GRADED TRACES AND IRREDUCIBLE REPRESENTATIONS OF AUT($A(\Gamma)$) ACTING ON GRADED $A(\Gamma)$ AND $A(\Gamma)^!$

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ABSTRACT OF THE DISSERTATION

Graded traces and irreducible representations of Aut$(A(\Gamma))$ acting on graded $A(\Gamma)$ and $A(\Gamma)^!$

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In this work we will study the structure of algebras $A(\Gamma)$ associated to directed, layered graphs. The algebras for which we find a decomposition are the algebras related to pseudo-roots of noncommutative polynomials and algebras associated to the Hasse graphs of polytopes, to the lattice of subspaces of a finite-dimensional vector space over a finite field, and to the complete layered graph. We will first find the filtration-preserving automorphism group of these algebras and develop methods of calculating the graded trace of an automorphism acting on the algebra. We will then find the multiplicities of the irreducible representations of $\mathrm{Aut}(A(\Gamma))$ acting on the homogeneous components of $A(\Gamma)$ and $A(\Gamma)^!$. The methods developed lead us to consider subalgebras of $\mathrm{gr}A(\Gamma)$.

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Chapter 1

Introduction

This research project began with the desire to better understand an algebra, called Q_n, related to factorizations of noncommutative polynomials of degree n. The factorization of noncommutative polynomials is an interesting and nontrivial problem that has been studied by I. Gelfand, S. Gelfand, Retakh, Serconek, and Wilson. A classical problem regarding generic monic polynomials over a (noncommutative) algebra with unity is to express the coefficients of a polynomial $P(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0 = (t - y_1)(t - y_2)\cdots(t - y_n)$ in terms of the right (or left) roots, where above y_n (resp. y_1) is a right (resp. left) root. Call all $y_1, ..., y_n$ pseudo-roots. The pseudo-roots can be written as rational expressions in a set of n right roots of the polynomial. Thus, the coefficients of $P(t)$ can be written in terms of the right roots (sums and products of pseudo-roots). The algebra Q_n describes the factorizations of $P(t)$ and is the universal algebra of pseudo-roots [GRSW].

We can relate the factorizations of polynomials to algebras associated with directed, layered graphs, and study properties of such algebras via this description. Q_n can be associated with the lattice of subsets of a set of n right roots of a polynomial of degree n and hence with the lattice of subsets of $\{1, ..., n\}$, $\mathcal{L}_{[n]}$ (as a directed, layered graph) (see [GRSW, GGRW, RSW3]). Q_n has a natural grading, and the automorphism group $Aut(Q_n)$ of Q_n acts on each homogeneous component of grQ_n. Because $Aut(\mathcal{L}_{[n]})$ is finite and we will take Q_n to be over a field of characteristic zero, each homogeneous component can be written as a direct sum of irreducible $Aut(\mathcal{L}_{[n]})$-modules. This work finds such a decomposition for Q_n and its dual.

This question of decomposition is also interesting for more general algebras, called $A(\Gamma)$, associated with directed, layered graphs. The algebra $A(\Gamma)$ is generated by the edges in the graph Γ. The relations are defined by associating to each path in Γ a polynomial with coefficients in the free algebra on the set of edges and requiring that the polynomials for two paths connecting the same pair of vertices are equal in $A(\Gamma)$. This family of algebras includes $A(\Gamma)$ for a number of interesting special cases of Γ such as: $\mathcal{L}_{[n]}$ (so $A(\Gamma) = Q_n$), graphs Γ whose automorphism group is the dihedral group on n elements, graphs Γ whose automorphism groups are Coxeter groups, the complete layered graph, and the Hasse graph of the lattice of subspaces of a finite-dimensional vector space over a finite field.

The technique needed to decompose graded $A(\Gamma)$ is interesting in its own right. Furthermore, subalgebras of $A(\Gamma)$ that naturally (and necessarily) arise in studying this question have provided a rich source of examples of algebras associated with more general graphs.

Structure of the Paper

In this paper we will be considering directed graphs Γ satisfying certain hypotheses. There exists an algebra $A(\Gamma)$ over a field k, an associated graded algebra $\mathrm{gr}A(\Gamma)$, and a dual algebra $A(\Gamma)^!$ associated with each of these graphs. In Chapter 2 we will give some preliminaries on how these algebras are built from the graphs as well as a basis for $A(\Gamma)$. The definition of the dual algebra and of subalgebras that will play an integral role in what follows will be given in Chapter 3. We will go into more detail about these objects in Chapter 7.

The automorphism group of the graph injects into the automorphism group $\mathrm{Aut}A(\Gamma)$ of $A(\Gamma)$. Furthermore, the nonzero scalars inject into the automorphism group of the algebra. Thus, one is naturally led to ask how these automorphism groups are related. This question will be answered in Chapter 4.

A second question that we are led to is to describe the homogeneous components of $A(\Gamma)$ and $A(\Gamma)^!$. We will decompose the algebra and its dual into irreducible $\mathrm{Aut}(A(\Gamma))$-modules by calculating the graded trace generating functions. These graded trace generating functions are actually the graded dimensions of certain subalgebras of $\mathrm{gr}A(\Gamma)$. Hence, the technique for calculating the graded trace generating functions is to abstract the problem into finding the graded dimension of subalgebras. In Chapter 5 we will explain why this is true and how these graded dimensions are found.

In Chapters 6, 8, 9, and 10 we will define a variety of algebras associated with directed, layered graphs and find their graded trace generating functions and their decompositions. We will also consider the decomposition of the dual of these algebras.

Future Directions

There are several questions that have arisen while conducting this research that have yet to be answered. The first is to find a complete decomposition of the algebra associated to the Hasse graph of the lattice of subspaces of a finite-dimensional vector space over a finite field. Also, the search continues for more graphs that give rise to interesting algebras as well as known algebras which may be efficiently studied in this manner. Finally, the reader will notice that the algebras in this work have a property relating the graded dimension of the algebras and their duals (c.f. Section 6.5.3) that is related to Koszulity. We wish to explain why the algebras have this property and what it means for our algebras. It would be interesting to know what the class of algebras with this property is.

Chapter 2

Preliminaries

2.1 The Algebra $A(\Gamma)$

Certain algebras, denoted $A(\Gamma)$, associated to directed graphs were first defined by Gelfand, Retakh, Serconek, and Wilson [GRSW]. We recall the definitions of $A(\Gamma)$ and $\mathrm{gr}A(\Gamma)$ following the development found in [[RSW], §2]. Let k be a field and for any set W let $T(W)$ be the free associative algebra on W over k. Let $\Gamma = (V, E)$ be a directed graph where V is the set of vertices, E the set of edges, and there are functions $t, h : E \to V$ (tail and head of e). We say Γ is a layered graph if $V = \bigcup_{i=0}^{n} V_i$, $E = \bigcup_{i=0}^{n} E_i$, $t : E_i \to V_i$, and $h : E_i \to V_{i-1}$. If $v \in V_i$ ($e \in E_i$), we say the level of v, (respectively e) is i; denote this by $|v|$, (resp. $|e|$). We will assume throughout this paper that $V_0 = \{*\}$ and that for all $v \in V_+ = \bigcup_{i=1}^{n} V_i$, there exists at least one $e \in E$ such that $t(e) = v$. For each $v \in V_+$, fix some $e_v \in E$ with $t(e_v) = v$; call this a distinguished edge.

A path from $v \in V$ to $w \in V$ is a sequence of edges $\pi = \{e_1, ..., e_m\}$ such that $t(e_1) = v$, $h(e_m) = w$, and $t(e_{i+1}) = h(e_i), 1 \leq i < m$. We will say $t(\pi) = v$, $h(\pi) = w$, and the length of $\pi, l(\pi)$, is m. Write $v > w$ if there exists a path from v to w. For $\pi = \{e_1, ..., e_m\}$, define $e(\pi, k) := \sum_{1 \leq i_1 < ... < i_k \leq m} e_{i_1} \cdots e_{i_k}$. For each $v \in V$ there is a path $\pi_v = \{e_1, ..., e_m\}$, called the distinguished path, from v to * defined by $e_1 = e_v$, $e_{i+1} = e_{h(e_i)}$ for $1 \leq i < m$, and $h(e_m) = *$. When π_v is the distinguished path from v to *, we will write $e(v, k)$ in lieu of $e(\pi_v, k)$. Let R be the two-sided ideal of $T(E)$ generated by

$\{e(\pi_1, k) - e(\pi_2, k) : t(\pi_1) = t(\pi_2),\ h(\pi_1) = h(\pi_2),\ 1 \leq k \leq l(\pi_1)\}$.

Definition. $A(\Gamma) = T(E)/R$

Let $\hat{e}(v, k)$ denote the image in $A(\Gamma)$ of $e_1 \cdots e_k$. Finally say that (v, k) covers (w, l) if $v > w$ and $k = |v| - |w|$, write this as $(v, k) \gtrdot (w, l)$.

Theorem 2.1. *[[RSW],Thm 1] - Let $\Gamma = (V, E)$ be a layered graph, $V = \bigcup_{i=0}^{n} V_i$, $V_0 = \{*\}$. Then*

$$\mathcal{B}(\Gamma) := \{\hat{e}(v_1, k_1) \cdots \hat{e}(v_l, k_l) : l \geq 0, v_1, ..., v_l \in V_+, 1 \leq k_i \leq |v_i|, (v_i, k_i) \not\geqslant (v_{i+1}, k_{i+1})\}$$

is a basis for $A(\Gamma)$.

There is also a presentation of $A(\Gamma)$ as a quotient of $T(V_+)$ [[RSW2],§3]. Every edge may be expressed as a linear combination of distinguished edges, and the distinguished edge e_v may be identified with $v \in V_+$. Define $S_1(v) := \{w \in V_{|v|-1} : v > w\}$. A layered graph is uniform if for every $v \in V_j$, $j \geq 2$, every pair of vertices u, w in $S_1(v)$ satisfies $S_1(u) \cap S_1(w) \neq \emptyset$ ("diamond condition").

Proposition 2.2. *[[RSW2],Prop 3.5] Let Γ be a uniform layered graph. Then $A(\Gamma) \cong T(V_+)/R_V$ where R_V is the two-sided ideal generated by $\{v(u-w) - u^2 + w^2 + (u-w)x : v \in \bigcup_{i=2}^{n} V_i, u, w \in S_1(v), x \in S_1(u) \cap S_1(w)\}$.*

Remark: From now on we will just write $e(v, k)$ for $\hat{e}(v, k)$.

2.2 Associated Graded Algebra $grA(\Gamma)$

Next we will describe a filtration and grading on $A(\Gamma)$. Here we will also denote by V the span of V in $T(V)$, and by E the span of E in $T(E)$, when no confusion will arise. Let $W = \sum_{k \geq 0} W_k$ be a graded vector space (in our case $W = V$ or E). Then $T(W)$ is bigraded. One grading $T(W) = \sum_i T(W)_{[i]}$ is given by degree in the tensor algebra; i.e., $T(W)_{[i]} = span\{w_1 \cdots w_i : w_1, ..., w_i \in W\}$. The other grading is given by $T(W) = \sum_{i \geq 0} T(W)_i$ where $T(W)_i = \text{span}\{w_1 \cdots w_r : r \geq 0, w_j \in W_{l_j}, l_1 + ... + l_r = i\}$. The second grading induces an increasing filtration on $T(W)$:

$T(W)_{(i)} = \text{span}\{w_1 \cdots w_r : r \geq 0, w_j \in W_{l_j}, l_1 + \cdots + l_r \leq i\} = T(W)_0 + \cdots + T(W)_i$.

Because $T(W)_{(i)}/T(W)_{(i-1)} \cong T(W)_i$, $T(W)$ can be identified with its associated graded algebra. Define a map $gr : T(W)\backslash\{0\} \to T(W)\backslash\{0\} = grT(W)$ by $w = \sum_{i=0}^{k} w_i \mapsto w_k$

where $w_i \in T(W)_i$, $w_k \neq 0$. Of course, gr is not an additive map. [[RSW2],§2]

Lemma 2.3. *[[RSW2],Lemma 2.1] Let W be a graded vector space and I an ideal in $T(W)$. Then $gr(T(W)/(I)) \cong T(W)/(grI)$.*

Thus the associated graded algebra of $A(\Gamma)$, $grA(\Gamma)$, is isomorphic to $T(E)/grR$. The graded relations, grR, are that for paths $\pi_1 = \{e_1, ..., e_m\}$ and $\pi_2 = \{f_1, ..., f_m\}$, $\{e_1 \cdots e_k = f_1 \cdots f_k, 1 \leq k \leq m\}$ (the leading term of $e(v,k)$). Another way to consider this is that $e(v, k+l) - e(v,k)e(u,l)$ is in grR where $v > u, k = |v| - |u|$. Recalling the definition of $\mathcal{B}(\Gamma)$ from Theorem 2.1, we see that $\{gr(b) : b \in \mathcal{B}(\Gamma)\}$ is a basis for $grA(\Gamma)$.

Let us now look at our second description of $A(\Gamma)$ as isomorphic to $T(V_+)/R_V$.

Proposition 2.4. *[[RSW2],Prop 3.6] Let Γ be a uniform layered graph. Then $grA(\Gamma) \cong T(V_+)/grR_V$ where grR_V is generated by $\{v(u-w) : v \in \bigcup_{i=2}^{n} V_i, u, w \in S_1(v)\}$.*

Also, $A(\Gamma)_i = (T(E)_i + R)/R = (T(V_+)_i + R_V)/R_V$ and $A(\Gamma)_{(i)} = (T(E)_{(i)} + R)/R = (T(V_+)_{(i)} + R_V)/R_V$.

Chapter 3

The Dual $A(\Gamma)^!$ and the Subalgebra $A(\Gamma^\sigma)$

3.1 The Dual $A(\Gamma)^!$

Definition $(A^!)$**.** *[[BVW],§ 2] Let* $A = T(E)/(R)$, $R \subseteq E^{\otimes 2}$. *Then* $A^! = T(E^*)/(R^\perp)$ *where* E^* *is the dual vector space of* E *and* $R^\perp$ *is the annihilator of* R*; i.e.* $R^\perp = \{f \in (E^{\otimes 2})^* : f(x) = 0\,\forall x \in R\}$ *of* $(E^{\otimes 2})^*$ *where* $(E^{\otimes 2})^*$ *is canonically identified with* $E^{*\otimes 2}$*.*

Definition $(A(\Gamma)^!)$**.** *The dual of* $grA(\Gamma)$ *is* $A(\Gamma)^! := T(E^*)/(grR)^\perp$*.*

The dual element to the generator $e(v,k)$ in $A(\Gamma)$ will be denoted $e(v,k)^*$.

Proposition 3.1. $A(\Gamma)^!$ *has a presentation with generators* $\{e(v,1)^*\}$ *and relations* $\{e(v,1)^*e(u,1)^* : v \not> u\} \cup \{e(v,1)^* \sum_{v > u} e(u,1)^*\}$

We will give a proof of this proposition and a basis for the dual in Chapter 7.

3.2 The Subalgebra $A(\Gamma^\sigma)$

We will now define a subalgebra of $gr\,A(\Gamma)$. Let σ be an automorphism of the layered graph Γ; i.e. an automorphism that preserves each level of the graph. Define $\Gamma^\sigma := (V_\sigma, E_\sigma)$ where V_σ is the set of vertices $v \in V$ such that $\sigma(v) = v$ and E_σ is the set of edges that connect the vertices minimally. Here minimally means that there is an edge $e \in E_\sigma$ from v to w, v, $w \in V_\sigma$ if and only if $v \geq u \geq w$, $u \in V_\sigma$, implies $u = v$ or $u = w$.

Definition $(A(\Gamma^\sigma))$**.** *Define* $A(\Gamma^\sigma)$ *to be* $spanB_\sigma$*,*

$B_\sigma = \{e(v_1,k_1)\cdots e(v_l,k_l) : l \geq 0, v_1, ..., v_l \in V_\sigma \backslash *, 1 \leq k_i \leq |v_i|, (v_i,k_i) \not> (v_{k+1}, k_{i+1})\}$.

B_σ is, in fact, a basis. The elements are linearly independent because the set is the subset of a basis.

Theorem 3.2. *$A(\Gamma^\sigma)$ is a subalgebra of $grA(\Gamma)$.*

A presentation for $A(\Gamma^\sigma)$ is given by generators $G' = \{e'(v,k) : v \in V_\sigma, 1 \leq k \leq |v|\}$ and relations $R' = \{e'(v,k+l) - e'(v,k)e'(u,l) : v > u \in V_\sigma, k = |v| - |u|\}$.

Proof. Define $\phi : T(G') \to grA(\Gamma)$ by $\phi(e'(v,k)) = e(v,k)$. We have $\phi(T(G')) \supseteq A(\Gamma^\sigma)$ because elements of B_σ are formed from products of elements in G'.

In $A(\Gamma)$ we have

$$(*) \qquad e(v,k)e(u,l) - e(v,k+l) \equiv \sum_{\substack{i_0, i_{r+1} \geq 0, i_1, \ldots, i_r \geq 1 \\ i_0 < k, i_0 + \ldots + i_r \leq k \\ i_0 + \ldots + i_{r+1} = k+l}} (-1)^{r+1} e(v,i_0)e(u,i_1)\cdots e(u,i_{r+1})$$

mod R [[GRSW], p6]. However, the elements on the right-hand side are all of lower degree than those on the left-hand side [[GRSW], Lemma 2.2]. Note that the elements on the left-hand side have degree k+l and are in $(T(E)/R)_{[(k+l)|v|-(k+l)(k+l+1)/2]}$. Therefore, in $grA(\Gamma)$, the terms on the right-hand side are zero. Hence, we have $e(v,k)e(u,l) - e(v,k+l) \equiv 0$. Consequently $\phi(e'(v,k)e'(u,l)) = \phi(e'(v,k+l))$.

Let $b' = e'(v_1,k_1)\cdots e'(v_l,k_l)$ be a monomial in $T(G')$. In $T(G')/ < R' >$, we may replace every occurrence of $e'(v_i,k_i)e'(v_{i+1},k_{i+1})$ such that $(v_i,k_i) \gtrdot (v_{i+1},k_{i+1})$ in b' with $e'(v_i,k_i+k_{i+1})$. Thus $b' \equiv e'(v'_1,k'_1)\cdots e'(v'_l,k'_l)$ such that $(v'_i,k'_i) \not\gtrdot (v'_{i+1},k'_{i+1})$ in $T(G')/ < R' >$. Hence $\phi(b') \in A(\Gamma^\sigma)$, and so $\phi(T(G')) = A(\Gamma^\sigma)$.

By (*), $R' \subseteq \ker\phi$ and we have an induced surjective homomorphism $\phi' : T(G')/ < R' > \to A(\Gamma^\sigma)$.

Let $f = \sum k_i b'_i \in \ker\phi'$, where k_i is an element in the field and b'_i a monomial in $T(G')/ < R' >$. Then $0 = \phi'(f) = \sum k_i \phi'(b'_i) = \sum k_i b_i$ is a linear combination of basis elements in $A(\Gamma^\sigma)$. This implies that $k_i = 0 \forall i$ and so $f = 0$. Therefore, ϕ' is an isomorphism. □

We will write $e(v,k)$ for $e'(v,k)$ from now on.

Remark: There are two natural duals related to $A(\Gamma^\sigma)$. One is to take the dual of $A(\Gamma)$ and then look at fixed points. The other is to take the dual of the subalgebra $grA(\Gamma^\sigma)$. These two constructions are described in Chapter 7.

For x a basis element of $A(\Gamma)_{[i]}$, write $\sigma(x)$ as a linear combination of basis elements and say the coefficient of x in $\sigma(x)$ is α. Denote this value α by $t_\sigma(x)$. Then, for finite-dimensional $A(\Gamma)_{[i]}$, $Tr_\sigma(A(\Gamma)_{[i]}) = \sum_{x \in \text{basis}} t_\sigma(x)$. In this paper we will be looking at the trace of σ acting on $A(\Gamma)_{[i]}$ and $A(\Gamma)^!_{[i]}$.

Chapter 4

Automorphism Group of $A(\Gamma)$

Throughout this paper Aut(A) will denote the filtration-preserving automorphisms of the graded algebra A (see Chapter 2).

Lemma 4.1. *$Aut(A(\Gamma)) \supseteq k^* \times Aut(\Gamma)$*

Proof. Any automorphism $\tilde{\sigma}$ of the graph extends to an automorphism σ of $T(E)$. Since $\tilde{\sigma}$ preserves paths, σ preserves the ideal R defined in Section 2. Hence it induces an automorphism, again denoted by σ, of $A(\Gamma) = T(E)/R$. Also, for any scalar α, multiplication by α is an automorphism because the relations are homogeneous. Thus $Aut(A(\Gamma)) \supseteq k^* \times \mathrm{Aut}(\Gamma)$. $\square$

Let Γ be a graph with a unique minimal vertex at level 0, $V_0 = \{*\}$, whose vertices are labeled in the following manner. Label the vertices in level one by $\{v_1, ..., v_m\}$ and index those in level r, $2 \leq r \leq n$, by a subset of the power set of $\{1, ..., m\}$. Let the edges connect vertices by minimal containment of their indices. There is a path from v_A ($|v_A| > |v_i|$) to v_i if and only if $i \in A$.

Theorem 4.2. *Let Γ be a graph as described above. If Γ satisfies i) $|V_1| > 2$, ii) no two vertices have the same label, and iii) there are either zero or two paths between any two vertices which are two levels apart in Γ, then $Aut(A(\Gamma)) = k^* \times Aut(\Gamma)$, k the base field.*

Proof. By Lemma 4.1, $Aut(A(\Gamma)) \supseteq k^* \times \mathrm{Aut}(\Gamma)$. Any automorphism of the algebra must preserve the relations. Thus, for all subsets $B, C \subseteq A \subseteq \{1, ..., m\}$ such that $v_A, v_B, v_C \in \Gamma$, $|v_A| \geq 2$, and $|v_A| - |v_B| = |v_A| - |v_C| = 1, |v_A| - |v_{B\cap C}| = 2$ (i.e. $v_A, v_B, v_C, v_{B\cap C}$ form a diamond), the image of $v_A(v_B - v_C) - v_B^2 + v_C^2 + (v_B - v_C)v_{B\cap C}$ must equal zero. Consider first paths from level 2 to level 0. Because of our assumption that there are exactly two paths from each vertex in level two, $|A| = 2$ for $v_A \in V_2$. Let $\sigma \in Aut(A(\Gamma))$ and $v_{A_1}, ..., v_{A_{m_2}}$ be vertices in level 2; then $\sigma(v_{ij}) = a^{ij}_{A_1} v_{A_1} + \cdots + a^{ij}_{A_{m_2}} v_{A_{m_2}} + b^{ij}_1 v_1 + \cdots + b^{ij}_m v_m$ for all i, j such that $v_{ij} \in \Gamma$ and $\sigma(v_i) = c^i_1 v_1 + ... + c^i_m v_m$ for all i, where all coefficients are in k.

Now $\sigma(v_{ij}(v_i - v_j)) = \sigma(v_i^2 - v_j^2)$ implies

$(a^{ij}_{A_1} v_{A_1} + \cdots + a^{ij}_{A_{m_2}} v_{A_{m_2}} + b^{ij}_1 v_1 + \cdots + b^{ij}_m v_m)((c^i_1 - c^j_1)v_1 + \cdots + (c^i_m - c^j_m)v_m)$

$= (c^i_1 v_1 + \cdots + c^i_m v_m)^2 - (c^j_1 v_1 + \cdots + c^j_m v_m)^2.$

There are no $v'_A s$ with $|A| = 2$ on the right-hand side, and so we must use our relations to eliminate them from the left-hand side. Thus, every occurrence of v_{kl} must be followed by $v_k - v_l$; and hence, $c^i_k - c^j_k = -(c^i_l - c^j_l)$ and $c^i_m - c^j_m = 0$ if $m \neq k, l$. Therefore, if $a^{ij}_{kl} \neq 0$,

$$(c^i_1 - c^j_1)v_1 + \cdots + (c^i_m - c^j_m)v_m = (c^i_k - c^j_k)(v_k - v_l). \tag{4.1}$$

This has two consequences. First, at most one a^{ij}_{kl} can be nonzero. If all a^{ij}_{kl} were zero, the element $v_{i\,j} \notin (A(\Gamma))_{(1)}$ would be sent to an element in $(A(\Gamma))_{(1)}$, which we cannot allow because then σ would not be invertible. Thus a^{ij}_{kl} must be nonzero for exactly one $\{kl\}$. Let us denote this set by $\{\tau(i)\tau(j)\}$. Then $\sigma(v_{ij}) = a^{ij}_{\tau(i)\tau(j)} v_{\tau(i)\,\tau(j)} + b^{ij}_1 v_1 + \cdots + b^{ij}_m v_m$. If $\tau(ij) = \tau(kl)$, then $\sigma(v_{i\,j} - a^{ij}_{\tau(i)\tau(j)}(a^{kl}_{\tau(k)\tau(l)})^{-1} v_{k\,l}) \in (A(\Gamma))_{(1)}$; this implies that $\{ij\} = \{kl\}$. Thus τ is one-to-one, and so is in S_n.

A second consequence of (4.1) is that $c^i_r - c^j_r$ is zero if and only if $r \neq \tau(i), \tau(j)$.

We now have from (4.1): $(a^{ij}_{\tau(ij)} v_{\tau(i)\,\tau(j)} + b^{ij}_1 v_1 + ... + b^{ij}_m v_m)(c^i_{\tau(i)} - c^j_{\tau(i)})(v_{\tau(i)} - v_{\tau(j)}) = (c^i_1 v_1 + ... + c^i_m v_m)^2 - (c^j_1 v_1 + ... + c^j_m v_m)^2$. (Recall $c^i_{\tau(i)} - c^j_{\tau(i)} = -(c^i_{\tau(j)} - c^j_{\tau(j)})$.)

Let $z = \sum\limits_{r \neq \tau(i), \tau(j)} c^i_r v_r = \sum\limits_{r \neq \tau(i), \tau(j)} c^j_r v_r$. Then

$$\begin{aligned}
&(a^{ij}_{\tau(ij)} v_{\tau(i)\,\tau(j)} + b^{ij}_1 v_1 + ... + b^{ij}_m v_m)((c^i_{\tau(i)} - c^j_{\tau(i)})(v_{\tau(i)} - v_{\tau(j)}) \\
&= (c^i_{\tau(i)} v_{\tau(i)} + c^i_{\tau(j)} v_{\tau(j)} + z)^2 - (c^j_{\tau(i)} v_{\tau(i)} + c^j_{\tau(j)} v_{\tau(j)} + z)^2 \\
&= (c^i_{\tau(i)} v_{\tau(i)} + c^i_{\tau(j)} v_{\tau(j)})^2 - (c^j_{\tau(i)} v_{\tau(i)} + c^j_{\tau(j)} v_{\tau(j)})^2 + (c^i_{\tau(i)} v_{\tau(i)} + c^i_{\tau(j)} v_{\tau(j)})z \\
&+ z(c^i_{\tau(i)} v_{\tau(i)} + c^i_{\tau(j)} v_{\tau(j)}) - (c^j_{\tau(i)} v_{\tau(i)} + c^j_{\tau(j)} v_{\tau(j)})z - z(c^j_{\tau(i)} v_{\tau(i)} + c^j_{\tau(j)} v_{\tau(j)}).
\end{aligned} \tag{4.2}$$

On the left-hand side of (4.2), v_r, for $r \neq \tau(i), \tau(j)$, is never the second term of the product of two $v'_r s$. Hence, $(c^i_{\tau(i)} v_{\tau(i)} + c^i_{\tau(j)} v_{\tau(j)} - c^j_{\tau(i)} v_{\tau(i)} - c^j_{\tau(j)} v_{\tau(j)})z = 0$. This implies that either $c^i_{\tau(i)} = c^j_{\tau(i)}$ and $c^i_{\tau(j)} = c^j_{\tau(j)}$, which is a contradiction since $\sigma(v_i) \neq \sigma(v_j)$, or $z = 0$. Thus $z = 0$ and so $c^i_r = c^j_r = 0$ for all $r \neq \tau(i), \tau(j)$. Now we have

$$(a^{ij}_{\tau(ij)} v_{\tau(i)\,\tau(j)} + b^{ij}_1 v_1 + \cdots + b^{ij}_m v_m)(c^i_{\tau(i)} - c^j_{\tau(i)})(v_{\tau(i)} - v_{\tau(j)}) \equiv$$
$$a^{ij}_{\tau(ij)}(c^i_{\tau(i)} - c^j_{\tau(i)})(v^2_{\tau(i)} - v^2_{\tau(j)}) + (b^{ij}_1 v_1 + \cdots + b^{ij}_m v_m)(c^i_{\tau(i)} - c^j_{\tau(i)})(v_{\tau(i)} - v_{\tau(j)}) =$$
$$((c^i_{\tau(i)})^2 - (c^j_{\tau(i)})^2) v^2_{\tau(i)} + (c^i_{\tau(i)} c^i_{\tau(j)} - c^j_{\tau(i)} c^j_{\tau(j)})(v_{\tau(i)} v_{\tau(j)} + v_{\tau(j)} v_{\tau(i)}) + ((c^i_{\tau(j)})^2 - (c^j_{\tau(j)})^2) v^2_{\tau(j)}$$

Because the right-hand side is in the subspace generated by $v_{\tau(i)}, v_{\tau(j)}$, the left-hand side is as well. Therefore, only $b^{ij}_{\tau(i)}, b^{ij}_{\tau(j)}$ can be nonzero.

Let us write down what we know so far. For any i, j, $1 \leq i, j \leq n$, we have:

1) $\sigma(v_{i\,j}) = a^{ij}_{\tau(ij)} v_{\tau(i)\tau(j)} + b^{ij}_{\tau(i)} v_{\tau(i)} + b^{ij}_{\tau(j)} v_{\tau(j)}$

2) $\sigma(v_i) = c^i_{\tau(i)} v_{\tau(i)} + c^i_{\tau(j)} v_{\tau(j)}$

3) $\sigma(v_j) = c^j_{\tau(i)} v_{\tau(i)} + c^j_{\tau(j)} v_{\tau(j)}$

and

4) $(a^{ij}_{\tau(ij)} v_{\tau(i)\tau(j)} + b^{ij}_{\tau(i)} v_{\tau(i)} + b^{ij}_{\tau(j)} v_{\tau(j)})(c^i_{\tau(i)} - c^j_{\tau(i)})(v_{\tau(i)} - v_{\tau(j)}) \equiv$
$$a^{ij}_{\tau(ij)}(c^i_{\tau(i)} - c^j_{\tau(i)})(v^2_{\tau(i)} - v^2_{\tau(j)}) + (b^{ij}_{\tau(i)} v_{\tau(i)} + b^i_{\tau(j)} v_{\tau(j)})(c^i_{\tau(i)} - c^j_{\tau(i)})(v_{\tau(i)} - v_{\tau(j)}) =$$
$$((c^i_{\tau(i)})^2 - (c^j_{\tau(i)})^2) v^2_{\tau(i)} + (c^i_{\tau(i)} c^i_{\tau(j)} - c^j_{\tau(i)} c^j_{\tau(j)})(v_{\tau(i)} v_{\tau(j)} + v_{\tau(j)} v_{\tau(i)}) + ((c^i_{\tau(j)})^2 - (c^j_{\tau(j)})^2) v^2_{\tau(j)}$$

Applying (1) and (3) above to $\{ik\}$ (we are using here that $|V_1| > 2$) we find that $\sigma(v_i) = c^i_{\tau(i)} v_{\tau(i)} + c^i_{\tau(k)} v_{\tau(k)}$. Because $\sigma(v_i) \neq 0$, $c^i_{\tau(i)}$ and $c^i_{\tau(j)}$ cannot both be zero. Furthermore, $\tau(i)$, $\tau(j)$, and $\tau(k)$ are distinct, so $c^i_{\tau(i)} \neq 0$ and $c^i_{\tau(j)} = c^i_{\tau(k)} = 0$. Thus, $\sigma(v_i) = c^i_{\tau(i)} v_{\tau(i)}$.

Because we have $c^i_{\tau(i)} - c^j_{\tau(i)} = -(c^i_{\tau(j)} - c^j_{\tau(j)})$ and $c^i_{\tau(j)} = 0 = c^j_{\tau(i)}$, we see that $c^i_{\tau(i)}$ is independent of i; call this coefficient c. Thus,

$$a^{ij}_{\tau(ij)} c(v^2_{\tau(i)} - v^2_{\tau(j)}) + (b^{ij}_{\tau(i)} v_{\tau(i)} + b^{ij}_{\tau(j)} v_{\tau(j)}) c(v_{\tau(i)} - v_{\tau(j)}) = c^2(v^2_{\tau(i)} - v^2_{\tau(j)})$$
$$\Rightarrow b^{ij}_{\tau(i)} = b^{ij}_{\tau(j)} = 0 \text{ and } a^{ij}_{\tau(ij)} = c \text{ for all } \{ij\}.$$

What σ does on level one forces what happens on the levels above. We may compose σ with the automorphism that multiplies each element by $1/c$; call this composition $\hat{\sigma}$. We have shown that $\hat{\sigma}$ permutes the vertices in levels 1 and 2. Assume that $\hat{\sigma}$ permutes the vertices in levels less than or equal to $k-1$;i.e. $\hat{\sigma}(v_B) = v_{\tau(B)}$. Let $v_A, v_{A_1}, ..., v_{A_{m_k}} \in V_k$, $v_B, v_C, v_{B_1}, ..., v_{B_{m_{k-1}}} \in V_{k-1}$. Each vertex v_A in level k is present in at least one relation $v_A(v_B - v_C) - v^2_B + v^2_C + (v_B - v_C) v_{B\cap C} = 0$. Apply $\hat{\sigma}$ to this relation and we get $(a^A_{A_1} v_{A_1} + \cdots +$

$a^A_{A_{m_k}} v_{A_{m_k}} + b^A_{B_1} v_{B_1} + \cdots + b^A_{B_{m_k-1}} v_{B_{m_{k-1}}})(v_{\tau(B)} - v_{\tau(C)}) - v^2_{\tau(B)} + v^2_{\tau(C)} + (v_{\tau(B)} - v_{\tau(C)}) v_{\tau(B\cap C)}$ by the induction hypothesis. In order for this to equal 0, we must have that v_A goes to $v_{\tau(B)\cup\tau(C)} = v_{\tau(B\cup C)} = v_{\tau(A)}$. Thus, $\hat{\sigma}(v_A) = v_{\tau(A)}$ for all $v_A \in V$, and so $\tau \in$Aut(Γ).

Therefore, Aut$(A(\Gamma)) = k^* \times$ Aut(Γ). □

Consequently, the Aut$(A(\Gamma))$-submodules of $A(\Gamma)_{[i]}$ are precisely the Aut(Γ)-submodules. Since Aut(Γ) is finite we have that $A(\Gamma)_{[i]}$ is a completely reducible Aut$(A(\Gamma))$-module whenever characteristic $k = 0$.

Chapter 5

Graded trace generating functions

Pass to the associated graded algebra, gr$A(\Gamma)$. Let $\phi_1, ..., \phi_l$ denote all of the distinct irreducible representations of Aut$A(\Gamma)$ and let χ_j denote the character afforded by ϕ_j. Aut$A(\Gamma)$ acts on each $A(\Gamma)_{[i]}$, and so the completely reducible Aut$A(\Gamma)$-module $A(\Gamma)_{[i]}$ may be written as $\bigoplus_{j=1}^{l} m_{ij}\phi_j$. Basis $\mathcal{B}(\Gamma)$ of $A(\Gamma)$ is invariant under the automorphism σ. Therefore, the trace of σ on $grA(\Gamma)$, $Tr\sigma|_{A(\Gamma)}$, is the number of fixed basis elements.

Remark: $Tr\sigma$ is the dimension of the subalgebra $A(\Gamma^\sigma)$, which is not the same as the dimension of the fixed point space. The subalgebra $A(\Gamma^\sigma)$ described in Chapter 3 is the span of the set of fixed elements of the basis. On the other hand, the fixed point space is the span of the sums of orbits of σ. Averages over orbits are in the fixed point space, but not in the subalgebra.

The Hilbert series gives the graded dimension of an algebra; the coefficient of t^k is the k-th graded dimension (see Chapter 2 for the grading on our algebras). Write this as $H(t) = \sum dim(A(\Gamma)_{[k]})t^k$. We will be finding graded trace generating functions which generalize the idea of Hilbert series; the graded trace generating functions for $A(\Gamma)$ are Hilbert series for $A(\Gamma^\sigma)$. We will give two methods by which to find the graded trace generating functions for general $A(\Gamma)$. The first will be to essentially count "allowable" and "non-allowable" words - a generating function that gives the number of irreducible words in each grading in the subalgebra $A(\Gamma^\sigma)$. The second will generalize Theorem 2 in [RSW], which gives the Hilbert series for $A(\Gamma)$, to use on the subgraph Γ^σ and subalgebra $A(\Gamma^\sigma)$. These graded trace generating functions will be used to find the multiplicities of irreducible representations.

5.1 Method 1 - Counting fixed words:

The $Tr\sigma|_{A(\Gamma)_{[i]}}$ is the number of fixed basis elements of degree i. In other words, the number of sequences $(x_1, k_1), ..., (x_l, k_l)$ such that $1 \leq k_j \leq |x_j|$, $k_1 + \cdots + k_l = i$, $e(x_i, k_1) \cdots e(x_l, k_l)$ is irreducible, and $\sigma x_j = x_j \forall j$. Recall that $e(x_i, k)e(x_j, l)$ is reducible if there is a path from x_i to x_j and the level of x_i equals the level of x_j plus k.

Lemma 5.1. *Let X be a vector space with fixed basis $\mathcal{B}$. Let $Z = \{V \subseteq X : V$ subspace, $V = span(V \cap \mathcal{B})\}$. Then $Z(+, \cap)$ is a lattice isomorphic to the lattice of $\mathcal{P}(\mathcal{B})(\cap, \cup)$.*

Proof. The map $\phi : Z \to \mathcal{P}(\mathcal{B})$ defined by $\phi(V) = V \cap \mathcal{B}$ is a lattice isomorphism. □

Remark: The lattice of subsets of $\mathcal{B}$ is distributive, and so Z is distributive.

Theorem 5.2. *Let $X = \sum X_i$ be a graded vector space and let the basis $\mathcal{X}$ of X consist of homogeneous elements; $\mathcal{X} = \cup X_i$, where $\mathcal{X}_i = \mathcal{X} \cap X_i$. Then $T(X)$ is bi-graded with $T(X)_{i,j} = span\{x_{l_1} \cdots x_{l_i} : x_{l_k} \in \mathcal{X}_{l_k}, l_1 + ... + l_i = j\}$. Let $\mathcal{Y}$ be a finite set of quadratic monomials in $\mathcal{X}$, and define $Y =< \mathcal{Y} >\subseteq T(X)$. Let $|\cdot|$ denote the bi-graded dimension of the space. Then*

$$|T(X)/Y| = \frac{1}{1 - |X| + |Y| - |XY \cap YX| + |X^2Y \cap XYX \cap YX^2| - \cdots}.$$

Proof. Denote $T(X)/Y$ by A in this proof. The graded dimension of $T(X)$ is $\frac{1}{1-|X|}$. Because each generator in Y is a quadratic monomial, as vector spaces we can identify A with the subspace of $T(X)$ spanned by words not containing a subword in Y.

For $i \geq 0$ define $Y^{(i)} := \bigcap_{j=0}^{i} X^j Y X^{i-j}$; note that $Y^{(0)} = Y$. It will also be convenient to define $Y^{(-1)} := X$ and $Y^{(-2)} := k$. For $i \geq -2$, let

$T_i := T(X)Y^{(i)}/(T(X)YX^{i+2}T(X) \cap T(X)Y^{(i)})$. Also, define $T_{-3} := T(X)/(T(X)XT(X))$.

Now define a map $\phi_i : T_i \to T_{i-1}$ for $i \geq -1$ by $a + T(X)YX^{i+2}T(X) \cap T(X)Y^{(i)} \mapsto a + T(X)YX^{i+1}T(X) \cap T(X)Y^{(i-1)}$. Also, define $\phi_{-2} : T_{-2} \to T_{-3}$ by $a + T(X)YT(X) \mapsto a + T(X)XT(X)$. Because $Y^{(i)} = XY^{(i-1)} \cap YX^i \subseteq XY^{(i-1)}$, $T(X)Y^{(i)} \subseteq T(X)Y^{(i-1)}$. Also, $T(X)YX^{i+2}T(X) \subseteq T(X)YX^{i+1}T(X)$ since $X^{i+2}T(X) = X^{i+1}XT(X) \subseteq X^{i+1}T(X)$. Thus, ϕ_i is a well-defined map.

Next we will show that the sequence $\cdots \to T_j \to T_{j-1} \to \cdots \to T_{-2} \to T_{-3} \to 0$ is exact. For $i \geq 0$, $\phi_{i-1}(\phi_i(a+T(X)YX^{i+2}T(X)\cap T(X)Y^{(i)})) = a+T(X)YX^iT(X)\cap T(X)Y^{(i-2)}$. Now $a \in T(X)Y^{(i)} \subseteq T(X)Y^{(i-1)} \subseteq T(X)Y^{(i-2)}$. Also, $a \in T(X)Y^{(i)} =$ $T(X)(YX^i \cap \cdots \cap X^iY) \subseteq T(X)YX^i \subseteq T(X)YX^iT(X)$. Thus, the image is 0 and $\mathrm{im}\phi_i \subseteq \ker\phi_{i-1}$.

To show the other inclusion, note that $\ker\phi_{i-1} = \{a+T(X)YX^{i+1}T(X)\cap T(X)Y^{(i-1)} : a \in T(X)YX^iT(X)\cap T(X)Y^{(i-2)}\}$. For $a \in T(X)Y^{(i-1)}$ define $\bar{a} = a+T(X)YX^{i+1}T(X)\cap T(X)Y^{(i-1)}$. Assume that $\bar{a} \in \ker\phi_{i-1}$. Then $a \in T(X)YX^iT(X)$. We want that $a \in T(X)Y^{(i)}$. To show this first observe that $a \in T(X)Y^{(i-1)}\cap T(X)YX^iT(X) = T(X)Y^{(i-1)}\cap (T(X)YX^i + T(X)YX^{i+1}T(X))$. Let $\mathcal{B}$ be the set of all monomials of $\mathcal{X}$; $\mathcal{B}$ is a basis for $T(X)$ and $\mathcal{Y} \subseteq \mathcal{B}$. Consider the set of X^iYX^j. Now X^iYX^j is equal to the span of $\mathcal{X}^i\mathcal{Y}\mathcal{X}^j$, and so Lemma 5.1 applies. Therefore, the lattice generated by all the X^iYX^j is distributive. Hence, we have that $T(X)Y^{(i-1)} \cap (T(X)YX^i + T(X)YX^{i+1}T(X)) = T(X)Y^{(i-1)}\cap T(X)YX^i + T(X)Y^{(i-1)}\cap T(X)YX^{i+1}T(X)$. Since $T(X)Y^{(i-1)}\cap T(X)YX^i \subseteq T(X)Y^{(i)}$, $a \in T(X)Y^{(i)}$ and $\bar{a} \in \mathrm{im}\phi_i$. Therefore, $\mathrm{im}\phi_i = \ker\phi_{i-1}$ for $i \geq 0$.

Since ϕ_{-1} is homogeneous in degree, $\mathrm{im}\phi_{-1} = A \cap T(X)X = \ker\phi_{-2}$. Furthermore, the image of ϕ_{-2} is nonzero and maps to a one-dimensional space, and thus is surjective. Therefore, the sequence is exact.

Finally we will show that $T_i \cong AY^{(i)}$. We will do this by proving that $T(X)Y^{(i)} = AY^{(i)} \oplus T(X)YX^{i+2}T(X)\cap T(X)Y^{(i)}$. $T(X)Y^{(i)} = (A+T(X)YT(X))Y^{(i)} \subseteq AY^{(i)} + T(X)YT(X)Y^{(i)} \subseteq AY^{(i)} + T(X)YX^{i+2}T(X) \cap T(X)Y^{(i)}$. Also, $AY^{(i)} \cap T(X)YX^{i+2}T(X) \subseteq AX^{i+2} \cap T(X)YT(X)X^{i+2} \subseteq (A \cap T(X)YT(X))X^{i+2} = (0)$. Thus, our claim is proved and $T_i \cong AY^{(i)}$. In particular,

$T_0 = T(X)Y/(T(X)YX^2T(X) \cap T(X)Y) \cong AY$,

$T_{-1} = T(X)X/(T(X)YXT(X) \cap T(X)X) \cong AX$,

$T_{-2} = T(X)/T(X)YT(X) \cong A$, and

$T_{-3} = T(X)/T(X)XT(X) \cong k$. Hence, we can write our exact sequence as

$\cdots \to AY^{(j)} \to AY^{(j-1)} \to \cdots \to AY \to AX \to A \to k \to 0.$

Therefore, $1 = |k| = \sum(-1)^i|T_i| = |A| - |A||X| + \sum_{i\geq 0}(-1)^i|A||Y^{(i)}| =$ $|A|(1 - |X| + \sum_{i\geq 0}(-1)^i|Y^{(i)}|)$ implies that

$$|A| = \frac{1}{1 - |X| + \sum_{i\geq 0}(-1)^i|Y^{(i)}|}$$

$\square$

Define another grading on $A(\Gamma)$ by $A(\Gamma)_{[[k]]} = \text{span}\{e(v_1, i_1)\cdots e(v_k, i_k) : (v_j, i_j) \not> (v_{j+1}, i_{j+1})\}$. This induces an increasing filtration on $A(\Gamma)$, $A(\Gamma)_{((k))} = \text{span}\{e(v_1, i_1)\cdots e(v_j, i_j) : j \leq k, (v_j, i_j) \not> (v_{j+1}, i_{j+1})\}$. Thus as a vector space we can identify $A(\Gamma)$ with its associated graded algebra, $gr'A(\Gamma)$.

Lemma 5.3. *Define* $W(\Gamma^\sigma) = span\{e(v, k) : l \geq 0, v \in V_\sigma, 1 \leq k \leq |v|\}$ *and* $R(\Gamma^\sigma)$ *the two-sided ideal in* $T(W(\Gamma^\sigma))$ *generated by* $\tilde{R}(\Gamma^\sigma) = span\{e(v, k)e(u, l) : v > u \in V_\sigma, k = |v| - |u|\}$. *Then* $A(\Gamma^\sigma) = T(W(\Gamma^\sigma))/R(\Gamma^\sigma)$ *as a subalgebra of* $gr'A(\Gamma)$.

Proof. This follows from the definitions of $A(\Gamma^\sigma)$ and of $gr'A(\Gamma)$. $\square$

$W(\Gamma^\sigma)$ and $R(\Gamma^\sigma)$ satisfy the hypotheses for Theorem 5.2. Therefore,

$$|A(\Gamma^\sigma)| = \frac{1}{1 - |W(\Gamma^\sigma)| + |\tilde{R}(\Gamma^\sigma)| - |\tilde{R}(\Gamma^\sigma)W(\Gamma^\sigma) \cap W(\Gamma^\sigma)\tilde{R}(\Gamma^\sigma)| + \cdots}.$$

5.2 Method 2 - Generalizing the Hilbert Series Equation for $A(\Gamma)$:

Theorem 2 in [RSW] gives the Hilbert series of $A(\Gamma)$ as:

$$H(A(\Gamma), t) = \frac{1 - t}{1 + \sum_{v_1 > \cdots > v_l \geq *} (-1)^l t^{|v_1| - |v_l| + 1}}. \tag{5.1}$$

We would like to apply the equation $H(A(\Gamma), t)$ to the subalgebras created by our fixed points. Take the subgraph of Γ consisting of the points fixed by an automorphism σ. This generates a subalgebra of gr$A(\Gamma)$ in the way described in Chapter 3. Thus, we are using

Equation (5.1) with the additional condition that the vertices in the sum are fixed by $\sigma \in Aut(A(\Gamma))$; call this modified formula $Tr_\sigma(A(\Gamma), t)$.

Theorem 5.4. *Let Γ be a layered graph with unique minimal element $*$ of level 0 and σ an automorphism of the graph. Let Γ^σ be the subgraph of Γ with vertices being those fixed by σ (as described in Chapter 3). Denote the Hilbert series of the subalgebra $A(\Gamma^\sigma)$, which is the graded trace function of σ acting on $A(\Gamma)$, by $Tr_\sigma(A(\Gamma), t)$ (or $Tr_\sigma(t)$ when $A(\Gamma)$ is clear). Then*

$$Tr_\sigma(A(\Gamma), t) = \frac{1-t}{1 - t \displaystyle\sum_{\substack{v_1 > \cdots > v_l \geq * \\ v_1, \ldots, v_l \in V_\sigma}} (-1)^{l-1} t^{|v_1| - |v_l|}}. \tag{5.2}$$

Proof. Write $Tr(t)$ for $Tr_\sigma(A(\Gamma), t)$ in this proof. Let $v_1, ..., v_l, v, w \in V_\sigma$. Recall that the basis for $A(\Gamma^\sigma)$ is

$$\mathcal{B}_\sigma = \{e(v_1, k_1) \cdots e(v_l, k_l) : v_1, ...v_l \in V_\sigma, 1 \leq k_i \leq |v_i|, e(v_i, k_i) \not> e(v_{i+1}, k_{i+1})\}.$$

For $v \in (V_\sigma)_+$, define $C_v = \bigcup_{k=1}^{|v|} e(v, k)\mathcal{B}_\sigma$, $B_v = C_v \cap \mathcal{B}_\sigma$, $D_v = C_v \backslash B_v$. Then $\mathcal{B}_\sigma = \{*\} \cup \bigcup_{v \in (V_\sigma)_+} B_v$. Let $Tr_v = Tr_\sigma(B_v, t)$, the graded dimension of the span of B_v. Then $Tr(t) = 1 + \sum_{v \in (V_\sigma)_+} Tr_v(t)$. We also have $Tr_\sigma(C_v, t) = (t + ... + t^{|v|})Tr(t) = t(\frac{t^{|v|}-1}{t-1})Tr(t)$ and, because $D_v = \bigcup_{v > w > *} e(v, |v| - |w|)B_w$, $Tr_\sigma(D_v, t) = \sum_{v > w > *} t^{|v|-|w|} Tr_w(t)$. Thus,

$$Tr_v(t) = t\left(\frac{t^{|v|}-1}{t-1}\right) Tr(t) - \sum_{v > w > *} t^{|v|-|w|} Tr_w(t). \tag{5.3}$$

This equation may be written in matrix form. Put an order on V_σ, arrange the elements in decreasing order, and index the elements of vectors and matrices by this ordered set. Let $\vec{Tr}(t)$ denote the column vector with $Tr_v(t)$ in the v-position and 0 in the $*$-position. Let $\vec{s}$ denote the column vector with entry $t^{|v|} - 1$ in the v-position, let $\vec{1}$ denote the column vector having 1 as each entry, and let $\zeta(t)$ denote the matrix with the entry in the (v, w)-position being $t^{|v|-|w|}$ if $v \geq w$ and 0 otherwise. Then rewriting Equation 5.3 gives $\zeta(t)\vec{Tr}(t) = \frac{t}{t-1}\vec{s}Tr(t)$.

Now $\zeta(t) - I$ is a strictly upper triangular matrix, and so $\zeta(t)$ is invertible; $\zeta^{-1}(t) = I - (\zeta(t) - I) + (\zeta(t) - I)^2 - \cdots$. Thus the (v, w)-entry of $\zeta^{-1}(t)$ is $\displaystyle\sum_{v = v_1 > \cdots > v_l = w \geq *} (-1)^{l+1} t^{|v_1| - |v_l|}$.

Thus we can multiply $\zeta(t)\vec{Tr}(t) = \frac{t}{t-1}\vec{s}Tr(t)$ by ζ^{-1} and then multiply by $\vec{1}^T$ to obtain $\vec{1}^T\vec{Tr} = Tr(t) - 1 = \frac{t}{t-1}\vec{1}^T\zeta^{-1}(t)\vec{s}Tr(t)$. Solving for $Tr(t)$ we get

$$Tr(t) = \frac{t-1}{t-1-t\vec{1}^T\zeta^{-1}(t)\vec{s}} = \frac{1-t}{1-t\displaystyle\sum_{v_1>\cdots>v_l\geq *}(-1)^{l-1}t^{|v_1|-|v_l|}}.$$

$\square$

Remark 1: ζ is the standard zeta matrix and ζ^{-1} is the Möbius matrix when $t = 1$.

Remark 2: We will normally apply this method; although, both methods can theoretically be applied in all layered graph algebras.

In general, if σ and τ are conjugate, Γ^σ and Γ^τ are isomorphic (by the conjugation acting on the subscripts of vertices). Thus, it is enough to find the graded trace functions for any one representative of each conjugacy class.

Chapter 6

Decomposition of Two Algebras: $A(\Gamma_{D_n})$ and Q_n

6.1 Definition and Hilbert Series of Two Algebras

6.1.1 Hasse graph of an n-gon: Γ_{D_n}

A Hasse graph, or Hasse diagram, is a graph which represents a finite poset $\mathcal{P}$. The vertices in the graph are elements of $\mathcal{P}$ and there is an edge between x, $y \in \mathcal{P}$ if $x < y$ and there does not exist a $z \in \mathcal{P}$ such that $x < z < y$. Furthermore, the vertex for x, v_x, is in a lower level than that for y, v_y (if we talk about layers in the graph, $|v_x| = |v_y| - 1$).

Consider a polytope. We can put a partial order on the set of k-faces in the polytope by $x < y$ if x is an $(n-1)$-face, y is an n-face and x is a face of y. For example, if there is an edge e between v and w in the polytope, then $e > v, w$ and $e \not> u$ for all $u \neq v, w$.

Thus, the Hasse graph of an n-gon has one vertex in levels 0 and 3 and n vertices on levels 1 and 2. The top vertex is connected to all vertices in level 2 (all edges are in the 2-dimensional polygon), each vertex in level 2 is connected to the vertex directly below it and the one to that vertex's right, with wrapping around to the first vertex in level one for the last vertex in level two (each edge connects two adjacent vertices), and each vertex in level 1 is connected to the minimal vertex. Label the vertices by using subscripts in $\mathbb{Z}/(n)$. In level 1 call the vertices $w_1, ..., w_n$, call the vertices in level 2 $v_{12}, ..., v_{n1}$ (where the subscripts indicate to which vertices in level 1 the vertex is connected), and the top vertex is u. See Figure 6.1.

We consider the algebra $A(\Gamma_{D_n})$ determined by this graph. The construction of this algebra is described in [GRSW] (see Chapter 2). In brief (using the definition given in Proposition 2.2), the generators are the vertices and the relations are that two paths which have the same starting and ending vertices are equivalent. We can write these relations by:

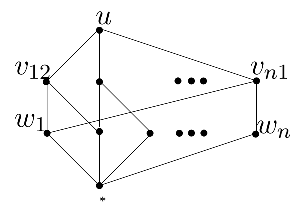

Figure 6.1: The Graph Γ_{D_n}

1) $v_{ii+1}(w_i - w_{i+1}) - w_i^2 + w_{i+1}^2$

2) $u(v_{ii+1} - v_{i+1i+2}) - v_{ii+1}^2 + v_{i+1i+2}^2 + (v_{ii+1} - v_{i+1i+2})w_{i+1}$, $1 \leq i \leq n$.

We will give here two bases for $A(\Gamma_{D_n})$, one for each of the two definitions of $A(\Gamma)$ given in Chapter 2. First we will give a basis in terms of the vertices (Proposition 2.2).

Proposition 6.1. *A basis $\mathcal{B}$ of $A(\Gamma_{D_n})$ consists of $*$ and the set of all words in u, $v_{i\,i+1}$, and w_i such that the following conditions on the words hold: the subword $v_{ii+1}w_j$ only occurs if $j \neq i+1$, the subword uv_{ii+1} only if $i = 1$, and $uv_{ii+1}w_j$ only if $i = j = 1$.*

Proof. We can describe a basis of monomials for $A(\Gamma_{D_n})$ using Bergman's Diamond Lemma [Berg]. Put a partial order on the generators such that $u > v_{i\,i+1} > w_j \forall i, j$ and $v_{i\,i+1} > v_{j\,j+1}$ and $w_i > w_j$ if $i > j$. Order monomials lexicographically. The reductions are $uv_{i+1\,i+2} \equiv uv_{i\,i+1} - v_{ii+1}^2 + v_{i+1i+2}^2 + (v_{ii+1} - v_{i+1i+2})w_{i+1}$, $1 \leq i \leq n-1$ and $v_{ii+1}w_{i+1} \equiv v_{ii+1}w_i - w_i^2 + w_{i+1}^2$, $1 \leq i \leq n$.

We need to find a complete list of reductions so that all ambiguities resolve. The only ambiguity will occur when we have a word that ends in v overlapping with one beginning with v; i.e. $uv_{i+1\,i+2}w_{i+2}$.

$(uv_{i+1\,i+2})w_{i+2} \equiv (uv_{ii+1} - v_{ii+1}^2 + v_{i+1i+2}^2 + (v_{ii+1} - v_{i+1i+2})w_{i+1})w_{i+2} \equiv uv_{ii+1}w_{i+2} - v_{ii+1}^2 w_{i+2} + v_{i+1i+2}(v_{i+1i+2}w_{i+1} - w_{i+1}^2 + w_{i+2}^2) + (v_{ii+1}w_i - w_i^2 + w_{i+1}^2)w_{i+2} - v_{i+1i+2}w_{i+1}w_{i+2} \equiv uv_{ii+1}w_{i+2} - v_{ii+1}^2 w_{i+2} + v_{i+1i+2}^2 w_{i+1} - v_{i+1i+2}w_{i+1}^2 + v_{i+1i+2}w_{i+1}w_{i+2} - w_{i+1}^2 w_{i+2} + w_{i+2}^3 + v_{ii+1}w_i w_{i+2} - w_i^2 w_{i+2} + w_{i+1}^2 w_{i+2} - v_{i+1i+2}w_{i+1}w_{i+2} = uv_{ii+1}w_{i+2} + v_{i+1i+2}^2 w_{i+1} - v_{ii+1}^2 w_{i+2} - v_{i+1i+2}w_{i+1}^2 + v_{ii+1}w_i w_{i+2} + w_{i+2}^3 - w_i^2 w_{i+2}$ and

$u(v_{i+1\,i+2}w_{i+2}) \equiv u(v_{i+1i+2}w_{i+1} - w_{i+1}^2 + w_{i+2}^2) \equiv (uv_{ii+1} - v_{ii+1}^2 + v_{i+1i+2}^2 + (v_{ii+1} - v_{i+1i+2})w_{i+1})w_{i+1} - uw_{i+1}^2 + uw_{i+2}^2 \equiv uv_{ii+1}w_i - uw_i^2 + uw_{i+1}^2 - v_{ii+1}(v_{ii+1}w_i - w_i^2 + w_{i+1}^2) +$

$v_{i+1i+2}^2 w_{i+1} + v_{ii+1} w_i w_{i+1} - w_i^2 w_{i+1} + w_{i+1}^3 - v_{i+1i+2} w_{i+1}^2 - uw_{i+1}^2 + uw_{i+2}^2 \equiv uv_{ii+1} w_i + uw_{i+2}^2 - uw_i^2 - v_{ii+1}^2 w_i + v_{ii+1} w_i^2 - v_{ii+1} w_i w_{i+1} + w_i^2 w_{i+1} - w_{i+1}^3 + v_{i+1i+2}^2 w_{i+1} + v_{ii+1} w_i w_{i+1} - w_i^2 w_{i+1} + w_{i+1}^3 - v_{i+1i+2} w_{i+1}^2 = uv_{ii+1} w_i + uw_{i+2}^2 - uw_i^2 + v_{i+1i+2}^2 w_{i+1} - v_{ii+1}^2 w_i - v_{i+1i+2} w_{i+1}^2 + v_{ii+1} w_i^2$.

Thus we need to add an additional reduction; namely, $uv_{ii+1} w_{i+2} \equiv uv_{ii+1} w_i + uw_{i+2}^2 - uw_i^2 + v_{ii+1}^2 w_{i+2} - v_{ii+1}^2 w_i - v_{ii+1} w_i w_{i+2} + v_{ii+1} w_i^2 - w_{i+2}^3 + w_i^2 w_{i+2}$. This does not create additional ambiguities since this reduction ends in w and we have no reductions which begin in w. Also, no reductions end in u. Thus, all ambiguities now resolve.

Therefore, by Bergman's Diamond Lemma, $A(\Gamma_{D_n})$ may be identified with the k-module of monomials which are irreducible under these reductions. Hence, $\mathcal{B}$ is a basis for $A(\Gamma_{D_n})$. $\square$

Next follows a basis in terms of edges (Thm 2.1).

Proposition 6.2. $\mathcal{B}' = \{e(x_1, k_1) \cdots e(x_l, k_l) : l \geq 0, x_1, ..., x_l \in \{u, v_{12}, ..., v_{n1}, w_1, ..., w_n\}$, $1 \leq k_i \leq |x_i|, (x_i, k_i) \not\gtrdot (x_{i+1}, k_{i+1})\}$ *is a basis for* $A(\Gamma_{D_n})$.

Proof. This follows directly from Theorem 2.1. $\square$

In the preliminaries we stated that the algebra is generated by distinguished edges and so we can identify the distinguished edges with the vertices which are their tails - e_v is identified with v. Thus $e(v, k)$ can be expressed as a product of k vertices (recall we are writing $e(v, k)$ in lieu of $\hat{e}(v, k)$), and so there is a correlation between the bases $\mathcal{B}$, $\mathcal{B}'$ as follows:

$e(u, 3) \leftrightarrow uv_{12} w_1$ $\quad e(u, 2) \leftrightarrow uv_{12}$ $\quad e(u, 1) \leftrightarrow u$

$e(v_{ii+1}, 2) \leftrightarrow v_{ii+1} w_i$ $\quad e(v_{ii+1}, 1) \leftrightarrow v_{ii+1}$ $\quad e(w_i, 1) \leftrightarrow w_i$

It is important to observe that in the associated graded algebra, $\sigma \in Aut(A(\Gamma))$ permutes the elements of gr$\mathcal{B}$ and gr$\mathcal{B}'$.

Recall that the Hilbert series of a graded algebra is $H(A, t) = \sum dim(A_{[k]}) t^k$.

Proposition 6.3. *The Hilbert series for* $A(\Gamma_{D_n})$ *is*

$$H(A(\Gamma_{D_n}), t) = \frac{1}{1 - (2n+1)t + (2n-1)t^2 - t^3} = \frac{1 - t}{1 - (2n+2)t + 4nt^2 - 2nt^3 + t^4}$$

Proof. We will give two proofs of this proposition. The first one uses Proposition 6.1 and induction to count basis elements. This will give us a recursion that can then be written as a generating function. The second method of proof uses Equation 5.1.

Method 1: By Proposition 6.1, there are $n(n-1)$ subwords of the form $v_{ii+1}w_j$ which can occur in an element of $\mathcal{B}$ and exactly one of the forms $uv_{i\,i+1}$ and $uv_{i\,i+1}w_j$. This means that there are n subwords of the form $v_{ii+1}w_j$ which cannot occur, $n-1$ of the form $uv_{i\,i+1}$, and n^2-1 of the form $uv_{i\,i+1}w_j$.

Let $d_k = \dim(A(\Gamma_{D_n})_{[k]})$.

We will proceed by induction.

- $d_0 = 1$
- $d_1 = 2n+1$: Every word of length one belongs to the basis since all reducible subwords are of length greater than one. A basis is: $\{u,\, v_{ii+1},\, w_i\}$.
- $d_2 = 4n^2 + 2n + 2$: There are $(2n+1)^2$ elements of length two and $2n-1$ of them are reducible. Hence, the dimension is $(2n+1)^2 - (2n-1) = 4n^2 + 2n + 2$. A basis is: $\{w_iu,\, w_iv_{j\,j+1},\, w_iw_j,\, uu,\, uv_{1\,2},\, uw_i,\, v_{ii+1}u,\, v_{ii+1}v_{jj+1},\, v_{ii+1}w_j\}$.

Use induction to determine d_k:

- If $x \in \mathcal{B}$ is a word of length $k-1$, then $w_ix \in \mathcal{B}$. Thus there are nd_{k-1} words of length k in $\mathcal{B}$ starting with w_i.
- If $x \in \mathcal{B}$ is a word of length $k-1$, then $v_{ii+1}x \in \mathcal{B}$ if and only if x does not begin with w_{i+1}. As determined in the previous bullet, there are nd_{k-2} basis elements starting with $w_j, 1 \le j \le n$ in degree $k-1$, and thus d_{k-2} of them beginning with w_{i+1}. Hence, for each i, there are $d_{k-1} - d_{k-2}$ possibilities for x. Therefore, there are $n(d_{k-1} - d_{k-2})$ words of length k of the form $v_{i\,i+1}x$.
- We will treat the case of words beginning with u in three cases. If $x \in \mathcal{B}$ of length $k-2$, $uux \in \mathcal{B}$ if and only if x does not begin with $v_{ii+1}, 2 \le i \le n$. There are $d_{k-1} - nd_{k-2} - n(d_{k-2} - d_{k-3})$ words beginning with u in degree $k-1$ (from previous bullets). Thus, there are that many words of the form $uux \in \mathcal{B}$. Next $uv_{12}x \in \mathcal{B}$ if

and only if x does not begin with $w_i, 2 \leq i \leq n$. Thus, there are $d_{k-2} - (n-1)d_{k-3}$ words of the form $uv_{12}x$. Finally, $uw_ix \in \mathcal{B}$ for all x. Thus, there are nd_{k-2} words of this form. This gives us a total of $d_{k-1} - 2nd_{k-2} + nd_{k-3} + d_{k-2} - (n-1)d_{k-3} + nd_{k-2} = d_{k-1} - (n-1)d_{k-2} + d_{k-3}$ words beginning with u.

Thus, $d_k = nd_{k-1} + n(d_{k-1} - d_{k-2}) + d_{k-1} - (n-1)d_{k-2} + d_{k-3}$
$= (2n+1)d_{k-1} - (2n-1)d_{k-2} + d_{k-3}$.

We can write this recurrence formula as a generating function following the method described by Wilf in [[Wilf],§1.2]. Let $H(t) = \sum_{i \geq 0} d_i t^i$ denote the generating function that we are trying to find. Let $d_{-2} = d_{-1} = 0, d_0 = 1$. Multiply both sides of the recursion by t^i and sum over $i \geq 0$. Then on the left-hand side we have $d_1 + d_2 t + d_3 t^2 + ... = \frac{H(t) - d_0}{t}$. And on the right hand side we have $(2n+1)H(t) - (2n-1)tH(t) + t^2 H(t)$. Solving for $H(t)$:

$H(t) - 1 = H(t)[(2n+1)t - (2n-1)t^2 + t^3] \Rightarrow$

$H(t)[1 - (2n+1)t + (2n-1)t^2 - t^3] = 1 \Rightarrow$

$H(t) = \frac{1}{1-(2n+1)t+(2n-1)t^2-t^3}$.

Method 2: Recall that the Hilbert series formula is

$$H(A(\Gamma), t) = \frac{1-t}{1 + \sum_{v_1 > \cdots > v_l \geq *} (-1)^l t^{|v_1| - |v_l| + 1}}.$$

In this example, the possible sequences indexing the sum are: $u, v_{ii+1}, w_i, *$, $u > v_{ii+1}$, $v_{ii+1} > w_i$, $v_{ii+1} > w_{i+1}$, $w_i > *$, $u > w_i$, $u > v_{ii+1} > w_i$, $u > v_{ii+1} > w_{i+1}$, $v_{ii+1} > *$, $v_{ii+1} > w_i > *$, $v_{ii+1} > w_{i+1} > *$, $u > *$, $u > v_{ii+1} > *$, $u > w_i > *$, $u > v_{ii+1} > w_i > *$, and $u > v_{ii+1} > w_{i+1} > *$. Thus, the coefficients of t, t^2, t^3, and t^4 are $-(2n+2)$, $n + 2n + n = 4n$, $n + n - 2n - 2n = -2n$, and $1 - 2n + 2n = 1$, respectively. The coefficient of t^k for $k \geq 5$ is zero.

Thus $H(A(\Gamma_{D_n}), t) = \frac{1-t}{1-(2n+2)t+4nt^2-2nt^3+t^4}$. $\square$

6.1.2 The Algebra Q_n

The algebras Q_n are the algebras associated with the lattice of subsets of $\{1, 2, ..., n\}$. Label the vertices in level i by $\{v_A : A \subseteq \{1, ..., n\}, |A| = i\}$. The lattice to which Q_4 is associated is shown in Figure 6.2 below. Their history and some properties are discussed in [GRSW].

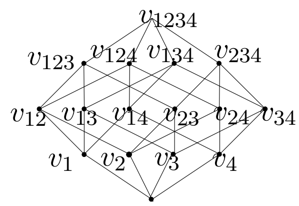

Figure 6.2: The lattice of subsets of $\{1, 2, 3, 4\}$

Following the definition given in Proposition 2.2, the generators of Q_n are the vertices $\{v_A : A \subseteq \{1, ..., n\}\}$ and the relations are

$$(*)\{v_A(v_{A\backslash i} - v_{A\backslash j}) - v^2_{A\backslash i} + v^2_{A\backslash j} + (v_{A\backslash i} - v_{A\backslash j})v_{A\backslash\{i,j\}} : A \subseteq \{1, ..., n\}, i, j \in A\}.$$

Furthermore,

Proposition 6.4. *$\mathcal{B}_{\mathcal{Q}} = \{e(v_{A_1}, k_1) \cdots e(v_{A_l}, k_l) : l \geq 0, A_1, ..., A_l \subseteq \{1, ..., n\}, 1 \leq k_i \leq |v_{A_i}| = |A_i|, (v_{A_i}, k_i) \not\triangleright (v_{A_{i+1}}, k_{i+1})\}$ is a basis for Q_n.*

Proof. This follows directly from Theorem 2.1. □

In [[RSW],Thm 3], Retakh, Serconek, and Wilson prove that

$$H(Q_n, t) = \frac{1 - t}{1 - t(2 - t)^n}$$

using Equation 5.1.

6.2 Automorphism Groups of $A(\Gamma_{D_n})$ and Q_n

Lemma 6.5. $Aut(\Gamma_{D_n}) = D_n$.

Proof. Any automorphism of the graph must preserve the set of vertices at each level and so acts on the set $\{w_1, ..., w_n\}$ of all n vertices in level 1. We may say $\sigma(w_i) = w_{\sigma(i)}$ (slightly abusing the use of σ). Thus we can think of an automorphism of the graph as being a permutation in S_n acting on the subscripts/labels of the vertices of level 1. This will uniquely determine what happens on higher levels; i.e. $\sigma(v_{ij}) = v_{\sigma(i)\,\sigma(j)}$. Labeling the vertices in level two by the vertices they are connected to in level one ensures that as long as the set of vertices in each level is preserved, the edges will be as well.

Recall that V_2 refers to the vertices in level two of the graph. Only permutations which send the set $V_2 = \{(i\,i+1) : 1 \leq i \leq n\}$ to itself are allowed. Clearly $r = (12...n)$ fixes V_2. We may replace σ by $r^i\sigma$ for some i and assume $\sigma(1) = 1$. Then $\sigma(12)$ is either (12) or $(1n)$, which implies either $\sigma(2) = 2$ (and thus $\sigma = id$) or $\sigma(2) = n$. In the latter case $\sigma = (2n)(3\,n-1)(4n-2)\cdots = s$. Thus r and s generate the automorphism group of Γ_{D_n}; this is the dihedral group on n elements, D_n. Note that these automorphisms may be viewed as reflections and rotations of the n-gon. $\square$

Theorem 6.6. *a) If $n \geq 3$, $Aut(A(\Gamma_{D_n})) = k^* \times D_n$, k the base field*

b) If $n = 2$,

$$Aut(A(\Gamma_{D_2})) \cong \{M \in GL(3,k) : M = \begin{bmatrix} c_1^1 + c_2^1 & c_1^2 - c_2^1 & c_2^1 - c_1^2 \\ 0 & c_1^1 & c_2^1 \\ 0 & c_1^2 & c_1^1 + c_2^1 - c_1^2 \end{bmatrix}, c_i^j \in k\,\forall i,j\}$$

Proof. a) By Lemma 6.5, $\mathrm{Aut}(\Gamma_{D_n}) = D_n$. It is clear by looking at the graph Γ_{D_n} (Figure 6.1) that for $n > 2$ Γ_{D_n} satisfies the conditions of Theorem 4.2. Therefore, $\mathrm{Aut}(A(\Gamma_{D_n})) = k^* \times D_n$.

b) In this case, Γ_{D_2} fails to satisfy condition (i) of Theorem 4.2; there are only two vertices on level 1. Consider the proof of Theorem 4.2. The proof is valid up until we apply (1) and (3) to $\{ik\}$ to find that $\sigma(v_i) = c^i_{\tau(i)}v_{\tau(i)} + c^i_{\tau(k)}v_{\tau(k)}$. In the case where $n = 2$,

$\tau(i)+1 = \tau(i-1)+2 = \tau(i-1)$ in $\mathbb{Z}/(2)$, so $c^i_{\tau(i)+1} = c^i_{\tau(i-1)}$ can be nonzero. Thus, w_i can go to a sum of multiples of w_1 and w_2; $\sigma(w_i) = c^i_1 w_1 + c^i_2 w_2$. Because we only have one vertex in level two, v_{12}, it can only go to a multiple of itself plus multiples of w_1 and w_2. Thus we can drop the sub and superscripts on a and the superscripts on b_i: $\sigma(v_{12}) = av_{12} + b_1 w_1 + b_2 w_2$. We can rewrite (4) in the proof of Theorem 4.2 as $a(c^1_1 - c^2_1)(w^2_1 - w^2_2) + (b_1 w_1 + b_2 w_2)(c^1_1 - c^2_1)(w_1 - w_2) = ((c^1_1)^2 - (c^2_1)^2)w^2_1 + (c^1_1 c^1_2 - c^2_1 c^2_2)(w_1 w_2 + w_2 w_1) + ((c^1_2)^2 - (c^2_2)^2)w^2_2$. We can conclude from this that $a + b_1 = c^1_1 + c^2_1$, $a + b_2 = c^1_2 + c^2_2$, and $-b_1(c^1_1 - c^2_1) = c^1_1 c^1_2 - c^2_1 c^2_2 = b_2(c^1_1 - c^2_1) \Rightarrow -b_1 = b_2$ (else $c^1_1 c^1_2 = c^2_1 c^2_2 \Rightarrow c^1_1 = c^2_1$ and $c^1_2 = c^2_2$, which is not possible). These imply that $2a = c^1_1 + c^1_2 + c^2_1 + c^2_2 \Rightarrow a = c^1_1 + c^1_2 \Rightarrow b_1 = c^2_1 - c^1_2 = -b_2$.

Write the element $rv_{12} + sw_1 + tw_2$ as the vector $[\ r \quad s \quad t\ ]$. Then a way to visualize what this automorphism group looks like is to consider the invertible transformation matrix M that sends $[\ r \quad s \quad t\ ] \mapsto [\ r \quad s \quad t\ ] * M$

$$M = \begin{bmatrix} c^1_1 + c^1_2 & c^2_1 - c^1_2 & c^1_2 - c^2_1 \\ 0 & c^1_1 & c^1_2 \\ 0 & c^2_1 & c^1_1 + c^1_2 - c^2_1 \end{bmatrix}$$

This matrix is conjugate to a triangular matrix and thus stabilizes a flag. The spaces M stabilizes can be found by solving $[\ r \quad s \quad t\ ] * M = \alpha[\ r \quad s \quad t\ ]$. M stabilizes the one-dimensional spaces $k\begin{bmatrix} 1 & -1 & -1 \end{bmatrix}$ and $k\begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$. $\square$

Denote the lattice of subsets of $\{1, ..., n\}$ by $\mathcal{L}_{[n]}$. In other words, $Q_n = A(\mathcal{L}_{[n]})$.

Lemma 6.7. *If $n \geq 3$, $Aut(\mathcal{L}_{[n]}) = S_n$.*

Proof. Any automorphism of the graph must preserve the set of vertices at each level and so acts on the set $\{v_1, ..., v_n\}$ of all n vertices in level 1; so, we may say $\sigma(v_i) = v_{\sigma(i)}$ (slightly abusing the use of σ). Thus we can think of an automorphism of the graph as being a permutation in S_n acting on the subscripts/labels of the vertices of level 1. This will uniquely determine what happens on higher levels; i.e. $\sigma(v_A) = v_{\sigma(A)}$. Labeling the vertices in levels two and higher by the vertices to which there is a path to in level one ensures that as long as the set of vertices in each level is preserved, the edges will be as well. Since for each subset of $\{1,...,n\}$ of cardinality i level i has a vertex labeled by that subset, every element of S_n is an automorphism of the graph. In other words, for every $\tau \in S_n$, τ will permute the vertices on each level. $\square$

Theorem 6.8. *If $n \geq 3$, $Aut(Q_n) = k^* \times S_n$.*

Proof. We can see that $\mathcal{L}_{[n]}$ satisfies conditions (i) and (ii) of Theorem 4.2 since each subset of $\{1,...,n\}$ occurs exactly once as a vertex in the lattice and for $n > 2$ there are more than 2 singleton subsets. Condition (iii) is satisfied because each vertex v_B directly below a vertex v_A is obtained by removing exactly one element from A. Thus for any vertex v_C two levels below v_A, $|C| = |A| - 2$. Say $C = A \backslash \{i,j\}$. There are only two ways to obtain C: first remove i then j or vice versa. Therefore, there are only two paths from v_A to v_C.

Therefore, by Lemma 6.7 and Theorem 4.2, $\mathrm{Aut}(Q_n) = k^* \times S_n$. $\square$

6.3 Graded Trace Generating Functions for $A(\Gamma_{D_n})$ and Q_n

Let us calculate the generating functions for our algebras. First of all, the graded trace of the identity acting on the algebra is the graded dimension of the algebra. We will derive it using the theorems above to show that we get the same result as the Hilbert series given earlier.

6.3.1 The Algebra $A(\Gamma_{D_n})$

We will now find $Tr_\sigma(A(\Gamma_{D_n}), t) = \frac{1}{1-(a_1t+a_2t^2+a_3t^3)}$ using Method 1 in Chapter 5. Notice that because Γ_{D_n} has three levels, $\tilde{R}(\Gamma^{id}_{D_n})^{(i)} = \{0\}$ for $i \geq 2$. $W(\Gamma^{id}_{D_n})$ has basis $\{e(u,3)$, $e(u,2)$, $e(u,1)$, $e(v_{i\,i+1},2)$, $e(v_{i\,i+1},1)$, $e(w_i,1)$, $1 \leq i \leq n\}$. Hence $|W(\Gamma^{id}_{D_n})| = (2n+1)t + (n+1)t^2 + t^3$.

The reducible words of degree 2 are $e(u,2)e(w_i,1)$, $e(u,1)e(v_{i\,i+1},2)$, $e(u,1)e(v_{i\,i+1},1)$, $e(v_{i\,i+1},1)e(w_i,1)$, $e(v_{i\,i+1},1)e(w_{i+1},1)$. As this set is a basis for $\tilde{R}(\Gamma^{id}_{D_n})$ we have $|\tilde{R}(\Gamma^{id}_{D_n})| = 3nt^2 + 2nt^3$.

The overlaps of reducible words are $e(u,1)e(v_{i\,i+1},1)e(w_i,1), e(u,1)e(v_{i\,i+1},1)e(w_{i+1})$. This set is a basis for $\tilde{R}(\Gamma^{id}_{D_n})W(\Gamma^{id}_{D_n}) \cap W(\Gamma^{id}_{D_n})\tilde{R}(\Gamma^{id}_{D_n})$ and so $|\tilde{R}(\Gamma^{id}_{D_n})W(\Gamma^{id}_{D_n}) \cap W(\Gamma^{id}_{D_n})\tilde{R}(\Gamma^{id}_{D_n})| = 2nt^3$.
Hence $a_3 = 1-(n+n)+(n+n) = 1$, $a_2 = (1+n)-(n+n+n) = 1-2n$, and $a_1 = 1+n+n = 2n+1$.
Thus, we have

$$Tr_{id}(A(\Gamma_{D_n}),t) = \frac{1}{1-((2n+1)t-(2n-1)t^2+t^3)},$$

which agrees with the earlier results.

Now since only u is fixed by r^i, $W(\Gamma^{r^i}_{D_n}) = \text{span}\{e(u,3),e(u,2),e(u,1)\}$ and $\tilde{R}(\Gamma^{r^i}_{D_n}) = \{0\}$. Hence,

$$Tr_{r^i}(A(\Gamma_{D_n}),t) = \frac{1}{1-(t+t^2+t^3)}.$$

If n is even, $W(\Gamma^{s}_{D_n})$ has basis $\{e(u,3)$, $e(u,2)$, $e(u,1)$, $e(v_{12},2)$, $e(v_{12},1)$, $e(v_{n/2+1\,n/2+2},2)$, $e(v_{n/2+1\,n/2+2},1)\}$ and $\tilde{R}(\Gamma^{s}_{D_n})$ has basis $\{e(u,1)e(v_{i\,i+1},2),\ e(u,1)e(v_{i\,i+1},1) : i = 1, \frac{n}{2}+1\}$. Also, $W(\Gamma^{rs}_{D_n})$ has basis $\{e(u,3),e(u,2),e(u,1),e(w_2,1),e(w_{n/2+2},1)\}$ and $\tilde{R}(\Gamma^{rs}_{D_n})$ has basis $\{e(u,2)e(w_2,1),\ e(u,2)e(w_{n/2+2},1)\}$. If n is odd, $W(\Gamma^{s}_{D_n})$ has basis $\{e(u,3),e(u,2),e(u,1),e(v_{12},2),e(v_{12},1),e(w_{(n+3)/2},1)\}$ and $\tilde{R}(\Gamma^{s}_{D_n})$ has basis $\{e(u,1)e(v_{12},2),\ e(u,1)e(v_{12},2),\ e(u,2)e(w_{(n+3)/2},1)\}$. Computing each separately we see that

$$Tr_{s}(A(\Gamma_{D_n}),t) = Tr_{rs}(A(\Gamma_{D_n}),t) = \frac{1}{1-(3t+t^2-t^3)}.$$

Using Method 2 we can get our graded trace generating functions by applying Equation (5.2) to the subalgebras of $A(\Gamma_{D_n})$. The automorphism r^i only fixes u and the minimal vertex (see Figure 6.3). Since we have two vertices, no vertices one or two levels apart, and one pair of vertices three levels apart (and only one path between them),

$$Tr_{r^i} = \frac{1-t}{1-(2t-t^4)} = \frac{1-t}{1-t(2-t^3)} = \frac{1}{1-(t+t^2+t^3)}.$$

The automorphism s acting on the algebra when n is even fixes the top vertex, the

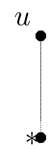

Figure 6.3: The subgraph $\Gamma^r_{D_n}$

minimal vertex, and two vertices on level two (v_{12} and $v_{n/2+1\,n/2+2}$)(see Figure 6.4). Similarly, when n is odd s fixes the top vertex, the minimal vertex, and one vertex on each of levels one and two (v_{12} and $w_{(n+3)/2}$). Finally, rs fixes the top vertex, the minimal vertex, and two vertices on level one (w_2 and $w_{n/2+2}$). Thus, in each case, there are 4 vertices, two edges of length one, and two of length two. For the coefficient of t^4, we have $u > *$, $u >$ vertex $> *$, and $u >$ vertex $> *$. Thus,

$$Tr_s(t) = Tr_{rs}(t) = \frac{1-t}{1-(4t-2t^2-2t^3+t^4)} = \frac{1-t}{1-t(2-t)(2-t^2)} = \frac{1}{1-(3t+t^2-t^3)}.$$

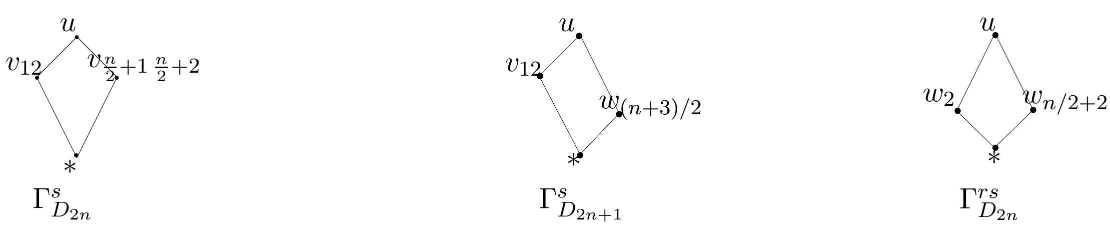

Figure 6.4: The subgraphs $\Gamma^\sigma_{D_n}$

The graded traces for the first few graded pieces are given in Example 6.1 below.

Example 6.1. $n = 4$

$Tr_\sigma|_{A(\Gamma_{D_4})_{[i]}}$ *is:*

i	*1*	r	...	r^m	s	rs
1	*9*	*1*	...	*1*	*3*	*3*
2	*74*	*2*	...	*2*	*10*	*10*
3	*604*	*4*	...	*4*	*32*	*32*
4	*4927*	*7*	...	*7*	*103*	*103*

6.3.2 The Algebra Q_n

Recall that V_σ denotes the set of vertices in Γ fixed by σ.

Theorem 6.9. *Let $\sigma \in S_n$ and $\sigma = \sigma_1 \cdots \sigma_m$ be its cycle decomposition. Denote the length of σ_j by i_j (so $i_j \geq 1$, $i_1 + \cdots + i_m = n$). Then*

$$Tr_\sigma(Q_n, t) = \frac{1-t}{1 - t\prod_{j=1}^{m}(2 - t^{i_j})}. \tag{6.1}$$

First of all, notice that when $\sigma = (1)$, this yields $H(Q_n, t)$ given above.

We will prove this using Equation 5.2 from Chapter 5:

$$Tr_\sigma(A(\Gamma), t) = \frac{1-t}{1 - t \sum\limits_{\substack{v_1 > \ldots > v_l \geq * \\ v_1, \ldots, v_l \in V_\sigma}} (-1)^{l-1} t^{|v_1| - |v_l|}}.$$

If $w \subseteq \{1, ..., n\}$ is σ-invariant, let $\|w\|$ be the number of σ-orbits in w. Also, let $\mathcal{O}_j$ denote the non-trivial orbit of σ_j.

The following Lemma and Corollary and their proofs are parallel to [[RSW],Lemma 2] and [[RSW],Corollary 1]. In the case where σ is the identity, they are the same.

Lemma 6.10. *Let $w \subseteq \{1, ..., n\}$ be fixed by σ. Then $\sum\limits_{w = w_1 \supset \cdots \supset w_l = \emptyset} (-1)^l = (-1)^{\|w\|+1}$ where the sum is over all chains of σ-fixed subsets of w.*

Proof. If $\|w\| = 1$, then $w = \mathcal{O}_j$ for some j. Thus, there are no fixed proper subsets of w, and we get that both sides are equal to 1. Assume the result holds for all sets with $\|\cdot\| < \|w\|$. Then $\sum\limits_{w \supset \cdots \supset w_l = \emptyset} (-1)^l = \sum\limits_{w \supset w_2 \supseteq \emptyset} \sum\limits_{w_2 \supset \cdots \supset w_l = \emptyset} (-1)^l = \sum\limits_{w \supset w_2 \supseteq \emptyset} (-1)^{\|w_2\|}$ by the induction assumption.

Now $\sum\limits_{w \supset w_2 \supseteq \emptyset} (-1)^{\|w_2\|} = \left(\sum\limits_{w \supseteq w_2 \supseteq \emptyset} (-1)^{\|w_2\|} \right) - (-1)^{\|w\|}$. Say $w = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_r$. Then the number of w_2 such that $w_2 \subseteq w$ and $\|w_2\| = i$ is $\binom{r}{i}$. Hence, we have $\sum\limits_{w \supset w_2 \supseteq \emptyset} (-1)^{\|w_2\|} = \left(\sum\limits_{w \supseteq w_2 \supseteq \emptyset} (-1)^{\|w_2\|} \right) - (-1)^{\|w\|} = \sum\limits_{i=0}^{r} \binom{r}{i} (-1)^i + (-1)^{\|w\|+1} = (-1)^{\|w\|+1}$, as desired, since the alternating sum of the binomial coefficients is zero. □

Corollary 6.11. *Let $\{1,...,n\} \supseteq v \supseteq w$ be fixed by σ. Then* $\sum_{v=v_1\supset v_2\supset\cdots\supset v_l=w}(-1)^l = (-1)^{\|v\|-\|w\|+1}$ *where the sum is over all chains of σ-fixed subsets.*

Proof. Let w' denote the complement of w in v. Sets u invariant under σ and satisfying $v \supseteq u \supseteq w$ are in one-to-one correspondence with the σ-invariant subsets of w' via the map $u \mapsto u \cap w'$. Thus, $\sum_{v=v_1\supset v_2\supset\cdots\supset v_l=w}(-1)^l = \sum_{w'=v'_l\supset\cdots\supset v'_1=\emptyset}(-1)^l = (-1)^{\|w'\|+1}$ by the lemma. Because $\|w'\| = \|v\| - \|w\|$, this gives us what we want. □

Proof of Theorem 6.9. By Corollary 6.11,

$$\sum_{v_1\supset\cdots\supset v_l\supseteq\emptyset}(-1)^l t^{|v_1|-|v_l|+1} = \sum_{\{1,...,n\}\supseteq v_1\supseteq v_l\supseteq\emptyset}(-1)^{\|v_1\|-\|v_l\|+1}t^{|v_1|-|v_l|+1}$$
$$= \sum_{\substack{w,v_l\\ w\cap v_l=\emptyset}}(-1)^{\|w\|+1}t^{|w|+1} = \sum_w 2^{m-\|w\|}(-1)^{\|w\|+1}t^{|w|+1}$$

The σ-invariant sets w are unions of σ-orbits. Write $a_j = 1$ if $\mathcal{O}_j$ is contained in w and $a_j = 0$ if not. Then the m-tuple $\{a_1, ..., a_m\}$ tells us which orbits are contained in w. We can then write $\sum_w 2^{m-\|w\|}(-1)^{\|w\|+1}t^{|w|+1}$ as $\sum_{a_1,...,a_m\in\{0,1\}}(-1)^{\sum a_j+1}2^{m-\sum a_j}t^{\sum(a_j i_j)+1}$. This equals

$$-t\sum_{a_1,...,a_m\in\{0,1\}}\prod_{j=1}^m(-1)^{a_j}2^{1-a_j}t^{a_j i_j}$$
$$= -t\prod_{j=1}^m\sum_{a_j=0}^1(-1)^{a_j}2^{1-a_j}t^{a_j i_j}$$
$$= -t\prod_{j=1}^m(2 - t^{i_j})$$

Therefore, we have (6.1). □

Example 6.2. *Here are the graded trace functions for Q_4:*

$$Tr_{(1)}(Q_4, t) = \frac{1-t}{1-t(2-t)^4} \qquad Tr_{(12)}(Q_4, t) = \frac{1-t}{1-t(2-t^2)(2-t)^2}$$

$$Tr_{(123)}(Q_4,t) = \frac{1-t}{1-t(2-t^3)(2-t)} \qquad Tr_{(12)(34)}(Q_4,t) = \frac{1-t}{1-t(2-t^2)^2}$$

$$Tr_{(1234)}(Q_4,t) = \frac{1-t}{1-t(2-t^4)}$$

6.4 Representations of Aut$(A(\Gamma))$ acting on $A(\Gamma)$

Now let us determine the multiplicities of the irreducible representations. Assume $A(\Gamma)_{[i]}$ is a completely reducible Aut(Γ)-module. Note that this is true in our examples. Fix n. Let the graded trace generating function be denoted by $Tr_\sigma(t) = \sum_i Tr_{\sigma,i}t^i$ where $Tr_{\sigma,i} = Tr\sigma|_{A(\Gamma)_{[i]}}$. Let ϕ be an irreducible representation of Aut(Γ) and $m_\phi(t) = \sum_i m_{\phi,i}t^i$ where $m_{\phi,i}$ is the multiplicity of ϕ in $A(\Gamma)_{[i]}$. Finally, let the matrix $C = [\chi_{\sigma\phi}]$ where $\chi_{\sigma\phi}$ is the trace of σ on the module which affords the irreducible representation ϕ; i.e. C is the character table of Aut(Γ).

Then, if we fix the degree, $Tr_{\sigma,i} = \sum_\phi \chi_{\sigma\phi}m_{\phi,i}$; so, we have $Tr_\sigma(t) = \sum_\phi \chi_{\sigma\phi}m_\phi(t)$. Write $\vec{Tr}(t) = [Tr_{\sigma_1}(t)...Tr_{\sigma_l}(t)]^T$ and $\vec{m}(t) = [m_{\phi_1}(t)...m_{\phi_l}(t)]^T$. Finally,

$$\vec{Tr}(t) = C^T\vec{m}(t) \implies \vec{m}(t) = (C^T)^{-1}\vec{Tr}(t).$$

6.4.1 Representations of Aut$(A(\Gamma_{D_n}))$ acting on $A(\Gamma_{D_n})$

Recall that the character table for D_n where $n = 2m$ is even is:

	1	r	...	r^j	...	r^m	s	rs
χ_{triv}	1	1	...	1	...	1	1	1
χ_{1-1}	1	1	...	1	...	1	-1	-1
χ_{-11}	1	-1	...	$(-1)^j$	...	$(-1)^m$	1	-1
χ_{-1-1}	1	-1	...	$(-1)^j$	...	$(-1)^m$	-1	1
χ_k	2	$2\cos(2\pi k/n)$	...	$2\cos(2\pi kj/n)$	...	$2\cos(2\pi km/n)$	0	0

where $(1 \le k \le m-1)$, $r = (12...n)$, and $s = (12)(3n)(4\,n-1)...(\frac{n}{2}+1\ \frac{n}{2}+2)$; so, $rs = (13)(4n)...(\frac{n}{2}+1\ \frac{n}{2}+3)$.

When $n = 2m+1$ is odd the character table is:

	1	r	...	r^j	...	r^m	s
χ_{triv}	1	1	...	1	...	1	1
χ_{1-1}	1	1	...	1	...	1	-1
χ_k	2	$2\cos(2\pi k/n)$	...	$2\cos(2\pi kj/n)$	...	$2\cos(2\pi km/n)$	0

where $(1 \leq k \leq m)$, $r = (12...n)$ and $s = (12)(3n)...(\frac{n+1}{2}\ \frac{n+5}{2})$.

Proposition 6.12. *Let $\vec{m}(t)$ be the vector of the graded multiplicities of the irreducible representations of D_n as described above. Set*
$a = \frac{1}{1-((2n+1)t-(2n-1)t^2+t^3)}$, $b = \frac{1}{1-(t+t^2+t^3)}$, *and* $c = \frac{1}{1-(3t+t^2-t^3)}$.

a) Let n be even. Then,

$$\vec{m}(t) = \begin{bmatrix} \frac{1}{2n}a + \frac{n-1}{2n}b + \frac{1}{2}c \\ \frac{1}{2n}a + \frac{n-1}{2n}b - \frac{1}{2}c \\ \frac{1}{2n}(a-b) \\ \frac{1}{2n}(a-b) \\ \frac{1}{n}(a-b) \\ \vdots \\ \frac{1}{n}(a-b) \end{bmatrix}$$

b) Let n be odd. Then,

$$\vec{m}(t) = \begin{bmatrix} \frac{1}{2n}a + \frac{n-1}{2n}b + \frac{1}{2}c \\ \frac{1}{2n}a + \frac{n-1}{2n}b - \frac{1}{2}c \\ \frac{1}{n}(a-b) \\ \vdots \\ \frac{1}{n}(a-b) \end{bmatrix}$$

This is obtained from deleting the third and fourth entries in the n is even case.

Proof. Multiply the transpose of the character table of D_n by $\vec{m}(t)$. The result is

$$\vec{Tr}(t) = \begin{bmatrix} a \\ b \\ \vdots \\ b \\ c \\ c \end{bmatrix}$$

as desired. $\square$

Notice that all of the representations are realized; and, with large multiplicity. The multiplicities for the first few degrees when $n = 4$ are given in Example 6.3.

Example 6.3. *The multiplicities of the representations in $A(\Gamma_{D_4})_{[i]}$ for $1 \leq i \leq 4$ are:*

	χ_{triv}	χ_{1-1}	χ_{-11}	χ_{-1-1}	χ_k
$m_{\phi,1}$	3	0	1	1	2
$m_{\phi,2}$	15	5	9	9	18
$m_{\phi,3}$	95	63	77	77	154
$m_{\phi,4}$	670	567	615	615	1230

6.4.2 Representations of S_n acting on Q_n

Unlike the $A(\Gamma_{D_n})$ case, we cannot write down one table giving all of the values in terms of the graded trace functions. However, we can give them in terms of the Frobenius formula. First, however, we will give an example.

Example 6.4. *Irreducible Representations for Q_4:*

The character table for S_4 is:

	(1)	(12)	(123)	(1234)	(12)(34)
χ_{triv}	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
χ_3	2	0	-1	0	2
χ_{reg}	3	1	0	-1	-1
$\chi_{sgn\otimes reg}$	3	-1	0	1	-1

Let $a = Tr_{(1)}(Q_4, t) = \frac{1-t}{1-t(2-t)^4}$ $\quad b = Tr_{(12)}(Q_4, t) = \frac{1-t}{1-t(2-t^2)(2-t)^2}$,
$c = Tr_{(123)}(Q_4, t) = \frac{1-t}{1-t(2-t^3)(2-t)}$ $\quad d = Tr_{(1234)}(Q_4, t) = \frac{1-t}{1-t(2-t^4)}$
$e = Tr_{(12)(34)}(Q_4, t) = \frac{1-t}{1-t(2-t^2)^2}$

Then, the multiplicities of the irreducible representations of S_4 *acting on* Q_4 *as sums of the graded trace generating functions are:*

$m_{triv} = \frac{1}{24}a + \frac{1}{4}b + \frac{1}{3}c + \frac{1}{4}d + \frac{1}{8}e$

$m_{sgn} = \frac{1}{24}a - \frac{1}{4}b + \frac{1}{3}c - \frac{1}{4}d + \frac{1}{8}e$

$m_3 = \frac{1}{12}a - \frac{1}{3}c + \frac{1}{4}e$

$m_{reg} = \frac{1}{8}a + \frac{1}{4}b - \frac{1}{4}d - \frac{1}{8}e$

$m_{sgn\otimes reg} = \frac{1}{8}a - \frac{1}{4}b + \frac{1}{4}d - \frac{1}{8}e$

The numerical values for the first few degrees are given below:

	χ_{triv}	χ_{sgn}	χ_3	χ_{reg}	$\chi_{sgn\times reg}$
$m_{\phi,1}$	4	0	1	3	0
$m_{\phi,2}$	26	1	17	36	13
$m_{\phi,3}$	219	54	239	434	273

We can also write the multiplicities in terms of Frobenius' formula. First we will give the notation used in the formula. Let C_i be a representative from the conjugacy class i and i_j be the number of j-cycles in i. Also, let λ be a partition of n (representing an irreducible representation), $\Delta(x) = \prod_{i<j}(x_i - x_j)$ and $P_j(x) = x_1^j + \cdots + x_k^j$ where k is at least the number of rows in λ. Set $l_1 = \lambda_1 + k - 1$, $l_2 = \lambda_2 + k - 2, ..., l_k = \lambda_k$. Finally, for $f(x) \in \mathbb{C}[x_1, ..., x_k]$ the scalar $f(x)_{l_1,...,l_k}$ is defined by $f(x) = \sum_{l_1,...,l_k} f(x)_{l_1,...,l_k} x_1^{l_1} x_2^{l_2} \cdots x_k^{l_k}$. Then Frobenius' formula says $\chi_\lambda(C_i) = [\Delta(x) \prod_j P_j(x)^{i_j}]_{l_1,...,l_k}$.

Proposition 6.13. *Let r denote the degree and λ_l the irreducible representation. Then*

$$m_{\lambda_l,r} = \left[\frac{1}{n!} \sum_{j=partition\ of\ n} \chi_{\lambda_l}(C_j)|\mathcal{C}(j)|Tr_{(j)}\right]_{(l_1,\ldots,l_k,r)}$$

Proof. Let

$$S = \begin{bmatrix} \chi_{\lambda_1}(C_1) & \cdots & \chi_{\lambda_1}(C_k) \\ \vdots & & \vdots \\ \chi_{\lambda_k}(C_1) & \cdots & \chi_{\lambda_k}(C_k) \end{bmatrix}$$

be the character table of S_n. By the orthogonality relations,

$$S^T S = D = \begin{bmatrix} \sum_i \chi_{\lambda_i}(C_1)^2 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sum_i \chi_{\lambda_i}(C_k)^2 \end{bmatrix}.$$

Thus

$$S^{-1} = D^{-1}S^T = \begin{bmatrix} \chi_{\lambda_1}(C_1)/\sum_i \chi_{\lambda_i}(C_1)^2 & \cdots & \chi_{\lambda_k}(C_1)/\sum_i \chi_{\lambda_i}(C_1)^2 \\ \vdots & & \vdots \\ \chi_{\lambda_1}(C_k)/\sum_i \chi_{\lambda_i}(C_k)^2 & \cdots & \chi_{\lambda_k}(C_k)/\sum_i \chi_{\lambda_i}(C_k)^2 \end{bmatrix}.$$

Because $\vec{m}(t)S = \vec{Tr}(t)$, $\vec{m}(t) = \vec{Tr}(t)S^{-1}$; and so,

$$m_{\lambda_l}(t) = \frac{\chi_{\lambda_l}(C_1)Tr_{\sigma_1}}{\sum_i \chi_{\lambda_i}(C_1)^2} + \cdots + \frac{\chi_{\lambda_l}(C_k)Tr_{\sigma_k}}{\sum_i \chi_{\lambda_i}(C_k)^2}.$$

However, $\sum_i \chi_{\lambda_i}(C_j)^2 = [S_n : \mathcal{C}(j)] = n!/|\mathcal{C}(j)|$, where $|\mathcal{C}(j)|$ is the size of the conjugacy class of partition j. Thus,

$$m_{\lambda_l,r} = \left[\frac{1}{n!} \sum_{j=\text{partition of n}} \chi_{\lambda_l}(C_j)|\mathcal{C}(j)|Tr_{(j)}\right]_{(l_1,\ldots,l_k,r)}$$

□

6.5 Representations of $\mathbf{Aut}(A(\Gamma))$ acting on $A(\Gamma)^!$

We will use the same methodology as for $A(\Gamma)$ to determine the irreducible representations that are realized in $A(\Gamma)^!$. See Chapter 3 for the definition of the dual. However, we will see that $Tr_\sigma(A(\Gamma)^!, t)$ has negative coefficients and so is not a generating function of a graded dimension, unlike in the case of $A(\Gamma)$.

6.5.1 Representations of $\mathbf{Aut}(A(\Gamma_{D_n}))$ acting on $A(\Gamma_{D_n})^!$

Proposition 6.14. *The set* $\{*,\, u^*,\, v^*_{ii+1},\, w^*_i\ 1 \le i \le n,\, u^*v^*_{ii+1}\ 1 \le i \le n-1,\, v^*_{ii+1}w^*_i\ 1 \le i \le n$*, and* $u^*v^*_{12}w^*_1\}$ *is a basis for the graded dual algebra* $A(\Gamma_{D_n})^!$.

Proof. The generators of $A(\Gamma_{D_n})^!$ are u^*, v^*_{ii+1}, w^*_i $1 \le i \le n$. In the associated graded algebra the relations are $v_{ii+1}(w_i - w_{i+1})$ and $u(v_{ii+1} - v_{i+1i+2})$. Thus, the relations in the dual are $u^{*2}, u^*w^*_i, v^*_{ii+1}u^*,\ w^*_iu^*, w^*_iw^*_j, v^*_{ii+1}v^*_{jj+1},\ w^*_iv^*_{jj+1}, u^*(v^*_{12} + \cdots + v^*_{n1}), v^*_{ii+1}w^*_j$ if $j \ne i, i+1, v^*_{ii+1}(w^*_i + w^*_{i+1})$. The elements in the graded dual follow. ∎

Now let us determine the trace on the graded pieces by seeing how each conjugacy class acts on the elements in the dual. As $A(\Gamma_{D_n})^! = A(\Gamma_{D_n})^!_{[0]} \oplus A(\Gamma_{D_n})^!_{[1]} \oplus A(\Gamma_{D_n})^!_{[2]} \oplus A(\Gamma_{D_n})^!_{[3]}$, there are only three degrees in the dual; so, we can calculate each independently.

Case 1: n=2m is even

The traces on the graded pieces are:

	1	r	...	r^j	...	r^m	s	rs
$Tr_{\sigma,1}$	2n+1	1	...	1	...	1	3	3
$Tr_{\sigma,2}$	2n-1	-1	...	-1	...	-1	-1	-1
$Tr_{\sigma,3}$	1	1	...	1	...	1	-1	-1

Now that we have the graded traces we can find the multiplicities of the representations by solving the system of equations: $\sum_{\phi} m_{\phi,i} * \chi_{\sigma\phi}(x) = Tr_{\sigma,i}(x)$, $x \in D_n$. They are:

	χ_{triv}	χ_{1-1}	χ_{-11}	χ_{-1-1}	χ_k
$m_{\phi,1}$	3	0	1	1	2
$m_{\phi,2}$	0	1	1	1	2
$m_{\phi,3}$	0	-1	0	0	0

Case 2: n=2m+1 is odd

The traces on the graded pieces are:

	1	r	...	r^j	...	r^m	s
$Tr_{\sigma,1}$	2n+1	1	...	1	...	1	3
$Tr_{\sigma,2}$	2n-1	-1	...	-1	...	-1	-1
$Tr_{\sigma,3}$	1	1	...	1	...	1	-1

The multiplicities are given below:

	χ_{triv}	χ_{1-1}	χ_k
$m_{\phi,1}$	3	0	2
$m_{\phi,2}$	0	1	2
$m_{\phi,3}$	0	-1	0

Notice that the graded traces and multiplicities are the same in both the even and odd cases.

These values give graded trace functions (in both even and odd cases) of:

$Tr_{(1)}(A(\Gamma_{D_n})^!, t) = 1 + (2n+1)t + (2n-1)t^2 + t^3$

$Tr_{r^i}(A(\Gamma_{D_n})^!, t) = 1 + t - t^2 + t^3$

$Tr_s(A(\Gamma_{D_n})^!, t) = Tr_{rs}(A(\Gamma_{D_n})^!, t) = 1 + 3t - t^2 - t^3$

6.5.2 Representations of S_n acting on $Q_n^!$

[[GGRSW],§6] determines a basis for $Q_n^!$ as follows:

Let $A \subseteq \{1, ..., n\}, B$ be the sequence $(b_1, ..., b_k)$, and $B' = \{b_1, ..., b_k\}$. Define $S(A : B) = s(A)s(A\backslash b_1) \cdots s(A\backslash b_1 \backslash ... \backslash b_k)$ where $s(A)$ is the image in $Q_n^!$ of the generator dual to $e(A, 1) \in Q_n$. Then $\mathcal{S} = \{S(A : B)|\min A \notin B \text{ and } b_1 > \cdots > b_k\} \cup \{\emptyset\}$ is a basis for $Q_n^!$. The relations in the associated graded dual are:

1) $s(A)\sum_{a \in A} s(A\backslash a) = 0, |A| \geq 2$

2) $s(A)s(A\backslash i)s(A\backslash i\backslash j) = -s(A)s(A\backslash j)s(A\backslash i\backslash j)$

3) $s(A)s(B) = 0$ if $B \nsubseteq A$ or $|B| \neq |A| - 1$.

As opposed to the case of Q_n, σ does not permute the basis elements of $Q_n^!$. Thus, it is not enough to count fixed basis elements to determine the trace. For each $S(A : B) \in \mathcal{S}$, we must write $\sigma S(A : B)$ as a linear combination of elements $\mathcal{S}$. Write this as $\sigma S(A : B) = S(\sigma A : \sigma B) = \displaystyle\sum_{S(C:D) \in \mathcal{S}} a_{\sigma A \sigma BCD} S(C : D)$. Then $\mathrm{Tr}_\sigma = \displaystyle\sum_{S(A:B) \in \mathcal{S}} a_{\sigma A \sigma BAB}$. We are going to get three possible values for a basis element's contribution to the trace: $-1, 0,$ or 1.

If B is σ-invariant, then let $l_B(\sigma)$ be the number of pairs i, j with $i < j$ and $\sigma b_i < \sigma b_j$. This is the length of σ restricted to B. If there exists $c \in B$ such that $\sigma(c) = minA$, then define $\sigma' := (c\, minA)\sigma$.

Recall that B' is the set $\{b_1, ..., b_k\}$.

Proposition 6.15.

$$a_{\sigma A \sigma B A B} = \begin{cases} (-1)^{l_B(\sigma)} & \textit{if } \sigma A = A, minA \notin \sigma B',\, \sigma B' = B' \\ (-1)^{l_B(\sigma')+1} & \textit{if } \sigma A = A, minA \in \sigma B' \\ & \textit{and for some } b \in B',\, \sigma(B' \backslash b) = B' \backslash minA \\ 0 & \textit{otherwise} \end{cases}$$

Proof. If $\sigma \in S_n$ and $\sigma B' = B'$, then by relation (2) above we have that

$$S(A : \sigma B) = (-1)^{l_B(\sigma)} S(A : B). \tag{6.2}$$

If min $A \in B'$, then by relation (1)

$$S(A : B) = -\sum_{c \in A \backslash B'} S(A : (b_1, ..., b_{k-1}, c)). \tag{6.3}$$

Let us break this down into parts.

(*) $a_{ABCD} = 0$ if $C \neq A$. This is true because no relation changes the first factor, $s(C)$, of $S(C : D)$.

(**) $a_{ABCD} = 0$ unless $B' = D'$ or $B' = (B' \cap D') \cup \{minA\}$. Only relation (1) can change which elements are removed, and that relation can only change one element.

Recall $\sigma S(A : B) = S(\sigma A : \sigma B) = \sum_{S(C:D) \in \mathcal{S}} a_{\sigma A \sigma B C D} S(C : D)$. We need to know the value of $a_{\sigma A \sigma B A B}$. By (*) and (**), this is 0 unless $\sigma A = A$ and $\sigma B' = B'$ or $\sigma B' = (B' \cap \sigma B') \cup \{minA\}$. If $\sigma A = A$ and $\sigma B' = B'$, then, by Equation 6.2, $a_{\sigma A \sigma B A B} = (-1)^{l_B(\sigma)}$. If $\sigma A = A$ and $\sigma B' = (B' \cap \sigma B') \cup \{minA\}$, write $\sigma(b_j) = minA$. Then $(b_j\, minA)\sigma B' = \sigma' B' = B'$. Thus, by Equation 6.3, $\sigma S(A : B) =$
$-S(A : (\sigma(b_1), ..., \hat{b_j}, ..., \sigma(b_k), b_j))+$other terms. And, again by Equation 6.2, $\sigma S(A : B) = (-1)^{l_B(\sigma')}(-S(A : B))$. Hence, $a_{\sigma A \sigma B A B} = (-1)^{l_B(\sigma)+1}$. ☐

Now that we know what each basis element contributes to the trace, we want to find $Tr_\sigma(Q_n^!, t)$.

Let us introduce some notation. For $\sigma \in S_n$, write $\sigma = \sigma_1 \cdots \sigma_m$, a product of disjoint cycles. Denote the orbits of σ by $\{\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_m\}$ and put an ordering on the orbits given by $\mathcal{O}_i < \mathcal{O}_j$ if the minimal element of $\mathcal{O}_i$ is less than that of $\mathcal{O}_j$. Say $\mathcal{O}_1 < \mathcal{O}_2 < \cdots < \mathcal{O}_m$. Let i_j be the size of $\mathcal{O}_j$ (equal to the length of σ_j).

Theorem 6.16.

$$Tr_\sigma(Q_n^!, t) = \frac{1 + t\prod_{k=1}^{m}(2 - (-t)^{i_k})}{1+t}$$

Proof. In order to prove the formula, we must take the sum over all $S(A : B) \in \mathcal{S}$ of $a_{\sigma A \sigma B A B}$, each of their contribution to the trace. We will do this in cases based on the value of the basis element's contribution to the trace.

Case 1: $\sigma A = A$, $\min A \notin \sigma B'$, $\sigma B' = B'$

Consider $B = \mathcal{O}_{r_1} \cup ... \cup \mathcal{O}_{r_l}$, where $r_1 < \cdots < r_l$. Because $B' \subseteq A$ and $\min A \notin \sigma B'$, A must contain all of $\mathcal{O}_{r_1}, ..., \mathcal{O}_{r_l}$ and at least one $\mathcal{O}_{r_0}$ such that $r_0 < r_1$ (so, $r_1 \neq 1$). Thus we must choose a nonempty subset of $\{\mathcal{O}_1, ..., \mathcal{O}_{r_1-1}\}$ and a subset A' of $\{\mathcal{O}_{r_1+1}, ..., \mathcal{O}_m\}$ such that $A' \cap \{\mathcal{O}_{r_1+1}, ..., \mathcal{O}_m\} = \emptyset$. This gives $2^{m-r_1-(l-1)}(2^{r_1-1} - 1) = 2^{m-l} - 2^{m-r_1-l+1}$ choices for A. Because $l_B(\sigma_{r_i}) = i_{r_i} - 1$, $l_B(\sigma) = \sum_{1 \le i \le l} l_B(\sigma_{r_i}) = i_{r_1} + \cdots + i_{r_l} - l$. Also, the degree of $S(A : B)$ is $i_{r_1} + \cdots + i_{r_l} + 1$. Thus the contribution toward Tr_σ for all such $S(A : B)$ given B is $d(B) = (-1)^{i_{r_1}+\cdots+i_{r_l}-l}[2^{m-l} - 2^{m-r_1-l+1}]t^{i_{r_1}+\cdots+i_{r_l}+1}$. Summing over all B in this case, we obtain

$$\sum_B d(B) = \sum_{\substack{2 \le r_1 < ... < r_l \le m \\ 1 \le l \le m-1}} (-1)^{i_{r_1}+\cdots+i_{r_l}-l}[2^{m-l} - 2^{m-r_1-l+1}]t^{i_{r_1}+\cdots+i_{r_l}+1}.$$

We will denote this by c_1 for ease of referencing later.

Case 2: $\sigma A = A$, $\min A \in \sigma B'$ and for some $b \in B'$, $\sigma(B' \backslash b) = B' \backslash \min A$

Fix B, say $B' \subset \mathcal{O}_{r_1} \cup ... \cup \mathcal{O}_{r_l}$. B' must contain all elements in $\{\mathcal{O}_{r_2}, ..., \mathcal{O}_{r_l}\}$ since B' and $\sigma B'$ can only differ by one element; and, that must occur in $\mathcal{O}_{r_1}$ because $\min A$ must be in $\mathcal{O}_{r_1}$ and cannot be in B'. Say $\sigma_{r_1} = (c_{i_{r_1}-1} ... c_1\, minA)$. Then B must also contain consecutive elements $\{c_1, ..., c_j\}$, $1 \le j \le i_{r_1} - 1$, in $\mathcal{O}_{r_1}$. If this were not the case, B' and $\sigma B'$ would differ by more than one element ($c_j \notin \sigma B$). Consider $\sigma' = (c_j\, minA)(minA\, c_{i_{r_1}-1} \cdots c_1)\sigma_{r_2} \cdots \sigma_{r_1} = (c_j\, minA)(minA\, c_1) \cdots (minA\, c_{i_{r_1}-1})\sigma_{r_2} \cdots \sigma_{r_1}$.

Then $l_B(\sigma') = \sum_{k=2}^{l}(i_{r_k} - 1) + j + 1$. Thus, by Proposition 6.15, the trace of σ acting on $S(A:B)$ is $(-1)^{j+i_{r_2}+\cdots+i_{r_l}-(l-1)+2} = (-1)^{j+i_{r_2}+\cdots+i_{r_l}-(l-1)}$.

Given B, A must contain $\{\mathcal{O}_{r_1}, ..., \mathcal{O}_{r_l}\}$ and may contain other orbits greater than $\mathcal{O}_{r_1}$. Thus, there are $2^{m-r_1-(l-1)}$ choices for A. Now there are $i_{r_1} - 1$ subsets $B \subset \mathcal{O}_{r_1} \cup \cdots \cup \mathcal{O}_{r_l}$. Putting this all together, given $\{\mathcal{O}_{r_1}, ..., \mathcal{O}_{r_l}\}$, $S(A:B)$ contributes a total of $2^{m-r_1-(l-1)} \sum_{j=1}^{i_{r_1}-1} (-1)^{j+i_{r_2}+\cdots+i_{r_l}-(l-1)} t^{j+i_{r_2}+\cdots+i_{r_l}+1}$ towards the graded trace function.

We will need to sum this over all $\{\mathcal{O}_{r_1}, ..., \mathcal{O}_{r_l}\}$ and multiply by $1+t$. This gives us

$$\sum_{\substack{1\le r_1<\cdots<r_l\le m \\ 1\le l\le m}} 2^{m-r_1-l+1}(-1)^{i_{r_2}+\cdots+i_{r_l}-l}\, t^{i_{r_2}+\cdots+i_{r_l}+2}$$
$$+ \sum_{\substack{1\le r_1<\cdots<r_l\le m \\ 1\le l\le m}} 2^{m-r_1-l+1}(-1)^{i_{r_1}+\cdots+i_{r_l}-l}\, t^{i_{r_1}+\cdots+i_{r_l}+1}$$

(notice that the sum over j is telescoping.) Let us label the first sum by c_2 and the second by c_3 for ease of referencing later.

Case 3: $B = \emptyset$.

Because $\sigma A = A$, $a_{\sigma A \sigma BAB} = 1$. Thus we have a contribution of $1 + (2^m - 1)t$ towards the graded trace. Multiplying by $1+t$ gives us $1 + 2^m t + (2^m - 1)t^2$.

If we sum over all possibilities for the traces and multiply by $1+t$, we have that $Tr_\sigma(Q_n^!, t) = 1 + 2^m t + (2^m - 1)t^2 + c_1 + c_1 t + c_2 + c_3$

Consider the following pieces of the expression.

First sum $(2^m - 1)t^2$ and the $l = 1$ terms of c_2.

$$(2^m - 1)t^2 + c_2|_{l=1} = (2^m - 1)t^2 + \sum_{1\le r_1\le m} 2^{m-r_1}(-1)^{-1}t^2$$
$$= t^2[(2^m - 1) - \sum_{r_1=1}^{m} 2^{m-r_1}] = t^2[(2^m - 1) - (2^{m-1+1} - 1)] = 0.$$

Next sum the remaining terms of c_2 (with $l > 1$) and tc_1 where we do a change of variables setting r_1 to r_2.

$c_1 t|_{r_1 \mapsto r_2} + c_2|_{l>1} =$

$$(2^{m-l+1} - 2^{m-r_2-l+1+1})(-1)^{i_{r_2}+\ldots+i_{r_l}-(l-1)} t^{i_{r_2}+\ldots+i_{r_l}+2}$$
$$+ \sum_{r_1=1}^{r_2-1} 2^{m-r_1-l+1}(-1)^{i_{r_2}+\ldots+i_{r_l}-l} t^{i_{r_2}+\ldots+i_{r_l}+2}$$
$$= (-1)^{i_{r_2}+\ldots+i_{r_l}-l+1}\, t^{i_{r_2}+\ldots+i_{r_l}+2}[2^{m-l+1} - 2^{m-r_2-l+2} - \sum_{r_1=0}^{r_2-2} 2^{m-r_1-l}]$$
$$= (-1)^{i_{r_2}+\ldots+i_{r_l}-l+1}\, t^{i_{r_2}+\ldots+i_{r_l}+2}[2^{m-l+1} - 2^{m-r_2-l+2} - [\sum_{r_1=0}^{m-l} 2^{r_1} - \sum_{r_1=0}^{m-r_2-l+1} 2^{r_1}]]$$
$$= (-1)^{i_{r_2}+\ldots+i_{r_l}-l+1}\, t^{i_{r_2}+\ldots+i_{r_l}+2}[2^{m-l+1} - 2^{m-r_2-l+2} - (2^{m-l+1}-1) + (2^{m-(r_2-1)-l+1}-1)]$$
$$= 0.$$

Finally sum c_1 and the terms of c_3 with $r_1 \neq 1$.

$$c_1 + c_3|_{r_1 \neq 1} = \sum_{\substack{1 \le r_1 < \ldots < r_l \le m \\ 1 \le l \le m-1}} (-1)^{i_{r_1}+\ldots+i_{r_l}-l} t^{i_{r_1}+\ldots+i_{r_l}+1}[2^{m-l} - 2^{m-r_1-l+1} - 2^{m-r_1-l+1}]$$
$$= \sum_{\substack{1 \le r_1 < \ldots < r_l \le m \\ 1 \le l \le m-1}} (-1)^{i_{r_1}+\ldots+i_{r_l}-l}\, t^{i_{r_1}+\ldots+i_{r_l}+1} 2^{m-l}.$$

The terms of c_3 with $r_1 = 1$ are: $\sum_{1 \le l \le m} (-1)^{i_1+\ldots+i_{r_l}-l}\, t^{i_1+\ldots+i_{r_l}+1} 2^{m-l}$.

Putting it all together we obtain:

$$1 + t[2^m + \sum_{\substack{1 \le r_1 < \ldots < r_l \le m \\ 1 \le l \le m}} (-1)^l 2^{m-l}(-t)^{i_{r_1}+\ldots+i_{r_l}}] = 1 + t\prod_{k=1}^{m}(2-(-t)^{i_k})$$

Therefore,

$$Tr_\sigma(Q_n^!, t) = \frac{1 + t\prod_{k=1}^{m}(2-(-t)^{i_k})}{1+t}$$

as desired. $\square$

Example 6.5. *Here are the graded trace functions for $Q_4^!$:*

$$Tr_{(1)}(Q_4^!, t) = \frac{1+t(2+t)^4}{1+t} = 1 + 15t + 17t^2 + 7t^3 + t^4$$

$$Tr_{(12)}(Q_4^!, t) = \frac{1 + t(2 - t^2)(2 + t)^2}{1 + t} = 1 + 7t + t^2 - 3t^3 - t^4$$

$$Tr_{(123)}(Q_4^!, t) = \frac{1 + t(2 + t^3)(2 + t)}{1 + t} = 1 + 3t - t^2 + t^3 + t^4$$

$$Tr_{(12)(34)}(Q_4^!, t) = \frac{1 + t(2 - t^2)^2}{1 + t} = 1 + 3t - 3t^2 - t^3 + t^4$$

$$Tr_{(1234)}(Q_4^!, t) = \frac{1 + t(2 - t^4)}{1 + t} = 1 + t - t^2 + t^3 - t^4$$

Now, to get the representations we do the same as in the case of the algebra. We have that $\vec{m}(t) = (S^T)^{-1}\vec{Tr}(t)$ and Proposition 6.13 are still true if you replace $Tr_\sigma(Q_n, t)$ with $Tr_\sigma(Q_n^!, t)$.

Example 6.6. *Irreducible Representations of S_4 acting on $Q_4^!$:*

There are only four degrees in the dual, so we can give all of the multiplicities:

	χ_{triv}	χ_{sgn}	χ_3	χ_{reg}	$\chi_{sgn\otimes reg}$
$m_{\phi,1}$	*4*	*0*	*1*	*3*	*0*
$m_{\phi,2}$	*0*	*0*	*1*	*3*	*2*
$m_{\phi,3}$	*0*	*1*	*0*	*0*	*2*
$m_{\phi,4}$	*0*	*1*	*0*	*0*	*0*

Notice that all of the representations are realized in at least one degree, but not in every. Also, each representation occurs with a much smaller multiplicity than in the algebra.

6.5.3 Koszulity property

An interesting property of quadratic algebras is Koszulity. One of many equivalent definitions of Koszulity is a lattice definition [F].

Definition (Koszul Algebra)**.** *[B] Let $A = (V, R)$ be a quadratic algebra where V is the span of the generators and R the span of the generating relations in $V \otimes V$. Then A is Koszul if the collection of subspaces $\{V^{\otimes i} \otimes R \otimes V^{\otimes n-i-2}, 0 \leq i \leq n-2\}$ generates a distributive lattice in $V^{\otimes n}$ for any n.*

One property of Koszul algebras is that the Hilbert series of the algebra and its dual are related by $H(A,t) * H(A^!,-t) = 1$. This property, however, is not equivalent to Koszulity. One can easily check that the analogous property holds for the graded trace functions that we found for $A(\Gamma)$ and its dual $A(\Gamma)^!$ in our two algebras. Namely, $Tr_\sigma(A(\Gamma),t) * Tr_\sigma(A(\Gamma)^!,-t)$=1 where σ is an element in the automorphism group of the algebra.

Chapter 7

More About the Subalgebra $A(\Gamma^\sigma)$

In this chapter we will define a subalgebra of the dual algebra $A(\Gamma)^!$ in Section 7.1 and define a dual of the subalgebra $A(\Gamma^\sigma)$ in Section 7.2. Also, we will show in Section 7.3 that the $A(\Gamma^\sigma)$ are isomorphic to algebras corresponding to generalized layered graphs as defined by Retakh and Wilson [RW]. While in succeeding chapters we will continue to find the graded trace of $A(\Gamma)^!$ and not the dimension of these dual subalgebras and subalgebra duals, these two objects are worth discussing because they naturally arise in our methods and lead to a wealth of examples.

First we will recall some definitions from Chapters 2, 3. The associated graded algebra $\mathrm{gr}A(\Gamma)$ is isomorphic to $T(E)/grR$ where grR is the two-sided ideal generated by $\{e(v,k+l)-e(v,k)e(u,l) : v>u, k=|v|-|u|\}$. If σ is an automorphism of the layered graph Γ, we define $\Gamma^\sigma := (V_\sigma, E_\sigma)$ where V_σ is the set of vertices $v \in V_\sigma$ such that $\sigma(v) = v$ and E_σ is the set of edges that connect the vertices minimally. Lastly, $A(\Gamma^\sigma)$ is the span$\{e(v_1,k_1)\cdots e(v_l,k_l) : l \geq 0, v_1,...,v_l \in V_\sigma \backslash *, 1 \leq k_i \leq |v_i|, (v_i,k_i) \not\gtrdot (v_{k+1},k_{i+1})\}$. It has generators $\{e(v,k) : v \in V_\sigma, 1 \leq k \leq |v|\}$ and relations $\{e(v,k+l)-e(v,k)e(u,l) : v > u \in V_\sigma, k=|v|-|u|\}$.

It will be convenient here to use a smaller generating set to define $A(\Gamma^\sigma)$. For $v \neq *$, $v \in V_\sigma$, let $l(v)$ denote the length of the longest edge with tail v. Also, fix a distinguished edge e_v such that the tail of e_v is v and the length of $e_v = l(v)$. Note that in $\Gamma = \Gamma^{id}$, $l(v) = 1$ for all v and we can choose the distinguished edge e_v to be the distinguished edge chosen in Chapter 2.

Proposition 7.1. *A set of generators for $A(\Gamma^\sigma)$ is $\mathcal{G}_\sigma = \{e(v,k) : 1 \leq k \leq l(v), v \in V_\sigma\}$.*

Proof. We need to show that we can get all $e(v,k)$ such that $1 \leq k \leq |v|$. There is a path from v to $*$. If there is no edge from v to $w \neq *$, then $l(v) = |v|$ and we are done. If there exists such a w, then consider the vertex w' such that the edge from v to w' is e_v. Then

$e(v,|v|) = e(v,l(v))e(w',|w'|)$, $e(v,|v|-1) = e(v,l(v))e(w',|w'|-1), ..., e(v,|v|-|w'|) = e(v,l(v))$; and, we already have those $e(v,k)$ such that $1 \leq k \leq l(v)$. By induction, we can do the same process with w'. Thus we have all $e(v,k)$ such that $1 \leq k \leq |v|$ as desired. $\square$

Let R_σ be the relations in $A(\Gamma^\sigma)$ using this new set of generators. Thus, $R_\sigma =$
$\{e(v,k)e(u,l) - e(v,k+l) : v \gtrdot u \in V_\sigma, \text{ and } k+l \leq l(v)\} \cup$
$\{e(v,k)e(u,l) - e(v,k')e(w,l') : v \gtrdot u, w \in V_\sigma, k+l = k'+l' > l(v)\}$.

Note that when σ is the identity, $\mathcal{G}_{id} = \{e(v,1)\}$ and
$R_{id} = \{e(v,1)e(u,1) - e(v,1)e(w,1) : v > u, w,\ 1 = |v|-|u| = |v|-|w|\}$.

Example 7.1. *a) $Q_3^{(1)}$: The generators for $Q_3^{(1)}$ (as a subalgebra of itself) are: $e(123,1)$, $e(12,1)$, $e(13,1)$, $e(23,1)$, $e(1,1)$, $e(2,1)$, $e(3,1)$. The relations are: $e(12,1)e(1,1) - e(12,1)e(2,1)$, $e(13,1)e(1,1) - e(13,1)e(3,1)$, $e(23,1)e(2,1) - e(23,1)e(3,1)$, $e(123,1)e(12,1) - e(123,1)e(13,1)$, $e(123,1)e(12,1) - e(123,1)e(23,1)$, $e(123,1)e(13,1) - e(123,1)e(23,1)$.*
b) $Q_3^{(12)}$ has presentation
$< e(123,1), e(123,2), e(12,1), e(12,2), e(3,1) | e(123,1)e(12,2) - e(123,2)e(3,1) >$.

7.1 Subalgebra $A(\Gamma)^{!\sigma}$ of $A(\Gamma)^!$

We will consider here the subalgebra of $A(\Gamma)^!$ which has as generators those of $A(\Gamma)^!$ fixed by an automorphism σ.

Write $v \gtrdot u$ for $(v,1) \gtrdot (u,1)$; i.e. $v > u$ and $|v| - |u| = 1$. Recall $S_1(v) = \{w : v > w, |v| - |w| = 1\}$. Define $M_v = \{u : |u| = 1, v > u\}$ and $\mathcal{I}(v_i, v_{i+2}) = \{w : v_i \gtrdot w \gtrdot v_{i+2}\}$. Define an order $\triangleright$ on the vertices in level 1 of Γ. This defines an order on the M_v by a lexicographical ordering on the sets. When drawing a graph, we will draw the vertices in increasing order from left to right. Now define a partial order $\succ$ on V. Say $v \succ w$ if $v > w$ or $|v| = |w|$ and $M_v \triangleright M_w$. Otherwise, they are incomparable.

We will now prove Proposition 3.1.

Proposition 3.1. *$A(\Gamma)^!$ has a presentation with generators $\{e(v,1)^*\}$ and relations*
$R^! = \{e(v,1)^* e(u,1)^* : v \not\gtrdot u\} \cup \{e(v,1)^* \sum_{v \gtrdot u} e(u,1)^*\}$.

Proof. Let $R_1 = \{e(v,1)e(u,1) : v \not\gtrdot u\}$ and $R_2 = \{e(v,1)^* \sum_{v \gtrdot u} e(u,1)^*\}$. Recall that $A(\Gamma)^!$ is defined to be $T(E^*)/(grR)^\perp = T(\mathcal{G}_{id}^*)/ < (R_{id})^\perp >$ with our new set of generators.

$\mathcal{G}_{id}^* = \{e(v,1)^*\}$ is a set of generators for the dual. The relations in the algebra are $R_{id} = \{e(v,1)e(u,1) - e(v,1)e(w,1) : v \gtrdot u, w\}$. Because $e(v,1)^*e(u,1) = \delta_{vu}$ and $(e(v,1)^*e(u,1)^*)(e(w,1)e(x,1)) = (e(v,1)^*e(w,1))(e(u,1)^*e(x,1))$, it is clear that $R^!$ annihilates R_{id}. We must show that $R^!$ is all of $R_{id}^\perp$. Assume $b = \sum e(v_i',1)^*e(u_i',1)^*$ is in $R_{id}^\perp$. This means that $(\sum e(v_i',1)^*e(u_i',1)^*)(e(v,1)e(u,1) - e(v,1)e(w,1)) = 0$ for all v, u, w such that $e(v,1)e(u,1) - e(v,1)e(w,1)$ in R_{id}. So, we must have that for every $v_i' \gtrdot u_i' \in b$ there exists j for every $(u,1)$ with $v_i' \gtrdot u$ such that $(u,1) = (u_j',1)$. Thus, $b \in R_1 \cup R_2$. □

Theorem 7.2. *$B^! = \{e(v_1,1)^* \cdots e(v_l,1)^* : l \geq 0, v_i \gtrdot v_{i+1}, v_{i+1} \neq max_\succ \mathcal{I}(v_i, v_{i+2})$ for $1 \leq i \leq l-1$, and $v_l \neq max_\succ S_1(v_{l-1})\}$ is a basis for $A(\Gamma)^!$.*

Proof. If we have $e(v_i,1)^*e(v_{i+1},1)^*e(v_{i+2},1)^*$ such that $v_{i+1} = \max_\succ \mathcal{I}(v_i, v_{i+2})$, then by R_3 we may write this as $-e(v_i,1)^*(\sum_{\substack{v_i \gtrdot u \gtrdot v_{i+2} \\ u \neq v_{i+1}}} e(u,1)^*)e(v_{i+2},1)^*$, which is a sum of elements in $B^!$. Likewise, if $v_l = \max_\succ S_1(v_{l-1})$, we can write $e(v_{l-1},1)^*e(v_l,1)^* =$
$-e(v_{l-1},1)^* \sum_{\substack{v_{l-1} \gtrdot u \\ u \neq v_l}} e(u,1)^*$. This again is a sum of elements in $B^!$. Thus $B^!$ spans $A(\Gamma)^!$.

Let $\vec{b} = \sum k_i b_i$ be a linear combination of elements in $B^!$. Assume that $\sum k_i b_i = 0$ and that not all k_i equal 0. $\vec{b}$ can only equal 0 if it is in $R^!$. Because each b_i is in $B^!$, $\vec{b} \notin R_1$. Also, for $\vec{b}$ to be in R_2 it must contain the sum of all elements in $\mathcal{I}(v_i, v_{i+2})$ or $S_1(v_{l-1})$; however, we removed the maximum element from each of these sets. Thus, $\vec{b} \neq 0$, and we have a contradiction. Therefore, $k_i = 0 \, \forall i$ and the elements of $B^!$ are linearly independent. □

Definition ($A(\Gamma)^{!\sigma}$). $A(\Gamma)^{!\sigma} := span\{e(v_1,1)^*e(v_2,1)^* \cdots e(v_l,1)^* : l \geq 0, v_1, ..., v_l \in V_\sigma, v_i \gtrdot v_{i+1}, v_{i+1} \neq max_\succ \mathcal{I}(v_i, v_{i+2})$ *for* $1 \leq i \leq l-1$, *and* $v_l \neq max_\succ S_1(v_{l-1})\} = spanB_\sigma^!$

$B_\sigma^!$ is a basis for $A(\Gamma)^{!\sigma}$ because the set is a subset of a basis.

Theorem 7.3. *$A(\Gamma)^{!\sigma}$ is a subalgebra of $A(\Gamma)^!$. $A(\Gamma)^{!\sigma}$ has a presentation with generators $\mathcal{G}_\sigma^* = \{e'(v,1)^* : v \in V_\sigma\}$ and relations $R_\sigma^! = \{e'(v,1)^*e'(u,1)^* : v, u \in V_\sigma, v \not\gtrdot u\} \cup \{e(v,1)^* \sum_{v \gtrdot u} e(u,1)^* : v, u \in V_\sigma\}$*

Proof. Define $\phi : T(\mathcal{G}_\sigma^*) \to A(\Gamma)^!$ by $\phi(e'(v,1)^*) = e(v,1)^*$. We have $\phi(T(\mathcal{G}_\sigma^*)) \supseteq A(\Gamma)^{!\sigma}$ because elements of $B_\sigma^!$ are formed from products of elements in $\mathcal{G}_\sigma^*$. In $A(\Gamma)^!$ we have $e(v,1)^*e(u,1)^* \equiv 0$ whenever $v \not> u$ and $e(v,1)^* \sum_{v>u} e(u,1)^* \equiv 0$. Consequently $\phi(e'(v,1)^*e'(u,1)^*) = 0$ and $\phi(e'(v,1)^* \sum_{v>u} e'(u,1)^*) = 0$. Thus, $R_\sigma^! \subseteq \ker\phi$.

Let $b' = e'(v_1,1)^* \cdots e'(v_l,l)^* \neq 0$ be a monomial in $T(\mathcal{G}_\sigma^*)$. In $T(\mathcal{G}_\sigma^*)/ < R_\sigma^! >$, we may replace every occurrence in b' of $e'(v_i,1)^*e'(v_{i+1},1)^*$ such that $v_i \not> v_{i+1}$ with 0. Also, if $v_{i+1} = \max_\succ \mathcal{I}(v_i, v_{i+2})$, then we may replace $e'(v_i,1)^*e'(v_{i+1},1)^*$ with $-e'(v,1)^* \sum_{\substack{u \neq v_{i+1} \\ v>u}} e(u,1)^*$. Likewise, if $v_l = \max_\succ S_1(v_{l-1})$, we replace $e'(v_{l-1},1)^*e'(v_l,1)^*$ with $e'(v_{l-1},1)^* \sum_{\substack{u \neq v_l \\ v>u}} e'(u,1)^*$. Thus $b' \equiv e'(v_1,1)^* \cdots e'(v_l,l)^*$ such that $v_i > v_{i+1}$, $v_{i+1} \neq \max_\succ \mathcal{I}(v_i, v_{i+2})$, and $v_l \neq \max_\succ S_1(v_{l-1})$ in $T(\mathcal{G}_\sigma^!)/ < R_\sigma^! >$. Hence $\phi(b') \in A(\Gamma)^{!\sigma}$, and so $\phi(T(\mathcal{G}_\sigma^*)) = A(\Gamma)^{!\sigma}$. Also, we have an induced surjective homomorphism $\phi' : T(\mathcal{G}_\sigma^*)/ < R_\sigma^! > \to A(\Gamma)^{!\sigma}$.

Let $f = \sum k_i b_i' \in \ker\phi'$, where k_i is an element in the field and b_i' a monomial in $T(\mathcal{G}_\sigma^*)/ < R_\sigma^! >$. Then $0 = \phi'(f) = \sum k_i \phi'(b_i') = \sum k_i b_i$ is a linear combination of basis elements in $A(\Gamma)^{!\sigma}$. This implies that $k_i = 0 \forall i$ and so $f = 0$. Therefore, ϕ' is an isomorphism. $\square$

We will now write $e(v,1)^*$ for $e'(v,1)^*$.

Example 7.2. *a)* $Q_3^{!(1)}$*: The basis of* $Q_3^{!(1)}$ *consists of the elements* $*, e(v,1)^* \forall v$, $e(12,1)^*e(1,1)^*$, $e(13,1)^*e(1,1)^*$, $e(23,1)^*e(2,1)^*$, $e(123,1)^*e(12,1)^*$, $e(123,1)^*e(13,1)^*$, *and* $e(123,1)^*e(12,1)^*e(1,1)^*$. *Thus its graded dimension is* $1 + 7t + 5t^2 + t^3$.

b) $Q_3^{!(12)}$ *has basis elements* $e(123,1)^*, e(12,1)^*, e(3,1)^*$, *and* $e(123,1)^*e(12,1)^*$. *Thus, its graded dimension is* $1 + 3t + t^2$. *Notice that this is not the same as the graded trace* $(1 + 3t - t^2 - t^3)$ *found earlier in the paper.*

7.2 The Dual $A(\Gamma^\sigma)^!$ of $A(\Gamma^\sigma)$

Definition $(A(\Gamma^\sigma)^!)$. $A(\Gamma^\sigma)^! := T(G_\sigma^*)/R_\sigma^\perp$.

Proposition 7.4. $R_\sigma^\perp = \{e(v,k)^*e(u,l)^* : (v,k) \not\gtrdot (u,l)\} \cup \{\sum_{u,k+l=r} e(v,k)^*e(u,l)^* : (v,k) \gtrdot (u,l), r > l(v)\} \cup \{e(v,r)^* + \sum_{u,k+l=r} e(v,k)^*e(u,l)^* : (v,k) \gtrdot (u,l), r \leq l(v)\}$.

Proof. Denote $\{e(v,k)^*e(u,l)^* : (v,k) \not\gtrdot (u,l)\}$ by $\mathcal{R}_1$, $\{\sum_{u,k+l=r} e(v,k)^*e(u,l)^* : (v,k) \gtrdot (u,l), r > l(v)\}$ by $\mathcal{R}_2$, and $\{e(v,r)^* + \sum_{u,k+l=r} e(v,k)^*e(u,l)^* : (v,k) \gtrdot (u,l), r \leq l(v)\}$ by $\mathcal{R}_3$. Recall that $R_\sigma = \{e(v,k)e(u,l) - e(v,k+l) : v \gtrdot u \in V_\sigma,\ k+l \leq l(v)\} \cup \{e(v,k)e(u,l) - e(v,k')e(w,l') : v \gtrdot u, w \in V_\sigma,\ k+l = k'+l' > l(v)\}$. $R_\sigma^\perp$ is the annihilator of these sets.

One can easily check that $\mathcal{R}_1, \mathcal{R}_2$, and $\mathcal{R}_3$ annihilate R_σ.

Assume $b = \sum e(v_i', k_i')^* e(u_i', l_i')^* + e(w_i', r_i')^*$ is in $R_{id}^\perp$. This means that
$(\sum e(v_i', k_i')^* e(u_i', l_i')^*)(e(v,k)e(u,l) - e(v,r)e(w,s)) = 0$ and
$(\sum e(v_i', k_i')^* e(u_i', l_i')^* + e(w_i', r_i')^*)(e(v,k)e(u,l) - e(v,k+l)) = 0$ for all v, u, w such that $e(v,k)e(u,l) - e(v,r)e(w,s)$ and $e(v,k)e(u,l) - e(v,k+l)$ are in R_σ. In the first case this means that for every $(v_i', k_i') \gtrdot (u_i', l_i')$ with $k_i' + l_i' > l(v_i)$ in b there exists j for every (u,l) with $(v_i', k) \gtrdot (u,l)$ and $k+l = k_i' + l_i'$ such that $(u,l) = (u_j', l_j')$ and $(v_j', k_j') = (v_i', k)$. Thus, $b \in R_1 \cup R_2$. In the second case this means that for each i such that $(v_i', k_i') \gtrdot (u_i', l_i')$, $v_i' = w_j'$ for some j and $k_i' + l_i' = r_j'$. Also, for each (w_i', r_i') such that there exists $(w_i', k_i') \gtrdot (u_i', r_i' - k_i')$, $e(w_j', k_i')^* e(u_i', r_i' - k_i')^*$ appears in the sum. Thus, $b \in R_1 \cup R_3$. Thus, $R_1 \cup R_2 \cup R_3$ is all of $R_\sigma^\perp$. ∎

Thus, we have a presentation for $A(\Gamma^\sigma)^!$ with generators $\{e(v,k)^* : v \in V_\sigma, 1 \leq k \leq l(v)\}$ and relations as given by $R_\sigma^\perp$.

Example 7.3. *a)* $Q_3^{(1)!}$*: This is the same as* $Q_3^{!(1)}$ *above because all of the elements in the dual are fixed by the identity.*

b) $Q_3^{(12)!}$ *has basis elements* $e(123,1)^*$, $e(123,2)^*$, $e(12,1)^*$, $e(12,2)^*$, $e(3,1)^*$, *and* $e(123,1)^*e(12,2)^*$. *Thus its graded dimension is* $1 + 3t + 2t^2 + t^3$.

We can see in these examples that $A(\Gamma)^{!\sigma} \ncong A(\Gamma^\sigma)^!$. Also, notice that while the graded traces of $A(\Gamma)$ and $A(\Gamma)^!$ satisfy the Koszulity property $Tr_\sigma(A(\Gamma), t) * Tr_\sigma(A(\Gamma)^!, -t) = 1$, the graded traces of $A(\Gamma^\sigma)$ and $A(\Gamma^\sigma)^!$ do not.

7.3 Generalized Layered Graphs

In this section we will show the correspondence between the subalgebras $A(\Gamma^\sigma)$ and algebras associated to generalized layered graphs as defined by Retakh and Wilson [RW].

A generalized layered graph $\Gamma_{gen} = (V, E_{gen})$ is a directed (layered) graph where the edges do not necessarily have length 1. Define the length of an edge from vertex v to w to be $l(e) = |v| - |w|$ and say the level of e is $|e| = |v|$. In this section Γ_{gen} will have a unique minimal vertex *. We will also choose to connect the vertices "minimally", where minimally means that there is an edge $e \in E_{gen}$ from v to w, if and only if $v \geq u \geq w$, implies $u = v$ or $u = w$ (although the relations will make it so that extra edges do not affect the algebra).

For each edge, define a polynomial $P_e(t) = (1 - c_1^e t + c_2^e t^2 - ... + (-1)^{l(e)} c_{l(e)}^e t^{l(e)})$ with central variable t associated with that edge. Note that when an edge has length 1 $P_e(t) = 1 - et$, which corresponds to what we had for $A(\Gamma)$ in Chapter 2.

Definition ($A(\Gamma_{gen})$)**.** *[RW] The algebra $A(\Gamma_{gen})$ associated with the generalized layered graph Γ_{gen} is defined by the following presentation. The generators of $A(\Gamma_{gen})$ are $c_1^e, ..., c_{l(e)}^e$ for all edges $e \in \Gamma_{gen}$. If the sequences of edges $e_1, ..., e_p$ and $f_1, ..., f_q$ define paths with the same origin and end, then they define a relation $P_{e_1}(t) \cdots P_{e_p}(t) = P_{f_1}(t) \cdots P_{f_q}(t)$. Pairs of paths with the same origin and end define all of the relations.*

By [RW] we can pick a distinguished path from $v = v_1$ to *. As in the subalgebra case, at each vertex v_i along the path let us always choose an edge e_{v_i} with tail v_i and length $l(v_i)$. Let $P_v(t) = (1 - c_1^{e_{v_1}} t + c_2^{e_{v_1}} t^2 - \cdots + (-1)^{l(e_{v_1})} c_{l(e_{v_1})}^{e_{v_1}} t^{l(e_{v_1})})(1 - c_1^{e_{v_2}} t + c_2^{e_{v_2}} t^2 - \cdots + (-1)^{l(e_{v_2})} c_{l(e_{v_2})}^{e_{v_2}} t^{l(e_{v_2})}) \cdots (1 - c_1^{e_{v_k}} t + c_2^{e_{v_k}} t^2 - ... + (-1)^{l(e_{v_k})} c_{l(e_{v_k})}^{e_{v_k}} t^{l(e_{v_k})})$ be the polynomial corresponding to the distinguished path from v to *. We can write

$P_v(t) = 1 + \sum_{j=1}^{|v|}(-1)^j \hat{e}(v,k)t^j$. Let U denote the set of all sequences of integers $(i_1, ..., i_k)$ such that $i_1 + \cdots + i_k = j$, $0 \le i_j \le l(e_{v_j})$. Then $\hat{e}(v,j) := \sum_{(i_1,...,i_k)\in U} c^v_{i_1} \cdots c^{v_k}_{i_k}$ [RW].

Theorem 7.5. *[[RW],Theorem 3.2] A basis for $A(\Gamma_{gen})$ is $B_{gen} = \{\hat{e}(v_1,k_1)\cdots\hat{e}(v_l,k_l) : l \ge 0, v_1, ..., v_l \in V\backslash *, 1 \le k_i \le |v_i|, (v_i,k_i) \not\triangleright (v_{i+1},k_{i+1})$ for $1 \le i \le l-1\}$*

In light of this, the following theorem should not surprise us.

Theorem 7.6. *The associated graded algebra $A(\Gamma^\sigma_{gen})$ generated by the subgraph Γ^σ as a generalized layered graph and the subalgebra $A(\Gamma^\sigma)$ defined in Definition 3.2 are isomorphic.*

Proof. It is clear that Γ^σ is a generalized layered graph. Recall that a basis for $A(\Gamma^\sigma)$ is $\{e(v_1,k_1)\cdots e(v_l,k_l) : l \ge 0, v_1, ..., v_l \in V_\sigma\backslash *, 1 \le k_i \le |v_i|, (v_i,k_i) \not\triangleright (v_{i+1},k_{i+1})$ for $1 \le i \le l-1\}$. Because we have one distinguished path for each vertex that we choose to be the same in both contexts, if we can show a correspondence between $e(v,k)$ and $\hat{e}(v,k)$, then the two algebras have isomorphic bases and thus we have proved our theorem.

The polynomial associated with e_v in Γ^σ is $(1 - e^v_1 t)\cdots(1 - e^v_{l(v)}t)$. For $1 \le k \le l(v)$, $e(v,k)$ is the coefficient of t^k in this polynomial: $e(v,k) = e_1 \cdots e_k$. Likewise, the polynomial associated with e_v in Γ^σ_{gen} is $1 - c^{e_v}_1 t + \cdots + (-1)^{l(v)} c^{e_v}_{l(v)} t^{l(v)}$. Thus, $\hat{e}(v,k) = c^{e_v}_k$. Hence, for an edge there is a correspondence $e(v,k) = e_1 \cdots e_k \leftrightarrow c^{e_v}_k = \hat{e}(v,k)$.

Now let us consider $e(v,k)$ for any $1 \le k \le |v|$. We claim that $e(v,k)$ is in correspondence with $c^v_{l(e_v)} \cdots c^{v_{i-1}}_{l(e_{v_{i-1}})} c^{v_i}_j$ where $l(e_{v_1}) + \cdots + l(e_{v_{i-1}}) + j = k$. We will induct on the number of edges in a distinguished path with tail v. The polynomial associated with a path is a product of the polynomials associated with each edge in the path. Thus, $e(v,k) = e(v,l(v))e(v_2,l(v_2))\cdots e(v_{i-1},l(v_{i-1}))e(v_i,j)$, where $l(v) + l(v_2) + \cdots l(v_{i-1}) + j = k$, in the associated graded algebra. By above we have that this equals $c^v_{l(e_v)} \cdots c^{v_{i-1}}_{l(e_{v_{i-1}})} c^{v_i}_j$, where again $l(e_{v_1}) + \cdots + l(e_{v_{i-1}}) + j = k$. However, this equals $\hat{e}(v,k)$ in the associated graded algebra. Therefore, there is an isomorphism between $\{e(v,k)\}$ and $\{\hat{e}(v,k)\}$, and the bases are isomorphic. ∎

Example 7.4. $Q_{3gen}^{(12)}$

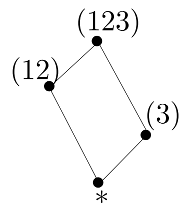

Figure 7.1: The generalized layered graph $\mathcal{L}_{[3]gen}^{(12)}$ associated with $Q_{3gen}^{(12)}$

We have $(1-a_1^{123}t)(1-b_1^{12}t+b_2^{12}t^2) = (1-c_1^{123}t+c_2^{123}t^2)(1-d_1^3t) \Rightarrow a_1+b+1 = c_1+d_1 \equiv a_1 = c_1$ *in the associated graded algebra,* $a_1b_1+b_2 = c_1d_1+c_2 \equiv a_1b_1 = c_2, a_1b_2 + c_2d_1$. *And so,* $a_1b_1 = e(123,1)e(12,1) = e(123,2) = c_2$ *and* $a_1b_2 = e(123,1)e(12,2) = e(123,2)e(3,1) = c_2d_1$.

Chapter 8

Examples of $A(\Gamma)$ Associated with Coxeter Groups

In this section we will give examples of $A(\Gamma)$ associated with more general Coxeter groups. We will consider the algebras associated with graphs whose automorphism groups are symmetry groups of regular polytopes. We have already seen two examples of this: S_n is the Weyl group of the type A_{n-1} root system and is the group of symmetries of the regular (n-1)-dimensional simplex and D_{2m} is the symmetry group of the m-gon. We will first give some definitions and general results. Then we will give the graded traces and decomposition into homogeneous components of the algebras associated to the Hasse graphs of the octahedron and the cube.

8.1 Definitions and Preliminary Theorems

Definition (Coxeter Group)**.** *[BB] A Coxeter group with identity e is defined by the presentation* $< s_1, s_2, ..., s_n | (s_i s_j)^{m_{ij}} = e >$ *where* $m_{ii} = 1$ *and* $m_{ij} \geq 2$ *for* $i \neq j$*.*

A Coxeter group can be associated with the graph having vertices $S = \{s_1, ..., s_n\}$ and an edge between s_i and s_j if and only if $m_{ij} \geq 3$. If $m_{ij} > 3$, the edge between s_i and s_j is labeled by that number. In the case where the graph has no branches, i.e. $m_{ij} > 2$ only if $j = i + 1$, then write the Coxeter group as $[m_{12}, m_{23}, ..., m_{n-1\,n}]$; see Figure 8.1. These Coxeter groups are the ones we will consider in what follows.

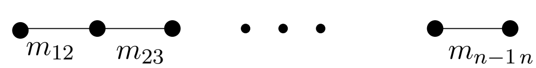

Figure 8.1: Coxeter Group $[m_{12}, m_{23}, ..., m_{n-1\,n}]$

Definition (Polytope)**.** *[MS] An abstract n-polytope* $\mathcal{P}$ *is a partially ordered set with properties (P1)-(P4) given below.*

The elements of $\mathcal{P}$ are called faces. Two faces F, G are incident if $F \geq G$ or $G \geq F$. A chain is a totally ordered subset of $\mathcal{P}$, and a flag is a maximal chain. For any two faces $F \geq G$, a section of $\mathcal{P}$ is $F/G := \{H : H \in \mathcal{P}, G \leq H \leq F\}$. $\mathcal{P}$ is connected if $n \leq 1$ or $n \geq 2$ and for any two proper faces F, G there exists a finite sequence of proper faces $F = H_0, H_1, ..., H_k = G$ such that each H_{i-1} and H_i are incident. If each section of $\mathcal{P}$ is connected, then $\mathcal{P}$ is strongly connected.

(P1 $\mathcal{P}$ contains a least face, F_{-1}, and a greatest face, F_n.

(P2) Each flag has length $n-1$.

(P3) $\mathcal{P}$ is strongly connected.

(P4) For each $i = 0, ..., n-1$, if F and G are incident faces of $\mathcal{P}$ of ranks $i+1$ and $i-1$, respectively, then there are exactly two i-faces H of $\mathcal{P}$ such that $F > H > G$.

Definition (Schläfli symbol)**.** *[MS] The Schläfli symbol of a polytope $\mathcal{P}$ is $\{p_1, ..., p_{n-1}\}$ where the p_i are defined as follows. Let F be an $(i-2)$-face and G an $(i+1)$-face of $\mathcal{P}$ which is incident with F. Then p_i is the number of i-faces of $\mathcal{P}$ in the section G/F. These numbers (in our case) do not depend on the choice of F and G since we will be working with regular polytopes.*

For example, the n-cube has Schläfli symbol $\{4, 3^{n-2}\}$. In a cube, symbol $\{4, 3\}$, four edges make up each face and 3 faces meet at each vertex.

Theorem 8.1 ([MS],Theorem 3A5)**.** *The symmetry group of a convex regular n-polytope is a Coxeter group. More precisely, if $\{p_1, ..., p_{n-1}\}$ is the Schläfli symbol of the polytope, then this Coxeter group is $[p_1, ..., p_{n-1}]$.*

For example, the n-dimensional hypercube and hyperoctahedron have symmetry groups the Coxeter groups $[4, 3^{n-2}]$ and $[3^{n-2}, 4]$, respectively, which are the Weyl groups of the special orthogonal group and sympletic groups, $B_n = C_n$. The icosahedron, $\{3, 5\}$, and dodecahedron, $\{5, 3\}$, have the Coxeter group $[3, 5] = H_3$ as their symmetry group.

Consider the Hasse graph of a n-dimensional polytope. In the Hasse graph of a polytope the faces of the same rank have vertices on the same level in the graph. There is an edge only between two vertices on adjacent levels, and this if and only if the corresponding faces are incident in the polytope. Thus we may label the vertices in level one by $\{v_1, ..., v_m\}$, where m is the number of vertices in the polytope, and the vertices in level r, $2 \leq r \leq n$ by v_A where $A = \{i_1, ..., i_s\} \subseteq \{1, ..., m\}$ and $v_A > v_{i_j}, \forall\, 1 \leq j \leq s$. Denote the algebra associated with the Hasse graph of the polytope $\{p_1, ..., p_{n-1}\}$ by $A(\Gamma_{\{p_1,...,p_{n-1}\}})$.

Lemma 8.2. *The automorphism group of* $\Gamma_{\{p_1,...,p_{n-1}\}}$ *is* $[p_1, ..., p_{n-1}]$.

Proof. Any automorphism of the graph must preserve the set of vertices at each level and so acts on the set $\{v_1, ..., v_m\}$ of all m vertices in level 1. We may say $\sigma(v_i) = v_{\sigma(i)}$. Thus we can think of an automorphism of the graph as being a permutation in S_n acting on the subscripts/labels of the vertices of level 1. This will uniquely determine what happens on higher levels; i.e. $\sigma(v_A) = v_{\sigma(A)}$. Labeling the vertices in levels two through n by the vertices they lie over in level one ensures that as long as the set of vertices in each level is preserved, the edges in the graph will be as well.

Preserving the set of vertices in each level is precisely given by the symmetries of the polytope. Symmetries of the polytope send i-faces to i-faces in a one-to-one fashion and preserve incidence. □

Theorem 8.3. *If* $\{p_1, ..., p_{n-1}\} \neq \{2\}$,
Aut($A(\Gamma_{\{p_1,...,p_{n-1}\}})$*)* $= k^* \times [p_1 = m_{12}, ..., p_{n-1} = m_{n-1\,n}]$, *$k$ the base field.*

Proof. $\mathrm{Aut}(\Gamma_{\{p_1,...,p_{n-1}\}}) = [p_1, ..., p_{n-1}]$ by Lemma 8.2. For $\{p_1, ..., p_{n-1}\} \neq \{2\}$, the number of vertices in level 1 is greater than 2. Clearly, no two faces have the same underlying vertex set in the polytope and so each vertex in the graph will have a unique label. Finally, by property (P4) in the definition of a polytope, there are either zero or two paths between any two vertices two levels apart. Thus, Theorem 4.2 applies. □

8.2 Decomposition of $A(\Gamma_{\{3,4\}})$ and $A(\Gamma_{\{4,3\}})$

We can find the graded trace generating functions and decomposition of $A(\Gamma_{\{p_1,...,p_{n-1}\}})$ as we did in Sections 6.3, 6.4 using the methods described in Chapter 5. In this section, we will do so for the algebras associated with the octahedron and the cube. Note that the octahedron and the cube are dual polytopes, and hence their Hasse graphs are upside-down images of each other. Thus, they will produce the same generating functions (easy to see by looking at the graphs of the subalgebras and using Method 2) and decompositions.

The Octahedron:

The octahedron has 6 vertices, 12 edges, and 8 faces. See Figure 8.2 below. In its Hasse graph label the vertices in level 1 by $v_1, ..., v_6$, those in level 2 by v_{12}, $v_{13}, v_{14}, v_{15}, v_{23}$, v_{24}, v_{26}, $v_{35}, v_{36}, v_{45}, v_{46}, v_{56}$, and those in level 3 by $v_{123}, v_{124}, v_{135}, v_{145}, v_{236}, v_{246}, v_{356}, v_{456}$.

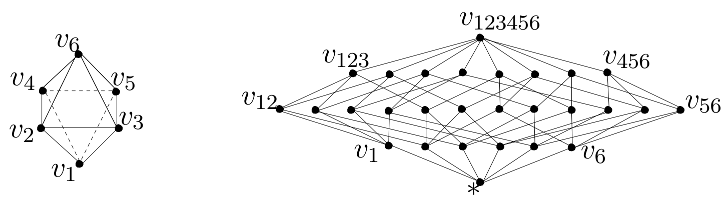

Figure 8.2: Octahedron, $\Gamma_{\{3,4\}}$

The Coxeter group $[3, 4]$ has presentation $< s_1, s_2, s_3 | s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_2 s_3)^4 = (s_1 s_3)^2 = e >$. Let $s_1 = (12)(56), s_2 = (13)(46), s_3 = (34)$. Use (1), (123)(654), (1265), (16)(25), (12)(56)(34), (16)(25)(34), (1265)(34), (123654), (16), and (12)(56) as the conjugacy class representatives of the Coxeter group.

Theorem 8.4. *The graded trace generating functions for $A(\Gamma_{\{3,4\}})$ are:*

$$a := Tr_{(1)}(A(\Gamma_{\{3,4\}}), t) = \frac{1-t}{1-t(28-62t+48t^2-14t^3+t^4)}$$

$$b := Tr_{(123)(654)} = Tr_{(1265)} = Tr_{(16)(25)}(A(\Gamma_{\{3,4\}}), t) = \frac{1-t}{1-t(2-t)(2-t^3)}$$

$$c := Tr_{(12)(56)(34)}(A(\Gamma_{\{3,4\}}), t) = \frac{1-t}{1-t(2-t^2)^2}$$

$$d := Tr_{(16)(25)(34)} = Tr_{(1265)(34)} = Tr_{(123654)}(A(\Gamma_{\{3,4\}}), t) = \frac{1-t}{1-t(2-t^4)}$$

$$e := Tr_{(16)}(A(\Gamma_{\{3,4\}}), t) = \frac{1-t}{1-t(10-12t+4t^3-t^4)}$$

$$f := Tr_{(12)(65)}(A(\Gamma_{\{3,4\}}), t) = \frac{1-t}{1-t(10-10t-4t^2+6t^3-t^4)}$$

Proof. You may use either Method 1 or 2; we will use Method 2: Equation 5.2

$$Tr_\sigma(A(\Gamma),t) = \frac{1-t}{1-t \sum_{\substack{v_1>\ldots>v_l\geq * \\ v_1,\ldots,v_l\in V_\sigma}} (-1)^{l-1} t^{|v_1|-|v_l|}}.$$

For this proof, write u for v_{123456}.

a) The possible sequences indexing the sum are: u, $v_{ijk}(8)$, $v_{ij}(12)$, $v_i(6)$, *, $u > v_{ijk}(8)$, $v_{ijk} > v_{i'j'}(24)$, $v_{ij} > v_{i'}(24)$, $v_i > *(6)$, $u > v_{ij}(12)$, $v_{ijk} > v_i(24)$, $v_{ij} > *(12)$, $u > v_{ijk} > v_{i'j'}(24)$, $v_{ijk} > v_{i'j'} > v_{i''}(48)$, $v_{ij} > v_{i'} > *(24)$, $u > v_i(6)$, $v_{ijk} > *(8)$, $u > v_{ijk} > v_{i'}(24)$, $u > v_{ij} > v_{i'}(24)$, $u > v_{ijk} > v_{i'j'} > v_{i''}(48)$, $v_{ijk} > v_{i'j'} > *(24)$, $v_{ijk} > v_{i'} > *(24)$, $v_{ijk} > v_{i'j'} > v_{i''} > *(48)$, $u > *(1)$, $u > v > *(26)$, $u > v_{ijk} > v_{i'j'} > *(24)$, $u > v_{ijk} > v_{i'} > *(24)$, $u > v_{ij} > v_{i'} > *(24)$, $u > v_{ijk} > v_{i'j'} > v_{i''} > *(48)$. The numbers after each sequence represent how many of this type there are. Thus, the coefficients of t, t^2, t^3, t^4, and t^5 are -1-8-12-6-1=-28, 8+24+24+6=62, 12+24+12-24-48-24=-48, 6+8-24-24+48-24-24+48=14, and 1-26+24+24+24-48=-1 respectively. The coefficient of t^k for $k \geq 6$ is 0.

b) The vertices fixed by (123)(654) are u, v_{123} and v_{456}. Those fixed by (1265) and (16)(25) are u, v_3 and v_4. We will give the sequences indexing the sum for the first, however, it is clear that the second gives the same result. The sequences are: $u, v_{123}, v_{456}, *$, $u > v_{123}, u > v_{456}, v_{123} > *$, $v_{456} > *, u > *$, $u > v_{123} > *, u > v_{456} > *$. Thus, the coefficients of t, t^2, t^3, t^4, and t^5 are -4, 2, 0, 2, $1-2=-1$, respectively. The coefficient of t^k for $k \geq 6$ is 0.

c) The vertices fixed by (12)(56)(34) are u, v_{12}, and v_{56}. The sequences indexing the sum are: $u, v_{12}, v_{56}, *$, $u > v_{12}, u > v_{56}, v_{12} > *, v_{56} > *, u > *, u > v_{12} > *, u > v_{56} > *$. Thus, the coefficients of t, t^2, t^3, t^4, and t^5 are -4, 0, 4, 0, $1-2=-1$, respectively. The coefficient of t^k for $k \geq 6$ is 0.

d) Only u is fixed by $(16)(25)(34), (1265)(34), (123654)$. Thus, we only have $u, *, u > *$. Hence, the coefficients of t, t^2, t^3, t^4, and t^5 are -2, 0, 0, 0, 1, respectively. The coefficient of t^k for $k \geq 6$ is 0.

e) The vertices fixed by (16) are $u, v_{23}, v_{24}, v_{35}, v_{45}, v_2, v_3, v_4, v_5$. Hence the possible sequences indexing the sum are $u, v_{ij}(4), v_i(4), *, v_{ij} > v_{i'}(8), v_i > *(4), u > v_{ij}(4), v_{ij} > *(4), v_{ij} > v_{i'} > *(8), u > v_i(4), u > v_{ij} > v_{i'}(8), u > *, u > v > *(8), u > v_{ij} > v_{i'} > *(8)$. Thus, the coefficients of t, t^2, t^3, t^4, and t^5 are -10, $8 + 4 = 12$, $4 + 4 - 8 = 0$, $4 - 8 = -4$, $1 - 8 + 8 = 1$, respectively. The coefficient of t^k for $k \geq 6$ is 0.

f) The vertices fixed by (12)(56) are $u, v_{123}, v_{124}, v_{356}, v_{456}, v_{12}, v_{56}, v_3, v_4$. Hence the possible sequences indexing the sum are $u, v_{ijk}(4), v_{ij}(2), v_i(2), *, u > v_{ijk}(4), v_{ijk} > v_{i'j'}(4), v_i > *(2), u > v_{ij}(2), v_{ijk} > v_{i'}(4), v_{ij} > *(2), u > v_{ijk} > v_{i'j'}(4), u > v_i(2), v_{ijk} > *(4), u > v_{ijk} > v_{i'}(4), v_{ijk} > v_{i'j'} > *(4), v_{ijk} > v_{i'} > *(4), u > *(1), u > v > *(8), u > v_{ijk} > v > *(8)$. Thus, the coefficients of t, t^2, t^3, t^4, and t^5 are -10, $4 + 4 + 2 = 10$, $2 + 4 + 2 - 4 = 4$, $2 + 4 - 4 - 4 - 4 = -6$, $1 - 8 + 8 = 1$, respectively. The coefficient of t^k for $k \geq 6$ is 0. $\square$

Now let us determine the multiplicities of the irreducible representations. Recall the following from Section 6.4. $A(\Gamma_{\{p_1,...,p_{n-1}\}})_{[i]}$ is a completely reducible $[p_1, ..., p_{n-1}]$-module. Let the graded trace generating function be denoted by $Tr_\sigma(t) = \sum_i Tr_{\sigma,i} t^i$ where $Tr_{\sigma,i} = Tr\sigma|_{A(\Gamma_{\{p_1,...,p_{n-1}\}})_{[i]}}$. Let ϕ be an irreducible representation of $[p_1, ..., p_{n-1}]$ and $m_\phi(t) = \sum_i m_{\phi,i} t^i$ where $m_{\phi,i}$ is the multiplicity of ϕ in $A(\Gamma_{\{p_1,...,p_{n-1}\}})_{[i]}$. Finally, let the matrix $C = [\chi_{\sigma\phi}]$ where $\chi_{\sigma\phi}$ is the trace of σ on the module which affords the irreducible representation ϕ; i.e. C is the character table of $[p_1, ..., p_{n-1}]$.

Then, if we fix the degree, $Tr_{\sigma,i} = \sum_\phi \chi_{\sigma\phi} m_{\phi,i}$; and so we have $Tr_\sigma(t) = \sum_\phi \chi_{\sigma\phi} m_\phi(t)$. Write $\vec{Tr}(t) = [Tr_{\sigma_1}(t)...Tr_{\sigma_l}(t)]^T$ and $\vec{m}(t) = [m_{\phi_1}(t)...m_{\phi_l}(t)]^T$. Finally,

$$\vec{Tr}(t) = C^T \vec{m}(t) \implies \vec{m}(t) = (C^T)^{-1} \vec{Tr}(t).$$

The character table of the Coxeter group $[3, 4]$ is given in Table 8.1[Wi].

Theorem 8.5. *Let $a, b, c, d, e,$ and f be as in Theorem 8.4. Then the multiplicities for the representations of $A(\Gamma_{\{3,4\}})$ are:*

$$\vec{m}(t) = \frac{1}{48}\begin{bmatrix} a+17b+6c+15d+3e+6f \\ a+5b-6c-3d-3e+6f \\ a+5b-6c+3d+3e-6f \\ a+17b+6c-15d-3e-6f \\ 2a-2b-6d+6e \\ 2a-2b+6d-6e \\ 3a+3b-6c+9d-3e-6f \\ 3a-9b+6c+3d+3e-6f \\ 3a-9b+6c-3d-3e+6f \\ 3a+3b-6c-9d+3e+6f \end{bmatrix}$$

Proof. We verify the assertion by multiplying the transpose of the character table of $[3, 4]$ by $\vec{m}(t)$. The result is

$$\vec{Tr}(t) = \begin{bmatrix} a & b & b & b & c & d & d & d & e & f \end{bmatrix}^T$$

as desired. $\square$

Definition (Dual Directed Graph)**.** *Let $\Gamma = (V, E)$. The dual directed graph is $\Gamma^{\uparrow} := (V, E)$ such that if $t(e) = v_1$ and $h(e) = v_2$ in Γ, then $t(e) = v_2$ and $h(e) = v_1$ in $\Gamma^{\uparrow}$.*

Proposition 8.6. *Algebras $A(\Gamma)$ and $A(\Gamma^{\uparrow})$ which are associated with dual directed graphs have the same graded trace generating functions and irreducible representations.*

Proof. The vertices and paths in Γ and $\Gamma^{\uparrow}$ are in one-to-one correspondence with each other. Thus, $A(\Gamma)$ and $A(\Gamma^{\uparrow})$ have the same graded trace generating functions by Theorem 5.4. The automorphism groups of Γ and $\Gamma^{\uparrow}$ are clearly the same. Hence, the character tables for the automorphism groups of the algebras are the same. Therefore, $A(\Gamma)$ and $A(\Gamma^{\uparrow})$ have the same irreducible representations. $\square$

The Cube:

The cube has 8 vertices, 12 edges, and 6 faces. In its Hasse graph, label the vertices in level one by $v_1, ..., v_8$, those in level two by $v_{12}, v_{13}, v_{15}, v_{24}, v_{26}, v_{34}, v_{37}, v_{48}, v_{56}, v_{57}, v_{68}, v_{78}$ and those in level three by $v_{1234}, v_{1256}, v_{1357}, v_{2468}, v_{3478}, v_{5678}$. The cube is dual to the octahedron, and hence their Hasse graphs are dual directed graphs. Therefore, by Proposition 8.6 $A(\Gamma_{\{3,4\}})$ and $A(\Gamma_{\{4,3\}})$ have the same graded trace generating functions and irreducible representations, except the labels (conjugacy class representatives) are different.

The Coxeter group $[4,3]$ has presentation $< s_1, s_2, s_3 | s_1^2 = s_2^2 = s_3^2 = (s_1s_2)^4 = (s_2s_3)^3 = (s_1s_3)^2 = e >$. Use (1), (235)(647), (1342)(5786), (14)(23)(58)(67), (18)(24)(36)(57), (18)(27)(36)(45), (1647)(2835), (18)(265734), (25)(47), and (15)(26)(37)(48) as the conjugacy class representatives of the Coxeter group, listed in the same order as for the character table of $[3,4]$.

Theorem 8.7. *The grading trace generating functions for $A(\Gamma_{\{4,3\}})$ are:*

$$a := Tr_{(1)}(A(\Gamma_{\{4,3\}}), t) = \frac{1-t}{1-t(28-62t+48t^2-14t^3+t^4)}$$

$$b := Tr_{(235)(647)} = Tr_{(1342)(5786)} = Tr_{(14)(23)(58)(67)}(A(\Gamma_{\{4,3\}}), t) = \frac{1-t}{1-t(2-t)(2-t^3)}$$

$$c := Tr_{(18)(24)(36)(57)}(A(\Gamma_{\{4,3\}}), t) = \frac{1-t}{1-t(2-t^2)^2}$$

$$d := Tr_{(18)(27)(36)(45)} = Tr_{(1647)(2835)} = Tr_{(18)(265734)}(A(\Gamma_{\{4,3\}}), t) = \frac{1-t}{1-t(2-t^4)}$$

$$e := Tr_{(25)(47)}(A(\Gamma_{\{4,3\}}), t) = \frac{1-t}{1-t(10-12t+4t^3-t^4)}$$

$$f := Tr_{(15)(26)(37)(48)}(A(\Gamma_{\{4,3\}}), t) = \frac{1-t}{1-t(10-10t-4t^2+6t^3-t^4)}$$

Proof. By Proposition 8.6, the graded trace generating functions are the same as in Theorem 8.4. To see that they correspond in the manner given, we consider the fixed vertices of the automorphisms. All permutations fix $v_{12345678}, *$. The permutation (235)(647) fixes v_1 and v_8, (1342)(5786) and (14)(23)(58)(67) fix v_{1234} and v_{5678}, (18)(24)(36)(57) fixes v_{24} and v_{57}, (18)(27)(36)(45), (1647)(2835), and (18)(265734) only fix $u, *$, (25)(47) fixes $v_1, v_3, v_6, v_8, v_{13}, v_{68}, v_{1256}, v_{3478}$, and (15)(26)(37)(48) fixes v_{15}, v_{26}, v_{37}, v_{48}, v_{1256}, v_{1357},

v_{2468}, v_{3478}. $\square$

In view of Proposition 8.6 and Theorem 8.7, the multiplicities for the representations of $A(\Gamma_{\{4,3\}})$ are given by Theorem 8.5.

8.3 Decomposition of $A(\Gamma_{\{3,4\}})^!$ and $A(\Gamma_{\{4,3\}})^!$

Recall that a basis for $A(\Gamma)^!$ is $\{e(v_1,1)^*e(v_2,1)^*\cdots e(v_l,1)^* : l \geq 0, v_i > v_{i+1}, v_{i+1} \neq \max_{\succ}\mathcal{I}(v_i,v_{i+2})$ for $1 \leq i \leq l-1,$ and $v_l \neq \max_{\succ} S_1(v_{l-1})\}$ by Theorem 7.2.

Therefore, we have the following Theorem.

Theorem 8.8. $A(\Gamma_{\{3,4\}})^!$ *has basis* $\{*, e(v_1,1)^*,\ e(v_2,1)^*,\ e(v_3,1)^*,\ e(v_4,1)^*,\ e(v_5,1)^*,$ $e(v_6,1)^*,\ e(v_{12},1)^*,\ e(v_{13},1)^*,\ e(v_{14},1)^*,\ e(v_{15},1)^*,\ e(v_{23},1)^*,\ e(v_{24},1)^*,\ e(v_{26},1)^*,$ $e(v_{35},1)^*,\ e(v_{36},1)^*,\ e(v_{45},1)^*,\ e(v_{46},1)^*,\ e(v_{56},1)^*,\ e(v_{123},1)^*,\ e(v_{124},1)^*,\ e(v_{135},1)^*,$ $e(v_{145},1)^*,\ e(v_{236},1)^*,\ e(v_{246},1)^*,\ e(v_{356},1)^*,\ e(v_{456},1)^*,\ e(v_{12},1)^*e(v_1,1)^*,$ $e(v_{13},1)^*e(v_1,1)^*,\ e(v_{14},1)^*e(v_1,1)^*,\ e(v_{15},1)^*e(v_1,1)^*,\ e(v_{23},1)^*e(v_2,1)^*,$ $e(v_{24},1)^*e(v_2,1)^*,\ e(v_{26},1)^*e(v_2,1)^*,\ e(v_{35},1)^*e(v_3,1)^*,\ e(v_{36},1)^*e(v_3,1)^*,$ $e(v_{45},1)^*e(v_4,1)^*,\ e(v_{46},1)^*e(v_4,1)^*,\ e(v_{56},1)^*e(v_5,1)^*,\ e(v_{123},1)^*e(v_{12},1)^*,$ $e(v_{123},1)^*e(v_{13},1)^*,\ e(v_{124},1)^*e(v_{12},1)^*,\ e(v_{124},1)^*e(v_{14},1)^*,\ e(v_{135},1)^*e(v_{13},1)^*,$ $e(v_{135},1)^*e(v_{15},1)^*,\ e(v_{145},1)^*e(v_{14},1)^*,\ e(v_{145},1)^*e(v_{15},1)^*,\ e(v_{236},1)^*e(v_{23},1)^*,$ $e(v_{236},1)^*e(v_{26},1)^*,\ e(v_{246},1)^*e(v_{24},1)^*,\ e(v_{246},1)^*e(v_{26},1)^*,\ e(v_{356},1)^*e(v_{35},1)^*,$ $e(v_{356},1)^*e(v_{36},1)^*,\ e(v_{456},1)^*e(v_{45},1)^*,\ e(v_{456},1)^*e(v_{46},1)v,\ e(u,1)^*e(v_{123},1)^*,$ $e(u,1)^*e(v_{124},1)^*,\ e(u,1)^*e(v_{135},1)^*,\ e(u,1)^*e(v_{145},1)^*,\ e(u,1)^*e(v_{236},1)^*,$ $e(u,1)^*e(v_{246},1)^*,\ e(u,1)^*e(v_{356},1)^*,\ e(u,1)^*e(v_{456},1)^*,\ e(v_{123},1)^*e(v_{12},1)^*e(v_1,1)^*,$ $e(v_{124},1)^*e(v_{12},1)^*e(v_1,1)^*,\ e(v_{135},1)^*e(v_{13},1)^*e(v_1,1)^*,\ e(v_{145},1)^*e(v_{14},1)^*e(v_1,1)^*,$ $e(v_{236},1)^*e(v_{23},1)^*e(v_2,1)^*,\ e(v_{246},1)^*e(v_{24},1)^*e(v_2,1)^*,\ e(v_{356},1)^*e(v_{35},1)^*e(v_3,1)^*,$ $e(v_{456},1)^*e(v_{45},1)^*e(v_4,1)^*,\ e(u,1)^*e(v_{123},1)^*e(v_{12},1)^*,\ e(u,1)^*e(v_{123},1)^*e(v_{13},1)^*,$ $e(u,1)^*e(v_{124},1)^*e(v_{14},1)^*,\ e(u,1)^*e(v_{135},1)^*e(v_{15},1)^*,\ e(u,1)^*e(v_{236},1)^*e(v_{26},1)^*,$ *and* $e(u,1)^*e(v_{123},1)^*e(v_{12},1)^*e(v_1,1)^*\}.$

Theorem 8.9. *The graded trace generating functions for $A(\Gamma_{\{3,4\}})^!$ are:*

$$a := Tr_{(1)}(A(\Gamma_{\{3,4\}})^!, t) = 1 + 27t + 35t^2 + 13t^3 + t^4$$

$$b := Tr_{(123)(654)} = Tr_{(1265)} = Tr_{(16)(25)}(A(\Gamma_{\{3,4\}})^!, t) = 1 + 3t - t^2 + t^3 + t^4$$

$$c := Tr_{(12)(56)(34)}(A(\Gamma_{\{3,4\}})^!, t) = 1 + 3t - 3t^2 - t^3 + t^4$$

$$d := Tr_{(16)(25)(34)} = Tr_{(1265)(34)} = Tr_{(123654)}(A(\Gamma_{\{3,4\}})^!, t) = 1 + t - t^2 + t^3 - t^4$$

$$e := Tr_{(16)}(A(\Gamma_{\{3,4\}})^!, t) = 1 + 9t + 3t^2 - 3t^3 - t^4$$

$$f := Tr_{(12)(65)}(A(\Gamma_{\{3,4\}})^!, t) = 1 + 9t + t^2 - 5t^3 - t^4$$

Proof. Recall that a presentation for $A(\Gamma_{\{3,4\}})^!$ is given by generators $\{e(v,1)^*\}$ and relations $\{e(v,1)e(u,1) : v \not> u\} \cup \{e(v,1)^* \sum_{v>u} e(u,1)^*\} \cup \{e(v,1)^*(\sum_{v>u>w} e(u,1)^*)e(w,1)^*\}$ (Proposition 3.1). Using these relations and the basis given above, we can calculate the graded trace functions. You can easily check the contribution to the trace of each basis element given below. We will write v for $e(v,1)^*$ in this proof.

a) Count the basis elements.

b) For (123)(456), the following are sent to a linear combination of basis elements involving themselves: $u, v_{123}, v_{456}, uv_{123}, v_{123}v_{12}v_1, uv_{123}v_{12}v_1$.

The following are sent to a linear combination involving their negatives: $v_{123}v_{12}, v_{456}v_{46}$.

The remainder of the basis elements are sent to a linear combination of basis elements not involving the one we started with.

c) The following are sent to a linear combination of basis elements involving themselves: u, v_{12}, v_{56}, $uv_{123}v_{12}v_1$.

The following are sent to a linear combination involving their negatives: $v_{12}v_1$, $v_{56}v_5$, uv_{356}, $uv_{123}v_{12}$.

The remainder of the basis elements are sent to a linear combination of basis elements not involving the one we started with.

d) For (16)(25)(34), the following are sent to a linear combination of basis elements involving themselves: $u, v_{123}v_{12}$.

The following are sent to a linear combination involving their negatives: $uv_{123}, v_{123}v_{12}$.

The remainder of the basis elements are sent to a linear combination of basis elements not involving the one we started with.

Example calculation: $uv_{123}v_{12} \mapsto uv_{456}v_{56} \equiv -uv_{356}v_{56} \equiv uv_{356}(v_{35}+v_{36}) \equiv -uv_{135}v_{35} - uv_{236}v_{36} \equiv uv_{135}(v_{13}+v_{15}) + uv_{236}(v_{23}+v_{26}) \equiv uv_{135}v_{15} + uv_{236}v_{26} - uv_{123}v_{13} - uv_{123}v_{23} \equiv uv_{135}v_{15} + uv_{236}v_{26} - uv_{123}v_{13} + uv_{123}v_{12} + uv_{123}v_{13}$.

e) The following are sent to a linear combination of basis elements involving themselves: u, v_{23}, v_{24}, v_{35}, v_{45}, v_2, v_3, v_4, v_5, $v_{23}v_2$, $v_{24}v_2$, $v_{35}v_3$, $v_{45}v_4$.

The following are sent to a linear combination involving their negatives: uv_{145}, $uv_{123}v_{13}$, $uv_{124}v_{14}$, $uv_{135}v_{15}$, $uv_{123}v_{12}v_1$.

The remainder of the basis elements are sent to a linear combination of basis elements not involving the one we started with.

f) The following are sent to a linear combination of basis elements involving themselves: u, v_{123}, v_{124}, v_{356}, v_{456}, v_{12}, v_{56}, v_3, v_4, $v_{123}v_{12}$, $v_{124}v_{12}$, uv_{123}, uv_{124}, uv_{356}, $uv_{123}v_{12}$.

The following are sent to a linear combination involving their negatives: $v_{56}v_5$, $v_{12}v_1$, $v_{124}v_{14}$, $v_{123}v_{13}$, $uv_{123}v_{13}$, $uv_{124}v_{14}$, $v_{123}v_{12}v_1$, $v_{124}v_{12}v_1$, $v_{356}v_{35}v_3$, $v_{456}v_{45}v_4$, $uv_{123}v_{12}v_1$.

The remainder of the basis elements are sent to a linear combination of basis elements not involving the one we started with. $\square$

Theorem 8.10. *The multiplicities for the representations of* $A(\Gamma_{\{3,4\}})^!$ *are:*

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}
$m_{\phi,1}$	*4*	*1*	*0*	*0*	*2*	*0*	*0*	*1*	*2*	*3*
$m_{\phi,2}$	*0*	*1*	*1*	*0*	*2*	*1*	*2*	*2*	*2*	*3*
$m_{\phi,3}$	*0*	*0*	*1*	*1*	*0*	*1*	*2*	*1*	*0*	*0*
$m_{\phi,4}$	*0*	*0*	*0*	*1*	*0*	*0*	*0*	*0*	*0*	*0*

Proof. We verify the assertion by multiplying the transpose of the character table of $[3,4]$ by $\vec{m}(t)$ for each graded piece. $\square$

We can write down a basis for $A(\Gamma_{\{4,3\}})^!$ by using Theorem 7.2. By Proposition 8.6,

$A(\Gamma_{\{4,3\}})^!$ will have graded trace generating functions given by Theorem 8.9 and multiplicities of irreducible representations given by Theorem 8.10 with the same correspondence of conjugacy classes as given above.

Notice that $A(\Gamma_{\{3,4\}})^!$ and $A(\Gamma_{\{4,3\}})^!$ also have the Koszulity property that $Tr_\sigma(A(\Gamma), t) * Tr_\sigma(A(\Gamma)^!, -t) = 1$, where σ is an element in the automorphism group of the algebra.

Table 8.1: Character table for the Coxeter groups $[3, 4]$ and $[4, 3]$

$[3,4]$	(1)	(123)(654)	(1265)	(16)(25)	(12)(56)(34)	(16)(25)(34)	(1265)(34)	(123654)	(16)	(12)(56)
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	1	-1	-1	1	-1	-1	1
χ_3	1	1	-1	1	-1	1	-1	1	1	-1
χ_4	1	1	1	1	1	-1	-1	-1	-1	-1
χ_5	2	-1	0	2	0	2	0	-1	2	0
χ_6	2	-1	0	2	0	-2	0	1	-2	0
χ_7	3	0	1	-1	-1	3	1	0	-1	-1
χ_8	3	0	-1	-1	1	-3	1	0	1	-1
χ_9	3	0	-1	-1	1	3	-1	0	-1	1
χ_{10}	3	0	1	-1	-1	-3	-1	0	1	1

Chapter 9

The Complete Layered Graph

A layered graph Γ is complete if for every i, $1 \leq i \leq n$, and for every $v \in V_i, w \in V_{i-1}$, there exists a unique edge e such that the tail of e is v and its head is w. A complete layered graph is determined up to isomorphism by the cardinalities of each level. [RSW] Let $\Gamma_{[r_1,...,r_n]}$ be the complete layered graph with a minimal vertex * and r_j vertices in level j, $1 \leq j \leq n$.

Lemma 9.1. *$Aut(\Gamma_{[r_1,...,r_n]}) = S_{r_1} \times \cdots \times S_{r_n}$.*

Proof. Label the vertices in level j of $\Gamma_{[r_1,...,r_n]}$ by $v_1, ..., v_{r_j}$. Any automorphism of the graph must preserve the set of vertices at each level and so acts on the set $\{v_1, ..., v_{r_j}\}$ of all r_j vertices in level j. We may say $\sigma(v_i) = v_{\sigma(i)}$. Thus we can think of an automorphism of the graph as being a permutation in $S_{r_1} \times \cdots \times S_{r_n}$ acting on the subscripts of the vertices of each level. Because the graph is complete, edges between vertices will automatically be preserved. Thus, we may permute the vertices in each level independently of the other levels. Therefore, the automorphism group of the graph is $S_{r_1} \times \cdots \times S_{r_n}$. ☐

Proposition 9.2. *$Aut(A(\Gamma_{[r_1,...,r_n]})) \supseteq k^* \times S_{r_1} \times \cdots \times S_{r_n}$, k the base field.*

Proof. This follows from Lemmas 4.1 and 9.1. ☐

We will consider the graded traces of $S_{r_1} \times \cdots \times S_{r_n}$ acting on $A(\Gamma_{[r_1,...,r_n]})$ because these graded trace generating functions are interesting generalizations of the Hilbert series of the algebra found in [RSW]. Also, we can use them to find some irreducible representations of the algebra.

Theorem 9.3. *Let m_j be the number of vertices in level j fixed by $\sigma_{r_j} \in S_{r_j}$. Let $\sigma = \sigma_{r_1} \times \cdots \times \sigma_{r_n} \in S_{r_1} \times \cdots \times S_{r_n}$. Then $Tr_\sigma(A(\Gamma_{[r_1,...,r_n]}), t) =$*

$$\frac{1-t}{1 - t\sum_{k=0}^{n}\sum_{j=k}^{n}(-1)^k m_j(m_{j-1}-1)(m_{j-2}-1)\cdots(m_{j-k+1}-1)m_{j-k}t^k}$$

Proof. We will prove this using Theorem 5.4. Because we have a complete layered graph, all that matters in calculating the graded trace is how many vertices on each level are fixed, not which ones. Also, there exists a path from every vertex in level i to every vertex in a lower level. Thus, the number of chains $v_1 > \cdots > v_l$ with $|v_i| = j_i, 1 \leq i \leq l$, is $m_{j_1} m_{j_2} \cdots m_{j_l}$. Therefore, the coefficient of t^{k+1} is the number of chains from level j to level $j-k$ where we may include from 0 to $k-1$ vertices, all from different levels, in the middle of the chain. Let $\vec{s}$ be a sequence of 0's and 1's of length $k-1$. Then the coefficient of t^{k+1} is

$$\begin{aligned}
\sum_{\substack{v_1 > \cdots > v_l \geq * \\ v_i \in V_{\sigma r_j} \\ |v_1| - |v_l| = k}} (-1)^l &= \sum_{j=k}^{n} -m_j \Big(\sum_{\vec{s}} (-1)^{\sum s_i} m_{j-1}^{s_1} m_{j-2}^{s_2} \cdots m_{j-k+1}^{s_{k-1}}\Big)(-m_{j-k}) \\
&= \sum_{j=k}^{n} (-m_j)(1 - m_{j-1})(1 - m_{j-2}) \cdots (1 - m_{j-k+1})(-m_{j-k}) \\
&= (-1)^{k+1} m_j (m_{j-1} - 1) \cdots (m_{j-k+1} - 1) m_{j-k}.
\end{aligned}$$

Hence, the theorem is proved. □

We will not give the irreducible representations because of the complexity of writing down the character table of $S_{r_1} \times \cdots \times S_{r_n}$. However, the methodology used in earlier chapters applies equally well here.

9.1 Dual Algebra $A(\Gamma_{[r_1,\ldots,r_n]})^!$

We will now give a basis for the dual algebra $A(\Gamma_{[r_1,\ldots,r_n]})^!$ and its graded trace generating functions.

Choose one vertex in each level $j, 1 \leq j \leq n$, and call it v_{r_j}.

Theorem 9.4. *A basis for* $A(\Gamma_{[r_1,\ldots,r_n]})^!$ *is* $\{e(v_1,1)^* e(v_2,1)^* \cdots e(v_l,1)^* \; : \; l \geq 0, v_i \gtrdot v_{i+1},$ *and for* $2 \leq i \leq l, v_i \neq v_{r_j} \; \forall j\}$.

Proof. This is true by Theorem 7.2. Because each vertex has a path to every vertex in level 1, we cannot put a partial order on the set of vertices. However, we can choose one vertex to be the "maximum" vertex that we do not allow. These are the vertices v_{r_j} we have chosen. $\square$

Theorem 9.5. *Let m_j be the number of vertices in level j fixed by $\sigma_{r_j} \in S_{r_j}$. Let $\sigma = \sigma_{r_1} \times \cdots \times \sigma_{r_n} \in S_{r_1} \times \cdots \times S_{r_n}$. Then $Tr_\sigma(A(\Gamma_{[r_1,\ldots,r_n]})^!, t) =$*

$$1 + t\sum_{j=1}^{n} m_j + \sum_{k=2}^{n}\sum_{j=k}^{n} m_j(m_{j-1}-1)(m_{j-2}-1)\cdots(m_{j-k+1}-1)t^k$$

Proof. In order for $\sigma e(v_1,1)^* \cdots e(v_l,1)^*$ to be a linear combination of basis elements containing a nonzero multiple of $\vec{e} = e(v_1,1)^* \cdots e(v_l,1)^*$, v_1 must be fixed by σ. If v_k is fixed for all $1 \leq k \leq l$, then the element's contribution to the trace is 1. If v_k is not fixed for some k, then the only way the basis element is going to contribute to the trace is if $\sigma(v_k) = v_{r_j}$, $k > 1$, v_k, v_{r_j} in the same level. Otherwise, $\vec{e}$ is sent to another basis element. In the case where $\sigma(v_k) = v_{r_j}$, we can use the relation $e(v_i,1)^* \sum_{v_j, v_i > v_j} e(v_j,1)^*$ to write $\sigma\vec{e}$ as a linear combination of basis elements which will include $(-1)\vec{e}$. Let $s_{\vec{e}}$ be the number of $v_k \in \vec{e}$ such that $\sigma(v_k) = v_{r_j}$, for the appropriate j. Note that $0 \leq s \leq n$, as at most one vertex from each level may go to v_{r_j}. If v_1 is fixed and $\sigma(v_k) = v_k$ or v_{r_j} for all $2 \leq k \leq l$, then we have shown that $\vec{e}$ contributes $(-1)^{s_{\vec{e}}}$ to the graded trace. Otherwise, the contribution is zero.

Let us consider the coefficient of t^k. This value is the contribution to the trace of basis elements of degree k: $e(v_{l_1},1)^* \cdots e(v_{l_k},1)^*$, $l_1 \geq k$. Because v_{l_1} must be fixed, there are m_{l_1} choices for $e(v_{l_1},1)^*$. We can then choose that s vertices in the basis element are fixed and the other $k-s$ of them are sent to v_{r_j}, the appropriate j for each. This gives us that the coefficient of t^k is

$$\sum_{j=k}^{n} m_j \sum_{\substack{j > j_1 > \cdots > j_s \geq 1 \\ 0 \leq s \leq k-1}} (-1)^{(k-1-s)} m_{j_1} \cdots m_{j_s}$$

$$= \sum_{j=k}^{n} m_j(m_{j-1}-1)(m_{j-2}-1)\cdots(m_{j-k+1}-1)$$

. $\square$

Observe that

$$
\begin{gathered}
(1+t\sum_{j=1}^{n} m_j + \sum_{k=2}^{n}\sum_{j=k}^{n} m_j(m_{j-1}-1)(m_{j-2}-1)\cdots(m_{j-k+1}-1)t^k)(1+t) \\
= 1+t\sum_{k=0}^{n}\sum_{j=k}^{n} m_j(m_{j-1}-1)(m_{j-2}-1)\cdots(m_{j-k+1}-1)m_{j-k}t^k \\
= Tr_\sigma(A(\Gamma_{[r_1,\ldots,r_n]}),-t)^{-1}(1+t).
\end{gathered}
$$

Thus, $A(L(n,q))$ also satisfies the Koszulity property that $Tr_\sigma(A(\Gamma),t)*Tr_\sigma(A(\Gamma)^!,-t)=1$, where σ is an element in the automorphism group of the algebra.

Chapter 10

Hasse Graph of the Lattice of Subspaces of a Finite-dimensional Vector Space over a Finite Field

Following the definition given in [RSW], we will denote by $L(n,q)$ the Hasse graph of the lattice of subspaces of an n-dimensional vector space V over the field F_q of q elements. Thus, the vertices of $L(n,q)$ are subspaces of F_q^n, the order relation $>$ is inclusion of subspaces $\supset$, the level $|U|$ of a subspace U is its dimension, and the unique minimal vertex $*$ is the zero subspace (0).

In [RSW], the authors found the Hilbert series of the algebra $A(L(n,q))$ to be

$$\frac{1-t}{1-t\sum_{m=0}^{n}\binom{n}{m}_q(1-t)(1-tq)\cdots(1-tq^{n-m-1})}.$$

We wish to find the graded trace generating functions as we have done for our other examples in order to find the decomposition of $A(L(n,q))$ into irreducible representations. However, in this case it is much more difficult to count chains of subspaces. Therefore, in this chapter we will give the graded traces for small n and results that enable one to find the graded traces for general q and specific n on a case-by-case basis.

Lemma 10.1. *$Aut(L(n,q)) \supseteq PGL_n(F_q)$.*

Proof. The automorphism group of V is $GL_n(F_q)$. An automorphism of V permutes subspaces of each dimension and maintains subspace containment. Thus, edges are preserved. Because each subspace is in the lattice, every $M \in GL_n(F_q)$ preserves the set of vertices at each level. Hence, the automorphism group of the Hasse graph of the lattice of subspaces contains $GL_n(F_q)$. The center of $GL_n(F_q)$, scalar multiples of the identity matrix, induces the identity automorphism on the graph. Therefore, to eliminate scalar multiples of automorphisms, we may quotient out by the center. Thus, the automorphism group of the graph contains $PGL_n(F_q)$. □

Proposition 10.2. *$Aut(A(L(n,q))) \supseteq k^* \times PGL_n(F_q)$.*

Proof. This follows from Lemmas 4.1 and 10.1. $\square$

We will consider the graded trace generating functions for $PGL_n(F_q)$ acting on $A(L(n,q))$.

Let $\sigma \in PGL_n(F_q)$. We may write $F_q^n = V_1 \oplus \cdots \oplus V_r$ where each V_i is an indecomposable σ-module and the minimum polynomial of $\sigma|_{V_i} = f_i(x)^{k_i}$ for some monic irreducible $f_i(x) \in F_q[x]$ and some $k_i \geq 1$. Then, relative to an appropriate basis, $\sigma|_{V_i}$ has matrix $\begin{bmatrix} C_i & J & 0 & \cdots & 0 \\ 0 & C_i & J & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & & C_i & J \\ 0 & \cdots & & 0 & C_i \end{bmatrix}$ where C_i is the companion matrix of $f_i(x)$ and $J = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}$ [[J],Chapter 3]. If $f_i(x)$ splits over F_q, then C_i is a 1-by-1 matrix and the above matrix is the Jordan canonical form of $\sigma|_{V_i}$.

We will deal primarily with those automorphisms which can be written in Jordan canonical form. Furthermore, we need only deal with the case where the matrix has a single eigenvalue (C_i is a 1-by-1 matrix and $C_i = C_j$ for all i, j). This follows from Proposition 10.4 below.

Definition $(P \times Q)$**.** *[S] Let P, Q be posets. The direct product of P and Q is the poset $P \times Q$ on the set $\{(x,y) : x \in P, y \in Q\}$ such that $(x,y) \leq (x',y')$ in $P \times Q$ if $x \leq x' \in P$ and $y \leq y' \in Q$.*

We draw the Hasse graph of $P \times Q$ by replacing each element x of P by a copy Q_x of Q and connect corresponding elements of Q_x and Q_y if x and y are connected in the Hasse graph of P [S]. See Figure 10.1 for an example of the Hasse graph of the direct product of two lattices of subspaces of V over F_q.

Definition (ζ)**.** *[S] The zeta function $\zeta(x,y)$ is defined by $\zeta(x,y) = 1$ for all $x \leq y \in P$. Otherwise it is 0. To the zeta function is associated a matrix ζ where the (x,y)-entry in the matrix is $\zeta(x,y)$. Note that it is a triangular, unipotent matrix.*

Definition (Möbius function)**.** *[S] The Möbius matrix μ is ζ^{-1}. The Möbius function $\mu(x,y)$ is the (x,y)-entry in μ.*

Proposition 10.3. *[[S],3.8.2] Let P, Q be locally finite posets and let $P \times Q$ be their direct product. If $(x,y) \leq (x',y') \in P \times Q$, then $\mu_{P\times Q}((x,y),(x',y')) = \mu_P(x,x')\mu_Q(y,y')$.*

We can create the zeta-matrix $\hat{\zeta}$ as in the proof of Theorem 5.4 where the (v,w)-entry is $t^{|v|-|w|}$ and the Möbius matrix $\hat{\mu}$ which is the inverse of $\hat{\zeta}$. Proposition 10.3 applies to $\hat{\mu}$ as well because $\hat{\zeta}_{P\times Q} = \hat{\zeta}_P \otimes \hat{\zeta}_Q$ and hence $\hat{\mu}_{P\times Q} = \hat{\mu}_P \otimes \hat{\mu}_Q$.

Proposition 10.4. *Let $T \in GL_n(F_q)$ be a matrix in Jordan canonical form with distinct eigenvalues $\lambda_1,..,\lambda_k$. Let $T_{\lambda_j} \in GL_{n_j}(F_q)$ be the submatrix of T containing only those blocks with eigenvalue λ_j. Finally say $\frac{1-t}{1-tf_{\lambda_j}}$ is the graded trace generating function of T_{λ_j} acting on $A(L(n_j,q))$. Then $Tr_T(A(L(n,q)),t) = \dfrac{1-t}{1-t\prod_{j=1}^{k} f_{\lambda_j}}$.*

Proof. Any subspace U invariant under T can be written as a direct product of indecomposable summands corresponding to blocks having the same eigenvalue. Thus, the subgraphs $L(n,q)^{T_{\lambda_j}}$ in $L(n,q)^T$ are disjoint; i.e. if $U \in L(n,q)^{T_{\lambda_j}}$, then $U \notin L(n,q)^{T_{\lambda_i}}\ \forall i \neq j$. Therefore, the Hasse graph $L(n,q)^T = L(n,q)^{T_{\lambda_1}} \times \cdots \times L(n,q)^{T_{\lambda_k}}$.

We need to show that $f_{\lambda_1\times\cdots\times\lambda_k} = f_{\lambda_1}\cdots f_{\lambda_k}$. Recall from the proof of Theorem 5.4 that $f_{\lambda_j} = \sum\limits_{v_1>\cdots>v_l\geq *} (-1)^{l+1} t^{|v_1|-|v_l|}$ and that the (v,w)-entry of $\hat{\mu}$ is $\sum\limits_{v=v_1>\cdots>v_l=w\geq *} (-1)^{l+1} t^{|v_1|-|v_l|}$. Let $\vec{1}$ denote the column vector having 1 as each entry. Consider $\vec{1}^T\hat{\mu}\vec{1}$; it is the sum of all the entries in $\hat{\mu}$. In other words, it is the sum of all chains from v to w for all vertices v,w in the graph. Thus, $\vec{1}^T\hat{\mu}\vec{1} = \sum\limits_{v_1>\cdots>v_l\geq *} (-1)^{l+1} t^{|v_1|-|v_l|} = f$.

Therefore, by Proposition 10.3 and the comment afterwards,

$$f_{\lambda_1\times\cdots\times\lambda_k} = \vec{1}^T_{L(n,q)^{T_{\lambda_1}}\times\cdots\times L(n,q)^{T_{\lambda_k}}}\hat{\mu}_{L(n,q)^{T_{\lambda_1}}\times\cdots\times L(n,q)^{T_{\lambda_k}}}\vec{1}_{L(n,q)^{T_{\lambda_1}}\times\cdots\times L(n,q)^{T_{\lambda_k}}}$$
$$= (\vec{1}^T_{L(n,q)^{T_{\lambda_1}}}\otimes\cdots\otimes\vec{1}^T_{L(n,q)^{T_{\lambda_k}}})(\hat{\mu}_{L(n,q)^{T_{\lambda_1}}}\otimes\cdots\otimes\hat{\mu}_{L(n,q)^{T_{\lambda_k}}})(\vec{1}_{L(n,q)^{T_{\lambda_1}}}\otimes\cdots\otimes\vec{1}_{L(n,q)^{T_{\lambda_k}}})$$
$$= (\vec{1}^T_{L(n,q)^{T_{\lambda_1}}}\hat{\mu}_{L(n,q)^{T_{\lambda_1}}}\vec{1}_{L(n,q)^{T_{\lambda_1}}})\cdots(\vec{1}^T_{L(n,q)^{T_{\lambda_k}}}\hat{\mu}_{L(n,q)^{T_{\lambda_k}}}\vec{1}_{L(n,q)^{T_{\lambda_k}}}) = f_{\lambda_1}\cdots f_{\lambda_k}.$$

□

Let $T \in GL_n(F_q)$ be a matrix in Jordan canonical form with exactly one eigenvalue λ. Let k be the number of its blocks. We want to determine the number of invariant subspaces of an n-dimensional vector space over F_q when acted upon by T.

Let $V = V_{[a]} = V_{a_1} \oplus \cdots \oplus V_{a_k}$ be indecomposable under T, where $[a] = a_1, ..., a_k$ are the sizes of the blocks in T. Let $W = W_{[i]} = W_{i_1} \oplus \cdots \oplus W_{i_{r'}} \oplus W_{i_{r'+1}} \oplus \cdots \oplus W_{i_r}$ be an invariant subspace of V where $[i] = i_1, ..., i_r$ are the sizes of its blocks. Say $i_1 \geq i_2 \geq \cdots \geq i_{r'} > 1$, $i_{r'+1} = \cdots = i_r = 1$, $r \leq k$. Let $V_{[a-1]} = V_{a_1-1} \oplus \cdots \oplus V_{a_k-1}$ and $W_{[i-1]} = W_{i_1-1} \oplus \cdots \oplus W_{i_{r'}-1}$. Denote the number of invariant subspaces $W_{[i]}$ of $V_{[a]}$ by $N_{[a,i]}$.

Proposition 10.5. *The number of invariant $W_{[i]} \subseteq V_{[a]}$ is $N_{[a,i]} = (q^{k-r})^{r'}\binom{k-r'}{r-r'}_q N_{[a-1,i-1]}$.*

Proof. Let $S = T - \lambda I$; note that S is a nilpotent matrix. Then $SV_{[a]} = V_{[a-1]} = V_{a_1-1} \oplus \cdots \oplus V_{a_k-1}$ and $SW_{[i]} = W_{[i-1]} = W_{i_1-1} \oplus \cdots \oplus W_{i_{r'}-1}$. The number of such SW in SV is given by induction, $N_{[a-1,i-1]}$. We need to multiply this induction by the number of W which map down to $W_{i_1-1} \oplus \cdots \oplus W_{i_{r'}-1}$. First of all you can add up to $r - r'$ W_1's because they will go to (0) under S. In other words, we can add on anything in $\ker(S)$. The kernel has one dimension from each of the k summands, but we must quotient out by the kernel of S^2 so that we do not double count; so, we can add on from a $k - r'$-dimensional space. Thus we are picking a $r - r'$-dimensional subspace from a $k - r'$-dimensional space, which gives us $\binom{k-r'}{r-r'}_q$ possibilities. Now fix one of these subspaces; i.e. fix W_1's space.

Now we need to pick the other W_i's with $i > 1$. Each W_i is generated by a cyclic vector; and, we can modify that vector by something that is annihilated by S. Thus we have $(q^{k-r})^{r'}$ choices. The $k-r$ is because we cannot modify by something we have already chosen and r' because we are only modifying the cyclic vectors in SW. In other words, there are q^{k-r} ways of modifying each $W_{i_j-1}, 1 \leq j \leq r'$. Therefore, the number of W which map down to SW is $(q^{k-r})^{r'} \binom{k-r'}{r-r'}_q$. $\square$

We will now give a description of $N_{[a,i]}$ in closed form. This will allow us to write an explicit formula for the number of invariant subspaces of V. Let the sizes of the blocks of T be $a_1 \geq a_2 \geq \cdots \geq a_k$ as above. Let k_j be the number of blocks of size greater than or equal to j, $1 \leq j \leq a_1$. Finally, let m_j the multiplicity of i_j; so, $W = m_1 W_{i_1} \oplus \cdots \oplus m_s W_{i_s}$.

Corollary 10.6. *The number of invariant subspaces W with the notation given above is*

$$\sum_{\substack{1 \leq i_j \leq a_{m_1+m_2+\ldots+m_j} \\ i_1 > i_2 > \cdots > i_s \\ m_1+\ldots+m_s = r}} \prod_{j=1}^{i_s-1} (q^{k_j-r})^r \cdot (q^{k_{i_s}-r})^{r-m_s} \binom{k_{i_s}-(r-m_s)}{m_s}_q \cdot$$

$$\prod_{j=i_s+1}^{i_{s-1}-1} (q^{k_j-(r-m_s)})^{r-m_s} \cdot (q^{k_{i_{s-1}}-(r-m_s)})^{r-m_s-m_{s-1}} \binom{k_{i_{s-1}}-(r-m_s-m_{s-1})}{m_{s-1}}_q \cdot$$

$$\cdots \prod_{j=i_2+1}^{i_1-1} (q^{k_j-m_1}) \cdot \binom{k_{i_1}}{m_1}_q$$

Proof. We are applying the induction process to Proposition 10.5. Because the formula only depends on k, r, r', the induction only changes when $S^l V_{a_i} = V_0$, $S^l W_{i_j} = W_1$, or $S^l W_{i_j} = W_0$ for some l, a_i, i_j. Hence, up until we have applied S^{i_s}, $i_j > 1 \forall j$ we have $r' = r$ and $(q^{k_1-r})^r N_{[a-1,i-1]} = (q^{k_1-r})^r \cdots (q^{k_{i_s-1}-r})^r N_{[a-i_s,i-i_s]}$. At the i_s stage in the induction, r is the same, but $\tilde{r'} = r - m_s$. Then, at the i_s+1 stage, $\tilde{r} = r - m_s$ and $\tilde{r'} = r - m_s$ until the i_{s-1} stage when $W_{i_{s-1}}$ is brought down to W_1. This process continues. $\square$

We will now give the graded trace generating functions for a variety of examples. Recall that from these examples we also have the generating functions for larger n when we have distinct eigenvalues by using Proposition 10.4. In Tables 10.1 and 10.2, let f be such that $Tr_T(A(L(n,q)),t) = \frac{1-t}{1-tf}$. The generating functions can be found using Theorem 5.2 or Theorem 5.4. The reader may easily check that the generating functions given are correct. The graphs for each example with a single eigenvalue are shown in Figure 10.1.

In Table 10.1 let U' be a 2-dimensional vector space over F_q, $V'_j = \text{span}\left\{\begin{bmatrix} 1 \\ j \end{bmatrix}\right\}$,

$V'_{q+1} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$. Also, let U be a three-dimensional vector space over F_q, $V_{jk} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ j \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ k \end{bmatrix}\right\}$, $V_i = \text{span}\left\{\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$, $V_{q+1} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$, $W_{ij} = \text{span}\left\{\begin{bmatrix} 1 \\ i \\ j \end{bmatrix}\right\}$, $W_i = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}\right\}$, and $W_{q+1} = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$. Assume $0 \leq i, j \leq q-1$ unless stated otherwise.

In Table 10.2 we will give the graded trace generating functions for some of the $n = 4$ cases and two general cases. We know the graded trace for all but one of the canonical forms for $n = 4$ based on the information in the above table and Proposition 10.4. Rather than listing the subspaces that are fixed, we will simply say how many of each type are fixed. These numbers are obtained from Proposition 10.5. This is enough information to calculate the graded trace; we will do the calculation for one example.

Example 10.1. *We will calculate the graded trace of* $\begin{bmatrix} a & 1 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}$ *acting on the 4-dimensional vector space* $V = V_2 \oplus V_1 \oplus V_1$ *using Proposition 10.5 and Theorem 5.4. First of all we have* $(q^{3-2})^1\binom{3-1}{2-1}_q * (q^{1-1})^0\binom{1-0}{1-0}_q = q^2 + q$ *3-dimensional invariant subspaces of form* $W_2 \oplus W_1$ *and* $(q^{3-3})^0\binom{3-0}{3-0}_q * 1 = 1$ *subspace of form* $W_1 \oplus W_1 \oplus W_1$. *We have* $(q^{3-1})^1\binom{3-1}{1-1}_q * (q^{1-1})^0\binom{1-0}{1-0}_q = q^2$ *2-dimensional invariant subspaces of the form* W_2 *and* $(q^{3-2})^0\binom{3-0}{2-0}_q * 1 = q^2+q+1$ *of the form* $W_1 \oplus W_1$. *And, there are* $(q^{3-1})^0\binom{3-0}{1-0}_q = q^2+q+1$ *1-dimensional invariant subspaces of the form* W_1.

We also need to know how many subspaces each subspace has in order to count the chains. Each subspace of the form $W_2 \oplus W_1$ *has* q *subspaces of the form* W_2, *1 of the form* $W_1 \oplus W_1$, *and* $q+1$ *of the form* W_1. *Each subspace of the form* $W_1 \oplus W_1 \oplus W_1$ *has* 0 *subspaces of the form* W_2, q^2+q+1 *of the form* $W_1 \oplus W_1$, *and* q^2+q+1 *of the form* W_1. *Lastly, each subspace of the form* W_2 *has 1 subspace and* $W_1 \oplus W_1$ *has* $q+1$.

Now we are ready to count the chains in the subgraph; i.e. calculate $\sum_{\substack{U_1 \supset \cdots \supset U_l \supseteq (0) \\ U_j \text{ invariant under } T}} (-1)^l t^{|U_1|-|U_l|+1}$. *The coefficient of* t *is the number of invariant subspaces, which is* $-(1+q^2+q+1+q^2+q^2+q+1+q^2+q+1+1) = -(4q^2+3q+5)$. *The coefficient of* t^2 *is the number of* $U_i \supset U_{i-1}$, *which is* $q^2+q+1+(q^2+q)(q+1)+1(q^2+q+1)+q^2(1)+(q^2+q+1)(q+1)+q^2+q+1 = 2q^3+8q^2+6q+4$. *For* t^3, *we count the number of* $U_i \supset U_{i-2}$ *minus* $U_i \supset U_{i-1} \supset U_{i-2}$. *This gives us* $[1(2q^2+q+1)+(q^2+q)(q+1)+1(q^2+q+1)+2q^2+q+1]-[1(q^2+q)(q+1)+1*1(q^2+q+1)+(q^2+q)q*1+(q^2+q)*1*(q+1)+1(q^2+q+1)(q+1)+q^2(1)+(q^2+q+1)(q+1)] = -(4q^3+4q^2+3q)$. *To find the coefficient of* t^4 *we consider the number of chains of the forms* $U_i \supset U_{i-3}$, $U_i \supset U_{i-1} \supset U_{i-3}$, $U_i \supset U_{i-2} \supset U_{i-3}$, *and* $U_i \supset U_{i-1} \supset U_{i-2} \supset U_{i-3}$. *There are* $[1(q^2+q+1)+q^2+q+1]-[1(q^2+q)(q+1)+1*1(q^2+q+1)+1q^2*1+1(q^2+q+1)(q+1)+(q^2+q)(q+1)+1(q^2+q+1)+(q^2+q)(q+1)+1(q^2+q+1)]-2*[(q^2+q)(q)(1)+(q^2+q)(1)(q+1)+1(q^2+q+1)(q+1)] = 2q^3$. *Finally, for the coefficient of* t^5, *we consider* $W_2 \oplus W_1 \oplus W_1 \supset (0)$, $W_2 \oplus W_1 \oplus W_1 \supset U_i \supset (0)$, $W_2 \oplus W_1 \oplus W_1 \supset U_i \supset U_j \supset (0)$, $W_2 \oplus W_1 \oplus W_1 \supset U_3 \supset U_2 \supset U_1 \supset (0)$. *This gives us* $1-[q^2+q+1+2q^2+q+1+q^2+q+1]+[(q^2+q)(q+1)+q^2+q+1+(q^2+q)(q+1)+q^2+q+1+q^2(1)+(q^2+q+1)(q+1)]-[(q^2+q)(q)+(q^2+q)(q+1)+(q^2+q+1)(q+1)] = 0$. *Because the graph has four levels, the coefficient of* $t^k, k \geq 6$, *is 0. Thus,* $Tr_T(A(L(n,q)),t) =$

$$\frac{1-t}{1-t((4q^2+3q+5)-(2q^3+8q^2+6q+4)t+(4q^3+4q^2+3q)t^2-2q^3t^3)}$$

We can see through this example that Propositions 10.4 and 10.5 enable us to calculate the graded trace for any matrix which can be put into Jordan canonical form acting on a finite-dimensional vector space over F_q. We do have a conjecture for a general formula for the graded trace in terms of T, as we have found for our other algebras.

In Figure 10.1 are the subgraphs $L(n,q)^T$ of $L(n,q)$ which have vertices fixed by T. The matrices T correspond to those given in Tables 10.1 and 10.2 and are labeled by $V = V_{a_1} \oplus \cdots \oplus V_{a_k}$ where a_j is the size of block j in T. The number following each subspace is the number of that type invariant under T. The number in parentheses along the edge from U to W ($U \supset W \in L(n,q)^T$) is the number of invariant subspaces with form W of each subspace with form U.

Table 10.1: $Tr_T(A(L(n,q)),t)$ for $n = 2,3$

Type of T	Fixes	$W = W_{i_1} \oplus \cdots \oplus W_{i_r}$	f
$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$	U' V'_j, V'_{q+1}	$W_1 \oplus W_1$ W_1	$(q+3) - 2(q+1)t$ $+qt^2$
$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$	U' V'_0, V'_{q+1}	$W_1 \oplus W_1$ W_1	$(2-t)^2$
$\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$	U' V'_0	W_2 W_1	$3 - 2t$
$\begin{bmatrix} 0 & -a_0 \\ 1 & -a_1 \end{bmatrix}$	U'		$2 - t^2$
$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$	U V_{jk}, V_i, V_{q+1} W_{ij}, W_i, W_{q+1}	$W_1 \oplus W_1 \oplus W_1$ $W_1 \oplus W_1$ W_1	$(2q^2 + 2q + 4)$ $-(q^3 + 4q^2 + 4q + 3)t$ $+2(q^3 + q^2 + q)t^2 - q^3t^3$
$\begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$	U V_{0k}, V_0 W_{0j}, W_{q+1}	$W_2 \oplus W_1$ $W_2, W_1 \oplus W_1$ W_1	$2(q+2) - (4q+3)t$ $+2qt^2$
$\begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$	U V_{00} W_{00}	W_3 W_2 W_1	$4 - 3t$
$\begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$	U V_{00}, V_0 W_{00}, W_{q+1}	$W_2 \oplus W_1$ W_2 W_1	$(3-2t)(2-t)$
$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$	U V_{00}, V_i, V_{q+1} W_{i0}, W_0, W_{q+1}	$W_1 \oplus W_1 \oplus W_1$ $W_1 \oplus W_1$ W_1	$[(q+3) - 2(q+1)t + qt^2]$ $*(2-t)$
$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$	U V_{00}, V_0, V_{q+1} W_{00}, W_0, W_{q+1}	$W_1 \oplus W_1 \oplus W_1$ $W_1 \oplus W_1$ W_1	$(2-t)^3$
$\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & -b_0 \\ 0 & 1 & -b_1 \end{bmatrix}$	U V_{q+1} W_{00}		$(2-t)(2-t^2)$
$\begin{bmatrix} 0 & 0 & -b_0 \\ 1 & 0 & -b_1 \\ 0 & 1 & -b_2 \end{bmatrix}$	U		$(2-t^3)$

Table 10.2: $Tr_T(A(L(n,q)),t)$ for $n=4$ and special cases

Type of T	$W = W_{i_1} \oplus \cdots \oplus W_{i_s}$	Number Fixed	f
$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}$	$W_1 \oplus W_1 \oplus W_1 \oplus W_1$	1	$(1-t)(1-tq)(1-tq^2)(1-tq^3)+$
	$W_1 \oplus W_1 \oplus W_1$	$1+q+q^2+q^3$	$(1+q+q^2+q^3)(1-t)(1-tq)(1-tq^2)+$
	$W_1 \oplus W_1$	$1+q+2q^2+q^3+q^4$	$(1+q+2q^2+q^3+q^4)(1-t)(1-tq)+$
	W_1	$1+q+q^2+q^3$	$(1+q+q^2+q^3)(1-t)+1$
$\begin{bmatrix} a & 1 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}$	$W_2 \oplus W_1 \oplus W_1$	1	
	$W_2 \oplus W_1, W_1 \oplus W_1 \oplus W_1$	$q^2+q,\ 1$	$(4q^2+3q+5)-(2q^3+8q^2+6q+4)t$
	$W_2, W_1 \oplus W_1$	$q^2,\ q^2+q+1$	$+(4q^3+4q^2+3q)t^2-2q^3t^3$
	W_1	q^2+q+1	
$\begin{bmatrix} a & 1 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{bmatrix}$	$W_2 \oplus W_2$	1	
	$W_2 \oplus W_1$	$q+1$	$(q^2+3q+5)-2(q^2+3q+2)t$
	$W_2, W_1 \oplus W_1$	$q^2+q,\ 1$	$+q(q+3)t^2$
	W_1	$q+1$	
$\begin{bmatrix} a & 1 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & a \end{bmatrix}$	$W_{n-1} \oplus W_1$	1	$((n+1)+(n-1)q)-$
	$W_j, W_{j-1} \oplus W_1$	$q(n-1)+1, n-2$	$(2(n-1)q+n)t+$
	$2 \leq j \leq n-1$		$(n-1)qt^2$
	W_1	$q+1$	
$\begin{bmatrix} a & 1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & a \end{bmatrix}$	W_n	1	$(n+1)-nt$
	W_i	1	
	$1 \leq i \leq n-1$		
$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{bmatrix}$	$W_1^a \oplus W_1^a \oplus W_1^b \oplus W_1^b$	1	
	$W_1^a \oplus W_1^a \oplus W_1^b,\ W_1^a \oplus W_1^b \oplus W_1^b$	$q+1, q+1$	$((q+3)-2(q+1)t+qt^2)^2$
	$W_1^a \oplus W_1^a, W_1^a \oplus W_1^b, W_1^b \oplus W_1^b$	$1, (q+1)^2, 1$	
	$W_1^a,\ W_1^b$	$q+1, q+1$	

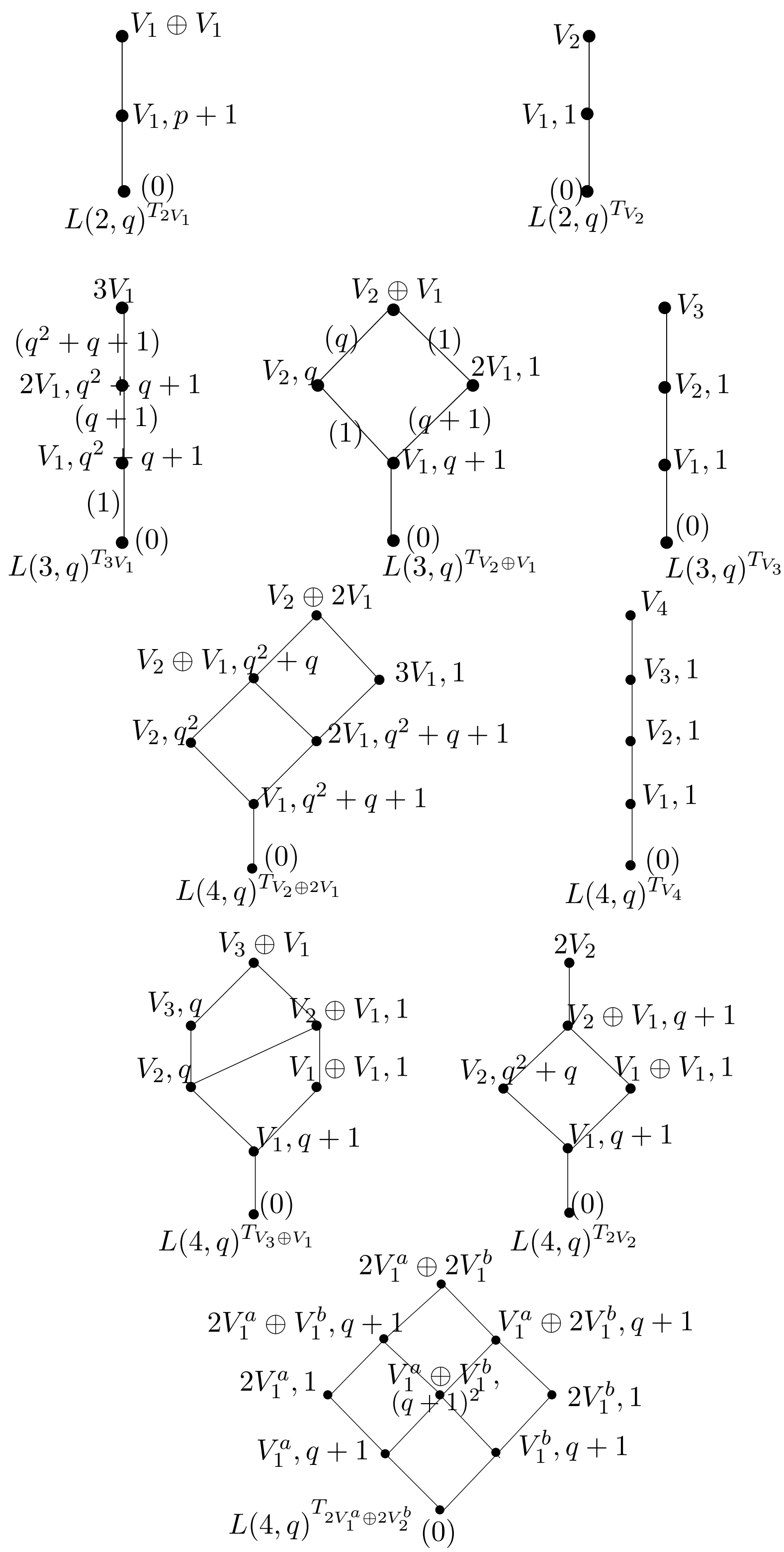

Figure 10.1: Subgraphs $L(n,q)^T$ of the Hasse graph of the lattice of subspaces of a finite-dimensional vector space over F_q

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