TOPICS IN STATISTICAL FINANCE

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ABSTRACT OF THE DISSERTATION

Topics in Statistical Finance

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This thesis is divided into three parts. The first part investigates the presence of long term dependence in stock price data via a permutation test based on the correlation structure of the underlying stock prices. These tests reveal the short term nature of stock price dependence structure. The second part extends Ramprasath and Singh(2007)'s 'statistical options' to define a group of American type options based on robust estimators of location. The payoff functions of these path dependent options are based on a new set of stochastic processes which are defined using various robust estimators of location. The asymptotic distributional behavior of these new processes is ascertained which in turn is used in pricing the options. Markov Chain Monte Carlo (MCMC) methods were used to compute the prices of the statistical options. The third part explores a stock price model parameter estimation problem and interprets a growth rate parameter.

Acknowledgements

Many people have contributed greatly in preparing this dissertation. This thesis would not have been possible without the valuable guidance, constant support and encouragement I received from Prof. Kesar Singh. I am deeply grateful for this opportunity to work with him and I sincerely thank him for everything. I would like to extend my heartfelt thanks to Prof. Larry Shepp, who constantly encouraged me in this entire effort. I have greatly benefited from his courses and from various discussions with him.

I thank Prof. Ren-Raw Chen for providing me with the stock price data and kindly agreeing to help me with the stock model estimation problem. I sincerely thank professors Cun-Hui Zhang and John Kolassa for supporting me in various ways and at various occasions during the last five years of my stay at Rutgers. I would like to thank Ramprasath Lakshminarasimhan, Sambuddha Roy, Rahul N.Pupala, Prasenjit Kapat, Suvajit Samanta and Pritam Ranjan for their warm friendship. Most importantly, I would like to thank my parents and family for their love and encouragement all these years.

Dedication

To my wife,

Krishna Chaitanya

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Chapter 1

Introduction

This thesis consists of three applications of statistical tools to finance. The first part of this thesis proposes a simple graphical method to detect the existence of long term dependence in the stock price data. The second part deals with a set of financial contracts called statistical options, first proposed by Ramprasath and Singh (2007). Various statistical options of the American type are introduced and their fair prices are obtained. The third part deals with a stock price model and its parameter estimation. A parameter denoted by θ is introduced and is interpreted as the stock price growth rate parameter. In the following three sections, the above three problems are introduced in greater detail.

1.1 Dependence in Stock Price Data

For many decades, researchers have been interested in the dependence structure of stock price time series. The presence of long term dependence in the stock price data has many important consequences. The ability to predict the future stock prices to some degree of accuracy allows an investor to make money with relatively low or no risk. This violates the basic assumption of 'no arbitrage'. Indeed many option trading firms exploit the dependence structure of stock prices in predicting the near future prices and make use of this information in their option purchasing/selling decisions.

The detection of long term dependence in time series data has been dealt with by many researchers. Some of the prominent works in this area include Hurst(1951), Mandelbrot(1971), Lo(1991), Willinger et al.(1991) and Robinson(1995). An important but simple concept in the theory of time series data is that of autocorrelation. A time series of a process $\{X_t, t > 0\}$ is a sequence of data, obtained periodically at consecutive time points. Autocorrelation is the correlation between the process at different time points. For example, autocorrelation of a time series Y between time points s and t is defined as the correlation between Y_s and Y_t. Also, autocorrelation at a lag value k is defined as the correlation between Y_t and Y_{t+k}. This current work proposes a permutation test based procedure to determine the presence of long range dependence in the stock price data. Suppose that the process $\{X_t, t > 0\}$ is a stock price process. Under both the geometric Brownian motion and the jump diffusion models, the log-returns of the process X_t are independent random variables. Moreover, if the log-returns are obtained at regular intervals of time, they are also identically distributed. In this work, the key observation is that, if a time series is a sequence of independent and identically distributed random variables, any permutation of the sequence is also a sequence of independent and identically distributed random variables and hence has zero autocorrelations at every lag. Therefore, any perumtation of the log-returns series retains the autocorrelation structure of the original log-returns time series. It is also important to note that, if the log-returns are not independent, a random permutation operation distorts the autocorrelation structure of the original log-returns sequence. Using these observations, a permutation test is obtained by comparing the autocorrelation structure of many randomly permuted versions of the log-returns time series with that of the original time series.

In our analysis, the daily closing prices of a stock price sequence are used to obtain the daily log-returns sequence. Under the geometric Brownian motion model, the log returns are independent normal random variables. The autocorrelation structure of these log-returns determines the dependence structure of the stock price sequence. A null hypothesis of no dependence in the log-returns series is formulated and the permutation procedure is applied. The details of the test procedure are explained in section 2.2. For all the stocks that were considered for the analysis, the test procedure revealed no presence of long term dependence. The results are shown in section 2.3.

1.2 Statistical Options

The concept of 'Statistical Options' was first introduced by Ramprasath and Singh (2007). (see also PhD thesis, Ramprasath (2005).) In this paper, robust estimators of location are used in proposing a new set of path dependent European type options, which protect the investor from sudden drops in stock prices. This current work extends the idea to the case of American type options. The details of these options are outlined in the following paragraphs.

A security option is a financial contract between a financial house and an investor. The investor is given the 'option' to 'exercise' the contract. For example, a European type call option lets the investor buy a security at a predetermined price K, on a predetermined date T. If the market price $X(T)$ of the security at time T is more than K, the investor gains the difference $X(T) - K$ by exercising the contract at time T and selling it in the market at the market price $X(T)$. In practice, the investor is paid $X(T) - K$, often called the 'option payoff' whenever the price $X(T)$ exceeds K. If the market price $X(T)$ falls below K, the investor gains nothing by exercising the contract and hence the option payoff is zero. In effect, the European call option provides an investor the option to execute the contract whenever the underlying security price exceeds the strike price K. Similarly a European Put option provides an investor with the option to sell the underlying security at the strike price (K) whenever the security price $X(T)$ falls below K. In general, European type options allow the investor to exercise the option only on a predetermined date called the exercise date. On the contrary, American type options allow the investor to exercise the option on any day between the onset of the option and the option expiry date. Apart from the aforementioned call and put options, there is a variety of options traded in the markets. For example, an Asian call option pays the investor $(\overline{X}(T) - K)^+$ where $\overline{X}(T)$ is the arithmetic average of the security prices between the onset of the option and the exercise date T. Such options are called path dependent options since their payoffs depend on the entire history of the underlying security price rather than on the security price at the exercise date. The premium an investor needs to pay for an option is called the option price. The fair value of an option price is often determined using the ‘no arbitrage’ principles. It is in general more difficult to price path dependent options due to the complicated structure of their payoff functions.

‘Statistical options’ considered in chapter 3 are also path dependent. These options are designed so as to protect the investor from significant market crashes. Of course, the investor would pay a reasonable premium for it. To understand the exact definitions of these payoffs, we review here, the statistical concept of robust estimators and their relation to the statistical option payoffs. In the statistical literature, robust estimators are characterized by their insensitivity to outliers or extreme observations. A classic estmator of a location parameter is the sample mean. Although sample mean often is the minimum variance estimator in the class of all unbiased estimators, it has a critical drawback. Sample mean is very sensitive to extreme observations in the data. The presence of a single outlier can drastically alter the sample mean. On the other hand, sample median is very insensitive to outliers. Since sample median is defined as the middle most observation of an ordered data set, it ignores all extreme observations of the data set. Other examples of robust location estimators are the Hodges-Lehmann estimator and the trimmed mean.

Under the geometric Brownian motion model for the security prices $X(t)$, the natural logarithm of the price process is a Weiner process with a constant drift and constant volatility. Let us denote this Weiner process by $W(t)$. Then, $W(t)$ can be written as the sum of n independent and identically distributed normal random variables for any n and for any $t > 0$. Under the jump diffusion model, the natural logarithm of the price process $X(t)$ also has independent and identically distributed increments. So, in either case, $\log(X(t))$ can be written as an IID sum of random variables. This sum of n IID variables can be viewed as n times the mean of n log returns of the process. Notice that the call option payoff increases with this mean log-returns. A sudden sharp drop in the security price corresponds to a large negative log returns term and this causes the decrease in a call option payoff. The motivation behind the statistical options lies in modifying this mean log returns term so as to protect the investor from this kind of decrease in the option payoff. The mean log-returns term is replaced by the median log-returns or some other statistically robust summary measure of log-returns so that, the payoff is not affected by large negative returns. In the rest of this thesis, a call option with this type of payoff is referred to as the statistical call option. To fix ideas, we introduce the following notation. Let the time period between 0 and T be equally divided into n parts. Let the log-returns be denoted by δ_i. Then we have,

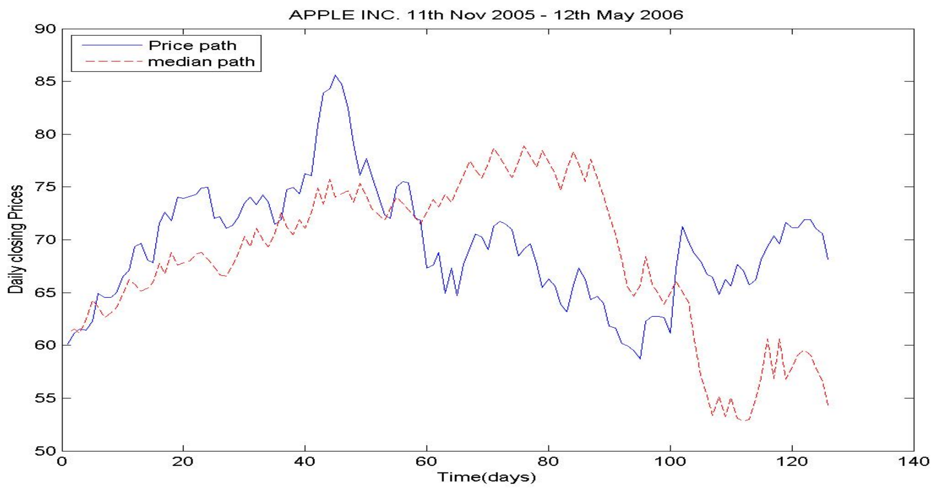
$$\delta_i = \log\left(\frac{X(t_i)}{X(t_{i-1})}\right), \tag{1.1}$$

where $t_i = \frac{iT}{n}, i = 0, 1, \ldots, n$. At time T, a median call option with strike price K pays off $(M_n(T) - K)^+$, where

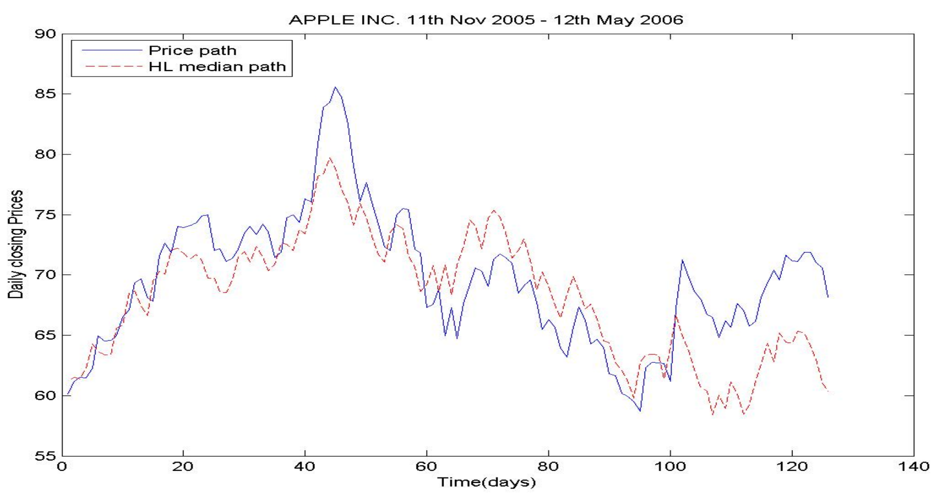
$$M_n(T) = X(0)\exp(n.median(\delta_1, \delta_2, \ldots, \delta_n)) \tag{1.2}$$

Notice that, the only difference between $M_n(T)$ and the usual call option payoff lies in the term $median(\delta_1, \delta_2, \ldots, \delta_n)$. One obtains the usual call option payoff by replacing the median in the above expression with mean. In chapter 3, we consider the stochastic process $M_n(t)$ and other similar processes for every $t \in [0, T]$. For a fixed n, the definition of $M_n(t)$ is clear for every $t \in t_i = iT/n : i = 0, 1, \ldots, n$. For values of t other than these, we define $M_n(t)$ by an interpolation of $M_n(t_k)$ and $M_n(t_{k+1})$ where, $t_k \leq t \leq t_{k+1}$. Now notice that such stochastic processes can be used in defining American type statistical options. For example, at time t, an American type median call option pays off $(M_n(t) - K)^+$. Thus, given a choice of robust estimator of location, one has a corresponding American type statistical option. The processes obtained by replacing the mean log-returns term with other robust location estimators are considered in chapter 3.

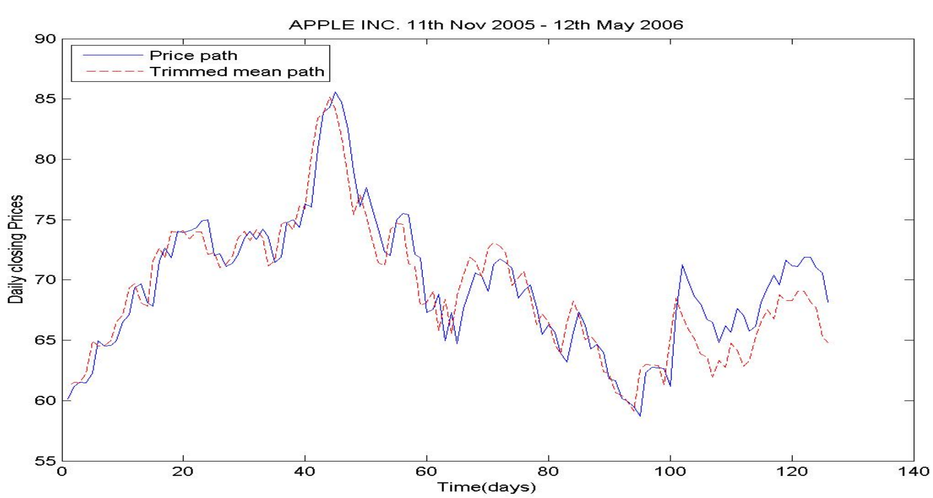
The sample pathwise behavior of the above processes is illustrated in the following three figures, where the darker curve represents the daily closing values of Apple Inc. over a period of six months. The dashed curves represent the median process $M_n(t)$, the HL estimator based process $HL_n(t)$ and the trimmed mean based process $TR_n(t)$ respectively. $HL_n(t)$ and $TR_n(t)$ are obtained by replacing median in equation (1.2) with the HL estimator and the trimmed mean respectively. See section 3.2 for rigorous definition of $HL_n(t)$ and $TR_n(t)$. Clearly, the trimmed mean path follows the stock price path more closely than do the HL median path and the median path.



(a) Median path.



(b) Hodges-Lehmann median path



(c) Trimmed mean path

Figure 1.1: Robust estimator based processes: Sample paths

Pricing the statistical options is the next major task. Historically, there are various methods to find the fair price of a security option. It is often very difficult to find an analytical solution to the option prices. It is all the more difficult when the options are path dependent. Due to the availability of computing power, various numerical methods are now available for option pricing. Binomial option pricing method, Finite difference methods, Markov Chain Monte Carlo methods, Bootstrap methods are a few of the popular option pricing methods. Various above mentioned numerical procedures are different in many ways, but all their methodologies depend on the 'no arbitrage principle'. A brief review of the above principle can be found in section 3.1 .

In pricing the statistical options, we rely on the limiting behavior of the above processes. It is proved in this thesis that if $X(t)$ is assumed to be a geometric Brownian motion process, the processes $M_n(t)$, $HL_n(t)$ and $TR_n(t)$ functionally converge in distribution to different geometric Brownian motion processes as n increases to infinity. Interestingly, if the underlying process $X(t)$ is a jump diffusion process, these processes converge to the same geometric Brownian motions whenever the robust estimators used in defining these processes trim outliers on both sides. We approximate the above processes with their asymptotic counterparts in pricing the statistical options. This approximation allows one to price American type options defined on the statistical assets. The pricing methods are discussed in great detail in chapter 4.

Notice that since the median of a group of observations can very well be less than the sample mean, the payoff of an American type median call option could be less than that of a vanilla call option. This is due to the fact that median also ignores the larger of the observations. The investor of a call option only needs to be protected from large drops in the stock price and an investor of a put option only needs to be protected from a sudden rise in the price. To protect the investor from any loss due to this two new statistical options are introduced. To ensure that the investor does not lose out on favorable price moves, one may want to modify the payoff by taking the maximum of the median option payoff with the vanilla option payoff. In this thesis, this option is referred to as the max type option. Another way could be to consider an estimator that ignores outliers only in one direction. A one sided trimmed mean is an example of such an estimator. For example, when considering a call option, one can replace the median in (1.2) by a lower trimmed mean which ignores a given fraction of the smallest of the observations. A call option with payoff based on this process always pays off more than the vanilla option does. In chapter 4, pricing methods for these variations of the statistical options are explained in more details. Several statistical option prices are computed and compared in sections 4.2 and 4.3.

1.3 Parameter estimation for a stock price model

In the third part of this thesis, a new stock price model is considered. This model is given by,

$$dX(t) = X^{\theta}(t)(\mu dt + \sigma dW(t)) \tag{1.3}$$

where $\theta \in [0,1]$, $\mu \in \mathbb{R}$ and $\sigma \in (0,\infty)$. The above model was proposed by Chen, Logan, Palmon and Shepp(2003). In this thesis, the parameter θ is of interest. Notice that $\theta = 0$ corresponds to the linear Brownian motion model of Bachelier(1900) and $\theta = 1$ corresponds to the geometric Brownian motion model of Black, Scholes and Merton(1973). In fact, Bachelier model corresponds to simple linear growth of stock value over time in the sense that, the mean value of the stock price is a linear function of time. Also, it is easy to note that the Black-Scholes-Merton model corresponds to continuously compounded growth of stock value over time, meaning that the expected value of the stock price under this model is an exponential function of time. The average value of the stock price is a monotone increasing function of θ and as θ increases to one, the above model corresponds to the famous Black-Scholes-Merton model. The current parameter estimation problem considers the estimation of θ with it's possible values in the interval [0,1].

The parameter θ is estimated by a kurtosis based method. Kurtosis is defined as the fourth standardized moment of a distribution. In symbols, we have,

$$\gamma = \frac{\mu_4}{\sigma^4} \tag{1.4}$$

where μ_4 is the fourth central moment and σ^4 is the square of variance. Kurtosis can be viewed as a measure of peakedness of a distribution in the sense that, a larger kurtosis value corresponds to heavy tails in a distribution. Since the stock prices are observed only at discrete time points, the differential terms in the stochastic differential equation 1.3 are replaced by their corresponding finite difference terms. For example, dt term at time $t = t_{k-1}$ is replaced by $\Delta t = t_k - t_{k-1}$ and $dW(t)$ is replaced by $\Delta W(t) = W(t_k) - W(t_{k-1})$. This method of discretization is often called the 'Euler discretization method'. Assuming model (1.3) and following this method, the modified returns $\frac{\Delta X(t)}{X(t)^\theta}$ are given by

$$\frac{\Delta X(t)}{X(t)^\theta} = \frac{(X(t + \Delta t) - X(t))}{X(t)^\theta}. \tag{1.5}$$

Notice that these modified returns follow a normal distribution for every t. Irrespective of the values of the parameters, normally distributed variables have a kurtosis value of three. This observation is key to our estimation procedure. We estimate the kurtosis of the modified returns for every $\theta \in [0, 1]$ and estimate θ by choosing the value that is closest to three. In all the stocks that we considered and for all values of $\theta \in [0, 1]$, the kurtosis values of the modified returns were always more than three, indicating a heavy tailed nature of the returns. Hence, the estimate of θ is the value which gives a minimum kurtosis value for the modified returns. Further details of the estimation procedure are given in section 5.2.

Chapter 2

Stock Prices Have Just A Few Days Of Memory! - A Correlation based Analysis.

2.1 Introduction

The presence of long range dependence or long term memory in the stock price data has been explored by many researchers over the past decades. The potential implications of long range dependence made these investigations both important and controversial. If there is substantial evidence of long term dependence in the stock price data (or any asset price data) the common assumption of "efficient markets" would be violated. By definition, an efficient market would not allow the investors to predict with accuracy, the future prices of a stock,ie, the past prices would not contain any information regarding the future prices. This requires the price changes to possess zero autocorrelations. If the stock prices indeed have long term dependence, the markets would become inefficient and this in turn makes the common asset pricing methodologies invalid.

A number of procedures have been developed over the past many years to detect long term dependence and many empirical tests were conducted on various stock market data. Harold Edwin Hurst (1951), an English hydrologist, first proposed the "Rescaled Range" R/S-statistic, which was later modified by Mandelbrot(1971). Lo (1991) proposed a test based on the modified R/S-statistic to overcome the sensitivity of Mandelbrot's R/S statistic to the presence of short range dependence in the data. Willinger et al.(1991) provide a detailed discussion on these procedures. Robinson (1995)considered semiparametric estimation of long term dependence. More recently, Berkes et al. (2006) devised a test discriminate between long term dependence and changes in the mean. Among empirical studies, Greene and Felietz (1977),Aydogen and Booth (1988), Lo(1991), Cheung et al (1993), Chow et al(1995) detected no presence of long term dependence while Crato(1994), Barkoulas and Baum (1996), Christodoulou-Volos and Siokos(2006) reported evidence of long term dependence.

This current study attempts to provide a new tool to detect the long-range / short-range dependence in the form of a permutation test based on the autocorrelations of the stock price returns. The methodology is graphical and easy to comprehend. It presents a whole profile of correlations, at a large number of lag values. This chapter is divided into four sections. Section-2 outlines the framework and describes the graphical test procedure. Section-3 describes the data and provides the results. Section-4 contains the concluding remarks.

2.2 Framework and the test procedure

Let X_t be the share price at time t, 0≤t≤T. For convenience, let the unit of time be one day. Also let the process

$$Y(t) = \log(X_t). \tag{2.1}$$

have stationary increments over the interval 0≤t≤T,

i.e., for any $\Delta > 0$,

$$Z_\Delta(t) = \log(X_{t+\Delta}) - \log(X_t) \tag{2.2}$$

has the same distribution for all $t > 0$.

So, in particular, $Z_1(0)$,$Z_1(1)$,$Z_1(2)$,...,$Z_1(T-1)$ are identically distributed random variables.

For future reference, we define the following processes which are often used in modeling the stock price data.

Black-Scholes Model:

$$S_1(t) = S_1(0)\exp(\mu t + \sigma W(t)), o \leq t \leq T \tag{2.3}$$

where $S_1(0)$ is the share price at time t=0, $\mu \in R$, $\sigma > 0$ and W(t) is a standard Brownian motion over the interval [0,T].

A Jump-Diffusion Model:

$$S_2(t) = S_2(0)\exp(\mu t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i), o \le t \le T \tag{2.4}$$

where $S_2(0)$ is the share price at time t=0, $\mu \in R$, $\sigma > 0$, W(t) is a standard Brownian motion over the interval [0,T] and N(t) is a Poisson process and Y_i are IID random variables.

Bachelier Model:

$$S_3(t) = S_3(0)(\mu t + \sigma W(t)), o \le t \le T \tag{2.5}$$

where $S_3(0)$ is the share price at time t=0, $\mu \in R$, $\sigma > 0$ and W(t) is a standard Brownian motion over the interval [0,T].

We quickly note that under the above two models, the process Z(t) is stationary and the variables has zero autocorrelations. Under the Black-Scholes model, $Z_1(0)$,$Z_1(1)$,$Z_1(2)$,...,$Z_1(T-1)$ are iid normal variables with mean μ and variance σ^2. Both these models satisfy the "Efficient Markets" assumptions and enable one to price various options and stock assets. This is due to the fact that the exponents in the above two models (ie, $\log(S_1(t)) - \log(S_1(0)$ and $\log(S_2(t)) - \log(S_2(0)$ respectively) fall under the larger class of processes called Levy Diffusions. Levy Diffusions are the processes with independent and identically distributed increments.

We considered daily and weekly data of closing prices for different stocks enlisted in NYSE. The exact details of the stock data is discussed in section 3.

Since $Z_1(t) = \log(X(t)) - \log(X(t-1))$ is a stationary process, $Z_1(t)$ have a common mean, say μ_z, for different t=1,2,...,T. And μ_z is unbiasedly estimated by

$$\overline{Z} = \frac{1}{T}\sum_{i=1}^{T} Z_1(i) \tag{2.6}$$

and hence

$$\widetilde{Z}(t) = Z_1(t) - \overline{Z}, t = 1, 2, \ldots, T \tag{2.7}$$

is a mean zero stationary process.

Under the hypothesis of short range memory, the autocorrelations r(t) of $\widetilde{Z}(t)$ must "die down to zero" quickly as the lag value increases. We compute the autocorrelations of $\widetilde{Z}(t)$ for different lags l=1,2,...,n. We considered lags of size up to 500 for the daily data and lags of size up to 100 for the weekly data. We notice that under the hypothesis of $\{\widetilde{Z}(t), t = 1, 2, \ldots, T\}$ being i.i.d random variables, the joint distribution is symmetric. Therefore, any random permutation of $\{\widetilde{Z}(t), t = 1, 2, \ldots, T\}$ has the same distribution and this fact forms the basis for the permutation test being deployed here.

Let us denote a random permutation of $\{\widetilde{Z}(t), t = 1, 2, \ldots, T\}$ by $\widetilde{Z}_P(t)$, indexed by the permutation P. Also let us denote by $\{r_P(l) : l = 1, 2, \ldots, n.\}$, the autocorrelations of the permuted sequence $\{\widetilde{Z}_P(t) : t = 1, 2, \ldots, n.\}$ at different lags $l = 1, 2, \ldots, n$. $r_P(l)$ denotes the empirical correlation obtained from the pairs
$(Z_P(0), Z_P(l))$,$(Z_P(1), Z_P(l+1))$,...,$(Z_P(T-l-l), Z_P(T-1))$. Since the sign of a correlation $r_P(l)$ is irrelevant to us, we replace $r_P(l)$ by it's absolute value $|r_P(l)|$. Now, we obtain a correlation curve by plotting the values of $|r_P(l)|$ in increasing order, ordered over different lags $l = 1, 2, \ldots, n$. We repeat this permutation process and simultaneously plot the correlation curves, obtained as above for a large number of random permutations. From now on, we call this set of simultaneously plotted curves a correlation band. The correlation curves may be viewed as a curve valued random variables or a stochastic process. The correlation band in essence described the conditional distribution of the correlation curve given the original data $\{Z_1(t), t = 1, 2, \ldots, T\}$ under the hypothesis of exchangeability.

Now under the short range memory hypothesis, the original sequence $\{r(l) : l = 1, 2, \ldots, n.\}$ of autocorrelations must form a correlation curve that lies well within the repeated permutation based correlation band as described above. We plotted these curves for different stocks over different time periods to graphically test the hypothesis. The results are shown in section 3.

To make the testing scheme rigorous at 5% level of significance, we deleted 5% of the correlation curves (out of the correlation band) which had highest values of the average

$$A_P = \frac{1}{n} \sum_{l=1}^{n} |r_P(l)|. \tag{2.8}$$

The choice of A_P for the purpose of deletion is ad-hoc, though logical. Other choices could be the median, the mean of the upper 100t% , just to name a few. Our findings are so unambiguous that the choice of trimming device would hardly matter for the conclusions.

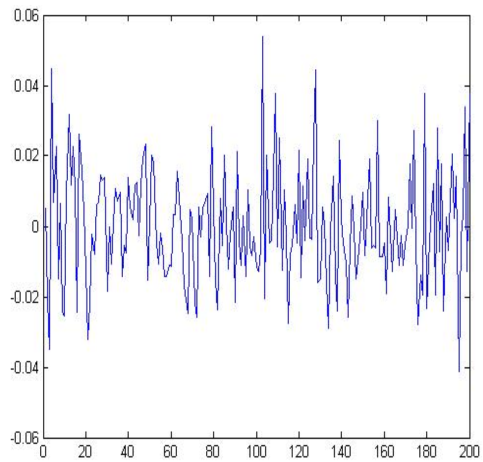
2.3 Data and Results

We obtained the daily and weekly data for various stocks enlisted in the New York Stock Exchange(NYSE). These stock data, the corresponding time periods and the data sizes are summarized in table 3.1.

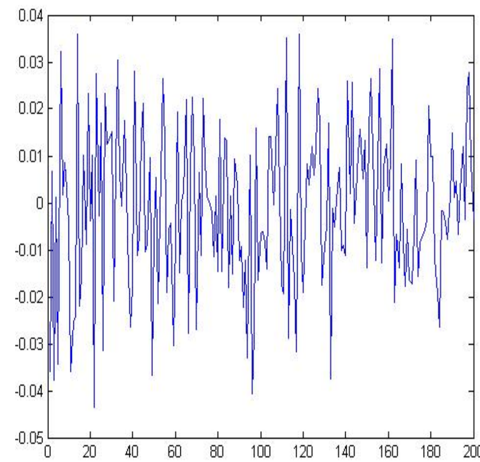
Stock	Begin Date	End Date	Data Size
Apple Inc.	2nd Jan 1985	12th May 2006	5391
Alberto-Culver Co.	2nd Jan 1985	12th May 2006	5390
Adobe Systems	14th Aug 1986	12th May 2006	4975
American Electric	2nd Jan 1985	12th May 2006	5389
Aetna Inc.	2nd Jan 1985	12th May 2006	5390
Allstate Corp.	3rd Jun 1993	12th May 2006	3262
Amazon.com Inc	2nd Jun 1997	12th May 2006	2253

Table 2.1: Stock data,time periods and data sizes.

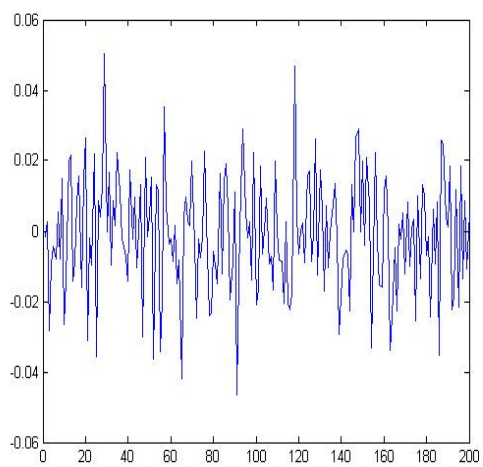
To begin our correlation analysis, we considered plotting the autocorrelations $\{r(l) : l = 1, 2, \ldots, n\}$ for the above data; where r(l), the autocorrelation with a lag value l is obtained as explained in section 2. The following figures 3.1 and 3.2 show the autocorrelations for different stocks for lags up to 200 days.



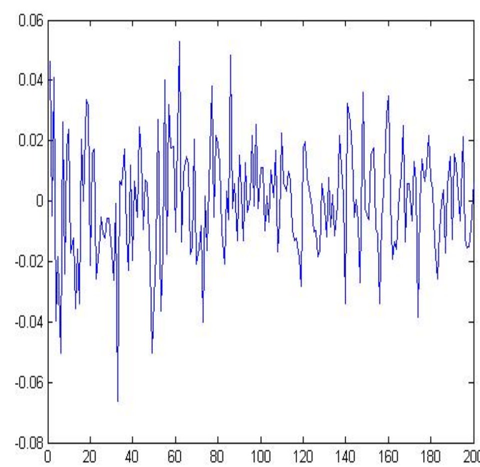
(a) Apple Inc.



(b) Alberto Culver

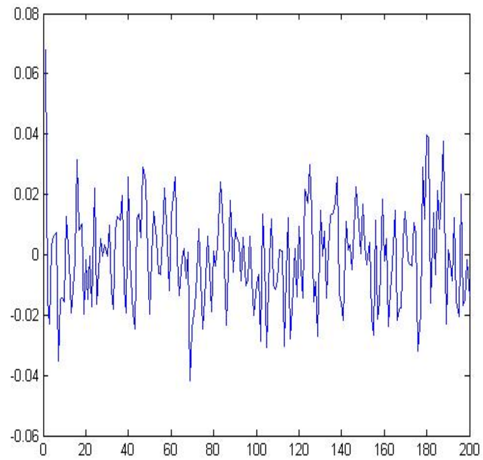


(c) Adobe Systems

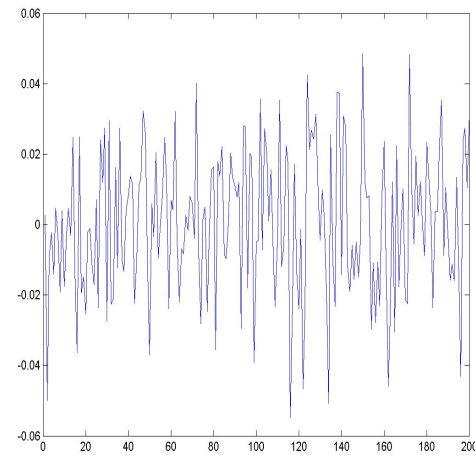


(d) American Electric

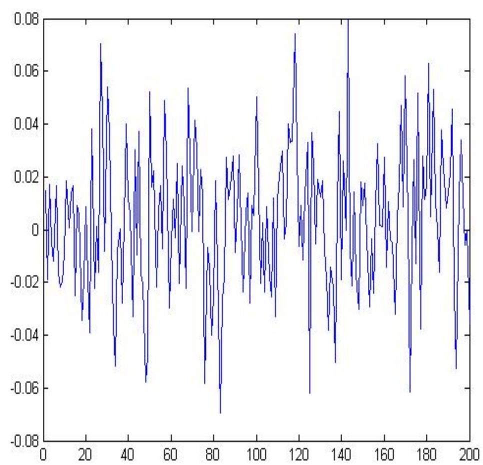
Figure 2.1: Autocorrelations



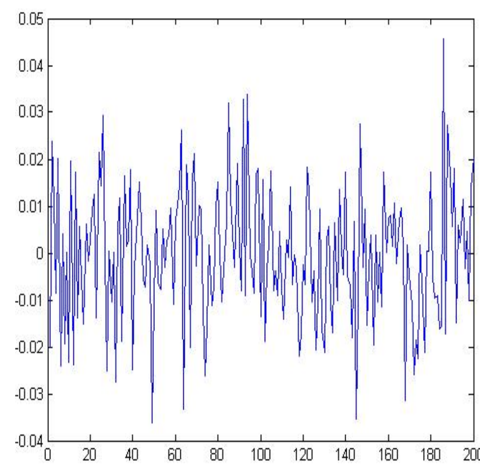
(a) Aetna Inc.



(b) Allstate Corp.



(c) Amazon.com Inc.



(d) Pseudo Normal data

Figure 2.2: Autocorrelations:Continued

From the above figures(a)-(g), it can be observed that the autocorrelations do not die down to zero as the lag value increases. To explore this phenomenon more, we simulated pseudo normal random data of size 5400 and computed the autocorrelations. These autocorrelations are plotted in figure 3.2(h). Note that if the stock price indeed follows the Black-Scholes model given by the equation(2.3), the autocorrelation plot would look like 3.2(h). We observe that these autocorrelations also oscillate around zero but do not tend to get closer to zero, as the lag value increases. Indeed, these autocorrelation plots do not provide enough evidence to reject the hypothesis of independent increments.

We now turn to the methodology explained in section 2 and plot the correlation bands for the stock price data. The following figures show the correlation bands and the actual correlation curves. The autocorrelation curves are shown in black to distinguish them from the correlation band.

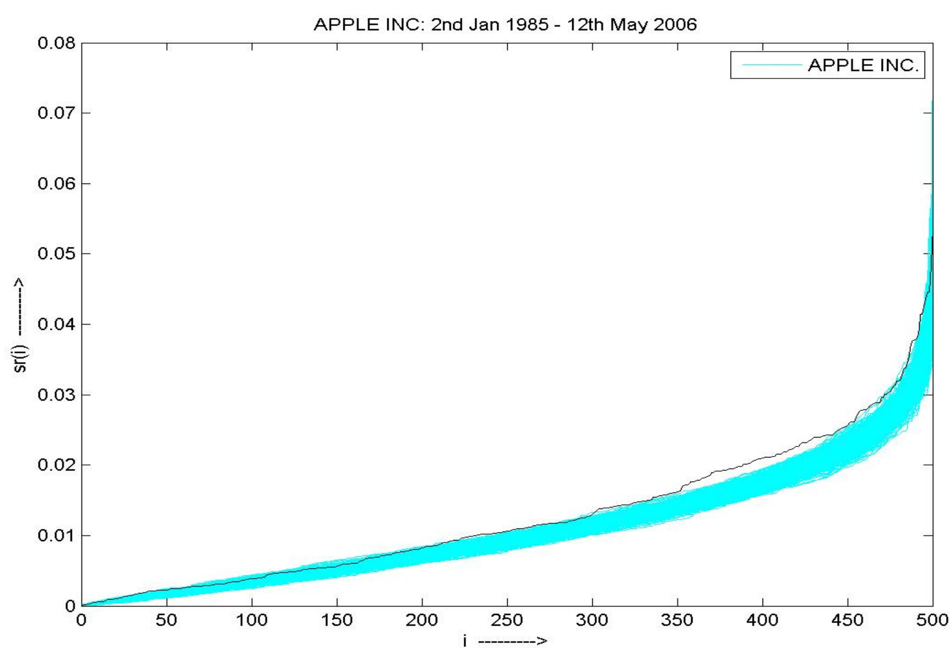


Figure 2.3: Correlation Band-Apple Inc

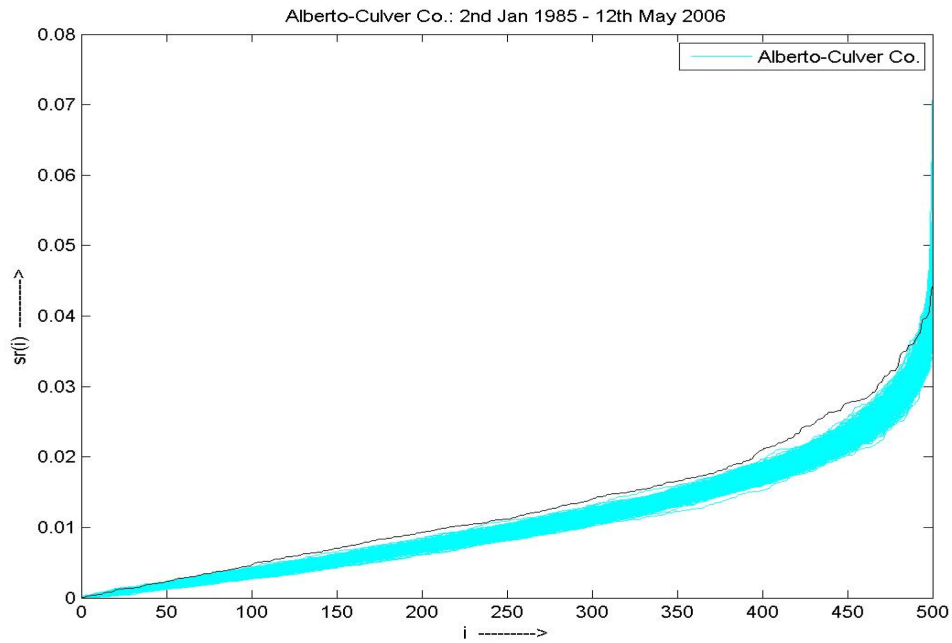


Figure 2.4: Correlation Band-Alberto Culver

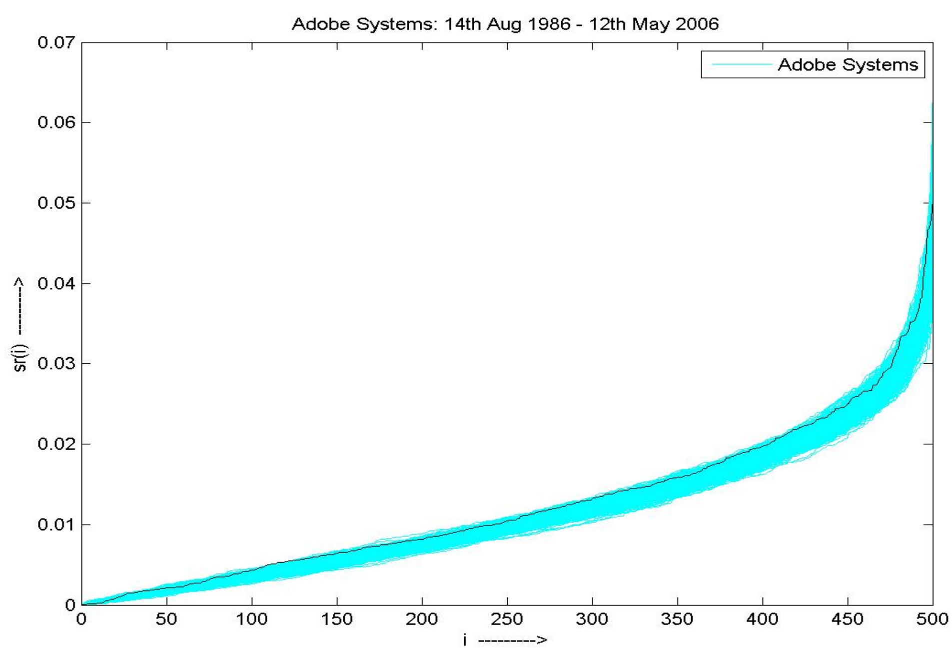


Figure 2.5: Correlation Band-Adobe Systems

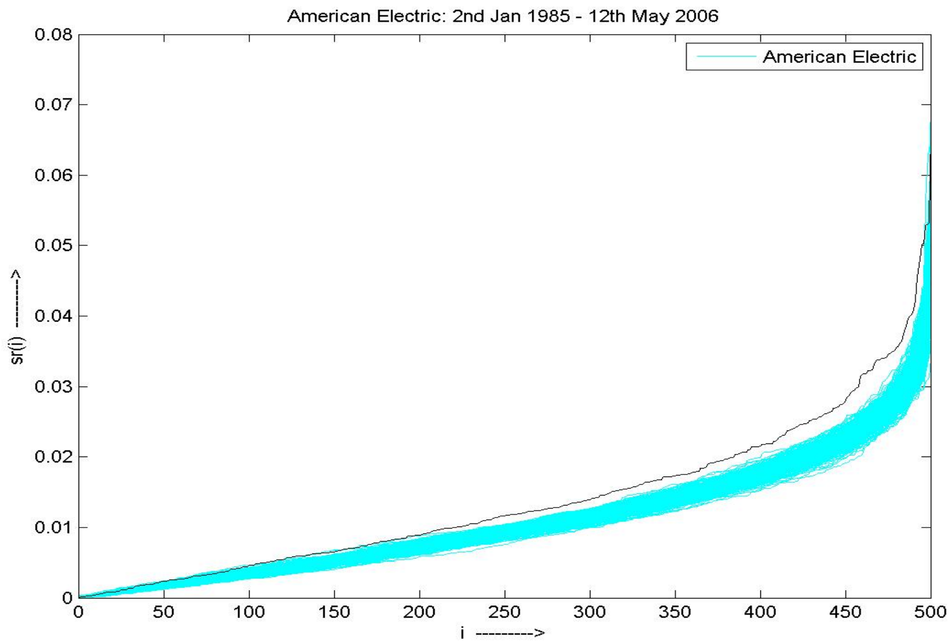


Figure 2.6: Correlation Band-American Electric

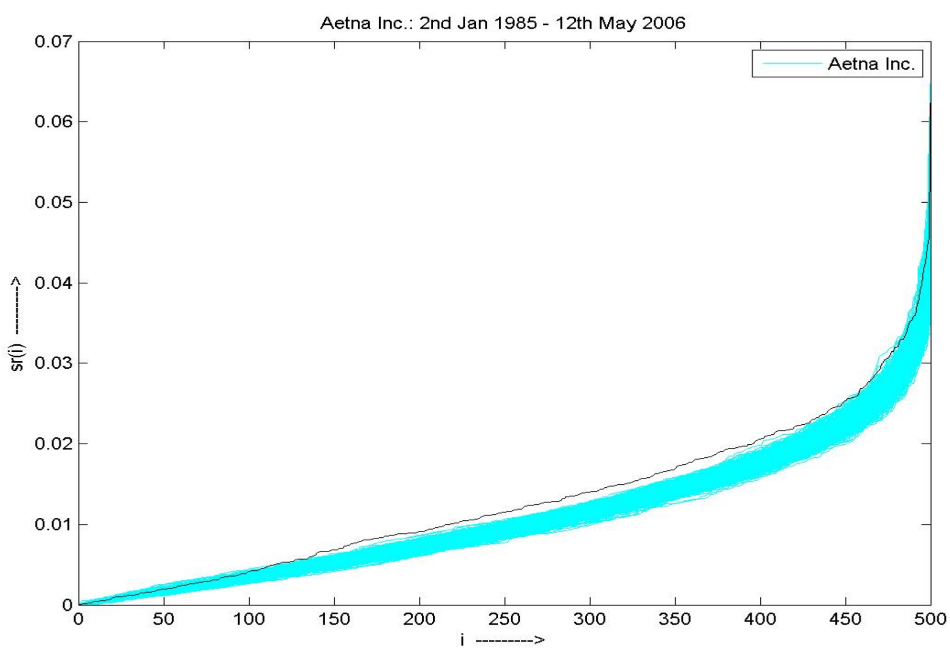


Figure 2.7: Correlation Band-Aetna Inc

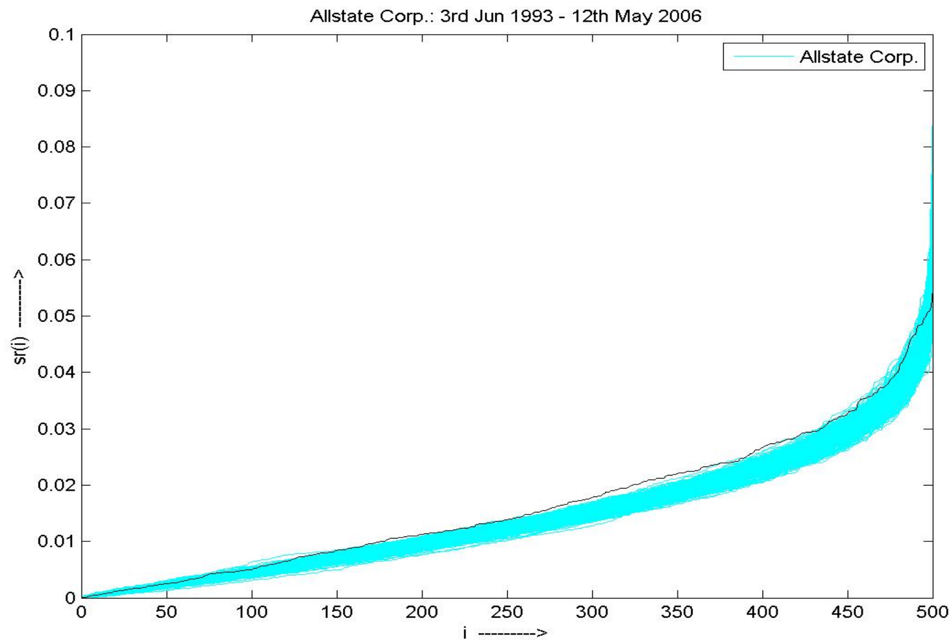


Figure 2.8: Correlation Band-Allstate Inc

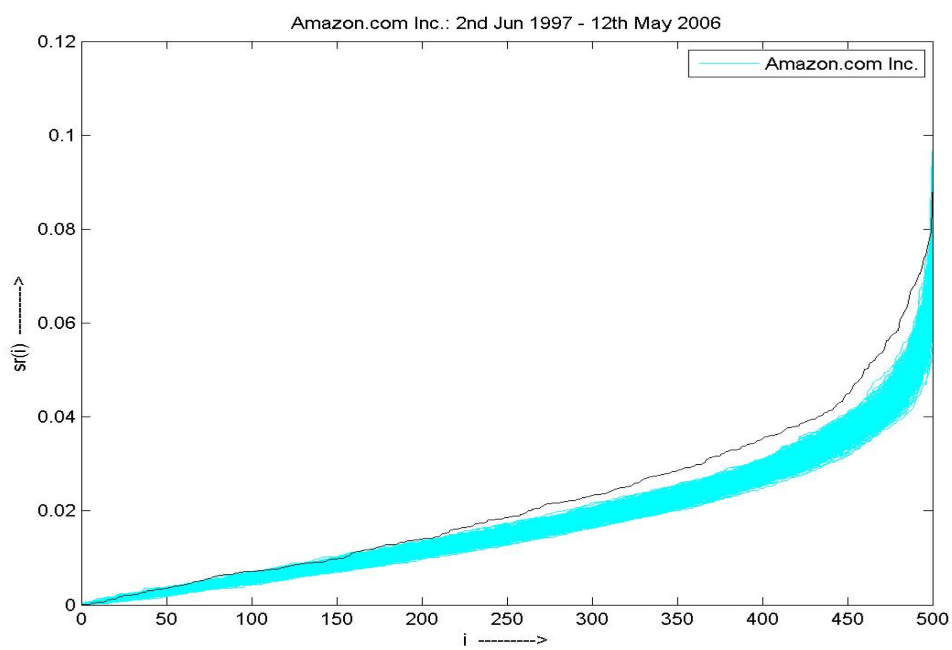


Figure 2.9: Correlation Band-Amazon.com Inc

We note that the correlation curve falls outside the correlation band for most of the above stocks. This is rather apparent in Alberto Culver, American Electric and Amazon stocks. This means that the hypothesis of independent increments is rejected. Now, we consider the data with it's alternative entries and we repeat the same graphical testing procedure on this data. If the correlation curve still falls outside the newly obtained correlation band, we proceeded to consider every third entry in the data set, thus forming a new data set with size approximately one thirds of the original data set, and so on... For all the stocks that we considered, the correlation curve came within the correlation band when every fourth entry was considered. We notice that, if the correlation curve falls well-within the correlation band when every k^{th} data entry is considered (say k=2,3,4), it means that the data is weakly dependent. We now present the correlation bands obtained using the above methodology.

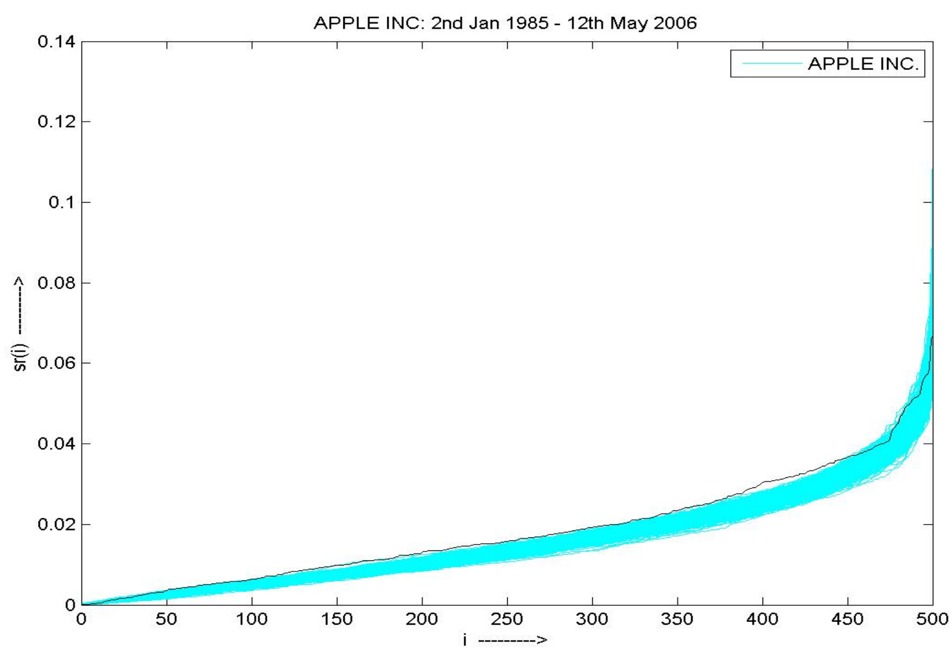


Figure 2.10: Correlation Band-Apple Inc. Alternate days

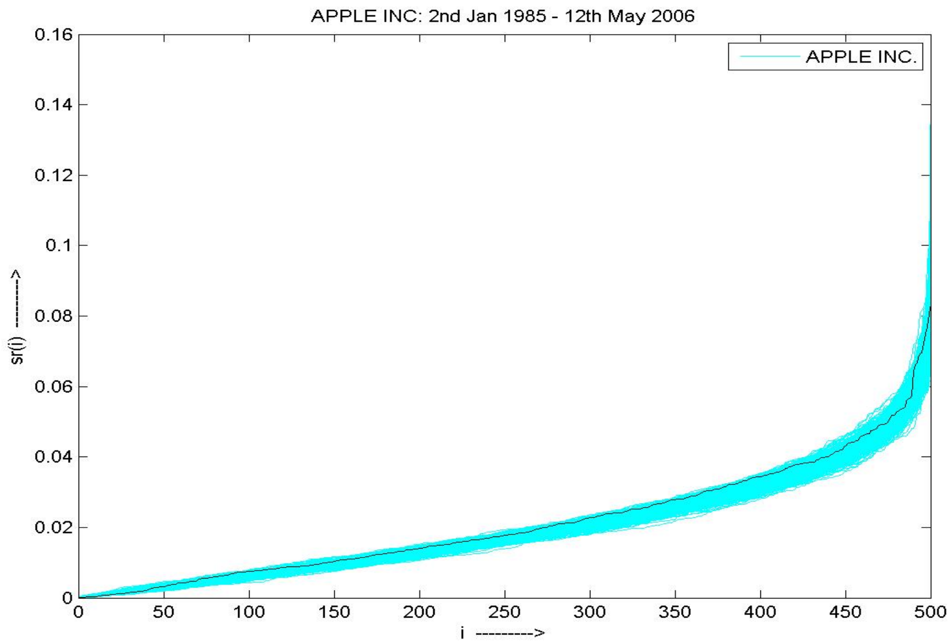


Figure 2.11: Correlation Band-Apple Inc.: Every third day

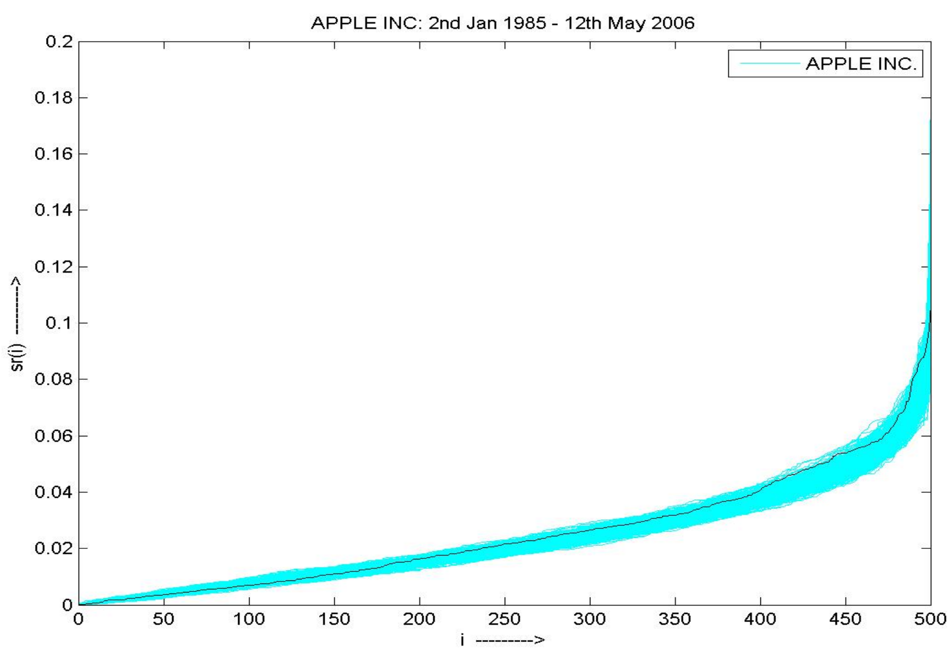


Figure 2.12: Correlation Band-Apple Inc.: Every fourth day

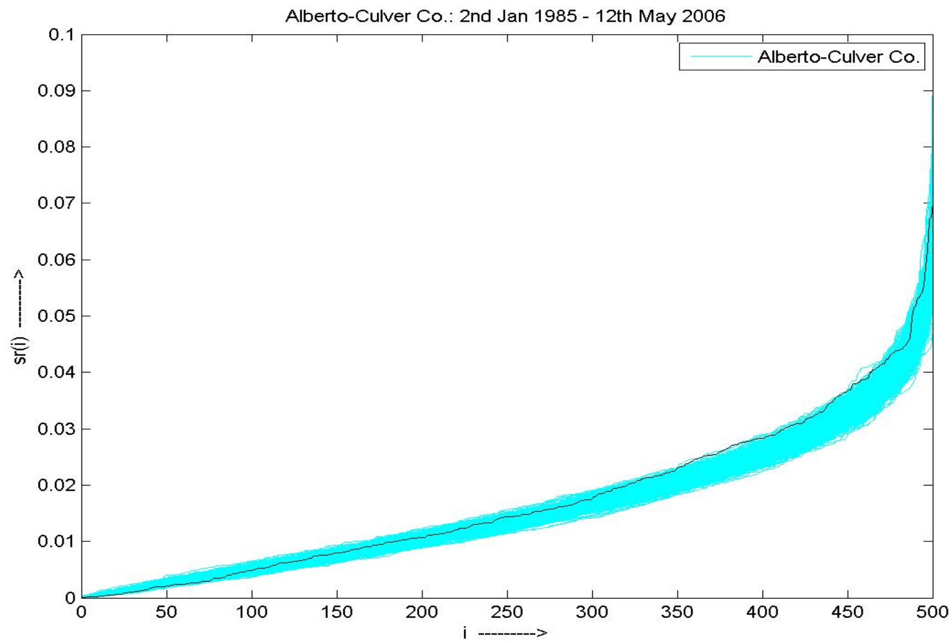


Figure 2.13: Correlation Band-Alberto Culver Alternate days

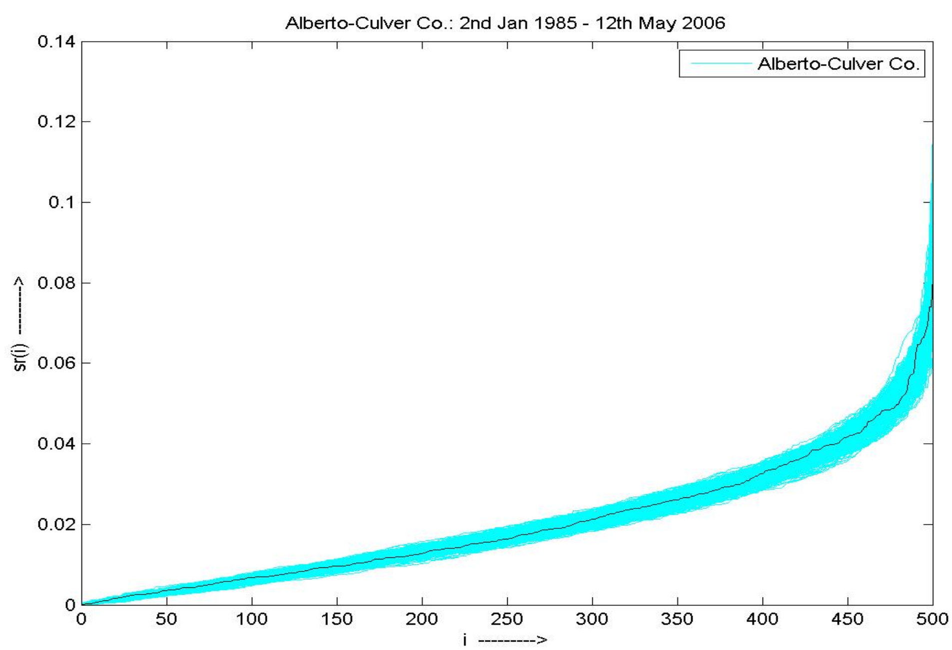


Figure 2.14: Correlation Band-Alberto Culver: Every third day

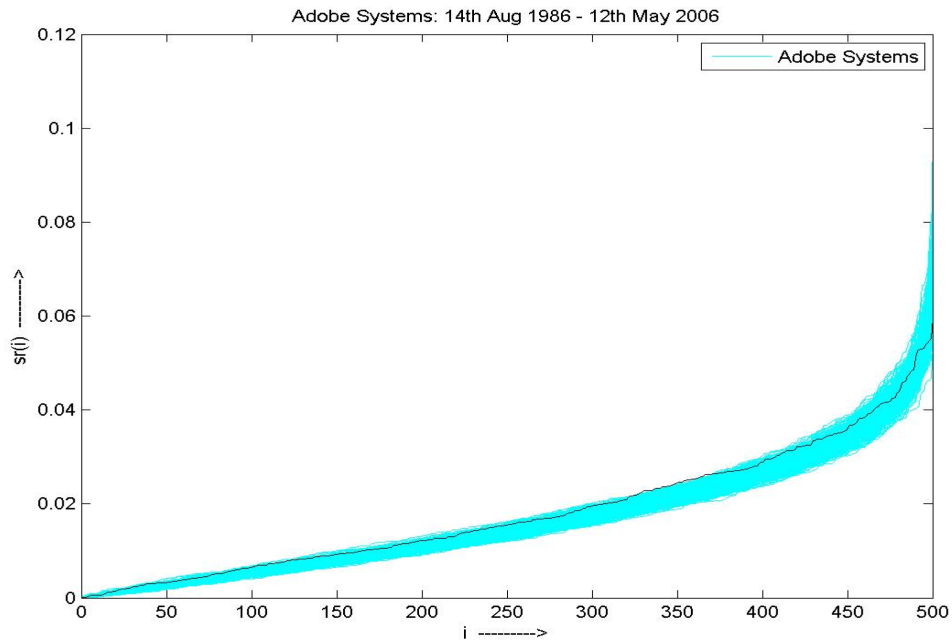


Figure 2.15: Correlation Band-Adobe Systems Alternate days

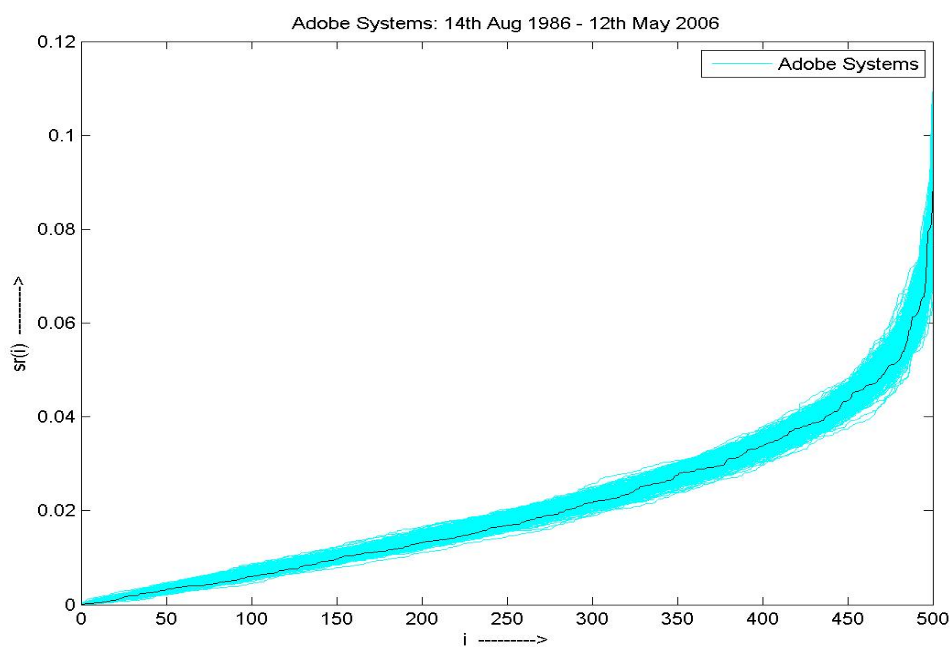


Figure 2.16: Correlation Band-Adobe Systems: Every third day

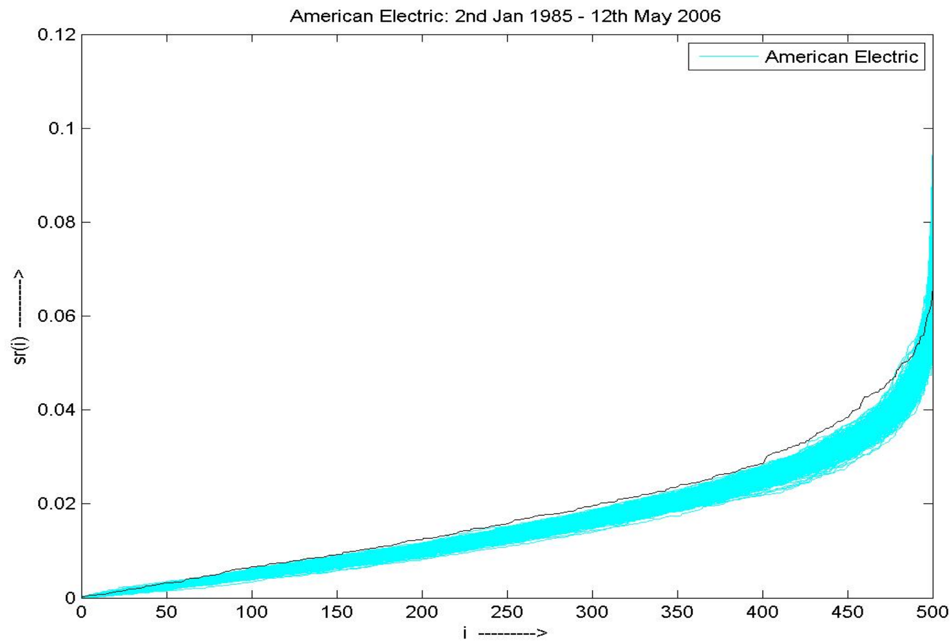


Figure 2.17: Correlation Band-American Electric Alternate days

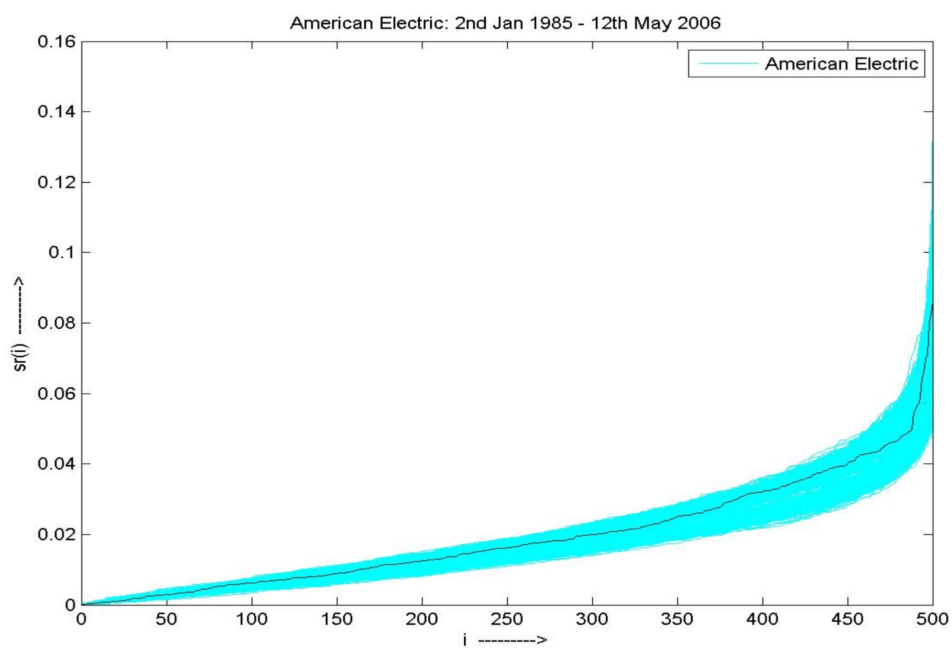


Figure 2.18: Correlation Band-American Electric: Every third day

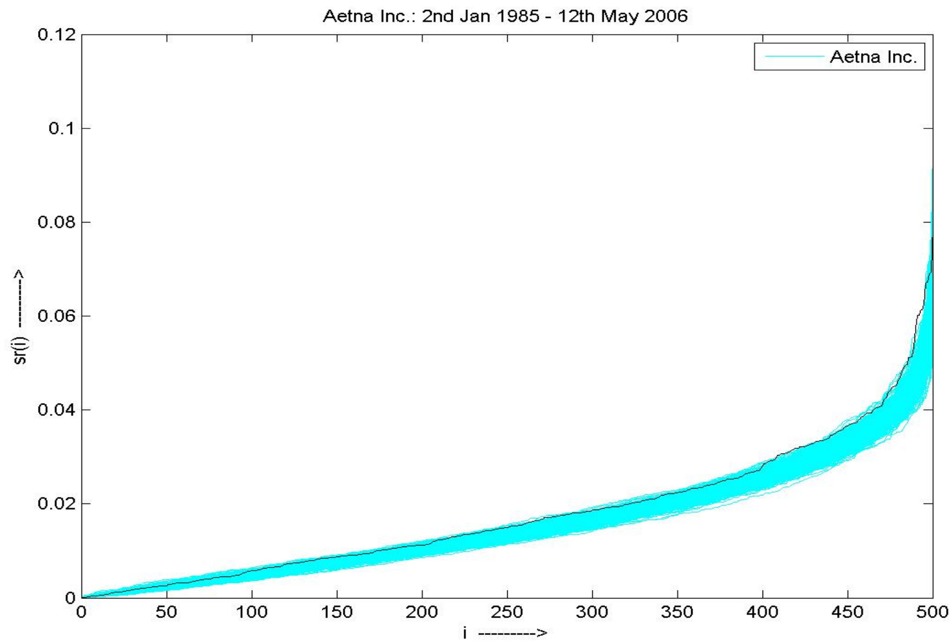


Figure 2.19: Correlation Band-Aetna Inc Alternate days

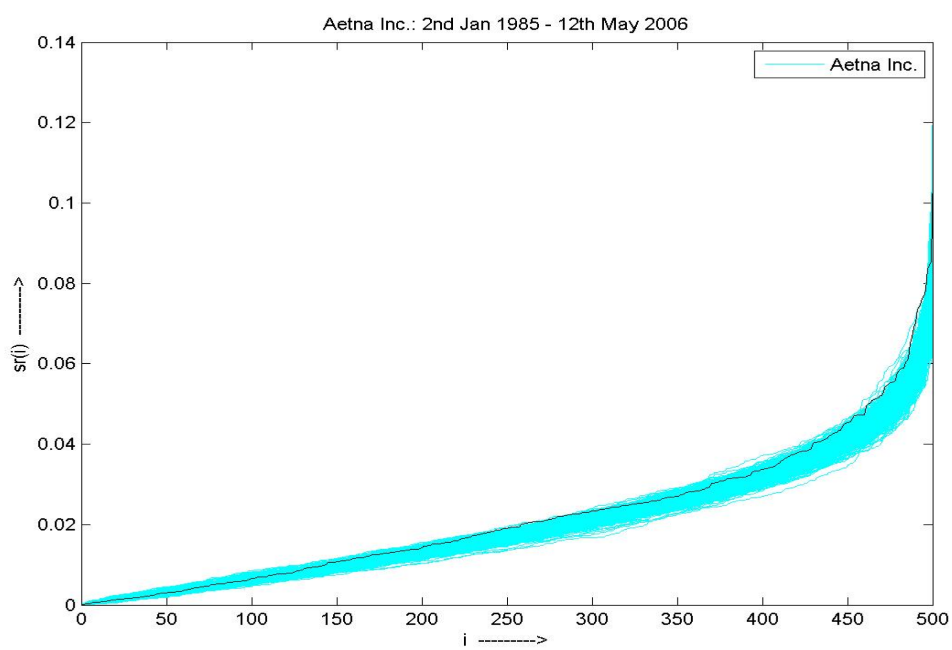


Figure 2.20: Correlation Band-Aetna Inc: Every third day

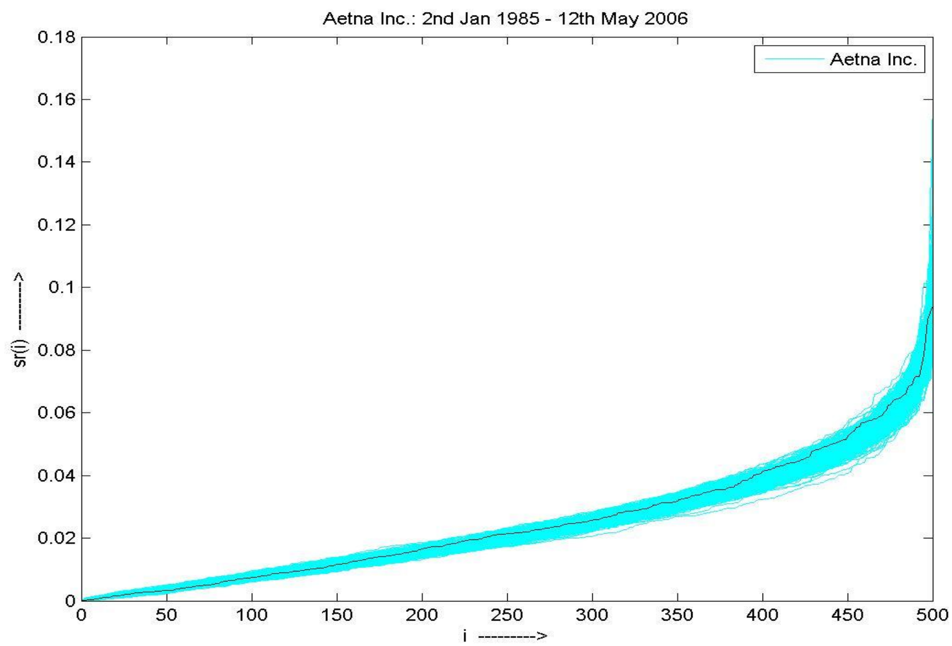


Figure 2.21: Correlation Band-Aetna Inc: Every fourth day

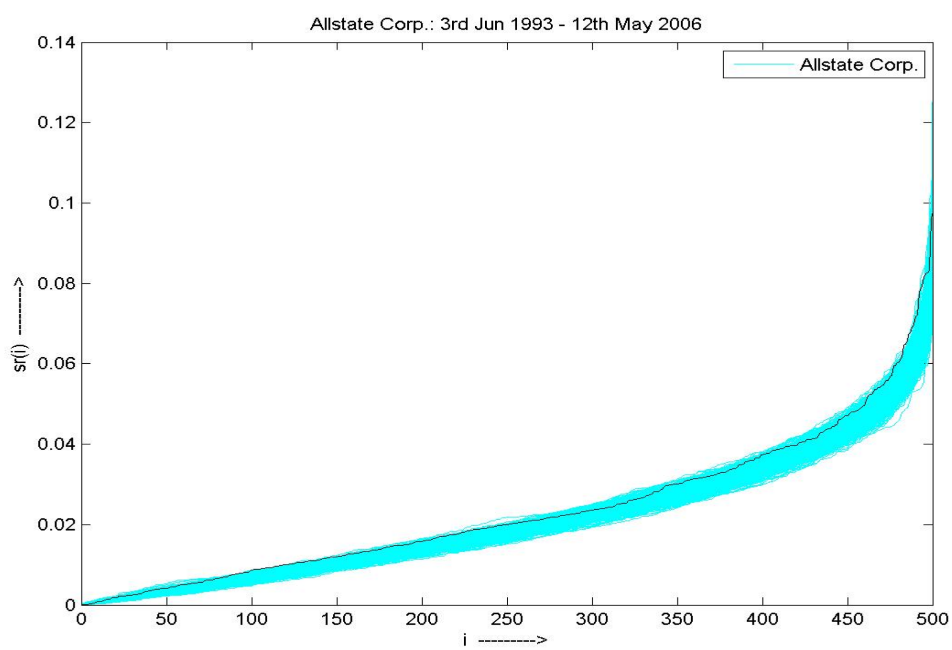


Figure 2.22: Correlation Band-Allstate Inc Alternate days

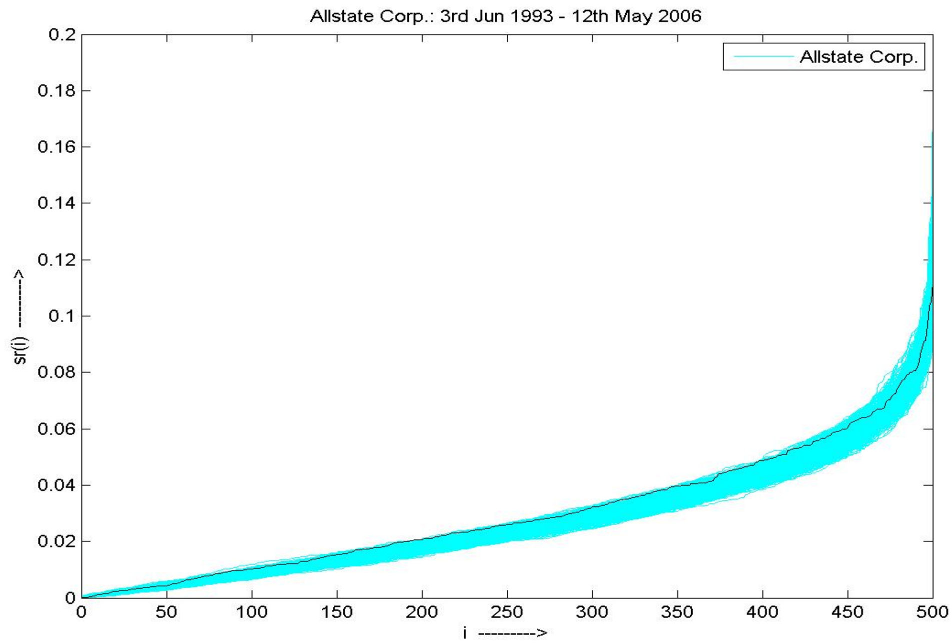


Figure 2.23: Correlation Band-Allstate Inc: Every third day

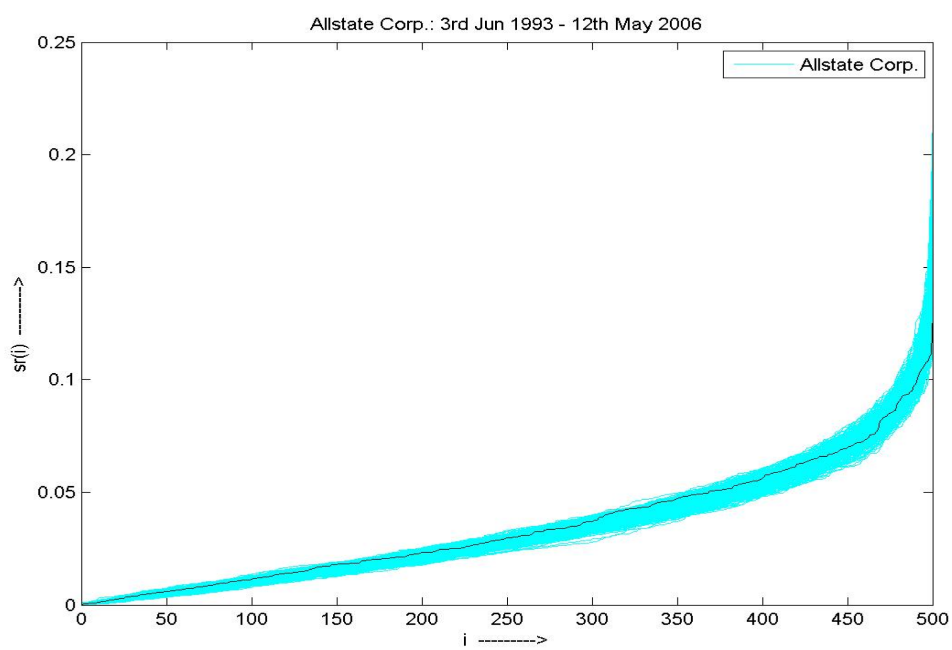


Figure 2.24: Correlation Band-Allstate Inc: Every fourth day

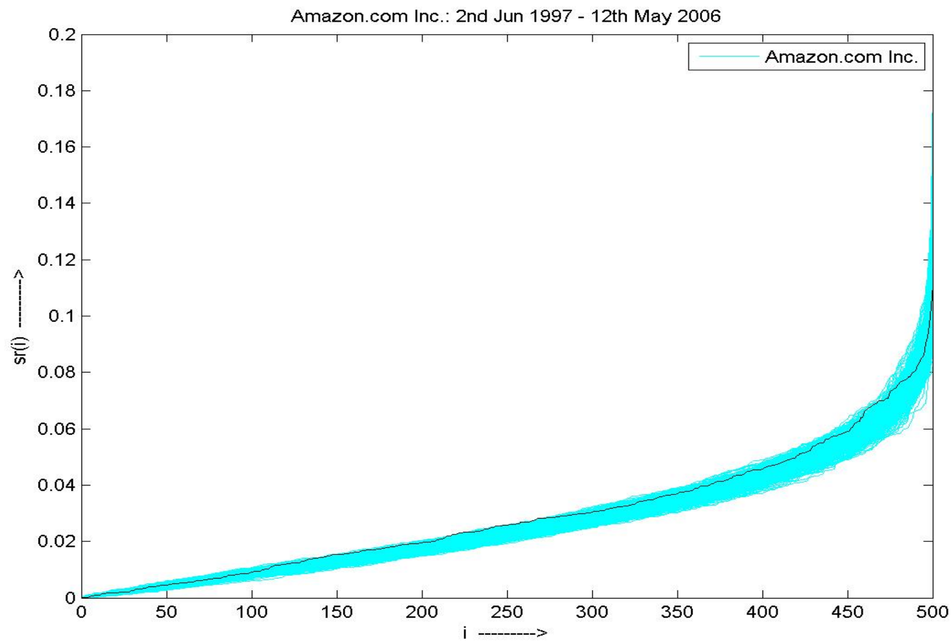


Figure 2.25: Correlation Band-Amazon.com Inc: Alternate days

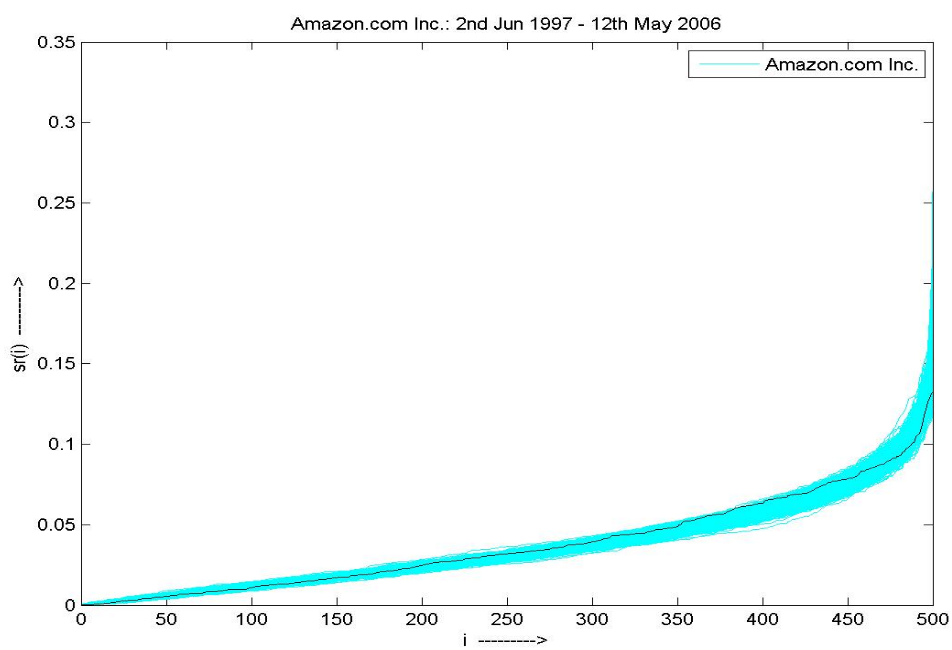


Figure 2.26: Correlation Band-Amazon.com Inc: Every third day

In all the above plots, as was mentioned in section 2, 5% of the correlation curves which had the highest value of the average (given by equation 2.8) were deleted from the correlation band. When every third day was considered, the correlation curve came well within the correlation band for most of the above stock returns data. We also considered the Bachelier model for which similar results were obtained except that the null hypothesis of independent increments was not accepted until when every fourth day was considered, i.e., The correlation curve came within the correlation band when every fourth day was considered. We then carried out similar analysis for the weekly data and found that the weekly stock data returns are not independent. But when alternate weeks were considered, the dependence vanished and the corresponding correlation curve fell within the correlation band.

2.4 Conclusions and Discussions

In this chapter, we applied our correlation band methodology to various stocks enlisted on New York Stock Exchange to detect any long-term dependence in the stock returns. The hypothesis of independent increments is rejected for all of the stocks. The methodology revealed that daily or weekly stock returns are weakly dependent and it detected little evidence of long-range dependence. The test methodology produced similar results under both Black-Scholes and Bachelier models proving it to be less model dependent.

Chapter 3

Statistical Assets and Derived Financial Contracts

3.1 Introduction

This chapter deals with a set of stock options called the Statistical options, first introduced by Ramprasath and Singh (2007). In their paper, a new class of European-type options, named statistical options, was developed to protect the buyer of a call option against a sudden drop in the stock price and the buyer of a put option against sudden rise in the stock price. This chapter extends the idea to American type options. For the sake of completeness, we next describe some of the basic concepts related to stock options.

A Stock option is a financial contract between a financial house and the buyer of the option. The financial house issues the contracts for a price. By selling the option to the buyer, the financial house enables the buyer to execute the option on a future date. As the name suggests, an option holder has the right to but is not obligated to execute the option. To consider a specific example, a European call option on a stock enables the buyer to purchase a share of the stock on a predetermined future date, at a predetermined price. This future date is called the exercise date and the predetermined price is called the strike price. The duration between the time of purchase of the option and the exercise date is called the lifetime of the option. A European type put option enables the buyer to sell the underlying stock on the exercise date at a predetermined strike price. Now if the stock price on the exercise date is more than the strike price, the call option buyer can execute the option at the strike price and sell it at the current market price to gain the difference of these two prices. On the other hand, if the stock price falls below the strike price, the call option becomes worthless. The net worth of an option on the exercise date is called the 'payoff' of the option. Most often, these options are available on a wide range of exercise dates and strike prices so as to cater to the diverse interests of the buyers. The price a buyer pays for an option varies with both the exercise time and the strike price. Determining the prices of various options is a major challenge for the subject of finance. Black and Scholes (1973) first derived option price formulae based on the 'no arbitrage' principle which says that a "fair" price would not allow the financial house or the investor to make risk free profits.

Apart from the European type options, there is a variety of stock options available to the investor. Most popular of them are the American type options. An American type option lets the investor exercise the option at any time point during the lifetime of the option. So, a holder of an American type call option has the right to buy the stock at any time during the lifetime of the option, paying the strike price. There is another popular class of options called the Asian options. An Asian call option pays the investor the average price of the stock during its lifetime. Since the payoff of Asian options depends on all the stock prices during the lifetime of the option, these are called 'path dependent' options. The statistical options proposed by Ramprasath and Singh (2006) are also path dependent. In this chapter, we propose extensions of these options to the American type and also deduce the fair prices to these American type options using 'no arbitrage' arguments. To put our work in perspective, we will revisit the concepts of Statistical options in section 2. Section 3 introduces the framework to our results and the results are stated in section 4. Some of the proofs for the results in section 4 are given in Appendix A.

3.2 Review and Motivation

Let $X(t)$ denote the share price at time t, $0 \leq t \leq T$. Let us assume the well known Black-Scholes model for the stock price, which is given by the stochastic differential equation(SDE),

$$dX(t) = X(t)(adt + \sigma dW(t)),\ 0 \leq t \leq T. \tag{3.1}$$

where $a \in \mathbb{R}$, $\sigma > 0$ and W(t) is a standard Brownian motion over the interval [0,T].

This above SDE is satisfied by the process,

$$X(t) = X(0)\exp(\mu t + \sigma W(t)),\ 0 \leq t \leq T, \tag{3.2}$$

where μ is given by $a - \sigma^2/2$ and $X(0)$ is the share value at time 0. In this model, μ is called the drift parameter and σ is called the volatility parameter.

We notice that $\mu t + \sigma W(t)$ has independent and identically distributed increments. For any $t > 0$ such that nt is an integer, if we break up the time period [0,t] into nt many equal time intervals, we can write $\mu t + \sigma W(t)$ as the sum

$$\mu t + \sigma W(t) = \delta_1 + \delta_2 + \ldots + \delta_{nt}, \tag{3.3}$$

where

$$\delta_i = \log(X(i/n)) - \log(X((i-1)/n)),\ i = 1, 2, \ldots, nt. \tag{3.4}$$

Notice that δ_i are IID normal variables with mean μ/n and variance σ^2/n. The sum in the right hand side of (3.3) can be viewed as nt times the average of nt many IID normal variables. Notice that if nT is an integer, we can replace the usual payoff $(X(T) - K)^+$ by $(X(0)\exp(nT.median(\delta_1, \delta_2, ..., \delta_{nT})) - K)^+$ so that the investor is protected against a sudden fall in the stock price.

This idea of using the median to protect the investor can be extended to other robust estimators of location. Thus, we can replace the mean by median, Hodges-Lehmann estimator or trimmed mean, just to name a few. The options thus obtained, by replacing the mean by a robust location estimator, are called statistical options. To emphasize the usefulness of Statistical options a little more, we plotted the sample path of a geometric Brownian motion along with the corresponding sample paths obtained by replacing the mean by the robust location estimators Median, HL estimator and the trimmed mean. These plots are shown in the figure below. These sample paths are simulated using a drift parameter $\mu = 0.25$ and a volatility parameter $\sigma = 0.6$ for a one year time period. The X-axis in each of these plots denotes time in days. Notice that, the median path always stayed above -0.2 while the actual Brownian path takes a minimum of -0.38. Also, the median path stayed above 0.2 for many more days than did the Brownian path. The HL and the trimmed mean paths follow the Brownian path more closely.

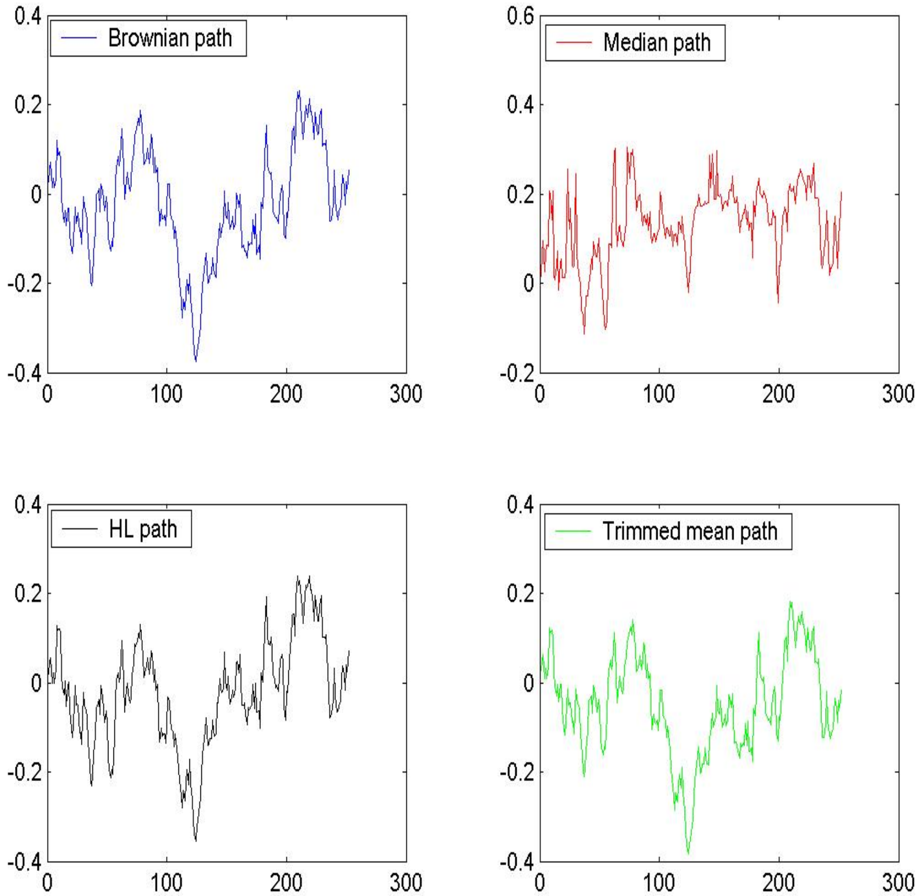


Figure 3.1: Simulated sample paths

We observe that in the case of an American call option, if the payoff at time t, $(X(t) - K)^+$ is replaced by $(X(0)\exp(nt.median(\delta_1, \delta_2, ..., \delta_{nt})) - K)^+$, the investor is still protected against a sudden fall in the stock price. Having observed this, we note that the use of robust estimators can be extended to the American options, thus allowing the investor to exercise the option at any time during its lifetime as well as protecting the investor from any sudden drops in the share price. Now if nt is not an integer, we can extend the payoff by linearly interpolating the median process. Hence, when nt is not an integer, the payoff is, for a suitable α,

$$(X(0)e^{\{\alpha([nt]+1)\text{median}(\delta_1,\delta_2,...,\delta_{[nt]+1})+(1-\alpha)[nt]\text{median}(\delta_1,\delta_2,...,\delta_{[nt]})\}} - K)^+. \qquad (3.5)$$

Once a suitable payoff function is chosen, the technical task is to price the stock option which gives the chosen payoff. Pricing European type statistical options was discussed in Ramprasath and Singh(2007). The pricing of American type Statistical options is discussed in the section 4. Section 5 extends the results stated in section 4 under a jump-diffusion model. The following section provides the framework that leads to the main results, which are stated in section 4.

3.3 Framework

For any $n \geq 1$, let us denote by $T_n = T_n(Z_1,\ Z_2, \ldots, Z_n)$, a statistic based on the IID variables $Z_1,\ Z_2, \ldots, Z_n$. Assume that T_n is affine equivariant in its arguments, i.e,

$$T_n(aZ_1 + b, aZ_2 + b, \ldots, aZ_n + b) = aT_n(Z_1, Z_2, \ldots, Z_n) + b, \qquad (3.6)$$

for any two real numbers a and b. Now, let us further assume that

$$\sqrt{n}(T_n - \theta) \to N(0, \nu^2). \qquad (3.7)$$

for some real θ and a $\nu^2 > 0$.

Also, define $T_0 = \theta$. Examples of T_n that are of interest to us include

Median Estimator

$$M_n = \text{median}(Z_1, Z_2, \ldots, Z_n), \tag{3.8}$$

Hodges-Lehmann Estimator

$$HL_n = \text{median}\{(Z_i + Z_j)/2 : i, j = 1, 2, \ldots, n.\}, \tag{3.9}$$

For an α in (0,1), let us define

Two-sided Trimmed mean Estimator

$$TR_n = \frac{1}{n - 2[n\alpha]} \sum_{i=[n\alpha]+1}^{n-[n\alpha]} Z_{(i)}, \tag{3.10}$$

One-sided Trimmed mean Estimators

Lower-sided Trimmed mean:

$$LTR_n = \frac{1}{n - [n\alpha]} \sum_{i=[n\alpha]+1}^{n} Z_{(i)}, \tag{3.11}$$

Upper-sided Trimmed mean:

$$UTR_n = \frac{1}{n - [n\alpha]} \sum_{i=1}^{n-[n\alpha]} Z_{(i)}, \tag{3.12}$$

where $Z_{(i)}$ is the i^{th} order statistic in $\{Z_1,Z_2,\ldots,Z_n\}$. It is easy to verify that HL_n,TR_n,LTR_n and UTR_n are all affine equivariant, thus satisfying the condition (3.6).

Let us assume that Z_1, Z_2,...,Z_n are IID N(0,1) variables, and from now on in this chapter, $T_n(Z_1,Z_2,\ldots,Z_n)$ is simply written as T_n. Now recall that

$$\sqrt{n}M_n \to N(0, \pi/2). \tag{3.13}$$

$$\sqrt{n}HL_n \to N(0, \pi/3), \tag{3.14}$$

$$\sqrt{n}TR_n \to N(0, f_\alpha), \tag{3.15}$$

$$\sqrt{n}\,(LTR_n - a) \to N(0, f_\alpha^{'}), \tag{3.16}$$

and

$$\sqrt{n}\,(UTR_n + a) \to N(0, f_\alpha^{'}), \tag{3.17}$$

where

$$a = \frac{\phi(\xi_\alpha)}{(1-\alpha)},$$

$$f_\alpha = \frac{1}{1-2\alpha} + \frac{2\xi_\alpha(\phi(\xi_\alpha) + \alpha\xi_\alpha)}{(1-2\alpha)^2} \text{ and}$$

$$f_\alpha^{'} = \frac{\alpha\xi_\alpha^2}{1-\alpha} + \frac{1}{1-\alpha} - \frac{\phi^2(\xi_\alpha)}{(1-\alpha)^2} + \frac{(1-2\alpha)\xi_\alpha\phi(\xi_\alpha)}{(1-\alpha)^2} - \frac{\alpha^2\xi_\alpha^2}{(1-\alpha)^2}.$$

Hence, M_n, HL_n, TR_n, LTR_n and UTR_n are all examples of statistics T_n satisfying (3.6) and (3.7).

For every t in [0,T] and every $n \geq 1$, let us define by $T_n(t)$ the process,

$$T_n(t) = X(0)\exp\left\{\alpha([nt]+1)T_{[nt]+1}(\delta_1, \delta_2, \ldots, \delta_{[nt]+1}) + (1-\alpha)[nt]T_{[nt]}(\delta_1, \delta_2, \ldots, \delta_{[nt]})\right\}, \tag{3.18}$$

where T_n is a statistic satisfying (3.6) and (3.7). For notational convenience, the same symbol T_n is used for both the statistic as well as the process. Since the statistic T_n has the vector $(\delta_1, \delta_2, \ldots, \delta_n)$ as its argument and the process T_n has the time variable t as its argument, the interpretation is self evident from the context.

Let us also define,

$$M_n(t) = X(0)\exp\left\{\alpha([nt]+1)M_{[nt]+1}(\delta_1, \delta_2, \ldots, \delta_{[nt]+1}) + (1-\alpha)[nt]M_{[nt]}(\delta_1, \delta_2, \ldots, \delta_{[nt]})\right\}, \tag{3.19}$$

$$HL_n(t) = X(0)\exp\left\{\alpha([nt]+1)HL_{[nt]+1}(\delta_1, \delta_2, \ldots, \delta_{[nt]+1}) + (1-\alpha)[nt]HL_{[nt]}(\delta_1, \delta_2, \ldots, \delta_{[nt]})\right\}, \tag{3.20}$$

$$TR_n(t) = X(0)\exp\left\{\alpha([nt]+1)TR_{[nt]+1}(\delta_1, \delta_2, \ldots, \delta_{[nt]+1}) + (1-\alpha)[nt]TR_{[nt]}(\delta_1, \delta_2, \ldots, \delta_{[nt]})\right\}. \tag{3.21}$$

and let us also define $LTR_n(t)$ and $UTR_n(t)$ in a similar fashion.

3.4 Results

The Theorem below is stated under the following two conditions.

Condition A1: Let g be a real function such that, for any $k > 0$, there exists a positive constant $c(k)$ such that

$$P(|T_n - \theta - \frac{1}{n}\sum_{i=1}^{n} g(Z_i)| > c(k)a_n) = O(n^{-k}),$$

where $a_n = o(\frac{1}{\sqrt{n}})$. Also assume that $E(g(Z_1)) = 0$ and $\mathrm{var}(g(Z_1))=\nu^2 < \infty$.

Condition A2:

$$max_{1\leq k\leq n}|T_k - \theta| = O_P(\log n).$$

Theorem 1: Under conditions A1 and A2, the process $T_n(t)/\exp(\theta\sigma\sqrt{n}t)$ converges weakly to a process $T(t)$ satisfying the stochastic differential equation

$$\frac{dT(t)}{T(t)} = \mu^{'} dt + \sigma\nu dW(t), \tag{3.22}$$

where $\mu^{'} = \mu + \sigma^2\nu^2/2$, with the initial condition $T(0) = X(0)$.

proof: Let us define,

$$\xi_n(t) = \frac{\sqrt{t}}{\nu}\left[\alpha\sqrt{[nt]+1}\left(T_{[nt]+1} - \theta\right) + (1-\alpha)\sqrt{[nt]}\left(T_{[nt]} - \theta\right)\right] \tag{3.23}$$

and

$$\eta_n(t) = \frac{\sqrt{t}}{\nu}\left[\frac{\alpha}{\sqrt{[nt]+1}}\sum_{i=1}^{[nt]+1} g(Z_i) + \frac{(1-\alpha)}{\sqrt{[nt]}}\sum_{i=1}^{[nt]} g(Z_i)\right]. \tag{3.24}$$

We will first show that as n tends to ∞, $sup_{0\leq t\leq T}|\xi_n(t) - \eta_n(t)|$ converges to zero in probability. But since, $\eta_n(t)$ converges in law to a standard Brownian motion by the classical functional CLT, it immediately follows that

$$\xi_n(t) \to W(t), 0 \le t \le T \qquad \text{as } n \to \infty. \tag{3.25}$$

Observe that, we can write $|\xi_n(t) - \eta_n(t)|$ as

$\frac{\sqrt{t}}{\nu}\left|\alpha\sqrt{[nt]+1}\left[T_{[nt]+1} - \theta - \frac{1}{[nt]+1}\sum_{i=1}^{[nt]+1} g(Z_i)\right] + (1-\alpha)\sqrt{[nt]}\left[T_{[nt]} - \theta - \frac{1}{[nt]}\sum_{i=1}^{[nt]} g(Z_i)\right]\right|$

and hence,

$|\xi_n(t) - \eta_n(t)|$

$$\le \frac{\sqrt{t}}{\nu} max\left\{\sqrt{[nt]+1}\left|T_{[nt]+1} - \theta - \frac{1}{[nt]+1}\sum_{i=1}^{[nt]+1} g(Z_i)\right|, \sqrt{[nt]}\left|T_{[nt]} - \theta - \frac{1}{[nt]}\sum_{i=1}^{[nt]} g(Z_i)\right|\right\}. \tag{3.26}$$

Now, we break up $sup_{0 \le t \le T}|\xi_n(t) - \eta_n(t)|$ into two parts using the equality

$$sup_{o \le t \le T}|\xi_n(t) - \eta_n(t)| = max\left\{sup_{o \le t < n^{\delta-1}}|\xi_n(t) - \eta_n(t)|, sup_{n^{\delta-1} \le t \le T}|\xi_n(t) - \eta_n(t)|\right\}. \tag{3.27}$$

Using equation (3.26),we have,

$$sup_{n^{\delta-1} \le t < T}|\xi_n(t) - \eta_n(t)| \le \frac{\sqrt{T}}{\nu} max_{n^\delta \le l \le nT}\sqrt{l}\left|T_l - \theta - \frac{1}{l}\sum_{i=1}^{l} g(Z_i)\right|. \tag{3.28}$$

Now notice that, using condition A1,

$P\left(max_{n^\delta \le l \le nT}\sqrt{l}|T_l - \theta - \frac{1}{l}\sum_{i=1}^{k} g(Z_i)| > c(k)\sqrt{n^\delta}a_{n^\delta}\right)$

$\le \sum_{l=n^\delta}^{nT} P\left(\sqrt{l}|T_l - \theta - \frac{1}{l}\sum_{i=1}^{k} g(Z_i)| > c(k)\sqrt{l}a_l\right)$

$\le (nT - n^\delta + 1)(n^\delta)^{-k} \to 0$ if $k > 1/\delta$ as $n \to \infty$ and that, $c(k)\sqrt{n^\delta}a_{n^\delta} \to 0$ as $n \to \infty$.

Combining this with equation (3.28) we have,

$sup_{n^{\delta-1} \le t < T}|\xi_n(t) - \eta_n(t)| \to 0$ in Probability as $n \to \infty$. Now, observe that,

$sup_{0 \le t \le n^{\delta-1}}|\xi_n(t) - \eta_n(t)| \le \frac{(n^{\delta-1})^{\frac{1}{2}}}{\nu} max_{1 \le l \le n^\delta}\sqrt{l}|T_l - \theta - \frac{1}{l}\sum_{i=1}^{l} g(Z_i)|$

which yields,

$sup_{0 \le t \le n^{\delta-1}}|\xi_n(t) - \eta_n(t)|$

$\leq \frac{(n^{\delta-1})^{\frac{1}{2}}}{\nu} max_{1\leq l\leq n^{\delta}} \sqrt{l}|T_l-\theta| + \frac{(n^{\delta-1})^{\frac{1}{2}}}{\nu} max_{1\leq l\leq n^{\delta}} \sqrt{l}|\sum_{i=1}^{l} g(Z_i)|$

$\leq C\delta \frac{(n^{\delta-1})^{\frac{1}{2}}}{\nu} n^{\delta/2} max_{1\leq l\leq n^{\delta}}|T_l-\theta| + \frac{(n^{\delta-1})^{\frac{1}{2}}}{\nu} n^{\delta/2} max_{1\leq l\leq n^{\delta}} \sum_{i=1}^{l}|g(Z_i)|$

$\leq C\delta \frac{(n^{\delta-1})^{\frac{1}{2}}}{\nu} n^{\delta/2} max_{1\leq l\leq n^{\delta}}|T_l-\theta| + \frac{(n^{\delta-1})^{\frac{1}{2}}}{\nu} n^{\delta/2} n^{\delta} max_{1\leq l\leq n^{\delta}}|g(Z_i)|$

$\leq C\delta \frac{(n^{\delta-1})^{\frac{1}{2}}}{\nu} n^{\delta/2}\log n + \frac{(n^{\delta-1})^{\frac{1}{2}}}{\nu} n^{2\delta}$ by condition A2 and the fact that $max_{1\leq i\leq n^{\delta}}|g(Z_i)| = O(n^{\delta/2})$ under the condition that $E(g^2(Z_1)) < \infty$.

Note that for any choice of $\alpha < 1/5$, the above expression converges to zero, as $n \to \infty$. Therefore,

$sup_{0\leq t\leq T}|\xi_n(t)-\eta_n(t)| = max\{\sup_{0\leq t\leq n^{\delta-1}}|\xi_n(t)-\eta_n(t)|, sup_{n^{\delta-1}\leq t\leq 1}|\xi_n(t)-\eta_n(t)|\}$ converges to zero as $n \to \infty$.

Now, by definition, and using the equivariance property of T_n, we have,

$$T_n(t) = X(0)\exp\left\{\mu\left(\frac{\alpha([nt]+1)+(1-\alpha)[nt]}{n}\right) + \sigma\left(\frac{\alpha([nt]+1)}{\sqrt{n}}T_{[nt]+1} + \frac{(1-\alpha)[nt]}{\sqrt{n}}T_{[nt]})\right)\right\}.$$

We will now proceed to show that $T_n(t)$ converges to a geometric Brownian motion, as claimed in part a). Note that the function $\frac{\alpha([nt]+1)+(1-\alpha)[nt]}{n}$ is bounded by T and it converges to t as n tends to ∞. On the other hand,
$\frac{\alpha([nt]+1)}{\sqrt{n}}T_{[nt]+1} + \frac{(1-\alpha)[nt]}{\sqrt{n}}T_{[nt]}$ can be written as

$$\left\{\frac{\alpha([nt]+1)}{\sqrt{n}}(T_{[nt]+1}-\theta) + \frac{(1-\alpha)[nt]}{\sqrt{n}}(T_{[nt]}-\theta)\right\} + \frac{\theta([nt]+\alpha)}{\sqrt{n}}. \tag{3.29}$$

In the above expression, the term in the braces can now be written as

$$\nu\xi_n(t) + \alpha\left(\frac{[nt]+1}{\sqrt{n}} - \sqrt{t}\sqrt{[nt]+1}\right)(T_{[nt]+1}-\theta) + (1-\alpha)\left(\frac{[nt]}{\sqrt{n}} - \sqrt{t}\sqrt{[nt]}\right)(T_{[nt]}-\theta).$$

(i) Note that if $nt < 1$,

$$\alpha\left(\frac{[nt]+1}{\sqrt{n}} - \sqrt{t}\sqrt{[nt]+1}\right)(T_{[nt]+1}-\theta) + (1-\alpha)\left(\frac{[nt]}{\sqrt{n}} - \sqrt{t}\sqrt{[nt]}\right)(T_{[nt]}-\theta)$$

reduces to $\alpha\left(\frac{1}{\sqrt{n}} - \sqrt{t}\right)(T_1(Z_1)-\theta)$ which is bounded by $\frac{2(T_1(Z_1)-\theta)}{\sqrt{n}}$ and hence uniformly converges to zero in probability as $n \to \infty$ and that

(ii) if $nt > 1$,

$$sup_{t>1/n}\left|\alpha\left(\frac{[nt]+1}{\sqrt{n}}-\sqrt{t}\sqrt{[nt]+1}\right)(T_{[nt]+1}-\theta)+(1-\alpha)\left(\frac{[nt]}{\sqrt{n}}-\sqrt{t}\sqrt{[nt]}\right)(T_{[nt]}-\theta)\right|$$

$$= sup_{t>1/n}\left|\alpha\frac{\sqrt{[nt]+1}}{\sqrt{n}}\frac{([nt]+1-nt)}{\sqrt{[nt]+1}+\sqrt{nt}}(T_{[nt]+1}-\theta)+(1-\alpha)\frac{\sqrt{[nt]}}{\sqrt{n}}\frac{([nt]-nt)}{\sqrt{[nt]}+\sqrt{nt}}(T_{[nt]}-\theta)\right|$$
$$\leq max_{1\leq k\leq nT}\left(\alpha\left|\frac{T_{k+1}-\theta}{\sqrt{n}}\right|+(1-\alpha)\left|\frac{T_k-\theta}{\sqrt{n}}\right|\right)$$
$$\leq max_{1\leq k\leq nT}\frac{|T_k-\theta|}{\sqrt{n}}$$

which converges to zero in probability by condition A2. Finally, notice that the difference between the second term in the equation (3.29) and $\theta\sqrt{nt}$ converges to zero in probability; i.e,

$$\frac{\theta([nt]+\alpha)}{\sqrt{n}}-\theta\sqrt{n}t = o_P(1).$$

Now the claim follows from the equation (3.25) and from the above observations (i) and (ii). **QED**.

In the following theorem we will extend the previous result to the multidimensional case. To state the result, let us assume that T_{n1},T_{n2},. . .,T_{nm} are statistics defined on the same probability space, satisfying the properties given by equations (3.6) and (3.7). Also, define the processes, $T_{n1}(t)$,$T_{n2}(t)$,. . .,$T_{nm}(t)$ as before, i.e,

$$T_{ni}(t) = X(0)\exp\left\{\alpha([nt]+1)T_{([nt]+1)i}(\delta_1,\delta_2,\ldots,\delta_{[nt]+1})+(1-\alpha)[nt]T_{([nt])i}(\delta_1,\delta_2,\ldots,\delta_{[nt]})\right\} \tag{3.30}$$

for each $i = 1, 2, \ldots, m$. Let us further assume the following three conditions.

Condition B1: For each i=1,2,...,n, there is a real function g_i such that, for any $k > 0$, there exists a real positive constant c_i satisfying

$$P(|T_{ni}-\theta_i-\frac{1}{n}\sum_{j=1}^{n}g_i(Z_j)| > c_i a_{ni}) = O(n^{-k}),$$

where $a_{ni} = o(\frac{1}{\sqrt{n}})$.

Condition B2: For each $i = 1, 2, \ldots, m$, $E(g_i(Z_1)) = 0$ and $Cov(g_i(Z_1), g_j(Z_1)) = \gamma_{ij}$ for each $i, j = 1, 2, \ldots, m$.

Condition B3: For each $i = 1, 2, \ldots, m$ we have

$$max_{1\leq k\leq n}|T_{ki}-\theta_i| = O_P(\log n).$$

Theorem 2: Under the above conditions B1-B3, the $\mathbb{R}^m$-valued process

$$(T_{n1}(t)/\exp(\theta_1\sigma\sqrt{n}t), T_{n2}(t)/\exp(\theta_2\sigma\sqrt{n}t), \ldots, T_{nm}(t)/\exp(\theta_m\sigma\sqrt{n}t))$$

converges weakly to a process satisfying the stochastic differential equation

$$\frac{\mathbf{dT(t)}}{\mathbf{T(t)}} = \boldsymbol{\mu}' dt + \sigma\Gamma^{1/2} d\mathbf{W(t)}, \tag{3.31}$$

with the initial condition $(T_1(0), T_2(0), \ldots, T_m(0)) = (X(0), X(0), \ldots, X(0))$ where $\mathbf{T(t)}$ is the process $(T_1(t), T_2(t), \ldots, T_m(t))$, $\frac{\mathbf{dT(t)}}{\mathbf{T(t)}}$ stands for $\left[\frac{dT_1(t)}{T_1(t)}, \frac{dT_2(t)}{T_2(t)}, \ldots, \frac{dT_m(t)}{T_m(t)}\right]$, $\boldsymbol{\mu}'$ is the m-dimensional vector $(\mu + \sigma^2\gamma_{11}/2, \mu + \sigma^2\gamma_{22}/2, \ldots, \mu + \sigma^2\gamma_{mm}/2)$, Γ is the m$\times$m matrix whose $(i,j)^{th}$ entry equals γ_{ij} and $\mathbf{W(t)}$ is the m-dimensional standard Brownian motion process $(W_1(t), W_2(t), \ldots, W_m(t))$.

proof: Let $\Theta = (\theta_1, \theta_2, \ldots, \theta_m)$. It follows from theorem 1 that for every i, $T_{ni}(t)/e^{\theta_i\sigma\sqrt{n}t}$ converges weakly to a geometric Brownian motion with drift parameter μ and volatility parameter γ_{ii}. It is easy to see that, $(\log(T_{n1}(t)), \log(T_{n2}(t)), \ldots, \log(T_{nm}(t))) - \sigma\sqrt{n}t\Theta$ converges weakly to a multivariate gaussian process. Since a gaussian process is well-defined by its mean and covariance functions it suffices to identify the asymptotic mean and covariance functions to determine the limiting process. The means are already given by Theorem 1. It remains to figure out the covariance structure. Notice that it follows from condition B1 that, for each $i = 1, 2, \ldots, m$,

$$T_{ni} = \theta_i + \frac{1}{n}\sum_{j=1}^{n} g_i(Z_j) + o_P(\frac{1}{\sqrt{n}}). \tag{3.32}$$

It then follows from the definition of $T_{ni}(t)$ and the above observation that

$$log(T_{ni}(t)) = \theta_i\sigma\sqrt{n}t + \frac{1}{\sqrt{n}}\sum_{j=1}^{[nt]} g_i(Z_j) + o_P(1). \tag{3.33}$$

Therefore, $(\log(T_{n1}(t)), \log(T_{n2}(t)), \ldots, \log(T_{nm}(t)))$ has the same asymptotic covariance structure as that of

$$\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{[nt]} g_1(Z_j), \frac{1}{\sqrt{n}}\sum_{j=1}^{[nt]} g_2(Z_j), \ldots, \frac{1}{\sqrt{n}}\sum_{j=1}^{[nt]} g_m(Z_j)\right)$$

and the result immediately follows from condition B2. **QED**

The rest of this section describes applications of the above theorems to various robust location estimators. We will prove that Condition A1 and Condition A2 of Theorem 1 are satisfied for all the location estimators defined earlier. Assuming for now that M_n, HL_n,TR_n,LTR_n and UTR_n do satisfy these two conditions, note that Theorem 1 can be used to price the American type statistical options with payoff functions suggested by the empirical processes $T_n(t)$ for each of the above choices of the statistics T_n. For example, an American type call option with payoff $(M_n(t)-K)^+$ can be priced using the standard American option pricing methods that assume Black-Scholes model. From corollary 1 below, the process $M_n(t)$ converges in law to a geometric Brownian motion with mean parameter μ and volatility $\sigma\sqrt{\pi/2}$. We can use this asymptotic result to approximately price the statistical option with payoff $(M_n(t)-K)^+$ and the approximation is quite accurate for large n. The details of these pricing methods and numerical results are presented in chapter 4. We now state the following corollaries which enable us to price various types of Statistical options. The proofs to these results are given in Appendix A.

Corollary 1:

(a) The process $M_n(t)$, $0 \leq t \leq T$ converges weakly to a process $M(t)$ given by $M(t) = X(0)\exp(\mu t + \sigma\sqrt{\frac{\pi}{2}}W(t))$ as n $\rightarrow \infty$.

(b) As n $\rightarrow \infty$, ($M_n(t)$,X(t)) , $0 \leq t \leq T$ converges weakly to the stochastic process $\widetilde{X}(0)\exp\{(\mu t, \mu t) + \sigma(W_1(t) + c_1 W_2(t), W_1(t))\}$ where $\widetilde{X}(0) = (X(0), X(0))$, W_1,W_2 are two independent Brownian Motions and $c_1 = \sqrt{\frac{\pi}{2} - 1}$.

Corollary 2:

(a) The process $HL_n(t)$, $0 \leq t \leq T$ converges weakly to a process $HL(t)$ given by $HL(t) = X(0)\exp(\mu t + \sigma\sqrt{\frac{\pi}{3}}W(t))$ as n $\rightarrow \infty$.

b) As n $\rightarrow \infty$, ($HL_n(t)$,X(t)) , $0 \leq t \leq T$ converges weakly to the stochastic process $\widetilde{X}(0)\exp\{(\mu t, \mu t) + \sigma(W_1(t) + c_2 W_2(t), W_1(t))\}$ where $\widetilde{X}(0) = (X(0), X(0))$, W_1,W_2 are two independent Brownian Motions and $c_2 = \sqrt{\frac{\pi}{3} - 1}$.

Corollary 3:

(a) The process $TR_n(t)$, $0 \leq t \leq T$ converges weakly to a process $TR(t)$ given by $TR(t) = X(0)\exp(\mu t + \sigma\sqrt{f_\alpha}W(t))$ as n $\rightarrow \infty$.

(b) As n $\rightarrow \infty$, ($TR_n(t)$,X(t)) , $0 \leq t \leq T$ converges weakly to the stochastic process $\widetilde{X}(0)\exp\{(\mu t, \mu t) + \sigma(W_1(t) + c_3 W_2(t), W_1(t))\}$ where $\widetilde{X}(0) = (X(0), X(0))$, W_1,W_2 are two independent Brownian Motions and $c_3 = \sqrt{f_\alpha - 1}$.

Corollary 4:

(a) The processes $LTR_n(t)/\exp(a\sigma\sqrt{n}t)$, $0 \leq t \leq T$ and $UTR_n(t)/\exp(-a\sigma\sqrt{n}t)$, $0 \leq t \leq T$ converge weakly to the same process $LTR(t)$ given by $LTR(t) = X(0)\exp(\mu t + \sigma\sqrt{f'_\alpha}W(t))$ as n $\rightarrow \infty$.

(b) As n $\rightarrow \infty$, the two processes ($LTR_n(t)/\exp(a\sigma\sqrt{n}t)$,X(t)) and ($UTR_n(t)/\exp(-a\sigma\sqrt{n}t)$,X(t)) $0 \leq t \leq T$ converge weakly to the same process $\widetilde{X}(0)\exp\{(\mu t, \mu t) + \sigma(W_1(t) + c_4 W_2(t), W_1(t))\}$ where $\widetilde{X}(0) = (X(0), X(0))$, W_1,W_2 are two independent Brownian Motions and $c_4 = \sqrt{f'_\alpha - 1}$.

We end this section with two application of the above results. Suppose that an investor purchases an American type median call option having a one year life time at a strike price K. Suppose that the stock does very well during the period and the stock value appreciates considerably. In such a situation, the investor would probably have profited more by purchasing a usual American call option on the stock instead of buying the Statistical option. To enable the investor to profit as much as he/she would have profited by purchasing a usual American type option, we propose an option which pays the maximum of the two prices; i.e, an option with payoff function, $max(X(t) - K, M_n(t) - K, 0)$. This option could be priced using the asymptotic result given by corollary 1(b). Since the joint process $(M_n(t), X(t))$ converges to the known process given by proposition1, we can price the above option under the limiting process.

Note that the statistics M_n, HL_n and TR_n have asymptotic mean $\theta = 0$ while the statistics LTR_n and UTR_n have nonzero asymptotic means a and -a where $a = \frac{\phi(\xi_\alpha)}{1-\alpha}$. We will use this property of the one-sided trimmed means to define a set of Statistical options that benefit the investor. Notice that at any time point t, $LTR_n(t)$ is always more than $TR_n(t)$. This is due to the fact that a two sided trimmed mean deletes both the largest and smallest $100\alpha\%$ order statistics from the average while the lower-sided trimmed mean deletes only the smallest $100\alpha\%$ order statistics, thus contributing larger summands to the average. For this reason, an American type call option with payoff $(LTR_n(t) - K)^+$ is more beneficial to the investor than an American call option with payoff $(TR_n(t) - K)^+$. Hence, whenever there is a big rise in the stock price, the buyer of a call option still benefits from it; but when there is a large dip in the stock price, the buyer is guarded against the loss due to the decrease in the price. In a similar fashion, an American type put option with payoff $(K - UTR_n(t))^+$ lets the investor gain from any dips in the stock price and also guards the investor against any large rises in the price. Both these options could be approximately priced using corollary 4. The exact details of these pricing methods will be discussed in full length in chapter 4.

3.5 Limit Theorems under a Jump Diffusion model

The occurrence of jumps in the stock prices and other asset prices is a rather common phenomenon in finance. For better understanding, let us say that a jump occurs when there is a 'large' price change during a 'small' period of time. It is difficult to explain the occurrence of such jumps using the Black-Scholes model; particularly so when the jumps occur persistently over time. To encompass the possibility of occurrence of sudden jumps in the asset prices, Merton(1976) first proposed the Jump-Diffusion model. For the reader's convenience, we restate Merton's Jump Diffusion model below.

Jump-Diffusion model:

$$X(t) = X(0)\exp\left(\mu t + \sigma W(t) + \sum_{j=1}^{N(t)} Y_j\right), o \leq t \leq T \tag{3.34}$$

where $X(0)$ is the share price at time t=0, $\mu \in R$, $\sigma > 0$, W(t) is a standard Brownian motion over the interval [0,T] and N(t) is a Poisson process with rate λ and Y_j are IID random variables. The jumps in the stock prices are explained by the random variables Y_j. The Poisson process N(t) determines the time points at which the jumps occur and the rate λ determines the frequency of the jumps. Ramprasath and Singh (2007) have dealt with the pricing of European type Statistical options under the jump diffusion model. This section extends their results to the American type statistical options. We establish a limit theorem, which will enable us to price the American type Statistical options under the Jump Diffusion model. As before, let us define for any $n \geq 1$,

$$\delta_i^J = log(X(i/n)) - log(X((i-1)/n)), i = 1, 2, \ldots, nT. \tag{3.35}$$

Notice that $\delta_i^J, i = 1, 2, \ldots, nT$ are IID random variables and that

$$\delta_i^J = \delta_i + \sum_{j=1}^{N(i/n)} Y_j - \sum_{j=1}^{N((i-1)/n)} Y_j. \tag{3.36}$$

For any statistic T_n which satisfies the conditions (3.6) and (3.7), let us define,

$$T_n^J(\boldsymbol{\delta}) = T_n(\delta_1^J, \delta_2^J, \ldots, \delta_n^J). \tag{3.37}$$

Now also define, for any $0 \leq t \leq T$ and $n \geq 1$, the process

$$T_n^J(t) = X(0) \exp\left\{\alpha([nt]+1)T_{[nt]+1}^J(\boldsymbol{\delta}) + (1-\alpha)[nt]T_{[nt]}^J(\boldsymbol{\delta})\right\}. \tag{3.38}$$

From now on in this section, for convenience, we will simply write $T_n(\boldsymbol{\delta})$ for $T_n(\delta_1, \delta_2, \ldots, \delta_n)$ and $T_n^J(\boldsymbol{\delta})$ for $T_n^J(\delta_1^J, \delta_2^J, \ldots, \delta_n^J)$. Since N(t) is assumed to be a Poisson process, N(T) is finite with probability one and it can be bounded by an integer M with a large probability. So, there are only finitely many i's for which δ_i^J is different from δ_i. We will show that, when T_n is the median, Hodges-Lehmann estimator or the Trimmed mean, for large n, $T_n(\boldsymbol{\delta})$ can be approximated by $T_n^J(\boldsymbol{\delta})$ with a large probability.

This observation leads us to the following result. We need condition C1 to state the theorem.

Condition C1: For any $\beta > 0$,

$$P\left(n\left|T_n^J(\boldsymbol{\delta}) - T_n(\boldsymbol{\delta})\right| > \beta\right) = o\left(\frac{1}{n}\right). \tag{3.39}$$

Theorem 3: Under conditions A1, A2 and C1, the process $T_n^J(t)/\exp(\theta\sigma\sqrt{n}t)$ converges weakly to a process T(t) satisfying the stochastic differential equation

$$\frac{dT(t)}{T(t)} = \mu^{'} dt + \sigma\nu dW(t), \tag{3.40}$$

with the initial condition that T(0)=X(0).

proof:

Notice that, given $\epsilon > 0$, there exist a $\delta > 0$ and a positive integer M such that,

$$P(N(\delta) > 0) < \epsilon/2. \tag{3.41}$$

For this reason, we have,

$$P(sup_{0<k\leq n\delta} k|T_k(\boldsymbol{\delta}) - T_k^J(\boldsymbol{\delta})| > 0) \leq P(N(\delta) > 0) \leq \epsilon/2. \tag{3.42}$$

Now using condition C1, for any $\beta > 0$,

$$P\left(n\left|T_n(\boldsymbol{\delta}) - T_n^J(\boldsymbol{\delta})\right| > \beta\right) = o(\frac{1}{n}).$$

Note that for any $t \in (\delta, T]$, we have,

$$P\left(nt\left|T_{[nt]}(\boldsymbol{\delta}) - T_{[nt]}^J(\boldsymbol{\delta})\right| > \beta\right) = o(\frac{1}{n}), \tag{3.43}$$

which implies that,

$P\left(sup_{\delta\leq t\leq T}[nt]\left|T_{[nt]}(\boldsymbol{\delta}) - T_{[nt]}^J(\boldsymbol{\delta})\right| > \beta\right) \leq P\left(max_{n\delta\leq k\leq nT} k\left|T_k(\boldsymbol{\delta}) - T_k^J(\boldsymbol{\delta})\right| > \beta\right)$
$\leq (nT - n\delta + 1)o(\frac{1}{n\delta}) \to 0$ as $n \to \infty$.

This observation, combined with equation (3.42) gives

$$P(max_{0\leq k\leq nT} k\left|T_k(\boldsymbol{\delta}) - T_k^J(\boldsymbol{\delta})\right| > \beta) \to 0, \text{ as } n \to \infty. \tag{3.44}$$

This observation implies that, the processes $T_n^J(t)/\exp(\theta\sigma\sqrt{n}t)$ and $T_n(t)/\exp(\theta\sigma\sqrt{n}t)$ converge to the same process. Now it follows directly from Theorem 1 that $T_n^J(t)/\exp(\theta\sigma\sqrt{n}t)$ converges to a process satisfying the stochastic differential equation given by (3.40).

QED.

We now proceed to extend the above result to the multidimensional case. As in Theorem 3, for $i = 1, 2, \ldots, m$, let $T_{ni}(\boldsymbol{\delta})$ be statistics defined on the vector $(\delta_1, \delta_2, \ldots, \delta_n)$. Let $T_{ni}^J(\boldsymbol{\delta})$ be defined as

$$T_{ni}^J(\boldsymbol{\delta}) = T_{ni}(\delta_1^J, \delta_2^J, \ldots, \delta_n^J). \tag{3.45}$$

To state Theorem 4, we need a condition to ensure that the statistics $T_{ni}^J(\boldsymbol{\delta})$ can be approximated by $T_{ni}(\boldsymbol{\delta})$.

Condition D1: For any $\beta > 0$ and for each $i = 1, 2, \ldots, m$,

$$P\left(n\left|T^J_{ni}(\boldsymbol{\delta}) - T_{ni}(\boldsymbol{\delta})\right| > \beta\right) = o\left(\frac{1}{n}\right). \tag{3.46}$$

Theorem 4: Under conditions B1,B2,B3 and D1, the process

$$\left(T^J_{n1}(t)/\exp(\theta_1\sigma\sqrt{n}t), T^J_{n2}(t)/\exp(\theta_2\sigma\sqrt{n}t), \ldots, T^J_{nm}(t)/\exp(\theta_m\sigma\sqrt{n}t)\right)$$

converges weakly to a process satisfying the stochastic differential equation

$$\frac{\mathbf{dT(t)}}{\mathbf{T(t)}} = \boldsymbol{\mu}'dt + \sigma\Gamma^{1/2}d\mathbf{W(t)}, \tag{3.47}$$

where Θ, $\boldsymbol{\mu}'$,Γ, $\mathbf{T(t)}$ and $\mathbf{W(t)}$ are as defined in theorem 2.

proof: Using conditions (3.32) and (3.43), it is easy to see that

$$(\log(T^J_{n1}(t)), \log(T^J_{n2}(t)), \ldots, \log(T^J_{nm}(t))) - \sigma\sqrt{n}t\Theta$$

converges weakly to a gaussian process. Hence, the limiting process is uniquely defined by the mean and covariance functions. The mean function is given by theorem 3. For the covariance, notice that using (3.44), the equation (3.34) is still valid with T_{ni} replaced T^J_{ni}. Hence, $(\log(T^J_{n1}(t)), \log(T^J_{n2}(t)), \ldots, \log(T^J_{nm}(t)))$ has the same covariance structure as

$$\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{[nt]} g_1(Z_j), \frac{1}{\sqrt{n}}\sum_{j=1}^{[nt]} g_2(Z_j), \ldots, \frac{1}{\sqrt{n}}\sum_{j=1}^{[nt]} g_m(Z_j)\right)$$

and the result follows from condition B2. **QED.**

We are now ready to apply theorems 3 and 4 to various examples of statistics T_n. As before, we will consider the statistics median, HL-estimator and the trimmed mean. To fix the notation, let us define,

$$M^J_n(\boldsymbol{\delta}) = median\{\delta^J_1, \delta^J_2, \ldots, \delta^J_n\}, \tag{3.48}$$

$$HL^J_n(\boldsymbol{\delta}) = median\{(\delta^J_i + \delta^J_j)/2, i, j = 1, 2, \ldots, n.\}, \tag{3.49}$$

and

$$TR^J_n(\boldsymbol{\delta}) = \frac{1}{n - 2[n\alpha]}\sum_{i=[n\alpha]+1}^{n-[n\alpha]} \delta^J_{(i)}. \tag{3.50}$$

Let us further define the processes

$$M_n^J(t) = X(0)\exp\left\{\alpha([nt]+1)M_{[nt]+1}^J(\boldsymbol{\delta}) + (1-\alpha)[nt]M_{[nt]}^J(\boldsymbol{\delta})\right\}, \tag{3.51}$$

$$HL_n^J(t) = X(0)\exp\left\{\alpha([nt]+1)HL_{[nt]+1}^J(\boldsymbol{\delta}) + (1-\alpha)[nt]HL_{[nt]}^J(\boldsymbol{\delta})\right\}, \tag{3.52}$$

and

$$TR_n^J(t) = X(0)\exp\left\{\alpha([nt]+1)TR_{[nt]+1}^J(\boldsymbol{\delta}) + (1-\alpha)[nt]TR_{[nt]}^J(\boldsymbol{\delta})\right\}. \tag{3.53}$$

It can be shown that the condition C1 is satisfied for all the three statistics $M_n^J(\boldsymbol{\delta})$, $HL_n^J(\boldsymbol{\delta})$ and $TR_n^J(\boldsymbol{\delta})$. We will prove this statement for the case of $M_n(t)$ and the remaining cases can be proved in a similar fashion. Using theorems 3 and 4, notice that part(a) of corollaries 1-3 are all valid when the processes $M_n(t)$,$HL_n(t)$ and $TR_n(t)$ are replaced by their jump diffusion model counterparts $M_n^J(t)$,$HL_n^J(t)$ and $TR_n^J(t)$. We formally state these results below.

Corollary 5:

(a) The process $M_n^J(t)$, $0 \le t \le T$ converges weakly to the process $M(t)$ given by $M(t) = X(0)\exp(\mu t + \sigma\sqrt{\frac{\pi}{2}}W(t))$ as n $\to \infty$.

(b) As n $\to \infty$, ($M_n^J(t)$,X(t)) , $0 \le t \le T$ converges weakly to the stochastic process $\widetilde{X}(0)\exp\left\{(\mu t, \mu t) + \sigma(W_1(t) + c_1 W_2(t), W_1(t) + \sum_{i=1}^{N(t)} Y_i)\right\}$ where $\widetilde{X}(0) = (X(0), X(0))$, W_1,W_2 are two independent Brownian Motions and $c_1 = \sqrt{\frac{\pi}{2} - 1}$.

Corollary 6:

(a) The process $HL_n^J(t)$, $0 \le t \le T$ converges weakly to the process $HL(t)$ given by $HL(t) = X(0)\exp(\mu t + \sigma\sqrt{\frac{\pi}{3}}W(t))$ as n $\to \infty$.

b) As n $\to \infty$, ($HL_n^J(t)$,X(t)) , $0 \le t \le T$ converges weakly to to the stochastic process $\widetilde{X}(0)\exp\left\{(\mu t, \mu t) + \sigma(W_1(t) + c_2 W_2(t), W_1(t) + \sum_{i=1}^{N(t)} Y_i)\right\}$ where $\widetilde{X}(0) = (X(0), X(0))$, W_1,W_2 are two independent Brownian Motions and $c_2 = \sqrt{\frac{\pi}{3} - 1}$.

Corollary 7:

(a) The process $TR_n^J(t)$, $0 \le t \le T$ converges weakly to the process $TR(t)$ given by

$TR(t) = X(0)\exp(\mu t + \sigma\sqrt{f_\alpha}W(t))$ as n $\to \infty$.

(b) As n $\to \infty$, ($TR_n^J(t)$,X(t)) , $0 \leq t \leq T$ converges weakly to to the stochastic process $\widetilde{X}(0)\exp\left\{(\mu t, \mu t) + \sigma(W_1(t) + c_3W_2(t), W_1(t) + \sum_{i=1}^{N(t)} Y_i)\right\}$ where $\widetilde{X}(0) = (X(0), X(0))$, W_1,W_2 are two independent Brownian Motions and $c_3 = \sqrt{f_\alpha - 1}$.

Chapter 4

Pricing derived Financial Contracts

4.1 Introduction:

This chapter addresses the problem of pricing various statistical options outlined in chapter 3. We will use the limit theorems proved in chapter 3 to price the statistical options. The theory of option pricing stands on the principle of 'no arbitrage'. In 1973, Fischer Black and Myron Scholes first developed the theory of option pricing using 'no arbitrage' arguments, in their seminal paper 'The pricing of options and corporate liabilities'. Robert C.Merton later expanded and unified the theory in his 'Theory of Rational Option Pricing'(1973). Their work has not only greatly influenced the future research in the area of asset pricing but also enabled the researchers and financial houses to price rather complex securities. We will now define various basic concepts of the theory and then briefly explain the option pricing methodology that follows from the no arbitrage theory.

A portfolio is a collection of assets held by an investor. The value of a portfolio at a given time t is the net value of all the assets of the portfolio at time t. A function $\{\theta(t) : t \geq 0\}$ is called a trading strategy if $\theta(t)$ is a portfolio for all $t \geq 0$. A trading strategy θ is said to be self-financing if for every $t \geq 0$, the value of $\theta(t)$ depends only on its asset prices at time t. In other words, a trading strategy is self-financing if no gains are withdrawn and no external funds are added. A self-financing trading strategy with initial value $V(0) = 0$ is said to be an arbitrage if for some fixed $t_0 > 0$; $V(t_0)$, the value of the strategy at time t_0, is nonnegative with probability one and $V(t_0)$ is strictly positive with positive probability. An arbitrage allows one to create wealth out of nothing and the existence of arbitrage contrasts economic equilibrium.

A price process $\{V(t) : t \geq 0\}$ is called attainable if there is a self-financing trading strategy such that at any time t, the value of the trading strategy equals $V(t)$. A positive process $\{D(t) : t \geq 0\}$ with $D(0) = 1$ is called a deflator if $\{D(t)/Z(t) : t \geq 0\}$ is a martingale for any attainable process $\{D(t) : t \geq 0\}$. If the process $\{D(t) : t \geq 0\}$ is a deflator then, by the martingale condition, we have for any $t > 0$,

$$V(0) = E\left(\frac{V(t)}{D(t)}\right). \tag{4.1}$$

It is easy to observe, since $D(t)$ is a positive process, that (4.1) precludes the existence of arbitrage. In fact, the converse is also true and the equivalence of 'no arbitrage' and the existence of a deflator is called the 'Fundamental theorem of option pricing'. Now we will derive a formula for option pricing under the assumption of no arbitrage. Let $D(t)$ be a deflator. Suppose that r is the risk free rate of interest. Then, $R(t) = e^{rt}$ is an attainable process and hence, the process $R(t)/D(t)$ is a positive martingale. Notice that $R(0)/D(0) = 1$ and so we can define a probability measure $P^{'}$ by the Radon-Nikodym derivative

$$\left(\frac{dP^{'}}{dP}\right)(t) = \frac{R(t)}{D(t)}, \text{ for all } t \in [0, T]. \tag{4.2}$$

The probability measure $P^{'}$ thus obtained is called the Risk-Neutral measure or Equivalent Martingale measure. Now notice that,

$$V(0) = E_P\left[\frac{V(t)}{D(t)}\right] = E_P\left[\left(\frac{R(t)}{D(t)}\right)\left(\frac{V(t)}{R(t)}\right)\right] = E_{P'}\left[\frac{V(t)}{R(t)}\right]. \tag{4.3}$$

Since the process $R(t) = e^{rt}$ is non-stochastic, we have,

$$V(0) = e^{-rt}E_{P'}\left[V(t)\right]. \tag{4.4}$$

Suppose now that $V(t)$ is the payoff of a derivative. Under the assumption of complete markets, the payoff $V(t)$ can be obtained by the value process of a trading strategy and hence is an attainable process. Therefore, the value of such a derivative at time zero is the discounted expected payoff under the Risk-Neutral measure $P^{'}$. This simple consequence of equation (4.4) simplifies the pricing of derivatives by a great extent. The task of derivative pricing reduces to computing the expected payoff under the Risk-Neutral measure.

Suppose that $X(t)$ is the share price of a stock at time t and r is the riskless rate of interest. Under the risk-neutral measure P', the discounted price process $e^{-rt}X(t)$ is a martingale and this condition imposes a relation on the parameter values. Under the Black-Scholes model (3.1), this condition implies that $a = r$ and under the Jump diffusion model (3.34), this yields the condition

$$\mu = r - \frac{\sigma^2}{2} - \lambda \left[E(e^{Y_1}) - 1 \right]. \tag{4.5}$$

For a European type call option or a European type put option, the equation (4.4) yields a closed form formula for the option price. American type options are more complex to price in the sense that a closed form solution is seldom obtained. This situation arises for many path dependent options. An option is said to be path dependent if its payoff depends on the price of the underlying asset not only at the time of option maturity but also at various time points during the life of the option. Examples of path dependent options include American type options, Asian options, Russian options, barrier options and lookback options. Clearly, all the statistical options defined in chapter 3 are path dependent. Path dependent options are more difficult to price because of their complex payoffs. Most path dependent options do not have closed form formulae for their prices and very often, the prices need to be computed by numerical procedures. In such situations. a computation intensive estimation procedure called Monte Carlo estimation comes to the rescue and lets one to estimate the option prices with great accuracy. This procedure advocates generating sample paths of the underlying process under the risk neutral measure and then taking the simple average of the payoffs obtained from each sample path to calculate the expectation in (4.4). This method is appealing for its wide generality. Option prices can be estimated using this procedure whenever it is feasible to simulate the sample paths of the underlying assets and to compute the payoffs based on the simulated sample paths. Monte Carlo estimators are often asymptotically unbiased and are also consistent. The interested reader is referred to Glasserman (2003) for a detailed discussion on Monte Carlo methods in finance.

So far in this discussion, we have assumed that the markets are complete and derived the equation (4.4) under this assumption. But in reality, the assumption of complete markets is not always satisfied. It is well known that perfect hedging is not possible due to the discrete nature of trading and transaction costs. This could give rise to the possibility of arbitrage opportunities. Indeed many financial firms have a 'statistical arbitrage' group whose task is to exploit any arbitrage opportunities present in the market and thus make profits for the firm. Arbitrage opportunities are often short lived and they quickly vanish in the presence of transaction costs. Statistical arbitrage groups rely on their huge computational power and algorithmic trading to profit from arbitrage opportunities. In spite of these facts, 'no arbitrage' option pricing theory provides one with benchmark prices for financial contracts. In the following sections, we will compute the no arbitrage prices of various statistical options defined in chapter 3. Section 2 discusses the pricing of American type statistical options based on the processes $M_n(t)$,$HL_n(t)$,$TR_n(t)$,$LTR_n(t)$ and $UTR_n(t)$ which are defined in chapter 3. Section 3 deals with the pricing of maximum type statistical options based on the above processes and the underlying stock price process.

4.2 Pricing American type Statistical options

Recall now the definitions of the statistical assets defined in chapter 3. Statistical assets were defined based on an underlying asset and various options were defined based on the statistical assets to cater to the diverse interests of investors. For instance, an American type call option on $M_n(t)$ reduces the investor's risk to sudden drops in the underlying stock. An American type call option on $LTR_n(t)$ lets the investor profit from any rise in the underlying stock prices, but guards the investor against sudden large drops in the stock price. Since $LTR_n(t)$ is defined based on the lower sided trimmed mean, the process only 'trims' large negative increments in the stock price. These options provide the investor with added insurance against sudden fall in the stock prices. This benefit usually comes with a cost and the investor needs to pay more for a call option on $M_n(t)$ or $LTR_n(t)$ than for a call option on the underlying stock process $X(t)$. Naturally, it is important to be able to price the statistical options fairly, so as to be able to sell them in the market. We will discuss the pricing methods in this section.

We will make use of the limit theorems proved in chapter 3 to price the statistical options. Recall the corollaries (1-3) which state that the law of the processes $M_n(t)$,$HL_n(t)$ and $TR_n(t)$ converge to that of a geometric Brownian motion as n tends to infinity. In pricing the Statistical options, we will use the limit law of geometric Brownian motion instead of the actual law. This approximation is quite reasonable for large n, in view of the convergence results given by corollaries(1-3) of chapter 3. Therefore, the problem reduces to pricing an American option on a stock following a Black Scholes model. Since geometric Brownian motion is a common model for stock prices, this approximation puts a vast literature on American type option pricing at our disposal. Unlike the European options, no closed form formulae are available for the American option prices. The price needs to be computed either through simulation procedures or by using an approximation method. The binomial tree method was used to compute the prices in tables 4.1 and 4.2 .

Suppose that the risk free rate of interest is 5%. Let the historical volatility be constant at 20% per annum. Let the time to maturity of the options, T be equal to one year. Suppose the stock provides a continuous dividend yield of 2%. Table 4.1 provides the American call option prices and table 4.2 gives the American Put option prices for the underlying processes $M_n(t)$,$HL_n(t)$ and $TR_n(t)$. The tables provide the prices when the current stock price is \$58 and \$66 for varying strike prices.

Stock price	Strike price	Median call	HL call	TR call	BLS call
58	50	11.12	10.41	10.38	10.35
58	54	8.57	7.69	7.65	7.60
58	58	6.48	5.47	5.42	5.36
58	62	4.78	3.73	3.69	3.63
58	66	3.45	2.48	2.43	2.38
58	70	2.47	1.58	1.54	1.50
66	50	17.86	17.46	17.44	17.43
66	54	14.73	14.10	14.07	14.04
66	58	11.89	11.05	11.02	10.97
66	62	9.45	8.39	8.35	8.29
66	66	7.38	6.22	6.17	6.10
66	70	5.66	4.45	4.40	4.33

Table 4.1: American call option prices on $M_n(t)$,$HL_n(t)$ and $TR_n(t)$

In the above table, the column "Median call" refers to the American call option price when the underlying stock follows $M(t)$, "HL call" refers to the call price when the underlying stock follows $HL(t)$ and "TR call" refers to the call price when the underlying stock follows $TR(t)$. Notice that the prices decrease in each row from left to right. This is due to the fact that the process $M(t)$ has more volatility than the process $HL(t)$ and the process $HL(t)$ has more volatility than $TR(t)$.

Stock price	Strike price	Median Put	HL Put	TR Put	BLS Put
58	50	1.88	1.16	1.13	1.09
58	54	3.20	2.29	2.25	2.21
58	58	5.00	3.98	3.93	3.87
58	62	7.23	6.22	6.17	6.12
58	66	9.91	9.02	8.98	8.93
58	70	12.98	12.30	12.28	12.25
66	50	0.74	0.34	0.32	0.31
66	54	1.44	0.79	0.77	0.74
66	58	2.44	1.59	1.55	1.50
66	62	3.87	2.80	2.75	2.69
66	66	5.69	4.53	4.47	4.41
66	70	7.90	6.73	6.67	6.61

Table 4.2: American put option prices on $M_n(t)$,$HL_n(t)$ and $TR_n(t)$

In table 4.2, 'Median Put' refers to the American Put price when the underlying stock follows $M(t)$, 'HL put' refers to the put price when the underlying stock follows $HL(t)$ and 'TR put' refers to the put price when the underlying stock follows $TR(t)$. Again as in Table 4.1, notice that the prices decrease in each row from left to right. Note that the above prices would not have altered had we considered the jump diffusion model (3.34) for the underlying stock price $X(t)$ instead of the Black-Scholes model given by (3.1). This follows from the corollaries (3.5)-(3.7).

We shall now consider pricing the American type options based on the statistical assets $LTR_n(t)$ and $UTR_n(t)$. Note that the investor would benefit with a larger pay-off either by purchasing a call option on $LTR_n(t)$ or by purchasing a put option on $UTR_n(t)$. In view of this observation, it is of interest to find the fair prices of these two options. Firstly, observe that, $LTR_n(t)$ explodes to infinity as n increases to infinity. For a large fixed n, let us write $LTR_n(t)$ as the product of $\exp(a\sigma\theta\sqrt{n}t)$ and $LTR_n(t)/\exp(a\sigma\theta\sqrt{n}t)$ and let us approximate the second term in the product by its limit $LTR(t)$. Hence for any large fixed n, the process $LTR_n(t)$ can be approximated by $\exp(a\sigma\theta\sqrt{n}t)LTR(t)$. We approximate the price of an American call option on $LTR_n(t)$ by the price of an American call option on the process $\exp(a\sigma\theta\sqrt{n}t)LTR(t)$. Table 4.3

gives the the call option prices obtained by this approximation. As before, let the risk-free rate of interest is 5% and let the volatility be equal to 20% per annum. Suppose that the time to maturity of the option is six months, the trimming level $\alpha = 5\%$ and that the stock provides a continuous dividend yield of 2%. Then, Table 4.3 gives the prices of an American call option on $LTR_n(t)$ using the above approximation.

Stock price	Strike price	LTR Call
58	50	14.31
58	54	11.03
58	58	8.03
58	62	5.46
58	66	3.45
58	70	2.03
66	50	22.17
66	54	18.74
66	58	15.36
66	62	12.12
66	66	9.14
66	70	6.54

Table 4.3: American call option prices on $LTR_n(t)$

Note that the these prices are quite higher when compared to the call option prices on $M_n(t)$. This is only to be expected because the payoff in this case is strictly higher than the payoff obtained from a call option on the process $M_n(t)$. Also, the prices strictly increase with the value of trimming level α. Finally, we notice that the price of a put option on $UTR_n(t)$ could be approximated by the price of an American put option on the process $\exp(-a\sigma\theta\sqrt{n}t)LTR(t)$. Table 4.4 provides the put option prices on the process $UTR_n(t)$ for the same parameter values as above.

4.3 Options on the maximum of two assets

This section deals with pricing of American options whose payoff equals the maximum of two American type option payoffs. As was discussed in chapter 3, a median call

Stock price	Strike price	UTR Put
58	50	2.15
58	54	4.43
58	58	7.60
58	62	11.36
58	66	15.47
58	70	19.76
66	50	0.47
66	54	1.35
66	58	2.96
66	62	5.45
66	66	8.65
66	70	12.38

Table 4.4: American put option prices on $UTR_n(t)$

option is profitable to the investor only when there are sharp decreases in the stock price. If the stock price is increasing over the lifetime of the option, a usual American option on the stock would payoff more than the median option would. To ensure the investor a payoff which is at least as much as that of a usual American option on the stock, we consider the max-type options. For instance, a max-type median call option pays $max(M_n(t)-K, X(t)-K, 0)$ at time t. Similarly, a median put option of the max type would pay $max(K-M_n(t), K-X(t), 0)$ at time t. One can define similar options using the processes $HL_n(t)$ and $TR_n(t)$.

Many researchers have successfully worked on pricing various financial securities with payoffs involving more than one asset. The max-type statistical options are examples of such securities. Johnson(1987), Kamrad and Ritchken(1991), Detemple, Feng and Tian(2003), Broadie and Detemple(1994) and Boyle, Ervine and Gibbs(1989) found various results on pricing options involving multiple assets. Note that as in section 2, we can use the limit theorems obtained in chapter 3 to price the max type options. For example, part (b) of corollary 1 states that the joint process $(M_n(t), X(t))$ converges in law to a two dimensional geometric brownian motion and we intend to use this approximation to price the max-type options. Now, the pricing problem reduces to pricing an

American type option on the maximum of two assets, each of which follow a geometric Brownian motion. As in the case of the usual American type options, there are no analytic formulae for the prices of max-type options and we rely on simulation methods to compute the prices. Broadie and Glasserman (1997) obtain an efficient Monte Carlo simulation method to price American type options. This method was used in the max-type option pricing and we will briefly describe this method below.

The price estimation method described by Broadie and Glasserman (1997) falls under the class of random tree methods. Two estimators called High estimator and Low estimator are obtained for the option price where high estimator has upward bias and low estimator has downward bias. Both the estimators are asymptotically unbiased and so, the simple average of these two serves as an asymptotically unbiased estimator for the option price. As the name suggests, a random tree method simulates a random tree of the underlying stock asset prices. Let the underlying asset price be $X(t)$ at time $t \in [0, T]$. Assume that $X(0) = X_0$. Suppose $t_0, t_1, t_2, \ldots, t_k$ are time points such that $0 = t_0 < t_1 < t_2 < \ldots < t_k = T$. We assume that the option could be exercise at any of these $(k + 1)$ many time points. Although an American option can be exercised at any time point during its lifetime, due to the discrete nature of trading, it is reasonable to assume that there are only finitely many exercise time points. Assume that the underlying stock follows a geometric Brownian motion and all the parameters of the model are specified. Also assume that there is a constant risk free rate of interest r. Given these values, one can simulate the asset prices at time points $t = t_0, t_1, t_2, \ldots, t_k$. We generate, under the risk neutral measure, m many replications of the asset price at time t_1 given that $X(t_0) = X_0$ and given each of these values of $X(t_1)$, we generate m many independent values of the asset price $X(t_2)$ and so on... Each of these asset prices obtained at each time point is called a node of the tree and each node is said to be branched into m many nodes. For example, a simulated value of $X(t_i)$ given the value of $X(t_{i-1})$ is a node at depth i. In this fashion, $X(t_i)$ is simulated k^i many times in the random tree. The total number of exercise time points is called the depth of the tree. Hence the depth of the tree in our case is k.

Notice that the value of the option at the terminal nodes (nodes at time $t = t_k$) is just the payoff of the option. The value of the option at a node at time $t = t_{k-1}$ is computed by taking the maximum of the option value by exercising at time t_{k-1} and the option value by not exercising at time t_{k-1}. Since the m paths originating from $X(t_{k-1})$ are only of depth one, the value of the option at the node $X(t_{k-1})$, by not exercising at time t_{k-1}, is the discounted average of the m payoffs at time t_k. The option is exercised at the node $X(t_{k-1})$ if the immediate payoff is more than the discounted expected payoff obtained by not exercising. Similarly, at each node, we determine whether to exercise the option or not by comparing the expected discounted payoff at that node to the immediate exercise payoff. Let $\{X(t_k i) : i = 1, 2, \ldots, m\}$ denote the m nodes originating from $X(t_{k-1})$. For each $i = 1, 2, \ldots, m$, let V_{ki} denote the value of the option at the node $X(t_k i)$. Then the value of a call option at the node $X(t_{k-1})$ is given by

$$max\left((X(t_{k-1}) - K)^+, e^{-r(t_k - t_{k-1})}\frac{1}{n}\sum_{i=1}^{n} V_{ki}\right).$$

In this case, since the option reached its maturity date, the value of the call option V_{ki} equals $(X(t_k i) - K)^+$. The value of the option at any given node is the maximum of the immediate payoff and the expected discounted payoff. Thus a value is assigned at each of the nodes. The value at the initial node X_0 is the option value at time zero and hence is the option price at time zero. This estimator of the option price is called the high estimator.

Unfortunately, the price obtained by the above method is not unbiased. At any time point t_i, the above method decides whether to exercise the option by comparing the average discounted value of the option by not exercising to the immediate exercise payoff. By doing this, the method is using information on the future in making the exercise decision. This unfair use of information on the future increases the option value at time zero and hence the method overestimates the price. This bias can be corrected by making the exercise decision using a subgroup of the m originating nodes and calculating the discounted expected payoff using only the rest of the nodes (which are not used to make exercise decision). Suppose that we use the first node $X(t_k 1)$ to decide whether to continue and the rest $m - 1$ nodes to compute the discounted expected payoff by not exercising the option at node $X(t_{k-1})$ i.e., we decide to exercise at $X(t_{k-1})$ if the payoff $(X(t_{k-1}) - K)^+$ is more than $e^{-r(t_k - t_{k-1})}(X(t_k1) - K)^+$. If the decision is made not to exercise the option, the value of the option at $X(t_{k-1})$ is the average discounted value of the nodes $\{X(t_ki) : i = 2, 3, \ldots, m\}$ This simple modification eliminates the high bias in the estimator but introduces a low bias. Since any of the m nodes $\{X(t_ki) : i = 1, 2, \ldots, m\}$ could be used to make the exercise decision, we repeat this procedure m-many times, each time using a different node to make the exercise decision. Thus, we obtain m-many option values at the node $X(t_{k-1})$. We define the average of these m values to be the low estimator at the node $X(t_{k-1})$. One can continue this procedure to obtain the low estimators at each node and the value at the node X_0 is the low estimator of the option price.

The following tables give the prices of the max-type call options based on $M_n(t)$, $HL_n(t)$ and $TR_n(t)$. For example, at time t, the American type max(median,mean) call option with strike price K pays $maximum(M_n(t) - K, X(t) - K, 0)$. Let current stock price be equal to \$58. As before, let the risk less rate of interest be equal to 5% per annum and let the continuous rate of dividends q=2%. Let the time to maturity be one year and the volatility be constant at 20%. The second and third columns of the following tables give the high and low estimators of the option prices respectively. In the computations, k was assumed to be equal to 3 and the number of replications m=10. The fourth and fifth columns give the standard errors in estimating the high and low estimators.

Strike price	High	Low	se(high)	se(low)
50	13.91	13.56	0.0431	0.0421
54	10.92	10.77	0.0429	0.0402
58	8.31	8.13	0.0369	0.0355
62	6.17	6.04	0.0320	0.0302
66	4.44	4.30	0.0254	0.0260
70	3.14	3.07	0.0211	0.0217

Table 4.5: Max(median,mean) call option prices

Strike price	High	Low	se(high)	se(low)
50	11.36	11.10	0.0372	0.0403
54	8.45	8.26	0.0314	0.0333
58	6.03	5.91	0.0282	0.0284
62	4.16	4.13	0.0237	0.0220
66	2.81	2.76	0.0193	0.0182
70	1.83	1.78	0.0138	0.0135

Table 4.6: Max(HL,mean) call option prices

Strike price	High	Low	se(high)	se(low)
50	11.10	10.85	0.0366	0.0358
54	8.20	8.00	0.0330	0.0319
58	5.88	5.79	0.0272	0.0273
62	4.04	3.97	0.0237	0.0226
66	2.67	2.63	0.0185	0.0184
70	1.71	1.70	0.0133	0.0123

Table 4.7: Max(TR,mean) call option prices

As earlier, notice that the max(median,mean) prices are higher than the max(HL,mean) prices which in turn are higher than the max(TR, mean) prices. This is due to the fact that the process $M_n(t)$ has more volatility than that of $HL_n(t)$ and $HL_n(t)$ in turn has more volatility than that of $TR_n(t)$.

Chapter 5

A Stock Price Model: Parameter Estimation

5.1 Introduction

In the year 1900, the French mathematician Louis Bachelier first proposed modeling the stock prices by linear Brownian motion. Bachelier developed a theory of option pricing based on the linear Brownian motion model for the stock prices. Later in 1965, Paul Samuelson proposed the geometric Brownian motion model for stock prices. In 1973, Black, Scholes and Merton developed option pricing theory under this model. Indeed, the no arbitrage option pricing theory developed by Black, Scholes and Merton completely revolutionized the field of derivative pricing forever and is still very influential in Academia as well as the finance industry.

Although both the above models have been extensively used for stock prices and other asset prices, they are fundamentally very different. It is well known that under the Bachelier model, stock price hits the value zero in finite time with probability one. So, the stock goes bankrupt for any choice of drift and volatility parameters. On the other hand, under the Black-Scholes model, the stock price is always strictly positive, thus precluding the possibility of a bankruptcy. In this chapter, we consider the model proposed by Chen, Logan, Palmon and Shepp(2003), of which both Bachelier and Black-Scholes models are special cases. This model is given by the stochastic differential equation

$$dX(t) = X^{\theta}(t)(\mu dt + \sigma dW(t)) \tag{5.1}$$

where $\theta \in [0, 1]$, $\mu \in \mathbb{R}$ and $\sigma \in (0, \infty)$. The above model degenerates to the Bachelier model for $\theta = 0$ and to the Black-Scholes model for $\theta = 1$. The case $\theta = 0$ corresponds to simple interest and the case $\theta = 1$ corresponds to continuously compounded interest in the sense that, at time t, the expected value of the stock price equals $X(0)\mu t$ if $\theta = 0$ and equals $X(0)e^{\mu t}$ if $\theta = 1$. This observation suggests the relation between the parameter θ and the growth rate of the stock price $X(t)$. Chen, Logan, Palmon and Shepp(2003) call θ, the capital productivity parameter. In this chapter, we concentrate on estimating and interpreting the parameter θ for various stocks enlisted on the New York Stock Exchange(NYSE).

Inference on continuous time variables, often modeled by diffusion processes is fundamentally different from the classical statistical inference due to the fact that each observation here corresponds to a real valued function of time, often referred to as a sample path, which is a random realization of the underlying diffusion process. Although each observation is a function of time, it is seldom possible to observe the whole sample path. Instead, the sample path values are obtained at various discrete time points. In view of this fact, classical parameter estimation methods are adapted to parameter estimation of diffusion processes by means of discretization methods. A discretization method approximates a continuous time model by a discrete time model. Since continuous data is seldom obtained, a discretization method plays a vital role in estimating the parameters of a continuous time process. Euler approximation is the simplest of such discretization methods. Although the entire price path cannot be observed, the discrete price values at various time points are often sufficient to accurately estimate various model parameters. The details of Euler approximation method are discussed in the next section.

Statistical inference of diffusion processes has been dealt by many researchers over the past decades. 'Statistical Inferences for diffusion type processes' by Basawa and Rao(1980) is a good starting point for the researcher interested in parameter estimation of diffusion processes. The review paper by Fan(2005) describes various nonparametric inference methods for financial time series. Anderson(1959), Billingsley(1961), Dacunha-Castelle and Florens-Zhmirou(1986), Lo(1988), Melino(1994), Pederson(1995), Duffie and Glynn(1997), Florens(1999), Zhang,Mykland and Ait-Sahalia(2005) discuss various estimation methods in the context of diffusion processes. More recently, the seminal paper by Ait-Sahalia(2002) considers the maximum likelihood estimation of diffusion process parameters and provides a rather accurate closed form approximation to the estimator. In this chapter, we consider estimation of the parameter θ of the model (5.1). The following section describes our method and discusses the results.

5.2 Estimation procedure

Let the stock price $X(t)$ follow the model (5.1). Suppose we have discrete data from the stock price $X(t)$ at time points $t = t_0, t_1, \ldots, t_n$. Euler Approximation method suggests replacing the differential terms at time t_{k-1}, dt and dW(t) of equation (5.1) by the differences $t_k - t_{k-1}$ and $W(t_k) - W(t_{k-1})$ respectively. This yields,

$$X(t_k)-X(t_{k-1}) = X^{\theta}(t_{k-1})\{\mu(t_k-t_{k-1})+\sigma(W(t_k)-W(t_{k-1}))\}, k = 1, 2, \ldots, n. \quad (5.2)$$

We assume that the process is observed at equal time intervals Δ. Therefore, suppose that the process is observed at time points $t = 0, \Delta, 2\Delta, \ldots, n\Delta$. Using the above observation, we observe that the modified returns

$$\frac{X(k\Delta) - X((k-1)\Delta)}{X^{\theta}((k-1)\Delta)}, k = 1, 2, \ldots, n \quad (5.3)$$

are independent normally distributed variables with mean $\mu\Delta$ and variance $\sigma^2\Delta$. Our estimation method makes use of a measure of peakedness of a variable, called kurtosis. The kurtosis of a variable Z is defined as,

$$\gamma = \frac{E((Z-\mu)^4)}{var^2(Z)}. \quad (5.4)$$

It is well known that a normal variable has a kurtosis value of three. Observe that, since the above modified returns are IID normal variables, the kurtosis of these variables should be equal to three. We estimate the parameter θ using this criterion, i.e, we estimate θ as that value for which the estimated kurtosis of the above variables is closest to three. The parameter θ was estimated for many daily stock data enlisted in the NYSE. The following table gives the θ values for various daily stock price data. The data size was typically around 5500.

Stock	Theta
Apple	1.000
Alberto Culver	0.688
Adbe Systems	0.952
Aetna	1.000
Amazon	0.844
Boeing Co.	1.000
Dell	0.918

Table 5.1: Theta estimates for various stocks

The following curves give, for each of the stocks from the above table, the kurtosis values of the modified returns for different values of θ between 0 and 1.

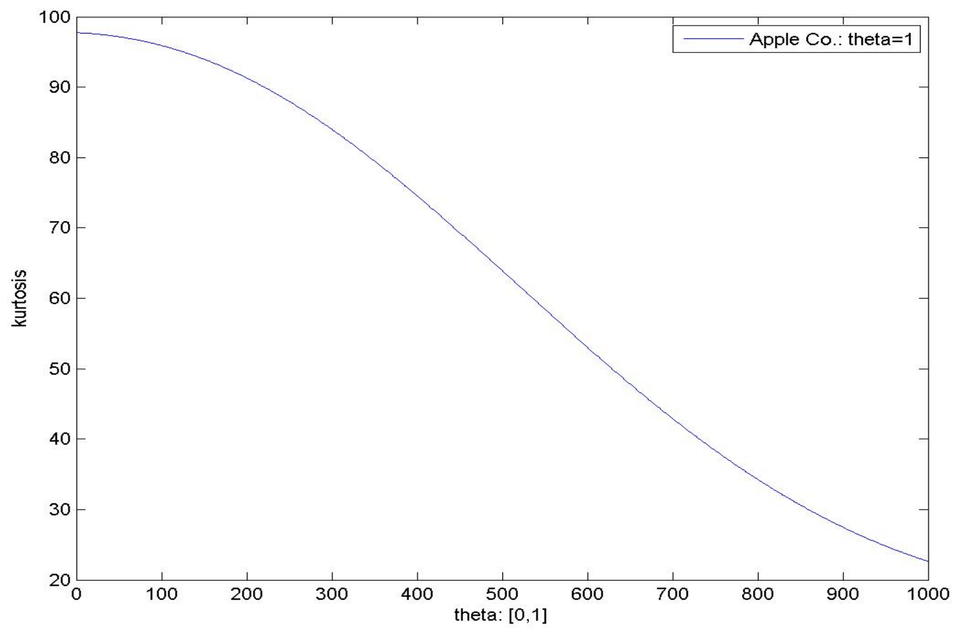


Figure 5.1: Kurtosis values-Apple Inc.

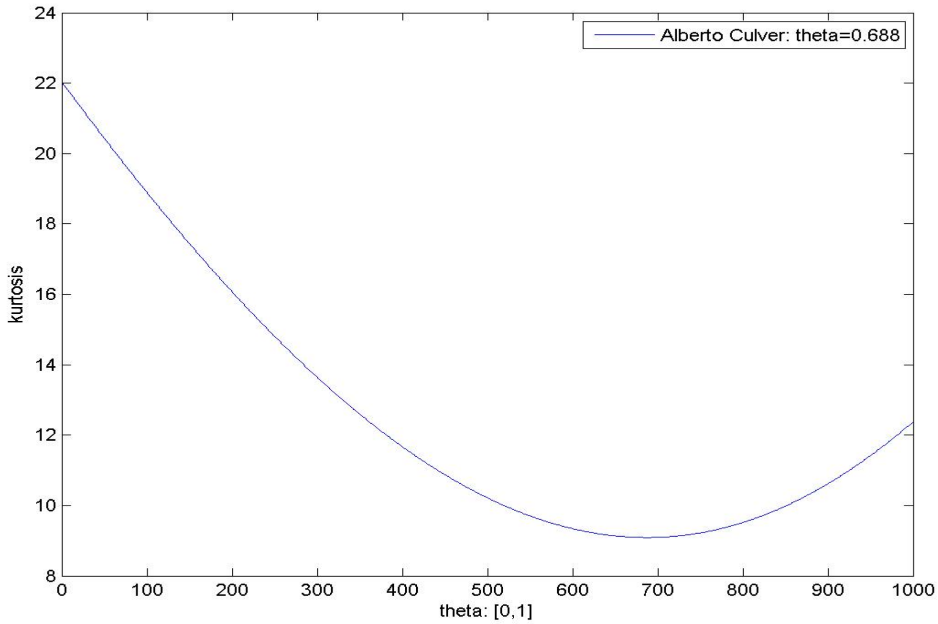


Figure 5.2: Kurtosis values-Alberto Culver

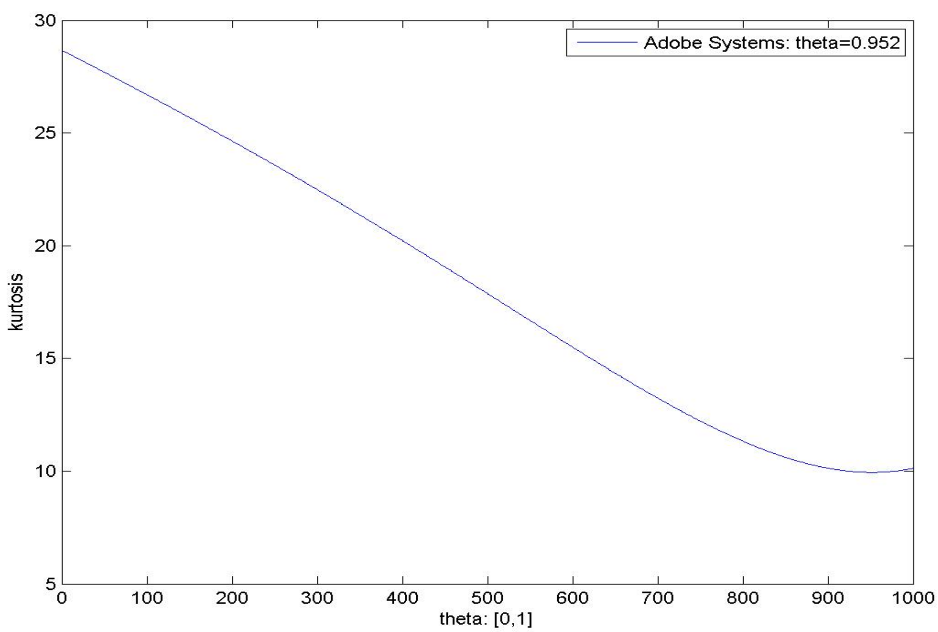


Figure 5.3: Kurtosis values-Adbe Systems

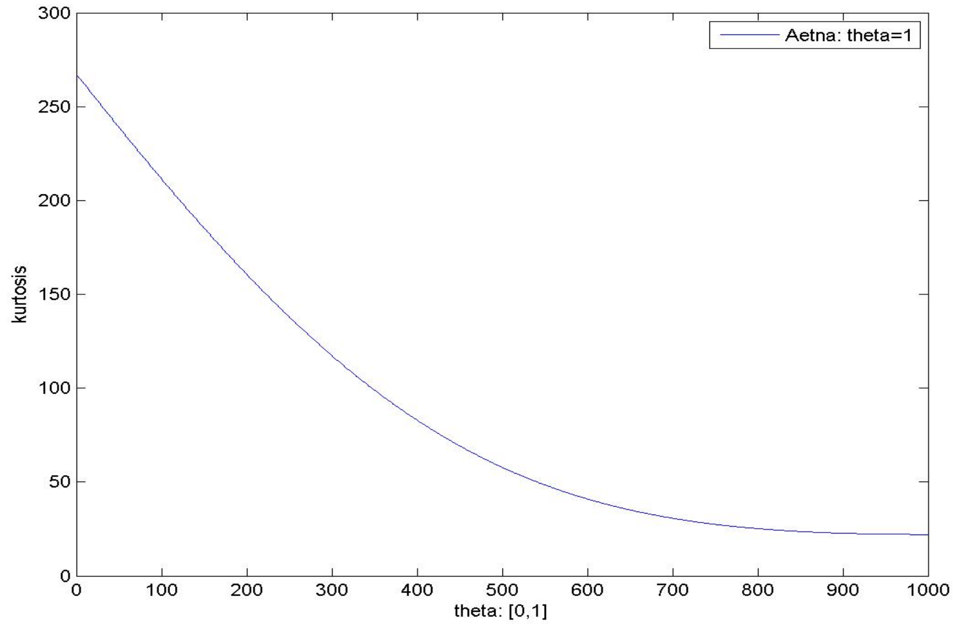


Figure 5.4: Kurtosis values-Aetna

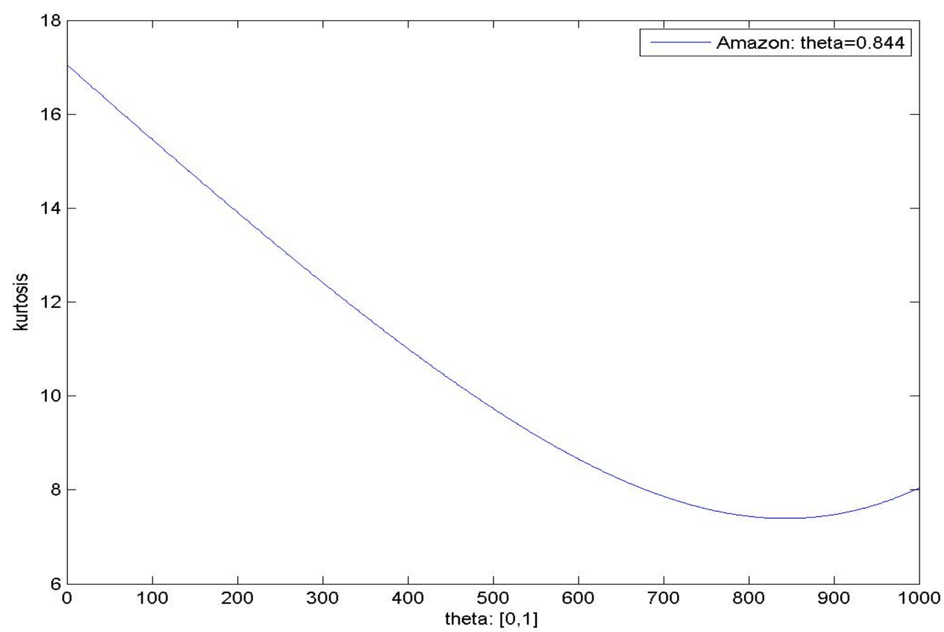


Figure 5.5: Kurtosis values-Amazon Inc.

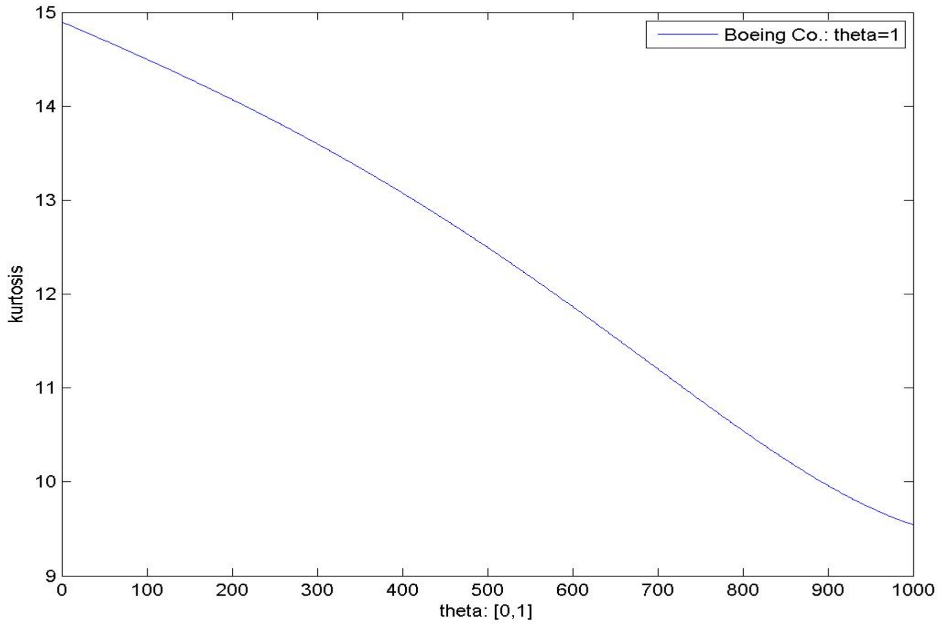


Figure 5.6: Kurtosis values-Boeing Co.

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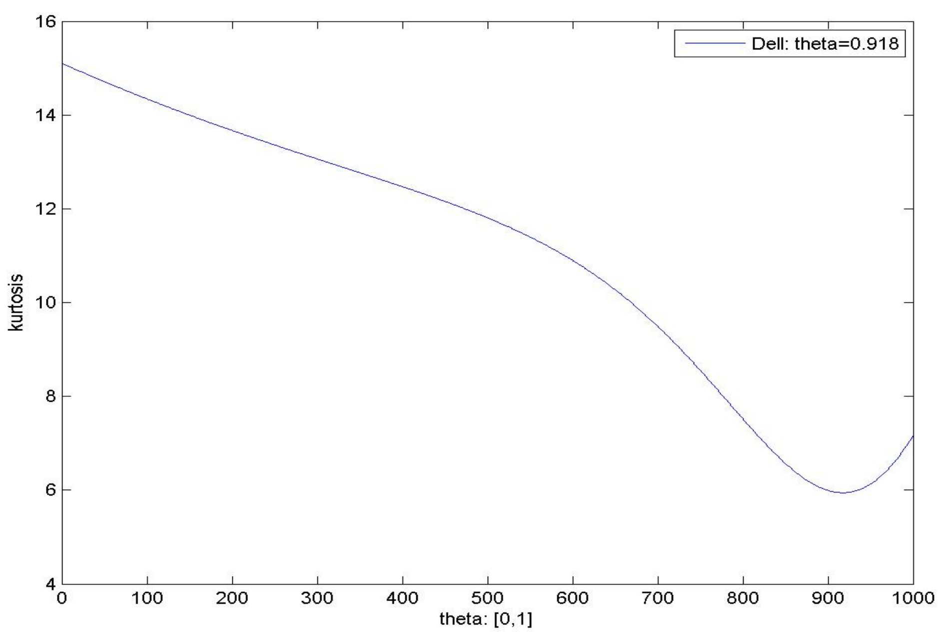

Figure 5.7: Kurtosis values-Dell

5.3 Conclusions:

We note that for some stocks, the above method yielded θ estimates that are significantly different from one. This suggests that the Black-Scholes-Merton model is not the best fit for all the stocks. The method described in this chapter relies only on the assumption that the modified return are normally distributed. The values of the other parameters μ and σ are hardly of any consequence in estimating the growth rate parameter θ.

Appendix A

Corollary 1 :

Proof : We show that Conditions A1 and A2 are satisfied for the case of M_n. By Bahadur's Quantile representation (1966), we have for any k,

$$P\left(\left|M_n + \frac{F_n(0) - \frac{1}{2}}{\phi(0)}\right| > c(k)n^{-\frac{3}{4}} \log n\right) = O(n^{-k}). \tag{A-1}$$

Equivalently,

$$P\left(\left|M_n - \frac{1}{n}\sum_{i=1}^{n} g(Z_i)\right| > cn^{-\frac{3}{4}} \log n\right) = O(n^{-k}), \tag{A-2}$$

where $g(Z_i)$ are defined as $\frac{(I(Z_i<0)-\frac{1}{2})}{\phi(0)}$, where ϕ is the standard normal density function. We also notice that E($g(Z_1)$)=0 and var($g(Z_1)$)=$\pi/2$. We will now show that condition B is satisfied.

Firstly, notice that, $\max_{1\leq k\leq n}|M_k| \leq \max_{1\leq k\leq n}|Z_k|$. We also have, $P(\max_{1\leq k\leq n}|Z_k| > c\log n) \leq n\left(P\left(|Z_1| > c\log n\right)\right)$ which converges to zero. Therefore, applying Theorem 1, we have, $M_n(t)$ converges weakly to the process $M(t)$ as given in (a). To prove part (b), observe that the statistic $A_n = \frac{1}{n}\sum_{i=1}^{n} Z_i$ satisfies conditions A1 and A2 with $\theta = 0$ and $g(Z_1) = Z_1$. Now using Theorem 2, the process $(M_n(t), A_n(t))$ converges weakly to the process $\widetilde{X}(0)\exp\left\{(\mu t, \mu t) + \sigma(W_1(t) + c_1W_2(t), W_1(t))\right\}$ as n $\rightarrow \infty$. The proof is complete observing that the process $A_n(t)$ converges weakly to $X(t)$.

To prove corollary 2, we will need the following lemmas. Let $X_1, X_2, \ldots, X_n$ be IID standard normal random variables. Now define,

$$F_n(x) = \frac{1}{n(n-1)}\sum\sum_{i\neq j} I\left(\frac{X_i + X_j}{2} \leq x\right),$$

$$F(x) = \Phi(\sqrt{2}x),$$

and

$$F_n^*(x) = \frac{1}{n}\sum_{i=1}^{n}\Phi(2x - X_i).$$

Also define, for any $x \in \mathbb{R}$,

$$R_n(x) = [F_n(x) - F(x)] - 2\left[F_n^*(x) - F(x)\right].$$

Lemma 1: Under this notation, we have

$$P(sup_{x\in\mathbb{R}}|R_n(x)| > cn^{-1}\log n) = O(n^{-k})$$

for any $k > 0$.

proof: We first show that, for any real number x and any $k > 0$,

$$P(|R_n(x)| > cn^{-1}\log n) = O(n^{-k}).$$

To prove this, we will state the following result by Arcones(1996).

Let $\{X_i : i \in \mathbb{N}\}$ be IID random variables. Let f be the kernel of a degenerate U-statistic with $\|f\|_\infty \leq c$. Then the inequality

$$P\left(n^{-m/2}\left|\sum_{I_m^n} f(X_{i_1}, X_{i_2}, \ldots, X_{i_k})\right| > t\right) \leq c_1 \exp(-c_2(t/c)^{2/m}) \qquad \text{(A-3)}$$

holds for any $t > 0$ where c_1, c_2 are positive constants depending only on m and I_m^n denotes the set $\{(i_1, i_2, \ldots, i_m) : i_j \in \mathbb{N}, 1 \leq i_j \leq n, i_j \neq i_k \text{ if } j \neq k\}$.

Now, let $f(X_1, X_2) = \left[I\left(\frac{X_1+X_2}{2} \leq x\right) - \Phi(\sqrt{2}x)\right] - 2\left[\Phi(2x - X_2) - \Phi(\sqrt{2}x)\right]$. Notice that

$$R_n(x) = \frac{1}{n(n-1)} \sum_{(i_1,i_2)\in I_2^n} f(X_{i_1}, X_{i_2}). \qquad \text{(A-4)}$$

Applying the above result to this f at $t = \frac{2k}{c_2}\log n$, we get

$$P\left(|R_n(x)| > n^{-1}\left(\frac{2k}{c_2}\log n\right)\right) \leq c_1 \exp\left(-c_2\frac{2k\log n}{2c_2}\right) = c_1 n^{-k}, \qquad \text{(A-5)}$$

for any x $\in \mathbb{R}$ and any $k > 0$, i.e, we have,

$$P(|R_n(x)| > cn^{-1}\log n) = O(n^{-k}) \qquad \text{(A-6)}$$

for any x $\in \mathbb{R}$ and for any $k > 0$. Define $A_n = \{\omega : |X_i(\omega)| > \log n \text{for some i=1,2,\ldots,n.}\}$. Notice that $P(A_n) = O(n^{-k})$ for any $k > 0$.

$P(sup_{x\in\mathbb{R}}|R_n(x)| > cn^{-1}\log n)$

$= P(sup_{x\in\mathbb{R}}|R_n(x)| > cn^{-1}\log n, A_n) + P(sup_{x\in\mathbb{R}}|R_n(x)| > cn^{-1}\log n, A_n^c)$

$\leq P(A_n) + P(sup_{x\in\mathbb{R}}|R_n(x)| > cn^{-1}\log n, A_n^c)$.

It suffices to show that,

$P(sup_{x\in\mathbb{R}}|R_n(x)| > cn^{-1}\log n, |X_i| \leq M, i = 1, 2, \ldots, n) = O(n^{-k})$.

Now, we break up the interval $[-\log n, \log n]$ into intervals of the form $[a_i, a_{i+1}]$, with $a_0 = 0$, $a_{-L} = a_L = \log n$ such that, $a_i - a_{i-1} = \frac{1}{n}$ for $i \in \{-L+1, -L+2, \ldots, -1.0, 1, 2, \ldots, L-1\}$. For $x \in [a_i, a_{i+1}]$, we have

$$R_n(a_{i-1}) - \frac{c}{n} \leq R_n(x) \leq R_n(a_i) + \frac{c}{n}.$$

Hence,

$P(sup_{|x|\leq\log n}|R_n(x)| > cn^{-1}\log n, A_n^c)$

$\leq \sum_{i=-L}^{L} P\left(sup_{x\in[a_i,a_{i+1}]}|R_n(x)| > cn^{-1}\log n, A_n^c\right)$

$\leq \sum_{i=-L}^{L} P\left(max\{|R_n(a_i)|, |R_n(a_{i+1})|\} > cn^{-1}\log n, A_n^c\right)$

$\leq 2\sum_{i=-L}^{L} P(|R_n(a_i)| > \frac{c}{2}n^{-1}\log n)$

$= O(n^{-k})$.

Now, we will show that

$$P(sup_{|x|\geq\log n}|R_n(x)| > cn^{-1}\log n, A_n^c) = O(n^{-k}).$$

If $x < -\log n$, $|X_i| \leq \log n, i = 1, 2, \ldots, n$, then $F_n(x) = 0$ and $F(x) = O(n^{-k})$ $1 \leq F_n^*(x) = \frac{1}{n}\sum_{i=1}^{n}\Phi(2x - X_i) = O(n^{-k})$ and hence, $|R_n(x)| = |F(x) - 2F_n^*(x)| = O(n^{-k})$. On the other hand, if $x > \log n$ and $|X_i| \leq \log n, i = 1, 2, \ldots, n$, we have $1 - F(x) = \Phi(-\sqrt{2}x) = O(n^{-k})$, $F_n(x) = 1$, and $1 - F_n^*(x) = O(n^{-k})$ and hence,

$|R_n(x)| = |1 + F(x) - 2F_n^*(x)|$

$= |2(1 - F_n^*(x)) - (1 - F(x))|$

$\leq 2|1 - F_n^*(x)| + |1 - F(x)|$

$= O(n^{-k})$.

Therefore,from the above observations, we have

$P(sup_x|R_n(x)| > cn^{-1}\log n, |X_i| \leq M, i = 1, 2, \ldots, n) = O(n^{-k})$.

Hence we have,

$$P(sup_x|R_n(x)| > cn^{-1}\log n) = O(n^{-k}) \text{ for any } k > 0.$$

Lemma 2: For any $k > 0$, there exists a $c(k) > 0$ such that

$$P(sup_{x\in\mathbb{R}}|F_n^*(x) - F(x)| > c(k)n^{-1/2}\log n^{1/2}) = O(n^{-k}).$$

proof: This proof is standard and is omitted here.

As a simple consequence of Lemmas 1 and 2, we have

Lemma 3: For any $k > 0$, there exists a $c(k) > 0$ such that

$$P(sup_{x\in\mathbb{R}}|F_n(x) - F(x)| > c(k)n^{-1/2}\log n^{1/2}) = O(n^{-k}).$$

Lemma 4: For any $k > 0$,

$$P(|HL_n - 0| > cn^{-1/2}\log n^{1/2}) = O(n^{-k}). \tag{A-7}$$

proof: In view of Lemma 3 and the fact that $F_n(HL_n) = F(0) \pm \frac{1}{n(n-1)}$,we have,

$$P(|F(0) - F(HL_n)| > cn^{-1/2}\log n^{1/2}) = O(n^{-k}).$$

Now note that,

$$|HL_n - 0| > cn^{-1/2}\log n^{1/2} \Rightarrow |F(HL_n - F(0))| > \frac{1}{2}c\sqrt{2\pi}n^{-1/2}\log n^{1/2}$$

for all large n and hence the claim.

Lemma 5: For any $k > 0$ and a suitable choice of a constant $c > 0$,

$$P(|\xi_n| > cn^{-1}\log n) = O(n^{-k})$$

where $\xi_n = -2[(F_n^*(HL_n) - F(HL_n)) - (F_n^*(0) - F(0))]$.

proof:

Note that

$\xi_n(x) = -2\left\{HL_n(F_n^{*'}(0) - F'(0)) + \frac{HL_n^2}{2}(F_n^{*''}(\omega) - F^{''}(\omega))\right\}$ for some ω.

$|F_n^{*''}(\omega) - F^{''}(\omega)| \leq K$ for some constant K.

$F_n^{*'}(0) - F^{'}(0) = \frac{2}{n}\sum_{i=1}^n \phi(-X_i) - \sqrt{2}\phi(0)$.

Noting that $E[F_n^{*'}(0)] = F^{'}(0)$, we observe that $F_n^{*'}(0) - F^{'}(0)$ is a centered IID mean and hence,

$P(|F_n^{*'}(0) - F^{'}(0)| > cn^{-1/2}\log n^{-1/2}) = O(n^{-k})$.

$P(|\xi_n| > cn^{-1}\log n)$

$\leq P(2|HL_n||F_n^{*'}(0) - F^{'}(0)| > cn^{-1}\log n) + P(kHL_n^2 > cn^{-1}\log n)$

$\leq P(|HL_n| > c_1 n^{-1/2}\log n^{-1/2}) + P(|F_n^{*'}(0) - F^{'}(0)| > c_2 n^{-1/2}\log n^{1/2}) + P(HL_n > c_3 n^{-1/2}\log n^{1/2})$

which is $O(n^{-k})$ for suitable choices of c,c_1,c_2 and c_3. Hence the claim.

Corollary 2:

proof: Notice that,

$\xi_n = -2[(F_n^*(HL_n) - F(HL_n)) - (F_n^*(0) - F(0))]$

$= \frac{1}{\sqrt{\pi}}\left[2\sqrt{\pi}(F(HL_n) - F_n^*(HL_n)) + \frac{2\sqrt{\pi}}{n}\sum_{i=1}^n(\Phi(-X_i) - \frac{1}{2})\right]$

$= \frac{1}{\sqrt{\pi}}\left[\sqrt{\pi}(F_n(HL_n) - F(HL_n)) + \sqrt{\pi}R_n(HL_n) + \frac{2\sqrt{\pi}}{n}\sum_{i=1}^n(\Phi(-X_i) - \frac{1}{2})\right]$

$= \frac{1}{\sqrt{\pi}}\left[\sqrt{\pi}HL_nF^{'}(0) + \sqrt{\pi}\frac{HL_n^2}{2}F^{''}(\omega) + \sqrt{\pi}R_n(HL_n) + \frac{2\sqrt{\pi}}{n}\sum_{i=1}^n(\Phi(-X_i) - \frac{1}{2})\right]$

$= \frac{1}{\sqrt{\pi}}\left[HL_n + \frac{2\sqrt{\pi}}{n}\sum_{i=1}^n(\Phi(-X_i) - \frac{1}{2})\right] + \frac{HL_n^2}{2}F^{''}(\omega) + R_n(HL_n)$.

Using lemmas 1,4,5 and the above equation, (noting that $F^{''}(\omega)$ is bounded) we have,

$$P(|HL_n + \frac{2\sqrt{\pi}}{n}\sum_{i=1}^n(\Phi(-X_i) - \frac{1}{2})| > cn^{-1}\log n) = O(n^{-k})$$

for any $k > 0$. Condition A2 of theorem 1 can be proved similarly as in corollary 1. To show part(b), we consider the joint process $(HL_n(t), A_n(t))$. By Theorem 2, this process weakly converges to the process $\widetilde{X}(0)\exp\{(\mu t, \mu t) + \sigma(W_1(t) + c_2W_2(t), W_1(t))\}$ as n $\to \infty$. Again as was observed in the proof of corollary 1, the proof is complete by noting that $A_n(t)$ weakly converges to $X(t)$.

Corollary 3:

proof:

For a weight function w, $0 \le w \le 1$, let us define

$$L_n(x) = \int_{-\infty}^{\infty} xw(F_n(x))dF_n(x).$$

and

$$L(x) = \int_{-\infty}^{\infty} xw(F(x))dF(x)$$

where $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \le x)$ and $F(x) = \Phi(x)$.

As can be observed from the above equations, an L-Statistic L_n is a weighted linear combination of order statistics. So, TR_n can be written as L_n for a suitable choice of weight function w. Let us also define ,

$$R_n = \left|(L_n - L) - n^{-1}\sum_{i=1}^{n} Z_i\right|$$

where

$$Z_i = \frac{1}{1-2\alpha}\int_{\alpha}^{1-\alpha} \frac{p - I(X_i \le \xi_p)}{\phi(\xi_p)} dp$$

for $i = 1, 2, \ldots, n$. Under the above notation, Singh(1981) shows that there exists a $c > 0$ depending on k, such that,

$$P(R_n > cn^{-1}\log n) \le n^{-k}.$$

This proves condition A1 and condition A2 follows from the observation that for any $\alpha \in (0,1)$,

$$|TR_n| \le max_{1\le k\le n}|Z_i| = O_P(\log n).$$

Hence, TR_n converges to a Geometric Brownian motion with mean parameter μ and volatility parameter $\sigma\sqrt{f_\alpha}$. For part(b), notice that it follows from Theorem 2 that the process $(TR_n(t), A_n(t))$ converges weakly to $\widetilde{X}(0)\exp\{(\mu t, \mu t) + \sigma(W_1(t) + c_3W_2(t), W_1(t))\}$. The result is immediate from the fact that $A_n(t)$ converges weakly to the process $X(t)$.

Corollary 4:

proof:

We will prove the result for the case of $LTR_n(t)$. The proof for the case of $UTR_n(t)$ is very similar and is left as an exercise for the interested reader. Notice that LTR_n is a trimmed mean and hence can be written as a L-statistic. Noticing this, condition A1 can be verified using the result by Singh(1981) stated in the previous proof. Condition A2 follows from the fact that,

$$|LTR_n| \leq max_{1\leq k\leq n}|Z_i| = O_P(\log n).$$

Finally, part(a) follows from the fact that the second term in equation (3.30), given by $\frac{\theta([nt]+\alpha)}{\sqrt{n}}$ is of the same order as $\theta\sqrt{n}t$ and . Part(b) can be easily verified as before.

Corollary 5:

proof:

We first notice that, part (a) is a direct consequence of condition C1. We will prove that, for any $r > 0$ and any $\beta > 0$, there is a sequence of positive real numbers $b_n \downarrow 0$ such that

$$P\left(n\left|T_n(\boldsymbol{\delta}) - T_n^J(\boldsymbol{\delta})\right| > \beta b_n\right) = O(n^{-r}). \tag{A-8}$$

Notice that this is a much stronger condition when compared to C1. To this end, we first notice that, since $N(T)$ is a Poisson random variable; for any given $\epsilon > 0$, there is a $K \in N$ such that for any $r > 0$,

$$P(N(T) > K\log n) = O(n^{-r}) \tag{A-9}$$

Now, let us define the set

$$A = \{\omega : N(T)(\omega) \leq K\log n\}. \tag{A-10}$$

We note that under the assumption that $N(T) \leq K\log n$, there are at the most $[K\log n]$ many Y_is which are nonzero and hence T_n^J lies between the $(\frac{1}{2}-\frac{k\log n}{n})^{th}$ and $(\frac{1}{2}+\frac{k\log n}{n})^{th}$ quantiles of the set $\{\delta_1, \delta_2, \ldots, \delta_n\}$. This observation yields,

$$\begin{aligned}
&P\left(n\left|T_n(\boldsymbol{\delta}) - T_n^J(\boldsymbol{\delta})\right| > \beta b_n\right) \\
&\leq P\left(n\left|T_n(\boldsymbol{\delta}) - T_n^J(\boldsymbol{\delta})\right| > \beta b_n, A\right) + P\left(n\left|T_n(\boldsymbol{\delta}) - T_n^J(\boldsymbol{\delta})\right| > \beta b_n, A^c\right) \\
&\leq P\left(\sqrt{n}\left|F_n^{-1}(\tfrac{1}{2}) - F_n^{-1}(\tfrac{1}{2} - \tfrac{K\log n}{n})\right| > \beta b_n\right) \\
&\quad + P\left(\sqrt{n}\left|F_n^{-1}(\tfrac{1}{2}) - F_n^{-1}(\tfrac{1}{2} + \tfrac{K\log n}{n})\right| > \beta b_n\right) + O(n^{-r})
\end{aligned}$$

where F_n is the empirical CDF of $Z_1, Z_2, \ldots, Z_n \sim N(0,1)$. It remains to show that the first two terms are $O(n^{-r})$ for any $r > 0$. This follows from the following two statements

(i) For any $t \in [\epsilon, 1-\epsilon]$ and for any $r > 0$ there is a $c > 0$ (depending on r) such that,

$$P\left(\left|F_n^{-1}(t) - F^{-1}(t) + \frac{F_n(t) - F(t)}{f(F^{-1}(t))}\right| > cn^{-3/4}\log n\right) = O(n^{-r}), \tag{A-11}$$

(ii) For any $t \in (0,1)$, for any $r > 0$ and for any real sequence $\beta_n = o(n^{-1/2}\log n^{1/2})$ there is a $c > 0$ (depending on r) such that,

$$P\left(|(F_n(t) - F(t)) - (F_n(t+\beta_n) - F(t+\beta_n))| > cn^{-3/4}\log n\right) = O(n^{-r}). \tag{A-12}$$

It is straight forward to see the claim by applying statement (i) at $t = 1/2$ and statement (ii) at $t = 1/2$ and $\beta_n = \pm\frac{K\log n}{n}$ and then combining these two. Now, it suffices to vindicate the above two statements. The first statement (i) follows from Bahadur's quantile representation (1966) and the second statement follows by observing that for any $s > 0$,

$P\left(|(F_n(t) - F(t)) - (F_n(t+\beta_n) - F(t+\beta_n))| > cn^{-3/4}\log n\right)$

$= P\left(\exp\{sn\,|(F_n(t) - F(t)) - (F_n(t+\beta_n) - F(t+\beta_n))|\} > \exp\{scn^{1/4}\log n\}\right)$

$\leq \exp(-scn^{1/4}\log n)M_\xi(s)$ (Markov Inequality)

where, $\xi = n\,|(F_n(t) - F(t)) - (F_n(t+\beta_n) - F(t+\beta_n))|$ and $M_\xi(s)$ is the moment generating function of the variable ξ at s. Note that ξ is the absolute value of a centered sample sum of indicator variables $I(t < X_i < t+\beta_n)$. Now observe that,

$$M_\xi(s) = \exp(\log(M_\xi(s))) \leq \exp(E(s\xi)) \leq \exp(s\sqrt{var(\xi)}) \leq \exp(s\sqrt{\log n}).$$

Therefore,

$P\left(|(F_n(t) - F(t)) - (F_n(t+\beta_n) - F(t+\beta_n))| > cn^{-3/4}\log n\right)$

$\leq \exp(-scn^{1/4}\log n + s\sqrt{\log n})$

which is $O(n^{-r})$ for any $r > 0$ for a suitable choice of s. Hence the claim.

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