ON THE QUESTIONS OF LOCAL AND GLOBAL WELL-POSEDNESS FOR THE HYPERBOLIC PDES OCCURRING IN SOME RELATIVISTIC THEORIES OF GRAVITY AND ELECTROMAGNETISM

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ABSTRACT OF THE DISSERTATION

On the Questions of Local and Global Well-Posedness for the Hyperbolic PDEs Occurring in Some Relativistic Theories of Gravity and Electromagnetism

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The two hyperbolic systems of PDEs we consider in this work are the source-free Maxwell-Born-Infeld (MBI) field equations and the Euler-Nordström system for gravitationally self-interacting fluids. The former system plays a central role in Kiessling's recently proposed self-consistent model of classical electrodynamics with point charges, a model that does not suffer from the infinities found in the classical Maxwell-Maxwell model with point charges. The latter system is a scalar gravity caricature of the incredibly more complex Euler-Einstein system. The primary original contributions of the thesis can be summarized as follows:

- We give a sharp non-local criterion for the formation of singularities in plane-symmetric solutions to the source-free MBI field equations. We also use a domain of dependence argument to show that 3-d initial data agreeing with certain plane-symmetric data on a large enough ball lead to solutions that form singularities in finite time. This work is an extension of a theorem of Brenier, who studied singularity formation in periodic plane-symmetric solutions.

- We prove well-posedness for the Euler-Nordström system with a cosmological constant (EN_κ) for initial data that are an H^N perturbation (not necessarily small) of a uniform, quiet fluid, for $N \geq 3$. The method of proof relies on the framework of energy currents that has been recently developed by Christodoulou. We turn to this method out of necessity: two common frameworks for showing well-posedness in H^N, namely symmetric hyperbolicity and strict hyperbolicity, do not apply to the EN_κ system, while Christodoulou's techniques apply to all hyperbolic systems derivable from a Lagrangian, of which the EN_κ system is an example.

- We insert the speed of light c as a parameter into the EN_κ system (and designate the family of systems EN_κ^c) in order to study the non-relativistic limit $c \to \infty$. Taking the formal limit in the equations gives the Euler-Poisson system with a cosmological constant (EP_κ). Using energy currents, we prove that for fixed initial data, as $c \to \infty$, the solutions to the EN_κ^c system converge uniformly on a spacetime slab $[0, T] \times \mathbb{R}^3$ to the solution of the EP_κ system.

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Dedication

To Mom and Dad, with love and appreciation.

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0.1 Introduction

The point of departure for this work is a nonlinear model of electromagnetism developed by Born and Infeld in the early 1930's (consult e.g. [5]) and which is now known as the Born-Infeld (BI) model. Different scientific communities have expressed renewed interest in the BI model for a variety of reasons, the most pertinent of which to the author's interests being that the theory features point charges with finite classical energy. This property distinguishes BI theory from many accepted "effective theories" of point charges and fields that suffer from divergence problems, including classical Maxwell-Maxwell[a] theory with point charge sources and quantum electrodynamics. This topic is explored in detail in the work [28], in which Kiessling proposes a self-consistent model of classical electrodynamics with point charges that does not suffer from divergence problems or require regularization/renormalization. Kiessling's model couples the Maxwell-Born-Infeld[b] (MBI) field equations to his newly-proposed relativistic guiding law for point charges whose guiding field satisfies a Hamilton-Jacobi PDE; the guiding law is truly a necessary addition to the theory, for the classical Lorentz force is ill-defined at the location of point charges, despite the finiteness of the electrostatic energy of point charges.

Because of the newness of Kiessling's theory, and because of the inherent complexity of nonlinear systems, many basic mathematical questions in his model are open. In fact, important questions concerning the source-free MBI field equations remain unanswered; i.e., even without a coupling of it to Kiessling's guiding law for point charges. The source-free MBI system is the subject matter of Chapter 1, which begins with a review of the structure of relativistic theories of source-free electromagnetism derivable from a Lagrangian. In particular, I discuss the classical Maxwell-Maxwell (MM) field equations and the MBI field equations. I then describe some special solutions to the source-free MBI system.

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[a]Section 1.1 contains a discussion of our use of the terminology "Maxwell-Maxwell" and "Maxwell-Born-Infeld."

[b]We refer to the field equations in the Born-Infeld model as the "MBI field equations" or "MBI system," while the term "BI model" is used to refer to the Born-Infeld model in its entirety.

The second topic I discuss in Chapter 1 is the formation of singularities in solutions to the source-free MBI field equations with smooth initial data. The physically relevant question is whether or not generic smooth initial data having finite energy launch solutions to the MBI system that form singularities in finite time. Some progress was made by Chae and Huh, who in [11] showed global existence for small initial data[c] by adapting a paper of Lindblad [35] that uses the null condition of Klainerman [31], [32] and Christodoulou [13]. The question of generic blowup for large data is not settled at this point. However, for periodic solutions with planar symmetry, Brenier [7] provides a sharp condition under which singularities form in finite time. These singularities are not shock waves: infinities already occur in the field variables themselves, rather than only in their gradients. I extend Brenier's criterion to the case of non-periodic plane-symmetric solutions in Theorem 1.3.1. Serre and Neves [41] have also contributed to the analysis of the MBI system by proving some general blowup results that hold for 2×2 totally linearly degenerate systems of conservation laws, of which a subsystem of the Augmented-Maxwell-Born-Infeld system is an example as discussed in Chapter 1.

Since therefore the evolution of solutions to the MBI system with planar symmetry is by now well-understood, it might seem that the next logical class of solutions to investigate is the class of spherically symmetric solutions. However, the curl of any radially-directed vectorfield vanishes, and a simple consequence of this fact is that there are no non-trivial spherical wave solutions to the source-free MBI system. Therefore, to make further progress on the question of blowup, one must confront either axially-symmetric solutions or the full 3-dimensional problem; both of these classes of solutions are significantly more complicated than solutions with planar symmetry. However, hyperbolic systems on spacetime manifolds exist that *do* feature spherically symmetric wave solutions, including models of so-called scalar gravity. Among the various possibilities, one of the putatively simplest such systems is formed by coupling Nordström's theory of gravity [43] to the relativistic Euler equations to form the Euler-Nordström (EN) system. We regard the analysis of solutions to the EN system as a primer for

[c]Chae and Huh considered initial data $\mathring{f}$ in the class C_0^∞ and showed that if ϵ is small enough, then the solution to the source-free MBI field equations with initial data $\epsilon\mathring{f}$ has a global solution. No explicit estimates of smallness constants were provided in this paper.

studying the question of blowup in the MBI field equations.

Before attempting a proof of blowup for spherically symmetric data for the EN system, it is natural to first study its Cauchy problem. Because the EN system is not symmetric hyperbolic or strictly hyperbolic, classes of systems for which the Cauchy problem is well-understood, alternate techniques are required to make progress on the analysis of its Cauchy problem. Fortunately, such techniques have recently been developed by Christodoulou [15], [16]. It turns out that these techniques are both rich in structure and versatile, so that the remainder of this thesis is dedicated to applying them to the EN system.

I now give a brief description of Nordström's theory before outlining the remaining chapters of the thesis. In 1913, the Finnish physicist Gunnar Nordström published a paper [43] in which he formulated one of the earliest consistent relativistic field theories of gravity. This theory was the result of an extensive correspondence between Nordström and Einstein. Interested readers may consult [44] for a detailed historical account of their exchanges. The historical significance of [43] is as follows. First of all, it contains for the first time an equation relating a purely geometric quantity, the Ricci scalar, to a purely physical quantity, the trace of the energy momentum tensor. Secondly, soon after its publication, Einstein and Fokker noted that the Lagrangian for the motion of test particles in Nordström's theory is the geodesic Lagrangian for a curved Lorentzian manifold with a conformally flat metric [22]. Two years later, Einstein published [20] "The Foundation of the General Theory of Relativity," the first systematic exposition of his theory of General Relativity, considered by many to be our gold standard theory of gravitation.

In view of these remarks, we may consider the EN system as a relativistic primer for studying the Euler-Einstein (EE) system, which is the coupling of Einstein's field equations to the relativistic Euler equations, and which is formidably complicated. This point of view is explored in [50] in which Shapiro and Teukolsky discuss numerical simulations of the EN system in the spherically symmetric case; they expect that the numerical schemes developed in their paper can be adapted to allow for the calculation of accurate wave forms in the EE model. Therefore, in addition to providing a starting point for studying 3-d blowup in the MBI field equations, the EN system also provides insight that might be useful for studying the EE system.

I begin Chapter 2 by describing the coupling of Nordström's theory to the relativistic Euler equations. The resulting theory of curved spacetime is shown to be mathematically equivalent to a theory in Minkowski spacetime that features conservation laws derivable from an energy momentum tensor. Additionally, I introduce a cosmological constant κ^2 into the system (and designate the system EN_κ) in order to study perturbations of a quiet uniform fluid, for without a cosmological constant, the only constant solutions are bound by the physically undesirable constraint[d] $\bar{\rho} = 3\bar{p}$, where the positive constants $\bar{\rho}$ and $\bar{p}$ denote the energy density and pressure respectively of the unperturbed, quiet, uniform fluid. Without the cosmological constant, this constraint places severe restrictions on the admissible equations of state. The situation is analogous to Einstein's addition of the cosmological constant to General Relativity in order to obtain non-trivial static universes [21].

After placing the EN_κ system in context in a hierarchy of existing models of self-gravitating fluids, I discuss the well-posedness of its Cauchy problem. I provide a detailed discussion of the geometry of the EN_κ system and apply the techniques recently developed by Christodoulou in [15] and [16] to generate energy currents. The energy currents provide L^2 estimates that are analogous to the estimates available in the theory of symmetric hyperbolic PDEs, a class of systems for which local existence is well-known. I provide a complete proof of local existence for perturbations of a quiet uniform fluid belonging to the Sobolev space H^N for the EN_κ system using the method of energy currents, where $N \geq 3$ is an integer. I then adapt some work by Kato to show that the solution depends continuously on the data, which completes the proof of well-posedness.

In Chapter 3, I introduce the speed of light c as a parameter into the EN_κ system; I denote the resulting system of equations by EN_κ^c. Taking a limit $c \to \infty$ in the EN_κ^c

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[d]Constant solutions to the EN_κ system also must satisfy a constraint, but the point is that with the addition of $\kappa > 0$, $\bar{p}$ can be chosen freely, and the constraint can be viewed as a restriction on the background potential $\bar{\Phi}$ that can be satisfied by arbitrary equations of state. A similar difficulty arises in the Euler-Poisson system with a cosmological constant, but unlike in the EN_κ system, $\bar{\Phi}$ is un-physical in the sense that only $\nabla^{(1)}\Phi$ appears in the equations for the evolution of the fluid. A more thorough discussion of this topic in the context of the EP_κ system can be found in [30].

system, one formally obtains the Euler-Poisson system with the cosmological constant κ^2 (EP_κ). The EN_κ^c system is hyperbolic for all finite positive c, while the EP_κ system is not. Therefore, $c \to \infty$ is a singular limit. Nevertheless, Christodoulou's techniques can be used to generate energy currents, even in the EP_κ case, that produce useful Sobolev estimates. I use these estimates to prove a theorem showing that for $N \geq 4$ and initial data in H^N, solutions to the EN_κ^c system converge in H^{N-1} to solutions of the EP_κ system as the speed of light c tends to infinity, thus vindicating the EN_κ^c system as a genuine relativistic generalization of the EP_κ system. A similar analysis is performed for the Euler-Einstein system in [45]. Therefore, we expect that in certain limiting situations, solutions to the EP, EN, and EE systems should share some qualitative and quantitative features.

Chapter 1

The Maxwell-Born-Infeld Electromagnetic Field Equations

In this chapter, we discuss the 3-d source-free Maxwell-Born-Infeld (MBI) electromagnetic field equations. After describing the structure of a relativistic electromagnetic theory, we discuss a few results concerning special solutions to the MBI system, most notably a blowup criterion for solutions with planar symmetry.

1.0.1 Notation

The electromagnetic theories discussed in this chapter take place in Minkoswki spacetime $\mathcal{M}$. Although we have included the parameter Born-Infeld parameter β in the MBI model, we work in units where the speed of light is unity. Furthermore, we work in a fixed Lorentz coordinate system, and for this preferred time-space splitting, we identify $t = x^0$ with time and $\mathbf{s} = (x^1, x^2, x^3)$ with space and use the notation (1.0.1.1) to denote the components of x relative to this fixed coordinate system:

$$x = (x^0, x^1, x^2, x^3). \tag{1.0.1.1}$$

The Minkwoski metric $\underline{g}$ has components $\underline{g}_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ in this coordinate system.

1.1 Relativistic Lagrangians and the Field Equations

In this section we discuss the structure of the class of electromagnetic field theories in $\mathcal{M}$ that are derivable from a relativistic Lagrangian. Let us begin by recalling that a relativistic Lagrangian for source-free electrodynamics in $\mathcal{M}$ can be written as a function $\mathcal{L}$ of the two invariants of the Faraday tensor, $|\mathbf{B}|^2 - |\mathbf{E}|^2$ and $\mathbf{E} \cdot \mathbf{B}$, where $\mathbf{E} : \mathcal{M} \to \mathbb{R}^3$ is the *electric field* and $\mathbf{B} : \mathcal{M} \to \mathbb{R}^3$ is the *magnetic induction*[a].

Varying the action

$$\int_{\mathcal{M}} \mathcal{L}(\mathbf{E}, \mathbf{B})\, d^4\mathbf{x} \tag{1.1.0.2}$$

with respect to $\mathbf{E}$ and $\mathbf{B}$ subject to the constraints[b]

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0 \tag{1.1.0.3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.1.0.4}$$

gives the equations

$$\partial_t \mathbf{D} = \nabla \times \mathbf{H} \tag{1.1.0.5}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{1.1.0.6}$$

where

$$\mathbf{D} \stackrel{\text{def}}{=} \frac{\partial \mathcal{L}}{\partial \mathbf{E}}(\mathbf{E}, \mathbf{B}) \tag{1.1.0.7}$$

$$\mathbf{H} \stackrel{\text{def}}{=} -\frac{\partial \mathcal{L}}{\partial \mathbf{B}}(\mathbf{E}, \mathbf{B}). \tag{1.1.0.8}$$

$\mathbf{D} : \mathcal{M} \to \mathbb{R}^3$ and $\mathbf{H} : \mathcal{M} \to \mathbb{R}^3$ are the *electric displacement* and *magnetic field* respectively[c].

Our reason for the use of the names “Maxwell-Maxwell” and “Maxwell-Born-Infeld” for the two systems discussed below, as opposed to “Maxwell” and “Born-Infeld,” is that equations (1.1.0.3) - (1.1.0.6) are already featured in Maxwell’s work, while the two systems (1.1.0.3) - (1.1.0.8) are distinguished by their constitutive relations (1.1.0.7), (1.1.0.8) which derive from their Lagrangians (1.2.0.12) and (1.3.1.1) respectively.

The Hamiltonian density $\mathcal{H}(\mathbf{D}, \mathbf{B})$ is given by taking the Legendre transformation of $\mathcal{L}$ with respect to $\mathbf{E}$:

$$\mathcal{H}(\mathbf{D}, \mathbf{B}) \stackrel{\text{def}}{=} \sup_{\mathbf{E}} \big(\mathbf{E} \cdot \mathbf{D} - \mathcal{L}(\mathbf{E}, \mathbf{B})\big). \tag{1.1.0.9}$$

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[a]Recall that $\mathbf{E}$ and $\mathbf{B}$ are (up to $-$ signs) components of the antisymmetric Faraday tensor $F^{\alpha\beta}$ and are thus subject to the associated transformation laws under coordinate changes.

[b]Equivalently, one may express the Lagrangian in terms of the 4-vector potential A^ν and perform unconstrained variations on A^ν.

[c]Many authors refer to $\mathbf{B}$ as “the magnetic field.”

We may recover $\mathbf{E}$ and $\mathbf{H}$ from $\mathcal{H}$ through the relations

$$\mathbf{E} = \frac{\partial \mathcal{H}}{\partial \mathbf{D}}(\mathbf{D}, \mathbf{B}), \tag{1.1.0.10}$$

$$\mathbf{H} = \frac{\partial \mathcal{H}}{\partial \mathbf{B}}(\mathbf{D}, \mathbf{B}). \tag{1.1.0.11}$$

1.2 Maxwell-Maxwell (MM) Electromagnetic Field Equations

As is well-known, the source-free Lagrangian density in classical MM electromagnetic field theory is given by

$$\mathcal{L}_M \stackrel{\text{def}}{=} \frac{|\mathbf{E}|^2 - |\mathbf{B}|^2}{2}. \tag{1.2.0.12}$$

In this case, (1.1.0.7) and (1.1.0.8) give

$$\mathbf{D} = \mathbf{E} \tag{1.2.0.13}$$

$$\mathbf{H} = \mathbf{B}. \tag{1.2.0.14}$$

Taking into account (1.1.0.3), (1.1.0.4), (1.1.0.5), and (1.1.0.6), the Maxwell-Maxwell system comprises the evolution equations

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{D} \tag{1.2.0.15}$$

$$\partial_t \mathbf{D} = \nabla \times \mathbf{B} \tag{1.2.0.16}$$

supplemented by the constraints

$$\nabla \cdot \mathbf{B} = 0 \tag{1.2.0.17}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{1.2.0.18}$$

which are propagated in time if satisfied by initial data.

It is easy to verify that "right-travelling" plane-symmetric fields of the form

$$\mathbf{D} = f(x_1 - t)\mathbf{e}_2 + g(x_1 - t)\mathbf{e}_3 \tag{1.2.0.19}$$

$$\mathbf{B} = -g(x_1 - t)\mathbf{e}_2 + f(x_1 - t)\mathbf{e}_3 \tag{1.2.0.20}$$

where $\mathbf{e}_i$ denotes the i^{th} standard basis vector in $\mathbb{R}^3$ and f, g are differentiable functions, are solutions to the MM system. Furthermore, $\mathbf{D} \cdot \mathbf{B} = 0$ and $|\mathbf{D}| = |\mathbf{B}|$ hold for solutions belonging to this family. We will make use of these facts in Section 1.3.2.

1.3 Maxwell-Born-Infeld (MBI) Electromagnetic Field Equations

1.3.1 Introduction to the MBI Field Equations

The Born-Infeld model was proposed by Born and Infeld in the 1930's (consult e.g. [5]), with [4] a precursor by Born, in an effort to eliminate the singularities present in classical Maxwellian electrodynamics with point charges. In [4], Born calculates the electromagnetic field created by a single stationary point charge and shows i) that the associated field energy is finite. If one also imposes the additional conditions ii) that the field equations are covariant under the Poincaré group, iii) that the field equations reduce to the Maxwell-Maxwell field equations in the weak field limit, iv) that the field equations are covariant under a Weyl (gauge) group, and (v) that solutions to the field equations are not birefringent[d], then as discussed in [28], one arrives at the unique one parameter family of models proposed by Born and Infeld. The uniqueness of the MBI family under these assumptions has enticed Kiessling to incorporate the MBI field equations into his proposed fundamental model for point charge electrodynamics [28], [29]. In this section, we discuss the source-free field equations and add to the body of known special solutions, and then elaborate upon some recent work by Brenier concerning the formation of singularities in the case of solutions with planar symmetry.

The Born-Infeld Lagrangian density is given by

$$\mathcal{L}_{BI}(\mathbf{E},\mathbf{B}) \stackrel{\text{def}}{=} \frac{1}{\beta^4} - \frac{1}{\beta^4}\big(1-\beta^4(|\mathbf{E}|^2-|\mathbf{B}|^2)-\beta^8(\mathbf{E}\cdot\mathbf{B})^2\big)^{1/2}, \tag{1.3.1.1}$$

where β is a parameter in the theory.

Equation (1.1.0.9) and omitted calculations give that the Born-Infeld Hamiltonian density is

$$\mathcal{H}_{BI}(\mathbf{D},\mathbf{B}) = \big(1+\beta^4(|\mathbf{B}|^2+|\mathbf{D}|^2)+\beta^8|\mathbf{P}|^2\big)^{1/2}-1, \tag{1.3.1.2}$$

where

$$\mathbf{P} \stackrel{\text{def}}{=} \mathbf{D}\times\mathbf{B}. \tag{1.3.1.3}$$

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[d] By this, we mean that in the "local rest frame" in which $\mathbf{E}$ and $\mathbf{B}$ are parallel, all of the characteristic speeds coincide in magnitude. More information on this topic can be found in [2] and [3].

Some omitted calculations using (1.1.0.10) and (1.1.0.11) give that

$$\mathbf{E} = \frac{\mathbf{D} + \beta^4 \mathbf{B} \times \mathbf{P}}{h_{BI}(\mathbf{D}, \mathbf{B})} \tag{1.3.1.4}$$

$$\mathbf{H} = \frac{\mathbf{D} - \beta^4 \mathbf{D} \times \mathbf{P}}{h_{BI}(\mathbf{D}, \mathbf{B})}, \tag{1.3.1.5}$$

where

$$h_{BI} \overset{\text{def}}{=} \mathcal{H}_{BI} + 1 = \big(1 + \beta^4(|\mathbf{B}|^2 + |\mathbf{D}|^2) + \beta^8 |\mathbf{P}|^2\big)^{1/2}. \tag{1.3.1.6}$$

By (1.1.0.3), (1.1.0.4), (1.1.0.5),(1.1.0.6), (1.3.1.4), and (1.3.1.5), the Maxwell-Born-Infeld system therefore comprises the evolution equations

$$\partial_t \mathbf{B} = -\nabla \times \Big(\frac{\mathbf{D} + \beta^4 \mathbf{B} \times \mathbf{P}}{h_{BI}(\mathbf{D}, \mathbf{B})} \Big) \tag{1.3.1.7}$$

$$\partial_t \mathbf{D} = \nabla \times \Big(\frac{\mathbf{D} - \beta^4 \mathbf{D} \times \mathbf{P}}{h_{BI}(\mathbf{D}, \mathbf{B})} \Big) \tag{1.3.1.8}$$

supplemented by the constraints

$$\nabla \cdot \mathbf{B} = 0 \tag{1.3.1.9}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{1.3.1.10}$$

which are propagated in time[e] if satisfied by initial data.

1.3.2 On MM Solutions that are Also MBI Solutions

Consider a solution $(\mathbf{E}, \mathbf{B}) = (\mathbf{D}, \mathbf{H})$ to the MM system that also has the properties $|\mathbf{E}| = |\mathbf{B}| = |\mathbf{D}| = |\mathbf{H}|$ and $\mathbf{E} \cdot \mathbf{B} = \mathbf{D} \cdot \mathbf{H} = 0$. For such solutions, a simple algebraic calculation gives that

$$\frac{\mathbf{D} + \beta^4 \mathbf{B} \times \mathbf{P}}{h_{BI}(\mathbf{D}, \mathbf{B})} = \mathbf{D} \tag{1.3.2.1}$$

$$\frac{\mathbf{D} - \beta^4 \mathbf{D} \times \mathbf{P}}{h_{BI}(\mathbf{D}, \mathbf{B})} = \mathbf{B}. \tag{1.3.2.2}$$

It follows that (1.3.1.7) and (1.3.1.8) are satisfied by $(\mathbf{D}, \mathbf{B})$, so that $(\mathbf{E}, \mathbf{B}) = (\mathbf{D}, \mathbf{H})$ is also a solution the MBI system. It has long since been observed by Schrödinger [48]

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[e] i.e., $\partial_t(\nabla \cdot \mathbf{B}) = -\nabla \cdot \nabla(\times \mathbf{E}) = 0$, and similarly for $\nabla \cdot \mathbf{D}$.

that monochromatic plane waves of the form

$$\mathbf{E} = \hat{\mathbf{E}}\exp(\omega t - k_j x^j) \tag{1.3.2.3}$$

$$\mathbf{B} = \hat{\mathbf{B}}\exp(\omega t - k_j x^j), \tag{1.3.2.4}$$

where $\hat{\mathbf{E}}, \hat{\mathbf{B}}$, and (k_1, k_2, k_3) are mutually perpendicular unit vectors, have the above properties, and therefore also solve the MBI system[f]. Furthermore, Kiessling notes that this result can be extended to include the case of a linearly polarized plane wave of arbitrary pulse shape (not necessarily monochromatic) in [28]. It has apparently gone unobserved in the literature that in fact any member of the family (1.2.0.19), (1.2.0.20), which includes pulses of arbitrary shape and polarization, is also a solution to the MBI system[g].

1.3.3 Criteria for Blowup of Solutions with Planar Symmetry

In this section, we extend Brenier's results [7] on the blowup of solutions to the MBI system featuring planar symmetry, that is, solutions $\mathbf{D}$ and $\mathbf{B}$ depending only on t and x_1. Of course, the globally-existing plane waves of the form discussed in Section 1.3.2 fail to meet the blowup criteria. This section is heavily influenced by Brenier's paper, to which I refer the reader for a more through discussion. Let us begin with the following lemma[h], which lists the continuity equations for the energy momentum tensor[i] associated to the $\mathcal{L}_{BI}$ Lagrangian:

Lemma 1.3.1. *Differentiable solutions to the source-free MBI system satisfy the conservation laws*

$$\partial_t h_{BI} + \partial^k P_k = 0 \tag{1.3.3.1}$$

$$\partial_t P_j + \partial^k \Big(\frac{P_k P_j - B_k B_j - D_k D_j}{h_{BI}} \Big) = \partial_j \Big(\frac{1}{h_{BI}} \Big) \qquad (j = 1, 2, 3). \tag{1.3.3.2}$$

[f] Schrödinger in fact showed that these monochromatic plane waves are solutions to *any* field equations derivable from a relativistic Lagrangian, provided that they reduce to the MM field equations in the weak field limit.

[g] It is also true "left-travelling" plane-symmetric solutions can be constructed using functions of the form $f(x_1 + t)$, $g(x_1 + t)$ and adjusting the minus signs. However, arbitrary superpositions of both left and right-moving plane-symmetric fields are not in general solutions to the MBI field equations.

[h] Here, $\partial^k = \partial_k$, and the repeated index k is summed from 1 to 3.

[i] We do not give an expression for the energy momentum tensor here.

Based on Lemma 1.3.1, Brenier's idea is to study the Augmented Maxwell-Born-Infeld (AMBI) system, in which the quantities h_{BI} and $\mathbf{P}$ are *not* constrained by the relations (1.3.1.3) and (1.3.1.6), but are instead treated as independent quantities. The 6×6 MBI system can be recovered from the 10×10 AMBI system by restricting the initial data to the "Born-Infeld" manifold, which is the submanifold of $\mathbb{R}^{10}$ on which the relations (1.3.1.3) and (1.3.1.6) hold. This augmentation lends many advantages to a study of the Cauchy problem. For example, calculations of the characteristic speeds are relatively easy in the AMBI system. Furthermore, Brenier shows that the AMBI system admits a smooth, strictly convex "entropy" function[j] of the state-space variables that is featured in an additional conservation law, which implies that the AMBI system is symmetrizable and hyperbolic [18]. The simplification of greatest relevance to our work here is that for solutions to the AMBI system with planar symmetry, there is a remarkable decoupling of the system (1.3.3.1), (1.3.3.2), from the remaining unknowns. The resulting system, which happens to be 2×2 because of the symmetry assumptions, is simple enough that one can understand its evolution through calculations involving the linear wave equation.

We now discuss the derivation of this decoupling and its consequences. By the $SO(3)$ covariance of the equations, it is sufficient to consider solutions that are functions of t and x_1, so that $\partial_2 = \partial_3 = 0$. By (1.3.1.7), (1.3.1.9), (1.3.1.8), and (1.3.1.10), we have that

$$\partial_t B_1 = \partial_1 B_1 = \partial_t D_1 = \partial_1 D_1 = 0, \tag{1.3.3.3}$$

so B_1 and D_1 are constants. Exactly following Brenier, we define for later use the constant

$$Z \stackrel{\text{def}}{=} (1 + B_1^2 + D_1^2)^{1/2}. \tag{1.3.3.4}$$

[j] The terminology "entropy" is used in this context because in some applications (although not in our work here!) the strictly convex function is, in the words of Dafermos [15], "intimately connected to the Second Law of thermodynamics."

From these facts, the decoupling of the system (1.3.3.1), (1.3.3.2) is immediate:

$$\partial_t h + \partial_1 P_1 = 0 \tag{1.3.3.5}$$

$$\partial_t P^1 + \partial_1 \Big(\frac{(P_1)^2 - Z^2}{h} = 0 \Big), \tag{1.3.3.6}$$

where h is the unconstrained variable in the AMBI system replacing h_{BI} and P_1 is also unconstrained[k] The remaining equations in the AMBI system are given by

$$\partial_t D_2 + \partial_1 \Big(\frac{B_3 + D_2 P_1 - D_1 P_2}{h} \Big) = 0 \tag{1.3.3.7}$$

$$\partial_t D_3 + \partial_1 \Big(\frac{-B_2 + D_3 P_1 - D_1 P_3}{h} \Big) = 0 \tag{1.3.3.8}$$

$$\partial_t B_2 + \partial_1 \Big(\frac{-D_3 + B_2 P_1 - B_1 P_2}{h} \Big) = 0 \tag{1.3.3.9}$$

$$\partial_t B_3 + \partial_1 \Big(\frac{D_2 + B_3 P_1 - B_1 P_3}{h} \Big) = 0 \tag{1.3.3.10}$$

$$\partial_t P_2 + \partial_1 \Big(\frac{P_1 P_2 - D_1 D_2 - B_1 B_2}{h} \Big) = 0 \tag{1.3.3.11}$$

$$\partial_t P_3 + \partial_1 \Big(\frac{P_1 P_3 - D_1 D_3 - B_1 B_3}{h} \Big) = 0. \tag{1.3.3.12}$$

If h and P_1 are known functions, the system (1.3.3.7) - (1.3.3.12) is linear symmetric hyperbolic in $(D_2, D_3, B_2, B_3, P_2, P_3)$. Consequently, if[l] $1/h, P_1 \in C_b^N([0,\infty) \times \mathbb{R})$ for a large enough N, then (1.3.3.7) - (1.3.3.12) is globally well-posed in the Sobolev space $H^N(\mathbb{R})$ through the energy principle for linear symmetric hyperbolic systems [36]. The question of global existence for plane-symmetric solutions to the MBI system is therefore reduced to studying the 2×2 system (1.3.3.5), (1.3.3.6), the solutions of which are characterized by the following theorem, which is an extension of Brenier's blowup criterion for periodic solutions with planar symmetry:

Theorem 1.3.1. *Let $\mathring{h}(x_1), \mathring{P}_1(x_1)$ denote C^1 initial data for the system* (1.3.3.5) (1.3.3.6) *constructed from C^1 plane-symmetric initial data $\mathring{\mathbf{D}}, \mathring{\mathbf{B}}$ for the MBI system. Then $\mathring{h}(x_1), \mathring{P}_1(x_1)$ launch a globally bounded C^1 solution (h, P_1) if and only if there*

[k] To avoid adding to the list of symbols, we do not introduce an additional symbol to denote the unconstrained variable P_1. Instead, we allow context to determine if P_1 is unconstrained or if it is determined through $\mathbf{D}$ and $\mathbf{B}$ by (1.3.1.3).

[l] Section 2.2 of Chapter 2 explains the notation we use for function spaces.

exists a constant μ *such that*

$$\sup_{x_1 \in \mathbb{R}} |\mathring{P}_1(x_1) - \mu \mathring{h}(x_1)| < Z. \tag{1.3.3.13}$$

Theorem 1.3.1 is of particular interest because of the fact that the MBI, AMBI, and 2×2 systems are *totally linearly degenerate.* Before proving the theorem, we will discuss the definition of and a conjecture surrounding totally linearly degenerate systems. In order to define "totally linearly degenerate," we begin by writing the system (1.3.3.5) (1.3.3.6) using standard matrix notation, in which $\mathbf{V} \stackrel{\text{def}}{=} (h, P_1)$ is the solution array:

$$A^0(\mathbf{V})\partial_t \mathbf{V} + A^1(\mathbf{V})\partial_1 \mathbf{V} = \mathbf{0}, \tag{1.3.3.14}$$

$$\tag{1.3.3.15}$$

where

$$A^0(\mathbf{V}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A^1(\mathbf{V}) = \begin{pmatrix} 0 & 1 \\ \big(Z^2 - (P_1)^2\big)/h^2 & 2P_1/h. \end{pmatrix}. \tag{1.3.3.16}$$

The characteristic subset[m] of $T_x^*\mathcal{M}$ for this 2×2 system is the set of (ξ_0, ξ_1) satisfying the following equation: $\det\big(\xi_0 A^0(\mathbf{V}) + \xi_1 A^1(\mathbf{V})\big) = 0$, where

$$\det = \xi_0^2 + 2\frac{\xi_0\xi_1 P_1}{h} + \frac{\xi_1^2(P_1^2 - Z^2)}{h^2} = \Big[\xi_0 + \xi_1\Big(\frac{P_1 - Z}{h}\Big)\Big]\Big[\xi_0 + \xi_1\Big(\frac{P_1 + Z}{h}\Big)\Big]. \tag{1.3.3.17}$$

The *characteristic speeds* are found by setting $|\xi_1| = 1$ and solving for ξ_0 in (1.3.3.17). Labelling the two speeds[n] as λ_- and λ_+, we have that

$$\lambda_- = \frac{P_1 - Z}{h} \tag{1.3.3.18}$$

$$\lambda_+ = \frac{P_1 + Z}{h}. \tag{1.3.3.19}$$

We now compute the right eigenvectors $\mathbf{V}_-, \mathbf{V}_+$ associated to λ_-, λ_+,

$$\mathbf{V}_- = \begin{pmatrix} 1 \\ \lambda_- \end{pmatrix}, \ \mathbf{V}_+ = \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix}, \tag{1.3.3.20}$$

---

[m]The term "characteristic subset" is defined and discussed in detail in Chapter 2.

[n]In the "local rest frame" in which $\mathbf{E}$ and $\mathbf{B}$ are parallel, it follows that $P_1 = 0$, and the characteristic speeds coincide in magnitude; this implies the "no birefringence" of the 2×2 system property mentioned in [2].

and calculate that

$$\nabla_{\mathbf{V}}\lambda_- = h^{-1}\begin{pmatrix} -\lambda_- \\ 1 \end{pmatrix}, \quad \nabla_{\mathbf{V}}\lambda_+ = h^{-1}\begin{pmatrix} -\lambda_+ \\ 1 \end{pmatrix}. \tag{1.3.3.21}$$

It therefore follows that

$$\mathbf{V}_- \cdot \nabla_{\mathbf{V}}\lambda_- = 0 = \mathbf{V}_+ \cdot \nabla_{\mathbf{V}}\lambda_+, \tag{1.3.3.22}$$

so that $\lambda_-(t, x_1)$ is constant along the integral curves of the vectorfield $\mathbf{V}_-$, and similarly for λ_+. The system (1.3.3.5), (1.3.3.6) is said to be *totally linearly degenerate* precisely because property (1.3.3.22) holds. It was conjectured in Chapter 3 of [36] that such systems should not form shock waves for smooth initial data and suggested that it might be reasonable to expect global solutions for smooth initial data. Nevertheless, Theorem 1.3.1 shows that L^∞ singularities can form in finite time, even if the data are smooth. This observation has been made independently in [41], in which Neves and Serre discuss the ill-posedness of the Caucy problem for totally linearly degenerate systems of conservation laws. We now return to the proof of Theorem 1.3.1.

Proof. We abbreviate $x \stackrel{\text{def}}{=} x_1$ throughout the proof. Since $\mathring{h}(x) \geq 1$ for data in the BI manifold, the formula

$$s = \int_0^{\mathring{X}(s)} \mathring{h}(y)dy \tag{1.3.3.23}$$

implicitly defines a C^1 diffeomorphism $s \to \mathring{X}(s)$ from $\mathbb{R}$ to $\mathbb{R}$. By (1.3.3.23), we have that

$$\mathring{h}(\mathring{X}(s)) = \frac{1}{\mathring{X}'(s)}. \tag{1.3.3.24}$$

We also define

$$\mathring{U}(s) = \frac{\mathring{P}_1(\mathring{X}(s))}{\mathring{h}(\mathring{X}(s))} \tag{1.3.3.25}$$

and solve the linear wave equation

$$-\partial_t^2 X(t,s) + Z^2\partial_x^2 X(t,s) = 0 \tag{1.3.3.26}$$

with initial conditions $X(t=0,s) = \mathring{X}(s), \partial_t X(t=0,s) = \mathring{U}(s)$, applying d'Alembert's formula to obtain

$$X(t,s) = \frac{1}{2}[\mathring{X}(s+Zt) + \mathring{X}(s-Zt)] + \frac{1}{2Z}\int_{s-Zt}^{s+Zt} \mathring{U}(s')ds', \tag{1.3.3.27}$$

$$\partial_s X(t,s) = \frac{1}{2}[\mathring{X}'(s+Zt) + \mathring{X}'(s-Zt)] + \frac{1}{2Z}[\mathring{U}(s+Zt) - \mathring{U}(s-Zt)]. \tag{1.3.3.28}$$

Simple calculations show that if condition $\partial_s X(t,s) > 0$ holds, then the formulas

$$h(t, X(t,s)) \stackrel{\text{def}}{=} \frac{1}{\partial_s X(t,s)} \tag{1.3.3.29}$$

$$P_1(t, X(t,s)) \stackrel{\text{def}}{=} h(t, X(t,s))\partial_t X(t,s) = \frac{\partial_t X(t,s)}{\partial_s X(t,s)} \tag{1.3.3.30}$$

implicitly define a solution to (1.3.3.5), (1.3.3.6) existing at the spacetime point $(t, X(t,s))$. With this fact and formulas (1.3.3.24) and (1.3.3.25) in mind, we observe that $X(t,\cdot)$ is a C^1 diffeomorphism from $\mathbb{R}$ to $\mathbb{R}$ (i.e. $\partial_s X(t,s) > 0$ for all $s \in \mathbb{R}$) for *all* t if and only if

$$\frac{\mathring{P}_1(\mathring{X}(s_1)) + Z}{\mathring{h}(\mathring{X}(s_1))} > \frac{\mathring{P}_1(\mathring{X}(s_2)) - Z}{\mathring{h}(\mathring{X}(s_2))} \text{ for all } s_1, s_2 \in \mathbb{R}. \tag{1.3.3.31}$$

We therefore define

$$\mu_- \stackrel{\text{def}}{=} \sup_{s\in\mathbb{R}} \frac{\mathring{P}_1(\mathring{X}(s)) - Z}{\mathring{h}(\mathring{X}(s))} \tag{1.3.3.32}$$

$$\mu_+ \stackrel{\text{def}}{=} \inf_{s\in\mathbb{R}} \frac{\mathring{P}_1(\mathring{X}(s)) + Z}{\mathring{h}(\mathring{X}(s))} \tag{1.3.3.33}$$

and consider separately the following 3 cases:

1. $\mu_+ > \mu_-$
2. $\mu_+ < \mu_-$.
3. $\mu_+ = \mu_-$.

Case 1 is equivalent to the existence of a $\mu \in \mathbb{R}$ such that (1.3.3.13) holds. In this case, $1/|\partial_s X(t,s)|$ is uniformly bounded on $\mathbb{R} \times \mathbb{R}$, so by formulas (1.3.3.29) and (1.3.3.30), there is a global solution $h(t,x), P_1(t,x)$ to (1.3.3.5), (1.3.3.6) that is uniformly bounded on $\mathbb{R} \times \mathbb{R}$.

In case 2, there exist numbers s_1, s_2 with

$$\mathring{U}(s_1) + Z\mathring{X}'(s_1) = \mathring{U}(s_2) - Z\mathring{X}'(s_2), \tag{1.3.3.34}$$

and $h(t,x)$ necessarily blows up in L^∞ at the spacetime point $(t,x) = \big((s_1 - s_2)/2Z, X((s_1 - s_2)/2Z, (s_1 + s_2)/2)\big)$. Consequently, the lifespan[o] of the solution is

$$\inf\left\{\frac{|s_1 - s_2|}{2Z} \;\middle|\; s_1, s_2 \in \mathbb{R} \text{ and } \frac{\mathring{P}_1(x_1) + Z}{\mathring{h}(x_1)} = \frac{\mathring{P}_1(x_2) - Z}{\mathring{h}(x_2)}\right\}, \tag{1.3.3.35}$$

where $x_i \stackrel{\text{def}}{=} \mathring{X}(s_i)$, for $i = 1, 2$.

Remark 1.3.1. It is a simple exercise to algebraically verify that case (2) never occurs for the global plane wave solutions defined by (1.2.0.19), (1.2.0.20).

In case 3, we may only conclude that there is a pair of sequences $\{r_n\}, \{s_n\}$ such that

$$\lim_{n\to\infty} \big[\mathring{U}(r_n) + Z\mathring{X}'(r_n)\big] - \big[\mathring{U}(s_n) - Z\mathring{X}'(s_n)\big] = 0. \tag{1.3.3.36}$$

We therefore have that

$$\lim_{n\to\infty} h(t_n, x_n) = \infty, \tag{1.3.3.37}$$

where

$$(t_n, x_n) \stackrel{\text{def}}{=} \big((r_n - s_n)/2Z, \mathring{X}((r_n - s_n)/2Z, (r_n + s_n)/2)\big), \tag{1.3.3.38}$$

so there is no global L^∞ bound for the solution, which may or may not exist globally. Case 3 can be further broken down into subcases depending on whether or not the sup or inf in (1.3.3.32), (1.3.3.33) is achieved; we leave these details to the interested reader. □

As a partial extension of Theorem 1.3.1, we state the following 3-d blowup result and sketch its proof:

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[o]By lifespan, we mean the smaller of the two numbers $\sup\{t \mid h(t,\cdot), P_1(t,\cdot) \text{ exist globally in space}\}$ and $\sup\{t \mid h(-t,\cdot), P_1(-t,\cdot) \text{ exist globally in space}\}$.

Corollary 1.3.2. *There exist finite energy solutions to the 3-d source-free MBI field equations with smooth initial data that form singularities in finite time.*

Proof. A computation gives that the characteristic speeds in the 3-d MBI model are bounded in magnitude by 1[p]. Therefore, two solutions agreeing on a ball of radius R at $t = 0$ must agree on a ball of radius $R - t$ at later times t. We may now define smooth, compactly supported initial data for the 3-d MBI system that agrees on a ball of radius R with a plane-symmetric solution that blows up at the origin at time t^* via theorem 1.3.1. By the above argument, if R is large enough, then the 3-d solution generated by these data must also blowup at the origin at time t^*. □

In order for Corollary 1.3.2 to be physically relevant, one must extend it to show blowup for a neighborhood of generic initial data; the blowup-producing data that agree with plane-symmetric data on a ball may be degenerate cases that are unstable under perturbations. This is a difficult open problem that we have in mind for future investigation.

[p]This computation is performed, for example, by Bialynicki-Birula [2] in a Lorentz frame in which **E** and **B** are parallel.

Chapter 2

The Euler-Nordström System with Cosmological Constant

This chapter is devoted to the application of Christodoulou's method of energy currents to the Cauchy problem for the Euler-Nordström (EN) system. Although most of the technical estimates involve energy currents, we use a separate argument inspired by Kato's work to complete the proof of continuous dependence on initial conditions.

2.1 Introduction

It is well-known that for symmetric hyperbolic systems of PDEs, an energy principle is available that implies well-posedness (local existence, uniqueness, and continuous dependence on initial data) for initial data belonging to an appropriate Sobolev space. Consult [17], [18], [23], [36], or [49] for the definition of a symmetric hyperbolic system and a detailed proof of local existence in this case. A full proof of well-posedness is difficult to locate in the literature, but Kato [27] supplies one using a very general setup that applies to symmetric hyperbolic systems in a Banach space. Additionally, for strictly hyperbolic (not necessarily symmetric) systems, well-posedness follows from the availability of a generalization of the energy principle for symmetric hyperbolic systems. For strictly hyperbolic systems, there are a variety of methods due to Petrovskii, Leray, Gårding, and Calderón for generating energy estimates; consult [17] or [34] for details on these methods.

We consider here the Cauchy problem for the Lorentz covariant Euler-Nordström (EN) system, which is a scalar caricature of the general covariant Euler-Einstein system describing a gravitationally self-interacting fluid. The EN system is a quasilinear hyperbolic system of PDEs that is not manifestly symmetric hyperbolic. Moreover, because of the repeated factors in the expression for $\mathcal{Q}(x;\cdot)$ in equation (2.5.1.7) below, it is not strictly hyperbolic. Therefore, well-posedness for the EN system does not follow from either of these two well-known frameworks.

Fortunately, alternate techniques recently developed by Christodoulou [15], which are applied to the study of relativistic fluids in Minkowski spacetime in particular in [16], offer a viable approach to studying the Cauchy problem for the EN system. The central advantage afforded by Christodolou's techniques, which provide energy currents for equations derivable from a Lagrangian, is that they bypass the physically artificial requirement of symmetry in the equations: even though the EN system is not manifestly symmetric, its energy currents allow for precisely the same energy estimates to be made as in the theory of symmetric hyperbolic systems. Once one has these estimates, the proof of well-posedness for the EN system mirrors the well known proof for symmetric hyperbolic systems. Our main goal is to use the method of energy currents to prove the following theorem (stated loosely here), which is divided into parts and stated rigorously in Section 2.7:

Theorem (Well-Posedness). *Let $N \geq 3$ be an integer. Assume that the initial data $\mathring{\mathbf{V}}$ for the EN system are an H^N perturbation of a constant background solution $\bar{\mathbf{V}}$. Then these data launch a unique solution $\mathbf{V}$ possessing the regularity property $\mathbf{V} - \bar{\mathbf{V}} \in C^0([0,T], H^N) \cap C^1([0,T], H^{N-1})$. Furthermore, the map from the initial perturbation $\mathring{\mathbf{V}} - \bar{\mathbf{V}}$ to $\mathbf{V} - \bar{\mathbf{V}}$ is a continuous map from an open subset of H^N into $C^0([0,T], H^N)$.*

While Christodoulou's methods are not the only techniques available for proving the well-posedness of the EN system, they are powerful and natural in the sense that they exploit the inherent geometry of the equations and apply to all physical equations of state[a]. In contrast, one may sometimes proceed by a change of state-space variables that renders the system symmetric hyperbolic. For example, Makino applies the symmetrizing technique to the Euler-Poisson equations in [37]. In fact, the EN system is symmetrizable under *some* equations of state if we assume isentropic conditions. This follows from the fact that the left-hand sides of (2.4.1.19) - (2.4.1.20) have the *symbolic*

[a]We list the hypotheses that a "physical" equation of state must satisfy in Section 2.3.3.

form[b] of the relativistic Euler equations, which, under *barotropic* equations of state (in which the pressure is a function of the energy density alone), are symmetrized in [38] and [39]. Further discussion of symmetrization appearing in the literature can be found in sections 2.3.1 and 2.3.2.[c] Yet the symmetrizing method is not without disadvantages: one must solve a formally over-determined system of equations to find the symmetrizing variables[d], and the resulting state-space variables, if they exist, may place un-physical and/or mathematically unappealing restrictions on the function spaces with which one would like to work. However, it should be noted that for certain equations of state, Makino's symmetrization allows one to prove local existence for a restricted class of compactly supported data, while the techniques applied here cannot yet handle such data due to singularities in the energy current (2.5.5.1) when the proper energy density ρ of the fluid vanishes.

2.2 Notation

We introduce here some notation that is used throughout this article, some of which is non-standard. We assume that the reader is familiar with standard notation for the L^p spaces and the Sobolev spaces H^k. Unless otherwise stated, the symbols L^p and H^k refer to $L^p(\mathbb{R}^3)$ and $H^k(\mathbb{R}^3)$ respectively.

2.2.1 Notation and Assumptions Regarding Spacetime

In the Euler-Poisson system with cosmological constant introduced below, we use $t \in \mathbb{R}$ to denote the time variable and $\mathbf{s} \in \mathbb{R}^3$ to denote the space variable. In the Euler-Einstein and EN systems (which we also equip with a cosmological constant below), we assume that spacetime is a 4-dimensional, time-orientable Lorentzian manifold $\mathcal{M}$ and

[b] By "symbolic form," we mean that the form of the left-hand sides of (2.4.1.19) - (2.4.1.20) is the same as that of the relativistic Euler equations as presented in [16]. However our symbols P and R represent the quantities defined in (2.4.1.5) and (2.4.1.6), whereas in the relativistic Euler equations, these symbols would represent the pressure and energy density respectively.

[c] The references given are far from exhaustive; we merely wish to provide the reader with some examples of the application of well-known techniques.

[d] Consult Chapter 3 of [18] for a discussion of symmetrization.

use the notation

$$x = (x^0, x^1, x^2, x^3) \tag{2.2.1.1}$$

to denote spacetime points. For the EN system with cosmological constant, we assume the existence of a global system of rectangular coordinates (an inertial frame), and for this preferred time-space splitting, we identify $t = x^0$ with time and $\mathbf{s} = (x^1, x^2, x^3)$ with space and use the notation (2.2.1.1) to denote the components of x relative to this fixed coordinate system.

2.2.2 Notation Regarding Differential Operators

For $\nu = 0, 1, 2, 3$, we use the notation ∂_ν to denote differentiation with the respect to x^ν, where x^ν is defined in (2.2.1.1). We sometimes write ∂_t in place of ∂_0.

If F is a scalar or finite-dimensional array-valued function on $\mathbb{R}^{1+3}$, then DF denotes the array consisting of all first-order spacetime partial derivatives (including the partial derivative with respect to time) of every component of F, while $\nabla^{(a)}F$ denotes the array of consisting of all a^{th} order *spatial* partial derivatives of every component of F; this should not be confused with ∇, which represents covariant differentiation.

2.2.3 Index Conventions

We adopt Einstein's convention that repeated Latin indices are summed from 1 to 3, while repeated Greek indices are summed from 0 to 3. Indices are raised and lowered using a relevant spacetime metric, which in our discussion below is either the Einstein metric $g_{\mu\nu}$ or the Minkowski metric $\underline{g}_{\mu\nu}$, depending on context.

2.2.4 Notation Regarding Norms and Function Spaces

If F is a scalar-valued or finite-dimensional array-valued quantity, we denote its Euclidean norm by $|F|$; we explain one exception to this rule just beneath equation (2.2.4.6). If such an F is a function on a subset $\mathcal{A}$ of $\mathbb{R}^3$ and $1 \leq p \leq \infty$, then we use the notation

$$\|F\|_{L^p(\mathcal{A})} \overset{\text{def}}{=} \left(\int_{\mathcal{A}} |F(\mathbf{s})|^p \, d^3\mathbf{s} \right)^{1/p} \qquad (1 \leq p < \infty) \tag{2.2.4.1}$$

$$\|F\|_{L^\infty(\mathcal{A})} \overset{\text{def}}{=} \inf\{b \in \mathbb{R} \mid \mu\{\mathbf{s} \in \mathcal{A} \mid |F(\mathbf{s})| \geq b\} = 0\}, \tag{2.2.4.2}$$

where μ denotes Lebesgue measure, to denote the usual L^p norm of F on the set $\mathcal{A}$.

If $\bar{\mathbf{V}}$ is a constant array, we use the notation

$$\|F\|_{L^p_{\bar{\mathbf{V}}}(\mathcal{A})} \overset{\text{def}}{=} \|F - \bar{\mathbf{V}}\|_{L^p(\mathcal{A})}, \tag{2.2.4.3}$$

and we denote the set of all Lebesgue measurable functions F such that $\|F\|_{L^p_{\bar{\mathbf{V}}}(\mathcal{A})} < \infty$ by $L^p_{\bar{\mathbf{V}}}(\mathcal{A})$. Unless we indicate otherwise, we assume that $\mathcal{A} = \mathbb{R}^3$ when the set $\mathcal{A}$ is not explicitly written.

If F is a map from $[0, T]$ into the normed function space X, we use the notation

$$||| \, F \, |||_{X,T} \overset{\text{def}}{=} \sup_{t \in [0,T]} \|F(t)\|_X. \tag{2.2.4.4}$$

We also use the notation $C^k([0, T], X)$ to denote the set of k-times continuously differentiable maps from $(0, T)$ into X that, together with their derivatives up to order k, extend continuously to $[0, T]$.

If $\mathcal{A} \subset \mathbb{R}^d$ (d frequently equals 3 or 10 in this article) and $\mathcal{A}$ is open, then $C^k_b(\bar{\mathcal{A}})$ denotes the set $k-$times continuously differentiable functions (either scalar or array-valued, depending on context) on $\mathcal{A}$ with bounded derivatives up to order k that extend continuously to the closure of $\mathcal{A}$. The norm of a function $F \in C^k_b(\bar{\mathcal{A}})$ is defined by

$$|F|_{k,\mathcal{A}} \overset{\text{def}}{=} \sum_{|\vec{\alpha}| \leq k} \sup_{z \in \mathcal{A}} |\partial_{\vec{\alpha}} F(z)|, \tag{2.2.4.5}$$

where $\partial_{\vec{\alpha}}$ represents differentiation with respect to the arguments z of F (which may be spacetime variables or state-space variables, depending on the context).

If $F(\mathbf{s})$ is a function on a subset $\mathcal{A}$ of $\mathbb{R}^3$, and N' is any integer, we use the notation

$$\|F\|_{H^{N'}(\mathcal{A})} \overset{\text{def}}{=} \Big(\sum_{|\vec{\alpha}| \leq N'} \|\partial_{\vec{\alpha}} F(\mathbf{s})\|^2_{L^2(\mathcal{A})} \Big)^{1/2} \tag{2.2.4.6}$$

to denote the usual $H^{N'}$ norm of F on the set $\mathcal{A}$. In (2.2.4.6), in keeping with the PDE convention, we are breaking with the convention that $|\vec{\alpha}|$ is the Euclidean norm, instead defining $|\vec{\alpha}| \overset{\text{def}}{=} \alpha^1 + \alpha^2 + \alpha^3$ if $\vec{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$.

Analogous to definition (2.2.4.3), we use the notation

$$\|F\|_{H^{N'}_{\bar{\mathbf{V}}}(\mathcal{A})} \overset{\text{def}}{=} \|F - \bar{\mathbf{V}}\|_{H^{N'}(\mathcal{A})}, \tag{2.2.4.7}$$

and we denote the set of all Lebesgue measurable functions F such that $\|F\|_{H^{N'}_{\bar{\mathbf{V}}}(\mathcal{A})} < \infty$ by $H^{N'}_{\bar{\mathbf{V}}}(\mathcal{A})$.

In the case $\mathcal{A} = \mathbb{R}^3$, we sometimes consider non-integer values of N', in which case we define $\|F\|_{H^{N'}}$ in the standard way using the Fourier transform of F.

If $F(t, \mathbf{s})$ is a function on $\mathbb{R}^{1+3}$, we write $\|F(t)\|_{H^{N'}(\mathcal{A})}$ to denote the $H^{N'}$ norm of the function $F(t, \cdot)$ on the set $\mathcal{A}$ with t held fixed, and similarly with the $H^{N'}$ norm replaced by the $H^{N'}_{\bar{\mathbf{V}}}$ norm, the L^p norm, or the $L^p_{\bar{\mathbf{V}}}$ norm.

We use the symbol $\mathcal{S}$ to denote the Schwartz functions on $\mathbb{R}^3$, the set of all C^∞ functions that together with all of their derivatives vanish at infinity faster than any power of $|\mathbf{s}|$.

2.2.5 Notation Regarding Operators

If X and Y are normed function spaces, then $\mathcal{L}(X, Y)$ denotes the set of bounded linear maps from X to Y. If $\mathcal{U} \in \mathcal{L}(X, Y)$, we denote the operator norm of $\mathcal{U}$ by

$$\|\mathcal{U}\|_{X,Y} \overset{\text{def}}{=} \sup_{x \in X} \frac{\|\mathcal{U}(x)\|_Y}{\|x\|_X}. \tag{2.2.5.1}$$

When $X = Y$, we use the notation $\mathcal{L}(X) \overset{\text{def}}{=} \mathcal{L}(X, X)$ and $\|\mathcal{U}\|_X \overset{\text{def}}{=} \|\mathcal{U}\|_{X,X}$.

If $\mathcal{U}(t, t')$ is an operator-valued map from the triangle $\triangle_T \overset{\text{def}}{=} \{0 \leq t' \leq t \leq T\}$ into $\mathcal{L}(X)$, then we adopt the notation

$$|||\, \mathcal{U}\, |||_{X,\triangle_T} \overset{\text{def}}{=} \sup_{(t,t') \in \triangle_T} \|\mathcal{U}(t, t')\|_X. \tag{2.2.5.2}$$

2.2.6 Notation Regarding Tempered Distributions

If F and G are tempered distributions (that is, continuous linear functionals on $\mathcal{S}$), we denote the Fourier transform of F by $\widehat{F}$, and the inverse Fourier transform of F by $F^\vee$. We use the notation

$$[F, G] \overset{\text{def}}{=} \int F^\vee \widehat{G}. \tag{2.2.6.1}$$

Here, if $F^{\vee}$ and $\widehat{G}$ are functions, the right-hand side is the usual integral over $\mathbb{R}^3$. If $F^{\vee}$ is a distribution and $G \in \mathcal{S}$, then the right-hand side is defined to be the action of the distribution $F^{\vee}$ on $\widehat{G}$, which agrees with the integral definition in the former case.

2.2.7 Notation Regarding Constants

We use the symbol C to denote a generic constant in the estimates below which is free to vary from line to line. If the constant depends on quantities such as real numbers N', subsets $\mathcal{A}$ of $\mathbb{R}^d$, functions F of the state-space variables, etc., that are peripheral to the argument at hand, we sometimes indicate this dependence by writing $C(N', \mathcal{A}, F)$, etc. We frequently omit the dependence of C on functions of the state-space variables below in order to conserve space, but we explicitly show the dependence when it is (in our judgment) illuminating. Occasionally, we shall use additional symbols such as $C_{\bar{\mathcal{O}}_2}, L, K$, etc., to denote constants that play a distinguished role in the discussion below.

2.3 Models in Context

The EN system is an intermediate model in between the Galilean covariant Euler-Poisson (EP) and the general covariant Euler-Einstein (EE) systems for self-gravitating classical fluids. Although it is the most fundamental of these models for self-gravitating Eulerian fluids, the EE system presents numerous technical difficulties that make a detailed analysis of the system's evolution, through either numerical or analytical methods, extremely difficult. In addition to the usual difficulties involved in studying quasilinear systems of PDEs, the EE system does not exhibit spherically-symmetric solutions featuring the propagation of gravitational waves.[e] Consequently, in order to study wave phenomena in the EE model, one must work with systems involving at least two spacelike independent variables and one timelike independent variable. Further complicating matters is that there is no known law of local conservation of gravitational field energy in General Relativity. Our main motivations for studying the EN system are to bridge

[e]This result is known as Birkhoff's Theorem.

the gap between the EP and the EE systems and to provide a special relativistic primer for studying the EE system.

Since it is based on Nordström's theory of gravity, it should be stressed that the EN system is physically wrong. However, since both the EN and the EE systems are relativistic generalizations of the EP system, we expect, at least in some limiting cases, that there are some qualitative similarities between solutions to the three systems. A rigorous result along these lines is stated and proved as Theorem 3.9.2. Furthermore, in [50], Shapiro and Teukolsky discuss numerical simulations of the EN system in the spherically symmetric case; they expect that the numerical schemes developed in their paper can be adapted to allow for the calculation of accurate wave forms in the EE model.

Before discussing the EN system in detail, we briefly recall the EP and EE systems, endowing both with a *cosmological constant*[f] denoted by κ^2. We also briefly discuss some local existence proofs for these systems in the case $\kappa = 0$, emphasizing their dependence on the symmetric hyperbolic setup or the method of Leray (strict) hyperbolicity.

We introduce a positive cosmological constant out of mathematical necessity: the EN system generally fails[g] to have non-zero constant solutions without it.[h] We emphasize the presence of the cosmological constant κ^2 in the models by referring to them as the EP_κ, EE_κ, and EN_κ systems; note that $\mathrm{EP}=\mathrm{EP}_0$, and similarly for the other two models.

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[f]We deviate from Einstein's notation; he denoted the cosmological constant by Λ.

[g]We provide some elaborating remarks on the existence of constant solutions in the EN_κ system in Section 0.1.

[h]This is similar to the reasoning that led Einstein to introduce the cosmological constant into General Relativity; he sought a static universe, and General Relativity without a cosmological constant features only Minkowski space as a static solution (see [21]).

2.3.1 The Euler-Poisson System with Cosmological Constant (EP_κ)

In units with Newton's universal gravitational constant equal to 1, the equations governing the dynamics in this case are

$$\partial_t \eta + v^k \partial_k \eta = 0 \tag{2.3.1.1}$$

$$\partial_t \rho + \partial_k (\rho v^k) = 0 \tag{2.3.1.2}$$

$$\rho \left(\partial_t v_j + v^k \partial_k v^j \right) + \partial_j p + \rho \partial_j \Phi = 0 \qquad (j = 1, 2, 3), \tag{2.3.1.3}$$

where

$$\Delta \Phi - \kappa^2 \Phi = 4\pi\rho \tag{2.3.1.4}$$

and

$$p = \mathcal{P}(\rho, \eta). \tag{2.3.1.5}$$

The unknowns in (2.3.1.1) - (2.3.1.4) are the cosmological Newtonian gravitational scalar potential $\Phi(t, \mathbf{s})$, and the state-space variables[i] mass density $\rho(t, \mathbf{s})$, velocity $\mathbf{v}(t, \mathbf{s}) = (v_1, v_2, v_3)$, pressure $p(t, \mathbf{s})$, and entropy density $\eta(t, \mathbf{s})$.[j] The equation that specifies p as a function $\mathcal{P}$ of ρ and η is known as the *equation of state.* This system of equations is discussed in [30], in which, under an *isothermal* equation of state ($p = c_s^2 \rho$, where the constant c_s denotes the speed of sound), Kiessling derives the Jeans dispersion relation that arises from linearizing (2.3.1.2) - (2.3.1.4) about a static state in which the background mass density $\bar{\rho}$ is non-zero, followed by taking the limit $\kappa \to 0$.[k]

In [37], Makino studies the Cauchy problem for the EP_0 system with compactly supported initial data belonging to an appropriate Sobolev space. He considers isentropic conditions ($\eta \equiv$ constant)[l] and an *adiabatic* equation of state ($p = A\rho^\gamma$, where

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[i]In the EP_κ system, Φ is not a state-space variable because it is uniquely determined by ρ under the assumption of appropriate decay conditions on Φ and ρ at infinity.

[j]We are influenced by Boltzmann's notation in denoting the entropy density by η.

[k]Entropy is not a relevant concept for the topics discussed Kiessling's paper, so equation (2.3.1.1) is not included in the system of equations he studies.

[l]Equation (2.3.1.1) is therefore not included in Makino's paper either.

A is a positive constant) under the mathematical assumption $1 < \gamma < 3$. After finding symmetrizing variables[m], he proves local existence.

2.3.2 The Euler-Einstein System with Cosmological Constant (EE$_\kappa$)

We work in units with Newton's universal gravitational constant and the speed of light both equal to 1. Given T, the energy-momentum tensor of the contemplated matter model, the gravitational spacetime with cosmological constant is determined by the Einstein field equations,

$$G_{\mu\nu} + \kappa^2 g_{\mu\nu} = 8\pi T_{\mu\nu} \qquad (0 \leq \mu, \nu \leq 3), \tag{2.3.2.1}$$

where G is the Einstein tensor of the spacetime metric g. As a consequence of (2.3.2.1), T has to satisfy the admissibility condition

$$\nabla_\mu T^{\mu\nu} = 0 \qquad (0 \leq \nu \leq 3), \tag{2.3.2.2}$$

where the ∇ denotes the covariant derivative induced by the *spacetime metric* g. Equation (2.3.2.2) follows from the twice contracted Bianchi identity, which implies that

$$\nabla_\mu G^{\mu\nu} = 0, \tag{2.3.2.3}$$

together with

$$\nabla_\lambda g^{\mu\nu} = 0 \qquad (0 \leq \lambda, \mu, \nu \leq 3), \tag{2.3.2.4}$$

which follows from the fact that ∇ is the Levi-Civita connection on spacetime.

For a perfect fluid model, the components of the energy-momentum tensor of matter read[n]

$$T^{\mu\nu} \stackrel{\text{def}}{=} (\rho + p) u^\mu u^\nu + p g^{\mu\nu}. \tag{2.3.2.5}$$

Here the scalar $\rho \geq 0$ is the *proper energy density*, the scalar $p \geq 0$ is the *pressure,* and the vector u is the *four-velocity,* a future-directed timelike vectorfield which is subject

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[m] Vanishing mass densities typically produce singularities in the expression for the energy, but Makino's choice of symmetrizing variables allows him to handle a class of compactly supported data.

[n] Consult [1] for a discussion on the form of the energy-momentum tensor for a perfect fluid.

to the normalization condition

$$g_{\mu\nu}u^\mu u^\nu = -1. \tag{2.3.2.6}$$

When g is given and T is defined by (2.3.2.5), equations (2.3.2.2) are 4 of the 5 Euler equations for a general-relativistic perfect fluid, the 5th Euler equation being given by

$$\nabla_\mu(nu^\mu) = 0, \tag{2.3.2.7}$$

where $n \geq 0$ is the *proper number density*. Equation (2.3.2.7) is a consequence of the variational formulation of fluid mechanics as described in [15]. We also introduce the thermodynamic variable $\eta \geq 0$, the *entropy density*.

In general, when both g and T are unknowns, (2.3.2.1), its consequence (2.3.2.2), together with (2.3.2.5), (2.3.2.6), and (2.3.2.7) form the EE_κ system[o] for u, ρ, p, g and η, up to closure, for instance by an equation of state relating ρ, p, and η. In the special case of barotropic fluids, the equation of state is a relationship between ρ and p alone, and the system is closed without considering (2.3.2.7). Local existence for barotropic perfect fluids has been discussed by several authors under various additional assumptions on the equation of state and initial data. For example, in [12], Choquet-Bruhat showed that the EE_0 system with barotropic perfect fluid sources forms a well-posed Leray-hyperbolic system, and in [47], Rendall adapted Makino's symmetrization (as discussed in section 2.3.1) of the EP_0 system to handle a subclass of compactly supported initial data for the EE_0 system under an adiabatic equation of state with $\gamma > 1$. Similar results are also proved in [6], in which Brauer and Karp write the equations as a symmetric hyperbolic system in harmonic coordinates.

2.3.3 The Euler-Nordström System with Cosmological Constant (EN_κ)

We base our discussion here on Calogero's derivation of the Vlasov-Nordström system [8].[p] Consult sections 2.2.1 and 2.2.3 for some remarks on our assumptions concerning

[o]Interested readers may also consult the first few pages of [14] for a short introduction to general-relativistic fluids.

[p]Each of the three Eulerian fluid models mentioned above has a kinetic theory counterpart. Collectively known as the Vlasov models, these diffeo-integral systems describe a particle density function f

spacetime and our use of index notation. As in the EE_κ model, we work in units with the speed of light and Newton's universal gravitational constant both equal to 1.

Like the EE_κ system, the EN_κ system subsumes equations (2.3.2.2), (2.3.2.5), (2.3.2.6), and (2.3.2.7), where $\rho \geq 0, p \geq 0, n \geq 0, \eta \geq 0$, and u are defined as in the EE_κ system. In contrast to the EE_κ model, we do *not* assume Einstein's field equations (2.3.2.1); instead we turn to Nordström's theory of gravity. We postulate that in our global rectangular coordinate system, the conformally flat metric is given by[q]

$$g_{\mu\nu} \stackrel{\text{def}}{=} e^{2\phi} \underline{g}_{\mu\nu}, \tag{2.3.3.1}$$

where ϕ is the *Nordström scalar potential*, and $\underline{g} = \mathrm{diag}(-1, 1, 1, 1)$ are the components of the Minkowski metric in the rectangular coordinate system. Following Nordström's lead [43], we also introduce the auxiliary energy-momentum tensor T_{aux} with components

$$T_{\text{aux}}^{\mu\nu} \stackrel{\text{def}}{=} e^{6\phi} T^{\mu\nu} \tag{2.3.3.2}$$

and postulate that ϕ is a solution to

$$\Box\phi - \kappa^2\phi = -\underline{g}_{\mu\nu} T_{\text{aux}}^{\mu\nu} = -e^{4\phi}(3p - \rho). \tag{2.3.3.3}$$

Note that $\Box\phi \stackrel{\text{def}}{=} -\partial_t^2\phi + \Delta\phi$ is the wave operator on flat spacetime applied to ϕ. The virtue of the postulate (2.3.3.3) is that it provides us with continuity equations for an energy-momentum tensor in Minkowski space which we label Θ and discuss below; see equations (2.4.1.8) and (2.4.1.9).

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on physical space $\times$ momentum space that evolves due to gravitational self-interaction. In particular, the EN_0 system is the Eulerian counterpart of the previously studied Vlasov-Nordström (VN) system (which does not feature a cosmological constant). See e.g., [8] or [9].

[q]Nordström's theory of gravity [43] belongs to the class of theories known as scalar metric theories of gravity. For theories in this class, gravitational forces are mediated by a scalar field (or "potential") ϕ that affects the spacetime metric. Furthermore, it is assumed that the effect of ϕ is to modify the otherwise flat metric by a scaling factor that depends on ϕ. Therefore, the physical metric in such a theory is given by $g_{\mu\nu} = \chi^2(\phi)\underline{g}_{\mu\nu}$, where $\underline{g}$ is the Minkowski metric. A metric of this form is said to be *conformally flat*. Strictly speaking, the scalar theory of gravity we study in this paper is not identical to the one published by Nordström in [43]. In his paper, Nordström makes the choice $\chi(\phi) = \phi$, while in our paper, we make the choice $\chi(\phi) = e^\phi$, a theory that appears as a homework exercise in the well-known text "Gravitation" by Misner, Thorne, and Wheeler [40]. See [8] or [19] concerning the significance of the choice $\chi(\phi) = e^\phi$, which has the property of *scale invariance* of the gravitational interaction. Also consult [46] for a discussion of scalar theories of gravity, including the two mentioned here.

As in the EP_κ and EE_κ models, we may close the EN_κ system by supplying an equation of state. The basic postulates we adopt for a physical equation of state are as follows (see e.g. [24]):

1. $\rho \geq 0$ is a function of $n \geq 0$ and $\eta \geq 0$.

2. $p \geq 0$ is defined by

$$p \stackrel{\text{def}}{=} n \left. \frac{\partial \rho}{\partial n} \right|_{\eta} - \rho, \tag{2.3.3.4}$$

   where the notation $|_{\cdot}$ indicates partial differentiation with $\cdot$ held constant.

3. A perfect fluid satisfies

$$\left. \frac{\partial \rho}{\partial n} \right|_{\eta} > 0, \left. \frac{\partial p}{\partial n} \right|_{\eta} > 0, \left. \frac{\partial \rho}{\partial \eta} \right|_{n} \geq 0 \text{ with `` = '' iff } \eta = 0, \tag{2.3.3.5}$$

   As a consequence, we have that σ, the speed of sound in the fluid, is always real:

$$\sigma^2 \stackrel{\text{def}}{=} \left. \frac{\partial p}{\partial \rho} \right|_{\eta} = \frac{\partial p / \partial n|_{\eta}}{\partial \rho / \partial n|_{\eta}} > 0. \tag{2.3.3.6}$$

4. We also demand that the speed of sound is less than the speed of light:

$$0 < \sigma < 1. \tag{2.3.3.7}$$

Remark 2.3.1. By (3.1.1.11), we can solve for σ and ρ as functions of p and η :

$$\sigma = \mathcal{S}(\eta, p) \tag{2.3.3.8}$$

$$\rho = \mathcal{R}(\eta, p). \tag{2.3.3.9}$$

As a typical example, we mention a *polytropic* equation of state, that is, an equation of state of the form (see e.g., [24])

$$\rho = n + \frac{A(\eta)}{\gamma - 1} n^{\gamma}, \tag{2.3.3.10}$$

where $1 < \gamma < 2$, and A is a positive, increasing function of η. In this case $p = An^{\gamma}$, $\partial p / \partial \rho|_{\eta}$ is increasing in ρ, and the speed of sound σ is bounded from above by $\sqrt{\gamma - 1}$.

Remark 2.3.2. We note here a curious discrepancy that arises when, for the polytropic equation of state under the isentropic condition $\eta \equiv \eta_0$, we consider the Newtonian limit, that is, the limit as the speed of light c goes to ∞. In dimensional units, (2.3.3.10) becomes $\rho = m_0 c^2 n + \frac{A_c(\eta)}{\gamma-1} n^\gamma$, and $p = A_c(\eta) n^\gamma$, where m_0 is the mass per fluid element, and $A_c(\eta)$ is a positive, increasing function of η indexed by the parameter c. The speed of sound squared is given by $\sigma^2 \overset{\text{def}}{=} c^2 \left.\frac{\partial p}{\partial \rho}\right|_\eta = \frac{\gamma c^2 A_c(\eta_0) n^{\gamma-1}}{c^2 m_0 + (\gamma/\gamma-1) A_c(\eta_0) n^{\gamma-1}}$. Assuming that $\lim_{c\to\infty} A_c(\eta_0) \overset{\text{def}}{=} A_\infty(\eta_0)$ exists, we may consider the Newtonian limit $c \to \infty$ of σ^2 and p, obtaining $\sigma^2 = \gamma m_0^{-1} A_\infty(\eta_0) n^{\gamma-1}$ and $p = A_\infty(\eta_0) n^\gamma$, Newtonian formulas that make mathematical sense and have physical interpretations for $1 \leq \gamma < \infty$. In the Newtonian case, $\gamma = 1$ corresponds to isothermal conditions, while $\gamma \to \infty$ yields the rigid body dynamics. However, for finite values of c, not all values of the parameter γ make mathematical or physical sense: there is a mathematical singularity in the formula for ρ at $\gamma = 1$. This is physically reasonable since isothermal conditions require the instantaneous transfer of heat energy. Thus, for finite c, the polytropic equations of state do not allow for the case corresponding to the instantaneous transfer of heat energy over finite distances, a feature which we find desirable in a relativistic model. Additionally, we have that $\lim_{n\to\infty} \sigma^2 = c^2(\gamma - 1)$, so that for $\gamma > 2$, there is a $\gamma-$dependent critical threshold for the number density above which the speed of sound exceeds the speed of light. Since larger values of γ correspond to "increasing rigidity" of the fluid, and the concept of rigidity violates the spirit of the framework of relativity, we are not surprised to discover that large values of γ may lead to superluminal sound speeds. However, we find ourselves at the moment unable to attach a physical interpretation to the fact that the mathematical borderline case is $\gamma = 2$.

We summarize this section by stating that equations (2.3.2.2), (2.3.2.5), (2.3.2.6), and (2.3.2.7), (3.1.1.1), (2.3.3.2), (2.3.3.3), (2.3.3.4), and (2.3.3.9) constitute the EN$_\kappa$ system.

2.4 Reformulation of the EN$_\kappa$ System, the Linearized EN$_\kappa$ System, and the Equations of Variation

Because it is mathematically advantageous, in this section we reformulate the EN$_\kappa$ system as a fixed-background theory in flat Minkowski space. This is a mathematical reformulation only; the "physical" metric in the EN$_\kappa$ system is g from (3.1.1.1) rather than the Minkowski metric $\underline{g}$. We also discuss the linearization of the EN$_\kappa$ system and the related equations of variation, systems that are central to the well-posedness arguments.

2.4.1 Reformulating the EN$_\kappa$ System

For the remainder of this chapter, indices are raised and lowered with the Minkowski metric, so for example, $\partial^\lambda \phi = \underline{g}^{\mu\lambda}\partial_\mu \phi$. To begin, we use the form of the metric (3.1.1.1) to compute that in our fixed rectangular coordinate system (see Section 2.2.1), the continuity equation (2.3.2.2) for the energy-momentum tensor (2.3.2.5) is given by

$$\begin{aligned} 0 = \nabla_\mu T^{\mu\nu} &= \partial_\mu T^{\mu\nu} + 6T^{\mu\nu}\partial_\mu\phi - e^{-2\phi} g_{\alpha\beta} T^{\alpha\beta}\partial^\nu\phi \\ &= \partial_\mu T^{\mu\nu} + 6T^{\mu\nu}\partial_\mu\phi - e^{-6\phi}\underline{g}_{\alpha\beta} T^{\alpha\beta}_{\text{aux}}\partial^\nu\phi \qquad (\nu = 0,1,2,3), \end{aligned} \tag{2.4.1.1}$$

where $T^{\mu\nu}_{\text{aux}}$ is given by (2.3.3.2). For this calculation we made use of the explicit form of the Christoffel symbols in our rectangular coordinate system:

$$\Gamma^{\alpha}_{\mu\nu} = \delta^\alpha_\nu \partial_\mu\phi + \delta^\alpha_\mu \partial_\nu \phi - \underline{g}_{\mu\nu}\underline{g}^{\alpha\beta}\partial_\beta\phi. \tag{2.4.1.2}$$

Under the postulate (2.3.3.3) for ϕ, (3.1.2.1) can be rewritten as

$$0 = e^{6\phi}\nabla_\mu T^{\mu\nu} = \partial_\mu\big(T^{\mu\nu}_{\text{aux}} + \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\underline{g}^{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi - \frac{1}{2}\underline{g}^{\mu\nu}\kappa^2\phi^2\big). \tag{2.4.1.3}$$

Equation (2.4.1.3) now illustrates the divergence-free energy-momentum tensor Θ mentioned in Section 2.3.3. Its components $\Theta^{\mu\nu}$ consist of the terms from (2.4.1.3) that are inside the parentheses; we are thus afforded with local conservation laws in Minkowski space.

To simplify the notation, we make the change of field variables (recalling equation (2.3.3.9) for the definition of the function $\mathcal{R}$)

$$U^\nu \stackrel{\text{def}}{=} e^\phi u^\nu \qquad\qquad (\nu = 0,1,2,3) \tag{2.4.1.4}$$

$$R \stackrel{\text{def}}{=} e^{4\phi}\rho = e^{4\phi}\mathcal{R}(p,\eta) \tag{2.4.1.5}$$

$$P \stackrel{\text{def}}{=} e^{4\phi} p \tag{2.4.1.6}$$

throughout the EN_κ system, noting that U is subject to the constraint

$$U^0 = (1 + U^k U_k)^{1/2}. \tag{2.4.1.7}$$

Following the above substitutions, Θ has components

$$\Theta^{\mu\nu} \stackrel{\text{def}}{=} (R+P)U^\mu U^\nu + P\underline{g}^{\mu\nu} + \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\underline{g}^{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi - \frac{1}{2}\underline{g}^{\mu\nu}\kappa^2\phi^2, \tag{2.4.1.8}$$

and (2.4.1.3) becomes

$$\partial_\mu \Theta^{\mu\nu} = 0 \qquad (\nu = 0,1,2,3). \tag{2.4.1.9}$$

We perform the same changes of variables in the equation (2.3.2.7) and expand the covariation differentiation in terms of coordinate derivatives and the Christoffel symbols (2.4.1.2), arriving at the equation

$$\partial_\mu \left(n e^{3\phi} U^\mu \right) = 0. \tag{2.4.1.10}$$

For our purposes below, we take as our equations the projections of (2.4.1.9) onto the orthogonal complement of U and in the direction of U. In this formulation, the form of the EN_κ system is that of the ordinary relativistic Euler equations (as presented in [16]) with inhomogeneous terms involving $D\phi$, supplemented by the linear Klein-Gordon equation (2.3.3.3) for ϕ. Thus, we introduce Π, the projection onto the orthogonal complement of U, given by

$$\Pi^{\mu\nu} \stackrel{\text{def}}{=} U^\mu U^\nu + \underline{g}^{\mu\nu}. \tag{2.4.1.11}$$

Considering first the projection of (2.4.1.9) in the direction of U, we remark that one may use (2.3.3.4) and (2.4.1.10) to conclude that for C^1 solutions, $U_\nu \partial_\mu \Theta^{\mu\nu} = 0$ is equivalent to

$$U^\mu \partial_\mu \eta = 0, \tag{2.4.1.12}$$

which implies that the entropy density η is constant along the integral curves of U.

The projection of (2.4.1.9) onto the orthogonal complement of U gives 4 equations, only 3 of which are independent:

$$(R+P)U^\mu \partial_\mu U^\nu + \Pi^{\mu\nu}\partial_\mu P = -(\Box\phi - \kappa^2\phi)\Pi^{\mu\nu}\partial_\mu \phi \qquad (\nu = 0,1,2,3). \tag{2.4.1.13}$$

By (2.3.3.9), (2.4.1.5) and (2.4.1.6), we may solve for R as a function $\mathfrak{R}$ of $P, \eta,$ and $\phi:$

$$R = \mathfrak{R}(\eta, P, \phi) \stackrel{\text{def}}{=} e^{4\phi}\mathcal{R}(\eta, e^{-4\phi}P). \tag{2.4.1.14}$$

We also introduce the nameless quantity Q and make use of (2.3.3.4), (2.3.3.6), (2.3.3.8), (2.3.3.9), (2.4.1.5), and (2.4.1.6) to express it as a function $\mathfrak{Q}$ of $P, \eta,$ and $\phi:$

$$\begin{aligned} Q = \mathfrak{Q}(P,\eta,\phi) \stackrel{\text{def}}{=} n\,\frac{\partial P}{\partial n}\bigg|_{\eta,\phi} &= \frac{\partial P}{\partial R}\bigg|_{n,\phi}\cdot n\,\frac{\partial R}{\partial n}\bigg|_{\eta,\phi} = e^{4\phi}\mathcal{S}^2(p,\eta)(\rho+p) \\ &= \mathcal{S}^2(\eta, e^{-4\phi}P,)[\mathfrak{R}(\eta,P,\phi)+P]. \end{aligned} \tag{2.4.1.15}$$

Then we use the chain rule together with (2.4.1.10), (2.4.1.12), and (2.4.1.15) to derive

$$U^\mu \partial_\mu P + Q\partial_\mu U^\mu = (4P - 3Q)U^\mu \partial_\mu \phi, \tag{2.4.1.16}$$

which we may use in place of (2.4.1.10).

Deleting the redundant equation from (3.1.2.21), using (2.4.1.7) to derive the relation

$$\partial_\lambda U^0 = \frac{U_k}{U^0}\partial_\lambda U^k, \tag{2.4.1.17}$$

and rewriting the linear Klein-Gordon equation for ϕ as an equivalent first order system, the working form of the EN_κ system that we adopt is

$$U^\mu \partial_\mu \eta = 0 \tag{2.4.1.18}$$

$$U^\mu \partial_\mu P + Q\frac{U_k}{U^0}\partial_0 U^k + Q\partial_k U^k = (4P - 3Q)U^\mu \psi_\mu \tag{2.4.1.19}$$

$$(R + P)U^\mu \partial_\mu U^j + \Pi^{\mu j}\partial_\mu P = (3P - R)\Pi^{\mu j}\psi_\mu \qquad (j = 1, 2, 3) \tag{2.4.1.20}$$

$$-\partial_0 \psi_0 + \partial^j \psi_j = \kappa^2 \phi + R - 3P \tag{2.4.1.21}$$

$$\partial_0 \psi_j - \partial_j \psi_0 = 0 \qquad (j = 1, 2, 3) \tag{2.4.1.22}$$

$$\partial_0 \phi = \psi_0. \tag{2.4.1.23}$$

Here, U^0, R, and Q are expressed in terms of the unknowns through the relations

$$U^0 = (1 + U^k U_k)^{1/2} \tag{2.4.1.24}$$

$$Q = \mathfrak{Q}(\eta, P, \phi) \tag{2.4.1.25}$$

$$R = \mathfrak{R}(\eta, P, \phi), \tag{2.4.1.26}$$

where the function $\mathfrak{Q}$ is defined in (2.4.1.15), and the function $\mathfrak{R}$ is defined in (2.4.1.14).

In our rewriting of the linear Klein-Gordon as a first order system, we treat $\psi_\nu \overset{\text{def}}{=} \partial_\nu \phi$ as separate unknowns for $\nu = 0, 1, 2, 3$. To simplify notation, we collect the unknowns $\mathbf{V}$ together into an array[r] given by

$$\mathbf{V} \overset{\text{def}}{=} (\eta, P, U^1, U^2, U^3, \phi, \psi_0, \psi_1, \psi_2, \psi_3) \tag{2.4.1.27}$$

and we refer to the first five components of $\mathbf{V}$ as

$$\mathbf{W} \overset{\text{def}}{=} (\eta, P, U^1, U^2, U^3). \tag{2.4.1.28}$$

2.4.2 Linearization and the Equations of Variation (EOV)

The standard techniques for proving well-posedness require the linearization of the EN_κ system around a known background solution, which we refer to as a "BGS." Each BGS $\widetilde{\mathbf{V}} : \mathcal{M} \to \mathbb{R}^{10}$ we consider is of the form $\widetilde{\mathbf{V}} = (\widetilde{\eta}, \widetilde{P}, \cdots, \widetilde{\psi}_2, \widetilde{\psi}_3)$. The resulting system is

[r] Although every array appearing in this work is a $q \times 1$ column vector, we write them as if they were row vectors to save space.

known as the *equations of variation* (EOV). Thus, given such a $\widetilde{\mathbf{V}}$ and inhomogeneous terms $f, g, \cdots, l^{(4)}$, we define the EOV by

$$\widetilde{U}^\mu \partial_\mu \dot{\eta} = f \tag{2.4.2.1}$$

$$\widetilde{U}^\mu \partial_\mu \dot{P} + \widetilde{Q}\frac{\widetilde{U}_k}{\widetilde{U}^0}\partial_0 \dot{U}^k + \widetilde{Q}\partial_k \dot{U}^k = g \tag{2.4.2.2}$$

$$(\widetilde{R} + \widetilde{P})\widetilde{U}^\mu \partial_\mu \dot{U}^j + \widetilde{\Pi}^{\mu j}\partial_\mu \dot{P} = h^{(j)} \qquad (j = 1, 2, 3) \tag{2.4.2.3}$$

$$-\partial_0 \dot{\psi}_0 + \partial^j \dot{\psi}_j = l^{(0)} \tag{2.4.2.4}$$

$$\partial_0 \dot{\psi}_j - \partial_j \dot{\psi}_0 = l^{(j)} \qquad (j = 1, 2, 3) \tag{2.4.2.5}$$

$$\partial_0 \dot{\phi} = l^{(4)}, \tag{2.4.2.6}$$

where

$$\widetilde{U}^0 \stackrel{\text{def}}{=} (1 + \widetilde{U}^k \widetilde{U}_k)^{1/2} \tag{2.4.2.7}$$

$$\widetilde{\Pi}^{\mu\nu} \stackrel{\text{def}}{=} \widetilde{U}^\mu \widetilde{U}^\nu + \underline{g}^{\mu\nu} \tag{2.4.2.8}$$

$$\widetilde{Q} \stackrel{\text{def}}{=} \mathfrak{Q}(\widetilde{\eta}, \widetilde{P}, \widetilde{\phi}) \tag{2.4.2.9}$$

$$\widetilde{R} \stackrel{\text{def}}{=} \mathfrak{R}(\widetilde{\eta}, \widetilde{P}, \widetilde{\phi}). \tag{2.4.2.10}$$

Here, the function $\mathfrak{Q}$ is defined in (2.4.1.15), and the function $\mathfrak{R}$ is defined in (2.4.1.14).

The unknowns are the components of $\dot{\mathbf{V}} \stackrel{\text{def}}{=} (\dot{\eta}, \dot{P}, \cdots, \dot{\psi}_2, \dot{\psi}_3)$, and we label the first five components of $\dot{\mathbf{V}}$ by $\dot{\mathbf{W}} \stackrel{\text{def}}{=} (\dot{\eta}, \dot{P}, \dot{U}^1, \dot{U}^2, \dot{U}^3)$.

Remark 2.4.1. We place parentheses around the superscripts of the inhomogeneous terms $h^{(j)}$ and $l^{(z)}$ in order to emphasize that we are merely labelling them, and that in general, they do not transform covariantly under changes of coordinates.

The EOV play multiple roles in this article. Except when discussing the space of variations $\dot{\mathbf{V}}$ as an abstract vector-space isomorphic to $\mathbb{R}^{10}$, we use the symbol $\dot{\mathbf{V}}$ to represent a quantity that solves the EOV. The quantity represented by $\dot{\mathbf{V}}$, the BGS $\widetilde{\mathbf{V}}$, and the inhomogeneous terms will vary from application to application, but we will always be clear about their definitions in the relevant sections.

In the case that we are discussing the linearization of the EN_κ system around a BGS $\widetilde{\mathbf{V}}$ (a situation which we took as our motivation for introducing the EOV), the inhomogeneous terms take the form

$$f = \mathfrak{F}(\widetilde{\mathbf{V}}) \stackrel{\text{def}}{=} 0 \tag{2.4.2.11}$$

$$g = \mathfrak{G}(\widetilde{\mathbf{V}}) \stackrel{\text{def}}{=} (4\widetilde{P} - 3\widetilde{Q})\widetilde{U}^{\mu}\widetilde{\psi}_{\mu} \tag{2.4.2.12}$$

$$h^{(j)} = \mathfrak{H}^{(j)}(\widetilde{\mathbf{V}}) \stackrel{\text{def}}{=} (3\widetilde{P} - \widetilde{R})\widetilde{\Pi}^{\mu j}\widetilde{\psi}_{\mu} \qquad (j = 1, 2, 3) \tag{2.4.2.13}$$

$$l^{(0)} = \mathfrak{L}^{(0)}(\widetilde{\mathbf{V}}) \stackrel{\text{def}}{=} \kappa^2\widetilde{\phi} + \widetilde{R} - 3\widetilde{P} \tag{2.4.2.14}$$

$$l^{(j)} = \mathfrak{L}^{(j)}(\widetilde{\mathbf{V}}) \stackrel{\text{def}}{=} 0 \qquad (j = 1, 2, 3) \tag{2.4.2.15}$$

$$l^{(4)} = \mathfrak{L}^{(4)}(\widetilde{\mathbf{V}}) \stackrel{\text{def}}{=} \widetilde{\psi}_0, \tag{2.4.2.16}$$

where $\mathfrak{F}, \mathfrak{G}, \cdots, \mathfrak{L}^{(4)}$ are functions of $\widetilde{\mathbf{V}}$.

It is quite important that the coordinate derivatives of solutions to (2.4.2.1) - (2.4.2.6) also satisfy (2.4.2.1) - (2.4.2.6) with different inhomogeneous terms. This may be seen by differentiating the equations and relegating all but the principal terms to the right-hand side. Similarly, the difference of two solutions to (2.4.2.1) - (2.4.2.6) also satisfies (2.4.2.1) - (2.4.2.6). Thus, the "$\cdot$" is a suggestive placeholder that will frequently represent "derivative" or "difference" depending on the application.

Notation. In reference to the inhomogeneous terms on the right-hand side of (2.4.2.11) - (2.4.2.16), we often use matrix notation including but not limited to

$$\mathbf{b} = (f, g, h^{(1)}, h^{(2)}, h^{(3)}) \tag{2.4.2.17}$$

$$\mathbf{l} = (l^{(0)}, l^{(1)}, l^{(2)}, l^{(3)}, l^{(4)}). \tag{2.4.2.18}$$

When it is convenient, we will use different matrix notation to refer to the inhomogeneous terms, but we always use the notation $f, g, \cdots, l^4$ to refer to the inhomogeneous terms in scalar form; our use of notation for the inhomogeneous terms will always be made clear in the relevant sections.

Terminology. If $\dot{\mathbf{V}}$ is a solution to the system (2.4.2.1) - (2.4.2.6), we say that $\dot{\mathbf{V}}$ is a solution to the EOV defined by the BGS $\widetilde{\mathbf{V}}$ with inhomogeneous terms $(\mathbf{b}, \mathbf{l})$.

When the EOV describe the linearization of the EN$_\kappa$ system around a given BGS $\widetilde{\mathbf{V}}$, in which case the inhomogeneous terms are given by (2.4.2.11) - (2.4.2.16), we say that $\dot{\mathbf{V}}$ is a solution to the linearization of the EN$_\kappa$ system around $\widetilde{\mathbf{V}}$.

Remark 2.4.2. Solutions $\mathbf{V}$ to the EN_κ system (2.4.1.18) - (2.4.1.23) are also solutions to the EOV defined by the BGS $\mathbf{V}$ with inhomogeneous terms given by the right-hand side of (2.4.1.18) - (2.4.1.23).

Notation. We will often find it advantageous to abbreviate the "upper half" of the various systems[s] in this work using matrix notation. For example, we sometimes write (2.4.2.1) - (2.4.2.3) as

$$A^\mu(\widetilde{\mathbf{V}})\partial_\mu \dot{\mathbf{W}} = \mathbf{b} \tag{2.4.2.19}$$

where each $A^\mu(\widetilde{\mathbf{V}})$ is a 5×5 matrix with entries that are functions of the BGS $\widetilde{\mathbf{V}}$, while $\mathbf{b}$ is defined by (2.4.2.17). For instance,

$$A^0(\widetilde{\mathbf{V}}) = \begin{pmatrix} \widetilde{U}^0 & 0 & 0 & 0 & 0 \\ 0 & \widetilde{U}^0 & \widetilde{Q}\widetilde{U}^1/\widetilde{U}^0 & \widetilde{Q}\widetilde{U}^2/\widetilde{U}^0 & \widetilde{Q}\widetilde{U}^3/\widetilde{U}^0 \\ 0 & \widetilde{\Pi}^{01} & (\widetilde{R}+\widetilde{P})\widetilde{U}^0 & 0 & 0 \\ 0 & \widetilde{\Pi}^{02} & 0 & (\widetilde{R}+\widetilde{P})\widetilde{U}^0 & 0 \\ 0 & \widetilde{\Pi}^{03} & 0 & 0 & (\widetilde{R}+\widetilde{P})\widetilde{U}^0 \end{pmatrix}, \tag{2.4.2.20}$$

and similarly for the $A^k(\widetilde{\mathbf{V}})$, for $k = 1, 2, 3$.

Remark 2.4.3. As is suggested by the above notation, we find it useful to view the matrices A^μ, for $\mu = 0, 1, 2, 3$, as functions of the *state-space* variables, writing "$A^\mu(\cdot)$" to emphasize this point of view. We state for emphasis that *the functions* $A^\mu(\cdot)$ *remain fixed throughout this chapter.*

Remark 2.4.4. A calculation gives that $\det\big(A^0(\widetilde{\mathbf{V}})\big) = -\widetilde{Q}(\widetilde{R}+\widetilde{P})^2(\widetilde{U}^0)^3$, and in the Cauchy problem studied below, this formula will ensure that A^0 is invertible.

2.5 The Geometry of the EN_κ System

In this section, we discuss the geometry of the characteristics of the EN_κ system and relate the geometry to the speeds of propagation.

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[s]We reserve the use of matrix notation for the "upper half" for two reasons. The first is that the "lower half" involves constant coefficient differential operators, so when differentiating the "lower half" equations, we don't have to worry about commutator terms, which are easily expressed using matrix notation as in (2.7.2.21), arising from differential operators acting on the coefficients. The second reason is that in Chapter 3, we study the "lower-half" in its original form as an inhomogeneous Klein-Gordon equation, but we will still use matrix notation for the "upper-half."

2.5.1 The Symbol and the Characteristic Subset of $T_x^*\mathcal{M}$

The *symbol* σ_ξ of the equations of variation at a given covector $\xi \in T_x^*\mathcal{M}$, the cotangent space of $\mathcal{M}$ at x, is a linear-algebraic operator (multiplication by a matrix) on the space of variations $\dot{\mathbf{V}}$. This operator is obtained by making the replacements $\partial_\lambda \mathcal{U} \longrightarrow \xi_\lambda \dot{\mathcal{U}}$ on the left-hand side of the system (2.4.2.1) - (2.4.2.6). Here, $\mathcal{U}$ stands for any of the unknowns. The *characteristic subset of the cotangent space* at x is defined to be the set of all covectors $\xi \in T_x^*\mathcal{M}$ such that σ_ξ has a nontrivial null space. Thus, ξ lies in the characteristic subset of $T_x^*\mathcal{M}$ iff the following *algebraic* system has non-zero solutions $\dot{\mathbf{V}} \subset \mathbb{R}^{10}$:

$$\widetilde{U}^\mu \xi_\mu \dot{\eta} = 0 \tag{2.5.1.1}$$

$$\widetilde{U}^\mu \xi_\mu \dot{P} + \widetilde{Q}\frac{\widetilde{U}_k}{\widetilde{U}^0}\xi_0 \dot{U}^k + \widetilde{Q}\xi_k \dot{U}^k = 0 \tag{2.5.1.2}$$

$$(\widetilde{R} + \widetilde{P})\widetilde{U}^\mu \xi_\mu \dot{U}^j + \widetilde{\Pi}^{\mu j}\xi_\mu \dot{P} = 0 \qquad (j = 1, 2, 3) \tag{2.5.1.3}$$

$$\xi_\mu \dot{\psi}^\mu = 0 \tag{2.5.1.4}$$

$$\xi_0 \dot{\psi}_j - \xi_j \dot{\psi}_0 = 0 \qquad (j = 1, 2, 3) \tag{2.5.1.5}$$

$$\xi_0 \dot{\phi} = 0. \tag{2.5.1.6}$$

The determinant of the linear operator σ_ξ at x, known as the *characteristic form* of the EOV and denoted by $\mathcal{Q}(x;\xi)$, is given by

$$\mathcal{Q}(x;\xi) \overset{\text{def}}{=} (\xi_0)^3 \left(\widetilde{U}^\lambda \xi_\lambda\right)^3 (\widetilde{h}^{-1})^{\mu\nu} \underline{g}^{\alpha\beta} \xi_\mu \xi_\nu \xi_\alpha \xi_\beta, \tag{2.5.1.7}$$

where $\widetilde{h}^{-1}$ is the *reciprocal acoustical metric* defined by

$$(\widetilde{h}^{-1})^{\mu\nu} \overset{\text{def}}{=} \widetilde{\Pi}^{\mu\nu} - \widetilde{\sigma}^{-2}\widetilde{U}^\mu \widetilde{U}^\nu = \underline{g}^{\mu\nu} - \left(\widetilde{\sigma}^{-2} - 1\right)\widetilde{U}^\mu \widetilde{U}^\nu, \tag{2.5.1.8}$$

$$\widetilde{\sigma} \overset{\text{def}}{=} \mathcal{S}(e^{-4\widetilde{\phi}}\widetilde{P}, \widetilde{\eta}), \tag{2.5.1.9}$$

and the function $\mathcal{S}$ is defined by (2.3.3.8). The characteristic subset of T_x^* is therefore equal to the level set

$$\{\xi \in T_x^*\mathcal{M} | \mathcal{Q}(x;\xi) = 0\}. \tag{2.5.1.10}$$

Consequently, ξ is an element of the characteristic subset of $T_x^*\mathcal{M}$ iff one of the following four conditions holds.

$$\xi_\mu \widetilde{U}^\mu = 0 \tag{2.5.1.11}$$

$$(\widetilde{h}^{-1})^{\mu\nu}\xi_\mu\xi_\nu = 0 \tag{2.5.1.12}$$

$$\underline{g}^{\mu\nu}\xi_\mu\xi_\nu = 0 \tag{2.5.1.13}$$

$$\xi_0 = 0. \tag{2.5.1.14}$$

Condition (2.5.1.11) defines a plane $P^*_{x,\widetilde{U}}$ in $T_x^*\mathcal{M}$, while conditions (2.5.1.12) and (2.5.1.13) define cones $C^*_{x,s(ound)}$ and $C^*_{x,l(ight)}$, respectively, in $T_x^*\mathcal{M}$. Condition (2.5.1.14) also defines a plane $P^*_{x,0}$ in $T_x^*\mathcal{M}$, and its presence is a consequence of our choice of $\partial_t\phi$ as a state-space variable in our rewriting of the linear Klein-Gordon equation as a first order system. We refer to (2.5.1.11) - (2.5.1.14) as the four *sheets* of the characteristic subset of $T_x^*\mathcal{M}$. Figure 2.1 illustrates the characteristic subset of $T_x^*\mathcal{M}$. In the illustration, we masquerade as if the domain of solutions to the EOV is R^{1+2}, with the vertical direction representing positive values of ξ_0.

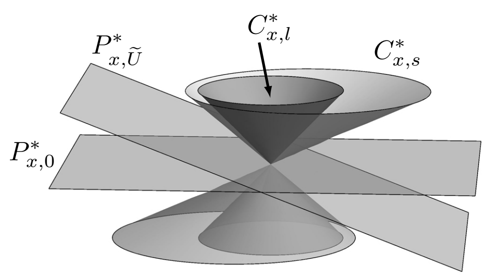

Figure 2.1: The Characteristic Subset of $T_x^*\mathcal{M}$

2.5.2 Characteristic Surfaces and the Characteristic Subset of $T_x\mathcal{M}$

A C^1 surface $S \subset \mathcal{M}$ given as a level set of a function φ is said to be a *characteristic surface* if at each point $x \in S$, the covector ξ with components $\xi_\nu = \partial_\nu\varphi$ for $\nu = 0, 1, 2, 3,$ is an element of the characteristic subset of $T_x^*\mathcal{M}$. It is well-known (consult e.g. [17]) that jump discontinuities in weak solutions can occur across characteristic surfaces, and that characteristic surfaces play a role in determining a domain of influence of a region of spacetime.

There is an alternative characterization of characteristic surfaces in terms of the duals of the sheets $P^*_{x,U}$, $P^*_{x,0}$, $C^*_{x,s}$, and $C^*_{x,l}$. The notion of duality we refer to is as follows (consult e.g. [17]): To each covector ξ in the characteristic subset of $T^*_x\mathcal{M}$ there corresponds the *null space* of ξ, which we denote by N_ξ. This 3-dimensional plane is a subset of $T_x\mathcal{M}$, the tangent space of $\mathcal{M}$ at x, and is described in coordinates as $N_\xi \stackrel{\text{def}}{=} \{X \in T_x\mathcal{M} | \xi_\mu X^\mu = 0\}$. We define the dual to a sheet of the characteristic subset of $T^*_x\mathcal{M}$ to be the envelope in $T_x\mathcal{M}$ generated by the N_ξ as ξ varies over the sheet. The *characteristic subset of the tangent space* at x is defined to be the union of the duals to the sheets (2.5.1.11) - (2.5.1.14). A calculation of the envelopes gives that the respective duals to (2.5.1.11), (2.5.1.12), (2.5.1.13), and (2.5.1.14) are the sets of $X \in T_x\mathcal{M}$ such that in our fixed rectangular coordinate system (see Section (2.2.1)),

$$X = \lambda\widetilde{U} \text{ for some } \lambda \in \mathbb{R} \tag{2.5.2.1}$$

$$\widetilde{h}_{\mu\nu}X^\mu X^\nu = 0 \tag{2.5.2.2}$$

$$\underline{g}_{\mu\nu}X^\mu X^\nu = 0 \tag{2.5.2.3}$$

$$X = \lambda(1,0,0,0) \text{ for some } \lambda \in \mathbb{R}, \tag{2.5.2.4}$$

where

$$\widetilde{h}_{\mu\nu} \stackrel{\text{def}}{=} \underline{g}_{\mu\nu} + (1-\widetilde{\sigma}^2)\widetilde{U}_\mu\widetilde{U}_\nu \tag{2.5.2.5}$$

is the *acoustical metric*, a non-degenerate quadratic form on the tangent space at x. The dual to $P^*_{x,\widetilde{U}}$, given by (2.5.2.1), is the linear span of $\widetilde{U}$, and the dual to the plane $P^*_{x,0}$, given by (2.5.2.4), is the linear span of $(1,0,0,0)$. The dual to $C^*_{x,s}$, given by (2.5.2.2) and labeled as $C_{x,s}$, is the sound cone in $T_x\mathcal{M}$, while the dual to $C^*_{x,l}$, given by (2.5.2.3) and labeled as $C_{x,l}$, is the light cone in $T_x\mathcal{M}$. We refer to these subsets of $T_x\mathcal{M}$ as the four sheets of the characteristic subset of the $T_x\mathcal{M}$ (noting that the degenerate cases (2.5.2.1) and (2.5.2.4) are lines rather than "sheets"). See Figure (2.2) for the picture in $\mathbb{R}^{1+2}$, where the vertical direction represents positive values of X^0.

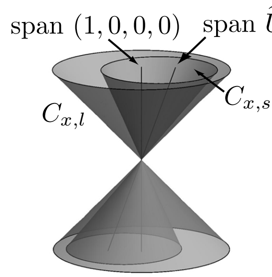

Figure 2.2: The Characteristic Subset of $T_x\mathcal{M}$

It follows from the above description that for each ξ belonging to a fixed sheet of the characteristic subset of $T_x^*\mathcal{M}$, N_ξ is tangent to the corresponding sheet of the characteristic subset of $T_x\mathcal{M}$. Therefore, we may equivalently define a characteristic surface as a surface S such that the tangent plane at each of its points x is tangent to any of the four sheets of the characteristic subset of $T_x\mathcal{M}$.

Remark 2.5.1. Note that $C_{x,s}$ lies inside $C_{x,l}$, but $C_{x,l}^*$ lies inside $C_{x,s}^*$.

2.5.3 Inner Characteristic Core, Strict Hyperbolicity, Spacelike Surfaces

The *inner characteristic core of the cotangent space* at x, denoted $\mathcal{I}_x^*$, is the subset of $T_x^*\mathcal{M}$ lying *strictly* inside the innermost sheet $C_{x,l}^*$. $\mathcal{I}_x^*$ comprises two components, and we refer to the component such that each covector ξ belonging to it has $\xi_0 > 0$ as the *positive component,* denoted by $\mathcal{I}_x^{*+}$:

$$\mathcal{I}_x^{*+} \stackrel{\text{def}}{=} \{\xi \in T_x^*\mathcal{M} | \xi_\mu \xi^\mu < 0 \text{ and } \xi_0 > 0\}. \tag{2.5.3.1}$$

A covector $\xi \in T_x^*\mathcal{M}$ is said to be *hyperbolic for $\mathcal{Q}$ at x* iff for any covector υ not parallel to ξ, $\mathcal{Q}(x;\lambda\xi+\upsilon)=0$ has real roots in λ, where $\mathcal{Q}$ is given in (2.5.1.7). The set of hyperbolic covectors at x is equal to $\mathcal{I}_x^*$; see Figure 2.1. A covector $\xi \in T_x^*\mathcal{M}$ is said to be *strictly hyperbolic for $\mathcal{Q}$ at x* iff for any covector υ not parallel to ξ, $\mathcal{Q}(x;\lambda\xi+\upsilon)=0$ has *distinct*[t] real roots in λ. As mentioned in Section 2.1, the EOV (and hence the EN$_\kappa$

[t]For PDEs derivable from a Lagrangian, the notion of hyperbolicity, characteristic subsets, etc., has

system) are (is) not strictly hyperbolic because of the repeated factors in the expression (2.5.1.7) for $\mathcal{Q}(x;\cdot)$.

A surface $S \subset \mathcal{M}$ is said to be *spacelike* (with respect to the light cone $C^*_{x,l}$) if at each $x \in S$, there is a covector ξ belonging to $\mathcal{I}^*_x$ such that the tangent plane to S at x is equal to N_ξ. Based on the discussion above, it follows that S is spacelike at x iff the tangent plane to S at x is the null space of a covector ξ that is hyperbolic for $\mathcal{Q}$ at x.

2.5.4 Speeds of Propagation

It is well-known that for first order symmetric hyperbolic systems, the speeds of propagation are locally governed by the characteristic subsets. For example, in the case that the characteristic subset of $T^*_x\mathcal{M}$ at each x includes an innermost sheet, the *domain of influence* of a spacetime point x' is contained in the interior of the forward conoid traced out by the set of all curves emanating from x' and remaining tangent to the sheets of the characteristic subsets of the $T_x\mathcal{M}$ that are dual to the innermost sheets of the characteristic subsets of the $T^*_x\mathcal{M}$ as the curve parameter varies; consult [34] for a detailed discussion of this fact.

We will later illustrate the occurrence of similar phenomena in the EN$_\kappa$ system. In this case, the innermost sheet at x is $C^*_{x,l}$, the dual of which is $C_{x,l}$, the light cone in $T_x\mathcal{M}$. Therefore, the forward conoid emanating from a spacetime point x' is the forward light cone in $\mathcal{M}$ with vertex at x'. Thus, one would expect that the fastest speed of propagation in the EN$_\kappa$ system is the speed of light. This claim is given rigorous meaning below in the uniqueness argument (see section 2.7.5) which shows, for example, that a solution that is constant in the Euclidean sphere of radius r centered at the point $\mathbf{s} \in \mathbb{R}^3$ at $t = 0$ remains constant in the Euclidean sphere of radius $r - t$ centered at $\mathbf{s}$ at time $t > 0$; see Remark 2.7.9.

We contrast this to the case of the ordinary relativistic Euler equations, in which there is no Klein-Gordon equation governing the propagation of gravitational waves at the speed of light, and the set $C^*_{x,l}$ does not belong to the characteristic subset of $T^*_x\mathcal{M}$.

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been generalized by Christodoulou [15] in a manner that allows one to handle characteristic forms that possess multiple roots.

The inner sheet at x in this case is $C^*_{x,s}$, the dual of which is $C_{x,s}$, the sound cone in $T_x\mathcal{M}$, and the methods applied below can be used to show that the fastest local speed of propagation is dictated by the sound cones $C_{x,s}$. This case is studied in detail in [15] and [16].

2.5.5 Energy Currents

The role of energy currents in the well-posedness proof is to replace the energy principle available for symmetric hyperbolic systems. After providing the definition of an energy current, we illustrate its two key properties, namely that it has the positivity property (2.5.5.2) below, and that its divergence is lower order in the variation $\dot{\mathbf{V}}$.

The Definition of an Energy Current

Given a variation $\dot{\mathbf{V}} : \mathcal{M} \rightarrow \mathbb{R}^{10}$ and a BGS $\widetilde{\mathbf{V}} : \mathcal{M} \rightarrow \mathbb{R}^{10}$ as defined in Section 2.4.2, we define the energy current to be the vectorfield $\dot{J}$ with components $\dot{J}^0$, $\dot{J}^j$, $j = 1, 2, 3$, in the global rectangular coordinate system given by

$$\begin{aligned}\dot{J}^0 &\stackrel{\text{def}}{=} \widetilde{U}^0\dot{\eta}^2 + \frac{\widetilde{U}^0}{\widetilde{Q}}\dot{P}^2 + 2\frac{\widetilde{U}_k\dot{U}^k}{\widetilde{U}^0}\dot{P} + (\widetilde{R}+\widetilde{P})\widetilde{U}^0\Big[\dot{U}^k\dot{U}_k - \frac{(\widetilde{U}_k\dot{U}^k)^2}{(\widetilde{U}^0)^2}\Big] \\ &\quad + \frac{1}{2}\big[(\dot{\phi})^2 + (\dot{\psi}_0)^2 + (\dot{\psi}_1)^2 + (\dot{\psi}_2)^2 + (\dot{\psi}_3)^2\big],\end{aligned} \tag{2.5.5.1}$$

$$\dot{J}^j \stackrel{\text{def}}{=} \widetilde{U}^j\dot{\eta}^2 + \frac{\widetilde{U}^j}{\widetilde{Q}}\dot{P}^2 + 2\dot{U}^j\dot{P} + (\widetilde{R}+\widetilde{P})\widetilde{U}^j\Big[\dot{U}^k\dot{U}_k - \frac{(\widetilde{U}_k\dot{U}^k)^2}{(\widetilde{U}^0)^2}\Big] - \dot{\psi}_0\dot{\psi}_j.$$

Notation. In an effort to avoid cluttering the notation, we sometimes suppress the direct dependence of $\dot{J}$ on $\dot{\mathbf{V}}$ and $\widetilde{\mathbf{V}}$ and instead emphasize the indirect dependence of $\dot{J}$ on $(t, \mathbf{s})$ through $\dot{\mathbf{V}}$ and $\widetilde{\mathbf{V}}$ by writing "$\dot{J}(t, \mathbf{s})$."

Terminology. We say that $\dot{J}$ is the energy current for the variation $\dot{\mathbf{V}}$ with coefficients defined by the BGS $\widetilde{\mathbf{V}}$.

Remark 2.5.2. The theory of hyperbolic PDEs derivable from a Lagrangian, and in particular the derivation of energy currents, is developed by Christodoulou in [15]. For readers interested in studying Christodoulou's techniques, we remark that the Lagrangian for (2.4.1.18) - (2.4.1.20) (the first 5 scalar equations of the EN_κ system) is expressed in the original variables as $\rho e^{4\phi}$. The energy current (2.5.5.1) is the sum of an energy current for the linear Klein-Gordon equation (which supplies the terms involving $(\dot{\phi})^2$ and $(\dot{\psi}_\nu)^2$) and an energy current used by Christodoulou in [16] to study the relativistic Euler equations without gravitational interaction.

The Positive Definiteness of $\xi_\mu \dot{J}^\mu$ for $\widetilde{P} > 0$ and $\xi \in \mathcal{I}_x^{*+}$

Given an energy current as defined by (2.5.5.1) and a covector $\xi \in T_x^*\mathcal{M}$, the quantity $\xi_\mu \dot{J}^\mu$ may be viewed as a quadratic form in the variations $\dot{\mathbf{V}}$ with coefficients defined by the BGS $\widetilde{\mathbf{V}}$. We emphasize this quadratic dependence on the variations by writing $\xi_\mu \dot{J}^\mu(\dot{\mathbf{V}}, \dot{\mathbf{V}})$. One of the two key features of the energy current is that $\widetilde{P} > 0$ and $\xi \in \mathcal{I}_x^{*+}$ together imply that the form $\xi_\mu \dot{J}^\mu(\dot{\mathbf{V}}, \dot{\mathbf{V}})$ is positive definite[u] in $\dot{\mathbf{V}}$:

$$\xi_\mu \dot{J}^\mu(\dot{\mathbf{V}}, \dot{\mathbf{V}}) > 0 \text{ if } \xi \in \{\zeta \in T_x^*(\mathcal{M}) \mid \zeta_\mu \zeta^\mu < 0 \text{ and } \zeta_0 > 0\} \text{ and } \dot{\mathbf{V}} \neq \mathbf{0}. \qquad (2.5.5.2)$$

This fact will allow us to use the form $\xi_\mu \dot{J}^\mu(\dot{\mathbf{V}}, \dot{\mathbf{V}})$ to estimate the L^2 norms of the variations, provided that we estimate the BGS $\widetilde{\mathbf{V}}$.

Remark 2.5.3. Although later in this article we make use of the fact that $\dot{\mathbf{V}}$ is a solution to the EOV, the inequality in (2.5.5.2) does not rely on this fact; it is an algebraic statement about $\xi_\mu \dot{J}^\mu(\dot{\mathbf{V}}, \dot{\mathbf{V}})$ viewed as a quadratic form on $\mathbb{R}^{10}$.

Remark 2.5.4. Note that according to the expression given in (2.5.5.1), the coefficients of the quadratic form $\xi_\mu \dot{J}^\mu(\dot{\mathbf{V}}, \dot{\mathbf{V}})$ depend smoothly on the BGS $\widetilde{\mathbf{V}}$.

The Divergence of the Energy Current

If the variations $\dot{\mathbf{V}}$ are solutions of the EOV (2.4.2.1) - (2.4.2.6) then we can compute $\partial_\mu \dot{J}^\mu$ and use the equations (2.4.2.1) - (2.4.2.6) for substitution to eliminate the terms containing the derivatives of $\dot{\mathbf{V}}$:

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[u] A direct verification of this fact can be done, for example, by calculating the eigenvalues of the matrix of the quadratic form $\xi_\mu \dot{J}^\mu(\cdot, \cdot)$. The eigenvalues depend on ξ and are positive whenever $\widetilde{P} > 0$ and $\xi \in \mathcal{I}_x^{+*}$.

$$\begin{aligned}\partial_\mu \dot{J}^\mu &= (\partial_\mu \widetilde{U}^\mu)\dot{\eta}^2 + \partial_\mu \left(\frac{\widetilde{U}^\mu}{\widetilde{Q}}\right)\dot{P}^2 + 2\partial_0\left(\frac{\widetilde{U}_k}{\widetilde{U}^0}\right)\dot{U}^k\dot{P} \\ &\quad + \partial_\mu[(\widetilde{R}+\widetilde{P})\widetilde{U}^\mu]\left[\dot{U}^k\dot{U}_k - \frac{(\widetilde{U}_k\dot{U}^k)^2}{(\widetilde{U}^0)^2}\right] - 2\widetilde{U}_k\dot{U}^k(\widetilde{R}+\widetilde{P})\left(\frac{\widetilde{U}^\mu}{\widetilde{U}^0}\right)\partial_\mu\left(\frac{\widetilde{U}_j}{\widetilde{U}^0}\right)\dot{U}^j \\ &\quad + 2\dot{\eta}f + 2\frac{\dot{P}g}{\widetilde{Q}} + 2\dot{U}_k h^{(k)} - 2\frac{\widetilde{U}_j h^{(j)}\widetilde{U}_k\dot{U}^k}{(\widetilde{U}^0)^2} - \ \dot{\psi}_0 l^{(0)} \ + \ \dot{\psi}_k l^{(k)} \ + \ \dot{\phi} l^{(4)}.\end{aligned} \tag{2.5.5.3}$$

That the right-hand side of (2.5.5.3) does not contain any derivatives of the variations is the second key property announced at the beginning of Section 2.5.5.

Remark 2.5.5. Given a spatial derivative multi-index $\vec{\alpha}$ and an energy current $\dot{J}$ as defined in (2.5.5.1) such that the variation $\dot{\mathbf{V}}$ is a solution of (2.4.2.1) - (2.4.2.6) with inhomogeneous terms $(\mathbf{b}, \mathbf{l})$, where $\mathbf{b}$ and $\mathbf{l}$ are defined by (2.4.2.17) and (2.4.2.18) respectively, we define the *higher-order energy current* $\dot{J}_{\vec{\alpha}}$ to be the energy current for the variation $\partial_{\vec{\alpha}}\dot{\mathbf{V}}$ with coefficients defined by the same BGS $\widetilde{\mathbf{V}}$. The variations $\partial_{\vec{\alpha}}\dot{\mathbf{V}}$ are solutions of (2.4.2.1) - (2.4.2.6) with inhomogeneous terms $(\mathbf{b}_{\vec{\alpha}}, \partial_{\vec{\alpha}}\mathbf{l})$, where $\mathbf{b}_{\vec{\alpha}}$ is defined in terms of $\mathbf{b}$ below through (2.7.2.20). Consequently, the expression for $\partial_\mu \dot{J}^\mu_{\vec{\alpha}}$ is given by taking the formula (2.5.5.3) for $\partial_\mu \dot{J}^\mu$ and making the replacements $\dot{\mathbf{V}} \to \partial_{\vec{\alpha}}\dot{\mathbf{V}}$ and $(\mathbf{b}, \mathbf{l}) \to (\mathbf{b}_{\vec{\alpha}}, \partial_{\vec{\alpha}}\mathbf{l})$.

2.6 Assumptions on the Initial Data

We now describe a class of initial data to which the energy methods for showing well-posedness can be applied. The Cauchy surface we consider is $\{(t, \mathbf{s}) \in \mathcal{M} \mid t = 0\}$.

2.6.1 An H^N Perturbation of a Quiet Fluid

The initial data for the EN_κ system are denoted by $\mathring{\mathbf{V}} = \mathring{\mathbf{V}}(\mathbf{s}) \stackrel{\text{def}}{=} (\mathring{\eta}, \mathring{P}, \mathring{U}^1, \cdots, \mathring{\psi}_3)$, where $\mathring{\psi}_j \stackrel{\text{def}}{=} \partial_j \mathring{\phi}$ for $j = 1, 2, 3$. We assume that the initial data $\mathring{\mathbf{V}}$ for the EN_κ system are constructed from initial data $(\mathring{\eta}, \mathring{p}, \mathring{u}^1, \cdots, \mathring{\psi}_3)$ in the original state-space variables $(\eta, p, u^1, \cdots, \psi^3)$ according to the substitutions (2.4.1.4), (2.4.1.5), and (2.4.1.6). Additionally, we assume that outside of the unit ball centered at the origin in the Cauchy surface[v]

$$\mathring{\mathbf{V}} \equiv \bar{\mathbf{V}} \overset{\text{def}}{=} (\bar{\eta}, \bar{P}, 0, 0, 0, \bar{\phi}, 0, 0, 0, 0), \tag{2.6.1.1}$$

where $\bar{\phi}$ is the unique solution to

$$\kappa^2 \bar{\phi} + e^{4\bar{\phi}} \left(\mathcal{R}(\bar{\eta}, \bar{p}) - 3\bar{p}\right) = 0, \tag{2.6.1.2}$$

$\bar{\eta}$ and $\bar{p}$ are positive constants denoting the initial entropy and pressure of the fluid outside of the unit ball, $\bar{P} \overset{\text{def}}{=} e^{4\bar{\phi}}\bar{p}$, and the function $\mathcal{R}$ is defined in (2.3.3.9). An initial state of this form is a perturbation of an infinitely extended quiet fluid, such that the perturbation is initially contained in the unit ball. Here we need the cosmological constant $\kappa^2 > 0$ in order to ensure that the EN_κ system has non-zero constant solutions of the form $\bar{\mathbf{V}}$.

Because the standard energy methods require that the initial data belong to a Sobolev space of high enough order, we assume that

$$\|\mathring{\eta} - \bar{\eta}\|_{H^N} + \|\mathring{p} - \bar{p}\|_{H^N} + \|\mathring{u}^k\|_{H^N} + \|\mathring{\phi} - \bar{\phi}\|_{H^{N+1}} + \|\mathring{\psi}_0\|_{H^N} < \infty, \tag{2.6.1.3}$$

where $N \in \mathbb{N}$ satisfies

$$N \geq 3. \tag{2.6.1.4}$$

Note that (2.6.1.3) implies that $\|\mathring{\psi}_j\|_{H^N} < \infty$ $(j = 1, 2, 3)$. By Proposition B.0.2 and Remark B.0.3, it follows from (2.6.1.3) that

$$\|\mathring{\mathbf{V}}\|_{H^N_{\bar{\mathbf{V}}}} < \infty. \tag{2.6.1.5}$$

2.6.2 The Admissible Subset of State-Space and the Uniform Positive Definiteness of $\dot{J}^0$

The Definition of the Admissible Subset of State-Space

In order to avoid studying the free boundary problem and in order to avoid singularities in the energy current, we assume that the initial pressure, energy density, and speed

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[v] This assumption is not necessary. It is sufficient to consider initial data $\mathring{\mathbf{V}}$ that differ from $\bar{\mathbf{V}}$ by a perturbation belonging to H^N, such that that $\mathring{\mathbf{V}}(\mathbb{R}^3)$ is contained in a compact subset of $\mathcal{O}$, where N is given by (2.6.1.4) and $\mathcal{O}$ is defined in Section 2.6.2. We make this assumption because it is useful for illustrating the speeds of propagation as discussed in Section 2.5.4, and because we plan to make use of this setup in future work.

of sound are uniformly bounded from below by a positive constant. According to our assumptions (3.1.1.11) on the equation of state, it is sufficient to consider initial data for the EN_κ system such that $\mathring{\mathbf{V}}(\mathbb{R}^3)$ is contained in a compact subset of the following open subset $\mathcal{O}$ of the state-space $\mathbb{R}^{10}$, the *admissible subset of state-space:*

$$\mathcal{O} = \{\mathbf{V} \in \mathbb{R}^{10} | \eta > 0, P > 0\}. \tag{2.6.2.1}$$

Thus, we assume that $\bar{\mathbf{V}} \in \mathcal{O}_1$ and $\mathring{\mathbf{V}}(\mathbb{R}^3) \subset \mathcal{O}_1$, where $\mathcal{O}_1$ is a precompact, open, convex set[w] with $\bar{\mathcal{O}}_1 \Subset \mathcal{O}$. We then fix a precompact, open, convex subset[x] $\mathcal{O}_2$ where $\bar{\mathcal{O}}_1 \Subset \mathcal{O}_2 \Subset \mathcal{O}$; our goal is to show the existence of a solution that remains in $\bar{\mathcal{O}}_2$ for short times.

The Uniform Positive Definiteness of $\dot{J}^0$

Most of the technical exposition below is devoted to obtaining control over $\|\dot{\mathbf{V}}(t)\|_{H^N}$, where $\dot{\mathbf{V}}$ is a solution to the EOV defined by a BGS $\widetilde{\mathbf{V}}$. Instead of trying to estimate $\|\dot{\mathbf{V}}(t)\|_{H^N}$ directly, it is advantageous to estimate $\|\dot{J}^0(t)\|_{L^1}$, where $\dot{J}$ is an energy current for $\dot{\mathbf{V}}$ with coefficients defined by the BGS $\widetilde{\mathbf{V}}$, since the divergence of $\dot{J}$ is lower order in $\dot{\mathbf{V}}$. $\|\dot{J}^0(t)\|_{L^1}$ can be used to estimate $\|\dot{\mathbf{V}}(t)\|_{L^2}^2$ from above and below provided that $\dot{J}^0$ is uniformly positive definite independent of the BGS $\widetilde{\mathbf{V}}$. More precisely, we claim that there exists $C_{\bar{\mathcal{O}}_2}$ with $0 < C_{\bar{\mathcal{O}}_2} < 1$ such that for any variation $\dot{\mathbf{V}}$ and any BGS $\widetilde{\mathbf{V}}$ contained in $\bar{\mathcal{O}}_2$, we have

$$C_{\bar{\mathcal{O}}_2} |\dot{\mathbf{V}}|^2 \leq \dot{J}^0(\dot{\mathbf{V}}, \dot{\mathbf{V}}) \leq \frac{1}{C_{\bar{\mathcal{O}}_2}} |\dot{\mathbf{V}}|^2. \tag{2.6.2.2}$$

To prove (2.6.2.2), recall that $\dot{J}$ is defined by (2.5.5.1) and note that $(1, 0, 0, 0) \in \mathcal{I}_x^+$ by (2.5.5.2). The uniform continuity of $\dot{J}$ (which we momentarily view as a function of $(\widetilde{\mathbf{V}}, \dot{\mathbf{V}})$) on the compact set $\bar{\mathcal{O}}_2 \times \{|\dot{\mathbf{V}}| = 1\}$ implies that there exists $C_{\bar{\mathcal{O}}_2}$ with $0 < C_{\bar{\mathcal{O}}_2} < 1$ such that (2.6.2.2) holds whenever $\widetilde{\mathbf{V}}(t, \mathbf{s}) \in \bar{\mathcal{O}}_2$ and $|\dot{\mathbf{V}}| = 1$. Since the inequalities in (2.6.2.2) are invariant under any rescaling of $\dot{\mathbf{V}}$, it follows that we may remove the restriction $|\dot{\mathbf{V}}| = 1$.

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[w] In practice, $\mathcal{O}_1$ can be chosen to be a large, open cube.

[x] We demand convexity because Proposition B.0.4 requires this hypothesis.

2.7 The Well-Posedness Theorems

In this section, we state and prove the two main theorems of the chapter. We have separated the proof of well-posedness into two theorems since the techniques used in proving each are different. Statements of the technical estimates involving the Sobolev-Moser calculus have been placed in the Appendix so as to not interrupt the flow of the main argument.

Theorem 2.7.1. ***(Local Existence and Uniqueness)*** *Let* $\mathring{\mathbf{V}}(\mathbf{s})$ *be initial data for the* EN_κ *system* (2.4.1.18) - (2.4.1.23) *that are subject to the conditions described in Section 2.6. Then there exists* $T > 0$ *such that* (2.4.1.18) - (2.4.1.23) *has a unique classical solution* $\mathbf{V}(t, \mathbf{s})$ *on* $[0, T] \times \mathbb{R}^3$ *of the form* $\mathbf{V} = (\eta, P, U^1, U^2, U^3, \phi, \partial_0\phi, \partial_1\phi, \partial_2\phi, \partial_3\phi)$ *with* $\mathbf{V}(0, \mathbf{s}) = \mathring{\mathbf{V}}(\mathbf{s})$*. The solution satisfies* $\mathbf{V}([0, T] \times \mathbb{R}^3) \subset \bar{\mathcal{O}}_2$*. Furthermore,* $\mathbf{V} \in C^0([0, T], H^N_{\bar{\mathbf{V}}}) \cap C^1([0, T], H^{N-1}_{\bar{\mathbf{V}}})$*, and* $\phi \in C^0([0, T], H^{N+1}_{\bar{\phi}}) \cap C^1([0, T], H^N_{\bar{\phi}})$
$\cap C^2([0, T], H^{N-1}_{\bar{\phi}})$.

Remark 2.7.1. In the proof, we often refer to the solution from Theorem 2.7.1 as $\mathbf{V}_{sol}$ for clarity.

Corollary 2.7.1. *The interval of existence* $[0, T]$ *supplied by the Theorem 2.7.1 depends only on the set* $\bar{\mathcal{O}}_2$ *from Section 2.6,* $\|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\bar{\mathbf{V}}}}$*, and the constant* Λ *chosen in* (2.7.2.7) *and* (2.7.2.8) *below. Here,* ${}^{(0)}\mathring{\mathbf{V}}$ *denotes the mollified initial data as described in Section 2.7.2. Furthermore, the set* $\bar{\mathcal{O}}_2$*, the mollified initial data* ${}^{(0)}\mathring{\mathbf{V}}$*, and constant* Λ *can be chosen to be independent of all initial data varying in a small* H^N *neighborhood of* $\mathring{\mathbf{V}}$*. Therefore, if we define* $B_y(\mathring{\mathbf{V}}) \overset{def}{=} \{\mathring{\widetilde{\mathbf{V}}} \in H^N_{\bar{\mathbf{V}}} \mid \|\mathring{\widetilde{\mathbf{V}}} - \mathring{\mathbf{V}}\|_{H^N} < y\}$*, then there exist* $\delta > 0$ *and* $T' > 0$ *(depending on* $\mathring{\mathbf{V}}$*) such that any initial data* $\mathring{\widetilde{\mathbf{V}}}$ *belonging to* $B_\delta(\mathring{\mathbf{V}})$ *launch a unique solution* $\widetilde{\mathbf{V}}$ *that exists on the common time interval* $[0, T']$ *and that has the property* $\widetilde{\mathbf{V}}([0, T'] \times \mathbb{R}^3) \subset \bar{\mathcal{O}}_2$.

Proof. The corollary follows from the proof of Theorem 2.7.1. See in particular Remark 2.7.3 and Remark 2.7.4 below. □

Corollary 2.7.2. *The norms* $|||\ \mathbf{V}\ |||_{H^N_{\bar{\mathbf{V}}},T}$ *and* $|||\ \partial_t\mathbf{V}\ |||_{H^{N-1},T}$ *of the solution from Theorem 2.7.1 depend only* $\bar{\mathcal{O}}_2$*,* $\|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\bar{\mathbf{V}}}}$*, and* Λ*. Furthermore, there exists a* $K > 0$

such that any initial data $\mathring{\widetilde{\mathbf{V}}}$ *belonging to the set* $B_\delta(\mathring{\mathbf{V}})$ *defined in Corollary 2.7.1 launch a unique solution* $\widetilde{\mathbf{V}}$ *that satisfies the uniform bound*

$$|||\ \widetilde{\mathbf{V}}\ |||_{H^N_{\mathbf{V}},T'}, |||\ \partial_t\widetilde{\mathbf{V}}\ |||_{H^{N-1},T'} < K(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\mathbf{V}}}, \delta). \tag{2.7.0.3}$$

Proof. The estimates for $|||\ \mathbf{V}\ |||_{H^N_{\mathbf{V}},T}$ and $|||\ \widetilde{\mathbf{V}}\ |||_{H^N_{\mathbf{V}},T'}$ follow from Corollary 2.7.1, Proposition 2.7.3, and the fact that the sequence of iterates $\{{}^{(m)}\mathring{\mathbf{V}}(t)\}$ constructed below converges strongly in $L^2_{\mathbf{V}}$ and weakly in $H^N_{\mathbf{V}}$ to $\mathbf{V}(t)$; consult [36] for the missing details. We then use the EN$_\kappa$ equations to solve for the time derivatives together with Proposition B.0.2 and Remark B.0.3 to obtain the estimates for
$|||\ \partial_t\mathbf{V}\ |||_{H^{N-1},T}$ and $|||\ \partial_t\widetilde{\mathbf{V}}\ |||_{H^{N-1},T'}$. □

Theorem 2.7.2. ***(Continuous Dependence on Initial Data.)*** *Let* $\mathring{\mathbf{V}}(\mathbf{s})$ *be initial data for the EN*$_\kappa$ *system* (2.4.1.18) - (2.4.1.23) *that are subject to the conditions described in Section 2.6, and let* $\mathbf{V}$ *be the solution existing on the time interval* $[0,T]$ *furnished by Theorem 2.7.1. Let* $B_\delta(\mathring{\mathbf{V}})$ *be as in Corollary 2.7.1. Let* $\{\mathring{\mathbf{V}}^m\} \subset B_\delta$ *be a sequence of initial data with* $\lim_{m\to\infty}\|\mathring{\mathbf{V}}^m - \mathring{\mathbf{V}}\|_{H^N} = 0$, *and let* $\mathbf{V}^m$ *denote the solution to* (2.4.1.18) - (2.4.1.23) *launched by* $\mathring{\mathbf{V}}^m$. *Then for all large* m, *the solutions* $\mathbf{V}^m$ *exist on* $[0,T]$, *and* $\lim_{m\to\infty}|||\ \mathbf{V}^m - \mathbf{V}\ |||_{H^N,T} = 0$.

Remark 2.7.2. It is unknown to the author whether or not the continuity statement from Theorem 2.7.2 can be strengthened to one of Lipschitz continuity or Hölder continuity. However, using Burger's equation $\partial_t u + u\partial_x u = 0$, Kato [27] provides a counter-example in which the map from the initial data $u_0 \in H^a(\mathbb{R})$ to the solution $u(t)$ is not Hölder continuous with any positive exponent in the H^a norm; such a counterexample is explicitly constructed for $a \geq 2$. On the other hand, inequality (2.7.5.37) below shows that for the EN$_\kappa$ system, the map from the initial data to the solution is a Lipschitz map from $H^N_{\mathbf{V}}$ into $C([0,T], H^{N-1}_{\mathbf{V}})$.

2.7.1 A Discussion of the Structure of the Proof of the Theorems

We prove local existence by following a standard method described in detail in Majda's book [36]: we construct a sequence of iterates $\{{}^{(m)}\mathbf{V}(t,\mathbf{s})\}$ that converges to the solution

$\mathbf{V}_{sol}(t,\mathbf{s})$. To construct the iterates, we first define a sequence of C^∞ initial data $\{{}^{(m)}\mathring{\mathbf{V}}\}$ such that ${}^{(m)}\mathring{\mathbf{V}} \Subset \mathcal{O}_2$ and $\lim_{m\to\infty} ||| \, {}^{(m)}\mathring{\mathbf{V}} - \mathring{\mathbf{V}} \, |||_{H^N} = 0$. The advantage of smoothing the data is that all of the iterates are C^∞, thus allowing us to work with classical derivatives during the approximation process. Then beginning with ${}^{(0)}\mathbf{V}(t,\mathbf{s}) \stackrel{\text{def}}{=} {}^{(0)}\mathring{\mathbf{V}}(\mathbf{s})$, we inductively define ${}^{(m+1)}\mathbf{V}(t,\mathbf{s})$ as the unique solution to the linearization of the EN$_\kappa$ system around ${}^{(m)}\mathbf{V}(t,\mathbf{s})$ with initial data ${}^{(m+1)}\mathbf{V}(0,\mathbf{s}) = {}^{(m+1)}\mathring{\mathbf{V}}(\mathbf{s})$. As a consequence of the theory of linear PDEs[y] (consult [17]), each iterate ${}^{(m)}\mathbf{V}$ is known to possess a classical solution on a slab $[0,T_m]\times\mathbb{R}^3$, on which it satisfies, for every real number N', ${}^{(m)}\mathbf{V} \in C^0([0,T_m], H^{N'}_{\mathbf{V}})$. Additionally, we require of each T_m that ${}^{(m)}\mathbf{V}([0,T_m]\times\mathbb{R}^3) \subset \bar{\mathcal{O}}_2$, which ensures that the sequence of proper energy densities is bounded from below by a uniform constant and therefore precludes singularities in the energy currents we use during the linearization process.

In order for the limiting function $\mathbf{V}_{sol}$ to be defined on a slab, it is obviously necessary that we show that the sequence of time values $\{T_m\}$ can be bounded from below by a positive constant T_*. To this end, we examine the EOV satisfied by ${}^{(m)}\mathbf{V} - {}^{(0)}\mathring{\mathbf{V}}$ and its partial derivatives, and we control the growth in T_* of $||| \, {}^{(m)}\mathbf{V} - {}^{(0)}\mathring{\mathbf{V}} \, |||_{H^N,T_*}$ uniformly in m using energy currents. According to the above paragraph and the Sobolev imbedding result $H^2(\mathbb{R}^3) \subset C^0_b(\mathbb{R}^3)$, it follows that if $||| \, {}^{(m)}\mathbf{V} - {}^{(0)}\mathring{\mathbf{V}} \, |||_{H^N,T_*}$ is small enough, uniformly in m, then T_* may be selected as a uniform lower bound on the T_m. Our detailed proof of the control of the terms $||| \, {}^{(m)}\mathbf{V} - {}^{(0)}\mathring{\mathbf{V}} \, |||_{H^N,T_*}$ is given in Proposition 2.7.3 below and uses the Sobolev-Moser calculus inequalities, which are refined versions of the fact that for $N' > \frac{3}{2}$, $H^{N'}(\mathbb{R}^3)$ is a Banach algebra. Their purpose is to control the L^2 norms of terms of a product form, based on known Sobolev regularity of each factor in the product. We state and prove the relevant Sobolev-Moser estimates in the Appendix.

The next step in the proof is to show that $\{{}^{(m)}\mathbf{V}(t)\}$ converges in $L^2_{\mathbf{V}}(\mathbb{R}^3)$ to a limiting function $\mathbf{V}_{sol}(t)$ for $t \in [0,T]$, with an appropriate choice of T satisfying $0 <$

[y]The exposition on linear theory in [17] makes use of the symmetric hyperbolic setup to obtain energy estimates for the linear systems. We may obtain similar energy estimates for the linearized EN$_\kappa$ equations by using energy currents of the form (2.5.5.1); the proof of Proposition 2.7.3 below illustrates the relevant techniques.

$T \leq T_*$. This is achieved by showing that $\sum_{m=0}^{\infty} ||| \, {}^{(m+1)}\mathbf{V}(t) - {}^{(m)}\mathbf{V}(t) \, |||_{L^2,T} < \infty$ if T is small enough. We show the convergence of the sum by studying the EOV satisfied by the differences ${}^{(m+1)}\mathbf{V} - {}^{(m)}\mathbf{V}$ of successive iterates and their partial derivatives. Again, we control $||| \, {}^{(m+1)}\mathbf{V}(t) - {}^{(m)}\mathbf{V}(t) \, |||_{L^2,T}$ using energy currents and the Sobolev-Moser calculus, and by making use of the established bound on $||| \, {}^{(m)}\mathbf{V} - {}^{(0)}\mathring{\mathbf{V}} \, |||_{H^N,T_*}$.

After establishing the existence of the limiting function $\mathbf{V}_{sol}$, we show that it is a classical solution to the EN$_\kappa$ system (2.4.1.18) - (2.4.1.23). The key idea stems from Lemma B.0.6, a standard Sobolev interpolation result that allows us to conclude without much additional effort that ${}^{(m)}\mathbf{V} \rightarrow \mathbf{V}_{sol}$ in $C^0([0,T], H^{N'}_{\mathbf{V}}(\mathbb{R}^3))$, for every real number N' such that $N' < N$. By Sobolev imbedding, $H^{5/2+\epsilon}(\mathbb{R}^3) \subset C^1_b(\mathbb{R}^3)$, and thus ${}^{(m)}\mathbf{V} \rightarrow \mathbf{V}$ in $C^0([0,T], C^1_b)$. Combining this fact with Lemma 2.7.8, a simple result from advanced calculus, we are able to pass to the limit in the linearized EN$_\kappa$ equations to conclude that $\mathbf{V}_{sol}$ is a classical solution.

To finish the existence aspect of the proof, we show the additional regularity result $\mathbf{V}_{sol} \in C^0([0,T], H^N_{\mathbf{V}}) \cap C^1([0,T], H^{N-1}_{\mathbf{V}})$. We first show that $\mathbf{V}_{sol} - \mathring{\mathbf{V}} \in C_{weak}([0,T], H^N)$, a function space that we define in Section 2.7.5, and then use the Hilbert space structure of of H^N together with a collection of norms on $H^N(\mathbb{R}^3)^{10}$ defined through energy currents to conclude the additional regularity stated in Theorem 2.7.1.

In Section 2.7.5 we show uniqueness and continuous dependence on initial data in the $H^{N'}$ norm for $N' < N$. The methods used in this argument are similar to the methods used to prove Proposition 2.7.3, so we provide fewer details. We consider the EOV satisfied by the difference of two solutions $\mathbf{V}$ and $\widetilde{\mathbf{V}}$ to the EN$_\kappa$ system. For $N' < N$, we use an appropriately defined energy current to bound the growth of $||| \, \mathbf{V} - \widetilde{\mathbf{V}} \, |||_{H^{N'},T}$ by a constant times exponential growth in T. The constant depends on the initial data and is shown converge to 0 as $\|\mathbf{V}(0) - \widetilde{\mathbf{V}}(0)\|_{H^{N'}} \rightarrow 0$, thus implying uniqueness and $H^{N'}$ continuous dependence on the initial data. Our proof of Theorem 2.7.1 is complete at the end of this section.

Our proof of Theorem 2.7.2 requires some machinery from the theory of evolution equations in a Banach space. The basic method is due to Kato [27], and most of the technical results we use in this section are merely quoted from his papers. We find it worthwhile to prove Theorem 2.7.2 because aside from Kato's work, we have had difficulty locating this result in the literature.

2.7.2 Preliminary Estimates on the Iterates

As described in Section 2.7.1, we produce a sequence of iterates $\{^{(m)}\mathbf{V}(t,\mathbf{s})\}$ that converges to the solution $\mathbf{V}_{sol}(t,\mathbf{s})$.

Smoothing the Initial Data

We begin by smoothing the initial data, which we assume are of the form described in Section 2.6, so that we can work with classical derivatives. Let $\chi(\mathbf{s})$ be a Friedrich's mollifier; i.e. $\chi \in C_c^\infty(\mathbb{R}^3)$, $\operatorname{supp}(\chi) \subset \{\mathbf{s}|\ |\mathbf{s}| \leq 1\}, \chi \geq 0$, and $\int \chi\ d^3\mathbf{s} = 1$. For $\epsilon > 0$, we set $\chi_\epsilon(\mathbf{s}) \stackrel{\text{def}}{=} \epsilon^{-3}\chi(\frac{\mathbf{s}}{\epsilon})$ and define $\chi_\epsilon \mathring{\mathbf{V}} \in C^\infty(\mathbb{R}^3)$ by

$$\chi_\epsilon \mathring{\mathbf{V}}(\mathbf{s}) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \chi_\epsilon(\mathbf{s}-\mathbf{s}')\mathring{\mathbf{V}}(\mathbf{s}')\, d^3\mathbf{s}'. \tag{2.7.2.1}$$

The following properties of such a mollification are well-known:

$$\lim_{\epsilon \to 0^+} \|\chi_\epsilon \mathring{\mathbf{V}} - \mathring{\mathbf{V}}\|_{H^N} = 0 \tag{2.7.2.2}$$

$$\exists \{\epsilon_0 > 0 \wedge C(\mathring{\mathbf{V}}) > 0\} \ni 0 < \epsilon < \epsilon_0 \Rightarrow \|\chi_\epsilon \mathring{\mathbf{V}} - \mathring{\mathbf{V}}\|_{H^0} \leq \epsilon \cdot C(\mathring{\mathbf{V}})\|\mathring{\mathbf{V}}\|_{H^1}. \tag{2.7.2.3}$$

We will choose below an ϵ_0 that is at least as small as the one in (2.7.2.3). Once chosen, for a given $m \in \mathbb{N}$, we define

$$\epsilon_m \stackrel{\text{def}}{=} 2^{-m}\epsilon_0 \tag{2.7.2.4}$$

$$^{(m)}\mathring{\mathbf{V}} \stackrel{\text{def}}{=} \chi_{\epsilon_m}\mathring{\mathbf{V}}, \tag{2.7.2.5}$$

$$^{(m)}\mathring{\mathbf{W}} \stackrel{\text{def}}{=} \chi_{\epsilon_m}\mathring{\mathbf{W}}, \tag{2.7.2.6}$$

where $\mathring{\mathbf{W}}$ denotes the first 5 components of $\mathring{\mathbf{V}}$.

By Sobolev imbedding, by the assumptions on the initial data $\mathring{\mathbf{V}}$, and by the mollification properties above, $\exists\{\Lambda > 0 \wedge \epsilon_0 > 0\}$ (at least as small as the ϵ_0 in (2.7.2.3)) such that

$$\|\mathbf{V} - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N} \leq \Lambda \Rightarrow \mathbf{V} \in \bar{\mathcal{O}}_2, \tag{2.7.2.7}$$

$$\|{}^{(m)}\mathring{\mathbf{V}} - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N} \leq C_{\bar{\mathcal{O}}_2}\frac{\Lambda}{2} \text{ holds for } m \geq 0, \tag{2.7.2.8}$$

where $C_{\bar{\mathcal{O}}_2}$ is defined in (2.6.2.2).

Remark 2.7.3. It is a standard result that if $\epsilon > 0$ and N' is any real number, then $\chi_\epsilon \mathring{\mathbf{V}} \in H^{N'}_{\mathring{\mathbf{V}}}(\mathbb{R}^3)$. We will make use of this remark below, for in the local existence proof, we will need to differentiate the equations (2.7.2.14) - (2.7.2.19) N times and utilize Sobolev estimates; since several terms from these undifferentiated equations already contain one derivative of the smoothed initial data, our estimates will involve $\|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\mathring{\mathbf{V}}}}$.

Remark 2.7.4. If we are considering initial data $\tilde{\mathring{\mathbf{V}}}$ in a small enough H^N neighborhood $\mathcal{N}$ of the initial data $\mathring{\mathbf{V}}$, we can use a *fixed* smoothed function ${}^{(0)}\mathring{\mathbf{V}}$ in place of each ${}^{(0)}\tilde{\mathring{\mathbf{V}}}$ in Proposition 2.7.3 below, and choose Λ to be uniform over the neighborhood. For what then enters into the proof of local existence for the initial data $\tilde{\mathring{\mathbf{V}}}$ are the quantities $\|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\mathring{\mathbf{V}}}}$ and $\|{}^{(m)}\tilde{\mathring{\mathbf{V}}} - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N}$, and the latter quantity is easily controlled by the inequality

$$\|{}^{(m)}\tilde{\mathring{\mathbf{V}}} - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N} \leq \|{}^{(m)}\tilde{\mathring{\mathbf{V}}} - \tilde{\mathring{\mathbf{V}}}\|_{H^N} + \|\tilde{\mathring{\mathbf{V}}} - \mathring{\mathbf{V}}\|_{H^N} + \|\mathring{\mathbf{V}} - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N}; \tag{2.7.2.9}$$

once we fix an appropriately chosen smoothed function ${}^{(0)}\mathring{\mathbf{V}}$ and a corresponding Λ satisfying (2.7.2.7), we may independently adjust the mollification of each $\tilde{\mathring{\mathbf{V}}}$ in $\mathcal{N}$ so that the right-hand side of (2.7.2.9) is $\leq C_{\bar{\mathcal{O}}_2}\Lambda/2$ for $m \geq 0$. This remark is relevant for Corollary 2.7.1 above.

Defining the Iterates

Consider the iteration scheme described in Section 2.7.1, and recall that it starts with the seed ${}^{(0)}\mathbf{V}(t,\mathbf{s}) \stackrel{\text{def}}{=} {}^{(0)}\mathring{\mathbf{V}}(\mathbf{s})$. Linear existence theory implies that each iterate ${}^{(m+1)}\mathbf{V}$

is a well-defined, smooth function with $\|^{(m+1)}\mathbf{V}(t) - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N} < \infty$ for $0 \le t \le T_m$. Here, by (2.7.2.7), $[0, T_m]$ is any time interval for which $|||\ {}^{(m)}\mathbf{V} - {}^{(0)}\mathring{\mathbf{V}}\ |||_{H^N, T_m} \le \Lambda$ holds. The components of the iterates are denoted by ${}^{(m)}\mathbf{V} = \left({}^{(m)}s, {}^{(m)}P, \cdots, {}^{(m)}\psi_3\right)$, and we use the notation ${}^{(m)}\mathbf{W} = \left({}^{(m)}s, {}^{(m)}P, {}^{(m)}U^1, {}^{(m)}U^2, {}^{(m)}U^3\right)$ to refer to the first five components of ${}^{(m)}\mathbf{V}$.

The Uniform Time Estimate

As discussed in Section 2.7.1, we show the existence of a fixed $T_* > 0$ such that $|||\ {}^{(m)}\mathbf{V} - {}^{(0)}\mathring{\mathbf{V}}\ |||_{H^N, T_*} \le \Lambda$ for $m \in \mathbb{N}$, thus ensuring that each iterate is defined for a uniform amount of time and remains inside of $\bar{\mathcal{O}}_2$. We state a slightly stronger version of this result as a proposition:

Proposition 2.7.3. *Let* Λ *be defined by* (2.7.2.7). *Then there exists* $T_* > 0$ *and* $L > 0$ *such that each of the iterates* ${}^{(m)}\mathbf{V}(t, \mathbf{s})$ *satisfies*

$$|||\ {}^{(m)}\mathbf{V} - {}^{(0)}\mathring{\mathbf{V}}\ |||_{H^N, T_*} \le \Lambda \tag{2.7.2.10a}$$

$$|||\ \partial_t\left({}^{(m)}\mathbf{V}\right)\ |||_{H^{N-1}, T_*} \le L. \tag{2.7.2.10b}$$

Proof. We proceed by induction, noting that ${}^{(0)}\mathbf{V}(t, \mathbf{s}) \overset{\text{def}}{=} {}^{(0)}\mathring{\mathbf{V}}(\mathbf{s})$ satisfies (3.8.2.1a) and (3.8.2.1b) with any $T_* > 0$ and any positive number L. We thus assume that ${}^{(m)}\mathbf{V}$ satisfies (3.8.2.1a) and (3.8.2.1b) without first specifying the values of T_* or L. At the end of the proof, we will show that we can choose such a T_* and an L, both independent of m, such that energy estimates imply the inductive step. To obtain the estimates stated in the proposition, it is convenient to work not with the iterates themselves, but with the difference between the iterate and the smoothed initial value. Thus, referring to the notation defined in (2.7.2.5) and (2.7.2.6) , for each $m \in \mathbb{N}$ we define

$$\dot{\mathbf{V}}(t, \mathbf{s}) \overset{\text{def}}{=} {}^{(m+1)}\mathbf{V}(t, \mathbf{s}) - {}^{(0)}\mathring{\mathbf{V}}(\mathbf{s}) \tag{2.7.2.11}$$

$$\dot{\mathbf{W}}(t, \mathbf{s}) \overset{\text{def}}{=} {}^{(m+1)}\mathbf{W}(t, \mathbf{s}) - {}^{(0)}\mathring{\mathbf{W}}(\mathbf{s}) \tag{2.7.2.12}$$

$$\widetilde{\mathbf{V}} \overset{\text{def}}{=} {}^{(m)}\mathbf{V}. \tag{2.7.2.13}$$

We have used the notation $\dot{\mathbf{V}}$ and $\widetilde{\mathbf{V}}$ suggestively: it follows from the definition of the iterates, definition (2.7.2.11), and definition (2.7.2.13) that $\dot{\mathbf{V}}$ is a solution to the EOV (2.4.2.1) - (2.4.2.6) defined by the BGS $\widetilde{\mathbf{V}}$ with initial data $\dot{\mathbf{V}}(0,\mathbf{s}) = {}^{(m+1)}\mathring{\mathbf{V}}(\mathbf{s}) - {}^{(0)}\mathring{\mathbf{V}}(\mathbf{s})$. Our notation (2.7.2.11) - (2.7.2.13) is therefore consistent with our notation for the EOV introduced in Section 2.4.2. Recalling also the notation (2.4.2.17) and (2.4.2.18) introduced in Section 2.4.2, the inhomogeneous terms in the EOV satisfied by $\dot{\mathbf{V}}$ are given by $(\mathbf{b},\mathbf{l}) = (f, g, \cdots, l^{(4)})$, where

$$f = -\widetilde{U}^k \partial_k [{}^{(0)}\mathring{\eta}] \tag{2.7.2.14}$$

$$g = -\widetilde{U}^k \partial_k [{}^{(0)}\mathring{P}] \ - \ \widetilde{Q}\partial_k [{}^{(0)}\mathring{U}^k] \ + \ (4\widetilde{P} - 3\widetilde{Q})\widetilde{U}^\mu \widetilde{\psi}_\mu \tag{2.7.2.15}$$

$$h^{(j)} = -(\widetilde{R} + \widetilde{P})[\widetilde{U}^k \partial_k][{}^{(0)}\mathring{U}^j] \ - \widetilde{\Pi}^{kj}\partial_k [{}^{(0)}\mathring{P}] + (3\widetilde{P} - \widetilde{R})\widetilde{\Pi}^{\mu j}\widetilde{\psi}_\mu \qquad (j = 1,2,3) \tag{2.7.2.16}$$

$$l^{(0)} = \kappa^2 \widetilde{\phi} + \widetilde{R} - 3\widetilde{P} - \partial^k [{}^{(0)}\mathring{\psi}_k] \tag{2.7.2.17}$$

$$l^{(j)} = \partial^j [{}^{(0)}\mathring{\psi}_0] \qquad (j = 1,2,3) \tag{2.7.2.18}$$

$$l^{(4)} = \widetilde{\psi}_0. \tag{2.7.2.19}$$

As explained in Section 2.4.2, for each *spatial* derivative multi-index $\vec{\alpha}$ with $0 \leq |\vec{\alpha}| \leq N$, we may differentiate the EOV with inhomogeneous terms $(\mathbf{b},\mathbf{l})$ to which $\dot{\mathbf{V}}$ is a solution, obtaining that $\partial_{\vec{\alpha}}\dot{\mathbf{V}}$ is also a solution to the EOV defined by the *same* BGS $\widetilde{\mathbf{V}}$ with inhomogeneous terms $(\mathbf{b}_{\vec{\alpha}}, \partial_{\vec{\alpha}}\mathbf{l})$. The inhomogeneous terms $\mathbf{b}_{\vec{\alpha}}$ are given by

$$\mathbf{b}_{\vec{\alpha}} \stackrel{\text{def}}{=} A^0 \partial_{\vec{\alpha}} \left((A^0)^{-1}\mathbf{b}\right) + \mathbf{k}_{\vec{\alpha}}, \tag{2.7.2.20}$$

where

$$\mathbf{k}_{\vec{\alpha}} \stackrel{\text{def}}{=} A^0 \left[(A^0)^{-1} A^k \partial_k (\partial_{\vec{\alpha}} \dot{\mathbf{W}}) - \partial_{\vec{\alpha}} \left((A^0)^{-1} A^k \partial_k \dot{\mathbf{W}}\right)\right] \tag{2.7.2.21}$$

for $0 \leq |\vec{\alpha}| \leq N$.

As discussed in Section 2.6.2, we will use energy currents to control $||| \ \dot{\mathbf{V}} \ |||_{H^N,T}$. We state here as a lemma the key differential inequality that allows us to proceed with our desired estimates.

Lemma 2.7.4. *Suppose* $r \geq T > 0$. *For* $0 \leq t \leq T$, *let* $\Sigma_{t,r-t} = \{x \in \mathcal{M} | x^0 = t, x^k x_k \leq r - t\}$ *denote the Euclidean sphere of radius* $r - t$ *centered at* $(t, 0, 0, 0)$ *in the flat hypersurface* $\{x^0 = t\}$, *and let* $M_{t,r} = \{x \in \mathcal{M} | 0 \leq x^0 \leq t, x^k x_k = r - x^0\}$ *denote the mantle of the past directed, truncated light cone with lower base* $\Sigma_{0,r}$ *and upper base* $\Sigma_{t,r-t}$ *(see Figure 2.3). Let* $\dot{\mathbf{V}}$ *be a solution to the EOV* (2.4.2.1) - (2.4.2.6) *defined by the BGS* $\widetilde{\mathbf{V}}$, *and assume that* $\widetilde{\mathbf{V}}([0,T] \times \mathbb{R}^3) \subset \bar{\mathcal{O}}_2$. *Let* $\dot{J}$ *be an energy current for the variation* $\dot{\mathbf{V}}$ *defined by the BGS* $\widetilde{\mathbf{V}}$, *and define* $\mathcal{E}(t;r) \overset{def}{=} \Big(\int_{\Sigma_{t,r-t}} \dot{J}^0(t, \mathbf{s}) \, d^3\mathbf{s} \Big)^{1/2}$. *Then*

$$2\mathcal{E}(t;r) \frac{d}{dt} \mathcal{E}(t;r) \leq \int_{\Sigma_{t,r-t}} \partial_\mu \dot{J}^\mu(t, \mathbf{s}) \, d^3\mathbf{s}. \tag{2.7.2.22}$$

Proof. By the Divergence Theorem, we have

$$\begin{aligned} \mathcal{E}^2(t;r) \overset{\text{def}}{=} \int_{\Sigma_{t,r-t}} \dot{J}^0(t, \mathbf{s}) \, d^3\mathbf{s} &= \int_{\Sigma_{0,r}} \dot{J}^0(0, \mathbf{s}) \, d^3\mathbf{s} \\ &- \int_{M_{t,r}} \langle \hat{n}(x), \dot{J}(x) \rangle_E \, d\mathcal{H}(x) \ + \int_{t'=0}^{t'=t} \Big(\int_{\Sigma_{t',r-t'}} \partial_\mu \dot{J}^\mu(t', \mathbf{s}) \, d^3\mathbf{s} \Big) \, dt'. \end{aligned} \tag{2.7.2.23}$$

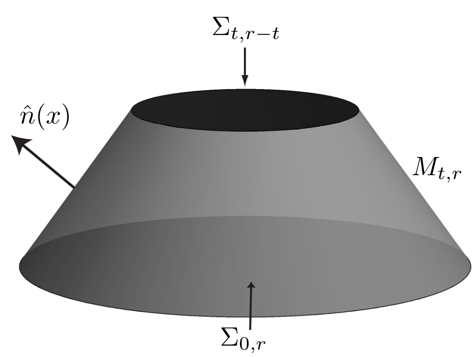

Figure 2.3: The Surfaces of Integration in Lemma 2.7.4

Here, $\hat{n}(x)$ is the Euclidean outer normal at $x \in M_{t,r}$ to the mantle of truncated cone, $\langle \hat{n}(x), \dot{J}(x) \rangle_E$ denotes the Euclidean inner product of $\hat{n}(x)$ and $\dot{J}(x)$ as vectors in $\mathbb{R}^4$, and $\mathcal{H}$ is the Hausdorff measure on the mantle of the cone. For each normal vector $\hat{n}(x)$, let ${}_{\hat{n}(x)}\xi$ denote the covector belonging to $T_x^*\mathcal{M}$ such that ${}_{\hat{n}(x)}\xi(X) = {}_{\hat{n}(x)}\xi_\mu X^\mu = \langle \hat{n}(x), X \rangle_E$ holds for every $X \in T_x\mathcal{M}$. By the positivity condition (2.5.5.2), covectors ξ belonging to $\mathcal{I}_x^+$ satisfy $\xi_\mu \dot{J}^\mu(\dot{\mathbf{V}}, \dot{\mathbf{V}}) > 0$ for all non-zero variations $\dot{\mathbf{V}}$. Since for each $x \in M_{t,r}$, the covector ${}_{\hat{n}(x)}\xi$ belongs to the boundary of $\mathcal{I}_x^+$, which is the positive component of the cone $C^*_{x,l}$, continuity in the variable ξ implies that $\langle \hat{n}(x), \dot{J}(x)\rangle_E \;= {}_{\hat{n}(x)}\xi_\mu \dot{J}^\mu(\dot{\mathbf{V}}, \dot{\mathbf{V}}) \geq 0$ holds for $x \in M_{t,r}$. Furthermore, if $t_1 < t_2$, then $M_{t_1,r} \subset M_{t_2,r}$. From these facts it follows that $-\int_{M_{t,r}} \langle \hat{n}(x), \dot{J}(x)\rangle_E \, d\mathcal{H}(x)$ is a decreasing function of t on $[0,T]$. Lemma 2.7.4 now follows from differentiating each side of (2.7.2.23) with respect to t and accounting for this decreasing term. Figure 2.3 illustrates the setup in $\mathbb{R}^{1+2}$, where the vertical direction represents positive values of t. $\square$

Returning to the proof of the proposition and recalling that we are using definitions (2.7.2.11) and (2.7.2.13) to define $\dot{\mathbf{V}}$ and $\widetilde{\mathbf{V}}$, we let $\dot{J}_{\vec{\alpha}}$ denote the energy current for the variation $\partial_{\vec{\alpha}}\dot{\mathbf{V}}$ defined by the BGS $\widetilde{\mathbf{V}}$. For notational convenience, we allow $\vec{\alpha}$ to take on the value $\vec{\mathbf{0}}$, in which case $\dot{J}_{\vec{\mathbf{0}}}$ is defined to be the energy current in the variation $\dot{\mathbf{V}}$ defined by the BGS $\widetilde{\mathbf{V}}$.

As in Lemma 2.7.4, we define for any $T_* > 0$ and $r > T_*$ the following functions of t on $[0, T_*]$:

$$\mathcal{E}_{\vec{\alpha}}(t;r) \stackrel{\text{def}}{=} \Big(\int\limits_{\Sigma_{t,r-t}} \dot{J}^0_{\vec{\alpha}}(t,\mathbf{s})\, d^3\mathbf{s} \Big)^{1/2}, \tag{2.7.2.24}$$

$$\mathcal{E}(t;r;N) \stackrel{\text{def}}{=} \Big(\sum_{0 \leq |\vec{\alpha}| \leq N} \mathcal{E}^2_{\vec{\alpha}}(t;r) \Big)^{\frac{1}{2}}. \tag{2.7.2.25}$$

Then with $C_{\bar{\mathcal{O}}_2}$ defined in (2.6.2.2), we have

$$C_{\bar{\mathcal{O}}_2}\mathcal{E}(t;r;N) \leq \|\dot{\mathbf{V}}(t)\|^2_{H^N(\Sigma_{t,r-t})} \leq \frac{1}{C_{\bar{\mathcal{O}}_2}}\mathcal{E}(t;r;N). \tag{2.7.2.26}$$

Additionally, by Lemma 2.7.4, we have the following inequality for $0 \leq t \leq T_*$:

$$2\mathcal{E}(t;r;N)\frac{d}{dt}\mathcal{E}(t;r;N) \leq \sum_{0 \leq |\vec{\alpha}| \leq N} \int\limits_{\Sigma_{t,r-t}} \partial_\mu \left(\dot{J}^\mu_{\vec{\alpha}}(t,\mathbf{s}) \right) d^3\mathbf{s}. \tag{2.7.2.27}$$

The technically cumbersome aspect of the proof of Proposition 2.7.3 is bounding the right-hand side of (2.7.2.27) by a constant times $\mathcal{E}(t;r;N) + \mathcal{E}^2(t;r;N)$, which then allows us to use Gronwall's inequality to exponentially bound from above the growth of $E(t;r;N)$ in t. We prove some of the technical points in lemmas 2.7.6 and 2.7.7 below, so as to not disrupt the main argument. The keys to proofs of lemmas 2.7.6 and 2.7.7 are Sobolev-Moser calculus inequalities, special versions of which are stated in the Appendix. In the following argument, $C = C(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\mathring{\mathbf{V}}}}, \Lambda, L)$. By Lemma 2.7.6, we have that

$$\sum_{0\leq|\vec{\alpha}|\leq N} \int_{\Sigma_{t,r-t}} \partial_\mu \left(\dot{J}^\mu_{\vec{\alpha}}(t,\mathbf{s}) \right) d^3\mathbf{s} \leq C \cdot \left[\|\dot{\mathbf{V}}(t)\|_{H^N(\Sigma_{t,r-t})} + \|\dot{\mathbf{V}}(t)\|^2_{H^N(\Sigma_{t,r-t})} \right] \quad (2.7.2.28)$$

$$\leq C \cdot \left[C^{-1/2}_{\bar{\mathcal{O}}_2} \mathcal{E}(t;r;N) + C^{-1}_{\bar{\mathcal{O}}_2} \mathcal{E}^2(t;r;N) \right],$$

where in the second inequality we have used (2.7.2.26). Combining (2.7.2.27) with (2.7.2.28), and applying Gronwall's lemma, we have for $0 \leq t \leq T_*$,

$$\mathcal{E}(t;r;N) \leq \left[\mathcal{E}(0;r;N) + C \cdot (2C_{\bar{\mathcal{O}}_2})^{-1/2} t \right] \cdot \left[\exp\big(C \cdot (2C_{\bar{\mathcal{O}}_2})^{-1} t \big) \right], \quad (2.7.2.29)$$

and consequently by (2.7.2.26),

$$\|\dot{\mathbf{V}}(t)\|_{H^N(\Sigma_{t,r-t})} \leq C_{\bar{\mathcal{O}}_2}{}^{-1} \left[\|\dot{\mathbf{V}}(0)\|_{H^N(\Sigma_{0,r})} + Ct \right] \cdot \left[\exp\big(Ct\big) \right]. \quad (2.7.2.30)$$

Letting $r \to \infty$, taking the sup over $t \in [0, T_*]$, and using (2.7.2.8), we have

$$|||\ \dot{\mathbf{V}}\ |||_{H^N, T_*} \leq C^{-1}_{\bar{\mathcal{O}}_2} \left[\|\dot{\mathbf{V}}(0)\|_{H^N} + CT_* \right] \cdot \exp\big(CT_*\big) \quad (2.7.2.31)$$

$$\leq \left[\Lambda/2 + C(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\mathring{\mathbf{V}}}}, \Lambda, L) T_* \right] \cdot \exp\big(C(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\mathring{\mathbf{V}}}}, \Lambda, L) T_* \big).$$

To make a viable choice of L,

we first *assume* that right-hand side of (2.7.2.31) is $\leq \Lambda$, (2.7.2.32)

which implies the inductive step (3.8.2.1a) for ${}^{(m+1)}\mathbf{V}$. Using assumption (2.7.2.32) as a hypothesis, Lemma 2.7.7 implies that there exists $L(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\mathring{\mathbf{V}}}}, \Lambda) > 0$ such that

$$|||\ \partial_t \big({}^{(m+1)}\mathbf{V} \big)\ |||_{H^{N-1}, T_*} \leq L. \quad (2.7.2.33)$$

For this fixed choice of L, we can implicitly solve for a $T_* > 0$ such that the right-hand side of inequality (2.7.2.31) is in fact $\leq \Lambda$, thus justifying the assumption (2.7.2.32) and the conclusion (2.7.2.33), thereby closing the induction argument.

□

Remark 2.7.5. Based on the above reasoning, we can remove the dependence of the constant C on L; i.e., for $T \leq T_*$, (2.7.2.31) becomes

$$||| \dot{\mathbf{V}} |||_{H^N,T} \leq \left[\Lambda/2 + CT\right] \cdot \exp(CT), \tag{2.7.2.34}$$

with $C = C(N, \bar{\mathcal{O}}_2, \|^{(0)}\mathring{\mathbf{V}}\|_{H_{\bar{\mathbf{V}}}^{N+1}}, \Lambda)$.

For later use, we now prove an important corollary to the proof of Proposition 2.7.3.

Corollary 2.7.5. *For every* $m \in \mathbb{N}$, ${}^{(m)}\mathbf{V} \in C^0([0,T_*], H_{\bar{\mathbf{V}}}^N)$;
i.e. $\lim_{t\to t_0} ||| \; {}^{(m)}\mathbf{V}(t) - {}^{(m)}\mathbf{V}(t_0) \; |||_{H^N} = 0$ *for* $t_0 \in (0, T_*)$, *and similarly for the one-sided limits at* $t_0 = 0$ *and* $t_0 = T$.

Proof. Since the linearized EN_κ system satisfied by ${}^{(m+1)}\mathbf{V}$ is time translation invariant and reversible in time, it is sufficient to prove the H^N right continuity of ${}^{(m+1)}\mathbf{V}(t)$. We will give the proof only at $t_0 = 0$, since the same argument applies for all $t_0 \in [0, T_*]$. To achieve the desired result, we need only to redefine $\dot{\mathbf{V}}(t,\mathbf{s}) \overset{\text{def}}{=} {}^{(m+1)}\mathbf{V}(t,\mathbf{s}) - {}^{(m+1)}\mathring{\mathbf{V}}(\mathbf{s})$, $\dot{\mathbf{W}}(t,\mathbf{s}) \overset{\text{def}}{=} {}^{(m+1)}\mathbf{W}(t,\mathbf{s}) - {}^{(m+1)}\mathring{\mathbf{W}}(\mathbf{s})$, so that $\dot{\mathbf{V}}(t=0) \equiv \mathbf{0}$, and repeat the proof of Proposition 2.7.3, making a few minor changes. We obtain the following bound, analogous to (2.7.2.34), for $T \in [0, T_*]$:

$$||| \dot{\mathbf{V}} |||_{H^N,T} \leq TC \cdot \exp\left(CT\right), \tag{2.7.2.35}$$

where $C = C(N, \bar{\mathcal{O}}_2, \|^{(m)}\mathring{\mathbf{V}}\|_{H_{\bar{\mathbf{V}}}^{N+1}}, \Lambda)$. The proof of Corollary 2.7.5 now easily follows.

□

We now state and prove the two technical lemmas quoted in the proof of Proposition 2.7.3.

Lemma 2.7.6. *Assume the hypotheses and notation of Proposition 2.7.3. In addition, assume that* $\|\partial_t \widetilde{\mathbf{V}}(t)\|_{H^{N-1}} \leq L$. *Then*

$$\|\partial_\mu \dot{J}^\mu_{\vec{\alpha}}(t)\|_{L^1(\Sigma_{t,r-t})} \leq C(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\bar{\mathbf{V}}}}, \Lambda, L)\left(\|\dot{\mathbf{V}}(t)\|_{H^N} + \|\dot{\mathbf{V}}(t)\|^2_{H^N}\right). \tag{2.7.2.36}$$

Proof. We use here the notation (2.7.2.11) and (2.7.2.13) from Proposition 2.7.3. Recall that $\partial_{\vec{\alpha}}\dot{\mathbf{V}}$ is a solution to the EOV defined by the BGS $\widetilde{\mathbf{V}}$ with inhomogeneous terms $(\mathbf{b}_{\vec{\alpha}}, \partial_{\vec{\alpha}}\mathbf{l})$, and that $\dot{J}_{\vec{\alpha}}$ is the energy current for $\partial_{\vec{\alpha}}\dot{\mathbf{V}}$ defined by the BGS $\widetilde{\mathbf{V}}$. Furthermore,

$$\|\widetilde{\mathbf{V}} - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N} \leq \Lambda \tag{2.7.2.37}$$

holds by the induction assumption from the proposition.

By (2.5.5.3) and Remark 2.5.5, the expression for $\partial_\mu \dot{J}^\mu_{\vec{\alpha}}$ consists of terms that are either precisely linear or precisely quadratic in the components of the variation $\partial_{\vec{\alpha}}\dot{\mathbf{V}}$. The coefficients of the quadratic variation terms are smooth functions with arguments $\widetilde{\mathbf{V}}$ and $D\widetilde{\mathbf{V}}$. Examining the particular form of these coefficients and using the fact that $\widetilde{\mathbf{V}}([0,T] \times \mathbb{R}^3) \subset \bar{\mathcal{O}}_2$, we see that their L^∞ norm is bounded by $C(\bar{\mathcal{O}}_2)\|D\widetilde{\mathbf{V}}\|_{L^\infty}$. By assumption, $\|\widetilde{\mathbf{V}} - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N} \leq \Lambda$ and $\|\partial_t \widetilde{\mathbf{V}}(t)\|_{H^{N-1}} \leq L$. Therefore, by Sobolev imbedding, $\|D\widetilde{\mathbf{V}}\|_{L^\infty} \leq C(N, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^N_{\bar{\mathbf{V}}}}, \Lambda, L)$. These facts imply that the $L^1(\Sigma_{t,r-t})$ norm of the terms involving the quadratic variations is bounded from above by $C(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^N_{\bar{\mathbf{V}}}}, \Lambda, L)\|\partial_{\vec{\alpha}}\dot{\mathbf{V}}\|^2_{L^2(\Sigma_{t,r-t})}$.

The coefficients of the linear variation terms are linear combinations of products of the components of the components of $(\mathbf{b}_{\vec{\alpha}}, \partial_{\vec{\alpha}}\mathbf{l})$, where $\mathbf{b}_{\vec{\alpha}}$ is defined in (2.7.2.20), with smooth functions, the arguments of which are the components of $\widetilde{\mathbf{V}}$. Since $\widetilde{\mathbf{V}}([0,T] \times \mathbb{R}^3) \subset \bar{\mathcal{O}}_2$, the smooth functions of $\widetilde{\mathbf{V}}$ are bounded in L^∞ by $C(\bar{\mathcal{O}}_2)$. Therefore, by the Cauchy-Schwarz integral inequality for L^2, the $L^1(\Sigma_{t,r-t})$ norm of the terms depending linearly on the variations is bounded from above by $C(\bar{\mathcal{O}}_2)\|(\mathbf{b}_{\vec{\alpha}}, \partial_{\vec{\alpha}}\mathbf{l})\|_{L^2}\|\partial_{\vec{\alpha}}\dot{\mathbf{V}}\|_{L^2(\Sigma_{t,r-t})}$. To complete the proof of (2.7.2.36), it remains to show that for $0 \leq |\vec{\alpha}| \leq N$, we have

$$\|(\mathbf{b}_{\vec{\alpha}}, \partial_{\vec{\alpha}}\mathbf{l})\|_{L^2} \leq C(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\bar{\mathbf{V}}}}, \Lambda)\big(1 + \|\dot{\mathbf{W}}\|_{H^N}\big). \tag{2.7.2.38}$$

The proof of (2.7.2.38) will follow easily from the propositions given in the Appendix.

Concerning ourselves with the $\|\mathbf{b}_{\vec{\alpha}}\|_{L^2}$ estimate first, we claim that the term $A^0\partial_{\vec{\alpha}}\big((A^0)^{-1}\mathbf{b}\big)$ from (2.7.2.20) satisfies

$$\|A^0\partial_{\vec{\alpha}}\big((A^0)^{-1}\mathbf{b}\big)\|_{L^2} \leq C(N,\bar{\mathcal{O}}_2,\|^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\bar{\mathbf{V}}}},\Lambda). \tag{2.7.2.39}$$

We repeat for clarity that $\mathbf{b} = (f,g,h^1,h^2,h^3)$, where the scalar-valued quantities $f,g,\cdots,h^3$ are defined in (2.7.2.14) - (2.7.2.16). Since $\|A^0(\widetilde{\mathbf{V}})\|_{L^\infty} \leq C(\bar{\mathcal{O}}_2)$, to prove (2.7.2.39), it suffices to control the L^2 norm of $\partial_{\vec{\alpha}}\big((A^0)^{-1}\mathbf{b}\big)$. Using Proposition B.0.2 in the Appendix (see also Remark B.0.3), with $(A^0)^{-1}$ playing the role of F in the proposition, and $\mathbf{b}$ playing the role of G, we have that

$$\|(A^0)^{-1}\mathbf{b}\|_{H^N} \leq C(N,\bar{\mathcal{O}}_2,\|^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\bar{\mathbf{V}}}},\Lambda)\|\mathbf{b}\|_{H^N}. \tag{2.7.2.40}$$

Furthermore, Proposition B.0.2 implies that

$$\|\mathbf{b}\|_{H^N} \leq C(N,\bar{\mathcal{O}}_2,\|^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\bar{\mathbf{V}}}},\Lambda). \tag{2.7.2.41}$$

Combining (2.7.2.40) with (2.7.2.41) proves (2.7.2.39).

We next claim that the $\mathbf{k}_{\vec{\alpha}}$ from (2.7.2.21) satisfy

$$\|\mathbf{k}_{\vec{\alpha}}\|_{L^2} \leq C(N,\bar{\mathcal{O}}_2,\|^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\bar{\mathbf{V}}}},\Lambda)\|\dot{\mathbf{W}}\|_{H^N}. \tag{2.7.2.42}$$

Again, since $\|A^0(\widetilde{\mathbf{V}})\|_{L^\infty} \leq C(\bar{\mathcal{O}}_2)$, to prove (2.7.2.42), it suffices to control the L^2 norm of $(A^0)^{-1}A^k\partial_k(\partial_{\vec{\alpha}}\dot{\mathbf{W}}) - \partial_{\vec{\alpha}}\Big((A^0)^{-1}A^k\partial_k\dot{\mathbf{W}}\Big)$. By Proposition B.0.5 (see also Remark B.0.5), with $(A^0)^{-1}A^k$ playing the role of F in the proposition, and $\partial_k\dot{\mathbf{W}}$ playing the role of G, we have that

$$\begin{aligned}\|(A^0)^{-1}A^k\partial_k(\partial_{\vec{\alpha}}\dot{\mathbf{W}}) - \partial_{\vec{\alpha}}\big((A^0)^{-1}A^k\partial_k\dot{\mathbf{W}}\big)\|_{L^2}&\\ \leq C(N,\bar{\mathcal{O}}_2,\|^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\bar{\mathbf{V}}}},\Lambda)\|\nabla^{(1)}\dot{\mathbf{W}}\|_{H^{N-1}},&\end{aligned} \tag{2.7.2.43}$$

from which (2.7.2.42) readily follows.

To finish the proof of (2.7.2.38), we will show that

$$\|\partial_{\vec{\alpha}}l^{(z)}\|_{L^2} \leq C(N,\bar{\mathcal{O}}_2,\|^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\bar{\mathbf{V}}}},\Lambda) \qquad (z=0,1,2,3,4). \tag{2.7.2.44}$$

For $l^{(1)},l^{(2)},l^{(3)}$, defined in (2.7.2.18), the claim is trivial. To estimate the component $l^{(0)}$, defined in (2.7.2.17), we first rewrite

$$l^{(0)} = \kappa^2(\widetilde{\phi}-\bar{\phi}) + (\widetilde{R}-\bar{R}) - 3(\widetilde{P}-\bar{P}) - \partial^k[^{(0)}\mathring{\psi}_k], \tag{2.7.2.45}$$

where $\bar{P} \overset{\text{def}}{=} e^{4\bar{\phi}}\bar{p}$ and $\bar{R} \overset{\text{def}}{=} e^{4\bar{\phi}}\mathcal{R}(e^{-4\bar{\phi}}\bar{P},\bar{\eta})$, the function $\mathcal{R}$ is defined in (2.4.1.5), and $\bar{p}$ and $\bar{\eta}$ are constants defined in Section 2.6. In equation (2.7.2.45), we have made use of (2.6.1.2), which is the assumption that $\kappa^2\bar{\phi} + \bar{R} - 3\bar{P} = 0$. Since

$$\|\kappa^2(\widetilde{\phi} - \bar{\phi})\|_{H^N} + 3\|(\widetilde{P} - \bar{P})\|_{H^N} + \|\partial^k[{}^{(0)}\mathring{\psi}_k]\|_{H^N} \leq C(\|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\bar{\mathbf{V}}}}, \Lambda), \qquad (2.7.2.46)$$

we only need to show that

$$\|\partial_{\vec{\alpha}}(\widetilde{R} - \bar{R})\|_{L^2} \leq C(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\bar{\mathbf{V}}}}, \Lambda). \qquad (2.7.2.47)$$

This follows immediately from definition (2.4.2.10), Proposition B.0.4, and Remark B.0.3. We omit the argument for $l^{(4)}$, defined in (2.7.2.19), since it is similar to the argument for $l^{(0)}$, and in fact simpler. This completes the proof of (2.7.2.44).

Inequality (2.7.2.38) now follows from combining (2.7.2.39), (2.7.2.42), and (2.7.2.44); this completes the proof of (2.7.2.36). □

Lemma 2.7.7. *Assume the hypotheses and notation of Proposition 2.7.3. Also assume the induction hypothesis* $|||\ {}^{(m)}\mathbf{V} - {}^{(0)}\mathring{\mathbf{V}}\ |||_{H^N,T_*} \leq \Lambda$ *from Proposition 2.7.3. Assume further that* $|||\ {}^{(m+1)}\mathbf{V} - {}^{(0)}\mathring{\mathbf{V}}\ |||_{H^N,T_*} \leq \Lambda$. *Then*

$$|||\ \partial_t\big({}^{(m+1)}\mathbf{V}\big)\ |||_{H^{N-1},T_*} \leq L(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\bar{\mathbf{V}}}}, \Lambda). \qquad (2.7.2.48)$$

Proof. By Remark 2.4.4, we may solve for $\partial_t\big({}^{(m+1)}\mathbf{W}\big)$:

$$\partial_t\big({}^{(m+1)}\mathbf{W}\big) = \big(A^0({}^{(m)}\mathbf{V})\big)^{-1}\big(\mathbf{b} - A^k({}^{(m)}\mathbf{V})\partial_k\big({}^{(m+1)}\mathbf{W}\big)\big), \qquad (2.7.2.49)$$

where the function $\mathbf{b}$ denotes the inhomogeneous terms from the linearized EN_κ equations satisfied by ${}^{(m+1)}\mathbf{W}$; i.e., $\mathbf{b}=\mathfrak{B}({}^{(m)}\mathbf{V})$, where

$$\mathfrak{B}(\cdot) \overset{\text{def}}{=} \Big(\mathfrak{F}(\cdot), \mathfrak{G}(\cdot), \mathfrak{H}^{(1)}(\cdot), \mathfrak{H}^{(2)}(\cdot), \mathfrak{H}^{(3)}(\cdot)\Big) \qquad (2.7.2.50)$$

is an array-valued function, the scalar-valued functions $\mathfrak{F}, \mathfrak{G}, \cdots, \mathfrak{H}^{(3)}$ are defined in (2.4.2.11) - (2.4.2.13), and the $A^\mu(\cdot)$ are defined in (3.3.0.16). See Remark 2.4.3 concerning our use of function notation here.

Using the hypotheses of the lemma, we apply Proposition B.0.2 (see also Remark B.0.3) to the right-hand side of (2.7.2.49), concluding that

$$|||\ \partial_t\big({}^{(m)}\mathbf{W}\big)\ |||_{H^{N-1},T_*} \leq L(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\bar{\mathbf{V}}}}, \Lambda). \qquad (2.7.2.51)$$

Likewise, an argument similar to the one used to prove (2.7.2.44) gives that

$$||| \, \partial_t\big({}^{(m)}\phi, {}^{(m)}\psi_0, {}^{(m)}\psi_1, {}^{(m)}\psi_2, {}^{(m)}\psi_3\big) \, |||_{H^{N-1},T_*} \leq L(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\bar{\mathbf{V}}}}, \Lambda). \tag{2.7.2.52}$$

Combining (2.7.2.51) and (2.7.2.52) proves (2.7.2.48). □

2.7.3 The L^2 Convergence of the Iterates

We now show that the sequence of iterates $\{{}^{(m)}\mathbf{V}\}$ converges in $L^2_{\bar{\mathbf{V}}}$ to function $\mathbf{V}_{sol}$. We accomplish this by making an appropriate choice of $T > 0$ such that

$$\sum_{m=1}^{\infty} ||| \, {}^{(m+1)}\mathbf{V} - {}^{(m)}\mathbf{V} \, |||_{L^2,T} < \infty. \tag{2.7.3.1}$$

This finiteness of this sum shows that for $0 \leq t \leq T$, the sequence of L^2 functions $\{{}^{(m)}\mathbf{V}(t) - \bar{\mathbf{V}}\}$ is Cauchy and therefore converges in L^2 to a function $\mathbf{V}_{sol}(t) - \bar{\mathbf{V}}$. We will use energy estimates to show the existence of a $T > 0$, an α with $0 < \alpha < 1$, and a sequence of non-negative reals β_m with $\sum_{m=1}^{\infty} \beta_m < \infty$ such that for $m \geq 2$,

$$||| \, {}^{(m+1)}\mathbf{V} - {}^{(m)}\mathbf{V} \, |||_{L^2,T} \leq \alpha \, ||| \, {}^{(m)}\mathbf{V} - {}^{(m-1)}\mathbf{V} \, |||_{L^2,T} + \beta_m, \tag{2.7.3.2}$$

from which the convergence of the series (2.7.3.1) follows.

To begin the proof of (2.7.3.2), we examine the system satisfied by the difference of two successive iterates; for $m \geq 2$, the following system of equations is satisfied:

$$A^\mu\big({}^{(m)}\mathbf{V}\big)\partial_\mu\big({}^{(m+1)}\mathbf{W} - {}^{(m)}\mathbf{W}\big) = \mathbf{b1} + \mathbf{b2} \tag{2.7.3.3}$$

$$-\partial_0\dot{\psi}_0 + \partial^j\dot{\psi}_j = \mathfrak{L}^{(0)}\big({}^{(m)}\mathbf{V}\big) - \mathfrak{L}^{(0)}\big({}^{(m-1)}\mathbf{V}\big) \tag{2.7.3.4}$$

$$\partial_0\dot{\psi}_j - \partial_j\dot{\psi}_0 = \mathfrak{L}^{(j)}\big({}^{(m)}\mathbf{V}\big) - \mathfrak{L}^{(j)}\big({}^{(m-1)}\mathbf{V}\big) \qquad (j = 1, 2, 3) \tag{2.7.3.5}$$

$$\partial_0\dot{\phi} = \mathfrak{L}^{(4)}\big({}^{(m)}\mathbf{V}\big) - \mathfrak{L}^{(4)}\big({}^{(m-1)}\mathbf{V}\big), \tag{2.7.3.6}$$

where

$$\mathbf{b}1 = \left[A^{\mu}\left({}^{(m-1)}\mathbf{V}\right) - A^{\mu}\left({}^{(m)}\mathbf{V}\right)\right]\partial_{\mu}\left({}^{(m)}\mathbf{W}\right) \tag{2.7.3.7}$$

$$\mathbf{b}2 = \mathfrak{B}\left({}^{(m)}\mathbf{V}\right) - \mathfrak{B}\left({}^{(m-1)}\mathbf{V}\right), \tag{2.7.3.8}$$

$\mathfrak{B}(\cdot)$ is the array-valued function defined in (2.7.2.50), the scalar-valued functions $\mathfrak{L}^{(0)}, \mathfrak{L}^{(1)}, \cdots, \mathfrak{L}^{(4)}$ are defined in (2.4.2.14) - (2.4.2.16), and the $A^{\mu}(\cdot)$ are defined in (3.3.0.16); see Remark 2.4.3 concerning our use of function notation here.

By inspection, we see that the difference of two successive iterates satisfies the equations of variation. Thus, consistent with our notation introduced in Section 2.4.2, we suggestively define

$$\dot{\mathbf{V}} \stackrel{\text{def}}{=} {}^{(m+1)}\mathbf{V} - {}^{(m)}\mathbf{V}. \tag{2.7.3.9}$$

Using definition (2.7.3.9) and referring to equations (2.7.3.3) - (2.7.3.6), we have that $\dot{\mathbf{V}}$ is a solution to the EOV defined by the BGS ${}^{(m)}\mathbf{V}$ with inhomogeneous terms $\left(\mathbf{b}1 + \mathbf{b}2, \mathfrak{L}({}^{(m)}\mathbf{V}) - \mathfrak{L}({}^{(m-1)}\mathbf{V})\right)$, where

$$\mathfrak{L}(\cdot) \stackrel{\text{def}}{=} \left(\mathfrak{L}^{(0)}(\cdot), \mathfrak{L}^{(1)}(\cdot), \mathfrak{L}^{(2)}(\cdot), \mathfrak{L}^{(3)}(\cdot), \mathfrak{L}^{(4)}(\cdot)\right) \tag{2.7.3.10}$$

is an array-valued function.

By combining the uniform bound on the H^N norms of the iterates obtained in Proposition 2.7.3, Sobolev imbedding (applied to the $\partial_{\mu}\left({}^{(m)}\mathbf{W}\right)$ term in $\mathbf{b}1$), and Taylor's theorem, we obtain the following bound on the inhomogeneous terms for $(t, \mathbf{s}) \in [0, T^*] \times \mathbb{R}^3$ (we suppress the dependence of the running constant C on $N, \bar{\mathcal{O}}_2, {}^{(0)}\mathring{\mathbf{V}}$, and Λ to avoid notational clutter):

$$\left|\left(\mathbf{b}1 + \mathbf{b}2, \mathfrak{L}({}^{(m)}\mathbf{V}) - \mathfrak{L}({}^{(m-1)}\mathbf{V})\right)\right| \leq C \cdot \left|{}^{(m)}\mathbf{V} - {}^{(m-1)}\mathbf{V}\right|. \tag{2.7.3.11}$$

If we now define $\dot{J}$ to be the energy current for the variation $\dot{\mathbf{V}}$ defined by the BGS ${}^{(m)}\mathbf{V}$, then by applying the uniform bound on the H^N norms of the iterates obtained in Proposition 2.7.3, Sobolev imbedding, and (2.7.3.11) to the formula (2.5.5.3) for the divergence of $\dot{J}$, we have, for $(t, \mathbf{s}) \in [0, T^*] \times \mathbb{R}^3$, the pointwise bound

$$\left|\partial_{\mu}\dot{J}^{\mu}\right| \leq C \cdot \left(|\dot{\mathbf{V}}|^2 + |\dot{\mathbf{V}}||{}^{(m)}\mathbf{V} - {}^{(m-1)}\mathbf{V}|\right), \tag{2.7.3.12}$$

and the following consequence of (2.7.3.12):

$$\|\partial_\mu \dot{J}^\mu(t)\|_{L^1} \leq C \cdot \Big(\|\dot{\mathbf{V}}(t)\|_{L^2}^2 + \|\dot{\mathbf{V}}(t)\|_{L^2} \|{}^{(m)}\mathbf{V}(t) - {}^{(m-1)}\mathbf{V}(t)\|_{L^2} \Big). \qquad (2.7.3.13)$$

Taking into account the initial condition $\dot{\mathbf{V}}(0,\mathbf{s}) = {}^{(m+1)}\mathring{\mathbf{V}}(\mathbf{s}) - {}^{(m)}\mathring{\mathbf{V}}(\mathbf{s})$, where ${}^{(m)}\mathring{\mathbf{V}}(\mathbf{s})$ is defined in (2.7.2.5), but otherwise omitting the details, we may argue as in the proof in Proposition 2.7.3 to conclude that for $0 \leq T \leq T_*$

$$||| \, \dot{\mathbf{V}} \, |||_{L^2,T} \leq \; C \cdot \exp\left(CT\right) \Big(T \, ||| \; {}^{(m)}\mathbf{V} - {}^{(m-1)}\mathbf{V} \; |||_{L^2,T} \; + \|{}^{(m+1)}\mathring{\mathbf{V}} - {}^{(m)}\mathring{\mathbf{V}}\|_{L^2} \Big). \qquad (2.7.3.14)$$

We then select $T \in (0, T_*]$ so that

$$\alpha \overset{\text{def}}{=} C(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\check{\mathbf{V}}}}, \Lambda) \exp\Big(C(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\check{\mathbf{V}}}}, \Lambda) T \Big) \, T < 1, \qquad (2.7.3.15)$$

noting that this choice of T and α are independent of m.

If we also define $\beta_m \overset{\text{def}}{=} T^{-1}\alpha \cdot \|{}^{(m+1)}\mathring{\mathbf{V}} - {}^{(m)}\mathring{\mathbf{V}}\|_{L^2}$, then (2.7.2.3), (2.7.2.4), and (2.7.2.6) imply that

$$\sum_{m=1}^{\infty} \beta_m < \infty. \qquad (2.7.3.16)$$

Recalling definition (2.7.3.9) and combining (2.7.3.14), (2.7.3.15), and (2.7.3.16) proves (2.7.3.2), which shows that the series (2.7.3.1) converges.

Remark 2.7.6. Equation (2.7.3.7) illustrates why we cannot replace the L^2 convergence in (2.7.3.1) with H^N convergence; the $\mathbf{b}1$ term already involves one derivative of ${}^{(m)}\mathbf{W}$, so its H^N norm can only be controlled in terms of $\|{}^{(m)}\mathbf{W}\|_{H^{N+1}_{\check{\mathbf{W}}}}$, which may be unbounded.

2.7.4 $\mathbf{V}_{sol}$ is a Classical Solution

Now that the iterates have been shown to converge, we will show that the limiting function is a classical solution to the EN$_\kappa$ system. The basic idea is that by using Sobolev estimates, the limiting function $\mathbf{V}_{sol}$ from Section 2.7.3 will be shown to inherit a certain degree of regularity from the iterates. Following this, we will show that the convergence is strong enough that we are able to take the pointwise limit of each term in the linearized EN_κ equations satisfied by the iterates, thus implying that $\mathbf{V}_{sol}$ is a classical solution. We will use the following elementary result from calculus, which we state here without proof.

Lemma 2.7.8. *Let $I \subset \mathbb{R}$ be an interval, and let ${}^{(m)}F : I \rightarrow \mathbb{R}$ be a sequence of C^1, real-valued functions. Assume that there are two functions F, G such that ${}^{(m)}F \rightarrow F$ pointwise and ${}^{(m)}F' \rightarrow G$ uniformly. Then F is differentiable and $F' = G$.*

Proof that $\mathbf{V}_{sol}$ is a Classical Solution

First observe that (2.7.3.1) implies

$$\lim_{m\rightarrow\infty} |||\ {}^{(m)}\mathbf{V} - \mathbf{V}_{sol}\ |||_{L^2,T} = 0. \tag{2.7.4.1}$$

By Proposition 2.7.3 and Lemma (B.0.6), we have that for $j, k \in \mathbb{N}$ and $0 \leq N' \leq N$,

$$\begin{aligned} |||\ {}^{(j)}\mathbf{V} - {}^{(k)}\mathbf{V}\ |||_{H^{N'},T} &\leq C(N')\ |||\ {}^{(j)}\mathbf{V} - {}^{(k)}\mathbf{V}\ |||_{L^2,T}^{1-N'/N} |||\ {}^{(j)}\mathbf{V} - {}^{(k)}\mathbf{V}\ |||_{H^N,T}^{N'/N} \\ &\leq C(N', \Lambda)\ |||\ {}^{(j)}\mathbf{V} - {}^{(k)}\mathbf{V}\ |||_{L^2,T}^{1-N'/N}\ . \end{aligned} \tag{2.7.4.2}$$

Thus, if $0 \leq N' < N$ and $t \in [0, T]$, the sequence $\{{}^{(m)}\mathbf{V}(t)\}$ is uniformly (in t) Cauchy in the $H^{N'}_{\mathbf{V}}$ norm. Therefore, by the completeness of $H^{N'}$, we have

$$\lim_{m\rightarrow\infty} |||\ {}^{(m)}\mathbf{V} - \mathbf{V}_{sol}\ |||_{H^{N'},T} = 0. \tag{2.7.4.3}$$

Combining (2.7.4.3) with Corollary 2.7.5 and using an $\frac{\epsilon}{3}$ argument, we have for $0 \leq N' < N$,

$$\mathbf{V}_{sol} \in C^0([0, T], H^{N'}_{\mathbf{V}}). \tag{2.7.4.4}$$

By choosing $\frac{5}{2} < N' < N$ and appealing to Sobolev imbedding, we have

$${}^{(m)}\mathbf{V} \rightarrow \mathbf{V}_{sol} \text{ in } C^0\left([0, T], C^1_b\right). \tag{2.7.4.5}$$

Applying (2.7.4.5) and Lemma (2.7.8) (replacing the $'$ from Lemma (2.7.8) with the partial derivative operator ∂_t) to the sequence of iterates ${}^{(m)}\mathbf{V}$, and using the facts (2.7.2.2), (2.7.2.4), and (2.7.2.6) concerning the sequence of initial values ${}^{(m)}\mathring{\mathbf{V}}$, we take the limit as $m \to \infty$ in the EOV satisfied by the ${}^{(m)}\mathbf{V}$ and conclude that $\mathbf{V}_{sol} = (\eta, P, U^1, U^2, U^3, \phi, \psi_0, \psi_1, \psi_2, \psi_3)$ is a classical solution to the EN$_\kappa$ system (2.4.1.18) - (2.4.1.23) with the desired initial data $\mathring{\mathbf{V}}$.

We furthermore claim that that the solution is of the form

$$\mathbf{V}_{sol} = (\eta, P, U^1, U^2, U^3, \phi, \partial_0\phi, \partial_1\phi, \partial_2\phi, \partial_3\phi), \tag{2.7.4.6}$$

implying that ϕ has continuous derivatives up to 2nd order. To see this, we first observe that equation (2.4.1.23) and statement (2.7.4.5) imply that for $j = 1, 2, 3$, we have $\partial_j\partial_0\phi \in C^0([0,T] \times \mathbb{R}^3)$, thus allowing us to interchange the derivatives ∂_0 and ∂_j. Coupling this fact with equation (2.4.1.22), it follows that on $[0,T] \times \mathbf{R}^3$

$$\partial_0(\psi_j - \partial_j\phi) = 0 \qquad (j = 1, 2, 3). \tag{2.7.4.7}$$

Since our assumptions on the initial data imply that $\psi_j(0, \mathbf{s}) - \partial_j(\phi(0, \mathbf{s})) = 0$, we conclude that on $[0,T] \times \mathbf{R}^3$

$$\psi_j(t, \mathbf{s}) = \partial_j\phi(t, \mathbf{s}). \tag{2.7.4.8}$$

Combining equation (2.4.1.27) with (2.7.4.8) proves (2.7.4.6).

2.7.5 Further Results on the Regularity of $\mathbf{V}_{sol}$

Showing that $\mathbf{V}_{sol}(t) - \mathring{\mathbf{V}} \in C^0([0,T], H^N)$

We have not yet shown the full statement of Theorem 2.7.1, in part because the regularity result given in (2.7.4.4) is valid only for $N' < N$. Since the EN$_\kappa$ system is time reversible and invariant under time translations, to show that

$$\mathbf{V}_{sol}(t) - \mathring{\mathbf{V}} \in C^0([0,T], H^N), \tag{2.7.5.1}$$

it is sufficient to show the H^N right continuity of $\mathbf{V}_{sol} - \mathring{\mathbf{V}}$ at $t = 0$. Observe that (2.7.5.1) is equivalent to $\mathbf{V}_{sol} \in C^0([0,T], H^N_{\mathring{\mathbf{V}}}(\mathbb{R}^3))$ and also to $\mathbf{V}_{sol}(t) - {}^{(0)}\mathring{\mathbf{V}} \in C^0([0,T], H^N(\mathbb{R}^3))$. Our proof of (2.7.5.1) will be accomplished in three stages. The first is to show that

$$\mathbf{V}_{sol} - \mathring{\mathbf{V}} \in L^\infty([0,T], H^N). \tag{2.7.5.2}$$

In our proof of (2.7.5.2), we make use of the following classical result concerning the duality of $H^{N'}$ and $H^{-N'}$.

Lemma 2.7.9. *Let $N' \in \mathbb{R}$. Then $H^{N'}$ and $H^{-N'}$ are dual through the pairing $[\cdot,\cdot]$.*

To initiate the proof of (2.7.5.2), let $G \in \mathcal{S}$. Then by Proposition (2.7.3) and Lemma 2.7.9, we have for $t \in [0,T]$:

$$\begin{aligned}
&|[\mathbf{V}_{sol}(t) - {}^{(0)}\mathring{\mathbf{V}}, G]| \leq |[{}^{(m)}\mathbf{V}(t) - {}^{(0)}\mathring{\mathbf{V}}, G]| + |[\mathbf{V}_{sol}(t) - {}^{(m)}\mathbf{V}(t), G]| \qquad (2.7.5.3)\\
&\leq C(N)\|{}^{(m)}\mathbf{V}(t) - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N}\|G\|_{H^{-N}} + C(N-1)\|\mathbf{V}_{sol}(t) - {}^{(m)}\mathbf{V}(t)\|_{H^{N-1}}\|G\|_{H^{-(N-1)}}\\
&\leq C(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\mathring{\mathbf{V}}}}, \Lambda)\|G\|_{H^{-N}} + C(N-1)\|\mathbf{V}_{sol}(t) - {}^{(m)}\mathbf{V}(t)\|_{H^{N-1}}\|G\|_{H^{-(N-1)}}.
\end{aligned}$$

Taking the $\limsup_{m\to\infty}$ of both sides of (2.7.5.3) and using (2.7.4.3), we have

$$|[\mathbf{V}_{sol}(t) - {}^{(0)}\mathring{\mathbf{V}}, G]| \leq C(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\mathring{\mathbf{V}}}}, \Lambda)\|G\|_{H^{-N}}. \qquad (2.7.5.4)$$

Therefore, by the density of $\mathcal{S}$ in H^{-N}, $[\mathbf{V}_{sol}(t) - {}^{(0)}\mathring{\mathbf{V}}, \cdot]$ extends to a continuous linear functional on H^{-N}. By Lemma 2.7.9, $\mathbf{V}_{sol}(t) - {}^{(0)}\mathring{\mathbf{V}} \in H^N(\mathbb{R}^3)$ and

$$\|\mathbf{V}_{sol}(t) - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N} \leq C(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\mathring{\mathbf{V}}}}, \Lambda), \qquad (2.7.5.5)$$

from which (2.7.5.2) follows.

The second stage of the proof of (2.7.5.1) is the proof of the claim

$$\mathbf{V}_{sol}(t) - \mathring{\mathbf{V}} \in C_{weak}([0,T], H^N), \qquad (2.7.5.6)$$

which means that for every $G \in H^{-N}(\mathbb{R}^3)$, $[\mathbf{V}_{sol}(t) - \mathring{\mathbf{V}}, G]$ is a continuous function of t on $[0,T]$. To prove (2.7.5.6), first fix $G \in H^{-N}(\mathbb{R}^3)$, and let $\{G_k\} \subset \mathcal{S}$ be a sequence of Schwartz functions such that $G_k \to G$ in H^{-N}. Then, by Lemma 2.7.9, (3.8.2.1a), and (2.7.5.5), we have

$$\begin{aligned}
&|[\mathbf{V}_{sol}(t) - {}^{(m)}\mathbf{V}(t), G]| \leq |[\mathbf{V}_{sol}(t) - {}^{(m)}\mathbf{V}(t), G - G_k]| + |[\mathbf{V}_{sol}(t) - {}^{(m)}\mathbf{V}(t), G_k]| \qquad (2.7.5.7)\\
&\leq C\|\mathbf{V}_{sol}(t) - {}^{(m)}\mathbf{V}(t)\|_{H^N}\|G - G_k\|_{H^{-N}} + C\|\mathbf{V}_{sol}(t) - {}^{(m)}\mathbf{V}(t)\|_{H^{N-1}}\|G_k\|_{H^{-(N-1)}}\\
&\leq C(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\mathring{\mathbf{V}}}}, \Lambda)\|G - G_k\|_{H^{-N}} + C\|\mathbf{V}_{sol}(t) - {}^{(m)}\mathbf{V}(t)\|_{H^{N-1}}\|G_k\|_{H^{-(N-1)}}.
\end{aligned}$$

By taking k large enough, we can make the first term on the righthand side of (2.7.5.7) arbitrarily small. For a fixed k, (2.7.4.3) implies that the second term on the right-hand side of (2.7.5.7) can be made arbitrarily small, independent of all $t \in [0, T]$, by taking m large enough. We thus conclude that

$$[{}^{(m)}\mathbf{V}(t) - \mathring{\mathbf{V}}, G] \to [\mathbf{V}_{sol}(t) - \mathring{\mathbf{V}}, G] \tag{2.7.5.8}$$

uniformly for $t \in [0, T]$ as $m \to \infty$. In particular, for each $t \in [0, T]$, we have

$${}^{(m)}\mathbf{V}_{sol}(t) - \mathring{\mathbf{V}} \rightharpoonup \mathbf{V}_{sol}(t) - \mathring{\mathbf{V}} \text{ weakly in } H^N \text{ as } m \to \infty, \tag{2.7.5.9}$$

a result which we will quote later. Now, by Lemma 2.7.9, we have

$$|[{}^{(m)}\mathbf{V}(t) - {}^{(m)}\mathbf{V}(t_0), G]| \leq C(N) \|{}^{(m)}\mathbf{V}(t) - {}^{(m)}\mathbf{V}(t_0)\|_{H^N} \|G\|_{H^{-N}}. \tag{2.7.5.10}$$

Combining Corollary 2.7.5 with (2.7.5.10), we conclude that for $m \in \mathbb{N}$,

$$[{}^{(m)}\mathbf{V}(t) - \mathring{\mathbf{V}}, G] \text{ is a continuous function of } t. \tag{2.7.5.11}$$

By (2.7.5.8) and (2.7.5.11), $[\mathbf{V}_{sol}(t) - \mathring{\mathbf{V}}, G]$, viewed as a function of t, is the uniform limit of continuous functions. Therefore,

$$[\mathbf{V}_{sol}(t) - \mathring{\mathbf{V}}, G] \text{ is a continuous function of } t, \tag{2.7.5.12}$$

which proves the claim (2.7.5.6).

The third stage in the proof of (2.7.5.1) relies on the following result from functional analysis.

Lemma 2.7.10. *If $\mathcal{H}$ is a Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$, and $\{w_j\} \subset \mathcal{H}$ is a sequence of vectors that converges weakly to $w \in \mathcal{H}$, then*

$$\|w\|_{\mathcal{H}} \leq \limsup_{j\to\infty} \|w_j\|_{\mathcal{H}} \tag{2.7.5.13}$$

and furthermore,

$$\lim_{j\to\infty} \|w - w_j\|_{\mathcal{H}} = 0 \iff \limsup_{j\to\infty} \|w_j\|_{\mathcal{H}} \leq \|w\|_{\mathcal{H}}. \tag{2.7.5.14}$$

Remark 2.7.7. Suppose $\mathcal{H}$ is equipped with two equivalent norms, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|'_{\mathcal{H}}$, defined through inner products $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$ respectively. Then the notion of weak convergence $w_m \rightharpoonup w$ is independent of the inner product; i.e., $\lim_{m\to\infty}\langle w_m, v\rangle = \langle w, v\rangle$ holds for all $v \in \mathcal{H} \iff \lim_{m\to\infty}\langle w_m, v\rangle' = \langle w, v\rangle'$ holds for all $v \in \mathcal{H}$. This is true because the equivalence of the two norms, together with the Cauchy-Schwarz inequality, imply that $\langle\cdot, v\rangle$ is a linear functional bounded in the norm $\|\cdot\|'_{\mathcal{H}}$, while $\langle\cdot, v\rangle'$ is a linear functional bounded in the norm $\|\cdot\|_{\mathcal{H}}$.

In light of Lemma 2.7.10, we now define norms on $H^N(\mathbb{R}^3)^{10}$ that are equivalent to the usual one. Given a BGS $\widetilde{\mathbf{V}}$ as defined in Section 2.4.2, we define the norm

$$\|\dot{\mathbf{V}}\|_{H^N,\widetilde{\mathbf{V}}(t)} \stackrel{\text{def}}{=} \Big(\sum_{0\leq|\vec{\alpha}|\leq N} \int_{\mathbb{R}^3} \dot{J}^0_{\vec{\alpha}}(t,\mathbf{s})\, d^3\mathbf{s}\Big)^{\frac{1}{2}}, \tag{2.7.5.15}$$

where $\dot{J}_{\vec{\alpha}}$ is the energy current for the variation $\partial_{\vec{\alpha}}\dot{\mathbf{V}}$ defined by the BGS $\widetilde{\mathbf{V}}$. If $\widetilde{\mathbf{V}}(t)\left(\mathbb{R}^3\right) \subset \bar{\mathcal{O}}_2$, then as a simple consequence of (2.6.2.2), we have that the norm in (2.7.5.15) is equivalent to the usual norm on $H^N(\mathbb{R}^3)^{10}$. Here, the notation $H^N(\mathbb{R}^3)^{10}$ is used to emphasize that $\dot{\mathbf{V}}$ is a $\mathbb{R}^{10}$-valued function on $\mathcal{M}$. This norm arises from an inner product on $H^N(\mathbb{R}^3)^{10}$ obtained from treating the $\dot{J}^0_{\vec{\alpha}}$ as a sesquilinear form in two variations $(\dot{\mathbf{V}}, \dot{\mathbf{Y}})$ rather than a quadratic form in $\dot{\mathbf{V}}$ in the obvious manner.

By (2.7.5.6), Remark 2.7.7, and Lemma 2.7.10, applied to the equivalent norms $\|\cdot\|_{H^N}$ and $\|\cdot\|_{H^N,\mathring{\mathbf{V}}}$, in order to prove (2.7.5.1), we only need to show that $\limsup_{t\to 0}\|\mathbf{V}_{sol}(t) - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N,\mathring{\mathbf{V}}} \leq \|\mathring{\mathbf{V}} - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N,\mathring{\mathbf{V}}}$. By Remark 2.5.4, definition (2.7.5.15), and (2.7.4.5), demonstrating this inequality is equivalent to showing

$$\limsup_{t\to 0}\|\mathbf{V}_{sol}(t) - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N,\mathbf{V}(t)} \leq \|\mathring{\mathbf{V}} - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N,\mathring{\mathbf{V}}}. \tag{2.7.5.16}$$

We prove (2.7.5.16) with the help of a modification to the proof of Proposition 2.7.3. Making use of definition (2.7.5.15), we take the limit $r \to \infty$ in (2.7.2.29) and arrive the following bound (in which we write $C = C(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\mathring{\mathbf{V}}}}, \Lambda)$ for notational ease), valid for $t \in [0, T]$:

$$\|{}^{(m+1)}\mathbf{V}(t) - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N,{}^{(m)}\mathbf{V}(t)} \leq \big(\|{}^{(m+1)}\mathring{\mathbf{V}} - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N,{}^{(m)}\mathring{\mathbf{V}}} + Ct\big)\cdot\exp\!\big(Ct\big). \tag{2.7.5.17}$$

By (2.7.2.2), (2.7.2.4), (2.7.2.6), (2.7.4.5), definition (2.7.5.15), and Remark 2.5.4, we have

$$\limsup_{m\to\infty}\big(\|^{(m+1)}\mathring{\mathbf{V}} - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N,{}^{(m)}\mathring{\mathbf{V}}} + Ct\big)\cdot\exp\big(Ct\big) \qquad (2.7.5.18)$$
$$= \big(\|\mathring{\mathbf{V}} - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N,\mathring{\mathbf{V}}} + Ct\big)\cdot\exp\big(Ct\big).$$

Using (2.7.5.9) and Remark 2.7.7, we apply Lemma 2.7.10 to the norm $\|\cdot\|_{H^N,\mathbf{V}_{sol}(t)}$, obtaining

$$\|\mathbf{V}_{sol}(t) - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N,\mathbf{V}_{sol}(t)} \leq \limsup_{m\to\infty}\|^{(m+1)}\mathbf{V}(t) - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N,\mathbf{V}_{sol}(t)}. \qquad (2.7.5.19)$$

By Remark 2.5.4, definition (2.7.5.15) and (2.7.4.5), we also have that

$$\limsup_{m\to\infty}\|^{(m+1)}\mathbf{V}(t) - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N,\mathbf{V}_{sol}(t)} = \limsup_{m\to\infty}\|^{(m+1)}\mathbf{V}(t) - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N,{}^{(m)}\mathbf{V}(t)}. \qquad (2.7.5.20)$$

Combining (2.7.5.19), (2.7.5.20), (2.7.5.17), and (2.7.5.18) (in this order), we obtain

$$\|\mathbf{V}_{sol}(t) - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N,\mathbf{V}_{sol}(t)} \leq \Big(\|\mathring{\mathbf{V}} - {}^{(0)}\mathring{\mathbf{V}}\|_{H^N,\mathring{\mathbf{V}}} + Ct\Big)\cdot\exp\left(Ct\right). \qquad (2.7.5.21)$$

Taking $\limsup_{t\to 0}$ of both sides of (2.7.5.21) proves (2.7.5.16), which concludes the proof of (2.7.5.1).

The Banach Space Differentiability of $\mathbf{V}_{sol}(t)$

We first show that

$$\partial_t\mathbf{V}_{sol} \in C^0([0,T], H^{N-1}(\mathbb{R}^3)). \qquad (2.7.5.22)$$

To prove, (2.7.5.22) we first solve for $\partial_t\mathbf{W}_{sol}$, where $\mathbf{W}_{sol}$ denotes the first 5 components of $\mathbf{V}_{sol}$, using the equations (2.4.1.18) - (2.4.1.20). We have that

$$\partial_t\mathbf{W}_{sol} = \big(A^0(\mathbf{V}_{sol})\big)^{-1}\Big[\mathbf{b}(\mathbf{V}_{sol}) - A^k(\mathbf{V}_{sol})\partial_k\left(\mathbf{W}_{sol}\right)\Big], \qquad (2.7.5.23)$$

where $\mathbf{b}(\mathbf{V}_{sol})$ denotes the right-hand side of (2.4.1.18) - (2.4.1.20) viewed as a function of $\mathbf{V}_{sol}$. From (2.7.5.23), (2.7.4.4), and Corollary B.0.3 in the Appendix, it immediately follows that $\partial_t\mathbf{W}_{sol} \in C^0([0,T], H^{N-1})$. Applying similar reasoning to $(\phi, \partial_t\phi, \partial_1\phi, \partial_2\phi, \partial_3\phi)$, we conclude (2.7.5.22).

To finish the proof of the local existence aspect of the theorem, we must justify the claims

$$\mathbf{V}_{sol} \in C^1([0,T], H^{N-1}_{\bar{\mathbf{V}}}), \tag{2.7.5.24}$$

$$\phi \in C^2([0,T], H^{N-1}_{\bar{\phi}}). \tag{2.7.5.25}$$

By statement (2.7.5.24), we mean that $\lim_{h\to 0}\|\frac{\mathbf{V}_{sol}(t+h)-\mathbf{V}_{sol}(t)}{h} - \partial_t \mathbf{V}_{sol}(t)\|_{H^{N-1}} = 0$ holds for every $t \in (0,T)$, and similarly for $h \to 0^+$ and $h \to T^-$ at $t=0$ and $t=T$ respectively; the analogous meaning is ascribed to statement (2.7.5.25). We give the proof of right differentiability of $\mathbf{V}_{sol}$ at $t=0$. For h small and positive, we have

$$\left\| \frac{\mathbf{V}_{sol}(h) - \mathbf{V}_{sol}(0)}{h} - \partial_t \mathbf{V}_{sol}(0) \right\|_{H^{N-1}} = \left\| \frac{1}{h}\int_0^h \partial_t \mathbf{V}_{sol}(t') - \partial_t \mathbf{V}_{sol}(0)\, dt' \right\|_{H^{N-1}} \tag{2.7.5.26}$$

$$\leq \frac{1}{h}\int_0^h \left\|\partial_t \mathbf{V}_{sol}(t') - \partial_t \mathbf{V}_{sol}(0)\right\|_{H^{N-1}}\, dt' \leq \sup_{t'\in[0,h]} \left\|\partial_t \mathbf{V}_{sol}(t') - \partial_t \mathbf{V}_{sol}(0)\right\|_{H^{N-1}}.$$

The equal sign above is justified since $\mathbf{V}_{sol}$ is a classical solution, while the first inequality is a theorem from the theory of integration of Banach space valued functions of t (see [52] for a discussion of integration in Banach spaces). Now we use (2.7.5.22) to conclude that the right-hand side of (2.7.5.26) goes to 0 as $h \to 0^+$; this proves the claim (2.7.5.24). Note that since (2.7.4.6) shows that $\partial_t\phi$ is one of the components of $\mathbf{V}_{sol}$, (2.7.5.24) implies that

$$\lim_{h\to 0}\left\| \frac{\partial_t\phi(t+h) - \partial_t\phi(t)}{h} - \partial_t^2\phi(t)\right\|_{H^{N-1}(\mathbb{R}^3)} = 0. \tag{2.7.5.27}$$

This justifies the claim (2.7.5.25).

Uniqueness and H^{N-1}–Continuous Dependence on Initial Data.

Let $\mathring{\mathbf{V}}$ denote initial data that launch a solution $\mathbf{V}$ of the EN_κ system as furnished by the existence aspect of Theorem 2.7.1. Let $\delta, B_\delta(\mathring{\mathbf{V}}), T'$, and $K(N, \bar{\mathcal{O}}_2, \|^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\bar{\mathbf{V}}}}, \delta)$ be as in corollaries 2.7.1 and 2.7.2. Assume that the initial data $\mathring{\widetilde{\mathbf{V}}}$ belong to B_δ, and let $\widetilde{\mathbf{V}}$ be a solution of the EN_κ system with initial data $\mathring{\widetilde{\mathbf{V}}}$ existing on the interval $[0,T']$ guaranteed by Corollary 2.7.1. We now define

$$\dot{\mathbf{V}} \overset{\text{def}}{=} \widetilde{\mathbf{V}} - \mathbf{V}. \tag{2.7.5.28}$$

It follows from definition (2.7.5.28) that $\dot{\mathbf{V}}$ is a solution to the EOV (2.4.2.1) - (2.4.2.6) defined by the BGS $\widetilde{\mathbf{V}}$ with inhomogeneous terms given by

$$f = (U^\mu - \widetilde{U}^\mu)\partial_\mu \eta \tag{2.7.5.29}$$

$$g = (U^\mu - \widetilde{U}^\mu)\partial_\mu P + \big[QU_k/U^0 - \widetilde{Q}\widetilde{U}_k/\widetilde{U}^0\big]\partial_0 U^k \tag{2.7.5.30}$$

$$+ (Q - \widetilde{Q})\partial_k U^k + (4\widetilde{P} - 3\widetilde{Q})\widetilde{U}^\mu\widetilde{\psi}_\mu - (4P - 3Q)U^\mu\psi_\mu$$

$$h^{(j)} = \big[(R+P)U^\mu - (\widetilde{R}+\widetilde{P})\widetilde{U}^\mu\big]\partial_\mu U^j + (\Pi^{\mu j} - \widetilde{\Pi}^{\mu j})\partial_\mu P \tag{2.7.5.31}$$

$$+ (3\widetilde{P} - \widetilde{R})\widetilde{\Pi}^{\mu j}\widetilde{\psi}_\mu - (3P - R)\Pi^{\mu j}\psi_\mu \tag{2.7.5.32}$$

$$l^{(0)} = \kappa^2(\widetilde{\phi} - \phi) + (\widetilde{R} - 3\widetilde{P}) - (R - 3P). \tag{2.7.5.33}$$

$$l^{(j)} = 0 \qquad (j = 1, 2, 3) \tag{2.7.5.34}$$

$$l^{(4)} = \widetilde{\psi}_0 - \psi_0, \tag{2.7.5.35}$$

and we denote them using the abbreviated notation $\mathbf{b}$ and $\mathbf{l}$ defined in (2.4.2.17) and (2.4.2.18).

By combining propositions B.0.2, Remark (B.0.3), and B.0.4 of the Appendix, (noting the particular manner in which the inhomogeneous terms depend on the difference of $\mathbf{V}$ and $\widetilde{\mathbf{V}}$) we have that

$$||| \, (\mathbf{b}, \mathbf{l}) \, |||_{H^{N-1},T'} \leq C(N, K, \bar{\mathcal{O}}_2, \Lambda) \, ||| \, \dot{\mathbf{V}} \, |||_{H^{N-1}_{\tilde{\mathbf{V}}},T'} \, . \tag{2.7.5.36}$$

Without providing details, we reason as in our proof of Proposition 2.7.3, using (2.7.5.36) in place of (2.7.2.41) and (2.7.2.44) to arrive at the following bound:

$$||| \, \dot{\mathbf{V}} \, |||_{H^{N-1}_{\tilde{\mathbf{V}}},T'} \leq C_{\bar{\mathcal{O}}_2}^{-1} \|\dot{\mathbf{V}}(0)\|_{H^{N-1}} \cdot \exp\left(CT'\right), \tag{2.7.5.37}$$

where $\dot{\mathbf{V}}(0) \overset{\text{def}}{=} \mathring{\widetilde{\mathbf{V}}} - \mathring{\mathbf{V}}$.

Inequality (2.7.5.37) is the analog of the first inequality in (2.7.2.31), but the "$\dot{\mathbf{V}}(0)+ CT_*$" term from (2.7.2.31) has been replaced with "$\dot{\mathbf{V}}(0)$" in (2.7.5.37). This change occurs because unlike the estimate (2.7.2.38) used in obtaining (2.7.2.31), the estimate (2.7.5.36), which is essential for obtaining (2.7.5.37), involves a *linear* bound in $||| \, \dot{\mathbf{V}} \, |||_{H^{N-1}_{\tilde{\mathbf{V}}},T'}$.

We now observe that (2.7.5.37) implies both the uniqueness statement in Theorem 2.7.1 and the Lipschitz-$H^{N-1}-$continuous dependence on initial conditions mentioned in Remark 2.7.2.

Remark 2.7.8. We cannot obtain an estimate analogous to (2.7.5.37) by using the $H^N_{\dot{\mathbf{V}}}$ norm in place of the $H^{N-1}_{\dot{\mathbf{V}}}$ norm; the inhomogeneous terms (2.7.5.29) - (2.7.5.35) already contain 1 derivative of $\mathbf{V}$, and therefore cannot be bounded in the H^N norm. However, for $N' < N$, we can obtain an estimate for the $H^{N'}_{\dot{\mathbf{V}}}$ norm by combining Proposition B.0.6, (2.7.5.37) and the uniform bound provide by the constant K. The inequality we obtain is

$$||| \dot{\mathbf{V}} |||_{H^{N'}_{\dot{\mathbf{V}}},T'} \leq C\|\dot{\mathbf{V}}(0)\|^{1-N'/N}_{H^{N'}} \cdot \exp\left(CT'\right), \tag{2.7.5.38}$$

where the constant C in (2.7.5.38) depends on $N', \bar{\mathcal{O}}_2$ and K.

In particular, by fixing $N' > 5/2$, we obtain through Sobolev imbedding that

$$||| \widetilde{\mathbf{V}} - \mathbf{V} |||_{C^1_b,T'} \to 0 \text{ as } \|\mathring{\widetilde{\mathbf{V}}} - \mathring{\mathbf{V}}\|_{H^N} \to 0 \tag{2.7.5.39}$$

a result which we will quote below.

Remark 2.7.9. The estimate (2.7.5.37) is a limiting version of the "conical" estimate

$$\|\dot{\mathbf{V}}(t)\|_{H^{N-1}(\Sigma_{t,r-t})} \leq C^{-1}_{\bar{\mathcal{O}}_2}\|\dot{\mathbf{V}}(0)\|_{H^{N-1}(\Sigma_{0,r})} \cdot \exp\left(CT\right), \tag{2.7.5.40}$$

where we are using notation defined in Lemma 2.7.4. A proof of (2.7.5.40) can be constructed using arguments similar to the ones used in our proof of (2.7.2.30). Inequality (2.7.5.40) shows that two solutions that agree on $\Sigma_{0,r}$ also agree on $\Sigma_{t,r-t}$. By translating the cone from Lemma 2.7.4 so that its lower base is centered at the spacetime point x, we may produce a translated version of the inequality. Thus, we observe that a *domain of dependence* for x is given by the solid backward light cone with vertex at x; i.e., the past (relative to x) behavior of a solution to the EOV outside of this cone does not influence behavior of the solution at x. Similarly, a *domain of influence* of x is the solid forward light cone with vertex at x; the behavior of a solution at x does not influence the future (relative to x) behavior of the solution outside of this cone. In [15], Christodoulou gives an advanced discussion of these and related topics for hyperbolic PDEs derivable from a Lagrangian.

This completes our proof of Theorem 2.7.1.

2.7.6 Proof of Theorem 2.7.2.

The Setup

Let $\{\mathring{\mathbf{V}}^m\}$ be the sequence of initial data from the hypotheses of Theorem 2.7.2 converging in $H^N_{\bar{\mathbf{V}}}$ to $\mathring{\mathbf{V}}$. By corollaries 2.7.1 and 2.7.2, for all large m, the initial data $\mathring{\mathbf{V}}^m$ and $\mathring{\mathbf{V}}$ launch unique solutions $\mathbf{V}^m$ and $\mathbf{V}$ respectively to (2.4.1.18) - (2.4.1.23) that exist on a common interval $[0,T']$ and that have the property $\mathbf{V}([0,T']\times\mathbb{R}^3), \mathbf{V}^m([0,T']\times\mathbb{R}^3) \subset \bar{\mathcal{O}}_2$. Furthermore, for all large m, with $K = K(N, \bar{\mathcal{O}}_2, \|{}^{(0)}\mathring{\mathbf{V}}\|_{H^{N+1}_{\bar{\mathbf{V}}}}, \delta)$, we have the uniform (in m) bounds

$$||| \ \mathbf{V} \ |||_{H^N_{\bar{\mathbf{V}}},T'}, ||| \ \partial_t\mathbf{V} \ |||_{H^{N-1},T'}, ||| \ \mathbf{V}^m \ |||_{H^N_{\bar{\mathbf{V}}},T'}, ||| \ \partial_t\mathbf{V}^m \ |||_{H^{N-1},T'} < \ K. \tag{2.7.6.1}$$

In this section, we will show that for all large $m, \mathbf{V}^m$ exists on $[0,T]$ and

$$\lim_{m\to\infty} ||| \ \mathbf{V}^m - \mathbf{V} \ |||_{H^N,T}= 0, \tag{2.7.6.2}$$

where $[0,T]$ is the interval of existence for $\mathbf{V}$.

The proof we give here is inspired by a similar proof given by Kato in [27]. We use results and terminology from the theory of abstract evolution equations in Banach spaces, an approach that streamlines the argument. We also freely use results from the theory of integration in Banach spaces; a detailed discussion of this theory may be found in [52]. We begin by rewriting the linearization of the EN$_\kappa$ system around $\mathbf{V}^m$ and $\mathbf{V}$ as abstract evolution equations in the Banach space $H^N(\mathbb{R}^3)^{10}$. In this form, the linearized systems are written as

$$\partial_t\mathbf{Z} + \mathcal{A}(\mathbf{V})\mathbf{Z} = \mathbf{f}(\mathbf{V}), \tag{2.7.6.3}$$

$$\partial_t\mathbf{Z} + \mathcal{A}(\mathbf{V}^m)\mathbf{Z} = \mathbf{f}(\mathbf{V}^m), \tag{2.7.6.4}$$

where $\mathbf{f}(\cdot)$ is a smooth function on $\mathcal{O}$. Here, the symbol $\mathbf{Z}$ stands for all 10 components of a solution to a linearized system, and the operator $\mathcal{A}(\cdot)$ is a first order spatial differential operator with coefficients that depend smoothly on its arguments. We state for clarity that the first 5 components of the inhomogeneous terms $\mathbf{f}(\mathbf{V}^m)$ are given by

$\left(A^0(\mathbf{V}^m)\right)^{-1} \cdot (\mathfrak{F}(\mathbf{V}^m), \mathfrak{G}(\mathbf{V}^m), \cdots, \mathfrak{H}^{(3)}(\mathbf{V}^m))^{Transpose}$, where the matrix-valued function $A^0(\cdot)$ is defined in (2.4.2.20) and the scalar-valued functions $\mathfrak{F}, \mathfrak{G}, \cdots \mathfrak{H}^3$ are defined in (2.4.2.11) - (2.4.2.13).

We will make use of the differential operator

$$\mathrm{S} \stackrel{\text{def}}{=} (1-\Delta)^{N/2}, \tag{2.7.6.5}$$

which is an isomorphism between H^N and L^2; i.e., $\mathrm{S} \in \mathfrak{L}(H^N, L^2)$ and $\mathrm{S}^{-1} \in \mathfrak{L}(L^2, H^N)$.

Technical Estimates

For certain function spaces X, there exist evolution operators

$$\mathcal{U}(t,t'),\ \mathcal{U}^m(t,t') : X \to X \tag{2.7.6.6}$$

defined on $\triangle_{T'} \stackrel{\text{def}}{=} \{0 \leq t' \leq t \leq T'\}$ that map solutions (belonging to the space X) of the corresponding *homogeneous* version of the linearized systems (2.7.6.3) and (2.7.6.4) at time t' to solutions at time t. The relevant spaces in our discussion are $X = L^2$ and $X = H^N$. In the following three lemmas, we describe the properties of the operators $\mathcal{U}(t,t')$ and $\mathcal{U}^m(t,t')$. Complete proofs are given in [25], [26], and [27]; rather than repeating them, we instead attempt to provide insight as to how one may prove them using the methods described in this paper.

Lemma 2.7.11. *$\mathcal{U}(\cdot,\cdot)$ and $\mathcal{U}^m(\cdot,\cdot)$ (for $m \geq 0$) are strongly-continuous maps from $\triangle_{T'}$ into $\mathfrak{L}(L^2) \cap \mathfrak{L}(H^N)$. Furthermore, there exists $C(K) > 0$ such that $|||\, \mathcal{U} \,|||_{L^2,\triangle_{T'}}$ $, |||\, \mathcal{U}^m \,|||_{L^2,\triangle_{T'}}, |||\, \mathcal{U} \,|||_{H^N,\triangle_{T'}}, |||\, \mathcal{U}^m \,|||_{H^N,\triangle_{T'}} < C(K)$.*

Remark 2.7.10. Lemma 2.7.11 is essentially a consequence of the fact that the uniform bound (2.7.6.1) for $\mathbf{V}$ and $\mathbf{V}^m$ allows for uniform Sobolev estimates to be made on L^2 (or H^N) norm of L^2 (or H^N) solutions to the linearized equations.

Remark 2.7.11. By Theorem 2.7.1, Corollary B.0.3, Remark B.0.3, and (2.6.1.2), the right-hand side (2.7.6.3) is an element of $C^0([0,T'], H^N)$. Given initial data $\mathring{\mathbf{Z}} \in H^N_{\mathbf{V}}$, it follows from Lemma 2.7.11 and standard linear theory (via Duhamel's principle) that there exists a unique solution $\mathbf{Z} \in C^0([0,T'], H^N_{\mathbf{V}})$ to (2.7.6.3) with initial data $\mathring{\mathbf{Z}}$. An analogous result holds for solutions to (2.7.6.4).

Lemma 2.7.12. *$\mathcal{U}^m(t,t')$ converges to $\mathcal{U}(t,t')$ strongly in $\mathcal{L}(L^2)$ as $m \to \infty$. Furthermore, the strong convergence is uniform on $\triangle_{T'}$.*

Remark 2.7.12. By smoothing the initial data, a solution $\mathbf{Z} \in C^0([0,T], L^2)$ to $\partial_t \mathbf{Z} + \mathcal{A}(\mathbf{V})\mathbf{Z} = 0$ can be realized as the limit (in the norm $|||\cdot|||_{L^2,T}$) of a sequence $\{\mathbf{Z}^m\} \subset C^0([0,T], H^N)$. Therefore, to prove Lemma 2.7.12, one only needs to check that given initial data $\mathring{\mathbf{Z}} \in H^N$, we have that

$$\lim_{m\to\infty} |||\,(\mathcal{U}^m(\cdot,0) - \mathcal{U}(\cdot,0))\,\mathring{\mathbf{Z}}\,|||_{L^2,T} = 0. \tag{2.7.6.7}$$

Based on Lemma 2.7.11 and (2.7.5.38), which shows that for $N' < N$ we have $\mathbf{V}^m \to \mathbf{V}$ in $C^0([0,T'], H^{N'}_{\bar{\mathbf{V}}})$, (2.7.6.7) easily follows from the method of energy currents.

Lemma 2.7.13. *There exist operator valued functions $\mathcal{B}, \mathcal{B}^m : [0,T'] \to \mathcal{L}(L^2)$ such that*

$$\mathrm{S}\mathcal{A}(\mathbf{V}(t))\mathrm{S}^{-1} = \mathcal{A}(\mathbf{V}(t)) + \mathcal{B}(t), \tag{2.7.6.8}$$

$$\mathrm{S}\mathcal{A}(\mathbf{V}^m(t))\mathrm{S}^{-1} = \mathcal{A}(\mathbf{V}^m(t)) + \mathcal{B}^m(t). \tag{2.7.6.9}$$

For for all t' with $0 \leq t' \leq T'$, $\mathcal{B}$ and $\mathcal{B}^m$ satisfy the estimates

$$|||\,\mathcal{B}^m - \mathcal{B}\,|||_{L^2,t'} \leq C(K)\,|||\,\mathbf{V}^m - \mathbf{V}\,|||_{H^N,t'} \tag{2.7.6.10}$$

$$\text{and}\ \ |||\,\mathcal{B}\,|||_{L^2,T'},\ |||\,\mathcal{B}^m\,|||_{L^2,T'} \leq C(K). \tag{2.7.6.11}$$

Lemma 2.7.14. *Let $\mathring{\mathbf{Z}}$ denote initial data in $H^N_{\bar{\mathbf{V}}}$, and let $\mathbf{Z} \in C^0([0,T'], H^N_{\bar{\mathbf{V}}})$ denote the unique solution to* (2.7.6.3) *with initial data $\mathring{\mathbf{Z}}$ furnished by Remark 2.7.11. Then* $\mathrm{S}(\mathbf{Z} - \bar{\mathbf{V}})$ *satisfies the Duhamel formula*

$$\mathrm{S}(\mathbf{Z}(t) - \bar{\mathbf{V}}) = \mathcal{U}(t,0)\mathrm{S}(\mathring{\mathbf{Z}} - \bar{\mathbf{V}}) - \int_0^t \mathcal{U}(t,t')\mathcal{B}(t')\mathrm{S}(\mathbf{Z}(t') - \bar{\mathbf{V}})\,dt' \tag{2.7.6.12}$$

$$+ \int_0^t \mathcal{U}(t,t')\mathrm{S}\mathbf{f}(\mathbf{V}(t'))\,dt'. \tag{2.7.6.13}$$

An analogous result holds for the linearization of the EN_κ system around $\mathbf{V}^m$.

Proof. We apply S to each side of the equation satisfied by $\mathbf{Z} - \bar{\mathbf{V}}$ and use Lemma 2.7.13 to arrive at the equation

$$\partial_t\left[\mathrm{S}(\mathbf{Z} - \bar{\mathbf{V}})\right] + \mathcal{A}(\mathbf{V})\mathrm{S}(\mathbf{Z} - \bar{\mathbf{V}}) = -\mathcal{B}(\mathbf{V})\mathrm{S}(\mathbf{Z} - \bar{\mathbf{V}}) + \mathrm{S}\mathbf{f}(\mathbf{V}). \tag{2.7.6.14}$$

Thus, $\mathrm{S}(\mathbf{Z}-\bar{\mathbf{V}})$ is a solution to the same linear equation that $\mathbf{Z}$ solves, except the inhomogeneous terms for $\mathrm{S}(\mathbf{Z}-\bar{\mathbf{V}})$ are given by the right-hand side of (2.7.6.14) and the initial data are given by $\mathrm{S}(\mathring{\mathbf{Z}}-\bar{\mathbf{V}})$. Equation (2.7.6.12) now follows from Duhamel's principle. $\square$

Proof

We will now demonstrate (2.7.6.2) by providing a proof of the equivalent statement

$$\lim_{m\to\infty} |||\, \mathrm{S}\left(\mathbf{V}^m-\mathbf{V}\right) |||_{L^2,T} = 0. \tag{2.7.6.15}$$

Lemma 2.7.14 gives the following equality, valid for $0 \le t \le T'$:

$$\begin{aligned} &\mathrm{S}\big(\mathbf{V}^m(t)-\mathbf{V}(t)\big) = \mathcal{U}^m(t,0)\mathrm{S}\big(\mathring{\mathbf{V}}^m-\mathring{\mathbf{V}}\big) + \big(\mathcal{U}^m(t,0)-\mathcal{U}(t,0)\big)\mathrm{S}(\mathring{\mathbf{V}}-\bar{\mathbf{V}}) \\ &+\int_0^t \mathcal{U}(t,t')\mathcal{B}(t')\mathrm{S}(\mathbf{V}(t')-\bar{\mathbf{V}})\,dt' - \int_0^t \mathcal{U}^m(t,t')\mathcal{B}^m(t')\mathrm{S}(\mathbf{V}^m(t')-\bar{\mathbf{V}})\,dt' \\ &+\int_0^t \mathcal{U}^m(t,t')\mathrm{S}\mathbf{f}(\mathbf{V}^m(t'))\,dt' - \int_0^t \mathcal{U}(t,t')\mathrm{S}\mathbf{f}(\mathbf{V}(t'))\,dt'. \end{aligned} \tag{2.7.6.16}$$

By Lemma 2.7.11, we have that

$$|||\, \mathcal{U}^m(t,0)\mathrm{S}\big(\mathring{\mathbf{V}}^m-\mathring{\mathbf{V}}\big) |||_{L^2,T'} \le C(\mathrm{S},K)\, |||\, \mathring{\mathbf{V}}^m-\mathring{\mathbf{V}} |||_{H^N,T'}\,. \tag{2.7.6.17}$$

We now rewrite the second line of (2.7.6.16) as

$$\begin{aligned} &\int_0^t (\mathcal{U}(t,t')-\mathcal{U}^m(t,t'))\mathcal{B}(t')\mathrm{S}(\mathbf{V}(t')-\bar{\mathbf{V}})\,dt' \\ &\quad+\int_0^t \mathcal{U}^m(t,t')\left(\mathcal{B}(t')-\mathcal{B}^m(t')\right)\mathrm{S}(\mathbf{V}(t')-\bar{\mathbf{V}})\,dt' \\ &\quad+\int_0^t \mathcal{U}^m(t,t')\mathcal{B}^m(t')\mathrm{S}\left(\mathbf{V}(t')-\mathbf{V}^m(t')\right)\,dt'. \end{aligned} \tag{2.7.6.18}$$

By (2.7.6.1), Lemma 2.7.11 and Lemma 2.7.13, for $0 \le t \le T_* \le T'$, the L^2 norms of the second term and third addends in (2.7.6.18) are each bounded from above by $C(\mathrm{S},K)T_*\, |||\, \mathbf{V}^m-\mathbf{V} |||_{H^N,T_*}$.

We similarly split the third line of (2.7.6.16) into two terms and use (2.7.6.1), Lemma 2.7.11, Lemma 2.7.13, Proposition B.0.4, and Remark B.0.4 to bound the L^2 norm of one of them from above by $C(\mathrm{S},K)T_*\, |||\, \mathbf{V}^m-\mathbf{V} |||_{H^N,T_*}$.

Combining these estimates with (2.7.6.17), we take the L^2 norm of each side of (2.7.6.16) followed by the sup over $t \in [0, T_*]$ to arrive at the inequality

$$||| \, \mathrm{S}\,(\mathbf{V}^m - \mathbf{V}) \, |||_{L^2,T_*} \leq C(\mathrm{S}, K) \|\mathring{\mathbf{V}}^m - \mathring{\mathbf{V}}\|_{H^N} + C(\mathrm{S}, K)T_* \, ||| \, \mathbf{V}^m - \mathbf{V} \, |||_{H^N,T_*} \tag{2.7.6.19}$$

$$+ \int_0^{T_*} \sup_{t\in[0,T_*]} \|(\mathcal{U}(t,t') - \mathcal{U}^m(t,t'))\mathcal{B}(t')\mathrm{S}(\mathbf{V}(t') - \bar{\mathbf{V}})\|_{L^2} \, dt'$$
$$+ \int_0^{T_*} \sup_{t\in[0,T_*]} \|(\mathcal{U}(t,t') - \mathcal{U}^m(t,t'))\mathrm{S}\mathbf{f}(\mathbf{V}(t'))\|_{L^2} \, dt'.$$

We now choose T_* small enough so that

$$C(\mathrm{S}, K)T_* \, ||| \, \mathbf{V}^m - \mathbf{V} \, |||_{H^N,T_*} \leq \frac{1}{2} \, ||| \, \mathrm{S}\,(\mathbf{V}^m - \mathbf{V}) \, |||_{L^2,T_*}, \tag{2.7.6.20}$$

from which it follows that

$$||| \, \mathrm{S}\,(\mathbf{V}^m - \mathbf{V}) \, |||_{L^2,T_*} \leq 2C(\mathrm{S}, K) \|\mathring{\mathbf{V}}^m - \mathring{\mathbf{V}}\|_{H^N} \tag{2.7.6.21}$$
$$+ 2\int_0^{T_*} \sup_{t\in[0,T_*]} \| \left(\mathcal{U}^m(t,t') - \mathcal{U}(t,t')\right) \mathcal{B}(t')\mathrm{S}(\mathbf{V}(t') - \bar{\mathbf{V}})\|_{L^2} \, dt'$$
$$+ 2\int_0^{T_*} \sup_{t\in[0,T_*]} \| \left(\mathcal{U}^m(t,t') - \mathcal{U}(t,t')\right) \mathrm{S}\mathbf{f}(\mathbf{V}(t'))\|_{L^2} \, dt'.$$

By (2.7.6.1), Lemma 2.7.11, Lemma 2.7.13, and Proposition B.0.2, the integrands in (2.7.6.21) are uniformly bounded on $[0, T_*]$. Furthermore, by Lemma 2.7.12, the integrands (viewed functions of t') converge to 0 pointwise as $m \to \infty$. Therefore, by the dominated convergence theorem, the two integrals in (2.7.6.21) converges to 0 as $m \to \infty$. Since we also have by hypothesis that $\lim_{m\to\infty} \|\mathring{\mathbf{V}}^m - \mathring{\mathbf{V}}\|_{H^N} = 0$, we conclude that

$$\lim_{m\to\infty} ||| \, \mathrm{S}\,(\mathbf{V}^m - \mathbf{V}) \, |||_{L^2,T_*} = 0. \tag{2.7.6.22}$$

To extend this argument to the interval $[0, 2T_*]$, let $\epsilon > 0$ and choose m_0 large enough so that $m \geq m_0$ implies that $||| \, \mathbf{V}^m - \mathbf{V} \, |||_{H^N,T_*} < \frac{\epsilon}{4C(\mathrm{S},K)}$. Starting from time T_*, we may argue as above to show that

$$\limsup_{m\to\infty} \sup_{t\in[T_*,2T_*]} \|\mathrm{S}\,(\mathbf{V}^m - \mathbf{V})\,\|_{H^N} \leq \frac{1}{2}\epsilon. \tag{2.7.6.23}$$

Thus, we can choose $m_1 \geq m_0$ such that $|||\ \mathrm{S}\,(\mathbf{V}^m - \mathbf{V})\ |||_{L^2,2T_*} \leq \epsilon$ when $m \geq m_1$. Continuing in this manner, we may inductively extend this argument to the interval $[0, T']$. We state for emphasis that the size of T_* required to satisfy the inequality (2.7.6.20) depends only on $C(\mathrm{S}, K)$. Consequently, the length of the time interval of extension T_* may be chosen to be the same at each step in the induction.

We now show that this argument can be extended to the entire interval $[0, T]$ on which $\mathbf{V}$ exists. Define

$$
\begin{aligned}
T_{max} \stackrel{\mathrm{def}}{=} \sup\{T' \mid \mathbf{V} \text{ and the } \mathbf{V}^m \text{ exist on the interval} \\
[0, T'] \text{ for all large } m \text{ and } \lim_{m \to \infty} |||\ \mathbf{V}^m - \mathbf{V}\ |||_{H^N, T'} = 0\}.
\end{aligned}
\tag{2.7.6.24}
$$

We will show that the assumption $T_{max} < T$ leads to a contradiction.

By Theorem 2.7.1 and Corollary 2.7.1, for each $t \in [0, T]$, there exist an H^N neighborhood $B_{\delta_t}(\mathbf{V}(t))$ of $\mathbf{V}(t)$ with positive radius δ_t and a $\Delta_t > 0$ such that initial data belonging to $B_{\delta_t}(\mathbf{V}(t))$ launch a unique solution that exists on the interval $[t, t + \Delta_t]$ (the term "initial" here refers to the time t). By continuity, $\mathbf{V}([0, T])$ is a compact subset of $H^N_{\mathbf{V}}$. Therefore, there exist $\delta > 0$ and $\Delta > 0$ such that initial data belonging to $B_\delta(\mathbf{V}(t))$ launch a unique solution that exists on the interval $[t, t + \Delta]$; we emphasize that δ and Δ are independent of t belonging to $[0, T]$.

The contradiction is now easily obtained. According to the above paragraph, initial data belonging to $B_\delta(\mathbf{V}(T_{max} - \frac{1}{2}\Delta))$ launch a solution that exists on the interval $[T_{max} - \frac{1}{2}\Delta, T_{max} + \frac{1}{2}\Delta]$. Furthermore, for all large m, $\mathbf{V}^m(T_{max} - \frac{1}{2}\Delta)$ is contained in $B_\delta(\mathbf{V}(T_{max} - \frac{1}{2}\Delta))$. Therefore, for all large m, $\mathbf{V}^m$ can be extended to a solution that exists on $[0, T_{max} + \frac{1}{2}\Delta]$. We can argue as before to show that $\lim_{m\to\infty} |||\ \mathbf{V}^m - \mathbf{V}\ |||_{H^N, T_{max}+\frac{1}{2}\Delta} = 0$. This contradicts the definition of T_{max} and completes the proof of Theorem 2.7.2.

Chapter 3

The Non-relativistic Limit of the Euler Nordström System with Cosmological Constant

In this chapter, we study the Newtonian limit of the family of Euler-Nordström systems indexed by the parameters κ and c (EN_κ^c), where κ^2 is the cosmological constant[a] and c is the speed of light. The limit $c \to \infty$ is singular since the EN_κ^c system is hyperbolic for all finite c, while the limiting system is not hyperbolic. Using Christodoulou's techniques [15] to generate energy currents, we develop Sobolev estimates and use them to prove the following theorem:

Theorem 3.0.3. *As the speed of light c tends to infinity, solutions to the EN_κ^c system converge uniformly on a spacetime slab to solutions of the EP_κ system.*

This theorem is stated and proved rigorously as Theorem 3.9.2. There is a great deal of overlap of this material with that of Chapter 2; for the sake of continuity of the discussion, some of the material has been repeated.

3.0.7 A List of Important Notational Remarks for Chapter 3

- In general, we use the notation for function spaces, differential operators, etc., described in Section 2.2 of Chapter 2.
- We use the convention $x^0 = t$, rather than the usual convention $x^0 = ct$, in our global coordinate system.
- The components of the symbols $\mathbf{V}$, $\mathbf{W}$, $\dot{\mathbf{W}}$, etc., have changed from Chapter 2 because of a change of variables involving the Newtonian velocity $\mathbf{v}$.

[a] The parameter $\kappa > 0$ is fixed throughout this chapter.

- N now denotes an integer that is greater than or equal to 4.

- $\mathcal{O}$ denotes an open, convex set contained in the *admissible subset of truncated state-space*, which is a subset of $\mathbb{R}^5$.

3.1 The Origin of the EN_κ^c System

In this section, we describe the origin of the EN_κ^c system, which was introduced in Section 2.3.3 in dimensionless units. Our discussion here is quite similar, the differences being that that we insert both the speed of light c and Newton's universal gravitational constant G into the system and perform a Newtonian change variables, which brings the system into the form (3.2.1.1) - (3.2.1.4). A similar analysis for the Vlasov-Nordström system is carried out in [10].

3.1.1 Deriving the Equations with c as a Parameter

We assume the existence of a global rectangular (inertial) coordinate system on the spacetime manifold $\mathcal{M}$. The components of the Minkowski metric and its inverse in this coordinate system are given by $\underline{g}_{\mu\nu} = \mathrm{diag}(-c^2, 1, 1, 1)$ and $\underline{g}^{\mu\nu} = \mathrm{diag}(-c^{-2}, 1, 1, 1)$ respectively. We adopt Nordström's postulate, namely that the *spacetime metric* is related to the Minkowski metric by a conformal scaling factor:

$$g_{\mu\nu} = e^{2\phi} \underline{g}_{\mu\nu}. \tag{3.1.1.1}$$

In (3.1.1.1), ϕ is the *Nordström scalar potential*, a dimensionless quantity.

We assume that a perfect fluid exists in $\mathcal{M}$, the energy momentum tensor of which has components $T^{\mu\nu}$ that read

$$T^{\mu\nu} = c^{-2}(\rho + p)u^\mu u^\nu + pg^{\mu\nu} = c^{-2}(\rho + p)u^\mu u^\nu + e^{-2\phi} p\underline{g}^{\mu\nu}, \tag{3.1.1.2}$$

where ρ is the *proper energy density* of the fluid, p is the *pressure*, and u is the *four-velocity*, which is subject to the normalization constraint

$$g_{\mu\nu} u^\mu u^\nu = e^{2\phi} \underline{g}_{\mu\nu} u^\mu u^\nu = -c^2. \tag{3.1.1.3}$$

The Euler equations for a perfect fluid are

$$\nabla_\mu T^{\mu\nu} = 0 \qquad (\nu = 0, 1, 2, 3) \tag{3.1.1.4}$$

$$\nabla_\mu (n u^\mu) = 0, \tag{3.1.1.5}$$

where n is the *proper number density* and ∇ denotes the covariant derivative induced by the spacetime metric g.

Nordström's theory [43] provides the following evolution equation for ϕ : we define an auxiliary momentum tensor

$$T^{\mu\nu}_{\text{aux}} \stackrel{\text{def}}{=} e^{6\phi} T^{\mu\nu} = c^{-2} e^{6\phi}(\rho + p) u^\mu u^\nu + e^{4\phi} p \underline{g}^{\mu\nu}, \tag{3.1.1.6}$$

and postulate that ϕ is a solution to

$$\Box\phi - \kappa^2\phi = -4\pi c^{-4} G e^{4\phi} \mathrm{tr}_g T = -4\pi c^{-4} G \underline{g}_{\mu\nu} T^{\mu\nu}_{\text{aux}} = 4\pi c^{-4} G e^{4\phi}(\rho - 3p). \tag{3.1.1.7}$$

Note that

$$\Box\phi \stackrel{\text{def}}{=} \underline{g}^{\mu\nu} \partial_\mu \partial_\nu \phi = -c^{-2} \partial_t^2 \phi + \triangle\phi \tag{3.1.1.8}$$

is the wave operator on flat spacetime applied to ϕ. The virtue of the postulate equation (3.1.1.7) is that it provides us with continuity equations (3.1.2.7) for an energy momentum tensor Θ in Minkowski space.

We close the system by supplying an equation of state, which may depend on c. A physical equation of state for a perfect fluid state satisfies the following criteria:

1. $\rho \geq 0$ is a function of $n \geq 0$ and $\eta \geq 0$:

$$\rho = \rho(n, \eta). \tag{3.1.1.9}$$

2. $p \geq 0$ is defined by

$$p = n \left. \frac{\partial \rho}{\partial n} \right|_\eta - \rho, \tag{3.1.1.10}$$

where the notation $|_\cdot$ indicates partial differentiation with $\cdot$ held constant.

3. A perfect fluid satisfies

$$\left.\frac{\partial\rho}{\partial n}\right|_{\eta} > 0,\ \left.\frac{\partial p}{\partial n}\right|_{\eta} > 0,\ \left.\frac{\partial\rho}{\partial\eta}\right|_{n} \geq 0 \text{ with `` = '' iff } \eta = 0. \tag{3.1.1.11}$$

   As a consequence, we have that σ, the speed of sound in the fluid, is always real for $\eta > 0$:

$$\sigma^2 \overset{def}{=} c^2 \left.\frac{\partial p}{\partial\rho}\right|_{\eta} = c^2 \frac{\partial p/\partial n|_{\eta}}{\partial\rho/\partial n|_{\eta}} > 0. \tag{3.1.1.12}$$

4. We also demand that the speed of sound is less than the speed of light:

$$0 < \sigma < c. \tag{3.1.1.13}$$

By (3.1.1.11), we can solve for σ^2 and $c^{-2}\rho$ as $c-$indexed functions $\mathcal{S}_c^2$ and $\mathcal{R}_\infty$ respectively of η and p :

$$\sigma^2 \overset{\text{def}}{=} \mathcal{S}_c^2(\eta, p) \tag{3.1.1.14}$$

$$c^{-2}\rho \overset{\text{def}}{=} \mathcal{R}_c(\eta, p). \tag{3.1.1.15}$$

Remark 3.1.1. Note that $c^{-2}\rho$ has the dimensions of mass density. As we will see in Section 3.4, $\lim_{c\to\infty} \mathcal{R}_c(\eta, p)$ will be identified with the Newtonian mass density.

Remark 3.1.2. We will make use of the following identity implied by equation (3.1.1.12):

$$\left.\frac{\partial\mathcal{R}_c}{\partial p}(\eta, p)\right|_{\eta} = \mathcal{S}_c^{-2}(\eta, p). \tag{3.1.1.16}$$

We summarize by stating that the equations (3.1.1.1) - (3.1.1.5), (3.1.1.7), (3.1.1.10), and (3.1.1.15) constitute the EN_κ^c system.

3.1.2 A Reformulation of the EN_κ^c System in Newtonian Variables

Following the discussion in Section 2.4, we will reformulate the EN_κ^c system as a fixed background theory in flat Minkowski space. We then introduce a Newtonian change of state-space variables and show that equations (3.2.1.1) - (3.2.1.4) are obtained.

To begin, we use the form of the metric (3.1.1.1) to compute that in our inertial coordinate system, the continuity equation (3.1.1.4) for the energy momentum tensor

(3.1.1.2) is given by :

$$\begin{aligned} 0 = \nabla_\mu T^{\mu\nu} &= \partial_\mu T^{\mu\nu} + 6T^{\mu\nu}\partial_\mu\phi - \underline{g}_{\alpha\beta}T^{\alpha\beta}\partial^\nu\phi \\ &= \partial_\mu T^{\mu\nu} + 6T^{\mu\nu}\partial_\mu\phi - e^{-6\phi}\underline{g}_{\alpha\beta}\underset{\text{aux}}{T^{\alpha\beta}}\partial^\nu\phi \qquad (\nu = 0,1,2,3), \end{aligned} \tag{3.1.2.1}$$

where $\underset{\text{aux}}{T^{\mu\nu}}$ is given by (3.1.1.6). For this calculation we made use of the explicit form of the Christoffel symbols in inertial coordinate system:

$$\Gamma^\alpha_{\mu\nu} = \delta^\alpha_\nu\partial_\mu\phi + \delta^\alpha_\mu\partial_\nu\phi - \underline{g}_{\mu\nu}\underline{g}^{\alpha\beta}\partial_\beta\phi. \tag{3.1.2.2}$$

Using the postulate equation (3.1.1.7) for ϕ, (3.1.2.1) can be rewritten as

$$0 = e^{6\phi}\nabla_\mu T^{\mu\nu} = \partial_\mu\Big[\underset{\text{aux}}{T^{\mu\nu}} + \frac{c^4}{4\pi G}\big(\partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\underline{g}^{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi - \frac{1}{2}\underline{g}^{\mu\nu}\kappa^2\phi^2\big)\Big]. \tag{3.1.2.3}$$

Recall that the coordinate derivative operators are raised and lowered with the Minkowski metric $\underline{g}$, so $\partial^\lambda\phi = \underline{g}^{\mu\lambda}\partial_\mu\phi$. Let us denote the terms from (3.1.2.3) that are inside the square brackets as $\Theta^{\mu\nu}$. Since the divergence of Θ vanishes, we are provided with local conservation laws in Minkowski space, and we regard Θ as an energy-momentum tensor. We also introduce the following state-space variables that play a mathematical role in the sequel:

$$R_c \overset{\text{def}}{=} c^{-2}\rho e^{4\phi} = e^{4\phi}\mathfrak{R}_c(\eta, p) \tag{3.1.2.4}$$

$$P \overset{\text{def}}{=} pe^{4\phi}. \tag{3.1.2.5}$$

Following this change of variables, the components of Θ read

$$\begin{aligned} \Theta^{\mu\nu} \overset{\text{def}}{=} &\big[R_c + c^{-2}P\big]e^{2\phi}u^\mu u^\nu + P\underline{g}^{\mu\nu} \\ &+ \frac{c^4}{4\pi G}\Big(\partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\underline{g}^{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi - \frac{1}{2}\underline{g}^{\mu\nu}\kappa^2\phi^2\Big), \end{aligned} \tag{3.1.2.6}$$

and we replace (3.1.1.4) with

$$\partial_\mu\Theta^{\mu\nu} = 0 \qquad (\nu = 0,1,2,3). \tag{3.1.2.7}$$

We also expand the covariation differentiation from (3.1.1.5) in terms of coordinate derivatives and the Christoffel symbols (3.1.2.2), arriving at the equation

$$\partial_\mu\big(ne^{4\phi}u^\mu\big) = 0. \tag{3.1.2.8}$$

Our goal is to obtain the system in EN_κ^c in the form (3.2.1.1) - (3.2.1.4) below. To this end, we project (3.1.2.7) onto the orthogonal complement of u and in the direction of u. Thus, we introduce the following rank 3 tensor Π, which has the following components in our inertial coordinate system:

$$\Pi^{\mu\nu} \stackrel{\text{def}}{=} c^{-2} e^{2\phi} u^\mu u^\nu + \underline{g}^{\mu\nu}. \tag{3.1.2.9}$$

Π is the projection onto the orthogonal complement of u :

$$\Pi^{\mu\nu} u^\lambda \underline{g}_{\lambda\mu} = 0 \qquad (\nu = 0, 1, 2, 3). \tag{3.1.2.10}$$

We now introduce the following Newtonian change of state-space variables[b]

$$v^j \stackrel{\text{def}}{=} \frac{u^j}{u^0} \qquad (j = 1, 2, 3) \tag{3.1.2.11}$$

$$\Phi \stackrel{\text{def}}{=} c^2 \phi, \tag{3.1.2.12}$$

where $\mathbf{v} = (v^1, v^2, v^3)$ is the Newtonian velocity and Φ is the Klein-Gordon-Newtonian potential. Relation (3.1.2.11) can be inverted to give

$$u^0 = e^{-\phi} \gamma_c \tag{3.1.2.13}$$

$$u^j = e^{-\phi} \gamma_c v^j, \tag{3.1.2.14}$$

where

$$\gamma_c(\mathbf{v}) \stackrel{\text{def}}{=} \frac{c}{(c^2 - |\mathbf{v}|^2)^{1/2}}. \tag{3.1.2.15}$$

Remark 3.1.3. We provide here a few comments on the Newtonian change of variables. Equation (3.1.2.11) provides the relationship between the Newtonian velocity and the four-velocity: If $x^\nu(t)$ is a curve in $\mathcal{M}$ parameterized by $x^0 = t$ and τ denotes the proper time parameter, then $v^j = \partial_t x^j = (\partial\tau/\partial t) \cdot u^j = u^j/u^0$.

Dimensional analysis correctly identifies the relationship (3.1.2.12) between the Nordström potential ϕ and Klein-Gordon-Newtonian potential Φ, and it justifies the identification of R_∞ from (3.2.2.1) - (3.2.2.4) with $\lim_{c\to\infty} \mathcal{R}_c(\eta, p)$ in (3.1.1.15) (assuming that this limit exists). Furthermore, these changes of variables can be justified

[b]As mentioned in Remark 3.1.1, equation (3.1.2.4) also represents a Newtonian change of variables.

through a formal expansion $\phi = \phi_{(0)} + c^{-2}\phi_{(1)} + \cdots$, $R_\infty = R_{(0)} + c^{-2}R_{(1)} + \cdots$ in equations (3.2.1.1) - (3.2.1.4). Equating the coefficients of powers of c^{-2} on each side of the equations implies the formal identification[c] $\phi_{(0)} = 0$, $(\Delta - \kappa^2)\phi_{(1)} = 4\pi G R_{(0)}$. It follows that $c^2\phi \approx \phi_{(1)} \approx \Phi$. A similar analysis for the Vlasov-Nordström system carried out in [10] .

Upon making the substitutions (3.1.2.11) - (3.1.2.12) and lowering an index with $\underline{g}$, the components of Π in our inertial coordinate system read (for $1 \leq j, k \leq 3$):

$$\Pi_0^0 \overset{\text{def}}{=} -c^{-2}\gamma_c^2|\mathbf{v}|^2 \tag{3.1.2.16}$$

$$\Pi_j^0 \overset{\text{def}}{=} c^{-2}\gamma_c^2 v^j \tag{3.1.2.17}$$

$$\Pi_0^j \overset{\text{def}}{=} -\gamma_c^2 v^j \tag{3.1.2.18}$$

$$\Pi_k^j \overset{\text{def}}{=} c^{-2}\gamma_c^2 v^j v_k + \delta_k^j. \tag{3.1.2.19}$$

Furthermore, we will also make use of the relation

$$\partial_\lambda \gamma_c = c^{-2}\gamma_c^3 v_k \partial_\lambda v^k \qquad (\lambda = 0, 1, 2, 3). \tag{3.1.2.20}$$

Considering first the projection of (3.1.2.7) in the direction of u, we remark that one may use (3.1.1.5) and (3.1.1.10) to conclude that for C^1 solutions, $u_\nu \partial_\mu \Theta^{\mu\nu} = 0$ is equivalent to the adiabatic condition (3.2.1.1).

We now project (3.1.2.7) onto the orthogonal complement of u, which, with the aid of (3.1.1.7), gives the three equations $\Pi_\nu^j \partial_\mu \Theta^{\mu\nu} = 0$, $j = 1, 2, 3$:

$$\begin{aligned} 0 = \Pi_\nu^j \partial_\mu \Theta^{\mu\nu} &= \Pi_\nu^j \big[R_c + c^{-2}P\big](e^\phi u^\mu)\partial_\mu(e^\phi u^\nu) + (\Pi_\nu^j \partial^\nu \phi)\frac{c^4}{4\pi G}(\Box\phi - \kappa^2\phi) \\ &= \Pi_\nu^j \big[R_c + c^{-2}P\big](e^\phi u^\mu)\partial_\mu(e^\phi u^\nu) + (\Pi_\nu^j \partial^\nu \Phi)(R_c - 3c^{-2}P). \end{aligned} \tag{3.1.2.21}$$

After making the substitutions (3.1.2.12), (3.1.2.13), (3.1.2.14), and (3.1.2.15), and using relation (3.1.2.20), it follows that for C^1 solutions, (3.1.2.21) is equivalent to (3.2.1.3).

---

[c] Upon expansion, the formal equation satisfied by $\phi_{(0)}$ is $(\Delta - \kappa^2)\phi_{(0)} = 0$, and from vanishing boundary conditions at infinity, we conclude that $\phi_{(0)} = 0$.

We also introduce the nameless quantity Q_c and make use of (3.1.1.10), (3.1.1.12), (3.1.1.14), (3.1.1.15), (3.1.2.4), and (3.1.2.5) to express it in the following form:

$$Q_c \stackrel{\text{def}}{=} n\,\frac{\partial P}{\partial n}\bigg|_{\eta,\phi} = \frac{\partial P}{\partial R}\bigg|_{n,\phi} \cdot n\,\frac{\partial R}{\partial n}\bigg|_{\eta,\phi} = \mathfrak{Q}_c(\eta, p, \Phi), \tag{3.1.2.22}$$

where

$$\mathfrak{Q}_c(\eta, p, \Phi) \stackrel{\text{def}}{=} \mathcal{S}_c^2(\eta, p)e^{4\Phi/c^2}[\mathcal{R}_c(\eta, p) + c^{-2}p] = \mathcal{S}_c^2(\eta, p)[R_c + c^{-2}P]. \tag{3.1.2.23}$$

Then we use the chain rule together with (3.1.1.5), (3.2.1.1), and (3.1.2.22) to derive

$$e^{\phi}u^{\mu}\partial_{\mu}P + Q_c\partial_{\mu}(e^{\phi}u^{\mu}) = (4P - 3Q_c)e^{\phi}u^{\mu}\partial_{\mu}\phi, \tag{3.1.2.24}$$

which we may use in place of (3.1.1.5). Upon making the substitutions (3.1.2.4), (3.1.2.5), (3.1.2.12), (3.1.2.13), and (3.1.2.14), and using the relation (3.1.2.20), it follows that for C^1 solutions, (3.1.2.24) is equivalent to (3.2.1.2).

3.2 The Formal Limit $c \to \infty$ of the EN_{κ}^{c} System

For convenience, in this section we list the final form of the EN_{κ}^{c} system as derived in sections 3.1 and 3.1.2. We also take the formal limit $c \to \infty$ to arrive at the EP_{κ} system and introduce the equations of variation (EOV_c).

3.2.1 A Recap of the EN_{κ}^{c} System

The EN_{κ}^{c} system is given by

$$\partial_t\eta + v^k\partial_k\eta = 0 \tag{3.2.1.1}$$

$$\begin{aligned}&\partial_t P + v^k\partial_k P + Q_c\partial_k v^k + c^{-2}\gamma_c^2 Q_c\big(v_k\partial_t v^k + v^k v_a\partial_k v^a\big) \\ &\quad = (4P - 3Q_c)\big[c^{-2}\partial_t\Phi + c^{-2}v^k\partial_k\Phi\big]\end{aligned} \tag{3.2.1.2}$$

$$\begin{aligned}&\gamma_c^2(R_c + c^{-2}P)\big[\partial_t v^j + v^k\partial_k v^j + c^{-2}\gamma_c^2(v^j v_a\partial_t v^a + v^j v^k v_a\partial_k v^a)\big] + \partial_j P \\ &\quad + c^{-2}\gamma_c^2(v^j\partial_t P + v^j v^k\partial_k P) = (3c^{-2}P - R_c)\big(\partial_j\Phi + \gamma_c^{-2}v^j[c^{-2}\partial_t\Phi + c^{-2}v^k\partial_k\Phi]\big)\end{aligned} \tag{3.2.1.3}$$

$$-c^{-2}\partial_t^2\Phi + \Delta\Phi - \kappa^2\Phi = 4\pi G(R_c - 3c^{-2}P), \tag{3.2.1.4}$$

where $j = 1, 2, 3$,

$$\gamma_c(\mathbf{v}) \stackrel{\text{def}}{=} \frac{c}{(c^2 - |\mathbf{v}|^2)^{1/2}} \tag{3.2.1.5}$$

$$R_c \stackrel{\text{def}}{=} e^{4\Phi/c^2}\mathcal{R}_c(\eta, p) \tag{3.2.1.6}$$

$$Q_c \stackrel{\text{def}}{=} \mathfrak{Q}_c(\eta, p, \Phi) \stackrel{\text{def}}{=} \mathcal{S}_c^2(\eta, p)e^{4\Phi/c^2}[\mathcal{R}_c(\eta, p) + c^{-2}p] \tag{3.2.1.7}$$

$$P \stackrel{\text{def}}{=} e^{4\Phi/c^2}p, \tag{3.2.1.8}$$

$\mathcal{S}_c(\eta, p)$ is the speed of sound, c denotes the speed of light, and the functions $\mathcal{R}_c$ and $\mathcal{S}_c$ derive from a $c-$indexed equation of state as discussed in Section 3.1. Note also the relationship between $\mathcal{R}_c$ and $\mathcal{S}_c$ given by (3.1.1.16). The variables $\eta, p, \mathbf{v} = (v^1, v^2, v^3)$, and Φ denote the entropy density, pressure, (Newtonian) velocity, and Klein-Gordon-Newtonian potential respectively. Section 3.4 contains a detailed discussion of the c-dependence of the EN_κ^c System.

3.2.2 The EP_κ System as a Formal Limit

Taking the formal limit $c \to \infty$ in the EN_κ^c system gives the Euler-Poisson system with a cosmological constant:

$$\partial_t \eta + v^k \partial_k \eta = 0 \tag{3.2.2.1}$$

$$\partial_t p + v^k \partial_k p + Q_\infty \partial_k v^k = 0 \tag{3.2.2.2}$$

$$\partial_t R_\infty + \partial_k (R_\infty v^k) = 0 \tag{3.2.2.2'}$$

$$R_\infty \big(\partial_t v_j + v^k \partial_k v^j\big) + \partial_j p = -R_\infty \partial_j \Phi \qquad (j = 1, 2, 3) \tag{3.2.2.3}$$

$$\Delta\Phi - \kappa^2 \Phi = 4\pi G R_\infty, \tag{3.2.2.4}$$

where

$$R_\infty \stackrel{\text{def}}{=} \mathcal{R}_\infty(\eta, p), \tag{3.2.2.5}$$

$$Q_\infty \stackrel{\text{def}}{=} \mathfrak{Q}_\infty(\eta, p) \stackrel{\text{def}}{=} \mathcal{S}_\infty^2(\eta, p) \cdot \mathcal{R}_\infty(\eta, p), \tag{3.2.2.6}$$

$\mathcal{R}_\infty(\eta, p)$ and $\mathcal{S}_\infty^2(\eta, p)$ are the limits as $c \to \infty$ of $\mathcal{R}_c(\eta, p)$ and $\mathcal{S}_c^2(\eta, p)$ respectively (see (3.4.3.1), (3.4.3.2), and (3.4.3.3)), and the quantity R_∞ is the mass density. Note that in Section 2.3.1, we used the symbol ρ instead of R_∞ to denote the mass density in the

EP_κ system, whereas throughout Chapter 3, ρ denotes the *proper* energy density in the EN_κ^c system. Equation (3.1.1.16) and (3.4.3.3) imply that $\partial \mathcal{R}_\infty(\eta, p)/\partial p = \mathcal{S}_\infty^{-2}(\eta, p)$, so by the chain rule, it follows that equations (3.2.2.2) and (3.2.2.2') are equivalent.

The solution to (3.2.2.4) is given by

$$\Phi(t, \mathbf{s}) = \bar{\Phi}_\infty - G \int_{\mathbb{R}^3} \left(\frac{e^{-\kappa|\mathbf{s}-\mathbf{s}'|}}{|\mathbf{s}-\mathbf{s}'|} \right) \left[\mathcal{R}_\infty(\eta(t, \mathbf{s}'), p(t, \mathbf{s}')) - \mathcal{R}_\infty(\bar{\eta}, \bar{p}) \right] d^3\mathbf{s}', \qquad (3.2.2.7)$$

where the constants $\bar{\Phi}_\infty, \bar{\eta}$, and $\bar{p}$, which are the values of Φ, η, and p respectively in a constant background state, are discussed in Section 3.6. The boundary conditions leading to this solution are that $\Phi(t, \cdot) - \bar{\Phi}_\infty$ vanishes at ∞, and we view $\Phi(t, \mathbf{s})$ as a (not necessarily small) perturbation of the constant potential $\bar{\Phi}_\infty$.

Remark 3.2.1. Consider the kernel $\mathcal{K}(\mathbf{s}) = e^{-\kappa|\mathbf{s}|}/|\mathbf{s}|$ appearing in (3.2.2.7). An easy computation gives that $\mathcal{K}(\mathbf{s}), \nabla^{(1)}\mathcal{K}(\mathbf{s}) \in L^1(\mathbb{R}^3)$. Therefore, a basic result from harmonic analysis (Young's inequality) implies that the map $f \to \mathcal{K} * f$, where $*$ denotes convolution, is a map from $H^j(\mathbb{R}^3)$ to $H^{j+1}(\mathbb{R}^3)$. From this fact and Remark B.0.4, it follows that $\Phi(t, \cdot) \in H^{N+1}_{\bar{\Phi}}(\mathbb{R}^3)$ whenever $\eta(t, \cdot), p(t, \cdot) \in H^N_{\bar{\eta}}(\mathbb{R}^3), H^N_{\bar{p}}(\mathbb{R}^3)$ respectively.

3.3 The Equations of Variation (EOV_c)

As in Section 2.4.2, the EOV_c are formed by linearizing the EN_κ^c system (EP_κ system in the case $c = \infty$) around a BGS $\widetilde{\mathbf{V}}$ of the form $\widetilde{\mathbf{V}} = (\widetilde{\eta}, \widetilde{P}, \widetilde{v}^1, \cdots, \widetilde{\Phi}_2, \widetilde{\Phi}_3)$. Given such a $\widetilde{\mathbf{V}}$ and inhomogeneous terms $f, g, \cdots, l$ we define the EOV_c by

$$\partial_t \dot{\eta} + \widetilde{v}^k \partial_k \dot{\eta} = f \qquad (3.3.0.8)$$

$$\partial_t \dot{P} + \widetilde{v}^k \partial_k \dot{P} + \widetilde{Q}_c \partial_k \dot{v}^k + c^{-2} \widetilde{\gamma}_c^2 \widetilde{Q}_c \big(\widetilde{v}_k \partial_t \dot{v}^k + \widetilde{v}^k \widetilde{v}_a \partial_k \dot{v}^a \big) = g \qquad (3.3.0.9)$$

$$\begin{aligned} \widetilde{\gamma}_c^2 (\widetilde{R}_c + c^{-2} \widetilde{P}) \big[\partial_t \dot{v}^j + \widetilde{v}^k \partial_k \dot{v}^j + c^{-2} \widetilde{\gamma}_c^2 (\widetilde{v}^j \widetilde{v}_a \partial_t \dot{v}^a + \widetilde{v}^j \widetilde{v}^k \widetilde{v}_a \partial_k \dot{v}^a) \big] \qquad (3.3.0.10) \\ + \partial_j \dot{P} + c^{-2} \widetilde{\gamma}_c^2 (\widetilde{v}^j \partial_t \dot{P} + \widetilde{v}^j \widetilde{v}^k \partial_k \dot{P}) = h^{(j)} \end{aligned}$$

$$-c^{-2} \partial_t^2 \dot{\Phi} + \Delta \dot{\Phi} - \kappa^2 \dot{\Phi} = l, \qquad (3.3.0.11)$$

where $\widetilde{\gamma}_c \overset{\text{def}}{=} c/(c^2 - |\widetilde{v}|^2)^{1/2}$, $\widetilde{R}_c \overset{\text{def}}{=} e^{4\widetilde{\Phi}/c^2} \mathcal{R}_c(\widetilde{\eta}, \widetilde{p})$, etc. The unknowns are the components of $\dot{\mathbf{W}} \overset{\text{def}}{=} (\dot{\eta}, \dot{P}, \dot{v}^1, \dot{v}^2, \dot{v}^3)$ and $\dot{\Phi}$.

Remark 3.3.1. We place parentheses around the superscripts of the inhomogeneous terms $h^{(j)}$ in order to emphasize that we are merely labelling them, and that in general, they do not transform covariantly under changes of coordinates.

We find it useful to analyze both the dependent variable p and the dependent variable P when discussing solutions to the above system, and we will make use of all four of the following arrays:

$$\mathbf{W} \stackrel{\text{def}}{=} (\eta, P, v^1, v^2, v^3) \tag{3.3.0.12}$$

$$\mathbf{V} \stackrel{\text{def}}{=} (\eta, P, v^1, v^2, v^3, \partial_t\Phi, \partial_1\Phi, \partial_2\Phi, \partial_3\Phi) \tag{3.3.0.13}$$

$$\boldsymbol{\mathcal{W}} \stackrel{\text{def}}{=} (\eta, p, v^1, v^2, v^3) \tag{3.3.0.14}$$

$$\boldsymbol{\mathcal{V}} \stackrel{\text{def}}{=} (\eta, p, v^1, v^2, v^3, \partial_t\Phi, \partial_1\Phi, \partial_2\Phi, \partial_3\Phi), \tag{3.3.0.15}$$

where $P = e^{4\Phi/c^2}p$. We also use notation such as $\widetilde{\mathbf{V}} = (\widetilde{\eta}, \widetilde{P}, \widetilde{p}^1, \cdots, \partial_3\widetilde{\Phi})$, where $\widetilde{P} \stackrel{\text{def}}{=} e^{4\widetilde{\Phi}/c^2}\widetilde{p}$, etc. Note that this notation disagrees with that of Chapter 2, in which the arrays $\mathbf{V}$ and $\mathbf{W}$ contained the components of the U^j rather than the Newtonian velocity $\mathbf{v}$. We may sometimes abuse terminology and refer to $\widetilde{\boldsymbol{\mathcal{V}}} = (\widetilde{\eta}, \widetilde{p}, \widetilde{v}^1, \cdots, \widetilde{\Phi}_2, \widetilde{\Phi}_3)$ as the BGS. When $c = \infty$, we may also refer to $\widetilde{\boldsymbol{\mathcal{W}}} = (\widetilde{\eta}, \widetilde{p}, \widetilde{v}^1, \widetilde{v}^2, \widetilde{v}^3)$ as the BGS, since the left-hand sides of (3.3.0.8) - (3.3.0.11) do not depend on $\widetilde{\Phi}$ in this case. Additionally, we may refer to the unknowns as $\dot{\boldsymbol{\mathcal{W}}} \stackrel{\text{def}}{=} (\dot{\eta}, \dot{p}, \dot{v}^1, \dot{v}^2, \dot{v}^3)$ when $c = \infty$.

As in Chapter 2, we sometimes write (3.2.1.1) - (3.2.1.3) as

$${}_c\mathcal{A}^\mu \partial_\mu \mathbf{W} = \mathbf{b}, \tag{3.3.0.16}$$

where each ${}_c\mathcal{A}$ is a 5×5 matrix with entries that are functions of $\boldsymbol{\mathcal{W}}, \Phi$, while $\mathbf{b} = (f, g, \cdots, h^{(3)})$ is the 5-component column array on the right-hand side of (3.2.1.1) - (3.2.1.3).

Remark 3.3.2. For conceptual clarity, we will always view ${}_c\mathcal{A}^\nu$ as a function of $\widetilde{\boldsymbol{\mathcal{W}}}, \widetilde{\Phi}$ and write "${}_c\mathcal{A}^\nu(\widetilde{\boldsymbol{\mathcal{W}}}, \widetilde{\Phi})$," as opposed to writing "${}_c\mathcal{A}^\nu(\widetilde{\mathbf{W}}, \widetilde{\Phi})$."

It is instructive to see the form of the ${}_c\mathcal{A}^\nu$, $\nu = 0, 1, 2, 3$, for we will later concern ourselves with their large-c asymptotic behavior. Abbreviating $\alpha_{(c)} \stackrel{\text{def}}{=} R_c + c^{-2}P$, $\beta_{(c)}^{(j)} \stackrel{\text{def}}{=} 1 + c^{-2}\gamma_c^2(v^j)^2$ we have

$$
{}_c\mathcal{A}^0(\boldsymbol{\mathcal{W}},\Phi) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & c^{-2}\gamma_c^2 Q_c v^1 & c^{-2}\gamma_c^2 Q_c v^2 & c^{-2}\gamma_c^2 Q_c v^3 \\
0 & c^{-2}\gamma_c^2 v^1 & \begin{array}{c}\gamma_c^2\alpha_{(c)} \\ \cdot\,\beta_{(c)}^{(1)}\end{array} & c^{-2}\gamma_c^4\alpha_{(c)}v^1v^2 & c^{-2}\gamma_c^4\alpha_{(c)}v^1v^3 \\
0 & c^{-2}\gamma_c^2 v^2 & c^{-2}\gamma_c^4\alpha_{(c)}v^2v^1 & \begin{array}{c}\gamma_c^2\alpha_{(c)} \\ \cdot\,\beta_{(c)}^{(2)}\end{array} & c^{-2}\gamma_c^4\alpha_{(c)}v^2v^3 \\
0 & c^{-2}\gamma_c^2 v^3 & c^{-2}\gamma_c^4\alpha_{(c)}v^3v^1 & c^{-2}\gamma_c^4\alpha_{(c)}v^3v^2 & \begin{array}{c}\gamma_c^2\alpha_{(c)} \\ \cdot\,\beta_{(c)}^{(3)}\end{array}
\end{pmatrix},
$$

$$
{}_\infty\mathcal{A}^0(\boldsymbol{\mathcal{W}}) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & R_\infty & 0 & 0 \\
0 & 0 & 0 & R_\infty & 0 \\
0 & 0 & 0 & 0 & R_\infty
\end{pmatrix},
$$

$$
{}_c\mathcal{A}^1(\boldsymbol{\mathcal{W}},\Phi) = \begin{pmatrix}
v^1 & 0 & 0 & 0 & 0 \\
0 & v^1 & Q_c\beta_{(c)}^{(1)} & c^{-2}\gamma_c^2 Q_c v^1 v^2 & c^{-2}\gamma_c^2 Q_c v^1 v^3 \\
0 & \beta_{(c)}^{(1)} & \begin{array}{c}\gamma_c^2\alpha_{(c)}v^1 \\ \cdot\,\beta_{(c)}^{(1)}\end{array} & \begin{array}{c}\gamma_c^2\alpha_{(c)}(v^1)^2 \\ \cdot\,c^{-2}\gamma_c^2 v^2\end{array} & \begin{array}{c}\gamma_c^2\alpha_{(c)}(v^1)^2 \\ \cdot\,c^{-2}\gamma_c^2 v^3\end{array} \\
0 & c^{-2}\gamma_c^2 v^1 v^2 & \begin{array}{c}\gamma_c^2\alpha_{(c)}(v^1)^2 \\ \cdot\,c^{-2}\gamma_c^2 v^2\end{array} & \begin{array}{c}\gamma_c^2\alpha_{(c)}v^1 \\ \cdot\,\beta_{(c)}^{(2)}\end{array} & \begin{array}{c}\gamma_c^2\alpha_{(c)}v^1 \\ \cdot\,c^{-2}\gamma_c^2 v^2 v^3\end{array} \\
0 & c^{-2}\gamma_c^2 v^1 v^3 & \begin{array}{c}\gamma_c^2\alpha_{(c)}(v^1)^2 \\ \cdot\,c^{-2}\gamma_c^2 v^3\end{array} & \begin{array}{c}\gamma_c^2\alpha_{(c)}v^1 \\ \cdot\,c^{-2}\gamma_c^2 v^3 v^2\end{array} & \begin{array}{c}\gamma_c^2\alpha_{(c)}v^1 \\ \cdot\,\beta_{(c)}^{(3)}\end{array}
\end{pmatrix},
$$

$$
{}_\infty\mathcal{A}^1(\boldsymbol{\mathcal{W}}) = \begin{pmatrix}
v^1 & 0 & 0 & 0 & 0 \\
0 & v^1 & Q_\infty & 0 & 0 \\
0 & 1 & R_\infty v^1 & 0 & 0 \\
0 & 0 & 0 & R_\infty v^1 & 0 \\
0 & 0 & 0 & & R_\infty v^1
\end{pmatrix},
$$

etc.

3.4 On the $c-$Dependence of the EN_κ^c System

3.4.1 Inequalities that Hold for All Large c

Let us begin by defining notation that will be heavily used throughout this chapter.

Notation. If A_c is a quantity that depends on the parameter c and X is any quantity, then we write $A_c \lesssim X$ if $A_c \leq X$ holds for all large c.

In addition to appearing directly as the term c^{-2}, the constant c appears in the EN_κ^c system (3.2.1.1) - (3.2.1.4) through 4 terms: $P = e^{4\Phi/c^2}p, \gamma_c = c/(c^2 - |\mathbf{v}|^2)^{1/2}, R_c = e^{4\Phi/c^2}\mathcal{R}_c(\eta, p)$, and $Q_c = \mathcal{S}_c^2(\eta, p)e^{4\Phi/c^2}[\mathcal{R}_c(\eta, p) + c^{-2}p]$. Because we want to recover the EP_κ system in the large c limit, the first obvious requirement we have is that the function $\mathcal{R}_c(\eta, p)$ has a limit $\mathcal{R}_\infty(\eta, p)$ as $c \to \infty$. For mathematical reasons, we will demand convergence in the norm $|\cdot|_{N+1,\mathfrak{K}}$ at a rate of order c^{-2}, where $\mathfrak{K}$ is a compact subset of state-space that depends on the initial data. Although a construction of $\mathfrak{K}$ from the initial data is rigorously carried out in Section 3.6.2, let us now provide a preliminary description: for given initial data, we will provide a compact set $\mathfrak{K}$ and a time interval $[0, T]$ so that for all large c, solutions $\boldsymbol{\mathcal{V}}_c = (\eta, p, v^1, \cdots, \partial_3\Phi)$ to the EN_κ^c system launched by the initial data exist on $[0, T] \times \mathbb{R}^3$ and satisfy $\boldsymbol{\mathcal{V}}_c([0, T] \times \mathbb{R}^3) \subset \mathfrak{K}'$. Intuitively, the aforementioned 4 terms should converge to $p, 1, R_\infty$, and Q_∞ respectively on $\mathfrak{K}'$. In this section, we make this intuition rigorous and specify what we mean by "converge."

3.4.2 Functions with $c-$Independent Properties

The main technical difficulty in this chapter is ensuring that the Sobolev estimates provided by the propositions appearing in Appendix B can be made independently of all large c. We introduce here some machinery that will allow us to easily discuss uniform-in-c estimates. Following this, we use this machinery to prove some preliminary lemmas that will be used in the proofs of Theorem 3.8.2 and Theorem 3.9.2.

Definition 3.4.1. Let $j \geq 2$, and let $q_1, q_2, \cdots, q_n$ be such that $q_i \in H^j_{\bar{q}_i}(\mathbb{R}^3)$, where the $\bar{q}_i$ are constants and the spaces $H^j_{\bar{q}_i}$ are defined in Section 2.2 of Chapter 2. Assume that the image set $\{(q_1(\mathbf{s}), q_2(\mathbf{s}), \cdots, q_n(\mathbf{s})) \mid \mathbf{s} \in \mathbb{R}^3\}$ is contained in the compact, convex set $\mathfrak{D}$ and that $(\bar{q}_1, \bar{q}_2, \cdots, \bar{q}_n) \in \mathfrak{D}$. We define $\mathcal{R}^j(c^{-k}; \mathfrak{D}; q_1, \cdots, q_n)$ to be the ring consisting of all expressions of the form $F_c(q_1, \cdots, q_n)$, where F_c is a $c-$indexed family of functions of class C_b^j on $\mathfrak{D}$ satisfying

$$|F_c|_{j,\mathfrak{D}} \lesssim c^{-k} C(\mathfrak{D}). \tag{3.4.2.1}$$

We emphasize that the constant $C(\mathfrak{D})$ is allowed to depend on the family F_c and the domain $\mathfrak{D}$, but within a given family and on a fixed domain, $C(\mathfrak{D})$ must be independent of all large c.

Remark 3.4.1. Although the q_i and $\bar{q}_i$ must satisfy the above criteria, we emphasize that *the q_i and $\bar{q}_i$ may depend on the parameter c, even though we do not explicitly indicate this dependence.* In our applications below, the q_i will be quantities related to solutions of the EN_κ^c system, and the $\bar{q}_i$ will typically be equal to the components of either (3.6.1.2), (3.6.1.10), or (3.6.1.11).

Remark 3.4.2. We repeatedly use the fact that $\mathcal{R}^j(c^{-k}; \mathfrak{D}; q_1, \cdots, q_n)$ is a ring without explicitly mentioning it; i.e., it is closed under products.

Remark 3.4.3. If $F_c \in \mathcal{R}^j(c^{-k}; \mathfrak{D}; q_1, \cdots, q_n)$ and for all large c, F_c doesn't vanish on $\mathfrak{D}$, then $1/F_c \in \mathcal{R}^j(c^{-k}; \mathfrak{D}; q_1, \cdots, q_n)$.

Definition 3.4.2. For $j \geq 2$, let $\mathcal{I}^j(c^{-k}; \mathfrak{D}; q_1, \cdots, q_n)$ denote the sub-ring contained in $\mathcal{R}^j(\mathfrak{D}; q_1, \cdots, q_n)$ consisting of all such expressions $F_c(q_1, \cdots, q_n)$ such that

$$\|F_c(q_1, \cdots, q_n)\|_{H^j} \leq c^{-k} C(\mathfrak{D}, \|q_1\|_{H^j_{\bar{q}_1}}, \cdots, \|q_n\|_{H^j_{\bar{q}_n}}) \tag{3.4.2.2}$$

holds for all $(q_1, \cdots, q_n)$ satisfying the hypotheses in Definition 3.4.1. The constant $C(\mathfrak{D})$ is allowed to depend on F_c and $\mathfrak{D}$, but it can only depend on the $q_1, \cdots, q_n$ through their $H^j_{\bar{q}_i}$ norms.

Notation. If $F_c \in \mathcal{I}^j(c^{-k}; \mathfrak{D}; q_1, \cdots, q_n)$, then we sometimes write

$$F_c(q_1, \cdots, q_n) = \mathcal{O}^j(c^{-k}; \mathfrak{D}; q_1, \cdots, q_n). \tag{3.4.2.3}$$

Remark 3.4.4. We employ the following abuse of notation: in writing "$F_c(q_1, \cdots, q_n) \in \mathcal{R}^j(c^{-k}; \mathfrak{D}; q_1, \cdots, q_n)$," we assume in addition to the decay on F_c given in (3.4.2.1) that the functions q_i satisfy the above hypotheses; i.e., we assume that $\mathfrak{D}$ is compact and convex, that there are constants $\bar{q}_i$ such that $q_i \in H^j_{\bar{q}_i}(\mathbb{R}^3)$, that the image set $\{\big(q_1(\mathbf{s}), q_2(\mathbf{s}), \cdots, q_n(\mathbf{s})\big) \mid \mathbf{s} \in \mathbb{R}^3\}$ is contained in $\mathfrak{D}$, and that $(\bar{q}_1, \bar{q}_2, \cdots, \bar{q}_n) \in \mathfrak{D}$. We employ a similar abuse of notation in writing "$F_c(q_1, \cdots, q_n) \in \mathcal{I}^j(c^{-k}; \mathfrak{D}; q_1, \cdots, q_n)$."

Remark 3.4.5. In the notation $\mathcal{R}(\cdots), \mathcal{I}(\cdots)$, and $\mathcal{O}(\cdots)$, we often omit the argument $\mathfrak{D}$. In this case, it is understood that there is an implied set $\mathfrak{D}$ that is to be inferred from context; frequently $\mathfrak{D}$ is to be inferred from L^∞ estimates on the q_i that follow from Sobolev imbedding. We omit the argument c^{-k} when $k = 0$. Furthermore, we have chosen to omit dependence on the constants $\bar{q}_i$ since their definitions will be clear from context. We will occasionally omit additional arguments when the context is clear.

Lemma 3.4.1. *If $j \geq 2$ and $F_c(q_1, \cdots, q_n) \in \mathcal{R}^j(c^{-k}; \mathfrak{D}; q_1, \cdots, q_n)$, then $F_c(q_1, \cdots, q_n) - F_c(\bar{q}_1, \cdots, \bar{q}_n) \in \mathcal{I}^j(c^{-k}; \mathfrak{D}; q_1, \cdots, q_n)$.*

Proof. Lemma 3.4.1 follows immediately from (B.0.2.17). □

Remark 3.4.6. Lemma 3.4.1 shows that if $F_c(\bar{q}_1, \cdots, \bar{q}_n) = 0$, then $F_c \in \mathcal{I}^j(c^{-k}; \mathfrak{D}; q_1, \cdots, q_n)$. In particular, if $\bar{q} = 0$, then any polynomial (of strictly positive degree) in q is an element of $\mathcal{I}^j(q)$.

Lemma 3.4.2. *Suppose that $j \geq 2$, $F_c \in \mathcal{R}^j(c^{-k_1}; \mathfrak{D}; q_1, \cdots, q_n)$, and $G_c \in \mathcal{I}^j(c^{-k_2}; \mathfrak{D}; q_1, \cdots, q_n)$. Then $\big(F_c \cdot G_c\big)(q_1, \cdots, q_n) \cdot \in \mathcal{I}^j(c^{-(k_1+k_2)}; \mathfrak{D}; q_1, \cdots, q_n)$.*

Proof. Lemma 3.4.2 follows immediately from (B.0.2.11). □

Remark 3.4.7. Lemma 3.4.2 shows in particular that for $k \geq 0$, $\mathcal{I}^j(c^{-k}; \mathfrak{D}; q_1, \cdots, q_n)$ is an ideal in $\mathcal{R}^j(\mathfrak{D}; q_1, \cdots, q_n)$.

Remark 3.4.8. If $k \geq 0$ and there is a fixed function $F_\infty \in \mathcal{R}^j(\mathfrak{D}; q_1, \cdots, q_n)$ such that $F_c - F_\infty \in \mathcal{R}^j(c^{-k}; \mathfrak{D}; q_1, \cdots, q_n)$, then it follows that $|F_c|_{j,\mathfrak{D}} \lesssim |F_\infty|_{j,\mathfrak{D}} + 1$, so that the family F_c is uniformly bounded in the norm $|\cdot|_{j,\mathfrak{D}}$ for all large c. A similar remark using the $\|\cdot\|_{H^j}$ norm applies if $F_\infty \in \mathcal{I}^j(\mathfrak{D}; q_1, \cdots, q_n)$ and $F_c - F_\infty \in \mathcal{I}^j(c^{-k}; \mathfrak{D}; q_1, \cdots, q_n)$.

Remark 3.4.9. As we change settings, it is sometimes useful to shift the point of view as to what are the arguments of a family $F_c(\cdots)$. For example, consider the expression $F_c = c^{-2}\partial_t\Phi$, where Φ is a solution variable in the EN_κ^c system depending on c through the initial data $\mathring{\mathbf{V}}_c$ and through the c dependence of the system itself. If it is known that $c^{-1}\|\partial_t\Phi\|_{H^3}$ is uniformly bounded by L for all large c, then we may write $F_c \in \mathcal{I}^3(c^{-1}; c^{-1}\partial_t\Phi)$ since $\|c^{-2}\partial_t\Phi\|_{H^3} \lesssim c^{-1}L$. If it also turns out that $\|\partial_t\Phi\|_{H^3}$ is uniformly bounded for all large c, then we may write $F_c \in \mathcal{I}^3(c^{-2}; \partial_t\Phi)$ or $F_c \in \mathcal{I}^3(c^{-1}; c^{-1}\partial_t\Phi) \cap \mathcal{I}^3(c^{-2}; \partial_t\Phi)$.

Remark 3.4.10. We are not always optimal in our estimates. For example, if $\mathfrak{D} \Subset \mathbb{R}$ contains the origin in its interior, we may on occasion write $F_c \in \mathcal{R}^4(\mathfrak{D}; \Phi)$ even if the stronger claim $F_c \in \mathcal{R}^4(\mathfrak{D}; c^{-2}\Phi)$ is true.

Lemma 3.4.3. *Suppose that* $j \geq 3$, $k_1 + k_2 = k_0$, *and that* $F_c(q_1, \cdots, q_n) \in \mathcal{R}^j(c^{-k_0}; \mathfrak{D}_1; q_1, \cdots, q_n)$. *Assume further that for* $1 \leq i \leq n, q_i \in C^0([0,T], H^j_{\bar{q}_i}) \cap C^1([0,T], H^{j-1}_{\bar{q}_i})$ *and that for all large* c, $c^{-k_2}\big(\partial_t q_1, \cdots, \partial_t q_n\big)([0,T] \times \mathbb{R}^3) \subset \mathfrak{D}_2$. *Then on* $[0,T]$, $\partial_t\Big(F_c(q_1, \cdots, q_n)\Big) \in \mathcal{I}^{j-1}(c^{-k_1}; \mathfrak{D}_1 \times \mathfrak{D}_2; q_1, \cdots, q_n, c^{-k_2}\partial_t q_1, \cdots, c^{-k_2}\partial_t q_n)$, *or equivalently,* $\partial_t\Big(F_c(q_1, \cdots, q_n)\Big) = \mathcal{O}^{j-1}(c^{-k_1}; \mathfrak{D}_1 \times \mathfrak{D}_2; q_1, \cdots, q_n, c^{-k_2}\partial_t q_1, \cdots, c^{-k_2}\partial_t q_n)$.

Proof. Lemma 3.4.3 follows from the chain rule and Remark B.0.3. □

Corollary 3.4.4. *Let* ∂_a *be a first-order spatial differential operator. Suppose that* $j \geq 3$, $k_1 + k_2 = k_0$, *and that* $F_c(q_1, \cdots, q_n) \in \mathcal{R}^j(c^{-k_0}; \mathfrak{D}_1; q_1, \cdots, q_n)$. *Assume that for all large* c, $c^{-k_2}\big(\partial_a q_1, \cdots, \partial_a q_n\big)([0,T] \times \mathbb{R}^3) \subset \mathfrak{D}_2$. *Then on* $[0,T]$, $\partial_a\Big(F_c(q_1, \cdots, q_n)\Big) \in \mathcal{I}^{j-1}(c^{-k_1}; \mathfrak{D}_1 \times \mathfrak{D}_2; q_1, \cdots, q_n, c^{-k_2}\partial_a q_1, \cdots, c^{-k_2}\partial_a q_n)$, *or equivalently,* $\partial_a\Big(F_c(q_1, \cdots, q_n)\Big) = \mathcal{O}^{j-1}(c^{-k_1}; \mathfrak{D}_1 \times \mathfrak{D}_2; q_1, \cdots, q_n, c^{-k_2}\partial_a q_1, \cdots, c^{-k_2}\partial_a q_n)$.

Proof. Corollary 3.4.4 also follows from the chain rule and Remark B.0.3. □

3.4.3 Application to the EN_κ^c System

We will now apply these lemmas to the EN_κ^c system. We begin by restating the hypotheses on the equation of state using our new notation:

Hypotheses on the $c-$Depedence of the Equation of State.

$$\mathcal{R}_c(\eta,p),\ \mathcal{R}_\infty(\eta,p) \in \mathcal{R}^{N+1}(\mathfrak{K}',\eta,p) \tag{3.4.3.1}$$

$$\mathcal{R}_c(\eta,p) - \mathcal{R}_\infty(\eta,p) \in \mathcal{R}^{N+1}(c^{-2};\mathfrak{K}',\eta,p), \tag{3.4.3.2}$$

where the set $\mathfrak{K}'$ was introduced at beginning of the section. As a simple consequence of (3.1.1.16), (3.4.3.1), and (3.4.3.2), we have that

$$\mathcal{S}_c^2(\eta,p) - \mathcal{S}_\infty^2(\eta,p) \in \mathcal{R}^{N}(c^{-2};\mathfrak{K},\eta,p). \tag{3.4.3.3}$$

We also assume that $\mathcal{R}_\infty(\eta,p)$ and $\mathcal{S}_\infty^2(\eta,p)$ are "physical" as defined in Section 3.1.2; i.e., we assume in particular that whenever $\eta,p > 0$, we have $0 < \mathcal{R}_\infty(\eta,p)$ and $0 < \mathcal{S}_\infty^2(\eta,p)$.

Remark 3.4.11. Hypothesis (3.4.3.2) can be weakened; we do not pursue this matter here since we are not striving for optimal results.

Lemma 3.4.5.

$$\gamma_c^2 - 1 \in \mathcal{R}^{N+1}(c^{-2};\mathfrak{K}';\mathbf{v}) \tag{3.4.3.4}$$

$$e^{4\Phi/c^2} - 1 \in \mathcal{R}^{N+1}(c^{-2};\mathfrak{K}';\Phi) \cap \mathcal{R}^{N}(c^{-1};c^{-1}\Phi) \tag{3.4.3.5}$$

$$R_c - R_\infty = e^{4\Phi/c^2}\mathcal{R}_c(\eta,p) - \mathcal{R}_\infty(\eta,p) \in \mathcal{R}^{N+1}(c^{-2};\mathfrak{K}';\boldsymbol{\mathcal{W}},\Phi) \tag{3.4.3.6}$$

$$\mathfrak{Q}_c(\eta,p,\Phi) - \mathfrak{Q}_\infty(\eta,p) \in \mathcal{R}^{N}(c^{-2};\mathfrak{K}',\boldsymbol{\mathcal{W}},\Phi) \tag{3.4.3.7}$$

$$\mathbf{W} - \boldsymbol{\mathcal{W}} \in \mathcal{I}^{N}(c^{-2};\mathfrak{K}';P,\Phi) \tag{3.4.3.8}$$

$$\begin{aligned}\partial_k\mathbf{W} - \partial_k\boldsymbol{\mathcal{W}} &\in \mathcal{I}^{N-1}(c^{-2};P,\partial_kP,\Phi,\partial_k\Phi) \\ &\cap \mathcal{I}^{N-1}(c^{-1};P,\partial_kP,c^{-1}\Phi,c^{-1}\partial_k\Phi).\end{aligned} \tag{3.4.3.9}$$

Proof. (3.4.3.4), and (3.4.3.5) are easy Taylor estimates. (3.4.3.6) follows from (3.4.3.1), (3.4.3.2), and (3.4.3.5). (3.4.3.7) then follows from follows from (3.1.2.23), (3.4.3.3), and (3.4.3.6). Since $P = e^{4\Phi/c^2}p$, (3.4.3.8) follows from (3.4.3.5), Lemma 3.4.2, and that the fact that $\boldsymbol{\mathcal{W}}$ and $\mathbf{W}$ differ only in that the second component of $\boldsymbol{\mathcal{W}}$ is p, while the second component of $\mathbf{W}$ is P. (3.4.3.9) then follows from (3.4.3.8) and Corollary 3.4.4. □

Lemma 3.4.6. *Let* ${}_c\mathcal{A}^\nu(\mathbf{\mathcal{W}}, \Phi),\ \nu = 0, 1, 2, 3,$ *denote the matrices introduced in Section 3.3. Then*

$$ {}_c\mathcal{A}(\mathbf{\mathcal{W}}, \Phi),\ {}_\infty\mathcal{A}^\nu(\mathbf{\mathcal{W}}) \in \mathcal{R}^N(\mathfrak{K}'; \mathbf{\mathcal{W}}) \tag{3.4.3.10} $$

$$ {}_c\mathcal{A}^\nu(\mathbf{\mathcal{W}}, \Phi) - {}_\infty\mathcal{A}^\nu(\mathbf{\mathcal{W}}) \in \mathcal{R}^N(c^{-2}; \mathfrak{K}'; \mathbf{\mathcal{W}}, \Phi) \tag{3.4.3.11} $$

$$ \big({}_c\mathcal{A}^0(\mathbf{\mathcal{W}}, \Phi)\big)^{-1} - \big({}_\infty\mathcal{A}^0(\mathbf{\mathcal{W}})\big)^{-1} \in \mathcal{R}^N(c^{-2}; \mathfrak{K}'; \mathbf{\mathcal{W}}, \Phi). \tag{3.4.3.12} $$

Proof. (3.4.3.10) follows from Hypothesis 3.4.3.1. (3.4.3.11) follows from Hypothesis 3.4.3.1, Hypothesis 3.4.3.2, and Lemma 3.4.5. (3.4.3.12) then follows from (3.4.3.11), the adjoint formula for the inverse of a matrix, and Remark 3.4.3. □

Lemma 3.4.7. *Let*
$\mathfrak{B}_\infty(\mathbf{\mathcal{W}}, \nabla^{(1)}\Phi) \overset{def}{=} \Big(0, 0, -\mathfrak{R}_\infty(\eta, p)\partial_1\Phi, -\mathfrak{R}_\infty(\eta, p)\partial_2\Phi, -\mathfrak{R}_\infty(\eta, p)\partial_3\Phi\Big)$ *denote the right-hand side of* (3.2.2.1), (3.2.2.2), (3.2.2.3), *and let* $\mathfrak{B}_c(\mathbf{\mathcal{W}}, \Phi, D\Phi)$ *denote the right-hand side* (3.2.1.1) - (3.2.1.3). *Then*

$$ \mathfrak{B}_c(\mathbf{\mathcal{W}}, \Phi, D\Phi) = \mathfrak{B}_\infty(\mathbf{\mathcal{W}}, \nabla^{(1)}\Phi) + F_c, \tag{3.4.3.13} $$

where

$$ F_c \in \mathcal{I}^N(c^{-2}; \mathbf{W}, \Phi, D\Phi) \cap \mathcal{I}^N(c^{-1}; \mathbf{W}, c^{-1}\Phi, c^{-1}\nabla^{(1)}\Phi, c^{-1}\partial_t\Phi). \tag{3.4.3.14} $$

Proof. Lemma 3.4.7 follows from Lemma 3.4.2 and Lemma 3.4.5. □

Example 3.4.1. As an enlightening example, we revisit the polytropic equation of state from Remark 2.3.2 of Chapter 2, which has a non-relativistic limit if we assume that $A_c, A_\infty \in \mathfrak{R}^{N+1}(\mathfrak{K}'; \eta)$, that $A_\infty > 0$ on $\mathfrak{K}'$, and that $A_c - A_\infty \in \mathfrak{R}^{N+1}(c^{-2}; \mathfrak{K}'; \eta)$. Some omitted calculations show that Hypotheses 3.4.3.1 and 3.4.3.2 then hold, and that

$$ R_c = e^{4\Phi/c^2}\mathfrak{R}_c(\eta, p) = \frac{m_0 P^{1/\gamma} e^{4\Phi/c^2(1-1/\gamma)}}{A_c^{1/\gamma}(\eta)} + \frac{P}{c^2(\gamma - 1)} \tag{3.4.3.15} $$

$$ Q_c = \mathfrak{Q}_c(\eta, p, \Phi) = \gamma P \tag{3.4.3.16} $$

$$ R_\infty = \mathfrak{R}_\infty(\eta, p) = \frac{m_0 p^{1/\gamma}}{A_\infty^{1/\gamma}(\eta)} \tag{3.4.3.17} $$

$$ Q_\infty = \mathfrak{Q}_\infty(\eta, p) = \gamma p. \tag{3.4.3.18} $$

3.5 Energy Currents

The role of energy currents is to replace the energy principle available for symmetric hyperbolic systems. Their two key properties are that for a fixed c, they are positive definite in the variations[d] when contracted against certain covectors, and that their divergence is lower order in the variations. In this section we define the energy currents, which exist even in the case $c = \infty$, and demonstrate their key properties. In Section 3.6.2, we will see that the positivity property is uniform for all large c.

3.5.1 The Definition of the Re-scaled Energy Current ${}_{(c)}\dot{\mathcal{J}}$

Given a variation $\dot{\mathbf{W}} : \mathcal{M} \to \mathbb{R}^5$ and a BGS[e] $\widetilde{\mathbf{V}} : \mathcal{M} \to \mathbb{R}^{10}$ as defined in Section 3.3, we define the energy current to be the vectorfield ${}_{(c)}\dot{\mathcal{J}}$ with components ${}_{(c)}\dot{\mathcal{J}}^0$, ${}_{(c)}\dot{\mathcal{J}}^j$, $j = 1, 2, 3$, in the global rectangular coordinate system given by

$$\begin{aligned}
{}_{(c)}\dot{\mathcal{J}}^0 &\overset{\text{def}}{=} \dot{\eta}^2 + \frac{\dot{P}^2}{\widetilde{Q}_c} + 2c^{-2}\widetilde{\gamma}_c^2(\widetilde{v}_k\dot{v}^k)\dot{P} + \widetilde{\gamma}_c^2(\widetilde{R}_c + c^{-2}\widetilde{P})\left[\dot{v}_k\dot{v}^k + c^{-2}\widetilde{\gamma}_c^2(\widetilde{v}_k\dot{v}^k)^2\right] \qquad (3.5.1.1)\\
{}_{(c)}\dot{\mathcal{J}}^j &\overset{\text{def}}{=} \widetilde{v}^j\dot{\eta}^2 + \frac{\widetilde{v}^j}{\widetilde{Q}_c}\dot{P}^2 + 2\left[\dot{v}^j + c^{-2}\widetilde{\gamma}_c^2\widetilde{v}^j\widetilde{v}_k\dot{v}^k\right]\dot{P}\\
&\quad + \widetilde{\gamma}_c^2(\widetilde{R}_c + c^{-2}\widetilde{P})\widetilde{v}^j\left[\dot{v}_k\dot{v}^k + c^{-2}\widetilde{\gamma}_c^2(\widetilde{v}_k\dot{v}^k)^2\right].
\end{aligned}$$

Note that energy currents can be defined when $c = \infty$. In this case, we have

$$\begin{aligned}
{}_{(\infty)}\dot{\mathcal{J}}^0 &\overset{\text{def}}{=} \dot{\eta}^2 + \frac{\dot{p}^2}{\widetilde{Q}_\infty} + \widetilde{R}_\infty\dot{v}_k\dot{v}^k \qquad (3.5.1.2)\\
{}_{(\infty)}\dot{\mathcal{J}}^j &\overset{\text{def}}{=} \widetilde{v}^j\dot{\eta}^2 + \frac{\widetilde{v}^j}{\widetilde{Q}_\infty}\dot{p}^2 + 2\dot{v}^j\dot{p} + \widetilde{R}_\infty\widetilde{v}^j\dot{v}_k\dot{v}^k.
\end{aligned}$$

The energy current (3.5.1.1) is very closely related to the energy current $\dot{J}$ in (2.5.5.1), where the following changes have been made. First, in the expression (3.5.1.1), we have dropped the terms from (2.5.5.1) corresponding to the variations of the potential $\dot{\Phi}$ and its derivatives, for we will bound these terms in a Sobolev norm using a separate argument. Second, the expression for ${}_{(c)}\dot{\mathcal{J}}$ is constructed using the velocity

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[d] The energy currents ${}_{(\infty)}\dot{\mathcal{J}}$ do not control the variations $\dot{\Phi}$ or $D\dot{\Phi}$; these terms are controlled through a separate argument.

[e] Recall that we also refer to $\widetilde{\mathbf{\mathcal{V}}}$ (and also $\widetilde{\mathbf{\mathcal{W}}}$ when $c = \infty$) as the BGS.

state-space variable $\mathbf{v}$ (3.1.2.11) and variations $\dot{\mathbf{v}}$, as opposed to the four-velocity components U^j and four-velocity variations $\dot{U}^j$ that appear in expression for $\dot{J}$. Finally, we remind the reader that the formula for ${}_{(c)}\dot{\mathcal{J}}^{\nu}$ is provided in the coordinate system with $x^0 = t$, whereas in the formula for $\dot{J}$, the implicit coordinate system is $x^0 = ct$, even though c was set equal to unity in Chapter 2.

Remark 3.5.1. Viewed as a quadratic form, ${}_{(\infty)}\dot{\mathcal{J}}^0$ is manifestly positive definite in the variations $\dot{\mathbf{W}}$ if $\widetilde{p} > 0$, for our fundamental assumptions on the equation of state give that $\widetilde{p} > 0 \implies \widetilde{R}_\infty > 0$. Furthermore, as a consequence of (3.4.3.1) and (3.4.3.2), for a fixed BGS $\widetilde{\mathbf{V}}$, the map $c \to {}_{(c)}\dot{\mathcal{J}}$ is continuous at $c = \infty$.

3.5.2 The Positive Definiteness of $\xi_{\mu(c)}\dot{\mathcal{J}}^{\mu}$ for $\xi \in \mathcal{I}_x^{s*+}$

As in Section 2.5.5, for ξ belonging to a certain subset of $T_x^*\mathcal{M}$, the quadratic form[f] $\xi_{\mu(c)}\dot{\mathcal{J}}^{\mu}(\dot{\mathbf{W}}, \dot{\mathbf{W}})$ is positive definite in $\dot{\mathbf{W}}$ if $\widetilde{P} > 0$. Since the energy current ${}_{(c)}\dot{\mathcal{J}}$ from (3.5.1.1) does not contain terms involving the variations of the potential $\dot{\Phi}$, $\xi_{\mu(c)}\dot{\mathcal{J}}^{\mu}(\dot{\mathbf{W}}, \dot{\mathbf{W}})$ is positive definite in $\dot{\eta}, \dot{P}, \dot{\mathbf{v}}$ for ξ belonging to $\mathcal{I}_x^{s*+}$, the interior of the positive component of the *sound* cone at x, which is *larger* than the light cone[g]. Expressed in coordinates, this statement reads

$$\xi_{\mu(c)}\dot{\mathcal{J}}^{\mu}(\dot{\mathbf{W}}, \dot{\mathbf{W}}) > 0 \text{ if } \dot{\mathbf{W}} > 0,\ \widetilde{P} > 0, \text{ and } \xi \in \mathcal{I}_x^{s*+}, \tag{3.5.2.1}$$

where $\mathcal{I}_x^{s*+} \overset{\text{def}}{=} \{\zeta \in T_x^*(\mathcal{M}) \mid (\widetilde{h}^{-1})^{\mu\nu}\zeta_\mu\zeta_\nu < 0 \text{ and } \zeta_0 > 0\}$, and $\widetilde{h}^{-1}$ is the reciprocal acoustical metric[h] with components that read

$$\widetilde{h}^{00} = -c^{-2} - \widetilde{\gamma}_c^2\big[\mathcal{S}_c^{-2}(\widetilde{\eta}, \widetilde{p}) - c^{-2}\big] \tag{3.5.2.2}$$

$$\widetilde{h}^{0j} = \widetilde{h}^{j0} = -\widetilde{\gamma}_c^2\big[\mathcal{S}_c^{-2}(\widetilde{\eta}, \widetilde{p}) - c^{-2}\big]\widetilde{v}^j \tag{3.5.2.3}$$

$$\widetilde{h}^{jk} = \delta^{jk} - \widetilde{\gamma}_c^2\big[\mathcal{S}_c^{-2}(\widetilde{\eta}, \widetilde{p}) - c^{-2}\big]\widetilde{v}^j\widetilde{v}^k \tag{3.5.2.4}$$

in the global rectangular coordinate system; recall that the function $\mathcal{S}_c$ is defined in 3.1.1.14. A proof of (3.5.2.1) can be given using the same methods as in our suggested

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[f] We write "$\xi_{\mu(c)}\dot{\mathcal{J}}^{\mu}(\dot{\mathbf{W}}, \dot{\mathbf{W}})$" to emphasize the point of view that $\xi_{\mu(c)}\dot{\mathcal{J}}^{\mu}$ is a quadratic form in $\dot{\mathbf{W}}$.

[g] Recall that the energy currents $\dot{J}$ defined in Chapter 2 were positive definite only on the light cone, since they also contained terms corresponding to the variations of $\dot{\phi}$ and its derivatives.

[h] The reciprocal acoustical metric was introduced using dimensionless variables in Section 2.5.1 of Chapter 2.

proof of (2.5.5.2). This fact allows us to use the form $\xi_{\mu(c)}\dot{\jmath}^{\mu}(\dot{\mathbf{W}},\dot{\mathbf{W}})$ to estimate the L^2 norms of the variations $\dot{\mathbf{W}}$, provided that we that we estimate the BGS $\widetilde{\mathbf{V}}$. We separately derive estimates for Sobolev norms of the terms $\dot{\Phi}$ and $\partial_t\dot{\Phi}$ in the EOV$_c$ using the lemmas in Appendix A.

Remark 3.5.2. For all large c, the covector with coordinates $(1,0,0,0)$ is an element of $\mathcal{I}_x^{s*+}$. Therefore, ${}_{(c)}\dot{\jmath}^{0}(\dot{\mathbf{W}},\dot{\mathbf{W}})$ is positive definite for all large c.

3.5.3 The Divergence of the Re-scaled Energy Current

As in Section 2.5.5, if the variations $\dot{\mathbf{W}}$ are solutions of the EOV$_c$ (3.3.0.8) - (3.3.0.10) then we can compute $\partial_{\mu(c)}\dot{\jmath}^{\mu}$ and use the equations (3.3.0.8) - (3.3.0.10) for substitution to eliminate the terms[i] containing the derivatives of $\dot{\mathbf{W}}$:

$$\begin{aligned}
&\partial_{\mu(c)}\dot{\jmath}^{\mu} = \left[\partial_t\left(\frac{1}{\widetilde{Q}_c}\right) + \partial_j\left(\frac{\widetilde{v}^j}{\widetilde{Q}_c}\right)\right]\dot{P}^2 \\
&+ 2c^{-2}\widetilde{\gamma}_c^2\dot{P}\left[\dot{v}^k\partial_t\widetilde{v}_k + \dot{v}^k\widetilde{v}_k\partial_j\widetilde{v}^j + \dot{v}^k\widetilde{v}^j\partial_j\widetilde{v}_k + 2c^{-2}\widetilde{\gamma}_c^2\dot{v}^k\widetilde{v}_k\left(\widetilde{v}_j\partial_t\widetilde{v}^j + \widetilde{v}^j\widetilde{v}_a\partial_j\widetilde{v}^a\right)\right] \\
&+ \left\{\partial_t\big[\widetilde{\gamma}_c^2\big(\widetilde{R}_c + c^{-2}\widetilde{P}\big)\big] + \partial_j\big[\widetilde{\gamma}_c^2\big(\widetilde{R}_c + c^{-2}\widetilde{P}\big)\widetilde{v}^j\big]\right\}\big[\dot{v}_k\dot{v}^k + c^{-2}\widetilde{\gamma}_c^2(\widetilde{v}_k\dot{v}^k)^2\big] \\
&+ 2c^{-2}\widetilde{\gamma}_c^4\left[\widetilde{R}_c + \frac{\widetilde{P}}{c^2}\right]\big[\widetilde{v}_k\dot{v}^k\dot{v}^j\partial_t\widetilde{v}_j + \widetilde{v}_k\dot{v}^k\dot{v}^a\widetilde{v}^j\partial_j\widetilde{v}_a + c^{-2}\widetilde{\gamma}_c^2(\widetilde{v}_k\dot{v}^k)^2\big(\widetilde{v}_j\partial_t\widetilde{v}^j + \widetilde{v}_a\widetilde{v}^j\partial_j\widetilde{v}^a\big)\big] \\
&+ 2\dot{\eta}f + 2\frac{\dot{P}}{\widetilde{Q}_c}g + 2\dot{v}_j h^{(j)}.
\end{aligned} \tag{3.5.3.1}$$

Remark 3.5.3. Equation 3.5.3.1 also holds in the case $c=\infty$, where $\widetilde{\gamma}_\infty \stackrel{\mathrm{def}}{=} 1$.

3.6 Assumptions on the Initial Data

In this section we describe a class of initial data for which our energy methods allow us to rigorously take the limit $c\to\infty$ in the EN$_\kappa^c$ system. The Cauchy surface we consider is $\{(t,\mathbf{s})\in\mathcal{M} \mid t=0\}$.

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[i]Showing this via a calculation is quite arduous. The lower-order divergence property is a generic feature of energy currents constructed in the manner described in [15].

3.6.1 An H^N Perturbation of a Quiet Fluid

Initial data for the EP_κ system are denoted by

$$\mathring{\mathbf{\mathcal{V}}}_\infty(\mathbf{s}) \stackrel{\text{def}}{=} (\mathring{\eta}, \mathring{p}, \mathring{v}^1, \mathring{v}^2, \mathring{v}^3, \mathring{\Phi}_\infty, \mathring{\Psi}_0, \mathring{\Psi}_1, \mathring{\Psi}_2, \mathring{\Psi}_3), \tag{3.6.1.1}$$

where $\mathring{\Psi}_0(\mathbf{s}) \stackrel{\text{def}}{=} \partial_t \Phi(0, \mathbf{s})$ and $\mathring{\Psi}_j \stackrel{\text{def}}{=} \partial_j \mathring{\Phi}_\infty(\mathbf{s})$. We assume that $\mathring{\mathbf{\mathcal{V}}}_\infty$ is an H^N perturbation of the constant state $\bar{\mathbf{\mathcal{V}}}_\infty$, where

$$\bar{\mathbf{\mathcal{V}}}_\infty \stackrel{\text{def}}{=} (\bar{\eta}, \bar{p}, 0, 0, 0, \bar{\Phi}_\infty, 0, 0, 0, 0), \tag{3.6.1.2}$$

$\bar{\eta}, \bar{p}$ are positive constants, and the constant $\bar{\Phi}_\infty$ is the unique solution to

$$\kappa^2 \bar{\Phi}_\infty + 4\pi G \mathcal{R}_\infty(\bar{\eta}, \bar{p}) = 0. \tag{3.6.1.3}$$

The constraint (3.6.1.3) must be satisfied in order for equation (3.2.2.4) to be satisfied by $\bar{\mathbf{\mathcal{V}}}_\infty$. By H^N perturbation, we mean that

$$\|\mathring{\mathbf{\mathcal{W}}}_\infty\|_{H^N_{\bar{\mathbf{\mathcal{W}}}_\infty}} < \infty, \tag{3.6.1.4}$$

where we use the notation $\mathring{\mathbf{\mathcal{W}}}_\infty$ and $\bar{\mathbf{\mathcal{W}}}_\infty$ to refer to the first 5 components of $\mathring{\mathbf{\mathcal{V}}}_\infty$ and $\bar{\mathbf{\mathcal{V}}}_\infty$ respectively. We note here that a further positivity restriction on the initial data is introduced in Section (3.6.2). Throughout this chapter, N is a fixed integer satisfying

$$N \geq 4. \tag{3.6.1.5}$$

Remark 3.6.1. We require $N \geq 4$ so that Corollary B.0.3 and Remark B.0.3 can be applied to conclude that $\partial_t^2 l \in C^0([0,T], H^{N-2}(\mathbb{R}^3))$, where l is defined in (3.8.2.12); this is a necessary hypothesis for Proposition A.0.5, which we use in our proof of Theorem 3.8.2.

Although we refer to $\mathring{\Phi}_\infty$ and $\mathring{\Psi}_\nu$, $\nu = 0, 1, 2, 3$, as "data," in the EP_κ system, these 5 quantities are determined by $\mathring{\eta}, \mathring{p}, \mathring{v}^1, \mathring{v}^2, \mathring{v}^3$ through the equations (3.2.2.2') and (3.2.2.4) together with vanishing conditions at infinity on $\mathring{\Phi}_\infty - \bar{\Phi}_\infty$ and $\mathring{\Psi}_0$:

$$\Delta \mathring{\Phi}_\infty - \kappa^2 (\mathring{\Phi}_\infty - \bar{\Phi}_\infty) = 4\pi G \big[\mathcal{R}_\infty(\mathring{\eta}, \mathring{p}) - \mathcal{R}_\infty(\bar{\eta}, \bar{p})\big] \tag{3.6.1.6}$$

$$\Delta \mathring{\Psi}_0 - \kappa^2 \mathring{\Psi}_0 = -4\pi G \partial_t \mathcal{R}_\infty(\eta, p)|_{t=0} = -4\pi G \partial_k \big(\mathcal{R}_\infty(\mathring{\eta}, \mathring{p}) \mathring{v}^k\big), \tag{3.6.1.7}$$

where the integral kernel from (3.2.2.7) can be used to compute $\mathring{\Phi}_\infty - \bar{\Phi}_\infty$ and $\mathring{\Psi}_0$. We will nevertheless refer to the array $\mathring{\mathbf{\mathcal{V}}}_\infty$ as the "data" for the EP_κ system.

Remark 3.6.2. Remark 3.2.1 implies that $\mathring{\Phi}_\infty \in H^{N+1}_{\bar{\Phi}_\infty}$.

We now construct data for the EN^c_κ system from $\mathring{\boldsymbol{\mathcal{V}}}_\infty$. Depending on which set of state-space variables we are working with, we denote the data for the EN^c_κ system by

$$\mathring{\boldsymbol{\mathcal{V}}}_c \stackrel{\text{def}}{=} (\mathring{\eta}, \mathring{p}, \mathring{v}^1, \mathring{v}^2, \mathring{v}^3, \mathring{\Phi}_c, \mathring{\Psi}_0, \mathring{\Psi}_1, \mathring{\Psi}_2, \mathring{\Psi}_3) \tag{3.6.1.8}$$

$$\text{or } \mathring{\mathbf{V}}_c \stackrel{\text{def}}{=} (\mathring{\eta}, e^{4\mathring{\Phi}_c/c^2}\mathring{p}, \mathring{v}^1, \mathring{v}^2, \mathring{v}^3, \mathring{\Phi}_c, \mathring{\Psi}_0, \mathring{\Psi}_1, \mathring{\Psi}_2, \mathring{\Psi}_3), \tag{3.6.1.9}$$

where unlike in the EP_κ case, $\mathring{\Phi}_c, \mathring{\Psi}_0, \mathring{\Psi}_1, \mathring{\Psi}_2$, and $\mathring{\Psi}_3$ are data in the sense that the system is under-determined if they are not prescribed. We choose the data $\mathring{\eta}, \mathring{p}, \mathring{v}^1, \mathring{v}^2, \mathring{v}^3, \mathring{\Psi}_0, \mathring{\Psi}_1, \mathring{\Psi}_2, \mathring{\Psi}_3$ for the EN^c_κ system to be the same as the data for the EP_κ system, but for technical reasons described below and indicated in (3.6.1.12) and (3.6.1.14), our requirement that there exists a constant background state typically constrains the datum $\mathring{\Phi}_c$ so that it differs from $\mathring{\Phi}_\infty$ by a small constant that vanishes as $c \to \infty$.

As in the EP_κ system, we assume that $\mathring{\boldsymbol{\mathcal{V}}}_c$ is an H^N perturbation of the constant state of the form (depending on which collection of state-space variables we are working with)

$$\bar{\boldsymbol{\mathcal{V}}}_c \stackrel{\text{def}}{=} (\bar{\eta}, \bar{p}, 0, 0, 0, \bar{\Phi}_c, 0, 0, 0, 0) \tag{3.6.1.10}$$

$$\text{or } \bar{\mathbf{V}}_c \stackrel{\text{def}}{=} (\bar{\eta}, \bar{P}_c, 0, 0, 0, \bar{\Phi}_c, 0, 0, 0, 0) \tag{3.6.1.11}$$

where $\bar{\eta}$ and $\bar{p}$ are the same constants appearing in $\bar{\boldsymbol{\mathcal{V}}}_\infty$, $\bar{\Phi}_c$ is the unique solution to

$$\kappa^2\bar{\Phi}_c + 4\pi G e^{4c^{-2}\bar{\Phi}_c}\left[\mathcal{R}_c(\bar{\eta}, \bar{p}) - 3c^{-2}\bar{p}\right] = 0, \tag{3.6.1.12}$$

and $\bar{P}_c \stackrel{\text{def}}{=} e^{4c^{-2}\bar{\Phi}_c}\bar{p}$. The constraint (3.6.1.12) must be satisfied in order for equation (3.2.1.4) to be satisfied by $\bar{p}, \bar{\eta}$, and $\bar{\Phi}_c$. Although the background potential $\bar{\Phi}_c$ for the EN^c_κ system is not in general equal to the background potential $\bar{\Phi}_\infty$ for the EP_κ system, it follows from the hypotheses (3.4.3.1) and (3.4.3.2) on the c-dependence of $\mathcal{R}_c$ that

$$\lim_{c\to\infty} \bar{\Phi}_c = \bar{\Phi}_\infty. \tag{3.6.1.13}$$

We now define the initial datum $\mathring{\Phi}_c$ appearing in the arrays (3.6.1.8) and (3.6.1.9) by

$$\mathring{\Phi}_c \stackrel{\text{def}}{=} \mathring{\Phi}_\infty - \bar{\Phi}_\infty + \bar{\Phi}_c, \tag{3.6.1.14}$$

which ensures that the deviation of $\mathring{\Phi}_c$ from the background potential $\bar{\Phi}_c$ matches the deviation of $\mathring{\Phi}_\infty$ from the background potential $\bar{\Phi}_\infty$. We denote the first 5 components of $\mathring{\boldsymbol{\mathcal{V}}}_c$, $\mathring{\mathbf{V}}_c$, $\bar{\boldsymbol{\mathcal{V}}}_c$, and $\bar{\mathbf{V}}_c$ by $\mathring{\boldsymbol{\mathcal{W}}}_c, \mathring{\mathbf{W}}_c, \bar{\boldsymbol{\mathcal{W}}}_c$, and $\bar{\mathbf{W}}_c$ respectively.

Remark 3.6.3. We could weaken the hypotheses by allowing the initial data for the EN_κ^c system to deviate from the data for the EP_κ system by an H^N perturbation that decays to 0 rapidly enough as $c \to \infty$. For simplicity, we will not pursue this line of thought here.

3.6.2 The Set $\mathfrak{K}$

The Construction of $\mathfrak{K}$

In order to avoid studying the free boundary problem and in order to avoid singularities in the energy current (3.5.1.1), we assume that the initial pressure, energy density, and speed of sound are uniformly bounded from below by a positive constant. According to our assumptions (3.1.1.11) on the equation of state and definition (3.1.2.23), to achieve this uniform bound, it is sufficient to make the following further assumption on the initial data: $\mathring{\boldsymbol{\mathcal{W}}}_\infty(\mathbb{R}^3)$ is contained in a compact subset of the following open subset $\mathcal{O}$ of the state-space $\mathbb{R}^5$, the *admissible subset of truncated state-space:*

$$\mathcal{O} = \{\boldsymbol{\mathcal{W}} \in \mathbb{R}^5 | \eta > 0, p > 0\}. \tag{3.6.2.1}$$

Therefore, we assume that $\mathring{\boldsymbol{\mathcal{W}}}_\infty(\mathbb{R}^3) \subset \mathcal{O}_1$ and $\bar{\boldsymbol{\mathcal{W}}}_\infty \in \mathcal{O}_1$, where $\mathcal{O}_1$ is a convex precompact open set[j] with $\bar{\mathcal{O}}_1 \Subset \mathcal{O}$. By slightly enlarging $\mathcal{O}_1$ if necessary, property (3.6.1.13) allows us to assume that for all large c, $\mathring{\mathbf{W}}_c(\mathbb{R}^3) \subset \mathcal{O}_1$ and $\bar{\mathbf{W}}_c \in \mathcal{O}_1$; also note that for all c, $\mathring{\boldsymbol{\mathcal{W}}}_\infty = \mathring{\boldsymbol{\mathcal{W}}}_c$. We then fix a convex[k] precompact open subset $\mathcal{O}_2$ with $\bar{\mathcal{O}}_1 \Subset \mathcal{O}_2 \Subset \mathcal{O}$.

We now address the variables $\big(\Phi, \partial_t\Phi, \partial_1\Phi, \partial_2\Phi, \partial_3\Phi\big)$. If *equalities* are achieved in the inequalities (3.8.2.1b) and (3.8.2.1d) below by solution variables $\big(\Phi, \partial_t\Phi, \partial_1\Phi, \partial_2\Phi, \partial_3\Phi\big)$, then definition (3.6.1.14) and Sobolev imbedding imply that

[j] In practice, $\mathcal{O}_1$ can be chosen to be a cube.

[k] Proposition B.0.4 requires the convexity of $\bar{\mathcal{O}}_2$.

there is a cube of the form $[-a,a]^5$ such that for all large c including $c=\infty$, $\big(\Phi, \partial_t\Phi, \partial_1\Phi, \partial_2\Phi, \partial_3\Phi\big)([0,T]\times\mathbb{R}^3) \subset [-a,a]^5$. Furthermore, the inequalities (3.8.2.1b) and (3.8.2.1d) are constructed in a manner such that for all large c including $c=\infty$, $\big(\mathring{\Phi}_c, \mathring{\Psi}_0, \mathring{\Psi}_1, \mathring{\Psi}_2, \mathring{\Psi}_3\big)(\mathbb{R}^3) \Subset [-a,a]^5$. We now define

$$\mathfrak{K} \stackrel{\text{def}}{=} \bar{\mathcal{O}}_2 \times [-a,a]^5 \tag{3.6.2.2}$$

and note that $\mathfrak{K}$ is convex, and that furthermore, for all large c including $c=\infty$, $\mathring{\mathbf{V}}_c(\mathbb{R}^3) \Subset \mathrm{Int}(\mathfrak{K})$ and $\bar{\mathbf{V}}_c \in \mathrm{Int}(\mathfrak{K})$. Our goal is to show that the solution $\mathbf{V}_c$ to (3.2.1.1) - (3.2.1.4) launched by the initial data $\mathring{\mathbf{V}}_c$ exists on a time interval $[0,T]$ that is independent of (all large) c and remains in $\mathfrak{K}$.

Remark 3.6.4. By slightly enlarging $\bar{\mathcal{O}}_2$, it easily follows from definitions (3.3.0.13) and (3.3.0.15) that there is a compact, convex set $\mathfrak{K}' \subset \mathbb{R}^{(10)}$ such that for all large c, $\mathbf{V} \in \mathfrak{K} \implies \mathcal{V} \in \mathfrak{K}'$. Additionally, we choose $\mathfrak{K}'$ so that $\mathring{\mathcal{V}}_c(\mathbb{R}^3) \Subset \mathrm{Int}(\mathfrak{K}')$ and $\bar{\mathcal{V}}_c \in \mathrm{Int}(\mathfrak{K}')$.

The Uniform-in-c Positive Definiteness of ${}_{(c)}\dot{\mathfrak{J}}^0$ on $\mathfrak{K}$

As described in Section 2.6.2, we will use the quantity $\|{}_{(c)}\dot{\mathfrak{J}}^0(t)\|_{L^1}$ to control $\|\dot{\mathbf{W}}(t)\|_{H^N}^2$, where ${}_{(c)}\dot{\mathfrak{J}}$ is an energy current for the variation $\dot{\mathbf{W}}$ with coefficients defined by a BGS $\widetilde{\mathbf{V}}$. Since we seek estimates that are uniform in c, it is important that ${}_{(c)}\dot{\mathfrak{J}}^0$ is uniformly positive definite independent of both the BGS $\widetilde{\mathbf{V}}$ and all large c. Let us now formulate this precisely as a lemma.

Lemma 3.6.1. *Let ${}_{(c)}\dot{\mathfrak{J}}$ be the energy current for the variation $\dot{\mathbf{W}}$ defined by the BGS $\widetilde{\mathbf{V}}$ as defined in Section 3.5.1. Assume that $\widetilde{\mathbf{V}}(t,\mathbf{s}) \in \mathfrak{K}$. Then there exists a constant $C_{\mathfrak{K}}$ with $0 < C_{\mathfrak{K}} < 1$ such that*

$$C_{\mathfrak{K}}|\dot{\mathbf{W}}|^2 \le {}_{(c)}\dot{\mathfrak{J}}^0(\dot{\mathbf{W}},\dot{\mathbf{W}}) \le C_{\mathfrak{K}}^{-1}|\dot{\mathbf{W}}|^2 \tag{3.6.2.3}$$

holds for all large c including $c=\infty$.

Proof. The proof we give involves a slight modification of our proof of (2.6.2.2). It is sufficient prove inequality (3.6.2.3) when $|\dot{\mathbf{W}}| = 1$ since it is invariant under any rescaling of $\dot{\mathbf{W}}$. Let $\widetilde{\mathcal{V}}$ be the array related to the array $\widetilde{\mathbf{V}}$ as in (3.3.0.13) and (3.3.0.15), and let $\mathfrak{K}'$ be the compact set defined in Remark 3.6.4. Recall that ${}_{(\infty)}\dot{\mathscr{J}}$ is defined in (3.5.1.2) and that ${}_{(\infty)}\dot{\mathscr{J}}^0$ is manifestly positive definite in the variations[1] $\dot{\mathbf{W}}$ if $\widetilde{p} > 0$. If we view ${}_{(\infty)}\dot{\mathscr{J}}^0$ as a function of $(\dot{\mathbf{W}}, \widetilde{\mathcal{W}})$, then by uniform continuity, there is a constant $0 < C(\mathfrak{K}') < 1$ such that $C(\mathfrak{K}')|\dot{\mathbf{W}}|^2 \leq {}_{(\infty)}\dot{\mathscr{J}}^0 \leq C(\mathfrak{K}')^{-1}|\dot{\mathbf{W}}|^2$ holds on the compact set $\{|\dot{\mathbf{W}}| = 1\} \times \mathfrak{K}'$. Furthermore, if we also view ${}_{(c)}\dot{\mathscr{J}}^0$ as a function of $(\dot{\mathbf{W}}, \widetilde{\mathcal{V}})$, then by Lemma 3.4.5 and (3.5.1.1), we have that ${}_{(c)}\dot{\mathscr{J}}^0 = {}_{(\infty)}\dot{\mathscr{J}}^0 + F_c \cdot |\dot{\mathbf{W}}|^2$, where $F_c \in \mathcal{R}^N(c^{-2}; \mathfrak{K}'; \widetilde{\mathcal{V}})$. (3.6.2.3) now easily follows: $C_{\mathfrak{K}}$ can be any positive number that is strictly smaller than $C(\mathfrak{K}')$. Note that it may be considered a slight abuse of notation that we label the constant "$C_{\mathfrak{K}}$" rather than "$C_{\mathfrak{K}'}$." $\square$

3.7 Smoothing the Initial Data

As in Section 2.7.2, we smooth the first 5 components $\mathring{\mathcal{W}}_\infty$ of the data $\mathring{\mathcal{V}}_\infty$ defined in (3.6.1.1) with the mollifier χ_ϵ, defining $\chi_\epsilon \mathring{\mathcal{W}}_\infty \in C^\infty$ by

$$\chi_\epsilon \mathring{\mathcal{W}}_\infty(\mathbf{s}) \overset{\text{def}}{=} \int_{\mathbb{R}^3} \chi_\epsilon(\mathbf{s} - \mathbf{s}') \mathring{\mathcal{W}}_\infty(\mathbf{s}') \, d^3\mathbf{s}'. \tag{3.7.0.4}$$

This is necessary because we will need to estimate the H^N norms of the right-hand sides of (3.8.2.6) - (3.8.2.8) below. Without smoothing, this would in general lead to infinite expressions. Note that we do not smooth the data $\mathring{\Phi}$ or $\mathring{\Psi}_0$.

The following property of such a mollification is well known:

$$\lim_{\epsilon \to 0^+} \|\chi_\epsilon \mathring{\mathcal{W}}_\infty - \mathring{\mathcal{W}}_\infty\|_{H^N} = 0. \tag{3.7.0.5}$$

We will choose below an $\epsilon_0 > 0$. Once chosen, we define the smoothed data $\chi_{\epsilon_0} \mathring{\mathcal{W}}_\infty$ by

$$^{(0)}\mathring{\mathcal{W}} \overset{\text{def}}{=} \big({}^{(0)}\mathring{\eta}, {}^{(0)}\mathring{p}, {}^{(0)}\mathring{\mathbf{v}}\big) \overset{\text{def}}{=} \chi_{\epsilon_0} \mathring{\mathcal{W}}_\infty, \tag{3.7.0.6}$$

$$^{(0)}\mathring{\mathbf{W}}_c \overset{\text{def}}{=} \big({}^{(0)}\mathring{\eta}, e^{4\mathring{\Phi}_c/c^2} \cdot {}^{(0)}\mathring{p}, {}^{(0)}\mathring{\mathbf{v}}\big), \tag{3.7.0.7}$$

where $\mathring{\Phi}_c$ is defined in (3.6.1.14). By Sobolev imbedding, the assumptions on the initial data $\mathring{\mathbf{W}}_c$, which are the first 5 components of the data $\mathring{\mathbf{V}}_c$ defined in (3.6.1.9), (3.4.3.5),

[1] To be consistent the notation used in formula (3.5.1.2), we "should" use the symbol $\dot{\mathcal{W}}$ to denote the variations appearing as arguments in ${}_{(\infty)}\dot{\mathscr{J}}$. However, for the purposes of this proof, there is no harm in identifying $\dot{\mathbf{W}} = \dot{\mathcal{W}}$ since these placeholder variables merely represent the arguments of ${}_{(\infty)}\dot{\mathscr{J}}$ when viewed as a quadratic form.

Lemma 3.4.2, and the mollification properties above, $\exists\{\Lambda_1 > 0 \wedge \epsilon_0 > 0\}$ (at least as small as the ϵ_0 in (**??**)) such that

$$\text{for all large } c,\ \|\mathbf{W} - {}^{(0)}\mathring{\mathbf{W}}_c\|_{H^N} \leq \Lambda_1 \Rightarrow \mathbf{W} \in \bar{\mathcal{O}}_2 \tag{3.7.0.8}$$

$$\|{}^{(0)}\mathring{\mathbf{W}}_c - \mathring{\mathbf{W}}_c\|_{H^N} \lesssim C_{\mathfrak{K}}\frac{\Lambda_1}{2}, \tag{3.7.0.9}$$

where $\bar{\mathcal{O}}_2$ and $\mathfrak{K}$ are defined in Section 3.6.2, and $C_{\mathfrak{K}}$ is from (3.6.2.3).

Remark 3.7.1. As in Remark 2.7.3, it is an important fact that $\|{}^{(0)}\mathring{\mathbf{V}}_c\|_{H^{N+1}_{\bar{\mathbf{V}}_c}}$ and $\|{}^{(0)}\mathring{\mathbf{V}}_c\|_{L^\infty}$ are uniformly bounded for all large c. Also see Remark 3.6.2 for a relevant comment.

3.8 Uniform-in-Time Local Existence for $\mathbf{EN}_\kappa^c$

In this section we prove the first important theorem of this chapter, namely that there is a uniform time interval $[0, T]$ on which solutions to the EN_κ^c system having the initial data $\mathring{\mathbf{V}}_c$ exist, as long as c is large enough.

3.8.1 Local Existence and Uniqueness for $\mathbf{EN}_\kappa^c$ Revisited

Let us recall the following local existence result provided by Theorem 2.7.1, in which it is not yet shown that T can be chosen independently of all large c.

Theorem (Local Existence for $\mathbf{EN}_\kappa^c$). *Let $\mathring{\mathbf{V}}_c(\mathbf{s})$ be initial data for the EN_κ^c system* (3.2.1.1) - (3.2.1.4) *that are subject to the conditions described in Section 3.6. Then for all large (finite) c, there exists $T_c > 0$ such that* (3.2.1.1) - (3.2.1.4) *has a unique classical solution $\mathbf{V}(t, \mathbf{s})$ on $[0, T_c] \times \mathbb{R}^3$ of the form*
$\mathbf{V} = (\eta, P, v^1, v^2, v^3, \Phi, \partial_t\Phi, \partial_1\Phi, \partial_2\Phi, \partial_3\Phi)$ *with $\mathbf{V}(0, \mathbf{s}) = \mathring{\mathbf{V}}_c(\mathbf{s})$. The solution satisfies* $\mathbf{V}([0, T_c] \times \mathbb{R}^3) \subset \mathfrak{K}$, *where the set $\mathfrak{K}$ is defined in Section 3.6.2. Furthermore,*
$\mathbf{V} \in C^0([0, T_c], H^N_{\bar{\mathbf{V}}_c}) \cap C^1([0, T_c], H^{N-1}_{\bar{\mathbf{V}}_c}) \cap C^2([0, T_c], H^{N-2}_{\bar{\mathbf{V}}_c})$ *and* $\Phi \in C^0([0, T_c], H^{N+1}_{\bar{\Phi}_c}) \cap C^1([0, T_c], H^N_{\bar{\Phi}_c}) \cap C^2([0, T_c], H^{N-1}_{\bar{\Phi}_c}) \cap C^3([0, T_c], H^{N-2}_{\bar{\Phi}_c})$.

Remark 3.8.1. Although it is not explicitly stated in Theorem 2.7.1, the fact that $\mathbf{V}$ is twice differentiable as a map from $[0, T_c]$ to $H^{N-2}_{\bar{\mathbf{V}}_c}$ follows from our assumption that

$N \geq 4$. Also, by Corollary B.0.3, we have that $p \in C^0([0,T_c], H^N_{\bar{p}}) \cap C^1([0,T_c], H^{N-1}_{\bar{p}}) \cap C^2([0,T_c], H^{N-2}_{\bar{p}})$, since $p = Pe^{-4\Phi/c^2}$.

Remark 3.8.2. The case $c = \infty$ is discussed separately in Theorem 3.9.1.

Remark 3.8.3. Strictly speaking, Theorem 2.7.1 was proved using the relativistic state-space variables U^ν. However, the form of the Newtonian change of variables made in sections 3.1 and 3.1.2 and Corollary B.0.3 allow us to conclude Sobolev regularity in one set of variables if the same regularity is known in the other set of variables.

Corollary 3.8.1. *Assume that* $5/2 < N' < N$ *and that* $\mathbf{V}(t,\mathbf{s})$ *is a solution to the* EN^c_κ *system having the regularity property* $\mathbf{V} \in L^\infty([0,T], H^N_{\bar{\mathbf{V}}_c}) \cap C^0([0,T], H^{N'}_{\bar{\mathbf{V}}_c}) \cap C^1([0,T], H^{N-1}_{\bar{\mathbf{V}}_c})$. *Then*

$$\mathbf{V} \in C^0([0,T], H^N_{\bar{\mathbf{V}}_c}). \tag{3.8.1.1}$$

Proof. We apply Theorem 3.8.1 to conclude that there exists $\epsilon > 0$ and a solution $\widetilde{\mathbf{V}} \in C^0([T-\epsilon,T], H^N_{\bar{\mathbf{V}}_c}) \cap C^1([T-\epsilon,T], H^{N-1}_{\bar{\mathbf{V}}_c})$ with $\widetilde{\mathbf{V}}(T) = \mathbf{V}(T)$. Furthermore, the uniqueness argument in Section 2.7.5 can be easily modified to show that solutions to the EN^c_κ system are unique in the class $C^0([T-\epsilon,T], H^{N'}_{\bar{\mathbf{V}}_c}) \cap C^1([0,T], H^{N-1}_{\bar{\mathbf{V}}_c})$. Therefore $\mathbf{V} \equiv \widetilde{\mathbf{V}}$ on their common slab of spacetime existence. □

3.8.2 The Uniform-in-Time Local Existence Theorem

We now state and prove the uniform time of existence theorem.

Theorem (Uniform-in-Time Existence). *Let* $\mathring{\mathbf{V}}_c(\mathbf{s})$ *be initial data for the* EN^c_κ *system that is subject to the conditions described in Section 3.6, and let* ${}^{(0)}\mathring{\mathbf{W}}_c$ *denote the smoothing of the first 5 components of* $\mathring{\mathbf{V}}_c$ *as described in Section 3.7. Let* $\mathfrak{K}$ *be the subset of* $\mathbb{R}^{10}$ *constructed from* $\mathring{\mathbf{V}}_c$ *as described in Section 3.6.2. Then there exists* $c_0 > 0$ *and* $T > 0$, *with* T *not depending on* c, *such that for* $c \geq c_0$, $\mathring{\mathbf{V}}_c$ *launches a unique classical solution* $\mathbf{V}$ *to* (3.2.1.1) - (3.2.1.4) *that exists on the slab* $[0,T] \times \mathbb{R}^3$ *and that has the properties* $\mathbf{V}(0,\mathbf{s}) = \mathring{\mathbf{V}}_c(\mathbf{s})$, $\mathbf{V}([0,T] \times \mathbb{R}^3) \subset \mathfrak{K}$. *The solution is of the form* $\mathbf{V} = (\eta, P, v^1, v^2, v^3, \Phi, \partial_t\Phi, \partial_1\phi, \partial_2\Phi, \partial_3\Phi)$ *and has the regularity property* $\mathbf{V} \in C^0([0,T], H^N_{\bar{\mathbf{V}}_c}(\mathbb{R}^3)) \cap C^1([0,T], H^{N-1}_{\bar{\mathbf{V}}_c}(\mathbb{R}^3))$, *and* $\Phi \in C^0([0,T], H^{N+1}_{\bar{\Phi}_c}(\mathbb{R}^3)) \cap$

$C^1([0,T], H^N_{\bar{\Phi}_c}(\mathbb{R}^3)) \cap C^2([0,T], H^{N-1}_{\bar{\Phi}_c}(\mathbb{R}^3))$. *Furthermore, with* $p = Pe^{-4\phi/c^2}$, *there exist constants* $\Lambda_1, \Lambda_2, L_1, L_2, L_3, L_4 > 0$ *such that*

$$||| \mathbf{W} - {}^{(0)}\mathring{\mathbf{W}}_c |||_{H^N,T} \lesssim \Lambda_1 \tag{3.8.2.1a}$$

$$||| \Phi - \mathring{\Phi}_c |||_{H^{N+1},T} \lesssim \Lambda_2 \tag{3.8.2.1b}$$

$$||| \partial_t \mathbf{W} |||_{H^{N-1},T} \lesssim L_1 \tag{3.8.2.1c}$$

$$||| \partial_t \Phi |||_{H^N,T} \lesssim L_2 \tag{3.8.2.1d}$$

$$||| \partial_t^2 \eta |||_{H^{N-2},T}, \ ||| \partial_t^2 p |||_{H^{N-2},T} \lesssim L_3 \tag{3.8.2.1e}$$

$$c^{-1} ||| \partial_t^2 \Phi |||_{H^{N-1},T} \lesssim L_4. \tag{3.8.2.1f}$$

Our proof has a lot in common with our proof of Proposition 2.7.3. For the sake of appearances, we suppress the dependence of the running constants on N and $\mathfrak{K}$ in our proof. We indicate dependence on the initial data $\|{}^{(0)}\mathring{\mathbf{W}}_c\|_{H^{N+1}_{\bar{\mathbf{W}}_c}}$, $\|\mathring{\Phi}_c\|_{H^{N+1}_{\bar{\Phi}_c}}$, and $\|\mathring{\Psi}_0\|_{H^N}$ by writing $C(id)$. By Remark 3.7.1, any constant $C(id)$ can be chosen to be independent of all large c. In our proof of Theorem 3.8.2, we also use an additional key ingredient, namely a continuation principle for Sobolev norm-bounded solutions:

Proposition 3.8.2. *Let* $\mathring{\mathbf{V}}_c(\mathbf{s})$ *be initial data for the* EN^c_κ *system* (3.2.1.1) - (3.2.1.4) *that are subject to the conditions described in Section 3.6, and let* $T > 0$. *Let* $\mathbf{V}$ *be the unique classical solution launched by* $\mathring{\mathbf{V}}_c(\mathbf{s})$, *and assume that* $\mathbf{V} \in C^0([0,T), H^N_{\bar{\mathbf{V}}_c}) \cap C^1([0,T), H^{N-1}_{\bar{\mathbf{V}}_c})$. *Assume that there are constants* $M_1, M_2 > 0$ *and a compact set* $\mathfrak{K}$ *such that the following three estimates hold for any* $T' \in [0,T)$:

1. $||| \mathbf{V} |||_{H^N_{\bar{\mathbf{V}}_c},T'} \leq M_1$
2. $||| \partial_t \mathbf{V} |||_{H^{N-1},T'} \leq M_2$
3. $\mathbf{V}([0,T'] \times \mathbb{R}^3) \subset \mathfrak{K}$.

Then

$$\mathbf{V} \in C^0([0,T], H^N_{\bar{\mathbf{V}}_c}) \cap C^1([0,T], H^{N-1}_{\bar{\mathbf{V}}_c}) \quad \textit{and} \quad \mathbf{V}([0,T] \times \mathbb{R}^3) \subset \mathfrak{K}. \tag{3.8.2.2}$$

Proof. We will first show that there exists a $\mathbf{V}^* \in H^N_{\bar{\mathbf{V}}_c}(\mathbb{R}^3)$ such that

$$\lim_{n\to\infty} \|\mathbf{V}(T_n) - \mathbf{V}^*\|_{H^{N-1}} = 0 \tag{3.8.2.3}$$

holds for any sequence $\{T_n\}$ of time values converging to T from below.

If $\{T_n\}$ is such a sequence, then hypothesis 2 implies that $\|\mathbf{V}(T_j) - \mathbf{V}(T_k)\|_{H^{N-1}} \leq M_2|T_j - T_k|$. By the completeness of H^{N-1}, there exists a $\mathbf{V}^* \in H^{N-1}$ such that (3.8.2.3) holds, and it is easy to check that $\mathbf{V}^*$ does not depend on the sequence $\{T_n\}$. By hypothesis 1, we also have that $\{\mathbf{V}(T_n)\}$ converges weakly in $H^N_{\bar{\mathbf{V}}_c}$ to $\mathbf{V}^*$ and that $\|\mathbf{V}^*\|_{H^N_{\bar{\mathbf{V}}}} \leq M_1$. We now fix a number N' with $5/2 < N' < N$. By Proposition B.0.6, we have that $\lim_{n\to\infty} \|\mathbf{V}(T_n) - \mathbf{V}^*\|_{H^{N'}} = 0$. Consequently, if we define $\mathbf{V}(T) \overset{\text{def}}{=} \mathbf{V}^*$, it follows that $\mathbf{V} \in L^\infty([0,T], H^N_{\bar{\mathbf{V}}_c}) \cap C^0([0,T], H^{N'}_{\bar{\mathbf{V}}_c}) \cap C^1([0,T], H^{N-1}_{\bar{\mathbf{V}}_c})$. The claim (3.8.2.2) now follows from Corollary 3.8.1 and continuity. $\square$

Outline of the Structure of the Proof of Theorem 3.8.2

The inequalities (3.8.2.1a) - (3.8.2.1f) are derived from a variety of a-priori energy estimates for the solution. As we will see, the difficulty is that each of the 6 estimates in (3.8.2.1a) - (3.8.2.1f) depends on one or more of the others, and so one must carefully arrange the order of the argument so that it is, heuristically speaking, of the form "Term 1 is *bounded* on $[0,T] \implies$ Term 2 is *bounded* on $[0,T] \cdots \implies$ Term n is *bounded* on $[0,T] \implies$ Term 1 is *small* on $[0,T]$ as long as T is small $\implies$ Term 2 is *small* on $[0,T]$ as long as T is small $\cdots \implies$ Term n is *small* on $[0,T]$ as long as T is small. This is carried out in detail in (3.8.2.19) - (3.8.2.25) below, but to preserve the continuity of the argument, we have placed all of the technical lemmas in a separate section. Once we have chosen T small enough to imply such a chain of a-priori estimates, we apply Proposition 3.8.2, which shows that a-priori smallness bounds on Sobolev norms of the solution allow one to continue it and therefore preclude the possibility of blowup on $[0,T]$. In the opinion of the author, the most interesting of the a-priori estimates is (3.8.2.25), in which $\|\partial_t\Phi\|_{H^N}$ is shown to be small based in part on the hypothesis that $c^{-1}\|\partial_t\Phi\|_{H^N}$ is small (and c can be large!); this estimate is essential to the argument.

Proof of Theorem 3.8.2

We begin our proof of Theorem 3.8.2 by introducing some notation. Let $\mathbf{V}$ denote the local in time solution to the EN_κ^c system (3.2.1.1) - (3.2.1.4) launched by the initial data $\mathring{\mathbf{V}}_c$ as furnished by Theorem 3.8.1. With $\mathbf{W}$ denoting the first 5 components of $\mathbf{V}$, we suggestively define

$$\dot{\mathbf{W}}(t,\mathbf{s}) \stackrel{\text{def}}{=} \mathbf{W}(t,\mathbf{s}) - {}^{(0)}\mathring{\mathbf{W}}_c(\mathbf{s}) \tag{3.8.2.4}$$

$$\dot{\Phi} \stackrel{\text{def}}{=} \Phi - \mathring{\Phi}_c, \tag{3.8.2.5}$$

where $\mathring{\Phi}_c$ is defined in (3.6.1.14) and ${}^{(0)}\mathring{\mathbf{W}}_c(\mathbf{s})$ is defined in (3.7.0.7).

It follows from the fact that $\mathbf{W}$ is a solution to (3.2.1.1) - (3.2.1.3) that $\dot{\mathbf{W}}$ is a solution to the EOV_c (3.3.0.8) - (3.3.0.10) defined by the BGS $\mathbf{V}$ with initial data $\dot{\mathbf{W}}(0,\mathbf{s}) = \mathring{\mathbf{W}}_c(\mathbf{s}) - {}^{(0)}\mathring{\mathbf{W}}_c(\mathbf{s})$. The inhomogeneous terms in the EOV_c satisfied by $\dot{\mathbf{W}}$ are given by $\mathbf{b} = (f, g, \cdots, h^{(3)})$, where for $j = 1, 2, 3$

$$f = -v^k \partial_k[{}^{(0)}\mathring{\eta}] \tag{3.8.2.6}$$

$$\begin{aligned} g = {} & (4P - 3Q_c)[\partial_t(c^{-2}\Phi) + v^k\partial_k(c^{-2}\Phi)] - v^k\partial_k[e^{4\mathring{\Phi}_c/c^2} \cdot {}^{(0)}\mathring{p}] \\ & - Q_c\partial_k[{}^{(0)}\mathring{v}^k] - c^{-2}\widetilde{\gamma}_c^2 Q_c v^k v_a \partial_k[{}^{(0)}\mathring{v}^a] \end{aligned} \tag{3.8.2.7}$$

$$\begin{aligned} h^{(j)} = {} & \big(3c^{-2}P - R_c\big)\big(\partial_j\Phi + \gamma_c^{-2} v^j[\partial_t(c^{-2}\Phi) + v^k\partial_k(c^{-2}\Phi)]\big) \\ & - \gamma_c^2(R_c + c^{-2}P)\big(v^k\partial_k[{}^{(0)}\mathring{v}^j] + c^{-2}\widetilde{\gamma}_c^2 v^j v^k v_a \partial_k[{}^{(0)}\mathring{v}^a]\big) \\ & - \partial_j[e^{4\mathring{\Phi}_c/c^2} \cdot {}^{(0)}\mathring{p}] - c^{-2}\widetilde{\gamma}_c^2 v^j v^k \partial_k[e^{4\mathring{\Phi}_c/c^2} \cdot {}^{(0)}\mathring{p}]. \end{aligned} \tag{3.8.2.8}$$

In order to show that the hypotheses of Proposition 3.8.2 are satisfied, we will need to estimate $\partial_{\vec{\alpha}}\dot{\mathbf{W}}$ in L^2. Therefore, we will study the equation that $\partial_{\vec{\alpha}}\dot{\mathbf{W}}$ satisfies: for $0 \leq |\vec{\alpha}| \leq N$, we differentiate the EOV_c defined by the BGS $\mathbf{V}$ with inhomogeneous terms $\mathbf{b}$ to which $\dot{\mathbf{W}}$ is a solution, obtaining that $\partial_{\vec{\alpha}}\dot{\mathbf{W}}$ satisfies[m]

$${}_c\mathcal{A}^\mu(\mathcal{W}, \Phi)\partial_\mu\big(\partial_{\vec{\alpha}}\dot{\mathbf{W}}\big) = \mathbf{b}_{\vec{\alpha}}, \tag{3.8.2.9}$$

where

$$\mathbf{b}_{\vec{\alpha}} \stackrel{\text{def}}{=} {}_c\mathcal{A}^0 \partial_{\vec{\alpha}}\left(({}_c\mathcal{A}^0)^{-1}\mathbf{b}\right) + \mathbf{k}_{\vec{\alpha}} \tag{3.8.2.10}$$

[m] Recall the convention stated in Remark 3.3.2.

and

$$\mathbf{k}_{\vec{\alpha}} \stackrel{\text{def}}{=} {}_c\mathcal{A}^0\big[({}_c\mathcal{A}^0)^{-1}{}_c\mathcal{A}^k\partial_k(\partial_{\vec{\alpha}}\dot{\mathbf{W}}) - \partial_{\vec{\alpha}}\big(({}_c\mathcal{A}^0)^{-1}{}_c\mathcal{A}^k\partial_k\dot{\mathbf{W}}\big)\big]. \tag{3.8.2.11}$$

Thus, $\partial_{\vec{\alpha}}\dot{\mathbf{W}}$ is a solution the EOV$_c$ defined by the *same* BGS $\mathbf{V}$ with inhomogeneous terms $\mathbf{b}_{\vec{\alpha}}$. Furthermore, $\dot{\Phi}$ is a solution to the EOV$_c$ (3.3.0.11), where

$$l \stackrel{\text{def}}{=} (\kappa^2 - \Delta)\mathring{\Phi}_c + 4\pi G(R_c - 3c^{-2}P). \tag{3.8.2.12}$$

We will return to these facts in Section 3.8.3 where we will use them in the proofs of some technical lemmas.

As an intermediate step, we will prove the following weaker version of (3.8.2.1d):

$$c^{-1} \,|||\, \partial_t\Phi \,|||_{H^N,T} \lesssim L_2'. \tag{3.8.2.1d'}$$

We now define the constants Λ_1, Λ_2, L_2', and L_4. We will then use a variety of energy estimates to define L_1, L_2, and L_3 in terms of these four constants and to show that (3.8.2.1a) - (3.8.2.1f) are satisfied if T is small enough. Λ_1 has been already defined in (3.7.0.8). To motivate our definitions of L_2', L_4, and Λ_2, see inequalities (A.0.2.4) and (A.0.2.6) of Lemma A.0.2 and inequality (A.0.2.20) of Corollary A.0.3, and let $C_0(\kappa)$ denote the constant that appears throughout the lemma and its corollary. By Lemma 3.8.8, we have that

$$C_0(\kappa)\big(c^{-1}\|\mathring{\Psi}_0\|_{H^N} + \|l(0)\|_{H^{N-1}}\big) \lesssim C_1(id,\kappa) \tag{3.8.2.13}$$

$$C_0(\kappa)\big(c\|l(0)\|_{H^{N-1}} + \|(\Delta - \kappa^2)\mathring{\Psi}_0 - \partial_t l(0)\|_{H^{N-2}} \lesssim C_2(id,\kappa). \tag{3.8.2.14}$$

Note also the trivial (and not optimal) estimate $c^{-2}C_0(\kappa)\|\mathring{\Psi}_0\|_{H^N} \lesssim 2C_0(\kappa)$. We thus define

$$\Lambda_2 \stackrel{\text{def}}{=} 2C_0(\kappa) \tag{3.8.2.15}$$

$$L_2' \stackrel{\text{def}}{=} 2C_1(id,\kappa) \tag{3.8.2.16}$$

$$L_4 \stackrel{\text{def}}{=} 2C_2(id,\kappa). \tag{3.8.2.17}$$

Let T_c^{max} be the maximal time for which the solution $\mathbf{V}_c$ exists, remains in $\mathfrak{K}$, and satisfies the estimates (3.8.2.1a), (3.8.2.1b), (3.8.2.1d'), and (3.8.2.1f); i.e.,

$$T_c^{max} \stackrel{\text{def}}{=} \sup\Big\{T|\mathbf{V} \in C^0([0,T], H^N_{\bar{\mathbf{V}}_c}(\mathbb{R}^3)) \cap C^1([0,T], H^{N-1}_{\bar{\mathbf{V}}_c}(\mathbb{R}^3)),$$
$$\mathbf{V}([0,T]\times\mathbb{R}^3) \subset \mathfrak{K}, \text{ and } (3.8.2.1a), (3.8.2.1b), \ (3.8.2.1d'), \text{and } (3.8.2.1f) \text{ hold}\Big\}. \tag{3.8.2.18}$$

Lemmas 3.8.16, 3.8.4, 3.8.9, 3.8.13, 3.8.11, and inequality (3.8.3.33) of Lemma 3.8.12 supply the following estimates which are valid for $0 \leq \tau < T_c^{max}$:

$$||| \, \dot{\mathbf{W}} \, |||_{H^N,\tau} \lesssim \big[\Lambda_1/2 + C(\Lambda_1,\Lambda_2,L_1,L_2')\tau\big] \cdot \exp\big(C(\Lambda_1,\Lambda_2,L_1,L_2')\tau\big) \tag{3.8.2.19}$$

$$||| \, \partial_t \mathbf{W} \, |||_{H^N,\tau} \lesssim L_1(\Lambda_1,\Lambda_2,L_2') \tag{3.8.2.20}$$

$$||| \, \partial_t^2 p \, |||_{H^{N-2},\tau}, ||| \, \partial_t^2 \eta \, |||_{H^{N-2},\tau} \lesssim L_3(\Lambda_1,\Lambda_2,L_1,L_2',L_4) \tag{3.8.2.21}$$

$$||| \, \dot{\Phi} \, |||^2_{H^{N+1},\tau} \lesssim (\Lambda_2)^2/4 + \tau \cdot C(\Lambda_1,\Lambda_2,L_1,L_2') + \tau^2 \cdot C(\Lambda_1,\Lambda_2,L_1,L_2',L_3,L_4) \tag{3.8.2.22}$$

$$c^{-1} \, ||| \, \partial_t \Phi \, |||_{H^N,\tau} \lesssim L_2'/2 + \tau C(\Lambda_1,\Lambda_2,L_1,L_2') \tag{3.8.2.23}$$

$$c^{-1} \, ||| \, \partial_t^2 \Phi \, |||_{H^{N-1},\tau} \lesssim L_4/2 + \tau \cdot C(\Lambda_1,\Lambda_2,L_1,L_2',L_3,L_4). \tag{3.8.2.24}$$

We apply the following sequence of reasoning to interpret the above inequalities: first L_1 in (3.8.2.20) is determined through the known constants Λ_1, Λ_2, and L_2'. Then L_3 in (3.8.2.21) is determined through the known constants $\Lambda_1, \Lambda_2, L_1, L_2'$ and L_4. Then the remaining constants $C(\cdots)$ in (3.8.2.19) - (3.8.2.24) are all determined through $\Lambda_1, \Lambda_2, L_1, L_2', L_3, L_4$.

We now choose T so that when $0 \leq \tau \leq T$, it algebraically follows that the right-hand sides of (3.8.2.19), (3.8.2.22), (3.8.2.23), and (3.8.2.24) are *strictly* less than $\Lambda_1, (\Lambda_2)^2, L_2'$, and L_4 respectively. Note that T may be chosen independently of (all large) c.

Following this, inequality (3.8.3.32) of Lemma 3.8.12 gives the bound

$$||| \, \partial_t \Phi \, |||_{H^N,\tau} \lesssim L_2(id,\Lambda_1,\Lambda_2,L_1,L_2')/2 + \tau C(\Lambda_1,\Lambda_2,L_1,L_2',L_3,L_4). \tag{3.8.2.25}$$

By shrinking T if necessary, we may assume that when $0 \leq \tau \leq T$, it algebraically follows that the right-hand side of (3.8.2.25) is *strictly* less than L_2. We now show that $T_c^{max} < T$ is impossible.

Assume that $T_c^{max} < T$. Then by Proposition 3.8.2, we have $\mathbf{V}_c \in C^0([0, T_c^{max}], H^N_{\bar{\mathbf{V}}_c}) \cap C^1([0, T_c^{max}], H^{N-1}_{\bar{\mathbf{V}}_c})$, and by continuity, inequalities (3.8.2.19) - (3.8.2.25) hold for $\tau = T_c^{max}$. Since the right-hand sides of (3.8.2.19), (3.8.2.20), (3.8.2.22) - (3.8.2.24), and (3.8.2.25) are *strictly* less than $\Lambda_1, (\Lambda_2)^2, L_2', L_4$, and L_2 respectively when $\tau = T_c^{max}$, Sobolev imbedding implies that $\mathbf{V}_c([0, T_c^{max}] \times \mathbb{R}^3)$ is contained in the *interior*[n] of $\mathfrak{K}$. Consequently, we may apply Theorem 3.8.1, taking T_c^{max} as the initial time and $\mathbf{V}_c(T_c^{max})$ as initial data, to produce an $\epsilon > 0$ such that $\mathbf{V}_c \in C^0([0, T_c^{max} + \epsilon], H^N_{\bar{\mathbf{V}}_c}) \cap C^1([0, T_c^{max} + \epsilon], H^{N-1}_{\bar{\mathbf{V}}_c})$ and $||| \, \dot{\mathbf{W}} \, |||_{H^N, T_c^{max}+\epsilon} \lesssim \Lambda_1$, $||| \, \dot{\Phi} \, |||^2_{H^{N+1}, T_c^{max}+\epsilon} \lesssim (\Lambda_2)^2$, $c^{-1} \, ||| \, \partial_t \Phi \, |||_{H^N, T_c^{max}+\epsilon} \lesssim L_2'$, and $c^{-1} \, ||| \, \partial_t^2 \Phi \, |||_{H^{N-1}, T_c^{max}+\epsilon} \lesssim L_4$. This contradicts the definition of T_c^{max} and completes the proof of Theorem 3.8.2.

3.8.3 Proofs of the Technical Lemmas

We now state and prove the technical lemmas quoted in the proof of Theorem 3.8.2. We will require some auxiliary lemmas along the way. Throughout this section, we assume the hypotheses and notation used in our proof of Theorem 3.8.2; i.e., $\mathbf{V}$ denotes the solution, $\mathbf{W}$ denotes its first 5 components, $\dot{\mathbf{W}}$ and $\dot{\Phi}_c$ are defined in (3.8.2.4) and (3.8.2.5) respectively, l is defined in (3.8.2.12), and so forth.

We also assume that $\tau \in [0, T_c^{max})$. By the definition of T_c^{max} and the set $\mathfrak{K}'$ defined in Remark 3.6.4, we have the following bounds:

$$||| \, \mathbf{W} - {}^{(0)}\mathring{\mathbf{W}}_c \, |||_{H^N, \tau} \lesssim \Lambda_1 \tag{3.8.3.1}$$

$$||| \, \Phi - \mathring{\Phi}_c \, |||_{H^{N+1}, \tau} \lesssim \Lambda_2 \tag{3.8.3.2}$$

$$c^{-1} \, ||| \, \partial_t \Phi \, |||_{H^N, \tau} \lesssim L_2'. \tag{3.8.3.3}$$

$$c^{-1} \, ||| \, \partial_t^2 \Phi \, |||_{H^{N-2}, \tau} \lesssim L_4. \tag{3.8.3.4}$$

$$\mathbf{V}([0, \tau] \times \mathbb{R}^3) \subset \mathfrak{K} \tag{3.8.3.5}$$

$$\boldsymbol{\mathcal{V}}([0, \tau] \times \mathbb{R}^3) \subset \mathfrak{K}'. \tag{3.8.3.6}$$

---

[n] Recall that in Section 2.6.2, $\mathfrak{K}$ was defined through Sobolev embedding based on the hypotheses that $\|\dot{\mathbf{W}}\|_{H^N} = \Lambda_1$, $\|\dot{\Phi}\|_{H^{N+1}} = \Lambda_2$, and $\|\partial_t \Phi\|_{H^N} = L_2$.

We make use of the bounds (3.8.3.1) - (3.8.3.6) in our arguments below without always explicitly mentioning that we are doing so.

Remark 3.8.4. By Sobolev imbedding, (3.8.3.3) and (3.8.3.4) imply L^∞ bounds on $c^{-1}\partial_t\Phi$. and $c^{-1}\partial_t^2\Phi$ respectively. As stated in Remark 3.4.5, we will make use of this implied L^∞ bound and the L^∞ bounds (3.8.3.5), (3.8.3.6) on $\mathbf{W}, \nabla^{(1)}\mathbf{W}, \boldsymbol{\mathcal{W}}, \nabla^{(1)}\boldsymbol{\mathcal{W}}, \Phi$, and $D\Phi$ without explicitly mentioning them.

Lemma 3.8.3.

$$(4\pi G)^{-1}l = \mathfrak{R}_\infty(\eta, p) - \mathfrak{R}_\infty(\mathring{\eta}, \mathring{p}) + F_c, \tag{3.8.3.7}$$

$$(4\pi G)^{-1}\partial_t l = \partial_t\big(\mathfrak{R}_\infty(\eta, p)\big) + G_c \tag{3.8.3.8}$$

$$(4\pi G)^{-1}\partial_t^2 l = \partial_t^2\big(\mathfrak{R}_\infty(\eta, p)\big) + H_c, \tag{3.8.3.9}$$

where

$$F_c \in \mathcal{I}^N(c^{-2}; \eta, p, \Phi) \cap \mathcal{I}^N(c^{-1}; \eta, p, c^{-1}\Phi) \tag{3.8.3.10}$$

$$\begin{aligned} G_c \in\ & \mathcal{I}^{N-1}(c^{-2}; \eta, p, c^{-1}\Phi, \partial_t\eta, \partial_t p, c^{-1}\partial_t\Phi) \\ & \cap \mathcal{I}^{N-1}(c^{-1}; \eta, p, c^{-1}\Phi, \partial_t\eta, \partial_t p, c^{-1}\partial_t\Phi) \end{aligned} \tag{3.8.3.11}$$

$$H_c \in \mathcal{I}^{N-2}(c^{-1}; \eta, p, c^{-1}\Phi, \partial_t\eta, \partial_t p, c^{-1}\partial_t\Phi, \partial_t^2\eta, \partial_t^2 p, c^{-1}\partial_t^2\Phi). \tag{3.8.3.12}$$

Proof. It follows from the discussion in Section 3.6 that

$$\begin{aligned} (4\pi G)^{-1}l = {} & \big[e^{4\Phi/c^2}\mathfrak{R}_c(\eta, p) - e^{4\bar{\Phi}_c/c^2}\mathfrak{R}_c(\bar{\eta}, \bar{p})\big] + 3c^{-2}\big(e^{4\bar{\Phi}_c/c^2}\bar{p} - e^{4\Phi/c^2}p\big) \\ & + \mathfrak{R}_\infty(\bar{\eta}, \bar{p}) - \mathfrak{R}_\infty(\mathring{\eta}, \mathring{p}). \end{aligned} \tag{3.8.3.13}$$

Therefore, (3.8.3.7) follows from Lemma 3.4.1, Lemma 3.4.2, and Lemma 3.4.5. (3.8.3.8) and (3.8.3.9) then follow from Remark 3.8.1 and Lemma 3.4.3. □

Lemma 3.8.4.

$$|||\ \partial_t\mathbf{W}\ |||_{H^{N-1},\tau} \lesssim C(\Lambda_1, \Lambda_2, L_2') \overset{def}{=} L_1(\Lambda_1, \Lambda_2, L_2'). \tag{3.8.3.14}$$

Proof. By using the EN_κ^c equations (3.2.1.1) - (3.2.1.3) to solve for $\partial_t\mathbf{W}$ and applying Lemma 3.4.2, (3.4.3.8), (3.4.3.9), Lemma 3.4.6, and Lemma 3.4.7, we have that

$$\begin{aligned} \partial_t\mathbf{W} &= \big({}_c\mathcal{A}^0(\boldsymbol{\mathcal{W}}, \Phi)\big)^{-1}[-{}_c\mathcal{A}^k(\boldsymbol{\mathcal{W}}, \Phi)\partial_k\mathbf{W} + \mathfrak{B}_c(\boldsymbol{\mathcal{W}}, \Phi, D\Phi)] \\ &= \big({}_\infty\mathcal{A}^0(\boldsymbol{\mathcal{W}})\big)^{-1}\big[-{}_\infty\mathcal{A}^k(\boldsymbol{\mathcal{W}})\partial_k\mathbf{W} + \mathfrak{B}_\infty(\boldsymbol{\mathcal{W}}, \nabla^{(1)}\Phi)\big] + F_c, \end{aligned} \tag{3.8.3.15}$$

where

$$F_c \in \mathcal{I}^{N-1}(c^{-2}; \mathbf{W}, \nabla^{(1)}\mathbf{W}, \Phi, D\Phi) \cap \mathcal{I}^{N-1}(c^{-1}; \mathbf{W}, \nabla^{(1)}\mathbf{W}, c^{-1}\Phi, c^{-1}D\Phi). \quad (3.8.3.16)$$

Lemma 3.8.4 now follows from Proposition B.0.2, the bounds (3.8.3.1) - (3.8.3.3) and the definition of $\mathcal{I}^{N-1}(c^{-1}; \mathbf{W}, \nabla^{(1)}\mathbf{W}, c^{-1}\Phi, c^{-1}D\Phi)$, which implies that $||| \ F_c \ |||_{H^{N-1},\tau} \lesssim c^{-1}C(\Lambda_1, \Lambda_2, L_2')$. □

Corollary 3.8.5.

$$\partial_t \boldsymbol{\mathcal{W}} = \big({}_\infty\mathcal{A}^0(\boldsymbol{\mathcal{W}})\big)^{-1}\big[-{}_\infty\mathcal{A}^k(\boldsymbol{\mathcal{W}})\partial_k \boldsymbol{\mathcal{W}} + \mathfrak{B}_\infty(\boldsymbol{\mathcal{W}}, \nabla^{(1)}\Phi)\big] + G_c, \quad (3.8.3.17)$$

where

$$G_c \in \mathcal{I}^{N-1}(c^{-2}; \mathbf{W}, \nabla^{(1)}\mathbf{W}, \Phi, D\Phi) \cap \mathcal{I}^{N-1}(c^{-1}; \mathbf{W}, \nabla^{(1)}\mathbf{W}, c^{-1}\Phi, c^{-1}D\Phi). \quad (3.8.3.18)$$

Proof. Observe that $\partial_t p = \partial_t P + (e^{4\Phi/c^2} - 1)\partial_t P - 4\partial_t \Phi/c^2 P$, and that $\partial_t P$ is the second component on the left-hand side of (3.8.3.15). The relation (3.8.3.17) now follows from (3.8.3.15), Lemma 3.4.2, (3.4.3.5), (3.4.3.9), and the fact that $\mathbf{W}$ and $\boldsymbol{\mathcal{W}}$ differ only in that the second component of $\mathbf{W}$ is P, while the second component of $\boldsymbol{\mathcal{W}}$ is p. □

Corollary 3.8.6.

$$||| \ \partial_t \boldsymbol{\mathcal{W}} \ |||_{H^{N-1},\tau} \lesssim C(\Lambda_1, \Lambda_2, L_2'). \quad (3.8.3.19)$$

Proof. Using the expression (3.8.3.17), the proof of (3.8.3.19) easily follows from the proof of Lemma 3.8.4. □

Corollary 3.8.7. *The following relation for the solution is valid on* $[0, T_c^{max})$ *:*

$$\partial_t p + v^k \partial_k p + \mathfrak{Q}_\infty(\eta, p)\partial_k v^k = F_c, \quad (3.8.3.20)$$

where

$$F_c \in \mathcal{I}^{N-1}(c^{-2}; \mathbf{W}, \nabla^{(1)}\mathbf{W}, \Phi, D\Phi) \cap \mathcal{I}^{N-1}(c^{-1}; \mathbf{W}, \nabla^{(1)}\mathbf{W}, c^{-1}\Phi, c^{-1}D\Phi). \quad (3.8.3.21)$$

Proof. Corollary 3.8.7 immediately follows from the fact that $\partial_t p$ is the second entry of $\partial_t \boldsymbol{\mathcal{W}}$ in the expression (3.8.3.17). □

Lemma 3.8.8. *There exists a constant $C(id) > 0$ such that*

$$\|l(0)\|_{H^N} \lesssim c^{-2}C(id) \tag{3.8.3.22}$$

$$\|(\Delta - \kappa^2)\mathring{\Psi}_0 - \partial_t l(0)\|_{H^{N-1}} \lesssim c^{-2}C(id). \tag{3.8.3.23}$$

Proof. The estimate (3.8.3.22) follows from inequality (3.8.3.7) at $t = 0$.

To obtain the estimate (3.8.3.23), first recall that according to the assumption (3.6.1.7) and the chain rule, we have that

$$\begin{aligned}(4\pi G)^{-1}(\kappa^2 - \Delta)\mathring{\Psi}_0 &= \partial_k\big(\mathcal{R}_\infty(\mathring{\eta},\mathring{p})\mathring{v}^k\big) \\ &= \frac{\partial \mathcal{R}_\infty}{\partial \eta}(\mathring{\eta},\mathring{p})\mathring{v}^k\partial_k\mathring{\eta} + \frac{\partial \mathcal{R}_\infty}{\partial p}(\mathring{\eta},\mathring{p})\mathring{v}^k\partial_k\mathring{p} + \mathcal{R}_\infty(\mathring{\eta},\mathring{p})\partial_k\mathring{v}^k.\end{aligned} \tag{3.8.3.24}$$

Furthermore, by (3.8.3.8), the chain rule, (3.2.1.1), (3.2.2.6), (3.8.3.20), and (3.1.1.16) in the case $c = \infty$, we have that

$$\begin{aligned}(4\pi G)^{-1}\partial_t l(0) = &- \frac{\partial \mathcal{R}_\infty}{\partial \eta}(\mathring{\eta},\mathring{p})\mathring{v}^k\partial_k\mathring{\eta} - \frac{\partial \mathcal{R}_\infty}{\partial p}(\mathring{\eta},\mathring{p})\mathring{v}^k\partial_k\mathring{p} \\ &- \mathcal{R}_\infty(\mathring{\eta},\mathring{p})\partial_k\mathring{v}^k + \mathcal{O}^{N-1}(c^{-2}; id).\end{aligned} \tag{3.8.3.25}$$

The estimate (3.8.3.23) now follows from (3.8.3.24) and (3.8.3.25).

$\square$

Lemma 3.8.9.

$$|||\ \partial_t^2\eta\ |||_{H^{N-2},\tau},\ |||\ \partial_t^2 p\ |||_{H^{N-2},\tau} \lesssim C(\Lambda_1,\Lambda_2,L_1,L_2',L_4) \overset{def}{=} L_3(\Lambda_1,\Lambda_2,L_1,L_2',L_4). \tag{3.8.3.26}$$

Proof. To obtain the bound for $\partial_t^2 p$, first isolate $\partial_t p$ in the expression (3.8.3.20), then differentiate with respect to t and apply Lemma 3.4.3 to conclude that

$$\partial_t^2 p = -\partial_t\big[v^k\partial_k p + \mathfrak{Q}_\infty(\eta,p)\partial_k v^k\big] + G_c, \tag{3.8.3.27}$$

where $G_c \in \mathcal{I}^{N-2}(c^{-1}; \mathbf{W}, \nabla^{(1)}\mathbf{W}, \partial_t\mathbf{W}, \nabla^{(1)}\partial_t\mathbf{W}, c^{-1}\Phi, c^{-1}D\Phi, c^{-1}\partial_t^2\Phi)$. We now use (3.8.3.1) - (3.8.3.4), the previously established bound (3.8.3.14) on $|||\ \partial_t\mathbf{W}\ |||_{H^{N-1},\tau}$, the previously established bound (3.8.3.19) on $|||\ \partial_t\mathcal{W}\ |||_{H^{N-1},\tau}$, and the definition of $\mathcal{I}^{N-2}(\cdots)$ to conclude the estimate (3.8.3.26) for $|||\ \partial_t^2 p\ |||_{H^{N-2},\tau}$.

The estimate for $\partial_t^2\eta$ is similar, and in fact much simpler: use equation (3.2.1.1) to solve for $\partial_t\eta$, then differentiate with respect to t and reason as above. $\square$

Lemma 3.8.10.

$$||| \ l \ |||_{H^N,\tau} \lesssim C(id, \Lambda_1, \Lambda_2) \tag{3.8.3.28}$$

$$||| \ \partial_t l \ |||_{H^{N-1},\tau} \lesssim C(\Lambda_1, \Lambda_2, L_1, L_2') \tag{3.8.3.29}$$

$$||| \ \partial_t^2 l \ |||_{H^{N-2},\tau} \lesssim C(\Lambda_1, \Lambda_2, L_1, L_2', L_3, L_4). \tag{3.8.3.30}$$

Proof. Recall that a convenient expression for l is given in (3.8.3.13). Lemma 3.8.10 follows from (3.8.3.1) - (3.8.3.4), Lemma 3.8.3, Lemma 3.8.4, and Lemma 3.8.9. □

Lemma 3.8.11.

$$c^{-1} \ ||| \ \partial_t \Phi \ |||_{H^N,\tau} \lesssim L_2'/2 + \tau C(\Lambda_1, \Lambda_2, L_1, L_2'). \tag{3.8.3.31}$$

Proof. (3.8.3.31) follows from definition (3.8.2.16), Lemma 3.8.8, inequality (3.8.3.29) of Lemma 3.8.10, and inequality (A.0.2.4) of Lemma A.0.2. □

Lemma 3.8.12.

$$\begin{aligned} ||| \ \partial_t \Phi \ |||_{H^N,\tau} &\lesssim C(id, \Lambda_1, \Lambda_2, L_1, L_2') + \tau C(\Lambda_1, \Lambda_2, L_1, L_2', L_3, L_4) \quad (3.8.3.32) \\ &\stackrel{def}{=} L_2(id, \Lambda_1, \Lambda_2, L_1, L_2')/2 + \tau C(\Lambda_1, \Lambda_2, L_1, L_2', L_3, L_4) \\ c^{-1} \ ||| \ \partial_t^2 \Phi \ |||_{H^{N-1},\tau} &\lesssim C(id) + \tau C(\Lambda_1, \Lambda_2, L_1, L_2', L_3, L_4) \quad (3.8.3.33) \\ &\stackrel{def}{=} L_4(id)/2 + \tau C(\Lambda_1, \Lambda_2, L_1, L_2', L_3, L_4). \end{aligned}$$

Proof. The estimate (3.8.3.32) follows from Lemma 3.8.8, inequalities (3.8.3.29) and (3.8.3.30) of Lemma 3.8.10, and inequality (A.0.2.25) of Proposition A.0.5. The estimate (3.8.3.33) follows from definition (3.8.2.17), Lemma 3.8.8, inequality (3.8.3.30) of Lemma 3.8.10, and inequality (A.0.2.6) of Lemma A.0.2. □

Remark 3.8.5. Inequality (3.8.3.32) is interesting in that it bounds $||| \ \partial_t \Phi \ |||_{H^N,\tau}$ from above by a quantity that depends on $||| \ c^{-1}\partial_t \Phi \ |||_{H^N,\tau}$ —hence the appearance of L_2' on the right-hand side of (3.8.3.32).

Lemma 3.8.13.

$$||| \ \dot{\Phi} \ |||^2_{H^{N+1},\tau} \lesssim \frac{(\Lambda_2)^2}{4} + \tau \cdot C(\Lambda_1, \Lambda_2, L_1, L_2') + \tau^2 \cdot C(\Lambda_1, \Lambda_2, L_1, L_2', L_3, L_4). \tag{3.8.3.34}$$

Proof. Inequality (3.8.3.34) follows from definition (3.8.2.15), (3.8.3.28), (3.8.3.32), and inequality (A.0.2.20) of Corollary A.0.3. □

Lemma 3.8.14. *Let* ${}_{(c)}\dot{\jmath}$ *be the energy current* (3.5.1.1) *for the variation* $\dot{\mathbf{W}}$ *defined by the BGS* $\mathbf{V}$, *and let* $\mathbf{b} \overset{def}{=} (f, g, \cdots, h^{(3)})$, *where* $f, g, \cdots, h^{(3)}$ *are the inhomogeneous terms from the* EOV_c *satisfied by* $\dot{\mathbf{W}}$ *that also appear in the expression* (3.5.3.1) *for the divergence of* ${}_{(c)}\dot{\jmath}$. *Then on* $[0, T_c^{max})$,

$$\|\partial_{\mu}{}_{(c)}\dot{\jmath}^{\mu}\|_{L^1} \lesssim C(\Lambda_1, \Lambda_2, L_1, L_2') \cdot \Big[\|\dot{\mathbf{W}}\|_{L^2}^2 + \|\dot{\mathbf{W}}\|_{L^2}\|\mathbf{b}\|_{L^2}\Big]. \tag{3.8.3.35}$$

Proof. Lemma 3.8.14 follows from the expression (3.5.3.1), Hypotheses (3.4.3.1) and (3.4.3.2), Lemma 3.4.5, Remark 3.4.8, the bounds (3.8.3.1) - (3.8.3.3), the bound (3.8.3.14), Sobolev imbedding, and the Cauchy-Schwarz inequality for integrals. □

We also state here the following corollary that will be used in the proof of Theorem 3.9.2. We do not give a proof, since it is similar to the proof of Lemma 3.8.14, and in fact, simpler: c does not enter into the estimates.

Corollary 3.8.15. *Let* $\mathcal{V}$ *be an* H^N *solution to the* EP_κ *system* (3.2.2.1) - (3.2.2.4) *existing on the interval* $[0, T]$, *and let* $\dot{\mathcal{W}}$ *be a solution to the* EOV_∞ (3.3.0.8) - (3.3.0.10) *defined by the BGS* $\mathcal{W}$ *with inhomogeneous terms* $\mathbf{b} = (f, g, \cdots, h^{(3)})$. *Let* ${}_{(\infty)}\dot{\jmath}$ *be the energy current* (3.5.1.2) *for the variation* $\dot{\mathcal{W}}$ *defined by the BGS* $\mathcal{W}$. *Then on* $[0, T]$,

$$\|\partial_{\mu}{}_{(\infty)}\dot{\jmath}^{\mu}\|_{L^1} \lesssim C(|||\, \mathcal{W}\, |||_{H^N_{\bar{\mathcal{W}}},T}, |||\, \partial_t \mathcal{W}\, |||_{H^{N-1},T}) \cdot \Big[\|\dot{\mathcal{W}}\|_{L^2}^2 + \|\dot{\mathcal{W}}\|_{L^2}\|\mathbf{b}\|_{L^2}\Big]. \tag{3.8.3.36}$$

Lemma 3.8.16.

$$|||\, \dot{\mathbf{W}}\, |||_{H^N,\tau} \lesssim \big[\Lambda_1/2 + C(\Lambda_1, \Lambda_2, L_1, L_2')\tau\big] \cdot \exp\big(C(\Lambda_1, \Lambda_2, L_1, L_2')\tau\big). \tag{3.8.3.37}$$

Proof. Our proof of Lemma 3.8.16 follows from a Gronwall estimate in the H^N norm of the variation $\dot{\mathbf{W}}$ defined in (3.8.2.4). Rather than directly estimating the H^N norm of $\dot{\mathbf{W}}$, we instead estimate the L^1 norm of ${}_{(c)}\dot{\jmath}^0_{\vec{\alpha}}$, where ${}_{(c)}\dot{\jmath}_{\vec{\alpha}}$ is the energy current for the variation $\partial_{\vec{\alpha}}\dot{\mathbf{W}}$ defined by the BGS $\mathbf{V}$. This is favorable because of property (3.5.2.1) and because by (3.5.3.1), the divergence of ${}_{(c)}\dot{\jmath}$ is lower order in $\dot{\mathbf{W}}$. We follow the method of proof of Proposition 2.7.3; the only difficulty is checking that our estimates are independent of all large c. We will show that for $0 \leq |\vec{\alpha}| \leq N$ and $t \in [0, T_c^{max})$, we have

$$\|\mathbf{b}_{\vec{\alpha}}\|_{L^2} \lesssim C(id, \Lambda_1, \Lambda_2, L_2')\big(1 + \|\dot{\mathbf{W}}\|_{H^N}\big), \tag{3.8.3.38}$$

where $\mathbf{b}_{\vec{\alpha}}$ is defined in (3.8.2.10). Let us assume (3.8.3.38) for the moment; we will provide a proof at the end of the proof of the lemma.

We now let ${}_{(c)}\dot{\mathcal{J}}_{\vec{\alpha}}$ denote the energy current (3.5.1.1) for the variation $\partial_{\vec{\alpha}}\dot{\mathbf{W}}$ defined by the BGS $\mathbf{V}$, and abbreviating $\dot{\mathcal{J}}_{\vec{\alpha}} \stackrel{\text{def}}{=} {}_{(c)}\dot{\mathcal{J}}_{\vec{\alpha}}$ to ease the notation, we define $\mathcal{E}(t) \geq 0$ by

$$\mathcal{E}^2(t) \stackrel{\text{def}}{=} \sum_{|\vec{\alpha}| \leq N} \int_{\mathbb{R}^3} \dot{\mathcal{J}}^0_{\vec{\alpha}}(t, \mathbf{s})\, d\mathbf{s}. \tag{3.8.3.39}$$

By (3.6.2.3) and the Cauchy-Schwarz inequality for sums, we have that

$$C_{\mathfrak{K}}\|\dot{\mathbf{W}}\|^2_{H^N} \lesssim \mathcal{E}^2(t) \lesssim C_{\mathfrak{K}}^{-1}\|\dot{\mathbf{W}}\|^2_{H^N}. \tag{3.8.3.40}$$

Then by definition (3.8.3.39), Lemma 3.8.14, (3.8.3.38), (3.8.3.40), we have

$$2\mathcal{E}\frac{d}{dt}\mathcal{E} = \sum_{|\vec{\alpha}| \leq N} \int_{\mathbb{R}^3} \partial_\mu \dot{\mathcal{J}}^\mu_{\vec{\alpha}}\, d^3\mathbf{s} \lesssim C(\Lambda_1, \Lambda_2, L_1, L_2') \cdot \big(\|\partial_{\vec{\alpha}}\dot{\mathbf{W}}\|^2_{L^2} + \|\partial_{\vec{\alpha}}\dot{\mathbf{W}}\|_{L^2}\|\mathbf{b}_{\vec{\alpha}}\|_{L^2}\big) \tag{3.8.3.41}$$

$$\lesssim C(id, \Lambda_1, \Lambda_2, L_1, L_2') \cdot \big(\|\dot{\mathbf{W}}\|^2_{H^N} + \|\dot{\mathbf{W}}\|_{H^N}\big) \lesssim C(id, \Lambda_1, \Lambda_2, L_1, L_2') \cdot \big(\mathcal{E}^2 + \mathcal{E}\big).$$

We now apply Gronwall's inequality to (3.8.3.41), concluding that
(with $C \stackrel{\text{def}}{=} C(id, \Lambda_1, \Lambda_2, L_1, L_2')$)

$$\mathcal{E}(t) \lesssim \big[\mathcal{E}(0) + C \cdot t\big] \cdot \exp(Ct). \tag{3.8.3.42}$$

Using (3.8.3.40) again, it follows from (3.8.3.42) that

$$\|\dot{\mathbf{W}}(t)\|_{H^N} \lesssim \big(C_{\mathfrak{K}}^{-1}\|\dot{\mathbf{W}}(0)\|_{H^N_{\mathring{\mathbf{W}}_c}} + Ct\big) \cdot \exp(Ct). \tag{3.8.3.43}$$

Recalling that $\dot{\mathbf{W}}(0) = \mathring{\mathbf{W}}_c - {}^{(0)}\mathring{\mathbf{W}}_c$ and taking into account inequality (3.7.0.9), the estimate (3.8.3.37) now follows.

It remains to show (3.8.3.38). Our proof is based on the Sobolev-Moser lemmas stated in Appendix B and the $c-$independent estimates of Section 3.4. We first claim that the term ${}_c\mathcal{A}^0\partial_{\vec{\alpha}}\big(({}_c\mathcal{A}^0)^{-1}\mathbf{b}\big)$ from (3.8.2.10) satisfies

$$\|{}_c\mathcal{A}^0\partial_{\vec{\alpha}}\big(({}_c\mathcal{A}^0)^{-1}\mathbf{b}\big)\|_{L^2} \leq C(\mathfrak{K}; id, \Lambda_1, \Lambda_2, L_2'), \tag{3.8.3.44}$$

where all of the estimates we derive in this section are valid on $[0, T_c^{max})$. Because Lemma 3.4.6 and (3.8.3.5) together imply that $\|{}_c\mathcal{A}^0(\mathbf{\mathcal{W}}, \Phi)\|_{L^\infty} \lesssim C(\mathfrak{K})$, it suffices to control the L^2 norm of $\partial_{\vec{\alpha}}\big(({}_c\mathcal{A}^0)^{-1}\mathbf{b}\big)$. Then using Lemma 3.4.6, (3.8.3.5), Proposition B.0.2, and Remark B.0.3 with $({}_c\mathcal{A}^0)^{-1}$ playing the role of F in the proposition, and $\mathbf{b}$ playing the role of G, we have that

$$\|({}_c\mathcal{A}^0)^{-1}\mathbf{b}\|_{H^N} \lesssim C(\mathfrak{K}; \Lambda_1, \Lambda_2)\|\mathbf{b}\|_{H^N}. \tag{3.8.3.45}$$

To estimate $\|\mathbf{b}\|_{H^N}$, we first split $\mathbf{b}$ into two terms:

$$\mathbf{b} = \mathfrak{B}_c(\mathbf{\mathcal{W}}, \Phi, D\Phi) + \mathfrak{I}_c(id, \mathbf{\mathcal{W}}, \Phi), \tag{3.8.3.46}$$

where $\mathfrak{B}_c$ is defined in Lemma 3.4.7 and the 5-component array $\mathfrak{I}_c$ contains the terms from (3.8.2.6) - (3.8.2.8) involving the smoothed initial data. By Lemma 3.4.2, Lemma 3.4.5, and (3.8.3.5), we have that

$$\mathfrak{I}_c \in \mathcal{I}^N(\mathfrak{K}; id, \mathbf{\mathcal{W}}, \Phi), \tag{3.8.3.47}$$

and from 3.8.3.47, (3.8.3.1), and (3.8.3.2), it follows that

$$\|\mathfrak{I}_c(id, \mathbf{\mathcal{W}}, \Phi)\|_{H^N} \lesssim C(\mathfrak{K}; id, \Lambda_1, \Lambda_2). \tag{3.8.3.48}$$

By Lemma 3.4.7, Proposition B.0.2, and (3.8.3.1) - (3.8.3.3), we have that

$$\|\mathfrak{B}_c(\mathbf{\mathcal{W}}, \Phi, D\Phi)\|_{H^N} \lesssim \|\mathfrak{B}_\infty(\mathbf{\mathcal{W}}, \nabla^{(1)}\Phi)\|_{H^N} + C(\mathfrak{K}; \Lambda_1, \Lambda_2, L_2') \lesssim C(\mathfrak{K}; \Lambda_1, \Lambda_2, L_2'). \tag{3.8.3.49}$$

Combining (3.8.3.46), (3.8.3.48) and (3.8.3.49), we have that

$$\|\mathbf{b}\|_{H^N} \lesssim C(\mathfrak{K}; id, \Lambda_1, \Lambda_2, L_2'). \tag{3.8.3.50}$$

Now (3.8.3.45) and (3.8.3.50) together imply (3.8.3.44).

We next claim that the $\mathbf{k}_{\vec{\alpha}}$ terms (3.8.2.11) satisfy

$$\|\mathbf{k}_{\vec{\alpha}}\|_{L^2} \lesssim C(\mathfrak{K}; id, \Lambda_1, \Lambda_2)\|\dot{\mathbf{W}}\|_{H^N}. \tag{3.8.3.51}$$

Since $\|_c\mathcal{A}^0(\boldsymbol{\mathcal{W}}, \Phi)\|_{L^\infty} \lesssim C(\mathfrak{K})$, to prove (3.8.3.51), it suffices to control the L^2 norm of $({}_c\mathcal{A}^0)^{-1}{}_c\mathcal{A}^k\partial_k(\partial_{\vec{\alpha}}\dot{\mathbf{W}}) - \partial_{\vec{\alpha}}\left(({}_c\mathcal{A}^0)^{-1}{}_c\mathcal{A}^k\partial_k\dot{\mathbf{W}}\right)$. By Lemma 3.4.6, (3.8.3.1), (3.8.3.2), (3.8.3.5), Proposition B.0.5, and Remark B.0.5, with $({}_c\mathcal{A}^0)^{-1}{}_c\mathcal{A}^k$ playing the role of F in the proposition, and $\partial_k\dot{\mathbf{W}}$ playing the role of G, we have that

$$\|({}_c\mathcal{A}^0)^{-1}{}_c\mathcal{A}^k\partial_k(\partial_{\vec{\alpha}}\dot{\mathbf{W}}) - \partial_{\vec{\alpha}}\big(({}_c\mathcal{A}^0)^{-1}{}_c\mathcal{A}^k\partial_k\dot{\mathbf{W}}\big)\|_{L^2} \lesssim C(\mathfrak{K}; id, \Lambda_1, \Lambda_2)\|\nabla^{(1)}\dot{\mathbf{W}}\|_{H^{N-1}}, \tag{3.8.3.52}$$

from which (3.8.3.51) readily follows. This concludes the proof of Lemma 3.8.16. □

A Corollary: $\mathbf{EN}_\kappa^c$ is Well-Approximated by $\mathbf{EP}_\kappa$ for Large c

Corollary 3.8.17. *For all large c, the solutions $\boldsymbol{\mathcal{V}} = (\boldsymbol{\mathcal{W}}, \Phi, D\Phi)$ from Theorem 3.8.2 satisfy*

$$_\infty\mathcal{A}^\mu(\boldsymbol{\mathcal{W}})\partial_\mu\boldsymbol{\mathcal{W}} = \mathfrak{B}_\infty(\boldsymbol{\mathcal{W}}, \nabla^{(1)}\Phi) + \mathfrak{E}1_c \tag{3.8.3.53}$$

$$\Delta(\Phi - \mathring{\Phi}_c) - \kappa^2(\Phi - \mathring{\Phi}_c) = 4\pi G[\mathcal{R}_\infty(\eta, p) - \mathcal{R}_\infty(\mathring{\eta}, \mathring{p})] + \mathfrak{E}2_c, \tag{3.8.3.54}$$

where

$$|||\ \mathfrak{E}1_c\ |||_{H^{N-1},T} \lesssim c^{-2}C(\mathfrak{K}; \Lambda_1, \Lambda_2, L_2) \tag{3.8.3.55}$$

$$|||\ \mathfrak{E}2_c\ |||_{H^{N-1},T} \lesssim c^{-1}C(\mathfrak{K}; \Lambda_1, \Lambda_2, L_4), \tag{3.8.3.56}$$

and T is from Theorem 3.8.2.

Remark 3.8.6. Note that the corollary is stated in terms of the state-space array $\boldsymbol{\mathcal{V}}$, and not in terms of $\mathbf{V}$.

Proof. The estimate (3.8.3.55) follows from Corollary 3.8.5 and the bounds (3.8.2.1a), (3.8.2.1b), and (3.8.2.1d). The estimate (3.8.3.56) follows from the fact that $\Delta(\Phi - \mathring{\Phi}_c) - \kappa^2(\Phi - \mathring{\Phi}_c) = c^{-2}\partial_t^2\Phi + l$, where l is defined in (3.8.3.13), together with (3.8.3.7), (3.8.2.1a), (3.8.2.1b), and (3.8.2.1f). □

3.9 The Non-relativistic Limit of the EN$_\kappa^c$ System

In this section, we state and prove our main theorem on the non-relativistic limit of the EN$_\kappa^c$ system. Readers who are interested in further examples of the analysis of singular limits in partial differential equations may consult [10], [33], or [45].

3.9.1 Local Existence for EP$_\kappa$

Before stating our main theorem, we briefly discuss local existence for the EP$_\kappa$ system.

Theorem (Local Existence for EP$_\kappa$). *Let* $\mathring{\mathbf{V}}_\infty$ *denote initial data for the EP$_\kappa$ system* (3.2.2.1) - (3.2.2.4) *that are subject to the conditions described in Section 3.6. Then there exists a* $T > 0$ *such that* (3.2.2.1) - (3.2.2.4) *has a unique classical solution* $\mathbf{V}_\infty(t, \mathbf{s})$ *on* $[0, T] \times \mathbb{R}^3$ *of the form* $\mathbf{V}_\infty = (\eta_\infty, p_\infty, \cdots, \partial_3\Phi_\infty)$ *with* $\mathbf{V}_\infty(0, \mathbf{s}) = \mathring{\mathbf{V}}_\infty(\mathbf{s})$*. The solution satisfies* $\mathbf{V}_\infty([0, T] \times \mathbb{R}^3) \subset \mathfrak{K}'$*, where the compact set* $\mathfrak{K}'$ *is defined in Section 3.6.2. Furthermore,* $\mathbf{V}_\infty \in C^0([0, T], H^N_{\bar{\mathbf{V}}_\infty}) \cap C^1([0, T], H^{N-1}_{\bar{\mathbf{V}}_\infty})$ *and* $\Phi \in C^0([0, T], H^{N+1}_{\bar{\Phi}_\infty}) \cap C^1([0, T], H^N_{\bar{\Phi}_\infty}) \cap C^2([0, T], H^{N-1}_{\bar{\Phi}_\infty})$.

Proof. Theorem 3.9.1 can be proved by adapting the method of energy currents: energy currents ${}_{(\infty)}\dot{\mathfrak{J}}$ can be used to control $\|\mathbf{W}_\infty(t)\|_{H^N_{\bar{\mathbf{W}}}}$, while $\|\Phi_\infty(t)\|_{H^{N+1}_{\bar{\Phi}_\infty}}$ can be controlled using an easy estimate on the operator $\Delta - \kappa^2$. Such methods are applied in the proof of the non-relativistic limit theorem below, so we don't provide a proof here. See also [37]. □

3.9.2 Statement and Proof of the Main Theorem

Theorem (Non-relativistic Limit). *Let* $\mathring{\mathbf{V}}_\infty$ *denote initial data for the EP$_\kappa$ system* (3.2.2.1) - (3.2.2.4) *that are subject to the conditions described in Section 3.6. Let* $\mathring{\mathbf{V}}_c$ *denote the corresponding initial data for the EN$_\kappa^c$ system* (3.2.1.1) - (3.2.1.4) *constructed from* $\mathring{\mathbf{V}}_\infty$ *as described in Section 3.6. Let* $\mathbf{V}_\infty \overset{def}{=} (\eta_\infty, p_\infty, v^1_\infty, \cdots, \partial_3\Phi_\infty)$ $\big(\mathbf{V}_c \overset{def}{=} (\eta_c, p_c, v^1_c, \cdots, \partial_3\Phi_c)\big)$ *denote the solution to the EP$_\kappa$ $\big($EN$_\kappa^c\big)$ system launched by* $\mathbf{V}_\infty$ *as furnished by Theorem 3.9.1 (Theorem 3.8.2). By Theorem 3.9.1 and Theorem 3.8.2, we may assume that for all large* c*,* $\mathbf{V}_\infty$ *and* $\mathbf{V}_c$ *exist on a common spacetime* *slab* $[0, T] \times \mathbb{R}^3$*, where* T *is the minimum of the two times from the conclusions of the theorems. Let* $\mathcal{W}_\infty$ *and* $\mathcal{W}_c$ *denote the first* 5 *components of* $\mathcal{V}_\infty$ *and* $\mathcal{V}_c$ *respectively. Then there exists a constant* $C > 0$ *such that*

$$||| \, \mathcal{W}_\infty - \mathcal{W}_c \, |||_{H^{N-1},T} \lesssim c^{-1} \cdot C \tag{3.9.2.1}$$

$$||| \, \big(\Phi_\infty - \bar{\Phi}_\infty\big) - \big(\Phi_c - \bar{\Phi}_c\big) \, |||_{H^N,T} \lesssim c^{-1} \cdot C \tag{3.9.2.2}$$

$$\lim_{c \to \infty} |\bar{\Phi}_\infty - \bar{\Phi}_c| = 0. \tag{3.9.2.3}$$

Remark 3.9.1. (3.9.2.1), (3.9.2.2), (3.9.2.3), and Sobolev imbedding imply that $\mathcal{W}_c \to \mathcal{W}_\infty$ uniformly and $\Phi_c \to \Phi_\infty$ uniformly on $[0, T] \times \mathbb{R}^3$ as $c \to \infty$.

Proof. Throughout the proof, we refer to the constants Λ_1, Λ_2, etc., from the conclusion of Theorem 3.8.2. We also refer to the set $\mathfrak{K}$ defined in Section 3.6.2. To ease the notation, we drop the subscripts c from the solution $\mathcal{V}_c$ and its first 5 components $\mathcal{W}_c$, setting $\mathcal{V} \overset{\text{def}}{=} \mathcal{V}_c$, $\mathcal{W} \overset{\text{def}}{=} \mathcal{W}_c$, etc. We then define

$$\dot{\mathcal{W}} \overset{\text{def}}{=} \mathcal{W}_\infty - \mathcal{W} \tag{3.9.2.4}$$

$$\dot{\Phi} \overset{\text{def}}{=} (\Phi_\infty - \bar{\Phi}_\infty) - (\Phi - \bar{\Phi}_c) = (\Phi_\infty - \mathring{\Phi}_\infty) - (\Phi - \mathring{\Phi}_c). \tag{3.9.2.5}$$

Our proof of Theorem 3.9.2 is similar to our proof of Lemma 3.8.16; we first use energy currents obtain a Gronwall estimate for the H^{N-1} norm of the variation $\dot{\mathcal{W}}$ defined in (3.9.2.4).

From definitions (3.9.2.4) and (3.9.2.5), it follows that $\dot{\mathcal{W}}, \dot{\Phi}$ are solutions to the following EOV_∞ defined by the BGS $\mathcal{W}_\infty$:

$${}_\infty\mathcal{A}^\mu(\mathcal{W}_\infty)\partial_\mu \dot{\mathcal{W}} = \mathbf{b} \tag{3.9.2.6}$$

$$(\Delta - \kappa^2)\dot{\Phi} = l, \tag{3.9.2.7}$$

where

$$\mathbf{b} = \mathfrak{B}_\infty(\mathcal{W}_\infty, \nabla^{(1)}\Phi_\infty) - \mathfrak{B}_\infty(\mathcal{W}, \nabla^{(1)}\Phi) + \big[{}_\infty\mathcal{A}^\mu(\mathcal{W}) - {}_\infty\mathcal{A}^\mu(\mathcal{W}_\infty)\big]\partial_\mu \mathcal{W} - \mathfrak{E}1_c, \tag{3.9.2.8}$$

$$l = 4\pi G\big[\mathcal{R}_\infty(\eta_\infty, p_\infty) - \mathcal{R}_\infty(\eta, p)\big] - \mathfrak{E}2_c, \tag{3.9.2.9}$$

$\mathfrak{B}_\infty$ is defined in Lemma 3.4.7, and $\mathfrak{E}1_c$, $\mathfrak{E}2_c$ are defined in Corollary 3.8.17. The initial condition satisfied by $\dot{\mathcal{W}}$ is

$$\dot{\mathcal{W}}(0) = \mathbf{0}. \tag{3.9.2.10}$$

Differentiating equation (3.9.2.6) with $\partial_{\vec{\alpha}}$, have that

$$_\infty\mathcal{A}^\mu(\mathcal{W}_\infty)\partial_\mu\big(\partial_{\vec{\alpha}}\dot{\mathcal{W}}\big) = \mathbf{b}_{\vec{\alpha}}, \tag{3.9.2.11}$$

where (suppressing the dependence of $_\infty\mathcal{A}^\nu$ on $\mathcal{W}_\infty$)

$$\mathbf{b}_{\vec{\alpha}} \overset{\text{def}}{=} {_\infty\mathcal{A}^0}\partial_{\vec{\alpha}}\left(({_\infty\mathcal{A}^0})^{-1}\mathbf{b}\right) + \mathbf{k}_{\vec{\alpha}} \tag{3.9.2.12}$$

and

$$\mathbf{k}_{\vec{\alpha}} \overset{\text{def}}{=} {_\infty\mathcal{A}^0}\big[({_\infty\mathcal{A}^0})^{-1}{_\infty\mathcal{A}^k}\partial_k(\partial_{\vec{\alpha}}\dot{\mathcal{W}}) - \partial_{\vec{\alpha}}\big(({_\infty\mathcal{A}^0})^{-1}{_\infty\mathcal{A}^k}\partial_k\dot{\mathcal{W}}\big)\big]. \tag{3.9.2.13}$$

We will show that for $0 \leq |\vec{\alpha}| \leq N-1$ and $t \in [0,T]$, we have that

$$\|\mathbf{b}_{\vec{\alpha}}\|_{L^2} \lesssim C(\mathfrak{K};\Lambda_1,\Lambda_2,L_1,L_2,|||\,\mathcal{W}_\infty\,|||_{H^N_{\bar{\mathcal{W}}_\infty},T}) \cdot \|\dot{\mathcal{W}}\|_{H^{N-1}} + c^{-1}C(\mathfrak{K};\Lambda_1,\Lambda_2,L_2,L_4). \tag{3.9.2.14}$$

Let us assume (3.9.2.14) for the moment and proceed as in Lemma 3.8.16: we let $_{(\infty)}\dot{\mathcal{J}}_{\vec{\alpha}}$ denote the energy current (3.5.1.2) for $\partial_{\vec{\alpha}}\dot{\mathcal{W}}$ defined by the BGS $\mathcal{W}_\infty$, and define $\mathcal{E}(t) \geq 0$ by

$$\mathcal{E}^2(t) \overset{\text{def}}{=} \sum_{|\vec{\alpha}|\leq N-1}\int_{\mathbb{R}^3}\dot{\mathcal{J}}^0_{\vec{\alpha}}(t,\mathbf{s})\,d^3\mathbf{s}, \tag{3.9.2.15}$$

where we have dropped the subscript (∞) on $\dot{\mathcal{J}}$ to ease the notation. By (3.6.2.3) and the Cauchy-Schwarz inequality for sums, we have that

$$C_{\mathfrak{K}}\|\dot{\mathcal{W}}\|^2_{H^{N-1}} \lesssim \mathcal{E}^2(t) \lesssim C_{\mathfrak{K}}^{-1}\|\dot{\mathcal{W}}\|^2_{H^{N-1}}. \tag{3.9.2.16}$$

Then by definition (3.9.2.15), Corollary 3.8.15, and (3.9.2.16), and with $C = C(\mathfrak{K};|||\,\mathcal{W}_\infty\,|||^N_{H_{\bar{\mathcal{W}}_\infty},T},|||\,\partial_t\mathcal{W}_\infty\,|||_{H^{N-1},T},\Lambda_1,\Lambda_2,L_1,L_2,L_4)$, we have

$$2\mathcal{E}\frac{d}{dt}\mathcal{E} \leq \sum_{|\vec{\alpha}|\leq N-1}\int_{\mathbb{R}^3}\partial_\mu\dot{\mathcal{J}}^\mu_{\vec{\alpha}}\,d^3\mathbf{s} \lesssim C\cdot\sum_{|\vec{\alpha}|\leq N-1}\left(\|\partial_{\vec{\alpha}}\dot{\mathcal{W}}\|^2_{L^2} + \|\partial_{\vec{\alpha}}\dot{\mathcal{W}}\|_{L^2}\|\mathbf{b}_{\vec{\alpha}}\|_{L^2}\right) \tag{3.9.2.17}$$

$$\lesssim C\cdot\|\dot{\mathcal{W}}\|^2_{H^{N-1}} + c^{-1}C\|\dot{\mathcal{W}}\|_{H^{N-1}} \lesssim C\cdot\mathcal{E}^2 + c^{-1}C\mathcal{E}.$$

Taking into account (3.9.2.10), which implies that $\mathcal{E}(0) = 0$, we apply Gronwall's inequality to (3.9.2.17), concluding that for $t \in [0, T]$,

$$\mathcal{E}(t) \lesssim c^{-1}C \cdot t \cdot \exp(Ct). \tag{3.9.2.18}$$

From (3.9.2.16) and (3.9.2.18), it follows that

$$||| \, \dot{\mathcal{W}} \, |||_{H^{N-1},T} \lesssim c^{-1}C \cdot T \cdot \exp(CT). \tag{3.9.2.19}$$

We now return to the proof of (3.9.2.14). To prove (3.9.2.14), we show only that the following bound holds for $t \in [0, T]$:

$$\|\mathbf{b}\|_{H^{N-1}} \lesssim C(\mathfrak{K}; \Lambda_1, \Lambda_2, L_1, L_2)\|\dot{\mathcal{W}}\|_{H^{N-1}} + c^{-1}C(\mathfrak{K}; \Lambda_1, \Lambda_2, L_2, L_4). \tag{3.9.2.20}$$

The remaining details, which we leave up to the reader, then follow as in the proof of Lemma 3.8.16. By Proposition B.0.4, we have that

$$\|\mathcal{R}_\infty(\eta_\infty, p_\infty) - \mathcal{R}_\infty(\eta, p)\|_{H^{N-1}} \lesssim C(\mathfrak{K})\|\dot{\mathcal{W}}\|_{H^{N-1}}, \tag{3.9.2.21}$$

and combining (3.8.3.56), (3.9.2.7), (3.9.2.9), (3.9.2.21), and Lemma A.0.4, it follows that

$$\|\dot{\Phi}\|_{H^N} \lesssim \|l\|_{H^{N-1}} \lesssim C(\mathfrak{K}) \cdot \|\dot{\mathcal{W}}\|_{H^{N-1}} + c^{-1}C(\Lambda_1, \Lambda_2, L_4). \tag{3.9.2.22}$$

Similarly, taking into account (3.9.2.22), we have that

$$\|\mathfrak{B}_\infty(\eta_\infty, p_\infty, \nabla^{(1)}\Phi_\infty) - \mathfrak{B}_\infty(\eta, p, \nabla^{(1)}\Phi)\|_{H^{N-1}} \lesssim C(\mathfrak{K}) \cdot (\|\dot{\mathcal{W}}\|_{H^{N-1}} + \|\nabla^{(1)}\dot{\Phi}\|_{H^{N-1}}) \tag{3.9.2.23}$$

$$\lesssim C(\mathfrak{K}) \cdot \|\dot{\mathcal{W}}\|_{H^{N-1}} + c^{-1}C(\Lambda_1, \Lambda_2, L_4).$$

Finally, by applying Corollary 3.4.4 to (3.4.3.8), by (3.4.3.9), by the bounds (3.8.2.1a) - (3.8.2.1d), and by Proposition B.0.4, we have that

$$\|\big[{}_\infty\mathcal{A}^\mu(\mathcal{W}) - {}_\infty\mathcal{A}^\mu(\mathcal{W}_\infty)\big]\partial_\mu \mathcal{W}\|_{H^{N-1}} \lesssim C(\mathfrak{K}; \Lambda_1, \Lambda_2, L_1, L_2)\|\dot{\mathcal{W}}\|_{H^{N-1}}. \tag{3.9.2.24}$$

Inequality (3.9.2.20) now follows from (3.8.3.55), (3.9.2.8), (3.9.2.23), and (3.9.2.24). The estimate (3.9.2.2) then follows from (3.9.2.19) and (3.9.2.22), while (3.9.2.3) is merely a restatement of (3.6.1.13). □

Appendix A

Inhomogeneous Linear Klein-Gordon Estimates

In this section, we collect together some standard energy estimates for the linear Klein-Gordon equation with an inhomogeneous term. We provide proofs for convenience. The most important issue here is making sure that the factors of c have been put in correctly!

Lemma A.0.1. *Let* $l \in C^0([0,T], H^N(\mathbb{R}^3))$ *and* $\mathring{\Psi}_0(\mathbf{s}) \in H^N(\mathbb{R}^3)$*, where* $N \in \mathbb{N}$*. Then there is a unique solution* $\dot{\Phi}(t,\mathbf{s}) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ *to the equation*

$$-c^{-2}\partial_t^2\dot{\Phi} + \Delta\dot{\Phi} - \kappa^2\dot{\Phi} = l \qquad \text{(A.0.2.1)}$$

with initial data $\dot{\Phi}(0,\mathbf{s}) = 0$, $\partial_t\dot{\Phi}(0,\mathbf{s}) = \mathring{\Psi}_0(\mathbf{s})$*. The solution has the property*
$\dot{\Phi} \in C^0([0,T], H^{N+1}(\mathbb{R}^3)) \cap C^1([0,T], H^N(\mathbb{R}^3))$.

Proof. This is a standard result; consult [51] for a proof. $\square$

Lemma A.0.2. *Assume the hypotheses of Lemma A.0.1. Assume further that that*

$\partial_t l \in C^0([0,T], H^{N-1}(\mathbb{R}^3))$ *and* $\partial_t^2 l \in C^0([0,T], H^{N-2}(\mathbb{R}^3))$. *Then there exists a constant* $C_0(\kappa) > 0$ *such that*

$$|||\, \dot{\Phi} \,|||_{H^{N+1},T} \leq C_0(\kappa)\big(c^{-1}\|\mathring{\Psi}_0\|_{H^N} + cT \,|||\, l \,|||_{H^N,T}\,\big) \tag{A.0.2.2}$$

$$|||\, \partial_t\dot{\Phi} \,|||_{H^N,T} \leq C_0(\kappa)\big(\|\mathring{\Psi}_0\|_{H^N} + c^2T \,|||\, l \,|||_{H^N,T}\,\big) \tag{A.0.2.3}$$

$$|||\, \partial_t\dot{\Phi} \,|||_{H^N,T} \leq C_0(\kappa)\big(\|\mathring{\Psi}_0\|_{H^N} + c\|l(0)\|_{H^{N-1}} + cT \,|||\, \partial_t l \,|||_{H^{N-1},T}\,\big) \tag{A.0.2.4}$$

$$|||\, \partial_t^2\dot{\Phi} \,|||_{H^{N-1},T} \leq C_0(\kappa)\big(c\|\mathring{\Psi}_0\|_{H^N} + c^2\|l(0)\|_{H^{N-1}} + c^2T \,|||\, \partial_t l \,|||_{H^{N-1},T}\,\big) \tag{A.0.2.5}$$

$$\begin{aligned} &|||\, \partial_t^2\dot{\Phi} \,|||_{H^{N-1},T} \\ &\quad \leq C_0(\kappa)\cdot\big(c^2\|l(0)\|_{H^{N-1}} + c\|(\Delta-\kappa^2)\mathring{\Psi}_0 - \partial_t l(0)\|_{H^{N-2}} + cT \,|||\, \partial_t^2 l \,|||_{H^{N-2},T}\,\big) \end{aligned} \tag{A.0.2.6}$$

$$\begin{aligned} &|||\, \partial_t^3\dot{\Phi} \,|||_{H^{N-2},T} \\ &\quad \leq C_0(\kappa)\cdot\big(c^3\|l(0)\|_{H^{N-1}} + c^2\|(\Delta-\kappa^2)\mathring{\Psi}_0 - \partial_t l(0)\|_{H^{N-2}} + c^2T \,|||\, \partial_t^2 l \,|||_{H^{N-2},T}\,\big). \end{aligned} \tag{A.0.2.7}$$

(A.0.2.8)

Proof. Because $\nabla^{(k)}\dot{\Phi}$ is a solution to the Klein-Gordon equation $-c^{-2}\partial_t^2\big(\nabla^{(k)}\dot{\Phi}\big) + \Delta\big(\nabla^{(k)}\dot{\Phi}\big) - \kappa^2\big(\nabla^{(k)}\dot{\Phi}\big) = \nabla^{(k)}l$, we will use standard energy estimates for the linear Klein-Gordon equation to estimate $|||\, \dot{\Phi} \,|||_{H^{N+1},T}$. Thus, for $0 \leq k \leq N$, we define $E_k(t) \geq 0$ by

$$E_k^2(t) \overset{\text{def}}{=} \|\kappa\nabla^{(k)}\dot{\Phi}(t)\|_{L^2}^2 + \|\nabla^{(k+1)}\dot{\Phi}\|_{L^2}^2 + \|c^{-1}\nabla^{(k)}\partial_t\dot{\Phi}(t)\|_{L^2}^2. \tag{A.0.2.9}$$

We now multiply each side the equation satisfied by $\nabla^{(k)}\dot{\Phi}$ by $-\nabla^{(k)}\partial_t\dot{\Phi}$, integrate over $\mathbb{R}^3$, and use Hölder's inequality to arrive at the following chain of inequalities:

$$\begin{aligned} E_k(t)\frac{d}{dt}E_k(t) = \frac{1}{2}\frac{d}{dt}\big(E_k^2(t)\big) &= \int_{\mathbb{R}^3}\big(-\nabla^{(k)}\partial_t\dot{\Phi}\big)\cdot\big(\nabla^{(k)}l\big)\,d\mathbf{s} \\ &\leq \|\nabla^{(k)}\partial_t\dot{\Phi}(t)\|_{L^2}\|\nabla^{(k)}l(t)\|_{L^2}, \end{aligned} \tag{A.0.2.10}$$

where $\big(-\nabla^{(k)}\partial_t\dot{\Phi}\big)\cdot\big(\nabla^{(k)}l\big)$ denotes the array-valued quantity formed by taking the component by component product of the two arrays $-\nabla^{(k)}\partial_t\dot{\Phi}$ and $\nabla^{(k)}l$.

If we now define $E(t) \geq 0$ by

$$E^2(t) \overset{\text{def}}{=} \left(\sum_{k=0}^{N} E_k^2(t)\right) = \kappa^2\|\dot{\Phi}(t)\|_{H^N}^2 + \|\nabla^{(1)}\dot{\Phi}(t)\|_{H^N}^2 + c^{-2}\|\partial_t\dot{\Phi}(t)\|_{H^N}^2, \tag{A.0.2.11}$$

it follows from (A.0.2.10) and the Cauchy-Schwarz inequality for sums that

$$E(t)\frac{d}{dt}E(t) = \frac{1}{2}\frac{d}{dt}(E^2(t)) \leq \|\partial_t\dot{\Phi}\|_{H^N}\|l(t)\|_{H^N} \leq cE(t)\|l(t)\|_{H^N}, \tag{A.0.2.12}$$

and so

$$\frac{d}{dt}E(t) \leq c\|l(t)\|_{H^N}. \tag{A.0.2.13}$$

Integrating (A.0.2.13) over time, we have the following inequality, valid for $t \in [0, T]$:

$$E(t) \leq E(0) + ct \ ||| \ l \ |||_{H^N,T} \ . \tag{A.0.2.14}$$

From the definition of $E(t)$ and the initial condition $\dot{\Phi} = 0$, we have that

$$\|\dot{\Phi}(t)\|_{H^{N+1}} \leq C(\kappa)E(t) \tag{A.0.2.15}$$

$$\|\partial_t \dot{\Phi}(t)\|_{H^N} \leq cE(t) \tag{A.0.2.16}$$

$$E(0) = c^{-1}\|\mathring{\Psi}_0\|_{H^N}. \tag{A.0.2.17}$$

Combining (A.0.2.14), (A.0.2.15), (A.0.2.16), and (A.0.2.17), and taking the sup over $t \in [0, T]$ proves (A.0.2.2) and (A.0.2.3).

To prove (A.0.2.4) - (A.0.2.7), we differentiate the Klein-Gordon equation with respect to t (twice to prove (A.0.2.6) and (A.0.2.7)) and argue as above, taking into account the initial conditions

$$\partial_t^2 \dot{\Phi}(0) = -c^2 l(0) \tag{A.0.2.18}$$

$$\partial_t^3 \dot{\Phi}(0) = c^2\big[(\Delta - \kappa^2)\mathring{\Psi}_0 - \partial_t l(0)\big]. \tag{A.0.2.19}$$

□

Corollary A.0.3. *Assume the hypotheses of Lemma A.0.2. Then*

$$||| \ \dot{\Phi} \ |||^2_{H^{N+1},T} \leq C_0(\kappa) \cdot \big(c^{-2}\|\mathring{\Psi}_0\|^2_{H^N} + T \cdot \ ||| \ \partial_t \dot{\Phi} \ |||_{H^N,T} \cdot \ ||| \ l \ |||_{H^N,T} \ \big). \tag{A.0.2.20}$$

Proof. Inequality (A.0.2.12) gives that $\frac{1}{2}\frac{d}{dt}(E^2(t)) \leq \|\partial_t \dot{\Phi}\|_{H^N}\|l(t)\|_{H^N}$. Taking into account (A.0.2.17), the proof of (A.0.2.20) easily follows. □

Lemma A.0.4. *Let $N \in \mathbb{N}$, and $\mathfrak{I} \in H^{N-1}(\mathbb{R}^3)$. Suppose that $\dot{\Phi} \in H^{N+1}(\mathbb{R}^3)$ and that $\Delta\dot{\Phi} - \kappa^2\dot{\Phi} = \mathfrak{I}$. Then*

$$\|\dot{\Phi}\|_{H^{N+1}(\mathbb{R}^3)} \leq C(N, \kappa)\|\mathfrak{I}\|_{H^{N-1}(\mathbb{R}^3)}. \tag{A.0.2.21}$$

Proof. For use in this argument, we define the Fourier transform through its action on integrable functions F by $\widehat{F}(\xi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} F(\mathbf{s})e^{-2\pi i\xi\cdot\mathbf{s}}\, d\mathbf{s}$. The following chain of inequalities uses standard results from Fourier analysis, including Plancherel's theorem:

$$\|\dot{\Phi}\|_{H^2}^2 \leq C\|(1+|2\pi\xi|^2)^2\widehat{\dot{\Phi}}\|_{L^2}^2 \leq C(\kappa)\int_{\mathbb{R}^3}(\kappa^2+|2\pi\xi|^2)^2|\widehat{\dot{\Phi}}(\xi)|^2\, d^3\xi \qquad \text{(A.0.2.22)}$$
$$= C(\kappa)\|(\kappa^2-\Delta)\dot{\Phi}\|_{L^2(\mathbb{R}^3)}^2 = C(\kappa)\|\mathfrak{I}\|_{L^2(\mathbb{R}^3)}^2,$$

and this proves (A.0.2.21) in the case $N=1$. To estimate L^2 norms of the k^{th} order derivatives of $\dot{\Phi}$ for $k \geq 1$, we differentiate the equation k times to arrive at the equation $\Delta\big(\nabla^{(k)}\dot{\Phi}\big) - \kappa^2\big(\nabla^{(k)}\dot{\Phi}\big) = \nabla^{(k)}\mathfrak{I}$, and argue as above to conclude that

$$\|\nabla^{(k)}\dot{\Phi}\|_{H^2}^2 \leq C(\kappa)\|\nabla^{(k)}\mathfrak{I}\|_{L^2}^2. \qquad \text{(A.0.2.23)}$$

Now we add the estimate (A.0.2.22) to the estimates (A.0.2.23) for $1 \leq k \leq N-1$ to conclude (A.0.2.21). $\square$

Remark A.0.2. The hypothesis $\dot{\Phi} \in H^{N+1}(\mathbb{R}^3)$ does not follow from the remaining assumptions. For example, consider $g(x) = e^x$. Then $g - \frac{d}{dx^2}g \in L^2(\mathbb{R})$, but $g \notin H^2(\mathbb{R})$.

Proposition A.0.5. *Assume the hypotheses of Lemma A.0.1. Assume further that that* $l \in C^0([0,T], H^N(\mathbb{R}^3))$, $\partial_t l \in C^0([0,T], H^{N-1}(\mathbb{R}^3))$, *and* $\partial_t^2 l \in C^0([0,T], H^{N-2}(\mathbb{R}^3))$. *Then*

$$|||\ \dot{\Phi}\ |||_{H^{N+1},T} \leq C(N,\kappa) \qquad \text{(A.0.2.24)}$$
$$\cdot\big(c^{-1}\|\mathring{\Psi}_0\|_{H^N} + \|l(0)\|_{H^{N-1}} + |||\ l\ |||_{H^{N-1},T} + T\ |||\ \partial_t l\ |||_{H^{N-1},T}\ \big)$$

and

$$|||\ \partial_t\dot{\Phi}\ |||_{H^N,T} \leq C(N,\kappa) \qquad \text{(A.0.2.25)}$$
$$\cdot\big(c\|l(0)\|_{H^{N-1}} + \|(\Delta-\kappa^2)\mathring{\Psi}_0 - \partial_t l(0)\|_{H^{N-2}} + |||\ \partial_t l\ |||_{H^{N-2},T} + T\ |||\ \partial_t^2 l\ |||_{H^{N-2},T}\ \big).$$

Proof. Define $\mathfrak{I} \stackrel{\text{def}}{=} l + c^{-2}\partial_t^2\dot{\Phi}$ and observe that $\dot{\Phi}$ is a solution to

$$\Delta\dot{\Phi} - \kappa^2\dot{\Phi} = \mathfrak{I}. \qquad \text{(A.0.2.26)}$$

By inequality (A.0.2.5) of Lemma A.0.2 and Lemma A.0.4, we have that

$$
\begin{aligned}
||| \dot{\Phi} |||_{H^{N+1},T} &\leq C(N,\kappa) ||| l + c^{-2}\partial_t^2 \dot{\Phi} |||_{H^{N-1},T} \qquad (A.0.2.27)\\
&\leq C(N,\kappa)\big(c^{-1}\|\mathring{\Psi}_0\|_{H^N} + \|l(0)\|_{H^{N-1}} + ||| l |||_{H^{N-1},T} + T ||| \partial_t l |||_{H^{N-1},T} \big),
\end{aligned}
$$

which proves (A.0.2.24).

Because $\partial_t \dot{\Phi}$ satisfies the equation $-c^{-2}\partial_t^3 \dot{\Phi} + \Delta\big(\partial_t \dot{\Phi}\big) - \kappa^2\big(\partial_t \dot{\Phi}\big) = \partial_t l$, we may use a similar argument to prove (A.0.2.25); we leave the simple modification, which makes use of (A.0.2.7), up to the reader. □

Appendix B

Sobolev-Moser Calculus

In this Appendix, we use notation that is as consistent as possible with our use of notation in the body of the paper. With the exception of Proposition B.0.6, which is a standard Sobolev interpolation inequality that we don't prove, the proofs we give are based on the following version of the Gagliardo-Nirenberg inequality [42], together with repeated use of Hölder's inequality and/or Sobolev imbedding:

Lemma B.0.1. *If $k, i \in \mathbb{N}$ with $0 \leq i \leq k$, and $\mathbf{V}$ is a scalar-valued or array-valued function on $\mathbb{R}^d$ with $\mathbf{V} \in L^\infty(\mathbb{R}^d)$ and $\|\nabla^{(k)}\mathbf{V}\|_{L^2(\mathbb{R}^d)} < \infty$, then*

$$\|\nabla^{(i)}\mathbf{V}\|_{L^{2k/i}} \leq C(k)\|\mathbf{V}\|_{L^\infty}^{1-\frac{i}{k}}\|\nabla^{(k)}\mathbf{V}\|_{L^2}^{\frac{i}{k}}. \qquad \text{(B.0.2.1)}$$

Proposition B.0.2. *Let $\mathcal{O}_2 \subset \mathbb{R}^n$ be an open set, and let $j, d \in \mathbb{N}$ with $j > \frac{d}{2}$. Let $\mathbf{V} : \mathbb{R}^d \to \mathbb{R}^n$ be an element of $H^j(\mathbb{R}^d)$, and assume that $\mathbf{V}(\mathbb{R}^d) \subset \bar{\mathcal{O}}_2$. Let $F \in C_b^j(\bar{\mathcal{O}}_2)$ be a $q \times q$ matrix-valued function and let $G \in H^j(\mathbb{R}^d)$ be a $q \times q$ ($1 \times q, q \times 1$) matrix-valued (array-valued) function. Then the $q \times q$ ($1 \times q, q \times 1$) matrix-valued (array-valued) function $(F \circ \mathbf{V})G$ is an element of $H^j(\mathbb{R}^d)$ and*

$$\|(F \circ \mathbf{V})G\|_{H^j(\mathbb{R}^d)} \leq C(j,d)|F|_{j,\bar{\mathcal{O}}_2}(1 + \|\mathbf{V}\|_{H^j(\mathbb{R}^d)}^j)\|G\|_{H^j(\mathbb{R}^d)}. \qquad \text{(B.0.2.2)}$$

Proof. We will assume for simplicity that $n = q = 1$ and since the general argument requires more burdensome notation but is similar. Let k be an integer with $0 \leq k \leq j$, and let $\vec{\alpha}$ be a spatial derivative multi-index with $|\vec{\alpha}| = k$. By the definition of $\|\cdot\|_{H^j}$, it is sufficient to show that

$$\partial_{\vec{\alpha}}\left((F \circ \mathbf{V})G\right) \in L^2(\mathbb{R}^d) \qquad \text{(B.0.2.3)}$$

for all such k and $\vec{\alpha}$. For notational simplicity, we do not distinguish between partial differentiation with respect to Cartesian coordinates, instead indicating each derivative by the symbol $|\partial|$. We thus write $|\partial|^k$ in place of $\partial_{\vec{\alpha}}$. By the product and Faà di Bruno (chain) rules,

$$|\partial|^k\left((F\circ\mathbf{V})G\right) = \sum_{a+b=k}\binom{a+b}{a}|\partial|^a(F\circ\mathbf{V})\cdot|\partial|^bG \tag{B.0.2.4}$$

and

$$|\partial|^a(F\circ\mathbf{V}) = \sum C(m_1,\cdots,m_a)(F^{(m_1+\cdots+m_a)}\circ\mathbf{V})\prod_{i=1}^{a}(|\partial|^i\mathbf{V})^{m_i}\big|_{m_i\neq 0}, \tag{B.0.2.5}$$

where the sum in (B.0.2.5) is taken over all non-negative integers $m_1, m_2, \cdots, m_a$ such that $\sum_{i=1}^a im_i = a$, $\binom{a+b}{a}$ is the binomial coefficient,
$C(m_1,\cdots,m_a) \overset{\text{def}}{=} a!/(m_1\cdots m_a!(1!)^{m_1}\cdots(a!)^{m_a})$, and $F^{(p)}$ denotes the p^{th} derivative of F.

We first consider (B.0.2.5) in the case $a=k$, where $k\geq 1$. We claim that $\prod_{i=1}^k(|\partial|^i\mathbf{V})^{m_i}\big|_{m_i\neq 0}\in L^2(\mathbb{R}^d)$. To see this, define $\gamma_i \overset{\text{def}}{=} \frac{k}{im_i}$. Then by Hölder's inequality and Lemma B.0.1 for $m_i\neq 0$,

$$\int_{\mathbb{R}^d}\prod_{i=1}^k(|\partial|^i\mathbf{V})^{2m_i}\,d\mathbf{x} \leq \prod_{i=1}^k\left[\int_{\mathbb{R}^d}(|\partial|^i\mathbf{V})^{2k/i}\,d\mathbf{x}\right]^{1/\gamma_i} = \prod_{i=1}^k \||\partial|^i\mathbf{V}\|_{L^{2k/i}}^{2m_i} \tag{B.0.2.6}$$

$$\leq C(j)\prod_{i=1}^k\|\mathbf{V}\|_{L^\infty}^{2m_i-2im_i/k}\||\partial|^k\mathbf{V}\|_{L^2}^{2im_i/k} = C(j)\left(\|\mathbf{V}\|_{L^\infty}^{-2+2\sum_{i=1}^k m_i}\right)\left(\||\partial|^k\mathbf{V}\|_{L^2}^2\right),$$

which proves the claim.

Since $F\in C_b^j\bar{\mathcal{O}}_2)$, it now easily follows from (B.0.2.5) and (B.0.2.6) that $|\partial|^k(F\circ\mathbf{V})\in L^2(\mathbb{R}^d)$ and

$$\||\partial|^k(F\circ\mathbf{V})\|_{L^2}\leq C(j)|\partial F/\partial\mathbf{V}|_{k-1,\bar{\mathcal{O}}_2}(1+\|\mathbf{V}\|_{L^\infty}^{k-1})(\||\partial|^k\mathbf{V}\|_{L^2}). \tag{B.0.2.7}$$

We also note that from the assumption $\widetilde{\mathbf{V}}(\mathbb{R}^d)\subset\bar{\mathcal{O}}_2$, (B.0.2.5), and Sobolev imbedding, it follows that

$$\|(F\circ\mathbf{V})\|_{L^\infty}\leq |F|_{j,\bar{\mathcal{O}}_2}. \tag{B.0.2.8}$$

We now show that each term in the summation in (B.0.2.4) is an element of $L^2(\mathbb{R}^d)$, and we argue by considering separately the cases $1\leq k$ and $k=0$. By Hölder's inequality, Lemma B.0.1, (B.0.2.8), (B.0.2.7), and Sobolev imbedding (in this order), we have for $1 \leq k \leq j$:

$$\||\partial|^a(F \circ \mathbf{V}) \cdot |\partial|^b G\|_{L^2} \leq \||\partial|^a(F \circ \mathbf{V})\|_{L^{2k/a}} \||\partial|^b G\|_{L^{2k/b}} \tag{B.0.2.9}$$
$$\leq C(j)\|F \circ \mathbf{V}\|_{L^\infty}^{1-a/k} \||\partial|^k(F \circ \mathbf{V})\|_{L^2}^{a/k} \|G\|_{L^\infty}^{1-b/k} \||\partial|^k G\|_{L^2}^{b/k}$$
$$\leq C(j)|F|_{j,\bar{\mathcal{O}}_2}(\|\mathbf{V}\|_{H^j} + \|\mathbf{V}\|_{H^j}^j)\|G\|_{H^j}.$$

For the case $k = 0$, we have

$$\|(F \circ \mathbf{V}) \cdot G\|_{L^2} \leq \|F \circ \mathbf{V}\|_{L^\infty} \|G\|_{L^2} \leq |F|_{j,\bar{\mathcal{O}}_2} \|G\|_{H^j}. \tag{B.0.2.10}$$

Statement (B.0.2.3) follows from (B.0.2.9) and (B.0.2.10). This demonstrates (B.0.2.2) and completes the proof of Proposition B.0.2. □

Corollary B.0.3. *Assume the hypotheses of Proposition B.0.2 with the following changes:* $\mathbf{V}, G \in C^0([0,T], H^j(\mathbb{R}^d))$. *Then the* $q \times q$ $(1 \times q, q \times 1)$ *matrix-valued (array-valued) function* $(F \circ \mathbf{V})G$ *is an element of* $C^0([0,T], H^j(\mathbb{R}^d))$.

Proof. We do not give details here, since the proof may be obtained without much additional work by modifying the proof of Proposition B.0.2. □

Remark B.0.3. We often make use of a slight modification of Proposition B.0.2 in which the assumption $\mathbf{V} \in H^j(\mathbb{R}^d)$ is replaced with the assumption $\mathbf{V} \in H^j_{\bar{\mathbf{V}}}(\mathbb{R}^d)$, where $\bar{\mathbf{V}} \in \mathbb{R}^n$ is a constant array. Under this modified assumption, the conclusion of Proposition B.0.2 may be modified as follows:

$$\|(F \circ \mathbf{V})G\|_{H^j} \leq C(j,d)|F|_{j,\bar{\mathcal{O}}_2}(1 + \|\mathbf{V}\|_{H^j_{\bar{\mathbf{V}}}}^j)\|G\|_{H^j}. \tag{B.0.2.11}$$

A similar modification can be made to Corollary B.0.3.

To prove (B.0.2.11), simply define $\widetilde{F}(\mathbf{Y}) \stackrel{\text{def}}{=} F(\bar{\mathbf{V}} + \mathbf{Y})$ and apply the proposition using $\widetilde{F}$ in place of F and using $\mathbf{Y} \stackrel{\text{def}}{=} \mathbf{V} - \bar{\mathbf{V}}$ in place of $\mathbf{V}$. A similar modification can be made to Corollary B.0.3.

Proposition B.0.4. *Let* $\mathcal{O}_2 \subset \mathbb{R}^n$ *be an open, convex set, and let* $j, d \in \mathbb{N}$ *with* $j > \frac{d}{2}$. *Let* $F \in C_b^j(\bar{\mathcal{O}}_2)$ *be a scalar or array-valued function. Let* $\mathbf{V}, \widetilde{\mathbf{V}} : \mathbb{R}^d \to \mathbb{R}^n$, *and* *assume that* $\mathbf{V}, \widetilde{\mathbf{V}} \in H^j(\mathbb{R}^d)$. *Assume further that that* $\mathbf{V}(\mathbb{R}^d), \widetilde{\mathbf{V}}(\mathbb{R}^d) \subset \bar{\mathcal{O}}_2$. *Then* $F \circ \mathbf{V} - F \circ \widetilde{\mathbf{V}} \in H^j(\mathbb{R}^d)$ *and*

$$\|F \circ \mathbf{V} - F \circ \widetilde{\mathbf{V}}\|_{H^j} \leq C(j, d, |F|_{j+1,\bar{\mathcal{O}}_2}, \|\mathbf{V}\|_{H^j}, \|\widetilde{\mathbf{V}}\|_{H^j}) \cdot \|\mathbf{V} - \widetilde{\mathbf{V}}\|_{H^j}. \tag{B.0.2.12}$$

Proof. The proof given here is a modification of the proof of Proposition B.0.2; we adopt the same notation as we did in this proposition. Let k be an integer with $0 \leq k \leq j$, and let $\vec{\alpha}$ be a spatial derivative multi-index with $|\vec{\alpha}| = k$. By the definition of $\|\cdot\|_{H^j}$, it is sufficient to show that

$$\|\partial_{\vec{\alpha}} \left(F \circ \mathbf{V} - F \circ \widetilde{\mathbf{V}}\right) \|_{L^2} \leq C(j, |F|_{j+1,\bar{\mathcal{O}}_2}, \|\mathbf{V}\|_{H^j}, \|\widetilde{\mathbf{V}}\|_{H^j}) \cdot \|\mathbf{V} - \widetilde{\mathbf{V}}\|_{H^j} \tag{B.0.2.13}$$

for all such k and $\vec{\alpha}$. We consider the quantity $\partial_{\vec{\alpha}} \left(F \circ \mathbf{V} - F \circ \widetilde{\mathbf{V}}\right)$, which can be written in the form

$$\sum C(m_1, m_2, \cdots, m_a) \left(X_0 X_1 \cdots X_p - \widetilde{X}_0 \widetilde{X}_1 \cdots \widetilde{X}_p\right) \tag{B.0.2.14}$$

using the expression (B.0.2.5). Here, we have broken up the each product in (B.0.2.5) into its linear factors, so that $p = m_1 + m_2 + \cdots m_a$. Furthermore, our convention is that X_0 denotes the factor $F^{(m_1+m_2+\cdots+m_a)} \circ \mathbf{V}$, and $\widetilde{X}_0$ denotes the factor $F^{(m_1+m_2+\cdots+m_a)} \circ \widetilde{\mathbf{V}}$. Therefore, the generic case is to consider is $X_0 = F^{(m_1+m_2+\cdots+m_a)} \circ \mathbf{V}, \widetilde{X}_0 = F^{(m_1+m_2+\cdots+m_a)} \circ \widetilde{\mathbf{V}}, X_1 = |\partial|^{j_1}\mathbf{V}, \widetilde{X}_1 = |\partial|^{j_1}\widetilde{\mathbf{V}}, \cdots, X_p = |\partial|^{j_r}\mathbf{V}, \widetilde{X}_p = |\partial|^{j_r}\widetilde{\mathbf{V}}$. We now make use of the algebraic identity

$$\prod_{i=0}^{p} X_i - \prod_{i=0}^{p} \widetilde{X}_i = (X_0 - \widetilde{X}_0) \prod_{i=1}^{p} X_i + \widetilde{X}_0 (X_1 - \widetilde{X}_1) \prod_{i=2}^{p} X_i + \cdots + \left(\prod_{i=0}^{p-1} \widetilde{X}_i\right) (X_p - \widetilde{X}_p). \tag{B.0.2.15}$$

We will establish the inequality (B.0.2.13) by showing that each of the $p+1$ products of $p+1$ linear factors on the right-hand side of (B.0.2.15) is bounded from above in the L^2 norm by the right-hand side of (B.0.2.13).

We consider as a special case the expression $(X_0 - \widetilde{X}_0) \prod_{i=1}^{p} X_i$. By Taylor's theorem (this is where we use the hypothesis that $\mathcal{O}_2$ is convex) and Sobolev imbedding, $\|X_0 - \widetilde{X}_0\|_{L^\infty} \leq |F|_{j+1,\bar{\mathcal{O}}_2} \|\mathbf{V} - \widetilde{\mathbf{V}}\|_{H^j}$. The L^2 norm of the remaining factors $\prod_{i=1}^{p} X_i$ can be bounded from above by $C(j, \|\mathbf{V}\|_{H^j})$ using the same reasoning as in (B.0.2.6) together with Sobolev imbedding. This establishes inequality (B.0.2.13) in this case.

We now discuss terms of the form $\left(\prod_{i=0}^{m} \widetilde{X}_i\right)(X_{m+1} - \widetilde{X}_{m+1})\left(\prod_{i=m+2}^{p} X_i\right)$. It suffices to consider the illuminating case $\widetilde{X}_0\widetilde{X}_1(X_2 - \widetilde{X}_2)\prod_{i=3}^{p} X_i$ since the remaining cases are similar. Because X_0 is bounded in L^∞ by $|F|_{0,\bar{\mathcal{O}}_2}$, it is sufficient to bound the L^2 norm of $\widetilde{X}_1(X_2 - \widetilde{X}_2)\prod_{i=3}^{p} X_i$ from above by the right-hand side of (B.0.2.13). This may be done using similar reasoning to that which we used in (B.0.2.6) (except that we now apply a version of Hölder's inequality to the *linear* factors rather than monomial factors) together with Sobolev imbedding; we leave the simple modification up to the reader. We remark that the $\|\mathbf{V} - \widetilde{\mathbf{V}}\|_{H^j}$ term from the righthand side of (B.0.2.13) is derived from the $(X_2 - \widetilde{X}_2)$ factor. $\square$

Remark B.0.4. As in Remark B.0.3, we may replace the hypotheses $\mathbf{V}, \widetilde{\mathbf{V}} \in H^j(\mathbb{R}^d)$ from Proposition (B.0.4) with the hypotheses $\mathbf{V}, \widetilde{\mathbf{V}} \in H^j_{\bar{\mathbf{V}}}(\mathbb{R}^d)$, where $\bar{\mathbf{V}}$ is a constant array, in which case the conclusion of the proposition is:

$$\|(F \circ \mathbf{V}) - (F \circ \widetilde{\mathbf{V}})\|_{H^j} \leq C(j, d, |F|_{j+1,\bar{\mathcal{O}}_2}, \|\mathbf{V}\|_{H^j_{\bar{\mathbf{V}}}}, \|\widetilde{\mathbf{V}}\|_{H^j_{\bar{\mathbf{V}}}}) \cdot \|\mathbf{V} - \widetilde{\mathbf{V}}\|_{H^j}. \quad \text{(B.0.2.16)}$$

Furthermore, a careful analysis of the special case $\widetilde{\mathbf{V}} = \bar{\mathbf{V}}$ gives the bound

$$\|F \circ \mathbf{V} - F \circ \bar{\mathbf{V}}\|_{H^j} \leq C(j, d)|F|_{j,\bar{\mathcal{O}}_2}(1 + \|\mathbf{V}\|^{j-1}_{H^j_{\bar{\mathbf{V}}}})(\|\mathbf{V}\|_{H^j_{\bar{\mathbf{V}}}}), \quad \text{(B.0.2.17)}$$

in which we require less regularity of F than we do in the general case. We leave these details to the reader.

Proposition B.0.5. *Assume the hypotheses of Proposition B.0.2 with the following two changes:*

1. *Assume* $j > \frac{d}{2} + 1$.

2. *Assume that* $G \in H^{j-1}(\mathbb{R}^d)$.

Let $k \in \mathbb{N}$ *with* $1 \leq k \leq j$, *and let* $\vec{\alpha}$ *be a spatial derivative multi-index with* $|\vec{\alpha}| = k$. *Then*

$$\begin{aligned} &\|\partial_{\vec{\alpha}}\left((F \circ \mathbf{V})G\right) - (F \circ \mathbf{V})\partial_{\vec{\alpha}} G\|_{L^2} \qquad \text{(B.0.2.18)} \\ &\leq C(j, d)|\partial F/\partial \mathbf{V}|_{j-1,\bar{\mathcal{O}}_2}(\|\mathbf{V}\|_{H^j} + \|\mathbf{V}\|^j_{H^j})\|G\|_{H^{j-1}}. \end{aligned}$$

Proof. We use here the same notation as we did in the proof of Proposition B.0.2. The quantity on the left-hand side of (B.0.2.18) that we must estimate in the L^2 norm is the same as the expression on the right-hand side of (B.0.2.4), except the term $(F\circ\mathbf{V})|\partial|^k G$ corresponding to $a=0, b=k$ is not present. We define $a'=a-1$ and $k'=k-1$ so that $a'+b=k'\leq j-1$, and we mirror the steps in (B.0.2.9) to obtain for $1\leq k'\leq j-1$:

$$\begin{aligned}
&\||\partial|^{a'}(|\partial|(F\circ\mathbf{V}))\cdot|\partial|^b G\|_{L^2} && \text{(B.0.2.19)}\\
&\leq C(j)\||\partial|(F\circ\mathbf{V})\|_{L^\infty}^{1-a'/k'}\||\partial|^k(F\circ\mathbf{V})\|_{L^2}^{a'/k'}\|G\|_{L^\infty}^{1-b/k'}\||\partial|^{k'}G\|_{L^2}^{b/k'}\\
&\leq C(j)|\partial F/\partial\mathbf{V}|_{j-1,\bar{\mathcal{O}}_2}(\|\mathbf{V}\|_{H^j}+\|\mathbf{V}\|_{H^j}^k)\|G\|_{H^{j-1}}.
\end{aligned}$$

For the case $k'=0$, we have

$$\||\partial|(F\circ\mathbf{V})\cdot G\|_{L^2}\leq\||\partial|(F\circ\mathbf{V})\|_{L^\infty}\|G\|_{L^2}\leq|\partial F/\partial\mathbf{V}|_{j-1,\bar{\mathcal{O}}_2}\|\mathbf{V}\|_{H^j}\|G\|_{H^{j-1}}. \tag{B.0.2.20}$$

Inequality (B.0.2.18) now follows easily from (B.0.2.19) and (B.0.2.20). □

Remark B.0.5. As in Remark B.0.3, we may replace the assumption $\mathbf{V}\in H^j(\mathbb{R}^d)$ in Proposition B.0.5 with the assumption $\mathbf{V}\in H^j_{\bar{\mathbf{V}}}(\mathbb{R}^d)$, where $\bar{\mathbf{V}}$ is a constant array, in which case we obtain

$$\begin{aligned}
&\|\partial_{\vec{\alpha}}((F\circ\mathbf{V})G)-(F\circ\mathbf{V})\partial_{\vec{\alpha}}G\|_{L^2} && \text{(B.0.2.21)}\\
&\leq C(j,d)|\partial F/\partial\mathbf{V}|_{j-1,\bar{\mathcal{O}}_2}(\|\mathbf{V}\|_{H^j_{\bar{\mathbf{V}}}}+\|\mathbf{V}\|_{H^j_{\bar{\mathbf{V}}}}^j)\|G\|_{H^{j-1}}.
\end{aligned}$$

Proposition B.0.6. *Let $N',N\in\mathbb{R}$ be such that $0\leq N'\leq N$, and assume $F\in H^N(\mathbb{R}^d)$. Then*

$$\|F\|_{H^{N'}}\leq C(N',d)\|F\|_{L^2}^{1-N'/N}\|F\|_{H^N}^{N'/N}. \tag{B.0.2.22}$$

References

[1] N. Andersson and G. L. Comer, *Relativistic fluid dynamics: Physics for many different scales [online article] [cited 3-5-08] http://relativity.livingreviews.org/Articles/lrr-2007-1/*, Living Reviews in Relativity, 10, pp. 1–83.

[2] I. Bialynicki-Birula, *Nonlinear electrodynamics: Variations on a theme by Born and Infeld*, Quantum Theory of Particles and Fields, (1983), pp. 31–48.

[3] G. Boillat, *Nonlinear electrodynamics: Lagrangians and equations of motion*, Journal of Mathematical Physics, 11 (1969), pp. 941–951.

[4] M. Born, *Modified field equations with a finite radius of the electron*, Nature, 132, p. 282.

[5] M. Born and L. Infeld, *Foundations of the new field theory*, in Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character, vol. 144, pp. 425–451.

[6] U. Brauer and L. Karp, *Local existence of classical solutions for the Einstein-Euler system using weighted Sobolev spaces of fractional order*, Comptes Rendus Mathematique, pp. 49–54.

[7] Y. Brenier, *Hydrodynamic structure of the augmented Born-Infeld equations*, Archive for Rational Mechanics and Analysis, pp. 65–91.

[8] S. Calogero, *Spherically symmetric steady states of galactic dynamics in scalar gravity*, Classical and Quantum Gravity, 20 (2003), pp. 1729–1741.

[9] ——, *Global classical solutions to the 3D Nordström-Vlasov system*, Communications in Mathematical Physics, 266 (2006), pp. 343–353.

[10] S. Calogero and H. Lee, *The non-relativistic limit of the Nordström-Vlasov system*, Communications in Mathematical Sciences, 2 (2004), pp. 19–34.

[11] D. Chae and H. Huh, *Global existence for small initial data in the Born-Infeld equations*, Journal of Mathematical Physics, 44 (2003), pp. 6132–6139.

[12] Y. F. (Choquet)-Bruhat, *Théorèmes d'existence en mécanique des fluides relativistes*, Bulletin de la Société Mathématique de France, 86 (1958), pp. 155–175.

[13] D. Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Communications on Pure and Applied Mathematics, 39 (1986), pp. 267–282.

[14] ——, *Self-gravitating relativistic fluids: A two-phase model*, Archive for Rational Mechanics and Analysis, 130 (1995), pp. 343–400.

[15] ——, *The Action Principle and Partial Differential Equations*, Princeton University Press, Princeton, NJ, 2000.

[16] ——, *The Formation of Shocks in 3-Dimensional Fluids*, European Mathematical Society, Zürich, Switzerland, 2007.

[17] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Volume II*, Interscience Publishers, New York London Sydney, 1966.

[18] C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Springer-Verlag, Berlin Heidelberg New York, 2000.

[19] T. Damour and G. Esposito-Farèse, *Tensor multi-scalar theories of gravitation*, Classical Quantum Gravity, 9 (1992), pp. 2093–2176.

[20] A. Einstein, *Die Nordströmsche Gravitationstheorie vom Standpunkt des absoluten Differenkalküls*, Annalen der Physik, 49 (1916), pp. 769–822.

[21] ——, *Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie*, Sitzungsberichte der Königlich Preußische Akademie der Wissenschaften (Berlin), 142-152 (1917), pp. 235–237.

[22] A. Einstein and A. D. Fokker, *Die Nordströmsche Gravitationstheorie vom Standpunkt des absoluten Differenkalküls*, Annalen der Physik, 44 (1914), pp. 321–328.

[23] K. O. Friedrichs, *Symmetric hyperbolic linear differential equations*, Communications on Pure and Applied Mathematics, 7 (1954), pp. 345–392.

[24] Y. Guo and S. Tahvildar-Zadeh, *Formation of singularities in relativistic fluid dynamics and in spherically symmetric plasma dynamics*, Contemporary Mathematics, 238 (1999), pp. 151–161.

[25] T. Kato, *Linear evolution equations of "Hyperbolic" type*, Journal of the Faculty of Science Section, The University of Tokyo, I, 17 (1970), pp. 241–258.

[26] ——, *Linear evolution equations of "Hyperbolic" type ii*, Journal of the Mathematical Society of Japan, 25 (1973), pp. 648–666.

[27] ——, *The Cauchy problem for quasi-linear symmetric hyperbolic systems*, Archive for Rational Mechanics and Analysis, 58 (1975), pp. 181–205.

[28] M. Kiessling, *Electromagnetic field theory without divergence problems. I. The Born legacy*, Journal of Statistical Physics, 116, pp. 1057–1122.

[29] ——, *Electromagnetic field theory without divergence problems. II. A least invasively quantized theory*, Journal of Statistical Physics, 116, pp. 1123–1159.

[30] ——, *The "Jeans swindle:" a true story—mathematically speaking*, Advances in Applied Mathematics, 31 (2003), pp. 132–149.

[31] S. Klainerman, *The null condition and global existence to nonlinear wave equations*, in Lectures in Applied Mathematics, vol. 23, American Mathematical Society, pp. 293–326.

[32] ——, *Global existence for nonlinear wave equations*, Communications on Pure and Applied Mathematics, 33 (1980), pp. 43–101.

[33] S. Klainerman and A. Majda, *Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids*, Communications on Pure and Applied Mathematics, 34 (1981), pp. 481–524.

[34] P. Lax, *Hyperbolic Partial Differential Equations*, American Mathematical Society, Providence, Rhode Island, 2006.

[35] H. Lindblad, *A remark on global existence for small initial data of the minimal surface equation in Minkoskian space time*, Proceedings of the American Mathematical Society, 132 (2004), pp. 1095–1102.

[36] A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Springer-Verlag, New York, 1984.

[37] T. Makino, *On a local existence theorem for the evolution equation of gaseous stars*, Patterns and Waves - Qualitative Analysis of Nonlinear Differential Equations, (1986), pp. 459–479.

[38] T. Makino and S. Ukai, *Local smooth solutions of the Relativistic Euler Equation*, Journal of Mathematics of Kyoto University, 35 (1995), pp. 105–114.

[39] ——, *Local smooth solutions of the Relativistic Euler Equation, ii*, Kodai Mathematical Journal, 18 (1995), pp. 365–375.

[40] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, W H Freeman, San Francisco, 1973.

[41] W. Neves and D. Serre, *Ill-posedness of the Cauchy problem for totally degenerate systems of conservation laws*, Electronic Journal of Differential Equations, 2005 (2005), pp. 1–25.

[42] L. Nirenberg, *On elliptic partial differential equations*, Annali della Scuola Normale Superiore di Pisa (Classe di Scienze), 13 (1959), pp. 115–162.

[43] G. Nordström, *Zur Theorie der Gravitation vom Standpunkt des Relativitätsprinzips*, Annalen der Physik, 42 (1913), pp. 533–554.

[44] J. D. Norton, *Einstein, Nordström and the early demise of scalar, Lorentz-covariant theories of gravitation*, Archive for History of Exact Sciences, 45 (1992), pp. 17–94.

[45] T. A. Oliynyk, *The Newtonian limit for perfect fluids*, Communications in Mathematical Physics, 276 (2007), pp. 131–188.

[46] F. Ravndal, *Scalar gravitation and extra dimensions. Proceedings of the Gunnar Nordstörm symposium on theoretical physics*, Commentationes Physico-Mathematicae, 166 (2004), pp. 151–164.

[47] A. D. Rendall, *The initial value problem for a class of general relativistic fluid bodies*, Journal of Mathematical Physics, 33 (1992), pp. 1047–1053.

[48] E. Schrödinger, *A new exact solution in non-linear optics (two-wave-system)*, Proceedings of the Royal Irish Academy. Section A. Mathematical and Physical Sciences, pp. 59–65.

[49] D. Serre, *Systems of Conservation Laws I: Hyperbolicity, Entropies, Shock Waves*, Cambridge University Press, Cambridge New York Melbourne, 1999.

[50] S. L. Shapiro and S. A. Teukolsky, *Scalar gravitation: A laboratory for numerical relativity*, Physical Review D, 47 (1993), pp. 1529–1540.

[51] C. D. Sogge, *Lectures on Nonlinear Wave Equations*, International Press Incorporated, Cambridge, MA, 1995.

[52] K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin Heidelberg New York, 1980.

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