab} V$ denote the stabilizer in $GL_n(\mathbb{Q}_p)$ of V. If $H \leq GL_n(\mathbb{Q}_p)$, then let $\operatorname{stab}_H V$ denote the stabilizer in H of V.

3.7.4. **Proposition**. *Let $n \geq 3$ and $k, l \leq n$, and suppose that $V \in Gr_k\mathbb{Q}_p^n$ and $W \in Gr_l\mathbb{Q}_p^n$. If $K \leq SL_n(\mathbb{Z}_p)$ is a subgroup of finite index and* $\operatorname{stab}_K V \subset \operatorname{stab} W$, *then $l = k$ and $W = V$.*

Proof. Since K are Zariski dense in $H = SL_n(\mathbb{Q}_p)$ (it is an open subgroup), we have that $\operatorname{stab}_H V \subset \operatorname{stab}_H W$. It is well-known that H acts *primitively* on each k-Grassmann space, *i.e.*, H acts transitively on $Gr_k\mathbb{Q}_p^n$ and the stabilizer in H of each point of $Gr_k\mathbb{Q}_p^n$ is a maximal subgroup of H. It follows immediately that we have $\operatorname{stab}_H V = \operatorname{stab}_H W$.

Now, it is not hard to see that V is uniquely determined by $\operatorname{stab}_H V$ and so $V = W$. (To prove this last fact directly, first consider $V_0 = \mathbb{Q}_p e_1 \oplus \cdots \oplus \mathbb{Q}_p e_k$. Then $\operatorname{stab}_H V_0$ consists of all matrices of the form:

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \in H$$

where 0 denotes the $(n-k) \times k$ zero matrix. It is routine to compute that V_0 is the unique subspace stabilized by this set. Since H acts transitively on $Gr_k\mathbb{Q}_p^n$, we can conclude that V is the unique subspace of $\mathbb{Q}_p^n$ stabilized by $\mathrm{stab}_H V$.) $\square$

Proof of Theorem 3.7.1. In the notation of Lemma 3.7.2, we have that $\phi = \epsilon \circ \chi_h \circ (-T)^i$. Since the center of $SL_n(\mathbb{Z})$ acts trivially on $Gr_l\mathbb{Q}_p^n$, we may suppose that ϵ is the identity map and that $\phi = \chi_h \circ (-T)^i$.

Having done so, ϕ clearly lifts to an automorphism Φ of $GL_n(\mathbb{Q}_p)$, again defined by the formula $\chi_h \circ (-T)^i$. For $i = 0, 1$, let K_i denote the closure in $SL_n(\mathbb{Z}_p)$ of Γ_i, so that X_i is a homogeneous K_i-space. Since $\Phi(K_0)$ is a compact group containing Γ_1, we have that $\Phi(K_0) \supset K_1$. By the same reasoning, we have $\Phi^{-1}(K_1) \supset K_0$, and so $\Phi(K_0) = K_1$. Hence by Lemma 3.5.1, after adjusting f on a null set, we may suppose that Φ makes f into an affine map.

Now, for all $k \in K_0$ we have:

$$kx = x \implies \Phi(k)f(x) = f(x)$$

and hence $\Phi(\mathrm{stab}_{K_0} x) \subset \mathrm{stab}_{K_1} f(x)$. Since Φ is either χ_h or $\chi_h \circ (-T)$, we have either $\mathrm{stab}_{K_1}(h^{-1}x) \subset \mathrm{stab}_{K_1} f(x)$ or $\mathrm{stab}_{K_1}(h^{-1}x^{\perp}) \subset \mathrm{stab}_{K_1} f(x)$. In the first case, we can apply Proposition 3.7.4 to conclude that $l = k$ and $f(x) = h^{-1}x$. In the second case we conclude that $l = n - k$ and $f(x) = h^{-1}x^{\perp}$. $\square$

Chapter 4

Superrigidity

We have said that in order to construct Borel incomparable equivalence relations, it is sometimes possible to adapt methods of constructing orbit inequivalent actions. In both settings, it is desirable that very different groups should give rise to very different orbit equivalence relations. Of course, this can only hold in special cases. Note for instance that by Dye's theorem (see Section 7 of [KM]), any ergodic measure preserving actions of two amenable groups are necessarily orbit equivalent.

Superrigidity theorems give hypotheses under which there is a hope that such collapsing does not occur. We take a moment to discuss the superrigidity phenomenon, which comes from the theory of discrete subgroups of semisimple groups. For us, it has come to have several closely related meanings, including but not limited to the following:

(a) A lattice in a Lie group is recognizable as such just from its structure as an abstract group.

(b) Given a measure preserving action of a lattice, its orbit structure remembers the ambient Lie group.

(c) Given a measure preserving action of a lattice, its orbit structure remembers the lattice and its action.

We emphasize that these statements are quite strong and hold only in restrictive senses. We have already used rigidity of type (a) in the proof of Lemma 3.7.2 to show that some homomorphisms defined on a lattice extend to the ambient Lie group. We have also explained that a type (b) result, namely Zimmer's Theorem 1.8.2, can be used to find incomparable countable Borel equivalence relations. In this chapter, we shall see several instances of rigidity results of type (c).

4.1 Cocycles

We begin with a brief interlude on Borel cocycles, wherein we will explain their importance in rigidity theory. If it is desired, the reader may skip Sections 4.1–4.3. In later sections we shall use only Corollary 4.3.3, the statement of which makes no mention of cocycles.

4.1.1. **Definition**. Let Γ and Λ be countable groups. If $\Gamma \curvearrowright X$, then a Borel function $\alpha : \Gamma \times X \to \Lambda$ is called a *(strict) cocycle* if it satisfies:

$$\alpha(\delta\gamma, x) = \alpha(\delta, \gamma x)\alpha(\gamma, x) \tag{4.1.2}$$

for all $\gamma, \delta \in \Gamma$ and all $x \in X$.

The property (4.1.2) is known as the *cocycle property*. If $\Gamma \curvearrowright X$ preserves a probability measure on X, it is more useful to consider the slightly weaker notion of a *measurable cocycle* in which (4.1.2) is only required to hold for almost every $x \in X$. For us, the terms "cocycle" and "Borel cocycle" will usually mean measurable cocycle.

The presence of cocycles in this subject is motivated by the following extremely important example. Again let $\Gamma \curvearrowright X$, and suppose additionally that Λ acts freely on the standard Borel space Y. Suppose that $f : X \to Y$ is a Borel homomorphism from E_Γ to E_Λ. Then since Λ acts freely, for every $x \in X$ and $\gamma \in \Gamma$ there exists a unique $\lambda \in \Lambda$ such that $f(\gamma x) = \lambda f(x)$. Hence, we may let:

$$\alpha(\gamma, x) := \text{the unique } \lambda \text{ such that } f(\gamma x) = \lambda f(x)$$

and it is easy to verify that this α satisfies the cocycle property (see Figure 4.1.3). We call α the cocycle *corresponding* to f.

Since we will be working with it a great deal, we shall make a few remarks about this example. First, without the hypothesis that Λ acts freely, we would still be able to choose a λ satisfying $f(\gamma x) = \lambda f(x)$. However, it is not possible in general to choose elements of Λ in a coherent (and Borel) fashion so as to satisfy the cocycle property. Second, the corresponding cocycle is in fact strict, and this is occasionally useful when applying superrigidity results.

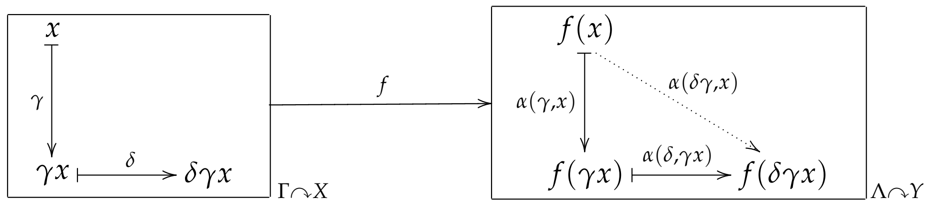

Figure 4.1.3. The cocycle α corresponds to f.

Now let α be the cocycle corresponding to a Borel homomorphism f from E_Γ to E_Λ, and consider the special case that there exists a homomorphism $\phi : \Gamma \to \Lambda$ which makes $(\phi, f) : \Gamma \curvearrowright X \longrightarrow \Lambda \curvearrowright Y$ into a permutation group homomorphism. Then since $f(\gamma x) = \phi(\gamma) f(x)$, we must always have that $\alpha(\gamma, x) = \phi(\gamma)$.

4.1.4. **Definition**. A cocycle is called *trivial* if it is independent of the second coordinate. More precisely, $\alpha : \Gamma \times X \to \Lambda$ is trivial iff $\alpha(\gamma, x) = \alpha(\gamma, x')$ for all $x, x' \in X$.

In this case, we can define a function $\phi(g) := \alpha(g, \cdot)$, and the cocycle property implies that ϕ is a homomorphism. Hence, if α corresponds to f, then α is trivial exactly when $f(\gamma x) = \phi(g) f(x)$ is a permutation group homomorphism.

We next seek to characterize when a Borel homomorphism can be "slightly perturbed" to become a permutation group homomorphism. Here, if E and F are Borel equivalence relations on X and Y and f is a Borel homomorphism from E to F, then we informally say that the Borel map f' is a *slight perturbation* of f if $f(x) \ F \ f(x')$ for all $x, x' \in X$. If F is induced by the action of Λ on Y, then f' is a slight perturbation of f iff there is a Borel function $b : X \to \Lambda$ such that $f'(x) = b(x) f(x)$. It turns out that this can be expressed in terms of cocycles.

4.1.5. **Definition**. Suppose that $\Gamma \curvearrowright X$ is a measure preserving action, and that $\alpha, \alpha' : \Gamma \times X \to \Lambda$ are cocycles. We say that α and α' are *cohomologous* iff there exists a Borel function $b : X \to \Lambda$ such that:

$$\alpha'(\gamma, x) = b(\gamma x)\alpha(\gamma, x) b(x)^{-1} \tag{4.1.6}$$

for every $\gamma \in \Gamma$ and almost every $x \in X$. (Since Γ is countable, we may write the quantifiers in either order.)

It is easy to verify that if α is the cocycle corresponding to the Borel homomorphism f, and if b and α' are as in Definition 4.1.5, then α' is (almost equal to) the cocycle corresponding to the function $f'(x) = b(x)f(x)$ (see Figure 4.1.7).

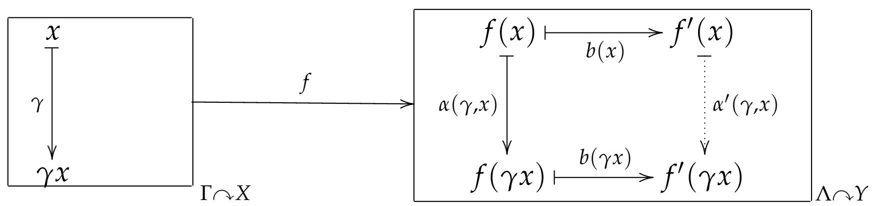

Figure 4.1.7. If f, f' differ only by multiples of Λ, then their corresponding cocycles α, α' are cohomologous.

Together with the preceding remarks, this leads to the key observation that f can be slightly perturbed (by a Borel function) to become a permutation group homomorphism iff its corresponding cocycle is cohomologous to a trivial cocycle. This explains the presence of cocycles in rigidity; the most general results give hypotheses under which an arbitrary cocycle α is equivalent to a trivial one.

4.2 Cocycle superrigidity theorems

Recall that Adams and Kechris used Zimmer's Theorem 1.8.2 to produce uncountably many incomparable countable Borel equivalence relations. We now give a more complete statement of the cocycle superrigidity theorem underlying this result. We first remark that the definition of a Borel cocycle can easily be extended to the case that the countable groups Γ, Λ are replaced by topological groups G, H.

4.2.1. **Theorem** (Theorem 3.4 of [AK], Theorem 5.2.5 of [Zim]). *Suppose that G is a connected simple Lie group of higher rank, and X carries a G-invariant ergodic probability measure. Let H be a noncompact connected simple real algebraic group and suppose that $\alpha : G \times X \to H$*

is a cocycle. If α is not cohomologous to a cocycle taking values in a proper algebraic subgroup of H, then α is cohomologous to a trivial cocycle.

This theorem can be used to prove, in special cases, that any Borel homomorphism f from E_Γ to E_Λ can be slightly perturbed to become a permutation group homomorphism. To have any hope of doing so using Zimmer's theorem, one must assume:

(a) Γ is a lattice in a connected simple higher rank Lie group G,

(b) X carries a Γ-invariant ergodic measure,

(c) Λ is contained in a noncompact connected simple real algebraic group H,

(d) Λ acts freely on Y.

We have seen in the previous section that clause (d) guarantees that one can define a cocycle $\alpha : \Gamma \times X \to \Lambda$ corresponding to f. Clauses (a) and (b) (together with the fact that the induced cocycle is strict) allow the construction of an *induced cocycle* $\hat{\alpha} : G \times \hat{X} \to H$, which is a canonical lifting of α to a cocycle on G. Here, $\hat{X}$ is a somewhat complicated space, and the action $G \curvearrowright \hat{X}$ contains many "twisted" copies of the action $\Gamma \curvearrowright X$. Zimmer's Theorem 4.2.1 can then be applied to the induced cocycle, and with some more technical results about cocycles, this yields a proof of Zimmer's Theorem 1.8.2.

Zimmer's theorem is no longer the only superrigidity result that is applicable to countable Borel equivalence relations. If E is a countable Borel equivalence relation, then E is said to be *essentially free* if there exists a free action $\Gamma \curvearrowright X$ of a countable group such that $E \sim_B E_\Gamma$. For many years it was unknown whether the universal countable Borel equivalence relation E_∞ was essentially free. This was recently solved by Thomas (see Corollary 3.10 of [Tho5]) using a superrigidity theorem of Popa.

In order to describe this result, we must first define the following canonical free action associated with a countable group Γ. Recall that the space 2^Γ of functions from Γ to $\{0,1\}$ carries a unique Γ-invariant measure, namely the product of the $\{\frac{1}{2},\frac{1}{2}\}$ measures. It is well-known that there exists a conull subset $(2)^\Gamma \subset 2^\Gamma$ on which Γ acts freely. The action $\Gamma \curvearrowright (2)^\Gamma$ is called the *Bernoulli* action associated with Γ.

4.2.2. **Theorem** (a special case of the main result in [Pop]). *Let S be an arbitrary countable group, and let $\Gamma = SL_3(\mathbb{Z}) \times S$. If Λ is an arbitrary group, then any cocycle $\alpha : \Gamma \times (2)^\Gamma \to \Lambda$ is cohomologous to a trivial cocycle.*

The full statement of Popa's theorem is more general, for instance $SL_3(\mathbb{Z})$ may be replaced by an arbitrary infinite Kazhdan group. However, we shall see that the true power of Popa's theorem lies in the fact that the *target* group Λ is arbitrary; there is no mention of linear groups or geometry.

For example, let us outline the argument that E_∞ isn't essentially free. Suppose, towards a contradiction, that $\Lambda \curvearrowright X$ is a free action and that the induced orbit equivalence relation E_Λ is universal countable. Choose any countable simple group S which does not embed into Γ, and let $\Gamma = SL_3(\mathbb{Z}) \times S$. Since E_Λ is universal countable, there exists a Borel reduction f from E_Γ to E_Λ. By Popa's theorem, the corresponding cocycle is cohomologous to a trivial cocycle. Hence, after deleting a null seubset of $(2)^\Gamma$, we can suppose that f comes from a permutation group homomorphism (ϕ, f). Clearly, $S \leq \ker \phi$ and so the Borel map f is S-invariant. Since S acts ergodically on $(2)^\Gamma$, this implies that f is constant on a conull subset of $(2)^\Gamma$, which is a contradiction.

There is another consequence of Popa's theorem that is worth mentioning, since it concerns torsion-free abelian groups. We have already seen that the isomorphism relation $\cong_n$ on the space of torsion-free abelian groups of rank n is not universal; since by Thomas's Theorem 1.9.1 we have that $\cong_n <_B \cong_{n+1}$ for all n. On the other hand, Thomas's result certainly implies that the isomorphism equivalence relation $\cong_{\text{fin}}$ on the space of *all* torsion-free abelian groups of finite rank is very complex, and so one might ask whether $\cong_{\text{fin}}$ is universal. In theorem 6.1 of [Tho5], Thomas used the fact that $\cong_{\text{fin}}$ is closely related to an essentially free equivalence relation together with Popa's theorem to answer this question in the negative.

4.3 Ioana's superrigidity theorem

In this section, we will introduce a recent cocycle superrigidity theorem of Adrian Ioana (see Chapter 3 of [Ioa]). As we have mentioned, Ioana's theorem is much more directly applicable to our needs than Zimmer's theorem. We shall also derive a straightforward corollary which will be used in the proofs of our main theorems.

For $i \in \mathbb{N}$, let $\Gamma \curvearrowright (X_i, \mu_i)$ and let $\rho_i : X_{i+1} \to X_i$ be a factor map (*i.e.*, a Γ-invariant measure preserving map). Then the corresponding *inverse limit* is a Γ-space (X, μ) together with factor maps $\pi_i : X \to X_i$ satisfying $\pi_i = \rho_i \circ \pi_{i+1}$ and the usual universal property associated with inverse limits.

4.3.1. **Definition**. If (X_i, μ_i) are finite Γ-spaces (with factor maps as above), then the inverse limit (X, μ) is called a *profinite* Γ-space.

4.3.2. **Theorem** (Theorem 3.3.2 of [Ioa]). *Suppose that Γ is a countable Kazhdan group, and let (X, μ) be a profinite Γ-space with corresponding factor maps $\pi_i : X \to X_i$. Suppose additionally that the action $\Gamma \curvearrowright (X, \mu)$ is ergodic and free. If $\alpha : \Gamma \times X \to \Lambda$ is a cocycle into an arbitrary countable group Λ, then there exists $i \in \mathbb{N}$ such that α is cohomologous to a cocycle $\Gamma \times X_i \to \Lambda$.*

More precisely, there exists $i \in \mathbb{N}$ and a cocycle $\alpha_i : \Gamma \times X_i \to \Lambda$ such that α is cohomologous to the cocycle α' defined by $\alpha'(g, x) = \alpha_i(g, \pi_i(x))$.

As with Zimmer's theorem, this is most useful in the case that α is a cocycle corresponding to a Borel homomorphism f from E_Γ to E_Λ, where E_Λ is the orbit equivalence relation induced by a free action of Λ.

4.3.3. **Corollary**. *Suppose that Γ is a countable Kazhdan group, and let (X, μ) be a free, ergodic profinite Γ-space. Let Λ be a countable group and let $\Lambda \curvearrowright Y$ be a free action. Suppose that f is a Borel homomorphism from E_Γ to E_Λ. Then there exists an ergodic component $\Gamma_0 \curvearrowright X_0$ for $\Gamma \curvearrowright X$ and a permutation group homomorphism $(\phi, f') : \Gamma_0 \curvearrowright X_0 \longrightarrow \Lambda \curvearrowright Y$ such that for all $x \in X_0$, we have that $f'(x)\ E_\Lambda\ f(x)$ (*i.e., *f' is a slight perturbation of f).*

Proof. Let $\alpha : \Gamma \curvearrowright X \to \Lambda$ be the cocycle corresponding to f. By Theorem 4.3.2, there exists a finite factor (X', μ') of (X, μ) (denote the projection map by π), and a Borel function $b : X \to \Lambda$ such that the adjusted cocycle:

$$\alpha'(g, x) := b(gx)\alpha(g, x)b(x)^{-1}$$

depends only on g and $\pi(x)$. Choose any $x_0 \in X'$, and let Γ_0 be the stabilizer of x_0 in

Γ. Clearly $\Gamma_0 \leq \Gamma$ is a subgroup of finite index, and by 3.1.2(ii) of [Ioa], $X_0 := \pi^{-1}(x_0)$ is an ergodic component for the action of Γ_0. Since π is constant on X_0, the restriction of α' to $\Gamma_0 \times X_0$ is independent of $x \in X_0$. It follows that $\phi(\gamma) := \alpha'(\gamma, \cdot)$ defines a homomorphism $\Gamma_0 \to \Lambda$, and so letting $f' = bf$ it is easily seen that (ϕ, f') satisfies our requirements. (Of course, since α' need only satisfy the cocycle identity (4.1.2) almost everywhere, one may need to delete a null set of X_0.) $\square$

4.4 An application to p-adic Grassmann spaces

Fix $n \in \mathbb{N}$ and a prime p. We shall now use Ioana's theorem to study some orbit equivalence relations on the k-Grassmann space $Gr_k\mathbb{Q}_p^n$. In this section and in future chapters, we shall use the following notation:

4.4.1. **Notation**. Let $E^k_{SL_n\mathbb{Z}}$ denote the orbit equivalence relation induced by the action of $SL_n(\mathbb{Z})$ on $Gr_k\mathbb{Q}_p^n$, and let $E^k_{GL_n\mathbb{Q}}$ denote the orbit equivalence relation induced by the action of $GL_n(\mathbb{Q})$ on $Gr_k\mathbb{Q}_p^n$.

Recall that we have already studied the equivalence relation $E^k_{SL_n\mathbb{Z}}$ in the context of homogeneous spaces. We shall use Corollary 4.3.3 in conjunction with the main result of the last chapter (Theorem 3.7.1) to establish the following Borel incomparability result.

4.4.2. **Theorem**. *Let $n \geq 4$ and $k, l \leq n$, and suppose that l is neither k nor $n-k$. Then $E^k_{GL_n\mathbb{Q}}$ is Borel incomparable with $E^l_{GL_n\mathbb{Q}}$.*

In the next chapter, we shall see that this theorem implies that the quasi-isomorphism relations on the spaces $R(n,p,k), R(n,p,l)$ are Borel incomparable. In order to establish Theorem 4.4.2, it will be necessary to prove the following stronger result:

4.4.3. **Theorem**. *Let $n \geq 4$ and $k, l < n$, and suppose that l is neither k nor $n-k$. Then $E^k_{SL_n\mathbb{Z}}$ (together with the Haar measure on $Gr_k\mathbb{Q}_p^n$) is $E^l_{GL_n\mathbb{Q}}$-ergodic.*

Theorem 4.4.2 follows immediately; any Borel reduction from $E^k_{GL_n\mathbb{Q}}$ to $E^l_{GL_n\mathbb{Q}}$ is clearly a weak Borel reduction from $E^k_{SL_n\mathbb{Z}}$ to $E^l_{GL_n\mathbb{Q}}$, and Theorem 4.4.3 implies that $E^k_{SL_n\mathbb{Z}} \not\leq^w_B E^l_{GL_n\mathbb{Q}}$.

Proof. Let $f : Gr_k\mathbb{Q}_p^n \to Gr_l\mathbb{Q}_p^n$ be a Borel homomorphism from $E^k_{SL_n\mathbb{Z}}$ to $E^l_{GL_n\mathbb{Q}}$, and suppose towards a contradiction that f does not map a conull set into a single $GL_n(\mathbb{Q})$-orbit. We first reduce the analysis to a situation where the hypotheses of Corollary 4.3.3 hold. Since $SL_n(\mathbb{Z})$ acts ergodically on $Gr_k\mathbb{Q}_p^n$, we need only argue that both $SL_n(\mathbb{Z}) \curvearrowright Gr_k\mathbb{Q}_p^n$ and $GL_n(\mathbb{Q}) \curvearrowright Gr_l\mathbb{Q}_p^n$ are free actions. While neither action is literally free, it will suffice to establish the following statements:

Claim (a). The action of $PSL_n(\mathbb{Z})$ on $Gr_k\mathbb{Q}_p^n$ is almost free.

Claim (b). The function f maps a (Haar) conull set into the free part of the action of $PGL_n(\mathbb{Q})$ on $Gr_l\mathbb{Q}_p^n$.

Here, if the countable group Γ acts on a standard Borel space X, then we let:

$$Fr(\Gamma \curvearrowright X) := \{x \in X : 1 \neq g \in \Gamma \implies gx \neq x\}$$

denote the *free part* of the action of Γ on X. If X carries a (not necessarily Γ-invariant) probability measure, then we say that $\Gamma \curvearrowright X$ is *almost free* if $Fr(\Gamma \curvearrowright X)$ is conull.

4.4.4. **Lemma** (essentially Lemma 5.1 of [Tho1]). *Suppose that $f : Gr_k\mathbb{Q}_p^n \to Gr_l\mathbb{Q}_p^n$ is a Borel homomorphism from $E^k_{SL_n\mathbb{Z}}$ to $E^l_{GL_n\mathbb{Q}}$. Then either f maps a conull set into a single $GL_n(\mathbb{Q})$-orbit, or there exists a conull $M \subset Gr_k\mathbb{Q}_p^n$ such that $f(M) \subset Fr(PGL_n(\mathbb{Q}) \curvearrowright Gr_l\mathbb{Q}_p^n)$.*

It is clear that the lemma establishes Claim (b); also Claim (a) follows by applying it in the case $l = k$ and f is the identity map on $Gr_k\mathbb{Q}_p^n$. Thomas stated Lemma 4.4.4 only in the case that $k = n - 1$, but he never used this detail in his argument. Due to its importance, we now summarize the proof.

Suppose, toward a contradiction, that f does not map a conull set into the free part for the action of $PGL_n(\mathbb{Q})$ on $Gr_l\mathbb{Q}_p^n$. Note that if there exists $1 \neq g \in PGL_n(\mathbb{Q})$ such that $gf(x) = f(x)$, then considering $f(x)$ as a linear subspace of $\bigwedge^l \mathbb{Q}_p^n$, we have that $f(x)$ is contained in an eigenspace for g. Using the ergodicity of the action of $SL_n(\mathbb{Z})$ on $Gr_k\mathbb{Q}_p^n$, one can argue that there exists a conull set of x such that $f(x)$ is contained in a *single* proper subspace $E < \bigwedge^l \mathbb{Q}_p^n$. By choosing an appropriate such E, one can argue that the group H of projective linear transformations induced on E by the action of $PGL_n(\mathbb{Q})$

does act freely on $Gr_1 E$. Hence Zimmer's superrigidity theorem can be applied, and a contradiction arises from the fact that H is contained in an algebraic group of strictly smaller dimension than that of $SL_n(\mathbb{R})$.

We now resume the proof of Theorem 4.4.3.

Claim. We may suppose that there exists an ergodic component $\Gamma_0 \curvearrowright X_0$ for the action $SL_n(\mathbb{Z}) \curvearrowright Gr_k \mathbb{Q}_p^n$ and a homomorphism $\phi : \Gamma_0 \to GL_n(\mathbb{Q})$ such that

$$(\phi, f) : \Gamma_0 \curvearrowright X_0 \longrightarrow GL_n(\mathbb{Q}) \curvearrowright Gr_l \mathbb{Q}_p^n$$

is a homomorphism of permutation groups.

In the proof, we will in fact produce a ϕ such that $\phi(\Gamma_0)$ is a subgroup of $SL_n(\mathbb{Z})$ of finite index.

Proof of claim. Using Claims (a) and (b) together, it is not difficult to see that we may apply Corollary 4.3.3 to suppose that there exists an ergodic component $\bar{\Gamma}_0 \curvearrowright X_0$ for $PSL_n(\mathbb{Z}) \curvearrowright Gr_k \mathbb{Q}_p^n$ and a homomorphism of permutation groups:

$$(\bar{\phi}, f) : \bar{\Gamma}_0 \curvearrowright X_0 \longrightarrow PGL_n(\mathbb{Q}) \curvearrowright Gr_l \mathbb{Q}_p^n$$

We wish to lift ϕ to a map $\Gamma_0 \to GL_n(\mathbb{Q})$, where Γ_0 is the preimage in $SL_n(\mathbb{Z})$ of $\bar{\Gamma}_0$.

First, suppose that $\bar{\phi}$ is not injective. In this case, by Margulis's theorem on normal subgroups (Theorem 8.1.2 of [Zim]), the kernel of $\bar{\phi}$ has finite index in $\bar{\Gamma}_0$. Hence, $\bar{\phi}$ has finite image and so passing to an ergodic subcomponent, we can suppose without loss of generality that $\bar{\phi} = 1$. This implies that f is $\bar{\Gamma}_0$-invariant and since $\bar{\Gamma}_0 \curvearrowright X_0$ is ergodic, f is almost constant. Hence, in this case f maps a conull set into a single $GL_n(\mathbb{Q})$-orbit, which is a contradiction.

Next, suppose that $\bar{\phi}$ is injective. In this case, we shall again make use of Margulis's results. The next lemma will be used in tandem with Lemma 3.7.2.

4.4.5. **Lemma**. *If $\Gamma_0 \leq SL_n(\mathbb{Z})$ is a finite index subgroup and $\phi : \Gamma_0 \to GL_n(\mathbb{Q})$ is a homomorphism, then there exists a finite index subgroup $\Lambda \leq \Gamma_0$ such that $\phi(\Lambda) \leq SL_n(\mathbb{Z})$ is a subgroup of finite index.*

Proof. Since Γ_0 is Kazhdan (see Theorem 1.5 of [Lub]), we have that $\Gamma_0' := [\Gamma_0, \Gamma_0]$ is a finite index subgroup of Γ_0 (see Corollary 1.29 of [Lub]). Now since $GL_n(\mathbb{Q})/SL_n(\mathbb{Q}) \cong \mathbb{Q}^\times$ is abelian, we have that:

$$\phi(\Gamma_0') \leq [GL_n(\mathbb{Q}), GL_n(\mathbb{Q})] \leq SL_n(\mathbb{Q})$$

(In fact, the latter $\leq$ is an equality.) Hence, replacing Γ_0 by Γ_0' if necessary, we may suppose without loss of generality that $\phi(\Gamma_0) \subset SL_n(\mathbb{Q})$. Repeating the proof of Lemma 3.7.2, after slightly adjusting ϕ if necessary, we may suppose that it extends to an automorphism of $SL_n(\mathbb{R})$ (the adjustment is by ϵ, in the notation of 3.7.2). It follows that $\phi(\Gamma_0)$ is again a lattice of $SL_n(\mathbb{R})$. Since $\phi(\Gamma_0) \subset SL_n(\mathbb{Q})$, by IX.4.14 of [Mar] we have that $\phi(\Gamma_0)$ is commensurable with $SL_n(\mathbb{Z})$. The lemma follows easily. $\square$

Although Lemma 4.4.5 has been stated so that it will be useful later on, for the present circumstances let us note that the same proof easily applies to the case of homomorphisms into $PGL_n(\mathbb{Q})$. In other words, replacing X_0 with a smaller ergodic component, we may suppose without loss of generality that $\bar{\phi}(\Gamma_0)$ is a subgroup of $PSL_n(\mathbb{Z})$ of finite index. Moreover, the proof of Lemma 3.7.2 shows that $\bar{\phi}$ lifts to a homomorphism $\phi : \Gamma_0 \to SL_n(\mathbb{Z})$. Since ϕ is a lifting, we easily obtain that $f(\gamma x) = \phi(\gamma) f(x)$ for all $\gamma \in \Gamma_0$, which completes the proof of the claim. $\dashv$

We now wish to maneuver into a situation where we can apply Theorem 3.7.1.

Claim. We may suppose that $f(X_0) \subset X_1$, where X_1 is an ergodic component for the action of $\phi(\Gamma_0)$ on $Gr_l \mathbb{Q}_p^n$.

Proof of claim. Let $Z_1, \ldots, Z_m$ be the ergodic components for the action of $\phi(\Gamma_0)$ on $Gr_l \mathbb{Q}_p^n$. Now, each $f^{-1}(Z_i)$ is Γ_0-invariant, and since $\Gamma_0 \curvearrowright X_0$ is ergodic, exactly one of the $f^{-1}(Z_i)$ is conull. Deleting a null subset of X_0, we may suppose that $f(X_0) \subset Z_i$, as desired. $\dashv$

Finally, we may apply Theorem 3.7.1 to conclude that $l = k$ or $l = n - k$, contradicting our initial hypothesis. This completes the proof of Theorem 4.4.3. $\square$

Chapter 5

The space of completed groups

In this chapter, we shall remind the reader of the definition of the completion of a p-local torsion-free abelian group, a notion which was used by Kurosh and Malcev in their classification of torsion-free abelian groups of finite rank. This will allow us to replace the space $R(n, p)$ of p-local torsion-free abelian groups of rank n with the more analytic space $\mathcal{M}(n, p)$ of $\mathbb{Z}_p$-submodules of $\mathbb{Q}_p^n$.

We shall then observe that each completed group M can be decomposed as the direct sum of a "vector space part" and a "lattice part." The map which sends M to its vector space part provides a relationship between $\mathcal{M}(n, p)$ and the k-Grassmann spaces $Gr_k\mathbb{Q}_p^n$. This will allow us to prove in Section 5.4 that the isomorphism relation on the space $R(n, p, k)$ (of p-local torsion-free abelian groups of divisible rank k) is Borel bireducible with an equivalence relation $E^k_{\cong}$ on $Gr_k\mathbb{Q}_p^n$ which is closely related to the orbit equivalence relation $E^k_{GL_n\mathbb{Q}}$.

Finally, in Section 5.5 we shall combine the work of this chapter with Theorem 4.4.3 to establish a large fragment of Theorem A, namely that the isomorphism relations on $R(n, p, k)$ and $R(n, p, l)$ are Borel incomparable whenever $l \neq k$ and $l \neq n - k$.

5.1 Completion of local torsion-free abelian groups

We begin by collecting several definitions and facts surrounding the notion of the completion of a local torsion-free abelian group. Nearly all of this section is gleaned from Section 93 of [Fuc]; some of the more technical proofs will be given in Appendix 7.1.

Recall that $A \in R(n)$ is p-local if it is q-divisible for all primes $q \neq p$, and that $R(n, p)$ denotes the subset of $R(n)$ consisting of just the p-local groups. For any such group A, we define its p-adic completion by:

$$\Lambda(A) := A \otimes \mathbb{Z}_p$$

That is, $\Lambda(A)$ is just the set of $\mathbb{Z}_p$-linear combinations of elements of A, where A is considered as a subset of $\mathbb{Q}_p^n$. (Note that we have slightly altered our notation from that of Section 2.1, where we wrote $\Lambda_p(A)$ rather than $\Lambda(A)$.) On $R(n,p)$, Λ takes values in the standard Borel space $\mathcal{M}(n,p)$ of $\mathbb{Z}_p$-submodules of $\mathbb{Q}_p^n$ with $\mathbb{Z}_p$-rank exactly equal to n. In fact, as we shall prove in Appendix 7.1, Λ is a $GL_n(\mathbb{Q})$-preserving *bijection* between $R(n,p)$ and $\mathcal{M}(n,p)$.

Suppose now that $M \in \mathcal{M}(n,p)$. In Appendix 7.1 (or see Exercise 93.3 of [Fuc]), we shall prove that M can be decomposed as a direct sum:

$$M = V_M \oplus L \tag{5.1.1}$$

where V_M is a vector subspace of $\mathbb{Q}_p^n$ and L is a free $\mathbb{Z}_p$-submodule of $\mathbb{Q}_p^n$. The vector subspace part V_M is uniquely determined by M, but there are many possible complementary submodules L. In any such decomposition, we will have that $\operatorname{rank} L + \dim V = n$.

5.1.2. **Definition**. For $A \in R(n,p)$, the *divisible rank* of A is the dimension of $V_{\Lambda(A)}$.

We let $R(n,p,k)$ denote the subspace of $R(n,p)$ consisting of those groups of divisible rank exactly k. Then we have that Λ is a $GL_n(\mathbb{Q})$-preserving bijection between $R(n,p,k)$ and the space $\mathcal{M}(n,p,k)$ consisting of those modules $M \in \mathcal{M}(n,p)$ with $\dim V_M = k$. Thus, we have the following result:

5.1.3. **Proposition**. *The isomorphism relation $\cong^k_{n,p}$ on $R(n,p,k)$ is Borel equivalent to the orbit equivalence relation induced by the action of $GL_n(\mathbb{Q})$ on $\mathcal{M}(n,p,k)$.*

5.2 Lattices in p-adic vector spaces

Let $M \in \mathcal{M}(n,p)$ be a $\mathbb{Z}_p$-submodule of $\mathbb{Q}_p^n$. In our study of $Gr_k\mathbb{Q}_p^n$, we have already given a great deal of attention to the vector space part V_M of M. In this section we begin to understand the complementary module L. Since L is a free $\mathbb{Z}_p$-submodule of $\mathbb{Q}_p^n$, it can be regarded as a lattice in its linear span W.

5.2.1. **Definition**. Let K be a discrete valuation field, and let R denote its ring of integers. If V is an l-dimensional vector space over K, then a *lattice* of V is a free R-submodule of V of rank exactly l. Equivalently, a lattice of V is the R-span of l linearly independent elements of V. We denote the set of lattices of V by $\mathcal{L}(V)$.

Observe that there are only countably many lattices in $\mathbb{Q}_p^l$. Indeed, by the discussion in Section II.1.1 of [Ser], since $\mathbb{Q}$ is a dense subfield of $\mathbb{Q}_p$ the map $L \mapsto L \otimes \mathbb{Z}_p$ is a bijection between the lattices of $\mathbb{Q}^l$ (with respect to the p-adic valuation on $\mathbb{Q}$) and the lattices of $\mathbb{Q}_p^l$. Hence, any lattice of $\mathbb{Q}_p^l$ may be expressed as the $\mathbb{Z}_p$-span of l linearly independent elements of $\mathbb{Q}^l$.

We shall say a great deal more about the structure of the space of lattices of $\mathbb{Q}_p^l$ in the next chapter.

5.3 Decomposition of the space of completed groups

Fix $V \in Gr_k\mathbb{Q}_p^n$ and let $M \in \mathcal{M}(n,p,k)$ be an arbitrary $\mathbb{Z}_p$-submodule of $\mathbb{Q}_p^n$ such that $V_M = V$. If W is any *complementary* subspace of V, meaning that $V \cap W = 0$ and $V \oplus W = \mathbb{Q}_p^n$, then M can always be written uniquely as $V \oplus L$ where $L < W$. Hence, the set:

$$\{M \in \mathcal{M}(n,p,k) : V_M = V\}$$

is in a natural bijection with the lattices of W.

Since there are only countably many lattices of W, the map $M \mapsto V_M$ that sends M to its vector space part is countable-to-one. By Exercise 18.14 of [Kec2], any countable-to-one Borel function between standard Borel spaces admits a Borel section. It follows easily that there exists a Borel bijection $f : Gr_k\mathbb{Q}_p^n \times \mathcal{L}(\mathbb{Q}_p^{n-k}) \to \mathcal{M}(n,p,k)$ satisfying $V_{f(V,L)} = V$ for all V, L.

In this section we will define a *particular* such bijection; among other things, this will provide us with a canonical embedding of $Gr_k\mathbb{Q}_p^n$ into $\mathcal{M}(n,p,k)$.

5.3.1. **Definition**. If $V \in Gr_k\mathbb{Q}_p^n$, let V^c be the unique complementary subspace of V spanned by basis vectors $e_{j_1}, \ldots, e_{j_{n-k}}$ and with the following properties:

(a) The $\mathbb{Z}_p$-span of $(V \cap \mathbb{Z}_p^n) \cup (\mathbb{Z}_p e_{j_1} \oplus \cdots \oplus \mathbb{Z}_p e_{j_{n-k}})$ is all of $\mathbb{Z}_p^n$.

(b) $\langle j_i \rangle$ is the lexicographically *greatest* sequence satisfying condition (a).

We call V^c the *canonical complementary subspace* of V.

We now discuss how to find and identify such a sequence $\langle j_i \rangle$; in particular we will show that V^c exists. The key is that we can write V as the column space of a $n \times k$ matrix A satisfying:

- Each row of the $k \times k$ identity matrix appears as a row of A (call these the *pivot* rows); and
- Every entry of A is in $\mathbb{Z}_p$.

(To obtain such a matrix, begin with an arbitrary matrix whose column space is V. Rescale the first column so that all entries are p-adic integers and at least one entry is 1. Then, use this 1 to zero out the other entries in its row. Repeat this for the second column, *etc*.) It is easily seen that the sequence j_i of indices of the *non-pivot* rows of A satisfies (a) above. Our requirement that $\langle j_i \rangle$ is lex-greatest amounts to the more natural assertion that the sequence of indices of the pivot rows of A is lex-least.

For example, we have already seen in equation (3.6.2) that if $V_0 = \mathbb{Q}_p e_1 \oplus \cdots \oplus \mathbb{Q}_p e_k$ then any $V \in (K_{p^t})V_0$ can be written as the column space of a matrix of the form $\begin{bmatrix} I_k \\ v \end{bmatrix}$, where the entries of v are in $\mathbb{Z}_p$. It follows that the canonical complementary subspace for V is $V^c = \mathbb{Q}_p e_{k+1} \oplus \cdots \oplus \mathbb{Q}_p e_n$.

Now, for $V \in Gr_k \mathbb{Q}_p^n$ and a lattice $L < \mathbb{Q}_p^{n-k}$, let $L[V]$ be the isomorphic copy of L inside V^c induced by following the obvious map (that is, the linear map defined by the abuse of notation $e_i \mapsto e_{j_i}$). We define the *adjoining* of V and L to be:

$$(V, L) := V \oplus L[V]$$

It is clear from our construction that the adjoining operation defines a Borel bijection:

$$(\cdot, \cdot) : Gr_k \mathbb{Q}_p^n \times \mathcal{L}(\mathbb{Q}_p^{n-k}) \to \mathcal{M}(n, p, k)$$

It may be necessary to explain why this map is Borel. First, it is possible to argue using the technique of Section 4 of [Tho3] that there exists a Borel map from $\mathcal{M}(n,p,k)$ to $(\mathbb{Q}_p^n)^k \times (\mathbb{Q}_p^n)^{n-k}$ which sends $V \oplus L$ to a sequence $v_1,\dots,v_k,w_1,\dots,w_{n-k}$, where $v_1,\dots,v_k$ is a $\mathbb{Q}_p$-basis for V and $w_1,\dots,w_{n-k}$ is a $\mathbb{Z}_p$-basis for L. Next, there is clearly a Borel map which given $v_1,\dots,v_k,w_1,\dots,w_{n-k}$ produces a basis $w_1',\dots,w_{n-k}'$ for the unique $L' \in \mathcal{L}(\mathbb{Q}_p^{n-k})$ such that $(V,L') = V \oplus L$. Finally, by Exercise 12.14 of [Kec2], there exists a Borel map $(\mathbb{Q}_p^n)^k \to Gr_k\mathbb{Q}_p^n$ sending a sequence of vectors to its linear span, and a Borel map $(\mathbb{Q}_p^n)^{n-k} \to \mathcal{L}(\mathbb{Q}_p^{n-k})$ sending a sequence of vectors to its $\mathbb{Z}_p$-span.

5.4 Relations on the completed groups

We now take up the task of expressing the isomorphism and quasi-isomorphism relations on $R(n,p,k)$ as relations on $Gr_k\mathbb{Q}_p^n$. If we are to have any hope of applying the results of Chapter 4, these relations should have something to do with $E^k_{GL_n\mathbb{Q}}$. The following result of Thomas already addresses the case of the quasi-isomorphism relation.

5.4.1. **Lemma** (Thomas). *The quasi-isomorphism relation $\sim^k_{n,p}$ on the space $R(n,p,k)$ is Borel bireducible with the orbit equivalence relation $E^k_{GL_n\mathbb{Q}}$ induced by the action of $GL_n(\mathbb{Q})$ on $Gr_k\mathbb{Q}_p^n$.*

Proof. By Theorem 4.3 of [Tho1], we have $A \sim^k_{n,p} B$ iff there exists $g \in GL_n(\mathbb{Q})$ such that $V_{\Lambda(B)} = g(V_{\Lambda(A)})$. Hence, the map $f(A) = V_{\Lambda(A)}$ is a Borel reduction from $\sim^k_{n,p}$ to $E^k_{GL_n\mathbb{Q}}$. Next, since Λ is bijective and $M \mapsto V_M$ is countable-to-one, we have that f is countable-to-one. By Exercise 18.14 of [Kec2], f admits a Borel section σ, and this map is clearly a Borel reduction from $E^k_{GL_n\mathbb{Q}}$ to $\sim^k_{n,p}$. $\square$

Thus, Theorem 4.4.2 translates to the following statement about the quasi-isomorphism relation, which is interesting in its own right.

5.4.2. **Corollary**. *Let $n \geq 4$ and let $k,l < n$ be such that l is neither k nor $n-k$. Then the quasi-isomorphism relations $\sim^k_{n,p}$ and $\sim^l_{n,p}$ are Borel incomparable.*

Our next target is the isomorphism equivalence relation, which we shall show is Borel bireducible with an equivalence relation that is *contained* in $E^k_{GL_n\mathbb{Q}}$. This will be accomplished by investigating the "canonical" copy of $Gr_k\mathbb{Q}_p^n$ in $\mathcal{M}(n,p,k)$. Letting L_0 denote the standard lattice $\mathbb{Z}_p^{n-k}$ of $\mathbb{Q}_p^{n-k}$, we put $Y_0 := \{(V, L_0) : V \in Gr_k\mathbb{Q}_p^n\}$. First, we have the following characterization:

5.4.3. **Proposition**. *Y_0 is precisely the orbit $(SL_n\mathbb{Z}_p)M_0$, where M_0 is the module:*

$$M_0 := (V_0, L_0) = (\mathbb{Q}_p e_1 \oplus \cdots \oplus \mathbb{Q}_p e_k) \oplus (\mathbb{Z}_p e_{k+1} \oplus \cdots \oplus \mathbb{Z}_p e_n)$$

Proof. We must show that for any $g \in SL_n(\mathbb{Z}_p)$, we have $g(V_0, L_0) \in Y_0$. Then, since the action of $SL_n(\mathbb{Z}_p)$ on $Gr_k\mathbb{Q}_p^n$ is transitive, Y_0 must be precisely the orbit $(SL_n\mathbb{Z}_p)(V_0, L_0)$.

Suppose first that $g(V_0, L_0)$ is of the form (V_0, L), in other words suppose that $gV_0 = V_0$. In this case, g acts on the quotient $\mathbb{Q}_p^n/V_0$ (in the basis represented by $e_{k+1}, \ldots, e_n$) via its $(n-k) \times (n-k)$ lower right-hand corner g^c. It follows easily that $g(V_0, L_0) = (V_0, g^c L_0)$. Since the entries of g^c lie in $\mathbb{Z}_p$, we clearly have $g^c L_0 = L_0$. Hence, $g(V_0, L_0) = (V_0, L_0)$ is an element of Y_0.

Now suppose that $g(V_0, L_0) = (V, L)$ is arbitrary. It suffices to show there exists $g_1 \in SL_n(\mathbb{Z}_p)$ such that $g_1(V_0, L) = (V, L)$, for then, $g_1^{-1}g(V_0, L_0) = (V_0, L)$ and we are in the previous case. Permuting the standard basis if necessary, we can suppose that $V = \operatorname{col}\begin{bmatrix} I_k \\ v \end{bmatrix}$ where the entries of v are in $\mathbb{Z}_p$. It follows easily that

$$g_1 := \begin{bmatrix} I_k & 0 \\ v & I_{n-k} \end{bmatrix}$$

satisfies our requirements. $\square$

While Y_0 is not invariant for the action of $GL_n(\mathbb{Q})$, the last proposition shows in particular that Y_0 is invariant for the action of the subgroup $SL_n(\mathbb{Z}_{(p)})$. However, it is not difficult to see that $\cong^k_{n,p}|_{Y_0}$ is not induced by the action of any subgroup of $GL_n(\mathbb{Q})$. Indeed, let $g = \operatorname{diag}(p, 1, \ldots, 1)$, where $\operatorname{diag}(d_1, \ldots, d_n)$ denotes the diagonal matrix with diagonal entries $a_{ii} = d_i$. One can easily find an $x \in Fr(PGL_n\mathbb{Q} \curvearrowright Gr_k\mathbb{Q}_p^n)$ such that $g(x, L_0) \in Y_0$. Hence any subgroup of $GL_n\mathbb{Q}$ which induces $\cong^k_{n,p}|Y_0$ must include some $h = ag$. But one can also find an $x \in Gr_k\mathbb{Q}_p^n$ such that $ag(x, L_0) \notin Y_0$.

5.4.4. **Proposition**. *Y_0 is a complete section for the orbit equivalence relation induced by the action of $GL_n\mathbb{Q}$ on $\mathcal{M}(n,p,k)$.*

Here, a subset $A \subset X$ is said to be a *complete section* for the equivalence relation E on X if A meets every E-class. A countable Borel equivalence relation is always bireducible with its restriction to any complete Borel section. Hence, using the obvious bijection of Y_0 with $Gr_k\mathbb{Q}_p^n$, we obtain that the orbit equivalence relation induced by the action of $GL_n(\mathbb{Q})$ on $\mathcal{M}(n,p,k)$ is bireducible with the following equivalence relation on $Gr_k\mathbb{Q}_p^n$:

5.4.5. **Definition**. *For $V, V' \in Gr_k\mathbb{Q}_p^n$, we define that $V\ E_{\cong}^k\ V'$ if there exists $g \in GL_n(\mathbb{Q})$ such that $(V, L_0) = g(V', L_0)$.*

Proof of Proposition 5.4.4. We must prove that for every $(V, L) \in \mathcal{M}(n,p,k)$, there exists $g \in GL_n(\mathbb{Q})$ and V' such that $g(V, L) = (V', L_0)$. (Of course, V' will be gV.) Permuting the standard basis if necessary, we may suppose that $V = \operatorname{col}\begin{bmatrix} I_k \\ v \end{bmatrix}$ where the entries of v are in $\mathbb{Z}_p$. Recall that L has a rational basis over $\mathbb{Z}_p$, and hence there exists $h \in GL_{n-k}(\mathbb{Q})$ such that $hL = L_0$. Now, choose a matrix j with rational entries such that the entries of $j + hv$ lie in $\mathbb{Z}_p$, and let:

$$g = \begin{bmatrix} I_k & 0 \\ j & h \end{bmatrix}$$

Then both V and $gV = \operatorname{col}\begin{bmatrix} I_k \\ j + hv \end{bmatrix}$ each have the canonical complementary subspace $V_1 = \mathbb{Q}_p e_{k+1} \oplus \cdots \oplus \mathbb{Q}_p e_n$. One now easily computes that $g(V, L) = (gV, hL) = (gV, L_0)$, as desired. $\square$

The following lemma summarizes the work of this section.

5.4.6. **Lemma**. *The equivalence relation $E_{\cong}^k$ on the space $Gr_k\mathbb{Q}_p^n$ satisfies the containments:*

$$E_{SL_n\mathbb{Z}}^k \ \subset\ E_{\cong}^k \ \subset\ E_{GL_n\mathbb{Q}}^k$$

Moreover, $E_{\cong}^k$ is Borel bireducible with the isomorphism relation $\cong_{n,p}^k$ on $R(n,p,k)$.

Proof. To see that $E_{SL_n\mathbb{Z}}^k \subset E_{\cong}^k$, suppose that $g \in SL_n(\mathbb{Z})$ and $gV = V'$. Then by Proposition 5.4.3, we have $g(V, L_0) = (V', L_0)$ and so $V\ E_{\cong}^k\ V'$. To see that $E_{\cong}^k \subset E_{GL_n\mathbb{Q}}^k$, notice that if $g \in GL_n(\mathbb{Q})$ and $g(V, L_0) = (V', L_0)$, then clearly $gV = V'$.

We have already remarked that by Proposition 5.4.4, the relation $E^k_{\cong}$ is Borel bireducible with the orbit equivalence relation induced by the action of $GL_n(\mathbb{Q})$ on $\mathcal{M}(n,p,k)$. But since Λ witnesses that the latter relation is Borel equivalent to $\cong^k_{n,p}$, we conclude that $E^k_{\cong}$ is Borel bireducible with $\cong^k_{n,p}$. $\square$

5.5 An application to torsion-free abelian groups

Using the results of the last section, we can already prove:

5.5.1. **Theorem A, case 1**. *Let $n \geq 4$ and $1 \leq k, l \leq n-1$, and suppose that $l \neq k$ and $l \neq n-k$. Then $\cong^k_{n,p}$ is Borel incomparable with $\cong^l_{n,p}$.*

Proof. By the second clause of Lemma 5.4.6, it suffices to prove that the relations $E^k_{\cong}$ and $E^l_{\cong}$ are Borel incomparable. So, suppose that $f : Gr_k\mathbb{Q}_p^n \to Gr_l\mathbb{Q}_p^n$ is a Borel reduction from $E^k_{\cong}$ to $E^l_{\cong}$. Using the containments described in Lemma 5.4.6, we clearly have that f is a weak Borel reduction from $E^k_{SL_n\mathbb{Z}}$ to $E^l_{GL_n\mathbb{Q}}$, but this contradicts Theorem 4.4.3. $\square$

This argument evidently relies on the fact that the orbit equivalence relations induced by the action of $GL_n(\mathbb{Q})$ on $Gr_k\mathbb{Q}_p^n$ and on $Gr_l\mathbb{Q}_p^n$ are highly incompatible. We remark that the same argument will not work in the case that $l = n-k$. For, we have:

5.5.2. **Proposition**. *The actions $GL_n(\mathbb{Q}) \curvearrowright Gr_k(\mathbb{Q}_p^n)$ and $GL_n(\mathbb{Q}) \curvearrowright Gr_{n-k}\mathbb{Q}_p^n$ are isomorphic as permutation groups. Indeed, the map $(-T, \perp)$ defined by $g \mapsto g^{-T}$ and $V \mapsto V^{\perp}$ is such an isomorphism.*

It follows also that the quasi-isomorphism relations $\sim^k_{n,p}$ on $R(n,p,k)$ and $\sim^{n-k}_{n,p}$ on $R(n,p,n-k)$ are Borel equivalent. In the next chapter, we shall see how to get around this difficulty.

Chapter 6

The proofs of the main theorems

We will begin this chapter with the definition of the type of a lattice, a notion which will help us understand the equivalence relations $E^k_{\cong}$ that we have defined on $Gr_k\mathbb{Q}^n_p$. We will then proceed with the first half of Theorem B, namely that if $n \geq 3$ and $k \leq n-2$, then the isomorphism relation $\cong^k_{n,p}$ on $R(n,p,k)$ is not Borel reducible to the quasi-isomorphism relation $\sim^k_{n,p}$ on $R(n,p,k)$. Actually, we shall prove instead that $E^k_{\cong} \not\leq_B E^k_{GL_n\mathbb{Q}}$, which is equivalent by Lemmas 5.4.6 and 5.4.1.

Moving on to Theorem A, first recall that we have already shown in Theorem 5.5.1 that $\cong^k_{n,p}$ and $\cong^l_{n,p}$ are Borel incomparable when $l \neq k$ and $l \neq n-k$. To show that $\cong^k_{n,p}$ is Borel incomparable with $\cong^{n-k}_{n,p}$ will require the additional technical step of replacing the space $\mathcal{M}(n,p,l)$ with a closely related space $\mathcal{M}^*(n,p,l)$ on which $PGL_n(\mathbb{Q})$ acts almost freely. We shall prove that the orbit equivalence relation $(\cong^l_{n,p})^*$ induced by the action of $GL_n(\mathbb{Q})$ on this space is Borel bireducible with the isomorphism relation $\cong^l_{n,p}$ on $R(n,p,l)$. Then, we will prove that $E^k_{\cong}$ is not Borel reducible to $(\cong^{n-k}_{n,p})^*$.

6.1 The type of a lattice

In this section, we introduce a key invariant on the space $\mathcal{L}(\mathbb{Q}^l_p)$ of lattices of $\mathbb{Q}^l_p$, which will be used in the proofs of Theorems A and B.

6.1.1. **Definition**. For a lattice $L \in \mathcal{L}(\mathbb{Q}^l_p)$, let A be any matrix such that L is equal to the $\mathbb{Z}_p$-span of the columns of A. Then the *type* of L, denoted $\mathrm{tp}(L)$, is the reduction modulo l of $\nu_p(\det A)$, where ν_p denotes the p-adic valuation on $\mathbb{Q}^*_p$.

We must first verify that $\mathrm{tp}(L)$ is independent of the choice of the matrix A. Indeed, if the $\mathbb{Z}_p$-span of the columns of A' is the same as that of A, then A' can be obtained from

A by column operations over $\mathbb{Z}_p$. Hence there exists $s \in GL_n(\mathbb{Z}_p)$ such that $As = A'$, and it follows that $\nu_p(\det A) = \nu_p(\det A')$. Similarly, one can easily deduce the following fact.

6.1.2. **Proposition**. *If $s \in GL_l(\mathbb{Q}_p)$ then* $\operatorname{tp}(sL) \equiv \nu_p(\det s) + \operatorname{tp}(L)$, *modulo l.*

In section 5.4, it will be necessary to work with the notion of the class of a lattice.

6.1.3. **Definition**. Let K and V be as in Definition 5.2.1. If L is a lattice of V, then its *class* Λ is the set of all scalar multiples $\{aL : a \in K^*\}$.

Proposition 6.1.2 easily implies that the type of a lattice depends only on its class. Hence, we may define the type of a lattice class as the type of any of its members.

6.1.4. **Remark**. In the case $l = 2$, there is a natural graph structure on the set of lattice classes. Join Λ and Λ' by an edge iff there are $L \in \Lambda$ and $L' \in \Lambda'$ such that L' is a maximal proper sublattice of L (or vice versa). The resulting graph is the $(p+1)$-regular tree and the types correspond to the colors in a 2-coloring of the tree. See the cover of the most recent printing of [Ser] for a picture in the case that $p = 2$.

6.2 The proof of Theorem B

We now attack the main case of Theorem B, which we recall was outlined in Section 2.3. The argument will closely follow the proof of Theorem 4.4.3.

6.2.1. **Theorem B, part 1**. *Suppose that $n \geq 3$, and let $k \leq n-2$. Then $E^k_{\cong} \not\leq_B E^k_{GL_n\mathbb{Q}}$.*

As mentioned in Section 2.2, this result also gives new examples of countable Borel equivalence relations $E \subset F$ such that $E \not\leq_B F$.

Proof. Suppose that $f : Gr_k\mathbb{Q}_p^n \to Gr_k\mathbb{Q}_p^n$ is a Borel reduction from $E^k_{\cong}$ to $E^k_{GL_n\mathbb{Q}}$. Then f is a weak Borel reduction from $E^k_{SL_n\mathbb{Z}}$ to $E^k_{GL_n\mathbb{Q}}$. Using the arguments in the proof of Theorem 4.4.3, we may suppose there is an ergodic component $\Gamma_0 \curvearrowright X_0$ for $SL_n(\mathbb{Z}) \curvearrowright Gr_k\mathbb{Q}_p^n$ and a homomorphism $\phi : \Gamma_0 \to GL_n(\mathbb{Q})$ such that $(\phi, f) : \Gamma_0 \curvearrowright X_0 \longrightarrow GL_n(\mathbb{Q}) \curvearrowright Gr_k\mathbb{Q}_p^n$ is a homomorphism of permutation groups.

By Lemma 4.4.5, we may replace $\Gamma_0 \curvearrowright X_0$ by an ergodic subcomponent to suppose that $\phi(\Gamma_0) \leq SL_n(\mathbb{Z})$ is a subgroup of finite index. Shortly, we shall argue that we can suppose that $f(X_0)$ is an ergodic component for the action of $\phi(\Gamma_0)$. However, since our argument is timing-sensitive, we must first reduce to the case that X_0 is especially simple. This will be used to simplify the computations at the end of the proof.

By Proposition 3.6.3, we may replace X_0 with an ergodic subcomponent to suppose that $\Gamma_0 \curvearrowright X_0$ is a principle congruence component. By definition this means that we may assume that $\Gamma_0 = \Gamma_{p^t}$ for some $t \geq 0$ and that X_0 is equal, modulo a null set, to a K_{p^t}-orbit.

Claim. We may suppose that the domain X_0 of f is equal, modulo a null set, to the *particular* ergodic component $Z_0 = (K_{p^t})V_0$.

Recall that the ergodic component Z_0 was described in equation (3.6.2).

Proof of claim. By Proposition 3.4.2, $SL_n(\mathbb{Z})$ acts transitively on the K_{p^t}-orbits. Hence, there exists $\gamma \in SL_n(\mathbb{Z})$ such that $\gamma Z_0 = X_0$, modulo a null set. Consider the map $f'(x) = f(\gamma x)$. By Lemma 5.4.6, $SL_n(\mathbb{Z})$ preserves $E^k_{\cong}$, and so we still have that f' is a Borel reduction from $E^k_{\cong}$ to $\cong^k_{n,p}$. Clearly, the domain of f' is as desired. Moreover, it is easily checked that (ϕ', f') is a permutation group homomorphism, where $\phi'(g) = \phi(\gamma g \gamma^{-1})$. Since $\gamma \in SL_n(\mathbb{Z})$, we have retained that $\phi'(\Gamma_0) \subset SL_n(\mathbb{Z})$. Hence, by replacing (ϕ, f) with (ϕ', f') the proof of the claim is complete. $\dashv$

Now, by the ergodicity of $\Gamma_0 \curvearrowright X_0$, we may delete a null subset of X_0 to suppose that $f(X_0)$ is an ergodic component for the action of $\phi(\Gamma_0)$. By Theorem 3.7.1, there exists $h \in GL_n(\mathbb{Q})$ such that $f(x) = hx$ for all $x \in X_0$.

Claim. We may suppose that $h = 1$ and so $f(x) = x$ for all $x \in X_0$.

Proof of claim. We want to replace $f(x) = hx$ with $f' = h^{-1}f$, but we must check that f' retains all properties of f that we have accumulated in previous claims. Since $h \in GL_n(\mathbb{Q})$, f' is still a Borel reduction from $E^k_{\cong}$ to $E^k_{GL_n\mathbb{Q}}$. Moreover, it is easily checked that (ϕ', f') is a permutation group homomorphism, where $\phi' = h^{-1}\phi h$. We can thus replace (ϕ, f) with (ϕ', f') to complete the proof of the claim. $\dashv$

Before proceeding to the final contradiction, we give a brief outline. If indeed $f(x) = x$ is a Borel reduction from the restriction to X_0 of $E^k_{\cong}$ to that of $E^k_{GL_n\mathbb{Q}}$, then whenever $x, gx \in X_0$ and $g \in GL_n(\mathbb{Q})$, we will have $x(E^k_{\cong})gx$. If we additionally suppose that $x, gx \in Z_0$, then by the observations of Section 5.3, x and gx each have the canonical complementary subspace $V_1 = \mathbb{Q}_p e_{k+1} \oplus \cdots \oplus \mathbb{Q}_p e_n$,. In order to obtain a contradiction, we shall choose the matrix g so that it acts nontrivially on V_1.

Turning to the details, let $g = \operatorname{diag}(1, \ldots, 1, p)$, where $\operatorname{diag}(d_1, \ldots, d_n)$ denotes the diagonal matrix with $a_{ii} = d_i$. Using equation (3.6.2) one easily checks that $gZ_0 \subset Z_0$, and since $X_0 = Z_0$ modulo a null set, we have that gX_0 is almost contained in X_0. Together with Lemma 4.4.4, this implies that we can choose $x \in Fr(PGL_n(\mathbb{Q}) \curvearrowright Gr_k\mathbb{Q}_p^n)$ in such a way that $x, gx \in X_0 \cap Z_0$. Then $x(E^k_{\cong})gx$, and so Definition 5.4.5 gives $h \in GL_n(\mathbb{Q})$ such that:

$$h(x, L_0) = (gx, L_0)$$

Now, $hx = gx$ and since we have chosen x so that it is not fixed by any element of $PGL_n(\mathbb{Q}) \setminus \{1\}$, there exists $a \in \mathbb{Q}_p^*$ such that $h = ag$. Since $x, gx \in Z_0$, each has the canonical complementary subspace $V_1 = \mathbb{Q}_p e_{k+1} \oplus \cdots \oplus \mathbb{Q}_p e_n$. Since h is diagonal, it clearly acts on V_1 via its $(n-k) \times (n-k)$ lower right-hand corner h^c. It follows that $h(x, L_0) = (hx, h^c L_0)$ and hence h^c stabilizes L_0. But it is readily seen that $\nu_p(\det h^c) \equiv 1 \mod (n-k)$, so Proposition 6.1.2 implies that h^c does not stabilize L_0, a contradiction. $\square$

6.3 The almost free space $\mathcal{M}^*$

In Section 6.4, we will prove that the isomorphism relation on $R(n, p, k)$ is not Borel reducible to the isomorphism relation on $R(n, p, n-k)$. In order to apply Corollary 4.3.3, it will be necessary to prove that $\cong^{n-k}_{n,p}$ is Borel bireducible with an orbit equivalence relation induced by an almost free action. We can realize $\cong^{n-k}_{n,p}$ as the orbit equivalence relation induced by the action of $GL_n(\mathbb{Q})$ on $\mathcal{M}(n, p, k)$, and in order to obtain a free action we should then consider the action of $PGL_n(\mathbb{Q})$. Unfortunately, $PGL_n(\mathbb{Q})$ does not act on $\mathcal{M}(n, p, k)$! Instead, we must work with the action of $PGL_n(\mathbb{Q})$ on the space $\mathcal{M}^*(n, p, l)$ of equivalence classes $[M] = \{aM : a \in \mathbb{Q}_p^*\}$, where $M \in \mathcal{M}(n,p,l)$.

6.3.1. **Proposition**. *The set $\mathcal{M}^*(n,p,l)$ is a standard Borel space, with its quotient Borel structure.*

Proof. It is sufficient to check that the equivalence relation E on $\mathcal{M}(n,p,l)$, which identifies M and M' iff $[M] = [M']$, is a smooth relation. First, recall that any $M \in \mathcal{M}(n,p,l)$ may be uniquely represented as $V \oplus L$, where L is contained in the canonical complementary subspace V^c of V. We then define $\sigma(V \oplus L) = V \oplus (p^t L)$, where t is the least integer such that $p^t L \subset V^c \cap \mathbb{Z}_p^n$. It is not hard to see, using the methods described at the end of Section 5.3, that σ is Borel. Clearly, σ is a Borel reduction from E to the equality relation on $\mathcal{M}(n,p,l)$. □

Now, we may let $(\cong_{n,p}^l)^*$ be the orbit equivalence relation induced by the action of $GL_n(\mathbb{Q})$ on $\mathcal{M}^*(n,p,l)$.

6.3.2. **Proposition**. *The equivalence relation $(\cong_{n,p}^l)^*$ is Borel bireducible with $\cong_{n,p}^l$.*

Proof. The map $M \mapsto [M]$ is clearly a Borel reduction from $\cong_{n,p}^l$ to $(\cong_{n,p}^l)^*$. On the other hand, the map σ described in the proof of Proposition 6.3.1 is clearly a Borel section of $M \mapsto [M]$, and it follows that σ is a Borel reduction from $(\cong_{n,p}^l)^*$ to $\cong_{n,p}^l$. □

The advantage of working with the space $\mathcal{M}^*(n,p,l)$ is that we can establish the following variant of Lemma 4.4.4.

6.3.3. **Lemma**. *Suppose that $f : Gr_k\mathbb{Q}_p^n \to \mathcal{M}^*(n,p,l)$ is a weak Borel reduction from $E_{SL_n\mathbb{Z}}^k$ to $(\cong_{n,p}^l)^*$. Then there exists a conull subset $M \subset Gr_k\mathbb{Q}_p^n$ such that:*

$$f(M) \subset Fr(PGL_n(\mathbb{Q}) \curvearrowright \mathcal{M}^*(n,p,l))$$

Proof. If $[M] = [M']$, it is clear that $V_M = V_{M'}$ (see the notation of (5.1.1)). Hence the map $[M] \mapsto V_M$ is well-defined, and we let $\bar{f} : Gr_k\mathbb{Q}_p^n \to Gr_l\mathbb{Q}_p^n$ denote the composition of this map with f. Clearly, $\bar{f}$ is a weak Borel reduction from $E_{SL_n\mathbb{Z}}^k$ to $E_{GL_n\mathbb{Q}}^{n-k}$. We claim that there exists a conull subset $M \subset Gr_k\mathbb{Q}_p^n$ such that:

$$\bar{f}(M) \subset Fr(PGL_n(\mathbb{Q}) \curvearrowright Gr_l\mathbb{Q}_p^n)$$

If this is not the case, then Theorem 4.4.4 implies that there exists a conull subset $M' \subset Gr_l\mathbb{Q}_p^n$ such that $\bar{f}(M')$ is contained in a single $GL_n(\mathbb{Q})$-orbit of $Gr_l\mathbb{Q}_p^n$. Since f is countable-to-one, it follows that $f(M')$ is contained in a countable set, contradicting that the Haar measure is nonatomic.

Now, for $x \in M$, we have that $\bar{f}(x) \in Fr(PGL_n(\mathbb{Q}) \curvearrowright Gr_l\mathbb{Q}_p^n)$. This means by definition that $1 \neq g \in PGL_n(\mathbb{Q})$ implies $gV_{f(x)} \neq V_{f(x)}$. Clearly, $gV_{f(x)} = V_{gf(x)}$, and so we have $V_{gf(x)} \neq V_{f(x)}$. It follows that $gf(x) \neq f(x)$, which means that $f(x) \in Fr(PGL_n(\mathbb{Q}) \curvearrowright \mathcal{M}^*(n,p,l))$, as desired. □

We extend the notion of type to $\mathcal{M}^*(n,p,l)$ by letting $\text{tp}[(V,L)] = \text{tp}(L)$. This is well-defined, as we have already observed that the type of L depends only on its class. The following fact is the last that we shall need in the proof of Theorem 6.4.2.

6.3.4. **Proposition**. *The group $SL_n(\mathbb{Z}_p)$ acts in a type-preserving fashion on $\mathcal{M}^*(n,p,l)$.*

Proof. We must show that whenever $g \in SL_n(\mathbb{Z}_p)$ and $g(V,L) = (V',L')$, we have that $\text{tp}(L) = \text{tp}(L')$. First suppose that $V = V' = V_0$, where this time $V_0 = \mathbb{Q}_pe_1 \oplus \cdots \oplus \mathbb{Q}_pe_l$. In particular, g fixes V_0. Letting g^c denote the $(n-l) \times (n-l)$ lower right-hand corner of g, we can argue as in the proof of Proposition 5.4.3 that $g(V_0, L) = (V_0, g^cL)$ and so $L' = g^cL$. But $g^c \in GL_{n-l}(\mathbb{Z}_p)$, and so Proposition 6.1.2 implies that $\text{tp}(L') = \text{tp}(L)$.

Also as in the proof of Proposition 5.4.3, this special case can be translated to establish the result in the general case. □

6.4 The proof of Theorem A

We are now ready to give a proof of the main case of Theorem A. Afterwards, we will tie up a couple of loose ends.

6.4.1. **Theorem A, case 2**. *Let $n \geq 3$ and suppose that $2 \leq k < n/2$. Then $\cong_{n,p}^k$ is Borel incomparable with $\cong_{n,p}^{n-k}$.*

This is an immediate consequence of the following slightly stronger result.

6.4.2. **Theorem**. *Suppose that $2 \leq k \neq n-k \leq n-1$. Then $E_{\cong}^k \not\leq_B^w (\cong_{n,p}^{n-k})^*$.*

Proof of Theorem 6.4.2. Suppose that $f : Gr_k\mathbb{Q}_p^n \to \mathcal{M}^*(n,p,n{-}k)$ is a weak Borel reduction from $E_{\cong}^k$ to $(\cong_{n,p}^{n-k})^*$. Then clearly f is a weak Borel reduction from $E_{SL_n\mathbb{Z}}^k$ to $(\cong_{n,p}^{n-k})^*$. Applying the arguments of Theorem 4.4.3 (and using Lemma 6.3.3 instead of Lemma 4.4.4 to define a cocycle), we may suppose that there exists an ergodic component $\Gamma_0 \curvearrowright X_0$ for $SL_n(\mathbb{Z}) \curvearrowright Gr_k\mathbb{Q}_p^n$, and a homomorphism $\phi : \Gamma_0 \to GL_n(\mathbb{Q})$ such that $(\phi, f) : \Gamma_0 \curvearrowright X_0 \longrightarrow GL_n(\mathbb{Q}) \curvearrowright \mathcal{M}^*(n,p,n{-}k)$ is a homomorphism of permutation groups.

Since the map $[M] \to V_M$ is $GL_n(\mathbb{Q}_p)$-preserving, the composition $\bar{f} : x \mapsto V_{f(x)}$ makes

$$(\phi, \bar{f}) : \Gamma_0 \curvearrowright X_0 \longrightarrow GL_n(\mathbb{Q}) \curvearrowright Gr_{n-k}\mathbb{Q}_p^n$$

into a homomorphism of permutation groups. By Lemma 4.4.5, we may replace $\Gamma_0 \curvearrowright X_0$ with an ergodic subcomponent to suppose that $\mathrm{im}(\phi) \subset SL_n(\mathbb{Z})$. We now have the following analogues to the claims we made in Theorem 6.2.1.

Claim. We may suppose that $\Gamma_0 = \Gamma_{p^t}$ is a principle congruence subgroup and that X_0 is equal, modulo a null set, to the particular component $Z_0 = (K_{p^t})V_0$. $\dashv$

Claim. We may suppose that (ϕ, f) is a permutation group homomorphism that makes $\bar{f}$ into an affine map $\bar{f}(x) = hx^{\perp}$. Adjusting f by h, we may suppose that $\bar{f}(x) = x^{\perp}$. $\dashv$

For the next claim, recall that we have defined that $\mathrm{tp}([M])$ is the type of any L such that $(V, L) \in [M]$.

Claim. We can suppose that there is a fixed $0 \le t < k$ such that $\mathrm{tp}(f(x)) = t$ for all $x \in X_0$.

Proof of claim. Let $F(\mathcal{M}^*(n,p,n{-}k))$ denote the standard Borel space of closed subsets of $\mathcal{M}^*(n,p,n{-}k)$ (considered with the quotient topology induced by the map $M \mapsto [M]$). Since $\phi(\Gamma_0) \subset SL_n(\mathbb{Z})$, we clearly have that the map $X_0 \to F(\mathcal{M}^*(n,p,n{-}k))$ given by:

$$x \mapsto (SL_n\mathbb{Z}_p)f(x)$$

is Γ_0-invariant. By ergodicity of $\Gamma_0 \curvearrowright X_0$, we can suppose that $f(X_0)$ is contained in a fixed $SL_n(\mathbb{Z}_p)$-orbit. Hence, the claim follows from Proposition 6.3.4. $\dashv$

Now, for any $z \in Z_0$, the canonical complementary subspace of z is $V_1 = \mathbb{Q}_p e_{k+1} \oplus \cdots \oplus \mathbb{Q}_p e_n$, and the canonical complementary subspace of $z^\perp$ is $V_0 = \mathbb{Q}_p e_1 \oplus \cdots \oplus \mathbb{Q}_p e_k$. In this argument, we shall choose an element $g \in GL_n(\mathbb{Q})$ which acts trivially on V_1, so it preserves the restriction to Z_0 of $E^k_{\cong}$; and nontrivially on V_0, so it fails to preserve lattice types on $f(X_0)$.

Proceeding, let $g = \mathrm{diag}(1/p, 1, \ldots, 1)$, where $\mathrm{diag}(d_1, \ldots, d_n)$ denotes the diagonal matrix with $a_{ii} = d_i$. Arguing as in the proof of Theorem 6.2.1, we can choose $x \in X_0$ such that $gx \in X_0$ and $x^\perp \in Fr(PGL_n(\mathbb{Q}) \curvearrowright Gr_{n-k}\mathbb{Q}_p^n)$. Now, both x and gx have the canonical complementary subspace $V_1 = \mathbb{Q}_p e_{k+1} \oplus \cdots \oplus \mathbb{Q}_p e_n$. Since g acts on V_1 by the identity map, we clearly have $g(x, L_0) = (gx, L_0)$, and so g witnesses that $x(E^k_{\cong})gx$. It follows that $f(x)(\cong^{n-k}_{n,p})^* f(gx)$. Since $(\cong^{n-k}_{n,p})^*$ is induced by the action of $GL_n(\mathbb{Q})$ on $\mathcal{M}^*(n, p, n-k)$, there exists $h \in GL_n(\mathbb{Q})$ such that $hf(x) = f(gx)$. It follows that $h\bar{f}(x) = \bar{f}(gx)$, so now:

$$hx^\perp = h\bar{f}(x) = \bar{f}(gx) = (gx)^\perp = g^{-T}x^\perp$$

Since we have chosen $x^\perp$ so that it is not fixed by any element of $PGL_n(\mathbb{Q})\backslash\{1\}$, there exists $a \in \mathbb{Q}_p^*$ such that $h = ag^{-T} = \mathrm{diag}(ap, a, \ldots, a)$ (see Figure 6.4.3 below).

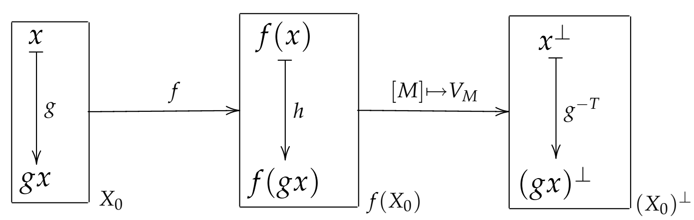

Figure 6.4.3. We have forced that an element of the form $h = ag^{-T}$ witnesses that $f(x)(\cong^{n-k}_{n,p})^* f(gx)$.

Finally, $x^\perp, (gx)^\perp$ each have the canonical complementary subspace $V_0 = \mathbb{Q}_p e_1 \oplus \cdots \oplus \mathbb{Q}_p e_k$. Letting h^c denote the *upper left*-hand corner of h, we have $\nu_p(h^c) \equiv 1 \mod k$. By Proposition 6.1.2, h^c acts in a type-*altering* fashion on $\mathcal{L}(V_0)$. But we have arranged for $\mathrm{tp}(f(x)) = \mathrm{tp}(f(gx))$, a contradiction. $\square$

The second part of Theorem B follows immediately.

6.4.4. **Theorem B, part 2**. *Let $n \geq 3$ and suppose that $k \leq n-2$. Then $E^k_{GL_n\mathbb{Q}} \not\leq^w_B E^k_{\cong}$.*

Proof. Recall that by Proposition 5.5.2 we have that $E^k_{GL_n\mathbb{Q}}$ is Borel bireducible with $E^{n-k}_{GL_n\mathbb{Q}}$ Hence, if $E^k_{GL_n\mathbb{Q}} \leq^w_B E^k_{\cong}$ then there exists a weak Borel reduction f from $E^{n-k}_{GL_n\mathbb{Q}}$ to $E^k_{\cong}$. Clearly, f is also a weak Borel reduction from $E^{n-k}_{\cong}$ to $E^k_{\cong}$, contradicting Theorem 6.4.2. $\square$

The keen-eyed reader will have noticed that Theorem A is as yet incomplete.

6.4.5. **Theorem A, case 3**. *We have* $E^1_{\cong} \not\leq_B E^{n-1}_{\cong}$.

Proof. By Theorem 4.4 of [Tho1], for groups $A, B \in R(n, p, n-1)$ we have that A is quasi-isomorphic to B iff A is isomorphic to B. In particular, $\cong^{n-1}_{n,p}$ is Borel bireducible with $\sim^{n-1}_{n,p}$, and it follows that $E^{n-1}_{\cong}$ is Borel bireducible with $E^{n-1}_{GL_n\mathbb{Q}}$. (Indeed, the case $k = n-1$ has been omitted from Theorem B for a good reason.) By Proposition 5.5.2, $E^{n-1}_{GL_n\mathbb{Q}}$ is Borel bireducible with $E^1_{GL_n\mathbb{Q}}$, and so we have established that the right-hand side $E^{n-1}_{\cong}$ is Borel bireducible with $E^1_{GL_n\mathbb{Q}}$. Hence, the result follows from Theorem B, part 1. $\square$

Chapter 7

Appendices

7.1 More on torsion-free abelian groups

In this section, we shall make Chapter 5 more self-contained by providing proofs for some of the abelian group theoretic facts that we have used. The content is taken almost entirely from [Fuc]. The author finds it a valuable contribution to rewrite and, in some cases, slightly clarify the proofs.

Recall that $R(n,p)$ denotes the set of p-local subgroups of $\mathbb{Q}^n$ of rank n. For $A \in R(n,p)$, recall that the completion of A is $\Lambda(A) = A \otimes \mathbb{Z}_p$. Here, $A \otimes \mathbb{Z}_p$ denotes the set of all $\mathbb{Z}_p$-linear combinations of elements of A, considered as a subset of $\mathbb{Q}_p^n$. The completion map Λ thus takes values in the space $\mathcal{M}(n,p)$ of $\mathbb{Z}_p$-submodules of $\mathbb{Q}_p^n$. We first verify that Λ is bijective.

7.1.1. **Proposition**. *The map Λ is a $GL_n(\mathbb{Q})$-preserving bijection from $R(n,p)$ onto $\mathcal{M}(n,p)$. The inverse map is $\sigma(M) = M \cap \mathbb{Q}^n$.*

Proof. That Λ is $GL_n(\mathbb{Q})$-preserving is clear. Injectivity is Lemma 93.1 of [Fuc]; surjectivity is Theorem 93.5 of the same. We reproduce the arguments.

Suppose first that $A \in R(n,p)$. It is clear that $A \subset \sigma\Lambda(A)$; we must show that $\sigma\Lambda(A) \subset A$. For any $a \in \sigma\Lambda(A)$, we have that $a \in \Lambda(A)$ and so we may express $a = z_1a_1 + \cdots + z_ma_m$ where $z_i \in \mathbb{Z}_p$ and $a_i \in A$. We may suppose without loss of generality that $n \leq m$ and $a_1, \ldots a_n$ are a basis for $\mathbb{Q}^n$, so write:

$$a = (z_1a_1 + \cdots + z_na_n) + (z_{n+1}a_{n+1} + \cdots + z_ma_m) \tag{7.1.2}$$

Next, since $a_{n+1}, \ldots, a_m$ depend on $a_1, \ldots, a_n$, we can assume without loss of generality that $z_{i+1}, \ldots, z_m$ are elements of $\mathbb{Z}_{(p)}$. To see this, write $a_{n+1} = q_1a_1 + \cdots q_na_n$ where

$q_i \in \mathbb{Q}$. Let p^t be the highest power of p in all the denominators of the q_i, and write $z_{n+1} = z_{n+1}^{\mathrm{lo}} + z_{n+1}^{\mathrm{hi}}$ where $z_{n+1}^{\mathrm{lo}} \in \mathbb{Z}_{(p)}$ and $p^t \mid z_{n+1}^{\mathrm{hi}}$. Now $z_{n+1}^{\mathrm{hi}} a_{n+1}$ can be folded into the first term of (7.1.2) and so z_{n+1} can be replaced by z_{n+1}^{lo}.

Since $a \in \mathbb{Q}^n$, we now have:

$$z_1 a_1 + \cdots z_n a_n = a - (z_{n+1} a_{n+1} + \ldots + z_m a_m) \in \mathbb{Q}^n$$

Since $a_1, \ldots, a_n$ also form a basis for $\mathbb{Q}_p^n$, it follows that $z_1, \ldots, z_n$ are rational. We have established that a is a $\mathbb{Z}_{(p)}$-linear combination of elements of A. Since A is p-local this implies $a \in A$. This proves that $A = \sigma\Lambda(A)$.

Suppose now that $M \in \mathcal{M}(n, p)$. It is clear that $\Lambda\sigma(M) \subset M$; we must show that $M \subset \Lambda\sigma(M)$. First, notice that M contains a lattice of $\mathbb{Q}_p^n$. By the discussion in Section II.1.1 of [Ser], any lattice of $\mathbb{Q}_p^n$ has a rational basis over $\mathbb{Z}_p$. In particular, M contains a full rank subset of $\mathbb{Q}^n$, and so any $m \in M$ can be expressed as a $\mathbb{Q}_p$-linear combination of elements of $\sigma(M)$. Using the fact that any p-adic number is the sum of a rational number and a p-adic integer, write $m = q + z$ where q is a rational combination of elements of $\sigma(M)$ and z is a p-adic integral combination of elements of $\sigma(M)$. By definition, $z \in \Lambda\sigma(M)$, and hence by the easy inclusion, $z \in M$. It follows that $q = m - z \in M$ as well. But already $q \in \mathbb{Q}^n$ and so $q \in \sigma(M)$. Hence, $m \in \Lambda\sigma(M)$. This proves that $M = \Lambda\sigma(M)$. □

We now work towards establishing some prerequisites to the definition of divisible rank. In what follows, remember that we are assuming that A is a p-local group. Define that a subset $\{a_1, \ldots, a_l\} \subset A \backslash \{0\}$ is *p-independent* if whenever $z_1, \ldots, z_l \in \mathbb{Z}$ and

$$z_1 a_1 + \cdots + z_l a_l \in pA$$

then $z_1, \ldots, z_l \in p\mathbb{Z}$. Now, $\{a_1, \ldots, a_l\}$ is said to be a *p-basis* of A if it is a maximal p-independent set. Notice that $\{a_1, \ldots, a_l\}$ is a p-basis iff its image in A/pA is a vector space basis over the finite field $\mathbb{F}_p$. Moreover, if P is the subgroup of A generated by $\{a_1, \ldots, a_l\}$, then it is easily seen that A/P is p-divisible. Since A was p-local already, A/P is divisible.

Now, since any divisible group factors as a direct sum of copies of $\mathbb{Q}$ and torsion, we have an exact sequence:

$$(7.1.3) \qquad 0 \longrightarrow P \longrightarrow A \longrightarrow \mathbb{Q}^{n-l} \oplus T \longrightarrow 0$$

where T is a torsion group. Now, it is well known that any torsion-free group is flat, and so we may tensor the sequence (7.1.3) with $\mathbb{Z}_p$ to obtain:

$$(7.1.4) \qquad 0 \longrightarrow L \longrightarrow \Lambda(A) \longrightarrow \mathbb{Q}_p^{n-l} \longrightarrow 0$$

where $L = P \otimes \mathbb{Z}_p$ is the free $\mathbb{Z}_p$ module generated by $a_1, \ldots, a_l$. (Note that $T \otimes \mathbb{Z}_p = 0$, since T contains no p-torsion. Indeed, it is easily checked that whenever $pa \in P$ we must have $a \in P$.)

7.1.5. **Proposition**. *The sequence* (7.1.4) *is split. Hence we have that:*

$$\Lambda(A) = V_A \oplus L$$

where V_A is a vector subspace of $\mathbb{Q}_p^n$ and L is a free $\mathbb{Z}_p$-module.

Proof. Let D be the divisible part of $\Lambda(A)$. We shall prove first that $L \cap D = 0$ and second that $L + D = \Lambda(A)$. For the first, it suffices to show that no element of L is infinitely p-divisible in $\Lambda(A)$. Suppose that $z_1, \ldots, z_l \in \mathbb{Z}_p$ and that:

$$z_1 a_1 + \cdots + z_l a_l \in p\Lambda(A)$$

is infinitely p-divisible. Dividing through by the common divisor of the z_i, we can suppose that some z_i is a unit. Now, write $z_i = z_i^0 + pw_i$ where $z_i^0 \in \{0, 1, \ldots p-1\}$ and $w_i \in \mathbb{Z}_p$. Then

$$z_1^0 a_1 + \cdots + z_l^0 a_l + p(w_1 a_1 + \cdots + w_l a_l) \in p\Lambda(A)$$

and it follows that $z_1^0 a_1 + \cdots + z_l^0 a_l \in p\Lambda(A)$ as well. But clearly $z_1^0 a_1 + \cdots + z_l^0 a_l \in A$, and so $z_1^0 a_1 + \cdots + z_l^0 a_l \in A \cap p\Lambda(A)$. Now, it is easily seen that $A \cap p\Lambda(A) \subset pA$. (Indeed, if $a \in A$ and $a/p \in \Lambda(A)$ then since $a/p \in \mathbb{Q}^n$ as well, by Proposition 7.1.1 we have $a/p \in A$.) Hence, we have $z_1^0 a_1 + \cdots + z_l^0 a_l \in pA$. Now $\{a_1, \ldots, a_l\}$ is a p-basis for A, so we have $p \mid z_i^0$ for all i and hence none of the $z_1, \ldots, z_l$ are units, a contradiction.

To see that $L + D = \Lambda(A)$, let $a \in \Lambda(A)$ be arbitrary. Since $\Lambda(A)/L$ is divisible, for any n we can write $a = p^n a_n + b_n$ where $a_n \in \Lambda(a)$ and $b_n \in L$. Since L is compact in the p-adic topology, there exists $\{n_i\} \subset \mathbb{N}$ such that $b_{n_i} \to b \in L$. Then letting $d = a - b \in \Lambda(A)$ we have $p^{n_i} a_{n_i} \to d$ and hence d is infinitely p-divisible in $\Lambda(A)$. Thus, we have written $a = b + d$ as an element of $L + D$. This concludes the proof. $\square$

Recall that the divisible rank of A was defined to be the dimension of V_A. Since V_A is the divisible part of $\Lambda(A)$, it depends only on A and not on the choice of p-basis. In particular, the divisible rank is well-defined. It also follows from this proposition that all p-bases of A have the same length, namely $l = n - \dim V_A$. This will allow us to verify the following proposition which gives an equivalent formulation of the divisible rank (initially stated just after Theorem A in Chapter 2).

7.1.6. **Proposition**. *Let A be a p-local torsion-free abelian group of finite rank. Then the divisible rank of A is precisely the maximum possible rank of a divisible quotient of A.*

Proof. It suffices to show that if $B \leq A$ and A/B is divisible then B contains a p-basis; it follows that the rank of divisible quotients A/B is bounded by k. Letting $b_1, \ldots, b_m$ be a p-basis for B, we shall show that the residues of $b_1, \ldots, b_m$ in A/pA form a spanning set. Then, by deleting elements from $b_1, \ldots, b_m$ we obtain a subset whose residues in A/pA form a basis, *i.e.*, a p-basis for A.

To see that $b_1, \ldots, b_m$ spans modulo p, we let $a \in A$ be arbitrary. Since A/B is divisible, there exist $a' \in A$ and $b \in B$ such that $a = pa' + b$. Since $B/\langle b_1, \ldots, b_m \rangle$ is already divisible, there exist $b' \in B$ and $z_1, \ldots, z_m \in \mathbb{Z}$ such that $b = pb' + \sum z_i b_i$. We now have that $a = pa' + pb' + \sum z_i b_i$, so that a is in the span of $b_1, \ldots, b_m$, modulo p. $\square$

7.2 Ergodic components of Grassmann space

In this appendix we explore the ergodic components for the homogeneous space $Gr_k \mathbb{Q}_p^n$ more thoroughly. We begin by considering the following alternative version of Theorem 3.7.1:

7.2.1. **Theorem**. *Suppose that $n \geq 3$ and let $k, l \leq n$. Let $\Gamma \curvearrowright X$ be an ergodic component for $SL_n(\mathbb{Z}) \curvearrowright Gr_k\mathbb{Q}_p^n$, and let $\Lambda \curvearrowright Y$ be an ergodic component for $SL_n(\mathbb{Z}) \curvearrowright Gr_l\mathbb{Q}_p^n$. If there exists a permutation group isomorphism $\Gamma \curvearrowright X \longrightarrow \Lambda \curvearrowright Y$, then $l = k$ or $l = n - k$.*

This result is of interest in itself. Using the superrigidity methods of this thesis, Theorem 7.2.1 can be used to give a proof of the following:

7.2.2. **Theorem**. *Suppose that $n \geq 3$ and let $k, l \leq n$ satisfy $l \neq k$ and $l \neq n - k$. Then the orbit equivalence relations induced by the actions $SL_n(\mathbb{Z}) \curvearrowright Gr_k\mathbb{Q}_p^n$ and $SL_n(\mathbb{Z}) \curvearrowright Gr_l\mathbb{Q}_p^n$ are Borel incomparable.*

This theorem is the Borel analog of Theorem 5.2.15(c) of [Zim], which states that the same actions are orbit inequivalent. In this section, we shall give a new proof of Theorem 7.2.1. The idea of the present argument is the following. If $\Delta' \leq \Delta \leq \Gamma$ are subgroups of finite index, then each ergodic component for the action of Δ on X breaks up into a finite union of ergodic components for the action of Δ'. Hence, a chain of finite index subgroups of Γ corresponds to a tree of ergodic subcomponents of X, the elements of the i^{th} level being the ergodic components for the action of the i^{th} subgroup. Roughly speaking, we shall show that the shape of this tree determines the set $\{k, n-k\}$.

It will be convenient to work with the principle congruence components, which were defined in Chapter 3. So for each m we shall count the number of ergodic components for the action on X of the principle congruence subgroup $\Gamma_m < SL_n(\mathbb{Z})$. Recall that the closure of Γ_m in $SL_n(\mathbb{Z}_p)$ is K_{p^t} where p^t is the highest power of p in M. By Proposition 3.4.2, the ergodic components for the action of Γ_m on $Gr_k\mathbb{Q}_p^n$ are just the K_{p^t}-orbits on $Gr_k\mathbb{Q}_p^n$. We begin with the general task of counting the number of K_{p^t} orbits on $Gr_k\mathbb{Q}_p^n$ for all t.

7.2.3. **Lemma**. *For $t > 0$, each K_{p^t}-orbit on $Gr_k\mathbb{Q}_p^n$ breaks up into $p^{k(n-k)}$ many $K_{p^{t+1}}$-orbits. Hence, every ergodic component for the action of Γ_m on $Gr_k\mathbb{Q}_p^n$ breaks up into $p^{k(n-k)}$ many ergodic components for the action of Γ_{pm}.*

For the record, we note the following consequence.

7.2.4. **Corollary**. *Let c_k denote the integer such that $Gr_k\mathbb{Q}_p^n$ breaks up into c_k many K_{p^1}-orbits. (In fact $c_k = |Gr_k\mathbb{F}_p^n|$, but we omit the proof.) Then there are exactly $c_k p^{k(n-k)t}$ many K_{p^t}-orbits on $Gr_k\mathbb{Q}_p^n$.*

We now begin the proof of Lemma 7.2.3. By Proposition 3.4.2, $SL_n(\mathbb{Z})$ acts transitively on the K_{p^t} orbits. Hence, it suffices to fix a base point $V_0 \in Gr_k\mathbb{Q}_p^n$ and work with the K_{p^t} orbit of V_0; as usual we arbitrarily choose $V_0 = \mathbb{Q}_p e_1 \oplus \cdots \oplus \mathbb{Q}_p e_k$. Observe that V_0 can be written as the column space $V_0 = \operatorname{col}\begin{bmatrix} I_k \\ 0 \end{bmatrix}$, where I_k denotes the $k \times k$ identity.

7.2.5. **Proposition**. *The orbit $(K_{p^t})V_0$ consists precisely of those subspaces of $\mathbb{Q}_p^n$ which can be written as $\operatorname{col}\begin{bmatrix} I_k \\ p^t r \end{bmatrix}$, where I_k is the $k \times k$ identity and r is a $(n-k) \times k$ matrix with entries in $\mathbb{Z}_p$.*

Proof. Clearly,

$$g = \begin{bmatrix} I_k & 0 \\ p^t r & I_{n-k} \end{bmatrix}$$

satisfies $gV_0 = \operatorname{col}\begin{bmatrix} I_k \\ p^t r \end{bmatrix}$ and $g \in K_{p^t}$. Hence, we need only show that *every* element of $(K_{p^t})V_0$ can be written in the desired form. Let $g \in K_{p^t}$, and divide g into blocks:

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A is $k \times k$. Then we have $gV_0 = \operatorname{col}\begin{bmatrix} A \\ C \end{bmatrix}$. Here, A is congruent modulo p^t to I_k, and C is congruent to 0 modulo p^t. In particular, A is invertible, and so we may right multiply by A^{-1} (which amounts to performing column operations) to obtain $gV_0 = \operatorname{col}\begin{bmatrix} I_k \\ CA^{-1} \end{bmatrix}$. Clearly CA^{-1} is again congruent to 0 modulo p^t and so we have expressed gV_0 in the desired form. $\square$

Now, consider the set of $p^{k(n-k)}$ elements of $(K_{p^t})V_0$ of the form:

$$\operatorname{col}\begin{bmatrix} I_k \\ p^t v \end{bmatrix}$$

where v is a $(n-k) \times k$ submatrix such that each entry of v lies in $\{0, 1, \ldots, p-1\}$. We claim that these form a system of representatives for the orbits of $K_{p^{t+1}}$ on $(K_{p^t})V_0$. This claim clearly follows from the following proposition:

7.2.6. **Proposition**. *Suppose that* $V = \mathrm{col}\begin{bmatrix} I_k \\ p^t v \end{bmatrix}$, *where all entries of* v *lie in* $\{0, 1, \ldots, p-1\}$. *Then the orbit* $(K_{p^{t+1}})V$ *consists precisely of those subspaces of* $\mathbb{Q}_p^n$ *which can be written*

$$\mathrm{col}\begin{bmatrix} I_k \\ p^t v + p^{t+1} r \end{bmatrix}$$

where the entries of r *lie in* $\mathbb{Z}_p$.

Proof. This time using the matrix

$$g = \begin{bmatrix} I_k & 0 \\ p^{t+1} r & I_{n-k} \end{bmatrix}$$

we clearly have:

$$gV = \mathrm{col}\begin{bmatrix} I_k \\ p^t v + p^{t+1} r \end{bmatrix}$$

Hence, we need only show that every element of $(K_{p^{t+1}})V$ can be written in the desired form. Let $g \in K_{p^{t+1}}$ be arbitrary, and again divide g into blocks:

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A is $k \times k$. Then we have:

$$gV = \mathrm{col}\begin{bmatrix} A + p^t B v \\ C + p^t D v \end{bmatrix}$$

Now, $A + p^t Bv$ is congruent modulo p^{t+1} to I_k, and $C + p^t Dv$ is congruent to $p^t v$ modulo p^{t+1}. So this time right-multiplying by $(A + p^t Bv)^{-1}$, we have $gV = \mathrm{col}\begin{bmatrix} I_k \\ C' \end{bmatrix}$ where again C' is congruent to $p^t v$ modulo p^{t+1}. This concludes the proof of Proposition 7.2.6. □

The proof of Lemma 7.2.3 is now complete.

Turning to the proof of Theorem 7.2.1, let us suppose that $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are as in the statement, and let $(\phi, f) : \Gamma \curvearrowright X \longrightarrow \Lambda \curvearrowright Y$ be an isomorphism of permutation groups. Shortly, we shall make use of Proposition 3.6.3 to pass to a principle congruence component contained in X. This next proposition tells us roughly that (ϕ, f) doesn't move principle congruence subgroups too far.

7.2.7. **Proposition**. *There exists* $D \in \mathbb{N}$, *depending only on* ϕ, *such that for any* $m \in \mathbb{N}$ *we have* $\Gamma_{Dm} \leq \phi(\Gamma_m)$ *and* $\Gamma_{Dm} \leq \phi^{-1}(\Gamma_m)$.

Proof. Recall that by 3.7.2, we can suppose that $\phi = \chi_h \circ (-T)^i$, where χ_h is conjugation by some $h \in GL_n(\mathbb{Q})$, $-T$ is transpose-inverse, and $i = 0, 1$. Suppose first that $\phi = \chi_h$. Let d be the least common multiple of all the denominators of entries in h and h^{-1}. We will show that $D = d^2$ meets our requirements.

Indeed, let us show that any $\gamma \in \Gamma_{d^2m}$ necessarily satisfies $\gamma \in \phi(\Gamma_m)$; the same goes for ϕ^{-1} by symmetry. Write $\gamma = I + d^2mB$ where I is the identity matrix and B is some matrix with integer entries. Then:

$$\begin{aligned}
\phi^{-1}(\gamma) &= I + d^2m\phi^{-1}(B) \\
&= I + d^2mhBh^{-1} \\
&= I + m(dh)B(dh^{-1}) \\
&= I + mB'
\end{aligned}$$

where B' again has integer entries. Hence $\phi^{-1}(\gamma) \in \Gamma_m$, as desired. The case $\phi = \chi_h \circ (-T)$ is similar. □

We next claim that (ϕ, f) is an isomorphism of *measure preserving* permutation groups. That is, we claim that f is actually a measure preserving map. To see this, let μ, ν denote the normalized Haar measures on X, Y, respectively. Then clearly Λ preserves $f_*(\mu)$, but recall that $\Lambda \curvearrowright Y$ is uniquely ergodic. It follows that $f_*(\mu) = \nu$, as desired. We have now proved the following result:

7.2.8. **Proposition**. *Let $\Delta' \leq \Delta \leq \Gamma$ be subgroups of finite index, and let $X_0 \subset X$ be an ergodic component for the action of Δ. Then $f(X_0)$ is an ergodic component for the action of $\phi(\Delta)$, and if X_0 breaks up into M ergodic subcomponents for the action of Δ', then $f(X_0)$ breaks up into M ergodic subcomponents for the action of $\phi(\Delta')$.*

To conclude the proof, fix a congruence subgroup $\Gamma_m \leq \Gamma$ and let t be arbitrary. By Proposition 7.2.7, we have the following chain of subgroups of Λ:

$$\phi(\Gamma_{D^2mp^t}) \leq \Gamma_{Dmp^t} \leq \Gamma_{Dm} \leq \phi(\Gamma_m)$$

By Lemma 7.2.3, each ergodic component for the action of Γ_{Dm} on Y breaks up into $p^{l(n-l)t}$ ergodic subcomponents for the action of Γ_{Dmp^t}. By Lemma 7.2.3 together with Proposition 7.2.8, each ergodic component for the action of $\phi(\Gamma_m)$ on Y breaks up into $Ap^{k(n-k)t}$ ergodic subcomponents for the action of $\phi(\Gamma_{D^2mp^t})$, where A is a constant depending only on D. Thus, we have:

$$p^{l(n-l)t} \leq Ap^{k(n-k)t}$$

But a similar argument using ϕ^{-1} gives:

$$p^{k(n-k)t} \leq Bp^{l(n-l)t}$$

Since these equations hold for all t, it follows that $k(n-k) = l(n-l)$, and hence that $k = l$ or $k = n - l$. This concludes the proof of Theorem 7.2.1.

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