WARPED THROAT GEOMETRIES AND LOW-ENERGY SPECTRUM OF CONFINING GAUGE THEORIES

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ABSTRACT OF THE DISSERTATION

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String theory on the warped deformed conifold, which in the low-energy limit was described by Klebanov and Strassler, is by now the only known consistent example of the supergravity dual of a four dimensional confining (supersymmetric) gauge theory. In this work bosonic supergravity excitations over the Klebanov-Strassler background are studied. The excitation correspond to the low-energy states of a dual $\mathcal{N} = 1$ supersymmetric gauge theory. Discovered states are distributed among seven supermultiplets, for which the gauge theory description is determined. This investigation is in particular motivated by an example of the low-energy spectrum in the pure glue gauge theory in the model that might be relevant for the new physics at the LHC.

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Dedication

To V. A. Kashaev, optimist of science

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Chapter 1
Introduction

1.1 Old Problems and New Methods

Modern high energy physics is well explored and understood up to a TeV energy scale. It is accepted that the Standard Model (SM) is a correct microscopic description of particle physics phenomena below that scale. The strong interactions in SM are governed by Quantum Chromodynamics (QCD), which has point-like quarks and gluons as fundamental (microscopic) fields. The coupling g_s of QCD is running with the scale and the theory becomes strongly coupled towards the infrared (IR) end of the renormalization group (RG) flow, i.e. at low energies. In the strong coupling regime, quarks and gluons are no longer observable particles in the physical spectrum as they confine to produce hadrons (mesons or baryons).

At strong coupling the dynamics of QCD becomes very complicated, since the standard perturbation theory does not make sense anymore. Therefore it is not straightforward to start from theory with quarks and gluons and derive the low-energy spectrum of hadrons. It is possible in principle by studying theory numerically on the lattice, but lattice means are still restricted at the moment and no results for observed quantities have been obtained yet. For theoretical calculations one often uses non-perturbative methods, which are typically semi-empirical.

The simplest example of non-perturbative calculation is evaluation of the pion leptonic decay rates. The amplitude of the process $\pi^- \to e^- + \tilde{\nu}_e$ can be written in terms of the following matrix element

$$\langle\, 0 \,|\, J_\mu \,|\, \pi^- \rangle = i p_\mu f_\pi ,$$

where J_μ is an effective non-perturbative operator, interacting with perturbative lepton current, that can create or destroy pions from the vacuum. Momentum of the pion p_μ represents the only possible Lorentz structure of the matrix element, while the value can be expressed in terms of a single constant f_π with dimension of mass. It is called pion decay constant and its numerical value, obtained from the experiment, is $f_\pi \simeq 130$ MeV. This value as well as pion mass m_π cannot be obtained independently from first principles, but it can be used to make the predictions of other observed quantities. For this such non-perturbative methods exist as soft pion theorems, sum rules etc. The range of application for such non-perturbative methods is usually limited.

A more general approach to study the low-energy physics is evaluation of the effective action, in which the microscopic degrees of freedom are integrated out and the fundamental fields are operators like J_μ. However, as was already mentioned, it is not straightforward. Some success was achieved for other examples of four dimensional gauge theories. In particular, supersymmetric gauge theories, although not yet realistic, are promising models for studying effective low-energy actions. Most famous examples are the full low-energy effective actions of $\mathcal{N} = 2$ Supersymmetric Yang-Mills theories (SYM) [1] and superpotentials of the $\mathcal{N} = 1$ SYM theories [2].

$\mathcal{N} = 1$ theories are more attractive from the practical point of view, but they are also more complicated, because less symmetry is involved. In the effective actions, supersymmetry only restricts the form of superpotentials, but kinetic terms, which are described by Kähler potential, are not specified. To find Kähler potential in $\mathcal{N} = 1$ theories one probably has to go beyond the methods of field theory.

Derivation of the $\mathcal{N} = 2$ and $\mathcal{N} = 1$ effective actions in the supersymmetric theories in [1], [2] can be done completely within a field theory formalism, however it was certainly inspired by progress in the string theory, where four dimensional effective theories were obtained via compactification of ten dimensional string theories on six dimensional supersymmetric manifolds, Calabi-Yau.[1] The reason, why the supersymmetric effective actions were derived first in the string theory, is that effective actions have a very simple and elegant geometric interpretation there.

An important ingredient of both stringy and field theoretic derivations of effective actions was duality. Indeed, canonically duality is a symmetry of a theory which interchanges the regimes with opposite values of coupling constant. For example, in theories of phase transition duality interchanges high and low temperature regimes. In the case of QCD duality interchanges the confined (strong coupling) and de-confined (weak coupling) phases of the theory. From the point of view of the action such symmetry acts as a Legendre transformation, while action is the generating functional for the transformation. The above ideas led to discovery of another interesting example of "holographic" duality, which is discussed in the next section.

1.2 Holographic Correspondence

The ideas of the holographic correspondence, namely a correspondence between the (five dimensional) gravity or string theory and the gauge theory on the four dimensional boundary, existed in physics for quite some time [5, 6, 7, 8]. However, it was not until 1997 when an explicit conjecture

[1]For example, see [3] and [4].

was made by J. Maldacena [9] about the duality of the type IIB string theory on the $AdS_5 \times S^5$ and the $SU(N)$ conformal $\mathcal{N} = 4$ SYM theory on the stack of N coincident D3-branes. This conjecture received the name of AdS/CFT correspondence.

The type IIB string theory is defined in ten dimensions. In order to obtain a five dimensional theory one needs to compactify it on some five dimensional manifold (S^5 in the case of AdS/CFT correspondence). In the low-energy limit this manifold will describe the intrinsic degrees of freedom of the gauge theory, while the dimension transverse to the four dimensional Minkowski space will measure the energy scale at which the theory is defined.

The conjecture was based on the symmetries existing in two theories. The conformal symmetry of the $\mathcal{N} = 4$ SYM is generated by the group $SO(4,2)$, while the only gravity solution with such group of isometries is the AdS_5 space. Also the isometries of the 5-sphere form the group $SO(6) \simeq SU(4)$, which is the $\mathcal{R}$-symmetry group of the $\mathcal{N} = 4$ SYM.

The following argument in favor of the conjecture was given in the review [10]. Consider the low-energy limit of the type IIB string theory in the flat space with a stack of N D3-branes. In the low energy limit ($\alpha' \to 0$) the massive excitations of the string theory decouple. Thus one can write an effective action for the massless excitations with massive ones integrated out. String theory has two type of fluctuations: the closed string that live in the bulk and the open strings that describe the modes of the theory on the D-branes. The massless spectrum of the closed type IIB strings is given by the fields of the type IIB supergravity, while the massless degrees of freedom of open strings on the stack of N D3-branes correspond to the $U(N)$ $\mathcal{N} = 4$ SYM. The effective action can be split into three parts:

$$S = S_{\text{bulk}} + S_{\text{branes}} + S_{\text{int}}.$$

The bulk action S_{bulk} represents the action of the type IIB supergravity with some higher derivative corrections, which can be ignored in the low-energy limit. The action S_{brane} is the Dirac-Born-Infeld of action on the brane, which reduces to the $\mathcal{N} = 4$ SYM action in the limit $\alpha' \to 0$. The bulk-brane interaction action S_{int} has a higher order in α'. Thus, in the low energy limit, the bulk theory and the theory on the D3-branes decouple.

On the other hand one can consider D3-branes as solitonic objects in the supergravity theory. There exist a solution describing the stack of D3-branes (see [11]):

$$\mathrm{d}s^2 = \left(1 + \frac{R^4}{r^4}\right)^{-1/2} \mathrm{d}x^\mu \mathrm{d}x_\mu + \left(1 + \frac{R^4}{r^4}\right)^{1/2} (\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_5^2), \tag{1.1}$$

where

$$R^4 = {\alpha'}^2 g_s N = {\alpha'}^2 \lambda. \tag{1.2}$$

The number N of the D3-branes comes from the flux of the self-dual 5-form F_5

$$\int_{S^5} F_5 = N. \tag{1.3}$$

There are two types of fluctuations on this background that one can consider. There long wave-length fluctuations in the bulk ($r \gg R$) and any excitations close to the horizon ($r \ll R$). It can be shown [6, 7] that at low energies the bulk modes decouple. In the near horizon limit the metric 1.1 becomes precisely the metric on the $AdS_5 \times S^5$.

In the two approaches, there is the same decoupled gravity theory in the bulk, but on the boundary we obtained two descriptions in the low energy limit. One corresponds to the $\mathcal{N} = 4$ SYM theory on the D-branes, while in the other one the string theory on $AdS_5 \times S^5$. Apparently two interpretations should correspond to the same theory.

Notice that the supergravity limit is valid when the curvature of the space is large. As can be seen from the formula (1.2) this is the case when the parameter $g_s N$ is large. However $g_s N$ is the 't Hooft coupling λ and large λ means the strong coupling regime for the gauge theory. This particularly means that it is not trivial to check the validity of AdS/CFT from the perturbative calculations in gauge theory.

In the works of S. Gubser, I. Klebanov and A. Polyakov [12] and E. Witten [13] the AdS/CFT conjecture was further refined by suggesting that

$$\langle e^{\int \mathrm{d}^4x \ \phi^0(x)\mathcal{O}} \rangle_{CFT} = Z_{\text{string}}[\phi(x^\mu, r)|_{r=\infty} = \phi_0(x)], \tag{1.4}$$

where on the left hand side one has a generating functional of the boundary conformal field theory with the sources ϕ_0, which are the boundary values for the bulk fields $\phi(x^\mu, r)$. The right hand side is represented by the full string partition function with the boundary conditions as above. When the non-conformal examples of the holographic correspondence will be considered below, we will also understand them in the sense provided by the formula (1.4).

Thus one can think of the boundary values of the supergravity fields as of the sources for the operators of the gauge theory

$$\int \mathrm{d}^4x \ \phi^0 \mathcal{O}. \tag{1.5}$$

Imagine that the operator $\mathcal{O}$ has a conformal dimension Δ. Then consider a massive scalar field ϕ that couples to $\mathcal{O}$ on the boundary. In the bulk, ϕ satisfies the equation

$$\Box_5 \phi - m_5^2 \phi = 0. \tag{1.6}$$

In the $AdS_5 \times S^5$ metric, the above equation reads

$$r^2 \partial_r^2 \phi + 5r\partial_r \phi - m_5^2 R^2 \phi + m_4^2 \frac{R^4}{r^2}\, \phi = 0. \tag{1.7}$$

Notice that four dimensional mass m_4 is not important at the boundary of AdS_5 $r \to \infty$. Denote k the asymptotic exponent of the solution to (1.7),

$$\phi \sim r^k, \qquad k = -2 + \sqrt{4 + m_5^2 R^2}. \tag{1.8}$$

Then the dimension of the operator, which couples to ϕ at the boundary is

$$\Delta = k + 4 = 2 + \sqrt{4 + m_5^2 R^2}. \tag{1.9}$$

The above formula gives the relationship between the five dimensional AdS_5 mass and operator dimension of the scalar field. In this work we will be interested in other spins as well. One can derive the following expressions for the dimensions:

$$\text{spin-1 field} \qquad \Delta = 2 + \sqrt{1 + m_5^2 R^2}, \tag{1.10}$$

$$\text{2-form field} \qquad \Delta = 2 + |m_5^2 R^2|, \tag{1.11}$$

$$\text{spin-2 field} \qquad \Delta = 2 + \sqrt{4 + m_5^2 R^2}. \tag{1.12}$$

In the non-conformal examples of the holographic correspondence the above expression will be modified in general. This is also the case in the quantum field theory, where the quantum dimension of the operators is different from its classical dimension. To find the classical dimension of the operators in that case one will need to find the limit in which the supergravity solution can be reduced to the AdS_5 geometry and than use the equations (1.9)-(1.12).

The most important property of the holographic correspondence in the form (1.4) is a possibility to evaluate the correlation functions of the gauge theory operators;

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle = \frac{\delta}{\delta \phi_1} \cdots \frac{\delta}{\delta \phi_n} Z_{\text{string}}[\phi_1 \ldots \phi_n]. \tag{1.13}$$

It is sufficient to work in the classical supergravity approximation to extract the information about the correlation functions in the strongly coupled regime of the gauge theory. However, the problem of this approach is that it will give the results valid only in the large N limit. In order to obtain the results for finite N, corrections to the supergravity approximation must be considered.

1.3 Summary and Structure of the Work

Present work summarizes recent investigations of the structure of lowest low-energy effective states of a supersymmetric gauge theory dual to the supergravity background, containing a warped deformed conifold. However, the following chapter 2 rather contains a motivation for such an investigation. The motivation is provided by a study of such states in a sector of particle physics that may be discovered at the TeV or higher energy scale [14].

In a particular example of the gauge/gravity correspondence considered below, the most interesting low-energy states to study are those made of gluons only. In real world such states exist in principle, but are very hard to observe experimentally. As further explained in the section 2.1 it is natural to expect a discovery of such states at higher energies as a part of new physics. The constituent gluons for these states will then be provided by a new gauge group, which is strongly coupled at a TeV scale. Details of a particular model motivated by a new physics expected at that scale are described in the sections 2.2 and 2.3. Within that model one can compute the decay rates of the glueball states to the SM particles. This is done in the section 2.4 of the next chapter, where the decay rates and some branching fractions are expressed in terms of unknown matrix elements.

The decay rates imply a certain pattern of events that should be observed experimentally if the new particles described in 2.2 indeed exist. Thus, there are two directions one can proceed with the information from the chapter 2. One can make the predictions of the decay rates and branching fractions of the possible new physical particles by obtaining estimations for the unknown matrix elements from the formalism of holographic correspondence. Conversely, if the glueball states were discovered, it would be possible to further test the holographic conjecture (1.4).

Chapter 3 contains a brief review of the gauge theories dual to a class of ten dimensional supergravity theories known as warped geometries. Most famous examples of warped geometries represent a space that is a product of Minkowski space and a conifold, a six dimensional Calabi-Yau manifold, which preserves one quarter of maximal supersymmetry[2]. In the section 3.2 reader can find a mathematical definition of the conifold as well as a more extended explanation of the importance of this particular manifold. The first example of a $\mathcal{N} = 1$ gauge theory dual to the supergravity on the conifold was considered by I. Klebanov and E. Witten [15] and is reviewed in the section 3.3. That theory is conformal and therefore is less interesting for practical purposes. Following the works of I. Klebanov with N. Nekrasov [16] and A. Tseytlin [17], section 3.4 describes how the four dimensional conformal invariance can be broken on the level of the supergravity solution. A complete non-conformal and non-singular solution was finally found by I. Klebanov and M. Strassler in [18]. It is reviewed in the section 3.5 and will be further called the Klebanov-Strassler (or KS) solution. The last section 3.6 of the chapter 3 is dedicated to a review of a generalization of the KS solution, which preserves physically important symmetries of the latter and was discovered by A. Butti et al. in [19].

Chapters 4 and 5 are dedicated to the main purpose of this work. They summarize the studies of the low-energy spectrum of the gauge theory dual to the KS solution, performed in the works [20],

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[2]This implies that the four dimensional gauge theory, dual to such a background, will contain only $\mathcal{N} = 1$ out of $\mathcal{N} = 4$ supersymmetries.

[21], [22], [23], [24], [25], [26] and [27]. In the section 4.1 the relationship between the physical states of the strongly coupled gauge theory and the perturbations of the dual supergravity background is explained using the example of the graviton multiplet dual to the supermultiplet of the stress-energy tensor. The term glueball will often be used to refer to both physical states and background fluctuations later in the text. The rest of the chapter 4 contains the study of a simple example.

Typically, the simplest background fluctuations to study is the spin two traceless fluctuation of the metric, which corresponds to the spin 2 states created by energy-momentum tensor. In the supersymmetric case, there is necessarily a vector fluctuation dual to the states created by the chiral current, which forms a bosonic half of the tensor supermultiplet together with the spin 2 fluctuation. This is explained in more detail in the section 4.2. Sections 4.3 and 4.4 contain derivations of the linearized supergravity equations for the tensor and vector fluctuations respectively. It is shown in the following section 4.5 that the fluctuations are related by supersymmetry. Also in that section the relation to the general results in the truncated five dimensional supergravities is investigated. All results are generalized to the extension of the KS background by [19]. Numerical study of the linearized equations is discussed in the section 4.6, where the spectrum of the spin 2 and spin 1 states is found.

Chapter 5 starts from introducing a classification of the glueballs in section 5.1, analogous to the standard particle J^{PC} classification. The role of C-conjugation on the supergravity background is played by a $\mathbb{Z}_2$ symmetry of the conifold. It is called $\mathcal{I}$-symmetry following [22] where this notation was introduced. Discussion in this work is restricted only to the states in the singlet sector of the $SU(2) \times SU(2)$ global symmetry, which is a symmetry of the conifold described in the sections 3.2 and 3.3.

In the section 5.2 the general $\mathcal{I}$-odd and $SU(2) \times SU(2)$ singlet ansatz for background fluctuations is presented. Also in this section we review the results of the work of M. Benna et al., where all possible (pseudo-) scalar fluctuations in this subsector were found, including the two zero-modes found by S. Gubser et al. in [22]. Similarly, section 5.3 studies all vector fluctuations in the $\mathcal{I}$-odd subsector. Study of the scalars in the $\mathcal{I}$-even subsector was performed by M. Berg et al. in [23] and [24]. Seven scalars were found. Here, in section 5.4 their results are reviewed.

Chapter 6 summarizes the investigations of two previous chapters. Numerical spectra of the linearized equations for different background fluctuations are presented in section 6.1. The lowest modes of each tower corresponding to every single glueball are depicted in the figure 6.1. Clearly many of the fluctuations have the same spectra as they describe the members of the same supermultiplets. Although supermultiplet structure is briefly discussed in parts of chapter 5, the complete analysis can be found in the section 6.2. An attempt is made in that section to identify all of the known $\mathcal{I}$-odd states with operators in the dual gauge theory. Directions for future investigations are discussed in the last section 6.3.

Chapter 2

Glueballs of Gauge Theory

2.1 New Physics

Physics beyond the Standard Model (SM) has been attracting physicist's attention for many decades already. Now physics community finds itself at a new frontier before the imminent start of the Large Hadron Collider (LHC). LHC will test a very interesting region of energies above a TeV scale, where the new physics is expected to be discovered to answer some puzzles of the current understanding.

There is no preferable model of physics above the TeV scale. Most of the minimal models introduced to solve major questions fail to elegantly fit some of the observed data. It might happen that physics beyond SM is non-minimal. In this respect it is natural to consider simple extensions of SM. For example, one can expect a discovery of a "hidden valley" [28] (a new yet invisible sector, which contains heavy flavor interacting via new gauge group), interacting with visible particles through SM interactions. In principle hidden sector can be embedded into other scenarios expected at the LHC. It is also motivated by stringy considerations [29].

In this chapter an example of the hidden sector will be considered, which contains a generation of heavy fermions X charged under some new $SU(n_v)$ gauge group as well as SM gauge group. Following M. Strassler and K. Zurek [28], here and below subscript or prefix "v" will signify that the object belongs to the hidden valley. The new gauge group is strongly coupled at some new scale $\Lambda_v \ll M_X \sim 1$ TeV. Therefore at the energies below TeV scale spectrum of the hidden sector is presented by v-colorless combinations of v-particles. In such case, interactions between the new sector and SM particles will proceed through higher dimensional operators induced by loops of heavy X-particles charged under both gauge groups.[1]

The novelty of the hidden-valley models is that the particles can be light with masses much below 1 TeV. Indeed, besides the heavy v-mesons and v-baryons there are also bound states with masses of order Λ_v, made of v-gluons only. Although these v-glueballs can be light enough, they

[1]Such model was first considered by L. B. Okun [30]. Possible discovery of the X-particles, or "quirks" at the LHC was also recently discussed by J. Kang and M. Luty in [31].

cannot be produced directly from the SM processes unlike the heavier X-particles. Effectively, there is a barrier between the SM and hidden valley, which must be penetrated in order to create them.

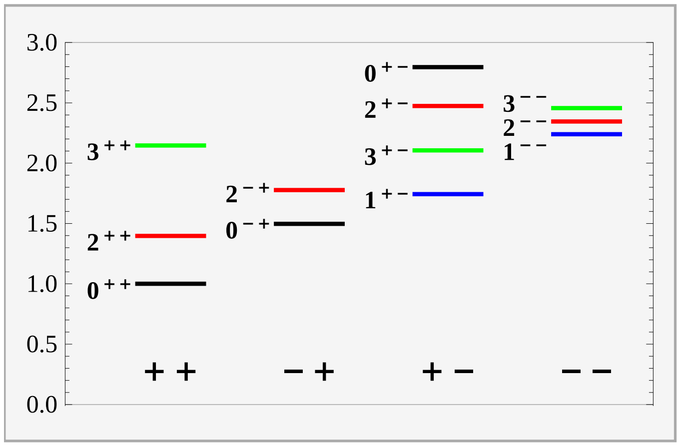

Figure 2.1: Spectrum of stable glueballs in pure glue $SU(3)$ theory [32].

The spectrum of low-energy states in the pure glue theory was found by C. Morningstar and M. Peardon [32] in a numerical simulation on the lattice. They considered $SU(3)$ gauge group and obtained the spectrum of the glueballs depicted on the figure 2.1. One can argue that for unitary groups the ratio of glueball masses is universal. The difference for other groups will be that some states from figure 2.1 will disappear. It will be assumed below that the hidden sector has the $SU(3)$ or another unitary group with a spectrum similar to that of $SU(3)$.

An interesting observation was made in [32] that internal decays within the system of twelve glueballs from the figure 2.1 are forbidden by masses, spins and PC quantum numbers. In the hidden valley model described above glueballs will be able to decay to another glueball radiating a SM particle or completely annihilate into SM content. Study of such decays will be the topic of the current chapter.

Interaction inducing the decays appears on the one-loop level:

$$\frac{1}{M^{D-4}}\,\mathcal{O}_s^{(D-d)}\mathcal{O}_v^{(d)}, \tag{2.1}$$

where $M \equiv M_X$ is the mass of the heavy particle in the loop. $\mathcal{O}_s^{(D-d)}$ represents SM part of the interaction of dimension $D-d$, while hidden-valley operator of dimension d is denoted $\mathcal{O}_v^{(d)}$. In the partial decay width, contribution of operators of higher dimension is suppressed by the powers of Λ/M. Indeed by simple dimensional analysis, $\Gamma^{(D)} \sim \Lambda^{2D-7}/M^{2D-8}$. Only lowest dimensional operators, which give the leading order contribution, will be considered below, which corresponds to $D = 8$.

2.2 The model and the Hidden sector

2.2.1 Model description

Consider adding to the SM a new gauge group G, with a confinement scale in the 10-1000 GeV range. We will refer to this sector as the "hidden valley", or the "v-sector" following [28]. What makes this particular confining hidden valley special is that it has no light charged matter; its only light fields are its gauge bosons, which we will call "hidden gluons" or "v-gluons". At low energy, confinement generates (meta)stable bound states, which we will call "v-glueballs", from the v-gluons. The SM is coupled to the hidden valley sector only through heavy fields X_r, in vector-like representations of both the SM and G, with masses of order the TeV scale. These states can be produced directly at the LHC, but because of v-confinement they cannot escape each other; they form a bound state which relaxes toward the ground state and eventually annihilates, often to v-glueballs. Thereafter, the v-glueballs decay, giving a potentially visible signal at the LHC.

For definiteness, we take the gauge group G to be $SU(n_v)$, and the particles X_r to transform as a fundamental representation of $SU(n_v)$ and in complete $SU(5)$ representations of the Standard Model, typically $\mathbf{5}+\bar{\mathbf{5}}$ and/or $\mathbf{10}+\overline{\mathbf{10}}$. We label the fields and their masses[2] as shown in table 2.1. In this work, we will calculate their effects as a function of m_r. The approximate global $SU(5)$

Field	$SU(3)$	$SU(2)$	$U(1)$	$SU(n_v)$	Mass
$X_{\bar{d}}$	$\mathbf{3}$	$\mathbf{1}$	$\frac{1}{3}$	$\mathbf{n_v}$	$m_{\bar{d}}$
X_ℓ	$\mathbf{1}$	$\mathbf{2}$	$-\frac{1}{2}$	$\mathbf{n_v}$	m_ℓ
$X_{\bar{u}}$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$-\frac{2}{3}$	$\mathbf{n_v}$	$m_{\bar{u}}$
X_q	$\mathbf{3}$	$\mathbf{2}$	$\frac{1}{6}$	$\mathbf{n_v}$	m_q
X_e	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{n_v}$	m_e

Table 2.1: The new fermions X_r that couple the hidden valley sector to the SM sector.

symmetry of the SM gauge couplings suggests that the masses $m_{\bar{d}}$ and m_ℓ should be roughly of the same order of magnitude, and similarly for the masses $m_q, m_{\bar{u}}, m_e$. It is often more convenient to express the answer as a function of the dimensionless parameters

$$\rho_r \equiv m_r/M, \tag{2.2}$$

where M will typically be chosen to be the mass of the lightest colored or colorless X_r particle.

Integrating out these heavy particles generates an effective Lagrangian $\mathcal{L}_{\text{eff}}$ that couples the

[2]In this work, we normalize hypercharge as $Y = T_3 - Q$, where T_3 is the third component of weak isospin.

v-gluons and the SM gauge bosons, which we will discuss in Sec. 2.3. The terms in the effective Lagrangian are of the form (2.1), with operators $\mathcal{O}_v^{(d)}$ constructed from the gauge invariant combinations[3] tr $\mathcal{F}_{\mu\nu}\mathcal{F}_{\alpha\beta}$ and tr $\mathcal{F}_{\mu\nu}\mathcal{F}_{\alpha\beta}\mathcal{F}_{\delta\sigma}$, contracted according to different irreducible representations of the Lorentz group.

The interactions in the effective action then allow the v-glueballs in figure 2.1, which cannot decay within the v-sector, to decay to final states containing SM particles and at most one v-glueball. This is analogous to the way that the Fermi effective theory, which couples the quark sector to the lepton sector, permits otherwise stable QCD hadrons to decay weakly to the lepton sector. As is also true for leptonic and semileptonic decays of QCD hadrons, our calculations for v-hadrons decaying into SM particles simplify because of the factorization of the matrix elements into a purely SM part and a purely hidden-sector part. To compute the v-glueball decays, we will only need the following factorized matrix elements, involving terms in the effective action of dimension eight:

$$\langle SM|\mathcal{O}_s^{(8-d)}|0\rangle\langle 0|\mathcal{O}_v^{(d)}|\Theta_\kappa\rangle, \tag{2.3}$$

$$\langle SM|\mathcal{O}_s^{(8-d)}|0\rangle\langle \Theta_{\kappa'}|\mathcal{O}_v^{(d)}|\Theta_\kappa\rangle\ . \tag{2.4}$$

Here d is the mass dimension of the operator in the v-sector, $\langle SM|$ schematically represents a state built from Standard Model particles, and $|\Theta_\kappa\rangle$ and $|\Theta_{\kappa'}\rangle$ refer to v-glueball states with quantum numbers κ, which include spin J, parity P and charge-conjugation C. We will see later that we only need consider $d = 4$ and 6; there are no dimension $D = 8$ operators in $\mathcal{L}_{\rm eff}$ for which $d = 5$, since there are no dimension-three SM operators to compensate.[4] The SM part $\langle SM|\mathcal{O}_s^{(8-d)}|0\rangle$ can be evaluated by the usual perturbative methods of quantum field theory, but a computation of the hidden-sector matrix elements $\langle 0|\mathcal{O}_v^{(d)}|\Theta_\kappa\rangle$ and $\langle\Theta_{\kappa'}|\mathcal{O}_v^{(d)}|\Theta_\kappa\rangle$ requires the use of non-perturbative methods.

2.2.2 Classification of glueball states

In this section we shall classify the nonvanishing hidden-sector matrix elements. A v-glueball state Θ_κ with quantum numbers J^{PC} is created by a corresponding operator $\mathcal{O}_v^{(d)}$ acting on the vacuum $|\,0\rangle$. Thus, finding all non-vanishing matrix elements, $\langle 0|\mathcal{O}_v^{(d)}|\Theta_\kappa\rangle$ and $\langle\Theta'_\kappa|\mathcal{O}_v^{(d)}|\Theta_\kappa\rangle$, is equivalent to finding how the operators in various Lorentz representations are projected onto states with given quantum numbers J^{PC}. Their classification was carried out in [33]. At mass dimension four there

[3]Here we represent the v-gluon fields as $\mathcal{F}_{\mu\nu} = \mathcal{F}^a_{\mu\nu}\,T^a$, where T^a denote the generators of the $SU(n_v)$ algebra with a common normalization tr $T^aT^b = \frac{1}{2}\,\delta^{ab}$.

[4]At higher loops, such operators could be generated from SM fermion bilinears, but at a level that does not affect our calculations.

are four different operators transforming in irreducible representations of the Lorentz group. These are shown in table 2.2. From now on, we denote the operators $\mathcal{O}_v^\xi$, where ξ runs over different irreducible operators $\xi = S, P, T, L, \cdots$.

Operator $\mathcal{O}_v^\xi$	J^{PC}
$S = \mathrm{tr}\ \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$	0^{++}
$P = \mathrm{tr}\ \mathcal{F}_{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu}$	0^{-+}
$T_{\alpha\beta} = \mathrm{tr}\ \mathcal{F}_{\alpha\lambda}\mathcal{F}_\beta{}^\lambda - \frac{1}{4}\, g_{\alpha\beta} S$	2^{++}, 1^{-+}, 0^{++}
$L_{\mu\nu\alpha\beta} = \mathrm{tr}\ \mathcal{F}_{\mu\nu}\mathcal{F}_{\alpha\beta} - \frac{1}{2}\,(g_{\mu\alpha}T_{\nu\beta} + g_{\nu\beta}T_{\mu\alpha} - g_{\mu\beta}T_{\nu\alpha} - g_{\nu\alpha}T_{\mu\beta})$ $- \frac{1}{12}\,(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})S + \frac{1}{12}\,\epsilon_{\mu\nu\alpha\beta}P$	2^{++}, 2^{-+}

Table 2.2: The dimension $d = 4$ operators, and the states that can be created by these operators [33]. We denote $\tilde{\mathcal{F}}_{\mu\nu} = \frac{1}{2}\,\epsilon_{\mu\nu\alpha\beta}\mathcal{F}^{\alpha\beta}$.

The study of irreducible representations of dimension-six operators is more involved. A complete analysis in terms of electric and magnetic gluon fields, $\vec{E}_a$ and $\vec{B}_a$, was also presented in [33], with a detailed description of the operators and the states contained in their spectrum. There are only two such operators of relevance for our work, which we denote $\Omega^{(1)}_{\mu\nu}$ and $\Omega^{(2)}_{\mu\nu}$ as shown in table 2.3. The other dimension-six operators simply cannot be combined with any SM operator to make a dimension-eight interaction.

Operator $\mathcal{O}_v^\xi$	J^{PC}
$\Omega^{(1)}_{\mu\nu} = \mathrm{tr}\ \mathcal{F}_{\mu\nu}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}$	1^{--}, 1^{+-}
$\Omega^{(2)}_{\mu\nu} = \mathrm{tr}\ \mathcal{F}^\alpha_\mu\mathcal{F}^\beta_\alpha\mathcal{F}_{\beta\nu}$	1^{--}, 1^{+-}

Table 2.3: The $d = 6$ operators. The states that can be created by these operators are shown.

Operators shown in tables 2.2 and 2.3 induce the dominant one-loop decay modes of the v-glueball states appearing in figure 2.1. In the $PC = ++$ sector, the lightest 0^{++} and 2^{++} v-glueballs will mostly decay directly to pairs of SM gauge bosons via S, T and L operators. Three-body decays $2^{++} \to 0^{++} + SM$, where SM stays for two SM gauge bosons, are also possible, although suppressed by phase space as well as other many-body decays. In the $PC = -+$ sector the lightest, states are the 0^{-+} and 2^{-+} v-glueballs. These will also decay predominantly to SM gauge boson pairs via P and L operators respectively. There are also C-changing $2^{-+} \to 1^{+-} + \gamma$ decays, induced by the $d = 6$ $D = 8$ operators (table 2.3). Similar C-changing decays are the only possible lowest order two-body decays in the $PC = +-$ sector, because C-conservation forbids decays to pairs of gauge bosons. In particular, 1^{+-}, the lightest v-glueball in that sector will decay to the lighter C-even states 0^{++}, 2^{++} and 0^{-+} by radiating a photon (or Z when it is possible kinematically). Same is true for the states in the $PC = --$ sector, with an exception that lightest 1^{--} v-glueball can annihilate to a pair of SM fermions through an off-shell photon or Z. The latter decay is induced by $\Omega^{\mu\nu}$ operators. We shall study the decays mentioned above in some detail. Regarding the 3^{++}, 3^{+-}, 3^{--}, 2^{+-}, 2^{--} and 0^{+-} v-glueball states, we will make only brief comments.

Of course the allowed decays and the corresponding lifetimes are dependent upon the masses of the v-glueballs. While the results of Morningstar and Peardon [32], understood as dimensionless in units of the confinement scale Λ, can be applied to any pure $SU(3)$ gauge sector, the glueball spectrum for $SU(4)$ or $SU(7)$ are not known. Fortunately, at least for $SU(n_v)$, the spectrum is expected to be largely independent of n_v. Still, the precise masses will certainly be different for $n_v > 3$, and this may have a substantive effect on v-glueball lifetimes and branching fractions. However, even more important is that for $n_v = 2$, or indeed for any $Sp(2n_v)$ gauge group, or for any $SO(2n_v+1)$ group, the operators Ω do not exist, as they are built from the d^{abc} symbol absent from such groups. The corresponding C-odd states are also absent. For $SO(2n_v)$ this is not quite true. The Ω $d = 6$ operators are present for $n_v = 3$, which is essentially equivalent to $SU(4)$. For general $SO(2n_v)$ the Ω operators become Pfaffian operators of dimension n_v. As suggested by [33] and as verified by [32], there is a correlation in the QCD spectrum and in the glueball spectrum between the dimension of an operator and the mass of the lightest corresponding state. For this reason we expect that for a pure $SO(2n_v)$ gauge theory with $n_v > 3$, the C-odd states are heavier than in figure 2.1 relative to the C-even states. We will address the importance of this in our conclusion. For the majority of this chapter, we will do calculations relevant for $SU(n_v)$, $n_v \geq 3$, for which figure 2.1 likely represents a good approximation to the true spectrum.

2.2.3 Matrix elements

To estimate the partial widths of v-glueball decays, our first step is to determine the v-sector matrix elements in v-glueball transitions. We have seen that the matrix elements are factorized into a purely SM part and a purely v-sector part. Therefore, we only need to calculate $\langle 0|\mathcal{O}_v^\xi|\Theta_\kappa\rangle$ and $\langle\Theta_{\kappa'}|\mathcal{O}_v^\xi|\Theta_\kappa\rangle$, where $|\Theta_\kappa\rangle$ and $|\Theta_{\kappa'}\rangle$ refer to v-glueball states with given quantum numbers and $\mathcal{O}_v^\xi$ is any of the operators in tables 2.2 and 2.3.

It is convenient to write the most general possible matrix element in terms of a few Lorentz invariant amplitudes or form factors. For the annihilation matrix elements we will write

$$\langle 0|\mathcal{O}_v^\xi|\Theta_\kappa\rangle = \Pi^\xi_{\kappa,\mu\nu\cdots}\mathbf{F}^\xi_\kappa, \tag{2.5}$$

where $\mathbf{F}^\xi_\kappa$ is the decay constant of the v-glueball Θ_κ and $\Pi^\xi_{\kappa,\mu\nu\cdots}$ is determined by the Lorentz representations of Θ_κ and $\mathcal{O}_v^\xi$. In table 2.4 we list $\Pi^\xi_{\kappa,\mu\nu\cdots}$ for each operator. The decay constant

$\mathbf{F}^{\xi}_{\kappa}$ depends on the internal structure of the v-glueball states and must be determined by non-perturbative methods, for instance, by numerical calculations of lattice gauge theory. In table 2.4 we have listed the values of $\mathbf{F}^{\xi}_{\kappa}$ known in $SU(3)$ Yang-Mills theory from the lattice calculations of [34]. The remainder are not known, except for those which vanish due to conservation laws, and our numerical results in later sections will consequently be subject to large uncertainties.

Likewise, the transition matrix elements $\langle\Theta_{\kappa'}|\mathcal{O}^{\xi}_{v}|\Theta_{\kappa}\rangle$ are of the form

$$\langle\Theta_{\kappa'}|\mathcal{O}^{\xi}_{v}|\Theta_{\kappa}\rangle = \Pi^{\xi}_{\kappa\kappa',\mu\nu\cdots}\mathbf{M}^{\xi}_{\kappa,\kappa'}, \tag{2.6}$$

where now $\mathbf{M}^{\xi}_{\kappa,\kappa'}$ is the transition matrix, which depends only on the transferred momentum. In table 2.5 we have listed $\Pi^{\xi}_{\kappa\kappa',\mu\nu\cdots}$ for the simplest cases considered later in this work. In general there are more Lorentz structures that can contribute to the transition elements, but although these may not vanish on general grounds, in our present context most of them either vanish or are suppressed. More details will follow in the section 2.4. Unfortunately none of these matrix elements are known from numerical simulation, so we will have to rely on estimates.

$\mathcal{O}^{\xi}_{v}$ (Θ_{κ})	$\Pi^{\xi}_{\kappa,\mu\nu\cdots}$	$\mathbf{F}^{\xi}_{\kappa}$
S (0^{++})	$\mathbf{1}$	$\mathbf{F}^{\mathbf{S}}_{\mathbf{0}^{++}} = 15.6 \pm 3.2$ GeV^3
P (0^{-+})	$\mathbf{1}$	$\mathbf{F}^{\mathbf{P}}_{\mathbf{0}^{-+}} = 8.6 \pm 1.3$ GeV^3
$T_{\alpha\beta}$ (0^{++})	$g_{\alpha\beta} - \frac{p_{\alpha}p_{\beta}}{p^2}$	0
$T_{\alpha\beta}$ (1^{-+})	$p_{\alpha}\epsilon_{\beta} + p_{\beta}\epsilon_{\alpha}$	0
$T_{\alpha\beta}$ (2^{++})	$\epsilon_{\alpha\beta}$	$\mathbf{F}^{\mathbf{T}}_{\mathbf{2}^{++}} = 0.52 \pm 0.19$ GeV^3
$L_{\mu\nu\alpha\beta}$ (2^{++})	$\epsilon_{\mu\alpha}\mathcal{P}_{\nu\beta} + \epsilon_{\nu\beta}\mathcal{P}_{\mu\alpha} - \epsilon_{\nu\alpha}\mathcal{P}_{\mu\beta} - \epsilon_{\mu\beta}\mathcal{P}_{\nu\alpha}$	$\mathbf{F}^{\mathbf{L}}_{\mathbf{2}^{++}}$
$L_{\mu\nu\alpha\beta}$ (2^{-+})	$(\epsilon_{\mu\nu\rho\sigma}\epsilon^{\sigma}{}_{\beta}p^{\rho}p_{\alpha} - \epsilon_{\mu\nu\rho\sigma}\epsilon^{\sigma}{}_{\alpha}p^{\rho}p_{\beta}$	$\mathbf{F}^{\mathbf{L}}_{\mathbf{2}^{-+}}$
	$+\epsilon_{\alpha\beta\rho\sigma}\epsilon^{\sigma}{}_{\nu}p^{\rho}p_{\mu} - \epsilon_{\alpha\beta\rho\sigma}\epsilon^{\sigma}{}_{\mu}p^{\rho}p_{\nu})/p^2$	
$\Omega^{(n)}_{\mu\nu}$ (1^{--})	$p_{\mu}\epsilon_{\nu} - p_{\nu}\epsilon_{\mu}$	$\mathbf{F}^{\mathbf{\Omega^{(n)}}}_{\mathbf{1}^{--}}$
$\Omega^{(n)}_{\mu\nu}$ (1^{+-})	$\epsilon_{\mu\nu\alpha\beta}(p^{\alpha}\epsilon^{\beta} - p^{\beta}\epsilon^{\alpha})$	$\mathbf{F}^{\mathbf{\Omega^{(n)}}}_{\mathbf{1}^{+-}}$

Table 2.4: Annihilation matrix elements. ϵ_{μ} and $\epsilon_{\mu\nu}$ are the polarization vectors of $1^{-+}, 1^{+-}$ and polarization tensor of $2^{++}, 2^{-+}$ respectively. $\mathcal{P}_{\alpha\beta} = g_{\alpha\beta} - 2p_{\alpha}p_{\beta}/p^2$.

Since we will consider v-glueballs with masses far above Λ_{QCD}, the SM part of the matrix element may always be treated perturbatively.[5] In all of our calculations, this merely requires a substitution $G_{\mu\nu} \leftrightarrow k_{\mu}\varepsilon_{\nu} - k_{\nu}\varepsilon_{\mu}$. For example, for a transition to two gauge bosons, we write

$$\langle k_1, \varepsilon^a_1; k_2, \varepsilon^b_2|\mathrm{tr}\ G_{\mu\nu}G_{\alpha\beta}|0\rangle = \frac{\delta^{ab}}{2}(k^1_{\mu}\varepsilon^1_{\nu} - k^1_{\nu}\varepsilon^1_{\mu})(k^2_{\alpha}\varepsilon^2_{\beta} - k^1_{\alpha}\varepsilon^2_{\beta}), \tag{2.7}$$

where $k^{1(2)}$, $\varepsilon^{1(2)}$ are the gauge-bosons' momenta and polarizations respectively. Later in the text

[5] We will do all our calculations at tree level; loop corrections are of course important for v-glueball decays to ordinary gluons.

$\mathcal{O}_v^\xi$ (Θ_κ $\Theta_{\kappa'}$)	$\Pi^\xi_{\kappa\kappa',\mu\nu\cdots}$	$\mathbf{M}^\xi_{\kappa\kappa'}$
P $(0^{-+}, 0^{++})$	$\mathbf{1}$	$\mathbf{M}^{\mathbf{P}}_{\mathbf{0}^+\mathbf{0}^-}$
P $(1^{--}, 1^{+-})$	$\epsilon^+\cdot\epsilon^- p^+\cdot p^- - \epsilon^+\cdot p^-\epsilon^-\cdot p^+$	$\mathbf{M}^{\mathbf{P}}_{\mathbf{1}^{--}\mathbf{1}^{+-}}$
$\Omega_{\mu\nu}$ $(1^{--}, 0^{++})$	$\Sigma_\mu\epsilon_\nu - \Sigma_\nu\epsilon_\mu$	$\mathbf{M}^{\mathbf{\Omega^{(n)}}}_{\mathbf{1}^{--}\mathbf{0}^{++}}$
$\Omega_{\mu\nu}$ $(1^{--}, 0^{-+})$	$\epsilon_{\mu\nu\alpha\beta}\Sigma^\alpha\epsilon^\beta$	$\mathbf{M}^{\mathbf{\Omega^{(n)}}}_{\mathbf{1}^{--}\mathbf{0}^{-+}}$
$\Omega_{\mu\nu}$ $(1^{+-}, 0^{-+})$	$\Sigma_\mu\epsilon_\nu - \Sigma_\nu\epsilon_\mu$	$\mathbf{M}^{\mathbf{\Omega^{(n)}}}_{\mathbf{1}^{+-}\mathbf{0}^{-+}}$
$\Omega_{\mu\nu}$ $(1^{+-}, 0^{++})$	$\epsilon_{\mu\nu\alpha\beta}\Sigma^\alpha\epsilon^\beta$	$\mathbf{M}^{\mathbf{\Omega^{(n)}}}_{\mathbf{1}^{+-}\mathbf{0}^{++}}$
$\Omega_{\mu\nu}$ $(2^{-+}, 1^{+-})$	$\Sigma_\mu\epsilon_{\nu\alpha}\epsilon^\alpha - \Sigma_\nu\epsilon_{\mu\alpha}\epsilon^\alpha$	$\mathbf{M}^{\mathbf{\Omega^{(n)}}}_{\mathbf{2}^{-+}\mathbf{1}^{+-}}$

Table 2.5: Transition matrix elements. Σ_μ stands for the half-sum of 4-momenta of initial and final particles.

we will sometimes use the following notation for the SM matrix elements

$$\langle SM|\mathcal{O}_s^\eta|0\rangle = h_\eta^{\mu\nu\cdots}, \tag{2.8}$$

where $h_\eta^{\mu\nu\cdots} = h_\eta^{\mu\nu\cdots}(k_1, k_2, \cdots)$ is a function of the momenta of the SM particles in the final state.

2.2.4 Physical states and anomaly

Before discussing the effective Lagrangian, we would like to comment on the relationship between operators and states in table 2.2. As explained in [33], when an operator $\mathcal{O}_v^\xi$ is conserved and the associated symmetry is not spontaneously broken, some states must decouple. That is the case of the 1^{-+} v-glueball, which is not actually present in the spectrum of $T_{\mu\nu}$. Indeed,

$$\langle 0|\, T_{\mu\nu}\,|\, 1^{-+}\rangle = (p_\mu\epsilon_\nu + p_\nu\epsilon_\mu)\mathbf{F}^{\mathbf{T}}_{\mathbf{1}^{-+}}, \tag{2.9}$$

but the conservation of $T_{\mu\nu}$ requires $\mathbf{F}^{\mathbf{T}}_{\mathbf{1}^{-+}} = 0$.

Representation analysis summarized in table 2.2 allows in principle to have the following matrix element

$$\langle 0|\, T_{\mu\nu}\,|\, 0^{++}\rangle = (ap^2 g_{\mu\nu} + bp_\mu p_\nu)\mathbf{F}^{\mathbf{T}}_{\mathbf{0}^{++}}, \tag{2.10}$$

where a and b are some functions of p^2. However is we want $T_{\mu\nu}$ to be conserved and traceless the above matrix element should vanish.

An important remark should be made here. We define $T_{\mu\nu}$ in such a way that it coincides with the stress-energy tensor $\Theta_{\mu\nu}$ on the classical level, where it is traceless and conserved. On the quantum level the operators are renormalized and $\Theta_{\mu\nu}$ is known to acquire a non-zero trace due to the conformal anomaly. Here $T_{\mu\nu}$ will denote the traceless part of the stress-energy tensor, while its trace part will renormalize the operator S.

To the effective action, which is discussed in the next section, operators S and $T_{\mu\nu}$ contribute as follows:

$$C_1\,\mathrm{tr}\,G_{\mu\nu}G^{\mu\nu}\,S^{(0)} + C_2\,\mathrm{tr}\,G^{\mu}{}_{\alpha}G^{\nu\alpha}\,\Theta^{(0)}_{\mu\nu}, \tag{2.11}$$

where superscript (0) stresses that the operators are not yet renormalized. After renormalization one needs to substitute S and $T_{\mu\nu}$ for $S^{(0)}$ and $\Theta^{(0)}_{\mu\nu}$ in the above expression, implying that

$$S =: S^{(0)} : + \frac{C_2}{4C_1} : \Theta^{\mu}{}_{\mu} :, \qquad \text{and} \qquad T_{\mu\nu} =: \Theta_{\mu\nu} : -\frac{1}{4}\,g_{\mu\nu} : \Theta^{\mu}{}_{\mu} :, \tag{2.12}$$

where the ellipsis mean the renormalized operators. These renormalized operators will be assumed in all amplitudes below.

2.3 Effective lagrangian

In this section we discuss the effective action $\mathcal{L}_{\text{eff}}$ linking the SM sector with the v-sector, and discuss the general form of the amplitudes controlling v-glueball decays. We will confirm that all the important decay modes are controlled by $D = 8$ operators involving the $d = 4$ and 6 operators listed in tables 2.2 and 2.3.

2.3.1 Heavy particles and the computation of $\mathcal{L}_{\text{eff}}$

The low-energy interaction of v-glueballs with SM particles is induced through a loop of heavy X-particles. In this section we present the one-loop effective Lagrangian that describes this interaction, to leading non-vanishing order in $1/M$, namely $1/M^4$, which we will see is sufficient for inducing all v-glueball decays. The relevant diagrams all have four external gauge boson lines, as depicted on figure 2.2. They give the amplitude for scattering of two v-gluons to two SM gauge bosons, of either strong (gluons g), weak (W and Z) or hypercharge (photon γ or Z) interactions (figure 2.2a) and scattering of three v-gluons to a γ or Z (figure 2.2b).

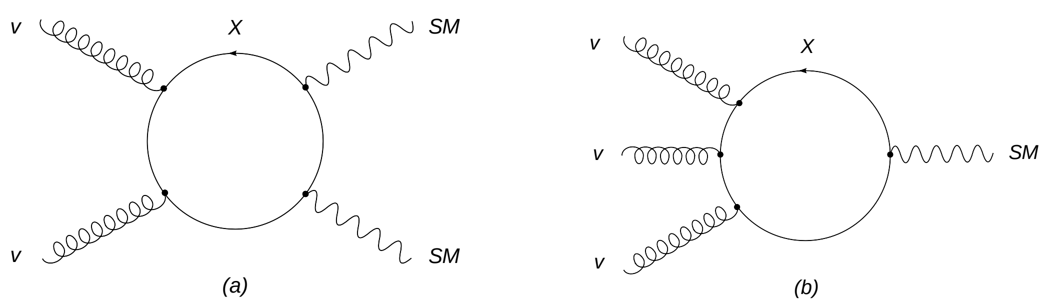

Figure 2.2: Diagrams contributing to the effective action

The dimension-eight operators appearing in the action can be found in studies of Euler-Heisenberg-like Lagrangians in the literature. Within the SM, effective two gluon - two photon, four gluon, and three gluon - photon vertices can be found in [35], [36] and [37] respectively. These results can be adapted for our present purposes.

We introduce now some notation, defining $G^1_{\mu\nu} \equiv B_{\mu\nu}$, $G^2_{\mu\nu} \equiv F_{\mu\nu}$ and $G^3_{\mu\nu} \equiv G_{\mu\nu}$, which are the field tensors of the $U(1)_Y$, $SU(2)$ and $SU(3)$ SM gauge groups. We denote their couplings g_i, $i = 1, 2, 3$, while g_v is the coupling of the new group $SU(n_v)$. In terms of the operators from tables 2.2 and 2.3, the effective Lagrangian reads

$$
\begin{aligned}
\mathcal{L}_{\text{eff}} = \frac{g_v^2}{(4\pi)^2 M^4} \Bigg(& \frac{g_1^2\chi_1}{60}\, S\, B_{\mu\nu}B^{\mu\nu} + \frac{g_2^2\chi_2}{60}\, S \,\text{tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{g_3^2\chi_3}{60}\, S\,\text{tr}\, G_{\mu\nu}G^{\mu\nu} \\
& + \frac{2g_1^2\chi_1}{45}\, P\, B_{\mu\nu}\tilde{B}^{\mu\nu} + \frac{2g_2^2\chi_2}{45}\, P\,\text{tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{2g_3^2\chi_3}{45}\, P\,\text{tr}\, G_{\mu\nu}\tilde{G}^{\mu\nu} + \\
& + \frac{11g_1^2\chi_1}{45}\, T_{\mu\nu}\, B^{\mu\rho}B^{\nu}{}_{\rho} + \frac{11g_2^2\chi_2}{45}\, T_{\mu\nu}\,\text{tr}\, F^{\mu\rho}F^{\nu}{}_{\rho} + \frac{11g_3^2\chi_3}{45}\, T_{\mu\nu}\,\text{tr}\, G^{\mu\rho}G^{\nu}{}_{\rho} - \\
& - \frac{g_1^2\chi_1}{9}\, L_{\mu\nu\rho\sigma}\, B^{\mu\nu}B^{\rho\sigma} - \frac{g_2^2\chi_2}{9}\, L_{\mu\nu\rho\sigma}\,\text{tr}\, F^{\mu\nu}F^{\rho\sigma} - \frac{g_3^2\chi_3}{9}\, L_{\mu\nu\rho\sigma}\,\text{tr}\, G^{\mu\nu}G^{\rho\sigma} + \\
& + \frac{7g_1^2\chi_1}{45}\, L_{\mu\nu\rho\sigma}\, B^{\mu\rho}B^{\nu\sigma} + \frac{7g_2^2\chi_2}{45}\, L_{\mu\nu\rho\sigma}\,\text{tr}\, F^{\mu\rho}F^{\nu\sigma} + \frac{7g_3^2\chi_3}{45}\, L_{\mu\nu\rho\sigma}\,\text{tr}\, G^{\mu\rho}G^{\nu\sigma} \Bigg) \\
& + \frac{g_v^3 g_1}{(4\pi)^2 M^4}\, \chi \left(\frac{14}{45}\, B^{\mu\nu}\Omega^{(1)}_{\mu\nu} - \frac{1}{9}\, B^{\mu\nu}\Omega^{(2)}_{\mu\nu} \right). \qquad (2.13)
\end{aligned}
$$

The coefficients χ_i and χ encode the masses of the heavy particles from table 2.1 and their couplings to the SM gauge groups. They are summarized in the table 2.6.

	χ , χ_i
χ_1	$\frac{1}{3\rho_{\bar{d}}^4} + \frac{1}{2\rho_l^4} + \frac{4}{3\rho_{\bar{u}}^4} + \frac{1}{6\rho_q^4} + \frac{1}{\rho_e^4}$
χ_2	$\frac{1}{\rho_l^4} + \frac{3}{\rho_q^4}$
χ_3	$\frac{1}{\rho_{\bar{d}}^4} + \frac{1}{\rho_{\bar{u}}^4} + \frac{2}{\rho_q^4}$
χ	$\frac{1}{\rho_{\bar{d}}^4} - \frac{1}{\rho_l^4} - \frac{2}{\rho_{\bar{u}}^4} + \frac{1}{\rho_q^4} + \frac{1}{\rho_e^4}$

Table 2.6: The coefficients χ sum over the SM charges of v-fermions running in the loop. The χ_i, $i = 1, 2, 3$, arise from the diagram in figure 2.2(a) with two external SM gauge bosons of group i, while χ is determined by the diagram 2.2(b) with a single hypercharge-boson on an external line. The ρ_a is defined in (2.2)

The effective Lagrangian (2.13) can be compactly written as

$$
\mathcal{L}_{\text{eff}} = \sum_{i=1}^{3} \sum_{\xi} \frac{g_v^{\frac{d_\xi}{2}} g_i^{4-\frac{d_\xi}{2}}}{(4\pi)^2 M^4}\, \Xi^i_\xi \mathcal{O}_s^{\eta(\xi)} \cdot \mathcal{O}_v^\xi \qquad (2.14)
$$

where the sum is over operators and different ways to contract Lorentz indices. The notation $\eta(\xi)$ is to make explicit that for each ξ there is only one SM operator $\mathcal{O}_s^\eta$ multiplying $\mathcal{O}_v^\xi$ in the effective lagrangian (see table 2.7).

The mass dimension of $\mathcal{O}_v^\xi$ is denoted d_ξ, and the Ξ_ξ^i are dimensionless coefficients given by

$$\Xi_\xi^i = \begin{cases} \chi^i C_\xi & d_\xi = 4 \\ \chi C_\xi & d_\xi = 6 \ . \end{cases} \tag{2.15}$$

The C_ξ are coefficients that depend only on the v-sector operators and the SM operator which with they are contracted; they are also givin in table 2.7.

$\mathcal{O}_v^\xi$	C_ξ	$\mathcal{O}_s^\eta \cdot \mathcal{O}_v^\xi$	$\mathcal{O}_v^\xi$	C_ξ	$\mathcal{O}_s^\eta \cdot \mathcal{O}_v^\xi$
S	$\frac{1}{60}$	$(\text{tr } G^i_{\mu\nu} G^{i\mu\nu})\ S$	T	$\frac{11}{45}$	$(\text{tr } G^{i\mu\lambda} G^{i\nu}_{\ \ \lambda})\ T_{\mu\nu}$
P	$\frac{2}{45}$	$(\text{tr } G^i_{\mu\nu} \tilde{G}^{i\mu\nu})\ P$	$\Omega^{(1)}$	$\frac{14}{45}$	$G^{1\mu\nu}\ \Omega^{(1)}_{\mu\nu}$
L_1	$-\frac{1}{9}$	$(\text{tr } G^{i\mu\nu} G^{i\alpha\beta})\ L_{\mu\nu\alpha\beta}$	$\Omega^{(2)}$	$-\frac{1}{9}$	$G^{1\mu\nu}\ \Omega^{(2)}_{\mu\nu}$
L_2	$\frac{7}{45}$	$(\text{tr } G^{i\mu\nu} G^{i\alpha\beta})\ L_{\mu\alpha\nu\beta}$			

Table 2.7: List of coefficients C_ξ and contractions of the operators $\mathcal{O}_v^\xi$ introduced in the tables 2.2 and 2.3. $G^i_{\mu\nu}$ represents the field tensor of the i^{th} SM group.

The coefficients χ and χ_i in table 2.6 yield the relative coupling of v-gluons to the electroweak-sector gauge bosons W_μ^i and B_μ for the $SU(2)$ and $U(1)_Y$ factors respectively. For applications it is convenient to convert these to the couplings to the photons γ, W and Z bosons. (These can be determined from the χ_i using the defining relations $A = B\cos\theta_W + W^3 \sin\theta_W$ and $Z = -B\sin\theta_W + W^3\cos\theta_W$.) We introduce the following coefficients

$$\begin{aligned} \chi_\gamma = \chi_1 + \chi_2/2, \qquad \chi_Z = \frac{\sin^4\theta_W \chi_1 + \cos^4\theta_W \chi_2/2}{\cos^2\theta_W}, \\ \chi_W = \chi_2, \qquad \chi_{\gamma Z} = \frac{\cos^2\theta_W \chi_2 - 2\sin^2\theta_W \chi_1}{\cos\theta_W}, \end{aligned} \tag{2.16}$$

We will often use these coefficients instead of χ_i in the effective Lagrangian (2.13), with a corresponding substitution of field tensors and couplings.

2.3.2 Decay amplitudes

Now, using (2.5), (2.8) and the couplings from (2.14), we obtain that the amplitude for a decay of a v-glueball into SM particles is given by

$$\begin{aligned} \mathcal{M} = \frac{g_v^{\frac{d_\xi}{2}} g_i^{4-\frac{d_\xi}{2}}}{(4\pi)^2 M^4}\, \Xi_\xi^i(\rho_{\bar{u}}, ..., \rho_e) \langle SM|\mathcal{O}_s^\eta|\,0\rangle\langle 0|\mathcal{O}_v^\xi|\,\Theta_\kappa\rangle \\ = \frac{g_v^{\frac{d_\xi}{2}} g_i^{4-\frac{d_\xi}{2}}}{(4\pi)^2 M^4}\, \Xi_\xi^i(\rho_{\bar{u}}, ..., \rho_e) f^i_{\xi,\eta}(p, q_1, q_2, ...) \mathbf{F}_\kappa^\xi \end{aligned} \tag{2.17}$$

where $f^i_{\xi,\eta}(p, q_1, q_2, ...) = h_\eta^{\mu\nu\cdots} \Pi^\xi_{\kappa,\mu\nu\cdots}$ encodes all the information about the matrix element that can be determined from purely perturbative computations and Lorentz or gauge invariance, and

$\mathbf{F}^{\xi}_{\kappa}$ is the v-glueball decay constant. See Eq. (2.15) for the definition of Ξ and Eq. (2.2) and table 2.6 for the definition of ρ.

Similarly, using (2.6), (2.8) and (2.14), the amplitude for the decay of a v-glueball into another v-glueball and SM particles reads

$$\mathcal{M} = \frac{g_v^{\frac{d_\xi}{2}} g_i^{4-\frac{d_\xi}{2}}}{(4\pi)^2 M^4}\, \Xi^i_\xi(\rho_{\bar{u}}, ..., \rho_e)\langle SM|\mathcal{O}^\eta_s|\,0\rangle\langle\Theta_{\kappa'}|\mathcal{O}^\xi_v|\,\Theta_\kappa\rangle$$
$$= \frac{g_v^{\frac{d_\xi}{2}} g_i^{4-\frac{d_\xi}{2}}}{(4\pi)^2 M^4}\, \Xi^i_\xi(\rho_{\bar{u}}, ..., \rho_e) f^i_{\kappa\kappa';\xi,\eta}(p, q_1, q_2, ...)\mathbf{M}^\xi_{\kappa\kappa'} \quad (2.18)$$

where now $\mathbf{M}^\xi_{\kappa\kappa'}$ is the glueball-glueball transition matrix and $f^i_{\kappa\kappa';\xi,\eta} = \Pi^\xi_{\kappa\kappa',\mu\nu\cdots}h^{\mu\nu\cdots}_\eta$.

2.4 Decay rates for lightest v-glueballs

In this section we will compute the decay rates for some of the v-glueballs in figure 2.1. In particular, we shall study decays of the 0^{++}, 2^{++}, 0^{-+}, 2^{-+}, 1^{+-} and 1^{--} v-glueballs. Since for this set of v-glueballs the combination of J and P quantum numbers is unique, we shall often omit the C quantum number from our formulas to keep them a bit shorter, referring simply to the 0^+, 2^+, 0^-, 2^-, 1^+ and 1^- states.

2.4.1 Light C-even sector decays

We first study decays of the C-even 0^{++}, 2^{++}, 0^{-+} and 2^{-+} v-glueballs, which can be created by dimension 4 operators. The first three have been studied in some detail in various contexts. The dominant decays of these states are annihilations $\Theta_\kappa \to G^a G^b$, where Θ_κ denotes a v-glueball state and G^a, G^b is a pair of SM gauge bosons: gg, $\gamma\gamma$, ZZ, W^+W^- or γZ. We will also consider radiative decays $\Theta_\kappa \to \Theta_{\kappa'} + \gamma/Z$, and three-body decays of the form $\Theta_\kappa \to G^a G^b \Theta'_\kappa$.

Annihilations are mediated by the dimension 4 operators in (2.3). In particular, we know from the previous discussion (see [33] and table 2.2 above) that 0^{++} can be annihilated (created) by the operator S. The 0^{-+} and 2^{-+} states are annihilated by the operators P and $L_{\mu\nu\alpha\beta}$ respectively. The tensor 2^{++} can be destroyed by both $T_{\mu\nu}$ and $L_{\mu\nu\alpha\beta}$.

Radiative two-body decays are induced by the dimension 6 operators in (2.4). However, the decays $\Theta_\kappa \to \Theta_{\kappa'} + \gamma/Z$ are forbidden if Θ_κ and $\Theta_{\kappa'}$ are both from the C-even subsector. For the spectrum in figure 2.1, appropriate for $n_v = 3$, the only kinematically allowed radiative decay is therefore $2^{-+} \to 1^{+-} + \gamma$ (or $1^{+-} + Z$ if glueball mass difference is sufficiently large). For $n_v > 3$, the glueball spectrum is believed to be quite similar to $n_v = 3$, but the close spacing between states implies that the ordering of masses might be altered, so that even this decay might be absent for larger n_v.

Decays of the 0^{++} state. The scalar state can be created or destroyed by the operator S. Then, according to a general discussion in the section 2.3, the amplitude of the decay of the scalar to two SM gauge bosons G^a and G^b is given by the expression

$$\frac{\alpha_i \alpha_v}{M^4}\,\chi_i\, C_S \langle G^a, G^b | \mathrm{tr}\; G_{\mu\nu} G^{\mu\nu} |\, 0\rangle\, \langle\, 0 | S |\, 0^{++}\rangle, \tag{2.19}$$

where α_i and χ_i encode the couplings of the bosons a and b of a SM gauge group i to the loop, introduced in section 2.3; see (2.14), (2.15) and table 2.6.

For the decay of the scalar to two gluons $\chi_s \equiv \chi_3$, (2.19) takes the form

$$\begin{aligned}\frac{\alpha_s \alpha_v}{M^4}\chi_s\, C_S \langle g_1^a g_2^b |\; \mathrm{tr}\; G_{\mu\nu} G^{\mu\nu} |\, 0\rangle\, \langle\, 0 |\, S |\, 0^{++}\rangle = \\ = \frac{\alpha_s \alpha_v}{M^4}\,\frac{\delta^{ab}}{2}\,\chi_s C_S \mathbf{F}^{\mathbf{S}}_{\mathbf{0}^{++}} 2(k^1_\mu \varepsilon^1_\nu - k^1_\nu \varepsilon^1_\mu)(k^{2\mu}\varepsilon^{2^\nu} - k^{2^\nu}\varepsilon^{2\mu}),\end{aligned} \tag{2.20}$$

where, according to our conventions, constant $\mathbf{F}^{\mathbf{S}}_{\mathbf{0}^{++}}$ denotes the matrix element $\langle 0|S|0^{++}\rangle$. The rate of the decay (accounting for a 1/2 from Bose statistics) is then given by

$$\Gamma_{0^+\to gg} = \frac{\alpha_s^2 \alpha_v^2}{16\pi M^8}\,(N_c^2 - 1)\chi_s^2 C_S^2 m_{0^+}^3 (\mathbf{F}^{\mathbf{S}}_{\mathbf{0}^{++}})^2. \tag{2.21}$$

The branching ratios for the decays to the photons, Z and $W^\pm$ are

$$\frac{\Gamma_{0^+\to\gamma\gamma}}{\Gamma_{0^+\to gg}} = \frac{1}{2}\,\frac{\alpha^2}{\alpha_s^2}\,\frac{\chi_\gamma^2}{\chi_s^2}, \tag{2.22}$$

$$\frac{\Gamma_{0^+\to ZZ}}{\Gamma_{0^+\to gg}} = \frac{1}{2}\,\frac{\alpha_w^2}{\alpha_s^2}\,\frac{\chi_Z^2}{\chi_s^2}\left(1 - 4\frac{m_Z^2}{m_{0^+}^2}\right)^{1/2}\left(1 - 4\frac{m_Z^2}{m_{0^+}^2} + 6\frac{m_Z^4}{m_{0^+}^4}\right), \tag{2.23}$$

$$\frac{\Gamma_{0^+\to\gamma Z}}{\Gamma_{0^+\to gg}} = \frac{1}{4}\,\frac{\alpha\alpha_w}{\alpha_s^2}\,\frac{\chi_{\gamma Z}^2}{\chi_s^2}\left(1 - \frac{m_Z^2}{m_{0^+}^2}\right)^3, \tag{2.24}$$

$$\frac{\Gamma_{0^+\to W^+W^-}}{\Gamma_{0^+\to gg}} = \frac{1}{4}\,\frac{\alpha_w^2}{\alpha_s^2}\,\frac{\chi_W^2}{\chi_s^2}\left(1 - 4\frac{m_W^2}{m_{0^+}^2}\right)^{1/2}\left(1 - 4\frac{m_W^2}{m_{0^+}^2} + 6\frac{m_W^4}{m_{0^+}^4}\right), \tag{2.25}$$

The coefficients χ used here were defined in Eq. (2.16). Factors of 1/2 in the above ratios come from the color factor $N_c^2 - 1 = 8$ and a difference in the normalization of abelian and non-abelian generators. An extra 1/2 is required if particles in the final state are not identical, such as W^+W^- and γZ.

Of course these are tree-level results. There will be substantial order-α_s corrections to the gg final state, so the actual lifetimes will be slightly shorter and the branching fractions to other final states slightly smaller than given in these formulas.

Decays of the 0^{-+} state. The decay of the pseudoscalar state 0^{-+} to two gauge bosons proceeds in a similar fashion. This decay is induced by the operator P:

$$\frac{\alpha_i \alpha_v}{M^4}\,\chi_i C_P \langle G^a, G^b|\,\mathrm{tr}\; G_{\mu\nu}\tilde{G}^{\mu\nu}|\,0\rangle\,\langle 0|\,P|0^{-+}\rangle. \tag{2.26}$$

The amplitude leads to the following two-gluon decay rate:

$$\Gamma_{0^-\to gg} = \frac{\alpha_s^2\alpha_v^2}{16\pi M^8}\,(N_c^2-1)\chi_s^2 C_P^2 m_{0^-}^3\,(\mathbf{F^P_{0^{-+}}})^2, \tag{2.27}$$

and the same branching fractions as for 0^{++}, except for the decays to ZZ and W^+W^-,

$$\frac{\Gamma_{0^-\to ZZ}}{\Gamma_{0^-\to gg}} = \frac{1}{2}\,\frac{\alpha_w^2}{\alpha_s^2}\,\frac{\chi_Z^2}{\chi_s^2}\,\left(1-4\frac{m_Z^2}{m_{0^-}^2}\right)^{3/2}, \tag{2.28}$$

$$\frac{\Gamma_{0^-\to W^+W^-}}{\Gamma_{0^-\to gg}} = \frac{1}{4}\,\frac{\alpha_w^2}{\alpha_s^2}\,\frac{\chi_W^2}{\chi_s^2}\,\left(1-4\frac{m_W^2}{m_{0^-}^2}\right)^{3/2}. \tag{2.29}$$

The 0^{-+} state can also decay to lower lying states by emitting a pair of gauge bosons, but these decays are suppressed. For instance, the amplitude for the decay of $0^{-+}\to 0^{++}gg$ is

$$\frac{\alpha_i \alpha_v}{M^4}\,\chi_i C_P \langle G^a, G^b\,|\,\mathrm{tr}\; G_{\mu\nu}\tilde{G}^{\mu\nu}|\,0\rangle\,\langle 0^{++}|\,P|0^{-+}\rangle. \tag{2.30}$$

The matrix element $\mathbf{M^P_{0^+0^-}} = \langle 0^{++}|P|0^{-+}\rangle$ is a function of the momentum transferred. Let us first treat it as approximately constant. Then we obtain the decay rate

$$\Gamma_{0^-\to 0^+ gg} = \frac{\alpha_s^2\alpha_v^2}{256\pi^3 M^8}\,(N_c^2-1)\chi_s^2 C_P^2 m_{0^-}^5\, f(a)(\mathbf{M^P_{0^+0^-}})^2, \tag{2.31}$$

where f is the dimensionless function of the parameter $a = m_{0^+}^2/m_{0^-}^2$,

$$f(a) = \frac{1}{12}(1-a^2)(1+28a+a^2) + a(1+3a+a^2)\ln a, \tag{2.32}$$

We plot f in figure 2.3; it falls rapidly from $1/12$ to 0, because of the rapid fall of phase space as the two masses approach each other. For the masses in figure 2.1, $a = 0.44$ and $f \approx 10^{-4}$. This is in addition to the usual $1/16\pi$ suppression of three-body decays compared to two-body decays. Thus the branching fraction for this decay is too small to be experimentally relevant, and our approximation that the matrix element is constant is inconsequential. This will be our general conclusion for three-body decays of the light v-glueball states. We will comment on the heavier v-glueball states later.

Decays of the 2^{++} state. Decays of the 2^{++} and 2^{-+} glueballs to two gauge bosons are induced by more than one operator in (2.13). In particular, the 2^{++} decays due to the $T_{\mu\nu}$ operator and two different contractions of the $L_{\mu\nu\alpha\beta}$ operators. This corresponds to the amplitude

$$\begin{aligned}&\frac{\alpha_i \alpha_v}{M^4}\,\chi_i \left[C_T \langle G^a, G^b|\,\mathrm{tr}\; G_{\mu\alpha}G_\nu{}^{\alpha}|\,0\rangle\,\langle 0|\,T^{\mu\nu}|2^{++}\rangle +\right.\\ &\qquad\left. + \left(C_{L_1}\langle G^a, G^b|\,\mathrm{tr}\; G_{\mu\nu}G_{\alpha\beta}|\,0\rangle + C_{L_2}\langle G^a, G^b|\,\mathrm{tr}\; G_{\mu\alpha}G_{\nu\beta}|\,0\rangle\right)\langle 0|\,L^{\mu\nu\alpha\beta}|2^{++}\rangle\right].\end{aligned} \tag{2.33}$$

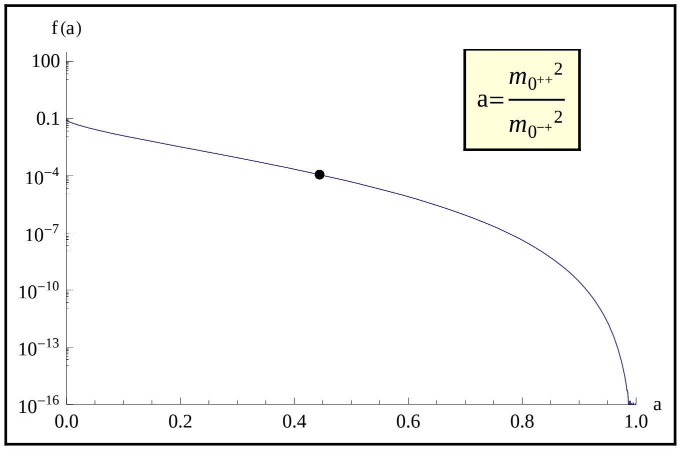

Figure 2.3: Kinematic suppression factor f(a). Point corresponds to a value of a taken for v-glueball masses from Morningstar and Peardon spectrum[32]

The width of the decay to two gluons is

$$\Gamma_{2^+\to gg} = \frac{\alpha_s^2\alpha_v^2}{160\pi M^8}\,(N_c^2-1)\chi_s^2 m_{2^+}^3 \left(\frac{1}{2}\,C_T^2(\mathbf{F^T_{2^{++}}})^2 + \frac{1}{3}\,(2C_{L_1}+C_{L_2})^2(\mathbf{F^L_{2^{++}}})^2\right)\ . \tag{2.34}$$

Here we used the following expressions for the matrix elements:

$$\langle 0|T^{\mu\nu}|2^{++}\rangle = \mathbf{F^T_{2^{++}}}\,\epsilon^{\mu\nu}, \tag{2.35}$$

$$\langle 0|L^{\mu\nu\alpha\beta}|2^{++}\rangle = \mathbf{F^L_{2^{++}}}\,[\mathcal{P}_{\mu\alpha}\epsilon_{\nu\beta} - \mathcal{P}_{\mu\beta}\epsilon_{\nu\alpha} + \mathcal{P}_{\nu\beta}\epsilon_{\mu\alpha} - \mathcal{P}_{\nu\alpha}\epsilon_{\mu\beta}]\,, \tag{2.36}$$

where $\mathcal{P}_{\alpha\beta}$ is defined in the caption to table 2.4.

The branching fraction for the decay to two photons is again similar to (2.22). For two Z bosons in the final state, the width of the decay is equal to

$$\Gamma_{2^+\to ZZ} = \frac{\alpha_w^2\alpha_v^2}{40\pi M^8}\,\chi_Z^2 m_{2^+}^3(1-4\zeta_2)^{1/2}\left(\frac{1}{2}\,C_T^2 f_T(\zeta_2)(\mathbf{F^T_{2^{++}}})^2 + \right.$$
$$\left.\frac{1}{3}\,(2C_{L_1}+C_{L_2})^2 f_L(\zeta_2)(\mathbf{F^L_{2^{++}}})^2 + \frac{20}{3}\,C_T(2C_{L_1}+C_{L_2})f_{TL}(\zeta_2)\mathbf{F^T_{2^{++}}}\mathbf{F^L_{2^{++}}}\right), \tag{2.37}$$

where f_T, f_L, f_{TL} are the following functions of the parameter $\zeta_2 = m_Z^2/m_{2^+}^2$.

$$f_T(\zeta_2) = 1-3\zeta_2+6\zeta_2^2\ ,\ \ f_L(\zeta_2) = 1+2\zeta_2+36\zeta_2^2\ ,\ \ f_{TL}(\zeta_2) = \zeta_2(1-\zeta_2)\ . \tag{2.38}$$

The decay to W^+W^- is obtained from Eq. (2.37) by substituting $\chi_Z \to \chi_W$, $m_Z \to m_W$ and multiplying by 1/2. For the γZ final state, the decay rate is

$$\Gamma_{2^+\to \gamma Z} = \frac{\alpha\alpha_w\alpha_v^2}{80\pi M^8}\,\chi_{\gamma Z}^2 m_{2^+}^3(1-\zeta_2)^3\left(\frac{1}{2}\,C_T^2 g_T(\zeta_2)(\mathbf{F^T_{2^{++}}})^2 + \right.$$
$$\left.\frac{1}{3}\,(2C_{L_1}+C_{L_2})^2 g_L(\zeta_2)(\mathbf{F^L_{2^{++}}})^2 + \frac{10}{3}\,C_T(2C_{L_1}+C_{L_2})\zeta_2\mathbf{F^T_{2^{++}}}\mathbf{F^L_{2^{++}}}\right), \tag{2.39}$$

where

$$g_T(\zeta_2) = 1 + \frac{1}{2}\,\zeta_2 + \frac{1}{6}\,\zeta_2^2, \qquad g_L(\zeta_2) = 1 + 3\zeta_2 + 6\zeta_2^2. \tag{2.40}$$

As in the case of the 0^{-+}, we can ignore the three-body transitions $2^{++} \to 0^{++} + gg$, etc.

Decays of the 2^{-+} **state**. The dominant decays of the 2^{-+} state occur due to the $L_{\mu\nu\alpha\beta}$ operator, which couples to the SM gauge groups in two different ways in (2.13). The amplitude for such decays is given by

$$\frac{\alpha_i \alpha_v}{M^4}\,\chi_i \left(C_{L_1}\langle G^a, G^b|\,\mathrm{tr}\;G_{\mu\nu}G_{\alpha\beta}|\,0\rangle + C_{L_2}\langle G^a, G^b|\,\mathrm{tr}\;G_{\mu\alpha}G_{\nu\beta}|\,0\rangle\right)\langle 0|\,L^{\mu\nu\alpha\beta}|2^{-+}\rangle. \tag{2.41}$$

The correct Lorentz structure that singles out the negative parity part of the operator $L_{\mu\nu\alpha\beta}$ is as follows:

$$\langle 0|L^{\mu\nu\alpha\beta}|2^{-+}\rangle = \mathbf{F}^{\mathbf{L}}_{\mathbf{2}^{-+}}\left(\epsilon_{\mu\nu\rho\sigma}\epsilon^{\sigma}{}_{\beta}n^{\rho}n_{\alpha} - \epsilon_{\mu\nu\rho\sigma}\epsilon^{\sigma}{}_{\alpha}n^{\rho}n_{\beta} + \epsilon_{\alpha\beta\rho\sigma}\epsilon^{\sigma}{}_{\nu}n^{\rho}n_{\mu} - \epsilon_{\alpha\beta\rho\sigma}\epsilon^{\sigma}{}_{\mu}n^{\rho}n_{\nu}\right), \tag{2.42}$$

where $n_\mu = p^\mu/m_{2^-}$ is a unit vector in the direction of the 4-momentum of the v-glueball.

The decay rate to two gluons is then given by

$$\Gamma_{2^-\to gg} = \frac{\alpha_s^2\alpha_v^2}{480\pi M^8}\,(N_c^2-1){\chi_s}^2 m_{2^-}^3 (2C_{L_1}+C_{L_2})^2(\mathbf{F}^{\mathbf{L}}_{\mathbf{2}^{-+}})^2 \tag{2.43}$$

and $\Gamma_{2^-\to\gamma\gamma}$ is provided by the same relation as (2.22). The widths of the decay to ZZ and γZ can be found from the ratios

$$\frac{\Gamma_{2^-\to ZZ}}{\Gamma_{2^-\to gg}} = \frac{1}{2}\,\frac{\alpha_w^2}{\alpha_s^2}\,\frac{\chi_Z^2}{\chi_s^2}\left(1-4\frac{m_Z^2}{m_{2^-}^2}\right)^{1/2}\left(1+2\frac{m_Z^2}{m_{2^-}^2}-24\frac{m_Z^4}{m_{2^-}^4}\right), \tag{2.44}$$

$$\frac{\Gamma_{2^-\to\gamma Z}}{\Gamma_{2^-\to gg}} = \frac{1}{4}\,\frac{\alpha\alpha_w}{\alpha_s^2}\,\frac{\chi_{\gamma Z}^2}{\chi_s^2}\left(1-\frac{m_Z^2}{m_{2^-}^2}\right)^{3}\left(1+3\frac{m_Z^2}{m_{2^-}^2}+6\frac{m_Z^4}{m_{2^-}^4}\right), \tag{2.45}$$

and the width for the decay to W^+W^- is again obtained by substituting in (2.44) $\chi_Z \to \chi_W$, $m_Z \to m_W$ and dividing the result by 2.

As before, we can neglect 3-body decays, but there is a 2-body radiative decay that we should consider. For the $SU(3)$ spectrum in [32] (and possibly all pure glue $SU(N)$, $N \geq 4$) 2^{-+} state is heavier than the lightest state in the C-odd sector, the pseudovector 1^{+-}. Thus, we need at least to consider the decay $2^{-+} \to 1^{+-} + \gamma$. This decay is induced by the second type of operators (table 2.3) in the effective action (2.13). The amplitude of the decay reads

$$\frac{eg_v^3}{(4\pi)^2M^4}\,\chi\langle\gamma|\,G^{\mu\nu}|\,0\rangle\,\left(C_{\Omega^{(1)}}\langle 1^{+-}|\,\Omega^{(1)}_{\mu\nu}|2^{-+}\rangle + C_{\Omega^{(2)}}\langle 1^{+-}|\,\Omega^{(2)}_{\mu\nu}|2^{-+}\rangle\right). \tag{2.46}$$

Unfortunately nothing quantitative is known about the matrix elements like $\langle 1^{+-}|\Omega^{(n)}_{\mu\nu}|2^{-+}\rangle$. In fact each contains multiple Lorentz structures, constructed out of polarization tensors ϵ_α, $\epsilon_{\beta\gamma}$

and momenta p and q of the 1^{+-} and 2^{-+} v-glueballs, times functions of the momentum transfer. Some simplification can be made if one takes into account the fact that masses of the v-glueballs are close. Still, the resulting computation cannot be converted to a numerical branching fraction at present.

We start from writing the general expression for the amplitude (2.46)[6]:

$$\langle\,\gamma|\,G^{\mu\nu}|\,0\rangle\,\langle\,1^{+-}|\,\Omega^{(n)}_{\mu\nu}|2^{-+}\rangle = 2\mathbf{M}^{\mathbf{\Omega^{(n)}}}_{\mathbf{2}^{-+}\mathbf{1}^{+-}}(k\cdot\Sigma\,\varepsilon^{\alpha}\epsilon_{\alpha\beta}\epsilon^{\beta}-\Sigma\cdot\varepsilon\,k^{\alpha}\epsilon_{\alpha\beta}\epsilon^{\beta})+$$
$$+\,2\mathbf{M}^{\mathbf{\Omega^{(n)}}\prime}_{\mathbf{2}^{-+}\mathbf{1}^{+-}}(k\cdot p\,\varepsilon\cdot\epsilon-k\cdot\epsilon p\cdot\varepsilon)\,\frac{p^{\alpha}\epsilon_{\alpha\beta}p^{\beta}}{m^2_{2^-}}+2\mathbf{M}^{\mathbf{\Omega^{(n)}}\prime\prime}_{\mathbf{2}^{-+}\mathbf{1}^{+-}}(k\cdot p\,\varepsilon^{\alpha}\epsilon_{\alpha\beta}p^{\beta}-p\cdot\varepsilon\,k^{\alpha}\epsilon_{\alpha\beta}p^{\beta})\,\frac{q\cdot\epsilon}{m^2_{2^-}}, \tag{2.47}$$

where $n=1,2$ and k, ε_α are the momentum and polarization of the Y boson, and $\Sigma_\mu=(p_\mu+q_\mu)/2$ is half the sum of the 4-momenta of the 1^{+-} and 2^{-+} states. All contributions of the terms proportional to primed form-factors[7] are suppressed by powers of $(m_{2^-}-m_{1^+})/(m_{2^-}+m_{1^+})\simeq 0.017$, so we may neglect them. Indeed, this is why we omitted the primed transition form-factors from table 2.5. Note, however, that if the mass splitting is larger for $n_v>3$, then there will be additional unknown quantities that should modify our result below.

We now find

$$\Gamma_{2^-\to1^++\gamma}=\frac{\alpha\alpha_v^3}{240\pi M^8}\,\chi^2\,\frac{(m^2_{2^-}-m^2_{1^+})^3}{m^5_{2^-}m^2_{1^+}}\,(3m^4_{2^-}+34m^2_{2^-}m^2_{1^+}+3m^4_{1^+})\left(\mathbf{M}^{\mathbf{\Omega}}_{\mathbf{2}^{-+}\mathbf{1}^{+-}}\right)^2. \tag{2.48}$$

Here we introduced the notation

$$\mathbf{M}^{\mathbf{\Omega}}_{\mathbf{2}^{-+}\mathbf{1}^{+-}}\equiv C_{\Omega^{(1)}}\mathbf{M}^{\mathbf{\Omega^{(1)}}}_{\mathbf{2}^{-+}\mathbf{1}^{+-}}+C_{\Omega^{(2)}}\mathbf{M}^{\mathbf{\Omega^{(2)}}}_{\mathbf{2}^{-+}\mathbf{1}^{+-}}. \tag{2.49}$$

Since we do not intend to evaluate the form-factors $\mathbf{M}^{\mathbf{\Omega^{(n)}}}_{\mathbf{2}^{-+}\mathbf{1}^{+-}}$, we shall not distinguish between them and use a collective notation, similar to (2.49), for them in the future.

The radiative decay of 2^{-+} to 1^{+-} can in principle occur through an emission of the Z boson. However this decay is even more difficult to analyze than the decay with photon emission considered above. Additional unknown form factors related to the finiteness of the Z mass would further reduce the predictive power of our computations. In fact such decay is unlikely to be very important for the discovery of v-glueballs, because its rate will lie somewhere between 0 and $\tan^2\theta_W\sim 20\%$ of the rate for decays to a photon. Moreover the Z boson will decay to the electrons and muons only 7% of the time, which further obstructs the registration of the original decay. Therefore, unless the form factors $\mathbf{M}(k^2)$ are enhanced at $k^2\simeq m_Z^2$, the decays to Z can be ignored.

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[6]Similar expressions for this and other matrix elements with spin 2 states were also used in [38] in a more general setup.

[7]Here we assume that the primed form-factors $\mathbf{M}$ are at most of the same order of magnitude as $\mathbf{M}^{\mathbf{\Omega^{(n)}}}_{\mathbf{2}^{-+}\mathbf{1}^{+-}}$.

Again we emphasize that in obtaining these results we made some assumptions and approximations, including $\Delta m \ll m_{2^+}$, and these results, especially the overall coefficient for radiative decay widths, might require generalization in other cases. However, we will adhere to these simplifying approximations in the other radiative decays computed below.

2.4.2 Decays of the vector and pseudovector

In the C-odd sector, the lightest states are the pseudovector 1^{+-} and vector 1^{--} v-glueballs. The lowest-dimension operators that can create or destroy 1^{--} and 1^{+-} v-glueballs are the $d = 6$ $\Omega_{\mu\nu}$ operators (table 2.3). Direct annihilation to non-abelian SM gauge bosons would require an operator in the effective action of dimension $D = 12$, and is hence negligible. Instead these operators, combined with a hypercharge field strength tensor to form an operator of dimension 8, induce radiative decays to C-even v-glueballs, and potentially, for the 1^{--} state, annihilation to SM fermions via an off-shell γ or Z. Three-body decays induced by dimension 4 operators such as S and P will be ignored; as before these three-body decays are heavily suppressed.

Radiative decays can proceed with a photon or Z emission, although the latter will not be considered. Generally decays to Z are described by the larger number of unknown form factors. This brings an additional uncertainty to any attempt to predict the decay widths and branching ratios, due to the lack of phenomenological data. Moreover, as we discussed in the case of the decay 2^{-+} to 1^{+-}, decays to Z are unlikely to be seen in the experiment on the first place.

Decays of the 1^{+-} state. Since 1^{+-} is the lightest v-glueball in the C-odd sector, its radiative decays are only to the lighter v-glueballs in the C-even sector.

According to the table 2.5, the amplitude of the decay $1^{+-} \to 0^{++} + \gamma$ is given by[8]

$$\frac{eg_v^3}{(4\pi)^2 M^4}\,\chi\langle\gamma|\,G^{\mu\nu}|\,0\rangle\,\langle 0^{++}|\,\Omega_{\mu\nu}|\,1^{+-}\rangle = \frac{eg_v^3}{(4\pi)^2 M^4}\,\chi\,2k_\mu\varepsilon_\nu\epsilon^{\mu\nu\alpha\beta}p_\alpha\epsilon_\beta\mathbf{M}^{\mathbf{\Omega}}_{\mathbf{1}^{+-}\mathbf{0}^{++}}, \tag{2.50}$$

where ε_μ and ϵ_μ are the polarization vectors of the photon and the pseudovector v-glueball respectively; p_μ is the 4-momentum of 1^{+-} . The Levi-Civita tensor enforces the final particles to be in a p-wave, as required by parity conservation. The decay rate of this process is

$$\Gamma_{1^+\to 0^+ +\gamma} = \frac{\alpha\alpha_v^3}{24\pi M^8}\,\chi^2\,\frac{(m_{1^+}^2 - m_{0^+}^2)^3}{m_{1^+}^3}\,(\mathbf{M}^{\mathbf{\Omega}}_{\mathbf{1}^{+-}\mathbf{0}^{++}})^2. \tag{2.51}$$

In the case of the decay to the pseudoscalar v-glueball $1^{+-} \to 0^{-+} + \gamma$, the amplitude is given by

$$\frac{eg_v^3}{(4\pi)^2 M^4}\,\chi\langle\gamma|\,G^{\mu\nu}|\,0\rangle\,\langle 0^{-+}|\,\Omega_{\mu\nu}|\,1^{+-}\rangle = \frac{eg_v^3}{(4\pi)^2 M^4}\,\chi 2k^\mu\varepsilon^\nu(\Sigma_\mu\epsilon_\nu - \Sigma_\nu\epsilon_\mu)\mathbf{M}^{\mathbf{\Omega}}_{\mathbf{1}^{+-}\mathbf{0}^{-+}}, \tag{2.52}$$

[8] Similar amplitudes are used in the studies of vector and pseudovector mesons. See for example [39, 40, 41].

where Σ_μ is the half-sum of the 4-momenta of 1^{+-} and 0^{-+}. The rate of the decay to pseudoscalar is then

$$\Gamma_{1^+\to 0^-+\gamma} = \frac{\alpha\alpha_v^3}{24\pi M^8}\,\chi^2\,\frac{(m_{1^+}^2 - m_{0^-}^2)^3}{m_{1^+}^3}\,(\mathbf{M}^{\mathbf{\Omega}}_{\mathbf{1}^{+-}\mathbf{0}^{-+}})^2. \quad (2.53)$$

The ratio of the decay rates to 0^{-+} and 0^{++} is

$$\frac{\Gamma_{1^+\to 0^-+\gamma}}{\Gamma_{1^+\to 0^++\gamma}} = \left(\frac{m_{1^+}^2 - m_{0^-}^2}{m_{1^+}^2 - m_{0^+}^2}\right)^3\left(\frac{\mathbf{M}^{\mathbf{\Omega}}_{\mathbf{1}^{+-}\mathbf{0}^{-+}}}{\mathbf{M}^{\mathbf{\Omega}}_{\mathbf{1}^{+-}\mathbf{0}^{++}}}\right)^2. \quad (2.54)$$

For the masses of glueballs in the figure 2.1, the factor involving the masses is about 0.39; the ratio of matrix elements is unknown, but if we guess that $\mathbf{M}^{\mathbf{\Omega}}_{\mathbf{1}^{--}\mathbf{0}^{\pm+}} \sim 1/\mathbf{F}^{\mathbf{S},\mathbf{P}}_{\mathbf{0}^{\pm+}}$, as would be true for pion emission, we would find this ratio to be slightly larger than 1. In any case there is no sign of a significant suppression of one rate relative to the other.

Finally, in the case of the decay to the tensor v-glueball, the amplitude $1^{+-} \to 2^{++} + \gamma$ contains two independent form factors, denoted $\mathbf{M}^{\mathbf{\Omega}}_{\mathbf{1}^{+-}\mathbf{0}^{-+}}$ and $\mathbf{M'}^{\mathbf{\Omega}}_{\mathbf{1}^{+-}\mathbf{0}^{-+}}$,

$$\begin{aligned}\frac{eg_v^3}{(4\pi)^2M^4}\,\chi\langle\,\gamma|\,G^{\mu\nu}|\,0\rangle\,\langle 2^{++}|\,\Omega_{\mu\nu}|\,1^{+-}\rangle &= \\ &= \frac{eg_v^3}{(4\pi)^2M^4}\,\chi 2k^\mu\varepsilon^\nu\epsilon_{\mu\nu\alpha\beta}\epsilon_{\beta\lambda}(\epsilon_\lambda p_\alpha \mathbf{M}^{\mathbf{\Omega}}_{\mathbf{1}^{+-}\mathbf{0}^{-+}} + \epsilon_\alpha p_\lambda \mathbf{M'}^{\mathbf{\Omega}}_{\mathbf{1}^{+-}\mathbf{0}^{-+}}) \quad (2.55)\end{aligned}$$

and the corresponding decay rate is

$$\begin{aligned}\Gamma_{1^+\to 2^++\gamma} = \frac{\alpha\alpha_v^3}{576\pi M^8}\,\chi^2\,&\frac{(m_{1^+}^2 - m_{2^+}^2)^3}{m_{1^+}^5 m_{2^+}^2}\,\left(3m_{2^+}^4 + 34m_{1^+}^2 m_{2^+}^2 + 3m_{1^+}^4\right)\times \\ &\times\left[\left(\mathbf{M}^{\mathbf{\Omega}}_{\mathbf{1}^{+-}\mathbf{2}^{++}} + \mathbf{M'}^{\mathbf{\Omega}}_{\mathbf{1}^{+-}\mathbf{2}^{++}} f(m_{1^+}, m_{2^+})\right)^2 + \left(\mathbf{M'}^{\mathbf{\Omega}}_{\mathbf{1}^{+-}\mathbf{2}^{++}}\right)^2 g(m_{1^+}, m_{2^+})\right], \quad (2.56)\end{aligned}$$

where f and g are the following functions of the v-glueball masses,

$$f(m_{1^+}, m_{2^+}) = \frac{\left(m_{2^+}^2 - m_{1^+}^2\right)\left(3m_{2^+}^2 + 7m_{1^+}^2\right)}{3m_{2^+}^4 + 34m_{1^+}^2 m_{2^+}^2 + 3m_{1^+}^4}, \quad (2.57)$$

$$g(m_{1^+}, m_{2^+}) = 12\,\frac{\left(m_{2^+}^2 - m_{1^+}^2\right)^2 m_{1^+}^2(6m_{2^+}^4 + 8m_{1^+}^2 m_{2^+}^2 + m_{1^+}^4)}{m_{2^+}^2\left(3m_{2^+}^4 + 34m_{1^+}^2 m_{2^+}^2 + 3m_{1^+}^4\right)^2}. \quad (2.58)$$

One can apply the above results to the case of Z emission in the limit, when mass of Z can be neglected compare to glueball mass difference. The decay rates will have the same form as above with the replacement $\alpha \to \alpha\tan^2\theta_W$.

Decays of the 1^{--} state. The decays of the vector v-glueball are similar to the decays of the pseudovector one with few additions. Contrary to the case of $1^{+}-$ v-glueball, the annihilation to a SM fermion-antifermion pair is possible for 1^{--}. First we consider the dominating radiative decays to light v-glueballs in the C-even sector. The decay to the scalar with photon emission

$1^{--} \to 0^{++} + \gamma$ is analogous to the decay $1^{--} \to 0^{++} + \gamma$; see table 2.5 and (2.53). Thus, its rate is

$$\Gamma_{1^- \to 0^+ + \gamma} = \frac{\alpha \alpha_v^3}{24\pi M^8}\, \chi^2\, \frac{(m_{1^-}^2 - m_{0^+}^2)^3}{m_{1^-}^3}\, (\mathbf{M}^{\mathbf{\Omega}}_{\mathbf{1}^{--}\mathbf{0}^{++}})^2. \tag{2.59}$$

The decay to the pseudoscalar, is analogous to the decay (2.51) and has the rate

$$\Gamma_{1^- \to 0^- + \gamma} = \frac{\alpha \alpha_v^3}{24\pi M^8}\, \chi^2\, \frac{(m_{1^-}^2 - m_{0^-}^2)^3}{m_{1^-}^3}\, (\mathbf{M}^{\mathbf{\Omega}}_{\mathbf{1}^{--}\mathbf{0}^{-+}})^2. \tag{2.60}$$

The amplitude of the decay to the 2^{++} state is similar to the amplitude (2.47) of the decay $2^{-+} \to 1^{+-} + \gamma$, although in this case the masses of glueballs are not close. Thus we cannot simply ignore the contribution of three additional form factors in the decay rate. In such situation we restrict ourselves to just demonstrating the general expression for the amplitude. The decay rate can in principle be computed, but will not be very useful.

$$\begin{aligned}
&\langle\, \gamma |\, G^{\mu\nu} |\, 0 \rangle\, \langle\, 2^{++} |\, \Omega^{(n)}_{\mu\nu} | 1^{--} \rangle = 2\mathbf{M}^{\mathbf{\Omega^{(n)}}}_{\mathbf{1}^{--}\mathbf{2}^{++}} (k \cdot \Sigma\, \varepsilon^{\alpha} \epsilon_{\alpha\beta} \epsilon^{\beta} - \Sigma \cdot \varepsilon\, k^{\alpha} \epsilon_{\alpha\beta} \epsilon^{\beta}) + \\
&+ 2\mathbf{M}^{\mathbf{\Omega^{(n)}}\prime}_{\mathbf{1}^{--}\mathbf{2}^{++}} (k \cdot p\, \varepsilon \cdot \epsilon - k \cdot \epsilon p \cdot \varepsilon)\, \frac{p^{\alpha} \epsilon_{\alpha\beta} p^{\beta}}{m_{1^-}^2} + 2\mathbf{M}^{\mathbf{\Omega^{(n)}}\prime\prime}_{\mathbf{1}^{--}\mathbf{2}^{++}} (k \cdot p\, \varepsilon^{\alpha} \epsilon_{\alpha\beta} p^{\beta} - p \cdot \varepsilon\, k^{\alpha} \epsilon_{\alpha\beta} p^{\beta}) \frac{q \cdot \epsilon}{m_{1^-}^2}.
\end{aligned}$$

The 1^{--} state is also massive enough to decay to 2^{-+} state. Such decay has an amplitude similar to the decay $1^{+-} \to 2^{++} + \gamma$ (2.55), which gives the decay rate

$$\begin{aligned}
\Gamma_{1^- \to 2^- + \gamma} = {} & \frac{\alpha \alpha_v^3}{576\pi M^8}\, \chi^2\, \frac{(m_{1^-}^2 - m_{2^-}^2)^3}{m_{1^-}^5 m_{2^-}^2} \left(3m_{2^-}^4 + 34 m_{1^-}^2 m_{2^-}^2 + 3m_{1^-}^4\right) \times \\
& \times \left[\left(\mathbf{M}^{\mathbf{\Omega}}_{\mathbf{1}^{--}\mathbf{2}^{-+}} + \mathbf{M}'^{\mathbf{\Omega}}_{\mathbf{1}^{--}\mathbf{2}^{-+}} f(m_{1^-}, m_{2^-}) \right)^2 + \left(\mathbf{M}'^{\mathbf{\Omega}}_{\mathbf{1}^{--}\mathbf{2}^{-+}} \right)^2 g(m_{1^-}, m_{2^-}) \right],
\end{aligned} \tag{2.61}$$

where functions f and g are defined by (2.57) and (2.58) respectively.

Now consider a decay of 1^{--} to SM fermion pairs through an off-shell γ or Z. For large m_{1^-} we can neglect the Z mass and treat the radiated particles as an off-shell hypercharge boson. The amplitude reads

$$\frac{\alpha g_v^3}{4\pi M^4}\, \frac{\chi}{\cos^2\theta_W}\, \langle l, \bar{l} | Y_L \bar{\psi}_L \gamma^{\mu} \psi_L + Y_R \bar{\psi}_R \gamma^{\mu} \psi_R |\, 0 \rangle\, \frac{1}{p^2}\, \langle 0 | p^{\nu} \Omega_{\nu\mu} | 1^{--} \rangle. \tag{2.62}$$

Here Y_L and Y_R are left and right hypercharges of the emitted fermions. For quarks a factor of 3 must be included to account for color. The width (ignoring the fermion masses) is given by

$$\Gamma_{1^- \to \bar{l}l} = \frac{\alpha^2 \alpha_v^3}{6M^8}\, \frac{\chi^2}{\cos^4\theta_W}\, (Y_L^2 + Y_R^2) m_{1^-} (\mathbf{F}^{\mathbf{\Omega}}_{\mathbf{1}^{--}})^2. \tag{2.63}$$

This result is valid for $m_{1^+} \gg m_Z$. For smaller m_{1^+} one must account for the non-zero Z mass through the substitution

$$\chi^2 \to \chi^2 \left(\cos^2\theta_W + \sin^2\theta_W \frac{m_{1^-}^2}{m_{1^-}^2 - m_Z^2} \right)^2, \tag{2.64}$$

which accounts for a finite mass of Z-boson. A quick check shows that this rate, while probably smaller than that of the radiative decays above, is not negligible. It is useful to write the ratio of this decay rate to one of the radiative decays considered above. For example,

$$\frac{\Gamma_{1^- \to \gamma^* \to f\bar{f}}}{\Gamma_{1^- \to 0^+ + \gamma}} = \frac{4\alpha}{\cos^2\theta_W}\,(Y_L^2 + Y_R^2)\left(\frac{m_{1^-}^2}{m_{1^-}^2 - m_{0^+}^2}\right)^3 \left(\frac{\mathbf{F}^{\mathbf{\Omega}}_{\mathbf{1}^{--}}}{\mathbf{M}^{\mathbf{\Omega}}_{\mathbf{1}^{--}\mathbf{0}^{++}}}\right)^2 . \tag{2.65}$$

In this chapter we considered the decays of the lightest states of the spectrum in the figure 2.1. The heavier states are less likely to be produced in the future experiment. The studies of their decay rates is more complicated and involve more unknown parameters. Even if such states were produced they would mostly decay to the lighter states, which we already considered.

To study the decay rates any further requires the data that should be obtained from an experiment or some calculations in the strong coupling regime of the gauge theory. Currently, neither glueballs, nor v-glueballs are discovered experimentally. The theoretical calculations are quite hard or even impossible in the case of generic matrix elements.[9] However, the holographic correspondence, discussed in the section 1.2, allows to do the strong coupling calculations in the limit of the large number of colors. Such results could be used for a qualitative analysis of the unknown matrix elements above. We are not going to pursue this goal in this work, but will rather make preliminary preparations for such a future investigation.

[9] In the QCD certain estimations can be obtained by employing non-perturbative methods, e.g. [42, 43, 44].

Chapter 3

Gauge/Gravity Correspondence. Warped Throat Geometries

3.1 Overview

In the section 1.2 of the introduction chapter we discussed the conjecture of the AdS/CFT correspondence, initially introduced in the work of J. Maldacena [9] and further developed by S. Gubser, I. Klebanov and A. Polyakov in [12] and E. Witten in [13]. The conjecture states the duality of the 10 dimensional type IIB supergravity theory on the $AdS_5 \times S^5$ space and $\mathcal{N} = 4$ conformal Supersymmetric Yang-Mills (SYM) theory on the four dimensional boundary of the AdS_5. The key feature of this identification is the possibility to gain an insight on the complicated strong coupling phase of the gauge theory from the relatively simple perturbative regime of string/supergravity theory. Many successes of such approach stimulated further efforts to find other examples of holographic correspondence. One of the most natural examples to study was a gravity dual of the $\mathcal{N} = 1$ SYM theory. This theory is simpler than QCD because of the supersymmetry, but in contrast with $\mathcal{N} = 4$ theory, it has a nontrivial strong coupling dynamics, known as confinement, which stands a challenging problem for quite a long time. In this chapter we will review the progress that was made in studies of $\mathcal{N} = 1$ gauge theories in the last decade.

One of the first models was suggested by I. Klebanov and E. Witten. In [15] they considered a supergravity configuration, consisted of N D3-branes in the space that preserved only one quarter of the maximal number of supersymmetries. Such space must have a vanishing Ricci curvature, i.e. be a six dimensional Calabi-Yau (CY) manifold. The conifold, a six dimensional generalization of the cone, was chosen as an example of a Ricci-flat manifold. The base of the conifold is the space $T^{1,1}$, which is isomorphic to $S^3 \times S^2$. In [15], the branes span the four dimensional boundary of AdS_5 and sit at the singular tip of the conifold (compare with AdS/CFT, where the background is $AdS_5 \times S^5$.) The gauge theory on the world-volume of the D3-branes in the $AdS_5 \times T^{1,1}$ background would be a $\mathcal{N} = 1$ superconformal theory with the gauge group $SU(N) \times SU(N)$.

To break the conformal invariance it was suggested to add some D5-branes, wrapped on the S^2 cycle of the conifold. Such branes are also called fractional D3-branes, since they are also situated at the tip, where S^2 sphere shrinks to zero size. Although they span the same world-volume as the normal D3-branes, they do not give an integer contribution to the flux through $T^{1,1}$. Instead they contribute an integer flux through the orthogonal S^3 sphere. An addition of M fractional branes changes the gauge group of the dual theory to $SU(N+M) \times SU(N)$, which breaks the conformal invariance [45], [16]. In particular, the latter work exhibits the relation between the logarithmic running of the coupling constant and the variation of the D3-brane flux, where the role of energy scale is played by the radial coordinate of the conifold.

A complete supergravity solution dual to the non-conformal $SU(N+M)\times SU(N)$ gauge theory was obtained by I. Klebanov and A. Tseytlin in [17]. The Klebanov-Tseytlin (KT) solution however has a metric that is singular at the tip of the conifold, which corresponds to the IR limit of the dual gauge theory. The origin of this singularity is in the logarithmic running of the effective D3-brane flux, which becomes negative in the IR, corresponding to a negative number of the D3-branes. It was conjectured that in the RG flow the gauge theory experiences a cascade of duality transitions in which the rank of gauge group factors repeatedly drops by M units. The cascade stops when the gauge group is $SU(2M) \times SU(M)$ or even simply $SU(M)$. Then the non-perturbative effects become essential. Those non-perturbative effects should resolve the singularity of the metric via deformation of the conifold.

The regular supergravity solution on the deformed conifold was found by I. Klebanov and M. Strassler in [18]. The Klebanov-Strassler (KS) solution coincides with the singular KT solution in the UV limit (large values of the conifold radius). The important difference is that the KS solution breaks the global $U(1)_{\mathcal{R}}$ symmetry down to $\mathbb{Z}_2$, which is a known non-perturbative effect in the dual theory, while the KT solution only breaks it to $\mathbb{Z}_{2N}$.

All of the examples considered above are symmetric with respect to the $SU(2)\times SU(2)$ rotations of the conifold. It was realized by S. Gubser, C. Herzog and I. Klebanov in [22] that there exist a one-parametric family of $SU(2) \times SU(2)$-symmetric solutions, growing from the KS solution, labeled by the expectation values of the baryonic operators in the dual gauge theory. This one-parametric family was later found by A. Butti et al. in [19].

Later in this chapter we will describe all supergravity solutions in greater detail.

3.2 Conifold

3.2.1 Supersymmetry

In the Klebanov-Witten (KW) holographic supergravity solution [15], the D3-branes were placed at the conical singularity of the 6-dimensional manifold Y_6. By a conical singularity here we understand the point of the manifold that has in its vicinity a metric of the form

$$\mathrm{d}s^2 = h_{mn}\mathrm{d}x^m\mathrm{d}x^n = \mathrm{d}r^2 + r^2 g_{ij}\,\mathrm{d}x^i\mathrm{d}x^j, \tag{3.1}$$

where g_{mn} is the metric on the 5-dimensional submanifold X_5. This metric has a particular property which explains the word "conical". Namely, there is a group of diffeomorphisms $r \to tr$ with $t \in \mathbb{R}^+$, which rescale the metric.

In order for the supergravity background to preserve one quarter of the maximal supersymmetry, the manifold Y_6 should be Ricci-flat. For the metric of the form (3.1) this tells us that X_5 should be an Einstein manifold. Indeed, for the five dimensional part of the six dimensional Ricci tensor $R^{(6)}_{mn}$ one has

$$R^{(6)}_{ij} = R^{(5)}_{ij} - (n-2)g_{ij} = 0.$$

So that the X_5 is Einstein manifold,

$$R^{(5)}_{ij} = (n-2)g_{ij}. \tag{3.2}$$

We are interested in the D-brane-like solutions of the supergravity [11]. In particular, the ones with the metric of the form

$$\mathrm{d}s^2 = H^{-1/2}(r)[-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2] + H^{1/2}(r)[\mathrm{d}r^2 + r^2 g_{ij}\,\mathrm{d}x^i\mathrm{d}x^j]. \tag{3.3}$$

This metric corresponds to the branes placed at the singular point of Y_6. In the near-horizon limit $r \to 0$ this solution behaves as $AdS_5 \times X_5$,

$$\begin{gathered} H(r) \simeq 1 + \frac{L^4}{r^4}, \qquad L^4 = 4\pi g_s N(\alpha')^2, \\ \mathrm{d}s^2 = L^2\left[\frac{r^2}{L^4}(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2) + \frac{\mathrm{d}r^2}{r^2} + g_{ij}\,\mathrm{d}x^i\mathrm{d}x^j\right], \end{gathered} \tag{3.4}$$

where L is the AdS_5 radius, g_s and α' are closed string coupling and tension respectively. According to the holographic prescription we call the theory on the world-volume of D-branes at the conical singularity the dual of type IIB theory on $AdS_5 \times X_5$.

Ricci-flatness is a necessary condition for the manifold to be supersymmetric. However it is not sufficient. For supersymmetry there should also exist covariantly constant spinors. To find them one has to solve the following equation on Y_6

$$\left(\partial_m + \frac{1}{4}\,\omega_m^{ab}\Gamma^{ab}\right)\epsilon = 0. \tag{3.5}$$

Here ω_m^{ab} is spin connection and Γ^{ab} is the commutator of two six dimensional gamma-matrices. In the metric (3.1) the equation becomes [46]

$$\left(\partial_i + \frac{1}{4}\,\omega_i^{jk}\Gamma^{jk} + \frac{1}{2}\,\Gamma_i^r\right)\epsilon = 0, \tag{3.6}$$

with $\Gamma_i^r = \Gamma_{is} n^s$, where n^s is the unit vector in the radial direction. The latter coincides with the covariant spinor equation for type IIB compactification on $AdS_5 \times X_5$ (the Γ_i^r term is responsible for the 5-form field strength effect). This tells us that the number of unbroken supersymmetries is the same as on the six-manifold Y_6. If Y_6 is a CY manifold, then there should be eight unbroken supercharges, which leaves a quarter of the original supersymmetry unbroken. Therefore upon the compactification on the CY manifold, one should end with the $\mathcal{N} = 1$ supersymmetric theory.

3.2.2 Conifold

Let us consider a concrete example of the manifold with a conical singularity. The conifold is defined by the equation

$$\sum_{n=1}^{4} z_n^2 = 0. \tag{3.7}$$

This equation has an obvious $SO(4)$ symmetry and also a rescaling symmetry $\mathbb{C}^* \simeq U(1) \times \mathbb{R}^+$. The $\mathbb{R}^+$ symmetry is the diffeomorphism, mentioned in the previous section, which refers to the "conical" nature of the conifold. The $U(1)$ symmetry will be identified with the $\mathcal{R}$-symmetry of the supersymmetric theory.

On any CY manifold there exist a holomorphic three-form, which in the case of the conifold is just

$$\Omega = \frac{dz_2 \wedge dz_3 \wedge dz_4}{z_1}. \tag{3.8}$$

The product $\Omega \wedge \bar{\Omega}$ gives the volume form on the Calabi-Yau. The holomorphic form should also transform as the volume form of the superspace $\mathrm{d}^2\theta$. If we assign charge one to the coordinates z_n under the $U(1)$ symmetry, then Ω will have charge two under those transformations, as expected. Thus, it is natural to identify the $U(1)$ symmetry with the $\mathcal{R}$-symmetry.

Let us now identify the space X_5. The latter can be obtained by deleting singular point from the conifold and factorizing over rescaling symmetry $\mathbb{R}^+$. The procedure is equivalent to intersecting the conifold with a unit sphere.

$$|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1. \tag{3.9}$$

The equations (3.7) and (3.9) define a five dimensional manifold. Let us show that the $SO(4)$ symmetry acts transitively on the intersection, namely, there are now fixed points under the action of the group. Take an arbitrary vector (z_1, z_2, z_3, z_4). By a $SO(4)$ rotation the real part of it can be transformed to

$$(z_1, z_2, z_3, z_4) \to (x_1, 0, 0, 0) + i(y_1, y_2, y_3, y_4).$$

Now the real part will be invariant under $SO(3) \subset SO(4)$. Using this $SO(3)$ we can rotate the imaginary part, so that we will end up with $(x_1 + iy_1, iy_2, 0, 0)$. Plugging this into the equations (3.7) and (3.9) we find

$$x_1 = \pm\frac{1}{\sqrt{2}}, \quad y_1 = 0, \quad y_2 = \pm\frac{1}{\sqrt{2}}.$$

The signs of x_1 and y_2 can be fixed by the remained transformations. We have so far shown that some particular point on the quotient can be obtained from an arbitrary point by SO(4) transformations. Inverting those transformations we get an arbitrary point from the given one. This means that the action of the SO(4) on the quotient is indeed transitive.

The stabilizer of the point $(1/\sqrt{2}, i/\sqrt{2}, 0, 0)$ is the subgroup $SO(2) \subset SO(4)$. Stabilizers of all other points are isomorphic due to transitivity. Therefore topologically the quotient is $SO(4)/SO(2) \simeq SU(2) \times SU(2)/U(1)$, or simply $S^3 \times S^2$. The coset space $SU(2) \times SU(2)/U(1)$ is denoted $T^{p,q}$ if $U(1)$ is generated by $p\sigma_L^3 + q\sigma_R^3$. In this particular case we are dealing with $p = q = 1$.

The metric on $T^{1,1}$ was found by P. Candelas and X. de la Ossa in [47]:

$$\mathrm{d}s_5^2 = \frac{1}{9}\,(\mathrm{d}\psi + \cos\theta_1\mathrm{d}\phi_1 + \cos\theta_2\mathrm{d}\phi_2)^2 + \frac{1}{6}\sum_{i=1}^{2}[\mathrm{d}\theta_i^2 + \sin^2\theta_i\mathrm{d}\phi_i^2], \tag{3.10}$$

where the angular coordinates are introduced as follows. One can parameterize two $SU(2)$ by two sets of Euler angles $\{\psi_i, \theta_i, \phi_i\}$, $i = 1, 2$. Factorization by a $U(1)$ in this case means that we need to identify the two ψ angles. Note that $\psi \in [0, 4\pi]$.

In the future we will be using the following basis of 1-forms on $T^{1,1}$:

$$\begin{gathered} g^1 = \frac{e^1 - e^3}{\sqrt{2}}, \qquad g^2 = \frac{e^2 - e^4}{\sqrt{2}}, \\ g^3 = \frac{e^1 + e^3}{\sqrt{2}}, \qquad g^4 = \frac{e^2 + e^4}{\sqrt{2}}, \\ g^5 = e^5, \end{gathered} \tag{3.11}$$

where

$$\begin{gathered} e^1 = -\sin\theta_1\mathrm{d}\phi_1, \qquad e^2 = \mathrm{d}\theta_1, \\ e^3 = \cos\psi\sin\theta_2\mathrm{d}\phi_2 - \sin\psi\mathrm{d}\theta_2, \\ e^4 = \sin\psi\sin\theta_2\mathrm{d}\phi_2 + \cos\psi\mathrm{d}\theta_2 \\ e^5 = \mathrm{d}\psi + \cos\theta_1\mathrm{d}\phi_1 + \cos\theta_2\mathrm{d}\phi_2. \end{gathered} \tag{3.12}$$

In terms of the g^i-basis, the metric on $T^{1,1}$ takes the form

$$\mathrm{d}s_5^2 = g_{ij}\,\mathrm{d}x^i\mathrm{d}x^j = \frac{1}{9}\,(g^5)^2 + \frac{1}{6}\sum_{i=1}^{4}(g^i)^2. \tag{3.13}$$

3.3 Conformal Case

3.3.1 Low Energy Conformal Gauge Theory

To establish what kind of gauge theory one will have after placing N D-branes on the conifold, it is useful to describe another representation of the manifold [15]. Changing variables

$$z_{1,3} \to w_{1,3} = \frac{z_{1,3} + z_{2,4}}{\sqrt{2}}, \qquad z_{2,4} \to w_{2,4} = i\frac{z_{1,3} - z_{2,4}}{\sqrt{2}}.$$

In the new variables the conifold will be defined by the equation

$$w_1 w_2 - w_3 w_4 = 0. \tag{3.14}$$

The general solution to this equation is

$$w_1 = A_1 B_1, \qquad w_2 = A_2 B_2, \qquad w_3 = A_1 B_2, \qquad w_4 = A_2 B_1. \tag{3.15}$$

Note, that the w_n remain unchanged under the following $U(1)$ transformations for A_k, B_l,

$$A_k \to \lambda A_k, \qquad B_l \to \lambda^{-1} B_l, \tag{3.16}$$

where $\lambda \in \mathbb{C}^*$. This is a representation which makes explicit the action of $SU(2)$ factors in $SO(4) \simeq SU(2) \times SU(2)$. One $SU(2)$ acts on the doublet (A_1, A_2), and another on (B_1, B_2). The $SU(2)$ invariants depend on the combinations $|A_1|^2 + |A_2|^2$ and $|B_1|^2 + |B_2|^2$. Adjusting the λ-parameter in the transformations (3.16), namely $s = |\lambda|$, one can set

$$|A_1|^2 + |A_2|^2 = |B_1|^2 + |B_2|^2. \tag{3.17}$$

This formula describes the product of two three-spheres far from the singularity. Fixing of s corresponds to factorizing out the rescaling symmetry $\mathbb{R}^+$. However, it still has the $U(1)$ invariance. Factorizing this out, one gets $SU(2) \times SU(2)/U(1)$.

Now consider $\mathcal{N} = 1$ $U(1)$ gauge theory, containing chiral multiplets A_k and B_l with charges $+1$ and -1 respectively under $U(1)$ and the charge $1/2$ under the $\mathcal{R}$-symmetry. Imposing the equation of motion on the auxiliary field D from the vector multiplet one obtains the algebraic definition of the moduli space of the theory. Indeed vanishing of D means

$$D = |A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = 0. \tag{3.18}$$

Here one also need to factorize over the $U(1)$ gauge transformations. Apparently the space of vacua (moduli) of this theory coincides with the manifold $T^{1,1}$, considered above, and the gauge theory describes the motion of the branes on the conifold. To have a pure gauge $U(1)$ theory on the world-volume one needs to introduce another $U(1)$ gauge factor, so that the overall gauge group would be $U(1) \times U(1)$, and the charges of the chiral fields $(1, -1)$ and $(-1, 1)$ respectively. The diagonal $U(1)$ subgroup of this product will leave the chiral fields invariant, and, thus, decouple from the chiral multiplet fields, which will acquire the vacuum expectation values. This will result in a $U(1)$ theory in the low-energy on the world volume of the brane.

The generalization to the non-abelian case is now straightforward. A_k and B_l become now fields in $(N, \bar{N})$ and $(\bar{N}, N)$ representations of $U(N)$. One can bring the matrices to the diagonal form in some basis. The eigenvalues will satisfy the D-flatness condition

$$|a_1^i|^2 + |a_2^i|^2 - |b_1^i|^2 - |b_2^i|^2 = 0, \quad i = 1, \ldots, N. \tag{3.19}$$

The group will be here broken down to $U(1)^N$, one $U(1)$ factor for each D-brane.

The theory does not admit a renormalizable superpotential. However, without the superpotential one will end with massless chiral supermultiplets, which will also include non-diagonal part of the matrices. The non-diagonal modes correspond to the motion of the branes in the space, transversal to the conifold. If the modes are massless, this transversal perturbations will cost no energy to create. This is something one would not generically expect. The superpotential needs to be introduced to make those modes massive.

The above theory has an $SU(2) \times SU(2)$ global symmetry and a $U(1)$ non-anomalous $\mathcal{R}$-symmetry. It is natural to try to construct a superpotential that respects these symmetries. The simplest superpotential, that satisfy this conditions is as follows,

$$W = \frac{\lambda}{2}\, \epsilon^{ij} \epsilon^{kl} \mathrm{tr} A_i B_k A_j B_l. \tag{3.20}$$

Since the chiral fields carry charge $1/2$ under the $\mathcal{R}$-symmetry, W has charge 2. This operator is exactly a marginal perturbation of the free conformal theory.

With the superpotential, the off-diagonal components receive mass, and only the diagonal ones remains massless, corresponding to the motion of N branes on the conifold. The vacuum configurations are those, with diagonal matrices A_k and B_l in some basis. The F-terms coming from the superpotential in this case automatically satisfy the F-flatness condition $F = 0$,

$$B_1 A_k B_2 - B_2 A_k B_1 = 0, \qquad A_1 B_l A_2 - A_2 B_l A_1 = 0. \tag{3.21}$$

The symmetry is broken down to $U(1)^N$.

The beta-functions of the U(1) factors in $U(N)$ are positive, and thus the former decouple in the IR, which means we are dealing with $SU(N) \times SU(N)$ gauge theory. From the point of view of either $SU(N)$ gauge factors, this is a supersymmetric $SU(N)$ gauge theory with $2N$ flavors. The latter flows to the superconformal fixed point in the IR. The same behavior is expected from the above theory. Since the superpotential is a marginal perturbation, there is actually a line of fixed points. This is consistent with the beta-function analysis. Moreover, this analysis confirms that (3.20) is the only possible marginal superpotential [15].

Another non-trivial consistency check might be the comparison of the $\mathcal{R}$-symmetries. The chiral fields in the gauge theory have the $\mathcal{R}$-charge 1/2. Consider an $\mathcal{R}$-symmetry transformation $e^{i\alpha}$, and choose $\alpha = \pi$. The fields A_k,B_l will transform into iA_k, iB_l. This is consistent with the representation (3.14) of the conifold. Indeed, the coordinates w_n have charge one under the $\mathcal{R}$-symmetry transformations, and hence A_k and B_l coordinates should be multiplied by i under them.

These arguments lead authors of [15] to the conjecture, that the type IIB theory on $AdS_5 \times T^{1,1}$ with N units of Ramond-Ramond (RR) flux through $T^{1,1}$ should be equivalent to the theory obtained by starting from the $SU(N) \times SU(N)$ gauge theory with the fields A_k and B_l in the $(N, \bar{N})$ and $(\bar{N}, N)$ representations, flowing to the IR fixed point, and then perturbing by the superpotential (3.20).

3.3.2 Parameter Matching

Following [15] let us briefly discuss the matching of the parameters (moduli) of the two theories. If the holomorphic scales of the two gauge factors are Λ_1 and $\tilde{\Lambda}_1$, then there are two dimensionless invariants in the gauge theory: $\lambda^2\Lambda_1\tilde{\Lambda}_1$ and $\tilde{\Lambda}_1/\Lambda_1$. The first invariant, which is the product of two scales corresponds to the sum of two complex gauge couplings $\tau_1 + \tau_2$

$$\frac{\mathrm{d}}{\mathrm{d}\log(\lambda\Lambda_1\tilde{\Lambda}_1)}\left(\tau_1(\Lambda_1) + \tau_2(\tilde{\Lambda}_1)\right) = \frac{\mathrm{d}}{\mathrm{d}\log\Lambda_1}\,\tau_1 + \frac{\mathrm{d}}{\mathrm{d}\log\tilde{\Lambda}_1}\,\tau_2.$$

The second invariant then apparently corresponds to the difference of two couplings $\tau_1 - \tau_2$. Here τ_1, τ_2 are defined as

$$\tau_i = \frac{\theta_i}{2\pi} + i\,\frac{4\pi}{g_i^2}. \tag{3.22}$$

On the string theory side there are moduli which arise due to compactification on $T^{1,1}$. These are the integrals of RR and Neveu-Schwartz (NS) NS-NS 2-forms over S^2. The matching between the moduli is as follows,

$$\frac{1}{g_1^2} + \frac{1}{g_2^2} \sim e^{-\phi}, \tag{3.23}$$

$$\frac{1}{g_1^2} - \frac{1}{g_2^2} \sim e^{-\phi}\left[\left(\int_{S^2} B_2\right) - \frac{1}{2}\right], \tag{3.24}$$

where ϕ and B_2 denote the background dilaton and NS-NS 2-form fields in the string theory.

3.4 Towards the Non-Conformal Theory

3.4.1 Type IIB supergravity

The action of the chiral $\mathcal{N} = 2$ ten dimensional supergravity (type IIB) has the property (unlike it's non-chiral counterpart, IIA theory) of being maximal supersymmetric theory, non-derivable from the higher dimensional theories with lower supersymmetry ($\mathcal{N} = 1$ eleven dimensional supergravity). Bosonic sector of this theory contains 1 complex scalar (dilaton), rank two symmetric tensor (metric), complex antisymmetric 2-form, and antisymmetric real 4-form with selfdual curvature. The fermionic sector consists of chiral complex gravitino and a chiral complex spinor - dilatino.

It turns out to be impossible to write down covariant action for type IIB supergravity. There is no way to construct the action for the self-dual five form so that the self-duality condition is a consequence of the equations of motion. It can only be imposed as an additional constraint.

The NS sector of type IIB supergravity contains the metric, the dilaton, and the NS 2-form B. The R sector consists of even-ranked forms C_0, C_2, and C_4 with the odd-ranked field strengths F_1, F_3, and F_5 respectively. According to this classification the ten dimensional type II theory action splits into three parts,

$$S_{IIB} = S_{NS} + S_R + S_{CS},$$

where S_{CS} is called Chern-Simons term, since it depends not only on the exterior derivatives of the form-potentials, but also on the potentials themselves. One might try to write the type IIB supergravity action in the form [48]:

$$\begin{aligned} S_{NS} &= \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-G} e^{-2\Phi} \left(R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} |H_3|^2 \right), \\ S_R &= -\frac{1}{4\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-G} \left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2} |\tilde{F}_5|^2 \right), \\ S_{CS} &= -\frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3, \end{aligned} \tag{3.25}$$

where κ_{10} is the ten dimensional gravity constant. In the above expressions the $\tilde{F}_3$ and $\tilde{F}_5$ are defined as follows,

$$\begin{aligned} \tilde{F}_3 &= F_3 - C_0 \wedge H_3, \\ \tilde{F}_5 &= F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3. \end{aligned} \tag{3.26}$$

The reason for introducing these new objects is that they have proper gauge transformations. The following equations of motion can be deduced from the above action (3.25)

$$\begin{aligned} \mathrm{d} * F_1 = -H_3 \wedge *\tilde{F}_3, \qquad \mathrm{d} * \tilde{F}_5 = H_3 \wedge \tilde{F}_3, \\ \mathrm{d} * \tilde{F}_3 = -H_3 \wedge *\tilde{F}_5, \\ \mathrm{d}(e^{-2\Phi} * H_3) - F_1 \wedge *\tilde{F}_3 - \tilde{F}_3 \wedge *\tilde{F}_5 = 0, \end{aligned} \tag{3.27}$$

There is also the equation for the dilaton:

$$\mathrm{d} * \mathrm{d}\Phi = \frac{1}{12}(e^{2\Phi}\tilde{F}_3 \wedge *\tilde{F}_3 - e^{-2\Phi}H_3 \wedge *H_3), \tag{3.28}$$

and the Einstein equation

$$\begin{aligned} R_{\mu\nu} = 2\kappa_{10}^2\partial_\mu\Phi\partial_\nu\Phi + \frac{\kappa_{10}^2}{2}\,F_\mu F_\nu + \frac{\kappa_{10}^2}{8}\,\tilde{F}_{\mu\lambda_1\lambda_2}\tilde{F}_\nu{}^{\lambda_1\lambda_2} - \frac{\kappa_{10}^2}{48}\,g_{\mu\nu}(\tilde{F}_3)^2 + \\ + \frac{\kappa_{10}^2}{8}\,e^{-2\Phi}H_{\mu\lambda_1\lambda_2}H_\nu{}^{\lambda_1\lambda_2} - \frac{\kappa_{10}^2}{48}\,g_{\mu\nu}e^{-2\Phi}(H_3)^2 + \frac{\kappa_{10}^2}{6}\,\tilde{F}_{\mu\lambda_1\lambda_2\lambda_3\lambda_4}\tilde{F}_\nu{}^{\lambda_1\lambda_2\lambda_3\lambda_4}. \end{aligned} \tag{3.29}$$

The $\tilde{F}_3$ and $\tilde{F}_5$ have non-trivial Bianchi identities

$$\mathrm{d}\tilde{F}_3 = F_1 \wedge H_3, \qquad \mathrm{d}\tilde{F}_5 = F_3 \wedge H_3. \tag{3.30}$$

The Bianchi identity and equation of motion for $\tilde{F}_5$ are consistent with the self-duality condition, however they do not imply it.

3.4.2 Breaking of Conformal Invariance

The theory, considered in the previous section was conformal. To break the conformal invariance it was proposed in [45] to add some fractional branes to the background. One can think of the fractional D3-branes as the regular D5-branes, wrapped on the S^2 of $T^{1,1}$.

In the conformal theory of [15] there were N units of F_5-form flux through the $T^{1,1}$,

$$\int_{T^{1,1}} F_5 = N. \tag{3.31}$$

After addition of M fractional D3-branes the flux of F_3 form through the S^3 of $T^{1,1}$ turns on:

$$\int_{S^3} F_3 = M. \tag{3.32}$$

Adding D5 branes wrapped on the S^2 breaks the symmetry between two $SU(N)$ factors in the gauge group, which now becomes $SU(N+M)\times SU(N)$. The chiral superfields A_k and B_l are now in the $(N+M, \bar{N})$ and $(\overline{N+M}, N)$ representations. The space of vacua will still be given by the equation (3.19), which tells that the D3-branes classically are constrained to move on the conifold. The superpotential (3.20) also remains unmodified.

Let us recall the beta-function calculations in the gauge theory. We have

$$\begin{aligned} \frac{d}{d\log\Lambda/\mu}\frac{8\pi^2}{g_1^2} &\sim 3(N+M) - 2N(1-\gamma_A-\gamma_B), \\ \frac{d}{d\log\Lambda/\mu}\frac{8\pi^2}{g_2^2} &\sim 3N - 2(N+M)(1-\gamma_A-\gamma_B), \end{aligned} \tag{3.33}$$

where γ_A and γ_B are anomalous dimensions of operators A and B. If $M=0$ then the theory is conformal if $\gamma_A=\gamma_B=1/4$. At nonzero M, however, the theory cannot be made conformal even if we assume that $\gamma_A \neq \gamma_B$. The theory undergoes a logarithmic RG-flow. This can be seen by taking the difference of two gauge couplings,

$$\frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} \sim M \log(\Lambda/\mu)[3 + 2(1-\gamma)]. \tag{3.34}$$

Here $\gamma = \gamma_A + \gamma_B = 1/2$ plus small corrections.

Let us reproduce this fact from the supergravity solution following [16]. It turns out that breaking of conformal invariance in the supergravity occurs due to the radial dependence of the NS-NS 2-form B_2.

One can try to solve the two-form supergravity equations assuming that the dilaton ϕ and the R-R scalar C_0 are constant and neglecting for the moment a back-reaction from the metric and the five-form field, which is a reasonable approximation in the limit $M/N \to 0$. From (3.27)[1]

$$\mathrm{d}(e^{-\phi} * H_3) = F_3 \wedge F_5. \tag{3.35}$$

To have M units of the F_3 flux through S^3, one has to set

$$F_3 = M\omega_3, \qquad \omega_3 = \frac{1}{2}\, g^5 \wedge (g^1 \wedge g^2 + g^3 \wedge g^4), \tag{3.36}$$

where g^i were defined in (3.11). For B_2 one should take the following ansatz:

$$B_2 = e^{\phi} f(r)\omega_2, \qquad \omega_2 = \frac{1}{2}\,(g^1 \wedge g^2 + g^3 \wedge g^4). \tag{3.37}$$

This gives the following expression for H_3:

$$H_3 = e^{\phi}\mathrm{d}f(r) \wedge \omega_2. \tag{3.38}$$

Since F_5 is a selfdual field, and $\int_{T1,1} F_5 = N$, it should have the form $F_5 \sim \mathrm{vol}(AdS_5) + \mathrm{vol}(T^{1,1})$. The equation (3.35) for B_2 takes the form

$$\frac{1}{\sqrt{g}}\,\partial_\mu(g^{\mu\nu}\sqrt{g}\partial_\nu B_2) \sim M\omega_2. \tag{3.39}$$

Evaluating this in the metric (3.4) with g_{ij} from (3.10), one gets

$$\frac{1}{r^3}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^5\,\frac{\mathrm{d}}{\mathrm{d}r}\,f(r)\right) \sim M, \tag{3.40}$$

which implies

$$f(r) \sim M \log\frac{r}{r_0}. \tag{3.41}$$

If one substitutes this result into (3.37) and take the integral over S_2, one will obtain

$$\int_{S^2} B_2 \sim Me^{\phi}\log\frac{r}{r_0}. \tag{3.42}$$

---

[1]From now on we omit the tildes for F_3 and F_5, and rescale the dilaton field $\phi = 2\Phi$ in the calculations.

Given the matching of the moduli of gauge theory and supergravity (3.24), this exactly the same result as the beta-function calculations (3.34). Thus the described matching of moduli indeed makes sense.

3.4.3 Complete UV Solution

The solution described above was valid only in the limit $M/N \to 0$, and did not take into account the back-reaction from the metric and the five-form. The complete solution of supergravity equations was found in [17].

In the KT solution the dilaton is set constant $e^{-\phi} \propto g_s$, and the same ansatz as before is chosen for B_2 and F_3:

$$F_3 = M\omega_3, \qquad B_2 = 3g_s M\omega_2 \log\frac{r}{r_0}, \tag{3.43}$$

$$H_3 = \mathrm{d}B_2 = 3g_s M\,\frac{1}{r}\,\mathrm{d}r \wedge \omega_2, \tag{3.44}$$

where ω_2 and ω_3 are defined above in (3.37) and (3.36). The factor 3 in the B_2 formula is related to the coefficients in the metric (3.13), and it reproduces the factor of 3 in the beta-function.

Note that

$$g_s *_6 F_3 = -H_3, \qquad g_s F_3 = *_6 H_3, \tag{3.45}$$

where $*_6$ is the Hodge dual with respect to the metric of the conifold. Multiplying this two equations one obtains

$$g_s^2 F_3^2 = H_3^2. \tag{3.46}$$

Substituting this into the dilaton equation of motion (3.28), one finds that $\phi = 0$. It also follows from (3.45) that $H_3 \wedge *_6 F_3 = 0$. This and the equation of motion for C_0 (3.27) imply that C_0 vanishes as well.

Following the calculation in [18], one can take the ansatz (3.3) and substitute it into the trace of the Einstein equation (3.29):

$$H^{-3/2}\nabla_6^2 H \sim g_s^2 F_3^2 + H_3^2 = 2g_s^2 F_3^2, \tag{3.47}$$

where ∇_6^2 is the Laplacian on the conifold. From the ansatz (3.43), it follows that $F_3^2 \sim M^2 r^{-6} H^{-3/2}$. Substituting this into (3.47) and solving the equation one should get

$$H(r) = \frac{R^4 + 2L^4(\log(r/r_0) + 1/4)}{r^4}, \tag{3.48}$$

where

$$R^4 = \frac{27}{4}\,g_s N\pi(\alpha')^2, \qquad L^2 = \frac{9}{4}\,g_s M\alpha'. \tag{3.49}$$

Let us redefine F_5 according to

$$F_5 = \mathrm{d}C_4 - \frac{1}{2}\,C_2 \wedge H_3 + \frac{1}{2}\,B_2 \wedge F_3 = \mathrm{d}C_4 - \mathrm{d}(C^2 \wedge B_2) + B_2 \wedge F_3 = \mathrm{d}\tilde{C}_4 + B_2 \wedge F_3. \quad (3.50)$$

In background considered the five-form F_5 acquires a radial dependence. Indeed $\omega_2 \wedge \omega_3 \sim \mathrm{vol}(T^{1,1})$, so one can write

$$F_5 = \mathcal{F}_5 + *\mathcal{F}_5, \qquad \mathcal{F}_5 = \mathcal{K}(r)\mathrm{vol}(T^{1,1}), \quad (3.51)$$

with

$$\mathcal{K}(r) = N + \frac{3}{2\pi}\,g_s M^2 \log r/r_0. \quad (3.52)$$

This represents the effective flux of F_5 through $T^{1,1}$, and at the UV scale $r = r_0$ coincides with the number of D3-branes. However, the number of colors of the gauge group or the number of D3-branes in the dual picture effectively changes with the RG flow. The flux of F_5 may completely disappear at some value of $r = \tilde{r}$,

$$\tilde{r} = r_0 \exp\left(-\frac{2\pi N}{3 g_s M^2}\right).$$

Such a behavior is related to the fact that $\int_{S^2} B_2$ is not a single-valued function in the supergravity solution. If one makes a small circle around a point r in the complex plane $r \to r e^{2\pi i}$, $\mathcal{K}(r) \to \mathcal{K}(r) - M$, which corresponds to dropping M units of 5-form flux.

In terms of the scale $\tilde{r}$,

$$\mathcal{K}(r) = \frac{3}{2\pi}\,g_s M^2 \log(r/\tilde{r}), \qquad H(r) = \frac{4\pi g_s}{r^4}\,[\mathcal{K}(r) + \frac{3}{8\pi}\,g_s M^2]. \quad (3.53)$$

This solution has a naked singularity at $r = r_s$, where $H(r_s) = 0$,

$$H(r) = \frac{R^4}{r^4}\log(r/r_s). \quad (3.54)$$

The singularity of the supergravity solution appears due to the fact that there are M units of F_3-form flux through the three-sphere. The flux does not depend on the radius of the sphere, therefore, to maintain the constant flux, the energy density F_3^2 must become infinite at zero radius.

The whole expression for the ten dimensional metric has the form

$$\mathrm{d}s^2 = \frac{r^2}{R^2\sqrt{\log(r/r_s)}}\,(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2) + \frac{R^2\sqrt{\log(r/r_s)}}{r^2}\,\mathrm{d}r^2 + R^2\sqrt{\log(r/r_s)}\mathrm{d}s^2_{T^{1,1}}. \quad (3.55)$$

This is the solution obtained in [17]. The meaning of this solution for $r \to \infty$ is an infinite RG cascade, in which N becomes bigger and bigger in the UV. For small r , as the theory flows to IR, the cascade must stop, since the negative N is physically nonsensical. Furthermore, the solution should be modified in the IR, to resolve the naked singularity in the metric. A solution that satisfies the above criteria was found in the paper [18].

3.5 Deformed Conifold

3.5.1 Duality Cascade

As explained in [18], the effective change of the 5-form flux, observed in the previous section, has the meaning of changing the number of colors in the dual gauge theory. From the point of view of the $SU(N)$ factor of the $SU(N+M) \times SU(N)$ gauge group this is a theory with $2(N+M)$ flavors. Dropping M units of 5-form flux very much resembles the phenomenon, known as Seiberg duality [49]. In the latter the theory with $SU(N_c)$ gauge group and $SU(N_f)$ flavor symmetry is dual to a theory with $SU(2N_c - N_f)$ gauge group with $2N_c$ flavors. This is indeed the case here. After the duality one ends with $SU(N) \times SU(N-M)$ theory. Another indication of a Seiberg-like duality is the nontrivial flux monodromy.

The superpotential (3.20) of the original theory is renormalized in the RG flow. The generic form of the renormalized superpotential is

$$W = \lambda_1 \epsilon^{ij} \epsilon^{kl} \mathrm{tr}\,(A_i B_k A_j B_l) F_1(I_1, J_1, R_1^{(s)}), \tag{3.56}$$

where $F_1(I_1, J_1, R_1^{(s)})$ is some function of global symmetry invariants[2]

$$I_1 = \lambda_1^{3M} \frac{\tilde{\Lambda}_1^{3N-2(N+M)}}{\Lambda_1^{3(N+M)-2N}} \,[\epsilon^{ij}\epsilon^{kl} \mathrm{tr}(A_i B_k A_j B_l)], \tag{3.57}$$

$$R_1^{(1)} = \frac{\epsilon^{ij}\epsilon^{kl}\mathrm{tr}[A_i B_k]\mathrm{tr}[A_j B_l]}{\epsilon^{ij}\epsilon^{kl}\mathrm{tr}(A_i B_k A_j B_l)}, \tag{3.58}$$

and the constant invariant made of the scales and the coupling,

$$J_1 = \lambda_1^{(N+M)+N} \Lambda_1^{3(N+M)-2N} \tilde{\Lambda}_1^{3N-2(N+M)}. \tag{3.59}$$

The last invariant plays the role similar to the dimensionless YM coupling τ.

The Seiberg dual gauge theory contains the fields a_i and b_i in fundamental and antifundamental representations of $SU(N)$, and also meson bilinears $M_{ij} = A_i B_j$ in the adjoint and singlet representation. The superpotential of the new theory is

$$W = \lambda_1 \epsilon^{ij}\epsilon^{kl} \mathrm{tr}(M_{ik} M_{jl}) F_1(I_1, J_1, R_1^{(s)}) + \frac{1}{\mu}\,\mathrm{tr} M_{ij}\, a_i b_j, \tag{3.60}$$

where μ plays the role of matching scale [50]. The meson fields M_{ij} are massive, and should be integrated out from the low energy spectrum, which leaves the superpotential

$$W = \lambda_2 \epsilon^{ij}\epsilon^{kl} \mathrm{tr}(a_i b_k a_j b_l) F_2(I_2, J_2, R_2^{(s)}), \tag{3.61}$$

---

[2] The superscript of the R_1 corresponds to different invariants, constructed in a way similar to the construction of $R_1^{(1)}$, but with the color and flavor indices contracted differently.

where the new parameters I_2, J_2 and $R_1^{(s)}$ are defined in the same wave as in the original theory. In fact the new $SU(N) \times SU(N-M)$ theory is very similar to the original one. It is also interesting to study matching of scales of two theories in more details. Let us denote the strong coupling scale of $SU(N_M)$ as $\tilde{\Lambda}_2$, and the strong coupling scale of $SU(N)$ as Λ_2. The matching conditions will be

$$\lambda_2 \sim \frac{\lambda_1}{\mu^2}, \tag{3.62}$$

and

$$\Lambda_1^{3(N+M)-2N} \tilde{\Lambda}_2^{3(N-M)-2N} \sim \mu^{2N} \sim \lambda_1^M \tilde{\Lambda}_1^{3N-2(N+M)} \lambda_2^{-M} \Lambda_2^{3N-2(N-M)}. \tag{3.63}$$

It is now straightforward to find the matching of the invariants. In particular,

$$J_2 \propto 1/J_1. \tag{3.64}$$

This behavior of the couplings is typical for the electric-magnetic duality transformation ($\tau \to -1/\tau$).

Such a "cascade" of dualities continues and after the k such transformations one of the gauge factors becomes $SU(N-kM)$. The cascade should stop at the point, where $N-kM$ becomes zero or negative. Note that in order to apply the Seiberg duality the theory should satisfy $N_f > N_c + 1$. Therefore at $N_f \simeq N_c$ a more careful analysis is required. What happens in fact in this regime is that the geometry is modified by non-perturbative effects. Following [18] we are going to first introduce the definition of a deformed conifold, which appears on the last step of the cascade. The reader can find a detailed review of the cascading RG flow in [51].

3.5.2 Deformation of the Conifold

The solution (3.55) obtained in [17] has a singularity in the IR ($r \to 0$). To avoid this singularity the solution needs to be deformed. The authors of [18] suggested the following modification of the conifold (3.7):

$$\sum_{n=1}^{4} z_n^2 = \epsilon^2, \tag{3.65}$$

where the singularity is removed by blowing-up of the S^3 cycle of $T^{1,1}$ at the tip of the conifold (see the figure 3.1).

Another argument in support of the statement, that the supergravity solution should be modified in the infrared, is the strong dynamics of the dual gauge theory. From the non-perturbative gauge theory dynamics one can expect a non-trivial structure of the space of vacua (appearance of M branches), breaking of $U(1)_{\mathcal{R}}$ symmetry down to $\mathbb{Z}_{2M}$, and further spontaneous breaking down to $\mathbb{Z}_2$.

A simple example considered in [18] was a study of the dynamics of a theory that starts from a single D3-brane, thus having $SU(M+1) \times SU(1) \simeq SU(M+1)$ gauge group. This theory contains the fields C_i and D_k in the fundamental and anti-fundamental representations, $i, k = 1, 2$. The superpotential of such theory is $W = \lambda \epsilon^{ij} \epsilon^{kl} C_i D_k C_j D_l$. The redefined gauge invariant fields $N_{ik} = C_i D_k$ would be similar to the coordinates z_n on the conifold in the following sense: one can look at z_n as at a matrix

$$z_{ij} = \frac{1}{\sqrt{2}} \sum_{n=1}^{4} \sigma_{ij}^{n} z_n. \tag{3.66}$$

In this variables the conifold equation (3.7) takes the form $\det z_{ij} = 0$. Now N_{ij}, being the product of fields C_i and D_j are equivalent to the coordinate matrix z_{ij}. The classical space of vacua of the theory is given by the equation $\det N_{ij} = 0$. However the quantum dynamics modifies the classical moduli space. The low energy theory can be studied in terms of these gauge invariant fields. The non-perturbative superpotential was first found by I. Affleck, M. Dine and N. Seiberg in [52],

$$W = \lambda \epsilon^{ij} \epsilon^{kl} N_{ik} N_{jl} + (M-1) \left[\frac{2\Lambda^{3M+1}}{\epsilon^{ij} \epsilon^{kl} N_{ik} N_{jl}} \right]^{\frac{1}{M-1}}. \tag{3.67}$$

The vacua of the superpotential are given by the equation

$$\left(\lambda - \left[\frac{2\Lambda^{3M+1}}{(\epsilon^{ij} \epsilon^{kl} N_{ik} N_{jl})^M} \right]^{\frac{1}{M-1}} \right) N_{ij} = 0. \tag{3.68}$$

Since $N_{ij} = 0$ gives infinity, the only solutions of this equation are

$$(\epsilon^{ij} \epsilon^{kl} N_{ik} N_{jl})^M = \frac{2\Lambda^{3M+1}}{\lambda^{M-1}}. \tag{3.69}$$

The solution obviously has M different branches, corresponding to M^{th} root of the r.h.s. of (3.69). There is the $\mathbb{Z}_{2M}$ symmetry that acts on the vacua by a phase $e^{2\pi i/M}$. This symmetry is spontaneously broken by a choice of particular vacuum. Moreover, now the vacuum solution is described by the equation

$$\det N_{ij} \equiv \frac{1}{2} \epsilon^{ij} \epsilon^{kl} N_{ik} N_{jl} = e^{2\pi i k/N} \left(\frac{\Lambda^{3M+1}}{(2\lambda)^{M-1}} \right)^{1/M}. \tag{3.70}$$

Comparing this with (3.65) one can conclude, that in the quantum theory D3-brane should live on the deformed conifold.

Another argument in favor of the deformed conifold mentioned by the authors of [18] is the pattern of $\mathcal{R}$-symmetry breaking. The original $U(1)$ symmetry rotates the coordinates z_n in (3.7) by a phase. This symmetry should be broken down to $\mathbb{Z}_{2M}$ and further to $\mathbb{Z}_2$. Indeed the deformation of the conifold (3.65) leaves only $\mathbb{Z}_2$ of $U(1)$ unbroken.

3.5.3 Supergravity Solution

The original $U(1)$ symmetry of the conifold (3.7) can be described as shifting of the angle ψ in the parametrization of the conifold by Euler angles, described in the section 3.2.2. In the case of the supergravity solution found in [17], the metric (3.55) is invariant under this $U(1)$. However, the flux of F_3, breaks this symmetry down to $\mathbb{Z}_{2N}$; see a further discussion in the section 4.2. This effect is related to the perturbative anomaly of the $U(1)_{\mathcal{R}}$ symmetry. It is the presence of only this partial breaking of the $\mathcal{R}$-symmetry signifies that the KT solution is only giving a dual of the perturbative regime. As was explained above we need to deform the conifold to break the $\mathbb{Z}_{2N}$ further down to $\mathbb{Z}_2$.

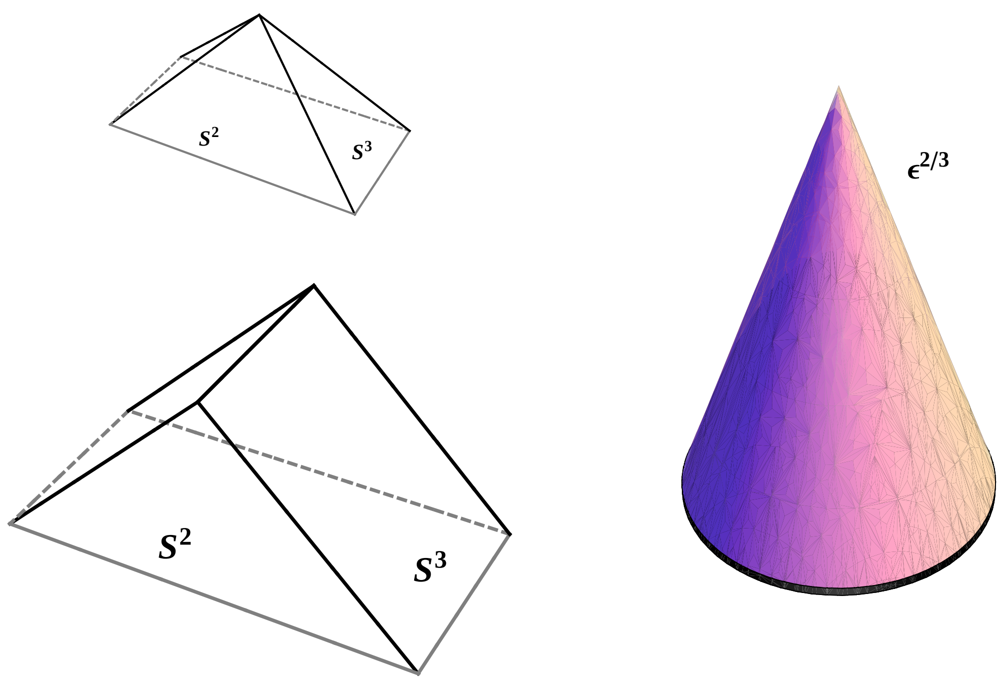

Figure 3.1: Deformation of the conifold. In the deformed conifold the S^3 is kept of a finite size at the tip, while the S^2 shrinks to a point (left). The deformation corresponds to smoothing out the sharp tip of the cone (right). The deformed cone is lower by $\epsilon^{2/3}$.

In the meantime one would like the new solution to smoothly interpolate between the singular conifold (3.7) in the UV, and the deformed conifold in the IR. The metric on the deformed conifold is known from the same work of P. Candelas and X. de la Ossa [47]:

$$\mathrm{d}s_6^2 = \frac{1}{2}\,\epsilon^{4/3}K(\tau)\Bigg[\frac{1}{3K^3(\tau)}\left(\mathrm{d}\tau^2 + (g^5)^2\right) + \cosh^2\left(\frac{\tau}{2}\right)\left[(g^3)^2 + (g^4)^2\right] + \\ + \sinh^2\left(\frac{\tau}{2}\right)\left[(g^1)^2 + (g^2)^2\right]\Bigg], \quad (3.71)$$

where

$$K(\tau) = \frac{(\sinh(2\tau) - 2\tau)^{1/3}}{2^{1/3}\sinh\tau} \tag{3.72}$$

and the basis g^i of one-forms on $T^{1,1}$ was defined earlier in (3.11). The old coordinate r has the following relation with the new coordinate τ

$$r^2 = \frac{3}{2}\,\epsilon^{4/3}\cosh^{2/3}\tau. \tag{3.73}$$

At large τ this becomes $r^3 \sim \epsilon^2 e^{\tau}$.

As one can see, at small τ the angular part of the metric behaves as

$$\mathrm{d}\Omega_3^2 = \frac{1}{2}\,\epsilon^{4/3}(2/3)^{1/3}\left[\frac{1}{2}(g^5)^2 + (g^3)^2 + (g^4)^2\right], \tag{3.74}$$

which corresponds to the S^3 cycle of $T^{1,1}$ staying of finite size, while the S^2 cycle is shrinking as

$$\mathrm{d}\Omega_2^2 = \frac{1}{8}\,\epsilon^{4/3}(2/3)^{1/3}\tau^2[(g^1)^2 + (g^2)^2]. \tag{3.75}$$

One can now see that the R-symmetry or the translational symmetry in ψ is indeed broken, since there is explicit dependence on ψ in the expression (3.71). However there is still a left-over $\mathbb{Z}_2$ related to the fact that ψ ia a double cover of the circle, $\psi \in [0, 4\pi]$.

Authors of [18] use the following ansatz for the 3-form:

$$F_3 = \frac{\alpha' M}{2}\left(g^5 \wedge g^3 \wedge g^4 + \mathrm{d}[F(\tau)(g^1 \wedge g^3 + g^2 \wedge g^4)]\right). \tag{3.76}$$

Indeed, if one imposes the following boundary conditions: $F(0) = 0$, $F(\infty) = 1/2$, such F_3 will reproduce the KT solution (3.36) at infinity and will be proportional to the S^3 volume at the origin.

For B_2, the following ansatz is taken:

$$B_2 = \frac{g_s M \alpha'}{2}\,[f(\tau) g^1 \wedge g^2 + k(\tau) g^3 \wedge g^4]. \tag{3.77}$$

This leads to the expression for $H_3 = \mathrm{d}B_2$,

$$H_3 = \frac{g_s M \alpha'}{2}\,[\mathrm{d}\tau \wedge (f' g^1 \wedge g^2 + k' g^3 \wedge g^4) + \frac{1}{2}\,(k - f) g^5 \wedge (g^1 \wedge g^3 + g^2 \wedge g^4)]. \tag{3.78}$$

Taking the external product of B_2 and F_3, one finds

$$\mathcal{F}_5 = B_2 \wedge F_3 = \frac{g_s M^2 (\alpha')^2}{4}\,\ell(\tau) g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge g^5, \tag{3.79}$$

where

$$\ell = f(1 - F) + kF. \tag{3.80}$$

As before, from $H_3 \wedge *_6 F_3 = 0$ and (3.27), follows the vanishing of R-R scalar, $C_0 = 0$. The dilaton can be set constant, namely

$$g_s^2 F_3^2 = H_3^2. \tag{3.81}$$

For the the metric the ansatz is similar to the regular D-brane metric (3.3) with the deformed conifold metric (3.71), as the six dimensional part,

$$\mathrm{d}s_{10}^2 = h^{-1/2}(\tau)(-(\mathrm{d}x^0)^2 + \mathrm{d}\vec{x}^2) + h^{1/2}(\tau)\mathrm{d}s_6^2. \tag{3.82}$$

Function $h(\tau)$, which parameterizes the metric is called the "warp"-factor and the class of metrics of the form (3.82) – warped metrics.

It is shown in [18] that the unknown functions in the ansatz satisfy the equations

$$\begin{aligned} f' &= (1-F)\tanh^2(\tau/2), \\ k' &= F\coth^2(\tau/2), \\ F' &= \frac{1}{2}(k-f), \end{aligned} \tag{3.83}$$

and

$$h' = -(g_s M\alpha')^2 2^{4/3}\epsilon^{-8/3}\frac{f(1-F)+kF}{K^2(\tau)\sinh^2\tau}. \tag{3.84}$$

The first three equations (3.83) form a closed system and can be solved first. The boundary conditions we need to satisfy are those, to match the solution (3.43) at large τ,

$$f \to \frac{\tau}{2}, \qquad k \to \frac{\tau}{2}, \qquad F \to \frac{1}{2}. \tag{3.85}$$

It is possible to find solution to (3.83) explicitly [18].

$$\begin{aligned} F(\tau) &= \frac{\sinh\tau - \tau}{2\sinh\tau}, \\ f(\tau) &= \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau - 1), \\ k(\tau) &= \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau + 1), \end{aligned} \tag{3.86}$$

Solution for h cannot be found explicitly. Instead we have an integral expression,

$$h(\tau) = (g_s M\alpha')^2 2^{2/3}\epsilon^{-8/3} I(\tau), \qquad I(\tau) = \int_\tau^\infty \mathrm{d}x \frac{x\coth x - 1}{\sinh^2 x}(\sinh(2x) - 2x)^{1/3}. \tag{3.87}$$

This solution has the following asymptotic. At large τ the integrand becomes $xe^{-4x/3}$, or

$$h \to \frac{3}{4}\,2^{1/3}\alpha\tau e^{-4\tau/3}, \tag{3.88}$$

which, together with identification (3.73) gives the old solution (3.55)

$$h \to \frac{\log r}{r^4}.$$

At small τ it behaves as

$$h \sim a_0 + a_1\tau^2 + \ldots, \tag{3.89}$$

approaching the constant value $a_0 \sim \alpha \sim (g_s M)^2$, instead of singularity of solution (3.55). This means that at small τ the geometry is approximately $R^{3,1}$ times deformed conifold.

$$\mathrm{d}s_{10}^2 \to a_0^{-1/2}(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2) + a_0^{1/2}\left(\frac{1}{2}\,\mathrm{d}\tau^2 + \mathrm{d}\Omega_3^2 + \frac{1}{4}\,\tau^2[(g^1)^2 + (g^2)^2]\right). \tag{3.90}$$

The radius of S^3 cycle stays finite at $\tau = 0$, of order $\sqrt{g_s M}$. The latter plays a role of 't Hooft coupling of $SU(M)$ gauge theory. As soon as the 't Hooft coupling is large, the curvature of the geometry is small, and supergravity approximation is reliable.

The solution, described above have a number of properties, which makes it a good candidate for the supergravity dual to the cascading $SU(N+M) \times SU(N)$ gauge theory in the case of a large parameter $g_s M$. This solution corresponds to the branes placed at the point on the deformed conifold, which does not have a singularity of the solution (3.55). The energy density of 3-form F_3^2 stays finite in the IR limit in this case. Moreover, in contrast with the solution (3.55), the deformed conifold solution breaks the $\mathcal{R}$-symmetry down to $\mathbb{Z}_2$ as we expect from strong dynamics of the gauge theory. Under appropriate circumstances the gauge theory at the bottom of the cascade is an $SU(M)$ theory. This is not however the desired pure $SU(M)$ $\mathcal{N} = 1$ SYM theory. As we will see later, the mesonic states made out of the superfields A_k and B_l are not heavy enough to be ignored. In fact, it was realized in [22] that there are even massless Goldstone bosons related to the expectation values of the operators that can be constructed from the A_k and B_l. Nevertheless, this is still a very interesting example of $\mathcal{N} = 1$ gauge theory to study.

3.6 Baryonic Branch

3.6.1 Baryonic Operators and Massless Modes

It was mentioned in the end of the previous section that the gauge theory dual to the the type IIB theory on the warped deformed conifold is not in the same universality class with pure gauge $SU(M)$ $\mathcal{N} = 1$ SYM theory. This question was addressed by S. Gubser, C. Herzog and I. Klebanov in [22], where the fate of the D1-branes in string theory on the conifold was investigated. It was shown that there exist a massless perturbation of the background that couples to the D1-branes. The D1-branes however do not have a dual interpretation in the IR regime of $SU(M)$ $\mathcal{N} = 1$ SYM theory.

In fact, it was earlier anticipated by O. Aharony [53] that the gauge theory dual to the supergravity on the conifold should be on the so-called "baryonic" branch [54] of the space of vacua. At the last step of the cascade, the theory should convert from $SU(2M) \times SU(M)$ to $SU(M)$ theory. From the point of view of $SU(2M)$ factor, the theory has equal number of colors and flavors. This means that apart from the mesonic operators $N_{kl} = A_k B_l$ there exist gauge invariant baryonic operators that can also take expectation values. The latter are constructed as follows:

$$\begin{aligned} \mathcal{B} &= \epsilon_{\alpha_1\alpha_2\ldots\alpha_{2M}} (A_1)_1^{\alpha_1} (A_1)_2^{\alpha_2} \ldots (A_1)_M^{\alpha_M} (A_2)_1^{\alpha_{M+1}} (A_2)_2^{\alpha_{M+2}} \ldots (A_2)_M^{\alpha_{2M}}, \\ \bar{\mathcal{B}} &= \epsilon^{\alpha_1\alpha_2\ldots\alpha_{2M}} (B_1)_{\alpha_1}^1 (B_1)_{\alpha_2}^2 \ldots (B_1)_{\alpha_M}^M (B_2)_{\alpha_{M+1}}^1 (B_2)_{\alpha_{M+2}}^2 \ldots (B_2)_{\alpha_{2M}}^M. \end{aligned} \tag{3.91}$$

In the theory after the cascade the superpotential is

$$W = \lambda (N_{ij})_\beta^\alpha (N_{kl})_\alpha^\beta \epsilon^{ik} \epsilon^{jl} + X(\det[(N_{ij})_\beta^\alpha] - \mathcal{B}\bar{\mathcal{B}} - \Lambda_{2M}^{4M}), \tag{3.92}$$

The vacua that preserve supersymmetry should satisfy $W = 0$, which implies $N = 0$ and apart from the point $X = 0$,

$$\mathcal{B}\bar{\mathcal{B}} = -\Lambda_{2M}^{4M}. \tag{3.93}$$

Since $\mathcal{B}$ and $\bar{\mathcal{B}}$ can acquire expectation number, the baryon number symmetry, $A_k \to e^{i\alpha} A_k$, $B_l \to e^{-i\alpha} B_l$ is spontaneously broken. The KS solution corresponds to the values $|\mathcal{B}| = |\bar{\mathcal{B}}| = \Lambda_{2M}^{2M}$. The baryonic branch has complex dimension one, because one can make a transformation that leaves (3.93) invariant,

$$\mathcal{B} \to i\xi \Lambda_{2M}^{2M}, \qquad \bar{\mathcal{B}} \to \frac{i}{\xi} \Lambda_{2M}^{2M}. \tag{3.94}$$

In contrast to the $U(1)_{\mathcal{R}}$ symmetry the baryon number symmetry is not anomalous. However, because it is broken spontaneously, there should be Goldstone bosons, corresponding to the motion along the branch. The authors of [22] argue that the pseudoscalar and the scalar massless modes, which they discover, are precisely the Goldstone bosons of the $U(1)_B$ spontaneous breaking.

Let us discuss these massless modes. As we mentioned in the section 1.2, the particles of the dual gauge theory corresponds to fluctuations of the supergravity background. In the work [22], the following fluctuations of the background are considered:

$$\begin{aligned} &\delta H_3 = 0, \qquad \delta F_3 = f_1(\tau) *_4 \mathrm{d}a + f_2(\tau) \mathrm{d}a \wedge \mathrm{d}g^5 + f_2' \mathrm{d}a \wedge \mathrm{d}\tau \wedge g^5, \\ &\qquad \delta F_5 = (1 + *)\delta F_3 \wedge B_2 = f_1(\tau)(*_4 \mathrm{d}a - \frac{\epsilon^{4/3}}{6K^2(\tau)} \mathrm{d}a \wedge \mathrm{d}\tau \wedge g^5) \wedge B_2. \end{aligned} \tag{3.95}$$

Here $a(x^0, x^1, x^2, x^3)$ denotes a pseudoscalar fluctuation of the background. f_1, f_2 are unknown functions of the coordinate τ and the subscript 4 of the Hodge dual operator denotes that the latter is taken with respect to the Minkowski metric. This is a consistent ansatz if the following equations are satisfied:

$$f_1(\tau) = 1, \qquad f_2(\tau) = \frac{2c}{K^2 \sinh^2 \tau} \int \mathrm{d}x\ h(x) \sinh^2 x, \tag{3.96}$$

where c is some constant and

$$\mathrm{d} *_4 \mathrm{d}a = 0. \tag{3.97}$$

In particular, the latter equation means that the pseudoscalar fluctuation a has zero four dimensional mass. Note that a does not depend on the radial variable τ. The τ-dependent generalization of the fluctuations (3.95) was latter found in [26].

Since we are dealing with the supergravity dual of a $\mathcal{N} = 1$ theory, the pseudoscalar mode should have a superpartner. The corresponding scalar mode were also discovered in the paper [22]. One needs to consider the ansatz

$$\delta B_2 = \chi\, \mathrm{d}g^5, \qquad \delta H_3 = \chi'\, \mathrm{d}\tau \wedge \mathrm{d}g^5, \qquad \delta F_3 = 0, \qquad \delta F_5 = 0, \qquad \delta G_{13} = \delta G_{24} = m. \tag{3.98}$$

where the scalar fluctuation of the background is described by the function χ and m that are functions of τ and four dimensional coordinates. Here G_{13} and G_{24} are components of the metric in the direction of the basis forms g^1, g^3 and g^2, g^4 respectively.

This ansatz is consistent if the following is satisfied.

$$\chi' = \frac{1}{2}\, g_s M z\, \frac{\sinh 2\tau - 2\tau}{\sinh^2 \tau}, \tag{3.99}$$

where $2^{-1/3}(\sinh 2\tau - 2\tau)^{1/3} h^{1/2}\, z = m$, and

$$\frac{\left(K^2 \sinh^2 \tau\, z'\right)'}{K^2 \sinh^2 \tau} = \left(\frac{2}{\sinh^2 \tau} + \frac{8}{9}\, \frac{1}{K^6 \sinh^2 \tau} - \frac{4}{3}\, \frac{\cosh \tau}{K^3 \sinh^2 \tau}\right) z. \tag{3.100}$$

The latter equation is solved by

$$z = \frac{c_1 \coth \tau + c_2 (\tau \coth \tau - 1)}{(\sinh 2\tau - 2\tau)^{1/3}}. \tag{3.101}$$

In this solution c_1 and c_2 are constants. For regular behavior at the origin, c_1 should be set to zero.

The scalar and pseudoscalar modes are the members of the same supermultiplet. Although it is obvious from the above derivation we will prove this in the chapter 5, where the generalizations of the above equations will be considered.

The massless pseudoscalar mode (3.97) mode corresponds to the phase transformation in (3.94), while the transformation corresponding to the scalar mode (3.101) change the absolute value of $\mathcal{B}$ and $\bar{\mathcal{B}}$. The presence of the scalar mode indicates that there should exist a one-parametric family of the solutions on the conifold. This family was indeed found by A. Butti et al. in [19]. The relationship between the parameter of the family and the condensate of baryonic operators was later studied by M. Benna, A. Dymarsky and I. Klebanov in [55].

3.6.2 One-Parametric Family of Solutions

The conifold possesses an $SO(4) \simeq SU(2) \times SU(2)$ group of isometries, which is apparent from the definitions (3.7) and (3.65). The KS solution respects this symmetry. However it is not the most general $SO(4)$-symmetric solution possible. G. Papadopoulos and A. Tseytlin considered in [56] the most general ansatz for the background solution respecting $SO(4)$ symmetry. Their ansatz can be written in the following way. Write the metric as

$$\mathrm{d}s_{10}^2 = e^{2A(\tau)}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \sum_{i=1}^{6} E_i^2, \tag{3.102}$$

where $A(\tau)$ is some unknown function of τ. Forms E_i can be expressed in terms of the basis 1-forms (3.12):

$$\begin{aligned} E_1 = e^{(x(\tau)+g(\tau))/2}e_1, \qquad E_2 = e^{(x(\tau)+g(\tau))/2}e_2, \\ E_3 = e^{(x(\tau)-g(\tau))/2}(e_3 - a(\tau)e_1), \qquad E_4 = e^{(x(\tau)-g(\tau))/2}(e_4 - a(\tau)e_2), \\ E_5 = e^{-(6p(\tau)+x(\tau))/2}\mathrm{d}\tau, \qquad E_5 = e^{-(6p(\tau)+x(\tau))/2}e_5, \end{aligned} \tag{3.103}$$

where the new unknown functions are $p(\tau)$, $x(\tau)$, $g(\tau)$ $a(\tau)$. For the supergravity form-fields introduce a $SO(4)$-invariant basis of 2-forms,

$$\begin{aligned} Y_1 = e_2 \wedge e_1, \qquad Y_2 = e_4 \wedge e_3, \\ Y_3 = e_3 \wedge e_1 + e_4 \wedge e_2, \\ Y_4 = e_4 \wedge e_1 - e_3 \wedge e_2. \end{aligned} \tag{3.104}$$

In this basis the 3-forms read

$$H_3 = \mathrm{d} \wedge [h_1(\tau) + \chi(\tau))Y_1 + (h_1(\tau) - \chi(\tau))Y_2 + h_2(\tau)Y_4], \tag{3.105}$$

$$F_3 = P\left[g^5 \wedge (Y_1 + Y_2 - b(\tau)Y_4) + \dot{b}(\tau)\mathrm{d}\tau \wedge Y_3\right], \tag{3.106}$$

$$F_5 = \mathcal{F}_5 + *\mathcal{F}_5, \qquad \mathcal{F}_5 = K(\tau)Y_1 \wedge Y_2 \wedge g^5. \tag{3.107}$$

where more new functions were introduced: h_1, h_2, χ, b and K. The primes denote the τ-derivative. Thus the $SO(4)$ symmetric supergravity background is described by eleven scalar functions, including the dilaton $\Phi(\tau)$. There is also a pseudoscalar RR field C, which can be set to zero:

$$C = 0. \tag{3.108}$$

The scalar functions of τ satisfy a complicated system of coupled second order ordinary differential equations. In [19] a system of first order equations that solves the equations of motions was found. Apart for few equations, this system is still complicated and can be solved only numerically. Below we present the equations of [19], omitting the tau dependence of the ansatz functions.

The following algebraic equations should be satisfied:

$$\begin{gathered} e^{2g} = -1 - a^2 - 2a\cosh\tau, \\ K = Q + 2P(h_1 + bh_2), \\ b = -\frac{\tau}{\sinh\tau}, \qquad h_1 = h_2\cosh\tau + Q, \end{gathered} \tag{3.109}$$

where P Q are constants that depend on M and N respectively. The first order equations read

$$\begin{gathered} \dot{h_2} = -\frac{(\tau - a^2\tau + 2a\tau\cosh\tau + a^2\sinh 2\tau)}{(1 + a^2 + 2a\cosh\tau)(t\coth\tau - 1)}h_2, \\ \dot{\chi} = \frac{a(1 + a\cosh\tau)(2\tau - \sinh 2\tau)}{(1 + a^2 + 2a\cosh\tau)(\tau\coth\tau - 1)}h_2, \\ \dot{A} = \frac{(\tau\coth\tau - 1)(\sinh 2\tau - 2\tau)}{16\sinh^2\tau}e^{2\Phi - 2x}, \\ e^{2\Phi} = \frac{2h_2\sinh\tau}{\eta(1 - \tau\coth\tau)}, \\ e^{2x} = \frac{(\tau\coth\tau - 1)^2(1 - \eta^2 e^2\Phi)}{4(1 + a\cosh\tau)}e^{2\Phi + 2g}. \end{gathered} \tag{3.110}$$

In the last two equations η is the integration constant, which can be fixed by the UV asymptotic. We will set it to one below.

The two following equations can be derived after introducing $v = e^{6p+2x}$:

$$\begin{aligned} \dot{a} &= -\frac{e^g(1 + a\cosh\tau)}{v\sinh\tau} - \frac{a\sinh\tau(\tau + a\sinh\tau)}{t\cosh\tau - \sinh\tau}, \\ \dot{v} &= -3ae^{-g}\sinh\tau - v\left[-a^2\cosh^3\tau + 2a\tau\coth\tau + a\cosh^2\tau(2 - 4\tau\coth\tau)\right. \\ &\quad \left. + \cosh\tau(1 + 2a^2 - (2 + a^2)\tau\coth\tau) + \tau\ \mathrm{csch}\tau\right] e^{-2g}/[\tau\cosh\tau - \sinh\tau]. \end{aligned} \tag{3.111}$$

This solution to the above system is parameterized by the integration constant. We will use the following parametrization below. In the IR $\tau \to 0$, one can find the expansion of the solution. The family parameter will be a coefficient in front of τ^2 in the a expansion:

$$a = -1 + \xi\tau^2 + O(\tau^4). \tag{3.112}$$

This ξ can take values in the open interval $(1/6, 5/6)$. At the boundaries of the interval one obtains another supergravity background known as Maldacena-Nuñez solution [57], which we did not review here. This limit however is singular. The KS solution corresponds to the point $\xi = 1/2$, where the above equations can be integrated.

Chapter 4

Glueballs of Gravity Dual Theories. Graviton Multiplet

4.1 Supercurrent

In the previous chapter we reviewed a construction of supergravity backgrounds dual to the vacua of a certain class of $\mathcal{N}=1$ gauge theories. From the backgrounds considered the Klebanov-Strassler (KS) solution [18] is of the most interest for particle physics, since the corresponding gauge theory has a non-trivial low-energy dynamics, known as confinement. That is a $\mathcal{N}=1$ supersymmetric $SU(N+M)\times SU(N)$ gauge theory with two pairs of the chiral superfields A_k and B_l, $k,l=1,2$, and the superpotential (3.20). Through a cascade of Seiberg-like dualities this theory flows to the IR, where cascade must stop after $k\simeq N/M$ steps. At the end of the cascade the strong-coupling dynamics modifies the space of vacua. M branches appear and the $\mathcal{R}$-symmetry spontaneously breaks down to $\mathbb{Z}_2$ by picking one of the vacua.

The dual gauge theory is a close relative of the pure $SU(M)$ $\mathcal{N}=1$ supersymmetric Yang-Mills (SYM) theory. However, as reviewed in the previous chapter, the two theories are not in the same universality class. The reason is that at the last step of the cascade the baryonic operators acquire expectation values, which means the theory is at the baryonic branch in the space of vacua. This is proven by the existence of massless excitations of the background, which are dual to the Goldstone bosons of the spontaneously broken baryon number symmetry [22].

Although the KS case neither describes QCD, nor even the $\mathcal{N}=1$ SYM theory, it is still a worthy case to study. In fact, this is the only known complete example of the holographic correspondence involving a four dimensional confining gauge theory. The purpose of this and the following chapters will be a study of the spectrum of lightest states in the KS dual gauge theory. In principle, the results obtained here are valid in the large N (M) approximation only. Nevertheless some details of the spectra can depend on N (M) very insignificantly, as it was also assumed in the separate investigation of chapter 2. We hope that the future studies will demonstrate that this is indeed the case.

In the chapter 1 we discussed the correspondence of supergravity fluctuations to the operators in a dual gauge theory. The particles are the poles in the two-point correlation functions of the operators. On the gravity side the poles correspond to the eigenvalues of the second order differential operator, which represents the supergravity equations linearized in the fluctuations subject to the boundary conditions. In simplest cases the supergravity fields are in one-to-one correspondence with the operators. Generically the supergravity fluctuations mix, which complicates the study of more sophisticated supergravity backgrounds, such as the KS solution.

In this chapter we will consider the simpler examples of the correspondence. We would like to remind the reader the relationship between the background metric and the stress energy tensor of a theory:

$$T_{\mu\nu} = \frac{2}{\sqrt{g}}\,\frac{\delta S}{\delta g^{\mu\nu}}.$$

This relation suggests that the stress energy tensor in the boundary theory will couple only to the metric fluctuations, but not to other fluctuations of the supergravity. We will show below that this is indeed true. The four dimensional traceless fluctuations of the metric decouple. Moreover the linearized equation for the metric perturbation is particularly simple.

In the gauge theory the operator $T_{\mu\nu}$ possess the following properties. It is conserved $\partial^\mu T_{\mu\nu} = 0$ and on the classical level it is also traceless $T^\mu_{\ \mu} = 0$. On the quantum level, the tracelessness is violated by the scale anomaly. In this case one typically separates the traceless and trace parts of $T_{\mu\nu}$ as we did in a similar situation in the section 2.2.4 of the chapter 2. In fact the trace part is related by the anomaly equations to other operators of the gauge theory like tr $F_{\mu\nu}F^{\mu\nu}$. One can also see this from the point of view of the supergravity side, where the traceless excitations of the metric nicely decouple, while the trace of the metric couples to other supergravity fluctuations [23].

It has been known for quite a while that in the $\mathcal{N}=1$ supersymmetric theories the operator $T_{\mu\nu}$ united with the $U(1)_{\mathcal{R}}$ current J^5_μ and superconformal current $S_{\mu\dot\alpha}$ to form a supermultiplet [58]. This supermultiplet can be written in the superfield notations.

$$V_\mu = J^5_\mu - \frac{i}{2}\,\theta\sigma^\nu\bar\theta\,T_{\nu\mu} + i\,\theta^2\,\partial_\mu\bar s - i\,\bar\theta^2\,\partial_\mu s + \frac{1}{4}\,\bar\theta^2\theta^2\left(2D_\mu + \Box J^5_\mu\right) + \text{fermions}, \tag{4.1}$$

where we omit the fermionic terms. Here s is an additional complex scalar field and D_μ is an auxiliary real vector field. From the point of view of four dimensional superalgebra, the supercurrent V_μ forms a reducible representation. To separate the irreducible representation, one typically imposes the constraints on the superfield. A generic constraint one can put for V_μ reads

$$D^\alpha V_{\alpha\dot\alpha} = \bar D_{\dot\alpha}\bar S. \tag{4.2}$$

This constraint is telling us that the irreducible representation is made by subtracting from V_μ the modes of the chiral superfield S. If one works out (4.2) in components one will see that S is made of the scalar s, trace of the stress-energy tensor $T^\mu_{\ \mu}$, divergence of the $U(1)_{\mathcal{R}}$ current $\partial^\mu J^5_\mu$ and two Weyl fermions, which we will ignore in the future as well as other fermionic components. Thus the irreducible representation contains traceless $\Theta_{\mu\nu} = T_{\mu\nu} - 1/4 g_{\mu\nu} T^\rho_{\ \rho}$, the transverse part of the $U(1)_{\mathcal{R}}$ current $\mathcal{J}^5_\mu$, $\partial^\mu \mathcal{J}^5_\mu = 0$ and the gamma-traceless part of the superconformal current. Notice that supersymmetry makes it manifest that the various traces are independent degrees of freedom.

We will further refer to the irreducible representation as to the "graviton" multiplet because it contains the traceless spin 2 field. The superfield S contains the parts of V_μ that represent the anomalies. It will thus be called the "anomaly" multiplet. In the conformal case, the scale, superconformal and $\mathcal{R}$ symmetries are anomaly-free. Therefore in that case the right hand side of (4.2) has to vanish.

On the gravity dual side the supersymmetry structure should be easily seen in the case of non-conformal theory like the KS solution. The fluctuations of the background fields should have the same spectrum of the four dimensional mass if they are in the same multiplet. Below we will see that the vector excitations of the background fields that are suspected to be dual to the $U(1)_{\mathcal{R}}$ current indeed share the same spectrum with the traceless four dimensional metric excitations. We will also demonstrate that the linearized equations for those excitations are related by a one dimensional Supersymmetric Quantum Mechanics (SQM) transformations, which are the remnants of the full ten dimensional supersymmetry transformations.

Comparison to other known results (e.g. [59]) shows that the above results can be generalized to the whole baryonic branch of solutions, which were reviewed in the end of the previous chapter. In next section we are going to remind the reader of the gravity dual interpretation of the $U(1)_{\mathcal{R}}$ anomaly and discuss the supergravity fluctuations dual to the operator $\mathcal{J}^5_\mu$.

4.2 Multiplets and Anomalies in the Dual Theory

To find the spectrum of the fluctuations dual to $\Theta_{\mu\nu}$, the traceless part of the conserved stress-energy tensor we will consider the transverse off-diagonal fluctuation of the background metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{4.3}$$

As we will demonstrate in the next section, this fluctuations satisfy the Laplace equation in the metric (3.82).

To find the fluctuations dual to $\mathcal{J}^5_\mu$, recall that $U(1)_{\mathcal{R}}$ transformations act as rotations along the conifold base $T^{1,1}$,

$$\psi \to \psi + \zeta,$$

where ψ is one of the angles on the base (see the explanation in the beginning of the section 3.5.3).

In the conformal case, the background is invariant under this symmetry, what results in a massless gauge field $\tilde{A}_\mu$. The KS background, as well as the backgrounds along the baryonic branch, breaks the $U(1)_\mathcal{R}$ symmetry already in the UV. The 2-form potential for the RR form F_3 has an explicit ψ dependence. In the UV limit

$$C_2 \simeq M\psi\,\omega_2,$$

where M is the flux of F_3 through the S^3 of $T^{1,1}$, and ω_2 is the ψ independent 2-form on $T^{1,1}$. Given that ψ itself is a double cover of the circle, C_2 breaks $U(1)_\mathcal{R}$ down to $\mathbb{Z}_{2M}$ in the UV. In the IR the metric has an explicit ψ dependence that breaks $\mathbb{Z}_{2M}$ further to $\mathbb{Z}_2$ in the full agreement with the gauge theory.

As a result, the corresponding fluctuation of the background acquires a five dimensional mass terms, which is not vanishing even in the UV region [60, 61]. The fluctuation in question modifies the metric along the ψ direction $g_{\mu\psi}$ and can be described by the perturbation of the 1-form $\mathrm{d}\psi$ by the "gauge" field $\tilde{A} = \tilde{A}_\mu\,\mathrm{d}x^\mu + \tilde{A}_t\,\mathrm{d}t$,

$$\mathrm{d}\psi \to \mathrm{d}\psi + \tilde{A}. \tag{4.4}$$

Since the dependence on the angles of the conifold is not important, we can restrict our attention to the five-dimensional theory. In the conformal case, in the absence of the 3-form fluxes, the five-dimensional vector field $\tilde{A}$ satisfies the equation for the massless vector

$$\mathrm{d} *_5 \mathrm{d}\tilde{A} = 0. \tag{4.5}$$

The longitudinal part of $\tilde{A}$ is not fixed by the equation (4.5) as it is a gauge degree of freedom. The corresponding symmetry is anomaly free. After adding the fluxes, the equation for $\tilde{A}$ can be brought to the form

$$\mathrm{d}(f *_5 \mathrm{d}(g\tilde{A})) + *_5\tilde{A} = 0, \tag{4.6}$$

with some background-dependent functions f and g. The longitudinal part of $\tilde{A}$ is no longer trivial and satisfies

$$\mathrm{d} *_5 \tilde{A} = 0\ . \tag{4.7}$$

For an observer in four dimensions, the five-dimensional no-source equation (4.7) is precisely the equation with an anomalous source

$$\partial^\mu \tilde{A}_\mu = \theta(\Lambda)\ , \tag{4.8}$$

where μ denotes the space-time indices. This holographic anomaly mechanism is discussed in more detail in [60].

The backgrounds we are interested in have a global $SU(2) \times SU(2)$ symmetry. Since we are interested in the uncharged sector, all fluctuations should be s-waves with respect to the directions along the base of the conifold. This is obvious for the four-dimensional metric fluctuations as we keep it angle-independent. In the case of the vector, it is more tricky. In fact we need to switch from the 1-form $\mathrm{d}\psi$ to the invariant extension $g^5 \to g^5 + \tilde{A}$, where g^5 was defined by (3.11). Apparently the shift of $\mathrm{d}\psi$ results in the same shift of g^5.

In the section 4.4 we derive the equation (4.6) for the transverse part of the vector fluctuation (4.4). The transverse component decouples from the longitudinal part $\tilde{A}_\mu = \partial_\mu \tilde{a}$ and from other supergravity fluctuations as expected from the discussion in the section 4.1. Unfortunately it is much more complicated to derive the equation for the longitudinal mode $\tilde{a}$. Most likely it couples in a complicated way to other supergravity fields as is the case with its superpartner, fluctuation of the metric trace $h^\mu{}_\mu$ according to the derivation by M. Berg, M. Haack and W. Mück in [23]. Coupling with different supergravity excitations will lead to some non-trivial right hand side of the equation (4.8). It is particularly interesting to find the supergravity expression for $\theta(\Lambda)$ and compare it with the gauge theory predictions.

4.3 Graviton Equations

In the current and the following sections we will be interested in the equations for the bosonic components of the gravity multiplet, the graviton $h_{\mu\nu}$ and the vector mode $\tilde{A}_\mu$. We start with a ten-dimensional analysis of the linearized supergravity equations for the graviton excitations, valid for any solution on the baryonic branch, and proceed with a derivation of the equations for the vector field in the KS background in the section 4.4.

The traceless symmetric perturbation of the metric is described by the five-dimensional Klein-Gordon equation for a minimal scalar coupled to the background [21, 62].[1] A straightforward check performed in [21] shows that this property holds for the whole baryonic branch.

Here we use the notations for the background solution that were introduced by G. Papadopoulos and A. Tseytlin (PT) [56]; see the definitions (3.102)-(3.107). In particular, $A(\tau)$ is equivalent to the warp factor in the KS case $e^{-2A} = h^{1/2}$. It should not be confused with the vector fluctuation of the metric $\tilde{A} = \tilde{A}_i \, \mathrm{d}x^i$. In the Einstein frame the equation for the fluctuation of the graviton $\delta\left(\mathrm{d}s^2\right) = e^{-2A} h_{\mu\nu} \, \mathrm{d}x^\mu \mathrm{d}x^\nu$ takes the form

$$\ddot{h}_{\mu\nu} + 2(\dot{x} - \dot{\Phi} + 2\dot{A})\dot{h}_{\mu\nu} - k^2 e^{-2A-6p-x} h_{\mu\nu} = 0, \tag{4.9}$$

[1] Note that the KS solution as reviewed in the previous chapter was written in the stringy frame, i.e. for the action (3.25). In order to write the canonical Klein-Gordon equation, one has to switch to the Einstein frame. The transformation between the two frames is given by the metric rescaling $g_E = e^{-\Phi/2} g_s$.

where k^2 is the square of the 4-momentum and the over dots stand for τ derivatives. The equation (4.9) is precisely the Klein-Gordon equation for the minimal scalar in the the baryonic branch backgrounds including the KS point.

To proceed to the explicit form of the equation (4.9) for the KS background one chooses

$$\begin{aligned} e^{-2A} &= h^{1/2}, \qquad & e^{6p+2x} &= \frac{3}{2}\left(\coth\tau - \tau\,\mathrm{csch}^2\tau\right), \\ e^{\Phi} &= e^{\Phi_0} = 1, \qquad & e^{2x} &= \frac{1}{16}\left(\sinh\tau\cosh\tau - \tau\right)^{2/3} h, \end{aligned} \tag{4.10}$$

where $h(\tau)$ is the warp factor of the metric (3.82), defined by (3.87).

With these assignments the equation takes the form familiar from [20],

$$\ddot{h}_{\mu\nu} + \frac{8}{3}\frac{\sinh^2\tau}{\sinh 2\tau - 2\tau}\,\dot{h}_{\mu\nu} - k^2\,\frac{h(\tau)\sinh^2\tau}{(\sinh 2\tau - 2\tau)^{2/3}}\,h_{\mu\nu} = 0. \tag{4.11}$$

In the last term we absorbed the numerical constants in the normalization of the momentum. It is also convenient to write the equation in the conventional Schrödinger form

$$(-\partial_\tau^2 + V_2(\tau))h_{\mu\nu} = 0, \tag{4.12}$$

with the effective potential $V_2(k^2,\tau)$ given by

$$V_2 = \frac{k^2\,h(\tau)\sinh^2\tau}{(\sinh 2\tau - 2\tau)^{2/3}} - \frac{8}{9}\,\frac{\sinh^4\tau}{(\sinh\tau\cosh\tau - \tau)^2} + \frac{4}{3}\,\frac{\sinh\tau\cosh\tau}{(\sinh\tau\cosh\tau - \tau)}\,. \tag{4.13}$$

4.4 Vector Mode

To find a supergravity excitation that corresponds to the transverse (conserved) part of the $U(1)_{\mathcal{R}}$ current J^5_μ, one should consider a special deformation along the angular direction $\partial/\partial\psi$ of the $T^{1,1} \simeq S^3 \times S^2$, as was discussed in the section 4.2. Following [25], we perturb the $SU(2) \times SU(2)$ invariant 1-form g^5 in the following way:

$$g^5 \to g^5 + 2\tilde{\beta}(\tau)\tilde{A}, \qquad \tilde{A} \equiv \tilde{A}_\mu \mathrm{d}x^\mu\ , \tag{4.14}$$

where $\tilde{A}$ is a 1-form describing the vector mode and $\tilde{\beta}(\tau)$ is yet unknown function of τ. Such a deformation leads to the following perturbation of the metric:

$$\mathrm{d}s^2 \to \mathrm{d}s^2 + 2\,l(\tau)\,g^5 \cdot \tilde{A}, \tag{4.15}$$

where we introduced $l = 2\tilde{\beta}e^{-6p-x}$ for the later convenience. This change of the metric will affect the Einstein equation as well as other equations of the type IIB supergravity. In particular one needs to modify the RR 5-form F_5 to preserve its self-duality:

$$\begin{aligned} \delta F_5 = -\beta\tilde{A}\wedge\mathrm{d}g^5\wedge\mathrm{d}g^5 + \beta\mathrm{d}\tilde{A}\wedge g^5\wedge\mathrm{d}g^5 + \beta e^{3p+x/2} *_5 \mathrm{d}\tilde{A}\wedge\mathrm{d}g^5 + \\ + 2e^{-2x}(\beta - \tilde{\beta}K)e^{-3p-x/2} *_5 \tilde{A}\wedge g^5. \end{aligned} \tag{4.16}$$

Here $\beta(\tau)$ is another function to be determined and $K = 4\dot{A}\, e^{2x}$ in the PT notations (3.107). This turns out to be a minimal ansatz required for the KS solution. One can show that there is no need to perturb the other type IIB fields if one is interested only in the four-dimensional transverse part of $\tilde{A}$.

The ansatz so far contains the unknown functions β, and $\tilde{\beta}$ or l, which can be fixed by the equations of motion. The Bianchi identity provides us with the following equations:

$$\mathrm{d}(\beta e^{3p+x/2} *_5 \mathrm{d}\tilde{A}) + 2e^{-2x}(\beta - \tilde{\beta}K)e^{-3p-x/2} *_5 \tilde{A} = 0, \tag{4.17}$$

and a simple equation for the function β,

$$\dot{\beta} = 0, \qquad \text{or} \qquad \beta = \beta_0. \tag{4.18}$$

To find the function $\tilde{\beta}(\tau)$, or $l(\tau)$, one should linearize the Einstein equation with the perturbation of the metric as in (4.15). The only nontrivial equation comes from the $\delta R_{\mu\psi}$ term. After certain simplifications one can write it in the form

$$\begin{aligned} &\partial_\tau^2 \tilde{A}_\mu + \left(2(\dot{l}/l) + 6\dot{p} + 3\dot{x} + 2\dot{A}\right)\partial_\tau \tilde{A}_\mu - k^2 e^{-2A-6p-x}\tilde{A}_\mu + \\ &\qquad + \left((\ddot{l}/l) + (\dot{l}/l)(6\dot{p} + 3\dot{x} + 2\dot{A}) - 2\dot{A}(6\dot{p} + \dot{x}) - 2e^{-12p-4x}\right)\tilde{A}_\mu = \\ &\qquad\qquad = \left(\frac{e^{-6p-x}}{24}\left(H_3^2 + F_3^2\right) - \frac{2\beta_0}{l}\, e^{-6p-5x}K + \frac{1}{2}\, e^{-4x}K^2\right)\tilde{A}_\mu. \end{aligned} \tag{4.19}$$

In the KS background the square of the 3-forms is given by

$$F_3^2 = H_3^2 = 3e^{6p-x}\,\frac{\tau^2 + 2\,\tau^2\cosh^2\tau - 6\tau\,\sinh\tau\cosh\tau + \cosh^2\tau - 2 + \cosh^4\tau}{\sinh^4\tau}\,. \tag{4.20}$$

If one now writes the equation (4.17) in components, taking into account (4.18) and the transversality condition $\partial^\mu \tilde{A}_\mu = 0$,

$$\begin{aligned} &\partial_\tau^2 \tilde{A}_\mu + (6\dot{p} + \dot{x} + 2\dot{A})\partial_\tau \tilde{A}_\mu - k^2 e^{-2A-6p-x}\tilde{A}_\mu + \\ &\qquad\qquad + \left(8\tilde{\beta}\dot{A}\, e^{-12p-2x}/\beta_0 - 2\, e^{-12p-4x}\right)\tilde{A}_\mu = 0, \end{aligned} \tag{4.21}$$

and compares it with the equation (4.19), one will find that two equations coincide only for

$$\beta_0 = 1, \qquad \text{and} \qquad l = e^{-x}. \tag{4.22}$$

Thus, the equation (4.21) with the solution (4.22) describes the transverse vector excitation of the KS supergravity solution. For computation of the mass spectrum it is worth writing (4.21) in terms of the explicit solution (4.10). We obtain the equation

$$\partial_\tau^2 \tilde{A}_\mu + \mathcal{P}(\tau)\,\partial_\tau \tilde{A}_\mu + \mathcal{Q}(\tau)\,\tilde{A}_\mu = 0, \tag{4.23}$$

with[2]

$$\mathcal{P}(\tau) = \frac{4}{3}\,\frac{\sinh^2\tau}{(\sinh\tau\cosh\tau-\tau)} - 2\coth\tau - \frac{\dot h}{h}\,, \tag{4.24}$$

$$\mathcal{Q}(\tau) \quad = \quad -\frac{k^2 h\sinh^2\tau}{(\sinh 2\tau - 2\tau)^{2/3}} \;-\; \frac{8}{9}\,\frac{\sinh^4\tau}{(\sinh\tau\cosh\tau-\tau)^2} \;-\; \frac{2}{3}\,\frac{\dot h\sinh^2\tau}{(\sinh\tau\cosh\tau-\tau)h}\,. \tag{4.25}$$

Again, one could write the above equation in the form (4.12) with the new effective potential $V_1(k^2,\tau)$,

$$\begin{aligned} V_1 = \frac{1}{2}\,\dot{\mathcal{P}} + \frac{1}{4}\,\mathcal{P}^2 - \mathcal{Q} = \frac{k^2 h\,\sinh^2\tau}{(\sinh 2\tau-2\tau)^{2/3}} - 1 + 2\coth^2\tau + \frac{1}{4}\,\frac{(\sinh 2\tau - 2\tau)^{4/3}}{h\sinh^4\tau} + \\ + \frac{3}{4}\,\frac{(\sinh 2\tau-2\tau)^{2/3}(\tau\coth\tau-1)^2}{h^2\sinh^4\tau} + \frac{2}{3}\,\frac{\tau\coth\tau - 1}{(\sinh 2\tau - 2\tau)^{2/3}h} - \\ - \frac{2(\sinh 2\tau-2\tau)^{1/3}(\tau\coth\tau-1)\coth\tau}{h\sinh^2\tau}. \end{aligned} \tag{4.26}$$

Closing this section we notice that the equation (4.23) presented here coincides with the equation derived by M. Krasnitz in the UV limit of the KS theory. The $\tau\to\infty$ limit of (4.23) is the same as the equation (4.30) of [61] with the assignment

$$W_\mu = -\frac{27}{hr^4}\,K_\mu,$$

and the change to the standard radial variable $r = e^{\tau/3}$.

4.5 Supersymmetry and 5d Approach

In this section we compare our findings with the results obtained in the effective five-dimensional models of gauge/gravity correspondence studied by O. DeWolfe et al. in [59] and show that the equations for the graviton and the vector mode are related by a Supersymmetric Quantum Mechanics (SQM) transformation. This allows us to extend the equation for the vector mode to the baryonic branch.

The authors of [59] systematically study the $\mathcal{R}$-symmetry invariant sector of fluctuations above the $\mathcal{N}=2$ backgrounds of the five-dimensional $\mathcal{N}=8$ gauged supergravity. Those also include the gravity multiplet, i.e. the traceless four-dimensional metric fluctuation and the vector fluctuation, dual to the conserved part of the $U(1)_{\mathcal{R}}$ current.

Although, as noticed by A. Ceresole and G. Dall'Agata in [63], the KS solution truncated to five dimensions would correspond to a more general $\mathcal{N}=2$ supergravity theory, it is nevertheless interesting to compare the results of the two approaches. In fact, in both cases, the unbroken

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[2]Here we use the same momentum normalization as in the equation (4.11).

supersymmetry is $\mathcal{N} = 2$ as we deal with the supergravity dual models of $\mathcal{N} = 1$ gauge theories. Therefore the results based on the on-shell supersymmetry can be applicable in both cases. Indeed, we find that SQM transformations that relate the equations for the graviton and the vector mode in the case of the KS background coincide with the supergravity transformations used in [59].

In five-dimensional theories one can use the gauge freedom to recast the background metric into the "kink" form

$$\mathrm{d}s_5^2 = \mathrm{d}q^2 + e^{2T(q)}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{4.27}$$

According to a general observation of [7, 64], the traceless graviton fluctuation $h_{\mu\nu}$ in five dimensions satisfies the equation for a scalar minimally coupled to the geometry (4.27),

$$\left(\partial_q^2 + 4T'\,\partial_q - e^{-2T}k^2\right)\, h_{\mu\nu} = 0. \tag{4.28}$$

Using the transformations of the effective $\mathcal{N} = 2$ supergravity of [59] one can transform the graviton $h_{\mu\nu}$ into its superpartner – vector field $\hat{B}_\mu$. As a result, the minimal scalar equation transforms into

$$\left(\partial_q\, e^{2T}\partial_q - k^2 + 2e^{2T}\,\frac{\partial^2 T}{\partial q^2}\right)\hat{B}_\mu = 0. \tag{4.29}$$

Here again k is a 4-momentum. We are going to show that $\tilde{A}_\mu$ of (4.23) and $\hat{B}_\mu$ are related by a simple field redefinition.

The approach of [59] uses the superpotential, what can be problematic for the backgrounds from the baryonic branch (except for the KS solution) since the corresponding superpotentials are not known. Therefore there is a concern that the equations obtained for the KS may not be applicable for the outer branch. Nevertheless, we notice that the equation itself is W-independent. This already suggests that it is actually valid for any background of the form (4.27). Below we will give an argument based on supersymmetry that the equation (4.29) can be applied to the whole baryonic branch.

Let us first show that the equation (4.29) is the same as the equation (4.23) after an appropriate field redefinition. One can think of the metric (4.27) as an effective metric obtained by truncation of the ten dimensional theory with the KS metric (3.82) in the PT form, taken in the Einstein frame,

$$\mathrm{d}s_{10}^2 = \left(e^{-6p-x}\mathrm{d}\tau^2 + e^{2A}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + g^{(5)}_{\alpha\beta}\mathrm{d}y^\alpha\mathrm{d}y^\beta\right)e^{-\Phi/2}. \tag{4.30}$$

The metric (4.27) is then

$$ds_5^2 = \left(e^{-6p-x}\mathrm{d}\tau^2 + e^{2A}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\right)\det{}^{1/3}(g^{(5)})e^{-4\Phi/3} = \\ \left(e^{-6p-x}\mathrm{d}\tau^2 + e^{2A}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\right)e^{-2p+x}e^{-4\Phi/3}, \tag{4.31}$$

what gives the following identification for the coordinate q and the function $T(q)$:

$$\frac{\mathrm{d}}{\mathrm{d}q} = e^{4p+2\Phi/3}\frac{\mathrm{d}}{\mathrm{d}\tau}, \qquad 2T = 2A - 2p + x - \frac{4}{3}\,\Phi. \tag{4.32}$$

Hence the equations for the graviton in ten and five dimensions coincide, because they are just minimal scalar equations.

The equation (4.29) in the PT notations takes the form

$$\partial_\tau^2 \hat{B}_\mu + (2\dot{p} + \dot{x} + 2\dot{A} - \frac{2}{3}\,\dot{\Phi})\,\partial_\tau \hat{B}_\mu - k^2 e^{-2A-6p-x}\hat{B}_\mu + \\ + \left((4\dot{p} + \frac{2}{3}\,\dot{\Phi})\,(2\dot{A} - 2\dot{p} + \dot{x} - \frac{4}{3}\,\dot{\Phi}) + 2\ddot{A} - 2\ddot{p} + \ddot{x} - \frac{4}{3}\,\ddot{\Phi}\right)\hat{B}_\mu = 0. \tag{4.33}$$

To compare this to (4.23), derived in KS, set $\Phi = 0$. To match the kinetic terms in two equations one should redefine the field $\hat{B}_\mu = e^{2p}\tilde{A}_\mu$. After redefinition one gets

$$\partial_\tau^2 \tilde{A}_\mu \;+\; (6\dot{p} \;+\; \dot{x} \;+\; 2\dot{A})\,\partial_\tau \tilde{A}_\mu \;-\; k^2 e^{-2A-6p-x}\tilde{A}_\mu \;+\; \left(2\dot{p}\,(6\dot{A} + 3\dot{x}) + 2\ddot{A} + \ddot{x}\right)\tilde{A}_\mu \;=\; 0, \tag{4.34}$$

which is precisely the equation (4.23) for the KS solution (4.10).

We can further reduce the five-dimensional equations (4.28) and (4.29) to one dimension by taking the square of momentum k^2 to be the eigenvalue $-m^2$. This will reduce the supersymmetry algebra to the Supersymmetric Quantum Mechanics with two differential operators Q_1 and Q_2 that relate the solutions of the two equations (4.28) and (4.29). These operators realize the effective transformations of the supersymmetry algebra that was studied in [59]. Indeed, there are operators Q_1 and Q_2, such that the equations

$$Q_1 Q_2 h_{\mu\nu} = -m^2 h_{\mu\nu} \qquad \text{and} \qquad Q_2 Q_1 \hat{B}_\mu = -m^2 \hat{B}_\mu \tag{4.35}$$

coincide with the equation for the graviton (4.11) and the equation for the vector mode (4.23) in the form (4.33). It is easy to show that the operators that satisfy (4.35) are

$$Q_1 = (\partial_q + 2T') = e^{4p+2\Phi/3}\left(\partial_\tau + 2\dot{A} - 2\dot{p} + \dot{x} - \frac{4}{3}\,\dot{\Phi}\right) \tag{4.36}$$

and

$$Q_2 = e^{2T}\partial_q = e^{2A+2p+x-2\Phi/3}\partial_\tau. \tag{4.37}$$

The operator Q_2 is precisely the operator from (73) of [59] that realizes an $\mathcal{N} = 2$ supergravity transformation relating $h_{\mu\nu}$ and $\hat{B}_\mu$.

To get a more conventional representation of the SQM here, one can change the coordinates to $\partial_q = e^{-T}\partial_u$ and bring the equations (4.28) and (4.29) to the form (4.12) by redefining the wave functions $h_{\mu\nu}$ and $\hat{B}_\mu$. Let us define an operator

$$Q = \begin{pmatrix} 0 & \partial_u - W \\ \partial_u + W & 0 \end{pmatrix} \tag{4.38}$$

with $W = -3T'/2$, that acts on the vector made of redefined wave functions ψ_h and ψ_B. According to the equations (4.28) and (4.29) the action of Q^2 is as follows

$$Q^2 \begin{pmatrix} \psi_h \\ \psi_B \end{pmatrix} = -m^2 \begin{pmatrix} \psi_h \\ \psi_B \end{pmatrix}. \tag{4.39}$$

Therefore Q^2 is analogous to the Hamiltonian of the SQM. Notice, however, that its eigenvalues are m^2, not m, because Q_1 and Q_2 correspond to the squares of the original supersymmetry transformations, i.e. Q_1, Q_2 are bosonic operators.

We see now that the equation (4.28) and (4.29) are related by supersymmetry transformation for any background (4.27). Since the minimal scalar equation describing the graviton is valid for the whole branch, the superpartner of the graviton (the transverse vector mode) satisfies the "superpartner" equation (4.29) for any background from the baryonic branch.[3]

In the next section we calculate the spectrum of both equations numerically for the backgrounds along the baryonic branch. Since the equations for the superpartners are significantly different the discrepancy between the masses can be used as an error estimate of the numerical method used in the calculation.

4.6 Numerical Analysis

In this section we present the results of the numerical studies of bound state spectra for the baryonic branch backgrounds. In our computations we will rely on the shooting technique. The spectrum of the minimal scalar equation (4.11) in the KS background was also studied numerically in [20, 21, 24] while the analytical approximation was employed in [62].

We start by comparing the KS spectra of the equations for graviton (4.11) and vector mode (4.23). Two fluctuations are related by supersymmetry and thus their masses should be the same. The spectrum is presented in the table 4.1. The eigenvalues match with those obtained by M. Krasnitz [20] with the WKB approximation. Comparing the numeric values of the masses of the spin-2 and vector particles in the table 4.1 one could estimate the error of the shooting technique in the KS case to be around 0.1%.

First few (up to ten) values of m^2 in the KS spectrum can be approximated with a good

[3]In general, there is a family of equations like (4.29) that are related to (4.28) by a supersymmetry transformation. Indeed, for a given W from (4.38), any $\hat{W}$ that satisfies $\hat{W}^2 + \hat{W}' = W^2 + W'$ gives rise to such an equation through (4.39). Nevertheless, the equation (4.29) is uniquely specified by a requirement that the effective potential V_1 is singular at $\tau = 0$. This is true because V_1 is singular in the KS case (4.26) and hence should be singular everywhere on the branch by continuity.

n	1	2	3	4	5	6	7	8	9
Graviton	1.764	4.002	7.143	11.19	16.16	22.03	28.83	36.54	45.16
Vector Mode	1.762	3.999	7.136	11.18	16.12	22.01	28.80	36.50	45.12

Table 4.1: The spectrum of m^2 for the gravity multiplet

accuracy by a quadratic fit

$$m_n^2 = 0.46\, n^2 + 0.86\, n + 0.46, \qquad n = 1, 2, 3, \ldots \tag{4.40}$$

We present the results of the fit and the masses on the figure 4.1(a). It is interesting that the fit (4.40) is close to the spectrum even for small n. The fitting formula (4.40) is proportional to $(n + n_0)^2$, where n_0 is close to one. This is consistent with the approximation of [62], where the eigenvalues were matched to zeroes of the Bessel functions, ubiquitous in the conical geometry. A similar result was obtained in [65] for the GPPZ [66] flow, where the exact spectrum was proportional to $(n + 1)^2$.

The fit (4.40) was found by minimizing the sum

$$\sum_{n=1}^{N} \left| m_n^2 - (c_2 n^2 + c_1 n + c_0) \right|^2 \tag{4.41}$$

for the few first states $N = 5, \ldots, 10$. With more points taken into account the least square fit would increase the accuracy of the highest coefficient c_2 by the price of a larger deviation from m_n^2 for small n. We found c_2 to be ~ 0.459 in the KS case. This number is in good agreement with the universal coefficient obtained by M. Berg, M. Haack and W. Mück in [24]. In their normalization the coefficient takes value $(3/4)^{2/3} h(0)\, c_2 \simeq 0.27$.

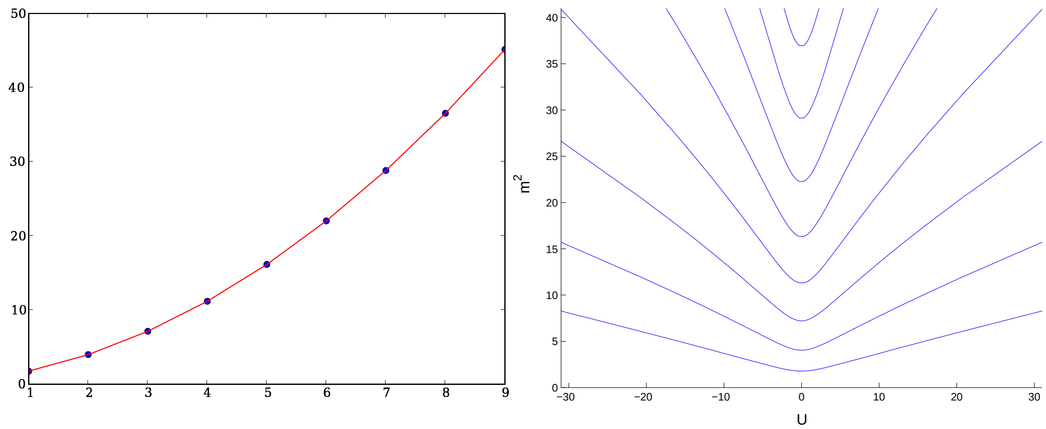
Remarkably, the coefficient c_2 does not depend on the details of the effective potential, but rather encodes information about the background geometry, namely, the combination $g^{00} g_{tt}$, which arises from the Laplace operator in five dimensions. Indeed, the WKB approach, applied in [20], gives

$$\int_0^{\tau^*} \mathrm{d}\tau \, \sqrt{-V_2(\tau)} \, \Big|_{k^2 = -m_n^2} = \frac{3}{4}\, \pi + (n-1)\pi, \tag{4.42}$$

where $V_2(\tau^*) = 0$. In the KS case V_2 is given by (4.13). For large n, and consequently large m_n, the k^2-independent term in V_2 can be dropped and we obtain an analytical expression for c_2 in the KS case

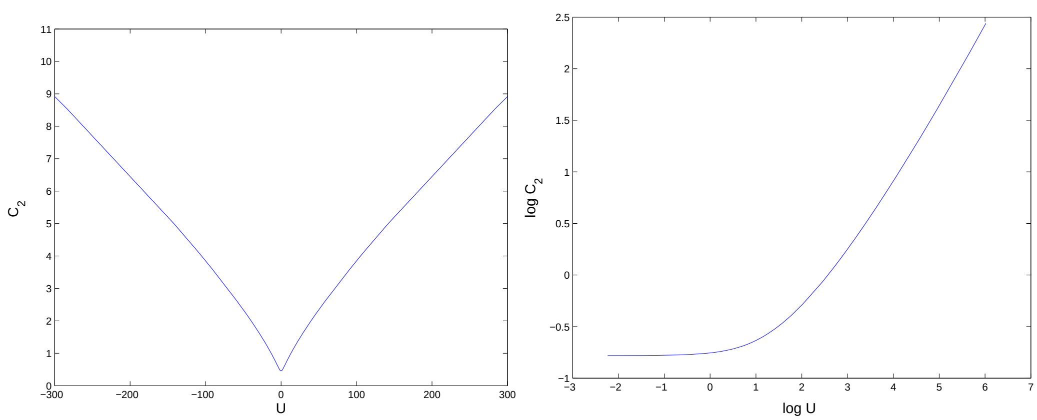
$$c_2 = \pi^2 \left[\int_0^\infty \mathrm{d}\tau \, \frac{\sqrt{h} \sinh \tau}{(\sinh 2\tau - 2\tau)^{1/3}} \right]^{-2} \sim 0.460\ . \tag{4.43}$$

Let us choose the coordinate U, introduced in [67], to parameterize the baryonic branch. To estimate the scale of the spectrum for a non-KS background we rewrite the potential (4.13) in



(a) **(b)**

Figure 4.1: **(a)** Values of m^2 for the graviton multiplet in KS for different quantum numbers n. **(b)** Extension of the spectrum to the baryonic branch parameterized by U.



(a) **(b)**

Figure 4.2: **(a)** c_2-coefficient as the function of U-parameter. **(b)** $\log c_2$ as the function of $\log U$.

terms of the PT ansatz [56], substituting k^2 for its eigenvalue $-m^2$:

$$V_2(m^2,\tau) = -\frac{m^2 e^{-2A+x}}{v} + \frac{2a\cosh\tau}{v}\, e^{-3g} - \frac{(a\cosh\tau+1)^2 + 2a^2\sinh^2\tau}{v^2}\, e^{-2g}, \tag{4.44}$$

where $a(\tau)$ is another function from the PT ansatz [56], $e^{2g} = -1 - a^2 - 2a\cosh\tau$, and $v = e^{6p+2x}$.

Although we cannot find the spectrum of m^2 analytically, we can estimate how it scales with the parameter U when we are significantly far from the origin of the branch. We start our analysis with the m^2-independent part of V_2, which only slightly varies as we increase U. Indeed, its leading UV ($\tau \to \infty$) asymptotic is U-independent:

$$V_2(0,\tau) = \frac{4}{9} - \frac{(5-2\tau)}{6}\, U^2 e^{-4\tau/3} + \ldots; \tag{4.45}$$

and $V_2(0,\tau)$ varies within a small range in the IR ($\tau = 0$):

$$V_2(0,\tau) = \frac{1}{4} - \frac{3}{5}\,\xi(1-\xi) + \mathcal{O}(\tau^2). \tag{4.46}$$

Here we remind that $\xi(U) \in (1/6 \ldots 5/6)$ (3.112), introduced in [19], is a function of U, which can also be used to parameterize the branch. It varies within the specified limits, and the point $\xi = 1/2$ corresponds to the KS solution. Hence $V_2(0,0) = 2/5$ for KS and $V_2(0,0)$ approaches $1/3$ for large U. The $V_2(0,\tau)$ is monotonic and therefore it can be approximated by a constant in the analysis below.

Unlike $V_2(0,\tau)$, the mass-dependent component $m^2\, e^{-2A+x} v^{-1}$ significantly depends on U. It monotonically changes from a finite value at zero to the zero value at infinity[4]

$$e^{-2A+x} v^{-1} = \frac{2^{1/3}3}{16}\,(4\tau - 1)e^{-2\tau/3} + ... \tag{4.47}$$

In general, the value at zero is a complicated function of U, $\xi(U)$ and $\Phi_0 = \Phi(U, \tau = 0)$. It can be simplified in the large U range by substituting the limiting value $\xi = 5/6$ and expressing Φ_0 in terms of U and ξ [67]: $e^{\Phi_0} \simeq 2^{3/2}3^{-1/4}U^{-3/4}$. This gives

$$e^{-2A+x} v^{-1} = \frac{2^{1/3}3}{2U}\left[1 - e^{2\Phi_0}\left(1 + \frac{2\tau^2}{9} + \frac{2\tau^4}{135} + ...\right)\right]. \tag{4.48}$$

	SUSY $D5$	Baryonic Condensate	Fundamental String	Glueballs	$D3, \bar{D}3$
α	0	$\alpha < 1$	$1/4$	$1/2$	$5/4$

Table 4.2: Scale behavior for large U: $T \sim U^{\alpha}$

The normalized solution to the equation (4.12) exist only if $V_2 < 0$ at the origin, which suggests that m^2 scales at least as U for large U. We can try to be more precise using the semiclassical approximation and express the n-th mass through the integral over the $V_2 < 0$ region, as we did above in (4.42). This integral can be roughly approximated as $\sqrt{-V_2(0)}\tau^* \sim m\tau^* U^{-1/2}$. The main complication is to estimate τ^*. Since $e^{2\Phi_0}$ from (4.48) is small, the perturbative expansion (4.48) suggests that τ^* increases with U until a point, where (4.48) is no longer reliable. At the same time the large τ asymptotic (4.47) is U-independent, what suggests that for large U the value of τ^* approaches a constant. Therefore we expect $m_n^2 \sim Un^2$ for sufficiently large U.

Numerical studies of the graviton multiplet spectrum on the baryonic branch shows the pattern depicted in the figure 4.1(b). Calculations confirm that the leading coefficient c_2 grows as U^{α}, where

[4]Here we use the normalization of the warped factor introduced by M. Krasnitz [20], what results in $e^{-2A} = 2^{1/3}3\sqrt{(e^{-2\Phi} - 1)U^{-1}}$.

α approaches 1 for large U (figure 4.2). As a final touch, we collect in the table 4.2 the known evidence about the U scaling parameter α for some non-perturbative objects on the baryonic branch.

Chapter 5

Vector and Scalar Mesons in KS

5.1 Global Symmetries

In the previous chapter we looked at the spectrum of the bosonic states in the graviton multiplet of the KS boundary gauge theory, which are the spin two and vector states. These corresponds to the poles in the two point correlation functions of the operators $\Theta_{\mu\nu}$ and $\mathcal{J}^5_\mu$, which are traceless part of the stress-energy tensor $T_{\mu\nu}$ and the conserved part of the $U(1)_\mathcal{R}$ current J^5_μ respectively. There are also two spin 3/2 fermionic state in this multiplet corresponding to the gamma-traceless part of the superconformal current, which we did not consider.

In this chapter we are going to consider more states from various $\mathcal{N} = 1$ supermultiplets. Recall that in the holographic approach the states (glueballs) in the gauge theory correspond to the eigenfunctions of the linearized supergravity equations with boundary conditions specified in [12, 13]. In general, this is a complicated system of equations, which is very hard to treat even by numerical methods. The most difficult task is to identify the eigenstates with gauge theory operators. To approach this problem one needs first of all to classify the states according to the symmetries of the theory.

One symmetry, which already helped us in the analysis of the glueball states, was supersymmetry. The fact that the vector excitation found in the previous chapter is indeed the dual of $\mathcal{J}^5_\mu$ was established once we proved that it has the same spectrum with the graviton. Another symmetry, which vastly simplifies our search is the global $SU(2) \times SU(2)$ symmetry of the dual gauge theory. All of the states that we have previously considered and are going to consider are singlets under the $SU(2) \times SU(2)$. The full spectrum is much reacher and contains other representations of $SU(2) \times SU(2)$, but such states will be ignored. One of the reasons of our ignorance is that the subsector of the non-singlet states does not contain the states of the pure gauge $\mathcal{N} = 1$ SYM. In terms of the basis 1-forms (3.11) we have the following invariant fluctuation at our disposal:

- metric: $\mathrm{d}x^\mu \cdot \mathrm{d}x^\nu$, $\mathrm{d}x^\mu \cdot \mathrm{d}\tau$, $\mathrm{d}x^\mu \cdot g^5$, $\mathrm{d}\tau^2$, $\mathrm{d}\tau \cdot g^5$, $(g^5)^2$, $(g^1)^2 + (g^2)^2$, $(g^3)^2 + (g^4)^2$ and $g^1 \cdot g^3 + g^2 \cdot g^4$;

- 1-forms: $\mathrm{d}x^\mu$, $\mathrm{d}\tau$, g^5;
- 2-forms: $g^1 \wedge g^2$, $g^3 \wedge g^4$, $g^1 \wedge g^3 + g^2 \wedge g^4$, $\mathrm{d}g^5 = g^2 \wedge g^3 - g^1 \wedge g^4$.

We will also classify the supergravity fluctuations according to the J^{PC} of their eigenstates. Besides it useful to determine the conformal dimensions of the corresponding operators as well as their $\mathcal{R}$-charges. This information is very helpful for the assigning of the glueballs to supermultiplets. The dimensions of the operators were discussed earlier in the section 1.2 so we will start from $\mathcal{R}$-charges.

Recall that on the KS background the $U(1)_\mathcal{R}$ symmetry is realized as the shifts of the angle ψ (see the discussion in the chapter 3). Fluctuations independent from ψ carry no charge under $U(1)_\mathcal{R}$. Such are

- $\mathcal{R} = 0$ metric fluctuations: $(g^5)^2$, $g^1 \cdot g^3 + g^2 \cdot g^4$, $(g^1)^2 + (g^2)^2 + (g^3)^2 + (g^4)^2$;
- $\mathcal{R} = 0$ 2-form fluctuations: $\mathrm{d}g^5$, $g^1 \wedge g^2 + g^3 \wedge g^4$.

The following are the forms that carry the charge $\mathcal{R} = \pm 2$ under $U(1)_\mathcal{R}$ respectively:

- $\mathcal{R} = \pm 2$ 2-form fluctuations: $(g^1 \wedge g^3 + g^2 \wedge g^4) \mp i(g^1 \wedge g^2 - g^3 \wedge g^4)$.

The space parity operation reflects the spacial part of the fluctuations along the Minkowskian directions and inverts the sign of all RR potentials. On the supergravity side the charge conjugation C acts by interchanging two S^2 spheres in the $T^{1,1}$, which amounts to the exchange of (θ_1, ϕ_1) and (θ_2, ϕ_2) simultaneously changing the signs of F_3 and H_3. This $\mathbb{Z}_2$ operation exchanges and conjugates the chiral superfields $(A_k, B_l) \leftrightarrow (\bar{B}_k, \bar{A}_l)$. All of the backgrounds, considered in the chapter 3 respect this symmetry except for the baryonic branch. In the work [22] this symmetry was called the $\mathcal{I}$-symmetry. We will also use this name below to stress its geometric origin. In the supergravity we have the following $SU(2) \times SU(2)$ singlet $\mathcal{I}$-even excitations:

- $\mathcal{I}$-even metric fluctuations: $(g^5)^2$, $(g^1)^2 + (g^2)^2$, $(g^3)^2 + (g^4)^2$;
- $\mathcal{I}$-even 1-form fluctuations: $\mathrm{d}\tau$, g^5
- $\mathcal{I}$-even 2-form fluctuations: $\mathrm{d}g^5$.

The following excitations are $\mathcal{I}$-odd:

- $\mathcal{I}$-odd metric fluctuations: $g^1 \cdot g^3 + g^2 \cdot g^4$;
- $\mathcal{I}$-odd 2-form fluctuations: $g^1 \wedge g^2$, $g^3 \wedge g^4$, $g^1 \wedge g^3 + g^2 \wedge g^4$.

According to the symmetries reviewed above, the graviton multiplet described in the previous chapter contains the 2^{++} tensor state and a pseudovector 1^{++}. In [23, 24], M. Berg, M. Haack and W. Mück have found a system of seven 0^{++} scalar excitations of the KS background. However we will start the review here from the $\mathcal{I}$-odd sector, where the analysis seems to be simpler. As one can notice consulting with the above analysis, the massless scalar and pseudoscalar found in [22] and reviewed in the section 3.6 belong to that sector. The ansatz for massless particles was extended to the massive case by M. Benna et al. in [26], where the massive tower of states above the massless scalar was found as well as an additional C-odd scalar and pseudoscalar states. Those three excitations are in fact the only possible massive (pseudo-) scalar particles in the $\mathcal{I}$-odd sector.

In the following sections we will write the most general $\mathcal{I}$-odd supergravity ansatz for spin 0 and spin 1 particles and classify all states that appear from it. Besides the three spin 0 states there are seven spin 1-states. Some of the (pseudo-) vector states are superpartners of the (pseudo-) scalars, as it was anticipated in [26]. There are also two pairs of vector and pseudovector states that form two new supermultiplets. We will return to a more detailed discussion of the supersymmetry in the next chapter.

5.2 $\mathcal{I}$-odd Supergravity Excitations. Scalars

We study the supergravity excitations over the KS background which are singlet with respect to the action of $SU(2) \times SU(2)$ on the conifold odd under the $\mathcal{I}$ operation. Note that the Hodge duality allows one to relate the p- and $(4-p)$-forms in four dimensions, and that is why the general ansatz can be written in terms of the zero, one and two-forms. It is also known that any form has a Hodge decomposition into the sum of an exact, co-exact and harmonic parts. Let us stress that we are looking for the massive excitations, i.e. all the four dimensional forms P_k in our ansatz satisfy

$$\Box_4 P_k = m^2 P_k \tag{5.1}$$

with some non-zero mass squared. It means that the harmonic part is absent from the decomposition (which is not generally the case for the 4-d massless modes). Therefore, any two-form P_2 can be written using the two vectors (one-forms) $\mathbf{M}$ and $\mathbf{N}$:[1]

$$P_2 \quad = \quad \mathrm{d}_4\mathbf{M} + *_4\mathrm{d}_4\mathbf{N}\,. \tag{5.2}$$

[1]Below we will use the boldface notation for the vector excitations.

Similarly, any vector $\tilde{\mathbf{N}}$ can be represented as a sum of an exact an a co-closed parts:

$$\mathbf{N} = d_4\chi + \tilde{\mathbf{N}}\,, \tag{5.3}$$

where

$$d_4 *_4 \tilde{\mathbf{N}} = 0\,. \tag{5.4}$$

These consideration reveal that all the $\mathcal{I}$-odd supergravity excitations over the KS background reduce to an ansatz containing only spin 0 and spin 1 excitations.

The most general scalar ansatz was considered by M.Benna et al. in [26]. Namely, there are the following two decoupled fluctuations,

$$\begin{aligned} \delta B_2 &= \chi(x,\tau)\,dg^5 + \partial_\mu\sigma(x,\tau)\,dx^\mu \wedge g^5\,, \\ \delta G_{13} = \delta G_{24} &= U(x,\tau)\,; \end{aligned} \tag{5.5}$$

and

$$\delta C_2 = \tilde{\chi}(x,\tau)\,dg^5 + \partial_\mu\tilde{\sigma}(x,\tau)\,dx^\mu \wedge g^5\,. \tag{5.6}$$

Recall that B_2 and C_2 (equivalently F_3 and H_3) change sign under $\mathcal{I}$. Therefore only $\mathcal{I}$-even 2-forms could have been used above. One could seemingly add the $\mathcal{I}$-odd excitations of F_5 to this ansatz,

$$\delta F_5 = (1+*)\big[d\tau \wedge (d_4 a \wedge g^1 \wedge g^2 + d_4 b \wedge g^3 \wedge g^4) \wedge g^5\big]\,; \tag{5.7}$$

or

$$\delta F_5 = (1+*)\big[d_4 c \wedge d\tau (g^1 \wedge g^3 + g^2 \wedge g^4) \wedge g^5\big]\,. \tag{5.8}$$

However, equations of motion would require the functions a, b and c to vanish identically.[2] After some redefinition of variables (see [26]) equations of motion for (5.5) become

$$z'' - \frac{2}{\sinh^2\tau}\,z + \tilde{m}^2\,\frac{I(\tau)}{K^2(\tau)}\,z = 2^{2/3}\tilde{m}\,K(\tau)\,w\,, \tag{5.9}$$

$$w'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\,w + \tilde{m}^2\,\frac{I(\tau)}{K^2(\tau)}\,w = 2^{2/3}\tilde{m}\,K(\tau)\,z\,; \tag{5.10}$$

and those for (5.6) become

$$\tilde{w}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\,\tilde{w} + \tilde{m}^2\frac{I(\tau)}{K(\tau)^2}\,\tilde{w} = 0\,. \tag{5.11}$$

In the chapter 6 we will return to the discussion of these spin 0 states.

[2] Note that this is not the case for the massless particles; e.g. [22].

The most general $SU(2)\times SU(2)$ singlet $\mathcal{I}$-odd vector excitations of the 3-form potentials reads

$$\mathbf{C}^{(1)}\wedge \mathrm{d}\tau + \mathbf{C}^{(2)}\wedge g^5 + *_4\mathrm{d}_4\mathbf{C}^{(3)}. \tag{5.12}$$

For the 5-form the most general vector perturbation is

$$\begin{aligned}(1+*)\,\Big[&\mathbf{F}^{(1)}\wedge \mathrm{d}\tau\wedge g^5\wedge g^1\wedge g^2 + \mathbf{F}^{(2)}\wedge \mathrm{d}\tau\wedge g^5\wedge g^3\wedge g^4 + \\ &+\mathbf{F}^{(3)}\wedge \mathrm{d}\tau\wedge g^5\wedge(g^1\wedge g^3+g^2\wedge g^4) + (\mathrm{d}_4\mathbf{F}^{(4)} + *_4\mathrm{d}_4\mathbf{F}^{(5)})\wedge g^5\wedge g^1\wedge g^2 + \\ &+(\mathrm{d}_4\mathbf{F}^{(6)} + *_4\mathrm{d}_4\mathbf{F}^{(7)})\wedge g^5\wedge g^3\wedge g^4 + (\mathrm{d}_4\mathbf{F}^{(8)} + *_4\mathrm{d}_4\mathbf{F}^{(9)})\wedge g^5\wedge(g^1\wedge g^3+g^2\wedge g^4)\Big]\ .\end{aligned} \tag{5.13}$$

General fluctuations (5.12) and (5.13) are considered below in the sections 5.3.1 and 5.3.2. The general ansatz decouples into two independent parts with three and four vector fluctuations correspondingly. Note that there are no $SU(2)\times SU(2)$ singlet $\mathcal{I}$-odd vector excitations of the metric. This way we find the complete spectrum of the $\mathcal{I}$-odd singlet supergravity excitations in the KS background.

5.3 $\mathcal{I}$-odd Vectors

In [26] it was suggested that the spin 0 excitations (5.5) and (5.6) have some vector superpartners. This suggestion was supported by the study of some vector ansatz in the KT [17] (large τ) limit; in which the scalar and vector particles shared the same equations of motion. In the present section we study the general massive $\mathcal{I}$-odd vector ansatz in the full KS background, and extract the exact equations for superpartners of the scalar particles found in [26]. Moreover we will find four new spin 1 particles that form two $j=1$ supermultiplets according to the classification in the book of J. Wess and J. Bagger [68].

5.3.1 Vector Dual to a Single Scalar Particle

Before we proceed, let us make a small digression about our conventions. We choose the names for the forms in the ansatz so as to possibly keep the similarity with notations used in the similar calculation for the KT limit in [26]. 1-forms (vectors) are shown in boldface. The four dimensional operations such as the Hodge star $*_4$ and Laplacian $\Box_4$ are performed w.r.t. the standard Minkowski metric (without the warp factor). As it was explained, the four dimensional one-forms are all divergence free:

$$\mathrm{d}_4 *_4 \mathbf{F} \quad = \quad 0\,. \tag{5.14}$$

The eigenvalue of the 4-Laplacian $\Box_4$ is m_4^2, however for compactness we shall express all our formulae in terms of the dimensionless combination $\tilde{m}^2$:

$$m_4^2 \;=\; \frac{3\,\epsilon^{4/3}}{2\cdot 2^{2/3}}\,\tilde{m}^2\,. \tag{5.15}$$

Derivation of the equations

In this section we are starting from an ansatz that is supposed to be a superpartner of the ansatz (5.6). Proposed deformations of the potentials are:

$$\begin{aligned} \delta B_2 &= *_4 \mathrm{d}_4\mathbf{H} + \mathbf{A}\wedge g^5\,, && (5.16)\\ \delta C_2 &= \mathbf{E}\wedge \mathrm{d}\tau\,, && (5.17)\end{aligned}$$

which give the following deformations of the 3-forms:

$$\begin{aligned} \delta H_3 &= \mathrm{d}_4\mathbf{A}\wedge g^5 - \mathbf{A}'\wedge \mathrm{d}\tau\wedge g^5 - \mathbf{A}\wedge \mathrm{d}g^5 - *_4\Box_4\mathbf{H} + *_4\mathrm{d}_4\mathbf{H}'\wedge\mathrm{d}\tau\,, && (5.18)\\ \delta F_3 &= \mathrm{d}_4\mathbf{E}\wedge\mathrm{d}\tau\,. && (5.19)\end{aligned}$$

Deformation of the 5-form is taken to be

$$\begin{aligned}\delta F_5 = (1+*)\big[&\mathrm{d}_4\mathbf{K}\wedge\mathrm{d}\tau\wedge g^1\wedge g^2 + \mathrm{d}_4\mathbf{L}\wedge\mathrm{d}\tau\wedge g^3\wedge g^4 +\\ &+\mathrm{d}_4\mathbf{M}\wedge(g^1\wedge g^3+g^2\wedge g^4)\wedge g^5 + \mathbf{N}\wedge\mathrm{d}\tau\wedge(g^1\wedge g^3+g^2\wedge g^4)\wedge g^5\big]\,. \end{aligned}\tag{5.20}$$

Linearized equations of motion for the ansatz functions are as follows. The Bianchi identity for F_5 at the linear order boils down to four independent equations when written in components:

$$\begin{aligned} \frac{1}{2}\mathbf{K} - \frac{1}{2}\mathbf{L} + \mathbf{M}' + \mathbf{N} &= -F'(\mathbf{A}+\mathbf{E})\,, && (5.21)\\ h\sqrt{-G}G^{55}\big((G^{11})^2\mathbf{K} + (G^{33})^2\mathbf{L}\big) &= \mathbf{H}\,, && (5.22)\\ h\sqrt{-G}(G^{33})^2G^{55}\Box_4\mathbf{L} - h^{1/2}\sqrt{-G}G^{11}G^{33}(G^{55})^2\mathbf{N} &= F\Box_4\mathbf{H}\,, && (5.23)\\ \Big[h\sqrt{-G}(G^{33})^2G^{55}\mathbf{L}\Big]' - h\sqrt{-G}G^{11}G^{33}G^{55}\mathbf{M} &= F\mathbf{H}'\,. && (5.24)\end{aligned}$$

Equations of motion for F_3 give the two equations:

$$\begin{aligned} -2h\sqrt{-G}G^{55}\Box_4\mathbf{E} &= 2(k-f)h^{1/2}\sqrt{-G}G^{11}G^{33}(G^{55})^2\mathbf{N} + \ell\,\Box_4\mathbf{H}\,, && (5.25)\\ \Big[2h\sqrt{-G}G^{55}\mathbf{E}\Big]' &= -2h\sqrt{-G}G^{55}\big(f'(G^{11})^2\mathbf{K} + k'(G^{33})^2\mathbf{L}\big) - \\ &\quad -2(k-f)h\sqrt{-G}G^{11}G^{33}G^{55}\mathbf{M} - \ell\,\mathbf{H}'\,. && (5.26)\end{aligned}$$

Another pair of equations appears from H_3 equation of motion:

$$\begin{aligned}&\Big[h^{1/2}\sqrt{-G}(G^{55})^2\mathbf{A}'\Big]' - 2h^{1/2}\sqrt{-G}G^{11}G^{33}\mathbf{A} + h\sqrt{-G}G^{55}\Box_4\mathbf{A} =\\ &\qquad\qquad = -2F'h^{1/2}\sqrt{-G}G^{11}G^{33}(G^{55})^2\mathbf{N}\,, \end{aligned}\tag{5.27}$$

$$\left[2h\sqrt{-G}G^{55}\mathbf{H}'\right]' + 2h^{3/2}\sqrt{-G}\Box_4\mathbf{H} = 2(1-F)\mathbf{K} + 2F\mathbf{L} + 4F'\mathbf{M} - \ell\,\mathbf{E}\,. \qquad (5.28)$$

No other supergravity equations contribute. In fact, some equations in the system (5.21)-(5.27) are algebraic and can be solved for the functions $\mathbf{E}$, $\mathbf{K}$, $\mathbf{L}$, $\mathbf{M}$ in terms of the functions $\mathbf{N}$ and $\mathbf{H}$. After doing so and slightly redefining $\mathbf{N}$,

$$\frac{G^{55}}{\sqrt{h}}\mathbf{N} \quad = \quad \Box_4\tilde{\mathbf{N}}\,, \qquad (5.29)$$

one can notice that, in terms of $\tilde{\mathbf{N}}$ and $\mathbf{H}$, equation (5.26) becomes an identity. Thus, there are only three independent second order differential equations for three unknown functions $\tilde{\mathbf{N}}$, $\mathbf{H}$ and $\mathbf{A}$. After the introduction of $\tilde{\mathbf{A}} = K^2\sinh\tau\,\mathbf{A}$ these equations take the form:

$$\tilde{\mathbf{N}}'' - \left(\frac{\cosh^2\tau+1}{\sinh^2\tau} + \frac{4\cdot 2^{1/3}(F')^2}{IK^2\sinh^2\tau}\right)\tilde{\mathbf{N}} + \tilde{m}^2\frac{I}{K^2}\tilde{\mathbf{N}} + \\ + F'\mathbf{H}' - \frac{2^{1/3}F'\ell}{IK^2\sinh^2\tau}\,\mathbf{H} + \frac{F'}{K^2\sinh\tau}\,\tilde{\mathbf{A}} = 0\,, \qquad (5.30)$$

$$\tilde{\mathbf{A}}'' - \frac{\cosh^2\tau+1}{\sinh^2\tau}\,\tilde{\mathbf{A}} + \tilde{m}^2\frac{I}{K^2}\tilde{\mathbf{A}} + \tilde{m}^2\frac{4\cdot 2^{1/3}F'}{K^2\sinh\tau}\,\tilde{\mathbf{N}} = 0\,, \qquad (5.31)$$

$$\mathbf{H}'' + \left(2\frac{\left(K\sinh\tau\right)'}{K\sinh\tau} + \frac{I'}{I}\right)\mathbf{H}' - \left(\frac{2^{1/3}\ell'}{IK^2\sinh^2\tau} + \frac{2^{2/3}\ell^2}{I^2K^4\sinh^4\tau}\right)\mathbf{H} + \tilde{m}^2\frac{I}{K^2}\,\mathbf{H} - \\ - \frac{4\cdot 2^{1/3}}{IK^2\sinh^2\tau}\left(F'\tilde{\mathbf{N}}\right)' - \frac{4\cdot 2^{2/3}F'\ell}{I^2K^4\sinh^4\tau}\tilde{\mathbf{N}} = 0\,. \qquad (5.32)$$

Analysis of the equations

It turns out that one can set $\tilde{\mathbf{N}} \equiv 0$, and then the remaining equations for $\mathbf{H}$ and $\tilde{\mathbf{A}}$ are equivalent. This reduces the system to just one equation, identical to that for the scalar particle. Let us stress that in this case the ansatz for δF_5 simplifies,

$$\delta F_5 \quad = \quad (1+*)\,\mathrm{d}_4\mathbf{H}\wedge H_3\,; \qquad (5.33)$$

which gives a natural generalization of the KT limit ansatz in [26] to the complete KS background (recall that in the KT limit $H_3 \sim \mathrm{d}\tau\wedge\omega_2$).

If one sets $\tilde{\mathbf{N}} = 0$, then the equation (5.30) becomes the first order equation:

$$\mathbf{H}' \quad = \quad \frac{2^{1/3}\ell}{IK^2\sinh^2\tau}\,\mathbf{H} - \frac{1}{K^2\sinh\tau}\,\tilde{\mathbf{A}}\,. \qquad (5.34)$$

Using this equation, one can eliminate the first and second derivatives of $\mathbf{H}$ from (5.32) and express $\mathbf{H}$ in terms of $\tilde{\mathbf{A}}$ and its derivative:

$$\mathbf{H} \quad = \quad \frac{\tilde{\mathbf{A}}' + \coth\tau\,\tilde{\mathbf{A}}}{\tilde{m}^2 I\sinh\tau}\,. \qquad (5.35)$$

When substituted back into (5.34), it yields the following equation for $\tilde{\mathbf{A}}$,

$$\tilde{\mathbf{A}}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\,\tilde{\mathbf{A}} + \tilde{m}^2\,\frac{I}{K^2}\,\tilde{\mathbf{A}} = 0\,; \tag{5.36}$$

and it exactly coincides with (5.31) for $\tilde{\mathbf{N}} = 0$. This is the same equation as (5.11), which describes the decoupled scalar fluctuation of the background form F_3. Thus, if we set $\tilde{\mathbf{N}} = 0$, the system of equations (5.30), (5.31), (5.32) describes a vector superpartner of the scalar particle.

The remaining two modes contained in the equations (5.30), (5.31) and (5.32) can be extracted if one chooses

$$\tilde{\mathbf{A}}' = -\tilde{m}^2\,\frac{\sqrt{I}}{K}\,\tilde{\mathbf{H}} - \coth\tau\,\tilde{\mathbf{A}}, \tag{5.37}$$

where $\tilde{\mathbf{H}} = \sqrt{I}K\sinh\tau\,\mathbf{H}$. In a similar way equation (5.35) was obtained, one can find a second first order equation that follows from (5.31) after imposing (5.37):

$$\tilde{\mathbf{H}}' = -\left(\log\frac{\sqrt{I}}{K\sinh\tau}\right)'\tilde{\mathbf{H}} + \frac{\sqrt{I}}{K}\,\tilde{\mathbf{A}} - \frac{2I'}{\sqrt{I}K^2\sinh\tau}\,\tilde{\mathbf{N}}. \tag{5.38}$$

One can show that with (5.37) and (5.38) the equation (5.32) is not independent and can be eliminated. Eliminating $\tilde{\mathbf{H}}$ from the above equations one obtains

$$\tilde{\mathbf{A}}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\,\tilde{\mathbf{A}} + \tilde{m}^2\,\frac{I}{K^2}\,\tilde{\mathbf{A}} - \frac{2\tilde{m}^2 I'}{K^3\sinh\tau}\,\tilde{\mathbf{N}} = 0\,, \tag{5.39}$$

$$\tilde{\mathbf{N}}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\,\tilde{\mathbf{N}} + \tilde{m}^2\,\frac{I}{K^2}\,\tilde{\mathbf{N}} - \frac{2^{-1/3}I'}{K^3\sinh\tau}\,\tilde{\mathbf{A}} = 0\,. \tag{5.40}$$

Then after the trivial rescaling and introduction of $\mathbf{X}_\pm = \tilde{\mathbf{A}} \pm \tilde{\mathbf{N}}$ these two equations decouple,

$$\mathbf{X}_\pm'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\,\mathbf{X}_\pm + \tilde{m}^2\,\frac{I}{K^2}\,\mathbf{X}_\pm \mp \frac{2^{5/3}\tilde{m}F'}{K^2\sinh\tau}\,\mathbf{X}_\pm = 0\,. \tag{5.41}$$

Later we are going to show that these particles are members of the two $j = 1$ multiplets and find their vector superpartners which share the same spectrum.

5.3.2 Vectors Dual to the Two Coupled Scalars

In this section we will find vector particles that are the superpartners of the scalar excitations (5.5). They will satisfy the same system of coupled equations‘(5.9) and (5.10). In addition we will find another pair of vector excitations that turn out to be the superpartners of the earlier discovered vectors $\mathbf{X}_\pm$, described by equations (5.41).

Derivation of the equations

We take the following deformations of the 3-form potentials:

$$\delta B_2 \;=\; \mathbf{J}\wedge \mathrm{d}\tau\,, \tag{5.42}$$

$$\delta C_2 \;=\; \mathbf{C}\wedge g^5 + *_4\mathrm{d}_4\mathbf{D}\,, \tag{5.43}$$

which lead to the deformations of the 3-forms:

$$\delta H_3 = d_4\mathbf{J}\wedge d\tau\,, \tag{5.44}$$

$$\delta F_3 = d_4\mathbf{C}\wedge g^5 - \mathbf{C}'\wedge d\tau\wedge g^5 - \mathbf{C}\wedge dg^5 - *_4\Box_4\mathbf{D} + *_4 d_4\mathbf{D}'\wedge d\tau\,, \tag{5.45}$$

We also consider an excitation of the 5-form:

$$\delta F_5 = (1+*)\big[\mathbf{F}\wedge d\tau\wedge g^1\wedge g^2\wedge g^5 + \mathbf{G}\wedge d\tau\wedge g^3\wedge g^4\wedge g^5 + d_4\mathbf{P}\wedge g^1\wedge g^2\wedge g^5 + d_4\mathbf{Q}\wedge g^3\wedge g^4\wedge g^5 + d_4\mathbf{R}\wedge d\tau\wedge(g^1\wedge g^3 + g^2\wedge g^4)\big]\,. \tag{5.46}$$

With the excitations (5.44) - (5.46), one obtains a set of five equations from the Bianchi identity for F_5:

$$-\frac{1}{2}\,h\sqrt{-G}G^{55}\Big((G^{11})^2\mathbf{P} - (G^{33})^2\mathbf{Q}\Big) + \Big(h\sqrt{-G}G^{11}G^{33}G^{55}\mathbf{R}\Big)' = F'\mathbf{D}', \tag{5.47}$$

$$-\Big(h^{1/2}\sqrt{-G}(G^{33})^2(G^{55})^2\mathbf{G}\Big)' + h\sqrt{-G}(G^{33})^2G^{55}\Box_4\mathbf{Q} = f'\Box_4\mathbf{D}, \tag{5.48}$$

$$-\Big(h^{1/2}\sqrt{-G}(G^{11})^2(G^{55})^2\mathbf{F}\Big)' + h\sqrt{-G}(G^{11})^2G^{55}\Box_4\mathbf{P} = k'\Box_4\mathbf{D}, \tag{5.49}$$

$$\mathbf{F} + \mathbf{P}' - \mathbf{R} = F\mathbf{J} + f'\mathbf{C}\,, \tag{5.50}$$

$$\mathbf{G} + \mathbf{Q}' + \mathbf{R} = (1-F)\mathbf{J} + k'\mathbf{C}\,, \tag{5.51}$$

a pair equations from the F_3 equation of motion:

$$\Big[h^{1/2}\sqrt{-G}(G^{55})^2\mathbf{C}'\Big]' - 2h^{1/2}\sqrt{-G}G^{11}G^{33}\mathbf{C} + h\sqrt{-G}G^{55}\Box_4\mathbf{C} = = h^{1/2}\sqrt{-G}(G^{55})^2\big(f'(G^{11})^2\mathbf{F} + k'(G^{33})^2\mathbf{G}\big)\,, \tag{5.52}$$

$$\Big[2h\sqrt{-G}G^{55}\mathbf{D}'\Big]' + 2h^{3/2}\sqrt{-G}\Box_4\mathbf{D} = 2k'\mathbf{P} + 2f'\mathbf{Q} + 4F'\mathbf{R} + \ell\mathbf{J}\,, \tag{5.53}$$

and two equations from equation of motion for H_3:

$$2h\sqrt{-G}G^{55}\Box_4\mathbf{J} = 2h^{1/2}\sqrt{-G}(G^{55})^2\big(F(G^{11})^2\mathbf{F} + (1-F)(G^{33})^2\mathbf{G}\big) + +\ell\Box_4\mathbf{D}\,, \tag{5.54}$$

$$\Big[2h\sqrt{-G}G^{55}\mathbf{J}\Big]' = 2h\sqrt{-G}G^{55}\big(F(G^{11})^2\mathbf{P} + (1-F)(G^{33})^2\mathbf{Q}\big) + +4F'h\sqrt{-G}G^{11}G^{33}G^{55}\mathbf{R} + \ell\mathbf{D}'\,. \tag{5.55}$$

As in the case of the previous ansatz, one of the equations is not independent and it is easy to demonstrate that any of the equations (5.47)-(5.49) or (5.54)-(5.55) can be eliminated. Thus, we obtain a system of eight equations for eight unknown forms. To write it in a more convenient way let us introduce the following parametrization:

$$\frac{G^{55}}{\sqrt{h}}\,\coth^2\frac{\tau}{2}\,\mathbf{F} \;=\; \coth\frac{\tau}{2}\,\Box_4\tilde{\mathbf{F}}\,, \tag{5.56}$$

$$\frac{G^{55}}{\sqrt{h}}\,\tanh^2\frac{\tau}{2}\,\mathbf{G} \;=\; \tanh\frac{\tau}{2}\,\Box_4\tilde{\mathbf{G}}\,. \tag{5.57}$$

We solve the algebraic equations for ansatz functions $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{J}$, which we express in terms of the newly defined functions $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$. The remaining four coupled second order differential equations are again most conveniently written in terms of the functions I, K, $\sinh\tau$ and their derivatives. This way we obtain a system

$$\tilde{\mathbf{F}}'' - \left[\frac{2}{\sinh^2\tau}+\frac{1}{2}\right]\tilde{\mathbf{F}} + \tilde{m}^2\frac{I}{K^2}\,\tilde{\mathbf{F}} + \frac{1}{2}\,\tilde{\mathbf{G}} + \left(\frac{1}{2}\,K^3\sinh\tau + 2^{-4/3}\frac{I'}{K}\right)(\mathbf{D}'-\mathbf{J}) = $$
$$= \frac{1}{2}\,K\tilde{\mathbf{C}} - \frac{2^{-4/3}I'}{K^3\sinh\tau}\,\tilde{\mathbf{C}}\,, \tag{5.58}$$

$$\tilde{\mathbf{G}}'' - \left[\frac{2}{\sinh^2\tau}+\frac{1}{2}\right]\tilde{\mathbf{G}} + \tilde{m}^2\frac{I}{K^2}\,\tilde{\mathbf{G}} + \frac{1}{2}\,\tilde{\mathbf{F}} + \left(\frac{1}{2}\,K^3\sinh\tau - 2^{-4/3}\frac{I'}{K}\right)(\mathbf{D}'-\mathbf{J}) = $$
$$= \frac{1}{2}\,K\tilde{\mathbf{C}} + \frac{2^{-4/3}I'}{K^3\sinh\tau}\,\tilde{\mathbf{C}}\,. \tag{5.59}$$

$$\tilde{\mathbf{C}}'' - \frac{\cosh^2\tau+1}{\sinh^2\tau}\,\tilde{\mathbf{C}} + \tilde{m}^2\frac{I}{K^2}\,\tilde{\mathbf{C}} = 2^{1/3}\,\tilde{m}^2K(\tilde{\mathbf{F}}+\tilde{\mathbf{G}}) - \tilde{m}^2\,\frac{I'}{K^3\sinh\tau}\,(\tilde{\mathbf{F}}-\tilde{\mathbf{G}})\,, \tag{5.60}$$

$$\mathbf{D}'' + \Big(\log(IK^2\sinh^2\tau)\Big)'\mathbf{D}' + \tilde{m}^2\frac{I}{K^2}\,\mathbf{D} =$$
$$= -\frac{(I'K^2\sinh^2\tau)'}{IK^2\sinh^2\tau}\,\mathbf{D} - \frac{I'}{I}\,\mathbf{J} + \frac{1}{IK^2\sinh^2\tau}\,\left(2^{1/3}K^3\sinh\tau(\tilde{\mathbf{F}}+\tilde{\mathbf{G}}) + \frac{I'}{K}(\tilde{\mathbf{F}}-\tilde{\mathbf{G}})\right)'\;; \tag{5.61}$$

where we introduced a new function $\tilde{\mathbf{C}} = K^2\sinh\tau\,\mathbf{C}$, $\tilde{m}$ is defined in (5.15) and $\mathbf{J}$ is expressed in terms of given functions as follows:

$$\mathbf{J} \;=\; -\frac{I'}{I}\,\mathbf{D} + \frac{2^{1/3}K}{I\sinh\tau}\,(\tilde{\mathbf{F}}+\tilde{\mathbf{G}}) + \frac{I'}{IK^3\sinh^2\tau}(\tilde{\mathbf{F}}-\tilde{\mathbf{G}})\,. \tag{5.62}$$

The form of the equations in (5.58)-(5.61) suggests that we introduce $\mathbf{B}_\pm = \tilde{\mathbf{F}} \pm \tilde{\mathbf{G}}$, so that the equations become

$$\mathbf{B}_+'' - \frac{2}{\sinh^2\tau}\,\mathbf{B}_+ + \tilde{m}^2\frac{I}{K^2}\,\mathbf{B}_+ + K^3\sinh\tau(\mathbf{D}'-\mathbf{J}) - K\tilde{\mathbf{C}} \;=\; 0\,, \tag{5.63}$$

$$\mathbf{B}_-'' - \frac{\cosh^2\tau+1}{\sinh^2\tau}\,\mathbf{B}_- + \tilde{m}^2\frac{I}{K^2}\,\mathbf{B}_- + 2^{-1/3}\frac{I'}{K}\,(\mathbf{D}'-\mathbf{J}) + \frac{2^{-1/3}I'}{K^3\sinh\tau}\,\tilde{\mathbf{C}} \;=\; 0\,, \tag{5.64}$$

$$\tilde{\mathbf{C}}'' - \frac{\cosh^2\tau+1}{\sinh^2\tau}\,\tilde{\mathbf{C}} + \tilde{m}^2\frac{I}{K^2}\,\tilde{\mathbf{C}} - 2^{1/3}\tilde{m}^2K\mathbf{B}_+ + \tilde{m}^2\,\frac{I'}{K^3\sinh\tau}\,\mathbf{B}_- \;=\; 0\,, \tag{5.65}$$

$$\mathbf{D}'' + \Big(\log(IK^2\sinh^2\tau)\Big)'\mathbf{D}' + \tilde{m}^2\frac{I}{K^2}\,\mathbf{D} + \frac{(I'K^2\sinh^2\tau)'}{IK^2\sinh^2\tau}\,\mathbf{D} \;=$$
$$-\frac{I'}{I}\,\mathbf{J} + \frac{1}{IK^2\sinh^2\tau}\,\left(2^{1/3}K^3\sinh\tau\,\mathbf{B}_+ + \frac{I'}{K}\,\mathbf{B}_-\right)'\;; \tag{5.66}$$

and

$$\mathbf{J} \;=\; -\frac{I'}{I}\,\mathbf{D} + \frac{2^{1/3}K}{I\sinh\tau}\,\mathbf{B}_+ + \frac{I'}{IK^3\sinh^2\tau}\,\mathbf{B}_-\,. \tag{5.67}$$

Analysis of the equations

The system of the equations (5.63)-(5.66) can be further reduced. In this section we find a linear transformation that decouples two equations. The remaining pair of coupled equations turns out to be the same as the system (47)-(48) from [26]. The decoupled equations coincide with the equations (5.41) for $\mathbf{X}_\pm$ found earlier.

First, we set

$$\mathbf{B}_- \;=\; 0\,, \tag{5.68}$$

then (5.64) implies

$$\mathbf{D}' - \mathbf{J} = -\frac{1}{K^2\sinh\tau}\,\tilde{\mathbf{C}}. \tag{5.69}$$

Differentiating this equation using (5.67) and plugging it into the equation (5.66), one gets, after eliminating $\mathbf{D}'$ via (5.69), a simple relation

$$\tilde{\mathbf{C}}' = \tilde{m}^2 I\sinh\tau\,\mathbf{D} - \coth\tau\,\tilde{\mathbf{C}}. \tag{5.70}$$

Note that differentiating (5.70) and then eliminating the derivatives of $\tilde{\mathbf{C}}$ from (5.65) we recover (5.69) (and therefore (5.66) as well). Thus, the constraint (5.68) singles out a consistent subsystem of the two equations:

$$\mathbf{B}_+'' - \frac{2}{\sinh^2\tau}\,\mathbf{B}_+ + \tilde{m}^2\frac{I}{K^2}\,\mathbf{B}_+ \;=\; 2K\tilde{\mathbf{C}}\,, \tag{5.71}$$

$$\tilde{\mathbf{C}}'' - \frac{\cosh^2\tau+1}{\sinh^2\tau}\,\tilde{\mathbf{C}} + \tilde{m}^2\frac{I}{K^2}\,\tilde{\mathbf{C}} \;=\; 2^{1/3}\tilde{m}^2 K\mathbf{B}_+\,. \tag{5.72}$$

After a trivial rescaling of variables it reproduces the scalar equations (5.9) and (5.10). Thus $\mathbf{B}_+$ and $\tilde{\mathbf{C}}$ describe the vector superpartners of the scalar excitations z and w discovered in [26].

To find the orthogonal pair of equations, one can just set

$$\mathbf{B}_+ \;=\; 0\,, \tag{5.73}$$

Equation (5.63) implies a first order constraint

$$\mathbf{D}' \;=\; -\frac{I'}{I}\,\mathbf{D} + \frac{I'}{IK^3\sinh^2\tau}\,\mathbf{B}_- + \frac{1}{K^2\sinh\tau}\,\tilde{\mathbf{C}}\,. \tag{5.74}$$

Using this equation one can eliminate the derivatives of $\mathbf{D}$ from (5.66) and get the relation

$$\tilde{\mathbf{C}}' = -\tilde{m}^2 I\sinh\tau\,\mathbf{D} - \coth\tau\,\tilde{\mathbf{C}}. \tag{5.75}$$

Note that after eliminating the $\tilde{\mathbf{C}}$ derivatives from (5.65) using this equation we recover (5.74) (and thus (5.63) and (5.66)). There remains a consistent subsystem of the two equations for $\mathbf{B}_-$ and $\tilde{\mathbf{C}}$:

$$\begin{aligned} \mathbf{B}_-'' - \frac{\cosh^2\tau+1}{\sinh^2\tau}\,\mathbf{B}_- + \tilde{m}^2\frac{I}{K^2}\,\mathbf{B}_- &= -\frac{2^{2/3}I'}{K^3\sinh\tau}\,\tilde{\mathbf{C}}\,, & (5.76)\\ \tilde{\mathbf{C}}'' - \frac{\cosh^2\tau+1}{\sinh^2\tau}\,\tilde{\mathbf{C}} + \tilde{m}^2\frac{I}{K^2}\,\tilde{\mathbf{C}} &= -\tilde{m}^2\,\frac{I'}{K^3\sinh\tau}\,\mathbf{B}_-\,. & (5.77)\end{aligned}$$

After a trivial rescaling of the variables these equations become

$$\begin{aligned} \mathbf{B}_-'' - \frac{\cosh^2\tau+1}{\sinh^2\tau}\,\mathbf{B}_- + \tilde{m}^2\frac{I}{K^2}\,\mathbf{B}_- &= -\frac{2^{1/3}\tilde{m}I'}{K^3\sinh\tau}\,\tilde{\mathbf{C}}\,, & (5.78)\\ \tilde{\mathbf{C}}'' - \frac{\cosh^2\tau+1}{\sinh^2\tau}\,\tilde{\mathbf{C}} + \tilde{m}^2\frac{I}{K^2}\,\tilde{\mathbf{C}} &= -\frac{2^{1/3}\tilde{m}I'}{K^3\sinh\tau}\,\mathbf{B}_-\,. & (5.79)\end{aligned}$$

Then equations for $\mathbf{Y}_\pm = \mathbf{B}_- \pm \tilde{\mathbf{C}}$ decouple,

$$\mathbf{Y}_\pm'' - \frac{\cosh^2\tau+1}{\sinh^2\tau}\,\mathbf{Y}_\pm + \tilde{m}^2\frac{I}{K^2}\,\mathbf{Y}_\pm \pm \frac{2^{1/3}\tilde{m}I'}{K^3\sinh\tau}\,\mathbf{Y}_\pm = 0\,. \qquad (5.80)$$

Equivalently,

$$\mathbf{Y}_\pm'' - \frac{\cosh^2\tau+1}{\sinh^2\tau}\,\mathbf{Y}_\pm + \tilde{m}^2\frac{I}{K^2}\,\mathbf{Y}_\pm \mp \frac{2^{5/3}\tilde{m}F'}{K^2\sinh\tau}\,\mathbf{Y}_\pm = 0\,. \qquad (5.81)$$

These equations exactly coincide with (5.41), which means that $\mathbf{X}_+$, $\mathbf{Y}_+$ and $\mathbf{X}_-$, $\mathbf{Y}_-$ are members of two new supermultiplets. Most natural guess would be to assign them to two $j=1$ "gravitino" multiplets, each containing a vector, an axial vector and fermions of spin 1/2 and 3/2. The two supergravity ansatzes, studied in this and the previous sections, plus the scalar ansatz cover all $\mathcal{I}$-odd modes of the KS background. Therefore there are no other bosonic modes to share the multiplets with $\mathbf{X}_\pm$ and $\mathbf{Y}_\pm$. This completes the derivation of the spectrum of the $SU(2)\times SU(2)$ singlet $\mathcal{I}$-odd excitations over the KS background.

Let us briefly summarize the results of the above investigation. With the states discussed here, we expect to have five different supermultiplets. Three of them contain spin 0 modes:

- Multiplet I. $j=1/2$. Contains the pseudoscalar $\tilde{w}$ (5.11) and the vector $\tilde{\mathbf{A}}$ (5.36).
- Multiplet II. $j=1/2$. Contains the lightest scalar eigenmode of (w,z) system (5.9)-(5.10) and the lightest pseudovector eigenmode of the $(\mathbf{B}_+,\tilde{\mathbf{C}})$ system (5.71)-(5.72).
- Multiplet III. $j=1/2$. Contains the heaviest scalar eigenmode of (w,z) system (5.9)-(5.10) and the heaviest pseudovector eigenmode of the $(\mathbf{B}_+,\tilde{\mathbf{C}})$ system (5.71)-(5.72).

The remaining two multiplets contain only spin 1 bosonic modes:

- Multiplet IV. $j=1$. Contains the vector-pseudoscalar pair $\mathbf{X}_+$ (5.41) and $\mathbf{Y}_+$ (5.81).
- Multiplet V. $j=1$. Contains the vector-pseudoscalar pair $\mathbf{X}_-$ (5.41) and $\mathbf{Y}_-$ (5.81).

5.4 Seven $\mathcal{I}$-even Scalars

In the final section of this chapter let us briefly review the works of M. Berg, M. Haack and W. Mück, in which they computed the mass spectra of the scalar fluctuations over the KS background [23, 24]. The result is based on the general $SU(2) \times SU(2)$ invariant scalar ansatz written by G. Papadopoulos and A. Tseytlin [56] for the warped deformed conifold. The ansatz contains eleven scalar functions of the variable τ; these are A, p, x, a, g, b, h_1, h_2, χ, K from the equations (3.102)-(3.107) and the dilaton Φ.

In terms of the scalar functions the supergravity equations are reduced to a set of algebraic and ordinary differential equations of a single variable τ. The authors of [56] find an effective one dimensional action, from which the equations for the scalar functions follow. However to find the four dimensional spectrum, the fluctuations should also depend on the space-time coordinates. The idea exploited in [23] was to embed the one dimensional action of [56] into the five dimensional supergravity. To validate the procedure, the authors of [23] have checked that the result is a consistent truncation of the ten dimensional theory to five dimensions. As a result one obtains a five dimensional gauged supergravity:

$$S = \int \mathrm{d}^5 x \, \sqrt{g} \left(-\frac{1}{4} \, R + G_{ab}(\varphi) \partial_i \varphi^a \partial^i \varphi^b + V(\varphi) \right) . \tag{5.82}$$

The field φ^a is a collective notations for the ten of the above eleven scalar except A. The fluctuation of the latter will be related to the fluctuations of the five dimensional metric, similar to the metric (4.27) in section 4.5. The sigma model metric $G_{ab}(\varphi)$ and the potential $V(\varphi)$ are given by the equations (3.13) and (3.14) from [23].

Not all of this scalar functions can be independent supergravity fluctuations. Some supergravity fluctuations imply algebraic relations on the scalar functions. In particular the equations

$$K = Q + 2P(h_1 + bh_2), \qquad \text{and} \qquad e^{2g} = -1 - a^2 - 2a \cosh \tau$$

can be used to eliminate the fluctuations of K and g. Also G_{ab} does not contain the kinetic term for χ and the latter can be integrated out. Thus we are left with the metric and the seven scalars.

The authors of [23] approach the problem with a large extent of generality. They derive the linearized equations for the action (5.82) in the case of a general supersymmetric background using the explicitly gauge invariant and covariant sigma model formalism. In this context supersymmetric means that the potential $V(\varphi)$ comes from some superpotential W,

$$W = \frac{1}{2} \, G^{ab} W_a W_b - \frac{4}{3} \, W^2,$$

where $W_a = \partial/\partial\varphi^a W$; and the following conditions are satisfied

$$\partial_q \varphi^a = G^{ab} W_b, \qquad \text{and} \qquad \partial_q T = -\frac{2}{3}\, W,$$

where T is the five dimensional warp factor and q is the radial coordinate in the metric (4.27).

The covariant sigma model approach gives the following equation for the gauge invariant basis of fluctuations ϕ^a of the seven scalars φ^a:

$$\left[\left(\delta^a{}_b D_q + W^a{}_{|b} - \frac{W^a W_b}{W} - \frac{8}{3}\, W \delta^a{}_b\right)\left(\delta^b{}_c D_q - W^a{}_{|b} + \frac{W^b W_c}{W}\right) + \delta^a{}_c e^{-2T} \Box\right] \phi^c = 0, \qquad (5.83)$$

where the covariant derivatives are determined in terms of the sigma model connection $\mathcal{G}^a_{bc}$

$$D_a \varphi_b \equiv \varphi_{b|a} = \partial_a \varphi_b - \mathcal{G}^c_{ab} \varphi_c, \qquad D_q \varphi^a = \partial_q \varphi^a + \mathcal{G}^a_{bc} W^b \varphi^c.$$

Studies of the metric fluctuations by this method gave the same result for the transverse traceless part as the earlier studied fluctuation $h_{\mu\nu}$ (4.9) in the chapter 4.

M. Berg et al. numerically computed the four dimensional mass spectrum of the seven scalars represented by the system of equations (5.83) [24]. However because of the complexity of this system its further analysis is challenging. In particular it is hard to understand the state-operator correspondence in this case. In the next chapter we will use the numerical results of the work [24] to see how they correlate with the multiplets that we discovered earlier and with the glueball spectrum of the $SU(3)$ theory [32] in the figure 2.1.

Chapter 6
Discussion

6.1 Numerical Spectra

In the chapters 4 and 5 we summarized the searches for different fluctuations of the Klebanov-Strassler supergravity solutions. In the two chapters the studies were going in two different directions. In chapter 4 we originally knew the operators of the gauge theory and found the fluctuations dual to them on the gravity side. In chapter 5 instead, we were rather looking at all possible excitations that respect certain symmetries and trying to understand which operators can stand behind them. In this chapter we will join the results of the two chapters and try to draw a general picture of the holographic correspondence in the KS case.

Let us start from collecting the results of numerical studies of the four dimensional spectra. The latter were computed in the works of different authors [20, 21, 24, 25, 26, 62], which used different conventions and normalizations. Here we will adopt the conventions of the work [26]. Those include the definition of the warp factor as in (3.87) and measuring four dimensional mass in units of $\tilde{m}$ (5.15).

In the case of the mass spectrum of a single equation it is convenient to use the shooting technique, as in the case of the graviton multiplet in section 4.6. As it was argued there, this method seems to give a good convergence. For a system of coupled equations the shooting method does not work as well. The determinant method, used in [24] and [26], seems to work better there. There are indications however that the determinant method also give a quite large numerical error for the lightest states in the spectrum. In particular, in the case of the shooting technique applied to a single equation, the spectrum is quadratic in n (principle quantum number) with a good accuracy. In the case of the determinant method applied to the seven scalars of [23], the deflection from the quadratic formula for the lightest states seems to be substantial.

In the table 6.1 we collect a few lowest mass eigenvalues for all known glueballs including the information about the numerical method. We also add an approximate quadratic formula in each case. If the states are in the same multiplet they have the same spectra. Therefore in the table we present the spectra of the multiplets, but not the individual states.

Multiplet	Method	1	2	3	4	5	$\tilde{m}_n^2$
Graviton	Shoot.	1.12	2.53	4.52	7.07	10.2	$0.29n^2 + 0.54n + 0.29$
I. $j = 1/2$	Shoot.	2.41	4.47	7.08	10.3	14.0	$0.29n^2 + 1.15n + 1.0$
II. $j = 1/2$	Determ.	0.129	0.703	1.76	3.33	5.43	$0.28n^2 + 0.14n + 0.36$
III. $j = 1/2$	Determ.	4.53	7.30	10.7	14.6	19.1	$0.29n^2 + 1.91n + 2.31$
IV. $j = 1$	Shoot.	3.01	5.20	7.96	11.3	15.2	$0.29n^2 + 1.31n + 1.44$
V. $j = 1$	Shoot.	1.89	3.83	6.31	9.34	12.9	$0.29n^2 + 1.02n + 0.63$
BHM I	Determ.	0.197	2.77	5.69	8.59	12.0	$0.29n^2 + 0.82n + 0.60$
BHM II	Determ.	0.455	3.53	5.98	8.77	12.1	$0.29n^2 + 0.99n - 0.46$
BHM III	Determ.	0.887	3.76	5.98	9.11	12.8	$0.29n^2 + 0.82n + 2.04$
BHM IV	Determ.	1.36	4.38	7.00	10.1	13.8	$0.29n^2 + 1.08n + 1.09$
BHM V	Determ.	1.73	4.44	7.08	10.2	13.8	$0.29n^2 + 1.19n + 0.36$
BHM VI	Determ.	2.06	4.71	7.20	10.3	14.1	$0.29n^2 + 1.22n + 0.69$
BHM VII	Determ.	2.49	4.71	7.59	11.1	15.1	$0.29n^2 + 1.15n + 2.7$

Table 6.1: The spectra of glueballs known in KS in units of $\tilde{m}^2$. First part of the table contains the spectra of the supermultiplets classified in the end of section 5.3. The second part contains the spectra of the seven scalars found by M. Berg, M. Haack and W. Mück [24].

The first half of the table 6.1 contains the spectra of glueballs computed in the full ten dimensional theory. Those contain the states of the graviton multiplet, studied in chapter 4 and the $\mathcal{I}$-odd states from sections 5.2 and 5.3. For the $\mathcal{I}$-odd glueballs we label supermultiplets according to the discussion in the end of the section 5.3.2. In the second half, we put the spectrum of the system of seven scalar states computed in the truncated five dimensional theory, which we briefly discussed in the last section of the previous chapter.

A few comments about the numerical spectra are in order. One first notices that the n^2 coefficient is practically the same (~ 0.29) for all glueballs in the table 6.1. This fact was acknowledged in the section 4.6, where it was derived from semiclassical analysis (in a slightly different normalization).

The mass spectrum of the gravity multiplet for the KS background can be approximated with a good accuracy by a simple quadratic formula (4.40), which is approximately

$$\tilde{m}_n^2 \simeq 0.29(n+1)^2. \tag{6.1}$$

Remarkably, in the five dimensional supergravity solution known as GPPZ background [66], which has some common properties with the ten dimensional KS case (see section 4.5) the spectrum of the graviton multiplet is exactly proportional to $(n+1)^2$; namely $m_n^2 = 4L^{-2}(n+1)^2$. One can then assume that some features of the spectra for certain glueballs do not crucially depend on the details of the background. Based on the exact result of the GPPZ case calculation for the mass spectrum of the anomaly multiplet S (4.2), $m_n^2 = 4L^{-2}(n+1)(n+2)$ [59, 65, 69], one can guess the answer for the KS case. In the units of $\tilde{m}^2$ the approximate formula reads

$$m_n^2 \simeq 0.29(n+1)(n+2), \qquad n = 1, 2, 3, \ldots$$

This is in fact close to one of the seven towers (BHM I in the table 6.1), described by the empirical formula

$$0.29n^2 + 0.82n + 0.6.$$

It would be interesting to confirm the matching between the trace of the metric and the lowest tower of the 7-particle system with a more rigorous approach.

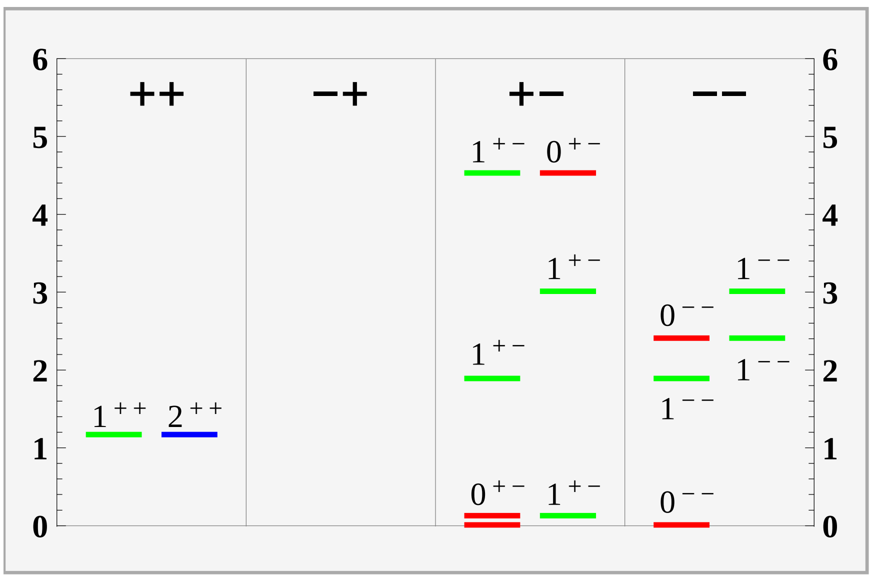


Figure 6.1: The spectrum of glueballs found so far. (Only the lightest, $n = 1$ modes of each tower are shown.)

In the figure 2.1 we plot the squared mass of the bosonic modes from the multiplets in the first half of the table 6.1. All states are classified by their J^{PC} quantum numbers. In the C-even sector we have only two states from graviton multiplet so far. The C-odd sector contains all possible massive $SU(2) \times SU(2)$ singlet states. Besides the five massive multiplets the are also the massless scalar and the pseudoscalar discussed in the last section of the chapter 3.

We do not add the BHM scalars in the figure 2.1. The lightest eigenvalues found in [24] might have been altered substantially by a numerical error. If one used instead the empirical formulas in table 6.1 to predict the $n = 1$ eigenvalues, one would find for BHM I $\tilde{m}_1^2 \simeq 1.7$, which is heavier than the mass of 2^{++} state, but lighter than the $1^{\pm -}$ states (except for the lightest pseudovector). This is consistent with our assumption that BHM I scalar gives the mass of the anomaly multiplet

S. Indeed the corresponding pseudoscalar state is between the 2^{++} and 1^{+-} states in the spectrum of the pure glue theory (figure 2.1).

6.2 Supermultiplets

We have not yet identified the $\mathcal{I}$-odd supergravity fluctuations studied in the chapter 5 with the operators of the dual gauge theory. At this time there are no systematic studies of this issue in the literature, primarily because of the complex mixing of the fluctuations in the KS case. Some results are known however from the works [22] and [26]. Here we will briefly review those results and speculate on the remaining modes of the $\mathcal{I}$-odd sector.

6.2.1 Baryonic Current Multiplets

In the section 3.6 we discussed the zero-modes of the KS background discovered in [22]. These zero-modes are associated with the spontaneous breaking of the baryon number symmetry. By analogy with the case of spontaneous chiral symmetry breaking in QCD, where the pions are the Goldstone modes of the broken symmetry, the massless pseudoscalar state in the figure 6.1 should be created by the (pseudovector) baryonic current operator J^B_μ. On the supergravity side the baryonic symmetry becomes gauged. There should exist a pseudovector fluctuation, dual to the baryonic current itself, that eats massless pseudoscalar and becomes massive [26]. As explained in [26] this pseudovector fluctuation is the lightest 1^{+-} glueball in the figure 6.1.

The massless scalar fluctuation forms a supermultiplet with the pseudoscalar. After the pseudovector eats the massless pseudoscalar, it takes the scalar particle to be a superpartner. The scalar equation of motion (5.9) predicts the classical dimension $\Delta = 2$ for the dual operator. Indeed there is such an operator in the gauge theory:

$$A\bar{A} - B\bar{B}. \tag{6.2}$$

The lower dimension of the operator, the lower the mass of the state. Therefore one should identify the lightest massive multiplet in the figure 6.1 with the vector multiplet of the baryonic current J^B_μ and the scalar operator (6.2).

Note that the J^B_μ has the classical dimension $\Delta = 3$. However the vector excitation that is the superpartner of the light scalar is described by the equation(5.71), which is the same as (5.9) and thus suggests the same dimension $\Delta = 2$. The resolve this puzzle we will study the simpler case of the massive multiplet that contains a massive pseudoscalar 0^{--} and a massive vector 1^{--}.

The pseudoscalar 0^{--} is described by the equation (5.11). The same equation is satisfied by the vector $\tilde{\mathbf{A}}$ (5.36), which suggest that they have the same dimension $\Delta = 5$. However in the representations of the superalgebra in four dimensions, bosonic superpartners of different spin should have dimensions different by one. The answer is that in fact a different fluctuation is the source for the superpartner of the 0^{--} according to the four dimensional superalgebra.

6.2.2 Dimensions and SQM

As it is explained in the section 4.5, supersymmetry transformations reduce to the effective one-dimensional Supersymmetric Quantum Mechanics transformations, which for two bosonic superpartners can have the following representation. There exist two first order differential operators Q_+ and Q_-, such that Q_+Q_- gives an equation for one superpartner and Q_-Q_+ for another one. The equations will be different but will share the same spectrum.

By an appropriate field and coordinate redefinition both equations can be brought to the Schrödinger form

$$Q_\pm Q_\mp \psi_\pm = \left(\partial_u^2 - V_\pm(u)\right)\psi_\pm = -m^2\psi_\pm,$$

where $Q_\pm = \partial_u \pm W$, $V_\pm = \pm W' + W^2$ and W is typically called a superpotential. If one equation is known, operators $Q_\pm$ can be found up to a constant. The latter can be fixed by requiring that one of the potentials $V_\pm$ should be non-singular at the origin. In particular, for the equation (5.11), the superpartner equation should be

$$\tilde{w}_s'' + \left(\frac{1}{2}\,\frac{I''}{I} - \frac{(K\sinh\tau)''}{K\sinh\tau} + \frac{I'}{I}\,\frac{(K\sinh\tau)'}{K\sinh\tau} - \frac{3}{4}\,\frac{I'^2}{I^2}\right)\tilde{w}_s + \tilde{m}^2\,\frac{I}{K^2}\,\tilde{w}_s = 0. \tag{6.3}$$

This equation can easily be obtained from our ansatz. Indeed, if one eliminates $\tilde{\mathbf{A}}$ from instead of $\mathbf{H}$ from the equations (5.34) and (5.35), equation (5.32) in the $\tilde{\mathbf{N}} = 0$ case takes the form

$$\mathbf{H}'' + \frac{(IK^2\sinh^2\tau)'}{IK^2\sinh^2\tau}\,\mathbf{H}' + \left(\frac{I''}{I} + 2\,\frac{I'}{I}\,\frac{(K\sinh\tau)'}{K\sinh\tau} - \frac{I'^2}{I^2}\right)\mathbf{H} + \tilde{m}^2\,\frac{I}{K^2}\,\mathbf{H} = 0. \tag{6.4}$$

After a function redefinition that eliminates $\mathbf{H}'$, $\tilde{\mathbf{H}} = \sqrt{I}K\sinh\tau\,\mathbf{H}$, one gets

$$\tilde{\mathbf{H}}'' + \left(\frac{1}{2}\,\frac{I''}{I} - \frac{(K\sinh\tau)''}{K\sinh\tau} + \frac{I'}{I}\,\frac{(K\sinh\tau)'}{K\sinh\tau} - \frac{3}{4}\,\frac{I'^2}{I^2}\right)\tilde{\mathbf{H}} + \tilde{m}^2\,\frac{I}{K^2}\,\tilde{\mathbf{H}} = 0, \tag{6.5}$$

which is the same as (6.3) and coincide with the predicted equation (69) from [26]. The dimension of the corresponding operator is $\Delta = 6$, in agreement with our expectations from supersymmetry. In the similar fashion the dimension of the lightest 1^{+-} state is $\Delta = 3$, while for the heaviest 1^{+-} it is $\Delta = 6$. In that case, however, similar analysis is obstructed by the fact that the fluctuations are coupled to each other.

No such problem arises in the case of gravitino multiplets containing a vector and a pseudovector. These two fluctuations should then have the same dimension. No surprise that they are described by the same equation.

6.2.3 Operators

For further understanding of the state-operator correspondence in the case of the spectrum in the figure 6.1 it us useful to look at the conformal example. A complete classification of the supergravity excitations and the operators for the theory on $AdS_5 \times T^{1,1}$ was given by A. Ceresole et al. in the work [70]. In the conformal case the multiplets, characterized by the representations of the superconformal group, are typically large. In transition to the non-conformal case the long multiplets can break into simpler parts, classified by representations of the four dimensional superalgebra. Here we will compare our short superalgebra multiplets with the superconformal ones and speculate on possible dual operators.

To start one can embed the graviton multiplet, described in the chapter 4 in the superconformal graviton multiplet of [70]. The modes of the superconformal multiplet are given in the table 2 of [70]. The modes of the graviton multiplet from figure 6.1 coincide with the bosonic modes of the "massless" shortened multiplet in that table. Although massless in this case refers to the 5-dimensional mass, the conformal graviton multiplet is also massless in the four dimensional sense, while the non-conformal one is massive. Therefore, the conformal graviton multiplet should "eat" another multiplet to acquire additional degrees of freedom to become the graviton multiplet of the figure 6.1. The conformal massless graviton multiplet is dual to the operator

$$J_{\alpha\dot{\alpha}} = \mathrm{tr}\ W_\alpha e^V \bar{W}_{\dot{\alpha}} e^{-V}, \tag{6.6}$$

which is classically the same as the operator (4.1).

Other interesting multiplets to look at are the type II and type IV gravitino multiplets in tables 4 and 6 of [70] respectively. The analysis of the dimensions and the $\mathcal{R}$-charges shows that the fluctuations from the $\mathcal{I}$-odd multiplets in the figure 6.1 (except for the multiplet of the baryonic current) can be embedded in these superconformal multiplets. In general the dimension of the shortened semi-long type IV gravitino multiplet is given by

$$\Delta = \frac{3}{2}k + \frac{9}{2},$$

which corresponds to the dimension of the lowest spin 1/2 component of the superfield. It also contains five dimensional (1/2,1/2) and (0,1) fields of dimension Δ+1/2 and (0,1) field of dimension

$\Delta+3/2$. For $k=0$ these are dimension 5 and 6 vector fields respectively, which in four dimensions break into dimension 5 (pseudo-) scalars and dimension 5 and 6 vectors of the figure 6.1.

Field	reps	Δ	R	Mode
a_μ	$(1/2,1/2)$	5	0	$\mathbf{C}^{(2)}$, $(\chi,\,\tilde{\chi})$
$b^{\pm}_{\mu\nu}$	$(1,0)$, $(0,1)$	5	∓ 2	$\mathbf{F}^{(1)}-\mathbf{F}^{(2)}$, $\mathbf{F}^{(3)}$
$a_{\mu\nu}$	$(1,0)$, $(0,1)$	6	0	$\mathbf{C}^{(3)}$

Field	reps	Δ	R	Mode
ϕ_μ	$(1/2,1/2)$	3	0	$\mathbf{F}^{(1)}+\mathbf{F}^{(2)}$
ϕ	$(0,0)$	2	0	U

Table 6.2: Shortened Gravitino Multiplets II, IV (left) and Vector Multiplet I (right) [70, 71]. Field notations are inherited from [72].

In the table 6.2 (left) we put the bosonic modes of the semi-long type IV gravitino multiplet of $\Delta = 9/2$ versus the fluctuations of the KS background with the same quantum numbers. Notice that the components of the superconformal multiplets in [70] are five dimensional fields. In particular, five dimensional field a_μ contains a four dimensional vector fluctuation $\mathbf{C}^{(2)}$ (5.12) and a four dimensional scalar χ from (5.5) (or $\tilde{\chi}$ from (5.6)). As follows from the table, we take the above fluctuations together with $\mathbf{F}^{(1)}-\mathbf{F}^{(2)}$, $\mathbf{F}^{(3)}$, and $\mathbf{C}^{(3)}$ as independent. The other forms in the general ansatz of section 5.2 can be expressed in terms of them.

To get the correct number of degrees of freedom we also need to employ another superconformal multiplet; namely a similar semi-long type II gravitino multiplet of [70], which in the $k=0$ case is only different by taking the dual Lorentz representation (e.g. $(0,1)\to(1,0)$) and inverting $\mathcal{R}$-charges. Altogether we have two spin 1 and spin 0 modes coming from the complex a_μ, two modes of opposite $\mathcal{R}$-charge from $b^{+}_{\mu\nu}$ and $b^{-}_{\mu\nu}$ and two spin 1 modes from complex $a_{\mu\nu}$. This indeed accounts for four massive $\mathcal{I}$-odd supermultiplets in the figure 6.1.

Notice that type II and type IV gravitino multiplets in [70] are constructed in terms of the complex two-form field $A = B_2 + iC_2$. This is a more convenient representation for the supersymmetry. In the chapter 5, where the general $\mathcal{I}$-odd ansatz was studied, we used a real representation, in which the fields are the eigenstates of parity P. The physical degrees of freedom corresponding to (5.5), (5.6), (5.16)-(5.20) and (5.42)-(5.46) are in fact linear combinations of the components from both type II and type IV gravitino multiplets. This is why we do not distinguish them in the table 6.2. More on the correspondence between the supergravity fluctuations and the components of the superconformal multiplets can be found in [27].

In general the semi-long type IV gravitino multiplet is dual to the operator

$$L^{2k}_{\dot{\alpha}} = \mathrm{tr}\; e^{V}\bar{W}_{\dot{\alpha}}e^{-V}W^{2}(AB)^{k}, \tag{6.7}$$

which for $k = 0$ does not contain A and B fields and is made out of super Yang-Mills fields only;

$$\mathcal{O} = \text{tr } e^V \bar{W}_{\dot{\alpha}} e^{-V} W^2. \tag{6.8}$$

This is very intriguing, since the above superfield seems to contain operators like tr $F_{\mu\nu}F^2$, which are responsible for $1^{\pm -}$ states in pure glue theory [32, 33].

We identified all states in the $\mathcal{I}$-odd sector, except for the lightest massive scalar and the pseudovector, with the operator (6.8) representing a five dimensional representation of the superconformal algebra. From the point of view of the four dimensional superalgebra this representation is reducible and breaks into four heavy supermultiplets in the figure 6.1. The remaining massive vector multiplet comes from a so-called Betti multiplet, which is a massless (in the five dimensional sense) type I vector multiplet (table 6.2). Operator that couples to the Betti multiplet on the boundary is

$$\mathcal{U} = \text{tr } Ae^V \bar{A} e^{-V} - \text{tr } Be^V \bar{B} e^{-V}. \tag{6.9}$$

It contains the scalar component (6.2). We observed in the chapter 5 that Betti multiplet couples to the heaviest vector multiplet with dimension $\Delta = 5$ scalar and $\Delta = 6$ pseudovector. It would be interesting to further understand the reason of such coupling.

6.3 Future Directions

In this section we would like to outline future directions that one can follow given the results discussed in this work. We started in the chapter 2 from a discussion of the glueballs as the bound states of the gauge bosons of a yet unobserved gauge group. One can expect them to be discovered as a part of new physics at the LHC.

The purpose of the chapter 2 was to compute the decay rates of the v-glueballs and branching ratios of various decay channels. The decay rates are expressed in terms of unknown matrix elements, which we have split into two categories of the decay constants **F** and the transition matrix elements **M**. Very little is known about this matrix elements in general, since they are related to the strong-coupling dynamics of the gauge theory. For practical purposes even estimations for the ratios of the matrix elements would be extremely useful.

The principle of the holographic correspondence allows one to do the calculations of the operator matrix elements in strongly coupled gauge theories. However there are also problems with this approach. Only few consistent examples are known of supergravity solutions dual to confining gauge theories. Even in the case of a consistent example one still typically has to content with the results in large N_c approximation of some supersymmetric theory. One can nevertheless obtain qualitative results in the cases, when they expected to have insignificant dependence on N_c or supersymmetry.

In particular, it would be interesting to check the assumption used in the chapter 2 that the mass ratio of the glueballs in the figure 2.1 does not crucially depend on N_c. Using the glueball solutions to the linearized supergravity equations, one could also find estimations for the ratios of the matrix elements of dual operators. The latter would be independent on N_c at least in the leading order.

The results of our investigation of the glueball spectra on the KS background look promising. We have found the states that correspond to the operators consisting only of the fields of the $\mathcal{N} = 1$ Supersymmetric Yang-Mills theory. Among those states, plotted in the figure 6.1 one can find 2^{++}, 1^{+-} and 1^{--} glueballs also present in the spectrum of the pure glue theory (figure 2.1). It would be extremely interesting to complete the figure 6.1 by finding the remaining $SU(2) \times SU(2)$-singlet states in the $++$ and $-+$ PC-sectors and by assigning them corresponding gauge theory operators.

This task is partially fulfilled by the works of M. Berg et al. [23] and [24], who found a set of seven 0^{++} states. Their approach seems to be solid, although the numerical results might be somewhat altered by the complexity of the system that they consider. One could try to check their results by computing the spectra of possible 0^{-+} and 1^{++} superpartners of those scalars. It is quite plausible though that in the case of the KS solution there are no other $SU(2) \times SU(2)$ singlet spin 1 states.

One can easily show that in the $\mathcal{I}$-even sector no spin 1 fluctuations of the 2-forms exist. Among the metric fluctuations one can only consider excitations of the form $\mathbf{A} \wedge g^5$ and $\mathbf{B} \wedge \mathrm{d}\tau$. The first one was already considered in the chapter 4 together with the relevant 5-form fluctuations, while the second seems to be discarded by the considerations of M. Berg et al.

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