

**EXPLICIT SOLUTIONS TO A PAIR OF CONTINUOUS
TIME STOCHASTIC CONTROL PROBLEMS**

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**A dissertation submitted to the
Graduate School—New Brunswick
Rutgers, The State University of New Jersey
in partial fulfillment of the requirements**

for the degree of

Doctor of Philosophy

Graduate Program in Statistics

Written under the direction of

Larry Shepp

and approved by

New Brunswick, New Jersey

May, 2008

ABSTRACT OF THE DISSERTATION

Explicit Solutions To A Pair Of Continuous Time Stochastic Control Problems

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We study a pair of continuous time stochastic control problems, arising in financial and engineering economics respectively. We first consider the optimal consumption and investment of a utility maximizing investor without an income. The optimal consumption and investment plan is derived and a new way of obtaining closed form expressions for these quantities is provided. We then consider a simple stochastic model for optimal extraction from a groundwater aquifer that has surprising features. A result concerning optimal policies that clarifies these features is proved.

Acknowledgements

This thesis represents the completion of four years of successful graduate study at the department of Statistics, Rutgers University. I was fortunate enough to come in contact with people who both helped me and taught me a lot, some of which I would like to take the opportunity to thank at this point. I would first of all like to thank my advisor Professor Larry Shepp for introducing me to research and for his tremendous support during my time at Rutgers. I would like to thank Professor Oded Palmon, at the Department of Finance and Economics, for his valuable suggestions during my work on this thesis. I would like to thank Professor Cun-Hui Zhang and Professor John Kolassa, graduate directors in the Department of Statistics, for their help and advice during my time in the graduate program. I would like to thank Professor Richard Gundy for his help and his suggestions. Finally, I would like to thank Professor Minge Xie at the Department of Statistics, and Professor Roger Nussbaum at the Department of Mathematics, for help related to my studies.

Decemeber 2007, Piscataway, New Jersey

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Chapter 1

Introduction

1.1 Stochastic Optimal Control

Let X be a set representing the possible states of some system. Fix $x_0 \in X$ and define inductively $x_{t+1} = z_t(x_t)$ where each z_t belongs to some collection \mathcal{A} of transformations of X . We may view the points x_0, \dots, x_T as the successive states of the system at times $0, 1, \dots, T$, with the transformation z_t applied at time t chosen by an “observer”. Let us assume that the observer carries certain preferences regarding the evolution and prefers large values of

$$g(x_0, x_1, \dots, x_T, z_0, \dots, z_{T-1})$$

where g is a given function. We call $z = (z_0, \dots, z_{T-1})$ a *policy* and the problem of how to optimally select a policy a *control problem*. This description is general enough to allow many examples; economic, physical, and engineering. It is often natural to let

$$g(x_0, x_1, \dots, x_T, z_0, \dots, z_{T-1}) = \sum_{j=0}^{T-1} u(x_j, z_j)$$

where u is some “utility” function.

The problem is solved via the dynamic programming principle of optimality [1], sometimes referred to as Bellman’s principle: *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.* Now define

$v_t(x)$ = total return starting in state x at time t using an optimal policy

Bellman’s principle says,

$$v_t(x) = \max_{z_0 \in \mathcal{A}} \{u(x, z_0) + v_{t-1}(z_0(x))\} \tag{1.1}$$

A function $\widehat{v}_t(x)$ is an upper bound on $v_t(x)$ if and only if

$$m_t = \sum_{j=0}^{t-1} u(x_j, z_j) + \widehat{v}_t(x_t)$$

is non-increasing for any z . If $\widehat{v}_t(x)$ is an upper bound and m_t is constant for some z^* , then $v_t(x) = \widehat{v}_t(x)$ for all t, x and z^* is optimal. To solve the control problem we guess $\widehat{v}_t(x)$ and show that it gives m_t the stated properties.

The purpose of control theory is to provide a framework in which decision problems can be studied mathematically. It is a fact that many real world decision problems involve uncertainty in the sense that, in addition to the actions of the observer, unknown quantities determine the evolution of the system. This uncertainty may be modeled by letting \mathcal{A} be a set of stochastic transformations, each taking the system into one of its states according to a given probability distribution¹. In this context a policy is a rule for selecting z_0 and $z_t, t \geq 1$, based on the random variables $(x_0, \dots, x_t, z_0, \dots, z_{t-1})$ ².

Example 1.1 Let $X = \mathbb{N}$ represent the wealth of an individual with initial wealth $x_0 \in X$. Define inductively $x_{t+1} = z_t(x_t) = x_t + b_t x_t - c_t$ where c_t represents consumption at time t and b_0, \dots, b_{T-1} are iid with positive expectation. Set

$$g(x_0, z) = \sum_{j=0}^{T-1} u(c_j)$$

where the utility $u(c_t)$ is some increasing function of c_t . In this model wealth fluctuates randomly, with the size of the fluctuations proportional to current wealth. If $x_s = 0$ for some $0 \leq s < T$ then $x_t = 0, t = s + 1, \dots, T$ and no additional consumption can take place. The observer faces the problem of how to organize his consumption to accumulate utility without exhausting his assets. \square

¹The distributions could be allowed to depend on both time and the current state. In this introductory chapter we do not address such issues in any detail. In order to keep the notation simple and intuitive, we make no reference to a probability space and hope that the reader is not bothered by questions of how to make a precise probabilistic description.

²In other words z_t should be measurable with respect to the σ -field generated by the random variables $x_0, \dots, x_t, z_0, \dots, z_{t-1}$.

For any policy z , $g(x_0, z)$ is a random variable. It is common practice to assume³ that the observer will attempt to maximize the expected value $E[g(x_0, z)]$. The *stochastic* control problem is to find z^* such that

$$\sup_z E[g(x_0, z)] = E[g(x_0, z^*)]$$

A function $\widehat{v}_t(x)$ is a bound on

$v_t(x) :=$ expected return starting in state x at time t using an optimal policy

if and only if the stochastic process M defined by $M_0 = \widehat{v}_0(x_0)$,

$$M_t = \sum_{j=0}^{t-1} u(x_j, z_j) + \widehat{v}_t(x_t), t = 1, \dots, T \quad (1.2)$$

satisfies

$$E[M_{t+1} | x_0, \dots, x_t, z_0, \dots, z_t] \leq M_t \quad (1.3)$$

for any policy z . Indeed, if (1.3) holds for any policy then properties of (conditional) expectation gives

$$\begin{aligned} E[g(x_0, z)] &= E\left[\sum_{j=0}^{T-1} u(x_j, z_j)\right] \\ &= E[M_T] \\ &= E[E[M_T | x_0, \dots, x_t, z_0, \dots, z_t]] \\ &\leq E[M_{T-1}] \\ &\leq \dots \\ &\leq E[M_0] = \widehat{v}_0(x_0) \end{aligned}$$

so that $\widehat{v}_0(x)$ is an upper bound on $v_0(x_0)$. If also

$$E[M_{t+1} | x_0, \dots, x_t, z_0^*, \dots, z_t^*] = M_t \quad (1.4)$$

$\forall t$ for some particular z^* then (with the inequalities above replaced by equalities)

$$E[g(x_0, z)] = \widehat{v}_0(x_0) \quad (1.5)$$

³The expected utility maxim can be derived from a small set of axioms [6].

which implies $\widehat{v}_0(x_0) = v_0(x_0)$ and that z^* is optimal.

We note that solving a stochastic control problem, as we have defined it, is equivalent to showing that the process M has certain martingale properties. We show that M is a super-martingale (1.3) for each policy, with the martingale property (1.4) satisfied for z^* to prove that z^* is optimal. We take the point of view that the only way to solve stochastic control problems is to use martingales.

1.2 Stochastic Control in Continuous Time

We use stochastic differential equations to describe the continuous time behavior of a system. The state at time t will be represented by an Ito process satisfying

$$\begin{aligned} dX_t &= \zeta(t, X_t, Z_t)dt + \eta(t, X_t, Z_t)dW_t, \\ X_0 &= x_0 \end{aligned} \tag{1.6}$$

or a generalization of such a process. We assume that we can choose the value of $Z_t \in G$ from some fixed set G of real numbers, with ζ, η real valued functions⁴ defined on $D := [0, \infty) \times \mathbb{R} \times G$. The value of Z_t must be based upon what has happened up to time t , which is expressed by the requirement that as a stochastic process, Z_t is adapted to the filtration generated by the Brownian motion. Depending on the problem, additional requirements on Z_t will be introduced. Let $U : D \rightarrow [-\infty, \infty)$ be the utility rate and let $0 < T \leq +\infty$ be a constant. Our objective is to find Z^* such that

$$V(x_0) = \sup_Z E \int_0^T U(u, X_u, Z_u)du = E \int_0^T U(u, X_u, Z_u^*)du$$

One approach to this problem, that we do not take, is to derive the Hamilton-Jacobi-Bellman equation which gives necessary conditions on $V(x)$ ([12], Theorem 5.1). We will seek conditions that a bound $\widehat{V}(x)$ on $V(x)$ needs to satisfy in order to endow a suitable analogue of (1.2) with the martingale properties described in the previous section. The Ito formula will be used as a basic tool, and we will make much use of that the Ito process (1.6) with $\zeta \leq 0$ is a super-martingale and a martingale if $\zeta = 0$.

⁴We do not specify conditions on ζ, η but simply assume that solutions to (1.3) exist at this point

1.3 Overview

We will study two continuous time stochastic optimal control problems. In Chapter 2 we consider the optimal consumption and investment of a utility maximizing retired investor without a steady income. In Chapter 3 we prove a result regarding optimal policies in a model for optimal groundwater extraction from an aquifer. We do not find it unnatural that the optimization problems that we consider come from Economics, the science of managing limited resources. We focus on simple models, ones that permit explicit solutions, and the insight that can be gained from the analysis of these models.

Chapter 2

Optimal Consumption and Investment

2.1 Background

Imagine a retired investor without income, with a wealth of x invested in one safe and one risky asset, e.g stock, at time $t = 0$. His goal is to find an investment and consumption plan to maximize the expected total utility, where utility is accumulated at a rate u , depending on time t , wealth and the rate of consumption, until his random time of departure, τ .

Given the amount of investors who find themselves in similar situations, a detailed study of the problem needs little further justification and its solution under realistic assumptions is both interesting and important. We shall study the problem under certain simplifying assumptions. First, we assume that the investors preferences are described by the utility function¹

$$u(t, z_t, x(t)) = \frac{1}{1 - \gamma} z_t^{1-\gamma}$$

where z_t is the consumption rate at time t . We use the geometric Brownian motion assumption regarding the time evolution of the price of the risky asset, a standard assumption in financial modeling. We consider a simple, frictionless market model: trading expenses are not taken into account, any amount of assets can be bought and sold at any time, the actions of the investor do not affect the market and so on. We also assume that the random time of exist τ is independent of the Brownian motion driving the price process of the risky asset, taking $p(t) = P(\tau > t)$ as a exogenously given, non-random function.

¹ u is known as the constant relative risk aversion utility function. The appendix includes some basic utility theory.

We begin by studying optimal consumption in the case when a fixed fraction of wealth is invested in risky assets. This problem is formulated in Section 2, and is explicitly solved in the following two sections. We show that the optimal consumption rate is a certain time-dependent multiple of the total wealth. An explicit expression for this optimal relative rate is provided. We show that this function may also be derived using the calculus of variations. We exploit this feature to solve the optimal consumption and investment problem, which is considered in Section 4. The main result is that the optimal fraction of wealth invested in the risky asset is a constant that is independent of time, wealth, as well as survival the probabilities. A discussion of this result can be found in the last section of this chapter.

2.2 Optimal Consumption

We study optimal consumption when a fixed fraction of wealth is invested in the risky asset. The price of the risky asset is modeled by a process S_t satisfying the stochastic differential equation

$$dS_t = S_t(\xi_1 dt + \xi_2 dW_t)$$

Assuming a rate of return r on safe investments, the wealth process X satisfies

$$\begin{aligned} dX_t &= X_t(\mu dt + \sigma dW_t) - Z_t dt, \\ X_0 &= x_0 > 0 \end{aligned} \tag{2.1}$$

where Z_t is the rate of consumption at time t , for some real constants $\mu, \sigma > 0$.

2.2.1 Problem Formulation

Below a short dictionary of terms is given.

p , survival probability function, smooth on $[0, T)$

$T := \sup\{t | p(t) > 0\} \leq +\infty$

$U_\gamma(z) := \frac{1}{1-\gamma} z^{1-\gamma}$, $0 < \gamma < 1$, utility from consumption at rate $z > 0$.

$\beta > 0$, subjective time preference factor of investor.

$X_t > 0$, wealth of investor at time $0 \leq t < T$.

$W_t, \mathcal{F} = \{\mathcal{F}_u\}_{u \leq t}$, standard Brownian motion and its associated filtration

Definition 2.1: If (2.1) admits positive solutions $X = \{X_t\}_{0 \leq t < T}$, $Z = \{Z_t\}_{0 \leq t < T}$ on some probability space then the process Z is called a consumption policy and we write $Z \in \mathcal{A}$.

Remark: What we really have in mind is

$$\mathcal{A} = \{((\Omega, \mathcal{B}, P), X, Z) : X_t = X_0 + \int_0^t (X_u \mu - Z_u) du + \int_0^t \sigma dW_u, \text{ a.s } P, \forall t\}$$

An element of \mathcal{A} is then a 4-tuple; a probability space, a Brownian motion on this space and two \mathcal{F} adapted processes X, Z for which (2.1) is satisfied. To keep the notation simple we stick to the more intuitive definition given above. Δ

Example 2.2 Given a probability space on which a Brownian motion is defined,

$$\begin{aligned} X_t &= x_0 e^{\mu t - \int_0^t f(u) du + \frac{\sigma^2}{2} W_t} \\ Z_t &= f(t) X_t = f(t) x_0 e^{\mu t - \int_0^t f(u) du + \frac{\sigma^2}{2} W_t} \end{aligned} \tag{2.2}$$

are positive solutions to (2.1) if f is a positive, non-random, function. This will turn out to be an interesting example. We use the notation “ $Z = fX$ ” to refer to this example. \square

To each policy Z corresponds a number

$$V_Z(x_0) = E \int_0^T e^{-\beta u} p(u) U_\gamma(Z_u) du \tag{2.3}$$

Definition 2.3 We say that Z is optimal if

$$V_Z(x_0) = V(x_0) := \sup_{Z \in \mathcal{A}} V_Z(x_0) \tag{2.4}$$

2.2.2 Dynamic Programming

The discussion in this section is heuristic. Rigorous arguments are postponed until the proofs of the results.

Let $V_t(x)$ denote the maximum amount of expected utility that an investor with wealth $X_t = x$ at time t can accumulate during $[t, T)$. A function $\widehat{V}_t(x)$ is an upper bound on $V_t(x)$ for all t, x if and only if

$$M_t = \widehat{V}_t(X_t)e^{-\beta t}p(t) + \int_0^t e^{-\beta u}p(u)U_\gamma(Z_u)du \quad (2.5)$$

is a super-martingale for any policy Z . If $\widehat{V}_t(x)$ is an upper bound and M is a martingale when Z^* is applied then this policy is optimal. This enables us to derive an equation that $\widehat{V}_t(x)$ should satisfy. By the Ito formula,

$$M_t = M_0 + \int_0^t \zeta(u, Z_u, X_u)e^{-\beta u}p(u)du \quad (2.6)$$

plus an Ito-integral where $\zeta(u, Z_u, x) =$

$$\left(-\beta + \frac{p'(u)}{p(u)}\right)\widehat{V}_u(x) + \frac{\partial}{\partial u}\widehat{V}_u(x) + \mu x\widehat{V}'_u(x) + \frac{\sigma^2 x^2}{2}\widehat{V}''_u(x) - \widehat{V}'_u(x)Z_u + \frac{1}{1-\gamma}Z_u^{1-\gamma} \quad (2.7)$$

So M is a super-martingale if ζ is less than or equal to zero for all values of t, x, Z_t . By treating Z_t as a real variable, regarding t, x as fixed, it is not difficult to show:

Lemma 2.4 $\zeta \leq 0$ if $\widehat{V}_t(x)$ satisfies

$$\left(-\beta + \frac{p'(t)}{p(t)}\right)\widehat{V}_t(x) + \frac{\partial}{\partial t}\widehat{V}_t(x) + \mu x\widehat{V}'_t(x) + \frac{\sigma^2 x^2}{2}\widehat{V}''_t(x) + \frac{\gamma}{1-\gamma}\widehat{V}'_t(x)^{1-\frac{1}{\gamma}} = 0 \quad (2.8)$$

and if in addition

$$Z_t = \widehat{V}'_t(x)^{-1/\gamma} \quad (2.9)$$

then $\zeta \equiv 0$.

Proof: See the appendix. \square

After some considerations that we do not give an account of, we make the ansatz $\widehat{V}_t(x) = \varphi(t)^{-\gamma} \frac{x^{1-\gamma}}{1-\gamma}$. If φ satisfies the ordinary differential equation

$$\varphi'(t) + \left(c_* - \frac{1}{\gamma} \frac{p'(t)}{p(t)}\right) \varphi(t) - \varphi(t)^2 = 0 \quad (2.10)$$

where

$$c_* = c_*(\beta, \mu, \sigma, \gamma) \equiv \frac{1}{\gamma} \left\{ \beta - \mu(1-\gamma) + \frac{\sigma^2}{2} \gamma(1-\gamma) \right\} \quad (2.11)$$

then $\widehat{V}_t(x)$ satisfies (2.8). Equation (2.10) has a unique² positive solution which tends to infinity as $t \rightarrow T$, namely

$$\varphi(t) = \frac{p(t)^{1/\gamma} e^{-c_*(\beta, \mu, \sigma, \gamma)t}}{\int_t^T p(u)^{1/\gamma} e^{-c_*(\beta, \mu, \sigma, \gamma)u} du}, \quad 0 \leq t < T \quad (2.12)$$

We have a candidate $\widehat{V}_t(x) = \varphi(t)^{-\gamma} \frac{x^{1-\gamma}}{1-\gamma}$ that satisfies (2.8) and therefore gives M the super-martingale property. If $Z = \varphi X$ then M is a martingale.

Remark: The martingale approach which is described and employed here is standard. We mention [7],[14] as examples of other problems that have been solved by this method. An approach that is specific to this problem, is described in Section 2.3 below.
 Δ

2.2.3 Results

We show that φX is optimal, where φ is given by (2.12), under the following realistic³ assumption.

²The transformation $\psi = 1/\phi$ gives a linear ode. The general solution to (2.10) is

$$\varphi(t) = \frac{p(t)^{1/\gamma} e^{-c_*(\beta, \mu, \sigma, \gamma)t}}{c_0 + \int_t^T p(u)^{1/\gamma} e^{-c_*(\beta, \mu, \sigma, \gamma)u} du}$$

³The assumption is satisfied, for example, if $p(T) = 0$ for some $T < \infty$

Assumption: The function p satisfies $p(t) \leq e^{-\lambda t}$ when t is large for some $\lambda > 0$ such that $c_*(\lambda + \beta, \mu, \sigma, \gamma) > 0$.

Theorem 2.5 φX is an optimal policy if φ is defined by (2.12).

Proof: For $Z \in \mathcal{A}$, we let

$$\begin{aligned} M_t &= \widehat{V}_t(X_t)e^{-\beta t}p(t) + \int_0^t e^{-\beta u}p(u)U_\gamma(Z_u)du, \\ \widehat{V}_t(x) &= \frac{\varphi(t)^{-\gamma}}{1-\gamma}x^{1-\gamma} \end{aligned} \quad (2.13)$$

By construction and Lemma 2.4, M_t is a super-martingale, so

$$\frac{1}{1-\gamma}E \int_0^t e^{-\beta u}Z_u^{1-\gamma}du \leq \widehat{V}_0(x_0) - E\widehat{V}(X_t)e^{-\beta t} \quad (2.14)$$

Our assumptions on $p(t)$ imply that $E\widehat{V}(X_t)e^{-\beta t} \rightarrow 0$, as $t \rightarrow T$, a fact which is given a detailed account of in the appendix, Lemma 4.1. By applying a convergence theorem, we obtain

$$\frac{1}{1-\gamma}E \int_0^T e^{-\beta u}Z_u^{1-\gamma}du \leq \widehat{V}_0(x_0) \quad (2.15)$$

and since Z was arbitrary, we must have

$$V(x_0) \leq \widehat{V}_0(x_0)$$

Equality holds in (2.14) if $Z = \varphi X$, in which case M is a martingale. We obtain, upon taking $t \rightarrow T$,

$$\frac{1}{1-\gamma}E \int_0^T e^{-\beta u}(\varphi(u)X_u)^{1-\gamma}du = \widehat{V}_0(x_0) \quad (2.16)$$

which gives the result. \square

Example 2.6 If $c_*(\beta + \lambda, \mu, \sigma, \gamma) > 0$ and $p(t) = e^{-\lambda t}$ then the function defined by (2.12) is $\varphi(t) = c_*$ for all t . In this case, $Z = c_*X$ is an optimal policy. On the other hand if $c_*(\beta + \lambda, \mu, \sigma, \gamma)$ is negative then the right hand side of (2.4) is $+\infty$. This is

because

$$X_t = x_0 e^{(\mu-c)t + \frac{\sigma^2}{2} W_t} \quad (2.17)$$

$$Z_t = cX_t = cx_0 e^{(\mu-c)t + \frac{\sigma^2}{2} W_t}$$

are positive solutions to (2.1) for any $c > 0$ and if $c = -\frac{\gamma}{1-\gamma}c_* > 0$, a direct computation gives $V_{cX}(x_0) = +\infty$. This illustrates that the assumption on p is needed for the control problem to be well defined. \square

2.2.4 Variational Calculus for a Class of Policies

We can write down an expression for $V_Z(x_0)$ if $Z = \varphi X$ for any positive smooth, deterministic function φ . This observation allows us to derive the optimal relative consumption rate using methods from the calculus of variations. Let \mathcal{C} denote the set of all (non-random) smooth, positive functions. As was pointed out in Example 2.1,

$$\begin{aligned} X_t &:= x_0 e^{\mu t - \int_0^t \varphi(u) du + \frac{\sigma^2}{2} W_t}, \\ Z_t &:= \varphi(t) X_t = \varphi(t) x_0 e^{\mu t - \int_0^t \varphi(u) du + \frac{\sigma^2}{2} W_t} \end{aligned} \quad (2.18)$$

provide positive solutions to (2.1) on any probability space on which a Brownian motion W is defined. Now, (the interchange of operations is justified by Fubini's Theorem)

$$\begin{aligned} V_{\varphi X}(x_0) &= E \int_0^T e^{-\beta t} p(t) U_\gamma(\varphi(t) X_t) dt \\ &= \frac{1}{1-\gamma} \int_0^T E e^{-\beta t} p(t) (\varphi(t) X_t)^{1-\gamma} dt \\ &= \frac{1}{1-\gamma} \int_0^T E x_0^{1-\gamma} e^{-\beta t + \log p(t) + (1-\gamma) \log \varphi(t) + (1-\gamma)(\mu t - \frac{\sigma^2}{2} t - \int_0^t \varphi(s) ds) + (1-\gamma)\sigma W_t} dt \\ &= \frac{x_0^{1-\gamma}}{1-\gamma} \int_0^T e^{-\gamma c_*(\beta, \mu, \sigma, \gamma)t + \log p(t) + (1-\gamma)(\log \varphi(t) - \int_0^t \varphi(s) ds)} dt \end{aligned} \quad (2.19)$$

Consider $\varphi \mapsto V_{\varphi X}(x_0)$ as a functional on \mathcal{C} . If $\bar{\varphi}$ is a maximum, and $\phi(t, \epsilon) = \bar{\varphi}(t) + \epsilon \eta(t)$ then we need, for any function η ,

$$\frac{\partial}{\partial \epsilon} V_{\phi(\cdot, \epsilon) X}(x_0) \Big|_{\epsilon=0} = 0$$

One can verify that this condition leads to the equation (2.10). An account of the details is not given here, having already established the main result. Nevertheless, we

will make use of the observation above to solve the optimal investment and consumption problem.

2.3 Optimal Investment and Consumption

We take on the problem of optimal consumption and investment by assuming that the investor can control σ and that the linear relationship $\mu = a\sigma + b$ holds for some fixed numbers a, b . We thereby assume that the investor can rebalance his portfolio without friction. For simplicity, we assume $p(T) = 0$ for some $T > 0$.

2.3.1 Problem Formulation

We consider non-negative, bounded controls σ of two variables. The wealth process is assumed to satisfy

$$\begin{aligned} dX_t &= (a\sigma(t, X_t) + b)X_t dt + \sigma(t, X_t)X_t dW_t - Z_t dt \\ X_0 &= x_0 \end{aligned} \tag{2.20}$$

Definition 2.7 If (2.20) admits positive solutions Z, X on a probability space, where Z is adapted to the filtration generated by the Brownian motion, we say that (σ, Z) is an admissible investment and consumption pair. The class of admissible pairs is denoted by \mathcal{A} .

Example 2.8 If f, σ are non-random functions then

$$\begin{aligned} X_t &:= x_0 e^{\int_0^t (a\sigma(u)+b)du - \int_0^t f(u)du - \frac{1}{2} \int_0^t \sigma^2(u)du + \int_0^t \sigma(u)dW_u} \\ Z_t &:= f(t)x_0 e^{\int_0^t (a\sigma(u)+b)du - \int_0^t f(u)du - \frac{1}{2} \int_0^t \sigma^2(u)du + \int_0^t \sigma(u)dW_u} \end{aligned} \tag{2.21}$$

provide positive solutions to (2.20) on a probability space on which a Brownian motion W is defined. We use the notation “ $(\sigma, Z) = (\sigma, fX)$ ” to refer to this example. \square

For each pair (σ, Z) corresponds a number

$$V_{\sigma, Z}(x_0) := E \int_0^T e^{-\beta t} U_\gamma(Z_t) dt$$

Definition 2.9 We say that (σ, Z) is an optimal investment and consumption pair if

$$V_{\sigma, Z}(x_0) = V_*(x_0) := \sup_{(\sigma, Z) \in \mathcal{A}} E \int_0^T e^{-\beta u} p(u) U_\gamma(Z_u) du \quad (2.22)$$

Finding an optimal investment and consumption pair is the principal goal of the remainder of this chapter.

2.3.2 A Preliminary Result

Example 2.8 provides a class of reasonable consumption and investment pairs. We denote by $\mathcal{A}_0 \subset \mathcal{A}$ the set of pairs of the form (σ, fX) where $\sigma \geq 0, f > 0$ are non-random.

Claim 2.10 The supremum

$$\sup_{(\sigma, Z) \in \mathcal{A}_0} V_{\sigma, fX}(x_0)$$

is attained by $\sigma \equiv a/\gamma, f = \varphi_\kappa$ where

$$\begin{aligned} \varphi_\kappa(t) &= \frac{p(t)^{1/\gamma} e^{-\kappa t}}{\int_t^T p(u)^{1/\gamma} e^{-\kappa u} du}, 0 \leq t < T, \\ \kappa(\beta, a, b, \gamma) &= \frac{1}{\gamma} \left(\beta - \frac{a^2}{2\gamma} - b(1 - \gamma) \right) \end{aligned} \quad (2.23)$$

To justify the claim, let σ, f be non-random functions. By using that $\int_0^t \sigma(u) dW_u$ is Gaussian with mean zero and variance $\int_0^t \sigma(u)^2 du$, we obtain $E e^{-\beta t} p(t) U_\gamma(f(t) X_t) =$

$$\begin{aligned} &= \frac{x_0^{1-\gamma}}{1-\gamma} e^{-\beta t} p(t) E e^{(1-\gamma) \{ \int_0^t \mu(u) du - \int_0^t f(u) du - \frac{1}{2} \int_0^t \sigma^2(u) du + \int_0^t \sigma(u) dW_u \}} \\ &= \frac{x_0^{1-\gamma}}{1-\gamma} e^{-\gamma \kappa t + \log p(t) + (1-\gamma) \{ \log f(t) + \int_0^t [-\frac{\gamma}{2} (\sigma(u) - a/\gamma)^2 - f(u)] du \}} \\ &\leq \frac{x_0^{1-\gamma}}{1-\gamma} e^{-\gamma \kappa t + \log p(t) + (1-\gamma) \{ \log f(t) + \int_0^t -f(u) du \}} \end{aligned} \quad (2.24)$$

and equality holds if and only if $\sigma(t) = a/\gamma$. Thus,

$$\sup_{(\sigma, f) \in \mathcal{A}_0} V_{\sigma, fX}(x_0) = \sup_f \frac{x_0^{1-\gamma}}{1-\gamma} \int_0^T e^{-\gamma \kappa t + \log p(t) + (1-\gamma) \{ \log f(t) + \int_0^t -f(u) du \}} dt$$

and comparing with Theorem 2.5 and equation (2.19) gives the result.

We show next that this investment and consumption pair is optimal.

2.3.3 Main Result

Showing that

$$M_t = \widehat{V}_t(X_t)e^{-\beta t}p(t) + \int_0^t e^{-\beta u}p(u)U_\gamma(Z_u)du, \quad (2.25)$$

$$\widehat{V}_t(x) := \varphi_\kappa(t)^{-\gamma} \frac{x^{1-\gamma}}{1-\gamma}$$

is a super-martingale and a martingale for $(\sigma, Z) = (\frac{a}{\gamma}, \varphi_\kappa X)$ proves that this is an optimal investment and consumption pair. Here φ_κ is defined by (2.23). By using the Ito-formula one finds that

$$M_t = \widehat{V}_0(x_0) = \int_0^t e^{-ru}p(u)\zeta(u, X_u, Z_u) + \int_0^t e^{-ru}p(u)\eta(u, X_u, Z_u)dW_u$$

where $\zeta(t, X_t, Z_t) =$

$$\frac{X_t^{1-\gamma}}{1-\gamma} \left\{ -\gamma c_*(\beta, a\sigma(t, X_t) + b, \sigma(t, X_t), \gamma) + \frac{p'(t)}{p(t)} - \gamma \frac{\varphi'(t)}{\varphi(t)} - \frac{(1-\gamma)}{X_t} Z_t + \varphi(t)^\gamma \frac{Z_t^{1-\gamma}}{X_t^{1-\gamma}} \right\} \quad (2.26)$$

Lemma 2.11 $\zeta \leq 0$ and $\zeta(t, x, \varphi_\kappa(t)x) = 0$ for all $t \geq 0, x > 0$.

The proof of the Lemma is found in the appendix.

Theorem 2.12 $(\frac{a}{\gamma}, \varphi_\kappa X)$ is an optimal investment and consumption pair.

Proof: Using the standard argument obtains the result. That $E\widehat{V}_t(X_t)e^{-\beta t}p(t) \rightarrow 0$ as $t \rightarrow T$ is Lemma 4.2 in the appendix. \square

Theorem 2.11 shows that the optimal risk-return relationship depends on the parameter γ , but is independent of current wealth and, more remarkably, the survival probability at any time $t > 0$. A comment about the implications of this result can be found in the next section.

2.4 Discussion of Results

Although the result that consumption should optimally be at a rate proportional to wealth is intuitively appealing our main result, Theorem 2.12, is contradicted what some investment managers propose to their clients. Typically the advice is to take less risk, but increase consumption, as one gets closer to the (random) horizon. The result has been obtained under similar assumptions by a number of authors. *It should be pointed out that the author derived the results presented in this chapter not knowing that other studies had considered similar formulations.* An early study of optimal multi-period investment strategies is Mossin [10] who characterizes the class of utility functions for which it is optimal to maximize utility one period at a time. It is shown that a constant portfolio policy is optimal if the utility function is in the crra class and the yield distributions in the different periods are iid. In a model with a risk-free and a risky asset, Samuelson (1969) studies the combined problem of optimal investment and consumption, with results similar to those of Mossin in that for crra utility functions the optimal portfolio decision is independent of time, wealth, and all consumption decisions. Hakansson [5] studies a problem which is similar to that of Samuelson, but also assumes a fixed non-capital income for the investor at the end of each period. It is shown explicitly that if the current value of future income is added to the actual wealth, one obtains the results of Samuelson; the optimal fraction of wealth invested in the risky asset decreases as the present value of future non-capital income decreases.

Merton initiates the study of optimal investment and consumption in continuous time in two seminal papers in 1969 [8] and 1971 [9]. Among the important more recent papers are Pliska (1986), Karatzas et al (1986), Karatzas et al (1987). Cox and Huang (1989) developed an important method to study investment problems and consider the constrained optimization problem where wealth is required to be positive. Although some studies ([4]) have considered the idea of using calculus of variations, the author knows of no other study in which the closed form optimal consumption policy is derived by such methods.

Chapter 3

Optimal Groundwater Extraction

3.1 Introduction

An aquifer has been described as *a saturated permeable geologic unit that can transmit significant quantities of water under ordinary hydraulic gradients*. In a *water table aquifer* the water table forms the upper boundary while some layer of clay or rock constitutes its lower boundary. These formations typically occur near the surface of the earth. We consider a model of such a formation, originally proposed and studied in [15], and the problem of extracting water to maximize revenue in a natural sense. It is assumed that the marginal revenue from extraction is a decreasing, concave function of the water depth. A continuous time stochastic model is used to describe the fluctuations in the water table level due to water recharge. Most realistic effects are ignored in this relatively simple model.

Some investigations indicated that an optimal policy may involve extraction when the water table level is in either of two disjoint intervals[2],[3]. This anomaly was studied and partially explained in [11], where it was suggested that the reason was the initial model assumption of reflection of the water table level at the aquifer's lower boundary. We shall here prove that given a simplified boundary condition, the optimal strategy is always of a simple and intuitive form. We thereby give the arguments presented in [11] a clarification.

3.2 Problem Formulation

The water table is modeled by a non-negative process $X = \{X_t\}_{t \geq 0}$, formally given by

$$X_t = x_0 + \mu t + \sigma W_t + Z_t \tag{3.1}$$

where the control process $Z = \{Z_t\}_{t \geq 0}$ represents the amount of water extracted. Large values of X_t correspond to low water level. The zero level represents the surface of the earth. Here μ is real and $\sigma > 0$.

Definition 3.1 Suppose that X, Z are non-negative processes on a probability space satisfying (3.1). If Z is non-decreasing, right-continuous with left-hand limits and adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ generated by the Brownian motion we call Z a *policy*. The set of policies is denoted \mathcal{A} .

We do not restrict ourselves to the case when X is an Ito-process. In particular, we do not wish to assume that $Z_t = \int_0^t \phi_u du$ for some adapted process ϕ . Being the sum of a square-integrable martingale and a finite variation process, X is a semi-martingale and we shall use a generalization of the Ito formula for such processes.

Example 3.3 Given a probability space (Ω, \mathcal{B}, P) on which a brownian motion W with associated filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is defined there exists for each $a \geq 0$ an adapted process $Z^{(a)}$ which satisfies the conditions of definition 3.1, with

- (i) $Z_0^{(a)} = (a - x_0)^+$
- (ii) $X_t = x_0 + \mu t + \sigma W_t + Z_t^{(a)} \geq a$ for $t \geq 0$
- (iii) $t \mapsto Z_t^{(a)}$ is a.s continuous on $(0, \infty)$ and constant on any interval $[c, d]$, such that $X_t > a, \forall t \in [c, d]$

The process $Z^{(a)}$ is the “local time push”¹ exerted at the threshold a . We call such

¹The intuitive interpretation is that an amount $(a - x_0)^+$ of water is instantaneously removed at time $t = 0$ followed by extraction at infinite rate when the water level reaches a so as to keep $X_t \geq a$ at all times $t > 0$.

a policy a *simple* policy. \square

Assumption 3.4: The marginal revenue function q defined for $x \geq 0$ is both concave and decreasing. We choose our units (length and money) so that q satisfies $q(0) = 1, q(1) = 0$.

Definition 3.5 A policy Z is called optimal if²

$$V_Z(x_0) = \sup_{Z \in \mathcal{A}} V_Z(x_0), \quad (3.2)$$

where $V_Z(x_0) := E_{x_0} \int_0^\infty e^{-ru} q(X_u) dZ_u$

The problem is therefore to maximize revenue, in a natural sense, using a discount rate of $r > 0$. The properties of the function q reflect an assumption of increasing pumping costs (e.g energy) when X is large. In the setup of Section 1.2,

$$U(t, X_t, Z_t) = e^{-rt} q(X_t) \frac{dZ}{dt}$$

(although we do not assume that Z_t with positive probability be a differentiable function of t).

3.3 Optimal Extraction May be Anomalous

Example 3.3 provides a class of policies that imply extraction at infinite rate when X is in $[0, a]$. Under the additional assumption that X is reflected at the lower boundary $x_B, x_B > 1$, there was evidence indicating that an optimal strategy need not be of this simple type. For a given smooth, concave marginal revenue function it was known that for certain parameter values one optimally extracts, at infinite rate, when X is in either of two disjoint intervals $[0, a_1], [a_2, a_3] \subset [0, x_B]$ ³. The investigation in [11] partially explained the reason for this anomaly although no rigorous results were presented. It was argued that the reflection at the lower boundary creates two different ways

²the integral is taken in the Riemann-Stieltjes sense

³It should be pointed out that the relative gain in payoff (compared to the best simple policy) is about 10^{-5} .

of making money: Besides extracting with high marginal revenue it is profitable to extract, with relatively low marginal revenue, close to the point at which X is reflected. It was also suggested that the anomaly somehow arises when the difference in payoff for policies of these two types is very small.

It will be proved in the next section that this phenomena cannot occur under the assumptions of section 3.2 if q is a smooth function. It is the author's opinion that this result clarifies the reason for the anomaly and that it should be kept in mind if one wants to extend the model.

3.4 Result

Theorem 3.5 *Given a smooth marginal revenue function q and the assumptions of Section 3.2 the optimal policy is of the simple form.*

The value of a for which $Z^{(a)}$ is optimal is (see below) the root in $[0, 1]$ of

$$\frac{q'(a)}{\gamma_-} - q(a) = 0$$

The function q is still assumed to satisfy the conditions of assumption 3.4. Thus,

$$\begin{aligned} (i) \quad & q(1) = 0, q(0) = 1 \\ (ii) \quad & q'(x) \leq 0 \\ (iii) \quad & q''(x) \geq 0 \end{aligned} \tag{3.3}$$

3.4.1 Dynamic Programming

The proof of the theorem follows the method from Chapter 1, used in Chapter 2. We look for a function \widehat{V} that gives the process

$$M_t = \widehat{V}(X_t)e^{-rt} + \int_0^t e^{-ru}q(X_u)dZ_u$$

the super-martingale property, with the martingale property satisfied for some $Z^{(a)}$. For a twice continuously differentiable function \widehat{V} , let

$$\begin{aligned}\mathcal{L}\widehat{V}(x) &:= -r\widehat{V}(x) + \mu\widehat{V}'(x) + \frac{\sigma^2}{2}\widehat{V}''(x), \\ \mathcal{N}\widehat{V}(x) &:= \widehat{V}'(x) + q(x)\end{aligned}\tag{3.4}$$

By the Ito-formula for semi-martingales (Theorem 32, [13]), using $dZ_t \cdot dW_t = 0$,

$$\begin{aligned}M_t &= \widehat{V}(x_0) + \int_0^t \mathcal{L}\widehat{V}(X_u)e^{-ru}du + \int_0^t \widehat{V}'(X_u)e^{-ru}dZ_u^c + \int_0^t e^{-ru}\widehat{V}'(X_u)\sigma dW_u \\ &\quad + \sum_{0 \leq u \leq t} [\widehat{V}(X_u) - \widehat{V}(X_{u-})]e^{-ru} + \int_0^t e^{-ru}q(X_u)dZ^c + \sum_{0 \leq u \leq t} \int_{X_{u-}}^{X_u} q(z)dz\end{aligned}\tag{3.5}$$

where the countable sum is taken over the discontinuity points of X , $X_{u-} := \lim_{\epsilon \rightarrow 0^+} X_{u-\epsilon}$ and dZ^c is the integral with respect to the continuous part of Z . If we write $\widehat{V}(X_u) - \widehat{V}(X_{u-}) = \int_{X_{u-}}^{X_u} \widehat{V}'(z)dz$ then we get $M_t =$

$$\widehat{V}(x_0) + \int_0^t \mathcal{L}\widehat{V}(X_u)e^{-ru}du + \int_0^t \mathcal{L}\widehat{V}(X_u)e^{-ru}dZ_u^c + \sum_{0 \leq u \leq t} e^{-ru} \int_{X_{u-}}^{X_u} \mathcal{L}\widehat{V}(z)dz\tag{3.6}$$

plus an Ito-integral.

We construct below a function \widehat{V} which is twice continuously differentiable and satisfies $\mathcal{L}\widehat{V}(x) = 0$, $x \in [a, \infty)$ and $\mathcal{N}\widehat{V}(x) = 0$, $x \in [0, a)$ for some a . The construction implies that M is a martingale if $Z = Z^{(a)}$. To show that this policy is optimal we then verify that $\mathcal{L}\widehat{V}(x), \mathcal{N}\widehat{V}(x) \leq x$ for all $x \geq 0$ so that, by (3.6), M is a super-martingale for any policy Z .

Let the constants $\gamma_- < 0 < \gamma_+$ be the roots of the equation

$$-r - \mu\gamma + \frac{\sigma^2}{2}\gamma^2 = 0\tag{3.7}$$

The value of a for which $Z^{(a)}$ is optimal will be given by

Definition 3.6 If $|\gamma_-| \leq |q'(0)|$ we let $a = 0$ and if $|q'(0)| < |\gamma_-|$ let a be the unique root in $[0, 1]$ of

$$f(a) = \frac{q'(a)}{\gamma_-} - q(a) = 0 \quad (3.8)$$

□

To see that f has exactly one root in $[0, 1]$ if $|q'(0)| < |\gamma_-|$ observe that $f(0) < -1$, $f(1) = \frac{q'(a)}{\gamma_-} \geq 0$, so a root exists. Furthermore,

$$f'(a) = \frac{q''(a)}{\gamma_-} - q'(a) > 0$$

so f is increasing on $[0, 1]$.

Definition 3.7 Let \widehat{V} be defined by

$$\begin{aligned} \widehat{V}(x) &:= -\frac{1}{\gamma_-^2}q'(a) + \int_x^a q(u)du, 0 \leq x \leq a \\ &= \frac{-1}{\gamma_-^2}q'(a)e^{\gamma_-(x-a)}, x > a \end{aligned} \quad (3.9)$$

□

By construction, \widehat{V} satisfies $\mathcal{L}\widehat{V}(x) = 0, x \in [a, \infty)$ and $\mathcal{N}\widehat{V}(x) = 0, x \in [0, a)$ and \widehat{V} is twice continuously differentiable. We have used the so called *principle of smooth fit* to construct \widehat{V} .

3.5 Proof Of Result

The following Lemma says that the function \widehat{V} gives M the super-martingale property. The proof is, although straightforward, rather involved and can be found in the appendix. The proof uses that q is concave.

Lemma 3.8 *If $\widehat{V}(x)$ is given by (3.9) then*

$$\begin{aligned}\mathcal{L}\widehat{V}(x) &:= -r\widehat{V}(x) + \mu\widehat{V}'(x) + \frac{\sigma^2}{2}\widehat{V}''(x) \leq 0 \\ \mathcal{N}\widehat{V}(x) &:= \widehat{V}'(x) + q(x) \leq 0\end{aligned}$$

for all values of $x \geq 0$.

Proof of Theorem 3.5: Let Z be a policy. Then

$$M_t := \widehat{V}(X_t)e^{-rt} + \int_0^t e^{-ru}q(X_u)Z_u$$

is a super-martingale by (3.6) and Lemma 3.8 so

$$\begin{aligned}E_x \int_0^t e^{-ru}q(X_u)dZ_u &= E_x M_t - E\widehat{V}(X_t)e^{-rt} \\ &\leq \widehat{V}(x) - E_x \widehat{V}(X_t)e^{-rt}\end{aligned}\tag{3.10}$$

Since \widehat{V} is bounded, taking $t \rightarrow \infty$ and applying monotone convergence on the left hand side gives

$$E_x \int_0^\infty e^{-ru}q(X_u)dZ_u \leq \widehat{V}(x)\tag{3.11}$$

This shows that $V(x) \leq \widehat{V}(x)$. If $Z = Z^{(a)}$ then, by our construction of \widehat{V} , M is a martingale and we have

$$E_x \int_0^\infty e^{-ru}q(X_u)dZ_u^{(a)} = \widehat{V}(x)$$

implying $V(x) \geq \widehat{V}(x)$. Thus, $V(x) = \widehat{V}(x)$ and $Z^{(a)}$ is optimal. \square

Chapter 4

Appendix

4.1 Some Basic Utility Theory

The expected utility maxim states that an economic agent, when faced with a choice between different alternatives, should choose the one with the greatest expected utility according to some utility function assigning a value of utility to each possible outcome. The expected utility maxim can be deduced from a small set of axioms. In other words, it can be shown that for an individual who obeys a few, basic rules, there exists a utility function such that the expected utility maxim holds (see Herstein and Milnor [6]). A utility function can be defined as an increasing and concave function $u : \mathbb{R} \rightarrow [-\infty, \infty)$. It is also common to assume that u is twice continuously differentiable. The (absolute) risk aversion function $a : \mathbb{R} \rightarrow (-\infty, \infty)$ is defined by

$$a(x) = -\frac{u''(x)}{u'(x)}$$

The absolute risk aversion function is positive and measures the degree of risk aversion (locally at x). The requirement that a utility function u be concave is equivalent to requiring that the corresponding risk aversion function be positive.

An important class of utility functions is the class of so-called HARA (hyperbolic absolute risk aversion) utility functions, which can be written in the form

$$u(x) = \frac{1-\gamma}{\gamma} \left(\frac{\beta x}{1-\gamma} + \eta \right)^\gamma, \quad \frac{\beta x}{1-\gamma} + \eta > 0 >$$

with $\beta > 0$ and $\gamma \in \mathbb{R} \setminus \{0, 1\}$. This class includes some widely used utility functions as special cases: If $\eta = 0$ and $\gamma < 1$, then u is a power utility function for $\gamma \neq 0$, and it is the logarithmic utility function in the limiting case $\gamma \rightarrow 0$. These are also called CRRA (constant relative risk aversion) utility functions. If $\eta = 1$, then u is exponential in the limiting case $\gamma \rightarrow -\infty$ we get $u(x) = -e^{-\beta x}$.

4.2 Verification of a few Lemmas

Lemma 4.1 *Let $Z \in \mathcal{A}$. If φ is defined by (23) then*

$$\lim_{t \rightarrow T} E\varphi(t)^{-\gamma} X_t^{1-\gamma} e^{-\beta t} p(t) = 0$$

Proof: First note that $P(X_t > v) \leq P(X'_t > v)$, $v > 0$ where $X'_t = x e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$ is the solution to (2.1) for $Z \equiv 0$. This implies

$$\begin{aligned} EX_t^{1-\gamma} e^{-\beta t} &\leq Ex^{1-\gamma} e^{((\mu - \frac{\sigma^2}{2})t + \sigma W_t)(1-\gamma)} e^{-\beta t} \\ &= x^{1-\gamma} e^{-\beta t + (\mu - \frac{\sigma^2}{2})t(1-\gamma) + \frac{\sigma^2(1-\gamma)^2}{2}t} \\ &= x^{1-\gamma} e^{-\gamma c_*(\beta, \mu, \sigma, \gamma)t} \end{aligned} \quad (4.1)$$

This gives us

$$\begin{aligned} E\varphi(t)^{-\gamma} X_t^{1-\gamma} e^{-\beta t} p(t) &\leq x^{1-\gamma} e^{-\gamma c_*(\beta, \mu, \sigma, \gamma)t} \varphi(t)^{-\gamma} p(t) \\ &= x^{1-\gamma} \left(\int_t^T p(u)^{1/\gamma} e^{-c_*(\beta, \mu, \sigma, \gamma)u} du \right)^\gamma \end{aligned} \quad (4.2)$$

which goes to zero as $t \rightarrow T$. \square

Proof of Lemma 2.4: We have

$$\zeta(t, x, z) = \frac{x^{1-\gamma}}{1-\gamma} \varphi(t)^{-\gamma} \left\{ -\gamma c_*(\beta, \mu, \sigma, \gamma) + \frac{p'(t)}{p(t)} - \gamma \frac{\varphi'(t)}{\varphi(t)} - \frac{(1-\gamma)}{x} z + \varphi(t)^\gamma \frac{z^{1-\gamma}}{x^{1-\gamma}} \right\} \quad (4.3)$$

for $t \geq 0$, $x > 0$, $z \geq 0$. We show that $\zeta(t, x, \varphi(t)x) = 0$ for all values of t, x and $\zeta(t, x, z) \leq 0$ for any t, x, z .

We have

$$\begin{aligned} \zeta(t, x, \varphi(t)x) &= \frac{x^{1-\gamma}}{1-\gamma} \varphi(t)^{-\gamma} \left\{ -\gamma c_*(\beta, \mu, \sigma, \gamma) + \frac{p'(t)}{p(t)} - \gamma \frac{\varphi'(t)}{\varphi(t)} + \gamma \varphi(t) \right\} \\ &= -\gamma \frac{x^{1-\gamma}}{1-\gamma} \varphi(t)^{-(\gamma+1)} \{ \varphi'(t) + h(t)\varphi(t) - \varphi(t)^2 \} \\ &= 0 \end{aligned} \quad (4.4)$$

since $\varphi(t)$ satisfies (21). Fix $0 < t < T, 0 < x$. Then

$$\begin{aligned}\frac{d}{dz}\zeta(t, x, z) &= \frac{x^{1-\gamma}}{1-\gamma}\varphi(t)^{-\gamma}\left\{-\frac{(1-\gamma)}{x} + (1-\gamma)\varphi(t)^\gamma\frac{z^{-\gamma}}{x^{1-\gamma}}\right\} \\ &= x^{1-\gamma}\varphi(t)^{-\gamma}\left\{-\frac{1}{x} + \varphi(t)^\gamma\frac{z^{-\gamma}}{x^{1-\gamma}}\right\}, \\ \frac{d^2}{dz^2}\zeta(t, x, z) &= -\gamma z^{-(1+\gamma)}\end{aligned}\tag{4.5}$$

We see that $z \mapsto \zeta(t, x, z)$ is a concave function of z for $z > 0$ with $\frac{d}{dz}\zeta(t, x, z)|_{z=x\varphi(t)} = \zeta(t, x, z)|_{z=x\varphi(t)} = 0$. It follows that $\zeta(t, x, z) \leq 0, z > 0$. Since t, x were arbitrary, $\zeta \leq 0$. \square

Proof of Lemma 2.11 Let $\varphi = \varphi_\kappa$ and suppose that $\sigma : [0, \infty) \times (0, \infty) \rightarrow [0, \infty)$.

Define $\zeta : [0, T) \times (0, \infty) \times (0, \infty) \rightarrow \mathbb{R}$ by $\zeta(t, x, z) \equiv$

$$\frac{x^{1-\gamma}}{1-\gamma}\varphi(t)^{-\gamma}\left\{-\gamma c_*(\beta, a\sigma(t, x) + b, \sigma(t, x), \gamma) + \frac{p'(t)}{p(t)} - \gamma\frac{\varphi'(t)}{\varphi(t)} - \frac{(1-\gamma)}{x}z + \varphi(t)^\gamma\frac{z^{1-\gamma}}{x^{1-\gamma}}\right\}\tag{4.6}$$

Then (i) $\zeta(t, x, z) \leq 0$ for any t, x, z . (ii) If $\sigma(t, x) \equiv a/\gamma, \zeta(t, x, \varphi(t)x) = 0$ for any t, x .

Proof: (ii) Since $\varphi(t)$ satisfies

$$\varphi'(t) + h_0(t)\varphi(t) - \varphi(t)^2 = 0\tag{4.7}$$

$$\text{with } h_0(t) = \kappa(\beta, a, b, \gamma) - \frac{1}{\gamma}\frac{p'(t)}{p(t)}$$

we have

$$\begin{aligned}\zeta(t, x, \varphi(t)x) &= \frac{x^{1-\gamma}}{1-\gamma}\varphi(t)^{-\gamma}\left\{-\gamma c_*(\beta, a^2/\gamma + b, a/\gamma, \gamma) + \frac{p'(t)}{p(t)} - \gamma\frac{\varphi'(t)}{\varphi(t)} + \gamma\varphi(t)\right\} \\ &= \frac{x^{1-\gamma}}{1-\gamma}\varphi(t)^{-\gamma}\left\{-\gamma\kappa(\beta, a, b, \gamma) + \frac{p'(t)}{p(t)} - \gamma\frac{\varphi'(t)}{\varphi(t)} + \gamma\varphi(t)\right\} \\ &= -\gamma\frac{x^{1-\gamma}}{1-\gamma}\varphi(t)^{-(\gamma+1)}\{\varphi'(t) + h_0(t)\varphi(t) - \varphi(t)^2\} \\ &= 0\end{aligned}\tag{4.8}$$

(i) Let t, x be arbitrary. Then

$$\begin{aligned} \frac{d}{dz}\zeta(t, x, z) &= \frac{x^{1-\gamma}}{1-\gamma}\varphi(t)^{-\gamma}\left\{-\frac{(1-\gamma)}{x} + (1-\gamma)\varphi(t)^\gamma\frac{z^{-\gamma}}{x^{1-\gamma}}\right\} \\ &= x^{1-\gamma}\varphi(t)^{-\gamma}\left\{-\frac{1}{x} + \varphi(t)^\gamma\frac{z^{-\gamma}}{x^{1-\gamma}}\right\}, \\ \frac{d^2}{dz^2}\zeta(t, x, z) &= -\gamma z^{-(1+\gamma)} \end{aligned} \tag{4.9}$$

We see that $\zeta(t, x, z)$ is a concave function of z for $z > 0$ with $\frac{d}{dz}\zeta(t, x, z)|_{z=x\varphi(t)} = \zeta(t, x, z)|_{z=x\varphi(t)} = 0$. It follows that $\zeta \leq 0$. \square

Lemma 4.2 *Let $(\sigma, Z) \in \mathcal{A}^*$. If φ_κ is defined by (33) then*

$$\lim_{t \rightarrow T} E\varphi_\kappa(t)^{-\gamma} X_t^{1-\gamma} e^{-\beta t} p(t) = 0$$

Proof: Since $Z_t > 0$ we have $P(X_t > v) \leq P(X'_t > v)$ where X'_t is the solution of the wealth process without consumption (a solution exists since σ is bounded). If we let $m(t) = EX'_t^{1-\gamma}$ then $m(t)$ is continuous in t . This gives

$$\begin{aligned} E\varphi_\kappa(t)^{-\gamma} X_t^{1-\gamma} e^{-\beta t} p(t) &\leq m(t)\varphi_\kappa(t)^{-\gamma} p(t) \\ &= m(t)e^{-\gamma\kappa(\beta, a, b, \gamma)t} \left(\int_t^T p(u)^{1/\gamma} e^{-\kappa(\beta, a, b, \gamma)u} du \right)^\gamma \end{aligned} \tag{4.10}$$

which goes to zero as $t \rightarrow T$. \square

Lemma 3.8: *The functions $\mathcal{N}\widehat{V}, \mathcal{L}\widehat{V}$ are ≤ 0 for all values of $x \geq 0$.*

We have

$$\begin{aligned} (i) \quad q(1) &= 0 \\ (ii) \quad q'(x) &\leq 0 \\ (iii) \quad q''(x) &\leq 0 \end{aligned} \tag{4.11}$$

and

Definition 3.3.6: If $|\gamma_-| \leq |q'(0)|$ we set $a = 0$ and if $|q'(0)| < |\gamma_-|$ let a be the unique root in $[0, 1]$ of

$$f(a) = \frac{q'(a)}{\gamma_-} - q(a) = 0 \quad (4.12)$$

and

Definition 3.3.7:

$$\begin{aligned} \widehat{V}(x) &:= -\frac{1}{\gamma_-^2}q'(a) + \int_x^a q(u)du, 0 \leq x \leq a \\ &= \frac{-1}{\gamma_-^2}q'(a)e^{\gamma_-(x-a)}, x > a \end{aligned} \quad (4.13)$$

Proof of Lemma 3.8:

By construction, $\mathcal{N}\widehat{V}(x) = 0, x \in [0, a]$. To show that $\mathcal{N}\widehat{V}(x) \leq 0$ it is therefore enough to show that $\mathcal{N}\widehat{V}(x) \leq 0$ for $x > a$, i.e that

$$\frac{-1}{\gamma_-}q'(a)e^{\gamma_-(x-a)} + q(x) \leq 0 \quad (4.14)$$

for x in (a, ∞) . This inequality is trivial for $x > 1$ since γ_- is negative and $q(x) = 0$. We have equality at $x = a$ so it is enough to show that the derivative of the left hand side is less than or equal to zero for x in $(a, 1]$ i.e

$$-q'(a)e^{\gamma_-(x-a)} + q'(x) \leq 0 \quad (4.15)$$

We have,

$$\begin{aligned} -q'(a)e^{\gamma_-(x-a)} + q'(x) &\leq (\text{ by (4.11) part (iii)}) \\ -q'(a)e^{\gamma_-(x-a)} + q'(a) &= \\ q'(a)(e^{\gamma_-(x-a)} - 1) &\leq 0 \end{aligned} \quad (4.16)$$

so $\mathcal{N}\widehat{V}(x) \leq 0$ holds for x in (a, ∞) and therefore for all $x \geq 0$.

We now show $\mathcal{L}\widehat{V}(x) = 0, x \geq 0$. By construction, $\mathcal{L}\widehat{V}(x) = 0, x \in (a, \infty)$. For $\mathcal{L}\widehat{V}(x) \leq 0$ to hold for all $x \geq 0$ it therefore is sufficient to show that

$$\frac{d}{dx}\mathcal{L}\widehat{V}(x) \geq 0$$

for $0 \leq x \leq a$, or

$$\begin{aligned} -r\widehat{V}'(x) - \mu\widehat{V}''(x) + \frac{\sigma^2}{2}\widehat{V}'''(x) = \\ rq(x) + \mu q'(x) - \frac{\sigma^2}{2}q''(x) \geq 0 \end{aligned} \quad (4.17)$$

By (4.11)(iii) we have

$$\begin{aligned} rq(x) + \mu q'(x) - \frac{\sigma^2}{2}q''(x) \geq \\ rq(x) + \mu q'(x) \end{aligned} \quad (4.18)$$

so it is enough to show that this quantity is less than or equal to zero. By (4.11)(ii) the inequality

$$rq(x) + \mu q'(x) \geq 0 \quad (4.19)$$

is trivial if $\mu \leq 0$ so we assume $\mu > 0$.

We have,

$$\begin{aligned} rq(x) + \mu q'(x) &\geq (\text{ by (4.11) part (ii)}) \\ rq(a) + \mu q'(a) &\geq (\text{ by (4.11) part (iii)}) \\ rq(a) + \mu q'(a) &= (\text{ by the definition of } a) \\ r\frac{q'(a)}{\gamma_-} + \mu q'(a) &= \\ -q'(a)\left(-r\frac{1}{\gamma_-} - \mu\right) &= (\text{ by the definition of } \gamma_-) \\ -q'(a)\left(-\frac{\sigma^2}{2}\gamma_-\right) &\geq 0 \end{aligned} \quad (4.20)$$

We have proved that $\mathcal{L}\widehat{V}(x) \leq 0$ for all $x \geq 0$. \square

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