PARAMETERIZATIONS OF TEICHMÜLLER SPACES OF SURFACES WITH BOUNDARY

BY REN GUO

A dissertation submitted to the

Graduate School—New Brunswick

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Feng Luo

and approved by

New Brunswick, New Jersey

May, 2008

ABSTRACT OF THE DISSERTATION

Parameterizations of Teichmüller spaces of surfaces with boundary

by Ren Guo

Dissertation Director: Feng Luo

The Teichmüller space of a surface with boundary is the space of all isotopy classes of hyperbolic metrics with totally geodesic boundary. Using the cosine law of a hyperbolic right-angled hexagon, Feng Luo introduced a continuous family of new coordinates of the Teichmüller space: the ψ_λ coordinate. He proved that for $\lambda \geq 0$, the image of the Teichmüller space under the ψ_λ coordinate is an open convex polytope independent of λ. In this dissertation, we verify Luo's conjecture that for $\lambda < 0$, the image of the Teichmüller space under the ψ_λ coordinate is a bounded open convex polytope.

Acknowledgements

First of all, I am deeply indebted to my advisor, Professor Feng Luo, for his support, encouragement and constant guidance in the last five years. And I am very grateful for his generosity of sharing ideas and the unpublished manuscripts with me. Without his supervision, this dissertation would have never been written.

I would like to thank Professor Steven Ferry, Professor Xiaojun Huang, Professor Peter Landweber, Professor Xiaochun Rong for their kind helps, encouragement and the excellent courses taught at Rutgers. I would like to thank the faculty and stuff of Mathematics Department at Rutgers for their help and support.

I would like to thank the committee members of the defence of my dissertation for their patience and careful reading.

I am also indebted to Professor Francis Bonahon for his kind hospitality and valuable suggestion during my visit of University of Southern California.

To Dr. Saul Schleimer, I would like to thank for his nice lectures and comments on my works and seminar talks.

I would like to thank the organizers of conferences and workshops I attended for providing me financial support.

To Dr. Zheng Huang, Dr. Liang Kong, Dr. Xiaobo Liu, I thank for their helpful conversations.

Many thanks go to my friends at Rutgers: Dr. Yongzhong Xu, Zhixiang Wang, Shi-Ting Bao, Biao Yin, Ming Shi, Yuan Zhang, Nan Li, Jin Wang, Yuan Yuan, who make my years in Rutgers full of happiness and excitement.

Last but not least, I would like to express indebtedness to my parents and my wife Yan Wang for their understandings, support and love throughout all these years.

Dedication

To people who lost the chance of free thinking and speaking.

Table of Contents

Abstract ii

Acknowledgements iii

Dedication iv

List of Figures vi

1. Introduction 1

1.1. Teichmüller spaces 1

1.2. Ideal triangulations 2

1.3. The length coordinate of Teichmüller spaces 4

1.4. The ψ_λ coordinate of Teichmüller spaces 5

1.5. Image of Teichmüller spaces 6

2. The ψ_λ coordinate of Teichmüller spaces 8

2.1. The derivative cosine law 8

2.2. Smooth embedding 12

3. Image of Teichmüller spaces 17

3.1. Degenerations of a hyperbolic hexagon 17

3.2. Determine the image 19

3.3. Examples 26

References 29

Vita 30

List of Figures

1.1. Examples of surfaces, ideal triangulations and associated dual graphs. . 3

1.2. Definition of the ψ_λ coordinate. 6

2.1. An hyperbolic right-angled hexagon. 8

3.1. (a) An example of an edge path. (b) An example of an edge cycle. . . . 20

3.2. Construction of a smooth curve from a set of edges of infinite lengths. . 22

3.3. Smooth graph G at a vertex of degree 1 (a), degree 2 (b), degree 3 (c). . 24

3.4. The image of the Teichmüller space of the ideally triangulated three-holed sphere in Figure 1.1 (a). 27

3.5. The image of the Teichmüller space of the ideally triangulated three-holed sphere in Figure 1.1 (b). 28

Chapter 1

Introduction

The goal of this dissertation is to investigate the parameterizations of the Teichmüller spaces of surfaces with non-empty boundary. One part of the dissertation has been written as the preprint [6].

1.1 Teichmüller spaces

Let S be a compact connected orientable surface of genus $g \geq 0$ with $n > 0$ boundary components. Then its Euler characteristic is $\chi(S) = 2 - 2g - n$. If $\chi(S) < 0$, it is well-known that there are hyperbolic metrics on the surface S such that each boundary component is totally geodesic. By a hyperbolic metric, we mean a Riemannian metric of constant curvature -1. To consider all the hyperbolic metrics on a surface, we need to introduce the concept of Teichmüller space.

Definition 1.1.1. *Two hyperbolic metrics d, d' on a surface S with geodesic boundary are equivalent if there exists an isometry between (S, d) and (S, d') such that the isometry is homotopic to the identity. The set of all equivalence classes of hyperbolic metrics with geodesic boundary on S is called the Teichmüller space of the surface, denoted by $\mathcal{T}(S)$.*

One of the fundamental results in the theory of Teichmüller space says that the Teichmüller space $\mathcal{T}(S)$ is homeomorphic to a cell. An interesting problem is to find a useful parametrization of $\mathcal{T}(S)$ which should reveal more properties of $\mathcal{T}(S)$. There are several known parameterizations of the Teichmüller spaces $\mathcal{T}(S)$. In particular, using pants decompositions of surfaces, Fenchel and Nielsen introduced the length-twist coordinate for $\mathcal{T}(S)$. For more details, please see the book by Imayoshi & Taniguchi [5].

Using the Bonahon-Thurston shearing cocycles, Bonahon [1] produced a parametrization of the Teichmüller space $\mathcal{T}(S)$. Analog to Penner's simplicial coordinate [11, 12] of the decorated Teichmüller space, Ushijima [13], Luo [8] introduced the simplicial coordinate of the Teichmüller space $\mathcal{T}(S)$. Recently Luo [9] introduced a continuous family of coordinates for $\mathcal{T}(S)$: the ψ_λ coordinate depending on a parameter $\lambda \in \mathbb{R}$. And the simplicial coordinate of $\mathcal{T}(S)$ in [13, 8] is in fact ψ_0.

When $\lambda \geq 0$, Luo [9] proved that the image of the Teichmüller space $\mathcal{T}(S)$ under the ψ_λ coordinate is an open convex polytope independent of λ. He conjectured [10] that for $\lambda < 0$, the image of the Teichmüller space $\mathcal{T}(S)$ under the ψ_λ coordinate is a bounded open convex polytope. In this dissertation, we verify this conjecture.

1.2 Ideal triangulations

Since the ψ_λ coordinate of the Teichmüller space $\mathcal{T}(S)$ is defined using a fixed ideal triangulation, first let us introduce the construction of an ideal triangulation of a surface with non-empty boundary.

Definition 1.2.1. *A colored hexagon is a hexagon with three non-pairwise adjacent edges labelled by red and the opposite edges labelled by black.*

Take a finite disjoint union of colored hexagons and identify all red edges in pairs by homeomorphisms. The quotient is a compact surface with non-empty boundary together with an ideal triangulation. The 2-cells in the ideal triangulation are quotients of the hexagons. The quotients of red edges are called the edges of the ideal triangulation while the quotients of black edges are called the B-arcs of the ideal triangulation. The following result is well-known.

Lemma 1.2.2. *Each compact connected orientable surface S of non-empty boundary and negative Euler characteristic admits an ideal triangulation. Furthermore, the number of edges in an ideal triangulation is $6g - 6 + 3n$, where g is the genus of S and n is the number of boundary components of S.*

Given an ideal triangulation, we can associate a dual graph to this ideal triangulation. A vertex of the dual graph corresponds to a colored hexagon. There is a dual

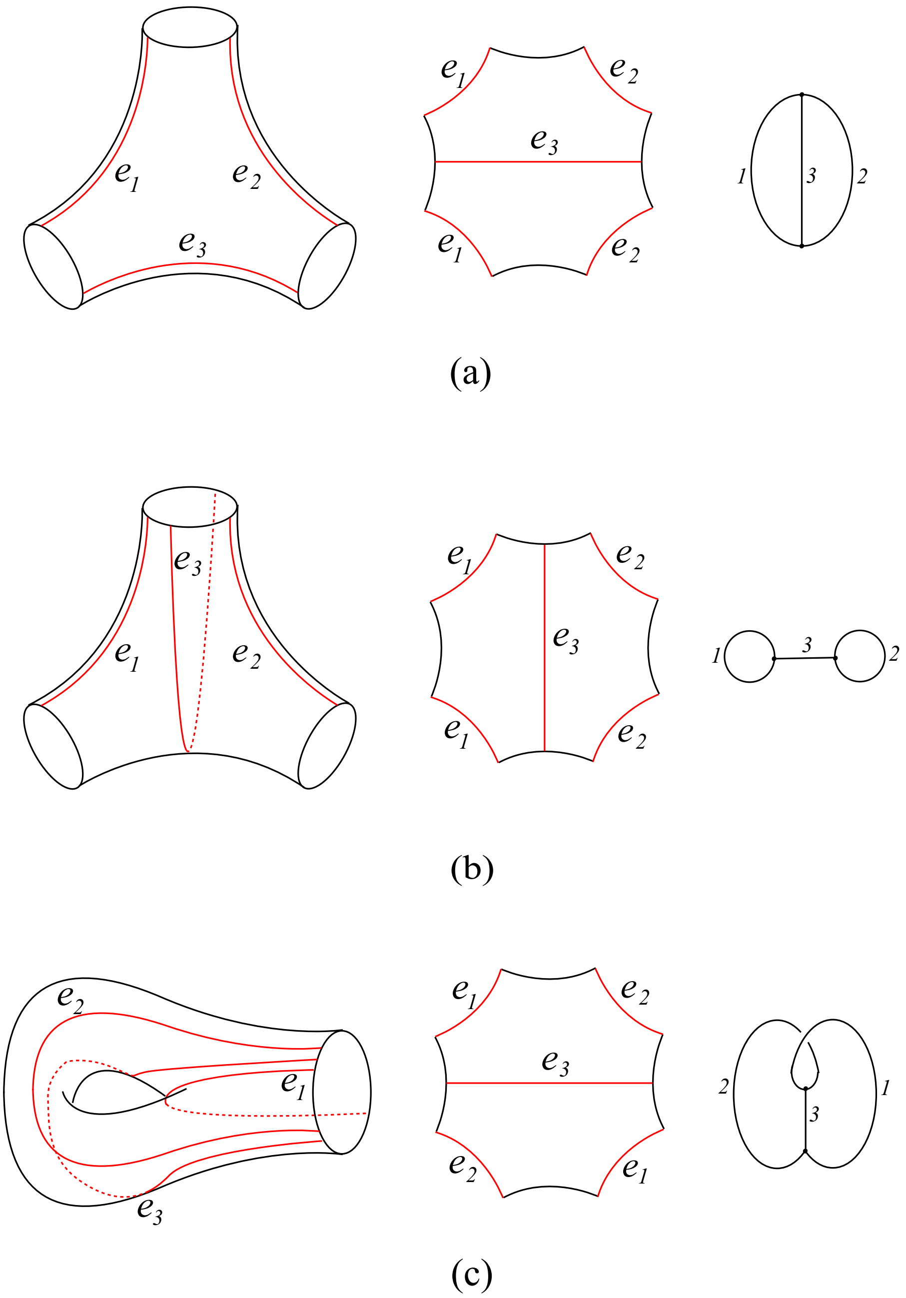

Figure 1.1: Examples of surfaces, ideal triangulations and associated dual graphs.

edge connecting two vertices if the two colored hexagon corresponding to the vertices share a common edge.

In Figure 1.1, some examples of surfaces, their ideal triangulations and associate dual graphs are given. In (a), the three-holed sphere is decomposed as a union of two colored hexagons, while in (b) another ideal triangulation of the three-holed sphere is given. In (c) the one-holed torus is decomposed as a union of two colored hexagons.

1.3 The length coordinate of Teichmüller spaces

The length coordinate of the Teichmüller space $\mathcal{T}(S)$ is the basis of the construction of the ψ_λ coordinate. We begin with the following lemma. See Buser [2] for a proof.

Lemma 1.3.1. *For any $l_1, l_2, l_3 \in \mathbb{R}_{>0}$, there exists a hyperbolic right-angled hexagon whose three non-pairwise adjacent edges have lengths l_1, l_2, l_3. Furthermore, the hexagon is unique up to isometry.*

Let (S, T) be an ideally triangulated surface with E the set of all edges. Let d be a hyperbolic metric with geodesic boundary on (S, T). For each edge $e \in E$, let e^* be the shortest geodesic arc homotopic to e relative to the boundary ∂S. Since the arc e^* is the shortest, it is an embedded arc and perpendicular to the boundary ∂S. Furthermore, if $e_1 \neq e_2 \in E$ are two different arcs, then the shortest geodesic arcs e_1^* and e_2^* are disjoint. We can see that the set $\{e^* | e \in E\}$ produces an ideal triangulation of S so that each colored hexagon is a hyperbolic right-angled hexagon.

Definition 1.3.2. *The length coordinate of a hyperbolic metric d on S is a vector $l^d \in \mathbb{R}^E_{>0}$ so that the value of l^d at an edge $e \in E$ is the length of the shortest geodesic arc e^* homotopic to e relative to the boundary ∂S.*

On the other hand, given a vector $l \in \mathbb{R}^E_{>0}$, we can produce a hyperbolic metric with totally geodesic boundary on S. This hyperbolic metric is constructed as follows. First by Lemma 1.3.1 we can realize each colored hexagon with red edges e_i, e_j, e_k as a hyperbolic right-angled hexagon whose red edges have lengths $l(e_i), l(e_j), l(e_k)$. Then a surface with a hyperbolic metric is obtained by gluing isometrically these hexagons along their red edges.

Thus we can see there is a correspondence between a hyperbolic metric and its length coordinate. In fact we have the following lemma due to Ushijima [13].

Lemma 1.3.3. *Suppose (S,T) is an ideally triangulated surface. The Teichmüller space $\mathcal{T}(S)$ can be parameterized by the length coordinate so that $\mathcal{T}(S)$ is identified with the space $\mathbb{R}^E_{>0}$.*

1.4 The ψ_λ coordinate of Teichmüller spaces

Let us begin by recalling the ψ_λ coordinate introduced by Luo [9]. Suppose (S,T) is an ideally triangulated surface. In a hyperbolic metric on (S,T) with geodesic boundary, each edge in the ideal triangulation T is isotopic (leaving the boundary of a surface fixed) to a shortest geodesic arc. Therefore each hexagon in T is isotopic to a hyperbolic right-angled hexagon. By Lemma 1.3.3, we identify a hyperbolic metric with its length coordinate. The ψ_λ coordinate of a hyperbolic metric $l \in \mathbb{R}^E_{>0} = \mathcal{T}(S)$ is a vector $\psi_\lambda \in \mathbb{R}^E$ so that the value of ψ_λ at an edge e is defined by

$$\psi_\lambda(e) = \int_0^{\frac{a+b-c}{2}} \cosh^\lambda(t)dt + \int_0^{\frac{a'+b'-c'}{2}} \cosh^\lambda(t)dt \tag{1.4.1}$$

where e is shared by two hyperbolic right-angled hexagons and a, b, c, a', b', c' are lengths of the B-arcs labelled as in Figure 1.2. Now consider the map

$$\Psi_\lambda : \mathcal{T}(S) = \mathbb{R}^E_{>0} \to \mathbb{R}^E$$
$$l \mapsto \psi_\lambda$$

sending a hyperbolic metric $l \in \mathbb{R}^E_{>0} = \mathcal{T}(S)$ to its ψ_λ coordinate.

The following theorem is proved in Luo [9]. The special case of $\lambda = 0$ is proved in Luo [8].

Theorem 1.4.1. *(Luo) Suppose (S,T) is an ideally triangulated surface. For any $\lambda \in \mathbb{R}$, the map $\Psi_\lambda : \mathcal{T}(S) \to \mathbb{R}^E$ is a smooth embedding. In particular, each hyperbolic metric with geodesic boundary on (S,T) is determined up to triangulation preserving isometry by its ψ_λ coordinate.*

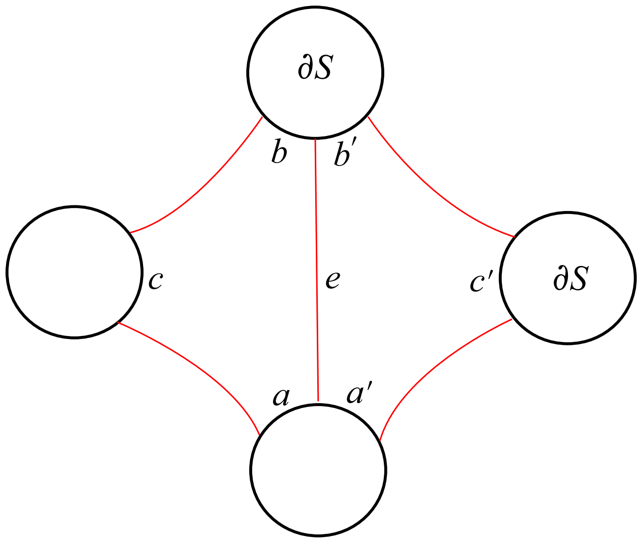

Figure 1.2: Definition of the ψ_λ coordinate.

Theorem 1.4.1 is proved by applying a variational principle. The action functional of this variational principle comes from the cosine law of a hyperbolic right-angled hexagon.

1.5 Image of Teichmüller spaces

We are interested in the image of the Teichmüller space $\mathcal{T}(S)$ under the ψ_λ coordinate. First let us introduce some definitions. An *edge path* $(H_0, e_1, H_1, ..., e_n, H_n)$ is a collection of hexagons and edges in an ideal triangulation so that two adjacent hexagons H_{i-1} and H_i sharing the edge e_i for $i = 1, ..., n$. An edge path $(H_0, e_1, H_1, ..., e_n, H_n)$ is an *edge cycle* if $H_0 = H_n$. For example see Figure 3.1. A *fundamental edge path (or fundament edge cycle)* is an edge path (or edge cycle) so that each edge in the ideal triangulation appears at most twice in the path (or cycle).

The following theorem proved in Luo [9] characterizes the image of Teichmüller space under the ψ_λ coordinate when $\lambda \geq 0$.

Theorem 1.5.1. *(Luo) Let $\lambda \geq 0$. For an ideal triangulated surface (S, T) with the set of edges E, $\Psi_\lambda(\mathcal{T}(S)) = \{z \in \mathbb{R}^E |$ for each fundamental edge cycle $(H_0, e_1, H_1, ..., e_n, H_n = H_0)$, $\sum_{i=1}^n z(e_i) > 0\}$. Thus $\Psi_\lambda(\mathcal{T}(S))$ is an open convex polytope independent of the* *parameter* $\lambda \geq 0$.

The main result in this dissertation is the verification of Luo's conjecture [10] that for $\lambda < 0$, the image of the Teichmüller space is a bounded open convex polytope.

Theorem 1.5.2. *Let $\lambda < 0$. For an ideal triangulated surface (S, T) with the set of edges E, $\Psi_\lambda(\mathcal{T}(S))$ is the set of points $z \in \mathbb{R}^E$ satisfying*

(1). $z(e) < 2\int_0^\infty \cosh^\lambda(t)dt$ for each edge e;

(2). $\sum_{i=1}^n z(e_i) > -2\int_0^\infty \cosh^\lambda(t)dt$ for each fundamental edge path $(H_0, e_1, H_1, ..., e_n, H_n)$;

(3). $\sum_{i=1}^n z(e_i) > 0$ for each fundamental edge cycle $(H_0, e_1, H_1, ..., e_n, H_n = H_0)$.

Thus $\Psi_\lambda(\mathcal{T}(S))$ is an open convex polytope so that

$$\Psi_{\lambda_1}(\mathcal{T}(S)) \subset \Psi_{\lambda_2}(\mathcal{T}(S)) \subset \Psi_0(\mathcal{T}(S))$$

for $\lambda_1 < \lambda_2 < 0$. The intersection $\cap_{\lambda=0}^{-\infty}\Psi_\lambda(\mathcal{T}(S))$ is empty.

By definition, a single edge e_i together with two hexagons adjacent to e_i consists an edge path or a single edge e_i together with only one hexagon adjacent to e_i consists an edge cycle. Thus the condition (2) requires that $z(e_i) > -2\int_0^\infty \cosh^\lambda(t)dt$ or the condition (3) requires that $z(e_i) > 0$. Therefor the image $\Psi_\lambda(\mathcal{T}(S))$ is contained in the box $(-c_\lambda, c_\lambda)^E \subset \mathbb{R}^E$, where $c_\lambda = 2\int_0^\infty \cosh^\lambda(t)dt$.

By the theorem above, we can see that the shape of the image of the Teichmüller space $\mathcal{T}(S)$ is completely determined by the combinatorics the dual graph of the ideal triangulation T of the surface S.

In the chapter 2, we introduce the derivative cosine law and present a proof of Theorem 1.4.1. In the chapter 3, a proof of Theorem 1.5.2 is given and images of the Teichmüller spaces of the simplest surfaces under the ψ_λ coordinate are explicitly described.

Chapter 2

The ψ_λ coordinate of Teichmüller spaces

In this chapter, we introduce the derivative cosine law of a hyperbolic right-angled hexagon and present a proof of Theorem 1.4.1. The cosine law of a hyperbolic right-angled hexagon relates the lengths of its edges. The derivative cosine law relates the differentials of the lengths of its edges. From the derivative cosine law, a closed differential one form is constructed. Therefore a convex function is defined and it is shown that the gradient of the convex function is the mapping induced by the ψ_λ coordinate.

2.1 The derivative cosine law

For a hyperbolic hexagon with three non-pairwise adjacent edges of lengths l_1, l_2, l_3 and opposite edges of lengths $\theta_1, \theta_2, \theta_3$ labelled in Figure 2.1, the following is the cosine law and the sine law.

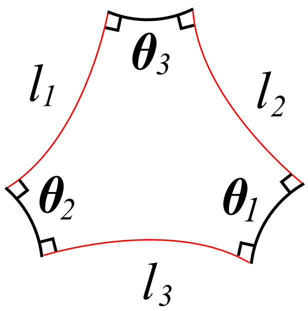

Figure 2.1: An hyperbolic right-angled hexagon.

Lemma 2.1.1 (The cosine law and the sine law). *For* $\{i,j,k\} = \{1,2,3\}$, *the following formulas hold:*

$$\cosh\theta_i = \frac{\cosh l_i + \cosh l_j \cosh l_k}{\sinh l_j \sinh l_k}, \tag{2.1.1}$$

$$\cosh l_i = \frac{\cosh\theta_i + \cosh\theta_j \cosh\theta_k}{\sinh\theta_j \sinh\theta_k}, \tag{2.1.2}$$

$$\frac{\sinh l_i}{\sinh\theta_i} = \frac{\sinh l_j}{\sinh\theta_j}. \tag{2.1.3}$$

The derivation of these formulas can be found in, for instance, Buser [2].

Let's recall the Gram matrix and the angle Gram matrix of a hyperbolic triangle. For a hyperbolic triangle with three edges of lengths l_1, l_2, l_3 and opposite inner angles $\theta_1, \theta_2, \theta_3$, its Gram matrix is

$$G_l := -\begin{pmatrix} -1 & \cosh l_3 & \cosh l_2 \\ \cosh l_3 & -1 & \cosh l_1 \\ \cosh l_2 & \cosh l_1 & -1 \end{pmatrix}$$

and its angle Gram matrix is

$$G_\theta := -\begin{pmatrix} -1 & \cosh\theta_3 & \cosh\theta_2 \\ \cosh\theta_3 & -1 & \cosh\theta_1 \\ \cosh\theta_2 & \cosh\theta_1 & -1 \end{pmatrix}.$$

By analogue, we define the Gram matrix and the angle Gram matrix of a hyperbolic right-angled hexagon as follows. For a hyperbolic right-angled hexagon with three non-pairwise adjacent edges of lengths l_1, l_2, l_3 and opposite edges of lengths $\theta_1, \theta_2, \theta_3$, its Gram matrix is

$$G_l := -\begin{pmatrix} -1 & \cosh l_3 & \cosh l_2 \\ \cosh l_3 & -1 & \cosh l_1 \\ \cosh l_2 & \cosh l_1 & -1 \end{pmatrix}$$

and its angle Gram matrix is

$$G_\theta := -\begin{pmatrix} -1 & \cosh\theta_3 & \cosh\theta_2 \\ \cosh\theta_3 & -1 & \cosh\theta_1 \\ \cosh\theta_2 & \cosh\theta_1 & -1 \end{pmatrix}.$$

Lemma 2.1.2. *For $\{i, j, k\} = \{1, 2, 3\}$, the following formulas hold:*

$$\det G_l = -(\sinh l_i \sinh l_j \sinh\theta_k)^2, \tag{2.1.4}$$

$$\det G_\theta = -(\sinh\theta_i \sinh\theta_j \sinh l_k)^2. \tag{2.1.5}$$

Proof. By the cosine law (2.1.1), we have

$$\begin{aligned}\sinh^2 l_i \sinh^2 l_j \sinh^2 \theta_k &= \sinh^2 l_i \sinh^2 l_j (\cosh^2 \theta_k - 1)\\ &= (\cosh l_k + \cosh l_i \cosh l_j)^2 - \sinh^2 l_i \sinh^2 l_j\\ &= (\cosh l_k + \cosh l_i \cosh l_j)^2 - (\cosh^2 l_i - 1)(\cosh^2 l_j - 1)\\ &= \cosh^2 l_k + 2\cosh l_i \cosh l_j \cosh l_k + \cosh^2 l_i + \cosh^2 l_j - 1\\ &= -\det G_l.\end{aligned}$$

By the symmetry of l's and θ's, we can see that the formula for $\det G_\theta$ is true. □

By Lemma 2.1.2 and the sine law (2.1.3), we can define

$$\begin{aligned}M :=& \frac{1}{\sqrt{-\det G_l}} \begin{pmatrix} \sinh l_1 & 0 & 0 \\ 0 & \sinh l_2 & 0 \\ 0 & 0 & \sinh l_3 \end{pmatrix}\\ =& \frac{1}{\sqrt{-\det G_\theta}} \begin{pmatrix} \sinh \theta_1 & 0 & 0 \\ 0 & \sinh \theta_2 & 0 \\ 0 & 0 & \sinh \theta_3 \end{pmatrix}.\end{aligned}$$

Lemma 2.1.3. $MG_l MG_\theta = I$.

Proof. This lemma is proved by direct computations involving the cosine law and the sine law (2.1.1), (2.1.2) and (2.1.3). In fact

$$
\begin{aligned}
&-G_\theta^{-1}\\
&=\begin{pmatrix} -1 & \cosh\theta_3 & \cosh\theta_2\\ \cosh\theta_3 & -1 & \cosh\theta_1\\ \cosh\theta_2 & \cosh\theta_1 & -1 \end{pmatrix}^{-1}\\
&=\frac{1}{-\det G_\theta}\\
&\begin{pmatrix} 1-\cosh^2\theta_1 & \cosh\theta_3+\cosh\theta_1\cosh\theta_2 & \cosh\theta_2+\cosh\theta_3\cosh\theta_1\\ \cosh\theta_3+\cosh\theta_1\cosh\theta_2 & 1-\cosh^2\theta_2 & \cosh\theta_1+\cosh\theta_2\cosh\theta_3\\ \cosh\theta_2+\cosh\theta_3\cosh\theta_1 & \cosh\theta_1+\cosh\theta_2\cosh\theta_3 & 1-\cosh^2\theta_3 \end{pmatrix}\\
&=\frac{1}{-\det G_\theta}\\
&\begin{pmatrix} -\sinh^2\theta_1 & \cosh l_3\sinh\theta_1\sinh\theta_2 & \cosh l_2\sinh\theta_3\sinh\theta_1\\ \cosh l_3\sinh\theta_1\sinh\theta_2 & -\sinh^2\theta_2 & \cosh l_1\sinh\theta_2\sinh\theta_3\\ \cosh l_2\sinh\theta_3\sinh\theta_1 & \cosh l_1\sinh\theta_2\sinh\theta_3 & -\sinh^2\theta_3 \end{pmatrix}\\
&=\frac{1}{-\det G_\theta}\\
&\begin{pmatrix} \sinh\theta_1 & 0 & 0\\ 0 & \sinh\theta_2 & 0\\ 0 & 0 & \sinh\theta_3 \end{pmatrix}\begin{pmatrix} -1 & \cosh l_3 & \cosh l_2\\ \cosh l_3 & -1 & \cosh l_1\\ \cosh l_2 & \cosh l_1 & -1 \end{pmatrix}\begin{pmatrix} \sinh\theta_1 & 0 & 0\\ 0 & \sinh\theta_2 & 0\\ 0 & 0 & \sinh\theta_3 \end{pmatrix}\\
&=M(-G_l)M.
\end{aligned}
$$

□

Let y_1, y_2, y_3 be three real-valued smooth functions of variables $x_1, x_2, x_3 \in \mathbb{R}$. Let $A = (\frac{\partial y_i}{\partial x_j})_{3\times 3}$ be the Jacobian matrix. Then the differentials dy_1, dy_2, dy_3 and dx_1, dx_2, dx_3 satisfy

$$
\begin{pmatrix} dy_1\\ dy_2\\ dy_3 \end{pmatrix} = A \begin{pmatrix} dx_1\\ dx_2\\ dx_3 \end{pmatrix}
$$

Lemma 2.1.4 (The derivative cosine law). *For a hyperbolic right-angled hexagon with three non-pairwise adjacent edges of lengths l_1, l_2, l_3 and opposite edges of lengths*

$\theta_1, \theta_2, \theta_3$, the differentials of $l's$ and $\theta's$ satisfy the following relations:

$$\begin{pmatrix} dl_1 \\ dl_2 \\ dl_3 \end{pmatrix} = MG_l \begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix} \tag{2.1.6}$$

$$\begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix} = MG_\theta \begin{pmatrix} dl_1 \\ dl_2 \\ dl_3 \end{pmatrix}. \tag{2.1.7}$$

Proof. We will establish either one of the derivative cosine law (2.1.6) or (2.1.7). The other one will be a consequence due to Lemma 2.1.3 or the symmetry of $l's$ and θ's. For simplicity, we prove (2.1.6).

From the cosine law (2.1.2), we have

$$\cosh l_i \sinh \theta_j \sinh \theta_k = \cosh \theta_i + \cosh \theta_j \cosh \theta_k.$$

By differentiating the two sides, we have

$$\begin{aligned} \sinh l_i \sinh \theta_j \sinh \theta_k \ dl_i = & \sinh \theta_i \ d\theta_i \\ & + (\sinh \theta_j \cosh \theta_k - \cosh l_i \cosh \theta_j \sinh \theta_k) \ d\theta_j \\ & + (\cosh \theta_j \sinh \theta_k - \cosh l_i \sinh \theta_j \cosh \theta_k) \ d\theta_k. \end{aligned}$$

After replacing $\cosh l_i$ by using the cosine law (2.1.2) and simplifying, we get

$$\begin{aligned} \sinh l_i \sinh \theta_j \sinh \theta_k \ dl_i = & \sinh \theta_i \ d\theta_i \\ & - \frac{\cosh \theta_i \cosh \theta_j + \cosh \theta_k}{\sinh \theta_j} \ d\theta_j \\ & - \frac{\cosh \theta_i \cosh \theta_k + \cosh \theta_j}{\sinh \theta_k} \ d\theta_k \\ = & - \sinh \theta_i (-d\theta_i + \cosh l_k \ d\theta_j + \cosh l_j \ d\theta_k). \end{aligned}$$

Then the formula (2.1.6) holds. □

2.2 Smooth embedding

For a hyperbolic hexagon with three non-pairwise adjacent edges of lengths l_1, l_2, l_3 and opposite edges of lengths $\theta_1, \theta_2, \theta_3$ labelled in Figure 2.1, for $\{i, j, k\} = \{1, 2, 3\}$, the *r-coordinate* is defined by

$$r_i = \frac{\theta_j + \theta_k - \theta_i}{2}.$$

The following lemma is derived in Luo [9].

Lemma 2.2.1 (The tangent law). *For $\{i, j, k\} = \{1, 2, 3\}$, the following formula holds:*

$$\tanh^2 \frac{l_i}{2} = \frac{\cosh r_j \cosh r_k}{\cosh r_i \cosh(r_i + r_j + r_k)}. \tag{2.2.1}$$

Proof. From the cosine law (2.1.2), we have

$$\begin{aligned}
\tanh^2 \frac{l_i}{2} &= \frac{\cosh l_i - 1}{\cosh l_i + 1} \\
&= \frac{\cosh \theta_i + \cosh \theta_j \cosh \theta_k - \sinh \theta_j \sinh \theta_k}{\cosh \theta_i + \cosh \theta_j \cosh \theta_k + \sinh \theta_j \sinh \theta_k} \\
&= \frac{\cosh \theta_i + \cosh(\theta_j - \theta_k)}{\cosh \theta_i + \cosh(\theta_j + \theta_k)} \\
&= \frac{\cosh \frac{\theta_i + \theta_j - \theta_k}{2} \cosh \frac{\theta_i - \theta_j + \theta_k}{2}}{\cosh \frac{\theta_i + \theta_j + \theta_k}{2} \cosh \frac{\theta_i - \theta_j - \theta_k}{2}} \\
&= \frac{\cosh r_k \cosh r_j}{\cosh(r_i + r_j + r_k) \cosh r_i}.
\end{aligned}$$

□

For any $\lambda \in \mathbb{R}$, we introduce two new variables

$$u_i = \int_1^{l_i} \tanh^{\lambda+1}(\frac{t}{2}) dt,$$

$$v_i = \int_0^{r_i} \cosh^{\lambda}(t) dt.$$

Corollary 2.2.2. *The differential one-form $\omega = \sum_{i=1}^{3} v_i \, du_i$ is closed. And the function $F(u_1, u_2, u_3) = \int \omega$ is strictly concave down.*

Proof. To show $d\omega = 0$, we need to check that there is a symmetric matrix A such that

$$\begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix} = A \begin{pmatrix} dv_1 \\ dv_2 \\ dv_3 \end{pmatrix}.$$

To show the function F is strictly concave down, we need to check that A is negative definite.

Since $d\theta_i = dr_j + dr_k$ for $\{i,j,k\} = \{1,2,3\}$, from the derivative cosine law (2.1.6), we have

$$\begin{pmatrix} dl_1 \\ dl_2 \\ dl_3 \end{pmatrix} = \frac{-1}{\sqrt{-\det G_l}} \begin{pmatrix} \sinh l_1 & 0 & 0 \\ 0 & \sinh l_2 & 0 \\ 0 & 0 & \sinh l_3 \end{pmatrix} \begin{pmatrix} -1 & \cosh l_3 & \cosh l_2 \\ \cosh l_3 & -1 & \cosh l_1 \\ \cosh l_2 & \cosh l_1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} dr_1 \\ dr_2 \\ dr_3 \end{pmatrix}.$$

Since $du_i = \tanh^{\lambda+1}\frac{l_i}{2}\, dl_i$ and $dv_i = \cosh^{\lambda} r_i\, dr_i$ for $i = 1,2,3$, we have

$$\begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix} = \frac{-1}{\sqrt{-\det G_l}} \begin{pmatrix} \tanh^{\lambda+1}\frac{l_1}{2}\sinh l_1 & 0 & 0 \\ 0 & \tanh^{\lambda+1}\frac{l_2}{2}\sinh l_2 & 0 \\ 0 & 0 & \tanh^{\lambda+1}\frac{l_3}{2}\sinh l_3 \end{pmatrix}$$
$$\begin{pmatrix} \cosh l_2 + \cosh l_3 & \cosh l_2 - 1 & \cosh l_3 - 1 \\ \cosh l_1 - 1 & \cosh l_3 + \cosh l_1 & \cosh l_3 - 1 \\ \cosh l_1 - 1 & \cosh l_2 - 1 & \cosh l_1 + \cosh l_2 \end{pmatrix}$$
$$\begin{pmatrix} \cosh^{-\lambda} r_1 & 0 & 0 \\ 0 & \cosh^{-\lambda} r_2 & 0 \\ 0 & 0 & \cosh^{-\lambda} r_3 \end{pmatrix} \begin{pmatrix} dv_1 \\ dv_2 \\ dv_3 \end{pmatrix}.$$

From (2.2.1), we have $\tanh\frac{l_i}{2}\cosh r_i = \tanh\frac{l_j}{2}\cosh r_j$. Thus A is symmetric and negative definite if and only if A_1 is symmetric and positive definite, where

$$A_1 = \begin{pmatrix} \tanh\frac{l_1}{2}\sinh l_1 & 0 & 0 \\ 0 & \tanh\frac{l_2}{2}\sinh l_2 & 0 \\ 0 & 0 & \tanh\frac{l_3}{2}\sinh l_3 \end{pmatrix} \begin{pmatrix} \cosh l_2 + \cosh l_3 & \cosh l_2 - 1 & \cosh l_3 - 1 \\ \cosh l_1 - 1 & \cosh l_3 + \cosh l_1 & \cosh l_3 - 1 \\ \cosh l_1 - 1 & \cosh l_2 - 1 & \cosh l_1 + \cosh l_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh l_1 - 1 & 0 & 0 \\ 0 & \cosh l_2 - 1 & 0 \\ 0 & 0 & \cosh l_3 - 1 \end{pmatrix} \begin{pmatrix} \cosh l_2 + \cosh l_3 & \cosh l_2 - 1 & \cosh l_3 - 1 \\ \cosh l_1 - 1 & \cosh l_3 + \cosh l_1 & \cosh l_3 - 1 \\ \cosh l_1 - 1 & \cosh l_2 - 1 & \cosh l_1 + \cosh l_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh l_1 - 1 & 0 & 0 \\ 0 & \cosh l_2 - 1 & 0 \\ 0 & 0 & \cosh l_3 - 1 \end{pmatrix} \begin{pmatrix} \frac{\cosh l_2 + \cosh l_3}{\cosh l_1 - 1} & 1 & 1 \\ 1 & \frac{\cosh l_3 + \cosh l_1}{\cosh l_2 - 1} & 1 \\ 1 & 1 & \frac{\cosh l_1 + \cosh l_2}{\cosh l_3 - 1} \end{pmatrix} \begin{pmatrix} \cosh l_1 - 1 & 0 & 0 \\ 0 & \cosh l_2 - 1 & 0 \\ 0 & 0 & \cosh l_3 - 1 \end{pmatrix}$$

which is obviously symmetric.

To show A_1 is positive definite, we only need to show that

$$A_2 = \begin{pmatrix} \frac{\cosh l_2 + \cosh l_3}{\cosh l_1 - 1} & 1 & 1 \\ 1 & \frac{\cosh l_3 + \cosh l_1}{\cosh l_2 - 1} & 1 \\ 1 & 1 & \frac{\cosh l_1 + \cosh l_2}{\cosh l_3 - 1} \end{pmatrix}$$

is positive definite.

Since the determinants of its 1×1 and 2×2 principal submatrices are positive. We need to check $\det A_2 > 0$. This is true due to the result of determinant of the Gram matrix (Lemma 2.1.2) and

$$A_2 = \begin{pmatrix} -1 & \cosh l_3 & \cosh l_2 \\ \cosh l_3 & -1 & \cosh l_1 \\ \cosh l_2 & \cosh l_1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\cosh l_1 - 1} & 0 & 0 \\ 0 & \frac{1}{\cosh l_2 - 1} & 0 \\ 0 & 0 & \frac{1}{\cosh l_3 - 1} \end{pmatrix}.$$

□

Proof of Theorem 1.4.1. By Lemma 1.3.3, we see that the Teichmüller space $\mathcal{T}(S)$ is identified with $\mathbb{R}^E_{>0}$. Precisely, a hyperbolic metric on an ideally triangulated surface (S, T) with geodesic boundary is identified with its length coordinate.

For a hyperbolic metric $l \in \mathbb{R}^E_{>0} = \mathcal{T}(S)$, define a map $U : \mathbb{R}^E_{>0} \to \mathbb{R}^E$ by

$$u(e) := \int_1^{l(e)} \tanh^{\lambda+1}(\frac{t}{2})dt.$$

Since U is a smooth embedding, to show $\Psi_\lambda : \mathcal{T}(S) \to \mathbb{R}^E$ is a smooth embedding, it is equivalent to show $\Psi_\lambda \circ U^{-1}$ is a smooth embedding, where $\Psi_\lambda \circ U^{-1}(u(e)) = \psi_\lambda(e)$.

Let's define a function $W = \sum_{\{i,j,k\}} F(u(e_i), u(e_j), u(e_k))$ where the sum runs over all the hexagons in the ideal triangulation and the function F is defined in Corollary 2.2.2. We can see that

$$\frac{\partial W}{\partial u(e)} = \psi_\lambda(e) = \Psi_\lambda \circ U^{-1}(u(e)).$$

This shows that $\Psi_\lambda \circ U^{-1}$ is the gradient of W. By Corollary 2.2.2, the function W is strictly concave down. Therefore $\Psi_\lambda \circ U^{-1}$ is an smooth embedding due the well-known fact:

Lemma 2.2.3. *If X is an open convex set in $\mathbb{R}^n$ and $f : X \to \mathbb{R}$ is smooth strictly convex, then the gradient $\nabla f : X \to \mathbb{R}^n$ is injective. Furthermore, if the Hessian of f is positive definite for all $x \in X$, then ∇f is a smooth embedding.*

□

Chapter 3

Image of Teichmüller spaces

3.1 Degenerations of a hyperbolic hexagon

In this section we always assume a hyperbolic right-angled hexagon has three non-pairwise adjacent edges of lengths l_1, l_2, l_3 and opposite edges of lengths $\theta_1, \theta_2, \theta_3$. Recall that the r-coordinate is defined as $r_i = \frac{\theta_j+\theta_k-\theta_i}{2}$.

We improve a lemma proved in Luo [9].

Lemma 3.1.1. *Consider r_i as a function of (l_1, l_2, l_3). We have $\lim_{l_i\to 0} r_i = \infty$ so that the convergence is uniform in (l_1, l_2, l_3).*

Proof. By the cosine law of a hyperbolic right-angled hexagon (2.1.1), we see that for $i \neq j \neq k \neq i$,

$$\begin{aligned}\cosh\theta_j &= \frac{\cosh l_j + \cosh l_i \cosh l_k}{\sinh l_i \sinh l_k}\\ &> \frac{\cosh l_i \cosh l_k}{\sinh l_i \sinh l_k}\\ &\geq \frac{\cosh l_i}{\sinh l_i}.\end{aligned}$$

Hence we have $\lim_{l_i\to 0}\theta_j = \infty$. Thus

$$\lim_{l_i\to 0}\frac{\cosh\theta_j}{\sinh\theta_j} = 1.$$

By symmetry we have

$$\lim_{l_i\to 0}\frac{\cosh\theta_k}{\sinh\theta_k} = 1.$$

On the other hand, by the cosine law we see that for $i \neq j \neq k \neq i$,

$$\cosh l_i - \frac{\cosh\theta_j\cosh\theta_k}{\sinh\theta_j\sinh\theta_k} = \frac{\cosh\theta_i}{\sinh\theta_j\sinh\theta_k} > \frac{2e^{\theta_i}}{e^{\theta_j+\theta_k}} = \frac{2}{e^{2r_i}}.$$

Since the left hand side converges to 0 as $l_i \to 0$, we have $\lim_{l_i\to 0} r_i = \infty$.

To show the convergence is uniform, by the tangent law (2.2.1), we have, for $i \neq j \neq k \neq i$,

$$\begin{aligned}\tanh^2 \frac{l_i}{2} &= \frac{\cosh r_j \cosh r_k}{\cosh r_i \cosh(r_i + r_j + r_k)} \\ &= \frac{1}{\cosh r_i} \cdot \frac{1}{(1 + \tanh r_j \tanh r_k) \cosh r_i + (\tanh r_j + \tanh r_k) \sinh r_i} \\ &\geq \frac{1}{\cosh r_i} \cdot \frac{1}{(1+1) \cosh r_i + (1+1)|\sinh r_i|} \\ &\geq \frac{1}{4\cosh^2 r_i}.\end{aligned}$$

It follows that

$$\cosh^2 r_i \geq \frac{1}{4\tanh^2 \frac{l_i}{2}}.$$

Thus r_i converges to ∞ uniformly. □

Lemma 3.1.2. *The following holds for some positive finite numbers f_1, f_2, f_3, f_4, f_5:*

(1). if (l_1, l_2, l_3) converges to (∞, f_1, f_2), then $(\theta_1, \theta_2, \theta_3)$ converges to (∞, f_3, f_4);

(2). if (l_1, l_2, l_3) converges to (∞, ∞, f_5), then θ_3 converges to 0;

(3). if (l_1, l_2, l_3) converges to (∞, ∞, ∞), then we can chose a subsequence of (l_1, l_2, l_3) such that at least two of θ_1, θ_2 and θ_3 converge to 0.

Proof. (1). The cosine law (2.1.1) says

$$\cosh \theta_1 = \frac{\cosh l_1 + \cosh l_2 \cosh l_3}{\sinh l_2 \sinh l_3}.$$

If $\lim(l_1, l_2, l_3) = (\infty, f_1, f_2)$, we have $\lim \cosh \theta_1 = \infty$, or $\lim \theta_1 = \infty$. Since $\lim \frac{\cosh l_1}{\sinh l_1} = 1$,

$$\lim \cosh \theta_2 = \lim \frac{\cosh l_2 + \cosh l_1 \cosh l_3}{\sinh l_1 \sinh l_3} = \frac{\cosh f_2}{\sinh f_2} > 1.$$

Thus $\lim \theta_2$ is a positive finite number. By symmetry $\lim \theta_3$ is a positive finite number.

(2). If $\lim(l_1, l_2, l_3) = (\infty, \infty, f_5)$, we have

$$\lim \cosh \theta_3 = \lim \frac{\cosh l_3 + \cosh l_1 \cosh l_2}{\sinh l_1 \sinh l_2} = \lim \frac{\cosh l_3}{\sinh l_1 \sinh l_2} + 1 = 1.$$

Thus $\lim \theta_3 = 0$.

(3). If $\lim(l_1, l_2, l_3) = (\infty, \infty, \infty)$, we have

$$\lim \cosh \theta_i = \lim \frac{\cosh l_i + \cosh l_j \cosh l_k}{\sinh l_j \sinh l_k} = \lim \frac{\cosh l_i}{\sinh l_j \sinh l_k} + 1$$

$$= \lim \frac{2e^{l_i}}{e^{l_j + l_k}} + 1 = \lim 2e^{l_i - l_j - l_k} + 1.$$

Since $\lim e^{l_i - l_j - l_k} e^{l_j - l_i - l_k} = \lim e^{-2l_k} = 0$, by taking subsequence of (l_1, l_2, l_3), we may assume $\lim e^{l_i - l_j - l_k}$ and $\lim e^{l_j - l_i - l_k}$ exist. Then one of $\lim e^{l_i - l_j - l_k}$ and $\lim e^{l_j - l_i - l_k}$ is 0. Hence at least two of $\lim \theta_1, \lim \theta_2$ and $\lim \theta_3$ are 0. □

3.2 Determine the image

Lemma 3.2.1. *If $a > 0$, then for any real number x, we have*

$$\int_0^{a+x} \cosh^\lambda(t)dt + \int_0^{a-x} \cosh^\lambda(t)dt > 0.$$

Proof. Let $f(a)$ be the function of the left hand side of the inequality. Since $f'(a) = \cosh^\lambda(a+x) + \cosh^\lambda(a-x) > 0$ and $f(0) = 0$, we have $f(a) > 0$ for $a > 0$. □

Proof of Theorem 1.5.2. We denote the polytope defined by the inequalities in condition (1), (2), (3) by P_λ. First we claim $\Psi_\lambda(\mathcal{T}(S)) \subset P_\lambda$. Indeed, fix a hyperbolic metric $l \in \mathbb{R}^E_{>0} = \mathcal{T}(S)$. For any edge e, let r, r' be the r-coordinates of the B-arcs facing e, then

$$\psi_\lambda(e) = \int_0^r \cosh^\lambda(t)dt + \int_0^{r'} \cosh^\lambda(t)dt < 2\int_0^\infty \cosh^\lambda(t)dt.$$

Thus the condition (1) holds.

Given an edge path $(H_0, e_1, H_1, ..., e_n, H_n)$, for $i = 1, ..., n-1$, let a_i be the length of the B-arc in H_i adjacent to e_i and e_{i+1}. Denote the lengths of the B-arcs in H_i facing e_i and e_{i+1} by b_i and c_i respectively as labelled in Figure 3.1 (a).

Then by definition

$$\psi_\lambda(e_1) = \int_0^r \cosh^\lambda(t)dt + \int_0^{\frac{a_1+c_1-b_1}{2}} \cosh^\lambda(t)dt,$$

where r is the r-coordinate of the B-arc in H_0 facing e_1. For $i = 2, ..., n-1$,

$$\psi_\lambda(e_i) = \int_0^{\frac{a_{i-1}+b_{i-1}-c_{i-1}}{2}} \cosh^\lambda(t)dt + \int_0^{\frac{a_i+c_i-b_i}{2}} \cosh^\lambda(t)dt,$$

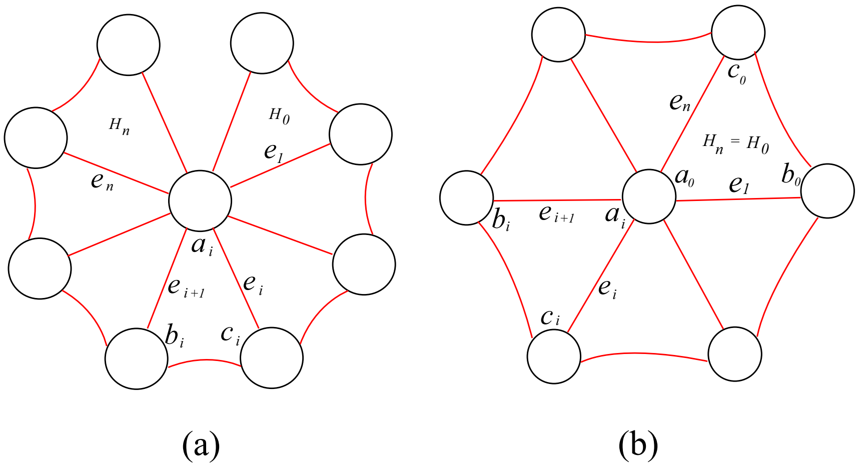

Figure 3.1: (a) An example of an edge path. (b) An example of an edge cycle.

$$\psi_\lambda(e_n) = \int_0^{\frac{a_{n-1}+b_{n-1}-c_{n-1}}{2}} \cosh^\lambda(t)dt + \int_0^{r'} \cosh^\lambda(t)dt,$$

where r' is the r-coordinate of the B-arc in H_n facing e_n.

Hence by Lemma 3.2.1,

$$\sum_{i=1}^{n} \psi_\lambda(e_i) = \sum_{i=1}^{n-1}(\int_0^{\frac{a_i+c_i-b_i}{2}} \cosh^\lambda(t)dt + \int_0^{\frac{a_i+b_i-c_i}{2}} \cosh^\lambda(t)dt)$$
$$+ \int_0^{r} \cosh^\lambda(t)dt + \int_0^{r'} \cosh^\lambda(t)dt$$
$$> \int_0^{r} \cosh^\lambda(t)dt + \int_0^{r'} \cosh^\lambda(t)dt$$
$$> 2\int_0^{-\infty} \cosh^\lambda(t)dt = -2\int_0^{\infty} \cosh^\lambda(t)dt.$$

Thus the condition (2) holds.

Furthermore, if $(H_0, e_1, H_1, ..., e_n, H_n = H_0)$ is an edge cycle, H_0 contains both e_1 and e_n. Let a_0 be the length of the B-arc in H_0 adjacent to e_1 and e_n, b_0, c_0 be the lengths of the B-arcs facing e_n and e_0 respectively as labelled in Figure 3.1 (b). Thus the r-coordinates are $r = \frac{a_0+b_0-c_0}{2}$ and $r' = \frac{a_0+c_0-b_0}{2}$. Hence

$$\sum_{i=1}^{n} \psi_\lambda(e_i) > \int_0^{r} \cosh^\lambda(t)dt + \int_0^{r'} \cosh^\lambda(t)dt > 0$$

by Lemma 3.2.1. Thus the condition (3) holds.

Now by Theorem 1.4.1, $\Psi_\lambda : \mathcal{T}(S) \to P_\lambda$ is an embedding. Therefore $\Psi_\lambda(\mathcal{T}(S))$ is open in P_λ. We only need to show it is also closed in P_λ. This will finish the proof since P_λ is connected.

Take a sequence $l^{(m)} \in \mathcal{T}(S)$ so that $\lim_{m\to\infty} \Psi_\lambda(l^{(m)}) = z \in P_\lambda$. By taking a subsequence, we may assume that $\lim_{m\to\infty} l^{(m)} \in [0,\infty]^E$ exists and the length of each B-arc converges into $[0,\infty]$. We only need to show that $\lim_{m\to\infty} l^{(m)} \in (0,\infty)^E = \mathcal{T}(S)$. This will finish the proof since $z = \Psi_\lambda(\lim_{m\to\infty} l^{(m)})$.

Suppose otherwise that there is an edge $e \in E$ so that $\lim_{m\to\infty} l^{(m)}(e) \in \{0,\infty\}$. We will discuss two cases.

Case 1, $\lim_{m\to\infty} l^{(m)}(e) = 0$ for some $e \in E$. Let H, H' be the hexagons sharing e and $r^{(m)}, r'^{(m)}$ be the r-coordinates of the B-arcs in H, H' facing e. Then by Lemma 3.1.1, $\lim_{m\to\infty} r^{(m)} \to \infty, \lim_{m\to\infty} r'^{(m)} \to \infty$. Then

$$z(e) = \lim_{m\to\infty} \left(\int_0^{r^{(m)}} \cosh^\lambda(t)dt + \int_0^{r'^{(m)}} \cosh^\lambda(t)dt \right) = 2\int_0^\infty \cosh^\lambda(t)dt.$$

This is impossible since $z \in P_\lambda$ must satisfy the condition (1).

Due to case 1, we can assume $\lim_{m\to\infty} l^{(m)} \in (0,\infty]^E$.

Case 2, $\lim_{m\to\infty} l^{(m)}(e) = \infty$ for some $e \in E$. The following construction is illustrated by the example in Figure 3.2. Define the subset

$$E_\infty = \{e \in E | \lim_{m\to\infty} l^{(m)}(e) = \infty\}.$$

We construct a graph G as a subgraph of the dual graph associated to the ideal triangulation as follows. A vertex of G is a hexagon with at least one edge in E_∞. There is a *dual-edge* in G joining two vertexes if and only if the two hexagons corresponding to the vertexes share an edge in E_∞. The degree of a vertex of the graph G can only be 1, 2 or 3. Actually a vertex of degree 1, 2 or 3 corresponds to the hexagon of type (1),(2) or (3) in Lemma 3.1.2 respectively.

We smooth the graph G at vertexes as follows. At a vertex of degree 1, we replace the small neighborhood of the vertex in G by a short smooth curve tangent to the unique dual-edge incident to the vertex as in Figure 3.3 (a). At a vertex v of degree of 2 or 3, every two dual-edges $\bar{e}_1, \bar{e}_2$ incident to v correspond to two edges e_1, e_2 in a

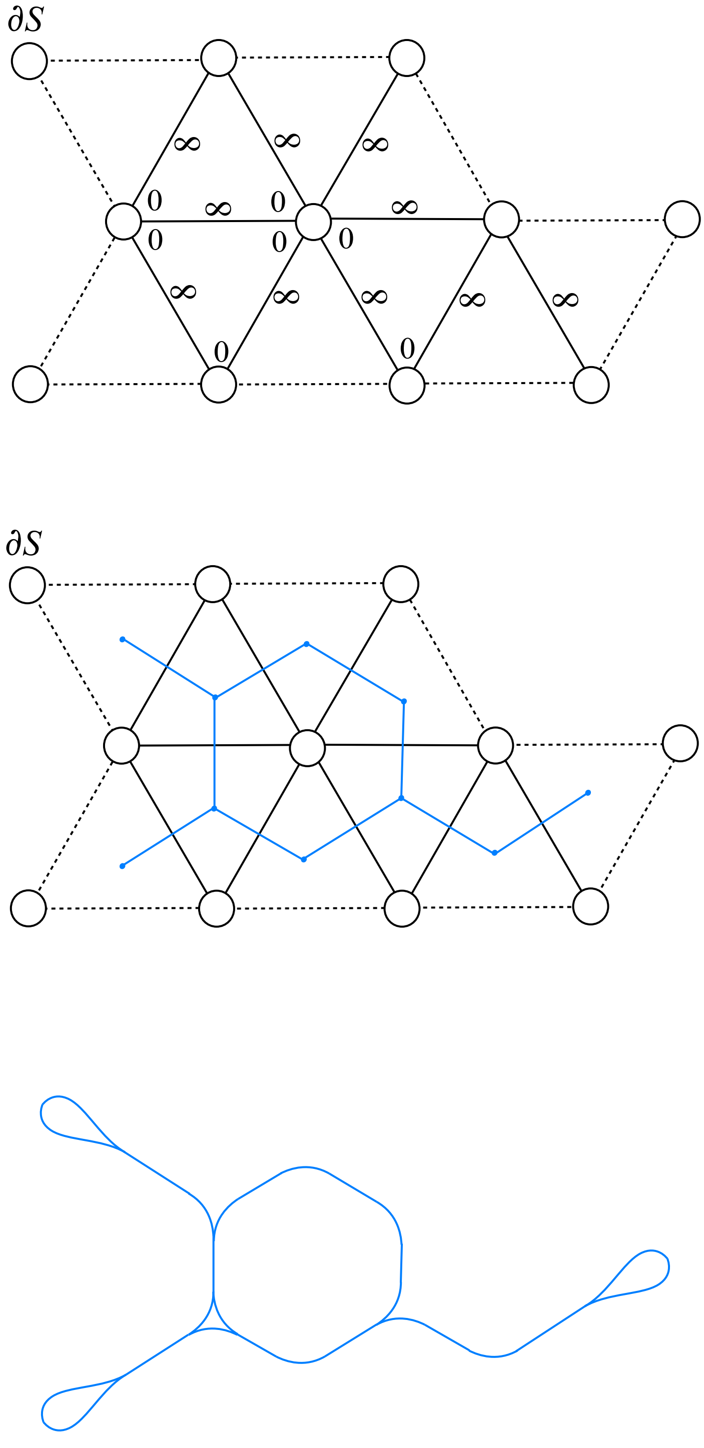

Figure 3.2: Construction of a smooth curve from a set of edges of infinite lengths.

hexagon. If the length of the B-arc adjacent to e_1, e_2 converges to 0, we replace the small neighborhood of the vertex v in G by a short smooth curve tangent to $\overline{e}_1, \overline{e}_2$. According to Lemma 3.1.2, every vertex of degree 2 can be smoothed as in Figure 3.3 (b) and there are two cases for a vertex of degree 3 according to the lengths of two or three B-arcs converge to 0 as in Figure 3.3 (c).

We denote by G' the graph smoothed at vertexes and the dual-edges of G' are the dual-edges of G. We claim that there exists a smooth closed curve in G' such that every dual-edge repeats at most twice in the closed curve. In fact, we give every dual-edge of G' an arbitrary orientation. Pick up any smooth closed curve in G' which may contains arbitrarily many dual-edges. If there exists a dual-edge $\overline{e}$ repeating with the same orientation in the closed curve, there is another smooth closed curve starting and ending at $\overline{e}$. By this procedure we can reduce the number of dual-edges in a closed curve. At last we obtain a smooth closed curve in G' such that every dual-edge repeats at most twice.

This smooth closed curve in G' corresponds a fundamental edge path or fundamental edge cycle in the ideal triangulation. First assume it is a fundamental edge path $(H_0, e_1, H_1, ..., e_n, H_n)$. Since the degree of the vertex corresponding to H_0 (or H_n) is 1, the lengths of other two edges other than e_1 (or e_n) converge to positive finite numbers in the sequence of metric $l^{(m)}$. By Lemma 3.1.2(1) the r-coordinate of the B-arc in H_0 (or H_n) facing e_1 (or e_n) converges to $-\infty$. By the construction of the edge path, the length of B-arc adjacent to e_i and e_{i+1} converges to 0 for $i = 1, ..., n-1$. We denote b_i, c_i the limit of lengths of B-arcs in H_i facing e_i, e_{i+1} respectively, see Figure 3.1(a).

Hence

$$z(e_1) = \int_0^{-\infty} \cosh^\lambda(t)dt + \int_0^{\frac{c_1-b_1}{2}} \cosh^\lambda(t)dt.$$

For $i = 2, ..., n-1$,

$$z(e_i) = \int_0^{\frac{b_{i-1}-c_{i-1}}{2}} \cosh^\lambda(t)dt + \int_0^{\frac{c_i-b_i}{2}} \cosh^\lambda(t)dt.$$

$$z(e_n) = \int_0^{\frac{b_{n-1}-c_{n-1}}{2}} \cosh^\lambda(t)dt + \int_0^{-\infty} \cosh^\lambda(t)dt.$$

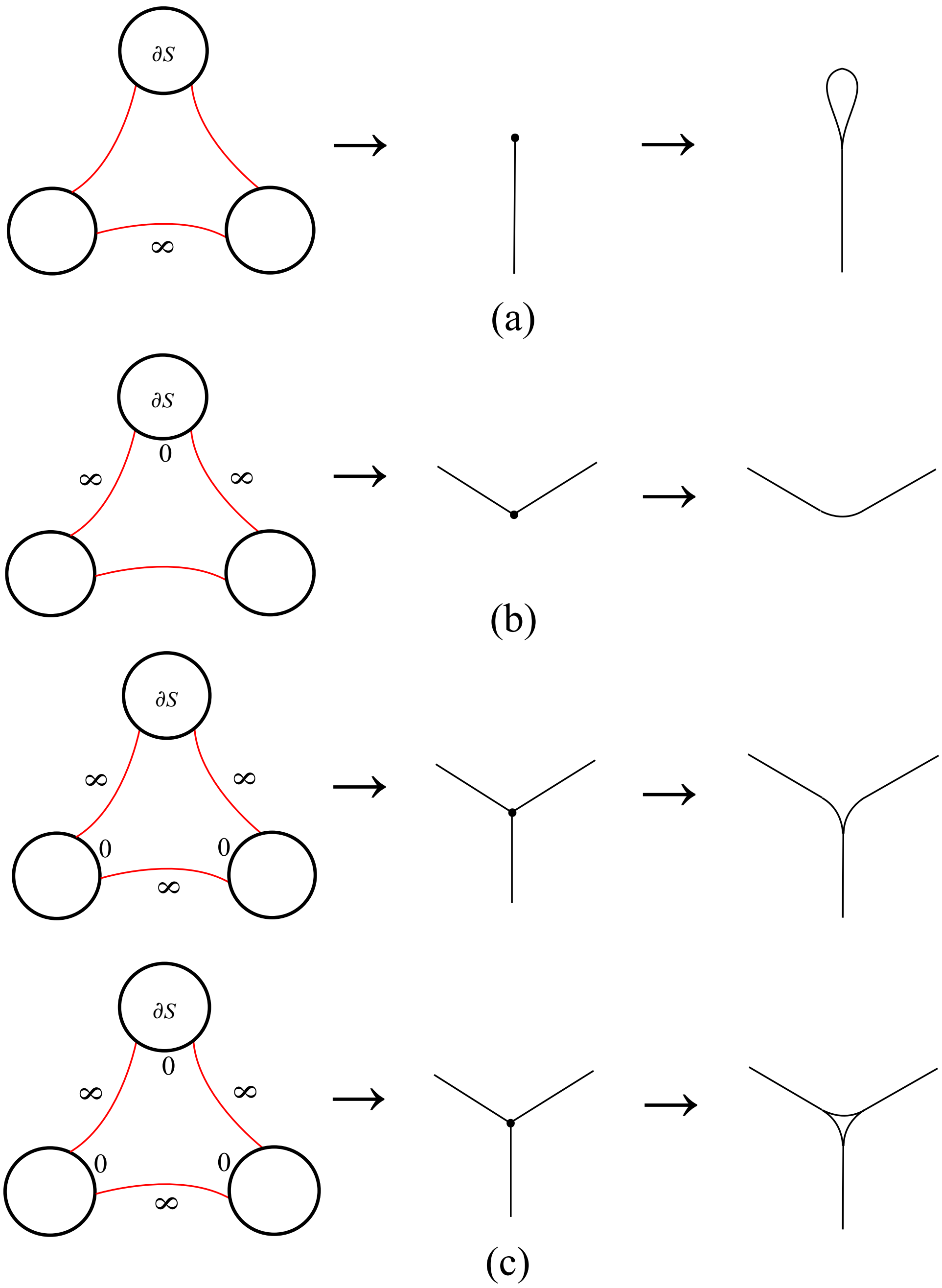

Figure 3.3: Smooth graph G at a vertex of degree 1 (a), degree 2 (b), degree 3 (c).

Therefore

$$\sum_{i=1}^{n} z(e_i) = 2\int_0^{-\infty} \cosh^\lambda(t)dt + \sum_{i=1}^{n-1}(\int_0^{\frac{c_i-b_i}{2}} \cosh^\lambda(t)dt + \int_0^{\frac{b_i-c_i}{2}} \cosh^\lambda(t)dt)$$
$$= -2\int_0^{\infty} \cosh^\lambda(t)dt.$$

This is impossible since $z \in P_\lambda$ must satisfy the condition (2).

If the smooth closed curve in G' corresponds to a fundamental edge cycle $(H_0, e_1, H_1, ..., e_n, H_n = H_0)$, the length of B-arc in H_0 adjacent to e_1 and e_n is 0. Denote b_0, c_0 the lengths of B-arcs facing e_n and e_0, see Figure 3.1(b). Thus

$$z(e_1) = \int_0^{\frac{b_0-c_0}{2}} \cosh^\lambda(t)dt + \int_0^{\frac{c_1-b_1}{2}} \cosh^\lambda(t)dt,$$

$$z(e_n) = \int_0^{\frac{b_{n-1}-c_{n-1}}{2}} \cosh^\lambda(t)dt + \int_0^{\frac{c_0-b_0}{2}} \cosh^\lambda(t)dt.$$

As in the case of fundamental edge path, for $i = 2, ..., n-1$,

$$z(e_i) = \int_0^{\frac{b_{i-1}-c_{i-1}}{2}} \cosh^\lambda(t)dt + \int_0^{\frac{c_i-b_i}{2}} \cosh^\lambda(t)dt.$$

Hence $\sum_{i=1}^{n} z(e_i) = 0$. This is impossible since $z \in P_\lambda$ must satisfy the condition (3).

We finish the proof of $\Psi_\lambda(\mathcal{T}(S)) = P_\lambda$. Since there are only finitely many fundamental edge paths or fundamental edge cycles in an ideal triangulation, $\Psi_\lambda(\mathcal{T}(S))$ is defined by finite many inequalities in condition (1), (2), (3). Thus it is a open convex polytope.

The statement $\Psi_{\lambda_1}(T(S)) \subset \Psi_{\lambda_2}(T(S)) \subset \Psi_0(T(S))$ for $\lambda_1 < \lambda_2 < 0$ is obvious since the function $\int_0^\infty \cosh^\lambda(t)dt$ is increasing in λ and it is ∞ when $\lambda \geq 0$.

Since $0 < \cosh^\lambda(t) < \cosh^{-1}(t)$ for $\lambda < -1$ and $\int_0^\infty \cosh^{-1}(t)dt < \infty$, by Lebesgue's dominated convergence theorem, we have

$$\lim_{\lambda\to-\infty} \int_0^{\infty} \cosh^\lambda(t)dt = \int_0^{\infty} \lim_{\lambda\to-\infty} \cosh^\lambda(t)dt = 0.$$

Thus the intersection $\cap_{\lambda=0}^{-\infty}\Psi_\lambda(\mathcal{T}(S))$ is the set of points $z \in \mathbb{R}^E$ satisfying $z(e) < 0$ for each edge e and $\sum_{i=1}^{n} z(e_i) > 0$ for each fundamental edge path $(H_0, e_1, H_1, ..., e_n, H_n)$. It is an empty set. □

3.3 Examples

In this section, the images of Teichmüller spaces of the simplest surfaces in Figure 1.1 under the ψ_λ coordinate are explicitly described.

In Figure 1.1 (a), the three-holed sphere $\Sigma_{0,3}$ is decomposed as a union of two colored hexagons. Denote this ideal triangulation by T. From the dual graph of T, it is easy to see that in this ideal triangulation the fundamental edge cycles are $(e_1, e_2), (e_2, e_3), (e_3, e_1)$ (without confusing we only indicate the edges not the hexagons in an edge cycle) and the fundamental edge pathes are $(e_1), (e_2), (e_3)$.

Therefore by Theorem 1.5.1, for any $\lambda \geq 0$, the image of the Teichmüller space $\Psi_\lambda(\mathcal{T}(\Sigma_{0,3}), T)$ is an open convex polytope defined by the following inequalities:

$$\begin{cases} z(e_1) + z(e_2) > 0 \\ z(e_2) + z(e_3) > 0 \\ z(e_3) + z(e_1) > 0 \end{cases}$$

This polytope $\Psi_\lambda(\mathcal{T}(\Sigma_{0,3}), T)(\lambda \geq 0)$ is an open cone with cone point $O = (0, 0, 0)$ bounded by three planes containing triangles OAB, OBC, OCA in Figure 3.4.

Let's denote $c_\lambda = 2\int_0^\infty \cosh^\lambda(t)dt$. By Theorem 1.5.2, for any $\lambda < 0$, the image of Teichmüller space $\Psi_\lambda(\mathcal{T}(\Sigma_{0,3}), T)$ is an open convex polytope defined by the following inequalities:

$$\begin{cases} c_\lambda > z(e_1) > -c_\lambda \\ c_\lambda > z(e_2) > -c_\lambda \\ c_\lambda > z(e_3) > -c_\lambda \\ z(e_1) + z(e_2) > 0 \\ z(e_2) + z(e_3) > 0 \\ z(e_3) + z(e_1) > 0 \end{cases}$$

This polytope $\Psi_\lambda(\mathcal{T}(\Sigma_{0,3}), T)(\lambda < 0)$ is drawn in Figure 3.4, where the box is $(-c_\lambda, c_\lambda)^3$ and this polytope is the open convex hull of the vertices

$$O = (0, 0, 0),$$

$$A = (-c_\lambda, c_\lambda, c_\lambda),$$

$$B = (c_\lambda, -c_\lambda, c_\lambda),$$

$$C = (c_\lambda, c_\lambda, -c_\lambda),$$

$$D = (c_\lambda, c_\lambda, c_\lambda).$$

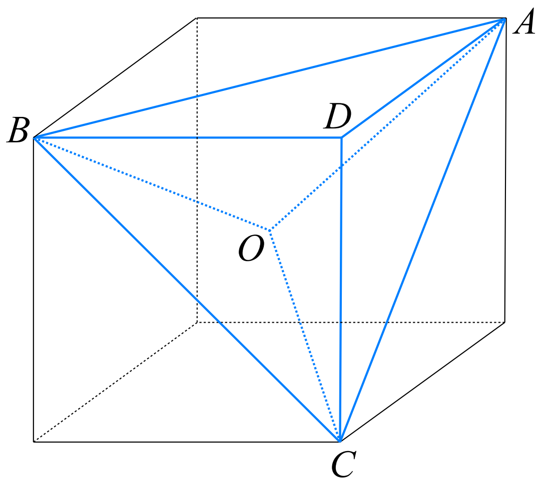

Figure 3.4: The image of the Teichmüller space of the ideally triangulated three-holed sphere in Figure 1.1 (a).

In Figure 1.1 (b), another ideal triangulation T' of the three-holed sphere $\Sigma_{0,3}$ is given. From the dual graph of T', it is easy to see that the fundamental edge cycles are $(e_1), (e_2), (e_1, e_3, e_2, e_3)$ and the fundamental edge path is (e_3).

Therefore by Theorem 1.5.1, for any $\lambda \geq 0$, the image of the Teichmüller space $\Psi_\lambda(\mathcal{T}(\Sigma_{0,3}), T')$ is an open convex polytope defined by the following inequalities:

$$\begin{cases} z(e_1) > 0 \\ z(e_2) > 0 \\ z(e_1) + z(e_2) + 2z(e_3) > 0 \end{cases}$$

This polytope $\Psi_\lambda(\mathcal{T}(\Sigma_{0,3}), T')(\lambda \geq 0)$ is an open cone with cone point $O = (0, 0, 0)$ bounded by three planes containing quadrilaterals $OCBF, OCAE, OFGE$ in Figure 3.5.

By Theorem 1.5.2, for any $\lambda \in \mathbb{R}$, the image of the Teichmüller space $\Psi_\lambda(\mathcal{T}(\Sigma_{0,3}), T')$

is an open convex polytope defined by the following inequalities:

$$\begin{cases} c_\lambda > z(e_1) > 0 \\ c_\lambda > z(e_2) > 0 \\ c_\lambda > z(e_3) > -c_\lambda \\ z(e_1) + z(e_2) + 2z(e_3) > 0 \end{cases}$$

This polytope $\Psi_\lambda(\mathcal{T}(\Sigma_{0,3}), T')(\lambda < 0)$ is drawn in Figure 3.5, where the box is $(-c_\lambda, c_\lambda)^3$ and this polytope is the open convex hull of the vertices

$$A = (0, c_\lambda, c_\lambda), E = (0, c_\lambda, -\frac{1}{2}c_\lambda),$$

$$B = (c_\lambda, 0, c_\lambda), F = (c_\lambda, 0, -\frac{1}{2}c_\lambda),$$

$$C = (0, 0, c_\lambda), O = (0, 0, 0),$$

$$D = (c_\lambda, c_\lambda, c_\lambda), G = (c_\lambda, c_\lambda, -c_\lambda).$$

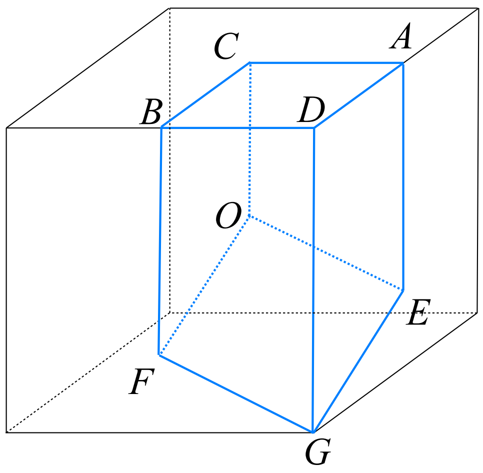

Figure 3.5: The image of the Teichmüller space of the ideally triangulated three-holed sphere in Figure 1.1 (b).

In Figure 1.1 (c), the one-holed torus $\Sigma_{1,1}$ is decomposed as a union of two colored hexagons. This ideal triangulation of $\Sigma_{1,1}$ and the ideal triangulation T of a three-holed sphere $\Sigma_{0,3}$ in (a) have same the dual graphs. Thus the image of their Teichmüller spaces under the ψ_λ coordinate are the same for any $\lambda \in \mathbb{R}$.

References

[1] F. Bonahon, *Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form*, Ann. Fac. Sci. Toulouse Math. (6) 5 (1996), no. 2, 233–297.

[2] Peter Buser, *Geometry and spectra of compact Riemann surfaces*, Progress in Mathematics, 106. Birkhäuser Boston, Inc., Boston, MA, 1992.

[3] Bennett Chow & Feng Luo, *Combinatorial Ricci flows on surfaces.* J. Differential Geom. 63 (2003), no. 1, 97–129

[4] Y. Colin de Verdiére, *Un principe variationnel pour les empilements de cercles.* Invent. Math., 104:655–669, 1991

[5] Y. Imayoshi & M. Taniguchi, *An introduction to Teichmüller spaces*, Translated and revised from the Japanese by the authors, Springer-Verlag, Tokyo, 1992.

[6] Ren Guo, *On parameterizations of Teichmüller spaces of surfaces with boundary*, arXiv:math.GT/0612221

[7] Feng Luo, *A characterization of spherical polyhedron surfaces*, J. Differential. Geom. 74 (2006), no. 3, 407-424.

[8] Feng Luo, *On Teichmüller space of surfaces with boundary*, Duke Math. J. Volume 139, Number 3 (2007), 463-482.

[9] Feng Luo, *Rigidity of polyhedral surfaces*, arXiv:math.GT/0612714.

[10] Feng Luo, *private communication.*

[11] R. C. Penner, *The decorated Teichmüller space of punctured surfaces*, Comm. Math. Phys. 113 (1987), no. 2, 299–339.

[12] R. C. Penner, *Decorated Teichmüller theory of bordered surfaces*, Comm. Anal. Geom. 12 (2004), no. 4, 793–820.

[13] Akira Ushijima, *A Canonical Cellular Decomposition of the Teichmüller Space of Compact Surfaces with Boundary*, Comm. Math. Phys. 201 (1999), no. 2, 305-326.

Vita

Ren Guo

May 2008 Ph.D. in Mathematics, Rutgers University, Piscataway, New Jersey

July 2003 M.S. in Mathematics, Beijing Normal University, Beijing, China

July 2000 B.S. in Mathematics, Beijing Normal University, Beijing, China

2007-2008 Graduate Assistant, Department of Mathematics, Rutgers University

2003-2007 Teaching Assistant, Department of Mathematics, Rutgers University

Publications

1. (with Feng Luo) *Rigidity of polyhedral surfaces, II.* arXiv:0711.0766v1
2. *A characterization of hyperbolic geometry among Hilbert geometry.* To appear in Journal of Geometry. arXiv:math.MG/0703184
3. *A note on circle patterns on surfaces.* Geom. Dedicata 125 (2007), 175–190.
4. *On parameterizations of Teichmüller spaces of surfaces with boundary.* arXiv:math.GT/0612221
5. *Geometric angle structures on triangulated surfaces.* Proc. Amer. Math. Soc. 135 (2007), 3005-3011.
6. (with Feng Luo) *On a Conjecture of Milnor about Volume of Simplexes.* To appear in J. Differential Geom. arXiv:math.GT/0510666
7. (with Hongzhu Gao) *Quotient spaces of anti-holomorphic involutions.* Sci. China Ser. A 48 (2005), no. 12, 1631–1636.
8. (with Hongzhu Gao) *Homology classes that can be represented by embedded spheres in rational surfaces.* Sci. China Ser. A 47 (2004), no. 3, 431–439.