

D-BRANES AND ORIENTIFOLDS IN CALABI-YAU COMPACTIFICATIONS

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ABSTRACT OF THE DISSERTATION

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We explore the dynamics of nonsupersymmetric D-brane configurations on Calabi-Yau orientifolds with fluxes. We show that supergravity D-terms capture supersymmetry breaking effects predicted by more abstract Π -stability considerations. We also investigate the vacuum structure of such configurations in the presence of fluxes. Based on the shape of the potential, we argue that metastable nonsupersymmetric vacua can be in principle obtained by tuning the values of fluxes.

We also develop computational tools for the tree-level superpotential of \mathbf{B} -branes in Calabi-Yau orientifolds. Our method is based on a systematic implementation of the orientifold projection in the geometric approach of Aspinwall and Katz. In the process we lay down some ground rules for orientifold projections in the derived category.

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Dedication

To Rocío, the love of my life, my friend and companion, my everything

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Chapter 1

Introduction

Particle physics has achieved phenomenal success in the second half of the twentieth century, resulting in the Standard Model $SU(3) \times SU(2) \times U(1)$, and the verification, to an astonishingly high precision, of many of its predictions. However, this theory does not accommodate the gravitational force, which is described classically by General Relativity. A framework uniting both theories is undoubtedly the most important challenge for physics in the 21st century. It is expected that such a theory would also predict all sorts of new and interesting phenomena.

In the past two decades or so, a promising candidate to such a Theory of Everything has emerged. String theory, which was originally invented as a model for the strong force in the late 1960's, turned out to naturally include a spin 2 massless particle: the graviton. It also miraculously does away with most of the infinities that pervade quantum field theories, which theoretically means we should be able to compute and make predictions in it. On the other hand, we discovered very early that string theory is of an incredible complexity. Thirty years after its first appearance, we still are not able to produce testable predictions out of it, but our understanding has advanced enormously and we may be close to making connection with experiment.

1.1 The basics

1.1.1 Strings and branes

The basic premise of string theory is that elementary particles are not point particles, but one-dimensional extended objects that propagate through spacetime. If the natural action for a relativistic point particle is the length of its world-line, then the action for a string should be the area of the surface swept by it – the so-called world-sheet. Quantization of this action yields a

2d conformal field theory, a QFT whose symmetries allow us to completely solve the theory. Its spectrum, however, contains only spacetime bosons, and we need to introduce supersymmetry on the world-sheet to obtain spacetime fermions.

Two consistency conditions impose severe constraints on the theory. First of all, cancellation of a certain anomaly, which breaks Lorentz invariance at the quantum level, requires the string to move in ten spacetime dimensions. Also, the existence of a tachyon (a field with negative squared mass) forces us to take a projection on the spectrum, known as the GSO projection. Incidentally, the latter has the added benefit of yielding spacetime supersymmetry.

Here the picture begins to get more complicated. There is some freedom in how to take the GSO projection, which leads to five different self-consistent string theories, called types I, IIA, IIB, heterotic $SO(32)$ and heterotic $E_8 \times E_8$. The last four are theories of closed strings only, while type I includes both open and closed strings.

In the mid nineties, we realized that objects of a solitonic character appear naturally in types I, IIA and IIB. These are objects extended in p space dimensions, and are thus called Dp -branes, or simply D-branes. One way of characterizing them is by saying that we introduce open strings in the theory, the endpoints of which are attached to these branes. These objects are the essential ingredient that allows us to say that all five string theories are but different descriptions of one and the same theory. The latter, which goes by the name of M-theory, lives in eleven dimensions.

Unfortunately, we still do not have a fundamental microscopic description of M-theory, so we usually study the different string theories to try to infer its physical properties.

1.1.2 Compactification

As pointed out above, gravitation is already built in string theory. In order to get the Standard Model, there are two things that we need to do. The first one is to “get rid of” the extra six dimensions, and the other one is to build the gauge group $SU(3) \times SU(2) \times U(1)$. We shall be concerned here with the first one mainly (for a review, see [1]). Although some of the ideas in this section apply to all five types of string theory, we concentrate on type II.

The idea of compactification, due to Theodor Kaluza [2] and Oskar Klein [3] in the 1920s, consists of curling up the extra dimensions and making them small. A good illustration of this construction is provided by a hair; from afar it seems to be one-dimensional, but if we look close enough we discover another dimension, wrapped up in a small circle.

In our case, we take six dimensions and make them into a compact manifold. In order to

have supersymmetry in four dimensions, this manifold is required to have a restricted $SU(3)$ holonomy group, meaning it is a particular kind of complex manifold, namely a Calabi–Yau threefold.

Each ten-dimensional field gives a tower of four-dimensional fields, most of which are massive and are thus integrated out in the low-energy limit. At this stage, 4d gauge fields show up, but are decoupled from matter. The introduction of fluxes of certain 10d antisymmetric tensor fields gives rise to charges for some matter fields, opening up the possibility of constructing the Standard Model gauge interactions.

The appearance of moduli is another of the problems of this procedure. Moduli are 4d massless scalar fields resulting from the dimensional reduction of the 10d fields and encode the structure of the compactification manifold. In the case of the quintic, one of the most used Calabi–Yau threefolds, there are 102 of them. Since we have not detected these particles in experiments, our model building has to somehow do away with them. Turning on fluxes generates a potential that gives some of them a mass, so that we can fix their values at the classical minimum. However, these are not enough to stabilize all of the moduli. Typically, the Kähler moduli describing the size of the internal space remain unaffected.

1.1.3 Orientifolds

There is yet a third problem that we have to tackle to get a realistic model. The compactification procedure as we have set it up cannot produce chiral matter. For that the theory has to be modded out by a certain symmetry: we need orientifolds.

To get to understand what an orientifold is we need two concepts. The first one is so-called world-sheet parity, which simply reverses the orientation of the string. On the other hand, if space-time has some discrete symmetry, then we can also mod out by it, obtaining what is called an orbifold. This operation involves only the target space. When the space-time symmetry is an involution (that is, it squares to the identity), we can combine both to obtain a theory that, away from fixed-points of the target space symmetry, looks just like the theory before the projection. It is around these fixed points, so-called orientifold planes, that new phenomena occur.

It turns out that orientifold planes are negatively charged with respect to some of the 10d antisymmetric tensor fields. The compactness of the internal manifold imposes a kind of Gauss’ law: we need the sum of these charges to vanish. The positively charged objects that allow us to do this are precisely D-branes.

We now have all the elements necessary for building a self-consistent model that may resemble the Standard Model: D-branes, fluxes and orientifolds.

1.2 A new compactification model

Appealing to certain non-perturbative effects, Kachru, Kallosh, Linde and Trivedi [4] were first able to also provide a model that fixes all moduli. In our research [5], which we discuss in chapter 2, we found a compactification setup in which all moduli are stabilized without having to resort to instanton contributions.

1.2.1 Moduli spaces and mirror symmetry

Remember from the last section that the moduli fields parametrize the geometry of the internal manifold. Let us describe in some detail the space in which these scalar fields take values.

For compactifications of type IIB, the moduli space has a direct product structure $\mathcal{M} \times \mathcal{K}$: the first factor describes the complex structure of the CY threefold, while the second contains the dilaton (which gives the string coupling) and another scalar field describing the size of the compact dimensions. They are therefore called the complex structure and Kähler moduli spaces, respectively.

Mirror symmetry relates type IIB string theory compactified on a certain CY manifold, to type IIA on another CY, called the mirror of the first one. The two factors of the moduli space are interchanged under this symmetry, so that \mathcal{M} is the Kähler moduli space of the IIA CY and \mathcal{K} the complex structure one.

Both \mathcal{M} and \mathcal{K} have two interesting limit points: the large volume or large complex structure limit, and the Landau–Ginzburg point. A low-energy description of string theory using supergravity is possible in the region around the large volume limit point (in \mathcal{K} for type IIB, or in \mathcal{M} for type IIA).

1.2.2 D-branes and supergravity D-terms

We begin with an $\mathcal{N} = 2$ CY compactification of type II string theory. After taking an orientifold projection, we get an $\mathcal{N} = 1$ compactification. Of course, only the moduli invariant under this symmetry remain in the spectrum, effectively reducing the moduli space to a subspace.

We then introduce a D-brane/anti D-brane pair. In type IIB, they wrap holomorphic curves in the CY and fill the whole of the four-dimensional spacetime. Choosing carefully these curves

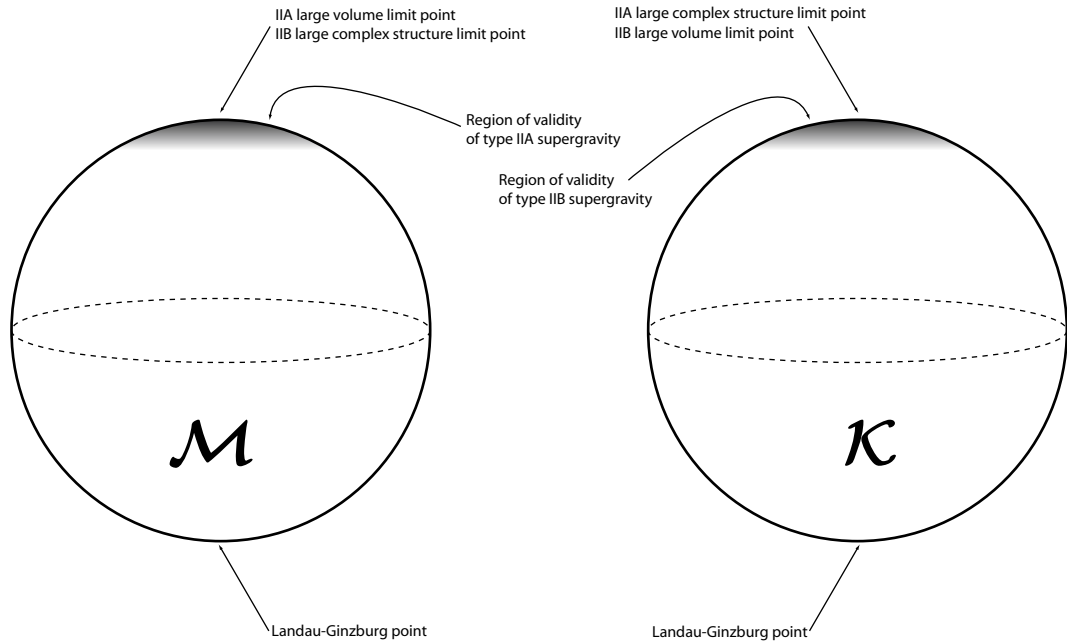


Figure 1.1: Schematic picture of the $\mathcal{N} = 2$ moduli space

curves yields the D-branes in a metastable configuration, meaning that they can only decay through tunneling effects. In terms of the 4d theory, this means that the open string moduli that enter into the picture are stabilized.

But the most important effect that these D-branes have is a SUSY breaking D-term in the 4d supergravity theory. In order for this supergravity formulation to make sense, we need the SUSY breaking to be small. In terms of type IIB, that forces us into the region close to the Landau–Ginzburg point in the Kähler moduli space \mathcal{K} . But then, we need to be also near the large complex structure limit point of \mathcal{M} , in which case the low-energy theory can be described using type IIA supergravity.

1.2.3 The D-brane landscape

The D-branes introduced in the last section also yield a superpotential interaction for the 4d theory, which adds to the superpotentials contributed by the fluxes.

Although the structure of the scalar potential given by superpotentials and D-terms is very complicated, we give some arguments that suggest that this setup effectively fixes all the moduli in the theory, yielding metastable vacua (either dS or AdS).

1.3 D-branes and categories

1.3.1 An algebraic structure on the space of open strings

Let us situate ourselves in type IIB compactified in a Calabi-Yau threefold. We will consider D-branes with all of their spatial dimensions along the compactified coordinates (so that they are point-particles from the four-dimensional point of view). In this setup, a D-brane is given by two pieces of data. Its location, or support, is a complex submanifold of the Calabi-Yau. The gauge theory on it is encoded in a holomorphic vector bundle over this submanifold. (Off-shell) open strings are then maps between these bundles. The connections with which the bundles come equipped yield a BRST operator Q on open strings, and so the on-shell open string states will be given by the cohomology of that operator.

Unfortunately, working directly with these vector bundles over subsets of the whole Calabi-Yau is not easy. The calculations can be carried out much easily, however, by writing these objects as a kind of linear combination of vector bundles on the whole space. This construction yields the derived category of coherent sheaves on the Calabi-Yau [6–8].

1.3.2 D-branes and orientifold projection

In the course of working in [5] we stumbled upon the problem of implementing orientifold projections in the framework described in the last section. Our research on this problem constitutes the article [9] and is discussed in chapter 3

Since orientifold projections involve both world-sheet parity and a space-time symmetry (a holomorphic involution σ), we need to study the effect of these on a D-brane. The geometric action is easy: it just pushes the bundle forward to its new location fiber by fiber.

World-sheet parity changes both the Chan-Paton factor, taking it to its transpose, and the connection on the bundle, which is essentially complex conjugation; the total effect of world-sheet parity then is to take the holomorphic vector bundle on a brane to its (graded) dual.

There is also a further action of this symmetry on the coupling of the D-brane gauge theory to the closed string theory in the bulk. Depending on the dimensionality of the fixed-point set of the involution σ , this coupling might change sign.

We can now mod out the theory by this symmetry by keeping only the invariant fields and calculate the superpotential of this projected theory. A method for computing this quantity was developed in [10] for the case of the unprojected theory. We argue that the superpotential of the orientifolded theory is just that of the original theory evaluated at invariant field configurations

and give several explicit examples.

Chapter 2

A D-Brane Landscape on Calabi-Yau Manifolds

Magnetized branes in toroidal IIB orientifolds have been a very useful device in the construction of semirealistic string vacua [11–21]. A very attractive feature of magnetized brane systems is Kähler moduli stabilization by D-term effects [22–28]. By turning on background fluxes, one can stabilize the complex structure moduli as well, obtaining an interesting distribution of isolated vacua in the string theory landscape. These are typically supersymmetric vacua because magnetized brane configurations are supersymmetric for special values of the toroidal moduli. Note however, that nonsupersymmetric vacua have also been found in [24, 26, 27] as a result of the interaction between D-term and nonperturbative F-term effects.

The purpose of the present chapter is to explore the landscape of magnetized brane configurations on Calabi-Yau manifolds. The starting point of this investigation is the observation that certain Calabi-Yau orientifolds exhibit a very interesting class of metastable D-brane configurations. As opposed to toroidal models, these brane configurations are not supersymmetric for any values of the moduli, but the supersymmetry breaking parameter is minimal for special values of the moduli. We investigate the dynamics of these brane configurations from the point of view of the low energy effective supergravity action. We compute the D-term contribution to the potential energy and show that it agrees with more abstract Π -stability considerations. A similar relation between supergravity D-terms and the perturbative part of Π -stability was previously found in [29]. Then we argue that the interplay between D-term effects and the flux superpotential can in principle give rise to a landscape of metastable nonsupersymmetric vacua. Note that different aspects of the open string landscape have been recently studied in [30–33].

In the absence of fluxes, our configurations admit two equivalent descriptions in terms of IIB and IIA compactifications respectively. These two descriptions are related by mirror symmetry. The IIB formulation is more convenient from a mathematical point of view since it allows very

explicit constructions for D-brane configurations in terms of holomorphic cycles. Moreover, the IIB formulation is also more convenient for describing the orientifold projection in terms of holomorphic involutions. These constructions can be translated in IIA language if we take for granted homological mirror symmetry as well as orientifold mirror symmetry [34–37]. However, the flux dynamics is under better control in the IIA picture in which case the system can be described as a large radius compactification. In the following we will use alternatively the IIB and the IIA picture keeping in mind that we will have to systematically use IIA variables once we address dynamical questions related to flux compactifications.

We will consider IIB orientifolds of Calabi-Yau manifolds with $h^{1,1} = 1$ which have only space-filling O3 planes. Our main example, described in detail in section 2.1, is an orientifold of the octic hypersurface in weighted projective space $WP^{1,1,1,1,4}$. The D-brane configuration consists of a D5-brane wrapping a holomorphic curve C and an anti-D5-brane wrapping the image curve C' under the orientifold projection. Both C, C' are rigid and do not intersect each other. We also turn on worldvolume $U(1)$ magnetic fluxes so that each brane has p units of induced D3-brane charge. This system has a IIA description in terms of D6-branes wrapping rigid special lagrangian cycles M, M' in the mirror octic.

Such configurations are obviously nonsupersymmetric, at least for generic values of the IIA complex structure moduli, since the branes wrapping the special lagrangian manifolds M, M' do not preserve the same amount of supersymmetry as the orientifold projection. The supersymmetry breaking parameter can be taken to be the phase difference between the central charges of these objects in the underlying $N = 2$ theory. This phase can be computed using standard Π -stability techniques, and depends only on the complex structure moduli of the $N = 2$ IIA theory. We will perform detailed computations for the octic orientifold example in section 2.2 and appendix 2.A. The outcome of these computations is that this system is not supersymmetric anywhere on the real subspace of the IIA complex structure moduli space preserved by the orientifold projection. However the supersymmetry breaking parameter reaches a minimum at the Landau-Ginzburg point. This is a new dynamical aspect which has not been encountered before in toroidal orientifolds.

In flat space we would expect this system to decay to a supersymmetric D-brane configuration. The dynamics is different on Calabi-Yau manifolds since the cycles M, M' are rigid, which means that the branes have no moduli. This can be viewed as a potential barrier in configuration space opposing brane–anti-brane annihilation. If the branes are sufficiently far apart, so that the open string spectrum does not contain tachyons and the attractive force is weak, we will obtain a metastable configuration. The system can still decay, but the decay has

to be realized by tunnelling effects.

Since the supersymmetry breaking phase is independent on IIA Kähler moduli, we can work near the large radius limit of the IIA compactification, where the dynamics is under control. In this regime, the theory has an effective four dimensional supergravity description, and the supersymmetry breaking effects are encoded in the D-term potential. In section 2.2 we compute the D-term effects and show that they agree with the Π -stability analysis.

Moduli stabilization in this system can be achieved by turning on IIA fluxes as in [37–45]. In section 2.3 we investigate the vacuum structure of the D-brane landscape. We analyze the shape of the potential energy, and formulate sufficient conditions for the existence of nonsupersymmetric metastable vacua. Then we argue that these conditions can be in principle satisfied by tuning the values of background fluxes. In principle this mechanism can give rise to either de Sitter or anti de Sitter vacua, providing an alternative to the existing constructions of de Sitter vacua [4, 22, 46–51] in string theory.

2.1 A Mirror Pair of Calabi-Yau Orientifolds

In this section we review some general aspects of Calabi-Yau orientifolds and present our main example. We will first describe the model in IIB variables and then use mirror symmetry to write down the low energy effective action in a specific region in parameter space.

Let us consider a $N = 2$ IIB compactification on a Calabi-Yau manifold X . Such compactifications have a moduli space $\mathcal{M}_h \times \mathcal{M}_v$ of exactly flat directions, where \mathcal{M}_h denotes the hypermultiplet moduli space and \mathcal{M}_v denotes the vector multiplet moduli space. It is a standard fact that \mathcal{M}_h must be quaternionic manifold whereas \mathcal{M}_v must be a special Kähler manifold. The dilaton field is a hypermultiplet component, therefore the geometry of \mathcal{M}_h receives both α' and g_s corrections. By contrast, the geometry of \mathcal{M}_v is exact at tree level in both α' and g_s . The hypermultiplet moduli space \mathcal{M}_h contains a subspace \mathcal{M}_h^0 parameterized by vacuum expectation values of NS-NS fields, the RR moduli being set to zero. At string tree level \mathcal{M}_h^0 has a special Kähler structure which receives nonperturbative α' corrections. These corrections can be exactly summed using mirror symmetry.

Given a $N = 2$ compactification, we construct a $N = 1$ theory by gauging a discrete symmetry of the form $(-1)^{\epsilon F_L} \Omega \sigma$ where Ω denotes world-sheet parity, F_L is left-moving fermion number and ϵ takes values 0, 1 depending on the model. $\sigma : X \rightarrow X$ is a holomorphic involution of X preserving the holomorphic three-form Ω_X up to sign

$$\sigma^* \Omega_X = (-1)^\epsilon \Omega_X.$$

We will take $\epsilon = 1$, which corresponds to theories with O3/O7 planes. In order to keep the technical complications to a minimum, in this paper we will focus on models with $h^{1,1} = 1$ which exhibit only O3 planes. More general models could be treated in principle along the same lines, but the details would be more involved.

According to [52], the massless spectrum of $N = 1$ orientifold compactifications can be organized in vector and chiral multiplets. For orientifolds with O3/O7 planes, there are $h_-^{2,1}$ chiral multiplets corresponding to invariant complex structure deformations of X , $h_+^{1,1}$ chiral multiplets corresponding to invariant complexified Kähler deformations of X , and $h_-^{1,1}$ chiral multiplets parameterizing the expectation values of the two-form fields $(B, C^{(2)})$. Moreover, we have a dilaton-axion modulus τ . Note that the real Kähler deformations of X are paired up with expectation values of the four-form field $C^{(4)}$ giving rise to the $h_+^{1,1}$ complexified Kähler moduli. Note also that for one parameter models i.e. $h^{1,1} = 1$, we have $h_-^{1,1} = 0$, hence there are no theta angles $(B, C^{(2)})$.

Mirror symmetry relates the IIB $N = 2$ compactification on X to a IIA $N = 2$ compactification on the mirror Calabi-Yau manifold Y . The complex structure moduli space \mathcal{M}_v of X is identified to the Kähler moduli space of Y . In particular, there is a special boundary point of \mathcal{M}_v – the large complex structure limit point (LCS) – which is mapped to the large radius limit point of Y . Therefore if the complex structure of the IIB threefold X is close to LCS point, we can find an alternative description of a large radius IIA compactification on Y . This is valid for any values of the Kähler parameters of X , including the region centered around the LG point, which is mapped to the LG point in the complex structure moduli space of Y .

Orientifold models follow the same pattern [34–37]. Orientifold mirror symmetry relates a Calabi-Yau threefold (X, σ) with holomorphic involution to a threefold (Y, η) equipped with an antiholomorphic involution η . As long as the holomorphic involution preserves the large complex limit of X , we can map the theory to a large radius IIA orientifold on Y which admits a supergravity description. At the same time, we can take the Kähler parameters of X close to the LG point, which is mapped to the LG point in the complex structure moduli space of Y . This is the regime we will be mostly interested in throughout this paper.

In this limit, the moduli space of the theory has a direct product structure [37]

$$\mathcal{M} \times \mathcal{K} \tag{2.1.1}$$

where \mathcal{M} is the IIA complex structure moduli space and \mathcal{K} parameterizes the Kähler moduli space of the IIA orientifold (Y, η) and the dilaton. We discuss a specific example in more detail below.

2.1.1 Orientifolds of Octic Hypersurfaces

Our example consists of degree eight hypersurfaces in the weighted projective space $WP^{1,1,1,1,4}$.

The defining equation of an octic hypersurface X is

$$P(x_1, \dots, x_5) = 0 \quad (2.1.2)$$

where P is a homogeneous polynomial of degree eight with respect to the \mathbb{C}^* action

$$(x_1, x_2, x_3, x_4, x_5) \rightarrow (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda^4 x_5).$$

This is a one-parameter model with $h^{1,1}(X) = 1$ and $h^{2,1}(X) = 149$.

In order to construct an orientifold model, consider a family of such hypersurfaces of the form

$$Q(x_1, \dots, x_4) + x_5(x_5 + \mu x_1 x_2 x_3 x_4) = 0 \quad (2.1.3)$$

where $Q(x_1, \dots, x_4)$ is a degree eight homogeneous polynomial, and μ is a complex parameter.

We will denote these hypersurfaces by $X_{Q,\mu}$. Consider also a family of holomorphic involutions of $WP^{1,1,1,1,4}$ of the form

$$\sigma_\mu : (x_1, x_2, x_3, x_4, x_5) \rightarrow (-x_3, -x_4, -x_1, -x_2, -x_5 - \mu x_1 x_2 x_3 x_4) \quad (2.1.4)$$

Note that a hypersurface $X_{Q,\mu}$ is invariant under the holomorphic involution σ_μ if and only if Q is invariant under the involution

$$(x_1, x_2, x_3, x_4) \rightarrow (-x_3, -x_4, -x_1, -x_2). \quad (2.1.5)$$

We will take the moduli space \mathcal{M} to be the moduli space of hypersurfaces $X_{Q,\mu}$ with Q invariant under (2.1.5). A similar involution has been considered in a different context in [53].

One can easily check that the restriction of σ_μ to any invariant hypersurface $X_{Q,\mu}$ has finitely many fixed points on $X_{Q,\mu}$ with homogeneous coordinates

$$\left(x_1, x_2, \pm x_1, \pm x_2, -\frac{\mu}{2} x_1 x_2 x_3 x_4 \right)$$

where (x_1, x_2) satisfy

$$Q(x_1, x_2, \pm x_1, \pm x_2) - \frac{\mu^2}{4} x_1^4 x_2^4 = 0.$$

Moreover the LCS limit point $\mu \rightarrow \infty$ is obviously a boundary point of \mathcal{M} . This will serve as a concrete example throughout this paper.

Mirror symmetry identifies the complexified Kähler moduli space \mathcal{M}_h^0 of the underlying $N = 2$ theory to the complex structure moduli space of the family of mirror hypersurfaces Y

$$x_1^8 + x_2^8 + x_3^8 + x_4^8 + x_5^2 - \alpha x_1 x_2 x_3 x_4 x_5 = 0 \quad (2.1.6)$$

in $WP^{1,1,1,4}/(\mathbb{Z}_8^2 \times \mathbb{Z}_2)$ [54–56]. At the same time the complex structure moduli space \mathcal{M}_v of octic hypersurfaces is isomorphic to the complexified Kähler moduli space of Y . Orientifold mirror symmetry relates the IIB orientifold (X, σ) to a IIA orientifold determined by (Y, η) where η is an antiholomorphic involution of Y .

For future reference, let us provide some details on the Kähler geometry of the moduli space following [37]. The tree level Kähler potential for \mathcal{K} in a neighborhood of the large complex structure is given by

$$K_{\mathcal{K}} = -\ln(\text{vol}(Y)). \quad (2.1.7)$$

This can be expanded in terms of holomorphic coordinates t^i $i = 1, \dots, h_+^{1,1}(Y)$ adapted to the large radius limit of Y [37].

The second factor \mathcal{M} parameterizes complex structure moduli of IIA orientifold and the dilaton. The corresponding moduli fields are [37] the real complex parameters of Y and the periods of three-form RR potential $C^{(3)}$ preserved by the antiholomorphic involution plus the IIA dilaton.

The antiholomorphic involution preserves the real subspace $\alpha = \bar{\alpha}$ of the $N = 2$ moduli space. This follows from the fact that the IIB B-field is projected out using the mirror map

$$B + iJ = \frac{1}{2\pi i} \ln(z) + \dots$$

where $z = \alpha^{-8}$ is the natural coordinate on the moduli space of hypersurfaces (2.1.6) near the LCS point.

According to [37] (section 3.3), the Kähler geometry of \mathcal{K} can be described in terms of periods of the three-form Ω_Y and the flat RR three-form C_3 on cycles in Y on a symplectic basis of invariant or anti-invariant three-cycles on Y with respect to the antiholomorphic involution. We will choose a symplectic basis of invariant cycles $(\alpha_0, \alpha_1; \beta^0, \beta^1)$ adapted to the large complex limit $\alpha \rightarrow \infty$ of the family (2.1.6). Using standard mirror symmetry technology, one can compute the corresponding period vector $(Z^0, Z^1; \mathcal{F}_0, \mathcal{F}_1)$ near the large complex structure limit by solving the Picard-Fuchs equation. Our notation is so that the asymptotic behavior of the periods as $\alpha \rightarrow \infty$ is

$$Z^0 \sim 1 \quad Z^1 \sim \ln(z) \quad \mathcal{F}_1 \sim (\ln(z))^2 \quad \mathcal{F}_0 \sim (\ln(z))^3.$$

Moreover, we also have the following reality conditions on the real axis $\alpha \in \mathbb{R}$

$$\text{Im}(Z^0) = \text{Im}(\mathcal{F}_1) = 0 \quad \text{Re}(Z^1) = \text{Re}(\mathcal{F}_0) = 0. \quad (2.1.8)$$

This reflects the fact that (α_0, β^1) are invariant and (α^1, β_0) are anti-invariant under the holomorphic involution. The exact expressions of these periods can be found in appendix 2.A. Note

that the reality conditions (2.1.8) are an incarnation of the orientifold constraints (3.45) of [37] in our model. In particular, the compensator field C defined in [37] is real in our case, i.e. the phase $e^{-i\theta}$ introduced in [37] equals 1.

The holomorphic coordinates on the moduli space \mathcal{M} are

$$\begin{aligned}\tau &= \frac{1}{2}\xi^0 + iC\text{Re}(Z^0) \\ \rho &= i\tilde{\xi}_1 - 2C\text{Re}(\mathcal{F}_1)\end{aligned}\tag{2.1.9}$$

where $(\xi^0, \tilde{\xi}_1)$ are the periods of the three-form field $C^{(3)}$ on the invariant three-cycles (α_0, β^1)

$$C^{(3)} = \xi^0 \alpha_0 - \tilde{\xi}_1 \beta^1.\tag{2.1.10}$$

Mirror symmetry identifies (τ, ρ) with the IIB dilaton and respectively orientifold complexified Kähler parameter [37], section 6.2.1. A priori, (τ, ρ) are defined in a neighborhood of the LCS, but they can be analytically continued to other regions of the moduli space. We will be interested in neighborhood of the Landau-Ginzburg point $\alpha = 0$, where there is a natural basis of periods $[w_2 \ w_1 \ w_0 \ w_7]^{tr}$ constructed in [55]. The notation and explicit expressions for these periods are reviewed in appendix 2.A. For future reference, note that the LCS periods (Z^0, \mathcal{F}_1) in equation (2.1.9) are related to the LG periods by

$$\begin{bmatrix} Z^0 \\ Z^1 \\ \mathcal{F}_1 \\ \mathcal{F}_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{3}{2} & -\frac{3}{2} & -\frac{1}{2} \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_2 \\ w_1 \\ w_0 \\ w_7 \end{bmatrix}\tag{2.1.11}$$

Note that this basis is not identical to the symplectic basis of periods computed in [55]; the later does not obey the reality conditions (2.1.8) so we had to perform a symplectic change of basis.

The compensator field C is given by

$$C = e^{-\Phi} e^{K_0(\alpha)/2}\tag{2.1.12}$$

where $e^\Phi = e^{\phi} \text{vol}(Y)^{-1/2}$ is the four dimensional IIA dilaton, and

$$\begin{aligned}K_0(\alpha) &= -\ln \left(i \int_Y \Omega_Y \wedge \bar{\Omega}_Y \right) \Big|_{\alpha=\bar{\alpha}} \\ &= -\ln [2 (\text{Im}(Z^1)\text{Re}(\mathcal{F}_1) - \text{Re}(Z^0)\text{Im}(\mathcal{F}_0))]\end{aligned}\tag{2.1.13}$$

is the Kähler potential of the $N = 2$ complex structure moduli space of Y restricted to the real subspace $\alpha = \bar{\alpha}$. The Kähler potential of the orientifold moduli space is given by [37]

$$\begin{aligned}K_{\mathcal{M}} &= -2 \ln \left(2 \int_Y \text{Re}(C\Omega_Y) \wedge * \text{Re}(C\Omega_Y) \right) \\ &= -2 \ln [2C^2 (\text{Im}(Z^1)\text{Re}(\mathcal{F}_1) - \text{Re}(Z^0)\text{Im}(\mathcal{F}_0))].\end{aligned}\tag{2.1.14}$$

Note that equations (2.1.9), (2.1.12) define $K_{\mathcal{M}}$ implicitly as a function of (τ, ρ) . The Kähler potential (2.1.14) can also be written as

$$K_{\mathcal{M}} = -\ln(e^{-4\Phi}) \quad (2.1.15)$$

where Φ is the four dimensional dilaton. Let us conclude this section with a discussion of superpotential interactions.

2.1.2 Superpotential Interactions

There are several types of superpotential interactions in this system, depending on the types of background fluxes. Since the theory has a large radius IIA description, it is natural to turn on even RR fluxes $F^A = F_2 + F_4 + F_6$ as well as NS-NS flux H^A on the manifold Y . In principle one can also turn on the zero-form flux F_0 as in [43, 44], but we will set it to zero throughout this paper. Note that for vanishing F_0 there is no flux contribution to the RR tadpole [43]. Therefore we will have to cancel the negative charge of the orientifold planes with background D-brane charge. This will have positive consequences for the D-brane landscape studied in section 2.3.

Even RR fluxes give rise to a superpotential for type IIA Kähler moduli of the form [37, 40, 57, 58]

$$W_{\mathcal{K}}^A = \int_Y F^A \wedge e^{-J_Y}, \quad (2.1.16)$$

where J_Y is the Kähler form of Y . The type IIA NS-NS flux is odd under the orientifold projection, therefore it will have an expansion

$$H^A = q_1 \alpha^1 - p^0 \beta_0. \quad (2.1.17)$$

According to [37], this yields a superpotential for the IIA complex structure moduli of the form

$$W_{\mathcal{M}}^A = -2p^0 \tau - iq_1 \rho. \quad (2.1.18)$$

In conclusion, in the absence of branes, we will have a total superpotential of the form

$$W = W_{\mathcal{M}}^A + W_{\mathcal{K}}^A. \quad (2.1.19)$$

In the presence of D-branes, there can be additional contributions to the superpotential induced by disc instanton effects. Such terms are very difficult to compute explicitly on compact Calabi-Yau manifolds. However, they are exponentially small at large volume, therefore we do not expect these effects to change the qualitative picture of the landscape.

Next, let us describe the brane configurations.

2.2 Magnetized Branes on Calabi-Yau Orientifolds

In this section we study the dynamics of certain nonsupersymmetric D-brane configurations in the absence of fluxes. These configurations admit two equivalent descriptions in terms of IIB and IIA variables respectively. In IIB language, we are dealing with magnetized D5-brane configurations wrapping rigid holomorphic curves in a Calabi-Yau threefold X . In the IIA language we have D6-branes wrapping special lagrangian cycles in the mirror manifold Y . Although the IIB language is more convenient for some practical purposes, all the results of this section can be entirely formulated in IIA language, with no reference to IIB variables. In particular we will show that for small supersymmetry breaking parameter, the system admits a low energy IIA supergravity description. This point of view will be very useful in next section, where we turn on IIA fluxes.

The world-sheet analysis of the brane system is based on Π -stability considerations in the underlying $N = 2$ theory [8, 59, 60]. The world-sheet aspects are captured by D-term effects in the IIA supergravity effective action. Similar computations have been performed for Type I D9-branes in [29], for IIB D3 and D7-branes on Calabi-Yau orientifolds in [61–65], and for D6-branes in toroidal models in [22–28]. In particular, a relation between the perturbative part of Π -stability (μ -stability) and supergravity D-terms has been found in [29]. D6-brane configurations in toroidal models have been thoroughly analyzed from the world-sheet point of view in [66, 67]. Earlier work on the subject in the context of rigid supersymmetric theories includes [68–71]. Our setup is in fact very similar to the situation analyzed in [69], except that we perform a systematic supergravity analysis. Finally, a conjectural formula for the D-term potential energy on D6-branes has been proposed in [72, 73] based on general supersymmetry arguments. We will explain the relation between their expression and the supergravity computation at the end of section 2.2.2. Let us start with the Π -stability analysis.

2.2.1 Π -stability and magnetized D-branes

From the world-sheet point of view, a wrapped D5-brane is described by a boundary conformal field theory which is a product between an internal CFT factor and a flat space factor. Aspects related to Π -stability and superpotential deformations depend only on the internal CFT part and are independent on the rank of the brane in the uncompactified four dimensions. For example the same considerations apply equally well to a IIB D5-brane wrapping C or to a IIA D2-brane wrapping the same curve. The difference between these two cases resides in the manner of describing the dynamics of the lightest modes in terms of an effective action on the

uncompactified directions of the brane. Since the D5-brane is space filling the effective action has to be written in terms of four dimensional supergravity as opposed to the D2-brane effective action, which reduces to quantum mechanics. Nevertheless we would like to stress that in both cases the open string spectrum and the dynamics of the system is determined by identical internal CFT theories; only the low energy effective description of these effects is different. Keeping this point in mind, in this section we proceed with the analysis of the internal CFT factor.

Although our arguments are fairly general, for concreteness we will focus on the octic hypersurface in $WP^{1,1,1,4}$. Other models can be easily treated along the same lines. Suppose we have a D5-brane wrapping a degree one rational curve $C \subset X$. Note that curvature effects induce one unit of spacefilling D3-brane charge as shown in appendix 2.A. In order to obtain a pure D5-brane state we have to turn on a compensating magnetic flux in the $U(1)$ Chan-Paton bundle

$$\frac{1}{2\pi} \int_C F = -1.$$

However for our purposes we need to consider states with higher D3-charge, therefore we will turn on $(p - 1)$ units of magnetic flux

$$\frac{1}{2\pi} \int_C F = p - 1$$

obtaining a total effective $D3$ charge equal to p . The orientifold projection will map this brane to a anti-brane wrapping $C' = \sigma(C)$ with $(-p - 1)$ units of flux, where the shift by 2 units is again a curvature effect computed in appendix 2.A.

We will first focus on the underlying $N = 2$ theory. Note that this system breaks tree level supersymmetry because the brane and the anti-brane preserve different fractions of the bulk $N = 2$ supersymmetry. The $N = 1$ supersymmetry preserved by a brane is determined by its central charge which is a function of the complexified Kähler moduli. The central charges of our objects are

$$Z_+ = Z_{D5} + pZ_{D3} \quad Z_- = -Z_{D5} + pZ_{D3} \quad (2.2.1)$$

where the label \pm refers to the brane and the anti-brane respectively. Z_{D5} is the central charge of a pure D5-brane state, and Z_{D3} is the central charge of a D3-brane on X . The phases of Z_+, Z_- are not aligned for generic values of the Kähler parameters, but they will be aligned along a marginal stability locus where $Z_{D5} = 0$. If this locus is nonempty, these two objects preserve identical fractions of supersymmetry, and their low energy dynamics can be described by a supersymmetric gauge theory. If we deform the bulk Kähler structure away from the $Z_{D5} = 0$ locus, we expect the brane world-volume supersymmetry to be broken. Ignoring

supergravity effects, this supersymmetry breaking can be modeled by Fayet-Iliopoulos couplings in the low energy gauge theory. We will provide a supergravity description of the dynamics in the next subsection. This effective description is valid at weak string coupling and in a small neighborhood of the marginal stability locus in the Kähler moduli space. For large deformations away from this locus the effective gauge theory description breaks down, and we would have to employ string field theory for an accurate description of D-brane dynamics.

Returning to the orientifold model, note that the orientifold projection leaves invariant only a real dimensional subspace of the $N = 2$ Kähler moduli space, because it projects out the NS-NS B -field. As explained in section 2.1.1, the IIB complexified Kähler moduli space can be identified with the complex structure moduli space of the family of mirror hypersurfaces (2.1.6). The subspace left invariant by the orientifold projection is $\alpha = \bar{\alpha}$.

Therefore it suffices to analyze the D-brane system along this real subspace of the moduli space. Note that orientifold $O3$ planes preserve the same fraction of supersymmetry as D3-branes. Therefore the above $D5 - \overline{D5}$ configuration would still be supersymmetric along the locus $Z_{D5} = 0$ because the central charges (2.2.1) are aligned with Z_{D3} . Analogous brane configurations have been considered in [74] for F-theory compactifications.

A bulk Kähler deformation away from the supersymmetric locus will couple to the world-volume theory as a D-term because this is a disc effect which does not change in the presence of the orientifold projection. This will be an accurate description of the system as long as the string coupling is sufficiently small and we can ignore higher order effects. Note that the $Z_{D5} = 0$ locus will generically intersect the real subspace of the moduli space along a finite (possibly empty) set.

To summarize the above discussion, the dynamics of the brane anti-brane system in the $N = 1$ orientifold model can be captured by D-term effects at weak string coupling and in a small neighborhood of the marginal stability locus $Z_{D5} = 0$ in the Kähler moduli space. Therefore our first concern should be to find the intersection between the marginal stability locus and the real subspace $\alpha = \bar{\alpha}$ of the moduli space. A standard computation performed in appendix 2.A shows that the central charges Z_{D3}, Z_{D5} are given by

$$Z_{D3} = Z^0 \quad Z_{D5} = Z^1.$$

in terms of the periods $(Z^0, Z^1; \mathcal{F}^1, \mathcal{F}^0)$ introduced in section 2.1.1. Then the formulas (2.2.1) become

$$Z_+ = pZ^0 + Z^1, \quad Z_- = pZ^0 - Z^1. \quad (2.2.2)$$

In appendix 2.A we show that the relative phase

$$\theta = \frac{1}{\pi} (\text{Im} \ln(Z_+) - \text{Im} \ln(Z_{D3})) \quad (2.2.3)$$

between Z_+ and Z_{D3} does not vanish anywhere on the real axis $\alpha = \bar{\alpha}$ and has a minimum at the Landau-Ginzburg point $\alpha = 0$. The value of θ at the minimum is approximatively $\theta_{min} \sim 1/p$. For illustration, we represent in fig 1. the dependence $\theta = \theta(\alpha)$ near the Landau-Ginzburg point for three different values of p , $p = 10, 20, 30$. Note that the minimum value of theta is $\theta_{min} \sim 0.12$, therefore we expect the dynamics to have a low energy supergravity description.

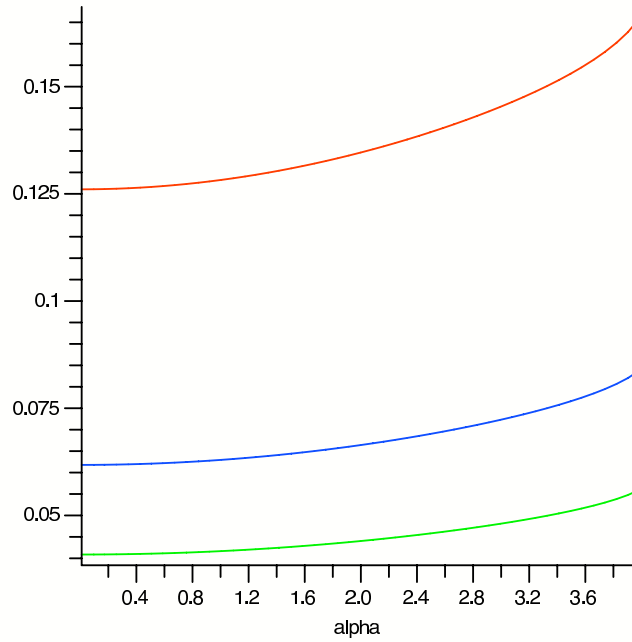


Figure 2.1: The behavior of the relative phase θ near the LG point for three different values of p . Red corresponds to $p = 10$, blue corresponds to $p = 20$ and green corresponds to $p = 30$.

It is clear from this discussion that the best option for us is to take the number p as high as possible subject to the tadpole cancellation constraints. This implies that there are no background D3-branes in the system, and we take p equal to the absolute value of the charge carried by orientifold planes. In fact configurations with background D3-branes would not be stable since there would be an attractive force between D3-branes and magnetized D5-branes. Therefore the system will naturally decay to a configuration in which all D3-branes have been converted into magnetic flux on D5-branes.

In order for the above construction to be valid, one has to check whether the D3-brane and D5-brane are stable BPS states at the Landau-Ginzburg point. This is clear in a neighborhood

of the large radius limit, but in principle, these BPS states could decay before we reach the Landau-Ginzburg point. For example it is known that in the $\mathbb{C}^2/\mathbb{Z}_3$ local model the D5-brane decays before we reach the orbifold point in the Kähler moduli space [75]. The behavior of the BPS spectrum of compact Calabi-Yau threefolds is less understood at the present stage. At best one can check stability of a BPS state with respect to a particular decay channel employing Π -stability techniques [8, 59, 60], but we cannot rigorously prove stability using the formalism developed in [76, 77]. In appendix 2.A we show that magnetized D5-branes on the octic are stable with respect to the most natural decay channels as we approach the Landau-Ginzburg point. This is compelling evidence for their stability in this region of the moduli space, but not a rigorous proof. Based on this amount of evidence, we will assume in the following that these D-branes are stable in a neighborhood of the Landau-Ginzburg point. Our next task is the computation of supergravity D-terms in the mirror IIA orientifold described in section 2.1.1

2.2.2 Mirror Symmetry and Supergravity D-terms

The above Π -stability arguments are independent of complex structure deformations of the IIB threefold X . We can exploit this feature to our advantage by working in a neighborhood of the LCS point in the complex structure moduli space of X . In this region, the theory admits an alternative description as a large volume IIA orientifold on the mirror threefold Y . The details have been discussed in section 2.1 of the present chapter. In the following we will use the IIA description in order to compute the D-term effects on magnetized branes.

Open string mirror symmetry maps the D5-branes wrapping C, C' to D6-branes wrapping special lagrangian cycles M, M' in Y . Since C, C' are rigid disjoint $(-1, -1)$ curves for generic moduli of X , M, M' must be rigid disjoint three-spheres in Y . The calibration conditions for M, M' are of the form

$$\mathrm{Im}(e^{i\theta}\Omega_Y|_M) = 0 \quad \mathrm{Im}(e^{-i\theta}\Omega_Y|_{M'}) = 0. \quad (2.2.4)$$

where Ω_Y is normalized so that the calibration of the IIA orientifold O6-planes has phase 1. The phase $e^{i\theta}$ in (2.2.4) is equal to the relative phase (2.2.3) computed above, and depends only on the complex structure moduli of Y . The homology classes of these cycles can be read off from the central charge formula (2.2.2). We have

$$[M] = p\beta^0 + \beta^1, \quad [M'] = p\beta^0 - \beta^1 \quad (2.2.5)$$

where $[M], [M']$ are cohomology classes on Y related to M, M' by Poincaré duality.

Taking into account $N = 1$ supergravity constraints, the D-term contribution is of the form

$$U_D = \frac{D^2}{2\text{Im}(g)} \quad (2.2.6)$$

where g is the holomorphic coupling constant of the brane $U(1)$ vector multiplet. The holomorphic coupling constant can be easily determined by identifying the four dimensional axion field a which has a coupling of the form

$$\int a F \wedge F \quad (2.2.7)$$

with the $U(1)$ gauge field on the brane. Such couplings are obtained by dimensional reduction of Chern-Simons terms of the form action.

$$\int C^{(3)} \wedge F \wedge F + C^{(5)} \wedge F$$

in the D6-brane world-volume action. Taking into account the expression (2.1.10) for $C^{(3)}$, dimensional reduction of the Chern-Simons term on the cycle M yields the following four-dimensional couplings

$$p \int \xi^0 F \wedge F + \int D^1 \wedge F. \quad (2.2.8)$$

Here ξ^0 is the axion defined in (2.1.10) and D^1 is the two-form field obtained by reduction of $C^{(5)}$

$$C^{(5)} = D^1 \wedge \alpha^1.$$

Equation (2.2.8) shows that the axion field a in (2.2.7) is ξ^0 . Then, using holomorphy and equation (2.1.9), it follows that the tree level holomorphic gauge coupling g must be

$$g = 2p\tau. \quad (2.2.9)$$

The second coupling in (2.2.8) is also very useful. The two-form field D^1 is part of an $N = 1$ linear multiplet L^1 whose lowest component is the real field $e^{2\Phi}\text{Im}(Z^1)$, where Φ is the four dimensional dilaton [37]. Moreover, one can relate L to the chiral multiplet ρ by a duality transformation which converts the second term in (2.2.8) into a coupling of the form

$$\int A_\mu \partial^\mu \tilde{\xi}_1.$$

The supersymmetric completion of this term determines the supergravity D-term to be [78–81]

$$D = \partial_\rho K_\rho. \quad (2.2.10)$$

Note that using equation (B.9) in [37], the D-term (2.2.10) can be written as

$$D = -2e^{2\Phi}\text{Im}(CZ^1) \quad (2.2.11)$$

where C is the compensator field defined in equation (2.1.12). Using equations (2.1.9) and (2.1.15), we can rewrite (2.2.11) as

$$\begin{aligned} D &= -2e^{K\kappa/2}\text{Im}(CZ^1) \\ &= -\frac{1}{C} \frac{\text{Im}(Z^1)}{\text{Im}(Z^1)\text{Re}(\mathcal{F}_1) - \text{Re}(Z^0)\text{Im}(\mathcal{F}_0)} \\ &= -\frac{1}{\text{Im}(\tau)} \frac{\text{Re}(Z^0)\text{Im}(Z^1)}{\text{Im}(Z^1)\text{Re}(\mathcal{F}_1) - \text{Re}(Z^0)\text{Im}(\mathcal{F}_0)}. \end{aligned} \quad (2.2.12)$$

Then, taking into account (2.2.9), we find the following expression for the D-term potential energy

$$U_D = \frac{1}{4p\text{Im}(\tau)^3} \left[\frac{\text{Re}(Z^0)\text{Im}(Z^1)}{\text{Im}(Z^1)\text{Re}(\mathcal{F}_1) - \text{Re}(Z^0)\text{Im}(\mathcal{F}_0)} \right]^2. \quad (2.2.13)$$

This is our final formula for the D-term potential energy.

In order to conclude this section, we would like to explain the relation between formula (2.2.13) and the II-stability analysis performed earlier in this section. Note that the II-stability considerations are captured by an effective potential in the mirror type IIA theory which was found in [72, 73]. According to [72, 73], the D-term potential for a pair of D6-branes as above should be given by

$$V_D = 2e^{-\Phi} \left(\left| \int_M \hat{\Omega}_Y \right| - \int_M \text{Re}(\hat{\Omega}_Y) \right) \quad (2.2.14)$$

where $\hat{\Omega}_Y$ is the holomorphic three-form on Y normalized so that

$$i \int_Y \hat{\Omega}_Y \wedge \bar{\hat{\Omega}}_Y = 1.$$

Recall that Φ denotes the four dimensional dilaton.

In the following we would like to explain that this expression is in agreement with the supergravity formula (2.2.13) for a small supersymmetry breaking angle $|\theta| \ll 1$. For large $|\theta|$ the effective supergravity description of the theory breaks down, and we would have to employ string field theory in order to obtain reliable results.

Note that one can write

$$\hat{\Omega}_Y = e^{K_0/2} \Omega_Y \quad (2.2.15)$$

where K_0 is has been defined in equation (2.1.13), and Ω_Y has some arbitrary normalization. The expression in the right hand side of this equation is left invariant under rescaling Ω_Y by a nonzero constant.

Formula (2.2.14) is written in the string frame. In order to compare it with the supergravity expression, we have to rewrite it in the Einstein frame. In the present context, the string metric has to be rescaled by a factor of $e^{2\phi}(\text{vol}(Y))^{-1} = e^{2\Phi}$ [82], hence the potential energy in the

Einstein frame is

$$V_D^E = 2e^{3\Phi} \left(\left| \int_M \widehat{\Omega}_Y \right| - \int_M \text{Re}(\widehat{\Omega}_Y) \right). \quad (2.2.16)$$

Taking into account equations (2.2.5) and (2.2.15) we have

$$\int_M \widehat{\Omega}_Y = e^{K_0/2} (p \text{Re}(Z^0) + i \text{Im}(Z^1)) = e^{K_0/2} Z_+$$

where Z_+ is the central charge defined in equation (2.2.1). For small values of the phase, $|\theta| \ll 1$, we can expand (2.2.16) as

$$V_D^E \sim e^{3\Phi} e^{K_0/2} \frac{\text{Re}(Z_0)}{p} \left[\frac{\text{Im}(Z^1)}{\text{Re}(Z^0)} \right]^2. \quad (2.2.17)$$

Now, using equations (2.1.9) and (2.2.11) in (2.2.6), we obtain

$$U_D = C e^{4\Phi} \frac{\text{Re}(Z^0)}{p} \left[\frac{\text{Im}(Z^1)}{\text{Re}(Z^0)} \right]^2 = e^{3\Phi} e^{K_0/2} \frac{\text{Re}(Z^0)}{p} \left[\frac{\text{Im}(Z^1)}{\text{Re}(Z^0)} \right]^2 \quad (2.2.18)$$

Therefore the supergravity D-term potential agrees indeed with (2.2.14) for very small supersymmetry breaking angle. This generalizes the familiar connection between Π -stability and D-term effects to supergravity theories.

2.3 The D-Brane Landscape

In this section we explore the IIA D-brane landscape in the mirror of the octic orientifold model introduced in section 2.1. We compute the total potential energy of the IIA brane configuration near the large radius limit in the Kähler moduli space. For technical reasons we will not be able to find explicit solutions to the critical point equations. However, given the shape of the potential, we will argue that metastable vacuum solutions are statistically possible by tuning the values of fluxes.

Throughout this section we will be working at a generic point in the configuration space where all open string fields are massive and can be integrated out. This is the expected behavior for D-branes wrapping isolated rigid special lagrangian cycles in a Calabi-Yau threefold. One should however be aware of several possible loopholes in this assumption since open string fields may become light along special loci in the moduli space.

In our situation, one should be especially careful with the open string-fields in the brane anti-brane sector. According to the Π -stability analysis in section 2.2, there is a tachyonic contribution to the mass of the lightest open string modes proportional to the phase difference θ . At the same time, we have a positive contribution to the mass due to the tension of the string stretching between the branes. In order to avoid tachyonic instabilities, we should work

in a region of the moduli space where the positive contribution is dominant. Since the cycles are isolated, the positive mass contribution is generically of the order of the string scale, which is much larger than the tachyonic contribution, since θ is of the order 0.05. Therefore we do not expect tachyonic instabilities in the system as long as the moduli are sufficiently generic.

This argument can be made slightly more concrete as follows. The position of the special lagrangian cycles M, M' in Y is determined by the calibration conditions (2.2.4), which are invariant under a rescaling of the metric on Y by a constant $\lambda > 1$. Such a rescaling would also increase the minimal geodesic distance between Y, Y' , which determines the mass of the open string modes. Therefore, if the volume of Y is sufficiently large, we expect the brane anti-brane fields to have masses at least of the order of the string scale.

Even if the open string fields have a positive mass, the system can still be destabilized by the brane anti-brane attraction force. Generically, we expect this not to be the case as long as the brane-brane fields are sufficiently massive since the attraction force is proportional to θ and it is also suppressed by a power of the string coupling. We can understand the qualitative aspects of the dynamics using a simplified model for the potential energy. Suppose that the effective dynamics of the branes can be described in terms of a single light chiral superfield Φ . Typically this happens when we work near a special point X_0 in the moduli space where the cycles M, M' are no longer rigid isolated supersymmetric cycles. Suppose these cycles admit a one-parameter space of normal deformations parameterized by a field Φ . Φ corresponds to normal deformations of the brane wrapping M , which are identified with normal deformations of the anti-brane wrapping M' by the orientifold projection. A sufficiently generic small deformation of X away from X_0 induces a mass term for Φ . Therefore we can model the effective dynamics of the system by a potential of the form

$$m(r - r_0)^2 + c \ln \left(\frac{r}{r_0} \right)$$

where r parameterizes the separation between the branes. The quadratic terms models a mass term for the open string fields corresponding to normal deformations of the branes in the ambient manifold. The second term models a typical two dimensional attractive brane anti-brane potential. The constant $c > 0$ is proportional to the phase θ and the string coupling g_s . Now one can check that if $c \ll mr_0$, this potential has a local minimum near $r = r_0$, and the local shape of the potential near this minimum is approximatively quadratic. In our case, we expect m, r_0 to be typically of the order of the string scale, whereas $c \sim g_s \theta \sim 10^{-2}$ therefore the effect of the attractive force is negligible.

Since it is technically impossible to make these arguments very precise, we will simply assume

that there is a region in configuration space where destabilizing effects are small and do not change the qualitative behavior of the system. Moreover, all open string fields are massive, and we can describe the dynamics only in terms of closed string fields. This point of view suffices for a statistical interpretation of the D-brane landscape. By tuning the values of fluxes, one can in principle explore all regions of the configuration space. The vacuum solutions which land outside the region of validity of this approximation will be automatically destabilized by some of these effects. Therefore there is a natural selection mechanism which keeps only vacuum solutions located at a sufficiently generic point in the moduli space.

Granting this assumption, we will take the configuration space to be isomorphic to the closed string moduli space $\mathcal{M} \times \mathcal{K}$ described in section 2.1.1. As discussed in section 2.1.2, we will turn on only RR fluxes $F^A = F_2 + F_4 + F_6$ and NS-NS flux H^A . In the presence of branes, the NS-NS flux H^A must satisfy the Freed-Witten anomaly cancellation condition [83], which states that the restriction of H^A to the brane world-volumes M, M' must be cohomologically trivial. Taking into account equations (2.1.17), (2.2.5), it follows that the integer q_1 in (2.1.17) must be set to zero. Therefore the superpotential does not depend on the chiral superfield ρ . This can also be seen from the analysis of supergravity D-terms in 2.2.2. The $U(1)$ gauge group acts as an axionic shift symmetry on ρ , therefore gauge invariance rules out any ρ -dependent terms in the superpotential [84]. The connection between the Freed-Witten anomaly condition and supergravity has been observed before in [41].

The total effective superpotential is then given by

$$W = \int_Y e^{J_Y} \wedge F^A - 2p^0 \tau. \quad (2.3.1)$$

In the presence of D-branes, the superpotential (2.3.1) can in principle receive disc instanton corrections. These corrections are exponentially small near the large radius limit, therefore they can be neglected.

The F-term contribution to the potential energy is

$$U_F = e^K \left(g^{i\bar{j}} (D_i W)(D_{\bar{j}} \bar{W}) + g^{a\bar{b}} (D_a W)(D_{\bar{b}} \bar{W}) - 3|W|^2 \right). \quad (2.3.2)$$

where i, j, \dots label complex coordinates on \mathcal{K} and $a, b = \rho, \tau$ label complex coordinates on \mathcal{M} . The D-term contribution is given by equation (2.2.13). We reproduce it below for convenience

$$U_D = \frac{1}{4p\text{Im}(\tau)^3} \left[\frac{\text{Re}(Z^0)\text{Im}(Z^1)}{\text{Im}(Z^1)\text{Re}(\mathcal{F}_1) - \text{Re}(Z^0)\text{Im}(\mathcal{F}_0)} \right]^2.$$

Since the moduli space of the theory is a direct product $\mathcal{K} \times \mathcal{M}$, the Kähler potential K in (2.3.2) is

$$K = K_{\mathcal{K}} + K_{\mathcal{M}}.$$

Note that we Kähler potentials $K_{\mathcal{K}}$, $K_{\mathcal{M}}$ satisfy the following noscale relations [37]

$$g^{i\bar{j}}\partial_i K_{\mathcal{K}}\partial_{\bar{j}} K_{\mathcal{K}} = 3 \quad g^{a\bar{b}}\partial_a K_{\mathcal{M}}\partial_{\bar{b}} K_{\mathcal{M}} = 4. \quad (2.3.3)$$

Using equations (2.1.9) and (2.1.14), we have

$$e^{K_{\mathcal{M}}} = \frac{1}{4\text{Im}(\tau)^4} \left[\frac{\text{Re}(Z^0)^2}{\text{Im}(Z^1)\text{Re}(\mathcal{F}_1) - \text{Re}(Z^0)\text{Im}(\mathcal{F}_0)} \right]^2.$$

Now we have a complete description of the potential energy of the system. Finding explicit vacuum solutions using these equations seems to be a daunting computational task, given the complexity of the problem. We can however gain some qualitative understanding of the resulting landscape by analyzing the potential energy in more detail.

First we have to find a convenient coordinate system on the moduli space \mathcal{M} . Note that the potential energy is an implicit function of the holomorphic coordinates (τ, ρ) via relations (2.1.9). One could expand it as a power series in (τ, ρ) , but this would be an awkward process. Moreover, the axion $\tilde{\xi}_1 = \text{Im}(\rho)$ is eaten by the $U(1)$ gauge field, and does not enter the expression for the potential. Therefore it is more natural to work in coordinates (τ, α) where α is the algebraic coordinate on the underlying $N = 2$ Kähler moduli space. As explained in section 2.1.1, α takes real values in the orientifold theory.

There is a more conceptual reason in favor of the coordinate α instead of ρ , namely α is a coordinate on the Teichmüller space of Y rather than the complex structure moduli space. Since in the II-stability framework the phase of the central charge is defined on the Teichmüller space, α is the natural coordinate when D-branes are present.

Next, we expand the potential energy in terms of (τ, α) using the relations (2.1.9). Dividing the two equations in (2.1.9), we obtain

$$\frac{\rho + \bar{\rho}}{\tau - \bar{\tau}} = 2i \frac{\text{Re}(\mathcal{F}_1)}{\text{Re}(Z^0)} \quad (2.3.4)$$

Let us denote the ratio of periods in the right hand side of equation (2.1.9) by

$$R(\alpha) = \frac{\text{Re}(\mathcal{F}_1)}{\text{Re}(Z^0)}. \quad (2.3.5)$$

Using equations (2.3.4) and (2.3.5), we find the following relations

$$\frac{\partial \alpha}{\partial \rho} = \frac{1}{2i} \frac{1}{\tau - \bar{\tau}} \left(\frac{\partial R}{\partial \alpha} \right)^{-1} \quad \frac{\partial \alpha}{\partial \tau} = -\frac{R}{\tau - \bar{\tau}} \left(\frac{\partial R}{\partial \alpha} \right)^{-1}. \quad (2.3.6)$$

Now, using the chain differentiation rule, we can compute the derivatives of the Kähler potential as functions of (τ, α) . Let us introduce the notation

$$V(\alpha) = \frac{\text{Im}(Z^1)\text{Re}(\mathcal{F}_1) - \text{Re}(Z^0)\text{Im}(\mathcal{F}_0)}{\text{Re}(Z^0)^2}.$$

Then we have

$$\begin{aligned}
\partial_\tau K_{\mathcal{M}} &= -\partial_{\bar{\tau}} K_{\mathcal{M}} = -\frac{2}{\tau - \bar{\tau}} \left[2 - R \frac{\partial_\alpha V}{V} (\partial_\alpha R)^{-1} \right] \\
\partial_\rho K_{\mathcal{M}} &= \partial_{\bar{\rho}} K_{\mathcal{M}} = \frac{i}{\tau - \bar{\tau}} \frac{\partial_\alpha V}{V} (\partial_\alpha R)^{-1} \\
\partial_{\tau\bar{\tau}} K_{\mathcal{M}} &= -\frac{2}{(\tau - \bar{\tau})^2} \left[2 - R \frac{\partial_\alpha V}{V} (\partial_\alpha R)^{-1} - R \partial_\alpha \left(R \frac{\partial_\alpha V}{V} (\partial_\alpha R)^{-1} \right) (\partial_\alpha R)^{-1} \right] \\
\partial_{\tau\rho} K_{\mathcal{M}} &= -\partial_{\rho\bar{\tau}} K_{\mathcal{M}} = -\frac{i}{(\tau - \bar{\tau})^2} \left[\frac{\partial_\alpha V}{V} (\partial_\alpha R)^{-1} + R \partial_\alpha \left(\frac{\partial_\alpha V}{V} (\partial_\alpha R)^{-1} \right) (\partial_\alpha R)^{-1} \right] \\
\partial_{\rho\bar{\rho}} K_{\mathcal{M}} &= \frac{1}{2(\tau - \bar{\tau})^2} \partial_\alpha \left(\frac{\partial_\alpha V}{V} (\partial_\alpha R)^{-1} \right) (\partial_\alpha R)^{-1}
\end{aligned} \tag{2.3.7}$$

Using equations (2.3.7), and the power series expansions of the periods computed in appendix A, we can now compute the expansion of the potential energy as in terms of (τ, α) . The D-term contribution takes the form

$$U_D = \frac{1}{p \text{Im}(\tau)^3} (0.03125 - 0.00178\alpha^2 + 0.00005\alpha^4 + \dots). \tag{2.3.8}$$

We will split the F-term contribution into two parts

$$U_F = U_F^{\mathcal{M}} + U_F^{\mathcal{K}}$$

where

$$\begin{aligned}
U_F^{\mathcal{K}} &= e^{K_{\mathcal{K}} + K_{\mathcal{M}}} (g^{i\bar{j}} (D_i W) (D_{\bar{j}} \bar{W}) - 3|W|^2) \\
U_F^{\mathcal{M}} &= e^{K_{\mathcal{K}} + K_{\mathcal{M}}} (g^{a\bar{b}} (D_a W) (D_{\bar{b}} \bar{W})).
\end{aligned}$$

We will also write the superpotential (2.3.1) in the form

$$W = W_0(z^i) + k\tau$$

where $k = -2p^0$. The factor $e^{K_{\mathcal{M}}}$ and the inverse metric coefficients $g^{a\bar{b}}$ can be expanded in powers of α using the equations (2.3.7) and formulas (2.A.5) in appendix 2.A. Using the noscale relations (2.3.3), we find the following expressions

$$U_F^{\mathcal{M}} = \frac{e^{K_{\mathcal{K}}}}{4 \text{Im}(\tau)^4} (0.0625 - 0.00357\alpha^2 + 0.00004\alpha^4 + \dots) \tag{2.3.9}$$

$$(g^{i\bar{j}} (\partial_i W_0) (\partial_{\bar{j}} \bar{W}_0) + g^{i\bar{j}} [(\partial_i W_0) (\partial_{\bar{j}} K_{\mathcal{M}}) (\bar{W}_0 + k\bar{\tau}) + (\partial_{\bar{j}} \bar{W}_0) (\partial_i K_{\mathcal{M}}) (W_0 + k\tau)])$$

$$\begin{aligned}
U_F^{\mathcal{K}} &= \frac{e^{K_{\mathcal{K}}}}{\text{Im}(\tau)^4} \left[\text{Im}(\tau)^2 (0.03125 - 0.00073\alpha^2 + 0.00001\alpha^4 + \dots) k^2 \right. \\
&\quad - \text{Im}(\tau) (0.03125 - 0.00178\alpha^2 + 0.00002\alpha^4 + \dots) (2k^2 \text{Im}(\tau) + 2k \text{Im}(W_0)) \\
&\quad \left. + (0.0625 - 0.00357\alpha^2 + 0.00004\alpha^4 + \dots) (k^2 \tau \bar{\tau} + k\tau \bar{W}_0 + k\bar{\tau} W_0 + |W_0|^2) \right]
\end{aligned} \tag{2.3.10}$$

Let us now try to analyze the shape of the landscape determined by the equations (2.3.8) and (2.3.9), (2.3.10). We rewrite the contribution (2.3.9) to the potential energy in the form

$$U_F^{\mathcal{M}} = \frac{e^{K\kappa}}{4\text{Im}(\tau)^4} (0.0625 - 0.00357\alpha^2 + 0.00004\alpha^4 + \dots) (P + kM\text{Re}(\tau) + kN\text{Im}(\tau)) \quad (2.3.11)$$

where

$$\begin{aligned} P &= g^{i\bar{j}}(\partial_i W_0)(\partial_{\bar{j}} \bar{W}_0) + g^{i\bar{j}}[(\partial_i W_0)(\partial_{\bar{j}} K_{\mathcal{M}}) \bar{W}_0 + (\partial_{\bar{j}} \bar{W}_0)(\partial_i K_{\mathcal{M}}) W_0] \\ M &= g^{i\bar{j}}[(\partial_i W_0)(\partial_{\bar{j}} K_{\mathcal{M}}) + (\partial_{\bar{j}} \bar{W}_0)(\partial_i K_{\mathcal{M}})] \\ N &= (-i)g^{i\bar{j}}[(\partial_i W_0)(\partial_{\bar{j}} K_{\mathcal{M}}) - (\partial_{\bar{j}} \bar{W}_0)(\partial_i K_{\mathcal{M}})] \end{aligned} \quad (2.3.12)$$

Then the α expansion of the F-term potential energy can be written as

$$U_F = U_F^{(0)} + \alpha^2 U_F^{(2)} + \dots$$

where

$$\begin{aligned} U_F^{(0)} &= 0.0156 \frac{e^{K\kappa}}{\text{Im}(\tau)^4} [P + k(N + 4\text{Im}(W_0))\text{Im}(\tau) + 2k^2\text{Im}(\tau)^2 \\ &\quad + 4|W_0|^2 + k(M + 8\text{Re}(W_0))\text{Re}(\tau) + 4k^2\text{Re}(\tau)^2] \end{aligned} \quad (2.3.13)$$

$$\begin{aligned} U_F^{(2)} &= -0.00178 \frac{e^{K\kappa}}{2\text{Im}(\tau)^4} [P + k(N + 4\text{Im}(W_0))\text{Im}(\tau) + 0.82k^2\text{Im}(\tau)^2 \\ &\quad + 4|W_0|^2 + k(M + 8\text{Re}(W_0))\text{Re}(\tau) + 4k^2\text{Re}(\tau)^2] \end{aligned} \quad (2.3.14)$$

The critical point equations resulting from (2.3.8), (2.3.9) and (2.3.10) are very complicated, and we will not attempt to find explicit solutions. We will try to gain some qualitative understanding of the possible solutions exploiting some peculiar aspects of the potential. Note that all contributions to the potential energy depend on even powers of α . Then it is obvious that $\alpha = 0$ is a solution to the equation

$$\partial_\alpha U = 0$$

where $U = U_D + U_F^{\mathcal{M}} + U_F^{\mathcal{K}}$. Moreover we also have

$$(\partial_i \partial_\alpha U)_{\alpha=0} = (\partial_\tau \partial_\alpha U)_{\alpha=0} = 0.$$

This motivates us to look for critical points with $\alpha = 0$. Then, the remaining critical point equations are

$$(\partial_i U)_{\alpha=0} = (\partial_\tau U)_{\alpha=0} = 0 \quad (2.3.15)$$

plus their complex conjugates.

The second order coefficient of α in the total potential energy is

$$U_F^{(2)} - 0.00178 \frac{1}{p\text{Im}(\tau)^3}. \quad (2.3.16)$$

Since the mixed partial derivatives are zero at $\alpha = 0$, in order to obtain a local minimum, the expression (2.3.16) must be positive. This is a first constraint on the allowed solutions to (2.3.15).

Next, let us examine the τ dependence of the potential for $\alpha = 0$ and fixed values of the Kähler parameters. Note that $U_F^{(0)}$ given by equation (2.3.13) is a quadratic function of the axion $\text{Re}(\tau)$. For any fixed values of $\text{Im}(\tau)$ and the Kähler parameters, this function has a minimum at

$$\text{Re}(\tau) = -\frac{8\text{Re}(W_0) + M}{8k}. \quad (2.3.17)$$

Therefore we can set $\text{Re}(\tau)$ to its minimum value in the potential energy, obtaining an effective potential for the Kähler parameters and the dilaton $\text{Im}(\tau)$. Then equations (2.3.13), (2.3.14) become

$$U_F^{(0)} = 0.0156 \frac{e^{K\kappa}}{\text{Im}(\tau)^4} \left[P + k(N + 4\text{Im}(W_0))\text{Im}(\tau) + 2k^2\text{Im}(\tau)^2 + 4|W_0|^2 - \frac{1}{16}(M + 8\text{Re}(W_0))^2 \right] \quad (2.3.18)$$

$$U_F^{(2)} = -0.00178 \frac{e^{K\kappa}}{2\text{Im}(\tau)^4} \left[P + k(N + 4\text{Im}(W_0))\text{Im}(\tau) + 0.82k^2\text{Im}(\tau)^2 + 4|W_0|^2 - \frac{1}{16}(M + 8\text{Re}(W_0))^2 \right] \quad (2.3.19)$$

Now let us analyze the dependence of $U_F^{(0)}$ on $\text{Im}(\tau)$. It will be more convenient to make the change of variables

$$u = \frac{1}{\text{Im}(\tau)}$$

since u is proportional to the string coupling constant. Then $U_F^{(0)}$ becomes a quartic function of the form

$$U_F^{(0)} = Au^2 - Bu^3 + Cu^4 \quad (2.3.20)$$

where

$$\begin{aligned} A &= 2k^2 \\ B &= -kN - 4k\text{Im}(W_0) \\ C &= P + 4|W_0|^2 - \frac{1}{16}(M + 8\text{Re}(W_0))^2. \end{aligned} \quad (2.3.21)$$

The behavior of this function for fixed values of the Kähler parameters is very simple. For positive A , this function has a local minimum away from the origin if and only if the following inequalities are satisfied

$$B > 0 \quad C > 0 \quad \text{and} \quad 9B^2 > 32AC. \quad (2.3.22)$$

The minimum is located at

$$u_0 = \frac{3B + \sqrt{9B^2 - 32AC}}{8C}. \quad (2.3.23)$$

Therefore, in order to construct metastable vacua, we have to find solutions to the equations (2.3.15) satisfying the inequalities (2.3.22). Moreover, we would like u_0 to be small in order to obtain a weakly coupled theory. The conditions (2.3.22) translate to

$$\begin{aligned} 9(N + 4\text{Im}(W_0))^2 &> 64 \left(P + 4\text{Im}(W_0)^2 - M\text{Re}(W_0) - \frac{M^2}{16} \right) > 0 \\ k(N + \text{Im}(W_0)) &< 0 \end{aligned} \quad (2.3.24)$$

where P, M, N are given by (2.3.12). This shows that we need a certain amount of fine tuning of the background RR fluxes in order to obtain a metastable vacuum. Note that in our construction the fluxes are not constrained by tadpole cancellation conditions, therefore we have a considerable amount of freedom in tuning the fluxes. Statistically, this improves our chances of finding a solution with the required properties.

Finally, note that we have to impose one more condition, namely the second order coefficient (2.3.16) in the α expansion of the potential should be positive. Assuming that we have found a solution of equations (2.3.15) which stabilizes u at the value $0 < u_0 < 1$, let us compute this coefficient as a function of (u_0, A, B, C) . Note that equation (2.3.19) can be rewritten as

$$U_F^{(2)} = 0.00178 \frac{e^{K\kappa}}{2} (Bu_0^3 - 0.4Au_0^2 - Cu_0^4). \quad (2.3.25)$$

Equation (2.3.23) yields

$$B = \frac{4}{3}Cu_0 + \frac{2}{3}\frac{A}{u_0} \quad (2.3.26)$$

Substituting (2.3.26) in (2.3.25), and adding the D-term contribution, the coefficient of α^2 becomes

$$0.00178 \left[e^{K\kappa} \left(\frac{2}{15}Au_0^2 + \frac{1}{6}Cu_0^4 \right) - \frac{1}{p}u_0^3 \right] \quad (2.3.27)$$

Since $C > 0$, a sufficient condition for (2.3.27) to be positive is

$$\frac{2p}{15}Ae^{K\kappa} > u_0 \quad \Rightarrow \quad \frac{4pk^2}{15} > u_0 \text{vol}(Y). \quad (2.3.28)$$

Here we have used

$$e^{K\mathcal{M}} = \frac{1}{\text{vol}(Y)}.$$

This condition reflects the fact that the F-term and D-term contributions to the potential energy must be of the same order of magnitude in order to obtain a metastable vacuum solution. If the volume of Y is too large, there is a clear hierarchy of scales between the two contributions, and the D-term is dominant. This would give rise to a runaway behavior along the direction of

α . On the other hand, we have to make sure that the volume of Y is sufficiently large so that the IIA supergravity approximation is valid. Therefore some additional amount of fine tuning is required in order to obtain a reliable solution.

In conclusion, metastable nonsupersymmetric vacua at $\alpha = 0$ can be in principle obtained by tuning the IIA RR flux $F^{(A)}$ and NS-NS flux $H^A = k\beta^0$ so that conditions (2.3.24), (2.3.28) are satisfied at the critical point. A more precise statement would require a detailed numerical analysis, which we leave for future work.

We would like to conclude this section with a few remarks.

(i) We have also restricted ourselves to singly wrapped magnetized D5-branes. One could in principle consider multiply wrapped D5-branes as long we can maintain the phase difference θ sufficiently small. If this is possible, we would obtain an additional nonperturbative contribution to the superpotential of the form

$$be^{-a\tau}.$$

Such terms may be also helpful in the fine tuning process.

(ii) Finally, note that we could also allow for a nonzero background value of the RR zero-form F_0 , which was also set to zero in this paper. Then, according to [43], there is an additional contribution to the RR tadpole cancellation condition of the form km_0 . If we choose k, m_0 appropriately, it follows that we can make the background D-brane charge p larger than the orientifold charge. In fact it seems that there is no upper bound on p , hence we could make the supersymmetry breaking D-term very small by choosing a large p . This may have important consequences for the scale of supersymmetry breaking in string theory.

(iii) Note that the vacuum construction mechanism proposed above can give rise to de Sitter or anti de Sitter vacua, depending on the values of fluxes. In particular, it is not subject to the no-go theorem of [85] because the magnetized branes give a positive contribution to the potential energy. In principle we could try to employ the same strategy in order to construct nonsupersymmetric metastable vacua of the F-term potential energy (2.3.2) in the absence of magnetized branes. Then we have several options for RR tadpole cancellation. We can turn on background F_0 flux as in [43] or local tadpole cancellation by adding background D6-branes. It would be interesting to explore these alternative constructions in more detail.

(iv) Since it is quite difficult to find explicit vacuum solutions, it would be very interesting to attempt a systematic statistical analysis of the distribution of vacua along the lines of [86–92].

(v) In our approach the scale of supersymmetry breaking is essentially determined by the total RR tadpole $p = |N_{O3}|$ of the orientifold model. While this tadpole is typically of the order

of 32 in perturbative models, it can reach much higher values in orientifold limits of F-theory. It would be very interesting to implement our mechanism in such an F-theory compactification, perhaps in conjunction with other moduli stabilization mechanism [4, 93–98]. Provided that the dynamics can still be kept under control, we would then obtain smaller supersymmetry breaking scales.

2.A II-Stability on the Octic and $N = 2$ Kähler Moduli Space

In this section we analyze the $N = 2$ Kähler moduli space and stability of magnetized branes for the octic hypersurface. Recall [55] that the mirror family is described by the equation

$$x_1^8 + x_2^8 + x_3^8 + x_4^8 + x_5^2 - \alpha x_1 x_2 x_3 x_4 x_5 = 0. \quad (2.A.1)$$

in $WP^{1,1,1,1,4}/(\mathbb{Z}_8^2 \times \mathbb{Z}_2)$. The moduli space of the mirror family can be identified with a sector in the α plane defined by

$$0 \leq \arg(\alpha) < \frac{2\pi}{8}.$$

The entire α plane contains eight such sectors, which are permuted by monodromy transformations about the LG point $\alpha = 0$. In this parameterization, the LCS point is at $\alpha = \infty$, and the conifold point is at $\alpha = 4$.

A basis of periods for this family has been computed in [55] by solving the Picard-Fuchs equations. For our purposes it is convenient to write the solutions to the Picard-Fuchs equations in integral form

$$\begin{aligned} \Pi_0 &= \frac{1}{2\pi i} \int ds \frac{\Gamma(1+8s)\Gamma(-s)}{\Gamma(1+s)^3\Gamma(1+4s)} e^{i\pi s} (\alpha)^{-8s} \\ \Pi_1 &= -\frac{1}{(2\pi i)^2} \int ds \frac{\Gamma(1+8s)\Gamma(-s)^2}{\Gamma(1+s)^2\Gamma(1+4s)} (\alpha)^{-8s} \\ \Pi_2 &= \frac{2}{(2\pi i)^3} \int ds \frac{\Gamma(1+8s)\Gamma(-s)^3}{\Gamma(1+s)\Gamma(1+4s)} e^{i\pi s} (\alpha)^{-8s} \\ \Pi_3 &= -\frac{1}{(2\pi i)^4} \int ds \frac{\Gamma(1+8s)\Gamma(-s)^4}{\Gamma(1+4s)} (\alpha)^{-8s}. \end{aligned} \quad (2.A.2)$$

as in [60]. All integrals in (2.A.2) are contour integrals in the complex s -plane. The contour runs from $s = -\epsilon - i\infty$ to $-\epsilon + i\infty$ along the imaginary axis and it can be closed either to the left or to the right. If we close the contour to the right, we obtain a basis of solutions near the LCS limit $\alpha = \infty$, while if we close the contour to the left, we obtain a basis of solutions near the LG point $\alpha = 0$. Near the large radius limit it is more convenient to write the solutions in terms of the coordinate $z = \alpha^{-8}$.

Note that there is a different set of solutions at the LG point [55] of the form

$$w_k(\alpha) = \Pi_0(e^{2\pi k i} \alpha), \quad k = 0, \dots, 7. \quad (2.A.3)$$

In particular we have an alternative basis $[w_2 \ w_1 \ w_0 \ w_7]^{tr}$ near $\alpha = 0$. The transition matrix between the two bases is

$$\begin{bmatrix} \Pi_0 \\ \Pi_1 \\ \Pi_2 \\ \Pi_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -1 & -2 & -1 \\ -1 & -\frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} w_2 \\ w_1 \\ w_0 \\ w_7 \end{bmatrix} \quad (2.A.4)$$

In section 2.1 we have used a third basis of periods $[Z^0 \ Z^1 \ \mathcal{F}_1 \ \mathcal{F}_0]^{tr}$ compatible with the orientifold projection. The relation between the orientifold basis and the LG basis $[w_2 \ w_1 \ w_0 \ w_7]^{tr}$ is given in equation (2.1.11). The power series expansion of the orientifold periods at the LG point is

$$\begin{aligned} \text{Re}(Z^0) &= -0.37941\alpha + 0.00541\alpha^3 + 0.00009\alpha^5 + \dots \\ \text{Im}(Z^1) &= -0.53656\alpha + 0.00766\alpha^3 - 0.00012\alpha^5 + \dots \\ \text{Re}(\mathcal{F}_1) &= 1.29538\alpha - 0.00317\alpha^3 - 0.00005\alpha^5 + \dots \\ \text{Im}(\mathcal{F}_0) &= 0.31431\alpha - 0.02615\alpha^3 + 0.00043\alpha^5 + \dots \end{aligned} \quad (2.A.5)$$

Now let us discuss some geometric aspects of octic hypersurfaces required for the Π -stability analysis. For intersection theory computations, it will be more convenient to represent X as a hypersurface in a smooth toric variety Z obtained by blowing-up the singular point of the weighted projective space $WP^{1,1,1,1,4}$. Z is defined by the following $\mathbb{C}^\times \times \mathbb{C}^\times$ action

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & u & v \\ 1 & 1 & 1 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array} \quad (2.A.6)$$

with forbidden locus $\{x_1 = x_2 = x_3 = x_4 = 0\} \cup \{u = v = 0\}$. The Picard group of Z is generated by two divisor classes η_1, η_2 determined by the equations

$$\eta_1 : x_1 = 0 \quad \eta_2 : v = 0. \quad (2.A.7)$$

The cohomology ring of Z is determined by the relations

$$\eta_2^4 = 64 \quad \eta_2(\eta_2 - 4\eta_1) = 0. \quad (2.A.8)$$

The total Chern class of Z is given by the formula

$$c(Z) = (1 + \eta_1)^4(1 - 4\eta_1 + \eta_2)(1 + \eta_2) \quad (2.A.9)$$

and the hypersurface X belongs to the linear system $|2\eta_2|$. Using the adjunction formula

$$c(X) = \frac{c(Z)}{(1 + 2\eta_2)} \quad (2.A.10)$$

one can easily compute

$$c_1(X) = 0 \quad c_2(X) = 22\eta_1^2 \quad \text{Td}(X) = 1 + \frac{11}{6}\eta_1^2. \quad (2.A.11)$$

Note that the divisor class $\eta_2 - 4\eta_1$ has trivial restriction to X , therefore the Picard group of X has rank one, as expected. A natural generator is η_1 , which can be identified with a hyperplane section of X in the weighted projective space $WP^{1,1,1,1,4}$. Then we will write the complexified Kähler class as $B + iJ = t\eta_1$. For future reference, note that we will denote by $E(p)$ the tensor product $E \otimes \mathcal{O}_X(p\eta_1)$ for any sheaf (or derived object) E on X .

Employing the conventions of [99], we will define the central charge of a D-brane E in the large radius limit to be

$$Z^\infty(E) = \int_X e^{B+iJ} \text{ch}(E) \sqrt{\text{Td}(X)}. \quad (2.A.12)$$

This is a cubic polynomial in t . Using the mirror map

$$t = \frac{\Pi_1}{\Pi_0} \quad (2.A.13)$$

and the asymptotic form of the periods

$$\begin{aligned} \Pi_1 &= t + \dots \\ \Pi_2 &= t^2 + t - \frac{11}{6} + \dots \\ \Pi_3 &= \frac{1}{6}t^3 - \frac{13}{12}t + \dots \end{aligned} \quad (2.A.14)$$

we can determine the exact expression of the period Z_E as a function of the algebraic coordinate α . The phase of the central charge is defined as

$$\phi(E) = -\frac{1}{\pi} \arg(Z(E)) \quad (2.A.15)$$

and is normalized so that it takes values $-2 < \phi(E) \leq 0$ at the large radius limit point.

As objects in the derived category $D^b(X)$, the magnetized branes are given by

$$\underline{\mathcal{O}_C}(p-1) \quad \underline{\mathcal{O}_{C'}}(-p-1)[1] \quad (2.A.16)$$

where C, C' are smooth rational curves on X conjugated under the holomorphic involution. Given a coherent sheaf E on X , we have denoted by \underline{E} the one term complex which contains E in degree zero, all other terms being trivial. In order to compute their asymptotic central charges using formula (2.A.12), we have to use the Grothendieck-Riemann-Roch theorem for the embeddings $\iota : C \rightarrow X$, $\iota' : C' \rightarrow X$. Since the computations are very similar, it suffices to present the details only for one of these objects, for example the first brane in (2.A.16).

Given a line bundle $\mathcal{L} \rightarrow C$, the Chern character of its pushforward $\iota_*(\mathcal{L})$ to X is given by

$$\text{ch}(\iota_*(\mathcal{L}))\text{Td}(X) = \iota_*(\text{ch}(\mathcal{L})\text{Td}(C)). \quad (2.A.17)$$

In our case (2.A.17) yields

$$\text{ch}_0(\iota_*(\mathcal{L})) = \text{ch}_1(\iota_*(\mathcal{L})) = 0 \quad \text{ch}_2(\iota_*(\mathcal{L})) = [C] \quad \text{ch}_3(\iota_*(\mathcal{L})) = (\deg(\mathcal{L}) + 1)[pt] \quad (2.A.18)$$

where $[C] \in H^{2,2}(X)$ denotes the Poincaré dual of C and $[pt] \in H^{3,3}(X)$ denotes the Poincaré dual of a point on X . The shift by 1 in $\text{ch}_3(\iota_*(\mathcal{L}))$ represents the contribution of the Todd class of C

$$\text{Td}(C) = 1 + \frac{1}{2}c_1(C)$$

to the right hand side of equation (2.A.17). From a physical point of view, this can be thought of as D3-brane charge induced by a curvature effect. Using formulas (2.A.12), (2.A.18) it is easy to compute

$$Z^\infty(\underline{\mathcal{O}}_C(p-1)) = t + p \quad Z^\infty(\underline{\mathcal{O}}_{C'}(-p-1)[1]) = -t + p. \quad (2.A.19)$$

The exact expressions for the central charges are

$$Z(\underline{\mathcal{O}}_C(p-1)) = \Pi_1 + p\Pi_0 \quad Z(\underline{\mathcal{O}}_{C'}(-p-1)[1]) = -\Pi_1 + p\Pi_0. \quad (2.A.20)$$

Taking into account the transition matrices (2.1.11), (2.A.4), it is clear that these formulas are identical with (2.2.2) in the main text. In order to study the behavior of their phases near the LG point, we have to rewrite the central charges (2.A.20) in terms of the basis $[w_2 \ w_1 \ w_0 \ w_7]^{tr}$ using the transition matrix (2.A.4). We find

$$\begin{aligned} Z(\underline{\mathcal{O}}_C(p-1)) &= \frac{1}{2}(w_2 + w_1 - w_0 - w_7) + pw_0 \\ Z(\underline{\mathcal{O}}_{C'}(-p-1)[1]) &= -\frac{1}{2}(w_2 + w_1 - w_0 - w_7) + pw_0. \end{aligned} \quad (2.A.21)$$

Note that the central charge of a single D3-brane is

$$Z(\underline{\mathcal{O}}_{pt}) = w_0. \quad (2.A.22)$$

Then, using the expansions (2.A.5) we can plot the relative phase

$$\theta = \phi(\underline{\mathcal{O}}_C(p-1)) - \phi(\underline{\mathcal{O}}_{pt}) \quad (2.A.23)$$

near the LG point, obtaining the graph in figure 1.

In the remaining part of this section, we will address the question of stability of magnetized brane configurations near the LG point. As explained below figure 1, we will analyze stability

with respect to the most natural decay channels from the geometric point of view. We will show below that the objects (2.A.16) are stable with respect to all such decay processes, which is strong evidence for their stability at the LG point. Since all these computations are very similar, it suffices to consider only one case in detail. For the other cases we will just give the final results.

Decay channels in the Π -stability framework are classified by triangles in the derived category [60]. In our case, the most natural decay channels are in fact determined by short exact sequences of sheaves. For example let us consider the following short exact sequence

$$0 \rightarrow \mathcal{J}_C(p-1) \rightarrow \mathcal{O}_X(p-1) \rightarrow \mathcal{O}_C(p-1) \rightarrow 0 \quad (2.A.24)$$

where \mathcal{J}_C is the ideal sheaf of C on X . The first two terms represent rank one D6-branes on X with lower D4 and D2 charges. All three terms are stable BPS states in the large volume limit. The mass of the lightest open string states stretching between the first two branes in the sequence (2.A.24) is determined by the relative phase

$$\Delta\phi = \phi(\underline{\mathcal{O}}_X(p-1)) - \phi(\underline{\mathcal{J}}_C(p-1)). \quad (2.A.25)$$

If $\Delta\phi < 1$, the lightest state in this open string sector is tachyonic, and these two branes will form a bound state isomorphic to $\underline{\mathcal{O}}_C(p-1)$ by tachyon condensation. In this case $\underline{\mathcal{O}}_C(p-1)$ is stable. If $\Delta\phi > 1$, the lightest open string state has positive mass, and it is energetically favorable for $\underline{\mathcal{O}}_C(p-1)$ to decay into $\underline{\mathcal{J}}_C(p-1)$ and $\underline{\mathcal{O}}_X(p-1)$. In this case $\underline{\mathcal{O}}_C(p-1)$ is unstable. Therefore we have to compute the phase difference $\Delta\phi$ as a function of α in order to find out if this decay takes place anywhere on the real α axis. For the purpose of this computation it is more convenient to denote $q = p-1$, and perform the calculations in terms of q rather than p .

We have

$$\begin{aligned} Z^\infty(\underline{\mathcal{O}}_X(q)) &= \int_X e^{(t+q)\eta_1} \sqrt{\text{Td}(X)} \\ &= \frac{1}{3}(t+q)^3 + \frac{11}{6}(t+q) \\ Z^\infty(\underline{\mathcal{J}}_C(q)) &= \int_X e^{(t+q)\eta_1} \sqrt{\text{Td}(X)} - Z^\infty(\underline{\mathcal{O}}_C(q)) \\ &= \frac{1}{3}(t+q)^3 + \frac{5}{6}(t+q) - 1. \end{aligned} \quad (2.A.26)$$

Using the asymptotic form of the periods (2.A.14) and formulas (2.A.26), we find the following expressions for the exact central charges

$$\begin{aligned} Z(\underline{\mathcal{O}}_X(q)) &= 2\Pi_3 + q\Pi_2 + (q^2 - q + 4)\Pi_1 + \left(\frac{1}{3}q^3 + \frac{11}{3}q\right)\Pi_0 \\ Z(\underline{\mathcal{J}}_C(q)) &= 2\Pi_3 + q\Pi_2 + (q^2 - q + 3)\Pi_1 + \left(\frac{1}{3}q^3 + \frac{8}{3}q - 1\right)\Pi_0 \end{aligned} \quad (2.A.27)$$

In terms of the LG basis of periods, these expressions read

$$\begin{aligned}
Z(\underline{\mathcal{O}}_X(q)) &= \left(\frac{1}{2}q^2 - \frac{1}{2}q\right)w_2 + \left(\frac{1}{2}q^2 - \frac{3}{2}q + 1\right)w_1 \\
&\quad + \left(\frac{1}{3}q^3 - \frac{1}{2}q^2 + \frac{13}{6}q - 1\right)w_0 + \left(-\frac{1}{2}q^2 - \frac{1}{2}q\right)w_7 \\
Z(\underline{\mathcal{I}}_C(q)) &= \left(\frac{1}{2}q^2 - \frac{1}{2}q - \frac{1}{2}\right)w_2 + \left(\frac{1}{2}q^2 - \frac{3}{2}q + \frac{1}{2}\right)w_1 \\
&\quad + \left(\frac{1}{3}q^3 - \frac{1}{2}q^2 + \frac{7}{6}q - \frac{3}{2}\right)w_0 + \left(-\frac{1}{2}q^2 - \frac{1}{2}q + \frac{1}{2}\right)w_7
\end{aligned} \tag{2.A.28}$$

Substituting the expressions (2.A.2) in (2.A.27), (2.A.28), we can compute the relative phase (2.A.25) at any point on the real axis in the α -plane except the conifold point $\alpha = 4$. The conifold point can be avoided following a circular contour of very small radius ϵ centered at $\alpha = 4$.

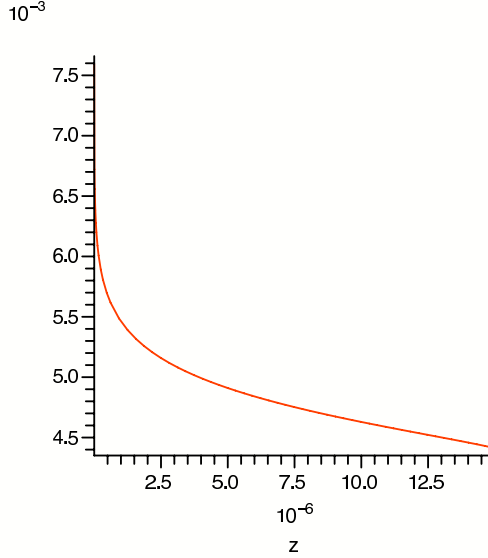


Figure 2.2: The behavior of the relative phase $\Delta\phi$ in the geometric phase for $p = 10$.

The graph in fig. 2 represents the dependence of $\Delta\phi$ as a function of $z = \alpha^{-8}$ in the large radius phase $0 < z < 4$ for $p = 10$. Note that it decreases monotonically from 0.0075 to 0.0044 as we approach the conifold point. Using formulas (2.A.28), we find that in the LG phase $0 < \alpha < 4$, $\Delta\Phi$ also decreases monotonically until it reaches the value 0.027 at the LG point. One can also calculate the values of $\Delta\phi$ along a small circular contour surrounding the conifold, confirming that it varies continuously in this region. Since $\Delta\phi < 1$, everywhere on the real axis, we conclude that the magnetized brane $\underline{\mathcal{O}}_C(q)$ is stable with respect to the decay channel (2.A.24).

The analysis of other decay channels is very similar. Another decay channel is given by the following short exact sequence

$$0 \rightarrow \mathcal{O}_D(-C)(q) \rightarrow \mathcal{O}_D(q) \rightarrow \mathcal{O}_C(q) \rightarrow 0 \quad (2.A.29)$$

where D is a divisor on X in the linear system η_1 containing C . Then, an analogous computation yields a similar variation of $\Delta\phi$ on the real axis, except that the maximum value is approximately 0.015 and it decreases monotonically to 0.008 at the LG point. Therefore the magnetized brane is also stable with respect to the decay (2.A.29). In principle there could exist other decay channels, perhaps described by more exotic triangles in the derived category. A systematic analysis would take us too far afield, so we will simply assume that the magnetized branes are stable at the LG point based on the evidence presented so far. A rigorous proof of stability is not within the reach of current Π -stability techniques.

Chapter 3

D-Brane Superpotentials in Calabi-Yau Orientifolds

D-branes in Type IIB orientifolds are an important ingredient in the constructions of string vacua. A frequent problem arising in this context is the computation of the tree-level superpotential for holomorphic D-brane configurations. This is an important question for both realistic model building as well as dynamical supersymmetry breaking.

Various computational methods for the tree-level superpotential have been proposed in the literature. A geometric approach which identifies the superpotential with a three-chain period of the holomorphic $(3,0)$ -form has been investigated in [100–104]. A related method, based on two-dimensional holomorphic Chern-Simons theory, has been developed in [105–108]. The tree-level superpotential for fractional brane configurations at toric Calabi-Yau singularities has been computed in [109–114]. Using exceptional collections, one can also compute the superpotential for non-toric del Pezzo singularities [115–118]. Perturbative disc computations for superpotential interactions have been performed in [119–121]. Finally, a mathematical approach based on versal deformations has been developed in [122].

A systematic approach encompassing all these cases follows from the algebraic structure of \mathbf{B} -branes on Calabi-Yau manifolds. Adopting the point of view that \mathbf{B} -branes form a triangulated differential graded category [6–8, 123–125] the computation of the superpotential is equivalent to the computation of a minimal A_∞ structure for the D-brane category [126–130].

This approach has been employed in the Landau-Ginzburg D-brane category [131–133], and in the derived category of coherent sheaves [10, 134]. These are two of the various phases that appear in the moduli space of a generic $\mathcal{N} = 2$ Type II compactification. In particular, Aspinwall and Katz [10] developed a general computational approach for the superpotential, in which the A_∞ products are computed using a Čech cochain model for the off-shell open string fields.

The purpose of the present chapter is to apply a similar strategy for D-branes wrapping holomorphic curves in Type II orientifolds. This requires a basic understanding of the orientifold projection in the derived category, which is the subject of section 3.1. In section 3.2 we propose a computational scheme for the superpotential in *orientifold* models. This relies on a systematic implementation of the orientifold projection in the calculation of the A_∞ structure.

We show that the natural algebraic framework for deformation problems in orientifold models relies on L_∞ rather than A_∞ structures. This observation leads to a simple prescription for the D-brane superpotential in the presence of an orientifold projection: one has to evaluate the superpotential of the underlying unprojected theory on invariant on-shell field configurations. This is the main conceptual result of the paper, and its proof necessitates the introduction of a lengthy abstract machinery.

Applying our prescription in practice requires some extra work. The difficulty stems from the fact that while the orientifold action is geometric on the Calabi-Yau, it is *not* naturally geometric at the level of the derived category. Therefore, knowing the superpotential in the original theory does not trivially lead to the superpotential of the orientifolded theory. To illustrate this point we compute the superpotential in two different cases. Both will involve D-branes wrapping rational curves, the difference will be in the way these curves are obstructed to move in the ambient space.

The organization of the paper is as follows. Section 2 reviews the construction of the categorical framework in which we wish to impose the orientifold projection, as well as how to do the latter. Section 3 describes the calculation of the D-brane superpotential in the presence of the projection. Finally, section 4 offers concrete computations of the D-brane superpotential for obstructed curves in Calabi-Yau orientifolds.

3.1 D-Brane Categories and Orientifold Projection

This section will be concerned with general aspects of topological **B**-branes in the presence of an orientifold projection. Our goal is to find a natural formulation for the orientifold projection in D-brane categories.

For concreteness, we will restrict ourselves to the category of topological **B**-branes on a Calabi-Yau threefold X , but our techniques extend to higher dimensions. In this case, the D-brane category is the derived category of coherent sheaves on X [6, 8]. In fact, a systematic off-shell construction of the D-brane category [123, 124] shows that the category in question is actually larger than the derived category. In addition to complexes, one has to also include

twisted complexes as defined in [135]. We will show below that the off-shell approach is the most convenient starting point for a systematic understanding of the orientifold projection.

3.1.1 Review of D-Brane Categories

Let us begin with a brief review of the off-shell construction of D-brane categories [123, 124, 135]. It should be noted at the offset that there are several different models for the D-brane category, depending on the choice of a fine resolution of the structure sheaf \mathcal{O}_X . In this section we will work with the Dolbeault resolution, which is closer to the original formulation of the boundary topological **B**-model [136]. This model is very convenient for the conceptual understanding of the orientifold projection, but it is unsuitable for explicit computations. In Section 4 we will employ a Čech cochain model for computational purposes, following the path pioneered in [10].

Given the threefold X , one first defines a differential graded category \mathcal{C} as follows

$$\begin{aligned} \text{Ob}(\mathcal{C}) &: \text{holomorphic vector bundles } (E, \bar{\partial}_E) \text{ on } X \\ \text{Mor}_{\mathcal{C}}((E, \bar{\partial}_E), (F, \bar{\partial}_F)) &= \left(\oplus_p A_X^{0,p}(\mathcal{H}om_X(E, F)), \bar{\partial}_{EF} \right) \end{aligned}$$

where we have denoted by $\bar{\partial}_{EF}$ the induced Dolbeault operator on $\mathcal{H}om_X(E, F)$ -valued $(0, p)$ forms.¹ The space of morphisms is a \mathbb{Z} -graded differential complex. In order to simplify the notation we will denote the objects of \mathcal{C} by E , the data of an integrable Dolbeault operator $\bar{\partial}_E$ being implicitly understood.

The composition of morphisms in \mathcal{C} is defined by exterior multiplication of bundle valued differential forms. For any object E composition of morphisms determines an associative algebra structure on the endomorphism space $\text{Mor}_{\mathcal{C}}(E, E)$. This product is compatible with the differential, therefore we obtain a differential graded associative algebra structure (DGA) on $\text{Mor}_{\mathcal{C}}(E, E)$.

At the next step, we construct the *shift completion* $\tilde{\mathcal{C}}$ of \mathcal{C} , which is a category of holomorphic vector bundles on X equipped with an integral grading.

$$\begin{aligned} \text{Ob}(\tilde{\mathcal{C}}) &: \text{pairs } (E, n), \text{ with } E \text{ an object of } \mathcal{C} \text{ and } n \in \mathbb{Z} \\ \text{Mor}_{\tilde{\mathcal{C}}}((E, n), (F, m)) &= \text{Mor}_{\mathcal{C}}(E, F)[n - m]. \end{aligned}$$

The integer n is the boundary ghost number introduced in [8]. Note that for a homogeneous element

$$f \in \text{Mor}_{\tilde{\mathcal{C}}}^k((E, n), (F, m))$$

¹ $\mathcal{H}om_X(E, F)$ is the sheaf Hom of E and F , viewed as sheaves.

we have

$$k = p + (m - n)$$

where p is the differential form degree of f . The degree k represents the total ghost number of the field f with respect to the bulk-boundary BRST operator. In the following we will use the notations

$$|f| = k, \quad c(f) = p, \quad h(f) = m - n.$$

The composition of morphisms in $\tilde{\mathcal{C}}$ differs from the composition of morphisms in \mathcal{C} by a sign, which will play an important role in our construction. Given two homogeneous elements

$$f \in \text{Mor}_{\tilde{\mathcal{C}}}((E, n), (E', n')) \quad g \in \text{Mor}_{\tilde{\mathcal{C}}}((E', n'), (E'', n''))$$

one defines the composition

$$(g \circ f)_{\tilde{\mathcal{C}}} = (-1)^{h(g)c(f)} (g \circ f)_{\mathcal{C}}. \quad (3.1.1)$$

This choice of sign leads to the graded Leibniz rule

$$\bar{\partial}_{EE''}(g \circ f)_{\tilde{\mathcal{C}}} = (\bar{\partial}_{E'E''}(g) \circ f)_{\tilde{\mathcal{C}}} + (-1)^{h(g)} (g \circ \bar{\partial}_{EE'}(f))_{\tilde{\mathcal{C}}}.$$

Now we construct a pre-triangulated DG category $\text{Pre-Tr}(\tilde{\mathcal{C}})$ of twisted complexes as follows

finite collections of the form

$$\begin{aligned} \text{Ob}(\text{Pre-Tr}(\tilde{\mathcal{C}})) : & \quad \{(E_i, n_i, q_{ji}) \mid q_{ji} \in \text{Mor}_{\tilde{\mathcal{C}}}^1((E_i, n_i), (E_j, n_j))\} \\ & \quad \text{where the } q_{ji} \text{ satisfy the Maurer-Cartan equation} \\ & \quad \bar{\partial}_{E_i E_j}(q_{ji}) + \sum_k (q_{jk} \circ q_{ki})_{\tilde{\mathcal{C}}} = 0. \end{aligned}$$

$$\text{Mor}_{\text{Pre-Tr}(\tilde{\mathcal{C}})}((E_i, n_i, q_{ji}), (F_i, m_i, r_{ji})) = \left(\bigoplus_{i,j} \text{Mor}_{\tilde{\mathcal{C}}}((E_i, n_i), (F_j, m_j)), Q \right)$$

where the differential Q is defined as

$$Q(f) = \bar{\partial}_{E_i F_j}(f) + \sum_k (r_{kj} \circ f)_{\tilde{\mathcal{C}}} - (-1)^{|f|} (f \circ q_{ik})_{\tilde{\mathcal{C}}}, \quad f \in \text{Mor}_{\tilde{\mathcal{C}}}((E_i, n_i), (F_j, m_j)).$$

$|f|$ is the degree of f in $\text{Mor}_{\tilde{\mathcal{C}}}((E_i, n_i), (F_j, m_j))$ from above. For each object, the index i takes finitely many values between 0 and some maximal value which depends on the object. Note that $Q^2 = 0$ because $\{q_{ji}\}, \{r_{ji}\}$ satisfy the Maurer-Cartan equation. Composition of morphisms in $\text{Pre-Tr}(\tilde{\mathcal{C}})$ reduces to composition of morphisms in $\tilde{\mathcal{C}}$.

Finally, the triangulated D-brane category \mathcal{D} has by definition the same objects as $\text{Pre-Tr}(\tilde{\mathcal{C}})$, while its morphisms are given by the zeroth cohomology under Q of the morphisms of $\text{Pre-Tr}(\tilde{\mathcal{C}})$:

$$\begin{aligned} \text{Ob}(\mathcal{D}) &= \text{Ob}(\text{Pre-Tr}(\tilde{\mathcal{C}})) \\ \text{Mor}_{\mathcal{D}}((E_i, n_i, q_{ji}), (F_i, m_i, r_{ji})) &= H^0 \left(Q, \text{Mor}_{\text{Pre-Tr}(\tilde{\mathcal{C}})}((E_i, n_i, q_{ji}), (F_i, m_i, r_{ji})) \right). \end{aligned} \quad (3.1.2)$$

The bounded derived category of coherent sheaves $D^b(X)$ is a full subcategory of \mathcal{D} . To see this consider the objects of the form (E_i, n_i, q_{ji}) such that

$$n_i = -i, \quad q_{ji} \neq 0 \Leftrightarrow j = i - 1. \quad (3.1.3)$$

Since $q_{ji} \in \text{Mor}_{\tilde{\mathcal{C}}}^1((E_i, n_i), (E_j, n_j))$, the second condition in (3.1.3) implies that their differential form degree must be 0. The Maurer-Cartan equation for such objects reduces to

$$\bar{\partial}_{E_i E_{i-1}} q_{i-1, i} = 0, \quad (q_{i-1, i} \circ q_{i, i+1})_{\tilde{\mathcal{C}}} = 0.$$

Therefore the twisted complex (E_i, n_i, q_{ji}) is in fact a complex of holomorphic vector bundles

$$\cdots \longrightarrow E_{i+1} \xrightarrow{q_{i, i+1}} E_i \xrightarrow{q_{i-1, i}} E_{i-1} \longrightarrow \cdots \quad (3.1.4)$$

We will use the alternative notation

$$\cdots \longrightarrow E_{i+1} \xrightarrow{d_{i+1}} E_i \xrightarrow{d_i} E_{i-1} \longrightarrow \cdots \quad (3.1.5)$$

for complexes of vector bundles, and also denote them by the corresponding Gothic letter, here \mathfrak{E} .

One can easily check that the morphism space (3.1.2) between two twisted complexes of the form (3.1.3) reduces to the hypercohomology group of the local Hom complex $\mathcal{H}om(\mathfrak{E}, \mathfrak{F})$

$$\text{Mor}_{\mathcal{D}}((E_i, n_i, q_{ji}), (F_i, m_i, r_{ji})) \simeq \mathbb{H}^0(X, \mathcal{H}om(\mathfrak{E}, \mathfrak{F})). \quad (3.1.6)$$

As explained in [125], this hypercohomology group is isomorphic to the derived morphism space $\text{Hom}_{D^b(X)}(\mathfrak{E}, \mathfrak{F})$. Assuming that X is smooth and projective, any derived object has a locally free resolution, hence $D^b(X)$ is a full subcategory of \mathcal{D} .

3.1.2 Orientifold Projection

Now we consider orientifold projections from the D-brane category point of view. A similar discussion of orientifold projections in matrix factorization categories has been outlined in [137].

Consider a four dimensional $N = 1$ IIB orientifold obtained from an $N = 2$ Calabi-Yau compactification by gauging a discrete symmetry of the form

$$(-1)^{\epsilon F_L} \Omega \sigma$$

with $\epsilon = 0, 1$. Employing common notation, Ω denotes world-sheet parity, F_L is the left-moving fermion number and $\sigma: X \rightarrow X$ is a holomorphic involution of X satisfying

$$\sigma^* \Omega_X = (-1)^\epsilon \Omega_X, \quad (3.1.7)$$

where Ω_X is the holomorphic $(3, 0)$ -form of the Calabi-Yau. Depending on the value of ϵ , there are two classes of models to consider [52]:

1. $\epsilon = 0$: theories with $O5/O9$ orientifolds planes, in which the fixed point set of σ is either one or three complex dimensional;
2. $\epsilon = 1$: theories with $O3/O7$ planes, with σ leaving invariant zero or two complex dimensional submanifolds of X .

Following the same logical steps as in the previous subsection, we should first find the action of the orientifold projection on the category \mathcal{C} , which is the starting point of the construction. The action of parity on the K-theory class of a D-brane has been determined in [138]. The world-sheet parity Ω maps E to the dual vector bundle E^\vee . If Ω acts simultaneously with a holomorphic involution $\sigma: X \rightarrow X$, the bundle E will be mapped to $\sigma^*(E^\vee)$. If the projection also involves a $(-1)^{F_L}$ factor, a brane with Chan-Paton bundle E should be mapped to an anti-brane with Chan-Paton bundle $P(E)$.

Based on this data, we define the action of parity on \mathcal{C} to be

$$P: E \mapsto P(E) = \sigma^*(E^\vee) \quad (3.1.8)$$

$$P: f \in \text{Mor}_{\mathcal{C}}(E, F) \mapsto \sigma^*(f^\vee) \in \text{Mor}_{\mathcal{C}}(P(F), P(E))$$

It is immediate that P satisfies the following compatibility condition with respect to composition of morphisms in \mathcal{C} :

$$P((g \circ f)_{\mathcal{C}}) = (-1)^{c(f)c(g)} (P(f) \circ P(g))_{\mathcal{C}} \quad (3.1.9)$$

for any homogeneous elements f and g . It is also easy to check that P preserves the differential graded structure, i.e.,

$$P(\bar{\partial}_{EF}(f)) = \bar{\partial}_{P(F)P(E)}(P(f)). \quad (3.1.10)$$

Equation (3.1.9) shows that P is not a functor in the usual sense. Since it is compatible with the differential graded structure, it should be interpreted as a functor of A_∞ categories [139]. Note however that P is “almost a functor”: it fails to satisfy the compatibility condition with composition of morphisms only by a sign. For future reference, we will refer to A_∞ functors satisfying a graded compatibility condition of the form (3.1.9) as *graded functors*.

The category \mathcal{C} does not contain enough information to make a distinction between branes and anti-branes. In order to make this distinction, we have to assign each bundle a grading, that is we have to work in the category $\tilde{\mathcal{C}}$ rather than \mathcal{C} . By convention, the objects (E, n) with n even are called branes, while those with n odd are called anti-branes.

We will take the action of the orientifold projection on the objects of $\tilde{\mathcal{C}}$ to be

$$\tilde{P}: (E, n) \mapsto (P(E), m - n) \quad (3.1.11)$$

where we have introduced an integer shift m which is correlated with ϵ from (3.1.7):

$$m \equiv \epsilon \pmod{2}. \quad (3.1.12)$$

This allows us to treat both cases $\epsilon = 0$ and $\epsilon = 1$ in a unified framework.

We define the action of P on a morphisms $f \in \text{Mor}_{\tilde{\mathcal{C}}}((E, n), (E', n'))$ as the following graded dual:

$$\tilde{P}(f) = -(-1)^{n'h(f)} P(f), \quad (3.1.13)$$

where $P(f)$ was defined in (3.1.8).² Note that the graded dual has been used in a similar context in [137], where the orientifold projection is implemented in matrix factorization categories.

With this definition, we have the following:

Proposition 3.1.1. *\tilde{P} is a graded functor on $\tilde{\mathcal{C}}$ satisfying*

$$\tilde{P}((g \circ f)_{\tilde{\mathcal{C}}}) = -(-1)^{|f||g|} (\tilde{P}(f) \circ \tilde{P}(g))_{\tilde{\mathcal{C}}} \quad (3.1.14)$$

for any homogeneous elements

$$f \in \text{Mor}_{\tilde{\mathcal{C}}}((E, n), (E', n')), \quad g \in \text{Mor}_{\tilde{\mathcal{C}}}((E', n'), (E'', n'')).$$

Proof. It is clear that P is compatible with the differential graded structure of $\tilde{\mathcal{C}}$ since the latter is inherited from \mathcal{C} .

Next we prove (3.1.14). First we have:

$$\begin{aligned} \tilde{P}((g \circ f)_{\tilde{\mathcal{C}}}) &= -(-1)^{n''h(g \circ f)} P((g \circ f)_{\tilde{\mathcal{C}}}) \quad \text{by (3.1.13)} \\ &= -(-1)^{n''h(g \circ f) + h(g)c(f)} P((g \circ f)_{\mathcal{C}}) \quad \text{by (3.1.1)} \\ &= -(-1)^{n''h(g \circ f) + h(g)c(f) + c(f)c(g)} (P(f) \circ P(g))_{\mathcal{C}} \quad \text{by (3.1.9)} \end{aligned}$$

On the other hand

$$\begin{aligned} (\tilde{P}(f) \circ \tilde{P}(g))_{\tilde{\mathcal{C}}} &= (-1)^{n'h(f) + n''h(g)} (P(f) \circ P(g))_{\tilde{\mathcal{C}}} \quad \text{by (3.1.13)} \\ &= (-1)^{n'h(f) + n''h(g)} (-1)^{h(P(f))c(P(g))} (P(f) \circ P(g))_{\mathcal{C}} \quad \text{by (3.1.1)} \end{aligned}$$

But

$$h(g \circ f) = h(f) + h(g), \quad h(P(f)) = h(f), \quad c(P(g)) = c(g).$$

²There is no a priori justification for the particular sign we chose, but as we will see shortly, it leads to a graded functor. A naive generalization of (3.1.8) ignoring this sign would not yield a graded functor.

Now (3.1.14) follows from

$$n''(h(f) + h(g)) - n'h(f) - n''h(g) = (n'' - n')h(f) = h(g)h(f)$$

and

$$|f||g| = (h(f) + c(f))(h(g) + c(g)).$$

□

The next step is to determine the action of P on the pre-triangulated category $\text{Pre-Tr}(\tilde{\mathcal{C}})$. We denote this action by \hat{P} . The action of \hat{P} on objects is defined simply by

$$(E_i, n_i, q_{ji}) \mapsto (P(E_i), m - n_i, \tilde{P}(q_{ji})) \quad (3.1.15)$$

Using equations (3.1.10) and (3.1.14), it is straightforward to show that the action of P preserves the Maurer-Cartan equation, that is

$$\bar{\partial}_{E_i E_j}(q_{ji}) + \sum_k (q_{jk} \circ q_{ki})_{\tilde{\mathcal{C}}} = 0 \Rightarrow \bar{\partial}_{P(E_j)P(E_i)}\tilde{P}(q_{ji}) + \sum_k (\tilde{P}(q_{ki}) \circ \tilde{P}(q_{jk}))_{\tilde{\mathcal{C}}} = 0,$$

since all q_{ji} have total degree one. Therefore this transformation is well defined on objects. The action on morphisms is also straightforward

$$\begin{aligned} f &\in \oplus_{i,j} \text{Mor}_{\tilde{\mathcal{C}}}((E_i, n_i), (F_j, m_j)) \\ &\mapsto \hat{P}(f) = \tilde{P}(f) \in \oplus_{i,j} \text{Mor}_{\tilde{\mathcal{C}}}((P(E_j), m - m_j), (P(E_i), m - n_i)). \end{aligned} \quad (3.1.16)$$

Again, equations (3.1.10), (3.1.14) imply that this action preserves the differential

$$Q(f) = \bar{\partial}_{E_i F_j}(f) + \sum_k (r_{kj} \circ f)_{\tilde{\mathcal{C}}} - (-1)^{|f|} (f \circ q_{ik})_{\tilde{\mathcal{C}}}$$

since $\{q_{ji}\}, \{r_{ji}\}$ have degree one. This means we have

$$\hat{P}(Q(f)) = \bar{\partial}_{P(F_j)P(E_i)}(\tilde{P}(f)) + \sum_k (\tilde{P}(q_{ik}) \circ \tilde{P}(f))_{\tilde{\mathcal{C}}} - (-1)^{|\tilde{P}(f)|} (\tilde{P}(f) \circ \tilde{P}(r_{kj}))_{\tilde{\mathcal{C}}} \quad (3.1.17)$$

For future reference, let us record some explicit formulas for complexes of vector bundles. A complex

$$\mathfrak{E}: \quad \cdots \longrightarrow E_{i+1} \xrightarrow{d_{i+1}} E_i \xrightarrow{d_i} E_{i-1} \longrightarrow \cdots$$

in which E_i has degree $-i$ is mapped to the complex

$$\hat{P}(\mathfrak{E}): \quad \cdots \longrightarrow P(E_{i-1}) \xrightarrow{\tilde{P}(d_i)} P(E_i) \xrightarrow{\tilde{P}(d_{i+1})} P(E_{i+1}) \longrightarrow \cdots \quad (3.1.18)$$

where $\tilde{P}(d_i)$ is determined by (3.1.13)

$$\tilde{P}(d_i) = (-1)^i \sigma^*(d_i^\vee)$$

and $P(E_i)$ has degree $i - m$. Applying P twice yields the complex

$$\widehat{P}^2(\mathfrak{E}): \quad \cdots \longrightarrow E_{i+1} \xrightarrow{\widetilde{P}^2(d_{i+1})} E_i \xrightarrow{\widetilde{P}^2(d_i)} E_{i-1} \longrightarrow \cdots \quad (3.1.19)$$

where

$$\widetilde{P}^2(d_i) = (-1)^{m+1} d_i.$$

Therefore \widehat{P}^2 is not equal to the identity functor, but there is an isomorphism of complexes $J: \widehat{P}^2(\mathfrak{E}) \rightarrow \mathfrak{E}$:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & E_{i+1} & \xrightarrow{\widetilde{P}^2(d_{i+1})} & E_i & \xrightarrow{\widetilde{P}^2(d_i)} & E_{i-1} \longrightarrow \cdots \\ & & \downarrow J_{i+1} & & \downarrow J_i & & \downarrow J_{i-1} \\ \cdots & \longrightarrow & E_{i+1} & \xrightarrow{d_{i+1}} & E_i & \xrightarrow{d_i} & E_{i-1} \longrightarrow \cdots \end{array} \quad (3.1.20)$$

where

$$J_i = (-1)^{(m+1)i} \chi \operatorname{Id}_{E_i}.$$

and χ is a constant. Notice that $J^{-1}: \mathfrak{E} \rightarrow \widehat{P}^2(\mathfrak{E})$, and that $\widehat{P}^4 = \operatorname{Id}_{D^b(X)}$ implies that also $J: \widehat{P}^2(\widehat{P}^2(\mathfrak{E})) = \mathfrak{E} \rightarrow \widehat{P}^2(\mathfrak{E})$. Requiring both to be equal constrains χ to be $(-1)^\omega$ with $\omega = 0, 1$. This sign cannot be fixed using purely algebraic considerations, and we will show in section 3.3 how it encodes the difference between SO/Sp projections. In functorial language, this means that there is an isomorphism of functors $J: \widehat{P}^2 \rightarrow \operatorname{Id}_{D^b(X)}$.

We conclude this section with a brief summary of the above discussion, and a short remark on possible generalizations. To simplify notation, in the rest of the paper we drop the decorations of the various P 's. In other words both \widehat{P} and \widetilde{P} will be denoted by P . Which P is meant will always be clear from the context.

1. The orientifold projection in the derived category is a graded contravariant functor $P: D^b(X) \rightarrow D^b(X)^{op}$ which acts on locally free complexes as in equation (3.1.18). Note that this transformation is closely related to the derived functor

$$\mathbf{L}\sigma^* \circ \mathbf{R}\mathcal{H}om(-, \mathcal{O}_X)[m].$$

The difference resides in the alternating signs $(-1)^i$ in the action of P on differentials, according to (3.1.18). From now on we will refer to P as a graded derived functor.

2. There is an obvious generalization of this construction which has potential physical applications. One can further compose P with an auto-equivalence \mathcal{A} of the derived category so that the resulting graded functor $P \circ \mathcal{A}$ is isomorphic to the identity. This would yield a new class of orientifold models, possibly without a direct geometric interpretation. The physical implications of this construction will be explored in a separate publication.

In the remaining part of this section we will consider the case of D5-branes wrapping holomorphic curves in more detail.

3.1.3 $O5$ models

In this case we consider holomorphic involutions $\sigma: X \rightarrow X$ whose fixed point set consists of a finite collection of holomorphic curves in X . We will be interested in D5-brane configurations supported on a smooth component $C \simeq \mathbb{P}^1$ of the fixed locus, that are preserved by the orientifold projection. We describe such a configuration by a one term complex

$$i_* V \quad (3.1.21)$$

where $V \rightarrow C$ is the Chan-Paton vector bundle on C , and $i: C \hookrightarrow X$ is the embedding of C into X .

Since $C \simeq \mathbb{P}^1$, by Grothendieck's theorem any holomorphic bundle V decomposes in a direct sum of line bundles. Therefore, for the time being, we take

$$V \simeq \mathcal{O}_C(a) \quad (3.1.22)$$

for some $a \in \mathbb{Z}$. We will also make the simplifying assumption that V is the restriction of a bundle V' on X to C , i.e.,

$$V = i^* V'. \quad (3.1.23)$$

This is easily satisfied if X is a complete intersection in a toric variety Z , in which case V can be chosen to be the restriction of bundle on Z .

In order to write down the parity action on this D5-brane configuration, we need a locally free resolution \mathfrak{E} for $i_* V = i_* \mathcal{O}_C(a)$. Let

$$\mathfrak{V}: \quad 0 \longrightarrow \mathcal{V}_n \xrightarrow{d_n} \mathcal{V}_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \mathcal{V}_1 \xrightarrow{d_1} \mathcal{V}_0 \longrightarrow 0 \quad (3.1.24)$$

be a locally free resolution of $i_* \mathcal{O}_C$.³ We take the degree of the term \mathcal{V}_k to be $(-k)$, for $k = 0, \dots, n$. Then the complex \mathfrak{E}

$$\mathfrak{E}: \quad 0 \longrightarrow \mathcal{V}_n(a) \xrightarrow{d_n} \mathcal{V}_{n-1}(a) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \mathcal{V}_1(a) \xrightarrow{d_1} \mathcal{V}_0(a) \longrightarrow 0 \quad (3.1.25)$$

is a locally free resolution of $i_* \mathcal{O}_C(a)$.

The image of (3.1.25) under the orientifold projection is the complex $\mathcal{P}(\mathfrak{E})$:

$$0 \longrightarrow \sigma^* \mathcal{V}_0^\vee(-a) \xrightarrow{-\sigma^* d_1^\vee} \sigma^* \mathcal{V}_1^\vee(-a) \xrightarrow{\sigma^* d_2^\vee} \cdots$$

³We usually underlined the 0th position in a complex.

$$\cdots \xrightarrow{(-1)^{n-1}\sigma^*d_{n-1}^\vee} \sigma^*\mathcal{V}_{n-1}^\vee(-a) \xrightarrow{(-1)^n\sigma^*d_n^\vee} \sigma^*\mathcal{V}_n^\vee(-a) \longrightarrow 0 \quad (3.1.26)$$

The term $\mathcal{V}_k^\vee(-a)$ has degree $k - m$.

Lemma 3.1.2. ⁴ *The complex (3.1.26) is quasi-isomorphic to*

$$i_*(V^\vee \otimes K_C)[m-2], \quad (3.1.27)$$

where $K_C \simeq \mathcal{O}_C(-2)$ is the canonical bundle of C .

Proof. As noted below (3.1.18), (3.1.26) is isomorphic to $\sigma^*(\mathfrak{E}^\vee)[m]$. Since C is pointwise fixed by σ , it suffices to show that the dual of the locally free resolution (3.1.24) is quasi-isomorphic to $i_*K_C[-2]$. The claim then follows from the adjunction formula:

$$i_*V = i_*(V \otimes \mathcal{O}_C) = i_*(i^*V' \otimes \mathcal{O}_C) = V' \otimes i_*\mathcal{O}_C \quad (3.1.28)$$

and the simple fact that $i^*(V'^\vee) = V^\vee$.

Let us compute $(i_*\mathcal{O}_C)^\vee$ using the locally free resolution (3.1.24). The cohomology in degree k of the complex

$$\mathfrak{V}^\vee: \quad 0 \rightarrow (\mathcal{V}_0)^\vee \rightarrow (\mathcal{V}_1)^\vee \rightarrow \cdots \rightarrow (\mathcal{V}_n)^\vee \rightarrow 0 \quad (3.1.29)$$

is isomorphic to the local Ext sheaves $\mathcal{E}xt_X^k(i_*\mathcal{O}_C, \mathcal{O}_X)$. According to [140, Chapter 5.3, pg 690] these are trivial except for $k = 2$, in which case

$$\mathcal{E}xt_X^2(\mathcal{O}_C, \mathcal{O}_X) \simeq i_*\mathcal{L},$$

for some line bundle \mathcal{L} on C .

To determine \mathcal{L} , it suffices to compute its degree on C , which is an easy application of the Grothendieck-Riemann-Roch theorem. We have

$$i_!(\mathrm{ch}(\mathcal{L})\mathrm{Td}(C)) = \mathrm{ch}(i_*\mathcal{L})\mathrm{Td}(X).$$

On the other hand, by construction

$$\mathrm{ch}_m(i_*\mathcal{L}) = \mathrm{ch}_m(\mathfrak{V}^\vee) = (-1)^m \mathrm{ch}_m(\mathfrak{V}) = (-1)^m \mathrm{ch}_m(i_*\mathcal{O}_C).$$

Using these two equations, we find

$$\deg(\mathcal{L}) = -2 \Rightarrow \mathcal{L} \simeq K_C.$$

⁴ We give an alternative derivation of this result in Appendix 3.4.1. That proof is very abstract, and hides all the details behind the powerful machinery of Grothendieck duality. On the other hand, we will be using the details of this lengthier derivation in our explicit computations in Section 3.3.

This shows that \mathfrak{V}^\vee has nontrivial cohomology i_*K_C only in degree 2.

Now we establish that the complex (3.1.29) is quasi-isomorphic to $i_*K_C[-2]$, by constructing such a map of complexes. Consider the restriction of the complex (3.1.29) to C . Since all terms are locally free, we obtain a complex of holomorphic bundles on C whose cohomology is isomorphic to K_C in degree 2 and trivial in all other degrees. Note that the kernel \mathcal{K} of the map

$$\mathcal{V}_2^\vee|_C \rightarrow \mathcal{V}_3^\vee|_C$$

is a torsion free sheaf on C , therefore it must be locally free. Hence \mathcal{K} is a sub-bundle of $\mathcal{V}_2^\vee|_C$. Since $C \simeq \mathbb{P}^1$, by Grothendieck's theorem both $\mathcal{V}_2^\vee|_C$ and \mathcal{K} are isomorphic to direct sums of line bundles. This implies that \mathcal{K} is in fact a direct summand of $\mathcal{V}_2^\vee|_C$. In particular there is a surjective map

$$\rho: \mathcal{V}_2^\vee|_C \rightarrow \mathcal{K}.$$

Since $H^2(\mathfrak{V}^\vee|_C) = K_C$ we also have a surjective map $\tau: \mathcal{K} \rightarrow K_C$. By construction then $\tau \circ \rho: \mathfrak{V}^\vee|_C \rightarrow K_C[-2]$ is a quasi-isomorphism. Extending this quasi-isomorphism by zero outside C , we obtain a quasi-isomorphism $\mathfrak{V}^\vee \rightarrow i_*K_C[-2]$, which proves the lemma. \square

Let us now discuss parity invariant D-brane configurations. Given the parity action (3.1.27) one can obviously construct such configurations by taking direct sums of the form

$$i_*V \oplus i_*(V^\vee \otimes K_C)[m-2] \tag{3.1.30}$$

with V an arbitrary Chan-Paton bundle. Note that in this case we have two stacks of D5-branes in the covering space which are interchanged under the orientifold projection.

However, on physical grounds we should also be able to construct a single stack of D5-branes wrapping C which is preserved by the orientifold action. This is possible only if

$$m = 2 \quad \text{and} \quad V \simeq V^\vee \otimes K_C. \tag{3.1.31}$$

The first condition in (3.1.31) fixes the value of m for this class of models. The second condition constrains the Chan-Paton bundle V to

$$V = \mathcal{O}_C(-1).$$

Let us now consider rank N Chan-Paton bundles V . We will focus on invariant D5-brane configurations given by

$$V = \mathcal{O}_C(-1)^{\oplus N}.$$

In this case the orientifold image $P(i_*V) = i_*(V^\vee \otimes K_C)$ is isomorphic to i_*V , and the choice of an isomorphism corresponds to the choice of a section

$$M \in \text{Hom}_C(V, V^\vee \otimes K_C) \simeq \mathcal{M}_N(\mathbb{C}). \quad (3.1.32)$$

where $\mathcal{M}_N(\mathbb{C})$ is the space of $N \times N$ complex matrices. We have

$$\begin{aligned} \text{Hom}_C(V, V^\vee \otimes K_C) &\simeq H^0(C, S^2(V^\vee) \otimes K_C) \oplus H^0(C, \Lambda^2(V^\vee) \otimes K_C) \\ &\simeq \mathcal{M}_N^+(\mathbb{C}) \oplus \mathcal{M}_N^-(\mathbb{C}) \end{aligned}$$

where $\mathcal{M}_N^\pm(\mathbb{C})$ denotes the space of symmetric and antisymmetric $N \times N$ matrices respectively. The choice of this isomorphism (up to conjugation) encodes the difference between SO and Sp projections. For any value of N we can choose the isomorphism to be

$$M = I_N \in \mathcal{M}_N^+(\mathbb{C}), \quad (3.1.33)$$

obtaining $SO(N)$ gauge group. If N is even, we also have the option of choosing the antisymmetric matrix

$$M = \begin{bmatrix} 0 & I_{N/2} \\ -I_{N/2} & 0 \end{bmatrix} \in \mathcal{M}_N^-(\mathbb{C}) \quad (3.1.34)$$

obtaining $Sp(N/2)$ gauge group. This is a slightly more abstract reformulation of [141]. We will explain how the SO/Sp projections are encoded in the derived formalism in sections 3.2 and 3.3.

3.1.4 $O3/O7$ Models

In this case we have $\epsilon = 1$, and the fixed point set of the holomorphic involution can have both zero and two dimensional components. We will consider the magnetized D5-brane configurations introduced in [5]. Suppose

$$i: C \hookrightarrow X \quad i': C' \hookrightarrow X$$

is a pair of smooth rational curves mapped isomorphically into each other by the holomorphic involution. The brane configuration consists of a stack of D5-branes wrapping C , which is related by the orientifold projection to a stack of anti-D5-branes wrapping C' . We describe the stack of D5-branes wrapping C by a one term complex i_*V , with V a bundle on C .

In order to find the action of the orientifold group on the stack of D5-branes wrapping C we pick a locally free resolution \mathfrak{E} for i_*V . Once again we choose the orientifold image is obtained by applying the graded derived functor P to \mathfrak{E} .

Applying Prop. 3.4.1, we have

Lemma 3.1.3. $\mathcal{P}(\mathfrak{E})$ is quasi-isomorphic to the one term complex

$$i'_*(\sigma^*(V^\vee) \otimes K_{C'})[m-2]. \quad (3.1.35)$$

It follows that a D5-brane configuration preserved by the orientifold projection is a direct sum

$$i_*V \oplus i'_*(\sigma^*(V^\vee) \otimes K_{C'})[m-2]. \quad (3.1.36)$$

The value of m can be determined from physical arguments by analogy with the previous case. We have to impose the condition that the orientifold projection preserves a D3-brane supported on a fixed point $p \in X$ as well as a D7-brane supported on a pointwise fixed surface $S \subset X$.

A D3-brane supported at $p \in X$ is described by a one-term complex $\mathcal{O}_{p,X}$, where $\mathcal{O}_{p,X}$ is a skyscraper sheaf supported at p . Again, using Prop. 3.4.1 one shows that $P(\mathfrak{V})$ is quasi-isomorphic to $\mathcal{O}_{p,X}[m-3]$. Therefore, the D3-brane is preserved if and only if $m = 3$.

If the model also includes a codimension 1 pointwise-fixed locus $S \subset X$, then we have an extra condition. Let V be the Chan-Paton bundle on S . We describe the invariant D7-brane wrapping S by $\mathfrak{L} \simeq i_*(V)[k]$ for some integer k , where $i: S \rightarrow X$ is the embedding.

Since S is codimension 1 in X , Prop. 3.4.1 tells us that

$$P(\mathfrak{L}) \simeq i_*(V^\vee \otimes K_S)[m-k-1]. \quad (3.1.37)$$

Therefore invariance under P requires

$$2k = m - 1 \quad V \otimes V \simeq K_S. \quad (3.1.38)$$

Since we have found $m = 3$ above, it follows that $k = 1$. Furthermore, V has to be a square root of K_S . In particular, this implies that K_S must be even, or, in other words that S must be spin. This is in agreement with the Freed-Witten anomaly cancellation condition [83]. If S is not spin, one has to turn on a half integral B -field in order to cancel anomalies.

Returning to the magnetized D5-brane configuration, note that an interesting situation from the physical point of view is the case when the curves C and C' coincide. Then C is preserved by the holomorphic involution, but not pointwise fixed as in the previous subsection. We will discuss examples of such configurations in section 3.3. In the next section we will focus on general aspects of the superpotential in orientifold models.

3.2 The Superpotential

The framework of D-brane categories offers a systematic approach to the computation of the tree-level superpotential. In the absence of the orientifold projection, the tree-level D-brane

superpotential is encoded in the A_∞ structure of the D-brane category [126–128, 130].

Given an object of the D-brane category \mathcal{D} , the space of off-shell open string states is its space of endomorphisms in the pre-triangulated category $\text{Pre-Tr}(\tilde{\mathcal{C}})$. This carries the structure of a \mathbb{Z} -graded differential cochain complex. In this section we will continue to work with Dolbeault cochains, and also specialize our discussion to locally free complexes \mathfrak{E} of the form (3.1.5). Then the space of off-shell open string states is given by

$$\text{Mor}_{\text{Pre-Tr}(\tilde{\mathcal{C}})}(\mathfrak{E}, \mathfrak{E}) = \oplus_p A^{0,p}(\mathcal{H}om_X(\mathfrak{E}, \mathfrak{E}))$$

where

$$\mathcal{H}om_X^q(\mathfrak{E}, \mathfrak{E}) = \oplus_i \mathcal{H}om_X(E_i, E_{i-q}).$$

Composition of morphisms defines a natural superalgebra structure on this endomorphism space [142], and the differential Q satisfies the graded Leibniz rule. We will denote the resulting DGA by $\mathcal{C}(\mathfrak{E}, \mathfrak{E})$.

The computation of the superpotential is equivalent to the construction of an A_∞ minimal model for the DGA $\mathcal{C}(\mathfrak{E}, \mathfrak{E})$. Since this formalism has been explained in detail in the physics literature [10, 128], we will not provide a comprehensive review here. Rather we will recall some basic elements needed for our construction.

In order to extend this computational framework to orientifold models, we have to find an off-shell cochain model equipped with an orientifold projection and a compatible differential algebraic structure. We made a first step in this direction in the previous section by giving a categorical formulation of the orientifold projection. In section 3.2.1 we will refine this construction, obtaining the desired cochain model.

Having constructed a suitable cochain model, the computation of the superpotential follows the same pattern as in the absence of the orientifold projection. A notable distinction resides in the occurrence of L_∞ instead of A_∞ structures, since the latter are not compatible with the involution. The final result obtained in section 3.2.2 is that the orientifold superpotential can be obtained by evaluating the superpotential of the underlying unprojected theory on invariant field configurations.

3.2.1 Cochain Model and Orientifold Projection

Suppose \mathfrak{E} is a locally free complex on X , and that it is left invariant by the parity functor. This means that \mathfrak{E} and $P(\mathfrak{E})$ are isomorphic in the derived category, and we choose such an isomorphism

$$\psi: \mathfrak{E} \rightarrow P(\mathfrak{E}). \tag{3.2.1}$$

Although in general ψ is not a map of complexes, it can be chosen so in most practical situations, including all cases studied in this paper. Therefore we will assume from now on that ψ is a quasi-isomorphism of complexes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & E_{m-i+1} & \xrightarrow{d_{m-i+1}} & E_{m-i} & \xrightarrow{d_{m-i}} & E_{m-i-1} \longrightarrow \cdots \\ & & \downarrow \psi_{m-i+1} & & \downarrow \psi_{m-i} & & \downarrow \psi_{m-i-1} \\ \cdots & \longrightarrow & P(E_{i-1}) & \xrightarrow{P(d_i)} & P(E_i) & \xrightarrow{P(d_{i+1})} & P(E_{i+1}) \longrightarrow \cdots \end{array} \quad (3.2.2)$$

We have written (3.2.2) so that the terms in the same column have the same degree since ψ is a degree zero morphism. The degrees of the three columns from left to right are $i-m-1$, $i-m$ and $i-m+1$. For future reference, note that the quasi-isomorphism ψ induces a quasi-isomorphism of cochain complexes

$$\psi_*: \mathcal{C}(P(\mathfrak{E}), \mathfrak{E}) \rightarrow \mathcal{C}(P(\mathfrak{E}), P(\mathfrak{E})), \quad f \mapsto \psi \circ f. \quad (3.2.3)$$

The problem we are facing in the construction of a viable cochain model resides in the absence of a natural orientifold projection on the cochain space $\mathcal{C}(\mathfrak{E}, \mathfrak{E})$. P maps $\mathcal{C}(\mathfrak{E}, \mathfrak{E})$ to $\mathcal{C}(P(\mathfrak{E}), P(\mathfrak{E}))$, which is not identical to $\mathcal{C}(\mathfrak{E}, \mathfrak{E})$. How can we find a natural orientifold projection on a given off-shell cochain model?

Since \mathfrak{E} and $P(\mathfrak{E})$ are quasi-isomorphic, one can equally well adopt the morphism space

$$\mathcal{C}(P(\mathfrak{E}), \mathfrak{E}) = \text{Mor}_{\text{Pre-Tr}(\tilde{\mathcal{C}})}(P(\mathfrak{E}), \mathfrak{E})$$

as an off-shell cochain model. As opposed to $\mathcal{C}(\mathfrak{E}, \mathfrak{E})$, this morphism space has a natural induced involution defined by the composition

$$\mathcal{C}(P(\mathfrak{E}), \mathfrak{E}) \xrightarrow{P} \mathcal{C}(P(\mathfrak{E}), P^2(\mathfrak{E})) \xrightarrow{J_*} \mathcal{C}(P(\mathfrak{E}), \mathfrak{E}) \quad (3.2.4)$$

where J is the isomorphism in (3.1.20). Therefore we will do our superpotential computation in the cochain model $\mathcal{C}(P(\mathfrak{E}), \mathfrak{E})$, as opposed to $\mathcal{C}(\mathfrak{E}, \mathfrak{E})$, which is used in [10].

This seems to lead us to another puzzle, since a priori there is no natural associative algebra structure on $\mathcal{C}(P(\mathfrak{E}), \mathfrak{E})$. One can however define one using the quasi-isomorphism (3.2.1). Given

$$f_{q,k}^p \in A^{0,p}(\mathcal{H}om_X(P(E_k), E_{m-k-q})) \quad g_{s,l}^r \in A^{0,r}(\mathcal{H}om_X(P(E_l), E_{m-l-s}))$$

we define

$$g_{s,l}^r \star_\psi f_{q,k}^p = \begin{cases} (-1)^{sp} g_{s,l}^r \cdot \psi_{m-k-q} \cdot f_{q,k}^p & \text{for } l = k + q \\ 0 & \text{otherwise.} \end{cases} \quad (3.2.5)$$

where \cdot denotes exterior multiplication of bundle valued differential forms.

With this definition, the map (3.2.3) becomes a quasi-isomorphism of DGAs. The sign $(-1)^{sp}$ in (3.2.5) is determined by the sign rule (3.1.1) for composition of morphisms in $\tilde{\mathcal{C}}$. This construction has the virtue that it makes both the algebra structure and the orientifold projection manifest. Note that the differential Q satisfies the graded Leibniz rule with respect to the product \star_ψ because ψ is a Q -closed element of $\mathcal{C}(P(\mathfrak{E}), \mathfrak{E})$ of degree zero.

Next we check two compatibility conditions between the involution (3.2.4) and the DGA structure.

Lemma 3.2.1. *For any cochain $f \in \mathcal{C}(P(\mathfrak{E}), \mathfrak{E})$*

$$J_*P(Q(f)) = Q(J_*P(f)). \quad (3.2.6)$$

Proof. Using equation (3.1.18), the explicit expression for the differential Q acting on a homogeneous element $f_{q,k}^p$ as above is

$$Q(f_{q,k}^p) = \bar{\partial}_{P(E_k)E_{m-k-q}}(f_{q,k}^p) + (d_{m-k-q} \circ f_{q,k}^p)_{\tilde{\mathcal{C}}} - (-1)^{p+q}(f_{q,k}^p \circ P(d_k))_{\tilde{\mathcal{C}}}.$$

According to equation (3.1.17), we have

$$\begin{aligned} P(Q(f_{q,k}^p)) &= \bar{\partial}_{P(E_{m-k-q})P^2(E_k)}(P(f_{q,k}^p)) + (P^2(d_k) \circ P(f_{q,k}^p))_{\tilde{\mathcal{C}}} \\ &\quad - (-1)^{|P(f)|}(P(f_{q,k}^p) \circ P(d_{m-k-q}))_{\tilde{\mathcal{C}}} \end{aligned} \quad (3.2.7)$$

The commutative diagram (3.1.20) shows that

$$J \circ P^2(d_k) = d_k \circ J.$$

Then, equation (3.2.7) yields

$$\begin{aligned} J_*P(Q(f_{q,k}^p)) &= \bar{\partial}_{P(E_{m-k-q})E_k}(J_*P(f_{q,k}^p)) + (d_k \circ J_*P(f_{q,k}^p))_{\tilde{\mathcal{C}}} \\ &\quad - (-1)^{|f|}(J_*P(f_{q,k}^p) \circ P(d_{m-k-q}))_{\tilde{\mathcal{C}}} \end{aligned}$$

which proves (3.2.6). \square

Lemma 3.2.2. *For any two elements $f, g \in \mathcal{C}(P(\mathfrak{E}), \mathfrak{E})$*

$$J_*P(g \star_\psi f) = -(-1)^{|f||g|} J_*P(f) \star_\psi J_*P(g). \quad (3.2.8)$$

Proof. Written in terms of homogeneous elements, (3.2.8) reads

$$J_*P(g_{s,l}^r \star_\psi f_{q,k}^p) = -(-1)^{(r+s)(p+q)} J_*P(f_{q,k}^p) \star_\psi J_*P(g_{s,l}^r) \quad (3.2.9)$$

where $l = k + q$. Using equations (3.1.13), (3.2.5) and the definition of (3.1.20) of J , we compute

$$\begin{aligned}
J_*P(g_{s,l}^r \star_\psi f_{q,k}^p) &= (-1)^{(m-s-l)(m+1)+\omega} (-1)^{(s+q)(m-s-l)+1} (-1)^{sp} \sigma^*(g_{s,l}^r \cdot \psi_{m-k-q} \cdot f_{q,k}^p)^\vee \\
&= (-1)^{(m-s-l)(m+1)+\omega} (-1)^{(s+q)(m-s-l)+1} (-1)^{sp} (-1)^{rp} \\
&\quad \sigma^*(f_{q,k}^p)^\vee \cdot \sigma^*(\psi_{m-k-q}^\vee) \cdot \sigma^*(g_{s,l}^r)^\vee \\
&= (-1)^{(m-s-l)(m+1)+\omega} (-1)^{(s+q)(m-s-l)+1} (-1)^{sp} (-1)^{rp} \\
&\quad (-1)^{(m-k-q)(m+1)+\omega} (-1)^{q(q+k-m)+1} (-1)^{(m-s-l)(m+1)+\omega} (-1)^{s(s+l-m)+1} \\
&\quad J_*P(f_{q,k}^p) \cdot \sigma^*(\psi_{m-k-q}^\vee) \cdot J_*P(g_{s,l}^r) \\
&= (-1)^{(r+s)(p+q)} J_*P(f_{q,k}^p) \star_\psi J_*P(g_{s,l}^r) = (-1)^{(r+s)(p+q)} (-1)^{qr} J_*P(f_{q,k}^p) \cdot \psi_l \cdot J_*P(g_{s,l}^r)
\end{aligned}$$

These expressions are in agreement with equation (3.2.9) if and only if ψ satisfies a symmetry condition of the form

$$J^*P(\psi_{m-l}) = -\psi_l \quad \Leftrightarrow \quad \sigma^*(\psi_{m-l})^\vee = (-1)^{(m+1)l+\omega} \psi_l \quad (3.2.10)$$

□

We saw in the last proof that compatibility of the orientifold projection with the algebraic structure imposes the condition (3.2.10) on ψ . From now on we assume this condition to be satisfied. Although we do not know a general existence result for a quasi-isomorphism satisfying (3.2.10), we will show that such a choice is possible in all the examples considered in this paper. We will also see that symmetry of ψ , which is determined by $\omega = 0, 1$ in (3.2.10), determines whether the orientifold projection is of type SO or Sp .

Granting such a quasi-isomorphism, it follows that the cochain space $\mathcal{C}(P(\mathfrak{E}), \mathfrak{E})$ satisfies all the conditions required for the computation of the superpotential, which is the subject of the next subsection.

3.2.2 The Superpotential

In the absence of an orientifold projection, the computation of the superpotential can be summarized as follows [129]. Suppose we are searching for formal deformations of the differential Q of the form

$$Q_{\text{def}} = Q + f_1(\phi) + f_2(\phi) + f_3(\phi) + \dots \quad (3.2.11)$$

where

$$f_1(\phi) = \phi$$

is a cochain of degree one, which represents an infinitesimal deformation of Q . The terms $f_k(\phi)$, for $k \geq 2$, are homogeneous polynomials of degree k in ϕ corresponding to higher order deformations. We want to impose the integrability condition

$$(Q_{\text{def}})^2 = 0 \quad (3.2.12)$$

order by order in ϕ . In doing so one encounters certain obstructions, which are systematically encoded in a minimal A_∞ model of the DGA $\mathcal{C}(P(\mathfrak{E}), \mathfrak{E})$. The superpotential is essentially a primitive function for the obstructions, and exists under certain cyclicity conditions.

In the orientifold model we have to solve a similar deformation problem, except that now the deformations of Q have to be invariant under the orientifold action. We will explain below that this is equivalent to the construction of a minimal L_∞ model.

Let us first consider the integrability conditions (3.2.12) in more detail in the absence of orientifolding. Suppose we are given an associative \mathbb{Z} -graded DGA (\mathcal{C}, Q, \cdot) , and let H denote the cohomology of Q . In order to construct an A_∞ structure on H we need the following data

(i) A \mathbb{Z} -graded linear subspace $\mathcal{H} \subset \mathcal{C}$ isomorphic to the cohomology of Q . In other words, \mathcal{H} is spanned in each degree by representatives of the cohomology classes of Q .

(ii) A linear map $\eta : \mathcal{C} \rightarrow \mathcal{C}[-1]$ mapping \mathcal{H} to itself such that

$$\Pi = \mathbb{I} - [Q, \eta] \quad (3.2.13)$$

is a projector $\Pi : \mathcal{C} \rightarrow \mathcal{H}$, where $[\ , \]$ is the graded commutator. Moreover, we assume that the following conditions are satisfied

$$\eta|_{\mathcal{H}} = 0 \quad \eta^2 = 0. \quad (3.2.14)$$

Using the data (i), (ii) one can develop a recursive approach to obstructions in the deformation theory of Q [129]. The integrability condition (3.2.12) yields

$$\sum_{n=1}^{\infty} [Q(f_n(\phi)) + B_{n-1}(\phi)] = 0 \quad (3.2.15)$$

where

$$B_0 = 0$$

$$B_{n-1} = \phi f_{n-1}(\phi) + f_{n-1}(\phi)\phi + \sum_{\substack{k+l=n \\ k, l \geq 2}} f_k(\phi)f_l(\phi), \quad n \geq 2$$

Using equation (3.2.13), we can rewrite equation (3.2.15) as

$$\sum_{n=1}^{\infty} [Q(f_n(\phi)) + ([Q, \eta] + \Pi)B_{n-1}(\phi)] = 0. \quad (3.2.16)$$

We claim that the integrability condition (3.2.15) can be solved recursively [129] provided that

$$\sum_{n=1}^{\infty} \Pi(B_{n-1}) = 0. \quad (3.2.17)$$

To prove this claim, note that if the condition (3.2.17) is satisfied, equation (3.2.16) becomes

$$\sum_{n=1}^{\infty} (Q(f_n(\phi)) + [Q, \eta]B_{n-1}(\phi)) = 0. \quad (3.2.18)$$

This equation can be solved by setting recursively

$$f_n(\phi) = -\eta(B_{n-1}(\phi)). \quad (3.2.19)$$

One can show that this is a solution to (3.2.19) by proving inductively that

$$Q(B_n(\phi)) = 0.$$

In conclusion, the obstructions to the integrability condition (3.2.15) are encoded in the formal series

$$\sum_{n=2}^{\infty} \Pi \left(\phi f_{n-1}(\phi) + f_{n-1}(\phi) \phi + \sum_{\substack{k+l=n \\ k,l \geq 2}} f_k(\phi) f_l(\phi) \right) \quad (3.2.20)$$

where the $f_n(\phi)$, $n \geq 1$, are determined recursively by (3.2.19).

The algebraic structure emerging from this construction is a minimal A_{∞} structure for the DGA (\mathcal{C}, Q) [143, 144]. [144] constructs an A_{∞} structure by defining the linear maps

$$\lambda_n : \mathcal{C}^{\otimes n} \rightarrow \mathcal{C}[2-n], \quad n \geq 2$$

recursively

$$\begin{aligned} \lambda_n(c_1, \dots, c_n) = & (-1)^{n-1} (\eta \lambda_{n-1}(c_1, \dots, c_{n-1})) \cdot c_n - (-1)^{n|c_1|} c_1 \cdot \eta \lambda_{n-1}(c_2, \dots, c_n) \\ & - \sum_{\substack{k+l=n \\ k,l \geq 2}} (-1)^r [\eta \lambda_k(c_1, \dots, c_k)] \cdot [\eta \lambda_l(c_{k+1}, \dots, c_n)] \end{aligned} \quad (3.2.21)$$

where $|c|$ denotes the degree of an element $c \in \mathcal{C}$, and

$$r = k + 1 + (l - 1)(|c_1| + \dots + |c_k|).$$

Now define the linear maps

$$\mathfrak{m}_n : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}[2-n], \quad n \geq 1$$

by

$$\begin{aligned} \mathfrak{m}_1 &= \eta \\ \mathfrak{m}_n &= \Pi \lambda_n. \end{aligned} \quad (3.2.22)$$

The products (3.2.22) define an A_∞ structure on $\mathcal{H} \simeq H$. If the conditions (3.2.14) are satisfied, this A_∞ structure is a minimal model for the DGA (\mathcal{C}, Q, \cdot) . The products \mathbf{m}_n , $n \geq 2$ agree up to sign with the obstructions $\Pi(B_n)$ found above.

The products \mathbf{m}_n determine the local equations of the D-brane moduli space, which in physics language are called F-term equations. If

$$\phi = \sum_{i=1}^{\dim(\mathcal{H})} \phi^i u_i$$

is an arbitrary cohomology element written in terms of some generators $\{u_i\}$, the F-term equations are

$$\sum_{n=2}^{\infty} (-1)^{n(n+1)/2} \mathbf{m}_n(\phi^{\otimes n}) = 0. \quad (3.2.23)$$

If the products are cyclic, these equations admit a primitive

$$W = \sum_{n=2}^{\infty} \frac{(-1)^{n(n+1)/2}}{n+1} \langle \phi, \mathbf{m}_n(\phi^{\otimes n}) \rangle \quad (3.2.24)$$

where

$$\langle \ , \ \rangle : \mathcal{C} \rightarrow \mathbb{C}$$

is a bilinear form on \mathcal{C} compatible with the DGA structure. The cyclicity property reads

$$\langle c_1, \mathbf{m}_n(c_2, \dots, c_{n+1}) \rangle = (-1)^{n|c_2|+1} \langle c_2, \mathbf{m}_n(c_3, \dots, c_{n+1}, c_1) \rangle.$$

Let us now examine the above deformation problem in the presence of an orientifold projection. Suppose we have an involution $\tau: \mathcal{C} \rightarrow \mathcal{C}$ such that the following conditions are satisfied

$$\begin{aligned} \tau(Q(f)) &= Q(\tau(f)) \\ \tau(fg) &= -(-1)^{|f||g|} \tau(g) \tau(f) \end{aligned} \quad (3.2.25)$$

As explained below equation (3.2.12), in this case we would like to study deformations

$$Q_{def} = Q + f_1(\phi) + f_2(\phi) + \dots$$

of Q such that

$$\tau(f_n(\phi)) = f_n(\phi) \quad (3.2.26)$$

for all $n \geq 1$.

In order to set this problem in the proper algebraic context, note that the DG algebra \mathcal{C} decomposes into a direct sum of τ -invariant and anti-invariant parts

$$\mathcal{C} \simeq \mathcal{C}^+ \oplus \mathcal{C}^-. \quad (3.2.27)$$

There is a similar decomposition

$$H = H^+ \oplus H^- \quad (3.2.28)$$

for the Q -cohomology.

Conditions (3.2.25) imply that Q preserves \mathcal{C}^\pm , but the associative algebra product is not compatible with the decomposition (3.2.27). There is however another algebraic structure which is preserved by τ , namely the graded commutator

$$[f, g] = fg - (-1)^{|f||g|} gf. \quad (3.2.29)$$

This follows immediately from the second equation in (3.2.25). The graded commutator (3.2.29) defines a differential graded Lie algebra structure on \mathcal{C} . By restriction, it also defines a DG Lie algebra structure on the invariant part \mathcal{C}^+ . In this context our problem reduces to the deformation theory of the restriction $Q^+ = Q|_{\mathcal{C}^+}$ as a differential operator on \mathcal{C}^+ .

Fortunately, this problem can be treated by analogy with the previous case, except that we have to replace A_∞ structures by L_∞ structures, see for example [128, 129, 145]. In particular, the obstructions to the deformations of Q^+ can be systematically encoded in a minimal L_∞ model, and one can similarly define a superpotential if certain cyclicity conditions are satisfied.

Note that any associative DG algebra can be naturally endowed with a DG Lie algebra structure using the graded commutator (3.2.29). In this case, the A_∞ and the L_∞ approach to the deformation of Q are equivalent [128] and they yield the same superpotential. However, the L_∞ approach is compatible with the involution, while the A_∞ approach is not.

To summarize this discussion, we have a DG Lie algebra on \mathcal{C} which induces a DG Lie algebra of Q . The construction of a minimal L_∞ model for \mathcal{C} requires the same data (i), (ii) as in the case of a minimal A_∞ model, and yields the same F-term equations, and the same superpotential. In order to determine the F-term equations and superpotential for the invariant part \mathcal{C}^+ we need again a set of data (i), (ii) as described above (3.2.13). This data can be naturally obtained by restriction from \mathcal{C} provided that the propagator η in equation (3.2.13) can be chosen compatible with the involution τ i.e

$$\tau(\eta(f)) = \eta(\tau(f)).$$

This condition is easily satisfied in geometric situations, hence we will assume that this is the case from now on. Then the propagator $\eta^+ : \mathcal{C}^+ \rightarrow \mathcal{C}^+[-1]$ is obtained by restricting η to the invariant part $\eta^+ = \eta|_{\mathcal{C}^+}$. Given this data, we construct a minimal L_∞ model for the DGL algebra \mathcal{C}^+ , which yields F-term equations and, if the cyclicity condition is satisfied, a superpotential W^+ .

Theorem 3.2.3. *The superpotential W^+ obtained by constructing the minimal L_∞ model for the DGL \mathcal{C}^+ is equal to the restriction of the superpotential W corresponding to \mathcal{C} evaluated on τ -invariant field configurations:*

$$W^+ = W|_{H^+}. \quad (3.2.30)$$

In the remaining part of this section we will give a formal argument for this claim. According to [145], the data (i), (ii) above equation (3.2.13) also determines an L_∞ structure on \mathcal{H} as follows. First we construct a series of linear maps

$$\rho_n: \mathcal{C}^{\otimes n} \rightarrow \mathcal{C}[2-n], \quad n \geq 2$$

by anti-symmetrizing (in graded sense) the maps (3.2.21). That is the recursion relation becomes

$$\begin{aligned} \rho_n(c_1, \dots, c_n) &= \sum_{\sigma \in Sh(n-1, 1)} (-1)^{n-1+|\sigma|} e(\sigma) [\eta \rho_{n-1}(c_{\sigma(1)}, \dots, c_{\sigma(n-1)}), c_{\sigma(n)}] \\ &\quad - \sum_{\sigma \in Sh(1, n)} (-1)^{n|c_1|+|\sigma|} e(\sigma) [c_1, \eta \rho_{n-1}(c_{\sigma(2)})] \\ &\quad - \sum_{\substack{k+l=n \\ k, l \geq 2}} \sum_{\sigma \in Sh(k, n)} (-1)^{r+|\sigma|} e(\sigma) [\eta \rho_k(c_{\sigma(1)}, \dots, c_{\sigma(k)}), \eta \rho_l(c_{\sigma(k+1)}, \dots, c_{\sigma(n)})] \end{aligned} \quad (3.2.31)$$

where $Sh(k, n)$ is the set of all permutations $\sigma \in S_n$ such that

$$\sigma(1) < \dots < \sigma(k) \quad \text{and} \quad \sigma(k+1) < \dots < \sigma(n)$$

and $|\sigma|$ is the signature of a permutation $\sigma \in S_n$. The symbol $e(\sigma)$ denotes the Koszul sign defined by

$$c_{\sigma(1)} \wedge \dots \wedge c_{\sigma(n)} = (-1)^{|\sigma|} e(\sigma) c_1 \wedge \dots \wedge c_n.$$

Then we define the L_∞ products

$$\mathfrak{l}_n: \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}$$

by

$$\mathfrak{l}_1 = \eta, \quad \mathfrak{l}_n = \Pi \rho_n. \quad (3.2.32)$$

One can show that these products satisfy a series of higher Jacobi identities analogous to the defining associativity conditions of A_∞ structures. If the conditions (3.2.14) is also satisfied, the resulting L_∞ structure is a minimal model for the DGL \mathcal{C} .

Finally, note that the A_∞ products (3.2.22) and the L_∞ products (3.2.32) are related by

$$\mathfrak{l}_n(c_1, \dots, c_n) = \sum_{\sigma \in S_n} (-1)^{|\sigma|} e(\sigma) \mathfrak{m}_n(c_{\sigma(1)}, \dots, c_{\sigma(n)}). \quad (3.2.33)$$

In particular, one can rewrite the F-term equations (3.2.23) and the superpotential (3.2.24) in terms of L_∞ products [128, 129].

The construction of the minimal L_∞ model of the invariant part \mathcal{C}^+ is analogous. Since we are working under assumption that the propagator η^+ is the restriction of η to \mathcal{C}^+ , it is clear that the linear maps $\rho_n^+(c_1, \dots, c_n)$ are also equal to the restriction $\rho_n|_{(\mathcal{C}^+)^n}$. The same will be true for the products \mathfrak{l}_n^+ , i.e.

$$\mathfrak{l}_n^+ = \mathfrak{l}_n|_{(H^+)^n}.$$

Therefore the F-term equations and the superpotential in the orientifold model can be obtained indeed by restriction to the invariant part.

Now that we have the general machinery at hand, we can turn to concrete examples of superpotential computations.

3.3 Computations for Obstructed Curves

In this section we perform detailed computations of the superpotential for D-branes wrapping holomorphic curves in Calabi-Yau orientifolds.

So far we have relied on the Dolbeault cochain model, which serves as a good conceptual framework for our constructions. However, a Čech cochain model is clearly preferred for computational purposes [10]. The simple prescription found above for the orientifold superpotential allows us to switch from the Dolbeault to the Čech model with little effort. Using the same definition for the action of the orientifold projection P on locally free complexes \mathfrak{E} , we will adopt a cochain model of the form

$$\mathcal{C}(P(\mathfrak{E}), \mathfrak{E}) = \check{C}(\mathfrak{U}, \mathcal{H}om_X(P(\mathfrak{E}), \mathfrak{E})) \quad (3.3.1)$$

where \mathfrak{U} is a fine open cover of X . The differential Q is given by

$$Q(f) = \delta(f) + (-1)^{c(f)} \mathfrak{d}(f) \quad (3.3.2)$$

where δ is the Čech differential, \mathfrak{d} is the differential of the local Hom complex and $c(f)$ is the Čech degree of f .

In order to obtain a well-defined involution on the complex (3.3.1), we have to choose the open cover \mathfrak{U} so that the holomorphic involution $\sigma: X \rightarrow X$ maps any open set $U \in \mathfrak{U}$ isomorphically to another open set $U_{s(\alpha)} \in \mathfrak{U}$, where s is an involution on the set of indices $\{\alpha\}$. Moreover, the holomorphic involution should also be compatible with intersections. That is, if $U_\alpha, U_\beta \in \mathfrak{U}$ are mapped to $U_{s(\alpha)}, U_{s(\beta)} \in \mathfrak{U}$ then $U_{\alpha\beta}$ should be mapped isomorphically to

$U_{s(\alpha)s(\beta)}$. Analogous properties should hold for arbitrary multiple intersections. Granting such a choice of a fine open cover, we have a natural involution J_*P acting on the cochain complex (3.3.1), defined as in (3.2.4).

According to the prescription derived in the previous section, the orientifold superpotential can be obtained by applying the computational scheme of [10] to invariant Q -cohomology representatives. Since the computation depends only on the infinitesimal neighborhood of the curve, it suffices to consider local Calabi-Yau models as in [10]. We will consider two representative cases, namely obstructed $(0, -2)$ curves and local conifolds, i.e., $(-1, -1)$ curves.

3.3.1 Obstructed $(0, -2)$ Curves in $O5$ Models

In this case, the local Calabi-Yau X can be covered by two coordinate patches (x, y_1, y_2) , (w, z_1, z_2) with transition functions

$$\begin{aligned} w &= x^{-1} \\ z_1 &= x^2 y_1 + x y_2^n \\ z_2 &= y_2. \end{aligned} \tag{3.3.3}$$

The $(0, -2)$ curve is given by the equations

$$C: \quad y_1 = y_2 = 0 \quad \text{resp.} \quad z_1 = z_2 = 0 \tag{3.3.4}$$

in the two patches. The holomorphic involution acts as

$$\begin{aligned} (x, y_1, y_2) &\mapsto (x, -y_1, -y_2) \\ (w, z_1, z_2) &\mapsto (w, -z_1, -z_2) \end{aligned} \tag{3.3.5}$$

This is compatible with the transition functions if and only if n is odd. We will assume that this is the case from now on. Using (3.1.31), the Chan-Paton bundles

$$V_N = \mathcal{O}_C(-1)^{\oplus N}. \tag{3.3.6}$$

define invariant D-brane configurations under the orientifold projection.

The on-shell open string states are in one-to-one correspondence with elements of the global Ext group $\text{Ext}^1(i_* V_N, i_* V_N)$. Given two bundles V, W supported on a curve $i: C \hookrightarrow X$, there is a spectral sequence [126]

$$E_2^{p,q} = H^p(C, V^\vee \otimes W \otimes \Lambda^q N_{C/X}) \Rightarrow \text{Ext}_X^{p+q}(i_* V, i_* W) \tag{3.3.7}$$

which degenerates at E_2 . This yields

$$\text{Ext}^1(i_* \mathcal{O}_C(-1), i_* \mathcal{O}_C(-1)) \simeq H^0(C, N_{C/X}) = \mathbb{C},$$

since $N_{C/X} \simeq \mathcal{O}_C \oplus \mathcal{O}_C(-2)$. Therefore a D5-brane with multiplicity $N = 1$ has a single normal deformation. For higher multiplicity, the normal deformations will be parameterized by an $(N \times N)$ complex matrix.

In order to apply the computational algorithm developed in section 3.2 we have to find a locally free resolution \mathfrak{E} of $i_*\mathcal{O}_C(-1)$ and an explicit generator of

$$\mathrm{Ext}^1(i_*\mathcal{O}_C(-1), i_*\mathcal{O}_C(-1)) \simeq \mathrm{Ext}^1(P(\mathfrak{E}), \mathfrak{E})$$

in the cochain space $\check{C}(\mathfrak{U}, \mathcal{H}om(P(\mathfrak{E}), \mathfrak{E}))$. We take \mathfrak{E} to be the locally free resolution from [10] multiplied by $\mathcal{O}_C(-1)$, i.e.,

$$\begin{array}{ccccccc} & & \mathcal{O}(-1) & & & \mathcal{O} & \\ & & \oplus & & & \oplus & \\ 0 \longrightarrow \mathcal{O}(-1) & \xrightarrow{\begin{pmatrix} y_2 \\ -1 \\ x \end{pmatrix}} & \mathcal{O} & \xrightarrow{\begin{pmatrix} 1 & y_2 & 0 \\ -x & 0 & y_2 \\ -y_2^{n-1} & -s & -y_1 \end{pmatrix}} & \mathcal{O} & \xrightarrow{(s \ y_1 \ y_2)} & \mathcal{O}(-1) \\ & & \oplus & & & \oplus & \\ & & \mathcal{O} & & & \mathcal{O}(-1) & \end{array} \quad (3.3.8)$$

The quasi-isomorphism $\psi: \mathfrak{E} \rightarrow P(\mathfrak{E})$ is given by

$$\begin{array}{ccccc} \mathcal{O}(-1) & \xrightarrow{\begin{pmatrix} 1 & y_2 & 0 \\ -x & 0 & y_2 \\ -y_2^{n-1} & -s & -y_1 \end{pmatrix}} & \mathcal{O}^{\oplus 2} & \xrightarrow{(s \ y_1 \ y_2)} & \mathcal{O}(-1) \\ \oplus & & \oplus & & \downarrow \begin{pmatrix} 0 \\ x \\ 1 \end{pmatrix} \\ \mathcal{O}^{\oplus 2} & & \mathcal{O}(-1) & & \mathcal{O}(1) \\ \downarrow \begin{pmatrix} 0 & x & 1 \end{pmatrix} & & \downarrow \begin{pmatrix} 0 & y_2^{n-1} & -x \\ -y_2^{n-1} & 0 & -1 \\ x & 1 & 0 \end{pmatrix} & & \\ \mathcal{O}(1) & \xrightarrow{\begin{pmatrix} s \\ y_1 \\ y_2 \end{pmatrix}} & \mathcal{O}^{\oplus 2} & \xrightarrow{\begin{pmatrix} 1 & -x & -y_2^{n-1} \\ -y_2 & 0 & s \\ 0 & -y_2 & y_1 \end{pmatrix}} & \mathcal{O}(1) \\ & & \oplus & & \oplus \\ & & \mathcal{O}(1) & & \mathcal{O}^{\oplus 2} \end{array} \quad (3.3.9)$$

Note that ψ satisfies the symmetry condition (3.2.10) with $\omega = 0$, which in this case reduces to

$$\sigma^*(\psi_{2-l})^\vee = (-1)^l \psi_l. \quad (3.3.10)$$

We are searching for a generator $\mathfrak{c} \in \check{C}(\mathfrak{U}, \mathcal{H}om(P(\mathfrak{E}), \mathfrak{E}))$ of the form $\mathfrak{c} = \mathfrak{c}^{1,0} + \mathfrak{c}^{0,1}$ for two homogenous elements

$$\mathfrak{c}^{p,1-p} \in \check{C}^p(\mathfrak{U}, \mathcal{H}om^{1-p}(P(\mathfrak{E}), \mathfrak{E})), \quad p = 0, 1.$$

The cocycle condition $Q(c) = 0$ is equivalent to

$$\begin{aligned} \mathfrak{d}c^{0,1} &= \delta c^{1,0} = 0 \\ Q(c^{0,1} + c^{1,0}) &= \delta c^{0,1} - \mathfrak{d}c^{1,0} = 0 \end{aligned} \quad (3.3.11)$$

A solution to these equations is given by

$$\begin{array}{ccccc} & & \mathcal{O}^{\oplus 2} & & \mathcal{O}(1) \\ & & \oplus & \longrightarrow & \oplus \\ \mathcal{O}(1) & \longrightarrow & \mathcal{O}(1) & & \mathcal{O}^{\oplus 2} \\ \downarrow \left(\begin{smallmatrix} x^{-1} & & \\ & 0 & \\ & & 0 \end{smallmatrix} \right)_{01} & & \downarrow \left(\begin{smallmatrix} 0 & 0 & & 0 \\ 0 & 0 & & 0 \\ 0 & 0 & -x^{-1} & y_2^{n-2} \end{smallmatrix} \right)_{01} & & \downarrow (x^{-1} \ 0 \ 0)_{01} \\ \mathcal{O}(-1) & & \mathcal{O}^{\oplus 2} & & \mathcal{O}(-1) \\ \oplus & \longrightarrow & \oplus & \longrightarrow & \mathcal{O}(-1) \\ \mathcal{O}^{\oplus 2} & & \mathcal{O}(-1) & & \end{array} \quad (3.3.12)$$

$$\begin{array}{ccc} & \mathcal{O}^{\oplus 2} & \\ & \oplus & \\ \mathcal{O}(1) & \longrightarrow & \mathcal{O}(1) \\ \downarrow \left(\begin{smallmatrix} 0 \\ -1 \\ 0 \end{smallmatrix} \right)_0 + \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \right)_1 & & \downarrow (0 \ 1 \ 0)_0 + (-1 \ 0 \ 0)_1 \\ \mathcal{O}^{\oplus 2} & & \mathcal{O}(-1) \\ \oplus & \longrightarrow & \mathcal{O}(-1) \\ \mathcal{O}(-1) & & \end{array} \quad c^{0,1} :=$$

These satisfy the symmetry conditions

$$J_* P(c^{p,1-p}) = -(-1)^\omega c^{p,1-p}, \quad p = 0, 1, \quad (3.3.13)$$

For multiplicity $N > 1$, we have the locally free resolution $\mathfrak{E}_N = \mathfrak{E} \otimes \mathbb{C}^N$. The quasi-isomorphism $\psi_N: \mathfrak{E}_N \rightarrow P(\mathfrak{E}_N)$ is of the form $\psi_N = \psi \otimes M$, where $M \in \mathcal{M}_N(\mathbb{C})$ is an $N \times N$ complex matrix. Note that ψ_N induces the isomorphism (3.1.32) in cohomology. Moreover, we have

$$\sigma^*(\psi_{N,m-l})^\vee = (-1)^{l+\omega} \psi_{N,l}.$$

Referring back to (3.3.10), we see that this last equation constrains the matrix M :

$$\omega = \begin{cases} 0, & \text{if } M = M^{tr} \\ 1, & \text{if } M = -M^{tr} \end{cases} \quad (3.3.14)$$

The first case corresponds to an $SO(N)$ gauge group, while the second case corresponds to $Sp(N/2)$ (N even). This confirms the correlation between the symmetry of ψ_N and the SO/Sp projection, as we alluded to after (3.2.10).

The infinitesimal deformations of the D-brane are now parameterized by a matrix valued field

$$\phi = \mathbb{C}(\mathfrak{c}^{1,0} + \mathfrak{c}^{0,1})$$

where $\mathbb{C} \in \mathcal{M}_N(\mathbb{C})$ is the $N \times N$ Chan-Paton matrix. Taking (3.3.13) into account, invariance under the orientifold projection yields the following condition on \mathbb{C}

$$\mathbb{C} = -(-1)^\omega \mathbb{C}^{tr}. \quad (3.3.15)$$

For $\omega = 1$, this condition does not look like the usual one defining the Lie algebra of $Sp(N/2)$ because we are working in a non-usual basis of fields, namely $\mathcal{C}(P(\mathfrak{E}_N), \mathfrak{E}_N)$. By composing with the quasi-isomorphism ψ_N , we find the Chan-Paton matrix in $\mathcal{C}(P(\mathfrak{E}_N), P(\mathfrak{E}_N))$ to be $M\mathbb{C}$. By performing a change of basis in the space of Chan-Paton indices, we can choose M to be

$$M = \begin{cases} \mathbf{1}, & \text{if } \omega = 0 \\ i \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, & \text{if } \omega = 1 \end{cases}$$

and so the Chan-Paton matrices satisfy the well-known conditions [141]

$$\begin{aligned} (M\mathbb{C})^{tr} &= -(M\mathbb{C}), & \text{for } \omega = 0, \\ (M\mathbb{C})^{tr} &= -M(M\mathbb{C})M, & \text{for } \omega = 1. \end{aligned}$$

The superpotential is determined by the A_∞ products (3.2.22) evaluated on ϕ . According to Theorem 3.2.3, the final result is obtained by the superpotential of the underlying unprojected theory evaluated on invariant field configurations. Therefore the computations are identical in both cases ($\omega = 0, 1$) and the superpotential is essentially determined by the A_∞ products of a single D-brane with multiplicity $N = 1$.

Proceeding by analogy with [10], let us define the cocycles

$$\mathbf{a}_p \in \check{C}^1(\mathfrak{U}, \mathcal{H}om^0(P(\mathfrak{E}), \mathfrak{E})) \quad \mathbf{b}_p \in \check{C}^1(\mathfrak{U}, \mathcal{H}om^1(P(\mathfrak{E}), \mathfrak{E}))$$

as follows

$$\begin{array}{ccccc}
 & & \mathcal{O}^{\oplus 2} & & \mathcal{O}(1) \\
 & & \oplus & \longrightarrow & \oplus \\
 \mathcal{O}(1) & \longrightarrow & \mathcal{O}(1) & & \mathcal{O}^{\oplus 2} \\
 \downarrow (0)_{01} & & \downarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -x^{-1}y_2^p \end{pmatrix}_{01} & & \downarrow (0)_{01} \\
 \mathcal{O}(-1) & & \mathcal{O}^{\oplus 2} & & \mathcal{O}(-1) \\
 \oplus & \longrightarrow & \oplus & \longrightarrow & \mathcal{O}(-1) \\
 \mathcal{O}^{\oplus 2} & & \mathcal{O}(-1) & &
 \end{array} \tag{3.3.16}$$

$$\begin{array}{ccc}
 & \mathcal{O}^{\oplus 2} & \\
 \mathcal{O}(1) & \longrightarrow & \oplus \\
 \downarrow \begin{pmatrix} 0 & 0 \\ x^{-1} & y_2^p \end{pmatrix}_{01} & & \downarrow \begin{pmatrix} 0 & -x^{-1}y_2^p & 0 \end{pmatrix}_{01} \\
 \mathcal{O}^{\oplus 2} & & \mathcal{O}(1) \\
 \oplus & \longrightarrow & \mathcal{O}(-1) \\
 \mathcal{O}(-1) & &
 \end{array}$$

One shows by direct computation that they satisfy the relations

$$\begin{aligned}
 \mathbf{b}_p &= Q(\mathbf{a}_{p-1}) \\
 \mathbf{b}_p &= \mathbf{c} \star_{\psi} \mathbf{a}_p + \mathbf{a}_p \star_{\psi} \mathbf{c}
 \end{aligned} \tag{3.3.17}$$

Moreover, we have

$$\begin{aligned}
 \mathbf{c} \star_{\psi} \mathbf{c} &= \mathbf{b}_{n-2} \\
 \mathbf{b}_p \star_{\psi} \mathbf{b}_p &= 0
 \end{aligned} \tag{3.3.18}$$

for any p . Therefore the computation of the A_{∞} products is identical to [10]. We find only one non-trivial product

$$\mathbf{m}_n(\mathbf{c}, \dots, \mathbf{c}) = -(-1)^{\frac{n(n-1)}{2}} \mathbf{b}_0. \tag{3.3.19}$$

If we further compose with \mathbf{c} we obtain

$$\mathbf{c} \star_{\psi} \mathbf{b}_0 := \begin{array}{c} \mathcal{O}(1) \\ \downarrow (x^{-1})_{01} \\ \mathcal{O}(-1) \end{array}$$

which is the generator of $\text{Ext}^3(i_*\mathcal{O}_C(-1), i_*\mathcal{O}_C(-1))$. Therefore we obtain a superpotential of the form

$$W = -(-1)^{\frac{n(n-1)}{2}} \mathbb{C}^{n+1}$$

where \mathbb{C} satisfies the invariance condition (3.3.15).

3.3.2 Local Conifold $O3/O7$ Models

In this case, the local Calabi-Yau threefold X is isomorphic to the crepant resolution of a conifold singularity, i.e., the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$. X can be covered with two coordinate patches (x, y_1, y_2) , (w, z_1, z_2) with transition functions

$$\begin{aligned} w &= x^{-1} \\ z_1 &= xy_1 \\ z_2 &= xy_2. \end{aligned} \tag{3.3.20}$$

The $(-1, -1)$ curve C is given by

$$x = y_1 = y_2 = 0 \quad w = z_1 = z_2 = 0 \tag{3.3.21}$$

and the holomorphic involution takes

$$\begin{aligned} (x, y_1, y_2) &\mapsto (-x, -y_1, -y_2) \\ (w, z_1, z_2) &\mapsto (-w, z_1, z_2). \end{aligned} \tag{3.3.22}$$

In this case we have an $O3$ plane at

$$x = y_1 = y_2 = 0$$

and a noncompact $O7$ plane at $w = 0$. The invariant D5-brane configurations are of the form $\mathcal{E}_n^{\oplus N}$, where

$$\mathcal{E}_n = i_*\mathcal{O}_C(-1+n) \oplus i_*(\sigma^*\mathcal{O}_C(-1-n))[1], \quad n \geq 1. \tag{3.3.23}$$

We have a global Koszul resolution of the structure sheaf \mathcal{O}_C

$$0 \longrightarrow \mathcal{O}(2) \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}} \mathcal{O}(1)^{\oplus 2} \xrightarrow{\begin{pmatrix} y_1 & y_2 \end{pmatrix}} \mathcal{O} \longrightarrow 0 \tag{3.3.24}$$

Therefore the locally free resolution of \mathcal{E}_n is a complex \mathfrak{E}_n of the form

$$\begin{array}{ccccccc} \sigma^*\mathcal{O}(1-n) & \xrightarrow{\begin{pmatrix} 0 \\ y_1 \\ y_2 \end{pmatrix}} & \mathcal{O}(1+n) & \xrightarrow{\begin{pmatrix} -y_2 & 0 & 0 \\ y_1 & 0 & 0 \\ 0 & y_2 & -y_1 \end{pmatrix}} & \mathcal{O}(n)^{\oplus 2} & \xrightarrow{\begin{pmatrix} y_1 & y_2 & 0 \end{pmatrix}} & \mathcal{O}(-1+n) \\ & & \oplus & & \oplus & & \\ & & \sigma^*\mathcal{O}(-n)^{\oplus 2} & & \sigma^*\mathcal{O}(-1-n) & & \end{array} \tag{3.3.25}$$

in which the last term to the right has degree 0, and the last term to the left has degree -3 .

The quasi-isomorphism $\psi: \mathfrak{E}_n \rightarrow P(\mathfrak{E}_n)$ is given by

$$\begin{array}{ccccccc}
 \sigma^* \mathcal{O}(1-n) & \xrightarrow{\begin{pmatrix} 0 \\ y_1 \\ y_2 \end{pmatrix}} & \mathcal{O}(1+n) & \xrightarrow{\begin{pmatrix} -y_2 & 0 & 0 \\ y_1 & 0 & 0 \\ 0 & y_2 & -y_1 \end{pmatrix}} & \mathcal{O}(n)^{\oplus 2} & \xrightarrow{(y_1 \ y_2 \ 0)} & \mathcal{O}(-1+n) \\
 \downarrow 1 & & \oplus & & \oplus & & \downarrow 1 \\
 & & \sigma^* \mathcal{O}(-n)^{\oplus 2} & & \sigma^* \mathcal{O}(-1-n) & & \\
 & & \downarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & & \\
 \sigma^* \mathcal{O}(1-n) & \xrightarrow{\begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix}} & \mathcal{O}(1+n) & \xrightarrow{\begin{pmatrix} y_2 & -y_1 & 0 \\ 0 & 0 & -y_2 \\ 0 & 0 & y_1 \end{pmatrix}} & \mathcal{O}(n)^{\oplus 2} & \xrightarrow{(0 \ y_1 \ y_2)} & \mathcal{O}(-1+n) \\
 & & \oplus & & \oplus & & \\
 & & \sigma^* \mathcal{O}(-n)^{\oplus 2} & & \sigma^* \mathcal{O}(-1-n) & &
 \end{array}
 \tag{3.3.26}$$

and satisfies $\sigma^*(\psi_{3-l})^\vee = \psi_l$, that is, the symmetry condition (3.2.10) with $\omega = 0$. The on-shell open string states $\text{Ext}_X^1(\mathfrak{E}_n, \mathfrak{E}_n)$ are computed by the spectral sequence (3.3.7):

$$\begin{aligned}
 \text{Ext}_X^1(\mathcal{O}_C(-1+n), \mathcal{O}_C(-1+n)) &= 0 \\
 \text{Ext}_X^1(\sigma^* \mathcal{O}_C(-1-n)[1], \sigma^* \mathcal{O}_C(-1-n)[1]) &= 0 \\
 \text{Ext}_X^1(\mathcal{O}_C(-1+n), \sigma^* \mathcal{O}_C(-1-n)[1]) &= \mathbb{C}^{4n} \\
 \text{Ext}_X^1(\sigma^* \mathcal{O}_C(-1-n)[1], \mathcal{O}_C(-1+n)) &= \mathbb{C}^{2n+1},
 \end{aligned}
 \tag{3.3.27}$$

where in the last two lines we have used the condition $n \geq 1$.

To compute the superpotential, we work with the cochain model $\check{C}(\mathfrak{U}, \mathcal{H}om(P(\mathfrak{E}_n), \mathfrak{E}_n))$. The direct sum of the above Ext groups represents the degree 1 cohomology of this complex with respect to the differential (3.3.2). The first step is to find explicit representatives for all degree 1 cohomology classes with well defined transformation properties under the orientifold projection. We list all generators below on a case by case basis.

$$a) \text{ Ext}^1(\sigma^* \mathcal{O}_C(-1-n)[1], \mathcal{O}_C(-1+n))$$

We have $2n+1$ generators $\mathbf{a}_i \in \check{C}^0(\mathfrak{U}, \mathcal{H}om^1(P(\mathfrak{E}_n), \mathfrak{E}_n))$, $i = 0, \dots, 2n$, given by

$$\mathbf{a}_i := x^i \mathbf{a}, \tag{3.3.28}$$

where

$$\begin{array}{ccccc}
 & & \sigma^* \mathcal{O}(-n)^{\oplus 2} & & \sigma^* \mathcal{O}(-1-n) \\
 & & \oplus & \longrightarrow & \oplus \\
 \sigma^* \mathcal{O}(1-n) & \longrightarrow & \mathcal{O}(1+n) & & \mathcal{O}(n)^{\oplus 2} \\
 \downarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & & \downarrow (1 \ 0 \ 0) \\
 \mathcal{O}(1+n) & & \mathcal{O}(n)^{\oplus 2} & & \\
 \oplus & \longrightarrow & \oplus & \longrightarrow & \mathcal{O}(-1+n) \\
 \sigma^* \mathcal{O}(-n)^{\oplus 2} & & \sigma^* \mathcal{O}(-1-n) & &
 \end{array} \tag{3.3.29}$$

Note that we have written down the expressions of the generators only in the U_0 patch.⁵ The transformation properties under the orientifold projection are

$$J_* P(\mathbf{a}_i) = -(-1)^{i+\omega} \mathbf{a}_i, \quad 0 \leq i \leq 2n. \tag{3.3.30}$$

$$b) \text{ Ext}^1(\mathcal{O}_C(-1+n), \sigma^* \mathcal{O}_C(-1-n)[1])$$

We have $4n$ generators $\mathbf{b}_i, \mathbf{c}_i \in \check{C}^1(\mathfrak{U}, \mathcal{H}om^0(P(\mathfrak{F}_n), \mathfrak{F}_n))$, $i = 1, \dots, 2n$ given by

$$\mathbf{b}_i := x^{-i} \mathbf{b}, \quad \mathbf{c}_i := x^{-i} \mathbf{c} \tag{3.3.31}$$

where

$$\begin{array}{ccc}
 \sigma^* \mathcal{O}(-n)^{\oplus 2} & & \sigma^* \mathcal{O}(-1-n) \\
 \oplus & \longrightarrow & \oplus \\
 \mathcal{O}(1+n) & & \mathcal{O}(n)^{\oplus 2} \\
 \downarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}_{01} & & \downarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}_{01} \\
 \mathcal{O}(1+n) & & \mathcal{O}(n)^{\oplus 2} \\
 \oplus & \longrightarrow & \oplus \\
 \sigma^* \mathcal{O}(-n)^{\oplus 2} & & \sigma^* \mathcal{O}(-1-n)
 \end{array} \tag{3.3.32}$$

⁵The expressions in the U_1 patch can be obtained using the transition functions (3.3.20) since the \mathbf{a}_i are Čech closed. They will not be needed in the computation.

$$\begin{array}{ccc}
\sigma^* \mathcal{O}(-n)^{\oplus 2} & \xrightarrow{\quad} & \sigma^* \mathcal{O}(-1-n) \\
\oplus & & \oplus \\
\mathcal{O}(1+n) & & \mathcal{O}(n)^{\oplus 2} \\
\downarrow \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{smallmatrix} \right)_{01} & & \downarrow \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right)_{01} \\
\mathcal{O}(1+n) & & \mathcal{O}(n)^{\oplus 2} \\
\oplus & \xrightarrow{\quad} & \oplus \\
\sigma^* \mathcal{O}(-n)^{\oplus 2} & & \sigma^* \mathcal{O}(-1-n)
\end{array} \tag{3.3.33}$$

The action of the orientifold projection is

$$J_* P(\mathbf{b}_i) = (-1)^{i+\omega} \mathbf{b}_i, \quad J_* P(\mathbf{c}_i) = (-1)^{i+\omega} \mathbf{c}_i. \tag{3.3.34}$$

For multiplicity $N \geq 1$, we work as in the last subsection, taking the locally free resolution $\mathfrak{E}_{n,N} = \mathfrak{E}_n \otimes \mathbb{C}^N$, together with the quasi-isomorphism $\psi_N: \mathfrak{E}_{n,N} \rightarrow P(\mathfrak{E}_{n,N})$; $\psi_N = \psi \otimes M$. Again, M is a symmetric matrix for $\omega = 0$ and antisymmetric for $\omega = 1$. A general invariant degree one cocycle ϕ will be a linear combination

$$\phi = \sum_{i=0}^{2n} A^i \mathbf{a}_i + \sum_{i=1}^{2n} (\mathbf{B}^i \mathbf{b}_i + \mathbf{C}^i \mathbf{c}_i) \tag{3.3.35}$$

where A^i, B^i, C^i are $N \times N$ matrices satisfying

$$(\mathbf{A}^i)^{tr} = -(-1)^{i+\omega} \mathbf{A}^i \quad (\mathbf{B}^i)^{tr} = (-1)^{i+\omega} \mathbf{B}^i \quad (\mathbf{C}^i)^{tr} = (-1)^{i+\omega} \mathbf{C}^i. \tag{3.3.36}$$

In the following we will let the indices i, j, k, \dots run from 0 to $2n$ with the convention $\mathbf{B}^0 = \mathbf{C}^0 = 0$.

The multiplication table of the above generators with respect to the product (3.2.5) is

$$\begin{aligned}
\mathbf{a}_i \star_\psi \mathbf{a}_j &= \mathbf{b}_i \star_\psi \mathbf{b}_j = \mathbf{c}_i \star_\psi \mathbf{c}_j = 0 \\
\mathbf{b}_i \star_\psi \mathbf{c}_j &= \mathbf{c}_i \star_\psi \mathbf{b}_j = 0.
\end{aligned} \tag{3.3.37}$$

The remaining products are all Q -exact:

$$\begin{aligned}
\mathbf{a}_i \star_\psi \mathbf{b}_j &= Q(f_2(\mathbf{a}_i, \mathbf{b}_j)) \\
\mathbf{b}_i \star_\psi \mathbf{a}_j &= Q(f_2(\mathbf{b}_i, \mathbf{a}_j))
\end{aligned}$$

as required in (3.2.15). Let us show a sample computation.

$$\begin{array}{ccc}
 & & \sigma^* \mathcal{O}(-n)^{\oplus 2} \\
 & \nearrow & \oplus \\
 \sigma^* \mathcal{O}(1-n) & \longrightarrow & \mathcal{O}(1+n) \\
 \downarrow x^{-i+j} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}_{01} & & \downarrow x^{-i+j} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}_{01} \\
 \mathcal{O}(1+n) & & \mathcal{O}(n)^{\oplus 2} \\
 \oplus & \longrightarrow & \oplus \\
 \sigma^* \mathcal{O}(-n)^{\oplus 2} & & \sigma^* \mathcal{O}(-1-n)
 \end{array} \tag{3.3.38}$$

For $j \geq i$,

$$\begin{array}{ccc}
 & & \sigma^* \mathcal{O}(-n)^{\oplus 2} \\
 & \nearrow & \oplus \\
 \sigma^* \mathcal{O}(1-n) & \longrightarrow & \mathcal{O}(1+n) \\
 \downarrow x^{-i+j} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}_0 & & \downarrow x^{-i+j} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}_0 \\
 \mathcal{O}(1+n) & & \mathcal{O}(n)^{\oplus 2} \\
 \oplus & \longrightarrow & \oplus \\
 \sigma^* \mathcal{O}(-n)^{\oplus 2} & & \sigma^* \mathcal{O}(-1-n)
 \end{array} \tag{3.3.39}$$

For $j < i$,

$$\begin{array}{ccc}
 & & \sigma^* \mathcal{O}(-n)^{\oplus 2} \\
 & \nearrow & \oplus \\
 \sigma^* \mathcal{O}(1-n) & \longrightarrow & \mathcal{O}(1+n) \\
 \downarrow (-1)x^{-i+j+1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_1 & & \downarrow (-1)x^{-i+j+1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}_1 \\
 \mathcal{O}(1+n) & & \mathcal{O}(n)^{\oplus 2} \\
 \oplus & \longrightarrow & \oplus \\
 \sigma^* \mathcal{O}(-n)^{\oplus 2} & & \sigma^* \mathcal{O}(-1-n)
 \end{array} \tag{3.3.40}$$

$$\begin{array}{ccc}
\sigma^* \mathcal{O}(-n)^{\oplus 2} & \xrightarrow{\quad} & \sigma^* \mathcal{O}(-1-n) \\
\oplus & & \oplus \\
\mathcal{O}(1+n) & & \mathcal{O}(n)^{\oplus 2} \\
\downarrow x^{i-j} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}_{01} & & \downarrow x^{i-j} (0 \ -1 \ 0)_{01} \\
\mathcal{O}(n)^{\oplus 2} & & \mathcal{O}(-1+n) \\
\oplus & \xrightarrow{\quad} & \\
\sigma^* \mathcal{O}(-1-n) & &
\end{array} \tag{3.3.41}$$

$\mathbf{a}_i \star \mathbf{b}_j =$

For $i \geq j$,

$$\begin{array}{ccc}
\sigma^* \mathcal{O}(-n)^{\oplus 2} & \xrightarrow{\quad} & \sigma^* \mathcal{O}(-1-n) \\
\oplus & & \oplus \\
\mathcal{O}(1+n) & & \mathcal{O}(n)^{\oplus 2} \\
\downarrow x^{i-j} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}_0 & & \downarrow x^{i-j} (0 \ -1 \ 0)_0 \\
\mathcal{O}(n)^{\oplus 2} & & \mathcal{O}(-1+n) \\
\oplus & \xrightarrow{\quad} & \\
\sigma^* \mathcal{O}(-1-n) & &
\end{array} \tag{3.3.42}$$

$f_2(\mathbf{a}_i, \mathbf{b}_j) =$

For $i < j$,

$$\begin{array}{ccc}
\sigma^* \mathcal{O}(-n)^{\oplus 2} & \xrightarrow{\quad} & \sigma^* \mathcal{O}(-1-n) \\
\oplus & & \oplus \\
\mathcal{O}(1+n) & & \mathcal{O}(n)^{\oplus 2} \\
\downarrow x^{i-j+1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}_1 & & \downarrow x^{i-j+1} (0 \ 1 \ 0)_1 \\
\mathcal{O}(n)^{\oplus 2} & & \mathcal{O}(-1+n) \\
\oplus & \xrightarrow{\quad} & \\
\sigma^* \mathcal{O}(-1-n) & &
\end{array} \tag{3.3.43}$$

$f_2(\mathbf{a}_i, \mathbf{b}_j) =$

Since all pairwise products of generators are Q -exact, it follows that the obstruction $\Pi(B_1(\phi)) = \Pi(\phi \star \phi)$ vanishes. Moreover, the second order deformation $f_2(\phi)$ is given by

$$f_2(\phi) = \sum_{i,j} (\mathbf{A}^i \mathbf{B}^j f_2(\mathbf{a}_i, \mathbf{b}_j) + \mathbf{B}^i \mathbf{A}^j f_2(\mathbf{b}_i, \mathbf{a}_j) + \mathbf{A}^i \mathbf{C}^j f_2(\mathbf{a}_i, \mathbf{c}_j) + \mathbf{C}^i \mathbf{A}^j f_2(\mathbf{c}_i, \mathbf{a}_j)). \tag{3.3.44}$$

Following the recursive algorithm discussed in section 3.2 we compute the next obstruction $\Pi(\phi \star f_2(\phi) + f_2(\phi) \star \phi)$. For this, we have to compute products of the form

$$\alpha_i \star f_2(\alpha_j, \alpha_k), \quad f_2(\alpha_j, \alpha_k) \star \alpha_i.$$

Again we present a sample computation in detail. For $i \geq j$,

$$\begin{array}{ccc}
 & & \sigma^* \mathcal{O}(-n)^{\oplus 2} \\
 & & \oplus \\
 \sigma^* \mathcal{O}(1-n) & \longrightarrow & \mathcal{O}(1+n) \\
 \downarrow x^{i-j+k} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}_0 & & \downarrow x^{i-j+k} (0 \ -1 \ 0)_0 \\
 \mathcal{O}(n)^{\oplus 2} & & \mathcal{O}(-1+n) \\
 \oplus & \longrightarrow & \\
 \sigma^* \mathcal{O}(-1-n) & &
 \end{array} \tag{3.3.45}$$

$-\mathbf{a}_k \star f_2(\mathbf{b}_j, \mathbf{a}_i) =$

and, for $i < j$,

$$\begin{array}{ccc}
 & & \sigma^* \mathcal{O}(-n)^{\oplus 2} \\
 & & \oplus \\
 \sigma^* \mathcal{O}(1-n) & \longrightarrow & \mathcal{O}(1+n) \\
 \downarrow (-1)^{n+1} x^{i-j+k-2n+1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_1 & & \downarrow (-1)^{n+2} x^{i-j+k-2n+1} (0 \ 1 \ 0)_1 \\
 \mathcal{O}(n)^{\oplus 2} & & \mathcal{O}(-1+n) \\
 \oplus & \longrightarrow & \\
 \sigma^* \mathcal{O}(-1-n) & &
 \end{array} \tag{3.3.46}$$

$-\mathbf{a}_k \star f_2(\mathbf{b}_j, \mathbf{a}_i) =$

For $k \geq j$,

$$\begin{array}{ccc}
 & & \sigma^* \mathcal{O}(-n)^{\oplus 2} \\
 & & \oplus \\
 \sigma^* \mathcal{O}(1-n) & \longrightarrow & \mathcal{O}(1+n) \\
 \downarrow x^{i-j+k} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_0 & & \downarrow x^{i-j+k} (0 \ 1 \ 0)_0 \\
 \mathcal{O}(n)^{\oplus 2} & & \mathcal{O}(-1+n) \\
 \oplus & \longrightarrow & \\
 \sigma^* \mathcal{O}(-1-n) & &
 \end{array} \tag{3.3.47}$$

$-f_2(\mathbf{a}_k, \mathbf{b}_j) \star \mathbf{a}_i =$

and, for $k < j$,

$$\begin{array}{ccc}
 & & \sigma^* \mathcal{O}(-n)^{\oplus 2} \\
 & & \oplus \\
 \sigma^* \mathcal{O}(1-n) & \longrightarrow & \mathcal{O}(1+n) \\
 \downarrow (-1)^{n-1} x^{i-j+k+1-2n} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}_1 & & \downarrow (-1)^n x^{i-j+k+1-2n} (0 \ -1 \ 0)_1 \\
 \mathcal{O}(n)^{\oplus 2} & & \mathcal{O}(-1+n) \\
 \oplus & \longrightarrow & \\
 \sigma^* \mathcal{O}(-1-n) & &
 \end{array} \tag{3.3.48}$$

$-f_2(\mathbf{a}_k, \mathbf{b}_j) \star \mathbf{a}_i =$

Then the third order products are the following. For $k < j \leq i$,

$$\begin{array}{ccc}
 & & \sigma^* \mathcal{O}(-n)^{\oplus 2} \\
 & & \oplus \\
 \sigma^* \mathcal{O}(1-n) & \longrightarrow & \mathcal{O}(1+n) \\
 \downarrow x^{i-j+k} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} & & \downarrow x^{i-j+k} (0 \ -1 \ 0) \\
 \mathcal{O}(n)^{\oplus 2} & & \mathcal{O}(-1+n) \\
 \oplus & \longrightarrow & \\
 \sigma^* \mathcal{O}(-1-n) & &
 \end{array} \tag{3.3.49}$$

$\mathbf{m}_3(\mathbf{a}_k, \mathbf{b}_j, \mathbf{a}_i) =$

and, for $i < j \leq k$,

$$\begin{array}{ccc}
 & & \sigma^* \mathcal{O}(-n)^{\oplus 2} \\
 & & \oplus \\
 \sigma^* \mathcal{O}(1-n) & \longrightarrow & \mathcal{O}(1+n) \\
 \downarrow x^{i-j+k} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & & \downarrow x^{i-j+k} (0 \ 1 \ 0) \\
 \mathcal{O}(n)^{\oplus 2} & & \mathcal{O}(-1+n) \\
 \oplus & \longrightarrow & \\
 \sigma^* \mathcal{O}(-1-n) & &
 \end{array} \tag{3.3.50}$$

According to [10], the corresponding terms in the superpotential can be obtained by taking products of the form $\mathfrak{m}_2(\mathfrak{m}_3(\alpha_i, \alpha_j, \alpha_k), \alpha_l)$, which take values in $\text{Ext}^3(\mathcal{E}_n, \mathcal{E}_n)$. For $k < j \leq i$ and $i - j + k - l = -1$ we have

$$\begin{array}{ccc}
 & & \sigma^* \mathcal{O}(-n)^{\oplus 2} \\
 & & \oplus \\
 \mathfrak{m}_2(\mathfrak{m}_3(\mathbf{a}_k, \mathbf{b}_j, \mathbf{a}_i), \mathbf{c}_l) = & \mathcal{O}(1+n) & \\
 & \downarrow x^{i-j+k-l} (0 \ 0 \ 1)_{01} & \\
 & \mathcal{O}(-1+n) &
 \end{array} \tag{3.3.51}$$

The expression obtained in the right hand side of equation (3.3.51) is a generator for

$$\text{Ext}^3(\sigma^* \mathcal{O}_C(-1-n)[1], \sigma^* \mathcal{O}_C(-1-n)[1]) = \mathbb{C}. \tag{3.3.52}$$

For $i < j \leq k$ and $i - j + k - l = -1$,

$$\begin{array}{ccc}
 & & \sigma^* \mathcal{O}(-n)^{\oplus 2} \\
 & & \oplus \\
 \mathfrak{m}_2(\mathfrak{m}_3(\mathbf{a}_k, \mathbf{b}_j, \mathbf{a}_i), \mathbf{c}_l) = & \mathcal{O}(1+n) & \\
 & \downarrow x^{i-j+k-l} (0 \ 0 \ -1)_{01} & \\
 & \mathcal{O}(-1+n) &
 \end{array} \tag{3.3.53}$$

Note that the expression in the right hand side of (3.3.53) is the same generator of (3.3.52)

multiplied by (-1) . The first product (3.3.51) gives rise to superpotential terms of the form

$$\mathrm{Tr}(C^l A^k B^j A^i)$$

with

$$(i + k) - (j + l) = -1, \quad k < j \leq i.$$

The second product (3.3.53) gives rise to terms in the superpotential of the form

$$-\mathrm{Tr}(C^l A^k B^j A^i)$$

with

$$(i + k) - (j + l) = -1, \quad i < j \leq k.$$

If we consider the case $n = 1$ for simplicity, the superpotential interactions resulting from these two products are

$$\begin{aligned} W = \mathrm{Tr} & (C^1 A^0 B^1 A^1 - C^1 A^1 B^1 A^0 + C^2 A^0 B^1 A^2 - C^2 A^2 B^1 A^0 \\ & + C^1 A^0 B^2 A^2 - C^1 A^2 B^2 A^0 + C^2 A^1 B^2 A^2 - C^2 A^2 B^2 A^1). \end{aligned} \quad (3.3.54)$$

3.4 An alternative derivation

In this appendix we give an alternative derivation of Lemma 3.1.2. This approach relies on one of the most powerful results in algebraic geometry, namely Grothendieck duality. Let us start out by recalling the latter. Consider $f: X \rightarrow Y$ to be a proper morphism of smooth varieties⁶. Choose $\mathcal{F} \in D^b(X)$ and $\mathcal{G} \in D^b(Y)$ to be objects in the corresponding bounded derived categories. Then one has the following isomorphism (see, e.g., III.11.1 of [146]):

$$\mathbf{R}f_* \mathbf{R}\mathcal{H}om_X(\mathcal{F}, f^! \mathcal{G}) \cong \mathbf{R}\mathcal{H}om_Y(\mathbf{R}f_* \mathcal{F}, \mathcal{G}). \quad (3.4.1)$$

Now it is true that $f^!$ in general is a complicated functor, in particular it is *not* the total derived functor of a classical functor, i.e., a functor between the category of coherent sheaves, but in our context it will have a very simple form.

The original problem that lead to Lemma 3.1.2 was to determine the derived dual, a.k.a, Verdier dual, of a torsion sheaf. Let $i: E \rightarrow X$ be the embedding of a codimension d subvariety E into a smooth variety X , and let V be a vector bundle on E . We want to determine $\mathbf{R}\mathcal{H}om_X(i_* V, \mathcal{O}_X)$. Using (3.4.1) we have

$$\mathbf{R}\mathcal{H}om_X(i_* V, \mathcal{O}_X) \cong i_* \mathbf{R}\mathcal{H}om_E(V, i^! \mathcal{O}_X), \quad (3.4.2)$$

⁶The Grothendieck duality applies to more general schemes than varieties, but we limit ourselves to the cases considered in this paper.

where we used the fact that the higher direct images of i vanish. Furthermore, since V is locally free, we have that

$$\mathbf{R}\mathcal{H}om_E(V, i^! \mathcal{O}_X) = \mathbf{R}\mathcal{H}om_E(\mathcal{O}_E, V^\vee \otimes i^! \mathcal{O}_X) = V^\vee \otimes i^! \mathcal{O}_X, \quad (3.4.3)$$

where V^\vee is the dual of V on E , rather than on X . On the other hand, for an embedding

$$i^! \mathcal{O}_X = K_{E/X}[-d], \quad (3.4.4)$$

where $K_{E/X}$ is the relative canonical bundle. Now if we assume that the ambient space X is a Calabi-Yau variety, then $K_{E/X} = K_E$. We can summarize this

Proposition 3.4.1. *For the embedding $i: E \rightarrow X$ of a codimension d subvariety E in a smooth Calabi-Yau variety X , and a vector bundle V on E we have that*

$$\mathbf{R}\mathcal{H}om_X(i_* V, \mathcal{O}_X) \cong i_* (V^\vee \otimes K_E)[-d]. \quad (3.4.5)$$

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WORK EXPERIENCE

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