GEOMETRIC FEATURES OF STRING THEORY AT LOW-ENERGY

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ABSTRACT OF THE DISSERTATION

Geometric features of string theory at low-energy

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In this thesis we study several differential-geometric aspects of the low energy limit of string theory. We focus on anomaly cancellation issues in M-theory on a manifold with boundary and background fluxes, and the computation of non-holomorphic quantities in Calabi-Yau compactifications. In the first chapter we introduce the motivation and the problems that we will study.

In the second chapter we show how the coupling of gravitinos and gauginos to fluxes modifies anomaly cancellation in M-theory on a manifold with boundary. Anomaly cancellation continues to hold, after a shift of the definition of the gauge currents by a local gauge invariant expression in the curvatures and E_8 fieldstrengths. We compute the first nontrivial correction of this kind.

In the last chapter, we introduce methods to determine the form of the effective four-dimensional field theory corresponding to compactifications of string theory. More precisely, we develop iterative methods for finding solutions to the Ricci flat equations on a Calabi-Yau variety, and to the hermitian Yang-Mills equation on stable holomorphic vector bundles, following ideas developed by Donaldson. Finally, we show how these techniques can be understood using the language of geometric quantization of Kähler manifolds, and suggest how one can use these ideas to explicitly construct additional geometric objects.

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Dedication

To Juliana.

Quotations

To do great work, you have to have a pure mind. You can think only about the mathematics. Everything else is human weakness.

Mikhail Gromov

Madness is to think of too many things in succession too fast, or of one thing too exclusively.

Voltaire

The soul is healed by being with children.

Fyodor Dostoevsky

Anyone who keeps the ability to see beauty never grows old.

Franz Kafka

Be that self which one truly is.

Søren Kierkegaard

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Chapter 1
Introduction

The particle physics phenomenology below the electroweak scale $M_{EW} \simeq 100\,GeV$, is successfully described by an effective field theory (EFT) which couples the Standard Model (SM) with General Relativity. However, the present framework cannot solve conceptual problems such as the *hierarchy problem*, the *cosmological constant problem* and more ambitiously, *quantum gravity*. Superstring theories are the best candidate quantum theories to unify gravity, Yang-Mills theory and fundamental matter, and furthermore to solve these fundamental problems.

Unfortunately, a precise mathematical understanding of string theory is still missing. We still do not know how the kinematics, dynamics and initial conditions of the theory are entangled; we do not even know if such a question can be answered. It has been shown mathematical evidence to support the existence of a unique theory –known as M-theory– which contains, in different limits, every other string theory. However, it is not clear if such conjectural theory yields a unique vacuum solution in which kinematics, dynamics and initial conditions are combined in such a way that the four dimensional physics that we know appears in the low energy limit.

One more practical way to study string theory is to follow a phenomenological approach. In such approach one integrates out all the string excitations that have very short wavelength (i.e. the very massive excitations), and formulates an effective field theory that describes the light (long wavelength) degrees of freedom. If one wishes to give a phenomenological description of the consequences of string theory for low-energy physics, it should not be necessary to describe explicitly what the massive states are doing. Such an effective description turns out to be useful not only for a phenomenological analysis, but even as framework for addressing certain theoretical issues, such as the occurrence of anomalies. One implication of this approach, is that it allows the formulation of a huge number (maybe infinity) of possible low energy limits of the theory; hence, in this framework one is just able to build models of particle physics, while neglecting the ultraviolet completion of the EFT. If this multiplicity of models is an apparent or a fundamental feature of the precise formulation of string/M-theory is still a matter of discussion.

In this thesis, we study some aspects of the geometrical structures that appear in the effective field theory approach to the low-energy limit of string theory.

In chapter 2, we consider anomaly cancellation in the supergravity description of M-theory on a manifold with boundary and background fluxes, [64]. We show how the coupling of gravitinos and gauginos to fluxes modifies anomaly cancellation in M-theory on a manifold with boundary. Although anomalies were originally understood as an ultraviolet effect, they can also be understood as an infrared effect; after all, the anomaly is a failure of gauge invariance that cannot be removed by adding any local counterterm to the effective action and therefore cannot depend on unknown modifications of the short distance physics. Anomaly cancellation in M-theory was discussed in the geometric framework in [42]. The ideas in chapter 2 begin to fill a gap left open in [42] and indeed left open in the entire literature on anomaly cancellation in 10- and 11-dimensional supergravities. Namely, in the past the coupling of gravitinos and gauginos to fluxes was omitted. In our analysis, such couplings are retained.

In chapter 3, we introduce some numerical methods to describe accurately many differential geometric objects that appear in the large volume limit of Calabi-Yau compactifications of string theory. The goal is to use these ideas to extract the form of the effective four-dimensional field theory corresponding to compactifications of string theory. For $N = 1$ supersymmetric compactifications [1] we know that the effective supergravity theory depends on the Kähler potential $K(\Psi, \Psi^\dagger)$, the superpotential $W(\Psi)$ and the gauge kinetic function $f(\Psi)$, where Ψ represents the chiral superfields surviving at low energies. These include both the charged matter superfields C and the singlet moduli superfields Φ. It is well known that W and f, being holomorphic, are under much better control than the real function K. In particular K is not protected by the standard non-renormalization theorems of $N = 1$ supersymmetry.

The standard way to extract the functional form of K, W and f at tree-level is by dimensionally reducing the original 10d theory, having carefully identified the appropriate 4d superfields in terms of the corresponding 10d geometrical quantities (such as Ricci-flat metric moduli, hermite-Yang-Mills connection moduli, etc.). This allows the determination of K, W and f as functions of the moduli fields and some of the matter fields. The importance of knowing the Kähler potential for the physical matter fields is clear: it is needed for correctly identifying the canonically normalized fields and therefore determines the structure of most of the observable physical quantities, such as the corresponding scalar potential, the Yukawa couplings, etc. In particular, the matter Kähler potential plays a crucial role in the determination of soft supersymmetry breaking terms.

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[1] Although the validity of our discussion is general, we will concentrate mostly on Calabi-Yau compactifications of heterotic string theory with gauge group $E_8 \times E_8$.

Summarizing, in chapter 3, we quickly review the basics on Calabi-Yau compactifications of the heterotic string, we show how to use geometric quantization to "discretize" the differential geometry of the Calabi-Yau threefold and how to use this to construct numerical algorithms that compute relevant geometrical quantities, such as Einstein metrics. We conclude by pointing out several applications of our techniques and future directions of research.

Chapter 2

Anomaly cancellation with background fluxes

2.1 Overview

M-theory on a manifold with boundary exhibits some extraordinary features, first noted by Horava and Witten [51, 52]. First among these features is a subtle anomaly cancellation, requiring the presence of an independent E_8 super-Yang-Mills multiplet (of either chirality) on each boundary component. In general, anomaly cancellation is best addressed in the geometric framework of determinant line bundles with connection [7, 6, 71, 65, 88, 4, 41]. For recent discussions see, for example [42, 63]. This framework is conceptually clear, is the best approach to cancellation of global anomalies, and is in any case the basis for the descent formalism. In a word, it states that the effective action after integrating out fermions must be a section of a geometrically trivialized line bundle, that is, a topologically trivial line bundle with a trivial connection.

The natural connection on a determinant line bundle for an operator D is a regularized version of $\mathrm{Tr}D^{-1}\delta D$. Therefore, including couplings to the flux results in a change in the connection on the determinant line bundle and hence in the curvature, i.e., it results in a change in the (local) anomaly. In [42] it was shown that if we omit these couplings then there is a canonical geometrical trivialization (termed there a canonical "setting of the integrand") of the line with connection $\mathcal{L}_{\mathrm{Fermi}} \otimes \mathcal{L}_{CS}$. Here the fermion effective action is a section of $\mathcal{L}_{\mathrm{Fermi}}$ while $\mathcal{L}_{CS}$ accomodates the Chern-Simons term. (See [30] for an in depth discussion of this line bundle and its connection). Including the couplings of the fermions to the fluxes spoils the geometrical trivialization. Nevertheless, as we show here, the curvature of $\mathcal{L}_{\mathrm{Fermi}} \otimes \mathcal{L}_{CS}$ is of the form $\mathcal{F} = \mathrm{d}A$ where A is a globally well-defined 1-form on the space of (gauge-equivalence classes of) bosonic fields. Moreover, A is of the form $\int_X I_{11}$ where I_{11} is local in the fields, and X is the 10-dimensional boundary. Physically this means that although there is a change in the anomaly polynomial I_{12}, it changes by dI_{11} where I_{11} is gauge invariant. There is still a physical consequence of this change - the change of connection needed to restore geometrical trivialization corresponds to a change of the definition of the gauge current. We give an explicit formula for this change, to lowest order in fluxes and in flat space below.

2.1.1 Organization

The organization of this chapter is as follows: section 2 contains a definition of the one-loop effective action in M-theory, taking into acount their couplings with the flux. We derive explicitly the contributions from the bulk and the boundary, and thus determine the line bundle $\mathcal{L}_{\text{Fermi}}$, where the exponentiated effective action is defined. In section 3, we analyze the geometry of this line bundle. The contribution from the boundaries yields a non-vanishing local curvature $\mathcal{F}_{\text{Fermi}} \in \Omega^2(\mathcal{T})$ for $\mathcal{L}_{\text{Fermi}}$. Here $\mathcal{T}$ is the space of (gauge inequivalent) bosonic field configurations. After including the contribution of $\mathcal{L}_{CS}$, the total curvature is a globally exact form $\mathcal{F} = \mathrm{d}A$. Thus, it is possible to obtain a geometrical trivialization by changing the connection. Similarly, the contribution from the bulk gives rise to possible $\mathbb{Z}_2$-holonomies for loops in $\pi_1(\mathcal{T})$, due to an ambiguity in the definition of the sign of the Rarita-Schwinger determinant [89, 42]. We show how the flux corrections do not alter the usual $\mathbb{Z}_2$ (or parity) anomaly cancellation mechanism. Section 4 provides explicit formulas for the curvature of the line bundle when the boundaries of Y are flat Euclidean space. We show how our calculations, based on heat kernel expansions and the descent formalism, confirm the general arguments given in section 3.

For completeness, we also study this local anomaly using Fujikawa's method, determining the flux correction to the gauge current as a gauge invariant 9-form in $\Omega^9(\mathbb{R}^{10})$. At the end of the thesis, we include two appendices: one states our Clifford algebra conventions, and the other briefly indicates the connection to supersymmetric quantum mechanics.

2.2 The one-loop effective action

In this section we sketch the gravitino partition function in the case of M-theory on a spin 11-dimensional manifold Y, which might have a nonempty boundary.

The supergravity multiplet consists of the metric g, a gravitino ψ, and a 3-form gauge potential with corresponding field strength G. The low energy limit of M-theory is described by 11-dimensional supergravity [28]. Here we focus on the quadratic part of the action for the gravitino

$$\frac{-1}{2}\int_Y \mathrm{vol}(g)\Big[\overline{\psi}_I\gamma^{IJK}D_J\psi_K + \frac{\ell^3}{96}\big(\overline{\psi}_I\gamma^{IJKLMN}\psi_N + 12\overline{\psi}^J\gamma^{KL}\psi^M\big)G_{JKLM}\Big] \tag{2.1}$$

with $I, J, \ldots$ worldindices, D_I the spin connection and ℓ the eleven dimensional Planck length. We are neglecting higher order terms in ψ_I. The local supersymmetry transformation for the gravitino up to 3-fermi terms, is

$$\delta\psi_I = D_I\epsilon + \frac{\ell^3}{288}(\gamma_I{}^{JKLM} - 8\delta_I^J\gamma^{KLM})G_{JKLM}\epsilon := \hat{D}_I\epsilon. \tag{2.2}$$

We will write (2.2) as $\delta\psi_I = \hat{D}_I\epsilon$, and will refer to $\hat{D}_I$ as the supercovariant derivative. We will abbreviate the action as

$$\int_Y \overline{\psi}R\psi. \tag{2.3}$$

Denote by $\mathbf{S}$ the spin bundle on Y. The generalized Rarita-Schwinger operator $R : \Gamma(S \otimes T^*Y) \to \Gamma(S \otimes T^*Y)$, fits into the complex

$$0 \to \Omega^0(\mathbf{S}) \xrightarrow{\hat{D}} \Omega^1(\mathbf{S}) \xrightarrow{R} \Omega^1(\mathbf{S}) \xrightarrow{\hat{D}^*} \Omega^0(\mathbf{S}) \to 0, \tag{2.4}$$

if we require the vanishing of $R \circ \hat{D}$. Furthermore, at the level of principal symbols the complex is exact so (2.4) defines an elliptic complex. To check the exactness of (2.4) at the level of symbols it is enough to work in flat space, thus if $\sigma_{\hat{D}}(k) = k \in T^*Y$ is the principal symbol associated to $\hat{D}$ and the symbol for R is $\sigma_R(k) = \gamma^{MNP}k_N$, then $\mathrm{Ker}(\sigma_R(k))$ consists of the elements $\mathbf{s}\sigma_{\hat{D}}(k)$ for a spinor $\mathbf{s}$.

The consistency condition $R \circ \hat{D} = 0$ requires that the equations of motion for the bosonic fields must be satisfied as we show below. Hence, if we write the equations of motion for the gravitino field as [28][1]

$$R\psi = \gamma^{MNP}\hat{D}_N\psi_P = 0, \tag{2.5}$$

we can write the condition $R \circ \hat{D} = 0$ as

$$R \circ \hat{D} = \gamma^{MNP}[\hat{D}_N, \hat{D}_P] = 0. \tag{2.6}$$

We can describe the bosonic configurations satisfying (2.6) by considering the seemingly simpler relation

$$\gamma^P[\hat{D}_N, \hat{D}_P] = 0. \tag{2.7}$$

We claim that (2.6) and (2.7) are equivalent. That (2.7) implies (2.5) follows from $\gamma^{MNP} = \gamma^M\gamma^N\gamma^P + g^{NP}\gamma^M - g^{MP}\gamma^N + g^{MN}\gamma^P$. To prove the converse observe that

$$0 = (g_{QM} + \frac{1}{11-2}\gamma_Q\gamma_M) \times (\gamma^{MNP}[D_N, D_P]) = 2\gamma^P[D_N, D_P]. \tag{2.8}$$

By a straightforward computation can express the condition (2.6) on the bosonic fields, using the relation (2.7) as follows

$$\gamma^N[\hat{D}_M, \hat{D}_N] = -\frac{\ell^3}{288}(D_{[N}G_{PQRS]})\gamma^{MNPQRS} + \frac{5\ell^3}{144}(D_{[M}G_{NPQR]})\gamma^{NPQR}$$

[1]Note that we are expressing the Rarita-Schwinger operator R in two equivalent ways.

$$
\begin{aligned}
&-\frac{\ell^3}{72}\Big(D^N G_{NPQR}+\frac{\ell^3}{4\times 288}G^{I_1\ldots I_4}G^{J_1\ldots J_4}\varepsilon_{I_1\ldots I_4 J_1\ldots J_4 PQR}\Big)g_{MT}\gamma^{TPQR}\\
&\quad+\frac{\ell^3}{12}\Big(D^N G_{NMPQ}+\frac{\ell^3}{4\times 288}G^{I_1\ldots I_4}G^{J_1\ldots J_4}\varepsilon_{I_1\ldots I_4 J_1\ldots J_4 MPQ}\Big)\gamma^{PQ}\\
&\quad-\frac{1}{2}\Big(\mathcal{R}_{MN}-\frac{\ell^6}{6}\Big(G_{MPQR}G_N^{\ PQR}-\frac{1}{12}g_{MN}G_{PQRS}G^{PQRS}\Big)\Big)\gamma^N=0. \qquad (2.9)
\end{aligned}
$$

Here, we expand (2.7) in terms of completely antisymmetrized products of gamma matrices (see appendix), hence (2.9) implies the following constraints for the bosonic fields

$$dG=0 \tag{2.10}$$

$$d\star G=-\frac{\ell^3}{2}G\wedge G \tag{2.11}$$

$$\mathcal{R}_{MN}=\frac{\ell^6}{6}\Big(G_{MPQR}G_N^{\ PQR}-\frac{1}{12}g_{MN}\star(G\wedge\star G)\Big) \tag{2.12}$$

where $\mathcal{R}_{MN}$ is the Ricci tensor. These are just the classical equations of motion of 11-dimensional supergravity.

2.2.1 The gravitino partition function

Since the local fermionic gauge symmetries of $n=11$ supergravity do not close into a super Lie algebra for off-shell bosonic backgrounds, we should in principle use the BV quantization procedure to get a correct gauge fixed action. Here we determine the gauge fixed action for backgrounds that satisfy (2.11), and (2.12). This allows us to use standard BRST procedures [55, 68] and simplifies the discussion considerably. Of course, it leaves an important gap in our treatment. Accordingly, we consider the gravitino partition function

$$\mathcal{Z}=\int_{\Omega^1(\mathbf{S})/\mathrm{Im}\hat{D}}[d\psi]e^{-\int_Y\overline{\psi}R\psi}, \tag{2.13}$$

It is useful to introduce the notation:

$$\not{G}=\gamma^{PQRN}G_{PQRN} \tag{2.14}$$

$$\not{G}_N=\gamma^{PQR}G_{PQRN}=-\gamma^{PQR}G_{NPQR} \tag{2.15}$$

$$\not{G}_{RN}=\gamma^{PQ}G_{PQRN}. \tag{2.16}$$

A direct calculation shows that

$$\gamma_M^{\ \ PQRN}G_{PQRN}=\gamma_M\not{G}-4\not{G}_M, \tag{2.17}$$

and therefore we can write (2.2) as

$$\hat{D}_M\epsilon=D_M\epsilon+\frac{\ell^3}{288}\gamma_M\not{G}\epsilon+\frac{\ell^3}{72}\not{G}_M\epsilon. \tag{2.18}$$

Since $\gamma^M\gamma_M = -11$ and $\gamma^M \not{G}_M = -\not{G}$, the associated supercovariant Dirac operator will be

$$\widehat{\not{D}} = \gamma^M \hat{D}_M = \not{D} - \frac{5\ell^3}{96}\not{G}. \tag{2.19}$$

Thus we can write the action (2.1) as

$$\frac{-1}{2}\int_Y \mathrm{vol}(g)\Big[\overline{\psi}_I\gamma^{IJK}D_J\psi_K + \frac{\ell^3}{96}\overline{\psi}_I(\gamma^{IK}\not{G} - 8\gamma^{[I}\not{G}^{K]} - 24\not{G}^{IK})\psi_K\Big]. \tag{2.20}$$

We now use the formal $BRST$ procedure to determine the gravitino gauge fixed action, and choose the gauge $s = \gamma\cdot\psi$ for an arbitrary spinor $s \in \Omega^0(\mathbf{S})$. This leaves unfixed zeromodes of the Dirac equation, constituting a finite dimensional space which we will deal with presently. Following standard procedure we write

$$\mathbf{1} = \int_{\Omega^0(\mathbf{S})^\perp}[\mathrm{d}\epsilon]\,\delta(s - \gamma^M(\psi_M + \hat{D}_M\epsilon))(\det{}'\widehat{\not{D}})^{-1} \tag{2.21}$$

with $\Omega^0(\mathbf{S})^\perp = \big(\mathrm{Ker}\,\widehat{\not{D}}\big)^\perp$ and where $\hat{D}_M$ and $\widehat{\not{D}}$ were defined in (2.18) and (2.34). We now insert (2.32) into

$$\int[\mathrm{d}\psi]e^{-\int_Y \overline{\psi}R\psi} \tag{2.22}$$

and divide by the volume of the supergauge group to obtain the gauge-fixed expression

$$\int[d\psi]\delta(s - \gamma\cdot\psi)(\det{}'\widehat{\not{D}})^{-1}e^{-\int_Y \overline{\psi}R\psi}. \tag{2.23}$$

Ghost fields are introduced by writing the determinant (2.32) in terms of commuting ghost ϵ and antighost β fields as

$$(\det{}'\widehat{\not{D}})^{-1} = \int[\mathrm{d}\beta][\mathrm{d}\epsilon]e^{-\int \overline{\beta}\widehat{\not{D}}\epsilon}, \tag{2.24}$$

the prime in the determinant denotes the omission of the null eigenvalues.

Furthermore we invoke the following algebraic identity for $\phi_M = \psi_M + \frac{1}{2}\gamma_M(\gamma\cdot\psi)$, which allows us to split the gauge fixed action as a sum of functionally independent quadratic terms, i.e. we have the relation

$$-\overline{\phi}\widehat{\not{D}}_{T^*Y}\phi = \overline{\psi}R\psi - \frac{1}{4}(n-2)\overline{(\gamma\cdot\psi)}\widetilde{\not{D}}(\gamma\cdot\psi), \tag{2.25}$$

where R was defined in (2.20) to be

$$R^{IK} = \gamma^{IJK}D_J + \frac{\ell^3}{96}(\gamma^{IK}\not{G} - 8\gamma^{[I}\not{G}^{K]} - 24\not{G}^{IK}), \tag{2.26}$$

while $\widetilde{\not{D}}$ and $\widehat{\not{D}}_{T^*Y}$ are uniquely fixed to be the generalized Dirac operators

$$\widetilde{\not{D}} = \not{D} + \frac{\ell^3}{288}\not{G} \tag{2.27}$$

and

$$\widehat{\not{D}}_{T^*Y} = \not{D}_{T^*Y} - \frac{\ell^3}{96}\not{G}. \tag{2.28}$$

Here, the subscript T^*Y denotes the coupling with the cotangent bundle of Y. The identity (2.25) is easy to check when we substitute the ϕ-field and the operators $\widehat{\not{D}}_{T^*Y}$ and $\widetilde{\not{D}}$ in it and use the following relations for $\not{G}$ and the gamma matrices

$$\gamma^M \not{G} - \not{G}\gamma^M = 8\not{G}^M \tag{2.29}$$

$$\gamma^M \not{G}^P + \not{G}^P \gamma^M = -6\not{G}^{MP} \tag{2.30}$$

$$\gamma^{IK} = \gamma^I \gamma^K + g^{IK}. \tag{2.31}$$

At this point, rather than setting $s = 0$ we average over $s = (\gamma \cdot \psi)$ using the expression

$$\mathbf{1} = \frac{1}{(\det{}' \widetilde{\not{D}})^{1/2}} \int_{(\Omega^0(\mathbf{S}))^\perp} [\mathrm{d}s]\, e^{-\int \overline{s}\widetilde{\not{D}} s}. \tag{2.32}$$

Formally, using (2.25) the gauge fixed partition function for the gravitino can be written as

$$\mathcal{Z}' = \frac{1}{(\det \widetilde{\not{D}})^{1/2}} \int [\mathrm{d}\psi][\mathrm{d}\beta][\mathrm{d}\epsilon] \exp\Bigg(- 2\pi \int_Y \mathrm{vol}(g) (\overline{\psi} \widehat{\not{D}}_{T^*Y} \psi - \overline{\beta} \widehat{\not{D}} \epsilon) \Bigg). \tag{2.33}$$

We still must fix the remaining global fermionic symmetries given by supercovariantly constant spinors. We will assume the procedure described in [42], eq. $(A.11)$ continues to hold. The net result is the following key statement. [2]

Let $\mathcal{T}$ denote the space of bosonic M-theory data on Y, i.e., the Riemannian metrics and G-fluxes, and introduce the fibration $\mathcal{Y} \to \mathcal{T}$ whose fiber is the spacetime manifold Y. This yields a family of operators $(\widehat{\not{D}}, \widetilde{\not{D}}, \widehat{\not{D}}_{T^*Y})$ built up fiberwise in $\mathcal{Y}$ through the geometric data parametrized by $\mathcal{T}$:

$$\widehat{\not{D}} = \not{D} - \frac{5\ell^3}{96}\not{G}, \tag{2.34}$$

$$\widetilde{\not{D}} = \not{D} + \frac{\ell^3}{288}\not{G}, \tag{2.35}$$

$$\widehat{\not{D}}_{T^*Y} = \not{D}_{T^*Y} - \frac{\ell^3}{96}\not{G}. \tag{2.36}$$

Then, the gravitino partition function $\exp(-\Gamma_{\text{gravitino}})$ is a section of the line bundle

$$\mathcal{L}_{\text{gravitino}} := \mathrm{Pfaff}\, \widehat{\not{D}}_{T^*Y} \otimes (\mathrm{Pfaff}\, \widetilde{\not{D}})^{-1} \otimes (\mathrm{Det}\, \widehat{\not{D}})^{-1} \to \mathcal{T}, \tag{2.37}$$

In fact, this is a line bundle with connection, as we will discuss below. In addition, the Chern-Simons term of M-theory is also a section of a line bundle with connection $\mathcal{L}_{CS} \to \mathcal{T}$, and hence

[2] We are unaware of an adequate treatment of the ghost zeromodes in the gravitino partition function in the literature. In our analysis we sidestep that issue and assume that the gravitino effective action is a section of the line bundle introduce below.

the M-theory measure is a section of

$$\mathcal{L}_{\text{gravitino}} \otimes \mathcal{L}_{CS} \to \mathcal{T} \tag{2.38}$$

2.2.2 Boundary contribution to the effective action

Let us now turn to the case where Y has a boundary. We denote by ∂Y_i the different connected components and by ω^∂ Clifford multiplication by the volume 10-form on the boundary. We follow closely the discussion of boundary conditions in [42]. We fix a spatial boundary condition for the spinor field Ψ, by imposing

$$\epsilon_i \Psi^\partial = \Psi^\partial \quad \text{with } \epsilon_i = i\omega^\partial \text{ or } \epsilon_i = -i\omega^\partial \tag{2.39}$$

at each connected component ∂Y_i. The presence of boundaries produces local anomalies in the theory.

The fermionic content at the boundary in the low energy description of M-theory comes from the restriction of the gravitino and the presence of gauginos. We generalize the discussion of Horava and Witten and attach an independent $N = 1$ super Yang-Mills multiplet with gauge group E_8 and chirality ϵ_i to each connected component of the boundary. According to [51], we should write the quadratic part of the action for the gauginos, as

$$S_i = -\frac{1}{4\pi\ell^6}\int_{\partial Y_i} \mathrm{vol}(g^\partial)\mathrm{tr}\left[\overline{\chi}\not{D}_{E_8}\chi - \frac{\ell^3}{24}\overline{\chi}^a\gamma(\iota_\nu G^\partial)\chi^a\right]. \tag{2.40}$$

The superscript $^\partial$ denotes restriction of the field on the boundary and ι_ν is the contraction with the unit outward normal vector field to ∂Y_i. As shorthand, we can write the action using the generalized Dirac operator

$$\hat{\not{D}}_{E_8} = \not{D}_{E_8} - \frac{\ell^3}{24}\gamma(\iota_\nu G^\partial), \tag{2.41}$$

so the exponentiated effective action for χ is section of the line bundle

$$\mathcal{L}_{\text{gaugino}} = \bigotimes_{\epsilon_i = -i\omega^\partial}\left(\mathrm{Pfaff}\,\hat{\not{D}}^{\partial Y_i}_{E_8}\right)\bigotimes_{\epsilon_i = i\omega^\partial}\left(\mathrm{Pfaff}\,\hat{\not{D}}^{\partial Y_i}_{E_8}\right)^{-1} \longrightarrow \mathcal{T}, \tag{2.42}$$

It is useful to decompose the boundary 4-form G, in its tangential and normal components

$$G^\partial = \nu^\flat \wedge \iota_\nu G^\partial + (1 - \nu^\flat \wedge \iota_\nu)G^\partial = G^\partial_N + G^\partial_T \tag{2.43}$$

with $\nu^\flat$ the 1-form dual to the unit normal vector field ν. Also, we introduce the local "torsion"

$$\mathbb{h} = -\frac{1}{24}\ell^3\iota_\nu G^\partial. \tag{2.44}$$

For the gravitino sector, we have to generalize the gauge fixed action $\exp(-\Gamma_{\mathrm{gravitino}})$ to the case $\partial Y \neq \emptyset$. To do this recall the relation between a Dirac-like operator on the boundary ∂Y_i and that in the bulk close to the boundary $\partial Y_i \times [0, \epsilon)$

$$\widehat{\not{D}} = \gamma^{\nu}(\partial_{\nu} - \widehat{\not{D}}^{\partial}) \tag{2.45}$$

where ν is the normal unit vector field to the boundary. Then, as $(\gamma^{\nu})^2 = -1$, the generalized Dirac operators that we have to study on the boundary are

$$\widehat{\not{D}}_{T^*Y} = \not{D}_{T^*Y} - \frac{\ell^3}{96}\not{G}, \quad \Rightarrow \quad \widehat{\not{D}}^{\partial}_{T^*Y} = \not{D}^{\partial}_{T^*Y} - \frac{\ell^3}{24}\gamma(\iota_{\nu}G_N + \nu^{\flat} \wedge G_T) \tag{2.46}$$

$$\widetilde{\not{D}} = \not{D} + \frac{\ell^3}{288}\not{G}, \quad \Rightarrow \quad \widetilde{\not{D}}^{\partial} = \not{D}^{\partial} + \frac{\ell^3}{72}\gamma(\iota_{\nu}G_N + \nu^{\flat} \wedge G_T) \tag{2.47}$$

$$\widehat{\not{D}} = \not{D} - \frac{5\ell^3}{96}\not{G}, \quad \Rightarrow \quad \widehat{\not{D}}^{\partial} = \not{D}^{\partial} - \frac{5\ell^3}{24}\gamma(\iota_{\nu}G_N + \nu^{\flat} \wedge G_T) \tag{2.48}$$

as $\iota_{\nu}G_N$ is a 3-form and $\nu^{\flat} \wedge G_T$ a 5-form, the operators $\widehat{\not{D}}^{\partial}_{T^*Y}$, $\widetilde{\not{D}}^{\partial}$ and $\widehat{\not{D}}^{\partial}$ anticommute with ω^{∂} and hence have a well defined index.

The restriction ψ^{∂} of the Rarita-Schwinger field $\psi \in \Omega^1(\mathbf{S})$ to ∂Y_i decomposes into tangential and normal components:

$$\psi^{\partial} = \psi^{\partial}_T + \psi^{\partial}_{\nu} \tag{2.49}$$

and their boundary conditions are given by the following definite choice of sign

$$\omega^{\partial}\psi^{\partial}_T = +i\psi^{\partial}_T \tag{2.50}$$

$$\omega^{\partial}\psi^{\partial}_{\nu} = -i\psi^{\partial}_{\nu}. \tag{2.51}$$

These boundary conditions imply that the gauge group must be restricted by

$$\omega^{\partial}\hat{\nabla}_T\epsilon^{\partial} = +i\hat{\nabla}_T\epsilon^{\partial} \tag{2.52}$$

$$\omega^{\partial}\hat{\nabla}_{\nu}\epsilon^{\partial} = -i\hat{\nabla}_{\nu}\epsilon^{\partial} \tag{2.53}$$

where

$$\hat{\nabla}_{\nu} = \nu^M \hat{D}_M \tag{2.54}$$

$$\hat{\nabla}_T = \hat{D}_M - \nu_M \hat{\nabla}_{\nu} \tag{2.55}$$

and $\hat{D}_M$ is the supersymmetric variation of the gravitino

$$\hat{D}_M = D_M + \frac{\ell^3}{288}\gamma_M \not{G} + \frac{\ell^3}{72}\not{G}_M. \tag{2.56}$$

We then choose boundary conditions on the other ghost β, so that $\widehat{\not{D}}^{\partial}$ is skew-adjoint:

$$\omega^{\partial}\hat{\nabla}_T\beta^{\partial} = -i\hat{\nabla}_T\beta^{\partial} \tag{2.57}$$

$$\omega^{\partial}\hat{\nabla}_{\nu}\beta^{\partial} = +i\hat{\nabla}_{\nu}\beta^{\partial} \tag{2.58}$$

The third ghost that comes from integrating over s (2.32), has the same boundary conditions as $\gamma \cdot \psi$:

$$\omega^{\partial}(s^{\partial}) = -is^{\partial}. \tag{2.59}$$

As the chiralities of ϵ^{∂} and β^{∂} are opposite, (2.52), and (2.57), lead to pfaffian line bundles which cancel, therefore $\widehat{\not{D}}^{\partial}$ does not appear in our analysis. On the other hand, as ψ_ν comes from a component of the Rarita-Schwinger field in $\Omega^1(\mathbf{S})$, it couples to

$$\widehat{\not{D}}^{\partial}_{\nu} = \not{D}^{\partial} - \frac{\ell^3}{96}\gamma(\iota_\nu G_N + \nu^\flat \wedge G_T) \tag{2.60}$$

and s couples to $\widetilde{\not{D}}^{\partial}$, as defined in (2.46). Therefore, according to the theorems stated in [42], the boundary contribution to the exponentiated effective action $\exp(-\Gamma_{\text{gravitino}})$ is section of

$$\mathcal{L}_{\text{gravitino}} = \bigotimes_{-i\omega^{\partial}} \Big[\big(\text{Pfaff}\,\widehat{\not{D}}^{\partial Y_i}_{T^*Y}\big)^{1/2} \otimes \big(\text{Pfaff}\,\widehat{\not{D}}^{\partial Y_i}_{\nu}\big)^{-1/2} \otimes \big(\text{Pfaff}\,\widetilde{\not{D}}^{\partial Y_i}\big)^{-1/2}\Big]$$

$$\bigotimes_{+i\omega^{\partial}} \Big[\big(\text{Pfaff}\,\widehat{\not{D}}^{\partial Y_i}_{T^*Y}\big)^{-1/2} \otimes \big(\text{Pfaff}\,\widehat{\not{D}}^{\partial Y_i}_{\nu}\big)^{+1/2} \otimes \big(\text{Pfaff}\,\widetilde{\not{D}}^{\partial Y_i}\big)^{+1/2}\Big] \to \mathcal{T} \tag{2.61}$$

where we are taking into account the contribution from every connected component of the boundary. Finally we have

$$\mathcal{L}_{\text{Fermi}} = \mathcal{L}_{\text{gaugino}} \otimes \mathcal{L}_{\text{gravitino}}. \tag{2.62}$$

In the following sections, we study the curvatures of the determinant line bundles associated to generalized Dirac operators. The G-dependent contributions to the curvature of (2.62) are given by terms constructed with the exterior derivatives $\mathrm{d}(\nu^\flat \wedge G_T)$ and $\mathrm{d}\iota_\nu G_N$. Now

$$\mathrm{d}(\nu^\flat \wedge G_T) = \mathrm{d}\nu^\flat \wedge G_T - \nu^\flat \wedge \mathrm{d}G_T = 0. \tag{2.63}$$

To see this, we work in the neighborhood of the boundary $\partial Y_i \times [0,\, \epsilon)$ such that $\mathrm{d}\nu^\flat = 0$. Also, as G_T is closed on the boundary we have $\mathrm{d}G_T = 0$. Thus we can neglect the contributions from $\nu^\flat \wedge G_T$, and just work with the local torsion $\mathbb{h}$ of (2.44).

2.2.3 Hořava-Witten reduction

It is useful to connect our formalism to the standard Hořava-Witten setup $Y = X \times [0,\, 1]$, used to describe the strongly coupled heterotic string with gauge group $E_8 \times E_8$, in its low energy limit.

The H flux of heterotic string theory is recovered from the M-theory data according to

$$\mathrm{H} = \int_{[0,\,1]} \mathrm{d}t G_{11\,MNP} \mathrm{d}x^M \wedge \mathrm{d}x^N \wedge \mathrm{d}x^P \tag{2.64}$$

with $t = x^{11}$ and $1 \leq M,\ N,\ P \leq 10$. On the other hand, using the decomposition of the G-flux in terms of tangential and normal components to the $11^{\rm th}$-coordinate

$$G = G_{11\,MNP} \mathrm{d}t \wedge \mathrm{d}x^M \wedge \mathrm{d}x^N \wedge \mathrm{d}x^P + G_{QRST} \mathrm{d}x^Q \wedge \mathrm{d}x^R \wedge \mathrm{d}x^S \wedge \mathrm{d}x^T = G_N + G_T \quad (2.65)$$

with the indices $M, N, \ldots$ running between 1 and 10. On the boundaries $\iota^*(G_T)$ at $t = 0, 1$ we have $\iota_t^* G_T = \mathrm{tr}\, F_t^2 - \frac{1}{2} \mathrm{tr}\, \mathcal{R}_t^2 \in \Omega^4(X)$ where $F_t,\ t = 0, 1$ is the curvature of the E_8 bundle on the boundary X_t. If we extend G_T as a family of closed forms on X then

$$0 = \mathrm{d}_{11} G = \Big(\mathrm{d}t \wedge \frac{\partial}{\partial t} + \mathrm{d}\Big)(G_N + G_T) = \mathrm{d}G_N + \mathrm{d}t \wedge \frac{\partial}{\partial t} G_T \quad (2.66)$$

(d and d_{11} are exterior derivatives on X and Y, respectively). Therefore, from (2.66)

$$\mathrm{d}G_N = -\mathrm{d}t \wedge \frac{\partial}{\partial t} G_T \quad (2.67)$$

Using (2.64) and (2.67) we recover the usual formula

$$\mathrm{d}\,\mathrm{H} = \mathrm{tr}\, F_1^2 + \mathrm{tr}\, F_2^2 - \mathrm{tr}\, \mathcal{R}^2 \quad (2.68)$$

Finally we would like to see how the interaction term

$$\Delta S = \frac{1}{96 \pi \ell^3} \int_X \mathrm{vol}(g_X) \mathrm{Tr}_{\mathbf{496}} \Big[\overline{\chi} \gamma(\mathrm{H}) \chi\Big], \quad (2.69)$$

in heterotic string theory, is recovered from the boundary interactions of M-theory (2.40)

$$\Delta S_i = \frac{1}{96 \pi \ell^3} \int_{\partial Y_i} \mathrm{vol}(g^\partial) \mathrm{Tr}_{\mathbf{248}} \Big[\overline{\chi}_i \gamma(\mathbb{h}) \chi_i\Big]. \quad (2.70)$$

with $i = 1,\ 2$ labeling the boundaries of the cylinder $X \times [0,\ 1]$. In the zeromode limit we have

$$\mathcal{L}_t G_N = 0, \quad \text{or} \quad \iota_t G_N = \iota_t G_N^{\partial_1} = \iota_t G_N^{\partial_2} = \mathrm{H} = \mathbb{h}_1 = \mathbb{h}_2 \quad (2.71)$$

i.e. G_N is t-independent and the non-trivial t-dependence of G comes from G_T. Therefore $\Delta S = \Delta S_1 + \Delta S_2$.

2.3 Setting the bosonic measure in the presence of fluxes

In this section we will describe a connection on the gravitino and gaugino line bundles and compute its curvature.

Without loss of generality, we can fix attention on one boundary component, and fix a chirality. We choose to study

$$\big(\mathrm{Pfaff}\, \hat{\not{D}}^{\partial}_{E_8}\big) \otimes \big(\mathrm{Pfaff}\, \widehat{\not{D}}^{\partial Y_i}_{T^*Y}\big)^{1/2} \otimes \big(\mathrm{Pfaff}\, \widehat{\not{D}}^{\partial Y_i}_{\nu}\big)^{-1/2} \otimes \big(\mathrm{Pfaff}\, \widetilde{\not{D}}^{\partial Y_i}\big)^{-1/2} \to \mathcal{T}^\partial, \quad (2.72)$$

where $\mathcal{T}^\partial$ is the space of bosonic fields on the boundary and the generalized Dirac operators in (2.72) are

$$\hat{\not{D}}^\partial_{E_8} = \not{D}^\partial_{E_8} + \gamma(\mathbb{h}), \tag{2.73}$$

$$\widehat{\not{D}}^\partial_{T^*Y} = \not{D}^\partial_{T^*Y} + \gamma(\mathbb{h}), \tag{2.74}$$

$$\widehat{\not{D}}^\partial_{\nu} = \not{D}^\partial_{\nu} + \gamma(\mathbb{h}), \tag{2.75}$$

$$\widetilde{\not{D}}^\partial = \not{D}^\partial - \frac{1}{3}\gamma(\mathbb{h}). \tag{2.76}$$

where $\gamma(\cdot)$ denotes Clifford multilication by elements in $\Omega^*(X)$, with $X := \partial Y$.

A natural choice of connection on the determinant and Pfaffian line bundles follows the discussion of Bismut and Freed [14, 13]. Working fiberwise in $\mathcal{X} \to \mathcal{T}^\partial$, we can define generalized Dirac operators $\hat{\not{D}}$ on X, as the ones which appear in the definition of the effective action, i.e., the operators (2.73), (2.74), (2.75) and (2.76). We now drop the superscript ∂ in the remainder of this section. The generalized Dirac operator

$$\hat{\not{D}} = \not{D} + \alpha_0\gamma(\mathbb{h}), \ \ \mathbb{h} \in \Omega^3(X), \tag{2.77}$$

(where α_0 is $\alpha_0 = 1, -1/3$ in the case of interest here) can be viewed as an odd endomorphism acting on the Hilbert bundle of spinors

$$\Omega^0(\mathbf{S}_+) \oplus \Omega^0(\mathbf{S}_-) \to \mathcal{T}^\partial, \tag{2.78}$$

where the subindices $+$ and $-$ denote the chirality of the spinor. In the Weyl basis $\hat{\not{D}}$ decomposes as

$$\hat{\not{D}} = \begin{pmatrix} 0 & \hat{\not{D}}_- \\ \hat{\not{D}}_+ & 0 \end{pmatrix},$$

Next, using a Riemannian structure on $\mathcal{T}^\partial$ we can then introduce a connection $\widetilde{\nabla}$ on the Hilbert bundle $\Omega(\mathbf{S}) \otimes \Lambda^*(\mathcal{T}^\partial) \to \mathcal{T}^\partial$. This connection allows us to study the geometry of the determinant line bundle where the effective action lives, i.e. given the Hilbert bundle (2.78) it is possible to define its associated determinant line bundle

$$\mathrm{Det}\hat{\not{D}}_+ \to \mathcal{T}^\partial, \tag{2.79}$$

which can be also written as[3]

$$\det\Omega^0(\mathbf{S}_+) \otimes \det(\Omega^0(\mathbf{S}_-)^\vee) \to \mathcal{T}^\partial. \tag{2.80}$$

[3]See [71] and [14], for a rigorous definition of such infinite dimensional bundles.

This line bundle has a natural connection on it which can be determined using heat kernel expansions [13]. More concretely, when restricted to a 2 dimensional submanifold $\Sigma \hookrightarrow \mathcal{T}^\partial$, one can compute its curvature as [14, 15]

$$\int_\Sigma \mathcal{F}(\mathrm{Det}\,\hat{\not{D}}_+ \to \mathcal{T}^\partial) = 2\pi i \int_{\pi^{-1}(\Sigma)} \Big[\mathrm{Tr}_s a_6(\hat{\not{\mathcal{D}}})\Big]_{(12)}, \tag{2.81}$$

with $\mathcal{F} \in \Omega^2(\Sigma)$ and $\pi\colon \mathcal{X} \to \mathcal{T}^\partial$ the defining fibration of the family, with fiber X. [4] In (2.81) we are using the heat kernel expansion

$$\mathrm{Tr}_s\big(\exp(-t\hat{\not{\mathcal{D}}}^2)\big) = \frac{\mathrm{Tr}_s a_0}{t^6} + \frac{\mathrm{Tr}_s a_1}{t^5} + \ldots + \mathrm{Tr}_s a_6 + \mathcal{O}(t), \tag{2.82}$$

where $\mathrm{Tr}_s(\,\cdot\,) = \mathrm{Tr}(\Gamma^{13}\cdot\,)$, and $\hat{\not{\mathcal{D}}}$ the generalized Dirac operator on the spin bundle of the 12 manifold $\pi^{-1}(\Sigma)$, defined as

$$\hat{\not{\mathcal{D}}} = \not{\mathcal{D}} + \alpha_0 \Gamma(\mathbb{h}), \tag{2.83}$$

with $\not{\mathcal{D}}$ the usual Dirac operator on $\pi^{-1}(\Sigma)$, $\mathbb{h} \in \Omega^3(X)$ and $\Gamma(\,\cdot\,)$ denotes the Clifford multiplication in the Clifford algebra Cliff(12).

This approach allows us to compute the curvature of the line bundle form the integral over two-dimensional submanifolds $\Sigma \hookrightarrow \mathcal{T}^\partial$.

2.3.1 Flux corrections to the line bundle's curvature

If $\mathrm{Tr}_s a_6(\hat{\not{D}})$ is the heat kernel coefficient associated to the generalized Dirac operator $\hat{\not{D}}$, the curvature of the physical line bundle which appears in M-theory (2.72) can be expressed as

$$\begin{aligned}\mathcal{F}(\mathcal{L}_{\text{gaugino}} \otimes \mathcal{L}_{\text{gravitino}} \otimes \mathcal{L}_{CS} \to \mathcal{T}^\partial) = \mathcal{F}(\mathcal{L}_{CS} \to \mathcal{T}^\partial) + \\ \frac{2\pi i}{4}\left[\int_X 2\mathrm{Tr}_s a_6(\hat{\not{D}}_{E_8}) + \mathrm{Tr}_s a_6(\widehat{\not{D}}^\partial_{T^*Y}) - \mathrm{Tr}_s a_6(\widehat{\not{D}}^\partial_\nu) - \mathrm{Tr}_s a_6(\widetilde{\not{D}}^\partial)\right]_{(2)},\end{aligned} \tag{2.84}$$

where $[\,\cdot\,]_{(2)}$ extracts the two-form part. Thus, evaluating the curvature of (2.72) is equivalent to computing certain heat kernel coefficients.

Without evaluating the heat kernel coefficients we can make the following observation just based on index theory. From [42] we know that the curvature (2.84) is zero for $\mathbb{h} = 0$. Since the flux can be turned on by a compact perturbation the curvature will be an exact 2-form on $\mathcal{T}^\partial$

$$\mathcal{F}(\mathcal{L}_{\text{gaugino}} \otimes \mathcal{L}_{\text{gravitino}} \otimes \mathcal{L}_{CS} \to \mathcal{T}^\partial) = \mathrm{d}A, \tag{2.85}$$

---

[4] The theorems of [13] and [15] that we use here were stated for families of ordinary Dirac operators and not generalized Dirac operators. However the argument using Eq.(1.56) of [13], as well as the identity Eq.(5.4) of [15] can be shown to extend to the case of generalized Dirac operators. One need only require some mild conditions on the generalized Dirac operators, which turn out to be compatible with the physics of our problem.

for some *globally well-defined* 1-form $A \in \Omega^1(\mathcal{T}^\partial)$. As we have said, $\mathcal{T}^\partial$ is the space of gauge inequivalent field configurations, that is, the base of the $\mathcal{G} := \mathrm{Diff}(Y) \times Aut(E)$ bundle

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathrm{Met}(Y) \times \mathcal{A} \xrightarrow{\pi} \mathcal{T}^\partial \longrightarrow 0, \tag{2.86}$$

with $\mathrm{Met}(Y)$ the space of Riemannian metrics on Y and $\mathcal{A}$ the affine space of E_8-gauge connections on the E_8 gauge bundle $E \to X$. We can write the 12-form I_{12}, used to define the curvature of the line bundle $\mathcal{F} = \int_X I_{12}$, as the exterior differential of a $\mathcal{G}$-equivariant 11-form $I_{11}(\mathcal{R}, F, G)$. Therefore, the descent formalism suggests that such flux corrections do not contribute to the anomaly.

In order to justify the above claim we proceed as follows. As we showed above, we can construct a generalized Dirac operator acting on the Hilbert bundle (2.78). If we now restrict to an arbitrary 2-dimensional famliy $\Sigma \subset \mathcal{T}^\partial$ then the index of this operator, which we will denote by $\mathrm{Index}\, \hat{\not{\mathcal{D}}}$ is given by

$$\mathrm{Index}\, \hat{\not{\mathcal{D}}} = \int_{\pi^{-1}(\Sigma)} \mathrm{Tr}_s a_6(\hat{\not{\mathcal{D}}}). \tag{2.87}$$

One the other hand, since $\hat{\not{\mathcal{D}}} = \not{\mathcal{D}} + \gamma(\mathbb{h})$ differ by a compact perturbation

$$\mathrm{Index}\, \hat{\not{\mathcal{D}}} = \mathrm{Index}\, \not{\mathcal{D}}. \tag{2.88}$$

Since this applies to arbitrary families Σ we learn that

$$\int_X \mathrm{Tr}_s a_6(\hat{\not{\mathcal{D}}}) = \int_X \mathrm{Tr}_s a_6(\not{\mathcal{D}}) + \mathrm{d}\alpha, \tag{2.89}$$

for some globally well-defined 1-form A on $\mathcal{T}^\partial$. However, since the heat kernel expression is a local expression in the fields we must have

$$\mathrm{Tr}_s a_6(\hat{\not{\mathcal{D}}}) = \mathrm{Tr}_s a_6(\not{\mathcal{D}}) + \mathrm{d}\alpha, \tag{2.90}$$

for some 11-form α, that becomes zero when $\mathbb{h} = 0$. In the next section we will verify this explicitly for the case of flat space to lowest order in $\mathbb{h}$.

2.3.2 The $\mathbb{Z}_2$-anomaly

As noted in [42] there is a natural real structure on the gravitino line bundle, respected by the Bismut-Freed connection, and hence the holonomy group is at most $\mathbb{Z}_2$. In fact, it can very well be equal to $\mathbb{Z}_2$. The coupling of the gravitino to the G-flux respects this real structure, and hence coupling to the G-flux cannot modify the $\mathbb{Z}_2$ anomaly cancellation. It will, however change the one-loop measure. Here we give an expression for that change.

We need to compute

$$\xi(\not{D}_{RS} + \ell^3 \Xi \cdot G) := \xi\Big(\not{D}_{T^*Y} - \frac{\ell^3}{96}\not{G}\Big) - \xi\Big(\not{D} + \frac{\ell^3}{288}\not{G}\Big) - 2\xi\Big(\not{D} - \frac{5\ell^3}{96}\not{G}\Big). \tag{2.91}$$

where ξ is the invariant appearing in the APS index theorem. We introduce a 1-parameter family of such operators by scaling $G \to tG$ and constructing the 12-dimensional operator:

$$\widehat{\mathcal{D}} = \sigma^2 \otimes \frac{\partial}{\partial t} + \sigma^1 \otimes \not{D} + \ell^3 t \sigma^1 \otimes \not{G}, \tag{2.92}$$

acting on spinors in the twelve-manifold $Z = Y \times \mathbb{R}$. In order to apply index theory we should think of the Dirac operator as

$$\widehat{\mathcal{D}} := \mathcal{D} + t\ell^3 \Gamma(\star G), \tag{2.93}$$

where $\Gamma(\star G)$ is the Clifford multiplication by $\star G \in \Omega^7(Z)$ in Cliff(12), $\star$ is the 11-dimensional Hodge operator defined on $\Omega^*(Y)$, and

$$\mathcal{D} = \sigma^2 \otimes \frac{\partial}{\partial t} + \sigma^1 \otimes \not{D} \tag{2.94}$$

is the Dirac operator in 12-dimensions. Then we have

$$\frac{\partial \xi(\not{D} + \ell^3 \not{G}_t)}{\partial t} dt = \int_Y \mathrm{Tr}_s\big(a_6(\widehat{\mathcal{D}})\big)_{(12)}, \tag{2.95}$$

with a_6 being the t-independent part of the heat kernel expansion for $\exp(-t\widehat{\mathcal{D}}^2)$. We can write the tensor products by the Pauli matrices in (2.92) as gamma matrices in 12 dimensions. The 12-form that we integrate on Y in (2.95) can be interpreted as the index density of $\widehat{\mathcal{D}}$, hence in order to extract information on the G-dependence of (2.91), we can use results from geometric index theory for families of operators $\widehat{\mathcal{D}}$, as we did in the case of the local anomaly.

The index of (2.93), is not modified by the presence of the G-flux, hence the flux-correction to the 12-form $\mathrm{Tr}_s\big(a_6(\widehat{\mathcal{D}})\big)_{(12)}$ will be

$$\mathrm{Tr}_s\big(a_6(\widehat{\mathcal{D}})\big)_{(12)} = \mathrm{Tr}_s\big(a_6(\mathcal{D})\big)_{(12)} + \ell^3 \mathrm{d}\varphi_{\mathcal{D}}(\star G, \mathcal{R}), \tag{2.96}$$

with $\int_Y \varphi_{\mathcal{D}}(\star G, \mathcal{R}) : \mathcal{T} \mapsto \mathbb{R}$ a well defined diffeomorphism-invariant function defined on the functional space of bosonic configurations. Adding the contributions of the various terms we obtain an expression of the form:

$$\xi\Big(\not{D}_{T^*Y} - \frac{\ell^3}{96}\not{G}\Big) - \xi\Big(\not{D} + \frac{\ell^3}{288}\not{G}\Big) - 2\xi\Big(\not{D} - \frac{5\ell^3}{96}\not{G}\Big) = \xi(\not{D}_{RS}) + \ell^3 \int_Y \varphi(\star G, \mathcal{R}). \tag{2.97}$$

Since $\varphi(\star G, \mathcal{R})$ is local and gauge invariant we see explicitly that the $\mathbb{Z}_2$ anomaly cancellation is unchanged.

2.4 Example: Eleven manifold with flat boundaries

Let $X := \partial Y = \mathbb{R}^{10}$ be flat 10-dimensional Euclidean space. Let $E \to X$ be the adjoint E_8-vector bundle, and $D_M = \partial_M + \mathbf{A}_M$ the gauge connection on E, i.e. $D_M : \Omega^0(\mathbf{S} \otimes E) \to \Omega^1(\mathbf{S} \otimes E)$. Thus the quadratic action for the gaugino is constructed through the generalized Dirac operator

$$\hat{\slashed{D}}_{E_8} = \gamma^M D_M + \gamma(\mathbb{h}) \tag{2.98}$$

where $\mathbb{h} = -\frac{\ell^3}{24} i_\nu G^\partial$ is the 3-form that comes from contracting the M-theory G-flux in the bulk Y, with the normal unit vector field to the boundary $\partial Y = X$ and

$$\gamma(\mathbb{h}) = \gamma^{M_1 M_2 M_3} \mathbb{h}_{M_1 M_2 M_3}. \tag{2.99}$$

We consider the fibration $\mathcal{X} \to \mathcal{T}^\partial$ encoding the family of geometric data on the fiber X, i.e. gauge connections and fluxes, and calculate the curvature of the Pfaffian line bundle Pfaff $\hat{\slashed{D}}_{E_8} \to \Sigma \hookrightarrow \mathcal{T}^\partial$ using (2.81) as follows

$$\mathcal{F}(\text{Pfaff } \hat{\slashed{D}}_{E_8} \to \Sigma \hookrightarrow \mathcal{T}^\partial) = \pi i \int_X \mathrm{Tr}_s a_6(\hat{\slashed{D}}_{E_8}) \tag{2.100}$$

where $\mathrm{Tr}_s a_6(\hat{\slashed{D}}_{E_8})$ is the t-independent finite part of the heat kernel expansion for

$$\mathrm{Tr}_s \exp(-t\hat{\slashed{D}}^2_{E_8}) \tag{2.101}$$

when $t \to 0$ while $t > 0$. $\mathrm{Tr}_s(\cdot) := \mathrm{Tr}(\gamma^{13}\cdot)$ means supertrace. In contrast to the case with zero flux, there are nonzero divergent terms in the $t \to 0$ expansion. However, these may be easily cancelled by gauge invariant counterterms, so we focus on the t-independent term.

2.4.1 Determining a_6 up to $\mathcal{O}(\mathbb{h}^2)$

Formally, we can expand $\mathrm{Tr}_s(a_6)$ as a series in $\mathbb{h}$:

$$\mathrm{Tr}_s(a_6) = \alpha_0(\mathbb{h}) + \alpha_1(\mathbb{h}) + \alpha_2(\mathbb{h}) + \alpha_3(\mathbb{h}) + \dots, \tag{2.102}$$

with $\alpha_i(\mathbb{h})$ a 2-form in $\mathcal{T}$ which scales homogeneously under scalings of the torsion, i.e. $\alpha_i(\lambda\mathbb{h}) = \lambda^i \alpha_i(\mathbb{h})$. For simplicity, we determine only the lowest correction $\alpha_1(\mathbb{h})$ to $\mathrm{Tr}_s(a_6)$.

In order to evaluate $\mathrm{Tr}_s a_6$, we are going to use known results on heat kernel expansions for generalized Laplacians of the type

$$\Delta = -(\nabla_N \nabla^N + V) \tag{2.103}$$

with $\nabla_N = \partial_N + Q_N$ a first order partial differential operator, $Q_N \mathrm{d}x^N$ a matrix of one-forms and V a scalar matrix. For such operators, the t-independent finite part of the heat kernel expansion for

$$\exp\big(-t(\nabla_N \nabla^N + V)\big) \tag{2.104}$$

has been calculated in flat space using different methods, see [40] and [84]. Thus we want to write $\hat{\not{D}}^2_{E_8}$ as an operator of the type (2.103). If we introduce the connection

$$\nabla_M = \partial_N + \mathbf{A}_N + 3\mathbb{h}_{NM_1M_2}\gamma^{M_1}\gamma^{M_2},$$

we find

$$\hat{\not{D}}^2_{E_8} = -\nabla_N \nabla^N + F_{MN}\gamma^{MN} + \partial_{M_1}\mathbb{h}_{M_2M_3M_4}\gamma^{M_1}\gamma^{M_2M_3M_4} + 4\mathbb{h}^{M_1M_2M_3}\mathbb{h}_{M_1M_2M_3}. \tag{2.105}$$

with $F = d\mathbf{A} + \mathbf{A}\wedge\mathbf{A}$, the curvature of the vector bundle $E \to X$. Hence, as $\nabla_N = \partial_N + \mathbf{A}_N + 3\mathbb{h}_{NM_1M_2}\gamma^{M_1}\gamma^{M_2}$, V in (2.103) is fixed to be

$$V = -F_{MN}\gamma^{MN} - \partial_{M_1}\mathbb{h}_{M_2M_3M_4}\gamma^{M_1}\gamma^{M_2M_3M_4} - 4\mathbb{h}^{M_1M_2M_3}\mathbb{h}_{M_1M_2M_3}. \tag{2.106}$$

Now, having written $\hat{\not{D}}^2_{E_8}$ as a generalized Laplacian, we can use the coefficient calculated in [40, 84],

$$a_6 = \frac{1}{6!}\Big[V^6 + 6V^2\nabla^N(V)V\nabla_N(V) + 4V^3\nabla^N(V)\nabla_N(V) + \mathcal{O}(V^4)\Big], \tag{2.107}$$

to evaluate the lowest order flux correction in $\mathrm{Tr}_s a_6(\hat{\not{D}}_{E_8})$, neglecting the $\mathcal{O}(\mathbb{h}^2)$ terms in V.

We now compute the contribution of every term in (2.107) as follows:

• $\mathrm{Tr}_s\big[V^6\big]$. The most obvious contribution is the leading term $\mathrm{Tr}(F^6)$. The first order contribution in $\mathbb{h}$ is

$$6\mathrm{Tr}_s\big[\partial_{M_1}\mathbb{h}_{M_2M_3M_4}\gamma^{M_1}\gamma^{M_2M_3M_4}\gamma(F)^5\big]. \tag{2.108}$$

As we are working with a 12 dimensional Clifford algebra, only the term proportional to $\gamma^{M_1M_2\ldots M_{12}}$ contributes to the supertrace in (2.108). Thus, we determine the contribution from (2.108) by studying the irreps of the rank 14 tensor

$$\partial_{M_1}\mathbb{h}_{M_2M_3M_4}F_{M_5M_6}\ldots F_{M_{13}M_{14}}, \tag{2.109}$$

defined in dimension 12 under the group $SO(12)$. Some of the symmetries of (2.109) under the permutation of indices are already known, for instance each curvature tensor $F_{M_iM_j}$ contributes antisymmetric couples M_i, M_j, also we know that $\mathbb{h}$ is a completely antisymmetric rank 3 tensor, etc. A detailed analysis along these lines, shows how just the symmetric part of M_1 with the triad

M_2, M_3 and M_4 in (2.109), gives a non zero contribution to the supertrace (2.108). Therefore, we find

$$6\mathrm{Tr}_s\big[\partial_{M_1}\mathbb{h}_{M_2M_3M_4}\gamma^{M_1}\gamma^{M_2M_3M_4}\gamma(F)^5\big] = 6(2-12)\partial^M\mathbb{h}_{MM_1M_2}\mathrm{d}x^{M_1}\mathrm{d}x^{M_2}\mathrm{Tr}(F^5). \tag{2.110}$$

• $\mathrm{Tr}_s\big[V^2\nabla^N(V)V\nabla_N(V)\big]$. Here, the $\mathbb{h}$ term can come from the ∇_N-derivative or from the matrix V. When it comes from the ($\nabla_N = D_N + 3\mathbb{h}_{NM_1M_2}\gamma^{M_1}\gamma^{M_2}$)-derivative, with $D_N = \partial_N + \mathbf{A}_N$ the usual gauge differential, we find

$$-6\mathrm{Tr}_s\big[\gamma(F)^2\cdot\mathbb{h}_{NM_1M_2}\gamma^{M_1M_2}\cdot\gamma(F)^2D^N(\gamma(F))\big], \tag{2.111}$$

which is the same as

$$-6\mathbb{h}_{M_1M_2M_3}\mathrm{d}x^{M_2}\mathrm{d}x^{M_3}\mathrm{Tr}\big(D^{M_1}(F)F^4\big). \tag{2.112}$$

If the $\mathbb{h}$-term comes from V, it cannot come from $\partial^N\mathbb{h}_{NM_1M_2}\gamma^{M_1}\gamma^{M_2}$ because we would combine less than 12 gamma matrices. Thus, up to global numerical factors we find

$$\mathrm{Tr}_s\big[(d\mathbb{h})_{M_1M_2M_3M_4}\gamma^{M_1M_2M_3M_4}\gamma(F)D^N(\gamma(F))\gamma(F)D_N(\gamma(F))\big] \tag{2.113}$$

and

$$\mathrm{Tr}_s\big[\gamma(F)^2D^N((d\mathbb{h})_{M_1M_2M_3M_4}\gamma^{M_1M_2M_3M_4})\gamma(F)D_N(\gamma(F))\big]. \tag{2.114}$$

• $\mathrm{Tr}_s\big[V^3\nabla^N(V)\nabla_N(V)\big]$. These terms are of the same type as in the previous case, just differing in the order of terms. For example, we find

$$\mathrm{Tr}_s\big[(d\mathbb{h})_{M_1M_2M_3M_4}\gamma^{M_1M_2M_3M_4}\gamma(F)^2D^N(\gamma(F))D_N(\gamma(F))\big], \tag{2.115}$$

$$-6\mathrm{Tr}_s\big[\gamma(F)^3\cdot\mathbb{h}_{NM_1M_2}\gamma^{M_1M_2}\cdot\gamma(F)D^N(\gamma(F))\big], \tag{2.116}$$

etc.

• $\mathrm{Tr}_s\big[\mathcal{O}(V^4)\big]$. It is easy to check that these terms only contribute to $\mathcal{O}(\mathbb{h}^2)$.

Thus, the terms above determined are the only ones that contribute to $\alpha_1(\mathbb{h})$ in the expansion of the line bundle curvature in "powers" of $\mathbb{h}$. Furthermore, we can group the terms in $\alpha_1(\mathbb{h})$ that scale as $\lambda^5\alpha_1(\mathbb{h})$ under scalings $F\mapsto\lambda\cdot F$, of the gauge connection curvature F by $\lambda\in\mathbb{R}$. This set of terms coming from (2.108), (2.112) and (2.116), can be written as

$$\mathrm{Tr}_s(V^6 + 6V^2\nabla^N(V)V\nabla_N(V) + 4V^3\nabla^N(V)\nabla_N(V)) =$$

$$-60\partial^M\mathbb{h}_{MPQ}\mathrm{d}x^P\mathrm{d}x^Q\mathrm{Tr}(F^5) - 60\mathbb{h}_{MPQ}\mathrm{d}x^P\mathrm{d}x^Q\mathrm{Tr}(D^M(F)F^4) + \ldots \tag{2.117}$$

where the $\ldots$ refer to terms with different scaling properties.

After evaluating the other supertraces, we find the forms $D_N(\mathrm{d}\mathbb{h})\wedge\mathrm{Tr}\big(D^N(F)F^3\big)$ and $\mathrm{d}\mathbb{h}\wedge$ $\mathrm{Tr}\big(FD_N(F)FD^N(F)\big)$ or $\mathrm{d}\mathbb{h}\wedge\mathrm{Tr}\big(F^2D_N(F)D^N(F)\big)$, depending if it comes from $\mathrm{Tr}_s\big[V^2\nabla^N(V)V\nabla_N(V)\big]$ or $\mathrm{Tr}_s\big[V^3\nabla^N(V)\nabla_N(V)\big]$.

Taking into account the numerical factors and recalling

$$\mathcal{F}(\mathrm{Pfaff}\,\hat{\not{D}}_{E_8}\to\mathcal{T}^\partial)=\pi i\int_X\mathrm{Tr}_s a_6(\hat{\not{D}}_{E_8}),\tag{2.118}$$

we obtain the curvature for the Pfaffian line bundle

$$\begin{aligned}\mathcal{F}(\mathrm{Pfaff}\,\hat{\not{D}}_{E_8}\to\mathcal{T}^\partial)=-\frac{\pi i}{6!}\int_X\Big(&\mathrm{Tr}(F^6)+60\partial^M\mathbb{h}_M\mathrm{Tr}(F^5)\\ &+60\mathbb{h}_M\mathrm{Tr}(D^M(F)F^4)-20D_N(\mathrm{d}\mathbb{h})\wedge\mathrm{Tr}\big(D^N(F)F^3\big)\\ &-12\mathrm{d}\mathbb{h}\wedge\mathrm{Tr}\big(FD_N(F)FD^N(F)\big)-18\mathrm{d}\mathbb{h}\wedge\mathrm{Tr}\big(F^2D_N(F)D^N(F)\big)\Big)+\mathcal{O}(\mathbb{h}^2),\end{aligned}\tag{2.119}$$

where $F=d\mathbf{A}+\mathbf{A}\wedge\mathbf{A}$ and $\mathbb{h}_N=\mathbb{h}_{NMP}\mathrm{d}x^M\wedge\mathrm{d}x^P$. According to our general discussion, we expect to be able to write the correction to the standard curvature $\mathrm{Tr}F^6$ as the total derivative of a gauge invariant local expression. In the next section we will check this explicitly.

2.4.2 Writing the flux corrections as total derivatives

The formula (2.119) allows us to compute the curvature of the M-theory line bundle $\mathcal{L}_G\to\mathcal{T}^\partial$. The curvature of the Chern-Simons bundle $\mathcal{L}_{CS}\to\mathcal{T}^\partial$ exactly cancels the $\mathbb{h}$-independent part of the curvature. Furthermore $\mathcal{L}_{\mathrm{gravitino}}\to\mathcal{T}^\partial$ does not contribute terms to $\alpha_0(\mathbb{h})$ nor $\alpha_1(\mathbb{h})$ in flat space. Therefore only the terms in (2.119) contribute to $\mathcal{F}(\mathcal{L}_G\to\mathcal{T}^\partial)$, up to $\mathcal{O}(\mathbb{h}^2)$.

For the first correction, we work with the set of terms

$$-60\partial^M\mathbb{h}_M\mathrm{Tr}(F^5)-60\mathbb{h}_M\mathrm{Tr}(D^M(F)F^4),\tag{2.120}$$

which have identical behavior under scalings of $\mathbb{h}$ and F. An obvious candidate to write (2.120) as a total divergence seems to be

$$-60\mathrm{d}\big(\mathbb{h}_N\mathrm{Tr}(F_{MP}\mathrm{d}x^PF^4)\big)\delta^{NM}\tag{2.121}$$

with δ^{NM} the Kronecker delta, which is the metric for the twelve dimensional space that we are dealing with. Expanding (2.121) we find

$$\mathrm{d}\big(\mathbb{h}_N\mathrm{Tr}(F_{PM}\mathrm{d}x^PF^4)\big)\delta^{NM}=\mathrm{d}\mathbb{h}_N\mathrm{Tr}(F_{PM}\mathrm{d}x^PF^4)\delta^{NM}+\mathbb{h}_N\mathrm{d}\mathrm{Tr}(F_{PM}\mathrm{d}x^PF^4)\delta^{NM}.\tag{2.122}$$

Furthermore, we can write the exterior differential of a trace over the color indices, as the trace of the covariant exterior differential, i.e.,

$$\mathrm{d}\mathrm{Tr}(F_{PM}\mathrm{d}x^PF^4)=\mathrm{Tr}\big(D(F_{PM}\mathrm{d}x^PF^4)\big)\tag{2.123}$$

with $D = \mathrm{d}x^N D_N \cdot = \mathrm{d}x^N(\partial_N \cdot + [\mathbf{A}_N, \cdot])$. The expression (2.123) holds because the trace of a commutator is zero. Therefore, recalling the Bianchi identity $DF = 0$, we get

$$\mathrm{Tr}\big(D(F_{PM}\mathrm{d}x^P F^4)\big) = \mathrm{Tr}\big(D(F_{PM}\mathrm{d}x^P)F^4\big) = \mathrm{Tr}\big(D_M(F)F^4\big), \tag{2.124}$$

where we have used the identity $D(F_{PM}\mathrm{d}x^P) = D_M(F)$, which follows from the antisymmetry under permutation of couples of indices in the rank 3 tensor $D_M F_{PQ}$. Using (2.124) we can write (2.121) as

$$\mathrm{d}\mathbb{h}_N \mathrm{Tr}(F_{PM}\mathrm{d}x^P F^4)\delta^{NM} + \mathbb{h}_N \mathrm{Tr}\big(D_M(F)F^4\big)\delta^{NM} \tag{2.125}$$

which is not yet clearly equal to (2.120), because the first term. To show how

$$\mathrm{d}\mathbb{h}_N \mathrm{Tr}(F_{PM}\mathrm{d}x^P F^4)\delta^{NM} = \partial^M \mathbb{h}_M \mathrm{Tr}(F^5), \tag{2.126}$$

we study the irreps of the rank 14 tensor

$$\partial_{M_1}\mathbb{h}_{M_2M_3M_4}\mathrm{Tr}(F_{M_5M_6}F_{M_7M_8}\ldots F_{M_{13}M_{14}}) \tag{2.127}$$

with the symmetries under the permutations of indices implicit in the L.H.S. of (2.126). These consist in the completely antisymmetry of the sets $\{M_2, M_3, M_4\}$ and $\{M_1, M_3, M_4, M_6, M_7, M_8, \ldots M_{14}\}$ and the completely symmetry of the couple M_2 and M_6 which is to be contracted with the symmetric tensor $\delta^{M_2M_6}$. Taken into account these constraints, we find that (2.127) defined on a twelve dimensional space lies already in a unique irreducible representation of $SO(12)$ on $(\mathbb{R}^{12})^{\otimes 14}$. This irreducible representation also implies the complete symmetry under permutations of the set of indices $\{M_1, M_2, M_6\}$. Therefore, we can write

$$\begin{aligned}&\partial_{M_1}\mathbb{h}_{M_2M_3M_4}\mathrm{Tr}(F_{M_5M_6}F_{M_7M_8}\ldots F_{M_{13}M_{14}})\mathrm{d}x^{M_1M_3M_4M_6M_7\ldots M_{14}} =\\ &\partial_{M_6}\mathbb{h}_{M_2M_3M_4}\mathrm{Tr}(F_{M_5M_1}F_{M_7M_8}\ldots F_{M_{13}M_{14}})\mathrm{d}x^{M_3M_4M_1M_6M_7\ldots M_{14}}\end{aligned} \tag{2.128}$$

that after contracting with $\delta^{M_2M_6}$ becomes identical to (2.126) as we wanted to prove. Hence

$$-60\partial^M \mathbb{h}_M \mathrm{Tr}(F^5) - 60\mathbb{h}_M \mathrm{Tr}(D^M(F)F^4) = -60\mathrm{d}\big(\mathbb{h}_N \mathrm{Tr}(F_{PM}\mathrm{d}x^P F^4)\delta^{NM}\big). \tag{2.129}$$

The second correction in (2.119) can be written as

$$-20D_N(\mathrm{d}\mathbb{h}) \wedge \mathrm{Tr}\big(D^N(F)F^3\big) = -20\partial_N(\mathrm{d}\mathbb{h}) \wedge \mathrm{Tr}\big(\partial^N(F)F^3\big), \tag{2.130}$$

because $D_N\mathrm{d}\mathbb{h} = \partial_N\mathrm{d}\mathbb{h} + [\mathbf{A}_N, \mathrm{d}\mathbb{h}]$ and $[\mathbf{A}_N, \mathrm{d}\mathbb{h}] = 0$. On the other hand, $\mathrm{Tr}([\mathbf{A}^N, F]F^3) = 0$, thus

$$-20\partial_N(\mathrm{d}\mathbb{h}) \wedge \mathrm{Tr}\big(\partial^N(F)F^3\big) = -5\partial_N(\mathrm{d}\mathbb{h}) \wedge \partial^N\mathrm{Tr}\big(F^4\big) = -\frac{5}{2}\Big[\partial_N\partial^N\Big(\mathrm{d}\mathbb{h} \wedge \mathrm{Tr}\big(F^4\big)\Big)$$

$$-\partial_N\partial^N\Big(\mathrm{d}\mathbb{h}\Big)\wedge\mathrm{Tr}\big(F^4\big)-\mathrm{d}\mathbb{h}\wedge\partial_N\partial^N\Big(\mathrm{Tr}\big(F^4\big)\Big)\Big], \tag{2.131}$$

or using the Hodge Laplacian $\partial_N\partial^N = \star\mathrm{d}\star\mathrm{d}+\mathrm{d}\star\mathrm{d}\star$ in Cartesian coordinates for the Euclidean space X, we write (2.131), as

$$-20\partial_N(\mathrm{d}\mathbb{h})\wedge\mathrm{Tr}\big(\partial^N(F)F^3\big) = -\frac{5}{2}\Big[\mathrm{d}\star\mathrm{d}\star\Big(\mathrm{d}\mathbb{h}\wedge\mathrm{Tr}\big(F^4\big)\Big)$$

$$-\mathrm{d}\star\mathrm{d}\star\Big(\mathrm{d}\mathbb{h}\Big)\wedge\mathrm{Tr}\big(F^4\big)-\mathrm{d}\mathbb{h}\wedge\mathrm{d}\star\mathrm{d}\star\Big(\mathrm{Tr}\big(F^4\big)\Big)\Big]. \tag{2.132}$$

The operator $\star\mathrm{d}\star\mathrm{d}$ never appears, because it always acts on closed forms.

Finally, the third and fourth correction in (2.119), can be written using the covariant Laplacian D_ND^N, as

$$-\mathrm{d}\mathbb{h}\wedge\mathrm{Tr}\big(12FD_N(F)FD^N(F)+18F^2D_N(F)D^N(F)\big)=$$

$$\mathrm{d}\mathbb{h}\wedge\mathrm{Tr}\big(6D_ND^N(F)F^3-3D_ND^N(F^4)+3D_ND^N(F^2)F^2\big). \tag{2.133}$$

Also, we can use a more transparent notation, using the covariant exterior derivative

$$D=\mathrm{d}x^N\wedge D_N=\mathrm{d}+[\mathbf{A},\,\cdot\,] \tag{2.134}$$

we can write the curvature F as $F=D^2$, and the covariant Laplacian as

$$D_ND^N=\star D\star D+D\star D\star \tag{2.135}$$

with $\star$ being the Hodge operator. Using the Bianchi identity $DF=0$, we rewrite (2.133) as

$$\mathrm{d}\mathbb{h}\wedge\mathrm{Tr}\big(6D_ND^N(F)F^3-3D_ND^N(F^4)+3D_ND^N(F^2)F^2\big)=$$

$$\mathrm{d}\mathbb{h}\wedge\mathrm{Tr}\big((6D\star D\star(F)F^3-3D\star D\star(F^4)+3D\star D\star(F^2)F^2\big). \tag{2.136}$$

Now note that

$$\mathrm{Tr}\big(D\star D\star(F^4)\big)=\mathrm{d}\mathrm{Tr}\big(\star D\star(F^4)\big)+\mathrm{Tr}\big([\mathbf{A},\star D\star(F^4)]\big)=\mathrm{d}\mathrm{Tr}\big(\star D\star(F^4)\big) \tag{2.137}$$

is an exact form. On the other hand, consider the 6-forms $\mathrm{Tr}\big(D\star D\star(F)F^2\big)$ and $\mathrm{Tr}\big(D\star D\star(F^2)F\big)$, and differentiate them twice

$$\mathrm{d}^2\mathrm{Tr}\big(D\star D\star(F)F^2\big)=\mathrm{d}\mathrm{Tr}\big(D[D\star D\star(F)F^2]\big)=$$

$$\mathrm{d}\mathrm{Tr}\big(\star D\star(F)F^3\big)=\mathrm{Tr}\big(D\star D\star(F)F^3\big) \tag{2.138}$$

and

$$\mathrm{d}^2\mathrm{Tr}\big(D\star D\star(F^2)F\big)=\mathrm{d}\mathrm{Tr}\big(D[D\star D\star(F^2)F]\big)=$$

$$\mathrm{dTr}\big(\star D\star(F^2)F^2\big)=\mathrm{Tr}\big(D\star D\star(F^2)F^2\big) \tag{2.139}$$

therefore, by construction (2.138) and (2.139) are zero. This means that we can write (2.136), as

$$\begin{aligned}\mathrm{d}\mathbb{h}\wedge\mathrm{Tr}\big(6D\star D\star(F)F^3-3D\star D\star(F^4)+3D\star D\star(F^2)F^2\big)=\\ -3\mathrm{d}\mathbb{h}\wedge\mathrm{dTr}\big(\star D\star(F^4)\big).\end{aligned} \tag{2.140}$$

Using the identities (2.129), (2.132) and (2.140), we can write the curvature of the M-theory line bundle as

$$\begin{aligned}\mathcal{F}(\mathcal{L}_G\to\mathcal{T}^{\partial})=-\frac{\pi i}{6!}\int_X\Big(60\mathrm{d}\big(\mathbb{h}_N\mathrm{Tr}(F_{PM}\mathrm{d}x^PF^4)\delta^{NM}\big)+\\ \frac{5}{2}\mathrm{d}\star\mathrm{d}\star\Big(\mathrm{d}\mathbb{h}\Big)\wedge\mathrm{Tr}\big(F^4\big)-\frac{5}{2}\mathrm{d}\star\mathrm{d}\star\Big(\mathrm{d}\mathbb{h}\wedge\mathrm{Tr}\big(F^4\big)\Big)-\frac{1}{2}\mathrm{d}\mathbb{h}\wedge\mathrm{d}\star\mathrm{d}\star\Big(\mathrm{Tr}\big(F^4\big)\Big)\Big)+\mathcal{O}(\mathbb{h}^2).\end{aligned} \tag{2.141}$$

This formula agrees with the results explained in section 3, where we claimed that the curvature of $\mathcal{L}_G\to\mathcal{T}^{\partial}$ is an exact form $\mathrm{d}A$, with A being a $\mathcal{G}$-equivariant one form on $\mathrm{Met}(Y)\times\mathcal{A}$. From (2.141), we can write A as:

$$\begin{aligned}A=-\frac{\pi i}{6!}\int_X\Bigg(60\big(\mathbb{h}_N\mathrm{Tr}(F_{PM}\mathrm{d}x^PF^4)\delta^{NM}\big)+\frac{5}{2}\star\mathrm{d}\star\Big(\mathrm{d}\mathbb{h}\Big)\wedge\mathrm{Tr}\big(F^4\big)\\ -\frac{5}{2}\star\mathrm{d}\star\Big(\mathrm{d}\mathbb{h}\wedge\mathrm{Tr}\big(F^4\big)\Big)-\frac{1}{2}\mathrm{d}\mathbb{h}\wedge\star\mathrm{d}\star\Big(\mathrm{Tr}\big(F^4\big)\Big)\Bigg)+\dots\end{aligned} \tag{2.142}$$

up to a globally exact form. The Hodge $\star$ depends on a metric on $\Sigma\hookrightarrow\mathcal{T}^{\partial}$. Note that there is a natural metric on $\mathcal{T}^{\partial}$, induced by the Riemannian metric itself.

2.4.3 Covariant form of the Anomaly

To get a better understanding of these flux corrections to the anomaly, it is instructive to calculate the contribution from the fluxes to the divergence of the gauge current using the gaussian cutoff proposed by Fujikawa. This approach to anomaly cancellation leads to the so-called covariant form of the anomaly. See [5, 11]. Fujikawa proposed to account for the local chiral anomaly from the variation of the measure $[d\chi][d\overline{\chi}]$ under the action of the gauge group in the path integral

$$\int[d\chi][d\overline{\chi}]\exp\big(\int_X\overline{\chi}\hat{\not{D}}\chi\big) \tag{2.143}$$

If $\{T_a\}$ is a basis for the Lie algebra of the gauge group $\mathcal{G}=\mathcal{E}_8$, then an infinitesimal gauge transformation can be expressed as $g=\mathbb{I}+\Lambda^aT_a+\mathcal{O}(\Lambda^2)$. We can compute

$$\frac{|d_{T_a}\det\hat{\not{D}}_{E_8}|}{|\det\hat{\not{D}}_{E_8}|}:=\mathrm{d}j_a=2i\,\mathrm{Tr}\Big[T_a\gamma^{11}\exp\big(-t\hat{\not{D}}^2_{E_8}\big)\Big], \tag{2.144}$$

where $j_a \in \Omega^9(X)$ is the gauge current.

Of course, $\mathrm{Tr}\big(T_a \gamma^{11} \exp\big(-t\hat{\not{D}}^2_{E_8}\big)\big)$ must be regulated, and we do so by taking

$$\mathrm{Tr}\Big[T_a \gamma^{11} \exp\big(-t\hat{\not{D}}^2_{E_8}\big)\Big],$$

where $t = 1/\Lambda$ should tend to zero. In stark contrast to the case without fluxes, the expression for dj_a has divergent terms for $t \to 0$. These divergent terms can be shown to be total covariant divergences of local gauge invariant expressions in the fields by a method explained below for the t-independent part of the heat kernel. Thus the current must be renormalized by adding these terms.

In order to evaluate the regulator independent part of the supertrace (2.144) we have to determine the heat kernel coefficient a_5. We can use again the results of [40], to calculate (2.144) up to second order in $\mathbb{h}$, i.e.

$$\mathrm{d}j_a = 2i\mathrm{Tr}\Big(T_a \gamma^{11} a_5(\hat{\not{D}}_{E_8})\Big) = \beta_0(\mathbb{h}) + \beta_1(\mathbb{h}) + \ldots, \tag{2.145}$$

where $\beta_k(\mathbb{h})$ are terms that scale homogeneously under dilations of $\mathbb{h}$, i.e. if λ is a real parameter then $\beta_k(\lambda\mathbb{h}) = \lambda^k \beta_k(\mathbb{h})$.

Therefore, the only terms in a_5 which contribute up to first order are

$$a_5 = \frac{1}{5!}\Big[V^5 + 2V\nabla_N(V)V\nabla^N(V) + 3V^2\nabla_N(V)\nabla^N(V) + \mathcal{O}(V^3)\Big]. \tag{2.146}$$

Doing a similar calculation as we did above for the heat kernel coefficient a_6, we find

$$\begin{aligned}\mathrm{d}j_a = \frac{2i}{5!}\mathrm{Tr}\Big(T_a F^5\Big) + \frac{4i}{5!}\Bigg[&20\partial^N \mathbb{h}_N \wedge \mathrm{Tr}(F^4) + 4\mathbb{h}_N \wedge \mathrm{Tr}(FD^N(F)F^2) + \\ 16\mathbb{h}_N \wedge \mathrm{Tr}(F^3 D^N(F)) + d\mathbb{h} \wedge \mathrm{Tr}\Big(&T_a D_N(F)FD^N(F) + 4T_a F D_N(F) D^N(F)\Big) + \\ \partial_N(d\mathbb{h}) \wedge \mathrm{Tr}\Big(&4T_a F^2 D^N(F) + T_a F D^N(F) F\Big)\Bigg] + \beta_2(\mathbb{h}) + \ldots\end{aligned} \tag{2.147}$$

where $D_N = \partial_N + \mathbf{A}_N$ is the gauge covariant derivative. The "Chern-Simons" terms, exactly cancel the expression $\frac{2i}{5!}\mathrm{Tr}\Big(T_a F^5\Big)$ in (2.147). We can then write the flux corrections to the anomalous divergence of the gauge current as the covariant exterior derivative of a gauge invariant 9-form $\Delta j(\mathbb{h})$:

$$\begin{aligned}D\Delta j = \frac{4i}{5!}\Bigg[&20\partial^N \mathbb{h}_N \wedge F^4 + 4\mathbb{h}_N \wedge FD^N(F)F^2 + 16\mathbb{h}_N \wedge F^3 \wedge D^N(F) + d\mathbb{h} \wedge \\ &\Big(D_N(F)FD^N(F) + 4FD_N(F)D^N(F)\Big) + \partial_N(d\mathbb{h}) \wedge \Big(4F^2 D^N(F) + FD^N(F)F\Big)\Bigg] + \ldots\end{aligned} \tag{2.148}$$

where we have written the expression as a Lie-algebra valued form.

We now show explicitly how this can be written as a total divergence of a gauge invariant quantity. For the three first terms in (2.148), one can show how

$$20\partial^N \mathrm{lh}_N F^4 + 4\mathrm{lh}_N F D^N(F)F^2 + 16\mathrm{lh}_N F^3 D^N(F) =$$

$$D\big(4\mathrm{lh}_N F F_{PM}\mathrm{d}x^P F^2 + 16\mathrm{lh}_N F^3 F_{PM}\mathrm{d}x^P\big)\delta^{NM}. \tag{2.149}$$

Expanding (2.149), using the identity $D_N(F) = D(F_{MN}\mathrm{d}x^M)$ gives us

$$D\big(4\mathrm{lh}_N F F_{PM}\mathrm{d}x^P F^2 + 16\mathrm{lh}_N F^3 F_{PM}\mathrm{d}x^P\big)\delta^{NM} = \big(4\mathrm{d}\mathrm{lh}_N F F_{PM}\mathrm{d}x^P F^2 +$$

$$16\mathrm{d}\mathrm{lh}_N F^3 F_{PM}\mathrm{d}x^P)\delta^{NM} + 4\mathrm{lh}_N F D^N(F)F^2 + 16\mathrm{lh}_N F^3 D^N(F), \tag{2.150}$$

thus, we have to prove the identity

$$20\partial^N \mathrm{lh}_N F^4 = \big(4\mathrm{d}\mathrm{lh}_N F F_{PM}\mathrm{d}x^P F^2 + 16\mathrm{d}\mathrm{lh}_N F^3 F_{PM}\mathrm{d}x^P)\delta^{NM}. \tag{2.151}$$

This can be achieved by analyzing the irreps of the rank 12 tensor

$$\partial_{M_1}\mathrm{lh}_{M_2M_3M_4}F_{M_5M_6}F_{M_7M_8}\ldots F_{M_{11}M_{12}}, \tag{2.152}$$

with antisymmetry under permutations of the sets of indices $\{M_2,\ M_3,\ M_4\}$ and $\{M_1,\ M_3,\ M_4,\ M_5,\ M_6,\ M_7,\ \ldots M_{12}\}$ and symmetry under permutations of the couple $\{M_2,\ M_8\}$. The only irreducible representation of $SO(10)$ in $(\mathbb{R}^{10})^{\otimes 12}$ which satisfies such properties under permutations of indices, also verifies the complete symmetry of the set $\{M_1,\ M_2$ and $M_8\}$, therefore we can prove the identity (2.151) by using the symmetry under permutations of M_1 and M_8, contracting M_2 and M_8 with the Kronecker delta $\delta^{M_2M_8}$ and contracting the other indices with their corresponding grassmann differentials. One should also consider the same argument with M_{12} playing the role of M_8 in order to achieve the full proof.

For the second set of terms, we realize that $D_N(F)FD^N(F) + 4FD_N(F)D^N(F)$ using the Laplacian $D_N D^N = \star D\star D + D\star D\star$. A short calculation yields

$$D_N(F)FD^N(F) + 4FD_N(F)D^N(F) = \frac{1}{2}D\star D\star(F^3) + \frac{3}{2}FD\star D\star(F^2)$$

$$-\frac{1}{2}D\star D\star(F^2)F - \frac{3}{2}FD\star D\star(F)F - 2F^2D\star D\star(F). \tag{2.153}$$

If again we use the identity $D_N(F) = D(F_{MN}\mathrm{d}x^M) := D(F_N)$, then we can write (2.148) as

$$D\Delta j = \frac{4i}{5!}\Big[D\big(4\mathrm{lh}_N F F_M F^2 + 16\mathrm{lh}_N F^3 F_M\big)\delta^{NM} + d\mathrm{lh}\wedge\Big(\frac{1}{2}D\star D\star(F^3) +$$

$$\frac{3}{2}FD\star D\star(F^2) - \frac{1}{2}D\star D\star(F^2)F - \frac{3}{2}FD\star D\star(F)F - 2F^2D\star D\star(F)\Big) +$$

$$\partial^N(d\mathbb{h}) \wedge \Big(4F^2 D(F_N) + FD(F_N)F\Big)\Bigg] + \dots \tag{2.154}$$

Finally, using the Bianchi identity $DF = 0$, it is easy to prove that

$$\Delta j = \frac{4i}{5!}\Bigg[\big(4\mathbb{h}_N F F_M F^2 + 16\mathbb{h}_N F^3 F_M\big)\delta^{NM} + d\mathbb{h} \wedge \Big(\frac{1}{2}\star D \star (F^3) +$$

$$\frac{3}{2}F \star D \star (F^2) - \frac{1}{2}\star D \star (F^2)F - \frac{3}{2}F \star D \star (F)F - 2F^2 \star D \star (F)\Big) +$$

$$\partial^N(d\mathbb{h}) \wedge \Big(4F^2 F_N + F F_N F\Big)\Bigg] + \dots . \tag{2.155}$$

This gives a non-trivial redefinition of the gauge current, by gauge invariant flux-dependent 9-forms $\Delta j(\mathbb{h})$.

Chapter 3

Differential-geometric characterization of Calabi-Yau compactifications

3.1 Motivation

The study of compactification of higher dimensional theories starts by making use of compactification manifolds with a good deal of symmetry, such as the torus, sphere, squashed spheres and so on. Such spaces have explicitly known metrics, allowing explicit solutions of the equations of motion, and explicit Kaluza-Klein reduction. However their high degree of symmetry tends to be a problem in trying to obtain models with the level of complexity of the Standard Model or its often-postulated extensions.

A way of increasing the complexity of the models one can obtain, is by using manifolds for which the relevant metrics are known to exist by general theorems, but for which explicit expressions are not known. The most famous examples are the Ricci-flat Kahler metrics conjectured to exist by Calabi and proven to exist by Yau [90]. In 1985, it was proposed by Candelas *et al* [23] that compactification of the heterotic string on a Calabi-Yau manifold could lead to quasi-realistic theories of particle physics, containing grand unified extensions of the Standard Model and low energy supersymmetry. Since then, other metrics of this type, such as G_2 holonomy metrics, have been used in quasi-realistic compactifications; see for example [1].

Over the subsequent years, many tricks were developed to bypass the difficulties posed by not knowing the compactification metric. These tricks began with the algebraic geometry behind the theorems of Yau and Donaldson-Uhlenbeck-Yau, and gradually evolved into entire branches of mathematical physics, such as topological string theory and special geometry. To drastically oversimplify, the general picture is that certain "protected" quantities in the four dimensional effective field theory, such as the superpotential in theories with four supercharges, and the prepotential in theories with eight supercharges, can be computed using techniques combining algebraic geometry with physical ideas. Other quantities, such as the Kahler potential in theories with four supercharges, cannot be computed directly. Since a good deal of important physics depends on the Kahler potential – precise values of particle masses, and the existence and stability of supersymmetry breaking vacua, this situation is not very satisfactory.

Almost all present knowledge about the Kahler potential in the EFT comes from studying expansions around more computable limits. The best known example is the case of $\mathcal{N} = 1$ compactifications which contain $\mathcal{N} = 2$ subsectors, such as heterotic $(2, 2)$ models, or type II on Calabi-Yau orientifolds. In these cases, there is a limit in which part of the $\mathcal{N} = 1$ Kahler potential becomes equal to that of the related $\mathcal{N} = 2$ theory, which is computable using special geometry. Other examples include the solvable orbifold or Gepner model limits, at which the entire Kahler potential is computable in principle using CFT techniques. However, it is not clear at present how representative such results are of the general case. Even a limited ability to compute in the general case would allow studying this question.

One completely general technique for addressing such problems is to compute the Ricci-flat metrics and related quantities numerically. Numerical methods are unavoidable in other areas of physics, beginning with such seemingly elementary problems as computing the spectrum of the helium atom or integrating Newton's equations for the three body problem in celestial mechanics; it would be surprising if string theory could avoid this. To bring string theory closer to a possible confrontation with real data, for example from collider physics, it may be valuable to develop these missing parts of the theory of compactification.

One can start in this direction by showing two things, [35, 36]. First, we review how to use existing mathematical techniques to numerically approximate metrics on Kahler manifolds, along lines recently developed by Donaldson [33]; we will explain the numerical methods in more detail and do some simple computations of terms in the EFT for compactification on a quintic Calabi-Yau 3-manifold. Second, we extend these mathematical techniques to hermitian Yang-Mills connections. It will be clear that these techniques could be pushed to compute higher order terms, metrics on moduli spaces, special Lagrangian submanifolds, etc.

Our direct inspirations are Donaldson's work [33] on numerical approximation of metrics, and of Wang [85] developing the corresponding mathematics for vector bundles.

Let us briefly explain the problem and survey some of the approaches one might take towards it, before beginning the detailed development in next section. Following [23], the derivation of the matter Lagrangian in a heterotic compactification on a Calabi-Yau X carrying a bundle V involves the following steps:

1. Find the Ricci-flat metric $g_{i\bar{j}}$ (with specified moduli) on X.

2. Find the hermitian Yang-Mills connection A_i on V.

3. Find the zero modes ψ^α of the Dirac operator. As is standard, on a Kahler manifold

this amounts to finding harmonic differential forms ψ valued in V, *i.e.* solutions of $0 = (\overline{\partial} + \overline{A})\psi = (\overline{\partial} + \overline{A})^* \psi$, where $*$ denotes the adjoint operator.

4. Find an orthonormal basis of forms ψ.

5. Compute the integrals over X of wedge products of these forms to get the superpotential.

The key step for us is (4). Existing methods for computing the superpotential, such as [47, 24], accomplish step (5) without needing the results of (1) and (2), by using unnormalized zero modes. This leads to a superpotential defined in terms of fields whose kinetic term is obtained from "some" unknown Kahler potential. To do better, we must either derive normalized zero modes in (4) for use in (5), or else take the zero modes used in (5) and compute their normalizations using the explicit metric from (1).

There seems to be no way of doing this without some knowledge of the Ricci-flat metric and thus the first step is to choose some approximation scheme for this metric. One's first thought might be to follow standard practice in numerical relativity, as done in [49], and introduce a six dimensional lattice which is a discrete approximation to the manifold X; in other words a position space approach. Taking the Kahler potential K as the basic dynamical variable, Einstein's equations reduce to the complex Monge-Ampere equation

$$\det(\partial\overline{\partial}K) = \Omega \wedge \overline{\Omega} \tag{3.1}$$

which can be solved by relaxation methods. One would then need to find similar lattice approximations for the connection on V and the zero modes.

An alternative approach, introduced by Donaldson [33], is to use geometric quantization to model the differential geometry of the compact manifold using finite dimensional data. One uses the overcomplete basis of coherent states to recover the smooth geometry in the semiclassical limit. Such basis of coherent states define a natural embedding of X into $\mathbb{P}^{N-1}$ provided by the N sections of an ample line bundle $\mathcal{L}^k$ (we will explain this in detail below). We then take as a candidate approximating metric on X the pull-back of a Fubini-Study metric on $\mathbb{P}^{N-1}$. Such a metric is defined by an $N \times N$ hermitian matrix. By suitably choosing this matrix we can try to make the associated Fubini-Study metric restrict to X in such a way that it gives a good approximation to the Ricci-flat metric on X.

A major advantage of this approach is that it avoids the complications and arbitrariness involved in choosing an explicit discretization of X; rather the entire approximation scheme follows from a single parameter k, the scale of the first Chern class of $\mathcal{L}$. Subsequent mathematical development reveals more structure which can be used to our advantage. For example, a very natural approximation to the Ricci-flat metric, which becomes exact as $k \to \infty$, is the so-called "balanced" metric. In a sense, to be described below, this is the metric for which the embedding of X into $\mathbb{P}^{N-1}$ has its center of mass at the "origin". It also satisfies a simple fixed point condition which can be used for relaxation, solving step (1).

Another advantage, which is key for the present application, is that Donaldson's method can be naturally extended to study holomorphic vector bundles on X. There is a standard relation between holomorphic connections and hermitian metrics, which we review in section 2, in which step (2) of the above prescription is turned into the problem of finding a hermitian-Einstein metric on a vector bundle. For illustrative purposes we will explicitly study hermitian-Einstein metrics on two spaces: complex projective space $\mathbb{P}^n$ and the Fermat quintic threefold.

The organization of this chapter is as follows. In Section 3.2 we review the basics of Calabi-Yau compactifications of the heterotic string and we review an example of a quasi-realistic compactification to show the degree of complexity involved in such models. In section 3.3 we provide an overview of the geometric background needed for our numerical construction (in particular we will describe Donaldson's approach for getting metrics of constant scalar curvature). In Section 3.4 we explain a numerical approximation to the hermitian Einstein metric on a holomorphic vector bundle by a simple adaptation of Donaldson's scheme, building on the mathematical work of Wang. In section 3.5 we describe some of the numerical methods involved in our computation; for instance we explain the numerical integration algorithm to approximate integrals on algebraic varieties. Finally, in section 3.6 we focus on several explicit examples and specific results.

3.2 Calabi-Yau compactifications of the heterotic string

Let us assume we are given a $10 = 4 + 6$ dimensional field theory. A compactification is then a 10-dimensional space-time which is topologically the product of a 4-dimensional space-time with an 6-dimensional manifold X, the compactification or "internal" manifold, carrying a Riemannian metric and with definite expectation values for all other fields in the 10d theory. These must solve the equations of motion, and preserve 4-dimensional Poincaré invariance.

The most general metric ansatz for a Poincaré invariant compactification is

$$G_{IJ} = \begin{pmatrix} f_W \ \eta_{\mu\nu} & 0 \\ 0 & G_{ij} \end{pmatrix},$$

where the tangent space indices are $0 \le I < 10$, $0 \le \mu < 4$, and $1 \le i \le 6$. Here $\eta_{\mu\nu}$ is the Minkowski metric, G_{ij} is a metric on X, and f_W is a real valued function on X called the "warp factor."

Contact with the SM requires finding compactifications to $d = 4$ with at most $\mathcal{N} = 1$ supersymmetry, because the SM includes chiral fermions, which are incompatible with $\mathcal{N} > 1$. Let us start with the $E_8 \times E_8$ heterotic string or "HE" theory [1].

Besides the metric, the other bosonic fields of the HE supergravity theory are a scalar Φ called the dilaton, Yang-Mills gauge potentials for the group $G \equiv E_8 \times E_8$, and a two-form gauge potential B (often called the "Neveu-Schwarz" or "NS" two-form) whose defining characteristic is that it minimally couples to the heterotic string world-sheet. We will need their gauge field strengths below: for Yang-Mills, this is a two-form F^a_{IJ} with a indexing the adjoint of $\mathfrak{G}$, and for the NS two-form this is a three-form H_{IJK}. Denoting the two Majorana-Weyl spinor representations of $SO(1,9)$ as S and C, then the fermions are the gravitino $\psi_I \in S \otimes V$, a spin 1/2 "dilatino" $\lambda \in C$, and the adjoint gauginos $\chi^a \in S$. We use Γ_I to denote Dirac matrices contracted with a "zehnbein," satisfying $\{\Gamma_I, \Gamma_J\} = 2G_{IJ}$, and $\Gamma_{IJ} = \frac{1}{2}[\Gamma_I, \Gamma_J]$ etc.

A local supersymmetry transformation with parameter ϵ is then

$$\delta\psi_I = D_I\epsilon + \frac{1}{8}H_{IJK}\Gamma^{JK}\epsilon \tag{3.2}$$

$$\delta\lambda = \partial_I\Phi\Gamma^I\epsilon - \frac{1}{12}H_{IJK}\Gamma^{IJK}\epsilon \tag{3.3}$$

$$\delta\chi^a = F^a_{IJ}\Gamma^{IJ}\epsilon. \tag{3.4}$$

We now assume $\mathcal{N} = 1$ supersymmetry. An unbroken supersymmetry is a spinor ϵ for which the left hand side is zero, so we seek compactifications with a unique solution of these equations.

For simplicity we discuss the case $H = 0$. Setting $\delta\psi_\mu$ in Eq. (3.2) to zero, we find that the warp factor f_W must be constant. The vanishing of $\delta\psi_i$ requires ϵ to be a covariantly constant spinor. For a six-dimensional space X to have a unique such spinor, it must have $SU(3)$ holonomy, in other words X must be a Calabi-Yau threefold. In the following we use basic facts about their geometry.

The vanishing of $\delta\lambda$ then requires constant dilaton Φ, while the vanishing of $\delta\chi^a$ requires the gauge field strength F to solve the hermitian Yang-Mills (HYM) equations,

$$F^{2,0} = F^{0,2} = G^{i\bar{j}}F_{i\bar{j}} = 0. \tag{3.5}$$

Choose such a bundle $E \to X$, where the HYM gauge connection lives; by the general discussion above, the commutant of H in G will be the automorphism group of the connection on E and thus the low energy gauge group of the resulting EFT. For example, since E_8 has a maximal

[1] This choice is made rather than HO because only in this case can we find the SM fermion representations as subrepresentations of the adjoint of the gauge group.

$E_6 \times SU(3)$ subgroup, if E has structure group $H = SL(3)$, there is an embedding such that the unbroken gauge symmetry is $E_6 \times E_8$, realizing one of the standard grand unified groups E_6 as a factor.

The choice of E is constrained by anomaly cancellation. This discussion modifies the Bianchi identity for H to

$$dH = \mathrm{Tr}\,(R \wedge R) - \frac{1}{30} \sum_a F^a \wedge F^a \tag{3.6}$$

where R is the matrix of curvature two-forms.

Thus, specifying a classical vacuum of a $\mathcal{N} = 1$ compactification, requires fixing a Ricci-flat metric on X and a hermite Yang-Mills gauge connection on $E \to X$. We explain in the next section how we approximate such objects, building on ideas of S. Donaldson, S. T. Yau and others [35, 36]. General arguments imply that these supersymmetric Minkowski solutions are stable, so the small fluctuations consist of massless and massive fields. Let us now discuss a few of the massless fields. Since the EFT has $\mathcal{N} = 1$ supersymmetry, the massless scalars live in chiral multiplets, which are local coordinates on a complex Kähler manifold.

First, the moduli of Ricci-flat metrics on X will lead to massless scalar fields: the complex structure moduli, which are naturally complex, and Kähler moduli, which are not. However, in string compactification the latter are complexified. Massless charged matter will arise from zero modes of the gauge field and its supersymmetric partner spinor χ^a. Substituting these zero modes into the ten-dimensional Yang-Mills action and integrating, one can derive the $d = 4$ EFT. For example, the cubic terms in the superpotential, usually called Yukawa couplings after the corresponding fermion-boson interactions in the component Lagrangian, are obtained from the cubic product of zero modes

$$\int_X \Omega \wedge \mathrm{Tr}\,(\phi_1 \wedge \phi_2 \wedge \phi_3)\,,$$

where Ω is the holomorphic three form on X, $\phi_i \in H^{0,1}(X, \mathrm{Rep}\,E)$ are the zero modes, and Tr arises from decomposing the E_8 cubic group invariant.

3.2.1 The Kähler potential

One can go further and solve the zero mode equations for the charged matter fields, by linearizing (3.5). Let us expand the 10d gauge connection around one solution to (3.5),

$$A^{(10)}(z,x) = \delta A_\mu(x) dx^\mu + A_i^{(6)} dz^i + \overline{A}_{\bar{j}}^{(6)} d\bar{z}^{\bar{j}} + \sum_p \phi_p(x) \delta_p A_i dz^i + \sum_{\bar{p}} \phi^*_{\bar{p}}(x) \overline{\delta}_{\bar{p}} \overline{A}_{\bar{j}} d\bar{z}^{\bar{j}}. \tag{3.7}$$

Here $\{\delta_p A^{(6)}\}$ is a basis of infinitesimal deformations of the connection $A^{(6)}$ which preserve the hermitian Yang-Mills equations (3.5), and $\phi_p(x)$ are scalar fields in $\mathbb{R}^4$. These will include both bundle moduli and charged matter fields, depending on the transformation properties of the corresponding deformation $\{\delta_p A^{(6)}\}_p$ under the unbroken gauge symmetry. Our explicit example in [37] consists of a $SU(3)$ bundle, leaving unbroken E_6 gauge symmetry. In this case, we are interested in deformations which transform as the fundamental and adjoint of $SU(3)$, which correspond to charged matter fields according to the decomposition of the E_8 adjoint under $SU(3) \times E_6$ as

$$248 = (1, 78) \oplus (8, 1) \oplus (3, 27) \oplus (\overline{3}, \overline{2}7).$$

Of course, by $\mathcal{N} = 1$ supersymmetry these scalar fields will come with partner fermions, corresponding to massless solutions of the Dirac equation on X. One could derive the supersymmetric effective field theory by working with either component; we will use the scalars and (3.7).

The kinetic term for the scalars ϕ_p is a sigma model kinetic term using the natural metric on the moduli space of solutions of the hermitian Yang-Mills equations. We can derive this physically by starting from the bosonic part of the ten-dimensional Yang-Mills action,

$$S_{YM} = \int_{X \times \mathbb{R}^4} \mathrm{dVol}^{(10)}\, \mathrm{Tr}\big(F_{IJ} F^{IJ}\big), \tag{3.8}$$

and performing dimensional reduction, to obtain

$$\begin{aligned} S_{kin}[\phi] &= \int_{X \times \mathbb{R}^4} \mathrm{dVol}^{(10)}\, \mathrm{Tr}\big(F_{i\mu} F_{\bar{j}\nu}\big) G^{i\bar{j}} \eta^{\mu\nu} && (3.9) \\ &= \sum_{p,\bar{q}} \int_X \mathrm{dVol}^{(6)}\, \mathrm{Tr}\big(\delta_p A_i,\, \overline{\delta}_{\bar{q}} A_{\bar{j}}\big) G^{i\bar{j}} \times \int_{\mathbb{R}^4} \mathrm{d}^4 x\, \partial_\mu \phi_p \partial^\mu \phi^*_{\bar{q}}, && (3.10) \end{aligned}$$

with $G_{i\bar{j}}$ the Ricci-flat metric on X, and $\mathrm{dVol}^{(6)} = \Omega \wedge \overline{\Omega}$ the compatible volume form.

Thus, to get canonically normalized 4d fields, we need to use an orthonormal basis of deformations in (3.7). Below we outline how this can be done.

3.2.2 Example: A quasi-realistic compactification

To illustrate the complexity involved in a quasi-realistic Calabi-Yau compactification of the heterotic string, we review the construction of [16], which describes a model of particle physics with similar matter content to the Minimal Supersymmetric Standard Model (MSSM). In [46], the author and collaborators described the Kähler moduli of the compactification and study the slope stability of the vector bundles involved. Note that these constructions are only a first step to determining the associated effective action of the 4d theory; this example indicates the gap of knowledge that we are trying to fill between the holomorphic data of the compactification –which is known and shown in this example– and the non-holomorphic data, which we want to determine. For more details, the reader can consult the references.

The Elliptic Calabi-Yau and its Kähler Cone

First, we briefly recall the construction of the Calabi-Yau threefold used in such heterotic standard model, following the reference [17]. Let $\widetilde{X}$ be the fiber product over $\mathbb{P}^1$ of two rational elliptic surfaces $\widetilde{X} = B_1 \times_{\mathbb{P}^1} B_2$, as in the diagram:

$$\begin{array}{ccccc} & & \widetilde{X} & & \\ & \pi_1 \swarrow & & \searrow \pi_2 & \\ B_1 & & \downarrow \pi & & B_2 \\ & \beta_1 \searrow & & \swarrow \beta_2 & \\ & & \mathbb{P}^1 & & \end{array}$$

This kind of Calabi-Yau threefolds were already studied by C. Schoen in [76]. The geometry of $\widetilde{X}$, is basically encoded in the geometry of the rational elliptic surfaces B_1 and B_2. Due to the phenomenological interest in finding threefolds which admit certain Wilson lines[2], the aim of [17] was to look for threefolds $\widetilde{X}$ such that $\mathbb{Z}_3 \times \mathbb{Z}_3 \subseteq \mathrm{Aut}(\widetilde{X})$. This search was achieved thanks to the existence of certain elliptic surfaces that admit an action of $\mathbb{Z}_3 \times \mathbb{Z}_3$ which can be characterized explicitly through a proper understanding of the Mordell-Weil group of B.

Following the Kodaira's classification of singular fibers, our elliptic surfaces B_1 and B_2 are characterized by three I_1 and three I_3 singular fibers. Such rational elliptic surfaces are described by one-dimensional families, that allow us to build fiber products $\widetilde{X}$, corresponding to smooth Calabi-Yau threefolds. Furthermore, $\widetilde{X}$ admits a free action of $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ and the quotient $X = \widetilde{X}/G$ is also a smooth Calabi-Yau threefold with fundamental group $\pi_1(X) = \mathbb{Z}_3 \times \mathbb{Z}_3$.

The threefold used in the description of this heterotic compactification is $X = \widetilde{X}/G$, although we will work with G-equivariant objects on $\widetilde{X}$. In the rest of this section we describe the G-invariant homology rings of B and $\widetilde{X}$, and their corresponding G-invariant Kähler cones (i.e. their ample cones, or spaces of polarizations).

For the homology of a surface B, we choose as set of generators: the 0-section σ, the generic fiber F, the 6 irreducible components of the three I_3 singular fibers that do not intersect the 0-section $\Theta_{1,1}, \Theta_{1,2}, \ldots \Theta_{3,1}, \Theta_{3,2}$ and the two sections generating the free part of the Mordell-Weil group[3] ξ and $\alpha_B \xi$. These generators are a basis for $H_2(B,\mathbb{Z}) \otimes \mathbb{Q}$, but adding the torsion generator of the Mordell-Weil group

$$\eta = \sigma + F - \frac{2}{3}\Big(\Theta_{1,1} + \Theta_{2,1} + \Theta_{3,1}\Big) - \frac{1}{3}\Big(\Theta_{1,2} + \Theta_{2,2} + \Theta_{3,2}\Big), \tag{3.11}$$

[2]I.e. flat line bundles with non-trivial holonomy.

[3]See Appendix A, for a complete description of the Mordell-Weil group of the elliptic surface.

we generate all $H_2(B,\mathbb{Z})$.

The intersection matrix of the homology generators is as follows:

$$\begin{pmatrix} \sigma \\ F \\ \Theta_{1,1} \\ \Theta_{2,1} \\ \Theta_{3,1} \\ \Theta_{1,2} \\ \Theta_{2,2} \\ \Theta_{3,2} \\ \xi \\ \alpha_B\xi \\ \eta \end{pmatrix}^T \cdot \begin{pmatrix} \sigma \\ F \\ \Theta_{1,1} \\ \Theta_{2,1} \\ \Theta_{3,1} \\ \Theta_{1,2} \\ \Theta_{2,2} \\ \Theta_{3,2} \\ \xi \\ \alpha_B\xi \\ \eta \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

The invariant homology under the action of $G = \mathbb{Z}_3 \times \mathbb{Z}_3$, is generated by

$$H_2(B,\mathbb{Z})^G = \mathrm{span}_{\mathbb{Z}}\Big\{F,\, t = -\sigma + \Theta_{2,1} + \Theta_{3,1} + \Theta_{3,2} + 2\xi + \alpha_B\xi + \eta - F\Big\}, \tag{3.12}$$

where t can be also expressed as the homological sum of three sections, i.e. $t = \xi + \alpha_B\xi + \eta \boxplus \xi$. The cohomology ring of X, can be expressed as

$$H^*(X,\,\mathbb{Q}) = H^*(\widetilde{X},\,\mathbb{Q})^G \tag{3.13}$$

using the G-invariant cohomology of $\widetilde{X}$. Hence

$$H^2(\widetilde{X},\,\mathbb{Q})^G = \left(\frac{H^2(B_1,\,\mathbb{Q}) \oplus H^2(B_2,\,\mathbb{Q})}{H^2(\mathbb{P}^1,\,\mathbb{Q})}\right)^G = \frac{H^2(B_1,\,\mathbb{Q})^G \oplus H^2(B_2,\,\mathbb{Q})^G}{H^2(\mathbb{P}^1,\,\mathbb{Q})}, \tag{3.14}$$

that due to (3.12), is the same as

$$H^2(X,\,\mathbb{Z}) = H^2(\widetilde{X},\,\mathbb{Z})^G = \mathrm{span}_{\mathbb{Z}}\Big\{\tau_1 = \pi_1^*(t_1),\, \tau_2 = \pi_2^*(t_2),\, \phi = \pi_1^*(F_1) = \pi_2^*(F_1)\Big\}, \tag{3.15}$$

where t_1 and t_2 (respectively, F_1 and F_2) are the t-classes (respectively, F-classes) defined in (3.12), corresponding to each surface B_1 and B_2. Using Poincaré duality, we know that $H^4(X,\,\mathbb{Q})$ is isomorphic to $H^2(X,\,\mathbb{Q})$, also $H^1(X,\,\mathbb{Z}) \simeq \pi_1(X) = \mathbb{Z}_3 \times \mathbb{Z}_3$ because the Hurewicz theorem, thus $H^1(X,\,\mathbb{Q}) = H^1(X,\,\mathbb{Z}) \otimes \mathbb{Q} = 0$.

The ring $H^*(\widetilde{X},\,\mathbb{Q})^G$ generated through the cup product of the generators (3.15), is homomorphic to

$$H^*(\widetilde{X},\,\mathbb{Q})^G = \mathbb{Q}[\tau_1, \tau_2, \phi]/\langle \phi^2,\, \phi\tau_1 = 3\tau_1^2,\, \phi\tau_2 = 3\tau_2^2\rangle, \tag{3.16}$$

with the top cohomology element being $\tau_1^2\tau_2 = \tau_1\tau_2^2 = 3\{\mathrm{pt.}\}$.

The Ample Cone of the Elliptic Surface

As first step to determine the Kähler cone on the threefold, we build the G-invariant ample cone of the rational elliptic surface through the Nakai's criterion. The set of ample classes is by definition the integral cohomology part of the Kähler moduli.

Using the Looijenga's classification of the effective curves in a rational elliptic surface [58], we know that the cone of effective classes in $H_2(B,\,\mathbb{Z})$ is generated by the following classes $e \in H_2(B,\,\mathbb{Z})$:

1) The *exceptional curves* $e^2 := -1$, i.e. every section of the elliptic fibration.

2) The *nodal curves* $e^2 := -2$, i.e. the irreducible components of the singular fibers.

3) The *positive classes*, i.e. the classes that live in the "future" side of the cone of $e^2 > 0$.

Nakai's criterion for surfaces says that a class s is ample if and only if $s \cdot s > 0$ and $e \cdot s > 0$ for every effective curve e. We will apply this criterion to the invariant classes $s = aF + bt$.

• Intersection of s with the *exceptional curves*. Although there is an infinite amount of *exceptional curves* or sections in the elliptic surface, we can characterize them completely thanks to our understanding of the Mordell-Weil group.

As it is explained in the appendix of [46], the representation of the Mordell-Weil group $E(K) \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3$ in $\mathrm{End}(H_2(B,\,\mathbb{Z}))$, has as generators: $(t_\xi)_*$, $(t_{\alpha_B\xi})_*$ and $(t_\eta)_*$. Thus, the homology of an arbitrary section can be expressed as

$$\big[\boxplus\, x\xi \boxplus y\alpha_B\xi \boxplus z\eta\big] = (t_\xi)_*^x (t_{\alpha_B\xi})_*^y (t_\eta)_*^z \sigma \tag{3.17}$$

where $\boxplus x\xi$ (respectively $\boxplus y\alpha_B\xi$ and $\boxplus z\eta$) means $\boxplus x\xi = \underbrace{\xi \boxplus \xi \boxplus \cdots \boxplus \xi}_{x}$.

Finding the Jordan canonical forms associated to $(t_\xi)_*$, $(t_{\alpha_B\xi})_*$ and $(t_\eta)_*$, allows us to expand (3.17) explicitly[4]. Hence, the intersections of the *exceptional curves* with the generators of the invariant homology are

$$F \cdot \big[\boxplus\, x\xi \boxplus y\alpha_B\xi \boxplus z\eta\big] = 1 \tag{3.18}$$

and

$$t \cdot \big[\boxplus\, x\xi \boxplus y\alpha_B\xi \boxplus z\eta\big] = x^2 + y^2 - xy - x. \tag{3.19}$$

[4]We exhibit the list of homology classes associated to the sections in the appendix of [46].

It is easy to check that x^2+y^2-xy-x, as a function $\mathbb{Z}\oplus\mathbb{Z}\to\mathbb{Z}$, is non-negative and becomes zero for $(x=0,\ y=0)$, $(x=1,\ y=0)$ and $(x=1,\ y=1)$. Therefore a G-invariant ample class $s=aF+bt$, has to verify

$$s\cdot\big[\boxplus\,0\xi\boxplus 0\alpha_B\xi\boxplus z\eta\big]=a>0, \tag{3.20}$$

and

$$s\cdot\big[\boxplus\,\infty\xi\boxplus\infty\alpha_B\xi\boxplus z\eta\big]=a+\infty b>0,\Rightarrow\ b>0. \tag{3.21}$$

• Intersection of s with the *nodal curves*. The nodal curves are identified with the irreducible components $\Theta_{i,j}$ of the singular fibers, thus their intersections with the invariant class $s=aF+bt$ give us

$$s\cdot\Theta_{i,j}=b>0. \tag{3.22}$$

Identical result to the inequality (3.21), derived above.

• Intersection of s with the *positive classes*. Let $\mathcal{K}^+(B)$ be the cone of positive classes in B, i.e. $\mathcal{K}^+(B)=\{e\in H_2(B,\mathbb{Z})|\,e\cdot e>0\}$. As $\mathcal{K}^+(B)$ is a convex set and we have to take intersections of elements in $\mathcal{K}^+(B)$ with invariant classes in $H_2(B,\mathbb{Z})^G$, only the intersection $\mathcal{K}^+(B)\cap H_2(B,\mathbb{Z})^G$ matters. From the intersection matrix of the homology generators, we know that the intersection matrix of the invariant homology $H_2(B,\mathbb{Z})^G$ is

$$\begin{pmatrix}F\\ t\end{pmatrix}^T\cdot\begin{pmatrix}F\\ t\end{pmatrix}=\begin{pmatrix}0&3\\3&1\end{pmatrix}$$

hence, we find

$$\mathcal{K}^+(B)\cap H_2(B,\mathbb{Z})^G:=\big\{e=xF+yt|\,6xy+y^2>0\big\}, \tag{3.23}$$

being the edges of such "future" cone F and $6t-F$. Furthermore, their intersections with our ample candidate $s=aF+bt$, give us the conditions

$$s\cdot F=(aF+bt)\cdot F=3b>0 \tag{3.24}$$

$$s\cdot(6t-F)=18a+6b-3b=18a+3b>0 \tag{3.25}$$

that do not constrain the inequalities (3.20), and (3.21).

Finally, as the cone generated by F and t is within $\mathcal{K}^+(B)\cap H_2(B,\mathbb{Z})^G$, the last Nakai's condition $s\cdot s>0$ or positivity of the Liouville's measure, is verified. Therefore, the G-invariant ample cone associated to the elliptic surface B is simply

$$\mathcal{K}(B)^G=\mathrm{span}_{\mathbb{Z}^+}\big\{F,\ t\big\}. \tag{3.26}$$

Ampleness in the Threefold

Once we have characterized the G-invariant ample cone on the rational surface, we can construct G-invariant ample classes on the threefold $\widetilde{X}$ as product of ample classes on the surfaces B_1 and B_2. In fact, the following proposition shows that the amples classes on $\widetilde{X}$ constructed in this way determine explicitely its G-invariant ample cone $\mathcal{K}(\widetilde{X})^G = \mathcal{K}(X)$.

Proposition 3.1 *The G-invariant ample cone of $\widetilde{X}$ is*

$$\mathcal{K}(\widetilde{X})^G = \mathrm{span}_{\mathbb{Z}^+}\big\{\tau_1, \tau_2, \phi\big\}. \tag{3.27}$$

Proof. If L_i is an ample class in B_i, then $\pi_1^* L_1 \otimes \pi_2^* L_2$ is an ample class in $\widetilde{X}$, hence $\mathcal{K}(\widetilde{X})^G$ contains the positive linear span of τ_1, τ_2 and ϕ.

To show the opposite inclusion, we apply Nakai's criterion to some effective classes. Let $H = a\tau_1 + b\tau_2 + c\phi$ be an ample class. If C_1 be the class of a fiber of π_1,

$$0 < H \cdot C_1 = 0a + 3b + 0c = 3b$$

Analogously, if C_2 is the class of a fiber of π_2, we obtain $a > 0$. Let $i : B_1 \times_{\mathbb{P}^1} B_2 \to B_1 \times B_2$. Let C be the class of $\sigma_1 \times_{\mathbb{P}^1} \sigma_2$, let c_1, c_2 be two integers with $c = c_1 + c_2$, and denote $[B_i]$ (respectively, [pt]) the class of B_i in $H^0(B_i, \mathbb{Z})$ (respectively, of a point in $H^4(B_i, \mathbb{Z})$).

$$\begin{aligned} 0 < H \cdot C &= i^*\big((at_1 + c_1 f_1) \otimes [B_2] + [B_1] \otimes (bt_2 + c_2 f_2)\big) \cdot i^*[\sigma_1 \otimes \sigma_2] \\ &= i^*\Big((at_1 + c_1 f_1)\sigma_1\,[\mathrm{pt}] \otimes \sigma_2 + \sigma_1 \otimes [\mathrm{pt}]\,(bt_2 + c_2 f_2)\sigma_2\Big) \\ &= i^*\Big(c_1\,[\mathrm{pt}] \otimes \sigma_2 + c_2\,\sigma_1 \otimes [\mathrm{pt}]\Big) \\ &= c_1 + c_2 = c \end{aligned} \tag{3.28}$$

♠

The vector bundles of the compactification

The rank-2 sub-bundle of the vector bundle $\mathcal{E}_{\mathrm{h}} \to X$ (the gauge group action on its fibers is the adjoint action of the hidden E_8 gauge group) defined through the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}}(2\tau_1 + \tau_2 - \phi) \longrightarrow \mathcal{H} \longrightarrow \mathcal{O}_{\widetilde{X}}(-2\tau_1 - \tau_2 + \phi) \longrightarrow 0, \tag{3.29}$$

is the vector bundle used on the hidden sector of the compactification, [16].

Now, we recall the construction of the visible bundle [16]. First it is defined an equivariant rank 2 vector bundle V_2 on B of trivial determinant given as nontrivial extension

$$0 \longrightarrow \mathcal{O}_B(-2F) \longrightarrow V_2 \longrightarrow \mathcal{I}_Z(2F) \longrightarrow 0, \tag{3.30}$$

with Z the scheme of 9 points, together with an equivariant structure on V_2 so that this extension is equivariant

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}}(-2\phi) \longrightarrow \pi_2^* V_2 \longrightarrow \mathcal{I}_\Theta(2\phi) \longrightarrow 0. \tag{3.31}$$

Here, Θ the lifting to $\widetilde{X}$ of Z by the second projection. Then, the visible rank 4 vector bundle V_4 of trivial determinant, is defined through the extension

$$0 \longrightarrow \mathcal{O}(-\tau_1 + \tau_2) \oplus \mathcal{O}(-\tau_1 + \tau_2) \longrightarrow V_4 \longrightarrow V_2(\tau_1 - \tau_2) \longrightarrow 0, \tag{3.32}$$

together with an equivariant structure making this extension equivariant, and general among such extensions.

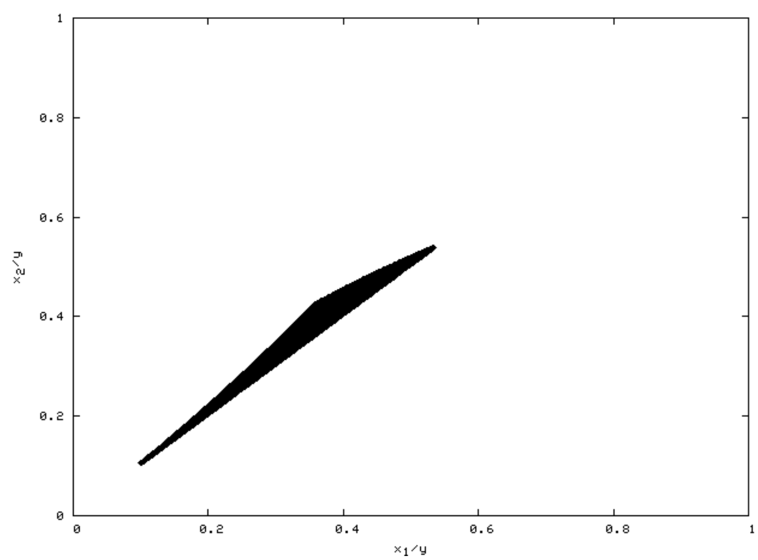

Figure 3.1: Polarizations which make V_4 stable.

The stability analysis made in [46] determined the set of ample classes that allow solutions to the Hermite Yang-Mills equations on these bundles. In the figure above we have plotted the region of the Kähler cone which satisfies the conditions that make the vector bundle V_4 stable.

In the rest of the thesis we explore how one can actually approximate solutions to the Ricci flat equations, the Hermite Yang-Mills equations and others, in compactifications like the one explained above.

3.3 Geometric quantization and balanced metrics

Using numerical methods to solve PDEs, involves finding natural discrete approximations of the space where the PDE is defined. In many important examples, the PDE to be solved is defined on some subset of the Euclidean space. In such a case, one can use an equally-spaced lattice to approximate functions on the Euclidean space as the finite collection of the values of the function at each node of the lattice. However, if one deals with a PDE defined on a general Calabi-Yau variety X there is not a natural way to introduce a lattice onto it. More abstractly, a "discretization" of X can be roughly defined as a sequence of vector spaces $\mathcal{H}_k$ labeled by a positive integer k (of increasing dimension as k increases), such that $C^\infty(X)$ is approximated by $\mathcal{H}_k$ when k is large enough.

When X is Kähler, there is a natural beautiful discrete approximation given by geometric quantization, as we explain in the following section.

3.3.1 Geometric quantization

Classical mechanics and geometric quantization have a beautiful formulation using the language of symplectic geometry, vector bundles, and operator algebras [8, 39]. In this language, symplectic manifolds X are interpreted as phase spaces, and spaces of smooth functions $C^\infty(X)$ as the corresponding classical observables.

Kähler quantization is understood far better than quantization on general symplectic manifolds; for this reason we only consider Kähler manifolds (which are symplectic manifolds endowed with a compatible complex structure). $(X,\, \mathcal{L}^{\otimes k})$ denotes a polarized Kähler manifold X with a very ample hermitian line bundle[5] $\mathcal{L}^{\otimes k}$, and $k \in \mathbb{Z}_+$ a positive integer. For simplicity, we consider X to be compact and simply connected. We work with a trivialization of $\mathcal{L}|_U \to U$, where $U \subset X$ is an open subset; we define $K(z, \overline{z})$ to be the associated Kähler potential and $e^{-kK(z,\overline{z})}$ the hermitian metric on $\mathcal{L}^{\otimes k} \to X$. If $\dim_{\mathbb{C}} Z = n$ and $\{z_i\}_{0<i\leq n}$ is a local holomorphic coordinate chart for the open subset $U \subset X$, we can write the Kähler metric on X and its compatible symplectic form as

$$ikg_{i\overline{\jmath}}dx^i \otimes d\overline{x}^{\overline{\jmath}} = k\omega_{i\overline{\jmath}}dz^i \wedge d\overline{z}^{\overline{\jmath}} = ik\frac{\partial}{\partial z^i}\frac{\partial}{\partial \overline{z}^{\overline{\jmath}}}K(z,\overline{z})dz^i \otimes d\overline{z}^{\overline{\jmath}}. \tag{3.33}$$

Classically, the space $(C^\infty(X),\, \omega)$ of observables has, in addition to a Lie algebra structure defined by the Poisson bracket

$$\{f,\, g\}_{PB} = \omega^{i\overline{\jmath}}(\partial_i f \overline{\partial}_{\overline{\jmath}} g - \partial_i g \overline{\partial}_{\overline{\jmath}} f), \quad f,\, g \in C^\infty(X),$$

the structure of a commutative algebra under pointwise multiplication,

$$(fg)(x) = f(x)g(x) = (gf)(x).$$

---

[5]In other words, $\mathcal{L}^{\otimes k}$ is an element of the Kähler cone associated to X.

Quantization can be understood as a non-commutative deformation of $C^\infty(X)$ parameterized by $\hbar$, with commutativity recovered when $\hbar = 0$. We will discuss the formalism of deformation quantization in the next section, although generally speaking, quantization refers to an assignment $T\colon f \to T(f)$ of classical observables to operators on some Hilbert space $\mathcal{H}$. When X is compact, the Hilbert space will be finite-dimensional with dimension $\dim \mathcal{H}_\hbar = \frac{\mathrm{vol}\, X}{\hbar^n} + O(\hbar^{1-n})$. The assignment T must satisfy the following requirements:

- Linearity, $T(af+g) = aT(f) + T(g), \quad \forall a \in \mathbb{C},\ f, g \in C^\infty(X)$.
- Constant map 1 is mapped to the identity operator Id, $T(1) = \mathrm{Id}$.
- If f is a real function, $T(f)$ is a hermitian operator.
- In the limit $\hbar \to 0$, the Poisson algebra is recovered $[T(f),\, T(g)] = i\hbar T(\{f,\, g\}_{PB}) + O(\hbar^2)$.

In geometric quantization the positive line bundle $\mathcal{L}^{\otimes k}$ is known as *prequantum line bundle*. The prequantum line bundle is endowed with a unitary connection whose curvature is the symplectic form $k\omega$ (which is quantized, i.e., $\omega \in H^2(X, \mathbb{Z})$). The *prequantum Hilbert space* is the space of L^2 sections

$$L^2(\mathcal{L}^{\otimes k}, X) = \left\{ s \in \Omega^0(\mathcal{L}^{\otimes k})\colon \quad \int_X h^k \langle s, \overline{s} \rangle \frac{\omega^n}{n!} < \infty \right\},$$

where h^k is the compatible hermitian metric on $\mathcal{L}^{\otimes k}$. The Hilbert space is merely a subspace of $L^2(\mathcal{L}^{\otimes k}, X)$, defined with the choice of a polarization on X. In the case of Kähler polarization, the split of the tangent space in holomorphic and anti-holomorphic directions, $TX = TX^{(1,0)} \oplus TX^{(0,1)}$, defines a Dolbeault operator on $\mathcal{L}^{\otimes k}$, $\overline{\partial}\colon \Omega^{(0)}(\mathcal{L}^{\otimes k}) \to \Omega^{(0,1)}(\mathcal{L}^{\otimes k})$. The Hilbert space $\mathcal{H}_k$ is only the kernel of $\overline{\partial}$, i.e., the space of holomorphic sections $H^0(X,\, \mathcal{L}^{\otimes k})$.

As a final remark, the quantization map T is not uniquely defined; there are different assignments of smooth functions on X to matrices on $\mathcal{H}_k$ that obey the same requirements stated above, giving rise to equivalent classical limits. For simplicity, we mention only the most standard ones [21]:

- *The Toeplitz map:*

  $$T(f)_{\alpha\overline{\beta}} = \int_X f(z, \overline{z}) s_\alpha(z) \overline{s}_{\overline{\beta}}(\overline{z}) h^k(z, \overline{z}) \frac{\omega(z, \overline{z})^n}{n!},$$

  with s_α a basis of sections for $\mathcal{H}_k$ and $s_\alpha(z)$ the corresponding evaluation of s_α at $z \in U \subset X$.

- *The geometric quantization map:* $Q(f) = iT\left(f - \frac{1}{2}\Delta f\right)$, with Δ the corresponding Laplacian on X.

We will work only with completely degenerated Hamiltonian systems (i.e. a constant Hamiltonian function on M); therefore the choice of quantization map will not be important. Rather we will study the semiclassical limit of the corresponding quantized system by determining the semiclassical vacuum states.

Coherent states and balanced metrics

As we described above, the geometric quantization picture is characterized by the prequantum line bundle, $\mathcal{L}^{\otimes k} \to X$, a holomorphic line bundle on X which is endowed with a $U(1)$ connection with Kähler 2-forms $k\omega$. As the positive integer k always appears multiplying the symplectic form, one can interpret $k^{-1} = \hbar$ as a discretized Planck's constant. Thus, according to this convention, the semiclassical appears in the limit $k \to \infty$.

In the local trivialization $U \subset X$, where $K(z, \overline{z})$ is the Kähler potential and $e^{-kK(z,\overline{z})}$ the hermitian metric on $\mathcal{L}^{\otimes k}|_U$, one can set the compatible Dolbeault operator to be locally trivial and write the covariant derivative as

$$\widetilde{\nabla} = dz^i(\partial_i - k\partial_i K) + d\overline{z}^{\overline{\imath}}\overline{\partial}_{\overline{\imath}},$$

where K is the yet undetermined Kähler potential on $\mathcal{L}$. One can also determine the associated unitary connection up to a $U(1)$ gauge transformation,

$$\nabla = dz^i(\partial_i + A_i) + d\overline{z}^{\overline{\imath}}(\overline{\partial}_{\overline{\imath}} - A_i^\dagger),$$

with $A_i = \sqrt{h^{-k}}\partial_i\sqrt{h^k}$, and $h = \exp(-K(z, \overline{z}))$.

As explained above, the Hilbert space $\mathcal{H}_k$ corresponds to the kernel of the covariant half-derivative $\nabla^{(0,1)} \colon \Omega^{(0)}(\mathcal{L}) \to \Omega^{(0,1)}(\mathcal{L})$, which are the holomorphic sections of $\mathcal{L}^{\otimes k}$

$$\mathcal{H}_k = H^0(X, \mathcal{L}^{\otimes k}) = \mathrm{span}_{\mathbb{C}} \left\{|s_\alpha\rangle\right\}_{\alpha=1}^N .$$

The dimension of the quantum Hilbert space is

$$N = \dim \mathcal{H}_k = \frac{1}{n!}\int_X c_1(\mathcal{L})^n k^n + \frac{1}{2(n-1)!}\int_X c_1(\mathcal{L})^{n-1} c_1(X) k^{n-1} + O(k^{n-2}). \tag{3.34}$$

We identify $|s_\alpha\rangle$ as the basis elements of $\mathcal{H}_k$. The *coherent state* localized at $x \in X$ can be defined (see [73]) on the trivialization $\mathcal{L}^{\otimes k}|_U \to U \subset X$ as the ray in $\mathbb{P}\mathcal{H}_k$ generated by

$$|\widetilde{\Omega}_x\rangle = \sum_\alpha s_\alpha(x)\exp(-kK(x, \overline{x})/2)|s_\alpha\rangle \in \mathcal{H}_k, \tag{3.35}$$

where $s_\alpha(x)\exp(-kK(x, \overline{x})/2)$ is the evaluation of the holomorphic section $|s_\alpha\rangle$ at the point $x \in U \subset X$, in the trivialization $\mathcal{L}^{\otimes k}|_U$. The coherent states are an overcomplete basis of $\mathcal{H}_k$, and obey the Parseval identity

$$\langle\zeta|\xi\rangle = \int_{X\hookrightarrow\mathbb{P}\mathcal{H}_k} \langle\zeta|\widetilde{\Omega}_x\rangle\langle\widetilde{\Omega}_x|\xi\rangle\frac{\omega^n(x,\overline{x})}{n!}, \quad \forall\zeta,\xi\in\mathcal{H}_k. \tag{3.36}$$

These points in $\mathbb{P}\mathcal{H}_k$ are independent of the trivialization, and they have the property of being localized at $x \in X$ with minimal quantum uncertainty. The *distortion function*, diagonal of the Bergman kernel, or expected value of the identity at x, $\rho(x,\overline{x})$ is defined as

$$\rho = \langle\widetilde{\Omega}_x|\widetilde{\Omega}_x\rangle = \sum_{\alpha,\beta}\overline{s}_{\overline{\alpha}}(\overline{x})s_\beta(x)\exp\left(-kK(x,\overline{x})\right)\langle s_\alpha|s_\beta\rangle, \tag{3.37}$$

which measures the relative normalization of the coherent states located at different points of X. Imposing $\rho(x,\overline{x}) = \langle\widetilde{\Omega}_x|\widetilde{\Omega}_x\rangle$ = constant, constrains the Kähler potential $K(x,\overline{x})$ to be a Fubini-Study Kähler potential:

$$K(x,\overline{x}) = \frac{1}{k}\log\left(\sum_{\alpha,\beta}\overline{s}_{\overline{\alpha}}(\overline{x})s_\beta(x)\langle s_\alpha|s_\beta\rangle\right). \tag{3.38}$$

One of the most important ingredients in the quantization procedure is the definition of the quantization map, $T : C^\infty(X) \to \mathrm{Herm}(\mathcal{H}_k)$. This maps classical observables, i.e. smooth real functions on the phase space X, to quantum observables, i.e., self-adjoint operators on the Hilbert space $\mathcal{H}_k$. If we work with an orthonormal basis $\langle s_\beta|s_\alpha\rangle = \delta_{\beta\alpha}$, the quantization condition

$$T(1_X) = \mathrm{Id} \in \mathcal{H}_k \otimes \mathcal{H}_k^*$$

implies that the embedding of the coherent states satisfies the *balanced* condition [33],

$$\delta_{\alpha\beta} = \langle s_\alpha|s_\beta\rangle = \sum_x\langle s_\alpha|\widetilde{\Omega}_x\rangle\langle\widetilde{\Omega}_x|s_\beta\rangle = \int_X \frac{\overline{s}_\alpha(\overline{x})s_\beta(x)}{\sum_\gamma|s_\gamma(x)|^2}\frac{\omega(x,\overline{x})^n}{n!}; \tag{3.39}$$

here, we have used the Parseval identity Eq. (3.36), and the Liouville's volume form on the phase space X, which can be written as

$$\frac{1}{n!}\omega(z,\overline{z})^n = \frac{1}{n!}\left[\overline{\partial}_{\overline{j}}\partial_i K(z,\overline{z})dz^i \wedge d\overline{z}^{\overline{j}}\right]^n.$$

In summary, in the geometric quantization of an algebraic Kähler manifold, the homogeneity of the distortion function $\langle\widetilde{\Omega}_x|\widetilde{\Omega}_x\rangle$ and the mapping of the constant function on X to the identity operator Id: $\mathcal{H}_k \to \mathcal{H}_k$, determines a unique metric on X known as *balanced metric*. In the semiclassical limit, $k \to \infty$, this sequence of balanced metrics approaches the Kähler-Einstein metric (if it exists) as explained below (see [31, 36]).

Emergence of classical geometry

For every k, the *balanced metric* has just been defined as result of setting $\langle\widetilde{\Omega}_x|\widetilde{\Omega}_x\rangle$ to be the constant function on X. In the semiclassical limit, $k \to \infty$, we can expand the distortion function in inverse powers of k (see [91])

$$\langle\widetilde{\Omega}_{x,k}|\widetilde{\Omega}_{x,k}\rangle \sim 1 + \frac{1}{2k}R + O(k^{-2}) + \dots, \tag{3.40}$$

and therefore the sequence of balanced metrics will converge to a metric of constant scalar curvature at $k = \infty$. For a Calabi-Yau manifold this is equivalent to a Ricci flat Kähler metric. It is interesting to note that if the identity matrix is identified with the quantum Hamiltonian, and the coherent states with the semiclassical states, the *balanced metric* can also be defined as the metric that yields a constant semiclassical vacuum energy $\langle\widetilde{\Omega}_{x,k}|\widetilde{\Omega}_{x,k}\rangle$, as a function of $x \in X$ and fixed k. In the next section, we show how one can compute such *balanced metrics* explicitly.

Other geometrical elements that one can recover naturally are the Lagrangian submanifolds with respect the Kähler-Einstein symplectic form. In the Kähler n-fold (X, ω), the level sets of n commuting functions $(f_1, f_2, \dots f_n)$ under the Poisson bracket

$$\{f_a, f_b\}_{PB} = \omega^{i\bar{j}}(\partial_i f_a \overline{\partial}_{\bar{j}} f_b - \partial_i f_b \overline{\partial}_{\bar{j}} f_a) = 0, \quad \forall\, a, b,$$

define a foliation by Lagrangian submanifolds. One can recover such commutation relations as the classical limit of n commuting self-adjoint operators on the Hilbert space $\mathcal{H}_k$, [21]:

$$\langle\widetilde{\Omega}_{x,k}|[\hat{f}_a, \hat{f}_b]|\widetilde{\Omega}_{x,k}\rangle \sim \frac{i}{k}\{f_a, f_b\}_{PB} + O(k^{-2}),$$

with $\langle\widetilde{\Omega}_{x,k}|\hat{f}_a|\widetilde{\Omega}_{x,k}\rangle \to f_a(x)$, and $|\widetilde{\Omega}_{x,k}\rangle$ the coherent state peaked at $x \in X$. Thus, one can approximate Lagrangian submanifolds by using n-tuples of commuting matrices for large enough k. One can impose further conditions, i.e. $\mathrm{Im}(\Omega)|_{SLag} = 0$, in order to describe special Lagrangian submanifolds. More precisely, we define the quantum operator

$$\begin{aligned}\mathcal{I}_{\overline{\alpha}\beta\overline{\alpha}_1\beta_1\cdots\overline{\alpha}_n\beta_n} = \frac{1}{2i}\int_X \frac{\omega^n}{n!}\overline{s}_{\overline{\alpha}} s_\beta \mathrm{e}^{-kK}\Big(&\Omega_{i_1\cdots i_n}\partial^{i_1}(\overline{s}_{\overline{\alpha}_1} s_{\beta_1}\mathrm{e}^{-kK})\cdots\partial^{i_n}(\overline{s}_{\overline{\alpha}_n} s_{\beta_n}\mathrm{e}^{-kK}) \\ &-\overline{\Omega}_{\bar{\imath}_1\cdots\bar{\imath}_n}\partial^{\bar{\imath}_1}(\overline{s}_{\overline{\alpha}_1} s_{\beta_1}\mathrm{e}^{-kK})\cdots\partial^{\bar{\imath}_n}(\overline{s}_{\overline{\alpha}_n} s_{\beta_n}\mathrm{e}^{-kK})\Big),\end{aligned} \tag{3.41}$$

with $\partial^i = g^{i\bar{j}}\overline{\partial}_{\bar{j}}$ and $\partial^{\bar{\imath}} = g^{\bar{\imath}j}\partial_j$. If $\mathrm{Herm}(\mathcal{H}_k)$ is the space of hermitian matrices in $\mathcal{H}_k$ and $\mathrm{Comm}(\oplus^n\mathrm{Herm}(\mathcal{H}_k))$ is the space of n mutually commuting tuples of hermitian matrices in $\mathcal{H}_k$, we can write the map as $\mathcal{I}\colon \mathrm{Comm}(\oplus^n\mathrm{Herm}(\mathcal{H}_k)) \to \mathrm{Herm}(\mathcal{H}_k)$. Therefore, one can use the kernel of $\mathcal{I}$ to approximate special Lagrangian submanifolds as the level sets of the n functions "$\langle\widetilde{\Omega}_{x,k}|\ker(\mathcal{I})|\widetilde{\Omega}_{x,k}\rangle$".

Also, one can generalize this quantum system by coupling the particle to a rank r holomorphic vector bundle $V \to X$. We will give more details of this generalization below, although we will say a few words. For instance, the system can be interpreted as a particle endowed with certain $U(r)$-charge. The associated quantum Hilbert space is $H^0(X, V \otimes \mathcal{L}^{\otimes k})$. One can also define an analogous set of coherent states and an associated distortion function. In the semiclassical limit, setting the generalized distortion function to be constant as a function of X gives rise to generalized balanced metrics, and therefore, to hermite-Yang-Mills metrics on $V \to X$ when $k^{-1} = 0$, [35].

Finally, as a technical comment, the *balanced metric* equations Eq. (3.39) and Eq. (3.38) can be explicitly solved for finite k, and its solutions used to approximate Ricci-flat metrics and hermitian Yang-Mills connections. A method to solve them involves the concepts of T-map and algebraic Monte-Carlo integration [33, 36], which can be applied whenever one has enough analytical control on the Kodaira's embeddings $X \hookrightarrow \mathbb{P}H^0(X, \mathcal{L}^{\otimes k})$. We explain more about these methods in the upcoming sections.

3.3.2 Balanced metrics

The balance metric construction goes back to [83] which provides a mathematical precise numerical scheme which is guaranteed to converge to it.

Let us phrase this construction in a way which can be used for an arbitrary manifold X. We choose a holomorphic line bundle $\mathcal{L}$ over X, with N global sections. Denote a complete basis of these as s_α, where $1 \le \alpha \le N$, and consider the map

$$i_k \colon X \longrightarrow \mathbb{P}^{N-1} \qquad i_k(Z_0, \dots, Z_n) = (s_1(Z), s_2(Z), \dots, s_N(Z)).$$

The geometric picture is that each point in our original manifold X (parameterized by the Z_i) corresponds to a point in $\mathbb{C}^N$ parameterized by the sections s_α. Since choosing a different frame for $\mathcal{L}$ would produce an overall rescaling $s_\alpha \to \lambda s_\alpha$, the overall scale is undetermined. Granting that $s_1(Z), s_2(Z), \dots, s_N(Z)$ do not vanish simultaneously, this gives us a map to $\mathbb{P}^{N-1}$.

The simplest example is to embed $\mathbb{P}^1$ using $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(k)$ into $\mathbb{P}^k$. In this case the map is

$$i_k(Z_0, Z_1) = (Z_0^k,\ Z_0^{k-1}Z_1,\ Z_0^{k-2}Z_1^2,\ \dots,\ Z_0 Z_1^{k-1},\ Z_1^k).$$

In general we want this map to be an embedding, *i.e.* that distinct points map to distinct points with non-vanishing Jacobian. In general, we can appeal to the Kodaira embedding theorem, which asserts that for positive $\mathcal{L}$ this will be true for all $\mathcal{L}^k$ for some $k \ge k_0$. For non-singular quintics, this is true for $\mathcal{O}_M(k)$ for all $k \ge 1$. As a point of language, the pair of a manifold X with a positive line bundle $\mathcal{L}$ is referred to as a *polarized manifold* $(X, \mathcal{L})$; the condition that this construction provides an embedding for some k is that $\mathcal{L}$ is *ample*.

Now, we consider the family (3.38) of Fubini-Study Kähler potentials, and rewrite them as

$$K_h = \log \left(\sum_{\alpha, \overline{\beta}} h^{\alpha\overline{\beta}} s_\alpha \overline{s}_{\overline{\beta}} \right)$$

or simply

$$K_h \equiv \log ||s||_h^2 \tag{3.42}$$

for short, where s_α plays the role of a degree k monomial. We now have an N^2-parameter family of Kähler potentials, and will seek a good approximating metric in this family. Just as before, this amounts to using the pull-back of a Fubini-Study metric from $\mathbb{P}^{N-1}$ as our trial metric.

Mathematically, the simplest interpretation of Eq. (3.42) is that it defines a hermitian metric on the line bundle $\mathcal{L} = \mathcal{O}_M(k)$. This is a sesquilinear map from $\overline{\mathcal{L}} \otimes \mathcal{L}$ to smooth functions $C^\infty(X)$, here defined by

$$(s, s') = e^{-K_h} \cdot \overline{s} \cdot s' = \frac{\overline{s} \cdot s'}{\sum_{\alpha, \overline{\beta}} h^{\alpha\overline{\beta}} s_\alpha \overline{s}_{\overline{\beta}}}.$$

The point is that a change of frame, which acts on our explicit sections as $s_\alpha \to \lambda s_\alpha$, cancels out of this expression.[6]

This metric allow us to define an inner product between the global sections:

$$G_{\alpha\overline{\beta}} = \langle s_\beta | s_\alpha \rangle = i \int_X \frac{s_\alpha \overline{s}_{\overline{\beta}}}{||s||_h^2} \, d\mathrm{vol}_X. \tag{3.43}$$

This is the "physical" inner product in a sense we will explain further below. Note that it depends on h in a nonlinear way, since h appears in the denominator.

Here $d\mathrm{vol}_X$ is a volume form on X, which has to be chosen. If X is Calabi-Yau, it is simplest to use $\Omega \wedge \overline{\Omega}$ as the volume form $d\mathrm{vol}_X$. If X is not Calabi-Yau, the standard choice of $d\mathrm{vol}_X$ is to take $d\mathrm{vol}_\omega = \omega^n/n!$, where ω is the Kahler metric derived from Eq. (3.42). This depends on h as well, so the expression is even more non-linear in h.

Thus, given h and a basis of global sections s_α, we could compute the matrix of inner products Eq. (3.43). Once we have it, we could make a linear redefinition, say $\widetilde{s} = G^{-1/2} s$, and go to a basis of orthonormal sections where

$$G_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}}. \tag{3.44}$$

[6]A possibly more familiar physics use of this is in $\mathcal{N} = 1$ supergravity: taking $K \to -K$ and $s \to W$, one gets the standard expression for the gravitino mass $e^K |W|^2$. In an example such as the flux superpotential, in which W is a sum of various terms s_α with constant coefficients, Eq. (3.42) also applies to give K.

On the other hand, Eq. (3.42) also implicitly defines a notion of orthonormal basis locally in the bundle, in which

$$h^{\alpha\overline{\beta}} = \delta^{\alpha\overline{\beta}}. \tag{3.45}$$

This is *a priori* different from Eq. (3.44); indeed we can freely postulate it when we write Eq. (3.42). However, if the two notions agree,

$$G_{\alpha\overline{\beta}} = (h^{-1})_{\alpha\overline{\beta}},$$

then we can go to a basis of sections in which

$$G_{\alpha\overline{\beta}} = h^{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}}. \tag{3.46}$$

In this case, the embedding of X in $\mathbb{P}^{N-1}$ using these sections is called *balanced*. More generally, we call a polarized manifold $(X, \mathcal{L}^k)$ balanced if such an embedding exists.

An equivalent definition of the balanced embedding is arrived at if we consider the function on X defined as

$$\rho(\omega)(x) = \sum_{\alpha,\overline{\beta}} (G^{-1})^{\alpha\overline{\beta}} (s_\alpha(x), \overline{s}_{\overline{\beta}}(x)) \tag{3.47}$$

or equivalently

$$\rho(\omega)(x) = \sum_{\alpha} ||s_\alpha(x)||^2$$

where the second sum is taken over an orthonormal basis in which $G = \delta_{\alpha\beta}$. X is balanced precisely when $\rho(\omega)(x)$ is the constant function.

Many theorems have been proven about balanced manifolds. Let us first recall the following theorem of Donaldson (Theorem 1 in [31]):

Theorem 3.3.1 *Suppose the automorphism group* $\mathrm{Aut}(X, \mathcal{L})$ *is discrete. If* $(X, \mathcal{L}^k)$ *is balanced, then the choice of basis in* $H^0(X, \mathcal{L}^k)$ *such that* $i_k(\mathcal{L})$ *is balanced is unique up to the action of* $U(N) \times \mathbb{R}^*$.

The condition on $\mathrm{Aut}(X, \mathcal{L})$, i.e., there are no continuous symmetries, is true for the quintic Q. This theorem then tells us that, if a metric h exists which gives a balanced embedding, it is unique up to scale.

Given a balanced embedding, one defines the *balanced metric* on X as the pullback of the Fubini-Study metric (3.42):

$$\omega_k = \frac{2\pi}{k} i_k^*(\omega_{FS}), \tag{3.48}$$

The cohomology class of the Kahler form $[\omega_k] = 2\pi c_1(\mathcal{L}) \in H^2(X)$ is independent of k. Using these definitions Donaldson proves that (Theorem 2 in [31]):

Theorem 3.3.2 *Suppose* $\mathrm{Aut}(X, \mathcal{L})$ *is discrete and* $(X, \mathcal{L}^k)$ *is balanced for sufficiently large* k. *If the metrics* ω_k *converge in the* C^∞ *norm to some limit* ω_∞ *as* $k \to \infty$, *then* ω_∞ *is a Kahler metric in the class* $2\pi c_1(\mathcal{L})$ *with constant scalar curvature.*

The constant value of the scalar curvature is determined by $c_1(X)$, and in particular for $c_1(X) = 0$ the scalar curvature is zero. Thus, the balanced metrics ω_k, in the large k limit, converge to the Ricci flat metric.

Therefore, if we can find the unique balanced metric for a given $\mathcal{L}$, it is a good candidate for approximating the Ricci flat metric on X. One may ask where the complex structure and Kahler moduli on which this Ricci flat metric depends, are put in. The complex structure enters implicitly through the basis for holomorphic sections s_α, as we will see in examples below. As for the Kahler class, recall that this is determined, up to scale, to be $2\pi c_1(\mathcal{L})$. Of course, the Ricci flatness condition is scale invariant, so the overall scale is irrelevant; however the point of this is that if $b^{1,1} > 1$, then by appropriately choosing $\mathcal{L}$ we choose a particular ray in the Kahler cone. This will not be relevant for our examples here but shows that in principle any Ricci-flat Kahler metric could be approximated in this way.

Finding the balanced metric

In [32, 33] Donaldson proposes a method to determine the hermitian metric h in Eq. (3.42), which will lead to a balanced metric. He defines the "T operator", which given a metric h computes the matrix G:

$$G_{\alpha\overline{\beta}} = T(h)_{\alpha\overline{\beta}} \equiv \frac{N}{\mathrm{vol}(X)} \int_X \frac{s_\alpha \overline{s}_{\overline{\beta}}}{||s||_h^2} \, \mathrm{dvol}_X \tag{3.49}$$

Now, suppose we find a fixed point of this operator,

$$T(h) = h.$$

Then, by a $GL(N)$ change of basis $s \to h^{-1/2} s$, we can bring h to the unit matrix, which will produce the balanced embedding.

The simplest way to find a fixed point of an operator is to iterate it. If the operator is contracting, this is guaranteed to work. In our case we have the following theorem [31, 75]:

Theorem 3.3.3 *Suppose that* $\mathrm{Aut}(X, L)$ *is discrete. If a balanced embedding exists then, for any initial* G_0 *hermitian metric, the sequence* $T^r(G_0)$ *converges to the balanced metric* G *as* $r \to \infty$.

Thus the T operator can be used to find approximate Ricci-flat metrics on Calabi-Yau manifolds, and more generally approximate constant scalar curvature Kahler metrics. In [33] Donaldson studies numerically explicit $\mathbb{P}^1$ and $K3$ examples. We will discuss some additional examples below.

Balanced metrics and constant scalar curvature

In this subsection we outline the reason why the limit of a family of balanced metrics has constant scalar curvature. This is the content of Theorem 3.3.2. This will be very useful later on, when we generalize the T-operator to vector bundles.

Note that the function $\rho(\omega)$ is independent of the choice of orthonormal basis, and remains unchanged if we replace h by a constant scalar multiple. Therefore, it is an invariant of the Kahler form. As discussed before, the balanced condition for (X, L^k) is equivalent to the existence of a metric ω_k such that $\rho(\omega_k)$ is a constant function on X. The asymptotic behavior of the "distorsion" function $\rho(\omega_k)$ as $k \to \infty$ for fixed ω has been studied in [83, 25, 91, 59]. Note that for any metric

$$\int_X \rho_k(\omega) = N = \dim H^0(X, L^k) = a_0 k^n + a_1 k^{n-1} + \cdots, \tag{3.50}$$

where the coefficients a_i can be determined using the Riemann-Roch formula. Note that a_0 is just the volume of X and

$$a_1 = \frac{1}{2\pi}\int_X S(\omega),$$

where $S(\omega)$ is the scalar curvature of ω. We will use the following result (Prop. 6 in [31]):

Proposition 3.3.4 *1. $\rho(\omega)$ has an asymptotic expansion as $k \to \infty$*

$$\rho_k(\omega) \sim A_0(\omega)k^n + A_1(\omega)k^{n-1} + \cdots$$

where $A_i(\omega)$ are smooth functions on X defined locally by ω. In particular,

$$A_0(\omega) = 1, \qquad A_1(\omega) = \frac{1}{2\pi} S(\omega).$$

2. The expansion holds uniformly in the C^∞ norm; in that for any $r, N > 0$

$$\left\| \rho_k(\omega) - \sum_{i=0}^{N} A_i(\omega) k^{n-i} \right\|_{C^r(X)} \leqslant K_{r,N,\omega} k^{n-N-1}$$

for some constants $K_{r,N,\omega}$.

Now assume that we are given balanced metrics ω_k converging to ω_∞. Then by the previous proposition

$$\left\| \rho_k(\omega_k) - k^n - \frac{1}{2\pi} S(\omega_k) k^{n-1} \right\|_{C^0(X)} \leqslant c k^{n-2}$$

for some constant c. Since ω_k is balanced $\rho_k(\omega_k)$ is constant: $\rho_k(\omega_k) = \dim H^0(X, L^k)/V$, and we can use (3.50) to find that

$$\left\| \frac{1}{V}(V k^n + a_1 k^{n-1} + \cdots) - k^n - \frac{1}{2\pi} S(\omega_k) k^{n-1} \right\|_{C^0(X)} \leqslant c k^{n-2},$$

or equivalently

$$\left\| \frac{2\pi}{V} a_1 - S(\omega_k) \right\|_{C^0(X)} = O(k^{-1})$$

Hence $S(\omega_\infty) = S_0 = \mathit{constant}$, where $S_0 = \frac{1}{V}\int_X S(\omega)$ is the mean curvature.

3.4 Balanced Hermitian metrics on stable holomorphic bundles

We are now ready to generalize the T-operator, which provided an approximation scheme for the constant curvature metric, to a "generalized T-operator" which can be used to find a solution of the Yang-Mills equations on a Calabi-Yau manifold X.

We briefly recall the argument that a solution of the Yang-Mills equations which preserves $\mathcal{N} = 1$ supersymmetry, must be hermitian Yang-Mills. First, the supersymmetry variation of the gaugino has to vanish,

$$\Gamma^{\mu\nu} F^a_{\mu\nu} \epsilon = 0,$$

where $F^a_{\mu\nu}$ is the Yang-Mills field strength, and ϵ is the covariantly constant spinor.

Going to complex coordinates $(i, \bar{i})$ and rewriting of the Clifford algebra as

$$\Gamma_i \to dz^i; \qquad \Gamma_{\bar{i}} \to \omega_{\bar{i}j}\ \partial^j,$$

this is equivalent to

$$F_{ij} = F_{\overline{ij}} = 0; \qquad \omega^{i\bar{j}} F_{i\bar{j}} = 0.$$

This is the particular case of the hermitian Yang-Mills equations with $\mathrm{TrF} = 0$. The general case replaces the last equation with

$$\omega^{i\bar{j}} F_{i\bar{j}} = c \cdot \mathbf{1}$$

for a constant c, determined by the first Chern class. For convenience we abbreviate this equation below as

$$\bigwedge F = c \cdot \mathbf{1}.$$

Next we review the relation between solutions of these equations, and holomorphic bundles carrying hermitian-Einstein metrics. In physics, one defines Yang-Mills theory in terms of a connection on a vector bundle with a fixed metric. First, a connection on a vector bundle can be described in terms of a connection one-form by choosing a frame for the bundle, say $e_a(x)$, and defining the covariant derivative as

$$D(v^a e_a) = (dv^a)e_a + v^a A^b_a e_b.$$

In physics, one usually takes the frame to be orthonormal, $(e_a, e_b) = \delta_{ab}$, and thus

$$(u, v) = (u^a e_a, v^b e_b) = (u^a)^* v^a, \tag{3.51}$$

where $*$ is complex conjugation.

The condition that the connection be compatible with the metric,

$$d(u, v) = (Du, v) + (u, Dv), \tag{3.52}$$

reduces to requiring the connection one-form to be anti-hermitian,

$$A^{(phys)}{}_i = -A^{(phys)}{}^\dagger_i. \tag{3.53}$$

In mathematics, one often considers a more general frame, for which the metric is a hermitian matrix,

$$(e_a, e_b) = H_{\overline{a}b}, \qquad H = H^\dagger. \tag{3.54}$$

Decomposing the positive definite hermitian matrix G as

$$H = h^\dagger h, \tag{3.55}$$

we see that the math and physics conventions differ by a complex gauge transformation: $u = h\, s$. This complex gauge transformation leads to a different form for the connection, according to the standard relation

$$\partial_i + A^{(math)}{}_i = h(\partial_i + A^{(phys)}{}_i)h^{-1}. \tag{3.56}$$

Now, equations of the form

$$F_{\overline{i}\overline{j}} = 0 \; \forall \; \overline{i}, \overline{j}$$

will be integrability conditions for the covariant derivatives. In particular, this equation has the general solution

$$\partial_{\overline{i}} + A^{(phys)}{}_{\overline{i}} = g^{-1} \overline{\partial}_{\overline{i}} g,$$

in other words the $\overline{D}$ covariant derivatives are obtained from the derivative $\overline{\partial}$ by a complex gauge transformation.

Thus, we can use Eq. (3.56) to bring the connection to the gauge $\overline{A}^{(math)} = 0$, at the cost of losing the simple metric Eq. (3.51) and Eq. (3.53). Actually, the covariant derivative is still compatible with the metric as in Eq. (3.52), we just have a non-trivial fiber metric H. The metric compatibility condition becomes

$$0 = \partial(u, v) = (\overline{\partial} u, v) + (u, Dv)$$

so

$$\partial H_{\overline{a}b} = H_{\overline{a}c} A^{(math)}{}^c_b$$

or equivalently

$$A^{(math)} = H^{-1} \partial H.$$

Conversely, if we are given a metric H, then we can use the inverse complex gauge transformation to bring the connection back to the unitary form. This leads to the formula

$$\overline{A}^{(phys)}{}_{\overline{i}} = h(\overline{\partial}_{\overline{i}} h^{-1}).$$

Using Eq. (3.53), we can get the entire connection, so the metric contains the same information as a connection satisfying $F^{(0,2)} = F^{(2,0)} = 0$. Thus we can rephrase the final equation on $F^{(1,1)}$, as a condition on the metric. It is simplest to write this in the "mathematical" gauge $\overline{A}^{(math)} = 0$, in which it is

$$c \cdot \mathbf{1} = \omega^{i\overline{j}} F_{i\overline{j}} = \omega^{\overline{j}i} \overline{\partial}_{\overline{j}} A^{(math)}{}_i = \omega^{\overline{j}i} \overline{\partial}_{\overline{j}} \left(H^{-1} \partial_i H \right). \tag{3.57}$$

A metric H satisfying this equation is a "hermitian-Einstein" metric. It is simply related to a hermitian Yang-Mills connection as above.

Finally, using the complex gauge transformation above, the standard physical inner product

$$\langle u | v \rangle \equiv \int_X (u^*)_a v^a \tag{3.58}$$

is equal to the natural inner product generalizing Eq. (3.43),

$$\langle u | v \rangle = \langle h\ s | h\ t \rangle \tag{3.59}$$

$$= i \int_X G_{a\overline{b}}\ \overline{s}^{\overline{b}}\ t^a\ \mathrm{dvol}_X. \tag{3.60}$$

3.4.1 Embeddings in Grassmannians and vector bundles

We now want to represent the hermitian metric $H_{a\overline{b}}$ in the same way as we did for line bundles, by introducing a complete basis of sections. Now an irreducible bundle E with $c_1 = 0$, and thus of interest for string compactification, will not have global sections. What we do instead is to make the same construction for $E(k) \equiv E \otimes \mathcal{L}^k$, which will have global sections. We can again think of these sections as a basis of polynomials approximating functions on which to base our numerical scheme.

Thus, consider a rank r vector bundle E, and suppose that $E(k)$ has N global sections. Choosing a local frame as above, a basis for these will be an N by r matrix z^a_α. This is defined up to a $GL(N)$ change of basis, and up to a $GL(r)$ change of frame. After making these identifications, such a matrix z defines a point in the Grassmannian $G(r,N)$ of r planes in $\mathbb{C}^N$.

Given a metric $H_{a\bar{b}}$ on the fibers of $E(k)$, we can define the matrix of inner products

$$G_{\alpha\overline{\beta}} = \langle z_\beta | z_\alpha \rangle$$

as above. Such a metric could be obtained by multiplying a metric $H^{(0)}$ on E, by one on $\mathcal{L}^k$ as defined earlier. Or, it might simply be an $r \times r$ hermitian matrix of functions (in each local frame) with appropriate transformation properties.

Now there is a natural set of metrics on $E(k)$ generalizing Eq. (3.42), again parameterized by an $r \times r$ matrix, defined by

$$(H^{-1})^{a\bar{b}} = g^{\alpha\overline{\beta}} z^a_\alpha (z^\dagger)^{\bar{b}}_{\overline{\beta}},$$

where the dagger is hermitian conjugation. Again, the approach will be to find a natural metric in this class which is a good approximation to the hermitian-Einstein metric. This will lead to a hermitian Yang-Mills connection on $E(k)$. But this is simply related to the hermitian Yang-Mills connection on E, because twisting by $\mathcal{L}^k$ only modifies the trace part of the field strength.

3.4.2 Generalized T-operator

We will now turn to a proposal for a generalized T-operator, which produces the hermitian-Einstein metric on a stable vector bundle. To begin with we use results by Wang about balanced metrics on such bundles [85].

We consider again a polarized n dimensional manifold $(X, \mathcal{L})$ and an irreducible holomorphic vector bundle E of rank r on X. Then by Kodaira embedding we know that for k sufficiently large, a basis z^a_α of the global sections of $E(k)$ will give rise to an embedding

$$X \xhookrightarrow{i} G(r, N).$$

Now Wang proves the following:

Theorem 3.4.1 *E is Gieseker stable iff there is an integer k_0 such that for $k > k_0$, the* k*th embedding given as above can be moved to a balanced place, i.e., there is a $g \in SL(N, \mathbb{C})$ which is unique up to left translation by $SU(N)$ such that:*

$$\frac{1}{V}\int_{g\cdot X} z(z^\dagger z)^{-1}z^\dagger \; dV = \frac{r}{N} I_{N\times N}.$$

We call the equation above the "balance equation." In the case that E is a line bundle, this definition reduces to that of a balanced embedding in $\mathbb{P}^{N-1}$.

Now, let h be a hermitian metric on $\mathcal{L}$ and H be a hermitian metric on E, and fix the Kähler form on X to be $\omega = \dfrac{i}{2\pi} Ric(h)$. Let vol denote the volume of (X, ω). Suppose $S_1, \ldots, S_N$ is an orthonormal basis of $H^0(X, E(k))$ with respect to the induced L^2 -metric $\langle .\, , . \rangle$. The Szegö kernel B_k is a generalization of the function $\rho(\omega)$ defined in Eq. (3.47). It is defined as the fiberwise homomorphism

$$B_k(x) = \sum_{i=1}^{N} \langle ., S_i(x) \rangle S_i(x) \colon E_x \to E_x.$$

This expression is independent of the choice of orthonormal basis.

Now the local form of Theorem 3.4.1 can be stated as follows (Corollary 1.1 of [85]):

Theorem 3.4.2 *E is Gieseker stable iff there is an integer k_0 such that for any $k > k_0$, we can find a metric $H^{(k)}$, which we will call the* balanced metric *on $E(k)$, such that the Szegö kernel satisfies the equation*

$$B_k(x) = \frac{\chi(k)}{V\ r} I_E$$

where I_E is the identity bundle morphism and $\chi(k)$ is the Hilbert polynomial of E with respect to the polarization $\mathcal{L}$.

The theorem tells us that if E is Gieseker stable then for large k there is a balanced metric $H^{(k)}$ on $E(k)$. Hence we will have a sequence of hermitian metrics $H_k := H^{(k)} \otimes h^{-k}$ on E. The importance of the balanced metric $H^{(k)}$ for physical applications follows from the following theorem:

Theorem 3.4.3 *Suppose E is Gieseker stable. If $H_k \to H_\infty$ in the C^∞ norm as $k \to \infty$, then the metric H_∞ solves the "weak hermitian-Einstein equation",*

$$\frac{i}{2\pi} \bigwedge F_{(E,H_\infty)} + \frac{1}{2} S(\omega) I_E = \left(\frac{deg(E)}{Vr} + \frac{\overline{s}}{2} \right) \cdot I_E \tag{3.61}$$

where $\bigwedge F_{(E,H_\infty)}$ is the contraction of the curvature form of E with ω, $S(\omega)$ is the scalar curvature of X and $\overline{s} := \frac{1}{V}\int_X S \frac{\omega^n}{n!}$. Conversely, suppose there is a hermitian metric H_∞ solving this equation, then $H_k \to H_\infty$ in C^r norm for any r.

To prove (3.61) one can work along the same lines as in the proof of Theorem 3.3.2, using Catlin's and Wang's results for the expansion of the Szegö kernel.

Proposition 3.4.4 *1. For fixed hermitian metrics H and h on E and $\mathcal{O}_X(1)$ respectively, there is an asymptotic expansion as $k \to \infty$*

$$B_k(H,h) \sim A_0(H,h)k^n + A_1(H,h)k^{n-1} + \cdots ,$$

where $A_i(H,h) \in \Gamma(\text{ End } E)$ are smooth sections defined locally by H. In particular,

$$A_0(H,h) = I_E, \quad A_1(H,h) = \frac{i}{2\pi} \bigwedge F(E, Ric(h)) + \frac{1}{2} S(\omega) \cdot I_E$$

2. The expansion holds uniformly in the C^∞ norm; in the sense that for any $r, N > 0$

$$\|B_k(H,h) - \sum_{i=0}^{N} A_i(H,h)k^{n-i}\|_{C^r} \leqslant K_{r,N,H,h} k^{n-N-1}$$

for some constants $K_{r,N,H,h}$.

Now we can repeat the steps of the argument outlined in Section 3.3.2. Under the assumption that $H_k \to H_\infty$ in C^∞ we find that for $r > 0$

$$\|B_k(H_k) - I_E k^n - \frac{i}{2\pi} \bigwedge F(E, Ric(h)) + \frac{1}{2} S(\omega) \cdot I_E k^{n-1}\|_{C^r} \leqslant C k^{n-2}$$

for some fixed constant C. By assumption $H^{(k)}$ is balanced, hence $B_k(H_k) = \chi(k)/rV I_E$. This implies that

$$\|\frac{i}{2\pi} \bigwedge F_{(E,H_\infty)} + \frac{1}{2} S(\omega) I_E - \left(\frac{deg(E)}{Vr} + \frac{\overline{s}}{2} \right) \cdot I_E \| = O(k^1).$$

Generalized T-operator

Using the strong analogy between the construction of metrics with constant Kahler curvature and metrics on stable bundles which obey the hermitian-Einstein equation, we propose the following generalized T-operator:

$$T(G) = \frac{N}{Vr} \int_X z(z^\dagger G^{-1} z)^{-1} z^\dagger \; dV, \tag{3.62}$$

where as before, z is an N by r matrix of holomorphic sections of E.

The relevance of this proposal follows from the following conjecture:

Conjecture 3.4.5 *If a balanced embedding $i \colon X \hookrightarrow G(r, N)$ exists, then for every hermitian $N \times N$ matrix G, the sequence $T^r(G)$ converges to a fixed point G_0 as $r \to \infty$. Using an orthonormal basis with respect to G_0, the embedding is balanced, and as outlined above, it provides an approximate solution for the corresponding hermitian-Einstein equation.*

This conjecture may require additional technical assumptions, such as the earlier one of $\mathrm{Aut}(X, E)$ being discrete. We have not attempted to prove it, but would hope that this can be done along the same lines as [31, 75].

In the following section we will numerically test the conjecture for several stable vector bundles on $\mathbb{P}^2$, and on the Fermat quintic in $\mathbb{P}^4$, and find that it works for these cases.

3.5 Numerical integration on Calabi-Yau varieties

In this work we develop numerical methods for approximating Ricci flat metrics on Calabi-Yau hypersurfaces based on these ideas. This also supplies a detailed analysis of the numerical methods used in [35]. We study the effectiveness of our approach in the example of a one parameter family of quintics in $\mathbb{CP}^4$. As we review in section 3.3, we work with a space of approximating metrics parameterized by an $N \times N$ hermitian matrix; the balanced metric is then the fixed point of the so-called "T map" on this space, defined in [33].

The main computational problem in implementing the T map numerically is the evaluation of a large number of integrals on the manifold. More precisely, given a Calabi-Yau n-fold X, with its corresponding holomorphic n-form $\Omega \in \Omega^{n,0}(X) = \Lambda^n (T^*X)^{1,0}$, and volume form $d\mu_\Omega = \Omega \wedge \overline{\Omega}$, one needs to compute integrals of the type

$$\int_X f \; d\mu_\Omega, \tag{3.63}$$

where $f \colon X \to \mathbb{C}$ is a smooth complex valued (but not holomorphic) function. Consequently in this section, we are devoted to developing a numerical approximation scheme to efficiently and accurately compute these integrals.

A second technical point, which is very valuable in simplifying these computations, is to take advantage of the discrete symmetries of the manifold. We discuss this in section 3.6.1 for a concrete example.

Our explicit numerical results appear in section 3.6.2, where we also provide a general discussion of the efficiency and accuracy of the algorithm, comparisons with alternatives, and suggestions for future work.

Before we begin, let us briefly set out the problem. Denote the Ricci flat metric on X (which is unique given a complex structure and Kähler class) as g_{RF}. We want to propose a set of approximating metrics g_h parameterized by parameters h, and give a numerical procedure to find the "best" approximation to g_{RF} within this set.

The criteria that a best approximation should satisfy include

1. **Accuracy**: we want to minimize the error $\epsilon = d(g_h, g_{RF})$, where d is some measure of the distance between the approximate and true metrics. A simple and natural choice for ϵ in the present context is to consider the function

$$\eta_h = \frac{\det \omega_h}{\Omega \wedge \overline{\Omega}} \tag{3.64}$$

on X, where ω_h is the Kähler form for g_h. For a Ricci flat metric, this will be the constant function. We then take [7]

$$\epsilon = 1 - \frac{\min_{x\in X}\ \eta_h(x)}{\max_{x\in X}\ \eta_h(x)}. \tag{3.65}$$

Of course, one could use other norms, such as $||\eta_h - \frac{1}{\mathrm{vol}\ X}\int \eta_h||_p$, or curvature integrals.

2. **Control**: we want an explicit bound on the error,

$$\epsilon(g_h, g_{RF}) < \epsilon_{max}, \tag{3.66}$$

depending on the parameters of the problem.

3. **Systematic improvement**: we would like to have a control parameter k, such that by increasing k, we can bring the error estimate ϵ_{max} down to any desired accuracy.

4. **Mathematical naturalness**. Our experience with string theory (and more generally in mathematics and physics) has been that in exploratory work such as this, rather than trying to incorporate all known aspects of a problem and find a "best" solution, we can learn far more by studying a well chosen simplification in depth. This favors a scheme in which one makes the smallest possible number of arbitrary or *ad hoc* choices not inherent in the original statement of the problem.

Of course the approximation should be efficiently computable as well. We will comment on these various aspects as they arise.

3.5.1 Basic setup

It is clear from the outset that analytic evaluation of the integrals appearing in the T-map (3.49) is not possible. On the other hand, if the integrands are smooth and relatively slowly varying functions, it will be possible to evaluate the integrals using Monte Carlo methods. This is clear for the sections themselves. Since h is positive definite, the denominator in (3.49) is strictly positive, mitigating (though not eliminating) the possibility of numerical blow-ups.

[7] We consider only compact varieties, hence both the minimum and the maximum are attained.

Let X be a compact Calabi-Yau n-fold,[8] with its corresponding holomorphic n-form $\Omega \in \Lambda^{n,0}(X)$. The volume form $\Omega \wedge \overline{\Omega}$ determines a natural measure $d\mu_\Omega$ on X in the sense that

$$\int_X f \, d\mu_\Omega = \int_X f \; \Omega \wedge \overline{\Omega}.$$

From now on we will not distinguish between a top form and the associated measure.

We can use $d\mu_\Omega$ to measure volumes. For an open set $\mathcal{U} \subset X$ the indicator or characteristic function $\mathbf{1}_\mathcal{U}$ is defined by

$$\mathbf{1}_\mathcal{U}(x) = \begin{cases} 1 & \text{if } x \in \mathcal{U} \\ 0 & \text{if } x \notin \mathcal{U}. \end{cases}$$

The measure of $\mathcal{U}$ is its volume

$$\mu_\Omega(\mathcal{U}) = \int_X \mathbf{1}_\mathcal{U} \, d\mu_\Omega = \mathrm{vol}(\mathcal{U}).$$

To do a Monte Carlo integration, one would ideally like to produce samples of points on X which are uniformly distributed according to the measure $d\mu_\Omega$. This means that for every sample of points $\{q_i \in X\}_{i=1}^{N_p}$, the expected number of points within each open subset $\mathcal{U} \subset X$ is

$$\sum_{i=1}^{N_p} \mathbf{1}_\mathcal{U}(q_i) = N_p \frac{\mu_\Omega(\mathcal{U})}{\mu_\Omega(X)}.$$

Using this, we can estimate integrals as finite sums:

$$\int_X f d\mu_\Omega \approx \mu_\Omega(X) \frac{1}{N_p} \sum_{i=1}^{N_p} f(q_i), \tag{3.67}$$

The statistical error of such an approximation is of order $1/\sqrt{N_p}$ times a quantity proportional to the mean of the $f(q_i)$'s. [81].

In practice, producing samples of points which are distributed according to the measure μ_Ω is not so easy. One way to overcome this problem is by producing points which are uniformly distributed according to another auxiliary measure, say $d\mu_A$. Let us assume that $d\mu_A$ is associated to the global top form A. The ratio $\Omega \wedge \overline{\Omega}/A$ is a global function on X, which we call the *mass function* m_A. At a point x it is defined to be the ratio of the two top forms evaluated at x :

$$m_A(x) = \frac{\Omega \wedge \overline{\Omega}(x)}{A(x)}. \tag{3.68}$$

In general this function is neither constant nor holomorphic.

While one could use this information to generate a sample distributed according to $d\mu_\Omega$ (*e.g.*, by rejection sampling or MCMC), it is simplest to explicitly put the mass function into

[8] Although most of what we present generalizes to varieties other than Calabi-Yau, we restrict attention to these spaces due to their importance in string theory.

the integrand. Thus, given a sample of points distributed according to $d\mu_A$, and the mass function, we can estimate (3.63) as

$$\int_X f d\mu_\Omega = \int_X f \, \frac{d\mu_\Omega}{d\mu_A} \, d\mu_A \approx \frac{\mu_\Omega(X)}{\sum m_j} \sum_{i=1}^{N_p} f(q_i) m(q_i), \tag{3.69}$$

The presence of the mass function increases the statistical error. On the other hand, the generic values of our mass function are order one, and this is a very mild penalty.

Rather than regarding the Monte Carlo as a way to estimate the original T-map, an alternate point of view is to regard the right-hand side of (3.90) as defining a new measure ν and a new T-map, leading to a new ν-balanced metric which approximates the desired $(\Omega \wedge \overline{\Omega})$-balanced metric. An advantage of this point of view is that in [33] it is shown that (under a very mild hypothesis on ν) the new T-map is contracting, and the new ν-balanced metric is unique. Thus, numerical pathologies will not enter at this stage, provided that we use the *same* sample of points throughout the computation of the balanced metric. This is also advantageous for efficiency reasons, so we always do this. One can then repeat the computation with different samples to estimate the statistical error.

3.5.2 Generating the sample

We now discuss how to efficiently generate points according to a known simple distribution. In this paper we restrict to the case of X a degree d hypersurface in $\mathbb{P}^{n+1}$. For definiteness let X be defined as the zero locus of the degree d homogenous polynomial f. The case of a complete intersection is a straightforward extension. Our main interest will be in $d = n + 2$, but we can be more general for the time being.

First, it is easy to generate random points distributed according to the Fubini-Study measure (for any h) in the ambient $\mathbb{P}^{n+1}$. We simply generate uniformly distributed points on S^{2n+3}, a standard numerical problem, and then mod out the overall phase.

Using this distribution, one approach to generating points on X would be to keep only the points that lie sufficiently close to X, in other words satisfy the defining equation of X with a given precision, and then use a root finding method (say Newton's method) to "flow" down to X. In essence this is a rejection-type algorithm. We implemented this strategy, but it has some problems. First, it is hard to control the emerging distribution on X (this depends on details of the root finding method). Second, it is an order of magnitude slower than the second method we are about to describe.

The approach we use starts by taking a pair of independently chosen random points $(X, Y) \in \mathbb{P}^{n+1} \times \mathbb{P}^{n+1}$, which define a random line in $\mathbb{P}^{n+1}$. By Bezout's theorem, a generic complex line in

$\mathbb{P}^{n+1}$ intersects X in precisely d points, and we take these d points with equal weight. Repeating this process M times generates some random distribution of $N_p = dM$ points.

One advantage of this approach is that finding all d roots of $f(z) = 0$ numerically is not much harder than finding one root. But the main advantage, as we show in Section 3.5.3 using results by Shiffman and Zelditch on zeroes of random sections, is that that the resulting points are distributed precisely according to the Fubini-Study measure restricted to X. The mass function (3.68) is then computable quite efficiently.

A possible disadvantage for some applications is that the resulting sample will have correlations between the points in each d-fold subset. For our purpose of Monte Carlo integration, this is not a problem, as (3.90) is the expectation value of a function of a single random variable, and does not see these correlations. If one were considering functions of several independent random variables, one would probably want to further randomize the sequence (say by permuting points between subsets) to remove these correlations.

3.5.3 Expected values of currents

Let us start with a smooth compact algebraic variety X, and an ample line bundle $\mathcal{L}$ on X. As reviewed in Section 3.3, this means that $\mathcal{L}^k$ defines an embedding i_k into projective space for any $k \geq k_0$, for some positive integer k_0 :

$$i_k \colon X \longrightarrow \mathbb{P}\,\mathrm{H}^0(X,\, \mathcal{L}^k)^*. \tag{3.70}$$

The idea is to consider *random* global sections of $\mathcal{L}^k$, distributed uniformly according to a natural measure, and look at the expected value of the zero locus that they cut out in X. For this it is convenient to use the Poincare dual formulation, where the divisor associated to a section becomes a form, and ask what is the expected value of the random forms. Shiffman and Zelditch answer this question, and the more general one when we intersect l such divisors, in full generality using the language of currents. This section is a brief review of some aspects of their work [79, 80]. For brevity we adapt their results to fit our needs, rather than reproducing them verbatim.

The space of global sections $\Gamma = \mathrm{H}^0(X,\, \mathcal{L}^k)$ is a complex vector space of dimension d_k. If we choose a basis for it, then it automatically defines a hermitian inner product, with respect to which the basis in question is orthonormal. Conversely, given a hermitian inner product $\langle\cdot,\cdot\rangle$ on Γ, there is an orthonormal basis $\mathcal{B} = \{s_1, \ldots, s_{d_k}\}$ on Γ. Now given $s \in \Gamma$, we can expand it in the basis $\mathcal{B}$, and the inner product induces a complex Gaussian probability measure on Γ:

$$d\gamma(s) = \frac{1}{\pi^m}\, e^{-||c||^2} d^{d_k} c\,, \qquad \text{where } s = \textstyle\sum_{j=1}^{d_k} c_j s_j \text{ and } ||c||^2 = \sum_{j=1}^{d_k} |c_j|^2\,. \tag{3.71}$$

Given a metric h on the line bundle $\mathcal{L}$, as explained in section 3.3, h defines a hermitian metric on Γ. This is the inner product that we are going to use on $\Gamma = \mathrm{H}^0(X, \mathcal{L}^k)$ throughout this section.

Given a random variable Y on the probability space $(\Gamma, d\gamma)$, the expected value of Y in the probability measure $d\gamma$ is

$$E(Y) = \int_\Gamma Y \, d\gamma. \tag{3.72}$$

We can think of the probability space $(\Gamma, d\gamma)$ in a slightly different way. Consider the unit sphere

$$\mathcal{S}\Gamma = \mathcal{S}\,\mathrm{H}^0(X, \mathcal{L}^k) = \{s \in \mathrm{H}^0(X, \mathcal{L}^k) \colon \langle s, s\rangle = 1\}.$$

The Gaussian probability measure on Γ restricts to the *uniform* measure $d\mu$ on $\mathcal{S}\Gamma$. The expected value of $Y|_{\mathcal{S}\Gamma}$ is $E(Y) = \int_{\mathcal{S}\Gamma} Y \, d\mu$. On the other hand, the uniform measure on the sphere $\mathcal{S}\Gamma$ descends to the Fubini-Study measure on the projectivization $\mathbb{P}\Gamma$. This alternative view will be very useful later on.

If we choose a section $s \in \Gamma = \mathrm{H}^0(X, \mathcal{L}^k)$, then there is a divisor Z_s associated to it, which, roughly speaking, is the zeros of s minus the poles of s. Since we work with $\mathcal{L}^k$ very ample, Z_s consists of only the zero locus of s. Given the probability measure $d\gamma$ on Γ, we can choose s randomly, and ask what is the expected value of the random variable Z (defined by $s \mapsto Z_s$). This same question can be asked in an equivalent form using Poincare duality. The Poincare dual of Z_s is a $(1,1)$ form T_s, and it is more convenient to work with forms in this context than to work with divisors. In general T_s is not a C^∞-form on X, but it can be given an explicit expression using the notion of currents, i.e., distribution valued forms.

Currents are defined as it is customary in the theory of distributions.[9] Let $\Omega_0^{p,q}(X)$ be the space of compactly supported C^∞ (p,q)-forms on X, and for now we assume that $\dim X = n$. The space of (p,q)-*currents* is the distributional dual of $\Omega_0^{n-p,n-q}(X)$: $\mathcal{D}^{p,q}(X) = \Omega_0^{n-p,n-q}(X)'$. An element of $\mathcal{D}^{p,q}(X)$ is a linear functional on $\Omega_0^{n-p,n-q}(X)$ which continuous in the C^∞ norm.

The usefulness of currents in our context stems from the fact that Poincare dual T_Y of an algebraic subvariety Y, defined by

$$\int_X T_Y \wedge \alpha = \int_Y \iota^*(\alpha), \qquad \text{for any } \alpha \in \Omega_0^{\dim Y, \dim Y}(X),$$

oftentimes has an explicit form in terms of currents ($\iota \colon Y \hookrightarrow X$ is the embedding). Let us focus on the case when Y is a hypersurface, or more generally the zero divisor of a section

[9] For a nice introduction to distributions and currents in algebraic geometry the reader can consult [48], Chapter 0 resp. Chapter 3.

$s \in \Gamma = \mathrm{H}^0(X, \mathcal{L}^k)$. In this case the current is given by the *Poincare-Lelong* formula:

$$T_s = \frac{i}{\pi}\,\partial\overline{\partial}\,\log\langle s, s\rangle \;\in \mathcal{D}^{1,1}(X).$$

T_s is also known as the zero current of s. Thus, the Poincare-Lelong formula induces a map

$$T\colon \mathbb{P}\,\mathrm{H}^0(X,\, \mathcal{L}^k) \longrightarrow \mathcal{D}^{1,1}(X), \qquad s \mapsto T_s.$$

As discussed earlier, the Fubini-Study measure makes $\mathbb{P}\,\mathrm{H}^0(X,\, \mathcal{L}^k)$ into a probability space, and we can also view T as a random variable. Since the currents form a linear space, we can inquire about their expected value in this probability measure

$$E(T) = \int_{\mathbb{P}\,\mathrm{H}^0(X,\,\mathcal{L}^k)} T_s\; d\mu_{FS}(s),$$

As it happens oftentimes in the theory of distributions, although T_s is not a C^∞ form, $E(T)$ is, and we have the following proposition:

Proposition 3.5.1 *([79, Lemma 3.1]) With X and $\mathcal{L}$ as above, for k sufficiently large so that Kodaira's map i_k associated to* $\mathrm{H}^0(X,\, \mathcal{L}^k)$*, as defined in* (3.70)*, is an embedding, the expected value of the random variable T representing the zero current is*

$$E(T_s) = \frac{1}{k}\, i_k^*\, \omega_k^{FS},$$

where ω_k^{FS} is the Fubini-Study 2-form on $\mathbb{P}\,\mathrm{H}^0(X,\, \mathcal{L}^k)$*, and i_k^* is pullback of forms (in this case restriction).*

This result generalizes to the case when we intersect several divisors, and this will be the case of main interest to us. Since intersection of subvarieties is Poincare dual to the wedge product, there is an obvious guess how Prop. 3.5.1 should generalize. Let $s_1, \ldots, s_m$ be sections of $\mathrm{H}^0(X,\, \mathcal{L}^k)$, and consider the zero current $T^{(m)}_{s_1,\ldots,s_m}$ associated to the set

$$Z_{s_1,\ldots,s_m} = \{x \in X\colon s_1(x) = \cdots = s_m(x) = 0\}.$$

It is obvious that if we consider any m linear combinations of $s_1, \ldots, s_m$ (which are themselves linearly independent), then they determine the same zero set $Z_{s_1,\ldots,s_m}$. Hence $Z_{s_1,\ldots,s_m}$ is really associated to the m-plane spanned by $s_1, \ldots, s_m$ in $\mathrm{H}^0(X,\, \mathcal{L}^k)$. Therefore the probability space is the Grassmannian of m-dimensional subspaces of $\mathrm{H}^0(X, \mathcal{L}^k)$, with its natural Haar measure $d\mu_{Haar}$ (a generalization of the Fubini-Study measure). So we can ask what is the expected value of this zero current, computed with $d\mu_{Haar}$.

Proposition 3.5.2 *([79, Lemma 4.3]) In the notation of Prop. 3.5.1, the expected value of the zero current $T^{(m)}$, associated to the simultaneous vanishing of m random sections of $\mathrm{H}^0(X,\, \mathcal{L}^k)$, distributed according the Haar measure of the corresponding Grassmannian, is*

$$E(T^{(m)}) = k^{m-1} \left(i_k^* \, \omega_k^{FS} \right)^m \tag{3.73}$$

Note in particular, that unlike in Prop. 3.5.1, the distribution is according to the Haar measure, but the final result still involves the Fubini-Study form on $\mathrm{H}^0(X,\, \mathcal{L}^k)$. This is in fact natural, given that an m-tuple of sections, each distributed according to Fubini-Study on $\mathrm{H}^0(X,\, \mathcal{L}^k)$, is the same thing as one m-plane, distributed according to Haar on the Grassmannian.

As usual, besides having expected values, random variables also have variance. The zero current $T^{(m)}$ is no different in this respect. Its variance has been computed recently in [80, Theorem 1.1]. In particular, it was shown that the ratio of the variance and the expected value goes to zero as k is increased

$$\frac{\left[\mathrm{Var}\left(T^{(m)}\right)\right]^{1/2}}{E\big(T^{(m)}\big)} \sim k^{-\frac{m}{2}-\frac{1}{4}}.$$

The expected zero current

We chose to work with an arbitrary positive line bundle $\mathcal{L} = \mathcal{O}_X(1)$ on X. The associated Kodaira embedding is precisely the defining one:

$$i_1 \colon X \hookrightarrow \mathbb{P}\,\mathrm{H}^0(X,\, \mathcal{O}_X(1))^*.$$

If we take n sections of $\mathcal{L} = \mathcal{O}_X(1)$, and look at their common zero locus, then by Bezout's theorem this is generically $\int_X c_1(\mathcal{L})^n/n!$ points (degenerations might occur). Therefore, considering random n-tuples of sections will give random $(\int_X c_1(\mathcal{L})^n/n!)$-tuples of points on X. But now we can tell how these points are distributed, provided that the sections were distributed according to the Fubini-Study measure on $\mathbb{P}\,\mathrm{H}^0(X,\, \mathcal{O}_X(1))$. Using Prop. 3.5.2 we know that the expected value of the zero current associated to the $\int_X c_1(\mathcal{L})^n/n!$ points of intersection is $\left(i_1^* \, \omega_{\mathbb{P}^{n+1}}^{FS}\right)^n$. This is an (n,n) form on X, and plugging it into (3.68) we obtain the mass formula

$$m(x) = \frac{\Omega \wedge \overline{\Omega}}{\left(i_1^* \, \omega_{\mathbb{P}^{n+1}}^{FS}\right)^n}(x). \tag{3.74}$$

Example (Application of the mass formula (3.74)).

We consider the elliptic curve in $\mathbb{CP}^2$ defined as the zero locus of the Weierstrass cubic polynomial

$$Z_2^2 Z_0 = 4Z_1^3 - 60 G_4(i) Z_1 Z_0^2$$

with $G_4(\tau)$ the Eisenstein series of index 4, evaluated at the complex parameter $\tau = i$. Such algebraic curve is the preimage under the Weierstrass map of the square lattice $\Gamma \subset \mathbb{C}$ generated by $\omega_1 = 1$ and $\omega_2 = i$, thus the embedding $\mathbb{C}/\Gamma \to \mathbb{P}^2$ is given by

$$z \mapsto (Z_0 = 1,\ Z_1 = \mathcal{P}(z),\ Z_2 = \mathcal{P}'(z))$$

with $\mathcal{P}(z)$ the Weierstrass function associated to the elliptic curve defined by the lattice Γ.

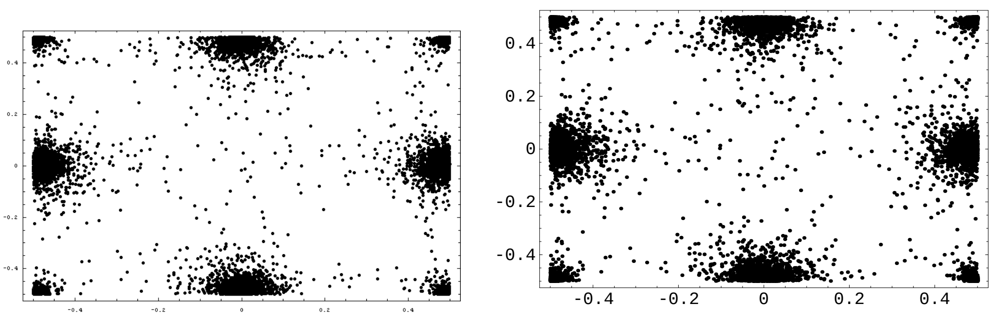

Figure 3.2: 10K points generated by random sections (left) and by a Monte Carlo simulation using the mass formula (right).

In Fig. 3.2 we divide the fundamental square $[1, 0] \times [0, 1] \subset \mathbb{C}$ that defines a coordinate chart for the elliptic curve, in a set of smaller regions given by a sub-lattice. On the left we plot more than 10K points generated by using random couples of sections. On the left, we compare our result by generating the same number of points through a Monte Carlo using the mass formula (3.74). Both results are equivalent as one expects.

3.6 Examples

3.6.1 Calabi-Yau hypersurfaces in projective spaces

In this section we put all the pieces together, and explicitly show how to build the numerical measure $\{q_i \in X,\, m(q_i)\}_{i=1}^{N_p}$ introduced in Section 3.5.1. We will focus on a smooth Calabi-Yau hypersurface X in $\mathbb{P}^{n+1}$ of degree $n+2$. Let X be given by the zero locus of the degree $n+2$ homogeneous polynomial f, and let $(Z_0, Z_1, \ldots Z_{n+1})$ be the homogeneous coordinates on $\mathbb{P}^{n+1}$. We denote the embedding by

$$i\colon X = \mathcal{Z}(f) \hookrightarrow \mathbb{P}^{n+1}.$$

Our approach is to generate random points on X using random lines on $\mathbb{P}^{n+1}$, by looking at the intersection of these random lines with X. We can view a random line as the intersection of n random hyperplanes. This allows us to compute the expected value of the corresponding zero current using the techniques of Section 3.5.3.

For computational purposes, designing an algorithm to generate points on X in such a fashion is straightforward. To generate a random line on $\mathbb{P}^{n+1}$ we generate two random points, which lie on the unit sphere $\mathrm{S}^{2n+3} \subset \mathbb{C}^{n+2}$, and are distributed uniformly on this sphere. For instance, to generate random points uniformly on S^{2n+3} we can start with the unit cube in $\mathbb{R}^{2n+4}$, i.e., $[-1,\,1]^{2n+4} \subset \mathbb{R}^{2n+4}$. Using a good quality random number generator we generate an uniform distribution of points in $[-1,\,1]^{2n+4}$. Now take *only* those points which fall within the unit disk D^{2n+4}, and then project them radially to the boundary $\partial\mathrm{D}^{2n+4} = \mathrm{S}^{2n+3}$.

The intersection of the random line with X can be computed by restricting the defining polynomial f to the line. As a result, computing the common zero locus reduces to solving for the roots of a polynomial of degree $n+2$ in one variable. We find numerically the $n+2$ roots using the Durand-Kerner algorithm [38, 56], which is a refinement of the multidimensional Newton's method applied to a polynomial. This whole approach turns out to be very efficient in practice, in that one can generate a million points on a quintic in a matter of seconds.

The numerical mass

Let us look at the two differential forms involved in (3.74). For this we first choose affine coordinates $w_a = Z_a/Z_0$, $i = 1, 2, \ldots, n+1$ on $\mathbb{P}^{n+1}$. The Fubini-Study 2-form on $\mathbb{P}^{n+1}$ is

$$\omega_{FS}^{\mathbb{P}^{n+1}} = \left[\frac{\sum \mathrm{d}w_a \wedge \mathrm{d}\overline{w}_a}{1 + \sum w_a \overline{w}_a} - \frac{\left(\sum \overline{w}_a \mathrm{d}w_a\right) \wedge \left(\sum w_a \mathrm{d}\overline{w}_a\right)}{\left(1 + \sum w_a \overline{w}_a\right)^2} \right]. \tag{3.75}$$

The pullback $\left(i_1^* \, \omega_{\mathbb{P}^{n+1}}^{FS}\right)^n \in \Omega_0^{n,n}(X, \mathbb{Z})$ is a top form on X. Let $x_1, \ldots, x_n$ be local coordinates on X. Then

$$\left(i_1^* \, \omega_{\mathbb{P}^{n+1}}^{FS}\right)_{i\bar{\jmath}} = \frac{\partial w_a}{\partial x_i} \, \left(\omega_{\mathbb{P}^{n+1}}^{FS}\right)_{a\bar{b}} \, \overline{\frac{\partial w_{\bar{b}}}{\partial x_{\bar{\jmath}}}}, \tag{3.76}$$

and $\left(i_1^* \, \omega_{\mathbb{P}^{n+1}}^{FS}\right)^n$ is proportional to the determinant of this matrix. For obvious reasons we need to evaluate this determinant. Let us outline how this can be done, paying attention to some of the numerical aspects.

The idea is to choose local coordinates on X that are convenient to work with. Let us start with the point P on X with homogenous coordinates Z_i. To minimize the numerical error we go to the affine patch where $|Z_i|$ is maximal. Without loss of generality let us assume that this happens for $i = 0$. The affine coordinates are $w_a = Z_a/Z_0$.

Let p be the affine form of f, i.e., $p(w) = f(1, w_1, w_2, \ldots, w_{n+1})$. This equation determines one of the w_a-s in term of the others, as an implicit function. Let us assume for the sake of this presentation that $\partial p/\partial w_{n+1}(P) \neq 0$. The implicit function theorem then tell us that in an open neighborhood of P w_{n+1} is a function of the remaining variables: $w_{n+1} = w_{n+1}(w_1, w_2, \ldots, w_n)$. This allows us to choose the coordinates $w_1, \ldots, w_n$ to be the local coordinates $x_1, \ldots, x_n$ on X.

This choice of coordinates is quite advantageous for computing (3.76). All we need is to compute $\partial w_{n+1}/\partial x_i$, as $\partial w_j/\partial x_i = \delta_{ij}$. This can be done algebraically, without explicitly solving the $p = 0$ equation. Namely, using the fact that

$$p(w_1, \ldots, w_n, w_{n+1}(w_1, \ldots, w_n)) \equiv 0$$

is the identically zero function, its derivative with respect to any w_i vanishes identically, for $i = 1, \ldots, n$. As a result we have that

$$\frac{\partial w_{n+1}}{\partial w_i}(P) = -\frac{\partial p}{\partial w_i}(P)/\frac{\partial p}{\partial w_{n+1}}(P). \tag{3.77}$$

For numerical stability one should always solve for the variable for which $|\partial p/\partial w_i(P)|$ is the largest.

The second differential form entering (3.74) is $\Omega \wedge \overline{\Omega}$. The holomorphic n-form $\Omega \in \Omega^{n,0}(X, \mathbb{C})$ can be represented using the Poincare residue map [48, Section 1.1]

$$\Omega = (-1)^{i-1} \frac{\mathrm{d}w_1 \wedge \mathrm{d}w_2 \ldots \wedge \widehat{\mathrm{d}w_i} \wedge \ldots \wedge \mathrm{d}w_n}{\partial p(w)/\partial w_i}. \tag{3.78}$$

where $\widehat{\mathrm{d}w_i}$ means the omission of $\mathrm{d}w_i$ in the wedge product.

These explicit expressions allow us to perform integrals numerically on elliptic curves, $K3$ surfaces and more interestingly, quintic 3-folds.

Symmetries

Suppose our Calabi-Yau X is preserved as a complex manifold by the action of a discrete group Γ. Then a Ricci flat metric whose Kähler class ω is preserved by Γ will also be Γ-invariant, because it is unique. As we will see in this section, the same statement applies to the balanced metrics as well.

A general hermitian N by N matrix has N^2 independent real coefficients. On the other hand, if the Calabi-Yau X has discrete symmetries, then we expect to find symmetry relations between the matrix elements of $T(h)$. Taking advantage of these relations can drastically reduce the size of the problem. In this section, we argue that these symmetry relations are respected by the balanced metric and the T-map, and explain how we used them in the examples of Section 3.6.2.

Next, let us review the symmetries of X defined as a hypersurface in $\mathbb{P}^{n+1}$ by the degree $n+2$ homogenous polynomial

$$f = \sum_{i=0}^{n+1} Z_i^{n+2} - (n+2)\psi \prod_{i=0}^{n+1} Z_i. \tag{3.79}$$

Here ψ controls the complex structure of the hypersurface. Using the fact that X is Calabi-Yau, the symmetry group is finite. To find the symmetries of X we consider two natural group actions on $\mathbb{P}^{n+1}$, and impose conditions such that these group actions descend to X.

We start with the abelian group

$$\bigoplus_{i=0}^{n+1} \mathbb{Z}_p \subset GL(n+2)$$

that acts by independently rescaling the $n+2$ homogenous coordinates $Z_i \mapsto \alpha_i Z_i$, where α_i are pth roots of unity. Since the projective coordinates are defined only up to overall rescaling, we have to mod out by the diagonal action $\triangle\mathbb{Z}_p$ and find the group

$$\bigoplus_{i=0}^{n+1} \mathbb{Z}_p/\triangle\mathbb{Z}_p \subset \mathbb{P}GL(n+2)$$

acting on $\mathbb{P}^{n+1}$. In order for this group to descend to X, it must leave the defining equation (3.79) invariant. In the Fermat case, that is $\psi = 0$, we set $p = n+2$ and find that the Calabi-Yau is invariant under

$$\bigoplus_{i=0}^{n+1} \mathbb{Z}_{n+2}/\triangle\mathbb{Z}_{n+2} \cong (\mathbb{Z}_{n+2})^{n+1}. \tag{3.80}$$

For non-vanishing ψ, the α_i have to obey the additional constraint $\prod_{i=0}^{n+1} \alpha_i = 1$. This shows that the symmetry group is a subgroup of $\mathbb{Z}_{n+2}/\triangle\mathbb{Z}_{n+2} \cong (\mathbb{Z}_{n+2})^{n+1}$, given by the kernel of the product map $(\alpha_0, \ldots, \alpha_{n+1}) \mapsto \prod_{i=0}^{n+1} \alpha_i$. We call this group Ab_{n+2}, and it is clear that there is an isomorphism $Ab_{n+2} \cong (\mathbb{Z}_{n+2})^n$. For example, in the case of the torus defined by our cubic in $\mathbb{P}^2$, $Ab_3 \cong \mathbb{Z}_3$. At the Fermat point this group is enhanced to $\mathbb{Z}_3^2$.

The second symmetry group we consider is the symmetric group on $n+2$ elements $\mathbb{S}_{n+2}$. This group acts by permuting the coordinates of $\mathbb{P}^{n+1}$. Since (3.79) is invariant under permutations, $\mathbb{S}_{n+2}$ is a symmetry of X as well.

To see how these actions on the coordinates of $\mathbb{P}^{n+1}$ induce an action on $\{s_\alpha\}$, the global sections of the line bundle $\mathcal{L}^k$ on X defining the embedding in $\mathbb{P}^{N-1}$, we can use some simple algebraic geometry. The fact that X is given by a hyperplane in $\mathbb{P}^{n+1}$ gives a natural way to parameterize the global sections of $\mathcal{L}^k = \mathcal{O}_X(k)$. We start with the short exact sequence (SES) defining X:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(-n-2) \xrightarrow{\cdot f} \mathcal{O}_{\mathbb{P}^{n+1}} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Tensoring with $\mathcal{O}_{\mathbb{P}^{n+1}}(k)$, and using the fact that $\mathrm{H}^1(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(k-n-2)) = 0$, for $k > 0$, we get another SES:

$$0 \longrightarrow \mathrm{H}^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(k-n-2)) \xrightarrow{\cdot f} \mathrm{H}^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(k)) \longrightarrow \mathrm{H}^0(X, \mathcal{L}^k) \longrightarrow 0 \quad (3.81)$$

which shows that the global sections of $\mathrm{H}^0(X, \mathcal{L}^k)$ can be parameterized by degree k monomials in $n+2$ variables modulo the ideal generated by f. Therefore the sections inherit an obvious group action.

In particular, we also find that

$$N = \dim \mathrm{H}^0(X, \mathcal{L}^k) = \binom{n+k+1}{k} - \binom{k-1}{k-n-2}.$$

In addition, note that the map $i_k \colon X \hookrightarrow \mathbb{P}^{N-1}$ factorizes

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[rr, bend right, "i_k"'] & \mathbb{P}^{n+1} \arrow[r, hook, "v"] & \mathbb{P}^{N-1} \end{tikzcd} \quad (3.82)$$

The second embedding, $v \colon \mathbb{P}^{n+1} \hookrightarrow \mathbb{P}^{N-1}$, is the (Veronese) map associated to the incomplete linear system on $\mathbb{P}^{n+1}$ induced by the complete linear system $|\mathcal{L}^k|$ on X.

We will now consider the consequences of these actions on the T-map and the sequence of hermitian matrices $\{T^l(h)\}_{l=1,2,\ldots}$. First we consider the action of Ab_{n+2}. We assume that $T^0(h)$ is invariant under the group action. This is a choice we can always make. Since Ab_{n+2} is an abelian group, its irreducible representations are one dimensional, and can be labeled by the characters. Each section s_α transforms under a character χ_α. The operator T is defined in terms of the sections, and the Ab_{n+2} will force some of the $T(h)_{\alpha\overline{\beta}}$ matrix elements to be zero. To better understand this we look at a toy example: the integral of an odd function a on $\mathbb{R}$. The group G in question is $\mathbb{Z}_2$, and acts on $\mathbb{R}$ by $x \mapsto -x$. $\mathbb{Z}_2$ has only one nontrivial representation, and being odd, a transform in this irrep. Now we have that

$$\int_{-\infty}^{+\infty} a(t)\, dt = \int_{+\infty}^{-\infty} a(-x)\, d(-x) = -\int_{-\infty}^{+\infty} a(x)\, dx,$$

where we used a change of variable $t = -x$. This implies that $\int_{-\infty}^{+\infty} a(x) dx = 0$.

More generally, in $\mathbb{R}^n$ for a function a, a group G, and an element $g \in G$, we can do the change of coordinates $t = g \cdot x$ and then

$$\int_X a(t)\, dV(t) = \int_{g \cdot X} a(g \cdot x) g^*\, dV(x) = \int_X a(g \cdot x)\, dV(x)\,. \quad (3.83)$$

Here we assumed the measure to be G-invariant.

Applying (3.83) for $G = Ab_{n+2}$, and using the fact that s_α transform as a character of $G = Ab_{n+2}$, it gives that

$$T(h)_{\alpha\overline{\beta}} = \frac{N}{\mathrm{vol}(X)} \int_X \frac{\chi_\alpha(u) s_\alpha \overline{\chi_{\overline{\beta}}(u)} \overline{s}_{\overline{\beta}}}{||s||_h^2}\, d\mu_\Omega = \chi_\alpha(u) \overline{\chi_{\overline{\beta}}(u)}\, T(h)_{\alpha\overline{\beta}}. \quad (3.84)$$

We used the fact that X is invariant under that action of Ab_{n+2}, and so is the measure $\Omega \wedge \overline{\Omega}$, and the denominator $||s||_h^2$. (The latter follows by induction from the initial choice of $T^0(h)$ being invariant.) In particular, if $\chi_\alpha(u)\overline{\chi_{\overline{\beta}}(u)} \neq 1$, for any $u \in Ab_{n+2}$, then the corresponding $T(h)_{\alpha\overline{\beta}}$ has to vanish. In our numerical routine we impose this vanishing condition on all the matrices $T^l(h)$.

A similar argument applies for $G = \mathbb{S}_{n+2}$. Since $\mathbb{S}_{n+2}$ is not abelian, and hence its generic irreducible representations are not one dimensional, this constraint does not result in vanishing rules, but rather sets a priori independent coefficients of $T(h)$ equal to each other. To see how $\mathbb{S}_{n+2}$ acts, recall from (3.81) that

$$\mathrm{H}^0(X, \mathcal{L}) \cong \mathrm{H}^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(1)) = \mathbb{C}^{n+2}$$

is the fundamental representation of $\mathbb{S}_{n+2}$, call it F. Then $\mathrm{H}^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(k))$ is the kth symmetric tensor power of F, $Sym^k F$, and by (3.81) $\mathrm{H}^0(X, \mathcal{L}^k)$ is a quotient of this. Now we can return to (3.83). Once again, we choose $T^0(h)$ to be invariant under $\mathbb{S}_{n+2}$, and then induction and (3.83) tell us which matrix elements of $T(h)$ equal each other.

Therefore, imposing the symmetries of both finite groups, the number of independent components of $T^l(h)$ (for any l) reduces significantly. To illustrate this we consider $k = 12$ on the quintic in $\mathbb{P}^4$, i.e., $n = 3$ (this was the largest k we computed). In this case $N = 1490$. This means that $T(h)$ is a hermitian matrix with 2,220,100 components. Taking into account the Ab_5 and $\mathbb{S}_5$ relations, one is reduced to computing 9800 components. This simplification speaks for itself.

3.6.2 Numerical results for the quintic hypersurface

In this section we present our explicit numerical results. The main object that we compute is the balanced metric associated to the embedding of the quintic threefold defined in (3.79). For definiteness we chose to work with $\psi = 0.1$, but also tested other values of ψ. We also considered the case of elliptic curves ($n = 1$) and K3 surfaces ($n = 2$). In all these cases we obtained results similar to the ones to be presented here.

To find the balanced metric we study the associated $N \times N$ matrix h_k for several values of k, from $k = 1$ to $k = 12$. We use h_k to construct the associated Kähler form ω_k on X, and check how well it approximates the Ricci flat metric. We do this in several ways.

First, one can study the function defined in (3.99)

$$\eta_k = \frac{\det\, \omega_k}{\Omega \wedge \overline{\Omega}} : X \longrightarrow \mathbb{R}.$$

For a good approximation g_k to the Ricci flat metric g_{RF} the function η_k is almost constant. We study the behavior of η_k statistically, by summing over all the regions of X, and also locally paying attention to certain special regions of the threefold.

Second, one can compute the Ricci tensor of ω_k. To check pointwise how close to zero the Ricci tensor is, we need a diffeomorphism invariant quantity. We chose to work with the Ricci scalar. We also perform this analysis for several values of k, and show how the Ricci scalars decrease pointwise with k.

Before presenting the results let us comment on the errors coming from Monte Carlo integration. We estimate them by computing the balanced metrics associated to different samples of points, and then looking at the mean and variance of each individual matrix element. Ideally, one would like to produce samples of points with minimal induced error. Constructions that reduce the standard deviation of the integrals are refinements to the theory of numerical integration presented here. Markov Chain Monte Carlo techniques, construction of lattices on Calabi-Yau varieties, and of quasi-random points on such manifolds are different approaches that one could consider.

Approximating volumes v.s. Calabi-Yau volume

Here, we consider the way the function

$$|\eta_k - 1_X|\colon X \longrightarrow \mathbb{R}_+, \qquad x \mapsto |\eta_k(x) - 1|$$

behaves on X. As argued earlier, we expect $|\eta_k - 1_X|$ to approach the constant zero function. One can study the deviation of $|\eta_k - 1_X|$ from the zero function by computing the integral

$$\sigma_k = \int_X |\eta_k - 1_X| \, d\mu_\Omega \,. \tag{3.85}$$

We compute this integral by our Monte Carlo method, which introduces an error, and this error can be estimated by

$$\delta\sigma_k = \frac{1}{\sqrt{N_p}} \left(\int_X \left(|\eta_k - 1_X| - \sigma \right)^2 \, d\mu_\Omega \right)^{1/2} , \tag{3.86}$$

where N_p is the number of points used to perform the Monte Carlo integration in (3.85).

In Fig. 3.3 we plot the values σ_k defined in (3.85) for $k = 3, \ldots, 12$. The error bars for each value are the corresponding standard deviations (3.86). We also see how the errors decrease, along with σ_k, for higher and higher k. The fit in Fig. 3.3 is a curve of type

$$\sigma_k = \frac{\alpha}{k^2} + \frac{\beta}{k^3} + \mathrm{O}\left(\frac{1}{k^4}\right),$$

as we expect from the theory.

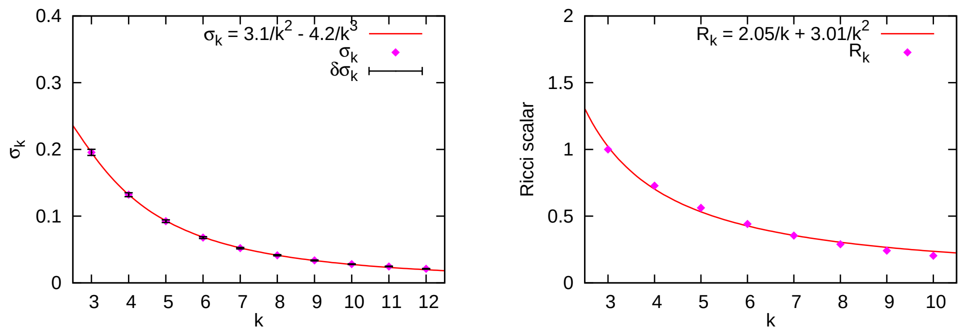

Figure 3.3: σ_k and Ricci scalars.

We can also study the local behavior of η_k by restricting it to a subspace. Given our quintic 3-fold, we consider the rational curve defined by

$$(Z_0 = z_0,\, Z_1 = -z_0,\, Z_2 = z_1,\, Z_3 = 0,\, Z_4 = -z_1), \tag{3.87}$$

where Z_i are homogeneous coordinates on $\mathbb{P}^4$, while $(z_0,\, z_1)$ are homogeneous coordinates for $\mathbb{P}^1$. This rational curve lies on every quintic defined by (3.79).

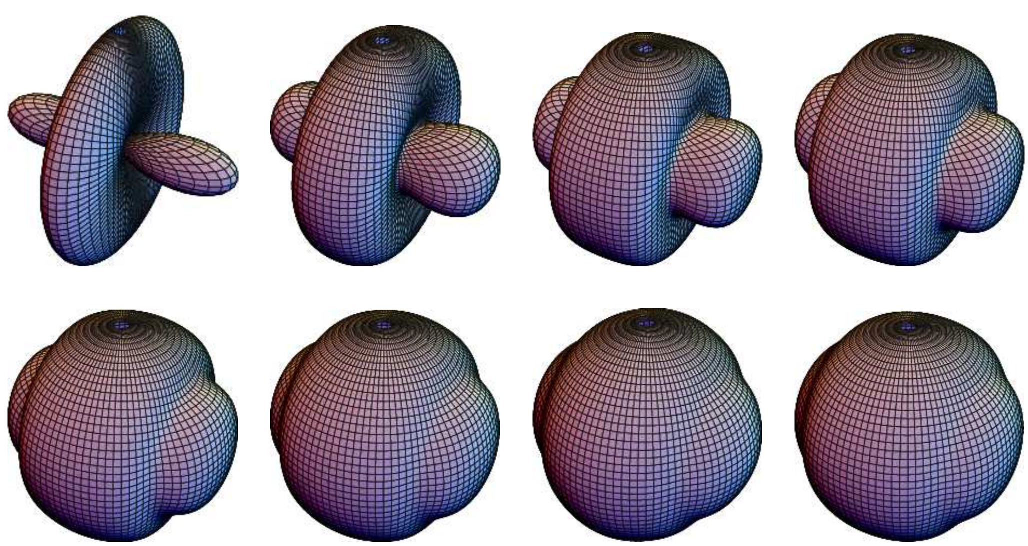

Figure 3.4: The values of η on the rational curve, for $k = 1, 3, 4, 5, 7, 9, 11$ and 12.

In Fig. 3.4 we plot the values of function η_k restricted to the rational curve defined above for 12 different values of k, ranging between 1 and 12. More concretely, given the embedding (3.87), we choose the local coordinate system on $\mathbb{P}^1$ defined by $t = z_1/z_0$, and take the stereographic projection of the t-plane. Using spherical coordinates $(\theta,\, \phi)$ on $\mathbb{P}^1 \equiv S^2$ we embed it into $\mathbb{R}^3$, by the parameterization

$$z_0 = \sin\theta\,\cos\phi, \qquad z_1 = \sin\theta\,\sin\phi + i\,\cos\theta.$$

In the radial direction of $\mathbb{R}^3$ we plot the function η_k. As expected, η_k approaches the constant function 1 as k increases.

3.6.3 Hermite-Einstein metric on the tangent bundle of $\mathbb{P}^n$

Let $\mathbb{P}^n$ be the complex projective space of dimension n, and $\{Z_i\}_{i=0}^{i=n}$ its homogeneous coordinates. We will work on the open set $Z_0 \neq 0$ and chose the local inhomogeneous coordinates $w_i = Z_i/Z_0$. The Fubini-Study metric on $\mathbb{P}^n$

$$g_{i\bar{j}} = \frac{1}{1+\sum_i |w_i|^2}\delta_{i\bar{j}} - \frac{w_i\overline{w}_j}{(1+\sum_i |w_i|^2)^2}.$$

is the unique maximally symmetric metric, with its group of Killing symmetries isomorphic to $U(n+1)$. In addition, this metric is Einstein, that is its Ricci tensor is proportional to the metric itself. Therefore its associated curvature tensor obeys the hermitian Yang-Mills equation. The Donaldson-Uhlenbeck-Yau theorem then implies that the tangent bundle of $\mathbb{P}^n$, $T\mathbb{P}^n$, is a rank n stable bundle on $\mathbb{P}^n$.[10] It follow from this that the balanced metric on the bundle $T\mathbb{P}^n$ must be the Fubini-Study metric.

To describe the tangent bundle $T\mathbb{P}^n$ we use the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \longrightarrow T\mathbb{P}^n \longrightarrow 0. \tag{3.88}$$

Here $\mathcal{O}_{\mathbb{P}^n}(1)$ denotes the hyperplane line bundle. After twisting the sequence by $\mathcal{O}_{\mathbb{P}^n}(k)$ and taking the cohomology we find the short exact sequence (SES)

$$0 \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k+1)^{\oplus(n+1)}) \longrightarrow H^0(\mathbb{P}^n, T\mathbb{P}^n(k)) \longrightarrow 0. \tag{3.89}$$

This gives an explicit description for $H^0(\mathbb{P}^n, T\mathbb{P}^n(k))$, which for sufficiently large k gives the embedding

$$\mathbb{P}^n \hookrightarrow \mathrm{G}(n, W) \tag{3.90}$$

where $W = H^0(\mathbb{P}^n, T\mathbb{P}^n(k))^*$, and $\mathrm{G}(n, W)$ is the Grassmannian of n-planes in W.

Based on (3.89), we choose to describe the global sections of $T\mathbb{P}^n(k)$ by an $n+1$ vector

$$(M_0, \ldots, M_n)$$

[10] The stability of $T\mathbb{P}^n$ also has purely algebraic proof.

where $\{M_i\}_{i=1}^n$ are arbitrary monomials of degree $k+1$ in the homogeneous coordinates Z_i, while M_0 is any degree $k+1$ monomial which does not contain an Z_0.

Now we show how to construct the embedding (3.89) for any $k \geq 0$. We start by choosing a frame $\{\hat{e}_i\}_{i=0}^n$ for the vector bundle $\mathcal{O}(k+1)^{\oplus(n+1)}$. This amounts to choosing a section for every one the $n+1$ summands. For simplicity we chose the same section in every summand. The Euler sequence (3.88) imposes the condition

$$\sum_{i=0}^{n} Z_i \hat{e}_i = 0.$$

Locally this gives a frame for $T\mathbb{P}^n$, if we solve for

$$\hat{e}_0 = -\sum_{i=1}^{n} \frac{Z_i}{Z_0} \hat{e}_i = -\sum_{i=1}^{n} w_i \hat{e}_i.$$

Expanding the global sections of $T\mathbb{P}^n(k)$ in the local frame $\{\hat{e}_i\}_{i=1}^n$ gives an $n \times \dim(W)$ matrix, which is the explicit realization of our embedding [48].

To illustrate the procedure consider $T\mathbb{P}^2(0)$. $\mathcal{O}_{\mathbb{P}^2}(1)$ has 3 global sections: Z_0, Z_1, Z_2. Choosing Z_0 to be the local frame in every summand of $\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3}$, and discarding the global section Z_0 from the first $\mathcal{O}_{\mathbb{P}^2}(1)$, we find the matrix

$$z = \begin{pmatrix} -w_1^2 & -w_1 w_2 & 1 & w_1 & w_2 & 0 & 0 & 0 \\ -w_1 w_2 & -w_2^2 & 0 & 0 & 0 & 1 & w_1 & w_2 \end{pmatrix} \tag{3.91}$$

For an initial hermitian metric G_0 on the vector space $W = H^0(\mathbb{P}^n, T\mathbb{P}^n(k))^*$, our generalized T-operator (3.70) gives the iterations

$$G_{m+1} = T(G_m) = \frac{\dim W}{n \operatorname{Vol}(\mathbb{P}^n)} \int_{\mathbb{P}^n} z (z^\dagger G_m^{-1} z)^{-1} z^\dagger \, dV.$$

We tested the converges of the T-map starting with $G_0 = I$ in the case $n = 2$ for $k = 1, \ldots, 5$. In all cases we converged to a given G_∞ for less than 10 iterations, with a precision of 0.1%.

The balanced metric $H^{(k)}$ on the vector bundle $T\mathbb{P}^n(k)$ induced by G_∞ is given by

$$H^{(k)} = (z^\dagger G_\infty^{-1} z)^{-1}. \tag{3.92}$$

Let h be Fubini-Study metric on the hyperplane bundle $\mathcal{O}_{\mathbb{P}^n}(1)$, that is the metric with constant scalar curvature. Then the metric

$$H_k := H^{(k)} \otimes h^{-k} = (z^\dagger G_\infty^{-1} z)^{-1} \otimes h^{-k}$$

is the balanced metric on $T\mathbb{P}^n$. Our numerical computations show that this is indeed the Fubini-Study metric on $T\mathbb{P}^n$, as explained earlier. The numerical agreement is within 0.5%. This provides the first non-trivial test of our conjecture.

3.6.4 A stable rank 3 bundle over $\mathbb{P}^2$

In this section we test our generalized T-operator on a rank 3 vector bundle V^* over $\mathbb{P}^2$. We first consider its dual V, defined by four linearly independent global sections $\{m_i\}$ of $\mathcal{O}_{\mathbb{P}^2}(2)$ through the SES

$$0 \longrightarrow V \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} \xrightarrow{m} \mathcal{O}_{\mathbb{P}^2}(2) \longrightarrow 0.$$

This bundle has moduli, which are implicitly determined by the choice of the sections $\{m_i\}$. Before choosing these, let us check stability, which does not depend on the specifics of this choice.

To check stability, we have to ensure that neither V nor $\wedge^2 V$ have destabilizing line bundles. Using the canonical isomorphism

$$\wedge^2 V = \det V \otimes V^*$$

we find the slopes

$$\mu(V) = -2/3, \;\; \mu(\wedge^2 V) = -4/3.$$

Since $Pic(\mathbb{P}^2) = \mathbb{Z}$, all line bundles are of the form $\mathcal{O}_{\mathbb{P}^2}(p)$ for some p. Hence it is sufficient to show that

$$H^0(\mathbb{P}^2, V) = 0, \quad H^0(\mathbb{P}^2, \wedge^2 V(1)) = 0.$$

The first fact follows from the defining sequence of V, if we assume that $\{m_i\}$ are linearly independent. To prove the second statement we use

$$H^0(\mathbb{P}^2, \wedge^2 V(1)) = H^0(\mathbb{P}^2, V^*(-1)) = H^2(\mathbb{P}^2, V(-2))^*.$$

Again, this statement follows easily from the defining sequence of V. Finally, stability of V implies stability for V^*.

We will now compute the hermitian Yang-Mills connection on V^* using our generalized T-operator. First observe that $V^*(k)$ fits into the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(k-2) \to \mathcal{O}_{\mathbb{P}^2}(k)^{\oplus 4} \to V^*(k) \to 0. \tag{3.93}$$

This leads to another SES

$$0 \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-2)) \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k)^{\oplus 4}) \to H^0(\mathbb{P}^2, V^*(k)) \to 0.$$

We can use this expression for an explicit parameterization of $H^0(\mathbb{P}^2, V^*(k))$.

For concreteness let us choose to be four global sections $\{m_i\}_{i=1}^4$ defining V to be

$$Z_1 Z_2, \; Z_0 Z_1, \; Z_0 Z_2, \; Z_0^2.$$

Now we choose a frame $\{\hat{e}_i\}$ for $\mathcal{O}_{\mathbb{P}^2}(k)^{\oplus 4}$. The defining equation (3.93) of $V^*(k)$ imposes the condition $\sum_i m_i \hat{e}_i = 0$, and gives a frame for V^*. Locally we can solve for $\hat{e}_0$, and working in inhomogeneous coordinates w_i we find that

$$\hat{e}_0 = -\frac{1}{w_2}\hat{e}_1 - \frac{1}{w_1}\hat{e}_2 - \frac{1}{w_1 w_2}\hat{e}_3.$$

Expanding the global sections of $H^0(\mathbb{P}^2, V^*(k))$ in the frame $\{\hat{e}_i\}_{i=1}^3$ gives a matrix, which is the embedding map.

We studied the convergence of our generalized T-operator numerically for $k = 2, 3$ and 4, and found that convergence was achieved for less than 10 iterations. As before, the metric on $V^*(k)$ is given by

$$H^{(k)} = (z^\dagger G_\infty^{-1} z)^{-1}, \tag{3.94}$$

while the corresponding metric on V^* is

$$H_k := H^{(k)} \otimes h^{-k} = (z^\dagger G_\infty^{-1} z)^{-1} \otimes h^{-k},$$

where h is again the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}^2}(1)$.

Since in this case the balanced metric on V^* is not a priori known, one needs a different approach, than used in the previous section for $T\mathbb{P}^2$, to test how close is the approximate balanced metric to satisfying the hermitian Yang-Mills equation. But this quite easy to do numerically once the balanced metric G_∞ is known, as all we need to do is to check how close we are to satisfying Eq. (3.57). In all cases considered Eq. (3.57) was satisfied to within 1% accuracy.

3.6.5 A rank 3 bundle on the Fermat quintic

In this section we turn to a much more complicated case than our previous examples, that of a stable rank 3 bundle on the Fermat quintic Q in $\mathbb{P}^4$:

$$Q: \qquad Z_0^5 + Z_1^5 + \cdots + Z_4^5 = 0. \tag{3.95}$$

Testing our generalized T-operator in this case necessitates knowledge of the Ricci flat metric on the Fermat quintic that computed above. In this example we consider the balanced metric for $k = 3$.

Balanced metric on the Fermat quintic

We consider the embedding of Q given by the complete linear system of cubics, $H^0(Q, \mathcal{O}_Q(3))$, whose complex projectivization is isomorphic to $\mathbb{P}^{34}$. The balanced metric will be the restriction of a Fubini-Study metric on $\mathbb{P}^{34}$.

In our computations we build 10 different samples of 100,000 points, which we use independently to iterate the T-map until convergence is reached, i.e., the sequence $\{T^r(G_0)\}_{r=0}$ obeys

$$||T^{r+1}(G_0) - T^r(G_0)|| < \epsilon.$$

In our simulations the fixed point of this discrete version of the T-map was reached after 15-20 iterations. Each weighted point set gave rise to a convergent sequence. The 10 different hermitian forms $\{G^e_\infty\}_{e=1}^{10}$ approximating the balanced metric in $\mathbb{P}H^0(Q, \mathcal{O}_Q(3))$ agree up to

$$\max\left[\frac{\sigma(G^e_\infty)}{|\langle G^e_\infty\rangle|}\right] \approx 0.9\%, \tag{3.96}$$

where $|\langle G^e_\infty\rangle|$ is the average matrix of the ten different outputs and $\sigma(G^e_\infty)$ is the standard deviation matrix. The ratio $\sigma(G^e_\infty)/|\langle G^e_\infty\rangle|$ is computed entry by entry, and the maximum is taken over all entries. We used the average $\langle G^e_\infty\rangle$ as approximation for the balanced metric on $H^0(Q, \mathcal{O}_Q(3))$.

Solution of the hermitian Yang-Mills equation

Here, we use the generalized T-operator to produce a hermitian Yang-Mills connection on a rank three stable vector bundle V on the Fermat quintic Q. We also implicitly test that the previously obtained balanced metric on Q indeed has vanishing Ricci curvature.

We define the rank three bundle V by the following SES

$$0 \longrightarrow \mathcal{O}_Q(-1) \xrightarrow{\beta} \mathcal{O}_Q^{\oplus 4} \longrightarrow V \longrightarrow 0.$$

β is given by four generic global sections of $\mathcal{O}_Q(1)$, which do not intersect on Q, hence V is indeed a vector bundle. In addition, the first Chern class of V is $c_1(V) = H$, hence V is not a simple twist of the tangent bundle of Q. It is a simple exercise to show that V is stable.

Once again, we use

$$0 \longrightarrow \mathcal{O}_Q(k-1) \longrightarrow \mathcal{O}_Q^{\oplus 4}(k) \longrightarrow V(k) \longrightarrow 0,$$

and its associated long exact sequence in cohomology

$$0 \longrightarrow H^0(Q, \mathcal{O}_Q(k-1)) \longrightarrow H^0(Q, \mathcal{O}_Q^{\oplus 4}(k)) \longrightarrow H^0(Q, V(k)) \longrightarrow 0,$$

to derive a frame for V and an explicit parameterization for the global sections. We choose $\beta = (Z_0, \ldots, Z_3)$. Using the frame $\{\hat{e}_i\}_{i=0}^4$ for $\mathcal{O}_Q^{\oplus 4}$, we also get a frame for V with the relation

$$\hat{e}_0 = -\sum_{i=1}^{3} w_i \hat{e}_i.$$

In this paper we restrict to the case $k=1$ for which $\dim H^0(Q,V(1))=19$. The coordinate matrix

$$z(w)=\begin{pmatrix} 1\ldots w_4 & 0 & 0 & -w_1^2 & -w_1w_2 & -w_1w_3 & -w_1w_4 \\ 0 & 1\ldots w_4 & 0 & -w_1w_2 & -w_2^2 & -w_3w_2 & -w_4w_2 \\ 0 & 0 & 1\ldots w_4 & -w_1w_3 & -w_2w_3 & -w_3^2 & -w_4w_3 \end{pmatrix} \tag{3.97}$$

gives the embedding into the Grassmannian $Q\hookrightarrow G(3,19)$.

Using the integration techniques described in the previous section, we iterate the generalized T-operator. We reach the fixed point after 12-15 iterations for several different samples of weighted points which approximate the analytical measure, allowing us to estimate the balanced metric for $H^0(Q,V(1))$ with an error of 1.1%.[11]

The metric on $V(1)$ is given by

$$H=(z^\dagger G_\infty^{-1}z)^{-1}, \tag{3.98}$$

To test the accuracy of this metric we evaluate the right hand side of the hermitian Yang-Mills equations (3.57). We find the mean to be

$$\langle\omega^{i\bar{j}}F_{i\bar{j}}\rangle=\frac{1}{\mathrm{Vol}(Q)}\int_Q\left(\omega^{i\bar{j}}F_{i\bar{j}}\right)d\mu_\Omega\approx 1.31\times\mathbf{I}_{3\times3}$$

with $\mathbf{I}_{3\times3}$ the 3×3 identity matrix. In our conventions the theoretical value of the constant is 4/3. The standard deviation of the individual matrix elements is

$$\sigma\left(\omega^{i\bar{j}}F_{i\bar{j}}\right)=\max\left[\sqrt{\frac{1}{\mathrm{Vol}(Q)}\int_Q\left(\omega^{i\bar{j}}F_{i\bar{j}}-\langle\omega^{i\bar{j}}F_{i\bar{j}}\rangle\right)^2d\mu_\Omega}\,\right]\approx 0.15,$$

where the square-root and the square are performed entry by entry. Therefore, $\omega^{i\bar{j}}F_{i\bar{j}}$ is a global constant on Q times the identity, within an error of $0.15/1.31\approx 11\%$. This implies that the hermitian Yang-Mills equation (3.57) is satisfied with this accuracy.

Testing the hermitian Yang-Mills equation provides an implicit test of Ricci flatness, since it is precisely the Ricci flat metric that is needed in the hermitian Yang-Mills equation. One expects much better results for metrics with $k>3$, as the ones that we constructed above.

Discussion

We will discuss further applications of these results elsewhere; here we discuss the advantages and limitations of this approach compared to others, for example position space methods [49].

[11] We estimate the errors using (3.96).

Our best approximation should minimize the error $\epsilon = d(g_h, g_{RF})$, where d is some measure of the distance between the approximate and true metrics. A simple and natural choice for ϵ in the present context is to consider the function

$$\eta_h = \frac{\det \omega_h}{\Omega \wedge \overline{\Omega}} \tag{3.99}$$

on X, where ω_h is the Kähler form for g_h. For a Ricci flat metric, this will be the constant function. We then take

$$\epsilon = 1 - \frac{\min_{x \in X} \ \eta_h(x)}{\max_{x \in X} \ \eta_h(x)}. \tag{3.100}$$

Of course, one could use other norms, such as $||\eta_h - \frac{1}{\mathrm{vol}\ X} \int \eta_h||_p$, or curvature integrals.

The runtime of a computation of the balanced metric can be approximated as

$$T = N_{it} \times N_p \times S^2,$$

where S is the number of independent sections (taking into account discrete symmetry), N_p is the number of points used in the Monte Carlo integration, and N_{it} is the number of iterations of the T-map required for convergence. Since convergence is exponential, this leads to a rough scaling with the accuracy as

$$T \sim \frac{\log \epsilon}{\epsilon^2} S^2.$$

The value of S required for a given accuracy depends on the symmetries and dimension. For the balanced metrics, we expect to need $k \sim 1/\sqrt{\epsilon}$; as discussed in section 2 this could probably be improved by choosing a different scheme if accuracy were paramount. For hypersurfaces in n complex dimensions, we then have $S \sim N \sim k^{n+1}$, leading to a rough overall scaling of $T \sim 1/\epsilon^{n+3}$. This might be compared with a (naive) $T \sim 1/\epsilon^{2n}$ for position space methods, so the two appear generally competitive. However, along with the other advantages we mentioned, we believe the approach we are discussing is far easier to program, and requires relatively little effort to adapt to different manifolds, and related problems such as hermitian Yang-Mills.

Since the sections s_α of $\mathcal{O}_X(k)$ are degree k polynomials, this basis is a simple type of Fourier or momentum space basis. Very roughly speaking, a degree k basis should be able to represent arbitrary structures on length scales down to $1/k$. They are particularly well suited for approximating smooth functions, as the Fourier coefficients of such a function fall off faster than any power of k (see the appendix of [33] for more precise statements). This is advantageous as the Ricci flat metric is smooth, suggesting that other approximation schemes could do better than $\epsilon \sim 1/k^2$.

On the other hand, in some limits (say a conifold limit) the metric can develop structure on small scales, which might not be well represented by a fixed k basis. This is also a problem for position space methods with a fixed lattice; there one deals with it by multi-scale methods, for example allowing the lattice spacing and structure to vary over the manifold. This is very powerful but also very intricate to program. In the present context, rather than increase k, one might look for analogous simplifications; either a multi-scale method which uses different k in different regions (or even some sort of wavelet-inspired method). Or, since we have many explicit expressions for Ricci flat metrics near singularities, it might be useful to develop a way to patch these solutions into the global approximate solutions we discussed.

On the computer code

Our numerics is based on code that has been written entirely in C++. Our experience shows that these computations must be done in a compiled language, rather than an interpreted one. We have made extensive use of the following Boost libraries: uBlas, random, bind and thread. These libraries are on par with Fortran code, due to implementation techniques using expression templates and template metaprograms. The computations were done on an Athlon 64 4800+ dual core machine, with 4GB memory. The computational time ranges from minutes, for low k, to hours, and eventually 2 days (for $k = 12$ on the quintic).

Appendices

Clifford algebras in dimensions 10, 11 and 12

In this appendix we summarize our conventions for the Clifford algebras Cliff(n). We follow the Clifford algebra multiplication convention

$$\{\gamma^M, \gamma^N\} = -2g^{MN}. \tag{3.101}$$

A natural basis for Cliff(n), is given by the set of matrices

$$\gamma^{M_1 M_2 \ldots M_p} = \gamma^{[M_1}\gamma^{M_2}\ldots\gamma^{M_p]} \quad p = 0, 1, \ldots n. \tag{3.102}$$

For n even, Cliff(n) is isomorphic to End$(\mathbf{S}) = \mathbf{S}^\vee \otimes \mathbf{S}$, the vector space of endomorphisms of the spinor bundle.

For n odd, there is a two to one correspondence between elements in Cliff(n) and elements in $\mathbf{S}^\vee \otimes \mathbf{S}$. This map between vector spaces is understood through the action of the volume element ω in Cliff(n), i.e. in local coordinates

$$\omega = \gamma^1\gamma^2\ldots\gamma^n = \frac{1}{n!}\varepsilon_{M_1 M_2 \ldots M_n}\gamma^{M_1}\gamma^{M_2}\ldots\gamma^{M_n} \tag{3.103}$$

verifies

$$\omega^2 = (-1)^{n(n+1)/2}\mathbf{1} \tag{3.104}$$

where $\mathbf{1}$ is the identity matrix in $\mathbf{S}^\vee \otimes \mathbf{S}$. As the volume element ω commutes with every element in Cliff(n) and the Clifford algebra is irreducible, Schur's lemma implies that ω must be represented by $\pm\mathbf{1}$. For $n = 11$, we choose $\omega = \mathbf{1}$, by convention. In local coordinates, Clifford multiplication by the volume form ω acts as a Hodge dual, that is, if H is a p-form and $\not{H}$ its associated Clifford multiplication

$$\gamma(H) = \not{H} = H_{M_1 M_2 \ldots M_p}\gamma^{M_1 M_2 \ldots M_p}, \tag{3.105}$$

then

$$\omega\gamma(H) = \gamma(\star H) \tag{3.106}$$

with $\star$ the Hodge star operator. Thus in odd dimensions, Clifford multiplication by a form and by its Hodge dual are represented by the same element in $\mathbf{S}^\vee \otimes \mathbf{S}$.

We will also use the relation between irreducible representations of Cliff$(2n)$ and Cliff$(2n-1)$, i.e. if γ^M is an irrep of Cliff$(2n-1)$, an irrep Γ^M for Cliff$(2n)$ is given by

$$\Gamma^M = \sigma^1 \otimes \gamma^M \qquad M = 1, \ldots, 2n-1 \tag{3.107}$$

$$\Gamma^{2n} = \sigma^2 \otimes \mathbf{1} \tag{3.108}$$

$$\Gamma^{2n+1} = \sigma^3 \otimes \mathbf{1} \tag{3.109}$$

where σ^i are the 2×2 Pauli matrices.

Heat kernel expansions and Quantum Mechanics

There are several algorithms to evaluate the trace $\mathrm{Tr}\big(\exp(-t\hat{\not{\mathcal{D}}}^2)\big)$. As we show in the main text of this thesis, the coefficients associated with the expansion of such a trace in powers of $t = 1/M^2$ determine the curvature of determinant line bundles and hence the anomalous divergence of the gauge current. Although there are explicit calculations of such expansions for flat space, see [40], we review here some of the techniques used to determine such coefficients, and explain qualitatively the one based on path integrals in supersymmetric quantum mechanics.

The main idea is to separate the interacting heat kernel

$$\langle x|K(t)|y\rangle = \langle x|\exp(-t\hat{\not{\mathcal{D}}}^2)|y\rangle, \tag{3.110}$$

as the product of the free heat kernel

$$\langle x|K_0(t)|y\rangle = \frac{1}{(4\pi t)^{n/2}}\exp\Big(-\frac{(x-y)^2}{4t}\Big) \tag{3.111}$$

with n the dimension of the x-space, and an interacting part H

$$H(x,y;t) = \sum_{k=0}^{\infty} a_k(x,y)t^k, \tag{3.112}$$

i.e., we compute (3.110) through the ansatz

$$\langle x|K(t)|y\rangle = \frac{1}{(4\pi t)^{n/2}}\exp\Big(-\frac{(x-y)^2}{4t}\Big)H(x,y;t). \tag{3.113}$$

There is a large variety of algorithms to calculate the coefficients a_k; they roughly fall in three categories:

- Recursive x-space algorithms based on recursive relations among different heat kernel coefficients [87].
- Nonrecursive algorithms based on the insertion of a momentum basis [44].

• The method of Zuk, based on graphical representations of the heat kernel coefficients [92].

If the supertrace of (3.113) is taken, we can evaluate the expansion using path integrals in quantum mechanics. We can follow the ideas of [2, 6] and [43], to determine the coefficients of the supertrace of the heat kernel expansion associated to a generalized Dirac operator $\hat{\not{\mathcal{D}}}$ in 12-dimensions, as the ones which appear in the definition of the curvature of the M-theory line bundle (2.72). Thus, given the operator $\hat{\not{\mathcal{D}}}$, the expansion

$$\mathrm{Tr}_s\big(\exp(-t\hat{\not{\mathcal{D}}}^2)\big) = \frac{\mathrm{Tr}_s a_0}{t^6} + \frac{\mathrm{Tr}_s a_1}{t^5} + \ldots + \mathrm{Tr}_s a_6 + \mathcal{O}(t), \tag{3.114}$$

can be determined through the partition function of a supersymmetric quantum mechanical model. The idea is to interpret (3.114) as the time evolution operator of a quantum mechanical system with Hamiltonian $H = \hat{\not{\mathcal{D}}}^2$, and calculate explicitly the expansion (3.114), through the path integral approach to quantum mechanics. A novelty introduced by the generalized Dirac operators is that there are coefficients $\mathrm{Tr}_s a_k$ with $k < 6$ which are not zero. This differs sharply from the super heat kernel expansions for standard Dirac operators, where the coefficients with inverse powers of t are known to be zero [12].

In the standard case the vanishing of the coefficients $\mathrm{Tr}_s a_k$ with $k < 6$ allows us to determine $\mathrm{Tr}_s a_6$ by evaluating the path integral in the limit $t \to 0$. In the case of generalized Dirac operators we find non-zero terms with inverse powers of t. Thus we have to be more careful and evaluate the path integral for a finite time interval instead of taking the limit $t \to 0$. Path integrals in supersymmetric quantum mechanics for a finite time interval were analyzed in detail by [19, 20], and used in [70], to determine index densities of generalized Dirac operators in 4-dimensions, which agree with the older result of [69].

The type of quantum mechanical theory that we consider, is a supersymmetric non-linear sigma model with target the 12-dimensional manifold $Z = \pi^{-1}(\Sigma)$ where $\pi : \mathcal{X} \to \mathcal{T}^\partial$ is the projection to the space of M-theory and $\Sigma \hookrightarrow \mathcal{T}^\partial$ is any surface where the curvature of $\mathrm{Det}\,\hat{\not{D}} \to \mathcal{T}^\partial$ is to be evaluated. Here, $\hat{\not{D}} = \not{D} + \gamma(\mathbb{h})$ stands for any of the chiral generalized Dirac operators that couple to the M-theory fermions at the boundary.

Therefore, let $\mathbb{R}^{1|1}$ denote the super Euclidean space with one even variable and one odd variable; i.e., $C^\infty(\mathbb{R}^{1|1}) = C^\infty(\mathbb{R}) \otimes \wedge^*(\mathbb{R})$. And let τ and θ be the natural even and odd variables, respectively. We consider a quantum theory of maps

$$X : \mathbb{R}^{1|1} \to Z \tag{3.115}$$

[12] In [45] E. Getzler calculates index densities for generalized Dirac operators. In his approach he introduces further scalings of the fluxes by the regulating parameter $t = 1/M^2$. The first non-vanishing term in his alternative expansion to (3.114) is t-independent. However such scalings of the field variables are not appropriate for our application.

where the action we take is

$$\mathcal{S}_{SQM} = -\frac{1}{2}\int_{\mathbb{R}^{1|1}} d\tau d\theta \left\{ g_{MN}(X)\frac{dX^M}{d\tau}DX^N + DX^{M_1}DX^{M_2}DX^{M_3}\mathbb{h}_{M_1M_2M_3}(X)\right\}, \quad (3.116)$$

with g_{MN} the metric tensor on Z and D is the superdifferential

$$D = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial\tau}. \quad (3.117)$$

The superfield that appears in (3.116), can be written in a local coordinate chart as

$$X^M = x^M + \theta\psi^M \quad (3.118)$$

where x is a local chart for Z. The supersymmetry transformations are generated by the supercharge operator Q

$$Q = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial\tau}, \quad (3.119)$$

with $\delta X^M = QX^M$. In quantizing (3.116), we construct the Hilbert space of the theory as the space $L^2(\mathbf{S}(Z))$, i.e. the space of L^2-sections of the spin bundle $\mathbf{S} \to Z$ tensored by the half-densities on Z. This space has a natural $\mathbb{Z}/2\mathbb{Z}$-grading induced by the chiral decomposition $\mathbf{S} = \mathbf{S}_+ \oplus \mathbf{S}_- \to Z$. Also, the quantum supercharge operator Q is:

$$Q = \hat{\not{\mathcal{D}}} = \not{\mathcal{D}} + \gamma(\mathbb{h}), \quad (3.120)$$

which acts naturally on the quantum Hilbert space $L^2(\mathbf{S}(Z))$.

Thus, the super heat kernel expansion for Q_+ can be expressed as the quantum mechanical partition function

$$\mathcal{Z} = \mathrm{Tr}_s\Big[\exp\big(-t(Q_-Q_+ + Q_+Q_-)/2\big)\Big] = \int [\mathrm{d}X]\exp(-\mathcal{S}_{SQM}), \quad (3.121)$$

where we take the action $\mathcal{S}_{SQM}$ defined in (3.116). More concretely, writing (3.116) in the field variables and recalling that the path integral matches with the left-hand-side of (3.121) iff the supercircle $X : S^{1|1} \to Z$ is chosen to be a supercircle of length t, we find [13]

$$\mathcal{S}_{SQM} = \frac{1}{t}\int_0^1 \mathrm{d}\tau\Big[\frac{1}{2}g_{MN}\frac{\mathrm{d}x^M}{\mathrm{d}\tau}\frac{\mathrm{d}x^N}{\mathrm{d}\tau} + \frac{1}{2}g_{MN}\psi^M\frac{D\psi^N}{D\tau} - \frac{1}{2}(\mathrm{d}\mathbb{h})_{MNOP}\psi^M\psi^N\psi^O\psi^P\Big], \quad (3.122)$$

where

$$\frac{D\psi^N}{D\tau} = \frac{d\psi^N}{d\tau} + \Gamma^N_{MQ}\dot{x}^M\psi^Q - 3\mathbb{h}^N_{MQ}\dot{x}^M\psi^Q \quad (3.123)$$

[13] See [70] for more details. There are terms of order $\mathcal{O}(t)$ which have to be included in the action, in order to make the path integral well defined for a finite time interval due to Weyl ordering ambiguities. Here, we just write out the classical expression derived by expanding (3.116) in the field variables.

One can use the background field approximation and expand the fields as classical fields plus quantum fluctuations

$$x^M = x_0^M + \delta x^M \tag{3.124}$$

$$\psi^Q = \psi_0^Q + \delta\psi^Q. \tag{3.125}$$

We are now ready to compute the path integral (3.122) via a loop expansion in the parameter t, with t playing the role of $\hbar$. We only need compute graphs of 12^{th} order in the background fermions ψ_0 in order to saturate the Grassmann integration. Due to the four-fermi interaction in (3.122), the tree level contribution after integrating $[\mathrm{d}^{12}\psi_0]$ yields terms of order $\mathcal{O}(t^{-3})$. For instance,

$$-\frac{1}{2^3 \cdot 3! t^3}(\mathrm{d}\mathbb{h})^3. \tag{3.126}$$

Thus, in order to extract the full $\mathcal{O}(t^0)$ contribution we should take into account up to four-loop diagrams which are of order $\mathcal{O}(t^{-3}) \times \mathcal{O}(t^3)$, since each loop order L contributes $\mathcal{O}(t^{L-1})$. In this formalism, it becomes clear how inverse powers of t appear in the expansion due to the presence of a non-vanishing $\mathbb{h}$-flux.

In other words, we have shown how the super heat kernel expansion will be of the type

$$\frac{1}{t^3}\mathrm{Tr}_s(a_3) + \frac{1}{t^2}\mathrm{Tr}_s(a_4) + \frac{1}{t}\mathrm{Tr}_s(a_5) + \mathrm{Tr}_s(a_6) + \ldots, \tag{3.127}$$

and we will have to evaluate up to four-loop Feynman diagrams, in order to determine the heat kernel coefficient $\mathrm{Tr}_s a_6$. Note that five-loop Feynman diagrams are at least of order $\mathcal{O}(t)$ and hence do not contribute to $\mathrm{Tr}_s a_6$.

Finally, we remark that if we had put in an appropriate extra scaling in $\mathbb{h}$, as is done in [45] we would have had no divergent terms for $t \to 0$ and would have obtained the index density

$$\int \hat{A} e^{\mathrm{d}\mathbb{h}}. \tag{3.128}$$

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Curriculum Vitæ

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