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THE MULTIPLIHEDRA IN LAGRANGIAN FLOER THEORY

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ABSTRACT OF THE DISSERTATION

The multiplihedra in Lagrangian Floer theory

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We apply the quilted Floer theory of Wehrheim and Woodward to families of quilted surfaces parametrized by the Stasheff multiplihedra. Our approach is modeled on the construction of the Fukaya category, which applies Floer theory to families of pointed Riemann surfaces parametrized by the associahedra. First, we show that the multiplihedra are realized as a moduli space of quilted disks. Using the quilted disks we define the moduli space of pseudoholomorphic quilted disks, which under suitable transversality assumptions are smooth manifolds. Then we prove a gluing theorem relating "broken" tuples of pseudoholomorphic quilted disks with boundaries of one-parameter familes of pseudoholomorphic quilted disks.

Acknowledgements

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Table of Contents

Chapter 1

Introduction

Lagrangian Floer homology was introduced by Floer in [2]. In essence it is a recipe for associating a homology group $HF(L_0, L_1)$ to a pair of Lagrangian submanifolds L_0, L_1 in a symplectic manifold. The chain groups are generated by points of intersection, and the Floer differential counts pseudoholomorphic strips connecting points of intersection. The Lagrangian submanifolds of M and associated Floer chain groups have the structure of an A_∞ category, whose main feature is a sequence of products,

$$\mu^n : CF(L_{n-1}, L_n) \otimes \ldots \otimes CF(L_0, L_1) \longrightarrow CF(L_0, L_n)$$

for $n \geq 1$, that satisfy a sequence of quadratic relations called the A_∞ associativity relations. The products μ^n are defined by counting pseudoholomorphic $n+1$-gons; the Floer differential ∂ is essentially μ^1. The main impediment to Lagrangian Floer homology is that the Floer differential can fail to satisfy $\partial^2 = 0$. Fukaya, Oh, Ohta and Ono [4] have shown that the obstruction can be encoded in a more general A_∞ structure, in which there is a notion of a μ^0. The A_∞ category encodes more information than the Floer homology groups alone, and it is called the Fukaya category of the symplectic manifold M.

As with any topological invariants, it is useful to know how they behave with respect to various notions of morphism at the level of the manifolds. Wehrheim and Woodward defined a *symplectic category* in [19], whose objects are symplectic manifolds and morphisms are sequences of Lagrangian correspondences. They developed a quilted version of Floer homology in which Lagrangian correspondences play a visible and canonical role, and established some functoriality properties of Floer homology with respect to Lagrangian correspondences. This thesis fits into a larger program of studying how their constructions extend to the chain level. One goal of the program is to show that Lagrangian correspondences induce A_∞ functors between Fukaya categories. The axioms for an A_∞ functor include a sequence of composition maps that satisfy a collection of quadratic relations called A_∞ *functor relations*. The results of this thesis are motivated by a proposed functor between Fukaya categories, associated to a Lagrangian correspondence between two manifolds. Its construction is based on counting pseudoholomorphic "quilted disks" with markings, and these are the objects studied in this thesis.

A quilted disk is simply a disk with an extra piece of data – an inner circle. Each $d \geq 1$ has an associated moduli space of quilted disks with $d+1$ markings, which we call $\mathcal{R}^{d,1}$; the elements of $\mathcal{R}^{d,1}$ are equivalence classes of tuples $(D, C, z_0, \ldots, z_d)$ where D is the unit disk, C is a circle contained in D and tangent to $z_0 \in \partial D$, and $z_0, \ldots, z_d$ is a configuration of $d+1$ distinct points on ∂D, in counterclockwise cyclic order. This moduli space has a compactification $\overline{\mathcal{R}}^{d,1}$ by semi-stable, nodal marked quilted disks, and the first part of the thesis is devoted to proving:

Theorem. *$\overline{\mathcal{R}}^{d,1}$ is homeomorhic to a compact, $(d-1)$-dimensional polytope, the Stasheff multiplihedron J_d.*

This is analogous to the fact that moduli space of marked disks, which are the domains behind μ^n in the Fukaya category, realize the associahedra.

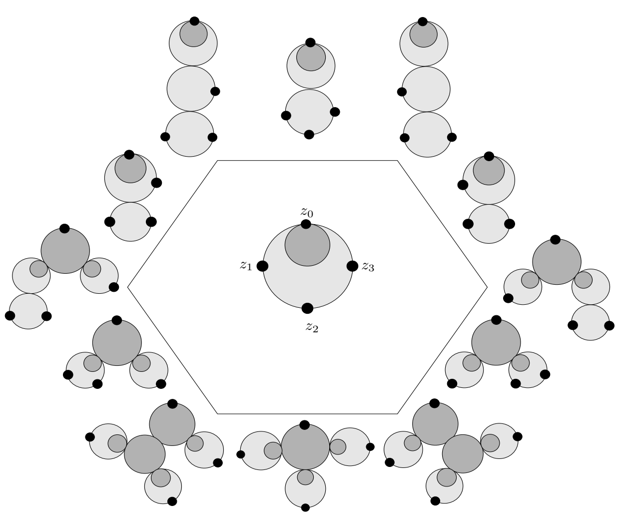

Figure 1.1: The compactified moduli space $\overline{\mathcal{R}}^{3,1}$, or multiplihedron J_3.

Following [19, Section 4.2], a quilted surface with striplike ends determines an elliptic boundary value problem once its boundary components are labeled with Lagrangian submanifolds, seams labeled by Lagrangian correspondences, and striplike ends labeled with generalized intersection points. Given such labeling data, a *pseudoholomorphic quilted disk* is a pair $(r, \underline{u})$ where $r \in \mathcal{R}^{d,1}$ parametrizes a quilted Riemann surface serving as the domain of a pseudoholomorphic quilt $\underline{u}$ with boundary values in the specified Lagrangians, and limits along the striplike ends given by the specified generalized intersections. Such pairs can be viewed as the intersection of a certain section of a Banach bundle with the zero section. Assuming transversality of this intersection, the moduli space of pseudoholomorphic quilted disks is a smooth manifold, which we call $\mathcal{M}_{d,1}(\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_d)$. Here $\underline{x}_0, \ldots, \underline{x}_d$ are the generalized intersection points prescribed for the striplike ends; the Lagrangian boundary conditions are suppressed from the notation. The underlying analysis is largely an extension of the analysis for pseudoholomorphic strips. It boils down to the Fredholm properties of certain linearized operators associated to a pseudoholomorphic quilted surface. When considering pseudoholomorphic quilted disks, the domains are *not* a fixed quilted surface, however their parameter space $\mathcal{R}^{d,1}$ is finite dimensional, and the effect of variations in the domain is to add a compact perturbation to the linearized operator corresponding to a fixed surface. The resulting linearized operators are still Fredholm, so the analytical techniques remain essentially unchanged.

The second part of this thesis is devoted to proving Theorem 6.1.1, which is a gluing theorem for certain "broken" pseudoholomorphic quilted disks; it is analogous to a gluing theorem for pseudoholomorphic "broken" marked disks which is behind the Fukaya category. Since the boundary strata of $\mathcal{R}^{d,1}$ also contain copies of moduli spaces of marked disks (a.k.a. associahedra), which we write as $\mathcal{R}^e$, the gluing statement also references moduli spaces of generalized pseudoholomorphic marked disks, written $\mathcal{M}_e(\underline{x}_0, \ldots, \underline{x}_d)$, which are pairs $(r, \underline{u})$ with $r \in \mathcal{R}^e$ parametrizing a quilted surface which is the domain for a quilted pseudoholomorphic map $\underline{u}$. For the sake of giving the flavor of the gluing statement, we state the theorem precisely here without having properly defined all terms in it. We warn that the meaning of *regular* used to describe a pair $(r, \underline{u})$ is not a topological meaning, but an analytical meaning which has to do with a certain linearized operator being surjective. The superscripts ^{0} and ^{1} for the moduli spaces denote their 0-dimensional and 1-dimensional components respectively, and $\mathcal{I}(\underline{L}, \underline{L}')$ denotes a set of generalized intersection points.

Theorem. *Let $\underline{x}_0 \in \mathcal{I}(\underline{L}_{0,AB}, \underline{L}_{d,AB})$ and for $i = 1, \ldots, d$ let $\underline{x}_i \in \mathcal{I}(\underline{L}_{i-1}, \underline{L}_i)$. Given either:*

1. *a regular pair*

$$\begin{aligned} (r_1, \underline{u}_1) &\in \mathcal{M}_{d-e+1,1}(\underline{x}_0, \underline{x_1} \ldots, \underline{x}_{i-1}, \underline{y}, \underline{x}_{i+e+1}, \ldots, \underline{x}_d)^0 \\ (r_2, \underline{u}_2) &\in \mathcal{M}_e(\underline{y}, \underline{x}_i, \underline{x}_{i+1}, \ldots, \underline{x}_{i+e})^0 \end{aligned}$$

   where $2 \le e \le d$, $1 \le i \le d - e$, and $\underline{y} \in \mathcal{I}(\underline{L}_{i-1}, \underline{L}_{i+e})$;

2. *or a regular $(k+1)$-tuple*

$$\begin{aligned} (r_0, \underline{u}_0) &\in \mathcal{M}_k(\underline{x_0}, \underline{y}_1, \ldots, \underline{y}_k)^0 \\ (r_1, \underline{u}_1) &\in \mathcal{M}_{d_1,1}(\underline{y}_1, \underline{x}_1, \ldots, \underline{x}_{d_1})^0 \\ (r_2, \underline{u}_2) &\in \mathcal{M}_{d_2,1}(\underline{y}_2, \underline{x}_{d_1+1}, \ldots, \underline{x}_{d_1+d_2})^0 \\ &\cdots \\ (r_k, \underline{u}_k) &\in \mathcal{M}_{d_k,1}(\underline{y}_k, \underline{x}_{d_1+\ldots+d_{(k-1)}+1}, \ldots, \underline{x}_{d_1+\ldots+d_{k-1}+d_k})^0 \end{aligned}$$

   where $d_1 + \ldots + d_k = d$, $d_i \ge 1$ for each i, and $\underline{y}_i \in \mathcal{I}(\underline{L}_{d_1+\ldots+d_{(i-1)}}, \underline{L}_{d_1+\ldots+d_i})$ (interpreting d_0 as 0);

3. *or a regular pair*

$$\begin{aligned} (r, \underline{u}) &\in \mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_{i-1}, \underline{y}, \underline{x}_{i+1}, \ldots, \underline{x}_d)^0 \\ \underline{v} &\in \widetilde{\mathcal{M}}_1(\underline{y}, \underline{x}_i)^0 \end{aligned}$$

   where $1 \le i \le d$, and $\underline{y} \in \mathcal{I}(\underline{L}_{i-1}, \underline{L}_i)$;

4. *or a regular pair*

$$\begin{aligned} \underline{v} &\in \widetilde{\mathcal{M}}_1(\underline{x}_0, \underline{y})^0 \\ (r, \underline{u}) &\in \mathcal{M}_{d,1}(\underline{y}, \underline{x}_1, \ldots, \underline{x}_d)^0 \end{aligned}$$

where $\underline{y} \in \mathcal{I}(\underline{L}_{0,AB}, \underline{L}_{d,AB})$,

there is an associated continuous gluing map

$$g : (R_0, \infty) \to \mathcal{M}_{d,1}(\underline{x}_0, \dots, \underline{x}_d)^1$$

defined for some $R_0 >> 0$*, such that* $g(R)$ *Gromov converges to the given pair/tuple as* $R \to \infty$*. Moreover, for sufficiently small* $\epsilon > 0$*, the gluing map surjects onto Gromov neighborhoods* U_ϵ *of the given broken pairs/tuples.*

1.1 Organization of the chapters

The next chapters are organized as follows. Chapter 2 sketches the background material that provides the larger context for this thesis, including Lagrangian Floer homology, the Fukaya category, and quilted Floer theory. It also outlines the construction of an A_∞ functor that motivates this research. Chapter 3 is a self-contained treatment of the associahedra and multiplihedra, from the point of view of moduli spaces of marked disks and their compactifications. The relationship between marked disks, the associahedra, and metric trees is well-established, but we present it in a way which generalizes to the quilted disks that are relevant to the functor construction. Chapter 3 defines quilts, following [19], and covers the construction of certain families of quilts that are used in the construction of the functor. Chapter 4 develops an analytical setting for doing Floer theory over the multiplihedra, which is completely analogous to the setting for Floer theory over the associahedra that is used to construct the Fukaya category. Chapter 5 then covers the gluing theorem, and presents the calculations and estimates that are needed for it.

1.2 What *isn't* covered

There are a number of issues that remain to be dealt with for the proposed A_∞ functor.

1. **Orientations and gradings.** The objects of the complete Fukaya category are really Lagrangian branes. These are Lagrangian submanifolds equipped with some additional information. The additional information allows one to define two

things: first, a way of orienting the moduli spaces used in the constructions, and second, a way of grading the Floer chain groups and defining the degrees of the composition maps. Orientations are important if one wants to be able to use coefficients from a field other than $\mathbb{Z}_2$. So to prove that the proposed A_∞ functor satisfies the A_∞ functor relations for more general coefficient fields, one needs information on orientations for the moduli spaces of pseudoholomorphic quilts used to define it, and how the orientations behave with respect to gluing maps.

2. **Strips of varying width** All strips considered in this thesis have width 1. In particular, all pseudoholomorphic quilted strips that we consider can be "folded" and thought of as ordinary pseudoholomorphic strips in a bigger, product manifold. In particular, results in the literature for ordinary pseudoholomorphic strips can be applied directly to these strips without needing any modification. Eventually, as in [19], one wants to allow varying widths and to study what happens as certain widths approach 0, which should correspond to taking geometric composition of Lagrangian correspondences.

3. **Functors for generalized Lagrangian correspondences, composing functors, and natural transformations** In order to fit in with the symplectic category picture of [19], there should also be an A_∞ functor associated to general sequences of Lagrangian correspondences between a pair of manifolds. This should fit in with composing the A_∞ functors associated to the individual Lagrangian correspondences in the sequence. Studying these requires studying moduli spaces of disks with multiple inner circles. Another issue is to define natural transformations between these A_∞ functors.

Chapter 2

Background

2.1 Outline of chapter

In this chapter we give an abbreviated treatment of Lagrangian Floer theory, which is the wider context for the material in this thesis. In Section 2.3 we sketch the construction of Lagrangian Floer homology, following the work of Floer [2] and Oh [12]. In Section 2.4 we give a simplified sketch of the construction of the Fukaya category, following Seidel's book [14]. In Section 2.5 we introduce the generalized construction of Floer homology using quilted surfaces that was developed by Wehrheim and Woodward [19]. Section 2.6 then describes the motivation behind the material in this thesis, in terms of A_∞ functors between Fukaya categories, together with a summary of the results of the thesis.

2.2 Symplectic preliminaries

Let (M, ω) be a symplectic manifold. This means the following: M is a smooth manifold, of even dimension $2n$, and ω is a 2-form on M with the properties

1. $d\omega = 0$, in which case we say ω is *closed*;

2. ω^n is a volume form, in which case we say that ω is *non-degenerate*.

Since M is a symplectic manifold, its tangent bundle $TM \longrightarrow M$ is a symplectic vector bundle. This is almost the same as being a complex vector bundle, and there is a cohomology class $c_1(TM) \in H^2(M; \mathbb{Z})$ called the *first Chern class* (see, for example, [9, p.74]).

Definition We say that (M,ω) is *monotone* if there is a constant $\tau > 0$ such that $[\omega] = \tau c_1(TM)$.

A *symplectomorphism* between two symplectic manifolds (M_1,ω_1) and (M_2,ω_2) is a diffeomorphism $\phi : M_1 \to M_2$ that preserves the symplectic form, i.e., $\phi^*\omega_2 = \omega_1$. The condition that ω_1 and ω_2 are non-degenerate forces $\dim M_1 = \dim M_2$.

A submanifold $L \subset M$ is called a *Lagrangian submanifold* if $\dim L = \frac{1}{2}\dim M = n$, and $\omega|_L \equiv 0$.

Example Let $\phi : (M,\omega) \to (M',\omega')$ be a symplectomorphism. Consider the product manifold $\mathbf{M} := M \times M'$ equipped with the symplectic form $\omega := -\mathrm{pr}_1^*\omega_1 + \mathrm{pr}_2^*\omega_2$, where pr_i denotes projection onto $M_i, i = 1,2$. Then the graph of ϕ,

$$\mathrm{gr}(\phi) := \{(x,\phi(x))\big|\ x \in M_1\}$$

is a Lagrangian submanifold of $(\mathbf{M},\omega)$, since for a pair $(\eta,\phi_*\eta),(\eta',\phi_*\eta') \in T_{(x,\phi(x))}\mathrm{gr}(\phi)$,

$$\begin{aligned} -\omega_1(\eta,\eta') + \omega_2(\phi_*\eta,\phi_*\eta') &= -\omega_1(\eta,\eta') + \phi^*\omega_2(\eta,\eta') \\ &= 0. \end{aligned}$$

Let Σ be a Riemann surface with boundary, (M,ω) a symplectic manifold and $L \subset M$ a Lagrangian submanifold. A map

$$u : \Sigma \to M, \quad u\big|_{\partial\Sigma} \subset L$$

determines a symplectic vector bundle $E \to \Sigma$ whose fibers are $E_z = T_{u(z)}M$ for $z \in \Sigma$, and a subbundle $F \subset E\big|_{\partial\Sigma}$ over the boundary $\partial\Sigma$ whose fibers are $F_z = T_{u(z)}L$ for $z \in \partial\Sigma$. For such a bundle pair (E,F) there is a well-defined *boundary Maslov index*, $\mu(E,F) \in \mathbb{Z}$ (see [10, Appendix C].). This index is induced from the Maslov index for Lagrangian subspaces in $(\mathbb{R}^{2n},\omega_0)$, where $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$ is the standard symplectic structure. The boundary Maslov index extends to a cohomology class $I \in H^2(M,L;\mathbb{Z})$, determining a homomorphism $I : \pi_2(M,L) \to \mathbb{Z}$. We write Σ_L for the positive generator of the image of the homomorphism I, and call it the *minimal Maslov number* of L.

Definition If (M, ω) is monotone, we say that a Lagrangian submanifold $L \subset M$ is *monotone* if for all $u \in \pi_2(M, L)$,

$$2A(u) = \tau I(u),$$

where $A(u) = \int u^*\omega$ is the symplectic action.

2.3 Lagrangian Floer homology

By the work of Floer [2] and Oh [12], to a pair of Lagrangian submanifolds L_0 and L_1 we can associate a homology $HF(L_0, L_1)$ with $\mathbb{Z}_2$ coefficients, under the following hypotheses:

A1 M is compact and monotone, with monotonicity constant $\tau > 0$.

A2 L_0 and L_1 are monotone, with minimal Maslov numbers at least 3.

A3 The image of either $\pi_1(L_0)$ or $\pi_1(L_1)$ is torsion in $\pi_1(M)$.

To define the homology $HF(L_0, L_1)$ under assumptions **A1, A2** and **A3** we need a few more auxiliary ingredients.

Definition A *Hamiltonian perturbation* for the pair L_0, L_1 is a choice of Hamiltonian $H_t \in C^\infty([0,1] \times M)$ such that if ϕ_t is the flow of the associated Hamiltonian vector field X_{H_t}, the submanifold $\phi_1(L_0)$ intersects L_1 transversally.

For dimension reasons, if $\phi_1(L_0)$ and L_1 intersect transversally, they must intersect at isolated points in M. Since M is compact there can only be finitely many such points. There is a one-to-one correspondence between the set of points $\phi_1(L_0) \cap L_1$ and the set of paths

$$\mathcal{I}(L_0, L_1) := \{x : [0,1] \to M \big| \dot{x} - X_{H_t}(x) = 0, x(0) \in L_0, x(1) \in L_1\}$$

which we call the set of *perturbed intersections of* L_0 *and* L_1 .

We are now able to define the Floer chain complex:

$$CF(L_0, L_1) := \bigoplus_{p \in \mathcal{I}(L_0, L_1)} \mathbb{Z}_2 \langle p \rangle. \tag{2.1}$$

The boundary operator $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$ is defined using *Floer trajectories*, which are types of pseudoholomorphic curves.

Definition An *almost-complex structure* on M is an endomorphism $J : TM \to TM$ such that $J^2 = -\mathrm{Id}$. We say that J is *compatible* with ω if $g_J(\cdot,\cdot) := \omega(\cdot, J\cdot)$ defines a Riemannian metric on M.

We denote the set of almost-complex structures on M by $\mathcal{J}$, and the subset of ω-compatible almost complex structures by $\mathcal{J}_\omega \subset \mathcal{J}$.

Fix a t-dependent, ω-compatible almost complex structure $J_t \in C^\infty([0,1], \mathcal{J}_\omega)$, and write $Z = \mathbb{R} \times i[0,1] \subset \mathbb{C}$ for the infinite strip with variables (s,t). *Floer's equation* for a map $u : Z \to M$ is

$$\left.\begin{aligned} &\partial_s u + J_t(\partial_t u - X_{H_t}) = 0 \\ &u(s,0) \subset L_0, \quad u(s,1) \subset L_1. \end{aligned}\right\} \tag{2.2}$$

Remark In the absence of the Hamiltonian perturbation term, Floer's equation reduces to the *pseudoholomorphic map equation* for the strip Z with boundary in the given Lagrangians,

$$\left.\begin{aligned} &\partial_s u + J_t \partial_t u = 0 \\ &u(s,0) \subset L_0, \quad u(s,1) \subset L_1. \end{aligned}\right\} \tag{2.3}$$

Definition The *energy* of a solution of (2.2) is the quantity

$$E(u) \quad = \quad \int_Z |\partial_s u|^2_{J_t}\, ds\; dt,$$

where $|\partial_s u|^2_{J_t} := \omega(\partial_s u, J_t \partial_s u)$.

If $E(u) < \infty$, a solution $u(s,t)$ converges to limits $x^\pm(t) = \lim_{s\to\pm\infty} u(s,t)$ that satisfy $\partial_t x^\pm(t) - X_{H_t}(x^\pm(t)) = 0$. Therefore, $x^\pm \in \mathcal{I}(L_0, L_1)$.

Let $\widetilde{\mathcal{M}}(x^-, x^+; L_0, L_1; J_t, H_t)$ be the set of finite energy solutions to Floer's equation, with asymptotic limits $x^\pm$ as $s \to \pm\infty$. There is a natural $\mathbb{R}$ action on the maps in $\widetilde{\mathcal{M}}(x^-, x^+; L_0, L_1; H_t, J_t)$ by translation in the s-variable, since the equations are independent of s.

Definition The *moduli space* of finite energy, J_t-holomorphic strips, with boundaries in L_0, L_1 and asymptotic limits $x^{\pm} \in \mathcal{I}(L_0, L_1)$ is the quotient

$$\mathcal{M}(x^-, x^+; L_0, L_1; H_t, J_t) := \widetilde{\mathcal{M}}(x^-, x^+; L_0, L_1; H_t, J_t)/\mathbb{R}. \tag{2.4}$$

Elements of this moduli space are called *trajectories.*

Theorem 2.3.1 (Floer, Oh)**.** *Under the assumptions* **A1, A2** *and* **A3***, there is a Baire second category subset* $\mathcal{J}_{reg} \subset C^{\infty}([0,1], \mathcal{J}_{\omega})$ *such that for every* $J_t \in \mathcal{J}_{reg}$*,*

1. *for every pair* $x^-, x^+ \in \mathcal{I}(L_0, L_1)$*, the moduli space* $\mathcal{M}(x^-, x^+; H_t, J_t)$ *is a smooth finite dimensional manifold,*
2. *the zero dimensional component* $\mathcal{M}(x^-, x^+; H_t, J_t)^0 \subset \mathcal{M}(x^-, x^+; H_t, J_t)$ *is compact,*
3. *the one-dimensional component* $\mathcal{M}(x^-, x^+; H_t, J_t)^1$ *has a compactification as manifold with boundary such that*

$$\partial\mathcal{M}(x^-, x^+; H_t, J_t)^1 \cong \bigcup_{y \in I(L_0, L_1)} \mathcal{M}(x^-, y; H_t, J_t)^0 \times \mathcal{M}(y, x^+; H_t, J_t)^0.$$

Remark The assumptions **A1, A2** and **A3** really come into play in proving the second and third statements of the theorem. Statements about compactness are based on Gromov compactness, which says that every sequence of trajectories with uniformly bounded energy contains a subsequence that converges to a tuple of pseudoholomorphic curves comprising "broken" trajectories and a finite number of pseudoholomorphic spheres and pseudoholomorphic disks which have "bubbled off". The monotonicity assumptions force the energy to be directly related to the index, which in turn is directly related to the dimensions of the moduli spaces. Any pseudoholomorphic spheres and disks must capture some minimal quantum of energy, and the assumption on the minimal Maslov number gives the minimal possible effect that the captured energy can have on lowering the index. If one is only considering the 0-dimensional and 1-dimensional components of the moduli spaces, the dimension cannot be lowered beyond zero, so the possibility of getting sphere and disk bubbles in the limits can be ruled out.

Definition For a pair of Lagrangian submanifolds L_0, L_1, a pair (H_t, J_t) such that $J_t \in \mathcal{J}_{reg}$ is called a *Floer datum* for L_0 and L_1.

Based on Theorem 2.3.1, define the *Floer differential* :

$$\begin{array}{rcl} \partial : CF(L_0, L_1) & \longrightarrow & CF(L_0, L_1) \\ \langle x \rangle & \mapsto & \displaystyle\sum_{y \in I(L_0, L_1)} \#\mathcal{M}(x, y)^0 \langle y \rangle \end{array}$$

where $\#\mathcal{M}(x, y)^0$ counts the number (mod 2) of Floer trajectories in the zero-dimensional components. To show that $\partial \circ \partial = 0$: by definition

$$\partial(\partial\langle x \rangle) = \sum_{y \in \mathcal{I}(L_0, L_1)} N_y \langle y \rangle,$$

where

$$N_y = \sum_{z \in \mathcal{I}(L_0, L_1)} \#\mathcal{M}(x, z)^0 \times \mathcal{M}(z, y)^0 \quad \mod 2.$$

It follows from part (c) of the theorem that this is precisely the count of the boundary points of the compactification of $\mathcal{M}(x, y)^1$, which as a 1-manifold with corners must have an even number of boundary points, so the count (mod 2) is 0.

Therefore the homology of $(CF(L_0, L_1), \partial)$ is well-defined,

$$HF(L_0, L_1) := \ker \partial / \mathrm{im} \partial. \tag{2.5}$$

Although the Floer chain group and boundary operator depend on the choices of (J_t, H_t), the homology obtained in the end is independent of the choices:

Theorem 2.3.2 (Floer, Oh)**.** *For any two choices of Floer data (H_t, J_t) and (H'_t, J'_t) for L_0 and L_1, there is a canonical isomorphism of Floer homology groups,*

$$\Phi_{H,J,H',J'} : HF(L_0, L_1; H_t, J_t) \xrightarrow{\cong} HF(L_0, L_1; H'_t, J'_t).$$

2.4 A_∞ categories and the Fukaya category

The main reference for this section is Seidel's book [14]. Let $\mathbb{K}$ be a field, from which all coefficients will be taken. In our applications we will generally use $\mathbb{Z}_2$.

Definition A *non-unital A_∞ category* $\mathcal{A}$ consists of a set of objects $Ob\ \mathcal{A}$, a graded vector space $\mathrm{Hom}_{\mathcal{A}}(X_0, X_1)$ for any pair of objects, and composition maps for $d \geq 1$,

$$\mu^d_{\mathcal{A}} : \mathrm{Hom}_{\mathcal{A}}(X_{d-1}, X_d) \otimes \ldots \otimes \mathrm{Hom}_{\mathcal{A}}(X_0, X_1) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X_0, X_d)[2-d]. \tag{2.6}$$

These composition maps should satisfy the A_∞ *associativity relations*,

$$\sum_{e,i} (-1)^* \mu^{d-e+1}(a_d, \ldots, a_{i+e+1}, \mu^e(a_{i+e}, \ldots, a_{i+1}), a_i, \ldots, a_1) = 0, \tag{2.7}$$

where $* = |a_1| + \ldots + |a_i| - i$. The summation is over all subsets that are of the form $[i+1, \ldots, i+e] \subset [1, \ldots, n]$ of size $1 \leq e \leq n$.

Remark It is helpful to think of the terms in (2.7) as being indexed by rooted trees with d leaves, and two vertices, as in Figure 2.1. The upper vertex represents applying μ^e to the inputs indexed $i+1, \ldots, i+e$, and the lower vertex represents applying μ^{d-e+1} to the output of μ^e and the remaining $d-e$ inputs.

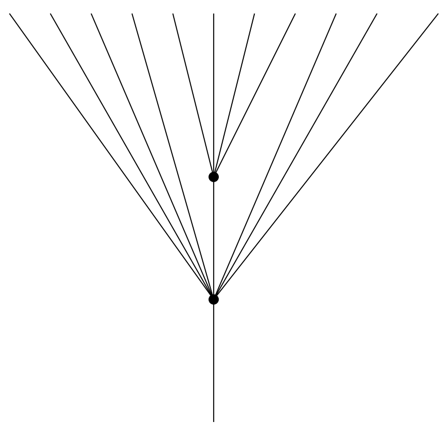

Figure 2.1: Tree indexing a term in the A_∞ associativity relations.

The Fukaya category is a construction in symplectic topology that produces non-unital A_∞ categories. It concerns underlying structure at the chain level of the homology groups defined in the previous section. In the following sketch of the construction of the Fukaya category, the issues of orientation and grading will not be addressed. The A_∞ structure that we describe is therefore that of an *ungraded* non-unital A_∞ category with coefficients in $\mathbb{Z}_2$:

Definition An *ungraded, non-unital*, A_∞ category $\mathcal{A}$ with coefficients in $\mathbb{Z}_2$ consists of a set of objects, *Ob* $\mathcal{A}$, a vector space $\mathrm{Hom}_{\mathcal{A}}(X_0, X_1)$ for any pair of objects, and composition maps for $d \geq 1$,

$$\mu^d_{\mathcal{A}} : \mathrm{Hom}_{\mathcal{A}}(X_{d-1}, X_d) \otimes \ldots \otimes \mathrm{Hom}_{\mathcal{A}}(X_0, X_1) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X_0, X_d). \tag{2.8}$$

These composition maps should satisfy the A_∞ *associativity relations*,

$$\sum_{e,i} \mu^{d-e+1}(a_d, \ldots, a_{i+e+1}, \mu^e(a_{i+e}, \ldots, a_{i+1}), a_i, \ldots, a_1) = 0, \tag{2.9}$$

where the summation is over all subsets $[i+1, \ldots, i+e] \subset [1, \ldots, n]$ of size $1 \leq e \leq n$.

Let (M, ω) be a compact, monotone symplectic manifold. Following [14], the *preliminary* version of the Fukaya category, $\mathrm{Fuk}(M)^{pr}$, is defined for a certain class of Lagrangian submanifolds of M. Recalling the assumptions **A1, A2** and **A3** of the previous section, we update **A3** to

A3* The image of $\pi_1(L)$ is torsion in $\pi_1(M)$.

This ensures that in a set of Lagrangian submanifolds satisfying **A1, A2** and **A3***, any pair of Lagrangian submanifolds in that set (we do not necessarily assume them to be distinct) will automatically satisfy **A1, A2** and **A3**.

Definition The preliminary Fukaya category $\mathrm{Fuk}(M)^{pr}$ is an ungraded A_∞ category whose objects are the Lagrangian submanifolds of M satisfying **A2, A3***, and for any pair of objects L, L', $\mathrm{Hom}(L, L') := CF(L, L')$. Higher composition maps μ^n for $n \geq 1$ are defined by

$$\begin{aligned} \mu^n : CF(L_{n-1}, L_n) \otimes \ldots \otimes CF(L_0, L_1) &\longrightarrow CF(L_0, L_n), \\ \langle p_n \rangle \otimes \ldots \otimes \langle p_1 \rangle &\mapsto \sum_{q \in \mathcal{I}(L_0, L_n)} \#\mathcal{M}(q, p_1, \ldots, p_n)^0 \ \langle q \rangle \end{aligned}$$

where for each $q \in \mathcal{I}(L_0, L_n)$, the number $\#\mathcal{M}(q, p_1, \ldots, p_n)^0$ is the count (mod 2) of elements in the zero-dimensional component of a moduli space of *inhomogeneous pseudoholomorphic* $n+1$*-gons*, $\mathcal{M}(q, p_1, \ldots, p_n)$.

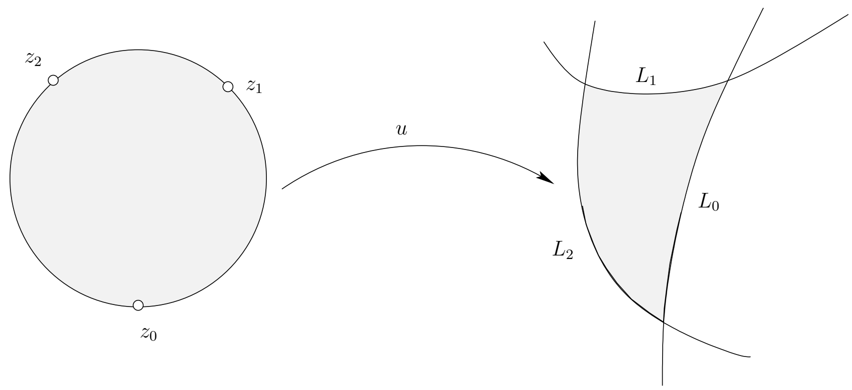

Figure 2.2: A holomorphic triangle used in the definition of μ^2.

When $n = 1$, the pseudoholomorphic 2-gons are the trajectories defined in (2.4), whose domain is always the strip $Z = \mathbb{R} \times i[0,1]$. For $n \geq 2$, the domains of the pseudoholomorphic $n+1$-gons are themselves part of the data. The domains are parametrized by a moduli space of of $n+1$-pointed disks, $\mathcal{R}^n$, which is a smooth manifold of dimension $d-2$ (see Chapter 2). The space $\mathcal{R}^n$ has a Deligne-Mumford type compactification by nodal pointed disks, $\overline{\mathcal{R}}^n$, which is homeomorphic to the n-th Stasheff polytope, or *associahedron* . The one dimensional component $\mathcal{M}_n(q, p_1, \dots, p_n)^1$ has a Gromov-type compactification by *broken pairs*, which combined with a gluing argument shows that the compactification $\overline{\mathcal{M}_n(q, p_1, \dots, p_n)}^1$ is a one-manifold with corners, with boundary

$$\begin{aligned} \partial \mathcal{M}_n(q, p_1, \dots, p_n)^1 \quad &\cong \quad \bigcup \mathcal{M}_{d-e+1}(q, p_1, \dots, p_i, y, p_{i+e+1}, \dots, p_n)^0 \\ &\quad \times \mathcal{M}_e(y, p_{i+1}, \dots, p_{i+e})^0. \end{aligned} \tag{2.10}$$

The union is over all i, e and all $y \in \mathcal{I}(L_i, L_{i+e})$. To see that the compositions μ^n satisfy the A_∞ associativity relations: the sum in (2.9) can be expressed as

$$\sum_{q \in \mathcal{I}(L_0, L_n)} N_q \langle q \rangle$$

where each coefficient N_q is

$$\sum (\#\mathcal{M}_{d-e+1}(q, p_1, \dots, p_i, y, p_{i+e+1}, \dots, p_n)^0)(\#\mathcal{M}_e(y, p_{i+1}, \dots, p_{i+e})^0) \tag{2.11}$$

where the sum is over i, e and $y \in \mathcal{I}(L_i, L_{i+e})$. By (2.10) this is the total number of boundary points of the compact 1-manifold with corners $\overline{\mathcal{M}_n(q, p_1, \dots, p_n)}^1$, which is an even number and hence (in $\mathbb{Z}_2$) 0.

2.5 Quilted Lagrangian Floer homology

In [19] Wehrheim and Woodward developed a generalized version of Lagrangian Floer theory, that allowed them to prove functoriality of Lagrangian Floer homology with respect to Lagrangian correspondences. A Lagrangian correspondence is a notion of morphism between symplectic manifolds, which includes symplectomorphisms but isn't as restrictive.

Definition A *Lagrangian correspondence* between (M_1, ω_1) and (M_2, ω_2) is a Lagrangian submanifold L of the product manifold $(M_1 \times M_2, \widetilde{\omega})$, where $\widetilde{\omega} = -\mathrm{pr}_1^*\omega_1 + \mathrm{pr}_2^*\omega_2$. We will often write $M_1^- \times M_2$ to denote this product symplectic manifold, where the negative sign in M_1^- represents the negative sign in $-\mathrm{pr}_1^*\omega_1$.

Example Let $\phi : (M, \omega_M) \to (N, \omega_N)$ be a symplectomorphism, so $\phi^*\omega_N = \omega_M$. Then the graph of ϕ,

$$\mathrm{gr}\phi := \{(x, \phi(x)) \Big| \; x \in M\}$$

is a Lagrangian correspondence between M and N, since it is a Lagrangian submanifold of $M \times N$ with respect to the symplectic form $-\mathrm{pr}_M\omega_M + \mathrm{pr}_N\omega_N$.

Definition A *generalized Lagrangian submanifold* $\underline{L}$ of a symplectic manifold M is a sequence of Lagrangian correspondences starting from a point and ending in M,

$$\underline{L} := \{pt \xrightarrow{L_{01}} M_1 \xrightarrow{L_{12}} M_2 \xrightarrow{L_{23}} \ldots \xrightarrow{L_{n-1,n}} M\}.$$

Example A Lagrangian submanifold of M is also a generalized Lagrangian submanifold,

$$pt \xrightarrow{L} M.$$

One would like to extend Floer homology to pairs of generalized Lagrangian submanifolds of M,

$$\begin{aligned} \underline{L} &= pt \xrightarrow{L_{-r}} M_{-r} \xrightarrow{L_{-r,-r+1}} \ldots M_{-1} \xrightarrow{L_{-1,0}} M, \\ \underline{K} &= pt \xrightarrow{L_s} M_s \xrightarrow{L_{s,s-1}} \ldots M_1 \xrightarrow{L_{1,0}} M. \end{aligned}$$

Two such sequences can be concatenated at M to get a cyclic sequence from a point to a point,

$$pt \xrightarrow{L_{-r}} M_{-r} \dots M_{-1} \xrightarrow{L_{-1,0}} M \xrightarrow{L_{0,1}} M_1 \dots M_s \xrightarrow{L_s} pt. \tag{2.12}$$

Consider the product symplectic manifold

$$\mathbf{M} := M_{-r}^{\sigma(r)} \times \dots \times M_{-1}^{-} \times M \times M_1^{-} \times \dots \times M_s^{\sigma(s)} \tag{2.13}$$

where $M_k^{\sigma(k)} = M_k$ if k is even, M_k^- if k is odd. The symplectic form is

$$\omega = (-1)^r \mathrm{pr}_{-r}^* \omega_r + \dots - \mathrm{pr}_{-1}^* \omega_1 + \mathrm{pr}_0^* \omega_0 - \mathrm{pr}_1^* \omega_1 + \dots + (-1)^s \mathrm{pr}_s^* \omega_s$$

The concatenated sequence (2.12) determines a pair of Lagrangian submanifolds of $\mathbf{M}$, by taking products over two alternating subsequences,

$$\mathbf{L_0} \;:=\; L_{-r} \times L_{-r+1,-r+2} \times L_{-r+3,-r+4} \times \dots \tag{2.14}$$

$$\mathbf{L_1} \;:=\; L_{-r,-r+1} \times L_{-r+2,-r+3} \times \dots \tag{2.15}$$

Example If $r = 2$ and $s = 1$, suppose we have

$$\begin{aligned} \underline{L} &= pt \xrightarrow{L_{-2}} M_{-2} \xrightarrow{L_{-2,-1}} M_{-1} \xrightarrow{L_{-1,0}} M, \\ \underline{K} &= pt \xrightarrow{L_1} M_1 \xrightarrow{L_{1,0}} M. \end{aligned}$$

Then $\mathbf{M} = M_{-2} \times M_{-1}^- \times M \times M_1^-$, and

$$\begin{aligned} \mathbf{L_0} &:= L_{-2} \times L_{-1,0} \times L_1 \\ \mathbf{L_1} &:= L_{-2,-1} \times L_{0,1}. \end{aligned}$$

Provided that $\mathbf{M}, \mathbf{L_0}$ and $\mathbf{L_1}$ satisfy **A1, A2** and **A3**, we can now appeal to the "classical" construction of Section 2.3, and define

$$HF(\underline{L}, \underline{K}) := HF(\mathbf{L}, \mathbf{L}'). \tag{2.16}$$

However, an alternative approach developed in [19] has the advantages of keeping the product structure clear.

Definition A Hamiltonian perturbation $\mathbf{H}_t \in C^\infty([0,1]\times\mathbf{M})$ for the Lagrangian submanifolds $\mathbf{L},\mathbf{L}'$ in $\mathbf{M}$ is said to be of *split type* if

$$\mathbf{H}_t = ((-1)^r H^{-r},\dots,-H^{-1},H^0,-H^1,\dots,(-1)^s H^s),$$

where $H^j \in C^\infty([0,1],M_j)$ for each j.

The perturbed intersections are the set

$$\mathcal{I}(\mathbf{L_0},\mathbf{L_1}) = \{\mathbf{y}:[0,1]\to\mathbf{M}|\dot{\mathbf{y}} = \mathbf{X_H}(\mathbf{y}),\ \mathbf{y}(0)\in\mathbf{L_0},\mathbf{y}(1)\in\mathbf{L_1}\}.$$

Since the perturbation is of split type, these are tuples $(y_{-r},\dots,y_{-1},y_0,y_1,\dots,y_s)$ where each $y_j:[0,1]\to M_j$ is a solution of $\dot{y_j} = X_{H_j}(y_j)$. (The Hamiltonian vector field X_{H_j} defined on M_j in terms of ω_j, is the same as the Hamiltonian vector field $X^-_{-H_j}$ on M_j^-, whose symplectic form is $-\omega_j$.) The condition that $\mathbf{y}(0)\in\mathbf{L_0},\mathbf{y}(1)\in\mathbf{L_1}$ says that

$$\begin{aligned}(y_{-r}(0),\dots,y_{-1}(0),y_0(0),y_1(0),\dots,y_s(0)) &\in L_{-r}\times L_{-r+1,-r+2}\times\dots\\ (y_{-r}(1),\dots,y_{-1}(1),y_0(1),y_1(1),\dots,y_s(1)) &\in L_{-r,-r+1}\times L_{-r+2,-r+3}\times\dots\end{aligned} \tag{2.17}$$

By [19, Proposition 3.4.3], Hamiltonian perturbations of split type are enough to achieve transverse intersection of the time one flow of $\mathbf{L_0}$ with $\mathbf{L_1}$.

Now we identify a tuple $\mathbf{y} = (y_{-r},\dots,y_{-1},y_0,y_1,\dots,y_s)\in\mathcal{I}(\mathbf{L_0},\mathbf{L_1})$ with a tuple

$$\underline{x} = (x_{-r-1} = pt, x_{-r},\dots,x_{-1},x_0,x_1,\dots,x_s,x_{s+1} = pt)$$

where for $k=-r,\dots,-1,0,1,\dots,s$,

$$x_k(t) = \begin{cases} y_k(1-t), & k \text{ odd}\\ y_k(t), & k \text{ even}\end{cases}. \tag{2.18}$$

Then the boundary condition (3.1) can be reformulated as

$$(x_i(1),x_{i+1}(0))\in L_{i,i+1} \text{ for all } i=-r-1,\dots,-1,0,1,\dots,s. \tag{2.19}$$

Definition The set of *generalized intersections* of $\underline{L}$ and $\underline{K}$ is

$$\mathcal{I}(\underline{L},\underline{K}) = \{\underline{x} = (pt,x_{-r},\dots,x_{-1},x_0,x_1,\dots,x_s,pt)\big|(2.18),(2.19)\}. \tag{2.20}$$

The generalized Floer chain group, using $\mathbb{Z}_2$ coefficients, is

$$CF(\underline{L}, \underline{K}) := \bigoplus_{\underline{x} \in \mathcal{I}(\underline{L}, \underline{K})} \mathbb{Z}_2 \langle \underline{x} \rangle.$$

To define the boundary operator, we set up a definition of generalized trajectories. We say that an almost complex structure $\mathbf{J}$ on $\mathbf{M}$ is of split type if

$$\mathbf{J} = ((-1)^r J_{-r}, \ldots, J_{-1}, J_0, J_1, \ldots, (-1)^s J_s)$$

where each $J_k \in C\infty([0,1], \mathcal{J}_{\omega_k})$ is a t-dependent almost complex structure on M_k, compatible with ω_k.

Definition A *quilted Floer trajectory* between $\underline{x}^+$, $\underline{x}^- \in \mathcal{I}(\underline{L}, \underline{K})$ is a tuple

$$\underline{u} = (u_{-r}, \ldots, u_{-1}, u_0, u_1, \ldots, u_s)$$

of maps $u_i : \mathbb{R} \times [0,1] \to M_i$ such that

1. $\lim_{s \to \pm\infty} (u_i(s,t), u_{i+1}(s,t)) = (x_i^{\pm}, x_{i+1}^{\pm})$
2. each u_i is a pseudoholomorphic map into M_i,
3. $(u_i(s,1), u_{i+1}(s,0)) \in L_{i,i+1}$ for all $s \in \mathbb{R}$.

The energy of a quilted trajectory $\underline{u}$ is a sum of the energies on individual strips,

$$\begin{aligned} E(\underline{u}) &= \sum_i \int_Z |\partial_s u_i|^2_{J_i} \, ds \, dt \\ &= \sum_i E(u_i). \end{aligned}$$

Let $\widetilde{\mathcal{M}}(\underline{x}^+, \underline{x}^-)$ denote the moduli space of quilted trajectories between $\underline{x}^+, \underline{x}^-$, with finite energy. There is an $\mathbb{R}$ action by translation in the s direction, since the equations are invariant under translation and the seam conditions transfer the action from strip to strip. Then

$$\mathcal{M}(\underline{x}^+, \underline{x}^-) := \widetilde{\mathcal{M}}(\underline{x}^+, \underline{x}^-)/\mathbb{R}$$

is the moduli space of quilted trajectories.

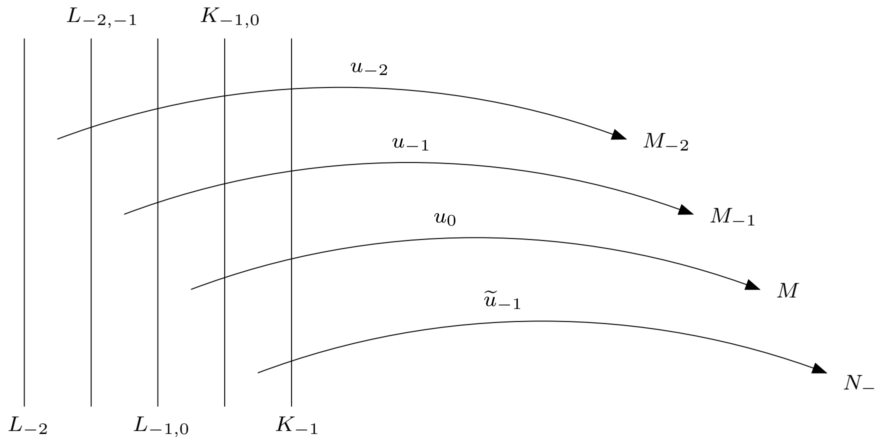

Figure 2.3: Domain of a quilted trajectory.

The construction of the Fukaya category also extends to generalized Lagrangian submanifolds. Define an ungraded A_∞ category $\mathrm{Fuk}^{\#} M^{pr}$, the *generalized preliminary Fukaya category*, as follows. The objects of $\mathrm{Fuk}^{\#} M^{pr}$ are generalized Lagrangian submanifolds of M, and $\mathrm{Hom}(\underline{L}, \underline{K}) = CF(\underline{L}, \underline{K})$. For $n \geq 2$ the higher compositions are defined with pseudoholomorphic quilted n-gons, $\mathcal{M}(\underline{x}_n, \ldots, \underline{x}_1, \underline{y})$. The domains of quilted n-gons are quilted surfaces corresponding to $n+1$ pointed disks, with additional strips attached to the boundary components. These quilted domains and their construction are described in more detail in Chapter 3. The important point is that these domains are still parametrized by the same moduli space as the ordinary Fukaya category; the quilted domains differ from the unquilted domains by some strips attached at the boundary, and when the widths of the attached strips are fixed the moduli space parametrizing the domains is unchanged. In particular, in generic conditions the moduli spaces are smooth finite dimensional manifolds, their zero dimensional components are compact, and their one dimensional components can be compactified, with a gluing argument showing that the composition maps

$$\begin{aligned} \mu^n : CF(\underline{L}_{n-1}, \underline{L}_n) \times \ldots \times CF(\underline{L}_0, \underline{L}_1) &\rightarrow CF(\underline{L}_0, \underline{L}_n) \\ (\langle \underline{x}_n \rangle, \ldots, \langle \underline{x}_1 \rangle) &\mapsto \sum_{\underline{y} \in \mathcal{I}(\underline{L}_0, \underline{L}_n)} \#\mathcal{M}(\underline{x}_n, \ldots, \underline{x}_1, \underline{y})_0 \langle \underline{y} \rangle \end{aligned}$$

satisfy the A_∞ associativity relations (2.7).

2.6 Motivation: A_∞ functors for Lagrangian correspondences

Definition A *non-unital* A_∞ *functor* between non-unital A_∞ categories $\mathcal{A}$ and $\mathcal{B}$ consists of a map $\Phi : Ob\mathcal{A} \to Ob\mathcal{B}$ on objects, together with a sequence of maps $\Phi^d, d \geq 1$,

$$\Phi^d : \mathrm{Hom}_{\mathcal{A}}(X_0, X_1) \otimes \ldots \otimes \mathrm{Hom}_{\mathcal{A}}(X_{d-1}, X_d) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(\Phi(X_0), \Phi(X_d))[1-d],$$

where the Φ^d fit together with the μ^j_A, μ^k_B to satisfy the A_∞ *functor relations*:

$$\sum_{i,j}(-1)^*\Phi^e(a_d, \ldots, a_{i+j+1}, \mu^j_A(a_{j+i}, \ldots, a_{i+1}), a_i, \ldots, a_1) = \tag{2.21}$$
$$\sum_{r, i_1+\ldots i_r = d} \mu^r_B(\Phi^{i_r}(a_d, \ldots, a_{d-i_r}), \ldots, \Phi^{i_1}(a_{i_1}, \ldots, a_1)),$$

where $* = |a_1| + \ldots + |a_i| - i$. An *ungraded* non-unital A_∞ functor between ungraded non-unital A_∞ categories is the same definition but with all references to gradings omitted.

Remark It is helpful to think of the terms in the sums in (2.21) as being indexed by certain types of *bicolored* rooted trees. Bicolored trees have a distinguished subset of *colored vertices* which separate the tree into two levels. In Figure 2.4 the colored vertices are those with the larger circles, and represent applying the functor to those inputs. The bicolored tree on the left represents a term on the left hand summation of (2.21) – the vertex above the colored vertex represents applying the composition map $\mu^j_{\mathcal{A}}$ in $\mathcal{A}$. The bicolored tree on the right indexes a term in the sum on the right hand side of (2.21) – the vertex below the colored vertices represents applying the composition map $\mu^r_{\mathcal{B}}$ in $\mathcal{B}$.

The motivation behind this thesis is the goal of associating to a Lagrangian correspondence, $L_{AB} \subset M_A^- \times M_B$, an A_∞ functor

$$\Phi_{AB} : \mathrm{Fuk}^\#(M_A) \to \mathrm{Fuk}^\#(M_B).$$

At the level of objects, Φ_{AB} concatenates L_{AB} to generalized Lagrangian submanifolds of M_A to give generalized Lagrangian submanifolds of M_B, i.e.,

$$(* \xrightarrow{L_{-r}} M_{-r} \xrightarrow{L_{-r,-r+1}} \ldots \xrightarrow{L_{-1,0}} M_A) \quad \mapsto \quad (* \xrightarrow{L_{-r}} M_{-r} \xrightarrow{L_{-r,-r+1}} \ldots \xrightarrow{L_{-1,0}} M_A \xrightarrow{L_{AB}} M_B).$$

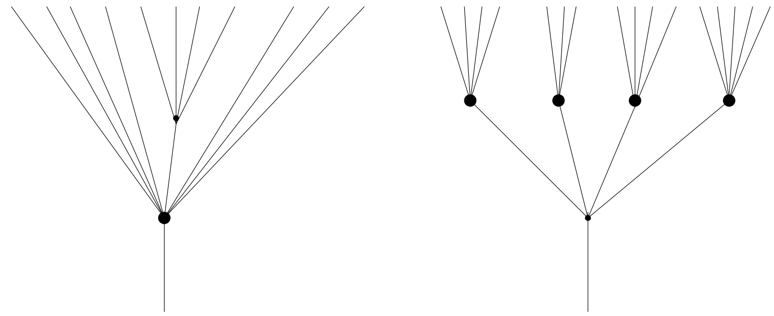

Figure 2.4: The two types of bicolored trees indexing the terms in the A_∞ functor relations.

We will write $\underline{L}_{AB}$ for the generalized Lagrangian submanifold of M_B that is obtained from $\underline{L}$ by concatenating L_{AB}.

For $d \geq 1$, the higher compositions are to be defined by

$$\begin{aligned}\Phi^d_{L_{AB}} : CF(\underline{L}_{d-1}, \underline{L}_d) \times \ldots \times CF(\underline{L}_0, \underline{L}_1) &\to CF(\Phi_{L_{AB}}(\underline{L}_0), \Phi_{L_{AB}}(\underline{L}_d)) \\ (\underline{x}_d, \ldots, \underline{x}_1) &\mapsto \sum_{\underline{y} \in \mathcal{I}(\underline{L}_{0,AB}, \underline{L}_{d,AB})} \#\mathcal{M}_{d,1}(\underline{x}_d, \ldots, \underline{x}_1, \underline{y})^0 \ \langle \underline{y} \rangle\end{aligned}$$

where $\underline{x}_i \in \mathcal{I}(\underline{L}_{i-1}, \underline{L}_i)$ for $i = 1, \ldots, d$, $\underline{y} \in \mathcal{I}(\underline{L}_{0,AB}, \underline{L}_{d,AB})$, and $\mathcal{M}_{d,1}(\underline{x}_d, \ldots, \underline{x}_1, \underline{y})^0$ is the zero-dimensional component of a moduli space of pseudoholomorhic marked quilted disks, converging at the marked points to generalized intersections $\underline{x}_d, \ldots, \underline{x}_1, \underline{y}$ (see Figure 2.5).

The quilted surfaces used in the construction of Φ^d are parametrized by a moduli space of $(d+1)$-marked disks with an interior circle, which we denote by $\mathcal{R}^{d,1}$. We will show in Chapter 3 that $\mathcal{R}^{d,1}$ has a compactification, $\overline{\mathcal{R}}^{d,1}$, by semistable nodal quilted disks with $(d+1)$ markings. The main result of Chapter 3 is that $\overline{\mathcal{R}}^{d,1}$ is homeomorphic to a bounded convex polytope of dimension $d-1$, the d-th *multiplihedron*. The multiplihedra are a family of Stasheff polytopes which were originally invented as CW complexes parametrizing A_∞ maps between A_∞ spaces. Chapter 4 constructs families of quilted surfaces with striplike ends that are parametrized by associahedra and multiplihedra.

Chapter 5 then sets up the analytic framework for doing Floer theory with surfaces parametrized by the associahedra and the multiplihedra. Let $\underline{L}_0, \ldots, \underline{L}_d$ be generalized

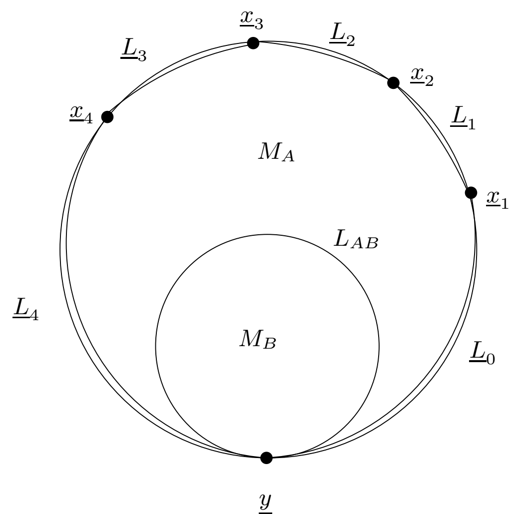

Figure 2.5: Marked quilted disks behind the moduli spaces $\mathcal{M}_{4,1}(\underline{x}_4, \underline{x}_3, \underline{x}_2, \underline{x}_1, \underline{y})$ which are used to define Φ^4_{AB}.

Lagrangian submanifolds of M_A. Given a Lagrangian correspondence L_{AB} between M_A and M_B, write $\underline{L}_{0,AB}$ for the generalized Lagrangian submanifold of M_B that is obtained from $\underline{L}_0$ by concatenating L_{AB}; similarly write $\underline{L}_{d,AB}$ for the concatenation of L_{AB} to $\underline{L}_d$. A $(d+1)$-tuple of generalized intersections

$$(\underline{x}_d, \ldots, \underline{x}_0) \in \mathcal{I}(\underline{L}_{d-1}, \underline{L}_d) \times \ldots \times \mathcal{I}(\underline{L}_0, \underline{L}_d)$$

determines a moduli space $\mathcal{M}_d(\underline{x}_d, \ldots, \underline{x}_0)$, while a $(d+1)$-tuple

$$(\underline{x}_d, \ldots, \underline{x}_1, \underline{y}) \in \mathcal{I}(\underline{L}_{d-1}, \underline{L}_d) \times \ldots, \times \mathcal{I}(\underline{L}_0, \underline{L}_1) \times \mathcal{I}(\underline{L}_{0,AB}, \underline{L}_{d,AB})$$

determines a moduli space $\mathcal{M}_{d,1}(\underline{x}_d, \ldots, \underline{x}_1, \underline{y})$. The main result of Chapter 6 is the gluing theorem in Theorem 6.1.1, which is needed to prove that the proposed Φ_{AB} is an A_∞ functor, satisfying the relations (2.21).

Chapter 3

Moduli of disks and Stasheff polytopes

3.1 Outline of chapter

The material in this chapter has already appeared in the preprint [8], and covers two moduli spaces and their compactifications: a moduli space of marked disks, which is behind the construction of the Fukaya category, and a moduli space of marked quilted disks, which is behind the construction of a proposed A_∞ functor. From the point of view of Floer theory, the purpose of this chapter is to establish that these compactified moduli spaces are homeomorphic to compact polytopes, and to get explicit descriptions of charts near the boundaries of these polytopes.

In the first half of this chapter, we prove that the compactified moduli spaces of marked disks are realizations of the Stasheff associahedra. The proof that we give is a slight modification of the proof in [5], which exploits an equivalence with a moduli space of metric trees. We also use the theory of toric varieties and moment maps to draw a direct connection between these moduli spaces and the convex hull realizations of the associahedra of Loday in [7].

The methods of the first half of the chapter generalize to the moduli space of marked *quilted* disks, which is the content of the second half of the chapter. We prove that the compactified moduli spaces of marked quilted disks are realizations of the Stasheff multiplihedra, exploiting an equivalence with a moduli space of *bicolored* metric trees. It was an open question until quite recently whether the multiplihedra could be realized as polytopes. This was answered by Forcey in [3], who produced a convex hull realization of the multiplihedra using a modification of the algorithm in [7]. We draw a direct connection between the moduli spaces of quilted disks and the convex hull realizations of [3] using the theory of toric varieties and moment maps.

3.2 The associahedron.

Let $d \geq 2$ be an integer. The d-th associahedron K_d is a CW-complex of dimension $d-2$ whose vertices correspond to the possible ways of parenthesizing d variables $x_1, \ldots, x_d$.

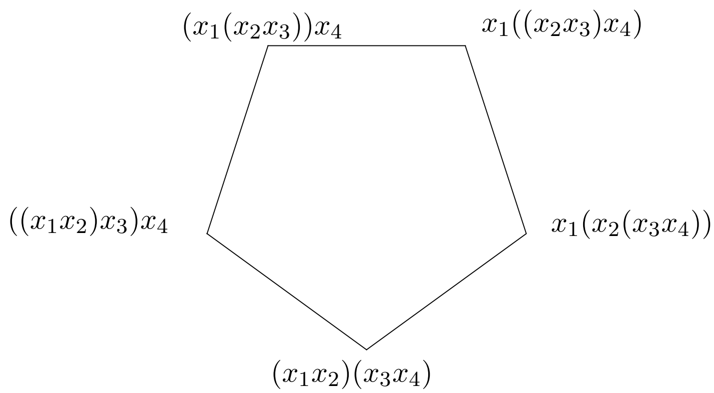

Figure 3.1: Vertices of K_4

Each facet of K_d is the image of an embedding

$$\phi_{i,e} : K_i \times K_e \to K_d, \quad i + e = d + 1 \tag{3.1}$$

corresponding to the expression $x_1 \ldots x_{i-1}(x_i \ldots x_{i+e})x_{i+e+1} \ldots x_d$.

The associahedra are defined by induction as follows. Let K_3 be the closed unit interval. Let $d > 3$ and suppose that we have constructed the associahedra K_i for $i \leq d-1$, together with the inclusions $K_i \times K_e \mapsto K_d$ corresponding to the facets of K_d. Stasheff [17] defines

$$L_d = \bigcup (K_i \times K_e)/\sim$$

where the union is over the facets of K_d, and the equivalence relation $\sim$ is defined by identifying the components in the image of the map

$$K_{i_1} \times K_{i_2} \times K_{i_3} \to (K_{i_1+i_2} \times K_{i_3}) \times (K_{i_1} \times K_{i_2+i_3}).$$

Define K_d to be the cone on L_d. The faces of K_d also correspond to rooted trees with $d+1$-branches and at least three edges meeting each vertex.

Alternatively the vertices correspond to triangulations of a regular $d+1$-gon. The edges of K_d correspond to changes of one bracketing, that is, changes of the tree by the move shown in Figure (3.3). The number of edges meeting any vertex is the number

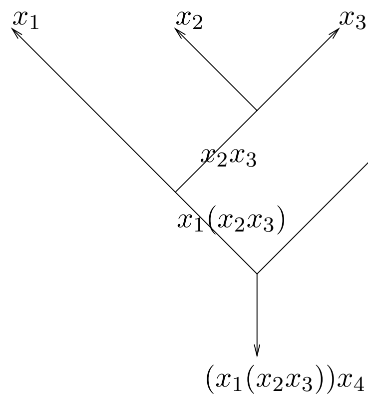

Figure 3.2: Tree corresponding to $(x_1(x_2x_3))x_4$

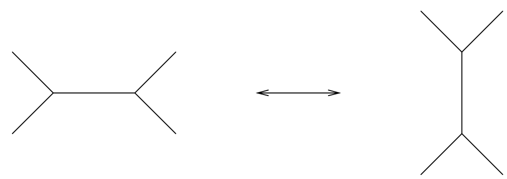

Figure 3.3: Move corresponding to an edge of the associahedron

$d-2$ of internal edges in the corresponding tree.

3.3 Moduli of semistable nodal disks

By *disk* we will always mean the unit disk in the complex plane,

$$D := \{z \in \mathbb{C} \big| |z| \le 1\} \subset \mathbb{C}$$

We will write ∂D for its boundary.

Definition Let $d \ge 0$. A *disk with* $(d+1)$ *markings* is a tuple $(D, z_0, \dots, z_d)$ where D is the unit disk, and $z_0, \dots, z_d$ are distinct points in ∂D, in counterclockwise order. Two marked disks $(D, z_0, \dots, z_d)$ and $(D', w_0, \dots, w_d)$ are *isomorphic* if there is a holomorphic isomorphism $\phi : D \longrightarrow D'$ such that $\phi(z_i) = w_i$ for $i = 0, \dots, d$. The *moduli space of* $(d+1)$*-marked disks*, $\mathcal{R}^d$, is the space of isomorphism classes of disks with $(d+1)$-markings.

The holomorphic isomorphisms of the unit disk consist of those fractional linear transformations which map the disk to itself. Since there is a unique fractional linear transformation that maps three given distinct points on ∂D to any other three distinct points on ∂D, the dimension of $\mathcal{R}^d$ is $d-2$.

$\mathcal{R}^d$ can be identified with a component of the real locus of the moduli space $M_{0,d+1}$ of Riemann spheres with $d+1$ marked points. When $n \geq 3$ the space $M_{0,n}$ is not compact, but has a Deligne-Mumford/Grothendieck-Knudsen compactification $\overline{M}_{0,n}$ by stable Riemann surfaces of genus zero with n marked points. This compactification is described in detail in [10, Appendix D], in an approach that uses cross-ratios. We follow this approach and describe the compactification of $\mathcal{R}^d$ by *nodal disks* with markings, $\overline{\mathcal{R}}^d$.

Definition A *nodal disk* is a tuple $(V, E, (D_\alpha)_{\alpha\in V}, (z_{\alpha\beta})_{\alpha E\beta})$ of the following data:

1. a finite set V, whose elements will be called *vertices*, and a set of *edges* $E \subset V \times V$ such that the pair (V, E) is a tree,
2. a disk D_α for each vertex $\alpha \in V$,
3. a nodal point $z_{\alpha\beta} \in \partial D_\alpha$ for each edge $(\alpha, \beta) \in E$.

The boundary of a nodal disk inherits an orientation from the orientations of the disk components with pairs of nodal points identified, $z_{\alpha\beta} \sim z_{\beta\alpha}$. A set of $d+1$ markings for a nodal disk $(V, E, (D_\alpha)_{\alpha\in V}, (z_{\alpha\beta})_{\alpha E\beta})$ is a set $\{z_0, \ldots, z_d\}$ of points on the boundary of the nodal disk in counterclockwise order, distinct from the nodal points.

Definition A nodal disk with $d+1$ markings is *semistable* if each disk component contains at least three nodal points or markings.

The combinatorial type of a nodal disk with $d+1$ markings is obtained from the tree (V, E) by adding semiinfinite edges associated to the markings. The resulting graph T has a distinguished vertex defined by the component containing the zeroth marking z_0. We call the semiinfinite edge labeled by z_0 the *root*, and the other semiinfinite edges labeled by $z_1, \ldots, z_d$ the *leaves*. Thus the combinatorial type of a nodal disk with $d+1$

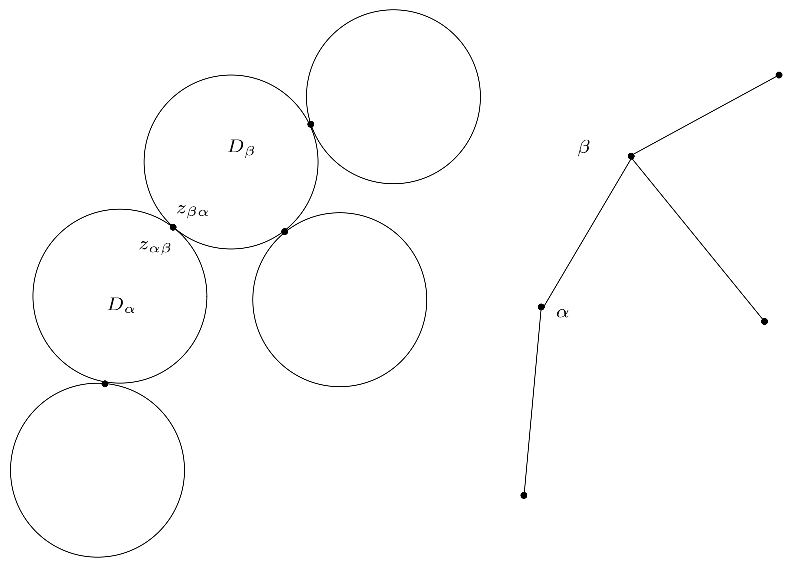

Figure 3.4: A nodal disk with 5 components.

markings is a rooted tree. If the marked nodal disk is semistable, then the valency of each vertex of T is at least 3; in this case we will say that T is stable.

Two nodal disks with $d+1$ markings are *isomorphic* if they have the same combinatorial type and there is a tuple $(\phi_\alpha)_{\alpha \in V}$ of holomorphic isomorphisms between corresponding disk components that preserve the nodal points and markings. Let $\mathcal{R}(T)$ denote the set of isomorphism classes of semistable nodal marked disks of combinatorial type T, and

$$\overline{\mathcal{R}}_d = \bigcup_T \mathcal{R}(T)$$

where T ranges over all stable rooted trees with d leaves. There is a canonical partial order on the combinatorial types, and we write $T' \leq T$ to mean that T' is obtained from T by contracting a subset of internal edges of T.

3.3.1 Topology via cross-ratios

This section is based on [10, Appendix D].

Definition The *cross-ratio* of four distinct points $w_1, w_2, w_3, w_4 \in \mathbb{C}$ is

$$\rho_4(w_1, w_2, w_3, w_4) = \frac{(w_2 - w_3)(w_4 - w_1)}{(w_1 - w_2)(w_3 - w_4)}$$

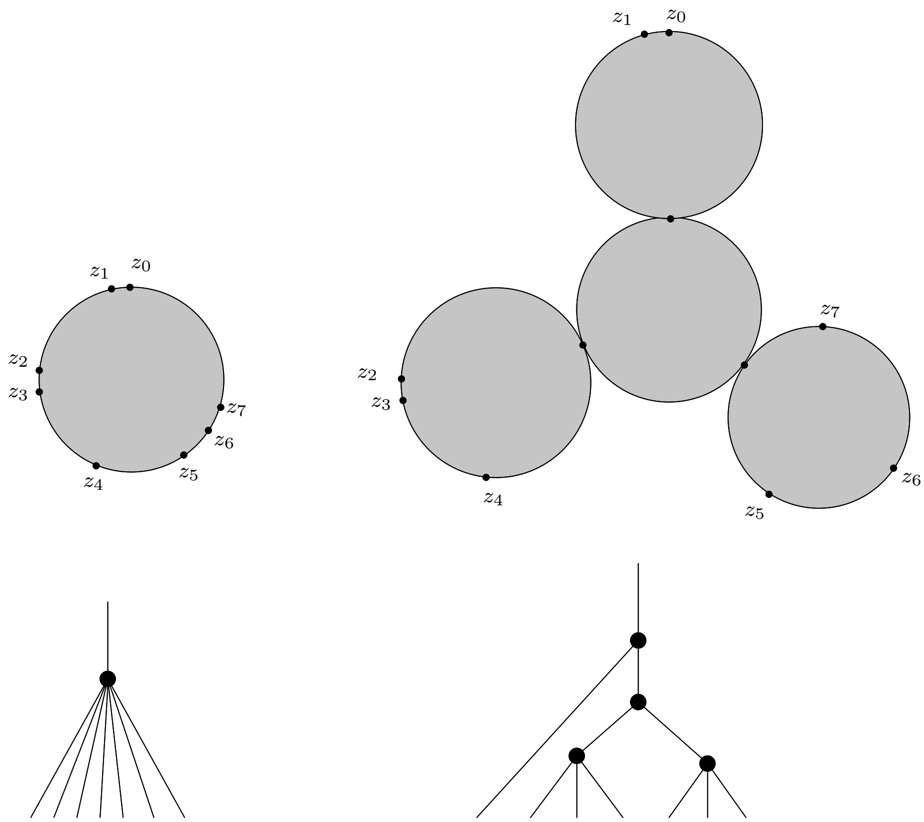

Figure 3.5: A marked disk in $\mathcal{R}^7$, and a marked nodal disk in $\overline{\mathcal{R}}^7$, with their combinatorial types drawn below.

and represents the image of w_4 under the fractional linear transformation that sends w_1 to 0, w_2 to 1, and w_3 to ∞.

ρ_4 is invariant under the action of $SL(2, \mathbb{C})$ on $\mathbb{C}$ by fractional linear transformations. By identifying $\mathbb{P}^1 \to \mathbb{C} \cup \{\infty\}$ and using invariance we obtain an extension of ρ_4 to $\mathbb{P}^1$, that is, a map

$$\rho_4 : \{(w_1, w_2, w_3, w_4) \in (\mathbb{P}^1)^4, \quad i \neq j \implies w_i \neq w_j\} \to \mathbb{C} - \{0\}.$$

ρ_4 naturally extends to the geometric invariant theory quotient

$$(\mathbb{P}^1)^4 /\!\!/ SL(2, \mathbb{C}) = \{(w_1, w_2, w_3, w_4), \text{ no more than two points equal}\}/SL(2, \mathbb{C})$$

by setting

$$\rho_4(w_1,w_2,w_3,w_4) = \left\{ \begin{array}{lll} 0 & \text{if} & w_2 = w_3 \text{ or } w_1 = w_4 \\ 1 & \text{if} & w_1 = w_3 \text{ or } w_2 = w_4 \\ \infty & \text{if} & w_1 = w_2 \text{ or } w_3 = w_4 \end{array} \right\} \tag{3.2}$$

and defines an isomorphism

$$\rho_4 : (\mathbb{P}^1)^4 /\!\!/ SL(2,\mathbb{C}) \to \mathbb{P}^1.$$

Definition Let C be a circle in the Riemann sphere $\mathbb{C}\cup\{\infty\}$. We will say that distinct points $z_1, z_2, \ldots, z_n$ on C are *in cyclic order* if they fall in that order on C. In particular, the compactified line $\mathbb{R} \cup \{\infty\}$ is a circle in the Riemann sphere, and distinct points $z_1, \ldots, z_n$ in $\mathbb{R}$ are in cyclic order if there is some cyclic permutation of them such that they are in strictly ascending or strictly descending order.

Let $\mathbb{R}^4_+ \subset \mathbb{R}^4$ denote the subset of distinct points $(w_1, w_2, w_3, w_4) \in \mathbb{R}^4$ in cyclic order. The restriction of ρ_4 to $\mathbb{R}^4_+$ takes values in $(-\infty, 0)$ and is invariant under the action of $SL(2,\mathbb{R})$ by fractional linear transformations. Hence it descends to a map

$$(\mathbb{R}^4)_+/SL(2,\mathbb{R}) \to (-\infty, 0).$$

Let D denote the unit disk, and identify $D\backslash\{-1\}$ with the half plane $\mathbb{H}$ by $z \mapsto 1/(z+1)$. Using invariance one constructs an extension

$$\rho_4 : (\partial D)^4_+/SL(2,\mathbb{R}) = \mathcal{R}^3 \to (-\infty, 0)$$

where $(\partial D)^4_+$ is the set of distinct points on ∂D in counterclockwise cyclic order, z_0, z_1, z_2, z_3. ρ_4 admits an extension to $\overline{\mathcal{R}}^3$ via (3.2) and so defines a homeomorphism

$$\rho_4 : \overline{\mathcal{R}}^3 \to [-\infty, 0].$$

For any distinct indices i, j, k, l the cross-ratio ρ_{ijkl} is the function

$$\rho_{ijkl} : \mathcal{R}^d \to \mathbb{R}, \quad [w_0, \ldots, w_d] \mapsto \rho_4(w_i, w_j, w_k, w_l).$$

Fix four distinct indices $0 \le i < j < k < l \le d$. We extend ρ_{ijkl} to $\overline{\mathcal{R}}^d$ as follows: for a combinatorial type T, let $T(ijkl)$ be the subtree whose ending edges are the semiinfinite

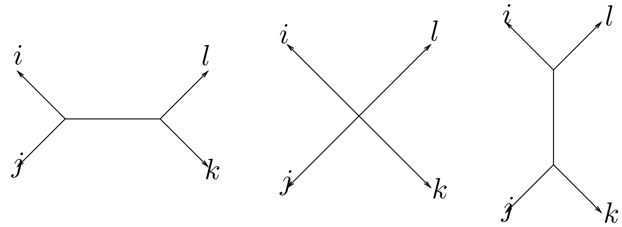

Figure 3.6: Cross-ratios by combinatorial type: for the first type, $\rho_{ijkl}(p) = -\infty$, for the second type $\rho_{ijkl}(p) \in (-\infty, 0)$, and for the third type $\rho_{ijkl}(p) = 0$.

edges i, j, k, l. The subtree $T(ijkl)$ is one of the three types shown in Figure 3.6. In the first resp. third case, we define

$$\rho_{ijkl}(p) = -\infty \quad \text{resp. } 0.$$

In the second case, let $\overline{w}_i, \overline{w}_j, \overline{w}_k, \overline{w}_l$ be the points on the component where the four branches meet and define

$$\rho_{ijkl}(p) = \rho(\overline{w}_i, \overline{w}_j, \overline{w}_k, \overline{w}_l).$$

The collection of functions ρ_{ijkl} restricted to indices $0 \leq i < j < k < l \leq d$ defines a map of sets

$$\rho_d : \overline{\mathcal{R}}^d \mapsto [-\infty, 0]^N$$

where $N = \begin{pmatrix} d+1 \\ 4 \end{pmatrix}$.

The combinatorial type T of a point $p \in \overline{\mathcal{R}}^d$ can be read off from $\rho_d(p)$ as follows. Note that if T is a tree, then removing any finite edge e of T separates T, and in particular the set $\mathrm{Edge}^\infty(T)$ of semiinfinite edges, into two components. The tree can be reconstructed from the set of such partitions of $\mathrm{Edge}^\infty(T)$. If some cross-ratio $\rho_{ijkl}(p)$ is ∞ (resp. 0), the semiinfinite edges i, j, k, l can be separated into pairs i, j and k, l (resp. i, l and j, k) by removing an edge (Figure 3.6). Thus the combinatorial structure of p is determined by which cross-ratios are 0 or $-\infty$. The positions of the marked points on each component with at least 4 markings or singularities can be reconstructed from the cross-ratios. Hence ρ_d is injective and so the pull-back ρ_d of the topology on the target defines on $\overline{\mathcal{R}}^d$ the structure of a compact Hausdorff topological space. The moduli space $\overline{\mathcal{R}}^4$ is shown in Figure 3.7.

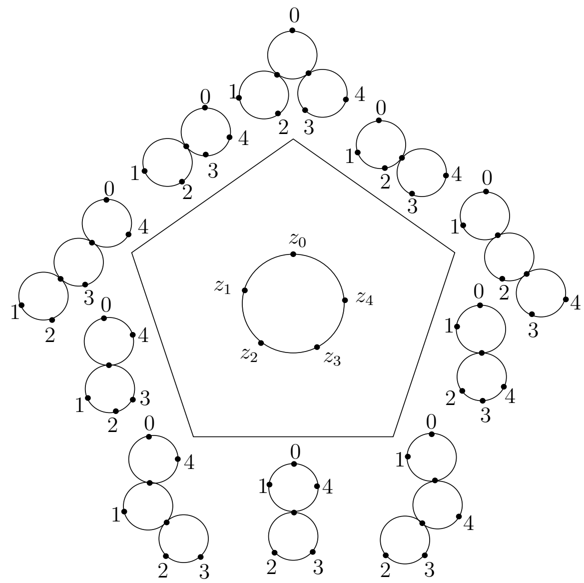

Figure 3.7: $\overline{\mathcal{R}}^4$. Drawing all nodal disks with z_0 at the top makes the vertical axis of symmetry clear.

Remark The map $\rho_{ijkl} : \overline{\mathcal{R}}^d \to \overline{\mathcal{R}}^3 \cong [-\infty, 0]$ is a special case of a type of *forgetful morphism* considered by Knudsen. More generally, for any subset $I \subset \{0, \ldots, n\}$ of size k we have a continuous map $\overline{\mathcal{R}}^d \mapsto \overline{\mathcal{R}}^k$ obtained by forgetting the positions of the markings z_i, $i \notin I$ and collapsing the unstable components. By definition the topology on $\overline{\mathcal{R}}^d$ is the one for which all forgetful morphisms are continuous, and $\overline{\mathcal{R}}^3 \cong [-\infty, 0]$.

3.3.2 Properties of the cross-ratio coordinates

The ρ_{ijkl} satisfy the following properties (see [10, Appendix D]), which all follow from elementary facts about cross-ratios. **(Invariance):** For all $p \in \overline{\mathcal{R}}^d$, and for all $\phi \in SL(2, \mathbb{R})$, $\rho_{ijkl}(\phi(p)) = \rho_{ijkl}(p)$.

(Symmetry): $\rho_{jikl} = \rho_{ijlk} = 1 - \rho_{ijkl}$, and $\rho_{ikjl} = \frac{\rho_{ijkl}}{\rho_{ijkl}-1}$.

(Normalization): $\rho_{ijkl} = \begin{cases} \infty, & \text{if } i = j \text{ or } k = l, \\ 1, & \text{if } i = k \text{ or } j = l, \\ 0, & \text{if } i = l \text{ or } j = k. \end{cases}$

(Recursion): As long as the set $\{1, \infty, \rho_{ijkl}, \rho_{ijkm}\}$ contains three distinct numbers, then

$$\rho_{jklm} = \frac{\rho_{ijkm} - 1}{\rho_{ijkm} - \rho_{ijkl}} \tag{3.3}$$

for any five pairwise distinct integers $i, j, k, l, m \in \{0, 1, \dots, d\}$. There is an equivalent version of this formula that will sometimes be more convenient to use; and that is that as long as the set $\{0, \infty, \rho_{mijk}, \rho_{mijl}\}$ contains three distinct numbers, then

$$1 - \rho_{jklm} = \frac{\rho_{mijk}}{\rho_{mijl}}. \tag{3.4}$$

3.3.3 Charts using cross-ratios.

The cross-ratios can be used to construct explicit coordinate charts which give $\overline{\mathcal{R}}^d$ the structure of a $(d-2)$ dimensional manifold-with-corners. A chart around a point in $p \in \overline{\mathcal{R}}^d$ is defined based on the combinatorial type T of p. If $|E|$ is the number of interior edges of T, the chart is a homeomorphism between $(-\infty, 0)^{d-2-|E|} \times (-\infty, 0]^{|E|}$ and the open subset

$$\overline{\mathcal{R}(T)} := \bigcup_{T' \leq T} \mathcal{R}(T')$$

i.e., all points in $\overline{\mathcal{R}}^d$ whose combinatorial type is obtained from T by contracting a subset of interior edges.

The charts using cross-ratios are based on the analogous charts in [10, Appendix D] for $\overline{M}_{0,n}$. We include their proof for the sake of completeness. For a combinatorial type T, a chart consists of

1. $n_v - 3$ cross-ratios for each vertex $v \in V$ that represents a disk component with n_v special points (i.e., marked points or nodal points).

2. a cross-ratio $\rho_{ijkl} = 0$ for each internal edge $e \in E$, where i, j, k and l are such that $\rho_{ijkl} = 0$ for any combinatorial type modeled on that edge.

This gives a total of

$$\begin{aligned} \sum_{v \in V} (n_v - 3) + |E| &= d + 1 + 2|E| - 3|V| + |E| \\ &= d + 1 + 3(|E| - |V|) \\ &= d - 2 \end{aligned}$$

coordinates, since the number of edges in a tree is one less than the number of vertices.

Theorem 3.3.1 (Theorem D.5.1 in [10]). *Let $p \in \overline{\mathcal{R}}^n$. Suppose p has combinatorial type T, and that $d-2$ cross-ratios have been chosen as prescribed by (a), (b) above. Then, in the open set $\overline{\mathcal{R}(T)}$, all cross-ratio coordinates are smooth functions of those in the chart. Hence $\overline{\mathcal{R}}^d$ is a smooth manifold-with-corners of real dimension $d-2$.*

Proof. Fix a combinatorial type T with $d+1$ leaves. To show that the $d-2$ coordinates described above form a chart in a neighborhood of p, we proceed by induction on the number of edges $|E|$ of the combinatorial type T.

If $|E| = 0$, T is the corolla with one vertex, a root and $d+1$ leaves, corresponding to the equivalence classes of the unit disk D with $d+1$ distinct marked points $z_0, \dots, z_d$ on ∂D. The $d-2$ cross-ratios

$$\{\rho_{0123}, \rho_{0124}, \dots, \rho_{012d}\}$$

form a chart, since an explicit formula for all other cross-ratios is

$$\rho_{ijkl} = \frac{(\rho_{012j} - \rho_{012k})(\rho_{012l} - \rho_{012i})}{(\rho_{012i} - \rho_{012j})(\rho_{012k} - \rho_{012l})}$$

with well-defined limits

$$\rho_{ijkl} = \begin{cases} \infty, & \text{if } i = j \text{ or } k = l, \\ 1, & \text{if } i = k \text{ or } j = l, \\ 0, & \text{if } i = l \text{ or } j = k. \end{cases}$$

So assume that the statement holds for all trees with strictly less than $|E|$ edges, and consider a combinatorial type T with $|E|$ edges. Fix an edge e joining vertices α and β. Removing the edge e splits T into two subtrees, one containing α and the other containing β. For each of these trees, put a semiinfinite edge where e was. Let T_A be the tree containing α, and T_B the one containing β. Relabeling indices if necessary we can assume that T_A has marked points $0, \dots, m+1$, and T_B has marked points $m, \dots, d$. Let ρ_A be the set of cross-ratios with indices in $\{0, \dots, m, m+1\}$, and let ρ_B be the set of cross-ratios with indices in $\{m, m+1, \dots, d\}$. By the inductive hypothesis, all cross-ratios in ρ_A are smooth functions of the cross-ratios in a chart for T_A, and all cross-ratios in ρ_B are smooth functions of the cross-ratios in a chart for T_B. Fix the cross-ratio $\rho_{0,m,m+1,d} = \infty$ to represent the edge e. Note that $\rho_{0,m,m+1,d} = \infty$ for all combinatorial types containing the edge e. We need to show that the chart for T_A, the chart for T_B, and $\rho_{0,m,m+1,d}$ together form a chart for T. It suffices to show that all cross-ratios with some indices less than or equal to m and other indices greater than or equal to $m+1$ are smooth functions of cross-ratios in ρ_A, ρ_B, and $\rho_{0,m,m+1,d}$. There are really only two cases to prove, namely

1. ρ_{ijkl} where $i, j \leq m$ and $k, l \geq m+1$, and

2. ρ_{ijkl} where $i, j, k \leq m$ and $l \geq m+1$,

since the only other case, $i \leq m$ and $j, k, l \geq m+1$, is dual to the latter. Since $\rho_{m+1,i,j,l} = 0$, applying Recursion formula (3.3),

$$\rho_{i,j,k,l} = \frac{\rho_{m+1,i,j,l} - 1}{\rho_{m+1,i,j,l} - \rho_{m+1,i,j,k}},$$

shows that it is enough to show that $\rho_{m+1,i,j,l}$ and $\rho_{m+1,i,j,k}$ are smooth functions of the chart coordinates. Note that if $k \leq m$, then $\rho_{m+1,i,j,k} \in \rho_A$, so is a smooth function of the chart coordinates. Therefore we only need to show that for $l > m+1$, the cross-ratio $\rho_{m+1,i,j,l}$ is a smooth function of the chart coordinates. But we also have that $\rho_{m,m+1,l,i} = 0$, so that the Recursion formula (3.3) holds,

$$\rho_{m+1,l,i,j} = \frac{\rho_{m,m+1,l,j} - 1}{\rho_{m,m+1,l,j} - \rho_{m,m+1,l,i}}.$$

So it is enough to prove that for all $i < m$ and all $l > m+1$, the cross-ratio $\rho_{m,m+1,l,i}$ is a smooth function of the chart coordinates.

To show this, first note that $\rho_{m,m+1,d,0} = 0$ so again by (3.3),

$$\rho_{m+1,d,0,l} = \frac{\rho_{m,m+1,d,l} - 1}{\rho_{m,m+1,d,l} - \rho_{m,m+1,d,0}}$$

and since $\rho_{m,m+1,d,l} \in \rho_B$ for $l > m+1$ we conclude that $\rho_{m+1,d,0,l}$ is a smooth function of the chart coordinates.

Next, $\rho_{d,m,l,0} = 1$ so by the variation (3.4) of the Recursion formula,

$$1 - \rho_{l,0,m+1,d} = \frac{\rho_{d,m,l,0}}{\rho_{d,m,l,m+1}}$$

$$\implies \quad \rho_{d,m,l,0} = \rho_{d,m,l,m+1}(1 - \rho_{l,0,m+1,d})$$

showing that $\rho_{d,m,l,0}$ is a smooth function of the chart coordinates.

Again using $\rho_{d,m,l,0} = 1$ and (3.4),

$$\begin{aligned} 1 - \rho_{m,l,m+1,0} &= \frac{\rho_{0,d,m,l}}{\rho_{0,d,m,m+1}} \\ \Longrightarrow \quad \rho_{m,l,m+1,0} &= 1 - \frac{\rho_{0,d,m,l}}{\rho_{0,d,m,m+1}} \end{aligned}$$

Next, $\rho_{m,m+1,0,l} = 1$ so by (3.4)

$$\begin{aligned} 1 - \rho_{0,l,i,m} &= \frac{\rho_{m,m+1,0,l}}{\rho_{m,m+1,0,i}} \\ \Longrightarrow \quad \rho_{0,l,i,m} &= 1 - \frac{\rho_{m,m+1,0,l}}{\rho_{m,m+1,0,i}}. \end{aligned}$$

Finally, $\rho_{m,l,i,m+1} = 1$ implies, by (3.4) again,

$$\begin{aligned} 1 - \rho_{i,m+1,0,m} &= \frac{\rho_{m,l,i,m+1}}{\rho_{m,l,i,0}} \\ \Longrightarrow \quad \rho_{m,l,i,m+1} &= \rho_{m,l,i,0}(1 - \rho_{i,m+1,0,m}) \end{aligned}$$

proving that $\rho_{m,l,i,m+1}$ is a smooth function of the chart coordinates.

$\square$

Remark The charts corresponding to maximal combinatorial types – that is, the binary trees – suffice to cover the whole moduli space, since all combinatorial types can be obtained from the maximal ones by contracting an edge. From now on we will think of each chart as being from a maximal tree. We can also assume that all cross-ratios in the chart are of the form ρ_{ijk0} or ρ_{0ijk}, since all edges in a binary tree can be given coordinates of that form. In other words, we can assume that all chart coordinates have been chosen relative to the root of the tree, where the root corresponds to the distinguished marked point z_0.

3.3.4 Charts using simple ratios.

We now describe an equivalent topology on $\mathcal{R}^d$ using coordinate charts based on ratios that we will call "simple ratios". The main advantage of simple ratios is that the relations between them are of a much simpler form than the relations between the cross-ratios. By choosing parametrizations such that $z_0 = \infty$, elements of $\overline{\mathcal{R}}^d$ are identified with configurations of d distinct points in $\mathbb{R}$,

$$-\infty < z_1 < z_2 < \ldots < z_d < \infty,$$

modulo translation and scaling. Set $X_i = z_{i+1} - z_i$. The coordinates $X_1, \ldots, X_{d-1}$ are invariant under translations, and scale simultaneously, so are effectively projective coordinates $(X_1 : \ldots : X_{d-1})$. The compactification depends on the values of ratios of the form X_i/X_j, which we call "simple ratios".

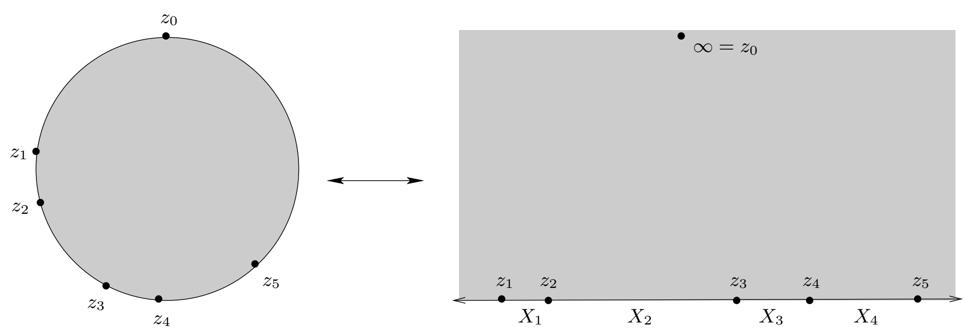

Figure 3.8: Projective coordinates $(X_1 : X_2 : X_3 : X_4)$ for a marked disk in $\mathcal{R}^5$.

Let T be a maximal tree representing a combinatorial type in $\overline{\mathcal{R}}^d$. Each pair of adjacent leaves in T, i and $i+1$ say, determines a unique vertex which we label v_i. Each edge in T is determined by a pair of vertices, say v_i and v_j. If v_i is closer to the root of the tree than v_j, we give the edge the coordinate X_j/X_i.

Example Figure 3.9 shows the two charts for a combinatorial type in $\overline{\mathcal{R}}^6$.

3.3.5 Equivalence of charts.

Each cross-ratio in a chart is a smooth function of the simple ratios, and is zero if and only if the corresponding simple ratio for that edge is zero. The vice-versa is also true, and the proof is much the same. By symmetry it suffices to consider the edge pictured in Figure 3.10, where an edge joins vertices v_r and v_s, and v_r is above v_s, and the cross-ratio allocated to the edge is ρ_{ijk0}. Parametrizing so that $z_0 = \infty$ we can write

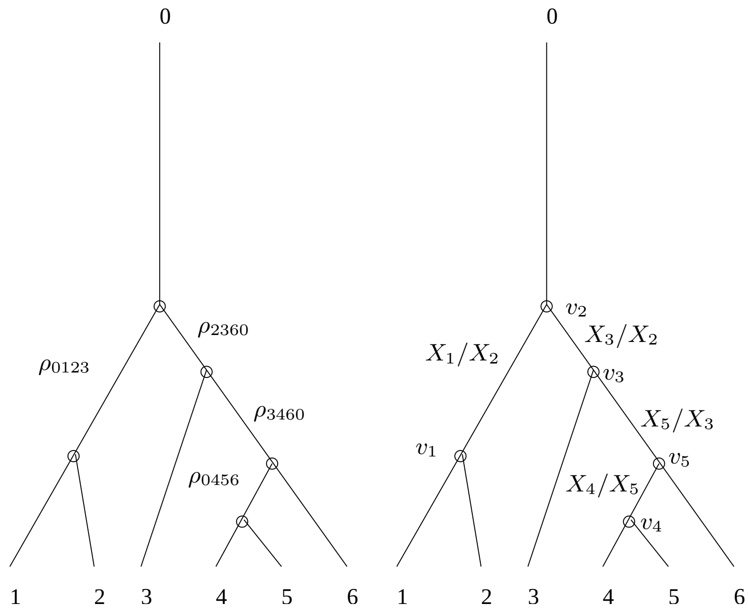

Figure 3.9: Two equivalent charts, one using cross-ratios and the other using simple ratios, for a combinatorial type in $\overline{\mathcal{R}}^6$.

$$\begin{aligned}
\rho_{ijk0} &= -\frac{z_j - z_k}{z_j - z_i} \\
&= -\frac{X_j + X_{j+1} + \ldots + X_s + \ldots + X_{k-1}}{X_i + X_{i+1} + \ldots + X_r + \ldots + X_{j-1}} \\
&= -\frac{X_s}{X_r}\left(\frac{X_j/X_s + X_{j+1}/X_s + \ldots + 1 + \ldots + X_{k-1}/X_s}{X_i/X_r + \ldots + 1 + \ldots + X_{j-1}/X_r}\right).
\end{aligned}$$

The ratios appearing in the big bracket are in general products of chart ratios corresponding to edges below v_r and v_s. The bracketed rational function is smooth for all positive non-zero ratios and continuous as ratios in the chart go to 0. Moreover $\rho_{ijk0} = 0$ if and only if $X_s/X_r = 0$.

Thus the cross-ratio charts define the same topology on $\overline{\mathcal{R}}^d$ as the simple ratio charts.

3.4 Metric ribbon trees

In this section we introduce the space of metric ribbon trees, attempting to remain as consistent as possible with the notation of Fukaya and Oh in [5], and Nadler and Zaslow in [11].

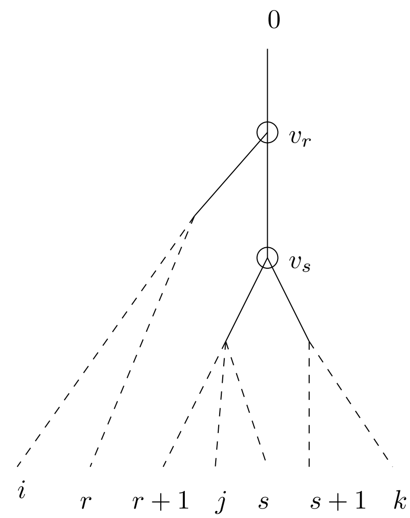

Figure 3.10: Comparing a cross-ratio with a simple ratio.

Definition A *based metric ribbon tree* is a quadruple (T, i, e_0, λ) of the following data:

1. T is a finite tree, with $d+1$ semi-infinite exterior edges labeled $e_0, \ldots, e_d$, and no vertex of valency 1 or 2. We denote by $E_{int}(T)$ the set of interior edges of T, which are all of finite length, and denote by $V_{int}(T)$ the set of vertices of T.

2. $i : T \hookrightarrow D \subset \mathbb{R}^2$ is an embedding of T in the closed unit disk such that the limit of the image of each the semi-infinite exterior edge e_j is a point in ∂D.

3. e_0 is a distinguished exterior edge of T, the *root*. The other exterior edges, $e_1, \ldots, e_d$ are called the *leaves*. The labeling $e_0, e_1, \ldots, e_d$ is consistent with their counterclockwise order on ∂D that comes from the embedding i.

4. $\lambda : E_{int}(T) \to \mathbb{R}_+$ is an *edge length map*, that assigns a non-negative real number to each interior edge.

Two based metric ribbon trees are *equivalent* if there is an isotopy of the closed disk which identifies all the data.

We use the same notation as in [5]: for a fixed integer $d > 2$ and a fixed root e_0, let $Gr(T)$ be the set of all maps $\lambda : E_{int}(T) \to \mathbb{R}_+$, and define

$$Gr_d = \bigcup_T Gr(T)$$

where T runs over all rooted trees with $d+1$ leaves, and no vertices of valency 1 or 2. We now define a topology on Gr_d. Assume that $\{\lambda_n\}_{n\in\mathbb{N}}$ is a sequence in $Gr(T)$, such that for each $e \in E_{int}(T)$ the sequence $\lambda_n(e)$ converges to $\lambda_\infty(e) \in [0,\infty)$. We define the limit of this sequence to be the quadruple (T', i', e'_0, λ') where

1. T' is the tree obtained from T by contracting all edges with $\lambda_\infty(e) = 0$. Thus, T' is the image of a surjective morphism of planar rooted trees, $p : T \to T'$.

2. i' is the embedding obtained from the embedding i by contracting the collapsed edges.

3. e_0 is the end-vertex in T' corresponding to the end vertex of the same label in T.

4. For each edge $e \in E_{int}(T')$, we have that $p^{-1}(e) \in E_{int}(T)$, so we define $\lambda'(e) = \lambda_\infty(p^{-1}(e))$.

Write $T' \leq T$ if T' is obtained from T by contracting a subset of its interior edges. With this notation, the closure of $Gr(T)$ in Gr_d is

$$\text{cl } Gr(T) = \bigcup_{T' \leq T} Gr(T').$$

With respect to this topology, Gr_d is Hausdorff and closed.

We introduce some relevant terminology about trees here: a *geodesic* between two vertices v, v' in a metric tree T is the (unique) path joining them of shortest total length; the total length of the geodesic is called the *distance* from v to v'. Moreover, the geodesic path is independent of the lengths assigned to the edges, and depends only on the combinatorics of the tree.

The following theorem is due to Stasheff ([17], [16]), and another proof was given by Fukaya and Oh [5]. The following proof is a modification of that in [5].

Theorem 3.4.1. *Gr_d is homeomorphic to $\mathbb{R}^{d-2}$.*

Proof. We show this by constructing a homeomorphism $\Theta : Gr_d \to \mathcal{R}^d$, and the result follows from the fact that $\mathcal{R}^d$ is homeomorphic to $\mathbb{R}^{d-2}$.

For each combinatorial type T, and $i = 1, \dots, d-1$, let v_i be the (unique) vertex of T at which the geodesic from e_i to e_0 first intersects the geodesic from e_{i+1} to e_0.

$$\begin{aligned}
\Theta_T : Gr(T) &\rightarrow \mathcal{R}^d \\
\lambda \in Gr(T) &\mapsto (X_1 : \dots : X_{d-1}) \in \mathbb{R}P^{d-2} \quad \text{where} \\
X_i &= e^{-\mathrm{dist}(v_i, e_0)}.
\end{aligned}$$

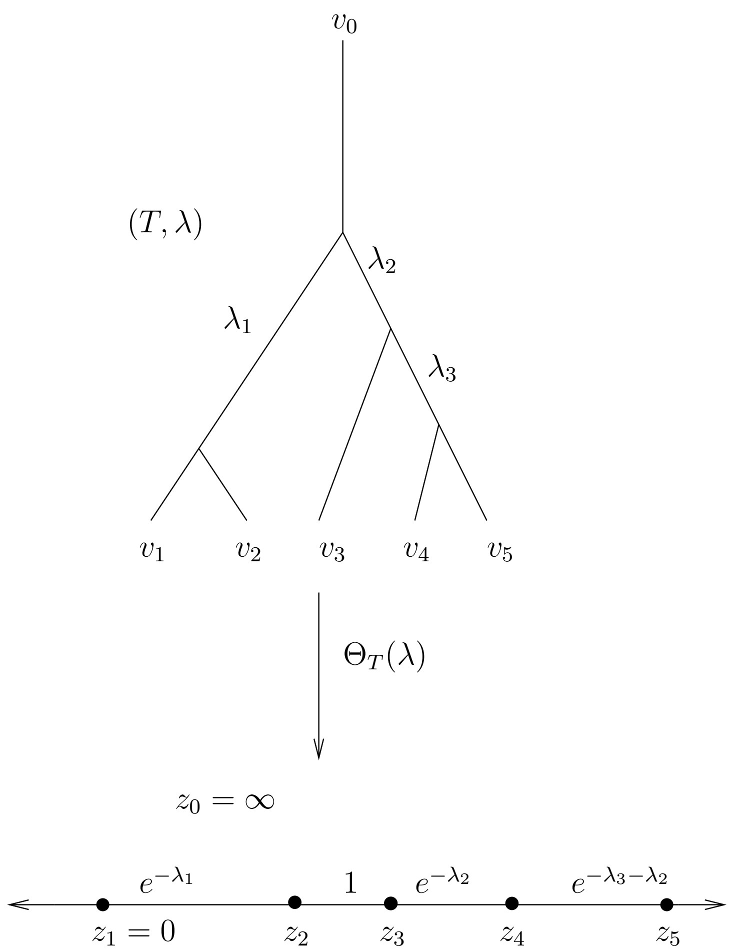

Figure 3.11: Identifying a metric ribbon tree in Gr_5 with a marked disk in $\mathcal{R}^5$.

The map Θ_T is a well-defined, continuous function of λ. It is injective since there is always a pair e_i, e_{i+1} such that the vertex v_i is adjacent to the root e_0, so that $X_i = z_{i+1} - z_i = 1$ throughout the image of $Gr(T)$. So if $[\Theta_T(\lambda)] = [\Theta_T(\lambda')]$, we must have $z_i(\lambda) = z_i(\lambda')$ for all i. But then one can show inductively on the lengths of paths from the root e_0, that $\lambda(e) = \lambda'(e)$ for all edges e in such a path, hence $\lambda = \lambda'$.

We also note that

$$\Theta_{T_1}(Gr(T_1)) \cap \Theta_{T_2}(Gr(T_2)) = \bigcup_{T<T_1,T_2} \Theta_T(Gr(T)).$$

To see that $\Theta : Gr_d \to \mathcal{R}^d$ is surjective, we show that given any collection of d distinct points $z_1 < z_2 < \ldots < z_d$, if we fix the parametrization so that $z_1 = 0$ and $\max(z_{i+1} - z_i) = 1$, then $(z_1, \ldots, z_d)$ is in the image of some $\Theta_T(Gr(T))$. To reconstruct the combinatorial type T, we start by partitioning the set $z_1, \ldots, z_d$ into disjoint subsets of consecutive points, $I_1, \ldots, I_k$ say, by placing a partition between z_{i+1} and z_i if $z_{i+1} - z_i = 1$. Now form a tree with root e_0, and k leaves $\tilde{e}_1, \ldots, \tilde{e}_k$ indexed by the subsets $I_1, \ldots, I_k$. For every subset I_j with at least 2 distinct points in it, we fix a dilation factor of e^{λ_j} so that $\max(z_{i+1} - z_i) = 1$ for $i \in I_j$. This in turn determines a partition of the set I_j, which determines leaves beneath $\tilde{e}_j$, and we set $\lambda(\tilde{e}_j) = \lambda_j$. Proceeding inductively in this way, one reconstructs the tree T and the edge lengths $\lambda \in Gr(T)$ such that $\Theta_T(\lambda) = (z_1, \ldots, z_d)$.

□

Remark The map Θ is really an identification of each cone $Gr(T)$ with a corresponding cone in $\mathbb{R}^{d-2}$, such that the identifications of cones along their boundaries in $\mathbb{R}^{d-2}$ are the same as the identifications along their boundaries in Gr_d, and the union of all the cones in $\mathbb{R}^{d-2}$ is all of $\mathbb{R}^{d-2}$. To see this explicitly, fix the parametrization of configurations in $\mathcal{R}^d$ so that, for example, $z_d - z_{d-1} = 1$ and $\mathcal{R}^d$ is spanned by the coordinates $X_1, \ldots, X_{d-2}$ where $X_i = z_{i+1} - z_i$ and all X_i's are positive. The images of the cones $Gr(T)$ are also cones in $\mathbb{R}^{d-2}_{>0}$, where the cones are centered at $(1, 1, \ldots, 1)$ and their union is $\mathbb{R}^{d-2}_{>0}$. The map $(x, y) \mapsto (\log x, \log y)$ identifies $\mathbb{R}^{d-2}_{>0}$ with $\mathbb{R}^{d-2}$, and maps cones to cones.

Example Consider the map $\Theta : Gr_4 \to \mathcal{R}^4$. Fix the parametrization of $\mathcal{R}^4$ so that $z_2 - z_1 = 1$, the map Θ is a subdivision of $\mathbb{R}^2_{>0}$ into five cells, see Figure 3.12.

Each cell $Gr(T)$ has a compactification by allowing the edge lengths to be infinite. This induces a compactification $\overline{Gr}_d$ of Gr_d. The map $\Theta : Gr_d \to \mathcal{R}^d$ extends to a map

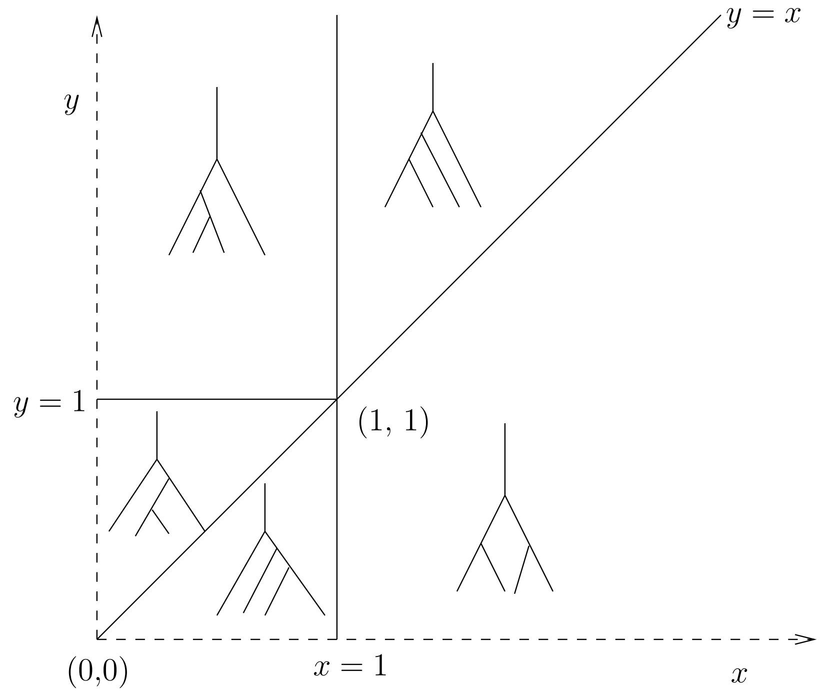

Figure 3.12: The images of the cones of Gr_4 in $\mathbb{R}^2_{>0}$. The map $(x, y) \mapsto (\log x, \log y)$ identifies them with cones in $\mathbb{R}^2$ whose union is all of $\mathbb{R}^2$.

$\Theta : \overline{Gr}_d \to \overline{\mathcal{R}}^d$ by taking limits in appropriate charts:

Theorem 3.4.2. $\Theta : \overline{Gr}_d \to \overline{\mathcal{R}}^d$ *is a homeomorphism.*

Proof. It follows from the observation that if λ is the edge-length in $Gr(T)$ assigned to an edge e, then in the chart $\mathcal{R}(T) \subset \overline{\mathcal{R}}^d$ the ratio X_i/X_j representing that same edge e has value $e^{-\lambda}$. Thus the image of a compactified cell $Gr(T)$ under the map Θ corresponds to the part of the chart $\mathcal{R}(T)$ for which the ratios in the chart take values in $[0, 1]$. The boundary of $\overline{Gr}_d$, in which some edge lengths are infinite, maps to the boundary of $\overline{\mathcal{R}}_d$, in which the corresponding ratios are zero.

$\square$

3.5 Toric varieties and moment polytopes.

We now show that $\overline{\mathcal{R}}^d$ can be identified with the non-negative part of an embedded toric variety in $\mathbb{C}P^{k-1}$, where $k(d) = C_{d-1}$ is the $(d-1)$-st Catalan number , which counts the number of binary planar rooted trees with d leaves. Each such tree determines a monomial in the $d-1$ variables $X_1, X_2, \ldots, X_{d-1}$. The weight vector of the monomial is read directly from the combinatorics of the tree, according to the algorithm given by Loday [7]. An immediate consequence of the theory of toric varieties (see for example [15], [6]) is that $\overline{\mathcal{R}}^d$ is homeomorphic to the moment polytope of the toric variety, which is the convex hull of the weight vectors.

A point in $\mathcal{R}^d$ can be identified with a configuration of d distinct, ordered points on $\mathbb{R}$, $z_1 < z_2 < \ldots < z_d$, modulo translations and scaling. We get translational invariance by considering the variables

$$X_1 = z_2 - z_1, X_2 = z_3 - z_2, \ldots, X_i = z_{i+1} - z_i, \ldots, X_{d-1} = z_d - z_{d-1},$$

and the X_i's are unique up to scalar multiplication, so are really projective coordinates,

$$\mathbf{X} = (X_1 : X_2 : \ldots : X_{d-1}).$$

To every binary, planar, rooted tree T, we associate a weight vector $\mathbf{N}_T \in \mathbb{Z}^{d-1}$, using Loday's recipe. Each pair of adjacent leaves in T, labelled i and $i+1$ say, determines a unique vertex, which we label v_i, in T. Let a_i be the number of leaves on the left side of v_i, and let b_i be the number of leaves on the right side of v_i. Then the weight vector is

$$\mathbf{N}_T = (a_1 b_1, \ldots, a_i b_i, \ldots, a_{d-1} b_{d-1}),$$

and the corresponding monomial is

$$\mathbf{X}^{\mathbf{N}_T} := \prod_{i=1}^{d-1} X_i^{a_i b_i}.$$

Example For the tree pictured in Figure 3.13, the weight vector is $(1, 8, 3, 1, 2)$, so the corresponding monomial is $X_1 X_2^8 X_3^3 X_4 X_5^2$.

Label the planar, binary rooted trees $T_1, \ldots, T_k$. We define a projective toric variety $V \subset \mathbb{C}P^{k-1}$ by

$$(X_1 : \ldots : X_{d-1}) \quad \mapsto \quad (\mathbf{X}^{\mathbf{N}_{T_1}} : \ldots : \mathbf{X}^{\mathbf{N}_{T_k}})$$

The entries in the weight vectors always sum to $d(d-1)/2$, so the monomials all have the same degree and the map is well-defined on the homogeneous coordinates.

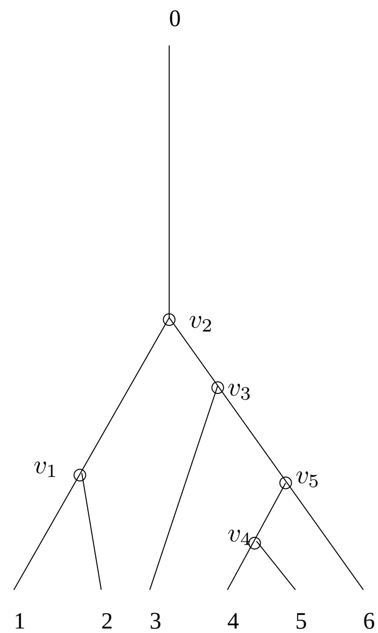

Figure 3.13: The labels on the leaves of a binary tree induce a labeling on the vertices.

Definition We say that two binary trees T and T' differ by a *flop* if there is one interior edge e of T and one interior edge e' of T' such that if e is contracted in T, and e' is contracted in T', the resulting trees are the same (see Figure 3.14).

Lemma 3.5.1. *Suppose that two maximal trees T and T' differ by a single flop across an edge e of T. Let R denote the simple ratio labeling the edge e in the chart determined by T. Then*

$$\frac{\mathbf{X}^{\mathbf{N}_{T'}}}{\mathbf{X}^{\mathbf{N}_T}} = R^m$$

for some integer $m > 0$. In general, for two trees T and T',

$$\frac{\mathbf{X}^{\mathbf{N}_{T'}}}{\mathbf{X}^{\mathbf{N}_T}} = R_{i_1}^{m_1} R_{i_2}^{m_2} \dots R_{i_r}^{m_r}$$

for some ratios $R_{i_1}, \dots, R_{i_r}$ in the ratio chart associated to T and positive integers $m_1, \dots, m_r$.

Proof. First let us consider the case of a single flop. Without loss of generality consider the situation in Figure 3.14. Say T is on the left, and T' is on the right, and the affected edges are in bold. The weight vectors $\mathbf{N}_T$ and $\mathbf{N}_{T'}$ are the same in all entries except

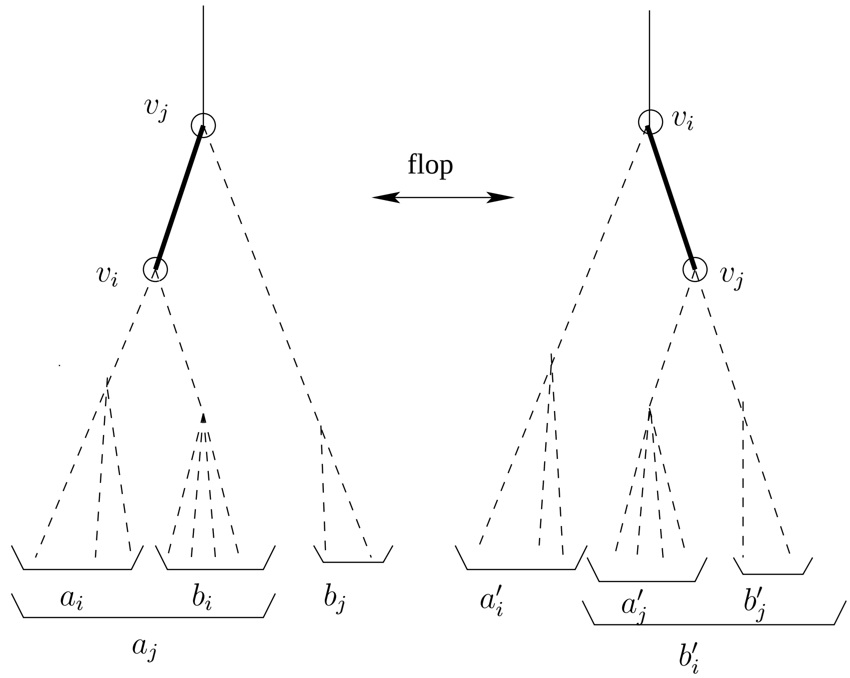

Figure 3.14: Two trees differ by a single flop if a single interior edge can be contracted in each of them to produce the same tree.

entries i and j, where

$$\begin{aligned}
(\mathbf{N}_T)_i &= a_i b_i, \\
(\mathbf{N}_T)_j &= a_j b_j = (a_i + b_i) b_j \\
(\mathbf{N}_{T'})_i &= a_i (b_i + b_j), \\
(\mathbf{N}_{T'})_j &= b_i b_j.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\mathbf{X}^{\mathbf{N}_{T'}}}{\mathbf{X}^{\mathbf{N}_T}} &= \frac{X_i^{a_i(b_i+b_j)} X_j^{b_i b_j}}{X_i^{a_i b_i} X_j^{(a_i+b_i)b_j}} \\
&= \frac{X_i^{a_i b_j}}{X_j^{a_i b_j}} \\
&= \left(\frac{X_i}{X_j}\right)^{a_i b_j}
\end{aligned}$$

and observe that X_i/X_j is the ratio labeling that edge of T.

The vertices are partially ordered by their positions in the tree; the effect of a flop on the partial ordering is a single change, $(v_i \leq v_j) \mapsto (v_j \leq v_i)$, in between a pair of vertices which are adjacent in the partial order. In general, every maximal tree is obtained from a *fixed* tree T by a sequence of independent flops – by independent we just mean that each flop involves a different pair of vertices.

To prove the general case we can do induction on the number of independent flops needed to get from a fixed maximal tree T, to any other maximal tree T'. We have already proved the base case, so consider a tree T' obtained after a sequence of $k+1$ flops. Write $\widetilde{T}$ for a tree which is k independent flops away from T and one flop away from T'. Suppose that the the final flop between $\widetilde{T}$ and T' is described by the change $(v_i \leq v_j) \mapsto (v_j \leq v_i)$ for a pair of adjacent vertices v_i and v_j. By the inductive hypothesis and the base step,

$$\begin{aligned}\frac{\mathbf{X}^{\mathbf{N}_{T'}}}{\mathbf{X}^{\mathbf{N}_T}} &= \frac{\mathbf{X}^{\mathbf{N}_{T'}}}{\mathbf{X}^{\mathbf{N}_{\widetilde{T}}}}\frac{\mathbf{X}^{\mathbf{N}_{\widetilde{T}}}}{\mathbf{X}^{\mathbf{N}_T}} \\ &= \left(\frac{X_i}{X_j}\right)^m R_{i_1}^{m_1} R_{i_2}^{m_2} \dots R_{i_r}^{m_r}\end{aligned}$$

for some positive integers $m_1, \dots, m_r$ and m, and some ratios $R_{i_1}, \dots, R_{i_r}$ in the chart for T. Since none of the previous flops involved the pair v_i and v_j, the partial order in the original tree T must have also had $v_i \leq v_j$, although they were possibly not adjacent in T. In any case, the ratio X_i/X_j is a product of ratios for the edges in the path from v_i to v_j. This completes the inductive step.

□

Theorem 3.5.2. $\overline{\mathcal{R}}^d$ *is homeomorphic to the non-negative part of the projective toric variety* V.

Proof. We show that each chart $\mathcal{R}(T_i)$ is identified with the non-negative part of the affine slice $V \cap \mathbb{A}_i$, where

$$\mathbb{A}_i = (\xi_1 : \xi_2 : \dots : \underbrace{1}_{i^{th}} : \dots : \xi_k).$$

We prove it for the the first affine piece. $V \cap \mathbb{A}_1$ consists of all points

$$(1 : \frac{\mathbf{X}^{\mathbf{N}_{T_2}}}{\mathbf{X}^{\mathbf{N}_{T_1}}} : \dots : \frac{\mathbf{X}^{\mathbf{N}_{T_k}}}{\mathbf{X}^{\mathbf{N}_{T_1}}})$$

where the ratios are allowed to be 0. Lemma 3.5.1 says that for any edge in T_1, with ratio say X_i/X_j, there is an entry $(X_i/X_j)^m$ in the slot belonging to the tree obtained by a flop of that edge. Note that for any positive integer m, the map $r \mapsto r^m$ is a homeomorphism for $r \in [0, \infty)$, which is the domain of the ratios in the chart $\mathcal{R}(T_1)$. The other entries are higher products of ratios in $\mathcal{R}(T_1)$ so depend smoothly on the chart $\mathcal{R}(T_1)$. Therefore $\mathcal{R}(T_1)$ is homeomorphic to the non-negative part of $V \cap \mathbb{A}_1$.

□

Corollary 3.5.3. *$\overline{\mathcal{R}}^d$ is homeomorphic to a polytope in $\mathbb{R}^{d-1}$ which is the convex hull of the weight vectors. In particular it is homeomorphic to the Stasheff associahedron K_d.*

Proof. The non-negative part of a projective toric variety constructed with weight vectors is homeomorphic, via the moment map, to the convex hull of the weight vectors (see, for example, [6], [15]), which is Loday's convex hull realization of the associahedron.

□

3.6 The multiplihedron.

Stasheff also introduced a family of CW-complexes called the *multiplihedra*, which play the same role for A_∞ maps as the associahedra play in the recognition principle for loop or A_∞ spaces. The d-th multiplihedron J_d is a complex of dimension $d-1$ whose vertices correspond to ways of bracketing d variables $x_1, \ldots, x_d$ and applying an operation, say f. The multiplihedron J_3 is the hexagon shown in Figure 3.15.

The facets of J_d are of two types. First, there are the images of the inclusions

$$J_{i_1} \times \ldots \times J_{i_j} \times K_j \to J_d$$

for partitions $i_1 + \ldots + i_j = d$, and secondly the images of the inclusions

$$J_{d-e+1} \times K_e \to J_d$$

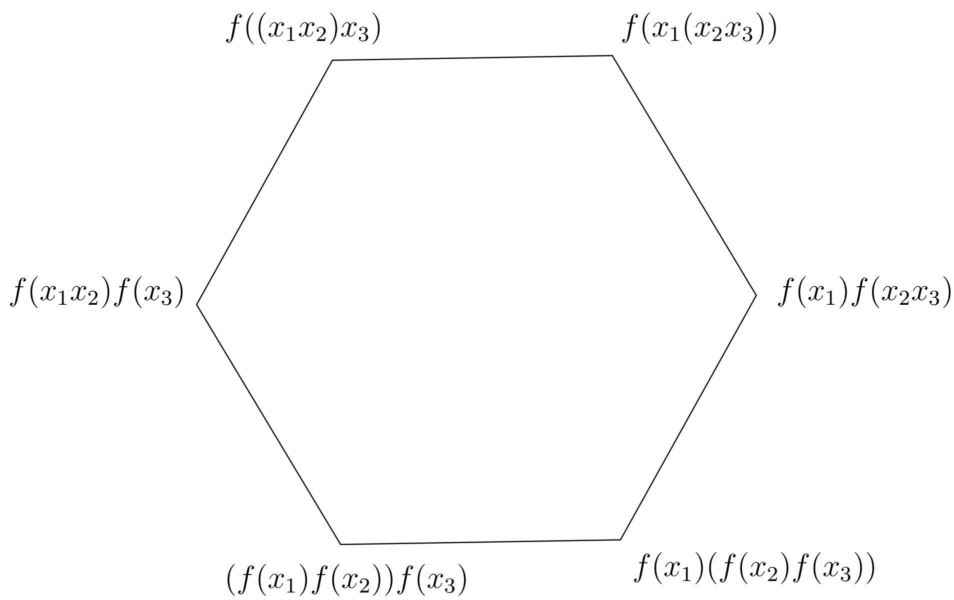

Figure 3.15: Vertices of J_3

for $2 \leq e \leq d$. One constructs the multiplihedron inductively starting from setting J_2 and K_3 equal to closed intervals.

Each vertex corresponds to a rooted tree with two types of vertices, the first a trivalent vertex corresponding to a bracketing of two variables and the second a bivalent vertex corresponding to an application of f.

Dualizing the rooted tree gives a triangulation of the $d + 1$-gon together with a partition of the two-cells into two types, dependening on whether they occur before or after a bivalent vertex in a path from the root.

The edges of J_d are of two types:

1. A change in bracketing

$$\dots x_{i-1}(x_i x_{i+1}) \dots \mapsto (x_{i-1}x_i)x_{i+1}$$

   (a flop of the type shown in Figure 3.3) or vice-versa

2. A move of the form

$$\dots f(x_i x_{i+1}) \dots \mapsto f(x_i)f(x_{i+1}) \dots$$

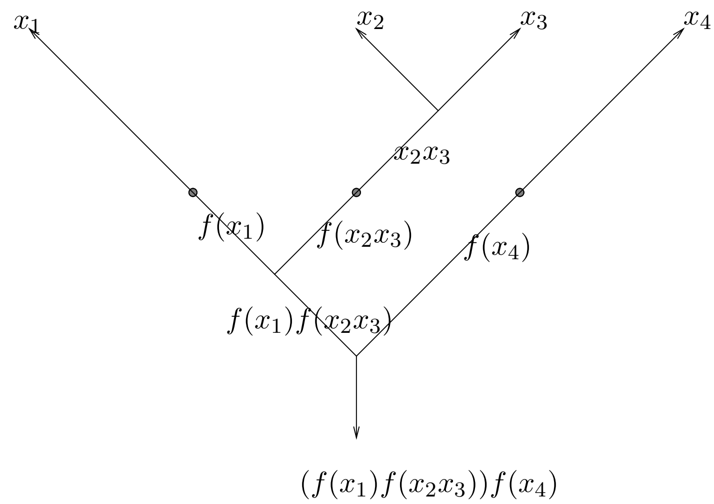

Figure 3.16: Tree for $(f(x_1)f(x_2x_3))f(x_4)$

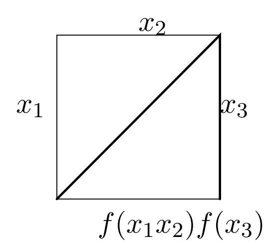

Figure 3.17: Triangulation corresponding to $f(x_1x_2)f(x_3)$

or vice versa, which corresponds to moving one of the bivalent vertices past a trivalent vertex, after which it becomes a pair of bivalent vertices, or vice-versa:

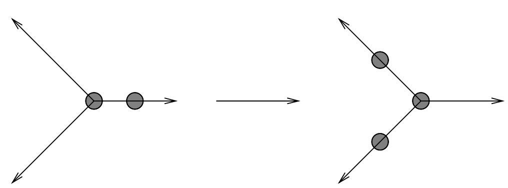

Figure 3.18: Splitting of bivalent vertices

3.7 Moduli spaces of quilted disks

In this section, a *quilted disk* refers to the unit disk $D \subset \mathbb{C}$ together with a circle $C \subset D$ (the *seam* of the quilt) tangent to a unique point in the boundary. Thus C divides the interior of D into two components. Given quilted disks (D_0, C_0) and (D_1, C_1), a morphism from (D_0, C_0) to (D_1, C_1) is a holomorphic isomorphism $D_0 \to D_1$ mapping C_0 to C_1.

Any quilted disk is isomorphic to the pair (D, C) where D is the unit disk in the complex plane and C the circle of radius $1/2$ passing through 1 and 0. Let $H \subset \mathbb{C}$ denote the upper half plane. Consider the map $D \setminus \{1\} \to H$ given by $z \mapsto -i/(z-1)$. The image of the circle C is the horizontal line L through i. Thus the automorphism group of (H, L) is the group $T \subset SL(2, R)$ of translations by real numbers.

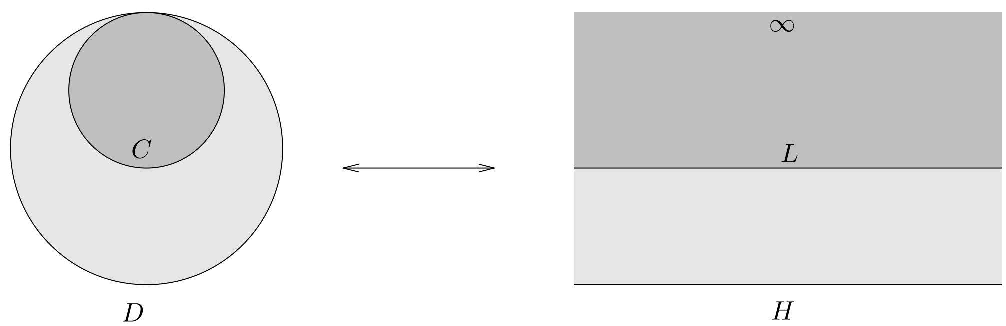

Figure 3.19: Quilted disk as a quilted half-plane

Definition Let $d \geq 1$. A *quilted disk with* $d+1$ *markings* is a tuple $(D, C; z_0, \ldots, z_d)$ where D is the unit disk, $z_0, \ldots, z_d \in \partial D$ are distinct marked points in counterclockwise order, and $C \subset D$ is a circle such that $C \cap \partial D = \{z_0\}$.

In other words, C is an inner circle tangent to z_0, of radius between 0 and 1. A morphism $(D_0, C_0; z_0, \ldots, z_d) \to (D_1, C_1; w_0, \ldots, w_d)$ is a holomorphic isomorphism $D_0 \to D_1$ mapping C_0 to C_1 and z_j to w_j for $j = 0, \ldots, d$. Let $\mathcal{R}^{d,1}$ be the set of isomorphism classes of $d+1$-marked quilted disks. We compactify $\mathcal{R}^{d,1}$ as follows. A *nodal* $(d+1)$-*quilted disk* S is a collection of quilted and unquilted marked disks, identified at pairs of points on the boundary. The combinatorial type of S is a graph Γ with two types of vertices, depending on whether the corresponding disk component is quilted or not. We require that

1. The combinatorial type of S is a tree.

2. Each unquilted disk component contains at least 3 singular or marked points.

3. Each quilted disk component is attached to only unquilted components;

4. The unique non-self-crossing path from the semi-infinite edge marked z_0 to the semi-infinite edge z_j crosses exactly one quilted vertex, for each $j = 1, \ldots, d$.

A nodal quilted disk is called *semistable* if

1. Each quilted disk component contains at least 2 singular or marked points;

2. Each unquilted disk component contains at least 3 singular or marked points.

Thus the automorphism group of any disk component is trivial, and from this one may derive that the automorphism group of any semistable $d+1$-marked nodal quilted disk is also trivial.

Remark The appearance of the two kinds of disks can be explained by the following bubbling considerations. Suppose that $(D_\alpha, C_\alpha; z_{0,\alpha}, \ldots, z_{d,\alpha})$ is a sequence in $\mathcal{R}^{d,1}$. For any sequence of rescalings $\varphi_\alpha : D_\alpha \to D_\alpha$ consider the following set of real numbers:

$$\operatorname{dist}(z_{\alpha,i}, z_{\alpha,j}), \operatorname{dist}(z_{\alpha,i}, C_\alpha), \operatorname{dist}(C_\alpha, \partial D_\alpha).$$

We say that a subset of $\{z_{\alpha,i}, C_\alpha\}$ of size at least 3 is *on the same scale* if after some sequence of re-scalings, the distances approach finite, non-zero values. An *admissible sequence of rescalings* is one for which some subset of size at least 3 is on the same scale. Two admissible sequences of rescalings are *equivalent* if the numbers above have the same limits. Each admissible sequence of rescalings defines a partition of the $\{z_{\alpha,i}\}$ according to which points have zero distance limit after the rescaling . That is, we say

$$z_{\alpha,i} \sim_{\varphi_\alpha} z_{\alpha,j}$$

if

$$\operatorname{dist}(\varphi_\alpha(z_{\alpha,i}), \varphi_\alpha(z_{\alpha,j})) \to 0, \quad \alpha \to \infty.$$

Each admissible sequence of rescalings gives rise to a bubble in the limit, of three kinds: Either $C_\alpha \to \partial D_\alpha$ in the limit, in which case we say that the resulting bubble has no interior circle (the circle is now at radius 1), or C_α approaches a circle of radius in $(0,1)$, in which case we say that the bubble *has interior circle*, or the radius approaches zero, in which case we say that the bubble *has no interior circle*. Thus the limiting sequence is a bubble tree, whose bubbles are of the two types discussed above.

Let $\overline{\mathcal{R}}^{d,1}$ denote the set of isomorphism classes of semistable $d+1$-marked nodal quilted disks.

Example $\overline{\mathcal{R}}^{3,1}$ is a hexagon, see Figure 3.20.

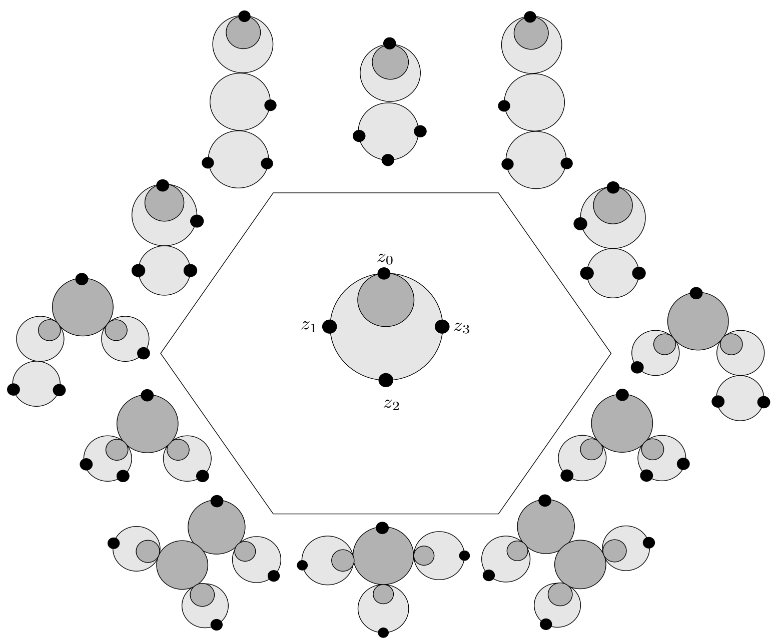

Figure 3.20: $\overline{\mathcal{R}}^{3,1}$

3.7.1 Topology via cross-ratios

We introduce a topology on $\overline{\mathcal{R}}^{d,1}$ via cross-ratios, as in the previous section for the moduli space of semistable nodal marked disks. Let D denote the unit disk, C a circle in D passing through a unique point z_0 and $z_1, z_2 \in D$ points in D such that z_0, z_1, z_2 are distinct. Let w be a point in C not equal to z_0. Writing $\mathrm{Im}(z)$ for the imaginary part of a complex number z, we define

$$\rho_{3,1}(D, C, z_1, z_2) = \mathrm{Im}(\rho(z_0, z_1, z_2, w))$$

$\rho_{3,1}$ is independent of the choice of w and invariant under the group of automorphisms of the disk and so defines a map

$$\rho_{3,1} : \mathcal{R}^{3,1} \to (0, \infty).$$

We extend $\rho_{3,1}$ to $\overline{\mathcal{R}}^{3,1}$ by setting $\rho_{3,1}(S) = 0$ if S is the 3-marked quilted nodal disk with three components, and $\rho_{3,1}(S) = \infty$ if S is the 3-marked nodal disk with two components. Thus $\rho_{3,1}$ extends to a bijection

$$\rho_{3,1} : \overline{\mathcal{R}}^{3,1} \to [0, \infty].$$

More generally, given $d \geq 3$ and a pair i, j of distinct, non-zero vertices, let Γ_{ij} denote the minimal connected subtree of Γ containing the semininfinite edges corresponding to z_i, z_j, z_0. There are three possibilities for Γ_{ij}, depending on whether the quilted vertex appears closer or further away than the trivalent vertex from z_0, or equals the trivalent vertex.

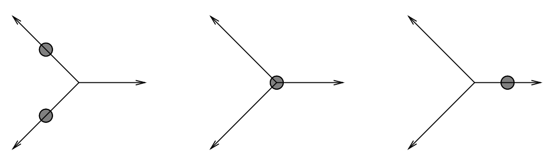

Figure 3.21: Tree types for J_3

If the first, resp. third case define $\rho_{ij}(S) = 0$ resp ∞. In the second case let (D, C) denote the disk component corresponding to the trivalent vertex, $w_i, w_j \in \partial D$ the points corresponding to the semiinfinite edges labelled z_i, z_j, and define

$$\rho_{ij}(S) = \rho_{3,1}(D, C, w_i, w_j).$$

In addition, for any four distinct indices i, j, k, l we have the cross-ratio

$$\rho_{ijkl} : \overline{\mathcal{R}}^{d,1} \to [0, \infty]$$

defined in the previous section, obtained by treating the quilted disk component as an ordinary component. Consider the map

$$\rho_{d1} : \overline{\mathcal{R}}^{d,1} \to \mathbb{R}^{(d+1)d(d-1)(d-2)/4! + d(d-1)/2}$$

obtained from all the cross-ratios. Given an element $S \in \overline{\mathcal{R}}^{d,1}$, the combinatorial type of S can be obtained from examining which cross-ratios are 0 or ∞. First, ignoring types of vertices the tree Γ can be obtained from the cross-ratios ρ_{ijkl}. The cross-ratios ρ_{ij} determine whether the trivalent vertex of the tree Γ_{ij} is on the same side of the quilted vertices as z_0 or not. In addition, the isomorphism class of each disk component of S is determined by the cross-ratios ρ_{ijkl} and ρ_{ij} with values in $(0, \infty)$. Thus the map $\rho_{d,1}$ is injective and we define the topology on $\overline{\mathcal{R}}^{d,1}$ by pulling back the topology on the codomain. Since the codomain is Hausdorff and compact, so is $\overline{\mathcal{R}}^{d,1}$.

Remark As before, the maps $\overline{\mathcal{R}}^{d,1} \to \overline{\mathcal{R}}^{4}$, $\overline{\mathcal{R}}^{d,1} \to \overline{\mathcal{R}}^{3,1}$ are special cases of forgetful morphisms constructed as follows. For any subset $I \subset \{0, \ldots, n\}$ of size k we have a forgetful morphism

$$\overline{\mathcal{R}}^{d,1} \mapsto \overline{\mathcal{R}}^{k}$$

obtained by forgetting the position of the circle and collapsing all unstable components. The map

$$\overline{\mathcal{R}}^{d,1} \mapsto \overline{\mathcal{R}}^{d}$$

deserves special mention. Its fiber over an element $S \in \overline{\mathcal{R}}^{d}$ consists of a union of point and intervals whose number is the maximal length of a path from z_0 to $z_i, i \neq 0$ in the combinatorial type of S (resp. minus one). Shown in Figure 3.22 is a fiber consisting of three points and two open intervals.

Similarly, for any subset $J \subset \{1, \ldots, n\}$ of size l we have a forgetful morphism

$$\overline{\mathcal{R}}^{d,1} \mapsto \overline{\mathcal{R}}^{l,1}$$

obtained by forgetting the positions of $z_j, j \notin J$ and collapsing all unstable disk components. By definition, the topology on $\overline{\mathcal{R}}^{d,1}$ is the minimal topology such that all forgetful morphisms are continuous and the topology on $\overline{\mathcal{R}}^{3,1} \cong [0, \infty]$, $\overline{\mathcal{R}}^{4} \cong [-\infty, 0]$ is induced by the cross-ratio.

3.7.2 Properties of the cross-ratio coordinates.

The properties of the coordinates ρ_{ijkl} are listed in Section 3.3.2, and remain the same. For the coordinates $\rho_{i,j}$ there are similar properties, whose proofs (which we omit) are

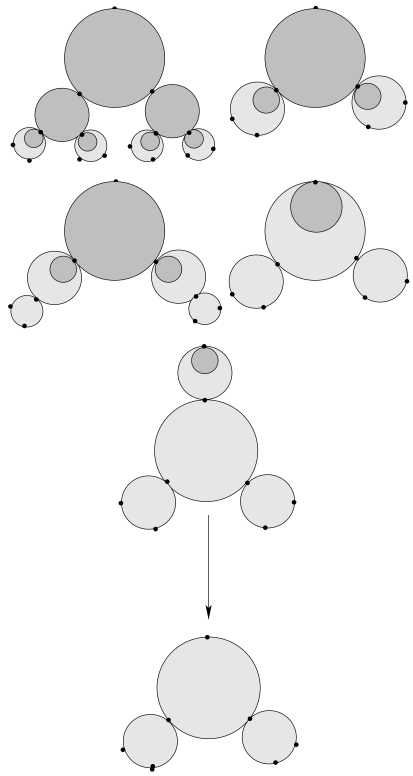

Figure 3.22: A fiber in $\overline{\mathcal{R}}^{4,1} \to \overline{\mathcal{R}}^{4}$. The projection forgets the inner circle and then collapses any unstable components to a point.

very minor modifications of the proofs for ρ_{ijkl}:

(Invariance): For all $p \in \overline{\mathcal{R}}^{d,1}$, and for all $\phi \in SL(2, \mathbb{R})$, $\rho_{i,j}(\phi(p)) = \rho_{i,j}(p)$.

(Symmetry): $\rho_{i,j} = -\rho_{j,i}$.

(Normalization): $\rho_{i,j} = \begin{cases} \infty, & \text{if } i \neq j \text{ and } L = \infty, \\ 0, & \text{if } i \neq j \text{ and } L = \mathbb{R}. \end{cases}$

(Recursion):

$$\rho_{i,k} = \frac{\rho_{i,j}}{\rho_{j,k}} \tag{3.5}$$

Finally, the two types of coordinate are related by

(Relations):

$$\rho_{j,k} = \frac{\rho_{i,j}}{-\rho_{ijk0}} \tag{3.6}$$

$$\rho_{i,k} = \frac{\rho_{i,j}}{1 - \rho_{ijk0}}. \tag{3.7}$$

To prove (3.6), pick a component of the nodal disk on which z_i, z_j and z_0 are distinct. Picking some $\zeta \in C$, without loss of generality we may set $z_i = 0, z_j = 1, z_0 = \infty, z_k =$

ρ_{ij0k} and $\zeta = \rho_{ij0\zeta}$. Then we have,

$$\begin{aligned}\rho_{jk0\zeta} &= \frac{(z_k - z_0)(\zeta - z_j)}{(z_j - z_k)(z_0 - \zeta)} \\ &= \frac{-(\rho_{ij0\zeta} - 1)}{1 - \rho_{ij0k}} \\ &= \frac{(\rho_{ij0\zeta} - 1)}{\rho_{ij0k} - 1} \\ &= \frac{\rho_{ij0\zeta} - 1}{-\rho_{ijk0}}\end{aligned}$$

Equating imaginary parts yields (3.6). The proof of (3.7) is almost identical.

3.7.3 Charts using cross-ratios

As in the case of the moduli space of nodal disks, one can use the cross-ratios to define local charts on the space of quilted disks, $\overline{\mathcal{R}}^{d,1}$. However, $\overline{\mathcal{R}}^{d,1}$ is *not* a manifold-with-corners. We say that a point $S \in \overline{\mathcal{R}}^{d,1}$ is a *singularity* if $\overline{\mathcal{R}}^{d,1}$ is not combinatorially a manifold with corners near S.

Example The first singular point occurs for $d = 4$. The vertex labeled by the expression $(f(x_1)f(x_2))(f(x_3)f(x_4))$ is adjacent to the expressions

$$f(x_1x_2)(f(x_3)f(x_4)), \quad (f(x_1)f(x_2))f(x_3x_4)$$

$$f(x_1)(f(x_2)(f(x_3)f(x_4))), \quad ((f(x_1)f(x_2))f(x_3))f(x_4)$$

and hence there are four edges coming out of the corresponding vertex. On the other hand, the dimension of $\mathcal{R}^{4,1}$ is 3, see Figure 3.23. Thus $\overline{\mathcal{R}}^{4,1}$ cannot be a manifold with corners (and therefore, not a simplicial polytope.)

Let $\Gamma_{d,1}$ be a combinatorial type in $\overline{\mathcal{R}}^{d,1}$. A cross-ratio chart associated to $\Gamma_{d,1}$ consists of

1. $n - 3$ cross-ratios for each disk component in $\Gamma_{d,1}$ that has n special features, where a special feature is either a marked point, a nodal point, or an inner circle of radius $0 < r < 1$;

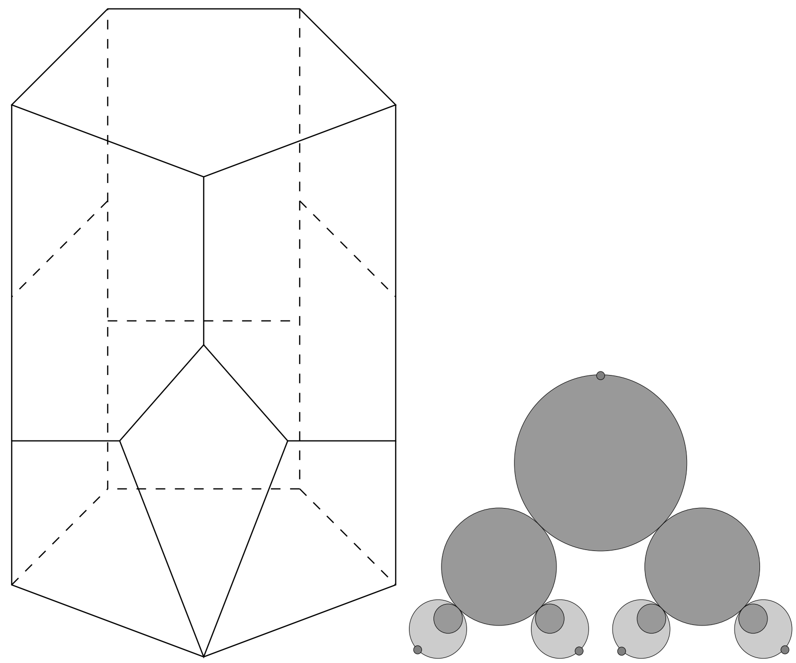

Figure 3.23: $\overline{\mathcal{R}}^{4,1}$, or "Chinese lantern". The singular point on the boundary, which has 4 edges coming out of it, corresponds to the nodal quilted disk at right.

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2. a coordinate $\rho_{abcd} = 0$ for each finite edge in Γ that is incident at each end to a trivalent vertex;

3. a coordinate $\rho_{a,b} = 0$ for each finite edge in Γ that is incident to a *bivalent* colored vertex from above, and a coordinate $1/\rho_{a,b} = 0$ for each finite edge in Γ that is incident to a bivalent colored vertex from below,

4. $k-1$ relations among these coordinates, where k is the number of colored vertices in Γ.

Rather than computing formulas for the $(k-1)$ relations between the cross-ratios in our chart, we will return to them in the next section after introducing an equivalent collection of charts on $\overline{\mathcal{R}}^{d,1}$, in which the relations are simpler.

Proposition 3.7.1. *Let $p \in \overline{\mathcal{R}}^{d,1}$. Suppose that p has combinatorial type $\Gamma_{d,1}$ with k colored vertices, and suppose that a set of chart coordinates has been chosen following (a), (b) and (c) above. Then, in a neighborhood of p, all cross-ratios $\rho_{i,j,k,l}$ and $\rho_{i,j}$ are smooth functions of those in the chart.*

Proof. First we prove that all cross-ratios of the form ρ_{ijkl} are smooth functions of those in the chart associated to $\Gamma_{d,1}$. Denote by Γ_d the combinatorial type in $\overline{\mathcal{R}}^d$ obtained from $\Gamma_{d,1}$ by forgetting colored vertices. Taking all cross-ratios of the form ρ_{ijkl} in the chart associated to $\Gamma_{d,1}$ *almost* gives a chart for Γ_d in the sense of Section 3.3.3. The only coordinates that might be missing for the chart are those corresponding to edges that have a bivalent colored vertex on them. For each bivalent vertex, we can assume that the lower edge has coordinate $\rho_{i,j} = \infty$ and the upper edge is either $\rho_{j,k} = 0$ or $\rho_{h,i} = 0$. Assuming the first case, relation (3.6) holds and

$$\rho_{i,j,k,0} = \frac{-\rho_{i,j}}{\rho_{j,k}},$$

expressing $\rho_{i,j,k,0}$ as a smooth function of the chart coordinates, and $\rho_{i,j,k,0}$ is a valid chart coordinate for the underlying edge in Γ_d. The other case is very similar, by relation (3.6),

$$\rho_{h,i,j,0} = \frac{-\rho_{h,i}}{\rho_{i,j}},$$

expresses $\rho_{h,i,j,0}$ as a smooth function of the chart coordinates, and $\rho_{h,i,j,0}$ is a valid chart coordinate for the underlying edge in Γ_d. Hence we get a valid chart for Γ_d. Now by Theorem 3.3.1 all cross-ratios of the form ρ_{abcd} are smooth functions of the chart coordinates.

Finally, relations (3.6) and (3.7) can be used to obtain all cross-ratios of the form $\rho_{a,b}$ from the cross-ratios of the form $\rho_{i,j}$ in the chart, and the appropriate ρ_{ijk0}'s. □

3.7.4 Charts using simple ratios.

As in Section 3.3.4 we describe an equivalent topology on $\mathcal{R}^{n,1}$ using coordinate charts based on "simple ratios". Choosing parametrizations such that $z_0 = \infty$, the elements of $\mathcal{R}^{n,1}$ can be identified with configurations of n distinct points $-\infty < z_1 < z_2 < \ldots < z_n < \infty$ in $\mathbb{R} \subset \mathbb{C}$, together with a horizontal line L in the upper half plane of $\mathbb{C}$, modulo transformations of the form $z \mapsto az + b$ for $a, b \in \mathbb{R}$ such that $a > 0$, i.e. dilation and translation. For such configurations define coordinates $(X_1, X_2, \ldots, X_n, Y)$ by $X_i = z_{i+1} - z_i$, and $Y = \mathrm{dist}(L, \mathbb{R})$. A transformation $z \mapsto az + b$ for $a, b \in \mathbb{R}$ sends

$$(X_1, X_2, \ldots, X_{n-1}, Y) \mapsto (aX_1, aX_2, \ldots, aX_{n-1}, aY),$$

so they are really projective coordinates, $(X_1 : X_2 : \ldots : X_{n-1} : Y)$.

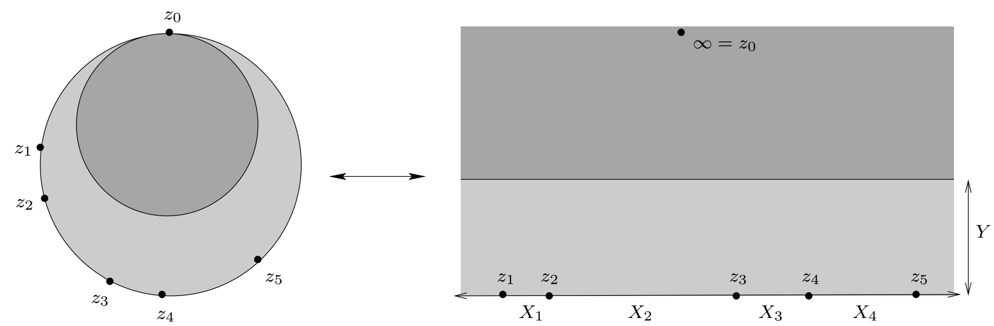

Figure 3.24: Projective coordinates $(X_1 : X_2 : X_3 : X_4 : Y)$ for a marked quilted disk in $\mathcal{R}^{5,1}$.

We construct new charts as follows. Each maximal bicolored tree has two types of edges: those that connect a pair of vertices v_i and v_j in the underlying graph, and those that connect a vertex v_i with either a colored vertex below it, or a colored vertex above it. For an edge of the first type: if the vertex v_i is below the vertex v_j, then the associated ratio is X_i/X_j. For an edge of the second type: if the vertex v_i is immediately below the colored vertex, then the associated ratio is Y/X_i; if the vertex v_i is immediately above the colored vertex, the associated ratio is X_i/Y. Write $\mathcal{R}(T)$ for the subset of $\mathcal{R}^{d,1}$ that is covered by a ratio chart corresponding to a maximal bicolored tree T.

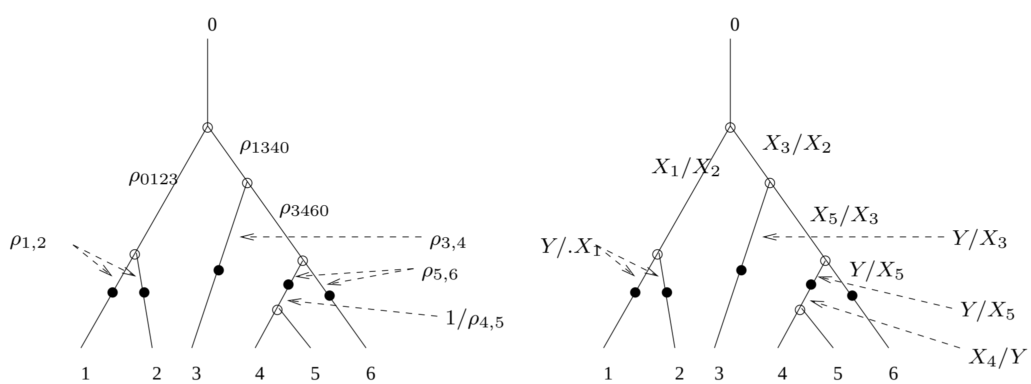

Figure 3.25: A cross-ratio chart and a simple-ratio chart for the same maximal bicolored tree.

3.7.5 Equivalence of charts

We show that each cross-ratio in the chart is a smooth function of the simple ratios, and is zero if and only if the corresponding simple ratio is zero. To show the other direction – that the simple ratios are smooth functions of the cross-ratios – is an argument that is similarly straightforward so we will not include it. We note again that if vertex v_i is below vertex v_j in the tree, then the ratio X_i/X_j is a product of the ratios representing the edges joining the vertex v_i to v_j. Hence the proof for a cross-ratio ρ_{ijk0} or ρ_{0ijk} follows from the proof given already in the case of ordinary trees. The only case that isn't already covered is that of a cross-ratio $\rho_{i,j}$ in the chart. Parametrizing so that $z_0 = \infty$ and writing Y for the height of the line with respect to this parametrization, consider an edge such as the one pictured in 3.26, where the cross-ratio assigned to it in the cross-ratio chart is $\rho_{i,j}$, while the simple ratio assigned to it is Y/X_r. Then we

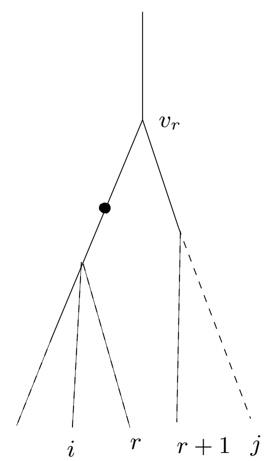

Figure 3.26: Comparing a cross-ratio $\rho_{i,j}$ with a ratio Y/X_r.

can write

$$\begin{aligned}
\rho_{i,j} &= \frac{Y}{z_j - z_i} \\
&= \frac{Y}{X_i + \ldots + X_r + \ldots + X_{j-1}} \\
&= \frac{Y}{X_r}\left(\frac{1}{\frac{X_i}{X_r} + \ldots + 1 + \ldots + \frac{X_{j-1}}{X_r}}\right)
\end{aligned}$$

where the ratios appearing in the big bracket are products of chart ratios corresponding to edges below v_r. The bracketed rational function is smooth and invertible for all positive non-zero ratios and it is continuous as ratios in the chart go to 0. Moreover $\rho_{i,j} = 0$ if and only if $Y/X_r = 0$. The other case, of a colored vertex above a regular vertex, is very similar so we omit it.

So the charts of simple ratios define the same topology on $\overline{\mathcal{R}}^{d,1}$ as the charts of cross-ratios.

3.8 Bicolored metric ribbon trees.

Definition A *bicolored metric ribbon tree* is a quintuple $(T, i, e_0, V_{col}, \lambda)$. T is a tree with semi-infinite exterior edges labeled $e_0, e_1, \ldots, e_d$, with e_0 distinguished as the "root" of the tree, while $e_1, \ldots, e_d$ are called the "leaves". The map $i : T \to D$ is an embedding of T into the unit disk such that the images of the exterior edges have a limit point on the boundary of D, with the limit points of $e_0, \ldots, e_d$ cyclically ordered following the counter clockwise orientation of ∂D. $V_{col} \subset V(T)$ is a collection of *colored vertices*, and $\lambda : E_{int}(T) \to \mathbb{R}_+$ is a map of edge lengths, subject to some conditions. Recall from Section 3.4 that the *geodesic* between two vertices v, v' in a metric tree T is the (unique) path joining them of shortest total length; the total length of the geodesic is called the *distance* from v to v'. The conditions on the data are:

1. In a geodesic from a leaf $e_i \in \{e_1, \ldots, e_d\}$ to the root e_0, exactly one vertex in the path is a colored vertex.

2. If a vertex $v \in V(T)$ is bivalent, then $v \in V_{col}$.

3. The edge length map λ is such that all colored vertices are the same distance from the root.

Example For the tree in Figure 3.27, an edge length map is subject to the relations

$$\begin{aligned}\lambda_1+\lambda_2+\lambda_3 &= \lambda_1+\lambda_2+\lambda_4\\ &= \lambda_1+\lambda_2+\lambda_5\\ &= \lambda_1+\lambda_6\\ &= \lambda_7.\end{aligned}$$

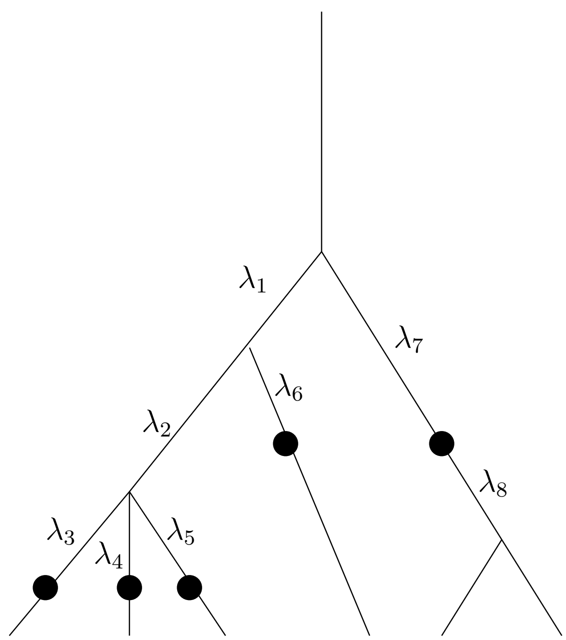

Figure 3.27: A bicolored ribbon tree. The relations on $\lambda_1,\ldots,\lambda_8$ imply that $\lambda_3=\lambda_4=\lambda_5, \lambda_3+\lambda_2=\lambda_6$, and $\lambda_6+\lambda_1=\lambda_7$.

For each tuple (T, V_{col}) satisfying conditions 1 and 2, we denote by $Gr(T, V_{col})$ the set of all maps λ satisfying condition 3, and then write

$$Gr_{k,1}=\bigcup_{(T,V_{col})} Gr(T,V_{col}).$$

We define a topology on $Gr(T, V_{col})$ as follows. Assume that a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of edge length maps converges for each interior edge e to a non-negative real number. In other words, $\lambda_n(e)\to\lambda_\infty(e)\in[0,\infty)$ for every $e\in E_{int}(T)$. Then, just as before, we define the limit to be the data $(T', i', V'_{col}, \lambda')$ given by

1. T' is the tree obtained from T by collapsing edges e for which $\lambda_\infty(e) = 0$. this defines a surjective morphism of bicolored based ribbon trees, $p : T \to T'$.

2. i' is the embedding obtained from i by contracting along collapsed edges.

3. $V'_{col} = p(V_{col})$

4. $\lambda'(e) = \lambda_\infty(p^{-1}(e))$, since every edge $e \in E_{int}(T')$ is the image of a unique edge in $E_{int}(T)$.

Proposition 3.8.1. *$Gr(T, V_{col})$ is a polyhedral cone in $\mathbb{R}^n$, where $n = |E_{int}| - |V_{col}| + 1$.*

Proof. There is an $\mathbb{R}_+$ action on $Gr(T, V_{col})$, given by $(\delta \cdot \lambda)(e) := \delta\lambda(e)$, so it is clearly a cone. The dimension follows from the fact that there are $|E_{int}|$ variables and $|V_{col}| - 1$ relations. The polyhedral structure can be seen by writing $|V_{col}| - 1$ variables as linear combinations of n independent variables. Then the condition that all $\lambda(e) \geq 0$ means that $Gr(T)$ is an intersection of half-spaces. □

Example In the example of Figure 3.27, $|E_{int}| = 8$, and $|V_{col}| = 5$. We can choose independent variables to be $\lambda_1, \lambda_2, \lambda_3, \lambda_8$, and express the remaining variables as

$$\begin{aligned}
\lambda_4 &= \lambda_3 \\
\lambda_5 &= \lambda_3 \\
\lambda_6 &= \lambda_2 + \lambda_3 \\
\lambda_7 &= \lambda_1 + \lambda_2 + \lambda_3.
\end{aligned}$$

Thus the space of admissible edge lengths is parametrized by points in the polyhedral cone that is the intersection of $\mathbb{R}^4_+$ (for the independent variables being non-negative) with the half-spaces $\lambda_4 \geq 0, \lambda_5 \geq 0, \lambda_6 \geq 0$ and $\lambda_7 \geq 0$.

Theorem 3.8.2. *$Gr_{k,1}$ is homeomorphic to $\mathcal{R}^{k,1}$, hence to $\mathbb{R}^{k-1}$.*

Proof. We define a homeomorphism $\Theta : Gr_{k,1} \to \mathcal{R}^{k,1}$.

Suppose first that the combinatorial type T has a single, bivalent, colored vertex. This is the case if and only if the colored vertex is adjacent to the root on one side, and a vertex of valency at least 3 on the other; call this vertex V. For $i = 1, \ldots, k-1$ let v_i be the unique vertex where the geodesic from e_i to V first intersects the geodesic from e_{i+1} to V. Set

$$z_0 = \infty, z_1 = 0, z_{i+1} - z_i = e^{-\operatorname{dist}(v_i, V)}.$$

If $\lambda \geq 0$ is the edge length of the edge between the colored vertex and V, we set the horizontal line to be $\Im(z) = e^{\lambda}$. For any other combinatorial type, set

$$z_0 = \infty, z_1 = 0, z_{i+1} - z_i = e^{-\operatorname{dist}(v_i, V)},$$

and the horizontal line to be

$$\operatorname{Im}(z) = e^{-\operatorname{dist}(V_{col}, V)},$$

which is independent of $v_{col} \in V_{col}$ because they are the same distance from the root. The continuity of Θ is clear, and injectivity and surjectivity follow as before. $\square$

Remark As before one can think of Θ as identifying the polyhedral cones in $Gr_{d,1}$ with cones in $\mathbb{R}^{d-1}$ in such a way that the boundaries match up as they should, and the union over the cones is $\mathbb{R}^{d-1}$.

Example Consider the case $d = 3$, where we have fixed the parametrization of the elements of $\mathcal{R}^{d,1}$ so that the height of the line L is 1. Let $x = z_2 - z_1$ and $y = z_3 - z_2$. The image of $\Theta : Gr_{3,1} \to \mathcal{R}^{3,1}$ subdivides $\mathbb{R}^2_{>0}$ into 6 regions, see Figure 3.29, each of which corresponds to a cone in $\mathbb{R}^2$ via the homeomorphism $(x, y) \mapsto (\log x, \log y)$.

There is a natural compactification of $Gr_{d,1}$ by allowing edges to have length ∞. The map Θ extends to the compactifications by taking limits in appropriate charts:

Proposition 3.8.3. *$\Theta : \overline{Gr_{d,1}} \to \overline{\mathcal{R}}^{d,1}$ is a homeomorphism.*

Proof. It follows from the observation that if λ is the length of an edge in $Gr(T)$, then $e^{-\lambda}$ is the value of the ratio corresponding to that edge. Thus the image of a cone $Gr(T)$ is contained in the image of the ratio chart $\mathcal{R}(T)$, and is identified directly with the subset of $\mathcal{R}(T)$ for which all ratios in the chart are restricted to lie in the interval $(0, 1]$. This identification passes to the limits as λ approaches ∞ and the corresponding ratio approaches 0. This shows that Θ is a homeomorphism between $\overline{\mathcal{R}}^{d,1}$ and $\overline{Gr}_{d,1}$. $\square$

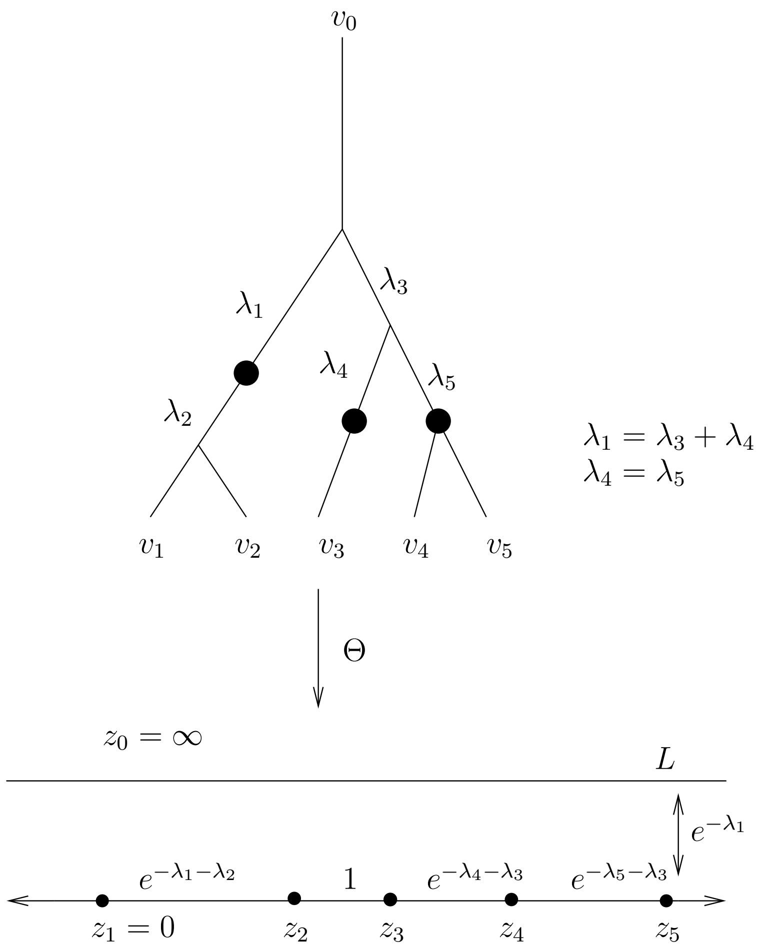

Figure 3.28: Identifying a bicolored metric ribbon tree in $Gr_{5,1}$ with a marked quilted disk in $\mathcal{R}^{5,1}$.

3.9 Toric varieties and moment polytopes.

We show that $\overline{\mathcal{R}}^{d,1}$ can be identified with the non-negative part of an embedded toric variety in $\mathbb{C}P^k$, where k is the number of maximal bicolored trees with d leaves. Each such tree determines a monomial in the d variables $X_1, X_2, \ldots, X_{d-1}, Y$. The weight vector of the monomial is read directly from the combinatorics of the tree, based on the algorithm of Forcey in [3] for a convex hull realization of the d-th multiplihedron. An immediate consequence is that $\overline{M}_{d,1}$ is homeomorphic to the moment polytope of the toric variety. Since the moment polytope is the convex hull of the weight vectors, this is a multiplihedron.

Recall from Section 3.7.4 that a point in $\mathcal{R}^{d,1}$ can be identified with a projective

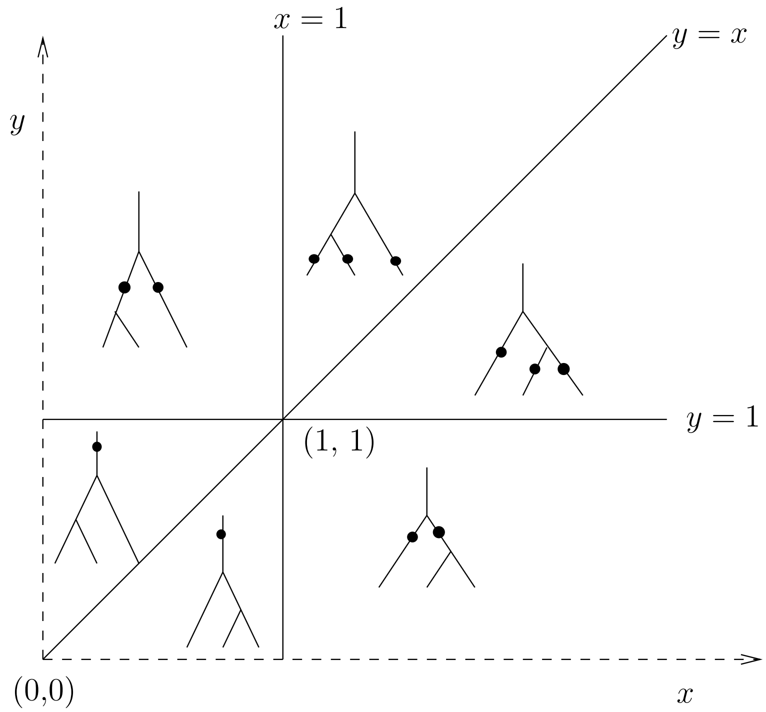

Figure 3.29: The image of the cones of $Gr_{3,1}$ in the moduli space $\mathcal{R}^{3,1}$.

coordinate

$$\mathbf{X} = (X_1 : X_2 : \ldots : X_{d-1} : Y),$$

by choosing a parametrization such that $z_0 = \infty$, and setting $X_i = z_{i+1} - z_i$ and Y to be the height of the line. To every maximal bicolored tree we associate a weight vector $\mathbf{N}_T \in \mathbb{Z}^d$, following the algorithm of [3].

Each pair of adjacent leaves in T, labelled i and $i+1$ say, determines a unique vertex, which we label v_i, in T. Let a_i be the number of leaves on the left side of v_i, and let b_i be the number of leaves on the right side of v_i. Let

$$\delta_i = \begin{cases} 0 & \text{if } v_i \text{ is below the level of the colored vertices, and} \\ 1 & \text{if } v_i \text{ is above the colored vertices.} \end{cases}$$

The weight vector is

$$\mathbf{N}_T = (a_1 b_1(1+\delta_1), \ldots, a_i b_i(1+\delta_i), \ldots, a_{d-1} b_{d-1}(1+\delta_{d-1}), -\sum_i \delta_i a_i b_i).$$

Example For the tree in Figure 3.30, the weight vector is $(2, 16, 6, 1, 4, -14)$, so the corresponding monomial is $X_1^2 X_2^{16} X_3^6 X_4 X_5^4 Y^{-14}$.

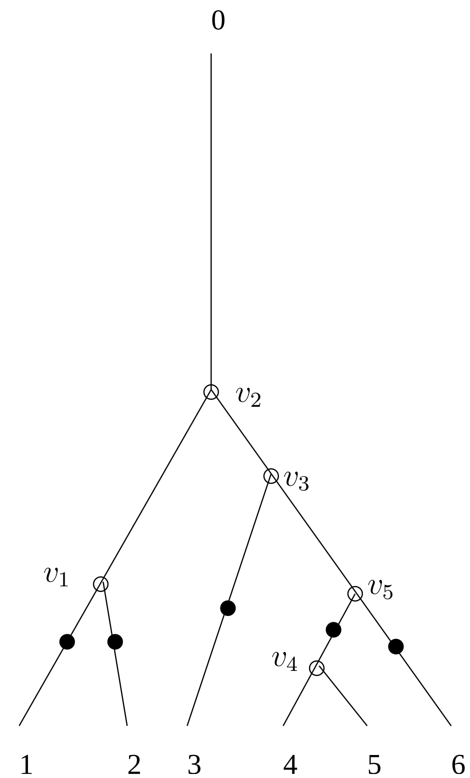

Figure 3.30: A maximal bicolored tree, whose weight vector is $(2, 16, 6, 1, 4, -14)$.

Label the maximal bicolored trees $T_1, \ldots, T_k$. We define a projective toric variety $V \subset \mathbb{C}P^{k-1}$ as the closure of the image of

$$(X_1 : \ldots : X_{d-1}) \mapsto (\mathbf{X}^{\mathbf{N}_{T_1}} : \ldots : \mathbf{X}^{\mathbf{N}_{T_k}}). \tag{3.8}$$

The entries in the weight vectors always sum to $d(d-1)/2$, so the monomials all have the same degree and the map is well-defined on the homogeneous coordinates.

Definition For a maximal bicolored tree T, we can define a *flop* of an interior edge e as in Definition 3.5, as long as the edge e is incident to a pair of trivalent vertices. We define a *fusion* move through an interior vertex v_i to be the move by which two colored vertices immediately below v_i become a single colored vertex immediately above v_i; the vice-versa we will call a *splitting* move. We will say that two maximal bicolored trees T and T' differ by a *basic move* if they differ by a flop, fusion, or splitting move.

The important point is that any maximal bicolored tree can be obtained from any other such tree by a sequence of basic moves.

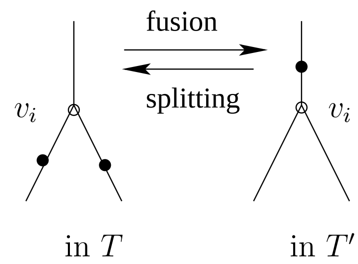

Figure 3.31: The other basic moves in a bicolored tree: fusion and splitting.

Lemma 3.9.1. *Suppose that two maximal bicolored trees T and T' differ by a single basic move involving an edge $e \in E(T)$. Let R denote the simple ratio labeling the edge e in the chart determined by T. Then*

$$\frac{\mathbf{X}^{\mathbf{N}_{T'}}}{\mathbf{N}_T} = R^m$$

for some integer $m > 0$. In general, for two trees T and T',

$$\frac{\mathbf{X}^{\mathbf{N}_{T'}}}{\mathbf{N}_T} = R_{i_1}^{m_1} R_{i_2}^{m_2} \ldots R_{i_r}^{m_r}$$

for some ratios $R_{i_i}, \ldots, R_{i_r}$ in the ratio chart associated to T, and positive integers $m_1, \ldots, m_r$.

Proof. Any maximal bicolored tree can be obtained from another by a sequence of basic moves. The first type, the flop, was dealt with in Lemma 3.5.1, and since the proof is practically the same we omit it here. For the other types of basic move, consider the situation in Figure 3.31, in which a colored vertex is below v_i in T, and above v_i in T'.

The weight vectors of T and T' are identical in all entries except for the i-th entry, which corresponds to the exponent of X_i, and the $n+1$-th entry, which corresponds to the exponent of Y:

$$\begin{aligned}
(\mathbf{N}_T)_i &= 2a_i b_i, \\
(\mathbf{N}_{T'} &= a_i b_i, \\
(\mathbf{N}_{T'})_{n+1} - (\mathbf{N}_T)_{n+1} &= -(0) - (-a_i b_i)
\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\mathbf{X}^{\mathbf{N}_{T'}}}{\mathbf{N}_T} &= \frac{X_i^{a_ib_i}Y^{-0}}{X_i^{2a_ib_i}Y^{-a_ib_i}}\\ &= \frac{Y^{a_ib_i}}{X_i^{a_ib_i}}\\ &= \left(\frac{Y}{X_i}\right)^{a_ib_i}\end{aligned}$$

where Y/X_i is the ratio labeling the two edges below v_i of T. The general case follows by induction on the number of basic moves.

$\square$

Theorem 3.9.2. $\overline{\mathcal{R}}^{d,1}$ *is homeomorphic to the non-negative part of the projective toric variety* V*.*

Proof. The proof proceeds just like the proof of Theorem 3.5.2. We show that each chart $\mathcal{R}(T_i)$ is identified with the non-negative part of the affine slice $V \cap \mathbb{A}_i$, where

$$\mathbb{A}_i = (\xi_1 : \xi_2 : \ldots : \underbrace{1}_{i^{th}} : \ldots : \xi_k).$$

We prove it for the the first affine piece. $V \cap \mathbb{A}_1$ consists of all points

$$(1 : \frac{\mathbf{X}^{\mathbf{N}_{T_2}}}{\mathbf{X}^{\mathbf{N}_{T_1}}} : \ldots : \frac{\mathbf{X}^{\mathbf{N}_{T_k}}}{\mathbf{X}^{\mathbf{N}_{T_1}}})$$

where the ratios are allowed to be 0. Lemma 3.5.1 says that for any edge in T_1, say with ratio R, there is an entry R^m in the slot belonging to the tree obtained by a flop of that edge. For any positive integer m, the map $r \mapsto r^m$ is a homeomorphism for $r \in [0, \infty)$, which is the domain of the ratios in the chart $\mathcal{R}(T_1)$. The other entries are higher products of ratios in $\mathcal{R}(T_1)$ so depend smoothly on the chart $\mathcal{R}(T_1)$. Therefore $\mathcal{R}(T_1)$ is homeomorphic to the non-negative part of $V \cap \mathbb{A}_1$. $\square$

Theorem 3.9.3. $\overline{\mathcal{R}}^{d,1}$ *is homeomorphic to the convex hull of the weight vectors in* $\mathbb{R}^d$*. Thus it is a* $(d-1)$*-dimensional polytope, homeomorphic to the Stasheff multiplihedron* J_d*.*

Proof. The non-negative part of a projective toric variety constructed with weight vectors is homeomorphic, via the moment map, to the convex hull of the weight vectors (see, for example, [6], [15]). The proof that it is homeomorphic to the Stasheff multiplihedron K_d is by induction on d: the one-dimensional spaces $\overline{\mathcal{R}}^{2,1}, \overline{\mathcal{R}}^{3}$, J_2 and K_3 are all compact and connected, and so homeomorphic. It suffices, therefore, to show that $\overline{\mathcal{R}}^{d,1}$ is the cone on its boundary. This is true since it is homeomorphic to a polytope. $\square$

Chapter 4

Quilts

4.1 Outline of chapter

There are two goals of this chapter. First, to set up the definition of pseudoholomorphic maps for general Riemann surfaces with striplike ends, and their quilted generalizations. Second, to construct families of quilted surfaces that are parametrized by the associahedra and multiplihedra, which will be used as domains for the Floer theory of Chapter 5.

4.2 Surfaces with striplike ends

Let S be a compact Riemann surface with boundary $\partial S \cong \cup_{j=1}^{k} S^1$. Suppose that the boundary is equipped with a collection of distinct marked points $\zeta_1, \ldots, \zeta_n \in \partial S$. A *striplike end* for a marked point $\zeta \in \partial S$ is a proper holomorphic embedding $\epsilon_\zeta : [0, \infty) \times [0, 1] \to S$ such that

1. $\epsilon_\zeta^{-1}(\partial S) = [0, \infty) \times \{0\} \cup [0, \infty) \times \{1\}$
2. $\lim_{s \to \infty} \epsilon_\zeta(s, t) = \zeta$, where the convergence is uniform in $t \in [0, 1]$.

We always assume that the images of striplike ends for distinct marked points are disjoint in S.

4.3 Quilted surfaces

Definition (Wehrheim-Woodward) A *quilted surface* $\underline{S}$ with strip-like ends consists of the following data:

1. A collection $\underline{S} = (S_k)_{k=1,\dots,m}$ of Riemann surfaces with boundary, equipped with strip-like ends. Let $\mathcal{E}(S_k)$ denote the set of boundary components of S_k.

2. A collection Σ of pairwise disjoint 2-element subsets

$$\sigma \subset \bigcup_{k=1}^{m} \{k\} \times \mathcal{E}(S_k),$$

and for each $\sigma = \{(k_\sigma, e_\sigma), (k'_\sigma, e'_\sigma)$, a real-analytic identification of the corresponding boundary components

$$\phi_\sigma : I_{k_\sigma, e_\sigma} \xrightarrow{\sim} I_{k'_\sigma, e'_\sigma}.$$

We call each pair $\sigma \in \Sigma$ a *seam* of $\underline{S}$. The identification map along a seam should be *compatible* with the strip-like ends: meaning that if a pair of striplike ends attached to each side of a seam are holomorphically identified with the half-infinite strips $[0,1] \times [0,\infty)$ and $[1,2] \times [0,\infty)$ such that the pair of boundary components in the seam mapping to $1 \times [0,\infty)$, then the seam identification is the identity map on $1 \times [0,\infty)$.

4.4 Pseudoholomorphic quilts

Let $\underline{S} = (S_k)_{1,\dots,m}$ be a quilted surface with strip-like ends, and for each component S_k let j_k be its complex structure. For each S_k fix a target symplectic manifold (M_k, ω_k). Write $\underline{M} = (M_k)_{1,\dots,m}$ for the collection of target manifolds.

Definition A *Lagrangian boundary condition* for $(\underline{S}, \underline{M})$ is a set $\underline{L}$ of Lagrangian submanifolds as labels of each boundary component and seam of $\underline{S}$: every boundary component of $\underline{S}$ that is a boundary component of S_k is labeled by a Lagrangian submanifold of M_k, and every seam identifying a boundary component of S_k with a boundary component of $S_{k'}$ is labeled by a Lagrangian correspondence $L_{(k,k')} \subset M_k^- \times M_{k'}$.

Write $I_{k,i}, i = 1, \dots, k_j$ for the boundary components of S_k that are not seams, and $L_{k,i} \subset M_k$ for their Lagrangian labels. For each seam $\sigma = \{(k_\sigma, e_\sigma), (k'_\sigma, e'_\sigma)\} \in \Sigma$, write $L_\sigma \subset M_{k_\sigma}^- \times M_{k'_\sigma}$ for its Lagrangian label.

For each of the symplectic manifolds $(M_k, \omega_k) \in \underline{M}$, let

1. $\mathcal{H}_k = C^\infty(M_k, \mathbb{R})$ be the space of Hamiltonian functions on M_k;

2. $\mathcal{J}_k$ be the space of all ω_k-compatible almost complex structures on M_k.

A *perturbation datum* is a pair of tuples $(\underline{K}, \underline{J}) = (K_k, J_k)_{k=1,\ldots,m}$ where $K_k \in \Omega^1(S_k, \mathcal{H}_k)$ and $J_k \in C^\infty(S_k, \mathcal{J}_k)$. Each K_k is a 1-form on S_k taking values in $\mathcal{H}_k$. Thus K_k determines a 1-form $Y_k = \Omega^1(S_k, C^\infty(TM_k))$ taking values in the space of Hamiltonian vector fields on M_k – for each $\xi \in TS_k$, $Y_k(\xi)$ is the Hamiltonian vector field associated to the Hamiltonian $K_k(\xi)$.

Let $\underline{u} = (u_k)_{k=1,\ldots,m}$ be a tuple of maps $u_k : S_k \to M_k$.

Definition The *inhomogeneous pseudoholomorphic quilt equation* for $\underline{u}$ is

$$\left.\begin{aligned} du_k(z) + J_k(z,u) \circ du_k(z) \circ j_k = Y_k(z,u) + J_k(z,u) \circ Y_k(z,u) \circ j_k \\ u_k\big|_{I_{k,i}} \subset L_{k,i}, \qquad u_{k_\sigma} \times u_{k'_\sigma}\big|_{(I_\sigma \times_{\phi_\sigma} I'_\sigma)} \subset L_\sigma \end{aligned}\right\} \tag{4.1}$$

for all $k = 1, \ldots, m$ and seams $\sigma \in \Sigma$.

4.5 Quilted strips

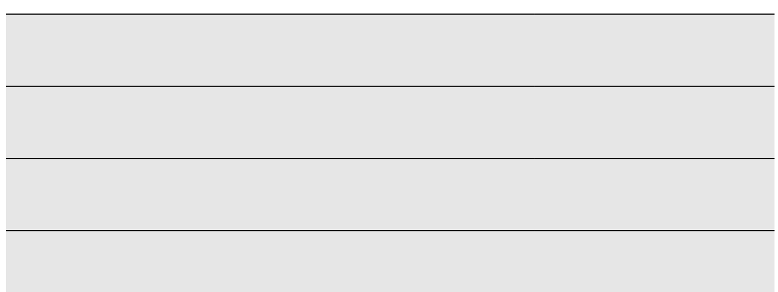

Figure 4.1: A quilted strip.

Let $Z = \mathbb{R} \times i[0,1] \subset \mathbb{C}$ denote the infinite strip with coordinates $s + it$. Denote its standard complex structure by j.

Definition A *quilted strip* is a tuple of strips, $\underline{Z} = (Z_k)_{1,\ldots,m}$, together with a tuple of holomorphic isomorphisms $\phi_k : Z_k \to \mathbb{R} \times i[k-1, k]$ such that $\mathbb{R} \times \{0\} \mapsto \mathbb{R} \times \{k-1\}$ and $\mathbb{R} \times \{1\} \mapsto \mathbb{R} \times \{k\}$. The seam maps are defined by the identity maps along the overlaps $\mathbb{R} \times i\{k\} = \phi_k(Z_k) \cap \phi_{k+1}(Z_{k+1})$, for $k = 1, \ldots, m-1$. A quilted strip $\underline{Z}$ with $k \geq 1$ components has $k-1$ seams and 2 boundary components.

Each strip has a one-dimensional family of automorphisms, corresponding to translation in the s variable. The seam conditions make an automorphism of a single strip extend to a simultaneous automorphism of all strips, so a quilted strip also has a one-dimensional family of automorphisms.

4.6 Quilted surfaces parametrized by the associahedron

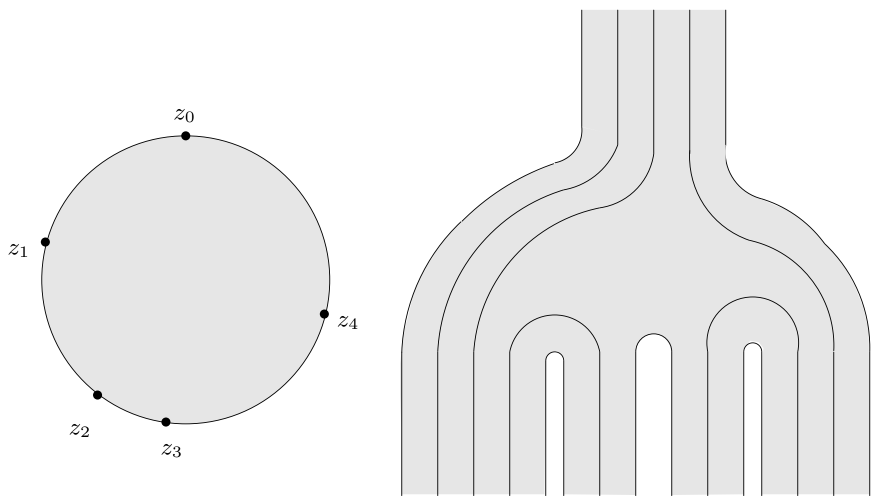

Figure 4.2: A quilted surface parametrized by a disk with markings, with striplike ends shaded.

Let $d \geq 2$. As in Chapter 3, we denote the d-th associahedron by $\overline{\mathcal{R}}^d$. The goal of this section is to construct, for each d, a smooth fiber bundle $\mathcal{S}^d \longrightarrow \mathcal{R}^d$ of quilted surfaces with striplike ends. We want these bundles to be consistent with gluing operations near the boundaries $\partial\overline{\mathcal{R}}^d$, in the following sense. Recall that the Deligne-Mumford compactification of $\mathcal{R}^d$ is the union

$$\overline{\mathcal{R}}^d = \bigcup_T \mathcal{R}^T,$$

where T ranges over all stable trees with d leaves, and $\mathcal{R}^T$ denotes all stable nodal marked disks of combinatorial type T. The boundary, $\partial\overline{\mathcal{R}}^d$, is the union over all combinatorial types T with at least one interior edge. For small gluing parameters, we want a combinatorial type T with interior edges to determine a gluing operation

$$\gamma_T : \bigtimes_{v \in V(T)} \mathcal{S}^{|v|-1} \times (0, \epsilon)^{|E_{int}(T)|} \longrightarrow \mathcal{S}^d$$

defined as follows. Each edge $e \in E_{int}(T)$ is incident to two vertices, v^- and v^+, where we write v^- for the vertex closer to the root. The edge e represents a striplike end $\epsilon_-(s,t)$ of $\mathcal{S}^{|v^-|-1}$ and a striplike end $\epsilon_+(s,t)$ of $\mathcal{S}^{|v^+|-1}$. Given a gluing parameter $\delta \in (0,1)$, with corresponding gluing length $R = R(\delta) := -\log \delta$, we truncate each of these striplike ends at $s = R$ and identify $\epsilon_-(R, 1-t) \sim \epsilon_+(R,t)$.

As a first step we construct a smooth fiber bundle $\mathcal{S}^d \longrightarrow \mathcal{R}^d$, where

- Each fiber $\mathcal{S}^d_r, r \in \mathcal{R}^d$ is a Riemann surface with boundary holomorphically isomorphic to the closed disk D with $d+1$ distinct boundary points removed.
- Each surface $\mathcal{S}^d_r$ is equipped with $d+1$ striplike ends.

We will construct the surfaces first with striplike ends, and then fix the complex structure on each surface such that the complex structure varies smoothly over $\mathcal{R}^d$, and coincides with the standard complex structure for the strip on the striplike ends.

Constructing the surfaces

We exploit the inductive nature of $\overline{\mathcal{R}}^d$. We proceed by induction on $d \geq 2$; at each step we will define the bundle over the boundary in terms of the lower strata, cover a neighborhood of the boundary using the gluing construction on the striplike ends, and finally interpolate over the interior.

Basis step: Fix a surface with three boundary components and three striplike ends for $\mathcal{S}^2$, as pictured in Figure 4.3.

Inductive step: Let $d > 2$. Assume that for $2 \leq e < d$ the bundles $\mathcal{S}^e \longrightarrow \mathcal{R}^e$ have been constructed. Use the gluing construction to define the bundle $\overline{\mathcal{S}}^d \longrightarrow \overline{\mathcal{R}}^d$ over an open neighborhood of $\partial\overline{\mathcal{R}}^d$. Observe that all surfaces already defined are diffeomorphic by orientation preserving diffeomorphisms that identify striplike ends. The space of such diffeomorphic 2-manifolds with boundary with fixed striplike ends is contractible, therefore the bundle $\mathcal{S}^d \longrightarrow \mathcal{R}^d$ extends over $\mathcal{R}^d$.

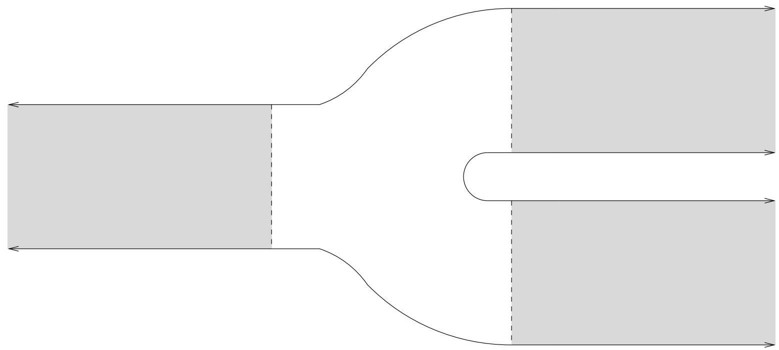

Figure 4.3: The basic surface $\mathcal{S}^2$ with striplike ends shaded.

Fixing the complex structures

Now that we have the surfaces, we fix the complex structure on each surface, again working by induction.

Basis step: For the surface $\mathcal{S}^2$, fix the complex structure on each of the striplike ends to be the standard complex structure for the strip. Since the boundary is smooth there is a tubular neighborhood of each boundary component diffeomorphic to a standard strip $[0, \delta) \times \mathbb{R}$ and for sufficiently small δ these tubular neighborhoods do not intersect for different boundary components. Moreover we assume that on the striplike ends these tubular neighborhoods are just ordinary strips. We take the complex structure to be the standard complex structure for the strip on those tubular neighborhoods; this ensures that the boundary is real-analytic. Finally, we can smoothly extend the complex structure over the remainder of the interior of $\mathcal{S}^2$ since the space of complex structures is contractible.

Inductive step: Assume that for $2 \leq e < d$ the complex structures on the fibers of the bundle $\overline{\mathcal{S}}^e \longrightarrow \overline{\mathcal{R}}^e$ have been determined. Use the gluing construction to determine the complex structure on the fibers over a neighborhood of the boundary. It remains to check that the complex structure can be smoothly extended over the bundle $\mathcal{S}^d \to \mathcal{R}^d$, in such a way that the boundary is real-analytic and the complex structure is the standard complex structure for the strip. We observe again that all surfaces are diffeomorphic by smoothly varying diffeomorphisms that preserve the striplike ends. So if we fix one of these surfaces as a base surface, call it S, we need to check that there is a smoothly varying family of tubular neighborhoods of the boundary and smoothly varying family of complex structures over the rest of the surface that coincide with the standard complex structure on the tubular neighborhood. But this is possible since the space of tubular neighborhoods of boundary components is contractible, and the space of complex structures is too. Since in dimension 2 all complex structures are integrable, so this construction produces a fiber bundle of Riemann surfaces with striplike ends.

Attaching strips

We obtain more general quilted surfaces parametrized by the associahedron by attaching strips to boundary components of the Riemann surfaces in the families defined for $d \geq 2$. By construction we have tubular neighborhoods of the boundary components which are holomorphically isomorphic to $\mathbb{R} \times i[0, \delta)$, where $\mathbb{R} \times i\{0\}$ is identified with the boundary, and δ is sufficiently small. We get a quilted surface by attaching a strip to the boundary component via the identification of the tubular neighborhood with the strip $\mathbb{R} \times [1, 1+\delta)$, and attaching the standard strip $\mathbb{R} \times i[0, 1]$ along the common seam, $\mathbb{R} \times i$.

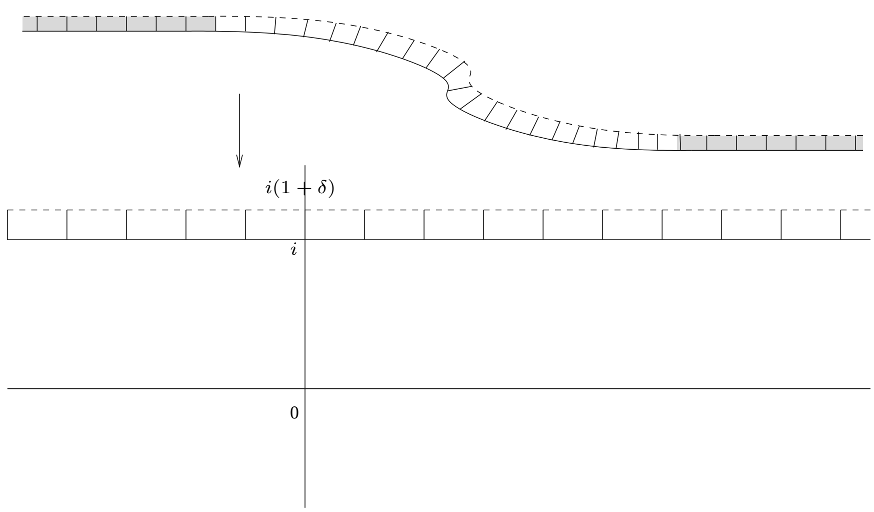

Figure 4.4: Attaching strips to boundary components.

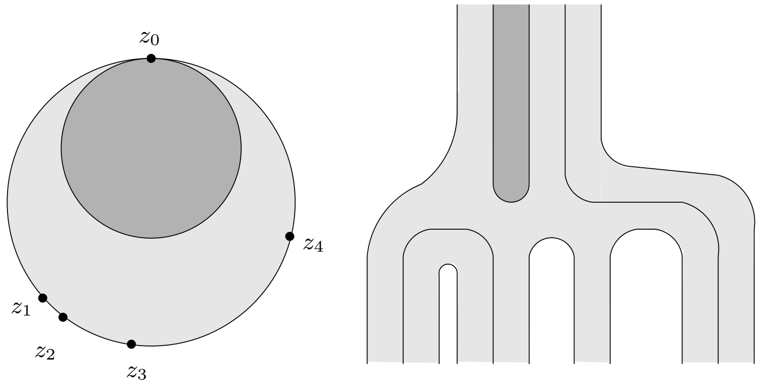

Figure 4.5: A quilted surface with striplike ends parametrized by a quilted marked disk.

4.7 Quilted surfaces parametrized by the multiplihedron

Let $d \geq 1$. As in Section 3.7, we denote the d-th multiplihedron by $\overline{\mathcal{R}}^{d,1}$. Our first goal is to construct a fiber bundle $\mathcal{S}^{d,1} \longrightarrow \mathcal{R}^{d,1}$, where

- Each fiber $\mathcal{S}^{d,1}_r$ is a contractible 2-manifold with $d+1$ boundary components, with an embedded 1-manifold (the seam).
- Each surface $\mathcal{S}^{d,1}_r$ is equipped with $d+1$ striplike ends, d of which are a single strip, and one of which is a quilted strip with three components.

We want these bundles to be consistent with gluing operations near the boundaries $\partial\overline{\mathcal{R}}^{d,1}$, in the following sense. Recall that the Deligne-Mumford-type compactification of $\mathcal{R}^{d,1}$ is

$$\overline{\mathcal{R}}^{d+1,1} = \bigcup_{\Gamma} \mathcal{R}^{\Gamma}$$

where Γ ranges over all stable bicolored trees with d leaves, and $\mathcal{R}^{\Gamma}$ denotes all nodal semi-stable quilted disks of combinatorial type Γ. The boundary, $\partial\overline{\mathcal{R}}^{d,1}$ is the union over all combinatorial types Γ with at least one interior edge. Recall from Section 3.8 that the gluing lengths, hence the gluing parameters, are required to satisfy a collection of gluing relations. In the framework of gluing lengths, the relations define a cone $\mathcal{G} \subset (0,\infty)^{|E_{int}(\Gamma)|}$. We call a tuple of gluing lengths *admissible* if it is in $\mathcal{G}$.

The colored vertices $V_{col} \subset V(\Gamma)$ divide the vertex set $V(\Gamma)$ into three disjoint sets; we write V_A (resp. V_B) for the vertices which are closer (resp. further) from the root than the vertices in V_{col}. From the previous section, fix two fiber bundles of surfaces parametrized by the associahedra:

1. $\mathcal{S}^d \longrightarrow \mathcal{R}^d$, where the fibers are Riemann surfaces;

2. $\mathcal{S}_A^d \longrightarrow \mathcal{R}^d$, where the fibers are quilted surfaces, corresponding to the Riemann surfaces in $\mathcal{S}^d$ with one strip attached to each boundary component.

We want a combinatorial type Γ with interior edges to determine a gluing operation

$$\gamma_\Gamma : \underset{v \in V_A}{\times} \mathcal{S}_A^{|v|-1} \underset{v \in V_B}{\times} \mathcal{S}^{|v|-1} \underset{v \in V_{col}}{\times} \mathcal{S}^{|v|-1,1} \times \mathcal{G} \longrightarrow \mathcal{S}^d$$

defined as follows. Each edge $e \in E_{int}(\Gamma)$ is incident to two vertices, v_- and v_+ where v_- is the vertex closer to the root, and determines an identification of two striplike ends. Then, given a gluing length for e, we can truncate striplike ends and identify along cuts as before.

To achieve this property we will again construct the surfaces first with their striplike ends, and then fix the complex structure on each surface.

Constructing the quilted surfaces

We exploit the inductive nature of $\overline{\mathcal{R}}^{d,1}$. We proceed by induction on $d \geq 1$, at each step defining the bundle over the boundary in terms of the lower strata, extending over a neighborhood of the boundary using the gluing construction on the striplike ends, and finally interpolating over the interior.

Basis step: Fix a surface with two boundary components, an embedded 1-manifold and three striplike ends as in Figure 4.6.

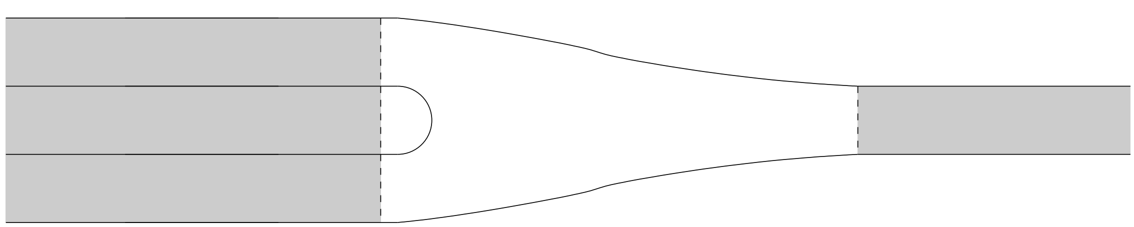

Figure 4.6: The basic surface $\mathcal{S}^{1,1}$ with striplike ends shaded.

Inductive step: Let $d > 1$. Assume that for $1 \leq e < d$ the bundles $\mathcal{S}^{e,1}$ have been constructed, and so have the bundles $\mathcal{S}^e \longrightarrow \mathcal{R}^e, \mathcal{S}^e_A \longrightarrow \mathcal{R}^e$ over the associahedra as in the previous section. Then the gluing construction allows us to define the bundle $\overline{\mathcal{S}}^d \longrightarrow \overline{\mathcal{R}}^d$ over an open neighborhood of $\partial\overline{\mathcal{R}}^d$. Observe that all surfaces defined so far are diffeomorphic to each other via orientation preserving diffeomorphisms that map the surface to surface and seam to seam, and are the identity map on the striplike ends. The space of such diffeomorphic 2-manifolds with boundary and seam with fixed striplike ends is contractible, therefore the bundle $\mathcal{S}^{d,1} \longrightarrow \mathcal{R}^{d,1}$ can be extended over the remainder of the interior of $\mathcal{R}^{d,1}$.

Fixing the complex structures

Now that we have the quilted surfaces, we fix the complex structure on each of their components, again working by induction.

Basis step: For the quilted surface $\mathcal{S}^{1,1}$, we first fix the complex structures on each strip component of the striplike ends to be the standard complex structure for the strip. Next, we choose a tubular neighborhood of each boundary component diffeomorphic to a standard strip $\mathbb{R} \times [0, \delta)$, and such that on the striplike ends it is exactly that strip. We also choose a tubular neighborhood of the seam that is diffeomorphic to $\mathbb{R} \times (-\delta, \delta)$ and is of exactly the form on the parts of the seam that are on the striplike end. We choose δ small enough that all these tubular neighborhoods are disjoint. We fix the complex structure on each of the tubular neighborhoods to be the standard complex structure on those strips; note that these are consistent with the standard complex structure on the striplike ends because we chose the tubular neighborhoods to coincide with standard strips of width δ on the striplike ends. This ensures that the boundary is real-analytic and the seam is a real-analytic submanifold. Finally, we can smoothly extend the complex structure over the remainder of the interiors of the components of $\mathcal{S}^{1,1}$ since the space of complex structures is contractible.

Inductive step: Assume that for $1 \leq e < d$ the complex structures on the fibers of the bundle $\mathcal{S}^e \longrightarrow \mathcal{R}^e$ have been fixed. By assumption the complex structures on the fibers of the bundles $\mathcal{S}^e \longrightarrow \mathcal{R}_e$ and $\mathcal{S}^e_A \longrightarrow \mathcal{R}^e$ constructed over the associahedra have been fixed. Since the complex structures are fixed over the striplike ends, the gluing construction determines the complex structures on the surfaces fibered over a neighborhood of the boundary $\mathcal{R}^{d,1}$. So one needs to check that the complex structures extend smoothly over the rest of the bundle $\mathcal{S}^{d,1} \longrightarrow \mathcal{R}^{d,1}$, in such a way that the striplike ends are always equipped with the standard complex structure, the boundary components are always real-analytic, the seam is always a real-analytic embedding. Observe that all surfaces in the bundle are diffeomorphic by smoothly varying diffeomorphisms that preserve the seam and the striplike ends. If we fix one of these surfaces with seam as a base surface, S, with embedded 1-submanifold C, we need to check that there is a smoothly varying family of tubular neighborhoods of the boundary and the seam, and a smoothly varying family of complex structures over the rest of the surface that on the tubular neighborhoods is the standard complex structure. But all this is possible because the space of tubular neighborhoods of boundary components and seam is contractible, and so is the space of complex structures. Moreover in dimension 2 all complex structures are integrable, so this construction produces a fiber bundle $\mathcal{S}^{d,1} \longrightarrow \mathcal{R}^{d,1}$ of quilted surfaces with striplike ends, in the sense of Section 4.3.

Attaching strips

We obtain more general quilted surfaces parametrized by the multiplihedron by attaching strips to the external boundary components in the families defined for $d \geq 1$. By construction we have tubular neighborhoods of these boundary components which are holomorphically isomorphic to $\mathbb{R} \times i[0, \delta)$, where $\mathbb{R} \times i\{0\}$ is identified with the boundary, and δ is sufficiently small. We get a quilted surface by attaching a strip to the boundary component via the identification of the tubular neighborhood with the strip $\mathbb{R} \times [1, 1+\delta)$, and attaching the standard strip $\mathbb{R} \times i[0, 1]$ along the common seam, $\mathbb{R} \times i$.

Chapter 5

Floer theory for families of quilted surfaces

5.1 Outline of chapter

The goal of this chapter is to describe the analytical framework for quilted Floer theory for the families of quilted surfaces that were explicitly constructed in the preceding chapter. The chapter is modeled on [14, Section 9], which describes the Floer theory for families behind the construction of the Fukaya category.

5.2 Inhomogeneous pseudo-holomorphic maps

Floer and perturbation data

To each striplike end of $\underline{S}$, with Lagrangian boundary conditions given by $\underline{L}$, we assign a *Floer datum*, which is a regular pair $(\underline{H}, \underline{J}) = ((H_k)_{k=1,\ldots,m}, (J_k)_{k=1,\ldots,m})$ of a Hamiltonian perturbation and an almost complex structure of split type.

Fix a collection of generalized Lagrangian submanifolds $\underline{L}_0, \ldots, \underline{L}_d$ of a symplectic manifold M_A. Given a Lagrangian correspondence L_{AB} between M_A and M_B, the compositions

$$\underline{L}_{0,AB} := \underline{L}_0 \circ L_{AB}$$
$$\underline{L}_{d,AB} := \underline{L}_d \circ L_{AB}$$

are generalized Lagrangian submanifolds of M_B.

Assign each pair $(\underline{L}_i, \underline{L}_{i+1})$ a Floer datum $(\underline{H}_i, \underline{J}_i)$, and assign $(\underline{L}_{0,AB}, \underline{L}_{d,AB})$ a Floer datum $(\underline{H}_{AB}, \underline{J}_{AB})$. Then fix a perturbation datum $(\underline{K}, \underline{J})$ which is compatible with the striplike ends and the Floer data along the striplike ends.

Fix a quilted surface $\underline{S}_{r_0}, r_0 \in \mathcal{R}^{d,1}$, which is labeled by the Lagrangians $\underline{L}_0, \ldots, \underline{L}_d$ and the Lagrangian correspondence L_{AB}. In a neighborhood U of r_0 all of the quilted surfaces are diffeormorphic, giving rise to a family of diffeomorphisms parametrized by points in U,

$$\Psi : U \times \mathcal{S}_{r_0} \mapsto \mathcal{S}\big|_U$$
$$\Psi(r, \cdot) : \mathcal{S}_{r_0} \xrightarrow{\cong} \mathcal{S}_r$$

which are "constant on the strip-like ends", meaning that

$$\Psi(r, \underline{\varepsilon}_\zeta(r_0, s, t)) = \underline{\varepsilon}_\zeta(r, s, t).$$

Each quilted surface $\underline{S}_r$ is equipped with complex structures $\underline{j}_{\mathcal{S}_r}$ which can be pulled back by Ψ_r to give a family of complex structures on $\underline{S}_{r_0}$ parametrized by r:

$$\underline{j}(r) := \Psi_r^*(\underline{j}_{\mathcal{S}_r})$$

Consistent perturbation data

We assume that a universal choice of perturbation data has been made, and that the universal choice is *consistent*, which means, in the language of [14, Section 9], that the following two conditions hold:

- There is a subset $U \subset \mathcal{R}$ where the gluing parameters are sufficiently small, such that the perturbation data obtained from gluing and the perturbation data in the family agree on the thin parts of the surfaces $\mathcal{S}_r, r \in U$.
- Let (K, J) be the first perturbation datum on $\mathcal{S}$ (i.e., obtained by gluing), and $(\overline{K}, \overline{J})$ its extension to the partial compactification $\overline{\mathcal{S}}$. Then the other datum (obtained by pullback from $\mathcal{S}^{d+1}$) also extends smoothly to $\overline{\mathcal{S}}$ and the extension agrees with $(\overline{K}, \overline{J})$ over the subset $\{0\}^{Ed^{int}(T)} \times \mathcal{R}^T \subset \overline{\mathcal{R}}$ where all the gluing parameters are zero.

The $\underline{K}$ in the perturbation datum is a smooth family of 1-forms on the fibers $\underline{S}_r$ which take values in the space of Hamiltonian functions on $\underline{M}$. Thus, a choice of $\underline{K}$ gives rise to a 1-form $\underline{Y}$ on each fiber $\underline{S}_r$, that takes values in the space of Hamiltonian vector fields on $\underline{M}$.

Bundles

Once we fix a collection of intersection points

$$(\underline{y}_0, \underline{x}_1, \ldots, \underline{x}_d) \in \mathcal{I}(\underline{L}_{0,AB}, \underline{L}_{d,AB}) \times \mathcal{I}(\underline{L}_0, \underline{L}_1) \times \ldots \times \mathcal{I}(\underline{L}_{d-1}, \underline{L}_d)$$

we can define a fiber bundle

$$\mathcal{B}_{\mathcal{S}\big|_U} \longrightarrow U$$

whose fiber over r is $C^\infty(\underline{S}_r, \underline{M}; \{\underline{L}\})$, the space of smooth maps from $\mathcal{S}_r$ to $\underline{M}$, satisfying the required boundary conditions on the seams of the quilt that involve the Lagrangians labeling the seams. There is another fiber bundle

$$\mathcal{E}_{\mathcal{S}\big|_U} \longrightarrow \mathcal{B}_{\mathcal{S}\big|_U},$$

whose fiber $\mathcal{E}_{(r,\underline{u})} = \Omega^{0,1,r}(\underline{S}_r, \underline{u}^*T\underline{M})$ of $(0,1)$ forms on $\mathcal{S}_r$ taking values in the pullback bundle $\underline{u}^*TM$, and the $(0,1)$ part is with respect to $\underline{J}(r, \underline{u})$ and $\underline{j}(r)$.

Definition The *inhomogeneous pseudo-holomorphic map equation* for $(r, \underline{u}) \in \mathcal{B}$ is

$$\begin{cases} d\underline{u}(z) + \underline{J}(r, \underline{u}, z) \circ d\underline{u}(z) \circ \underline{j}(r) = \underline{Y}(r, \underline{u}, z) + \underline{J}(r, \underline{u}, z) \circ \underline{Y}(r, \underline{u}, z) \circ \underline{j}(r) \\ \underline{u}(C) \subset L_C \text{ for all seams } C \text{ with label } L_C. \end{cases} \tag{5.1}$$

The compatibility of the perturbation datum with the Floer data along the striplike ends means that the above equation reduces to Floer's equation along the striplike ends. In particular, solutions with finite energy converge exponentially along the striplike ends to elements of

$$\mathcal{I}(\underline{L}_{0,AB}, \underline{L}_{d,AB}) \times \mathcal{I}(\underline{L}_0, \underline{L}_1) \times \ldots \times \mathcal{I}(\underline{L}_{d-1}, \underline{L}_d).$$

Definition Given a fixed set of intersection points $(\underline{y}_0, \underline{x}_1, \ldots, \underline{x}_d)$, the *moduli space of pseudoholomorphic quilted disks* $\mathcal{M}_{d,1}(\underline{y}_0, \underline{x}_1, \ldots, \underline{x}_d)$ is the set of finite energy solutions $(r, \underline{u})$ to (5.1) which converge along the strip-like ends labeled $\zeta_0, \ldots, \zeta_d$ to the respective intersections $(\underline{y}_0, \underline{x}_1, \ldots, \underline{x}_d)$.

Since equation (5.1) can be written as $(d\underline{u} - \underline{Y})^{0,1} = 0$, we will also abbreviate it as $(\overline{\partial} - \underline{\nu})(r, \underline{u}) = 0$, where $\overline{\partial}(r, \underline{u}) := (d\underline{u})^{0,1}$ and $\underline{\nu}(r, \underline{u}) := (\underline{Y}(r, \underline{u}))^{0,1}$, with the $0,1$ taken with respect to $\underline{J}(r, \underline{u}, z)$ and $\underline{j}(r)$. The operator $\overline{\partial} - \underline{\nu}$ defines a section $\mathcal{B} \to \mathcal{E}$ and solutions of (5.1) correspond to its intersection with the zero-section of $\mathcal{E}$.

5.3 Local trivializations

We now define local trivializations of the bundle $\mathcal{E}_{\mathcal{S}}\big|_U \longrightarrow \mathcal{B}_{\mathcal{S}}\big|_U$ in a small neighborhood of a point $(r_0, \underline{u}_0) \in \mathcal{B}$. This requires identifying fibers that are close to each other.

We will use local exponential maps with respect to Riemannian metrics on the manifolds to get the trivialization. In order for the exponential maps to preserve the Lagrangian boundary conditions, we need metrics that make the Lagrangian submanifolds in the boundary conditions totally geodesic.

The main ingredient in the construction of such metrics is Frauenfelder's lemma, which we quote here from [10, Lemma 4.3.3]. We say that a Lagrangian submanifold L is *totally real* with respect to an almost-complex structure J on M if at every $p \in L$, $T_pL \cap JT_pL = 0$. In particular, if J is an almost-complex structure that is compatible with ω, and L is a Lagrangian submanifold of M, then L is totally real for J. This follows from the fact that if $\xi, \eta \in T_pL$, then using the metric g_J induced by ω and J, $g_J(\xi, J\eta) = \omega(\xi, J^2\eta) = -\omega(\xi, \eta) = 0$ as L is Lagrangian.

Lemma 5.3.1 (Frauenfelder)**.** *Let (M, J) be an almost complex manifold and $L \subset M$ be a totally real submanifold with $2 \dim L = \dim M$. Then there exists a Riemannian metric $g = \langle \cdot, \cdot \rangle$ on M such that*

(i) $\langle J(p)v, J(p)w \rangle = \langle v, w \rangle$ for $p \in M$ and $v, w \in T_pM$,

(ii) $J(p)T_pL$ is the orthogonal complement of T_pL for every $p \in L$,

(iii) L is totally geodesic with respect to g.

The statement of (iii) is that L is totally geodesic with respect to the Levi-Civita connection ∇ of g. However we will also need the associated Hermitian connection $\widetilde{\nabla}$, defined by

$$\widetilde{\nabla}_v X := \nabla_v X - \frac{1}{2} J(\nabla_v J)X, \tag{5.2}$$

so we need to know that L is totally geodesic with respect to $\widetilde{\nabla}$.

Lemma 5.3.2. *Let (M, J) be an almost complex manifold, L a Lagrangian submanifold of M, and g a metric satisfying (i)-(iii) of Lemma 5.3.1. Let ∇ be the Levi-Civita* *connection of g, and $\widetilde{\nabla}$ the complex linear connection (5.2). Then L is totally geodesic with respect to $\widetilde{\nabla}$.*

Proof. We show that L is totally geodesic with respect to $\widetilde{\nabla}$ by showing that for every $p \in L$, and every $X, Y \in T_pL$, $\widetilde{\nabla}_X Y \in T_pL$. By definition,

$$\widetilde{\nabla}_X Y \quad = \quad \nabla_X Y - \frac{1}{2} J(\nabla_X J)Y.$$

where ∇ is the Levi-Civita connection of g. By assumption, L is totally geodesic with respect to ∇, therefore $\nabla_X Y \in T_pL$ for all $p \in L$ and all $X, Y \in T_pL$. So it is enough to show that $J(\nabla_X J)Y \in T_pL$. Since the orthogonal complement of T_pL is JT_pL, it reduces to showing that for all $X, Y, Z \in T_pL$, $J(\nabla_X J)Y \perp JZ$.

$$\begin{aligned} g(J(\nabla_X J)Y, JZ) &= g((\nabla_X J)Y, Z) \\ &= d(g(JY, Z))(X) - g(J\nabla_X Y, Z) - g(JY, \nabla_X Z) \\ &= 0 \end{aligned}$$

since all three terms in the penultimate line are zero. Hence, L is totally geodesic with respect to $\widetilde{\nabla}$. $\square$

Let $\mathcal{S} \longrightarrow \mathcal{R}$ be shorthand for either $\mathcal{S}^d \longrightarrow \mathcal{R}^d$ or $\mathcal{S}^{d,1} \longrightarrow \mathcal{R}^{d,1}$, the fiber bundles of quilted surfaces defined in section 4.3. Let $\underline{M} = (M_j)_{j=1,\ldots,k}$ be a collection of target manifolds for the quilted surfaces in $\mathcal{S}$, and let $\underline{L}$ be a collection of Lagrangian boundary conditions for the boundary components and the seams.

A point in $\mathcal{S}$ is a pair (r, z), where $r \in \mathcal{R}$ and $z \in \mathcal{S}_r$. Let us denote the set of Riemannian metrics on M_j by $R(M_j)$, and for a given almost-complex structure J on M_j we write $R(M_j, J)$ for the set of Riemannian metrics on M_j for which J is skew-adjoint.

There is a fiber bundle $RM \longrightarrow \mathcal{S}$, with fibers $RM_{(r,z)} = R(\underline{M}, \underline{J}(r, z))$ where $\underline{J}$ is the almost-complex structure on $\underline{M}$ from the fixed choice of perturbation data. More explicitly, the quilt bundle $\mathcal{S} \longrightarrow \mathcal{R}$ consists of a finite set of bundles $S_j \longrightarrow \mathcal{R}$ of Riemann surfaces with boundary, where $j = 1, \ldots, k$, together with seam identifications. For each j there is a bundle $R(M_j, J_j) \longrightarrow S_j$ of Riemannian metrics for which the almost-complex structure J_j is skew-adjoint.

Definition A *compatible metric* $\underline{g}$ for the pair $(\underline{M}, \underline{J})$ is a section $\underline{g} : \mathcal{S} \to RM$, that is, a tuple of sections $g_j : S_j \to R(M_j, J_j)$, where for each $r \in \mathcal{R}, z \in S_j(r)$, $g_j(r, z)$ is a Riemannian metric on M_j that is $J(r, z)$-invariant. We say that $\underline{g}$ is *consistent* with the Lagrangian boundary conditions $\underline{L}$, the striplike ends, and the boundary strata of $\mathcal{R}$, if it has the following properties.

1. For every true boundary component $I \subset S_j$ of $\mathcal{S}$, with Lagrangian label $L \subset M_j$, $g_j\big|_I$ makes L totally geodesic.

2. For every seam $\sigma = ((k, I_e), (k', I_{e'})) \in \Sigma$, with identification $\phi_\sigma : I_{k,e} \longrightarrow I_{k',e'}$, and Lagrangian label $L_{k,k'} \subset M_k^- \times M_{k'}$, the product metric $g_k \times g_{k'}\big|_{I_e \times_{\phi_\sigma} I_{e'}}$ on $M_k^- \times M_{k'}$ makes $L_{k,k'}$ totally geodesic.

3. On the striplike ends the metrics are independent of the parameter space $\mathcal{R}$, and are constant in the s direction.

4. Near the boundary of $\mathcal{R}^d$ (resp. $\mathcal{R}^{d,1}$), the metric is obtained from the metrics on the components making up the boundary strata, by gluing along striplike ends.

Lemma 5.3.3. *Consistent compatible metrics exist.*

Proof. The proof uses the recursive nature of the bundle $\mathcal{S} \longrightarrow \mathcal{R}$ to define the metrics by induction. We break it up into three steps.

Step 1: Consistent compatible metrics exist for quilted strips.

Proof of Step 1: Take the strips to be $Z_j = \mathbb{R} \times [j-1, j]$ for $j = 1, \ldots, k$. For each Z_j write (M_j, ω_j) for the target symplectic manifold, $J_j(t), t \in [0, 1]$ for the t-dependent almost-complex structure. Write L_1, L_k and $L_{j-1,j}$ for $j = 1, \ldots, k$ for the Lagrangian boundary conditions.

By assumption, L_1 is a Lagrangian submanifold of M_1 which is totally real with respect to $J_1(0)$, L_k is a Lagrangian submanifold of M_k which is totally real with respect to $J_k(1)$, and for $k = 2, \ldots, k$, the Lagrangian correspondences $L_{k-1,k}$ are totally real submanifolds of $M_{k-1}^- \times M_k$ with respect to the almost complex structure $J_{k-1}(1) \times J_k(0)$. By Lemma 5.3.1, for each of these Lagrangian submanifolds there exists a metric making them totally geodesic; let us call these metrics $\widetilde{g}_1, \widetilde{g}_k$ and $\widetilde{g}_{j-1,j}$ for $j = 1, \ldots, k$.

We illustrate how to get a consistent and compatible tuple $(g_j)_{j=1,\ldots,k}$ by explaining how to construct g_1; constructing the other g_j's uses identical lines of reasoning. On M_1, we have two metrics g_1 and $\mathrm{pr}_1^* g_{1,2}$, where $\mathrm{pr}_1 : M_1^- \times M_2 \to M_1^-$ is the projection map. The space of Riemannian metrics on M is contractible so we can fix a smooth path $g_1 : [0,1] \to R(M_1)$ such that $g_1(0) = \widetilde{g}_1$ and $g_1(1) = \widetilde{g}_{1,2}$. We can even make it a smooth path of $J(t)$-invariant metrics by replacing $g_1(t) \mapsto \frac{1}{2}(g_1(t) + J(t)^T g_1(t) J(t))$; note that this does not alter the metrics at $t = 0$ and $t = 1$, since they are already $J_1(0)$ and $J_1(1)$ invariant, respectively.

Step 2: Consistent compatible metrics exist for the bundles $\mathcal{S}^{1,1} \longrightarrow \mathcal{R}^{1,1}$ and $\mathcal{S}^2 \longrightarrow \mathcal{R}^2$.

Proof of Step 2: First we fix, by step 1, a choice of compatible metric on each striplike end, which depends on the almost complex structures in the Floer datum. Second, we extend the choice of metric smoothly over the boundary components, as follows. Note that the complement of the striplike ends on a boundary component I is compact. On that compact subset, the complex structures $J(z, r)$ in the perturbation datum vary smoothly over $z \in I$ and $r \in \mathcal{R}$, interpolating between the complex structures at each striplike end. It is an important point that for a fixed Lagrangian submanifold, the metrics of Lemma 5.3.1 can be chosen to depend smoothly on the almost-complex structure J, and for each almost complex structure J the set of metrics satisfying (i)-(iii) is convex. Using these facts we can choose metrics $g(r, z)$ satisfying (i) - (iii) that are parametrized by $z \in I$ and $r \in \mathcal{R}$, and interpolate between the metrics already chosen at each striplike end. Once the metrics are chosen for each boundary and seam component, use the convexity of Riemannian metrics to extend the choice of metrics smoothly over the interiors of the surfaces. Any smooth family of Riemannian metrics $g(r, z), r \in \mathcal{R}, z \in \mathcal{S}_r$ can be smoothly converted into a family of $J(r, z)$-invariant Riemannian metrics, by replacing $g(r, z)$ with $\frac{1}{2}(g(r, z) + J(r, z)^T g(r, z) J(r, z))$. Note that this map is the identity map on metrics that are already $J(r, z)$ invariant; in particular, this map is the identity map on the metrics chosen for the boundary and seam components, which are invariant by (i).

Step 3: Consistent compatible metrics exist for all $\mathcal{S}^{d,1} \longrightarrow \mathcal{R}^{d,1}$ for $d \geq 1$, and for all $\mathcal{S}^d \longrightarrow \mathcal{R}^d$ for $d \geq 2$.

Proof of Step 3: This is an inductive argument on d, with the base case covered by Step 2. Having constructed families of metrics for $\mathcal{R}^d$ or $\mathcal{R}^{d,1}$ for $d \leq k$, the gluing operation determines compatible metrics in a neighborhood of the boundary of $\mathcal{R}^{k+1}$ or $\mathcal{R}^{k+1,1}$. To extend over the rest of the interior of $\mathcal{R}^{k+1}$ or $\mathcal{R}^{k+1,1}$, again use the fact that the metrics required to make the Lagrangians totally geodesic depend smoothly on the choice of almost complex structure J, and in the absence of the geodesic condition the space of Riemannian metrics is convex. One can therefore choose smooth extensions that aren't necessarily $J(r,z)$ invariant but smoothly interpolate between the metrics chosen on the boundary, then use the map $g(r,z) \mapsto \frac{1}{2}(g(r,z) + J(r,z)^T g(r,z) J(r,z))$ to make each metric $J(r,z)$-invariant. As noted earlier this map does not affect the metrics chosen on the boundary, which are already $J(r,z)$-invariant by (i). $\square$

Let ∇ be the Levi-Civita connection associated to the metric $\underline{g}$. ∇ has to be understood componentwise, on the different manifolds that constitute $\underline{M}$. Let $\widetilde{\nabla}$ be the associated Hermitian connection, as defined in (5.2).

Let M be one of the symplectic manifolds in the collection $\underline{M}$, and $x \in M$, $\xi \in T_xM$. We write

$$\Phi_x(\xi) : T_xM \to T_{\exp_x(\xi)}M$$

to denote parallel transport along the geodesic $\exp_x(\lambda\xi), \lambda \in [0,1]$, with respect to the Hermitian connection $\widetilde{\nabla}$ on M. By Lemma 5.3.2, the Lagrangians in the boundary conditions are totally geodesic with respect to the Hermitian connections $\widetilde{\nabla}$ belonging to the metrics parametrized by boundary components and seams of the quilted surfaces.

Since $\overline{\mathcal{R}}^{d,1}$ is a $d-1$ dimensional convex polytope, we can fix any metric on it and use it to define a local exponential map $\exp_{r_0} : T_{r_0}U \to U$ in a neighborhood U of r_0. Write $\underline{S} := \mathcal{S}_{r_0}$ for the quilted surface corresponding to the parameter r_0. Then all maps $\underline{u} : \underline{S} \to \underline{M}$ which are sufficiently close to $\underline{u}_0$ can be expressed as $\underline{u} = \exp_{\underline{u}_0}(\underline{\xi})$ for some $\underline{\xi} \in \Omega^0(\underline{S}, \underline{u}_0^* T\underline{M})$. Define a map

$$\begin{aligned} \Phi_{\underline{u}}(\underline{\xi}) : \Omega^{0,1}_{\underline{S}(r)}(\underline{S}(r), \underline{u}^* T\underline{M}) &\longrightarrow \Omega^{0,1}_{\underline{S}(r)}(\underline{S}(r), \exp_{\underline{u}}(\underline{\xi})^* T\underline{M}) \\ \beta(z) &\mapsto \Phi_{u(z)}(\xi(z))\beta(z) \end{aligned}$$

where $\Phi^r_{u(z)}(\xi(z))$ is the map from $T_{u(z)}M$ to $T_{\exp_{u(z)}(\xi(z))}M$ given by parallel transport along the geodesic $\tau \mapsto \exp_{u(z)}(\tau\xi(z))$ with respect to the Hermitian connection $\widetilde{\nabla}$. We emphasize here that the Hermitian connection $\widetilde{\nabla}$ depends on $r \in \mathcal{R}, z \in \mathcal{S}_r$ and $\underline{u}(z) \in \underline{M}$.

Now define another map

$$\Phi_r(\rho) : \Omega^{0,1}_{\mathcal{S}_r}(\mathcal{S}_r, \underline{u}^* TM) \;\; \rightarrow \;\; \Omega^{0,1}_{\mathcal{S}_{\exp_r(\rho)}}(\mathcal{S}_{\exp_r(\rho)}, \underline{u}^* T\underline{M})$$

by defining its inverse to be the projection

$$\begin{aligned} \Phi_r(\rho)^{-1} : \Omega^{0,1}_{\mathcal{S}_{\exp_r(\rho)}}(\mathcal{S}_{\exp_r(\rho)}, \underline{u}^* T\underline{M}) &\rightarrow \Omega^{0,1}_{\mathcal{S}_r}(\mathcal{S}_r, \underline{u}^* T\underline{M}) \\ \beta(z) &\mapsto \frac{1}{2}\left(\beta(z) + J(r,u,z) \circ \beta(z) \circ j(r)\right). \end{aligned}$$

(Note that β, by assumption, is a $(0,1)$-form with respect to $J(\exp_r(\rho), u, z)$ and $j(\exp_r(\rho))$, i.e. it satisfies the identity $\beta(z) = \frac{1}{2}\left(\beta(z) + J(r,u,z) \circ \beta(z) \circ j(\exp_r(\rho))\right)$.) We observe that $\Phi_u(\xi)^{-1}$ commutes with $\Phi_r(\rho)^{-1}$ – for if $\psi \in L^p(S_{\exp_r(\rho)}, \Omega^{0,1} \otimes_J \exp_u(\xi)^* TM)$, then

$$\begin{aligned} \Phi_u(\xi)^{-1}\Phi_r(\rho)^{-1}\psi &= \Phi_u(\xi)^{-1}\left(\frac{1}{2}(\psi + J(r, \exp_u(\xi)) \circ \psi \circ j(r))\right) \\ &= \frac{1}{2}\left(\Phi_u(\xi)^{-1}\psi + \Phi_u(\xi)^{-1}(J(r, \exp_u(\xi)) \circ \psi \circ j(r))\right) \\ &= \frac{1}{2}\left(\Phi_u(\xi)^{-1}\psi + J(r,u) \circ \Phi_u(\xi)^{-1}\psi \circ j(r)\right) \\ &= \Phi_r(\rho)^{-1}\Phi_u(\xi)^{-1}\psi. \end{aligned}$$

Composing $\Phi_{\underline{u}}(\underline{\xi})$ and $\Phi_r(\rho)$ gives the local trivialization

$$\Phi_{\mathcal{S},r,\underline{u}}(\rho, \underline{\xi}) = \Phi_r(\rho) \circ \Phi_u(\xi) : \Omega^{0,1}(\mathcal{S}_r, \underline{u}^* T\underline{M}) \rightarrow \Omega^{0,1}(\mathcal{S}_{\exp_r(\rho)}, \exp_{\underline{u}}(\underline{\xi})^* T\underline{M}),$$

and hence the connection that we will use. We denote this connection by $\overline{\nabla}$.

Let $U \subset \mathcal{B}$, and consider a section $\psi : U \to \mathcal{E}\big|_U$. For $(r,u) \in U$ and $(\rho, \xi) \in T_{(r,u)}U$, let us abbreviate $r_\lambda := \exp_r(\lambda\rho), u_\lambda := \exp_u(\lambda\xi)$. Then,

$$\begin{aligned}
\overline{\nabla}_{(\rho,\xi)}\psi(r,u) &:= \frac{d}{d\lambda}\Big|_{\lambda=0} \Phi_{\mathcal{S},r,u}(\lambda\rho,\lambda\xi)^{-1}\psi(r_\lambda,u_\lambda) \\
&= \frac{d}{d\lambda}\Big|_{\lambda=0} \Phi_u(\lambda\xi)^{-1}\Phi_r(\lambda\rho)^{-1}\psi(r_\lambda,u_\lambda) \\
&= \frac{d}{d\lambda}\Big|_{\lambda=0} \Phi_u(\lambda\xi)^{-1}\frac{1}{2}\left(\psi(r_\lambda,u_\lambda) + J(r,u_\lambda)\circ\psi(r_\lambda,u_\lambda)\circ j(r)\right) \\
&= \frac{1}{2}\left(\widetilde{\nabla}_\xi\psi + \partial_\rho\psi + J(r,u)\circ[\widetilde{\nabla}_\xi\psi + \partial_\rho\psi]\circ j(r)\right) \\
&= \frac{1}{2}\left(\widetilde{\nabla}_\xi\psi + J(r,u)\circ\widetilde{\nabla}_\xi\psi\circ j(r)\right) \\
&\quad + \frac{1}{2}\left(\partial_\rho\psi + J(r,u)\circ\partial_\rho\psi\circ j(r)\right)
\end{aligned} \tag{5.3}$$

where $\widetilde{\nabla}$ is the Hermitian connection on M. Note that for each $(r,u) \in U$, $\psi(r,u)$ is a $(0,1)$-form with respect to $J(r,u)$ and $j(r)$, i.e. satisfies the identity

$$\psi(r,u) = \frac{1}{2}\left(\psi(r,u) + J(r,u)\circ\psi(r,u)\circ j(r)\right.$$

Hence

$$\begin{aligned}
\widetilde{\nabla}_\xi\psi &= \widetilde{\nabla}_\xi\frac{1}{2}\left(\psi(r,u) + J(r,u)\circ\psi(r,u)\circ j(r)\right) \\
&= \frac{1}{2}\left(\widetilde{\nabla}_\xi\psi + \widetilde{\nabla}_\xi(J\circ\psi)\circ j(r)\right) \\
&= \frac{1}{2}\left(\widetilde{\nabla}_\xi\psi + J\circ\widetilde{\nabla}_\xi\psi\circ j(r)\right)
\end{aligned}$$

Putting this into equation 5.3 gives an explicit expression for $\overline{\nabla}$,

$$\overline{\nabla}_{(\rho,\xi)}\psi(r,u) = \widetilde{\nabla}_\xi\psi + \frac{1}{2}\left(\partial_\rho\psi + J(r,u)\circ\partial_\rho\psi\circ j(r)\right). \tag{5.4}$$

5.4 The operator $\mathcal{F}_{\mathcal{S},r,u}$.

Let $(\rho,\underline{\xi}) \in T_{(r,\underline{u})}\mathcal{B} \cong T_r\mathcal{R} \times \Omega^0(\mathcal{S}_r, \underline{u}^*T\underline{M})$ and define

$$\mathcal{F}_{(r,\underline{u})}(\rho,\underline{\xi}) := \Phi_{\mathcal{S},r,\underline{u}}(\rho,\underline{\xi})^{-1}(\overline{\partial} - \underline{\nu})(\exp_r\rho, \exp_{\underline{u}}\underline{\xi})$$

Introduce the notation

$$\begin{aligned}
\widetilde{\mathcal{F}}_{(r,\underline{u})}(\rho,\underline{\xi}) &:= \Phi_{\mathcal{S},r,\underline{u}}(\rho,\underline{\xi})^{-1}\overline{\partial}(\exp_r\rho, \exp_{\underline{u}}\underline{\xi}) \\
\mathcal{P}_{(r,\underline{u})}(\rho,\underline{\xi}) &:= \Phi_{\mathcal{S},r,\underline{u}}(\rho,\underline{\xi})^{-1}\underline{\nu}(\exp_r\rho, \exp_{\underline{u}}\underline{\xi})
\end{aligned}$$

so that we can write

$$\mathcal{F}_{(r,\underline{u})}(\rho,\underline{\xi}) = \widetilde{\mathcal{F}}_{(r,\underline{u})}(\rho,\underline{\xi}) - \mathcal{P}_{(r,\underline{u})}(\rho,\underline{\xi}) \tag{5.5}$$

($\mathcal{P}$ stands for "perturbation"). Denote the corresponding linearized operators by

$$\begin{aligned} D_{\mathcal{S},r,\underline{u}} &:= d\mathcal{F}_{r,u}(0,0) \\ \widetilde{D}_{\mathcal{S},r,\underline{u}} &:= d\widetilde{\mathcal{F}}_{r,u}(0,0) \\ P_{\mathcal{S},r,\underline{u}} &:= d\mathcal{P}_{r,u}(0,0) \end{aligned}$$

which are related by

$$D_{\mathcal{S},r,\underline{u}} = \widetilde{D}_{\mathcal{S},r,\underline{u}} - P_{\mathcal{S},r,\underline{u}}.$$

5.5 The linearized operator $D_{\mathcal{S},r,u}$.

We will again use the notation $\underline{u}_\lambda := \exp_{\underline{u}}(\lambda\underline{\xi})$ and $r_\lambda := \exp_r(\lambda\rho)$, to calculate explicit formulas for the linearized operators defined in the previous section.

$$\begin{aligned} D_{\mathcal{S},r,\underline{u}}(\rho,\underline{\xi}) &:= d\mathcal{F}_{r,\underline{u}}(0,0)(\rho,\underline{\xi}) \\ &= \overline{\nabla}_\lambda(\overline{\partial}(r_\lambda,u_\lambda) - \nu(r_\lambda,u_\lambda))\big|_{\lambda=0} \\ &= \widetilde{\nabla}_\lambda(\overline{\partial}(r,u_\lambda) - \nu(r,u_\lambda))\big|_{\lambda=0} \\ &\quad + \frac{1}{2}\left[\partial_\lambda(\overline{\partial}(r_\lambda,u) - \nu(r_\lambda,u))\big|_{\lambda=0}\right. \\ &\qquad \left. +J(r,u)\circ\partial_\lambda(\overline{\partial}(r_\lambda,u) - \nu(r_\lambda,u))\big|_{\lambda=0}\circ j(r)\right]. \end{aligned}$$

Define

$$\begin{aligned} D_u^{(r)}(\xi) &:= \widetilde{\nabla}_\lambda(\overline{\partial}(r,u_\lambda) - \nu(r,u_\lambda))\big|_{\lambda=0} \\ D_r^{(u)}(\rho) &:= \frac{1}{2}\left[\partial_\lambda(\overline{\partial}(r_\lambda,u) - \nu(r_\lambda,u))\big|_{\lambda=0}\right. \\ &\qquad \left. +J(r,u)\circ\partial_\lambda(\overline{\partial}(r_\lambda,u) - \nu(r_\lambda,u))\big|_{\lambda=0}\circ j(r)\right]. \end{aligned}$$

With this notation we can write $D_{\mathcal{S},r,u}(\rho,\xi) = D_u^{(r)}(\xi) + D_r^{(u)}(\rho)$. We similarly define

$$\begin{aligned} \widetilde{D}_u^{(r)}(\xi) &:= \widetilde{\nabla}_\lambda\overline{\partial}(r,u_\lambda)\big|_{\lambda=0} \\ \widetilde{D}_r^{(u)}(\rho) &:= \frac{1}{2}\left[\partial_\lambda\overline{\partial}(r_\lambda,u)\big|_{\lambda=0} + J(r,u)\circ\partial_\lambda\overline{\partial}(r_\lambda,u)\big|_{\lambda=0}\circ j(r)\right] \\ P_u^{(r)}(\xi) &= \widetilde{\nabla}_\lambda\nu(r,u_\lambda)\big|_{\lambda=0} \\ P_r^{(u)}(\rho) &= \frac{1}{2}\left[\partial_\lambda\nu(r_\lambda,u)\big|_{\lambda=0} + J(r,u)\circ\partial_\lambda\nu(r_\lambda,u)\big|_{\lambda=0}\circ j(r)\right] \end{aligned}$$

and with this notation we have the identities

$$\begin{aligned} \widetilde{D}_{\mathcal{S},r,u}(\rho,\xi) &= \widetilde{D}_u^{(r)}(\xi) + \widetilde{D}_r^{(u)}(\rho), \text{ and} \\ P_{\mathcal{S},r,u}(\rho,\xi) &= P_u^{(r)}(\xi) + P_r^{(u)}(\rho). \end{aligned}$$

In the following computations, projection onto the $(0,1)$ component is always understood to be with respect to $J(r,u)$ and $j(r)$.

$$\begin{aligned} \widetilde{D}_u^{(r)}(\xi) &= \widetilde{\nabla}\overline{\partial}(r,u_\lambda)\Big|_{\lambda=0} \\ &= \widetilde{\nabla}\frac{1}{2}(du_\lambda + J(r,u_\lambda)\circ du_\lambda \circ j(r))\Big|_{\lambda=0} \\ &= \frac{1}{2}(\widetilde{\nabla} du_\lambda + J(r,u_\lambda)\circ \widetilde{\nabla} du_\lambda \circ j(r))\Big|_{\lambda=0} \\ &= \frac{1}{2}(\nabla du_\lambda + J(r,u_\lambda)\circ \nabla du_\lambda \circ j(r))\Big|_{\lambda=0} \\ &\quad - \frac{1}{4}(J(r,u_\lambda)\nabla_\lambda J(r,u_\lambda)du_\lambda + J(r,u_\lambda)\circ J(r,u_\lambda)\nabla_\lambda J(r,u_\lambda)\circ du_\lambda)\Big|_{\lambda=0} \\ &= \frac{1}{2}(\nabla\xi + J(r,\underline{u})\circ\nabla\underline{\xi}\circ j(r)) - \frac{1}{2}J(r,\underline{u})(\nabla_\xi J)(r,\underline{u})\partial_J(r,\underline{u}) \\ &= [\nabla\xi]^{0,1} - \frac{1}{2}J(\underline{u})[(\nabla_\xi J)(\underline{u})d\underline{u}]^{0,1} \end{aligned}$$

where ∇ is the Levi-Civita connection on M, $\partial_J(\underline{u}) = \frac{1}{2}(d\underline{u} - J\circ d\underline{u}\circ j(r))$.

$$\begin{aligned} \widetilde{D}_r^{(u)}(\rho) &:= \frac{d}{d\lambda}\Phi_r(\rho)^{-1}\overline{\partial}(r_\lambda,u)\Big|_{\lambda=0} \\ &= \frac{d}{d\lambda}[\overline{\partial}(r_\lambda,u)]^{0,1}\Big|_{\lambda=0} \\ &= \left[\frac{d}{d\lambda}\overline{\partial}(r_\lambda,u)\Big|_{\lambda=0}\right]^{0,1} \\ &= \left[\frac{d}{d\lambda}\frac{1}{2}\left(du + J(r_\lambda,u)\circ du\circ j(r_\lambda)\right)\Big|_{\lambda=0}\right]^{0,1} \\ &= \frac{1}{2}[\partial_\rho J\circ du\circ j(r) + J\circ du\circ\partial_\rho j]^{0,1}. \end{aligned}$$

$$\begin{aligned} P_u^{(r)}(\xi) &= \widetilde{\nabla}_\lambda\nu(r,u_\lambda)\Big|_{\lambda=0} \\ &= \widetilde{\nabla}_\lambda\frac{1}{2}(Y(r,u_\lambda) + J(r,u_\lambda)\circ Y(r,u_\lambda)\circ j(r))\Big|_{\lambda=0} \\ &= \frac{1}{2}(\widetilde{\nabla}_\lambda Y(r,u_\lambda)\Big|_{\lambda=0} + J(r,u)\circ\widetilde{\nabla}_\lambda Y(r,u_\lambda)\Big|_{\lambda=0}\circ j(r)) \\ &= \frac{1}{2}(\nabla_\lambda Y(r,u_\lambda)\Big|_{\lambda=0} + J(r,u)\circ\nabla_\lambda Y(r,u_\lambda)\Big|_{\lambda=0}\circ j(r)) \\ &\quad - \frac{1}{4}(J\circ\nabla_\lambda J(r,u_\lambda)\Big|_{\lambda=0}\circ Y + J\circ J\circ\nabla_\lambda J(r,u_\lambda)\Big|_{\lambda=0}\circ j(r)) \\ &= [\nabla_\xi Y]^{0,1} - \frac{1}{2}[J\circ\nabla_\xi J\circ Y]^{0,1}. \end{aligned}$$

$$
\begin{aligned}
P_r^{(u)}(\rho) &:= \frac{d}{d\lambda}\Phi_r(\rho)^{-1}\nu(r_\lambda,u)\Big|_{\lambda=0} \\
&= [\frac{d}{d\lambda}\frac{1}{2}(Y(r_\lambda,u)+J(r_\lambda)\circ Y(r_\lambda,u)\circ j(r_\lambda))\Big|_{\lambda=0}]^{0,1} \\
&= [\partial_\rho Y]^{0,1}+\frac{1}{2}[\partial_\rho J\circ Y\circ j+J\circ Y\circ \partial_\rho j]^{0,1}.
\end{aligned}
$$

5.6 Banach spaces and norms.

We need the map $\mathcal{F}_{\mathcal{S},r,u}$ to be between Banach spaces, so we need to define the Sobolev completions

$$
\begin{aligned}
\Omega^0(\mathcal{S}_r,\underline{u}^*T\underline{M}) &\longrightarrow W^{1,p}(\mathcal{S}_r,u^*TM) \\
\Omega^{0,1}(\mathcal{S}_r,\underline{u}^*T\underline{M}) &\longrightarrow L^p(\mathcal{S}_r,\Lambda^{0,1}\otimes_J u^*TM).
\end{aligned}
$$

Let S be a surface, with volume form dvol_S. For $u:S\to\mathbb{R}$, the norms on $W^{1,p}(S)$ and $L^p(S)$ are defined by

$$
\|u\|_{W^{1,p}} := \left(\int_S |u|^p+|du|^p \ \mathrm{dvol}_S\right)^{1/p} \tag{5.6}
$$

$$
\|u\|_{L^p} := \left(\int_S |u|^p \ \mathrm{dvol}_S\right)^{1/p} \tag{5.7}
$$

To define corresponding norms for sections $\xi\in\Omega^0(\mathcal{S}_r,\underline{u}^*T\underline{M})$, we need to fix a metric and a connection. On the bundle $\underline{u}^*T\underline{M}\to S$, for each $z\in S$ there is an almost-complex structure $J(z)$ which is compatible with the symplectic form ω, so defines a metric g_J on TM,

$$
|\xi(z)| := \omega_{u(z)}(\xi(z),J(z)\xi(z)) := g_z(\xi(z),\xi(z))
$$

and a corresponding Levi-Civita connection ∇. Define

$$
\begin{aligned}
\|\xi\|_{W^{1,p}(S,u^*TM)} &:= \left(\int_S |\xi|^p+|\nabla\xi|^p \ \mathrm{dvol}_S\right)^{1/p} \\
\|\xi\|_{L^p(S,u^*TM)} &:= \left(\int_S |\xi|^p \ \mathrm{dvol}_S\right)^{1/p}
\end{aligned}
$$

Remark (C.f. [10, Remark 3.5.1.]) If $\xi \in W^{1,p}(\mathcal{S}_r, u^*TM)$, then the scalar function $|\xi(z)| \in W^{1,p}(S)$. This is because

$$\begin{aligned} d\sqrt{|\xi|^2+\varepsilon} &= d\sqrt{g_z(\xi,\xi)+\varepsilon} \\ &= \frac{1}{2\sqrt{g_z(\xi,\xi)+\varepsilon}} dg_z(\xi,\xi) \\ &= \frac{1}{2\sqrt{g_z(\xi,\xi)+\varepsilon}} (2g_z(\nabla\xi,\xi)+(\nabla g)(\xi,\xi)). \end{aligned}$$

Since $J(z)$ is independent of s along the striplike ends, $|\nabla g|$ can be uniformly bounded by a constant c. So pointwise,

$$|d\sqrt{|\xi|^2+\varepsilon}| \leq \frac{1}{2\sqrt{g_z(\xi,\xi)+\varepsilon}} (2|\nabla\xi||\xi|+c|\xi|^2).$$

In the limit as $\varepsilon \to 0$,

$$|d|\xi|| \leq |\nabla\xi| + (c/2)|\xi|.$$

Hence

$$\begin{aligned} \|d|\xi|\|_{L^p(S,\mathbb{R})} &\leq \|\nabla\xi\|_{L^p(S,u^*TM)} + (c/2)\|\xi\|_{L^p(S,u^*TM)} \\ &\leq \|\xi\|_{W^{1,p}(S,u^*TM)} + (c/2)\|\xi\|_{W^{1,p}(S,u^*TM)} \\ &= (1+(c/2))\|\xi\|_{W^{1,p}(S,u^*TM)}. \end{aligned}$$

Therefore we can write:

$$\begin{aligned} \left(\||\xi|\|_{W^{1,p}(S,\mathbb{R})}\right)^p &:= \left(\||\xi|\|_{L^p(S,\mathbb{R})}\right)^p + \left(\||d|\xi||\|_{L^p(S,\mathbb{R})}\right)^p \\ &\leq \left(\|\xi\|_{W^{1,p}(S,u^*TM)}\right)^p + \left((1+c/2)\|\xi\|_{W^{1,p}(S,u^*TM)}\right)^p \\ &= (1+(1+c/2)^p)\left(\|\xi\|_{W^{1,p}(S,u^*TM)}\right)^p \end{aligned}$$

hence

$$\||\xi|\|_{W^{1,p}(S,\mathbb{R})} \leq (1+(1+c/2)^p)^{1/p}\|\xi\|_{W^{1,p}(S,u^*TM)}$$

where c depends only on the choice of perturbation datum.

5.7 Gromov convergence

Definition Consider a sequence $\{(r_n, \underline{u}_n)\}_{n=1}^{\infty} \subset \mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_d)$.

1. For $2 \le e \le d-1$, we say that the sequence *Gromov converges to the broken pair*

$$\begin{aligned} (r_1, \underline{u}_1) &\in \mathcal{M}_{d-e+1,1}(\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_i, \underline{y}, \underline{x}_{i+e+1}, \ldots, \underline{x}_d) \\ (r_2, \underline{u}_2) &\in \mathcal{M}_e(\underline{y}, \underline{x}_{i+1}, \ldots, \underline{x}_{i+e}) \end{aligned}$$

   if

   - $r_n \longrightarrow r_1 \#_0 r_2$ in the topology of $\mathcal{R}^{d,1}$ near the boundary point $r_1 \#_0 r_2 \in \partial \mathcal{R}^{d,1}$,
   - $E(\underline{u}_n) \longrightarrow E(\underline{u}_1) + E(\underline{u}_2)$,
   - $\underline{u}_n$ converges uniformly on compact subsets of $\mathcal{S}_{r_1}$ to $\underline{u}_1$, and converges uniformly on compact subsets of $\mathcal{S}_{r_2}$ to $\underline{u}_2$.

2. For $1 \le s_1, \ldots, s_k \le d-1$ such that $s_1 + \ldots + s_k = d$, we say the sequence *Gromov converges to the broken tuple*

$$\begin{aligned} (r_0, \underline{u}_0) &\in \mathcal{M}_k(\underline{x}_0, \underline{y}_1, \ldots, \underline{y}_k) \\ (r_1, \underline{u}_1) &\in \mathcal{M}_{s_1,1}(\underline{y}_1, \underline{x}_1, \ldots, \underline{x}_{s_1}) \\ (r_2, \underline{u}_2) &\in \mathcal{M}_{s_2,1}(\underline{y}_2, \underline{x}_{s_1+1}, \ldots, \underline{x}_{s_1+s_2}) \\ &\ldots \\ (r_k, \underline{u}_k) &\in \mathcal{M}_{s_k,1}(\underline{y_k}, \underline{x}_{d-s_k+1}, \ldots, \underline{x}_d) \end{aligned}$$

   if

   - $r_n \longrightarrow r_0 \#_0 (r_1, \ldots, r_k) \in \partial \mathcal{R}^{d,1}$ in the topology of $\mathcal{R}^{d,1}$ near the boundary,
   - $E(u_n) \longrightarrow E(\underline{u}_0) + E(\underline{u}_1) + \ldots + E(\underline{u}_k)$,
   - $\underline{u}_n$ converges uniformly on compact subsets of $\mathcal{S}_{r_j}$ to $\underline{u}_j$, for $j = 0, \ldots, k$.

3. For $i \in \{1, \ldots, d\}$, we say that the sequence *Gromov converges to the broken pair*

$$\begin{aligned} (r, \underline{u}) &\in \mathcal{M}_{d,1}(\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_{i-1}, \underline{y}, \underline{x}_{i+1}, \ldots, \underline{x}_d) \\ \underline{v} &\in \widetilde{\mathcal{M}}_1(\underline{y}, \underline{x}_i) \end{aligned}$$

if

- $r_n \to r$ in $\mathcal{R}^{d,1}$, where r is in the interior of $\mathcal{R}^{d,1}$,
- $E(\underline{u}_n) \longrightarrow E(\underline{u}) + E(\underline{v})$,
- $\underline{u}_n$ converges uniformly on compact subsets of $\mathcal{S}_r$ to $\underline{u}$, and there is a sequence $\tau_n \in \mathbb{R}$ (shift parameters) such that if (s,t) denote coordinates on the strip $\mathbb{R} \times [0,1]$, and $\epsilon_i : \mathbb{R}_{\geq 0} \times [0,1] \to \mathcal{S}_r$ is the i-th striplike end of $\mathcal{S}_r$, then the sequence of shifted maps $\underline{u}_n(\epsilon_i(s+\tau_n,t))$ converges uniformly on compact subsets of $\mathbb{R} \times [0,1]$ to a fixed parametrization of the Floer trajectory $\underline{v}$.

4. We say that the sequence *Gromov converges to the broken pair*

$$\begin{aligned} (r,u) &\in \mathcal{M}_{d,1}(\underline{y}, \underline{x}_1, \ldots, \underline{x}_d) \\ \underline{v} &\in \widetilde{\mathcal{M}}_1(\underline{x}_0, \underline{y}) \end{aligned}$$

if

- $r_n \to r$ in $\mathcal{R}^{d,1}$, where r is in the interior of $\mathcal{R}^{d,1}$,
- $E(\underline{u}_n) \longrightarrow E(\underline{u}) + E(\underline{v})$,
- $\underline{u}_n$ converges uniformly on compact subsets of $\mathcal{S}_r$ to $\underline{u}$, and there is a sequence $\tau_n \in \mathbb{R}$ (shift parameters) such that if (s,t) denote coordinates on the strip $\mathbb{R} \times [0,1]$, and $\epsilon_0 : \mathbb{R}_{\geq 0} \times [0,1] \to \mathcal{S}_r$ is the 0-th striplike end of $\mathcal{S}_r$, then the sequence of shifted maps $\underline{u}_n(\epsilon_0(s+\tau_n,t))$ converges uniformly on compact subsets of $\mathbb{R} \times [0,1]$ to a fixed parametrization of the Floer trajectory $\underline{v}$.

The latter two cases are also described as *a Floer trajectory breaking off.*

5.8 Gromov neighborhoods

We now define what we call *Gromov neighborhoods* of a broken quilt of Type 1, 2 or 3. For $\epsilon > 0$, we will define a subset $U_\epsilon \subset \mathcal{B}^{d,1}$. Under these definitions, a sequence $(r_\nu, \underline{u}_\nu) \in \mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_d)$ will Gromov converge to a broken quilt if, and only if, given $\epsilon > 0$ there is a ν_0 such that $(r_\nu, \underline{u}_\nu) \in U_\epsilon$ for all $\nu \geq \nu_0$.

Type 1

Let $(r_1, \underline{u}_1)$ and $(r_2, \underline{u}_2)$ be a broken pair,

$$\begin{aligned}(r_1, \underline{u}_1) &\in \mathcal{M}_{d-e+1,1}(\underline{x}_0, \underline{x_1} \dots, \underline{x}_{i-1}, \underline{y}, \underline{x}_{i+e+1}, \dots, \underline{x}_d)^0 \\ (r_2, \underline{u}_2) &\in \mathcal{M}_e(\underline{y}, \underline{x}_i, \underline{x}_{i+1}, \dots, \underline{x}_{i+e})^0.\end{aligned}$$

A small neighborhood of the point $r_1 \#_0 r_2 \in \partial\mathcal{R}^{d,1}$ is of the form

$$U \cong U_1 \times U_2 \times [0, \epsilon)$$

where $U_1 \subset \mathcal{R}^{d-e+1,1}$ is a neighborhood of r_1, and $U_2 \subset \mathcal{R}^e$ is a neighborhood of r_2, and the interval $[0, \epsilon)$ represents the gluing parameter. Recall that a gluing parameter δ corresponds to a gluing length $R(\delta) = -\log(\delta)$.

Fix a metric on $U_1 \subset \mathcal{R}^{d-e+1,1}$ and a metric on $U_2 \subset \mathcal{R}^e$, and define a metric topology on $U \cong U_1 \times U_2 \times [0, \epsilon)$ by

$$\mathrm{dist}_U(r_1 \#_\delta r_2, r_1' \#_{\delta'} r_2') := \sup\{\mathrm{dist}_{U_1}(r_1, r_1'), \mathrm{dist}_{U_2}(r_2, r_2'), |\delta - \delta'|\}.$$

By the construction of the surface bundles (Sections 4.6 and 4.7, Chapter 4), we can suppose that the neighborhood U is sufficiently small that the corresponding neighborhoods $U_1 \subset \mathcal{R}^{d-e+1,1}$ and $U_2 \subset \mathcal{R}^e$ are also small enough that all surfaces in the bundles over them are diffeomorphic to each other by diffeomorphisms preserving the striplike ends. Write $\mathcal{S}_{r_1 \#_\delta r_2} = \mathcal{S}^\delta_{r_1} \cup \mathcal{S}^\delta_{r_2} / \sim$, where $\mathcal{S}^\delta_{r_i}$ represents the truncation of $\mathcal{S}_{r_i}$ along the prescribed striplike end at $s = R(\delta)$, and $\sim$ is the identification of the two truncated surfaces along the cuts.

Definition Let $\epsilon > 0$ be given. Define a *Gromov neighborhood* $U_\epsilon \subset \mathcal{B}^{d,1}$ of the pair $(r_1, \underline{u}_1), (r_2, \underline{u}_2)$ as follows: $(r, \underline{u}) \in U_\epsilon$ if

- $r = \widetilde{r}_1 \#_\delta \widetilde{r}_2 \in U$ with $\mathrm{dist}_U(\widetilde{r}_1 \#_\delta \widetilde{r}_2, r_1 \#_0 r_2) < \epsilon$,
- $|E(u_1) + E(u_2) - E(u)| < \epsilon$,
- $\mathrm{dist}_{\underline{M}}(\underline{u}(z), \underline{u}_1(z)) < \epsilon$ for all $z \in \mathcal{S}^\delta_{r_1}$,

- $\text{dist}_{\underline{M}}(\underline{u}(z), \underline{u}_2(z)) < \epsilon$ for all $z \in \mathcal{S}^{\delta}_{r_2}$.

  (The metrics on the target manifolds $\underline{M}$ are those induced by their symplectic forms $\underline{\omega}$ and the choice of compatible almost complex structures $\underline{J} = \underline{J}(z)$.)

Type 2

Let $(r_0, \underline{u}_0), \ldots, (r_k, \underline{u}_k)$ be a broken tuple of the form

$$\begin{aligned}
(r_0, \underline{u}_0) &\in \mathcal{M}_k(\underline{x}_0, \underline{y}_1, \ldots, \underline{y}_k) \\
(r_1, \underline{u}_1) &\in \mathcal{M}_{s_1,1}(\underline{y}_1, \underline{x}_1, \ldots, \underline{x}_{s_1}) \\
(r_2, \underline{u}_2) &\in \mathcal{M}_{s_2,1}(\underline{y}_2, \underline{x}_{s_1+1}, \ldots, \underline{x}_{s_1+s_2}) \\
&\ldots \\
(r_k, \underline{u}_k) &\in \mathcal{M}_{s_k,1}(\underline{y_k}, \underline{x}_{d-s_k+1}, \ldots, \underline{x}_d).
\end{aligned}$$

A small neighborhood of the point $r_1 \#_0 r_2 \in \partial \mathcal{R}^{d,1}$ is of the form

$$U \cong U_0 \times U_1 \times \ldots U_k \times [0, \epsilon)$$

where $U_0 \subset \mathcal{R}^k$ is a neighborhood of r_0, $U_i \subset \mathcal{R}^{s_i,1}$ is a neighborhood of r_i for $i = 1, \ldots, k$, and the interval $[0, \epsilon)$ represents the gluing parameter. Fixing a metric on $U_0, \ldots, U_i$ determines a metric topology on $U \cong U_1 \times U_2 \times [0, \epsilon)$ by

$$\text{dist}_U(r_0 \#_\delta(r_1, \ldots, r_k), r'_0 \#_{\delta'}(r'_1, \ldots, r'_k)) := \sup\{\text{dist}_{U_0}(r_0, r'_0), \ldots, \text{dist}_{U_k}(r_k, r'_k), |\delta - \delta'|\}.$$

Taking the neighborhood U to be sufficiently small we can assume that all surfaces parametrized by $U_0, \ldots, U_k$ are diffeomorphic via diffeomorphisms that are constant on the striplike ends. Write $\mathcal{S}_{r_0} \#_\delta(r_1, \ldots, r_k) = \mathcal{S}^\delta_{r_0} \cup \mathcal{S}^\delta_{r_1} \cup \ldots \cup \mathcal{S}^\delta_{r_k} / \sim$, where each $\mathcal{S}^\delta_{r_i}$ is the truncation of the surface $\mathcal{S}_{r_i}$ along the prescribed striplike end at $s = R(\delta) = -\log(\delta)$, and $\sim$ is the identifications of the surfaces along the truncated ends.

Definition Let $\epsilon > 0$ be given. Define a *Gromov neighborhood* $U_\epsilon \subset \mathcal{B}^{d,1}$ of the tuple $(r_0, \underline{u}_0), \ldots, (r_k, \underline{u}_k)$ as follows: $(r, \underline{u}) \in U_\epsilon$ if

- $r = \widetilde{r_0} \#_\delta \{\widetilde{r_1}, \ldots, \widetilde{r_k}\} \in U$ with $\text{dist}_U(r, r_0 \#_0 \{r_1, \ldots, r_k\}) < \epsilon$,

- $|E(\underline{u}_0) + E(\underline{u}_1) + \ldots + E(\underline{u}_k) - E(\underline{u})| < \epsilon$,
- $\mathrm{dist}_{\underline{M}}(\underline{u}(z), \underline{u}_i(z)) < \epsilon$ for all $z \in \mathcal{S}_{r_i}^{\delta}$, $i = 0, \ldots, k$.

Type 3

Let $(r_0, \underline{u}_0) \in \mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_{i-1}, \underline{y}, \underline{x}_{i+1}, \ldots, \underline{x}_d)$ be a pseudoholomorphic quilted disk, and let $\underline{v} \in \widetilde{\mathcal{M}}_1(\underline{y}, \underline{x}_i)$ be a quilted Floer trajectory. We fix a parametrization of the Floer trajectory $\underline{v} : \mathbb{R} \times [0,1] \to \underline{M}$. As in previous sections we write $\epsilon_i : [0,\infty) \times [0,1] \to \mathcal{S}_r$ for the i-th striplike end, and Z_i for the image of this striplike end in $\mathcal{S}_r$.

Definition Let $\epsilon > 0$ be given, and define $R(\epsilon) = -\log(\epsilon)$. Define a *Gromov neighborhood* $U_\epsilon \subset \mathcal{B}^{d,1}$ of the pair $(r_0, \underline{u}_0), \underline{v}$ as follows: $(r, \underline{u}) \in U_\epsilon$ if

- $\mathrm{dist}_{\mathcal{R}^{d,1}}(r, r_0) < \epsilon$,
- $|E(\underline{u}_0) + E(\underline{v}) - E(\underline{u})| < \epsilon$,
- for $z \in \mathcal{S}_{r_0}^{R(\epsilon)}$, $\mathrm{dist}_{\underline{M}}(\underline{u}(z), \underline{u}_0(z)) < \epsilon$,
- there exists some $\tau \geq 2R(\epsilon)$ such that $\mathrm{dist}_{\underline{M}}(\underline{u}(\epsilon_i(s+\tau, t), \underline{v}(s,t)) < \epsilon$ for $(s,t) \in [-R(\epsilon), R(\epsilon)] \times [0,1]$.

Chapter 6

Gluing

6.1 Outline of chapter

Definition We say that a parametrized pseudoholomorphic quilt $(r, \underline{u})$ is *regular* if the linearized operator $D_{\mathcal{S},r,\underline{u}}$ is surjective. Similarly we say that a generalized Floer trajectory $\underline{v}$ is *regular* if the associated linearized operator $D_{\underline{v}}$ is surjective.

The goal of this chapter is to prove the following gluing theorem.

Theorem 6.1.1. *Let $\underline{x}_0 \in \mathcal{I}(\underline{L}_{0,AB}, \underline{L}_{d,AB})$ and for $i = 1, \ldots, d$ let $\underline{x}_i \in \mathcal{I}(\underline{L}_{i-1}, \underline{L}_i)$. Given either:*

1. *a regular pair*

$$\begin{aligned} (r_1, \underline{u}_1) &\in \mathcal{M}_{d-e+1,1}(\underline{x}_0, \underline{x_1} \ldots, \underline{x}_{i-1}, \underline{y}, \underline{x}_{i+e+1}, \ldots, \underline{x}_d)^0 \\ (r_2, \underline{u}_2) &\in \mathcal{M}_e(\underline{y}, \underline{x}_i, \underline{x}_{i+1}, \ldots, \underline{x}_{i+e})^0 \end{aligned}$$

   where $2 \leq e \leq d$, $1 \leq i \leq d-e$, and $\underline{y} \in \mathcal{I}(\underline{L}_{i-1}, \underline{L}_{i+e})$;

2. *or a regular $(k+1)$-tuple*

$$\begin{aligned} (r_0, \underline{u}_0) &\in \mathcal{M}_k(\underline{x_0}, \underline{y}_1, \ldots, \underline{y}_k)^0 \\ (r_1, \underline{u}_1) &\in \mathcal{M}_{d_1,1}(\underline{y}_1, \underline{x}_1, \ldots, \underline{x}_{d_1})^0 \\ (r_2, \underline{u}_2) &\in \mathcal{M}_{d_2,1}(\underline{y}_2, \underline{x}_{d_1+1}, \ldots, \underline{x}_{d_1+d_2})^0 \\ &\ldots \\ (r_k, \underline{u}_k) &\in \mathcal{M}_{d_k,1}(\underline{y}_k, \underline{x}_{d_1+\ldots+d_{(k-1)}+1}, \ldots, \underline{x}_{d_1+\ldots+d_{k-1}+d_k})^0 \end{aligned}$$

   where $d_1 + \ldots + d_k = d$, $d_i \geq 1$ for each i, and $\underline{y}_i \in \mathcal{I}(\underline{L}_{d_1+\ldots+d_{(i-1)}}, \underline{L}_{d_1+\ldots+d_i})$ (interpreting d_0 as 0);

3. or a regular pair

$$\begin{aligned}(r, \underline{u}) &\in \mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_{i-1}, \underline{y}, \underline{x}_{i+1}, \ldots, \underline{x}_d)^0 \\ \underline{v} &\in \widetilde{\mathcal{M}}_1(\underline{y}, \underline{x}_i)^0\end{aligned}$$

where $1 \leq i \leq d$, and $\underline{y} \in \mathcal{I}(\underline{L}_{i-1}, \underline{L}_i)$;

4. or a regular pair

$$\begin{aligned}\underline{v} &\in \widetilde{\mathcal{M}}_1(\underline{x}_0, \underline{y})^0 \\ (r, \underline{u}) &\in \mathcal{M}_{d,1}(\underline{y}, \underline{x}_1, \ldots, \underline{x}_d)^0\end{aligned}$$

where $\underline{y} \in \mathcal{I}(\underline{L}_{0,AB}, \underline{L}_{d,AB})$,

there is an associated continuous gluing map

$$g : (R_0, \infty) \to \mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_d)^1$$

defined for some $R_0 >> 0$, such that $g(R)$ Gromov converges to the given pair/tuple as $R \to \infty$. Moreover, for sufficiently small $\epsilon > 0$, the gluing map surjects onto Gromov neighborhoods U_ϵ of the given broken pairs/tuples.

The gluing map is an application of the Implicit Function Theorem, and the strategy of proof is standard. To be able to apply the implicit function theorem, the main steps are the following:

1: For gluing lengths $R > 0$, define a *pre-glued curve*, (r_R, u_R).

2: Compute that $\|(\overline{\partial} - \nu)(r_R, u_R)\|_{0,p} \leq \varepsilon(R)$ where $\varepsilon(R) \to 0$ as $R \to \infty$.

3: Show that $D_{\mathcal{S},r_R,u_R}$ is surjective, and construct a right inverse Q_R by first constructing an approximate right inverse, T_R, with the same image.

4: Show there is a uniform bound $\|Q_R\| \leq C$ for sufficiently large R.

5: Show that for each R, the function $\mathcal{F}_{\mathcal{S},r_R,u_R}$ satisfies a quadratic estimate

$$\|d\mathcal{F}_{\mathcal{S},r_R,u_R}(\rho, \xi) - D_{\mathcal{S},r_R,u_R}\| \leq c(|\rho| + \|\xi\|_{W^{1,p}}) \tag{6.1}$$

with the constant c independent of R.

These steps are the content of Sections 6.2 through 6.6 respectively. In Section 6.7 these ingredients are used to define, with the help of an implicit function theorem, a gluing map associated to each regular tuple, and we show that the image of the gluing map is contained in the one-dimensional component of the relevant moduli space of quilted disks, $\mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_d)^1$. In Section 6.8 we show that the gluing map is surjective, in the sense that if a pseudoholomorphic quilted disk $(r, \underline{u}) \in \mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_d)^1$ is in a sufficiently small Gromov neighborhood of the broken tuple, then it is in the image of the gluing map associated to that tuple.

6.2 Pregluing

There are three types of pre-gluing to consider - two types arise from the two types of facet of codimension one in the boundary of $\overline{\mathcal{R}}^{d,1}$ (corresponding to whether the inner circle has bubbled through or not), and the other type arises from a Floer trajectory breaking off.

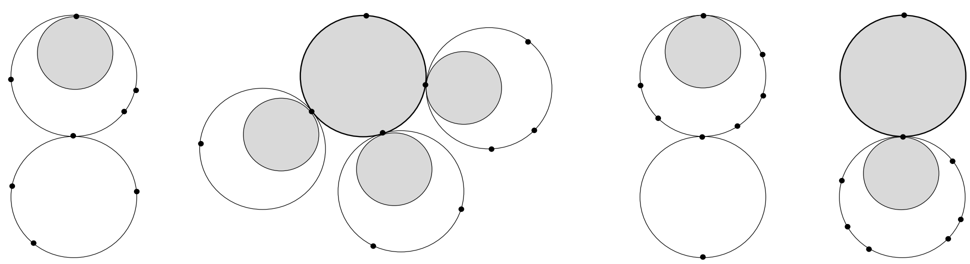

Figure 6.1: Three cases of gluing – the last two cases both correspond to a Floer trajectory bubbling off.

Definition Given a *gluing parameter* $\delta > 0$, define $R(\delta) := -\log(\delta)$ to be the *gluing length* corresponding to δ. So $R(\delta) \to \infty$ as $\delta \to 0$.

Type 1

Assume that we have a regular pair

$$\begin{aligned}
(r_1, \underline{u}_1) &\in \mathcal{M}_{d_1,1}(\underline{x}_0, \ldots, \underline{x}_{i-1}, \underline{y}, \underline{x}_{i+1}, \ldots, \ \ldots \underline{x}_{d_1})^0, \text{ where } i \in \{1, \ldots, d\}, \\
(r_2, \underline{u}_2) &\in \mathcal{M}_{d_2}(\underline{y}, \underline{z}_1, \ldots, \underline{z}_{d_2})^0.
\end{aligned}$$

The surface parametrized by r_1 is a quilted disk, and the surface parametrized by r_2 is a marked, unquilted disk. A marked point labeled ζ^- on r_1 is identified with a marked point ζ^+ on r_2, identifying the pair $r_1 \# r_2$ with a nodal quilted disk. Along the strip-like ends labelled by $\zeta^{\pm}$, $\underline{u}_1$ and $\underline{u}_2$ converge exponentially to $\underline{y}$.

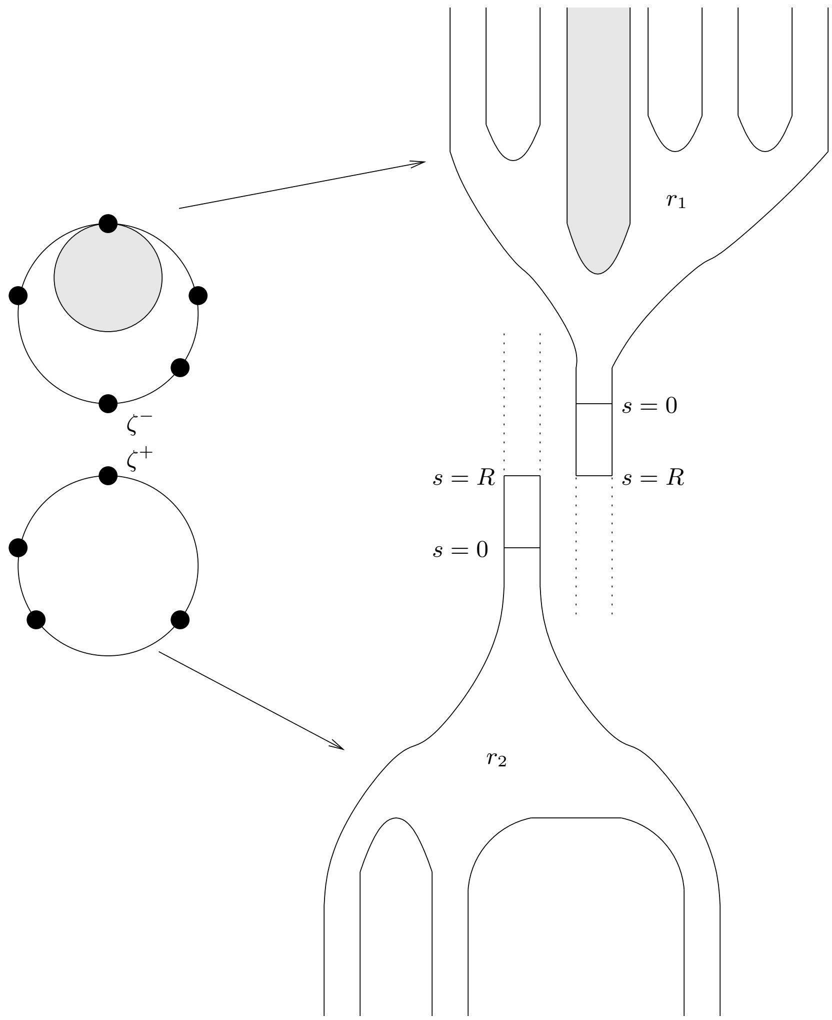

Figure 6.2: Case 1 of pregluing

The quilted surface $r_R := r_1 \#_R r_2$:

Truncate the surface $\mathcal{S}_{r_1}$ along the striplike end labeled by ζ^- at $s = R$, and truncate the surface $\mathcal{S}_{r_2}$ along the striplike end labeled by ζ^+ at $s = R$, then identify the two truncated surfaces along $s = R$. Explicitly, one identifies $\epsilon_{\zeta^+}(R, t) \sim \epsilon_{\zeta^-}(R, 1-t)$; see Figure 6.2.

Let $z = (s,t)$ denote the variables on the striplike end of $\mathcal{S}_{r_1}$, where $s \in [0,\infty)$ and $t \in [0,1]$. For $s >> 0$, we know that $\underline{u}_1(s,t)$ is very close to $\underline{y}(t) \in \mathcal{I}(\underline{L}, \underline{L}')$.

For such s we can locally model $\underline{u}_1(s,t)$ on a local trivialization around neighborhood of 0 in $T_{\underline{y}(t)}\underline{M}$, using exponential maps coming from a family of metrics on M.

As in Section 5.3, fix a tuple of t-dependent metrics $(g_k(t))_{k\in -r,\ldots,s}$ parametrized by $t \in [0,1]$, such that:

1. For each k and for each t, $g_k(t)$ is a metric on M_k, and $g_k(t)(J_k(t)\eta, J_k(t)\xi) = g_k(t)(\eta,\xi)$ for all $\eta,\xi \in T_pM_k, p \in M_j$.
2. For each k, $g_k(1) \times g_{k+1}(0)$ determines a metric on $M_k \times M_{k+1}$ and is such that $g_k(1) \times g_{k+1}(0) : T_{y_k(1)\times y_{k+1}(0)}L_{k,k+1} \to L_{k,k+1}$.
3. L_{-r} is totally geodesic with respect to $g_{-r}(0)$, L_s is totally geodesic with respect to $g_s(1)$, and for $k = -r,\ldots,s-1$, the Lagrangian correspondence $L_{k,k+1}$ is totally geodesic with respect to $g_k(1) \times g_{k+1}(0)$.

For very large $s >> 0$, define $\xi_1(s,t) \in T_{\underline{y}(t)}\underline{M}$ by

$$\exp_{\underline{y}(t)}(\xi_1(s,t)) = \underline{u}_1(s,t)$$

where exp is the tuple of exponential maps, $\exp_k : [0,1] \times T_{y_k(\cdot)}M_k \to M_k$, each of which comes from the Levi-Civita connection on M_k associated to the metric $g_k(t)$. Similarly, we can define $\xi_2(s,t) \in T_{\underline{y}(t)}\underline{M}$, for very large s, by

$$\exp_{\underline{y}(t)}(\xi_2(s,t)) = \underline{u}_2(s,t).$$

Now let $\beta : \mathbb{R} \to [0,1]$ be a smooth cut-off function such that $\beta(s) = 1, s \leq -1$, and $\beta(s) = 0, s \geq 0$ (see Figure 6.3).

We introduce a pair of intermediate approximate pseudoholomorphic quilted surfaces $(r_i, \underline{u}_i^R)$ for $i = 1,2$ which are defined as follows.

$$\underline{u}_i^R(z) = \begin{cases} \underline{u}_1(z), & z \in \mathcal{S}_{r_i} \setminus \epsilon_{\zeta^\pm}(s \geq R/2) \\ \exp_{\underline{x}}(\beta(s-R/2)\xi_i(s,t)), & R/2 - 1 \leq s \leq R/2 \\ \underline{y}(t), & s \geq R/2. \end{cases}$$

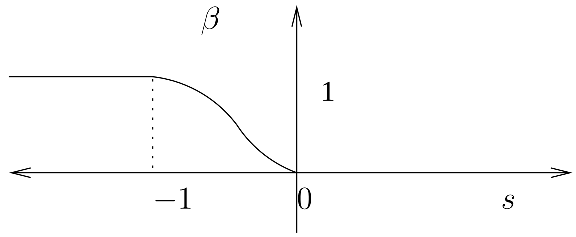

Figure 6.3: Cutoff function used to define $\underline{u}_1^R$ and $\underline{u}_2^R$.

Then the pre-glued map $u_1\#_R u_2 : \mathcal{S}_{r_1\#_R r_2} \to \underline{M}$ is defined by

$$u_1\#_R u_2(z) = \begin{cases} u_1^R(z), & z \in \mathcal{S}_{r_1} \setminus \epsilon_{\zeta^-}((R,\infty)\times[0,1]) \\ u_2^R(z), & z \in \mathcal{S}_{r_2} \setminus \epsilon_{\zeta^+}((R,\infty)\times[0,1]), \end{cases}$$

see Figure 6.4.

Type 2

Assume that we have a collection

$$\begin{aligned}
(r_0,u_0) &\in \mathcal{M}_k(\underline{x}_0, \underline{y}_1, \ldots, \underline{y}_k)^0 \\
(r_1,u_1) &\in \mathcal{M}_{d_1,1}(\underline{y}_1, \underline{z}_1, \underline{z}_2, \ldots, \underline{z}_{d_1})^0 \\
(r_2,u_2) &\in \mathcal{M}_{d_2,1}(\underline{y}_2, \underline{z}_{d_1+1}, \underline{z}_{d_1+2}, \ldots, \underline{z}_{d_1+d_2})^0 \\
&\ldots \\
(r_k,u_k) &\in \mathcal{M}_{d_k,1}(\underline{y}_k, \underline{z}_{d_1+d_2+\ldots+d_{k-1}+1}, \underline{z}_{d_1+d_2+\ldots+d_{k-1}+2}, \ldots, \underline{z}_{d_1+d_2+\ldots d_{k-1}+d_k})^0
\end{aligned}$$

of regular pseudoholomorphic quilted surfaces $(r_1,\underline{u}_1),\ldots,(r_k,\underline{u}_k)$ together with a regular pseudoholomorphic quilted surface $(r_0,\underline{u}_0)$. We label the striplike ends of $\mathcal{S}_{r_0}$ with $\zeta^{(0)},\zeta^{(1)},\ldots,\zeta^{(k)}$, where the map $\underline{u}_0 : \mathcal{S}_{r_0} \to \underline{M}$ converges to $\underline{y}_i$ along the striplike end labeled by $\zeta^{(i)}$ for $i = 1,\ldots,k$, and converges to $\underline{x}_0$ along the striplike end labeled by $\zeta^{(0)}$. For $i = 1,\ldots,k$, each quilted surface $\mathcal{S}_{r_i}$ has a distinguished strip-like end along which the map $\underline{u}_i : \mathcal{S}_{r_i} \to \underline{M}$ converges to $\underline{y}_i$, and we label this striplike end by $\eta_i^{(0)}$.

The quilted surface $r_R := r_0\#_R\{r_1,\ldots,r_k\}$:

Truncate the surface $\mathcal{S}_{r_0}$ along each of the striplike ends labeled $\zeta^{(1)},\ldots,\zeta^{(d)}$ at $s = R$, and truncate the surfaces $\mathcal{S}_{r_1},\ldots,\mathcal{S}_{r_k}$ along their respective striplike ends labeled

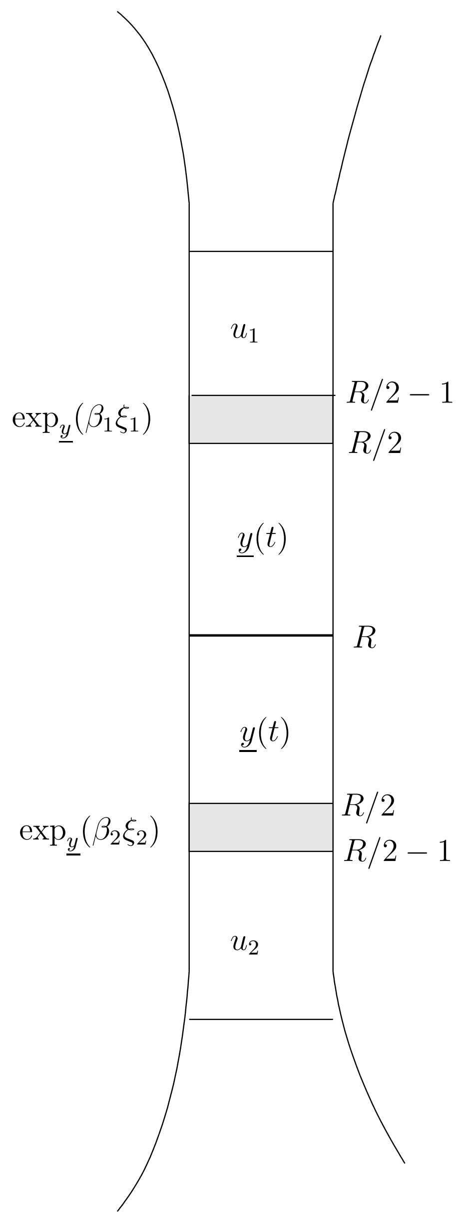

Figure 6.4: Gluing u_1 and u_2 along the neck.

$\eta_1^{(0)}, \ldots, \eta_k^{(0)}$ at $s = R$, then identify truncations labeled by pairs $\zeta^{(i)}, \eta_i^{(0)}$ along $s = R$; see Figure 6.5.

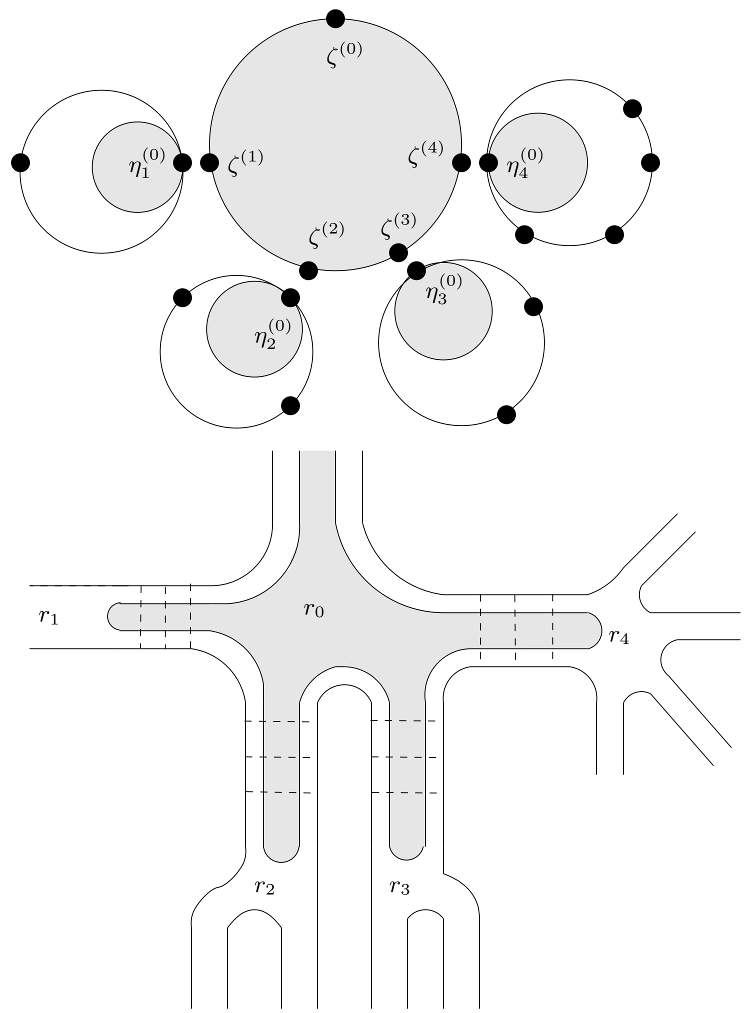

Figure 6.5: Pregluing in Case 2.

The approximate pseudoholomorphic map

Let $z = (s, t)$ denote variables for the striplike ends. We know that for $s >> 0$, the map $\underline{u}_0(\epsilon_{\zeta^{(i)}}(s,t))$ lands in a normal neighborhood of $\underline{y}_i(t)$. For such s we can define $\xi_i(s,t) \in T_{\underline{y}_i(t)}\underline{M}$ by

$$\exp_{\underline{y}_i(t)}(\xi_i(s,t)) = \underline{u}_0(\epsilon_{\zeta^{(i)}}(s,t)).$$

Similarly, for $i = 1, \ldots, k$ and sufficiently large s we can define $\xi_i^{(0)}(s,t) \in T_{\underline{y}_i(t)}\underline{M}$ by

$$\exp_{\underline{y}_i(t)}(\xi_i^{(0)}(s,t)) = \underline{u}_i(\epsilon_{\eta_i^{(0)}}(s,t)).$$

We introduce a collection of intermediate approximate pseudoholomorphic quilted surfaces $(r_0, \underline{u}_0^R), (r_1, \underline{u}_1^R), \ldots, (r_k, \underline{u}_k^R)$ which are defined as follows. The map $\underline{u}_0^R : \mathcal{S}_{r_0} \to \underline{M}$ is defined piecewise by

$$\underline{u}_0^R(z) = \begin{cases} \underline{u}_0(z), & z \in \mathcal{S}_{r_0} \setminus \bigcup_{i=1}^{k} \epsilon_{\zeta(i)}(s \geq R/2 - 1) \\ \exp_{\underline{y}_i(t)}(\beta(s - R/2)\xi_i(s,t)), & z = \epsilon_{\zeta(i)}(s,t), s \in [R/2 - 1, R/2] \\ \underline{y}_i(t), & z = \epsilon_{\zeta(i)}(s,t), s \geq R/2. \end{cases}$$

For $i = 1, \ldots, k$ the intermediate maps $\underline{u}_i^R : \mathcal{S}_{r_i} \to \underline{M}$ are defined by

$$\underline{u}_i^R(z) = \begin{cases} \underline{u}_i(z), & z \in \mathcal{S}_{r_i} \setminus \epsilon_{\eta_i^{(0)}}(s \geq R/2 - 1) \\ \exp_{\underline{y}_i(t)}(\beta(s - R/2)\xi_i(s,t)), & z = \epsilon_{\eta_i^{(0)}}(s,t), s \in [R/2 - 1, R/2] \\ \underline{y}_i(t), & z = \epsilon_{\eta_i^{(0)}}(s,t), s \geq R/2. \end{cases}$$

Then the pre-glued map $u_0 \#_R \{u_1, \ldots, u_k\} : \mathcal{S}_{r_0 \#_R \{r_1, \ldots, r_k\}} \to \underline{M}$ is defined by

$$u_0 \#_R \{u_1, \ldots, u_k\}(z) = \begin{cases} \underline{u}_0^R(z), & z \in \mathcal{S}_{r_0} \setminus \bigcup_{i=1}^{k} \epsilon_{\zeta(i)}(s \geq R) \\ \underline{u}_i^R(z), & z \in \mathcal{S}_{r_i} \setminus \epsilon_{\eta_i^{(0)}}(s \geq R). \end{cases}$$

Type 3

(A Floer trajectory breaks off.) Assume that we have a pair

$$\begin{aligned} (r_1, u_1) &\in \mathcal{M}_{d,1}(\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_d)^0 \\ v &\in \mathcal{M}(\underline{x}_i, \underline{y})^0 \end{aligned}$$

where (r_1, u_1) is a regular pseudoholomorphic quilted disk, and v is a regular Floer trajectory. Assume without loss of generality that $\underline{v} : \mathbb{R} \times [0,1] \to \underline{M}$ is such that

$$\begin{aligned} \lim_{s \to -\infty} v(s,t) &= \underline{x}_i(t) \\ \lim_{s \to \infty} v(s,t) &= \underline{y}(t), \end{aligned}$$

and let ζ label the striplike end of $\mathcal{S}_{r_1}$ for which $\lim_{s \to \infty} \underline{u}_1(\epsilon_\zeta(s,t)) = \underline{x}_i(t)$. The Floer trajectory $\underline{v}$ is defined only up to an $\mathbb{R}$ translation, but we can fix a parametrization of $\underline{v}$ to work with.

The quilted surface

In this case, $r_R = r_1$.

The approximate pseudoholomorphic map

For $s >> 0$ we know that $\underline{u}_1(\epsilon_\zeta(s,t))$ and $v(-s,t)$ are exponentially close to $\underline{x}_i(t)$. For such s, define $\xi(s,t), \eta(-s,t) \in T_{\underline{x}_i(t)}\underline{M}$ by the conditions that

$$\begin{aligned}
\exp_{\underline{x}_i(t)}(\xi(s,t)) &= \underline{u}_1(\epsilon_\zeta(s,t)), \\
\exp_{\underline{x}_i(t)}(\eta(-s,t)) &= \underline{v}(-s,t).
\end{aligned}$$

Using the same cutoff function β as used in all previous cases, define an approximate pseudoholomorphic quilted surface $(r_1, \underline{u}_1^R)$ and an approximate pseudoholomorphic trajectory $\underline{v}^R$ as follows.

$$\begin{aligned}
\underline{u}_1^R(z) &= \begin{cases} \underline{u}_1(z), & z \in \mathcal{S}_{r_1} \setminus \epsilon_\zeta(s \geq R/2 - 1) \\ \exp_{\underline{x}_i(t)}(\beta(s - R/2)\xi(s,t)), & z = \epsilon_\zeta(s,t), s \in [R/2 - 1, R/2] \\ \underline{x}_i(t), & z = \epsilon_\zeta(s,t), s \geq R/2. \end{cases} \\
\underline{v}^R(s,t) &= \begin{cases} \underline{v}(s - 2R, t), & s \geq 3R/2 + 1 \\ \exp_{\underline{x}_i(t)}(\beta(-s + 3R/2)\eta(s - 2R, t)), & s \in [3R/2, 3R/2 + 1] \\ \underline{x}_i(t), & s \leq 3R/2. \end{cases}
\end{aligned}$$

Then the preglued curve u_R is defined by

$$\underline{u}_R(z) = \begin{cases} \underline{u}_1^R(z), & z \in \mathcal{S}_{r_1} \setminus \epsilon_\zeta(s \geq R) \\ \underline{v}^R(s,t), & z \in \epsilon_\zeta(s \geq R). \end{cases}$$

6.3 Estimates for the preglued curves.

Proposition 6.3.1. *For sufficiently large $R_0 \geq 0$, there is a monotone decreasing function $\epsilon : [R_0, \infty) \to [0, \infty)$ such that*

$$\|(\overline{\partial} - \nu)(r_R, u_R)\|_{0,p} \leq \epsilon(R)$$

and $\epsilon(R) \to 0$ as $R \to \infty$.

Proof. We estimate it for each of the three types of pregluing separately.

Type 1

Let $(\widetilde{J}, \widetilde{K})$ denote the *approximate* perturbation datum on the preglued surface $\mathcal{S}_{r_R}$ that is inherited from the perturbation data (J_1, K_1) and (J_2, K_2) for $\mathcal{S}_{r_1}$ and $\mathcal{S}_{r_2}$ respectively via the pregluing procedure. Let $(J(r_R), K(r_R))$ be the perturbation datum on $\mathcal{S}_{r_R}$ that comes from the universal choice of perturbation data over the family of quilted surfaces parametrized by the multiplihedron. In general $(\widetilde{J}, \widetilde{K})$ and $(J(r_R), K(r_R))$ are not the same, but the assumption of consistency implies that for large values of R, the data agree on the "thin" part of $\mathcal{S}_{r_R}$, while on the complement of the thin part, which consists of two compact components coming from $\mathcal{S}_{r_1}$ and $\mathcal{S}_{r_2}$, $(J(r_R), K(r_R))$ converges uniformly to (J_1, K_1) or (J_2, K_2) respectively as $R \to \infty$.

Given that

$$\begin{aligned}
\overline{\partial}(r_R, u_R) - \nu(r_R, u_R) &= \frac{1}{2}(Du_R + J(r_R) \circ Du_R \circ j(r_R)) \\
&\quad -\frac{1}{2}(Y(r_R) + J(r_R) \circ Y(r_R) \circ j(r_R))
\end{aligned}$$

we will estimate this on different parts of the preglued surface $\mathcal{S}_{r_R}$.

Consider a striplike end $Z \subset \mathcal{S}_{r_R}$. It corresponds to a striplike end on either $\mathcal{S}_{r_1}$ or $\mathcal{S}_{r_2}$, that was *not* truncated in the pregluing step; let us denote this striplike end by Z too, where $Z \subset \mathcal{S}_{r_i}$ for $i = 1$ or 2. Since $j(r_R, z), J(r_R, u(z), z)$ and $Y(r_R, z)$ are independent of R and are the same as the corresponding data on $\mathcal{S}_{r_1}$,

$$\begin{aligned}
(\overline{\partial} - \nu)(r_R, u_R)\Big|_Z &= (\overline{\partial} - \nu)(r_1, u_1)\Big|_Z \\
&= 0
\end{aligned}$$

since we assumed that $(\overline{\partial} - \nu)(r_1, u_1) = 0$. This argument proves that $(\overline{\partial} - \nu)(r_R, u_R)$ is zero on *all* of the striplike ends of the glued surface $\mathcal{S}_{r_R}$.

Next, for $i = 1, 2$ let S_i denote the complement of the striplike ends on $\mathcal{S}_{r_i}$, and let S_i also denote its image in $\mathcal{S}_{r_R}$ after pregluing. Note that S_i is compact, and that

$u_R\Big|_{S_i} = u_i\Big|_{S_i}$. Hence,

$$\begin{aligned}(\overline{\partial} - \nu)(r_R, u_R)\Big|_{S_i} &= (\overline{\partial} - \nu)(r_R, u_i)\Big|_{S_i} \\ &= \frac{1}{2}[Du_i + J(r_R, u_i)Du_i j(r_R)]\Big|_{S_i} \\ &\quad -\frac{1}{2}[Y(r_R, u_i) + J(r_R, u_i)Y(r_R, u_i)j(r_R)]\Big|_{S_i}.\end{aligned}$$

Since we know that $(\overline{\partial} - \nu)(r_i, u_i) = 0$, and that $j(r_R), J(r_R, u_i)$ and $Y(r_R, u_i)$ converge uniformly to $j(r_i), J(r_i, u_i)$ and $Y(r_i, u_i)$ on S_i, it follows that for sufficiently large R there is a monotone decreasing function $\epsilon_1(R) \to 0$ as $R \to \infty$ such that the uniform pointwise estimate

$$|(\overline{\partial} - \nu)(r_R, u_R)(z)| \leq \epsilon_1(R)$$

holds for all $z \in S_i$.

Finally, consider the neck of $\mathcal{S}_{r_R}$ along which the pregluing took place. Let Z_1 and Z_2 denote the striplike ends of $\mathcal{S}_{r_1}$ and $\mathcal{S}_{r_2}$ that were truncated along $s = R$; and by slight abuse of notation let Z_1 and Z_2 also denote the images of the truncations after pregluing. By symmetry it is enough to consider what happens on Z_1. There are three regions to consider,

$$0 \leq s \leq R/2 - 1$$

$$R/2 - 1 \leq s \leq R/2$$

$$R/2 \leq s \leq R.$$

On the region $0 \leq s \leq R/2 - 1$, $(\overline{\partial} - \nu)(r_R, u_R) = (\overline{\partial} - \nu)(r_1, u_1) = 0$.

On the region $R/2 \leq s \leq R$, we have that $u_R(s, t) = x_0(t)$, and so

$$Dx_0(t) + J(t, x_0(t)Dx_0(t)j - X_{H_t} - J(t, x_0(t))X_{H_t}j \quad = \quad 0$$

because $\partial_s x_0(t) = 0$ and $\partial_t x_0(t) = X_{H_t}$ together imply that $Dx_0(t) - X_{H_t} = 0$.

On the region $R/2 - 1 \leq s \leq R/2$, since $u_1(s, t)$ converges exponentially to $\underline{y}(t)$ as $s \to \infty$, we know that $|\xi(s, t)|$ becomes exponentially small in s. Therefore, $|\beta(s)\xi_1(s, t)|$ is also exponentially small in s. Now, $D\underline{y}(t) - X_{H_t}(\underline{y}(t)) = 0$, so

$$\begin{aligned}|D\exp_{\underline{y}(t)}(\beta(s)\xi_1(s, t)) - X_{H_t}(\exp_{\underline{y}(t)}(\beta(s)\xi_1(s, t))| &\leq |D\exp_{\underline{y}(t)}(\beta(s)\xi_1(s, t)) - D\underline{y}(t)| \\ &\quad +|X_{H_t}(\underline{y}(t) - X_{H_t}(\exp_{\underline{y}(t)}(\beta(s)\xi_1(s, t)))|\end{aligned}$$

and since $\exp_{\underline{y}(t)}(\beta(s)\xi_1(s,t))$ is exponentially close to $\underline{y}(t)$, there is a monotone decreasing function $\epsilon_2(R) \to 0$ as $R \to \infty$ such that

$$|D\exp_{\underline{y}(t)}(\beta(s)\xi_1(s,t)) - X_{H_t}(\exp_{\underline{y}(t)}(\beta(s)\xi_1(s,t))| \leq \epsilon_2(R)$$

uniformly in t for all $s \geq R/2 - 1$.

Using all these estimates, we have

$$\begin{aligned}
\left(\int_{\mathcal{S}_{r_R}} |(\overline{\partial} - \nu)(r_R, u_R)|^p \ \mathrm{dvol}_{\mathcal{S}_{r_R}}\right)^{1/p} &\leq \left(\int_{S_1} |(\overline{\partial} - \nu)(r_R, u_R)|^p \ \mathrm{dvol}_{S_1}\right)^{1/p} \\
&\quad + \left(\int_{S_2} |(\overline{\partial} - \nu)(r_R, u_R)|^p \ \mathrm{dvol}_{S_2}\right)^{1/p} \\
&\quad + \left(\int_0^1 \int_{R/2-1}^{R/2} |(\overline{\partial} - \nu)(r_R, u_R)|^p \ ds\ dt\right)^{1/p} \\
&\leq \epsilon_1(R)\mathrm{vol}(S_1)^{1/p} + \epsilon_1(R)\mathrm{vol}(S_2)^{1/p} + \epsilon_2(R).
\end{aligned}$$

Here vol_{S_i} is the volume of the compact subset $S_i \subset \mathcal{S}_{r_i}$ with respect to a fixed volume form on $\mathcal{S}_{r_i}$.

Type 2

The estimate for Type 2 of the pregluing construction is very similar to that of Type 1. Repeating the same arguments on the corresponding parts of $\mathcal{S}_{r_R}$ leads to an estimate

$$\begin{aligned}
\|\mathcal{F}_{\mathcal{S},r_R,u_R}(0,0)\|_{0,p,R} &= \left(\int_{\mathcal{S}_{r_R}} |(\overline{\partial} - \nu)(r_R, u_R)|^p \ \mathrm{dvol}_{\mathcal{S}_{r_R}}\right)^{1/p} \\
&\leq \sum_{i=0}^{k} \left(\int_{S_i} |(\overline{\partial} - \nu)(r_R, u_R)|^p \ \mathrm{dvol}_{S_i}\right)^{1/p} \\
&\quad + \sum_{i=1}^{k} \left(\int_0^1 \int_{R/2-1}^{R/2} |(\overline{\partial} - \nu)(r_R, u_R)|^p \ ds\ dt\right)^{1/p} \\
&\leq \epsilon_1(R) \sum_{i=0}^{k} \mathrm{vol}(S_i)^{1/p} + k\epsilon_2(R).
\end{aligned}$$

Type 3

This type of pregluing is slightly different from the previous two types. Let Z denote the striplike end of $\mathcal{S}_{r_1}$ on which the pregluing takes place. The perturbation data is fixed, so on the complement of Z we have that

$$(\overline{\partial}-\nu)(r_1,u_R)=(\overline{\partial}-\nu)(r_1,u_1)=0.$$

Thus $(\overline{\partial}-\nu)(r_1,u_R)$ is not supported away from Z.

For $z=(s,t)$ such that $s\in[0,R/2-1]$,

$$(\overline{\partial}-\nu)(r_1,u_R)=(\overline{\partial}-\nu)(r_1,u_1)=0$$

so $(\overline{\partial}-\nu)(r_1,u_R)$ is not supported here either.

For $z=(s,t)$ such that $s\in[R/2,3R/2]$,

$$\begin{aligned}(\overline{\partial}-\nu)(r_1,u_R) &= (\partial_t\underline{y}(t)-X_{H_t}(\underline{y}(t))^{0,1}\\ &= 0\end{aligned}$$

since by assumption, $\underline{y}(t)$ is the Hamiltonian flow of H_t.

For $z=(s,t)$ such that $s\in[3R/2+1,\infty)$,

$$(\overline{\partial}-\nu)(r_1,u_R)=\overline{\partial}v(s-2R,t)-X_{H_t}(v(s-2R,t)=0$$

as v is a Floer trajectory.

Therefore $(\overline{\partial}-\nu)(r_1,u_R)$ is only supported on the part of Z where $s\in[R/2-1,R/2]$ or $s\in[3R/2,3R/2+1]$. On the first part, for sufficiently large R, $|\xi(s,t)|$ is exponentially small in R when $s\geq R/2-1$, and so for the same reasons as in the previous two calculations we can find a monotone decreasing $\epsilon_2(R)\to 0$ as $R\to\infty$ such that

$$|D\exp_{\underline{y}(t)}(\beta(s-R/2)\xi(s,t))-X_{H_t}(\exp_{\underline{y}(t)}(\beta(s-R/2)\xi(s,t))|\leq\epsilon_2(R)$$

uniformly in t, whenever $s \in [R/2-1, R/2]$. Similarly, for sufficiently large R, $|\eta(s - 2R, t)|$ is exponentially small and we can find an $\epsilon_2(R)$ as above such that

$$|D\exp_{\underline{y}(t)}(\beta(-s+3R/2)\eta(s-2R,t)) - X_{H_t}(\beta(-s+3R/2)\eta(s-2R,t)| \leq \epsilon_3(R)$$

uniformly in t, whenever $s \in [3R/2, 3R/2+1]$.

Putting all these estimates together we get that

$$\begin{aligned}
\|\mathcal{F}_{\mathcal{S},r_1,u_R}(0,0)\|_{0,p,R} &= \left(\int_{\mathcal{S}_{r_1}} |(\overline{\partial}-\nu)(r_R,u_R)|^p \ \mathrm{dvol}_{\mathcal{S}_{r_R}}\right)^{1/p} \\
&\leq \left(\int_0^1 \int_{R/2-1}^{R/2} |(\overline{\partial}-\nu)(u_R)|^p \ ds\ dt\right)^{1/p} \\
&\quad + \left(\int_0^1 \int_{3R/2}^{3R/2+1} |(\overline{\partial}-\nu)(u_R)|^p \ ds\ dt\right)^{1/p} \\
&\leq \epsilon_2(R) + \epsilon_3(R).
\end{aligned}$$

□

6.4 Constructing a right inverse

We construct an approximate right inverse for the linearized operator of the preglued surface and curve, then show that it is sufficiently close to being a right inverse that an actual right inverse can be obtained from it via a convergent power series. We treat the three types of pregluing separately.

Type 1

Let

$$\begin{aligned}
(r_1, u_1) &\in \mathcal{R}^{d_1+1,1} \times (\mathcal{B}_{\mathcal{S}_{r_1}})_{u_1} \\
(r_2, u_2) &\in \mathcal{R}^{d_2+1} \times (\mathcal{B}_{\mathcal{S}_{r_2}})_{u_2}
\end{aligned}$$

be such that the linearized operators

$$\begin{aligned} D_{\mathcal{S},r_1,u_1} : T_{r_1}\mathcal{R}^{d_1+1,1} \times W^{1,p}(\mathcal{S}_{r_1}, u_1^*TM) &\to L^p(\mathcal{S}_{r_1}, \Lambda^{0,1} \otimes_J u_1^*TM) \\ D_{\mathcal{S},r_2,u_2} : T_{r_2}\mathcal{R}^{d_2+1} \times W^{1,p}(\mathcal{S}_{r_2}, u_2^*TM) &\to L^p(\mathcal{S}_{r_2}, \Lambda^{0,1} \otimes_J u_2^*TM) \end{aligned}$$

defined using perturbation data (J_1, K_1) and (J_2, K_2) respectively, are surjective and Fredholm, with index zero.

Our goal is to show that $D_R := D_{\mathcal{S},r_1\#_R r_2,u_1\#_R u_2}$, defined with perturbation datum $(J(r_R), K(r_R))$, is also Fredholm and surjective, with a right inverse Q_R that is uniformly bounded for sufficiently large R.

Recall the intermediate functions

$$\begin{aligned} u_1^R : \mathcal{S}_{r_1} &\to M \\ z &\mapsto u_1\#_R u_2(z) \text{ for } z \in r_1^R, \\ &\quad\ \underline{y}(t) \text{ for } z \in r_2^R, \end{aligned}$$

and

$$\begin{aligned} u_2^R : r_2 &\to M \\ z &\mapsto u_1\#_R u_2(z) \text{ for } z \in r_2^R, \\ &\quad\ \underline{y}(t) \text{ for } \ z \in r_1^R. \end{aligned}$$

For large R, the functions $u_i^R, i = 1, 2$ are $W^{1,p}$-small perturbations of u_i. Note that the perturbation data $(J(r_R), K(r_R))$ are given by the Floer data along the neck of $\mathcal{S}_{r_R}$. Denote by $(J_i(r_R), K_i(r_R))$ the perturbation datum on $\mathcal{S}_{r_i}$ given by the restriction of $(J(r_R), K(r_R))$ to the truncation of $\mathcal{S}_{r_i}$, extended trivially over the rest of the striplike end of $\mathcal{S}_{r_i}$ by the given Floer data. The assumption of consistency implies that for large R, the data $(J_i(r_R), K_i(r_R))$ is a compact perturbation of (J_i, K_i).

The properties of being Fredholm and surjective are stable under $W^{1,p}$-small and compact perturbations, hence for sufficiently large R the pairs (r_i, u_i^R) are regular with respect to $(J_i(r_R), K_i(r_R))$. Let $Q_{i,R}$ denote a right inverse for the linearized operator $D_{\mathcal{S},r_i,u_i^R}$. We observe that $Q_{i,R}$ is actually a left *and* right inverse, since by assumption $\ker D_{\mathcal{S},r_i,u_i^R}$ has dimension 0, so $D_{\mathcal{S},r_i,u_i^R}$ is an isomorphism.

We return to the linearized operator

$$D_R : T_{r_R}\mathcal{R}^{d_1+d_2,1} \times W^{1,p}(\mathcal{S}_{r_R}, u_R^*TM) \to L^p(\mathcal{S}_{r_R}, \Lambda^{0,1} \otimes_J u_R^*TM)$$

and will now construct an approximate inverse

$$T_R : L^p(\mathcal{S}_{r_R}, \Lambda^{0,1} \otimes_J u_R^*TM) \to T_{r_R}\mathcal{R}^{d_1+d_2,1} \times W^{1,p}(\mathcal{S}_{r_R}, u_R^*TM).$$

Let $\eta \in L^p(\mathcal{S}_{r_R}, \Lambda^{0,1} \otimes_J u_R^*TM)$. Set

$$\begin{aligned} \eta_1(z) &= \eta(z),\ z \in r_1^R, \\ & \quad 0,\ z \in r_1 \setminus r_1^R \end{aligned}$$

and

$$\begin{aligned} \eta_2(z) &= \eta(z),\ z \in r_2^R, \\ & \quad 0,\ z \in r_2 \setminus r_2^R \end{aligned}$$

The abrupt cut-off does not matter because the norms involving η_1 and η_2 are L^p norms, so involve no derivatives. We now have

$$\begin{aligned} \eta_1 &\in L^p(\mathcal{S}_{r_1}, \Lambda^{0,1} \otimes_J (u_1^R)^*TM), \\ \eta_2 &\in L^p(\mathcal{S}_{r_2}, \Lambda^{0,1} \otimes_J (u_2^R)^*TM). \end{aligned}$$

Using $Q_{1,R}$ and $Q_{2,R}$ we get

$$\begin{aligned} Q_{1,R}\eta_1 =: (\tau_1, \xi_1) &\in T_{r_1}\mathcal{R}^{d_1+1,1} \times W^{1,p}(\mathcal{S}_{r_1}, (u_1^R)^*TM) \\ Q_{2,R}\eta_2 =: (\tau_2, \xi_2) &\in T_{r_2}\mathcal{R}^{d_2+1} \times W^{1,p}(\mathcal{S}_{r_2}, (u_2^R)^*TM). \end{aligned}$$

Our final step is to glue these into a single element of $T_{r_R}\mathcal{R}^{d_1+d_2,1} \times W^{1,p}(\mathcal{S}_{r_R}, u_R^*TM)$. For large R, local gluing charts near the boundary of $\mathcal{R}^{d_1+d_2,1}$ give an isomorphism

$$T_{r_R}\mathcal{R}^{d_1+d_2,1} \cong T_{r_1}\mathcal{R}^{d_1+1,1} \oplus T_{r_2}\mathcal{R}^{d_2+1} \oplus T_R\mathbb{R}$$

(the last component represents the gluing parameter). Using this isomorphism we get a well-defined element

$$\tau_1 \#_R \tau_2 := (\tau_1, \tau_2, 0) \in T_{r_R}\mathcal{R}^{d_1+d_2,1}.$$

Now fix a smooth cut-off function $\beta : \mathbb{R}_{\geq 0} \to [0,1]$ such that $\beta(s) = 1$ for $s \geq 1$ and $\beta(s) = 0$ for $s \leq 1/2$, and $0 \leq \dot{\beta} \leq 2$. Let $\beta_R(s) := \beta(s/R)$.

$$\xi_1 \#_R\, \xi_2(z) := \begin{cases} \xi_1(z) & \text{if } z \in \mathcal{S}_{r_1} \setminus Z_1 \\ \xi_1(s,t) + \beta_R(s)\xi_2(2R-s,1-t) & \text{if } z \in Z_1, s \in [0,R] \\ \beta_R(s)\xi_1(2R-s,1-t) + \xi_2(s,t) & \text{if } z \in Z_2, s \in [0,R] \\ \xi_2(z) & \text{if } z \in S_{r_2} \setminus Z_2. \end{cases}$$

To simplify notation, define $\beta_{1,R} : \mathcal{S}_{r_R} \to \mathbb{R}$ by

$$\beta_{1,R}(z) := \begin{cases} 1 & \text{if } z \in \mathcal{S}_{r_1} \setminus Z_1, \\ 1 & \text{if } z \in Z_1, s \in [0,R], \\ \beta_R(s), & \text{if } z \in Z_2, s \in [0,R], \\ 0 & \text{if } z \in \mathcal{S}_{r_2} \setminus Z_2 \end{cases}$$

and make a corresponding definition of $\beta_{2,R}$. With this notation, we can write $\xi_1 \#_R \xi_2 = \beta_{1,R}\xi_1 + \beta_{2,R}\xi_2$. We define

$$T_R\eta := (\tau_1 \#_R \tau_2, \xi_1 \#_R \xi_2).$$

Now we need to check that for all $\eta \in (\mathcal{E}_{\mathcal{S}_{r_R}})_u$, for sufficiently large R,

$$\|D_{\mathcal{S},r_R,u_R} T_R\eta - \eta\|_{0,p,R} \leq \frac{1}{2}\|\eta\|_{0,p,R}.$$

Now,

$$\begin{aligned} D_{\mathcal{S},r_R,u_R} T_R\eta - \eta &= D_{\mathcal{S},r_R,u_R}(\tau_1 \#_R \tau_2, \xi_1 \#_R \xi_2) - \eta \\ &= D^{u_R}_{r_R}\tau_1 \# \tau_2 + D^{r_R}_{u_R}\xi_1 \#_R \xi_2 - \eta. \end{aligned}$$

By construction $D^{u_R}_{r_R}$ and $D^{u_1^R}_{r_1}$ agree on the support of τ_1, which is $\mathcal{S}_{r_1} \setminus Z_1$. Similarly $D^{u_2^R}_{r_2}$ agrees with $D^{u_R}_{r_R}$ on the support of τ_2 which is $\mathcal{S}_{r_2} \setminus Z_2$. Hence,

$$D^{u_R}_{r_R}\tau_1 \#_R \tau_2 = D^{u_1^R}_{r_1}\tau_1 + D^{u_2^R}_{r_2}\tau_2.$$

Since $D^{r_R}_{u_R}$ and $D^{r_1}_{u_1^R}$ agree on the support of $\beta_{1,R}\xi_1$, we may write

$$\begin{aligned} D^{r_R}_{u_R}\beta_{1,R}\xi_1 &= D^{r_1}_{u_1^R}\beta_{1,R}\xi_1 \\ &= (\partial_s\beta_{1,R})\xi_1 + \beta_{1,R}D^{r_1}_{u_1^R}\xi_1, \\ D^{r_R}_{u_R}\beta_{2,R}\xi_2 &= D^{r_2}_{u_2^R}(\beta_{2,R})\xi_2 \\ &= (\partial_s\beta_{2,R})\xi_2 + \beta_{2,R}D^{r_2}_{u_2^R}\xi_2. \end{aligned}$$

By construction, $D^{u_i}_{r_i}\tau_i + D^{r_i}_{u_i^R}\xi_i = D^{r_i}_{u_i^R}Q_{i,R}\eta_i = \eta_i$, where $Q_{i,R}$ is the inverse of $D_{\mathcal{S},r_i,u_i^R}$. So,

$$\begin{aligned}
D_{\mathcal{S},r_R,u_R}T_R\eta - \eta &= D^{u_R}_{r_R}\tau_1\#_R\tau_2 + D^{r_R}_{u_R}\xi_1\#_R\xi_2 - \eta \\
&= D^{u_1^R}_{r_1}\tau_1 + D^{u_2^R}_{r_2}\tau_2 + D^{r_R}_{u_R}(\beta_{1,R}\xi_1 + \beta_{2,R}\xi_2) - \eta \\
&= D^{u_1^R}_{r_1}\tau_1 + D^{u_2^R}_{r_2}\tau_2 + D^{r_1}_{u_1^R}\beta_{1,R}\xi_1 + D^{r_2}_{u_2^R}\beta_{2,R}\xi_2 - \eta \\
&= D^{u_1^R}_{r_1}\tau_1 + D^{u_2^R}_{r_2}\tau_2 + (\partial_s\beta_{1,R})\xi_1 + \beta_{1,R}D^{r_1}_{u_1^R}\xi_1 + (\partial_s\beta_{2,R})\xi_1 \\
&\qquad\qquad + \beta_{2,R}D^{r_2}_{u_2^R}\xi_2 - \eta \\
&= \beta_{1,R}(D^{u_1}_{r_1}\tau_1 + D^{r_1}_{u_1^R}\xi_1) + \beta_{2,R}(D^{u_2}_{r_2}\tau_2 + D^{r_2}_{u_2^R}\xi_2) \\
&\qquad\qquad + (\partial_s\beta_{1,R})\xi_1 + (\partial_s\beta_{2,R})\xi_2 - \eta \\
&= \beta_{1,R}\eta_1 + \beta_{2,R}\eta_2 + (\partial_s\beta_{1,R})\xi_1 + (\partial_s\beta_{2,R})\xi_2 - \eta \\
&= \eta_1 + \eta_2 - \eta + (\partial_s\beta_{1,R})\xi_1 + (\partial_s\beta_{2,R})\xi_2 \\
&= (\partial_s\beta_{1,R})\xi_1 + (\partial_s\beta_{2,R})\xi_2
\end{aligned}$$

as the support of η_i is precisely where $\beta_{i,R} = 1$, and $\eta_1 + \eta_2 = \eta$. We can find a $c > 0$ and an $R_0 \geq 0$ such that operator norms $\|Q_{i,R}\| \leq c$ for all $R \geq R_0$. Therefore

$$\|\xi_i\|_{1,p,R} \leq \|Q_{i,R}\eta_i\|_R \leq c\|\eta_i\|_{0,p,R}$$

so we can estimate

$$\begin{aligned}
\|D_{\mathcal{S},u_R,r_R}T_R\eta - \eta\|_{0,p,R} &= \|(\partial_s\beta_{1,R})\xi_1 + (\partial_s\beta_{2,R})\xi_2\|_{0,p,R} \\
&\leq 2/R(\|\xi_1\|_{0,p,R} + \|\xi_2\|_{0,p,R}) \\
&\leq \frac{2c}{R}\|\eta\|_{0,p,R}
\end{aligned}$$

and $2c/R \leq 1/2$ for sufficiently large R.

Type 2

The construction for Type 2 is essentially the same as for Type 1, and essentially the same calculations go through. Assuming that

$$\begin{aligned}
(r_0, u_0) &\in \mathcal{M}_d(\underline{y}_0, \underline{y}_1, \ldots, \underline{y}_k)_0 \\
(r_1, u_1) &\in \mathcal{M}_{d,1}(\underline{y}_1, \underline{x}_1, \ldots, \underline{x}_{i_1})_0 \\
&\ldots \\
(r_k, u_k) &\in \mathcal{M}_{d,1}(\underline{y}_k, \underline{x}_{i_1+\ldots+i_{k-1}+1}, \ldots, \underline{x}_d)_0
\end{aligned}$$

are all regular, then for a gluing length R we form the preglued surface and map, abbreviating

$$(r_R, u_R) := (r_0 \#_R (r_1, \ldots, r_k), u_0 \#_R (u_1, \ldots, u_k)).$$

For $i = 0, \ldots, k$, denote by $(J_i(r_R), K_i(r_R))$ the perturbation datum on $\mathcal{S}_{r_i}$ given by the restriction of $(J(r_R), K(r_R))$ to the truncation of $\mathcal{S}_{r_i}$, extended trivially over the rest of the striplike end of $\mathcal{S}_{r_i}$ by the given Floer data. The assumption of consistency implies that for large R, the data $(J_i(r_R), K_i(r_R))$ is a compact perturbation of (J_i, K_i). Also, by construction the intermediate maps u_i^R are $W^{1,p}$ small perturbations of u_i. The properties of being Fredholm and surjective are stable under $W^{1,p}$-small and compact perturbations, hence for sufficiently large R, $(r_0, u_0^R), \ldots, (r_k, u_k^R)$ are regular with respect to $(J_i(r_R), K_i(r_R))$. The linearized operators $D_{\mathcal{S}, r_i, u_i^R}$ are surjective with zero dimensional kernel hence are isomorphisms, so let $Q_{i,R}$ be the inverse of $D_{\mathcal{S}, r_i, u_i^R}$. For convenience, for each i we will denote by $\mathcal{S}_{r_i}^R$ the truncation of $\mathcal{S}_{r_i}$ that appears in the pre-glued surface $\mathcal{S}_{r_R}$, denote by $Z_1, \ldots, Z_k$ the striplike ends of $\mathcal{S}_{r_1}, \ldots, \mathcal{S}_{r_k}$ that are truncated in the pre-gluing process, and denote by $Z_1', \ldots, Z_k'$ the corresponding striplike ends of the surface $\mathcal{S}_{r_0}$.

We construct an approximate right inverse T_R for the linearized operator D_R as follows:

Let $\eta \in L^p(\mathcal{S}_{r_R}, \Lambda^{0,1} \otimes_J u_R^* TM)$. Set

$$\begin{aligned} \eta_0(z) &= \begin{cases} \eta(z), & z \in \mathcal{S}^R_{r_0} \\ 0, & \text{else} \end{cases} \\ &\dots \\ \eta_k(z) &= \begin{cases} \eta(z), & z \in \mathcal{S}^R_{r_k}, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

The abrupt cut-offs do not matter because the norms involving $\eta_0, \dots, \eta_k$ are just L^p norms. We now have

$$\begin{aligned} \eta_0 &\in L^p(\mathcal{S}_{r_0}, \Lambda^{0,1} \otimes_J (u_0^R)^* TM), \\ &\dots \\ \eta_k &\in L^p(\mathcal{S}_{r_k}, \Lambda^{0,1} \otimes_J (u_k^R)^* TM) \end{aligned}$$

and define

$$\begin{aligned} (\tau_0, \xi_0) &:= Q_{0,R}(\eta_0) \in T_{r_0}\mathcal{R}^{k+1} \times W^{1,p}(\mathcal{S}_{r_0}, (u_0^R)^* TM) \\ &\dots \\ (\tau_k, \xi_k) &:= Q_{k,R}(\eta_k) \in T_{r_k}\mathcal{R}^{d_k+1,1} \times W^{1,p}(\mathcal{S}_{r_k}, (u_k^R)^* TM). \end{aligned}$$

The final step is to glue these $k+1$ things together to get a single element of $T_{r_R}\mathcal{R}^{d_1+d_2,1} \times W^{1,p}(\mathcal{S}_{r_R}, u_R^* TM)$. For large R, r_R is near the boundary of $\mathcal{R}^{d_1+\dots+d_k,1}$, where local charts identify

$$T_{r_R}\mathcal{R}^{d_1+\dots+d_k,1} \cong T_{r_0}\mathcal{R}^{k+1} \oplus T_{r_1}\mathcal{R}^{d_1+1,1} \oplus \dots \oplus T_{r_k}\mathcal{R}^{d_k+1,1} \oplus T_R\mathbb{R},$$

the last component coming from the gluing parameter. With this identification set

$$\tau_0 \#_R(\tau_1, \dots, \tau_k) := (\tau_0, \tau_1, \dots, \tau_k, 0).$$

Fix a smooth cutoff function $\beta : \mathbb{R}_{\geq 0} \to [0,1]$ such that $\beta(s) = 1$ for $s \geq 1$ and $\beta(s) = 0$ for $s \leq 1/2$, and $0 \leq \dot{\beta} \leq 2$. Let $\beta_R(s) := \beta(s/R)$. Define $\beta_{0,R} : \mathcal{S}_{r_R} \to [0,1]$ by the condition that

$$\beta_{0,R}(z) = \begin{cases} 1, & z \in \mathcal{S}^R_{r_0}, \\ \beta_R(s), & z \in Z_i, s \in [0,R], i = 1, \dots, k, \\ 0, & \text{else,} \end{cases}$$

and for each $i \in \{1, \dots, k\}$, let

$$\beta_{i,R}(z) = \begin{cases} 1, & z \in \mathcal{S}^R_{r_i}, \\ \beta_R(s), & z \in Z'_i \subset \mathcal{S}_{r_0}, s \in [0, R], \\ 0, & \text{else.} \end{cases}$$

Then write

$$\xi_0 \#_R \ (\xi_1, \dots, \xi_k) := \sum_{i=0}^{k} \beta_{i,R}\xi_i.$$

We define the approximate inverse by

$$T_R\eta := (\tau_0 \#_R(\tau_1, \dots, \tau_k), \xi_0 \#_R(\xi_1, \dots, \xi_k).$$

The construction leads to the estimate

$$\begin{aligned} D_{\mathcal{S},r_R,u_R} T_R\eta - \eta &= D^{u_R}_{r_R}\tau_0\#(\tau_1, \dots, \tau_k) + D^{r_R}_{u_R}\xi_0\#(\xi_1, \dots, \xi_k) - \eta \\ &= \sum_{i=0}^{k} D^{u_i^R}_{r_i}\tau_i + \sum_{i=0}^{k} D^{r_i}_{u_i^R}\beta_{i,R}\xi_i - \eta \\ &= \sum_{i=0}^{k} D^{u_i^R}_{r_i}\tau_i + \sum_{i=0}^{k} \beta_{i,R} D^{r_i}_{u_i^R}\xi_i + \sum_{i=0}^{k}(\partial_s\beta_{i,R})\xi_i - \eta \\ &= \sum_{i=0}^{k} \beta_{i,R}(D^{u_i^R}_{r_i}\tau_i + D^{r_i}_{u_i^R}\xi_i) + \sum_{i=0}^{k}(\partial_s\beta_{i,R})\xi_i - \eta \\ &= \sum_{i=0}^{k} \beta_{i,R}\eta_i - \eta + \sum_{i=0}^{k}(\partial_s\beta_{i,R})\xi_i \\ &= \sum_{i=0}^{k}(\partial_s\beta_{i,R})\xi_i. \end{aligned}$$

Since $(\tau_i, \xi_i) = Q_{i,R}\eta_i$ there is a $c > 0$ such that

$$\|\xi_i\|_{0,p,R} \leq \|Q_{i,R}\eta_i\|_{1,p,R} \leq c\|\eta_i\|_{0,p,R},$$

for all i. Combine everything into a total estimate

$$\begin{aligned} \|D_{\mathcal{S},u,r}T\eta - \eta\|_{0,p,R} &\leq \sum_{i=0}^{k} \|\dot{\beta}/R \ \xi_i\|_{0,p,R} \\ &\leq \|\partial_s\beta_{i,R}\|_\infty \sum_{i=0}^{k} \|\xi_i\|_{0,p,R} \\ &\leq \frac{2}{R}\sum_{i=0}^{k} c\|\eta_i\|_{0,p,R} \\ &\leq \frac{2c}{R}\|\eta\|_{0,p,R} \end{aligned}$$

and for sufficiently large R, $2c/R \leq 1/2$.

Type 3

Assume that

$$\begin{aligned} (r_1, u_1) &\in \mathcal{M}_{d,1}(\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_d)^0 \\ v &\in \mathcal{M}(\underline{x}_i, \underline{y})^0 \end{aligned}$$

where (r_1, u_1) is a regular pseudoholomorphic quilted disk, and v is a regular Floer trajectory. For large R, the intermediate maps (r_1, u_1^R) and v^R are $W^{1,p}$-small perturbations of (r_1, u_1) and v respectively, and since being Fredholm and surjective are properties that are stable under small perturbations, it follows that (r_1, u_1^R) and v^R are also regular for sufficiently large R. Let $Q_{1,R}$ be the inverse of $D_{\mathcal{S},r_1,u_1^R}$ (which is invertible), and let Q_{v^R} be a right inverse for D_{v^R} respectively, whose image is the L^2 orthogonal complement of the kernel of D_{v^R}. Let Σ denote the infinite strip $\mathbb{R} \times [0,1]$, and label by Z the striplike end of $\mathcal{S}_{r_1}$ on which the pregluing takes place. Construct an approximate right inverse

$$T_R : L^p(\mathcal{S}_{r_1}, \Lambda^{0,1} \otimes_J u_R^* TM) \to T_{r_1}\mathcal{R}^{d+1,1} \times W^{1,p}(\mathcal{S}_{r_1}, u_R^* TM)$$

of $D_{\mathcal{S},r_1,u_R}$ as follows. Let $\eta \in L^p(\mathcal{S}_{r_1}, \Lambda^{0,1} \otimes_J u_R^* TM)$. Set

$$\begin{aligned} \eta_1(z) &= \eta(z),\ z \in \mathcal{S}_{r_1}^R, \\ & \quad 0, \text{else}, \end{aligned}$$

and $\eta_v = 1 - \eta_1$. Then $\eta_1 \in L^p((\mathcal{S}_{r_1}, \Lambda^{0,1} \otimes_J (u_1^R)^* TM)$, and $\eta_v \in L^p(\Sigma, \Lambda^{0,1} \otimes_J (v^R)^* TM)$, and we set

$$\begin{aligned} Q_{1,R}\eta_1 &:= (\tau_1, \xi_1) \in T_{r_1}\mathcal{R}^{d+1,1} \times W^{1,p}(\mathcal{S}_{r_1}, (u_1^R)^* TM) \\ Q_{v^R}\eta_v &:= \xi_2 \in W^{1,p}(\Sigma, (v^R)^* TM). \end{aligned}$$

We need to glue ξ_1 and ξ_2 together to get an element of $W^{1,p}(\mathcal{S}_{r_1}, (u_1 \#_R v)^* T\underline{M})$. We fix a smooth cut-off function $\beta : \mathbb{R}_{\geq 0} \to [0,1]$ such that $\beta(s) = 0$ for $s \leq 1/2$ and

$\beta(s) = 1$ for $s \geq 1$, and $0 \leq \dot{\beta} \leq 2$. Let $\beta_R(s) := \beta(s/R)$. Define $\beta_{1,R} : \mathcal{S}_{r_1} \to [0,1]$ by

$$\begin{aligned} \beta_{1,R}(z) &= 1, z \in \mathcal{S}_{r_1} \setminus Z, \\ & \quad 1 - \beta_R(s - R/2), z = (s,t) \in Z, \end{aligned}$$

and

$$\begin{aligned} \beta_{2,R}(z) &= 0, z \in \mathcal{S}_{r_1} \setminus Z, \\ & \quad \beta_R(s), z = (s,t) \in Z. \end{aligned}$$

Putting

$$\xi_1 \#_R \xi_2 := \beta_{1,R}\xi_1 + \beta_{2,R}\xi_2$$

we define $T_R\eta := (\tau_1, \xi_1 \#_R \xi_2)$. Now,

$$\begin{aligned} D_{\mathcal{S},r_1,u_R} T_R \eta - \eta &= D_{\mathcal{S},r_1,u_R}(\tau_1, \xi_1 \#_R \xi_2) - \eta \\ &= D_{r_1}^{u_1^R} \tau_1 + D_{u_1^R}^{r_1} \beta_{1,R}\xi_1 + D_{v^R}\beta_{2,R}\xi_2 - \eta \\ &= D_{r_1}^{u_1^R} \tau_1 + (\partial_s \beta_{1,R})\xi_1 + \beta_{1,R} D_{u_1^R}^{r_1} \xi_1 + (\partial_s \beta_{2,R})\xi_2 + \beta_{2,R} D_{v^R}\xi_2 - \eta \\ &= \beta_{1,R}(D_{r_1}^{u_1^R} \tau_1 + D_{u_1^R}^{r_1} \xi_1) + \beta_{2,R} D_{v^R}\xi_2 - \eta + (\partial_s \beta_{1,R})\xi_1 + (\partial_s \beta_{2,R})\xi_2 \\ &= (\partial_s \beta_{1,R})\xi_1 + (\partial_s \beta_{2,R})\xi_2, \end{aligned}$$

and choosing a $c > 0$, $R_0 \geq 0$ such that the operator norms $\|Q_{1,R}\| \leq c$, $\|Q_{v^R}\| \leq c$ for all $R \geq R_0$, we have an estimate

$$\begin{aligned} \|D_{\mathcal{S},r_1,u_R} T_R \eta - \eta\|_{0,p,R} &\leq \frac{2}{R}(\|\xi_1\|_{0,p,R} + \|\xi_2\|_{0,p,R}) \\ &\leq \frac{2}{R}(c\|\eta_1\|_{0,p,R} + c\|\eta_2\|_{0,p,R}) \\ &= \frac{2c}{R}\|\eta\|_{0,p,R} \end{aligned}$$

so $2c/R \leq 1/2$ for sufficiently large R.

We obtain an actual right inverse $Q_R = T_R(D_R T_R)^{-1}$ using a power series

$$\begin{aligned} (D_R T_R)^{-1} &= (\mathrm{Id} + (D_R T_R - \mathrm{Id}))^{-1} \\ &= \sum_{k=0}^{\infty} (-1)^k (D_R T_R - \mathrm{Id})^k, \end{aligned}$$

which is convergent for large R because of the estimate $\|D_R T_R - \mathrm{Id}\| \leq 1/2$.

6.5 Uniform bound on the right inverse

By construction, the image of $Q_R = T_R(D_R T_R)^{-1}$ is the same as the image of T_R. The operator norm of $(D_R T_R)^{-1}$ can be uniformly estimated, by the results of the previous section. Thus it remains to find a uniform bound for $\|T_R\|$ in order to get a uniform bound $\|Q_R\|$.

Type 1

T_R is a composition of operations. The initial cut-offs define a map

$$\triangle_1 \times \triangle_2 : L^p(\mathcal{S}_{r_R}, u_R^* TM) \to L^p(\mathcal{S}_{r_1}, (u_1^R)^* TM) \times L^p(\mathcal{S}_{r_2}, (u_2^R)^* TM)$$

where the norm on the product $L^p(\mathcal{S}_{r_1}, (u_1^R)^* TM) \times L^p(\mathcal{S}_{r_2}, (u_2^R)^* TM)$ is the sum of the norms. So by construction, the operator norm $\|\triangle_1 \times \triangle_2\| = 1$. The next step is the map $Q_{1,R} \times Q_{2,R}$ whose domain is the product $L^p(\mathcal{S}_{r_1}, (u_1^R)^* TM) \times L^p(\mathcal{S}_{r_2}, (u_2^R)^* TM)$ and range is the product $T_{r_1}\mathcal{R} \times W^{1,p}(\mathcal{S}_{r_1}, (u_1^R)^* TM) \times T_{r_2}\mathcal{R} \times W^{1,p}(\mathcal{S}_{r_2}, (u_2^R)^* TM)$. The operator norm is estimated $\|Q_{1,R} \times Q_{2,R}\| \le \|Q_{1,R}\| + \|Q_{2,R}\| \le 2c$ where c is a uniform bound on the operator norms of $Q_{1,R}, Q_{2,R}$ for large R. (Such a uniform bound c exists because for large R, $Q_{1,R}$ and $Q_{2,R}$ converge to Q_1 and Q_2, the respective inverses of $D_{\mathcal{S},r_1,u_1}$ and $D_{\mathcal{S},r_2,u_2}$.) The final step in the construction of T_R uses an operator

$$\beta_1 \times \beta_2 : W^{1,p}(\mathcal{S}_{r_1}, (u_2^R)^* TM) \times W^{1,p}(\mathcal{S}_{r_2}, (u_2^R)^* TM) \to W^{1,p}(\mathcal{S}_{r_R}, (u_R)^* TM)$$

defined using the cut-off functions $\beta_{1,R}$ and $\beta_{2,R}$, which by construction satisfies

$$\|(\beta_1 \times \beta_2)(\xi_1, \xi_2)\|_{1,p,R} \le \|\xi_1\|_{1,p,R} + \|\xi_2\|_{1,p,R}$$

where the right hand side is the norm of (ξ_1, ξ_2) on the product $W^{1,p}(\mathcal{S}_{r_1}, (u_2^R)^* TM) \times W^{1,p}(\mathcal{S}_{r_2}, (u_2^R)^* TM)$. Hence $\|\beta_1 \times \beta_2\| = 1$. Putting everything together we get that $\|T_R\| \le 2c$.

Type 2

By almost identical arguments as above we can conclude that $\|T_R\| \le (k+1)c$, where c is a uniform bound for large R on the operator norms $\|Q_{i,R}\|$, for $i = 0, 1, \ldots, k$.

Type 3

By almost identical arguments as above we can conclude that $\|T_R\| \leq 2c$, where c is a uniform bound for large R on the operator norms $\|Q_{1,R}\|$ and $\|Q_{v,R}\|$.

6.6 Quadratic estimate

The goal of this section is to establish a quadratic estimate,

$$\|d\mathcal{F}_{\mathcal{S},r,u}(\rho,\xi) - D_{\mathcal{S},r,u}\| \leq c(\|\xi\|_{W^{1,p}} + |\rho|)$$

where the norm on the left is the operator norm. The $W^{1,p}$ norm on the right will depend on a choice of volume form on S, and in the proof of the gluing theorem it will be a different volume form for different values of the gluing parameter R. (N.B. The definition of the operator $\mathcal{F}_{\mathcal{S},r,u}$ doesn't depend in any way on the volume form on S.) The one thing to ensure is that the constant c in the estimate does not depend on anything that might vary with the gluing parameter R.

We assume that we are working in a local trivialization of the fiber bundle $\mathcal{S} \to \mathcal{R}$, in a neighborhood $U \subset \mathcal{R}$ of r. We will write $S := \mathcal{S}_r$. Recall that in this neighborhood of r, all the fibers $\mathcal{S}_{r'}$ are diffeomorphic to S, and so the varying almost complex structures can be pulled back to S, as can the perturbation data.

Definition For a fixed volume form on S, define the constant

$$c_p(\mathrm{dvol}_S) := \sup_{0 \neq f \in C^\infty(S) \cap W^{1,p}(S)} \frac{\|f\|_{L^\infty}}{\|f\|_{W^{1,p}}}.$$

.

That such a constant exists follows from the embedding statements of Appendix A. Moreover, it follows from Theorem A.0.8 that c_p is uniformly bounded for all surfaces in a given family $\mathcal{S}^{d,1}$ or $\mathcal{S}^d$.

Proposition 6.6.1 (c.f. Proposition 3.5.3 in [10])**.** *Let $p > 2$ and fix a quilted surface S with strip-like ends, and let $S^{thick(thin)}$ denote the thick (resp. thin) part of a thick-thin decomposition. Then, for every constant $c_0 > 0$, there exists a constant $c > 0$ such* *that the following holds for every volume form* dvol_S *on* S *such that* $c_p(\mathrm{dvol}_S) \le c_0$ *and* $\mathrm{vol}(S^{thick}) \le c_0$*. If* $u \in W^{1,p}(S, M)$*,* $\xi \in W^{1,p}(S, u^*TM)$*,* $r \in \mathcal{R}$*, and* $\rho \in T_r\mathcal{R}$ *satisfy*

$$\|du\|_{L^p} \le c_0, \ \|\xi\|_{L^\infty} \le c_0, \ |\rho| \le c_0, \tag{6.2}$$

then

$$\|d\mathcal{F}_{\mathcal{S},r,u}(\rho, \xi) - D_{\mathcal{S},r,u}\| \le c(\|\xi\|_{W^{1,p}} + |\rho|). \tag{6.3}$$

Here $\|\cdot\|$ *denotes the operator norm on the space of bounded linear operators from* $T_r\mathcal{R} \times W^{1,p}(S, u^*TM)$ *to* $L^p(S, \Lambda^{0,1} \otimes_J u^*TM)$*.*

Proof. Given $x \in M$, $\xi \in T_xM$, $r \in \mathcal{R}$ and $\rho \in T_r\mathcal{R}$, define linear maps

$$\begin{aligned} E_x(\xi) : T_xM &\rightarrow T_{\exp_x \xi}M \\ \tilde{\xi} &\mapsto \frac{d}{d\lambda}\Big|_{\lambda=0} \exp_x(\xi + \lambda\tilde{\xi}) \\ E_r(\rho) : T_r\mathcal{R} &\rightarrow T_{\exp_r \rho}\mathcal{R} \\ \tilde{\rho} &\mapsto \frac{d}{d\lambda}\Big|_{\lambda=0} \exp_r(\rho + \lambda\tilde{\rho}) \end{aligned}$$

and a bilinear map

$$\begin{aligned} \Psi_x(\xi) : T_xM \times T_xM &\rightarrow T_{\exp_x \xi}M \\ (\tilde{\xi}, \eta) &\mapsto \frac{d}{d\lambda}\Big|_{\lambda=0} \Phi_x(\xi + \lambda\tilde{\xi})\eta. \end{aligned}$$

Since M and $\mathcal{R}$ are compact, there is a constant c_1 such that

$$\begin{aligned} |E_r(\rho)\tilde{\rho}| &\le c_1|\tilde{\rho}| \\ |E_x(\xi)\tilde{\xi}| &\le c_1|\tilde{\xi}| \\ |\Psi_x(\xi)(\tilde{\xi}, \eta)| &\le c_1|\xi||\tilde{\xi}||\eta| \end{aligned}$$

for all $x \in M$, $r \in \mathcal{R}$, and for all $\xi \in T_xM$ with $|\xi| \le c_0$, and $\rho \in T_r\mathcal{R}$ with $|\rho| \le c_0$; the last inequality is because the bilinear form is 0 when $\xi = 0$. W define another bilinear map, for each $z \in S$, by

$$\begin{aligned} \Psi_{r,z}(\rho) : T_r\mathcal{R} \times (T_z^*S)^{0,1}_{j(r)} \otimes_{J(r)} T_{u(z)}M &\rightarrow (T_z^*S)^{0,1}_{j(\exp_r \rho)} \otimes_{J(\exp_r \rho)} T_{u(z)}M \\ (\rho, \eta \otimes \xi) &\mapsto \frac{d}{d\lambda}\Big|_{\lambda=0} \Phi_r(\rho + \lambda\tilde{\rho})(\eta \otimes \xi). \end{aligned}$$

We can also assume that for the same constant c_1 as above, we have

$$|\Psi_{r,z}(\rho;\tilde{\rho},\eta\otimes\xi)| \quad\leq\quad c_1|\rho||\tilde{\rho}||\eta\otimes\xi|,$$

for all $z \in S$, $r \in \mathcal{R}$, and all $\rho \in T_r\mathcal{R}$ such that $|\rho| \leq c_0$; this inequality is possible because when $\rho = 0$, the corresponding bilinear form is 0.

Observe that on the "thin" parts of S, the complex structures j and J are independent of $r \in \mathcal{R}$. Thus $\Phi_r(\rho+\lambda\tilde{\rho})$ is the identity for all λ, and

$$\begin{aligned}\frac{d}{d\lambda}\Big|_{\lambda=0}\Phi_r(\rho+\lambda\tilde{\rho})(\eta\otimes\xi) &= \frac{d}{d\lambda}\Big|_{\lambda=0}\eta\otimes\xi \\ &= 0,\end{aligned}$$

that is, $\Psi_{r,z}(\rho) \equiv 0$ on S^{thin}.

Now covariantly differentiate the identity

$$\Phi_{\mathcal{S},r,u}(\rho+\lambda\tilde{\rho},\xi+\lambda\tilde{\xi})\mathcal{F}_{\mathcal{S},r,u}(\rho+\lambda\tilde{\rho},\xi+\lambda\tilde{\xi}) = \overline{\partial}_J(\exp_r(\rho+\lambda\tilde{\rho}),\exp_u(\xi+\lambda\tilde{\rho}))$$

with respect to λ, at $\lambda = 0$, using the connection $\overline{\nabla}$, to get

$$\begin{aligned}\overline{\nabla}_\lambda\Phi_{\mathcal{S},r,u}(\rho+\lambda\tilde{\rho},\xi+\lambda\tilde{\xi})\mathcal{F}_{\mathcal{S},r,u}(\rho,\xi)\Big|_{\lambda=0} &+ \Phi_{\mathcal{S},r,u}(\rho,\xi)d\mathcal{F}_{\mathcal{S},r,u}(\rho,\xi)(\tilde{\rho},\tilde{\xi}) \\ &= D_{\mathcal{S},\exp_r\rho,\exp_u\xi}(E_r(\rho)\tilde{\rho},E_x(\xi)\tilde{\xi})\end{aligned}\tag{6.4}$$

For typesetting reasons we now abbreviate

$$F := \mathcal{F}_{\mathcal{S},r,u}(\rho,\xi), u_\xi := \exp_u(\xi), r_\rho := \exp_r(\rho)$$

in order to calculate

$$\begin{aligned}\overline{\nabla}_\lambda\Phi_{\mathcal{S},r,u}(\rho+\lambda\tilde{\rho},\xi+\lambda\tilde{\xi})F\Big|_{\lambda=0} &= \overline{\nabla}_\lambda\Phi_r(\rho+\lambda\tilde{\rho})\Phi_u(\xi+\lambda\tilde{\xi})F\Big|_{\lambda=0} \\ &= \Phi_r(\rho)\widetilde{\nabla}_\lambda\Phi_u(\xi+\lambda\tilde{\xi})F\Big|_{\lambda=0} \\ &\quad + \Phi_u(\xi)\frac{d}{d\lambda}\Big|_{\lambda=0}\frac{1}{2}\left[\Phi_r(\rho+\lambda\tilde{\rho})F\right. \\ &\quad \left.+ J(r_\rho,u)\circ\Phi_r(\rho+\lambda\tilde{\rho})F\circ j(r_\rho)\right] \\ &= \Phi_r(\rho)\Psi_u(\xi;\tilde{\xi},F) + \Phi_u(\xi)\frac{1}{2}[\Psi_r(\rho;\tilde{\rho},F) \\ &\quad +J(r_\rho,u)\circ\Psi_r(\rho;\tilde{\rho},F)\circ j(r_\rho)].\end{aligned}$$

Thus we can rewrite equation (6.4) as

$$\begin{aligned}\Phi_r(\rho)\Psi_u(\xi;\tilde{\xi},F) + \ \Phi_u(\xi)\frac{1}{2}[\Psi_r(\rho;\tilde{\rho},F) + J(r_\rho,u)\circ\Psi_r(\rho;\tilde{\rho},F)\circ j(r_\rho)]\\ + \Phi_{\mathcal{S},r,u}(\rho,\xi)d\mathcal{F}_{\mathcal{S},r,u}(\rho,\xi)(\tilde{\rho},\tilde{\xi}) = D_{\mathcal{S},r_\rho,u_\xi}(E_r(\rho)\tilde{\rho},E_x(\xi)\tilde{\xi}).\end{aligned}$$

Therefore we have

$$\begin{aligned} d\mathcal{F}_{\mathcal{S},r,u}(\rho,\xi)(\tilde{\rho},\tilde{\xi}) - D_{\mathcal{S},r,u}(\tilde{\rho},\tilde{\xi}) \quad = \quad & -\ \Phi_{\mathcal{S},r,u}(\rho,\xi)^{-1}\Phi_r(\rho)\Psi_u(\xi;\tilde{\xi},F)\\ & -\ \Phi_{\mathcal{S},r,u}(\rho,\xi)^{-1}\Phi_u(\xi)\frac{1}{2}[\Psi_r(\rho;\tilde{\rho},F)\\ & +\ J(r_\rho,u)\circ\Psi_r(\rho;\tilde{\rho},F)\circ j(r_\rho)]\\ & +\ \Phi_{\mathcal{S},r,u}(\rho,\xi)^{-1}D_{\mathcal{S},r_\rho,u_\xi}(E_r(\rho)\tilde{\rho},E_x(\xi)\tilde{\xi})\\ & -D_{\mathcal{S},r,u}(\tilde{\rho},\tilde{\xi})\end{aligned}$$

which simplifies to

$$\begin{aligned} d\mathcal{F}_{\mathcal{S},r,u}(\rho,\xi)(\tilde{\rho},\tilde{\xi}) \quad - \quad & D_{\mathcal{S},r,u}(\tilde{\rho},\tilde{\xi}) = -\ \underbrace{\Phi_u(\xi)^{-1}\Psi_u(\xi;\tilde{\xi},F)}_{A}\\ & -\ \underbrace{\Phi_r(\rho)^{-1}\frac{1}{2}[\Psi_r(\rho;\tilde{\rho},F) + \ J(r_\rho,u)\circ\Psi_r(\rho;\tilde{\rho},F)\circ j(r_\rho)]}_{B}\\ & +\ \underbrace{\Phi_{\mathcal{S},r,u}(\rho,\xi)^{-1}D_{\mathcal{S},r_\rho,u_\xi}(E_r(\rho)\tilde{\rho},E_x(\xi)\tilde{\xi}) - D_{\mathcal{S},r,u}(\tilde{\rho},\tilde{\xi})}_{C}.\end{aligned}$$

We now estimate A, B and C in turn.

Estimating A: Pointwise, since $\Phi_u(\xi)^{-1}$ is an isometry, we can write

$$\begin{aligned}|\Phi_u(\xi)^{-1}\Psi_u(\xi;\tilde{\xi},F)| \quad &= \quad |\Psi_u(\xi;\tilde{\xi},F)|\\ &\leq \quad c_1|\xi||\tilde{\xi}||F|.\end{aligned}$$

Now

$$\begin{aligned} F \quad &:= \quad \mathcal{F}_{\mathcal{S},r,u}(\rho,\xi)\\ &= \quad \Phi_r(\rho)^{-1}\Phi_u(\xi)^{-1}\frac{1}{2}[(du_\xi - Y(r_\rho,u_\xi)) + J(r_\rho,u_\xi)\circ(du_\xi - Y(r_\rho,u_\xi)\circ j(r_\rho)],\end{aligned}$$

so we can write

$$\begin{aligned} |F| \quad &= \quad |\Phi_r(\rho)^{-1}\Phi_u(\xi)^{-1}\frac{1}{2}[(du_\xi - Y(r_\rho,u_\xi)) + J(r_\rho,u_\xi)\circ(du_\xi - Y(r_\rho,u_\xi)\circ j(r_\rho)]|\\ &\leq \quad |du_\xi - Y(r_\rho,u_\xi)|\\ &\leq \quad |du_\xi| + |Y(r_\rho,u_\xi)|.\end{aligned}$$

There is a constant c_2 depending only on c_0 and the choice of compatible almost-complex structures in the perturbation datum (J,K), such that

$$|d\exp_u \xi| \leq c_2(|du| + |\nabla\xi|)$$

whenever $\|du\|_{L^p} \leq c_0$ and $\|\xi\|_{L^\infty} \leq c_0$. There is also a constant c_3 such that

$$|Y(r,x)| \leq c_3$$

for all $r \in \mathcal{R}$, $x \in M$, and all metrics $|\cdot|$ on TM induced by the almost complex structures in the perturbation datum (J,K). Hence,

$$\begin{aligned}
\|\Phi_u(\xi)^{-1}\Psi_u(\xi;\tilde{\xi},F)\|_{L^p} &\leq c_1c_2\|\xi\|_{L^\infty}\|\tilde{\xi}\|_{L^\infty}\|du\|_{L^p} + \ c_1c_2\|\xi\|_{L^\infty}\|\tilde{\xi}\|_{L^\infty}\|\nabla\xi\|_{L^p} \\
&\quad + c_3c_1\|\xi\|_{L^p}\|\tilde{\xi}\|_{L^\infty} \\
&\leq c_1c_2c_0\|\xi\|_{L^\infty}\|\tilde{\xi}\|_{L^\infty} + c_0c_1\|\tilde{\xi}\|_{L^\infty}\|\nabla\xi\|_{L^p} \\
&\quad +c_1c_3c_0\|\xi\|_{L^p}\|\tilde{\xi}\|_{L^p} \\
&\leq c_1c_2c_0^3\|\xi\|_{W^{1,p}}\|\tilde{\xi}\|_{W^{1,p}} + c_0^2c_1c_2\|\tilde{\xi}\|_{W^{1,p}}\|\xi\|_{W^{1,p}} \\
&\quad +c_0c_1c_3\|\xi\|_{W^{1,p}}\|\tilde{\xi}\|_{W^{1,p}} \\
&\leq (c_1c_2c_0^3 + c_0^2c_1c_2 + c_0c_1c_3)(|\rho| + \|\xi\|_{W^{1,p}})(|\tilde{\rho}| + \|\tilde{\xi}\|_{W^{1,p}}).
\end{aligned}$$

Estimating B: Pointwise,

$$|\Phi_r(\rho)^{-1}\frac{1}{2}[\Psi_r(\rho;\tilde{\rho},F) + \ J(r_\rho,u)\circ\Psi_r(\rho;\tilde{\rho},F)\circ j(r_\rho)]| \leq |\Psi_r(\rho;\tilde{\rho},F)|$$

so it suffices to estimate $|\Psi_r(\rho;\tilde{\rho},F)|$. Recall that $F = \mathcal{F}_{\mathcal{S},r,u}(\rho,\xi)$. We have

$$\begin{aligned}
|\Psi_r(\rho;\tilde{\rho},F)| &= |\Psi_r(\rho;\tilde{\rho},F)| \\
&\leq c_1|\rho||\tilde{\rho}||F| \\
&\leq c_1|\rho||\tilde{\rho}|(|du_\xi| + |Y(r_\rho,u_\xi)|) \\
&\leq c_1c_2|\rho||\tilde{\rho}|(|du| + |\nabla\xi|) + c_1|\rho||\tilde{\rho}||Y(r_\rho,u_\xi)|.
\end{aligned}$$

Recalling that $\Psi_r(\rho)$ vanishes on S^{thin}, and that $\mathrm{vol}(S^{thick}) < c_0$, we get

$$\begin{aligned}
\|\Psi_r(\rho;\tilde{\rho},F)\|_{L^p} &\leq c_1c_2|\rho||\tilde{\rho}|\|du\|_{L^p} + c_1c_2|\rho||\tilde{\rho}|\|\nabla\xi\|_{L^p} \\
&\quad +c_1|\rho||\tilde{\rho}|\|Y(r_\rho,u_\xi)\|_\infty(\mathrm{vol}(S^{thick}))^{1/p} \\
&\leq c_0c_1c_2|\rho||\tilde{\rho}| + c_0c_1c_2|\tilde{\rho}|\|\xi\|_{W^{1,p}} + c_1c_3c_0|\rho||\tilde{\rho}| \\
&\leq (2c_0c_1c_2 + c_1c_3)(|\rho| + \|\xi\|_{W^{1,p}})(|\tilde{\rho}| + \|\tilde{\xi}\|_{W^{1,p}}).
\end{aligned}$$

Estimating C: Using the splitting $D_{\mathcal{S},r_\rho,u_\xi} = D^{(r_\rho)}_{u_\xi} + D^{(u_\xi)}_{r_\rho}$, write

$$\begin{aligned}
&\Phi_{\mathcal{S},r,u}(\rho,\xi)^{-1} D_{\mathcal{S},r_\rho,u_\xi}(E_{r,u}(\rho,\xi)(\tilde\rho,\tilde\xi)) - D_{\mathcal{S},r,u}(\tilde\rho,\tilde\xi)\\
&\quad= \underbrace{[\Phi_{\mathcal{S},r,u}(\rho,\xi)^{-1} D^{(r_\rho)}_{u_\xi} E_u(\xi)(\tilde\xi) - D^{(r)}_u(\tilde\xi)]}_{C(a)}\\
&\quad+\underbrace{[\Phi_{\mathcal{S},r,u}(\rho,\xi)^{-1} D^{(u_\xi)}_{r_\rho} E_r(\rho)(\tilde\rho) - D^{(u)}_r(\tilde\rho)]}_{C(b)}
\end{aligned}$$

and estimate $C(a)$ and $C(b)$ separately.

To estimate $C(a)$, write

$$\begin{aligned}
\Phi_{\mathcal{S},r,u}(\rho,\xi)^{-1} D^{r_\rho}_{u_\xi} E_u(\xi)(\tilde\xi) - D^{(r)}_u(\tilde\xi) &= \Phi_r(\rho)^{-1}[\Phi_u(\xi)^{-1} D^{r_\rho}_{u_\xi} E_u(\xi)(\tilde\xi) - D^{(r_\rho)}_u(\tilde\xi)]\\
&+ [\Phi_r(\rho)^{-1} D^{(r_\rho)}_u(\tilde\xi) - D^{(r)}_u(\tilde\rho)]\\
&= \Phi_r(\rho)^{-1}[\Phi_u(\xi)^{-1} \widetilde{D}^{r_\rho}_{u_\xi} E_u(\xi)(\tilde\xi) - \widetilde{D}^{(r_\rho)}_u(\tilde\xi)]\\
&- \Phi_r(\rho)^{-1}[\Phi_u(\xi)^{-1} P^{(r_\rho)}_{u_\xi} E_u(\xi)(\tilde\xi) - P^{(r_\rho)}_u(\tilde\xi)]\\
&+ [\Phi_r(\rho)^{-1} \widetilde{D}^{(r_\rho)}_u(\tilde\xi) - \widetilde{D}^{(r)}_u(\tilde\rho)]\\
&- [\Phi_r(\rho)^{-1} P^{(r_\rho)}_u(\tilde\xi) - P^{(r)}_u(\tilde\rho)].
\end{aligned}$$

Therefore we have, pointwise,

$$\begin{aligned}
\underbrace{|\Phi_{\mathcal{S},r,u}(\rho,\xi)^{-1} D^{r_\rho}_{u_\xi} E_u(\xi)(\tilde\xi) - D^{(r)}_u(\tilde\xi)|}_{C(a)} &\le \underbrace{|\Phi_u(\xi)^{-1} \widetilde{D}^{r_\rho}_{u_\xi} E_u(\xi)(\tilde\xi) - \widetilde{D}^{(r_\rho)}_u(\tilde\xi)|}_{C(a)(i)}\\
&+ \underbrace{|\Phi_u(\xi)^{-1} P^{(r_\rho)}_{u_\xi} E_u(\xi)(\tilde\xi) - P^{(r_\rho)}_u(\tilde\xi)|}_{C(a)(ii)}\\
&+ \underbrace{|\Phi_r(\rho)^{-1} \widetilde{D}^{(r_\rho)}_u(\tilde\xi) - \widetilde{D}^{(r)}_u(\tilde\xi)|}_{C(a)(iii)}\\
&+ \underbrace{|\Phi_r(\rho)^{-1} P^{(r_\rho)}_u(\tilde\xi) - P^{(r)}_u(\tilde\xi)|}_{C(a)(iv)}.
\end{aligned}$$

Estimating $C(a)(i)$: Since $\Phi_u(\xi)$ is pointwise an isometry, we can estimate

$$|\widetilde{D}^{r_\rho}_{u_\xi} E_u(\xi)(\tilde\xi) - \Phi_u(\xi)\widetilde{D}^{(r_\rho)}_u(\tilde\xi)|$$

instead. The calculation takes place on a fixed fiber $\mathcal{S}_{r_\rho}$, so there is no risk of ambiguity if we now omit the superscript r_ρ from the notation. Using formula (5.6) for $\widetilde{D}_{u_\xi}$, we have the identity

$$\widetilde{D}^{r_\rho}_{u_\xi} E_u(\xi)(\tilde{\xi}) - \Phi_u(\xi)\widetilde{D}^{(r_\rho)}_u(\tilde{\xi}) = [\nabla(E_u(\xi)\tilde{\xi}) - \Phi_u(\xi)\nabla\tilde{\xi}]^{0,1}$$
$$- \frac{1}{2}J(\underline{u})[(\nabla_{E_u(\xi)\tilde{\xi}}J)(u_\xi)du_\xi - \Phi_u(\xi)(\nabla_{\tilde{\xi}}J)(u)du]^{0,1}$$

where the $(0,1)$ is with respect to $J(r_\rho, u)$ and $j(r_\rho)$. The first of these terms can be estimated pointwise by

$$\begin{aligned}
|\nabla(E_u(\xi)\tilde{\xi}) - \Phi_u(\xi)\nabla\tilde{\xi}| &\leq |(E_u(\xi) - \Phi_u(\xi))\tilde{\xi}| + |\nabla(E_u(\xi)\tilde{\xi}) - E_u(\xi)\nabla\tilde{\xi}| \\
&\leq c_4|\xi||\tilde{\xi}| + \|\nabla E_u(\xi)\||\tilde{\xi}| \\
&= c_4|\xi||\tilde{\xi}| + c_5(|du||\xi| + |\nabla\xi|)|\tilde{\xi}|.
\end{aligned}$$

Taking the L^p norm of both sides, we get

$$\begin{aligned}
\|\nabla(E_u(\xi)\tilde{\xi}) - \Phi_u(\xi)\nabla\tilde{\xi}\|_{L^p} &\leq c_4\|\xi\|_{L^\infty}\|\tilde{\xi}\|_{L^p} + c_5\|du\|_{L^p}\|\xi\|_{L^\infty}\|\tilde{\xi}\|_{L^\infty} \\
&\quad + c_5\|\nabla\xi\|_{L^p}\|\tilde{\xi}\|_{L^\infty} \\
&\leq (c_0c_4 + c_0^3c_5 + c_0c_5)\|\xi\|_{W^{1,p}}\|\tilde{\xi}\|_{W^{1,p}} \\
&\leq (c_0c_4 + c_0^3c_5 + c_0c_5)(|\rho| + \|\xi\|_{W^{1,p}})(|\tilde{\rho}| + \|\tilde{\xi}\|_{W^{1,p}}).
\end{aligned}$$

To estimate the other term, we have

$$\begin{aligned}
|(\nabla_{E_u(\xi)\tilde{\xi}}J)(u_\xi)du_\xi &- \Phi_u(\xi)(\nabla_{\tilde{\xi}}J)(u)du| \\
&\leq |(\nabla_{E_u(\xi)\tilde{\xi}}J)(u_\xi)du_\xi - (\nabla_{E_u(\xi)\tilde{\xi}}J)(u_\xi)\Phi_u(\xi)du| \\
&\quad + |(\nabla_{E_u(\xi)\tilde{\xi}}J)(u_\xi)\Phi_u(\xi)du - \Phi_u(\xi)(\nabla_{\tilde{\xi}}J)(u)du| \\
&\leq |(\nabla_{E_u(\xi)\tilde{\xi}}J)(u_\xi)||du_\xi - \Phi_u(\xi)du| \\
&\quad + |((\nabla_{E_u(\xi)\tilde{\xi}}J)(u_\xi)\Phi_u(\xi) - \Phi_u(\xi)(\nabla_{\tilde{\xi}}J)(u))(du)| \\
&\leq \|\nabla J\|_{L^\infty}|E_u(\xi)\tilde{\xi}|c_6(|du||\xi| + |\nabla\xi|) \\
&\quad + c_7|\xi||\tilde{\xi}||du| \\
&\leq c_8(|\tilde{\xi}||du||\xi| + |\tilde{\xi}||\nabla\xi| + |\xi||\tilde{\xi}||du|).
\end{aligned}$$

Taking the L^p norm of both sides gives

$$\begin{aligned}\|(\nabla_{E_u(\xi)\tilde{\xi}}J)(u_\xi)du_\xi - \Phi_u(\xi)(\nabla_{\tilde{\xi}}J)(u)du\|_{L^p} &\leq c_8(\|\tilde{\xi}\|_{L^\infty}\|du\|_{L^p}\|\xi\|_{L^\infty}\\ &+\|\tilde{\xi}\|_{L^\infty}\|\nabla\xi\|_{L^p} + \|\xi\|_{L^\infty}\|\tilde{\xi}\|_{L^\infty}\|du\|_{L^p})\\ &\leq (2c_0^3c_8 + c_0c_8)\|\tilde{\xi}\|_{W^{1,p}}\|\xi\|_{W^{1,p}}\\ &\leq (2c_0^3c_8 + c_0c_8)(|\rho| + \|\xi\|_{W^{1,p}})(|\tilde{\rho}| + \|\tilde{\xi}\|_{W^{1,p}}).\end{aligned}$$

Estimating $C(a)(ii)$: We can write

$$\begin{aligned}P_{u_\xi}E_u(\xi)(\tilde{\xi}) - \Phi_u(\xi)P_u(\tilde{\xi}) &= [\nabla_{E_u(\xi)(\tilde{\xi})}Y - \Phi_u(\xi)\nabla_{\tilde{\xi}}Y]^{0,1}\\ &- \frac{1}{2}J[\nabla_{E_u(\xi)(\tilde{\xi})}JY - \Phi_u(\xi)(\nabla_{\tilde{\xi}}J)Y]^{0,1}.\end{aligned}$$

The quantity $\nabla_{E_x(\xi)(\tilde{\xi})}Y - \Phi_x(\xi)\nabla_{\tilde{\xi}}Y$ is linear in $\tilde{\xi}$, and is zero when $\xi = 0$. Y depends smoothly on $r \in \mathcal{R}$, and $z \in S$; but $\mathcal{R}$ is compact, and Y is constant in the s direction (direction of infinite length) of the striplike ends of S, so effectively the z-dependence of Y is only over a compact set. Thus there is a constant $c_9 > 0$ such that

$$|\nabla_{E_x(\xi)(\tilde{\xi})}Y - \Phi_x(\xi)\nabla_{\tilde{\xi}}Y| \leq c_9|\xi||\tilde{\xi}|$$

holds for all $Y = Y(r, z)$, and all $x \in M$, $\xi \in T_xM$ such that $|\xi| \leq c_0$.

For $x \in M$, $\xi, \tilde{\xi} \in T_xM$ and $\eta \in T_{\exp_x \xi}M$, the quantity $\nabla_{E_x(\xi)(\tilde{\xi})}J\eta - \Phi_x(\xi)(\nabla_{\tilde{\xi}}J)\eta$ is linear in η and $\tilde{\xi}$, and zero when $\xi = 0$. J depends smoothly on $r \in \mathcal{R}$, and $z \in S$; $\mathcal{R}$ is a compact set, and J is constant in the s direction of the strip-like ends, so effectively the z dependence of J is only over a compact set. Thus there is a constant c_{10} such that

$$|\nabla_{E_x(\xi)(\tilde{\xi})}J\eta - \Phi_u(\xi)(\nabla_{\tilde{\xi}}J)\eta| \leq c_{10}|\xi||\tilde{\xi}||\eta|$$

for all $x \in M$, and $\xi \in T_xM$ such that $|\xi| \leq c_0$. Putting the pointwise estimates together gives

$$\begin{aligned}\|P_{u_\xi}E_u(\xi)(\tilde{\xi}) - \Phi_u(\xi)P_u(\tilde{\xi})\|_{L^p} &\leq c_9\|\xi\|_{L^p}\|\tilde{\xi}\|_{L^\infty} + c_{10}\|\xi\|_{L^\infty}\|\tilde{\xi}\|_{L^\infty}\|Y\|_{L^p}\\ &\leq c_{11}\|\xi\|_{W^{1,p}}\|\tilde{\xi}\|_{W^{1,p}}\\ &\leq c_{11}(|\rho| + \|\xi\|_{W^{1,p}})(|\tilde{\rho} + \|\tilde{\xi}\|_{W^{1,p}}).\end{aligned}$$

Estimating $C(a)(iii)$: We have

$$
\begin{aligned}
\Phi_r(\rho)^{-1}\widetilde{D}_u^{(r_\rho)}(\tilde{\xi}) - \widetilde{D}_u^{(u)}(\tilde{\xi}) &= [\widetilde{D}_u^{(r_\rho)}(\tilde{\xi})]^{0,1} - \widetilde{D}_u^{(r)}(\tilde{\xi}) \\
&= \left[\frac{1}{2}(\nabla\tilde{\xi} + J(r_\rho, u)\circ\nabla\tilde{\xi}\circ j(r_\rho))\right]^{0,1} \\
&\quad - \left[\frac{1}{2}(\nabla\tilde{\xi} + J(r, u)\circ\nabla\tilde{\xi}\circ j(r))\right]^{0,1} \\
&\quad - \left[\frac{1}{2}J(r_\rho)\frac{1}{2}((\nabla_{\tilde{\xi}}J)(r_\rho)du + J(r_\rho)\circ(\nabla_{\tilde{\xi}}J)(r_\rho)du\circ j(r_\rho))\right]^{0,1} \\
&\quad + \left[\frac{1}{2}J(r)\frac{1}{2}((\nabla_{\tilde{\xi}}J)(r) + J(r)\circ(\nabla_{\tilde{\xi}}J)(r)du\circ j(r))\right]^{0,1} \\
&= \left[\frac{1}{2}(J(r_\rho, u)\circ\nabla\tilde{\xi}\circ j(r_\rho) - J(r, u)\circ\nabla\tilde{\xi}\circ j(r))\right]^{0,1} \\
&\quad - \left[\frac{1}{2}J(r_\rho)\frac{1}{2}((\nabla_{\tilde{\xi}}J)(r_\rho)du + J(r_\rho)\circ(\nabla_{\tilde{\xi}}J)(r_\rho)du\circ j(r_\rho))\right. \\
&\quad \left. - \frac{1}{2}J(r)\frac{1}{2}((\nabla_{\tilde{\xi}}J)(r)du + J(r)\circ(\nabla_{\tilde{\xi}}J)(r)du\circ j(r))\right]^{0,1}
\end{aligned}
$$

and therefore a pointwise estimate

$$
|\Phi_r(\rho)^{-1}\widetilde{D}_u^{(r_\rho)}(\tilde{\xi}) - \widetilde{D}_u^{(u)}(\tilde{\xi})| \quad \leq \quad c_{12}|\rho||\nabla\tilde{\xi}| + c_{13}|\rho||\tilde{\xi}||du|.
$$

Taking the L^p norm of both sides gives

$$
\begin{aligned}
\|\Phi_r(\rho)^{-1}\widetilde{D}_u^{(r_\rho)}(\tilde{\xi}) - \widetilde{D}_u^{(u)}(\tilde{\xi})\|_{L^p} &\leq c_{12}|\rho|\|\nabla\tilde{\xi}\|_{L^p} + c_{13}|\rho|\|\tilde{\xi}\|_{L^\infty}\|du\|_{L^p} \\
&\leq c_{12}|\rho|\|\tilde{\xi}\|_{W^{1,p}} + c_0^2 c_{13}|\rho|\|\tilde{\xi}\|_{W^{1,p}} \\
&\leq (c_{12} + c_0^2 c_{13})(|\rho| + \|\xi\|_{W^{1,p}})(|\tilde{\rho}| + \|\tilde{\xi}\|_{W^{1,p}}).
\end{aligned}
$$

Estimating $C(a)(iv)$:

$$
\begin{aligned}
&\Phi_r(\rho)^{-1}P_u^{(r_\rho)}(\tilde{\xi}) - P_u^{(r)}(\tilde{\xi}) \\
&\quad = \left[\frac{1}{2}((\nabla_{\tilde{\xi}}Y)(r_\rho) + J(r_\rho)(\nabla_{\tilde{\xi}}Y)(r_\rho)j(r_\rho))\right]^{0,1} \\
&\qquad - \left[\frac{1}{2}((\nabla_{\tilde{\xi}}Y)(r) + J(r)\circ(\nabla_{\tilde{\xi}}Y)(r)\circ j(r))\right]^{0,1} \\
&\qquad - \left[\frac{1}{2}J(r_\rho)\frac{1}{2}((\nabla_{\tilde{\xi}}J)(r_\rho)Y(r_\rho) + J(r_\rho)\circ(\nabla_{\tilde{\xi}}J)(r_\rho)(Y(r_\rho)\circ j(r_\rho))\right]^{0,1} \\
&\qquad + \left[\frac{1}{2}J(r)\frac{1}{2}((\nabla_{\tilde{\xi}}J)(r)Y(r) + J(r)\circ(\nabla_{\tilde{\xi}}J)(r)Y(r)\circ j(r))\right]^{0,1} \\
&\quad = \left[\frac{1}{2}((\nabla_{\tilde{\xi}}Y)(r_\rho) + J(r_\rho)(\nabla_{\tilde{\xi}}Y)(r_\rho)j(r_\rho))\right. \\
&\qquad \left. - \frac{1}{2}((\nabla_{\tilde{\xi}}Y)(r) + J(r)\circ(\nabla_{\tilde{\xi}}Y)(r)\circ j(r))\right]^{0,1} \\
&\qquad - \left[\frac{1}{2}J(r_\rho)\frac{1}{2}((\nabla_{\tilde{\xi}}J)(r_\rho)Y(r_\rho) + J(r_\rho)\circ(\nabla_{\tilde{\xi}}J)(r_\rho)Y(r_\rho)\circ j(r_\rho))\right. \\
&\qquad \left. - \frac{1}{2}J(r)\frac{1}{2}((\nabla_{\tilde{\xi}}J)(r)Y(r) + J(r)\circ(\nabla_{\tilde{\xi}}J)(r)Y(r)\circ j(r))\right]^{0,1}
\end{aligned}
$$

We can find a constant $c_{14} > 0$ such that a pointwise estimate

$$
|\Phi_r(\rho)^{-1}P_u^{(r_\rho)}(\tilde{\xi}) - P_u^{(r)}(\tilde{\xi})| \quad \leq \quad c_{14}|\rho||\tilde{\xi}|
$$

holds for all $r \in \mathcal{R}$ and $\rho \in T_r\mathcal{R}$ with $|\rho| \leq c_0$. Hence

$$
\begin{aligned}
\|\Phi_r(\rho)^{-1}P_u^{(r_\rho)}(\tilde{\xi}) - P_u^{(r)}(\tilde{\xi})\|_{L^p} &\leq c_{14}|\rho|\|\tilde{\xi}\|_{L^p} \\
&\leq c_{14}(|\rho| + \|\xi\|_{W^{1,p}})(|\tilde{\rho}| + \|\tilde{\xi}\|_{W^{1,p}}).
\end{aligned}
$$

Now it only remains to estimate $C(b)$. We can write

$$
\begin{aligned}
\Phi_{\mathcal{S},r,u}(\rho,\xi)^{-1}D_{r_\rho}^{(u_\xi)}E_r(\rho)(\tilde{\rho}) - D_r^{(u)}(\tilde{\rho}) &= \Phi_u(\xi)^{-1}[\Phi_r(\rho)^{-1}D_{r_\rho}^{(u_\xi)}E_r(\rho)(\tilde{\rho}) - D_r^{(u_\xi)}(\tilde{\rho})] \\
&+ [\Phi_u(\xi)^{-1}D_r^{(u_\xi)}(\tilde{\rho}) - D_r^{(u)}(\tilde{\rho})].
\end{aligned}
$$

Thus, splitting the operator D into its components $\widetilde{D}$ and P, and recalling that

$\Phi_u(\xi)$ is pointwise an isometry, we may estimate

$$
\begin{aligned}
\underbrace{|\Phi_{\mathcal{S},r,u}(\rho,\xi)^{-1}D_{r_\rho}^{(u_\xi)}E_r(\rho)(\tilde\rho)-D_r^{(u)}(\tilde\rho)|}_{C(b)} &\leq \underbrace{|\Phi_r(\rho)^{-1}\widetilde{D}_{r_\rho}^{(u_\xi)}E_r(\rho)(\tilde\rho)-\widetilde{D}_r^{(u_\xi)}(\tilde\rho)|}_{C(b)(i)}\\
&+ \underbrace{|\Phi_r(\rho)^{-1}P_{r_\rho}^{(u_\xi)}E_r(\rho)(\tilde\rho)-P_r^{(u_\xi)}(\tilde\rho)|}_{C(b)(ii)}\\
&+ \underbrace{|\Phi_u(\xi)^{-1}\widetilde{D}_r^{(u_\xi)}(\tilde\rho)-\widetilde{D}_r^{(u)}(\tilde\rho)|}_{C(b)(iii)}\\
&+ \underbrace{|\Phi_u(\xi)^{-1}P_r^{(u_\xi)}(\tilde\rho)-P_r^{(u)}(\tilde\rho)|}_{C(b)(iv)}.
\end{aligned}
$$

Estimating $C(b)(i)$: We can estimate it pointwise by

$$
\begin{aligned}
|\Phi_r(\rho)^{-1}\widetilde{D}_{r_\rho}^{(u_\xi)}E_r(\rho)(\tilde\rho)-\widetilde{D}_r^{(u_\xi)}\tilde\rho| &\leq |(\Phi_r(\rho)^{-1}-\mathrm{Id})\widetilde{D}_{r_\rho}^{(u_\xi)}E_r(\rho)(\tilde\rho)|\\
&\quad +|\widetilde{D}_{r_\rho}^{(u_\xi)}E_r(\rho)(\tilde\rho)-\widetilde{D}_r^{(u_\xi)}\tilde\rho|
\end{aligned}
$$

There is a constant c_{15} such that the first term can be estimated pointwise by

$$
\begin{aligned}
|(\Phi_r(\rho)^{-1}-\mathrm{Id})\widetilde{D}_{r_\rho}^{(u_\xi)}E_r(\rho)(\tilde\rho)| &\leq c_{15}|\rho||du_\xi||\tilde\rho|\\
&\leq c_2c_{15}|\rho|(|du|+|\nabla\xi|)|\tilde\rho|
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\|(\Phi_r(\rho)^{-1}-\mathrm{Id})\widetilde{D}_{r_\rho}^{(u_\xi)}E_r(\rho)(\tilde\rho)\|_{L^p} &\leq c_2c_{15}|\rho|\|du\|_{L^p}|\tilde\rho|+c_2c_{15}|\rho|\|\nabla\xi\|_{L^p}|\tilde\rho|\\
&\leq 2c_0c_2c_{15}(|\rho||\tilde\rho|+\|\xi\|_{W^{1,p}}|\tilde\rho|)\\
&\leq 2c_0c_2c_{15}(|\rho|+\|\xi\|_{W^{1,p}})(|\tilde\rho|+\|\tilde\xi\|_{W^{1,p}}).
\end{aligned}
$$

To estimate the other term, we use the explicit formula

$$
\begin{aligned}
\widetilde{D}_r^{(u)}(\rho) = \frac{1}{4}((\partial_\rho J)(r)\circ du\circ j(r) - J(r)\circ(\partial_\rho J)(r)\circ du\\
+J(r)\circ du\circ(\partial_\rho j)(r) - du\circ(\partial_\rho J)(r)\circ j(r))
\end{aligned}
$$

to directly compute that

$$\begin{aligned}
4|\widetilde{D}_{r_\rho}^{(u_\xi)} E_r(\rho)(\tilde{\rho}) - \widetilde{D}_r^{(u_\xi)} \tilde{\rho}| &\leq |(\partial_{E_r(\rho)\tilde{\rho}} J)(r_\rho) \circ du_\xi \circ j(r_\rho) - (\partial_{\tilde{\rho}} J)(r) \circ du_\xi \circ j(r_\rho)| \\
&+ |J(r_\rho)(\partial_{E_r(\rho)\tilde{\rho}} J)(r_\rho) - J(r)(\partial_{\tilde{\rho}} J)(r)||du_\xi| \\
&+ |J(r_\rho) \circ du_\xi \circ (\partial_{E_r(\rho)\tilde{\rho}} j)(r_\rho) - J(r) \circ du_\xi \circ (\partial_{\tilde{\rho}} j)(r)| \\
&+ |du_\xi||(\partial_{E_r(\rho)\tilde{\rho}} j)(r_\rho) j(r_\rho) - (\partial_{\tilde{\rho}})(r) j(r)| \\
&\leq c_{16}|du_\xi||\rho||\tilde{\rho}| \\
&\leq c_2 c_{16}(|du| + |\nabla\xi|)|\rho||\tilde{\rho}|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\widetilde{D}_{r_\rho}^{(u_\xi)} E_r(\rho)(\tilde{\rho}) - \widetilde{D}_r^{(u_\xi)} \tilde{\rho}\|_{L^p} &\leq \frac{1}{4} c_2 c_{16}\|du\|_{L^p}|\rho||\tilde{\rho}| + \frac{1}{4} c_2 c_{16}\|\nabla\xi\|_{L^p}|\rho||\tilde{\rho}| \\
&\leq \frac{1}{2} c_0 c_2 c_{16}(|\rho||\tilde{\rho}| + \|\xi\|_{W^{1,p}}|\tilde{\rho}|) \\
&\leq c_0 c_2 c_{16}(|\rho| + \|\xi\|_{W^{1,p}})(|\tilde{\rho}| + \|\tilde{\xi}\|_{W^{1,p}}).
\end{aligned}$$

This completes the estimate for $C(b)(i)$.

Estimating $C(b)(ii)$: We can estimate it pointwise by

$$\begin{aligned}
|\Phi_r(\rho)^{-1} P_{r_\rho}^{(u_\xi)} E_r(\rho)(\tilde{\rho}) - P_r^{(u_\xi)} \tilde{\rho}| &\leq |(\Phi_r(\rho)^{-1} - \mathrm{Id}) P_{r_\rho}^{(u_\xi)} E_r(\rho)(\tilde{\rho})| \\
&\quad + |P_{r_\rho}^{(u_\xi)} E_r(\rho)(\tilde{\rho}) - P_r^{(u_\xi)} \tilde{\rho}|.
\end{aligned}$$

The first term can be estimated pointwise by

$$\begin{aligned}
|(\Phi_r(\rho)^{-1} - \mathrm{Id}) P_{r_\rho}^{(u_\xi)} E_r(\rho)(\tilde{\rho})| &\leq c_{17}|\rho||P_{r_\rho}^{(u_\xi)} E_r(\rho)(\tilde{\rho})| \\
&\leq c_{17}|\rho|(|(\partial_{E_r(\rho)\tilde{\rho}} Y)(r_\rho, u_\xi)| \\
&\quad + \frac{1}{2}|(\partial_{E_r(\rho)\tilde{\rho}} J)(r_\rho, u_\xi) \circ Y(r_\rho, u_\xi)| \\
&\quad + \frac{1}{2}|Y(r_\rho, u_\xi) \circ (\partial_{E_r(\rho)\tilde{\rho}} j)(r_\rho)|).
\end{aligned}$$

By construction, Y, J and j do not depend on $r \in \mathcal{R}$ along the striplike ends. Moreover they are consistent with the strip-like ends, so in particular they do not depend on $r \in \mathcal{R}$ on the images of the finite rectangles that come from the gluing procedure. Using the language of Seidel's book, the striplike ends and the finite rectangles arising from the gluing procedure constitute the "thin" part of the "thick-thin decomposition" of the surface; our observation is that Y, J and j do not depend on $r \in \mathcal{R}$ for $z \in S^{\text{thin}}$. Denote by $S^{\text{thin}} \subset S$ the thin part of this decomposition, noting that is a compact subset. Hence we can find constants a_1, a_2 and a_3 such that

$$|Y(r,x,z)| \leq a_1,\ |(\partial_\rho j)(r,z)| \leq a_2|\rho|,\ |(\partial_\rho J)(r,x)| \leq a_3|\rho|$$

for all $r \in \mathcal{R}, x \in M$ and $z \in S$, yielding an estimate

$$\begin{aligned}
\|(\Phi_r(\rho)^{-1} - \mathrm{Id})P_{r_\rho}^{(u_\xi)}E_r(\rho)(\tilde{\rho})\|_{L^p} &\leq c_{17}c_1|\rho||\tilde{\rho}|(\mathrm{vol}(S^{\text{thin}}))^{1/p} \\
&\quad + \frac{1}{2}c_{17}c_1a_3a_1|\rho||\tilde{\rho}|(\mathrm{vol}(S^{\text{thin}}))^{1/p} \\
&\quad + \frac{1}{2}c_1c_{17}a_1a_2|\rho||\tilde{\rho}|(\mathrm{vol}(S^{\text{thin}}))^{1/p} \\
&\leq c_{18}|\rho||\tilde{\rho}| \\
&\leq c_{18}(|\rho| + \|\xi\|_{W^{1,p}})(|\tilde{\rho}| + \|\tilde{\xi}\|_{W^{1,p}}).
\end{aligned}$$

Treating the other term is very similar: using the fact that

$$P_{r_\rho}^{(u_\xi)}E_r(\rho)(\tilde{\rho}) - P_r^{(u_\xi)}\tilde{\rho}$$

is linear in $\tilde{\rho}$, and 0 when $\rho = 0$, and its support is only on S^{thin}, we get an estimate of the form

$$\begin{aligned}
\|P_{r_\rho}^{(u_\xi)}E_r(\rho)(\tilde{\rho}) - P_r^{(u_\xi)}\tilde{\rho}\|_{L^p} &\leq c_{19}|\rho||\tilde{\rho}|(\mathrm{vol}(S^{\text{thin}})^{1/p} \\
&\leq c_{20}(|\rho| + \|\xi\|_{W^{1,p}})(|\tilde{\rho}| + \|\tilde{\xi}\|_{W^{1,p}}).
\end{aligned}$$

Estimating $C(b)(iii)$: We have pointwise that

$$\begin{aligned}
|\Phi_u(\xi)^{-1}\widetilde{D}_r^{(u_\xi)}(\tilde{\rho}) - \widetilde{D}_r^{(u)}(\tilde{\rho})| &\leq \frac{1}{2}|(\partial_{\tilde{\rho}}J)(u_\xi) \circ du_\xi \circ j - \Phi_u(\xi)(\partial_{\tilde{\rho}}J)(u) \circ du_\xi \circ j| \\
&\quad + \frac{1}{2}|J(u_\xi) \circ du_\xi \circ \partial_{\tilde{\rho}}j - \Phi_u(\xi) \circ J(u) \circ du \circ \partial_{\tilde{\rho}}j| \\
&\leq \frac{1}{2}|(\partial_{\tilde{\rho}}J)(u_\xi)||du_\xi - \Phi_u(\xi)du| \\
&\quad + \frac{1}{2}|(\partial_{\tilde{\rho}}J)(u_\xi) - (\partial_{\tilde{\rho}}J)(u)||du| \\
&\quad + \frac{1}{2}|du_\xi - \Phi_u(\xi)du||\partial_{\tilde{\rho}}j| \\
&\leq c_{21}|\tilde{\rho}|(|\nabla\xi| + |\xi||du|) \\
&\quad + c_{22}|\tilde{\rho}||\xi||du| \\
&\quad + c_{23}(|\nabla\xi| + |\xi||du|)|\tilde{\rho}|.
\end{aligned}$$

Taking L^p norms gives

$$\begin{aligned}
\|\Phi_u(\xi)^{-1}\widetilde{D}_r^{(u_\xi)}(\tilde{\rho}) - \widetilde{D}_r^{(u)}(\tilde{\rho})\|_p &\leq c_{21}|\tilde{\rho}|(\|\nabla\xi\|_p + \|\xi\|_\infty\|du\|_p) + c_{22}|\tilde{\rho}|\|\xi\|_\infty\|du\|_p \\
&\quad + c_{23}(\|\nabla\xi\|_p + \|\xi\|_\infty\|du\|_p)|\tilde{\rho}| \\
&\leq c_{24}(|\rho| + \|\xi\|_{1,p})(|\tilde{\rho}| + \|\tilde{\xi}\|_{1,p}).
\end{aligned}$$

Estimating $C(b)(iv)$: By definition of P_r^u we have that

$$\begin{aligned}
\Phi_u(\xi)P_r^{(u_\xi)}(\tilde{\rho}) - \Phi_u(\xi)P_r^{(u)}(\tilde{\rho}) &= [(\partial_{\tilde{\rho}}Y)(u_\xi) - \Phi_u(\xi)\partial_{\tilde{\rho}}Y(u)]^{0,1} \\
&\quad + \frac{1}{2}[(\partial_{\tilde{\rho}}J)(u_\xi)\circ Y(u_\xi)\circ j - \phi_u(\xi)(\partial_{\tilde{\rho}}J)(u)\circ Y(u)\circ j \\
&\quad + J(u_\xi)\circ Y(u_\xi)\circ\partial_{\tilde{\rho}}j - \phi_u(\xi)J(u)\circ Y(u)\circ\partial_{\tilde{\rho}}j]^{0,1}.
\end{aligned}$$

This leads to pointwise estimates

$$\begin{aligned}
|\Phi_u(\xi)P_r^{(u_\xi)}(\tilde{\rho}) - \Phi_u(\xi)P_r^{(u)}(\tilde{\rho})| &\leq |(\partial_{\tilde{\rho}}Y)(u_\xi) - \Phi_u(\xi)\partial_{\tilde{\rho}}Y(u)| \\
&\quad + \frac{1}{2}|(\partial_{\tilde{\rho}}J)(u_\xi)\circ Y(u_\xi) - \phi_u(\xi)(\partial_{\tilde{\rho}}J)(u)\circ Y(u)| \\
&\quad + \frac{1}{2}|Y(u_\xi) - \Phi_u(\xi)Y(u)||\partial_{\tilde{\rho}}j|.
\end{aligned}$$

The first term can be estimated by

$$|(\partial_{\tilde{\rho}}Y)(u_\xi) - \Phi_u(\xi)\partial_{\tilde{\rho}}Y(u)| \leq c_{25}|\xi||\tilde{\rho}|$$

for some c_{25} that depends only on the choice of perturbation datum (J,K). Now since $\Phi_u(\xi)J(u) = J(u_\xi)\Phi_u(\xi)$, we can write $\Phi_u(\xi)(\partial_{\tilde{\rho}}J)(u)\circ Y(u) = (\partial_{\tilde{\rho}}J)(u_\xi)\circ\Phi_u(\xi)Y(u)$, so the second term can be estimated by

$$\begin{aligned}
&|(\partial_{\tilde{\rho}}J)(u_\xi)\circ Y(u_\xi) - \Phi_u(\xi)(\partial_{\tilde{\rho}}J)(u)\circ Y(u)| \\
&\qquad = |(\partial_{\tilde{\rho}}J)(u_\xi)\circ Y(u_\xi) - (\partial_{\tilde{\rho}}J)(u_\xi)\circ\Phi_u(\xi)Y(u)| \\
&\qquad \leq |(\partial_{\tilde{\rho}}J)(u_\xi)||Y(u_\xi) - \Phi_u(\xi)Y(u)| \\
&\qquad \leq c_{26}|\tilde{\rho}||\xi|
\end{aligned}$$

where c_{26} again depends only on the choice of perturbation datum (J,K), that holds for all ξ such that $|\xi| \leq c_0$. Taking the L^p norm leads to the estimate

$$\|(\partial_{\tilde{\rho}}J)(u_\xi)\circ Y(u_\xi) - \Phi_u(\xi)(\partial_{\tilde{\rho}}J)(u)\circ Y(u)\|_p \leq c_{26}(|\rho| + \|\xi\|_{1,p})(|\tilde{\rho}| + \|\tilde{\xi}\|_{1,p}).$$

For the last term, there is a pointwise estimate

$$|Y(u_\xi) - \Phi_u(\xi)Y(u)||\partial_{\tilde{\rho}} j| \quad \le \quad c_{27}|\xi||\tilde{\rho}|$$

that holds for all ξ such that $|\xi| \le c_0$, where c_{27} depends only on the choice of perturbation datum and the family of complex structures $j(r), r \in \mathcal{R}$.

Taking the L^p norm leads to an estimate

$$\|(Y(u_\xi) - \Phi_u(\xi)Y(u))(\partial_{\tilde{\rho}} j)\|_p \le c_{28}(|\rho| + \|\xi\|_{1,p})(|\tilde{\rho}| + \|\tilde{\xi}\|_{1,p}).$$

□

6.7 The gluing map

To define the gluing map, we use an infinite dimensional implicit function theorem, quoted here from [10, Appendix A].

Theorem 6.7.1 (Theorem A.3.4 in [10])**.** *Let X and Y be Banach spaces, $U \subset X$ an open subset of X, and $f : U \to Y$ a continuously differentiable map. Suppose that for $x_0 \in U$, $D := df(x_0) : X \to Y$ is surjective, and has a bounded right inverse $Q : Y \to X$, and that $\delta > 0, C > 0$ are constants such that $\|Q\| \le C$, $B_\delta(x_0) \subset U$, and*

$$\|x - x_0\| \le \delta \implies \|df(x) - D\| \le \frac{1}{2C}.$$

Suppose that $x_1 \in X$ satisfies $\|x_1 - x_0\| \le \frac{\delta}{8}$, and $\|f(x_1)\| \le \frac{\delta}{4C}$. Then there exists a unique $x \in X$ such that

$$f(x) = 0, \quad x - x_1 \in \mathrm{im} Q, \quad \textit{and} \quad \|x - x_0\| \le \delta.$$

Moreover, $\|x - x_1\| \le 2C\|f(x_1)\|$.

We apply it to our situation as follows. For each R, take a local trivialization of a small neighborhood of the preglued curve (r_R, u_R). Identify

$$\begin{aligned} X &:= T_{r_R}\mathcal{R} \times W^{1,p}(\mathcal{S}_{r_R}, u_R^* TM) \\ Y &:= L^p(\mathcal{S}_{r_R}, \Lambda^{0,1} \otimes_{J_R} u_R^* TM) \\ f &:= \mathcal{F}_{\mathcal{S}, r_R, u_R} \\ x_0 &:= (0, 0). \end{aligned}$$

For $\delta \leq 1/(2Cc)$, the estimate (6.1) says that whenever $(|\rho| + \|\xi\|_{1,p}) \leq \delta$ then

$$\|d\mathcal{F}_{\mathcal{S},r_R,u_R}(\rho,\xi) - D_{\mathcal{S},r_R,u_R}\| \leq c(|\rho| + \|\xi\|_{W^{1,p}}) \leq c\delta \leq 1/(2C)$$

For sufficiently large R the constants C and c are independent of R, thus δ can also be chosen to be independent of R. So now let $x_1 := (0,0)$. Then

$$\begin{aligned} \|\mathcal{F}_{\mathcal{S},r_R,u_R}(0,0)\|_{0,p} &= \|(\overline{\partial} - \nu)(r_R,u_R)\|_{0,p} \\ &\leq c\ \epsilon(R). \end{aligned}$$

For sufficiently large R,

$$\begin{aligned} \|\mathcal{F}_{\mathcal{S},r_R,u_R}(0,0)\|_{0,p} &\leq c\ \epsilon(R) \\ &\leq \frac{\delta}{4C}. \end{aligned}$$

Thus the hypotheses of the Implicit Function Theorem are satisfied, so there exists a unique $(\rho_R,\xi_R) \in T_{r_R}\mathcal{R} \times W^{1,p}(\mathcal{S}_{r_R}, u_R^*TM)$ such that

$$\mathcal{F}_{\mathcal{S},r_R,u_R}(\rho_R,\xi_R) = 0$$

$$(\rho_R,\xi_R) \in \mathrm{im} Q_R$$

$$|\rho_R| + \|\xi_R\|_{W^{1,p}} \leq \delta.$$

Now $\mathcal{F}_{\mathcal{S},r_R,u_R}(\rho_R,\xi_R) = 0$ if and only if $(\exp_{r_R}\rho_R, \exp_{u_R}\xi_R) \in \mathcal{M}_{d,1}(\underline{x}_0,\ldots,\underline{x}_d)$. Thus we can define a continuous *gluing map*

$$\begin{aligned} g : [R_0,\infty) &\to \mathcal{M}_{d,1}(\underline{x}_0,\ldots,\underline{x}_d) \\ R &\mapsto (\exp_{r_R}\rho_R, \exp_{u_R}\xi_R) \end{aligned} \tag{6.5}$$

The implicit function theorem also tells us that

$$|\rho_R| + \|\xi_R\|_{W^{1,p}} \leq 2C\|\mathcal{F}_{\mathcal{S},r_R,u_R}(0,0)\|_{0,p} \leq 2C\epsilon(R) \to 0$$

as $R \to \infty$. In particular, the glued curves $g(R)$ converge to the preglued curves $(r_R, \underline{u}_R)$, hence as $R \to \infty$ they Gromov converge to the same limiting broken tuple.

For a gluing length $R >> 0$, write (r_R, u_R) for the corresponding preglued curve, and $(\widetilde{r}_R, \widetilde{u}_R)$ for the corresponding glued curve.

In a local trivialization about the preglued curve (r_R, u_R), the implicit function theorem implies that the moduli space of pseudoholomorphic quilts in a neighborhood of (r_R, u_R) is modeled on a complement of im Q_R. In particular, to show that the image of the gluing map is contained in the 1-dimensional component of the moduli space of pseudoholomorphic quilted disks, it's enough to check that im Q_R has codimension 1. Since ker D_R is a complement to imQ_R, an equivalent statement is that $\dim\ \ker\ D_R = 1$. By construction, the right inverse Q_R has the same image as the approximate right inverse T_R. We will prove that:

Proposition 6.7.2. *The images of the gluing maps are contained in the one-dimensional component of the moduli space,* $\mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_d)^1$

by proving that:

Lemma 6.7.3. *For each of the three types of gluing construction, with approximate right inverse* T_R*,*

$$\text{codim im } T_R = \dim\ \ker\ D_R = 1.$$

Proof. We prove Lemma 6.7.3 for each type of gluing construction in turn. Since the constructions behave somewhat differently, the line of proof will be as follows: for Types 1 and 2, we will prove that codim im $T_R = 1$ by finding an explicit description of im T_R and a one-dimensional complement. For Type 3, we will prove the existence of an isomorphism $\ker\ D_{\underline{v}} \cong \ker\ D_{\mathcal{S},r_1,\underline{u}_R}$; then, since $\dim\ \ker\ D_{\underline{v}} = 1$, it will follow that $\dim\ \ker\ D_{\mathcal{S},r_1,\underline{u}_R} = 1$ too.

Type 1

Suppose that $(\rho, \xi) \in T_{r_R}\mathcal{R}^{d,1} \times W^{1,p}(\mathcal{S}_{r_R}, (u_R)^*TM)$. Let $\beta_1 + \beta_{1,2} + \beta_2 = 1$ be a smooth partition of unity on $\mathcal{S}_{r_R}$ such that the support of β_1 is on the part of $\mathcal{S}_{r_R}$ that comes from the truncation of $\mathcal{S}_{r_1}$ on the neck at $s = R$, the support of β_2 is on the part of $\mathcal{S}_{r_R}$ that comes from the truncation of $\mathcal{S}_{r_2}$ on the neck at $s = R$, and the support of $\beta_{1,2}$ is on the subset of the neck of $\mathcal{S}_{r_R}$ corresponding to $R/2 \leq s \leq R$ in $\mathcal{S}_{r_1}$ as well as the corresponding piece from $\mathcal{S}_{r_2}$. For $\xi \in W^{1,p}(\mathcal{S}_{r_R}, (u_R)^*TM)$, it is clear that

$$\begin{aligned}
\beta_1\xi &\in W^{1,p}(\mathcal{S}_{r_1}, (u_1^R)^*TM), \\
\beta_2\xi &\in W^{1,p}(\mathcal{S}_{r_2}, (u_2^R)^*TM), \\
\beta_{1,2}\xi &\in W^{1,p}(\mathcal{S}_{r_1}, (u_1^R)^*TM) \cap W^{1,p}(\mathcal{S}_{r_2}, (u_2^R)^*TM).
\end{aligned}$$

For $\rho \in T_{r_R}\mathcal{R}^{d,1}$, we use local charts near the boundary to write $\rho = \rho_g + \rho_1 + \rho_2$, where $\rho_1 \in T_{r_1}\mathcal{R}, \rho_2 \in T_{r_2}\mathcal{R}$ and $\rho_g \in \mathbb{R}$ represents the component in the direction of the gluing parameter. Thus

$$(\rho, \xi) = (\rho_g, 0) + (\rho_1, \beta_1\xi) + (0, \beta_{1,2}\xi) + (\rho_2, \beta_2\xi)$$

and the result follows if we can show that the final three terms on the right hand side are all in $\mathrm{im}T_R$. The cases of $(\rho_1, \beta_1\xi)$ and $(\rho_2, \beta_2\xi)$ have identical proofs, so we prove it for the first case.

Consider $D_{\mathcal{S},r_R,u_R}(\rho_1, \beta_1\xi) \in L^p(\mathcal{S}_{r_R}, (u_R)^*TM)$. It is supported on the image of the truncation $\mathcal{S}_{r_1}\backslash\{s > R\}$ and so it can be identified with image of $D_{\mathcal{S},r_1,u_1^R}(\rho_1, \beta_1\xi)$. Note that the operators $D_{\mathcal{S},r,u}$ preserve basepoints on the surface $\mathcal{S}_r$; so that the support of $\xi \in W^{1,p}(\mathcal{S}_r, u^*TM)$ on $\mathcal{S}_r$ is the same as the support of $D_{\mathcal{S},r,u}(\rho, \xi) \in L^p(\mathcal{S}_r, u^*TM)$. Applying this fact we can conclude that the support of $D_{\mathcal{S},r_R,u_R}(\rho_1, \beta_1\xi)$ is precisely the support of the cut-off function used in the construction of T_R, and therefore

$$\begin{aligned}
T_R D_{\mathcal{S},r_R,u_R}(\rho_1, \beta_1\xi) &= Q_1 D_{\mathcal{S},r_1,u_1^R}(\rho_1, \beta_1\xi) \\
&= (\rho_1, \beta_1\xi)
\end{aligned}$$

where the last equality follows from the standing assumptions that $D_{\mathcal{S},r_1,u_1^R}$ is surjective and has trivial kernel, so that it is an isomorphism; as such its right-inverse Q_1 is a left-inverse too. Hence, $(\rho_1, \beta_1\xi) \in \mathrm{im}T_R$.

For the other piece, consider $D_{\mathcal{S},r_R,u_R}(0, \beta_{1,2}\xi)$. The support of $\beta_{1,2}\xi$ is on the part of the neck where all three of the operators $D_{\mathcal{S},r_R,u_R}, D_{\mathcal{S},r_1,u_1^R}$ and $D_{\mathcal{S},r_2,u_2^R}$ coincide, i.e.,

$$\begin{aligned}
D_{\mathcal{S},r_R,u_R}(0, \beta_{1,2}\xi) &= D_{\mathcal{S},r_1,u_1^R}(0, \beta_{1,2}\xi) \\
&= D_{\mathcal{S},r_2,u_2^R}(0, \beta_{1,2}\xi).
\end{aligned}$$

By assumption both $D_{\mathcal{S},r_1,u_1^R}$ and $D_{\mathcal{S},r_2,u_2^R}$ are isomorphisms (being surjective with trivial kernel). Writing $\eta = D_{\mathcal{S},r_R,u_R}(0,\beta_{1,2}\xi)$, recall that in the construction of T_R, one uses a cut-off function to write $\eta = \eta_1+\eta_2$, where the supports of η_1 and η_2 intersect only at the truncation line $s = R$, and then the right inverse Q_1 is used on η_1 and the right inverse Q_2 is used on η_2. We know that $Q_1\eta = (0,\xi)$ and $Q_2\eta = (0,\xi)$. The operator T_R is defined by $T_R\eta = Q_1\eta_1 + Q_2\eta_2$. So it is enough to show that $Q_1\eta_1 = Q_2\eta_1$. So suppose for the sake of contradiction that $Q_1\eta_1 - Q_2\eta_1 \neq 0$. If so, applying $D_{\mathcal{S},r_1,u_1^R}$, we could write

$$\begin{aligned} D_{\mathcal{S},r_1,u_1^R}(Q_1\eta_1 - Q_2\eta_1) &= D_{\mathcal{S},r_1,u_1^R}Q_1\eta_1 - D_{\mathcal{S},r_1,u_1^R}Q_2\eta_1 \\ &= D_{\mathcal{S},r_1,u_1^R}Q_1\eta_1 - D_{\mathcal{S},r_2,u_2^R}Q_2\eta_1 \\ &= \eta_1 - \eta_1 \\ &= 0, \end{aligned}$$

contradicting the assumption that $\ker D_{\mathcal{S},r_1,u_1^R} = 0$. Hence, $T_R\eta = (0,\beta_{1,2}\xi)$. In summary, we have:

1. If $(\rho,\xi) \in T_{r_R}\mathcal{R}^{d,1} \times W^{1,p}(\mathcal{S}_{r_R},(u_R)^*TM)$ is such that $\rho = (0,\rho_1,\rho_2) \in \mathbb{R} \times T_{r_1}\mathcal{R}^{d-e+,1} \times T_{r_2}\mathcal{R}^e \cong T_{r_R}\mathcal{R}^{d,1}$, then $(\rho,\xi) \in \text{im } T_R$,
2. If $(\rho,0) \in T_{r_R}\mathcal{R}^{d,1} \times W^{1,p}(\mathcal{S}_{r_R},(u_R)^*TM)$ is such that $\rho = (\rho_g,0,0) \in \mathbb{R} \times T_{r_1}\mathcal{R}^{d-e+,1} \times T_{r_2}\mathcal{R}^e \cong T_{r_R}\mathcal{R}^{d,1}$, then $(\rho,0) \notin \text{im } T_R$.

Since these elements span $T_{r_R}\mathcal{R}^{d,1} \times W^{1,p}(\mathcal{S}_{r_R},(u_R)^*TM)$, it follows that

$$\text{im } T_R = \{(\rho,\xi)\big|\rho_g = 0\}. \tag{6.6}$$

Type 2

Suppose that $(\rho,\xi) \in T_{r_R}\mathcal{R}^{d,1} \times W^{1,p}(\mathcal{S}_{r_R},(u_R)^*TM)$.

Let $\beta_0 + \beta_1 + \ldots + \beta_k + \beta_{0,1} + \ldots + \beta_{0,k} = 1$ be a smooth partition of unity on $\mathcal{S}_{r_R}$ such that for $i = 0,\ldots,k$, the support of β_i is on the part of $\mathcal{S}_{r_R}$ that comes from the truncation of $\mathcal{S}_{r_i}$ on the neck at $s = R$, and the support of $\beta_{0,i}$ is on the subset of the neck of $\mathcal{S}_{r_R}$ corresponding to $R/2 \le s \le R$ in $\mathcal{S}_{r_0}$ as well as the corresponding piece from $\mathcal{S}_{r_i}$. Then for $\xi \in W^{1,p}(\mathcal{S}_{r_R}, (u_R)^*TM)$, it is clear that

$$
\begin{aligned}
\beta_0\xi &\in W^{1,p}(\mathcal{S}_{r_0}, (u_0^R)^*TM), \\
\beta_1\xi &\in W^{1,p}(\mathcal{S}_{r_1}, (u_1^R)^*TM), \\
&\dots \\
\beta_k\xi &\in W^{1,p}(\mathcal{S}_{r_k}, (u_k^R)^*TM), \\
\beta_{0,1}\xi &\in W^{1,p}(\mathcal{S}_{r_0}, (u_0^R)^*TM) \cap W^{1,p}(\mathcal{S}_{r_1}, (u_1^R)^*TM), \\
&\dots \\
\beta_{0,k}\xi &\in W^{1,p}(\mathcal{S}_{r_0}, (u_0^R)^*TM) \cap W^{1,p}(\mathcal{S}_{r_k}, (u_k^R)^*TM).
\end{aligned}
$$

For $\rho \in T_{r_R}\mathcal{R}^{d,1}$, we use local charts near the boundary to write $\rho = \rho_0+\rho_1+\ldots+\rho_k+\rho_g$, where for $i = 0, \ldots, k$, $\rho_i \in T_{r_i}\mathcal{R}$, and $\rho_g \in \mathbb{R}$ represents the component of the tangent vector in the direction of the gluing parameter. Then as in the previous case we can write

$$(\rho, \xi) = (\rho_g, 0) + (\rho_0, \beta_0\xi) + (\rho_1, \beta_1\xi) + \ldots + (\rho_k, \beta_k\xi) + (0, \beta_{0,1}\xi) + \ldots (0, \beta_{0,k}\xi)$$

and the result follows if all terms except the first on the right hand side of the above expression are in $\text{im}T_R$. The same argument as used for Type 1 proves that all terms on the right except for $(\rho_g, 0)$ are in the image of T_R, and the result follows. In summary:

1. If $(\rho, \xi) \in T_{r_R}\mathcal{R}^{d,1} \times W^{1,p}(\mathcal{S}_{r_R}, (u_R)^*TM)$ is such that $\rho_g = 0$, then $(\rho, \xi) \in \text{im } T_R$,
2. If $(\rho, 0) \in T_{r_R}\mathcal{R}^{d,1} \times W^{1,p}(\mathcal{S}_{r_R}, (u_R)^*TM)$ is such that $\rho = \rho_g$, then $(\rho, 0) \notin \text{im } T_R$.

Since these elements span $T_{r_R}\mathcal{R}^{d,1} \times W^{1,p}(\mathcal{S}_{r_R}, (u_R)^*TM)$, it follows that

$$\text{im } T_R = \{(\rho, \xi) \big| \rho_g = 0\}. \tag{6.7}$$

Type 3

In this case we have glued $(r_1, u_1) \in \mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{y}, \ldots, \underline{x}_d)^0$ to a Floer trajectory $v \in \widetilde{M}(\underline{y}, \underline{x_i})^0$. Since the complement of im T_R is ker D_R, our goal is to show that the vector space ker D_R as the same dimension as ker D_v, which by assumption is 1.

From this it would follow that the image of the gluing map lies in a one dimensional component of the moduli space.

To show that the finite dimensional vector spaces ker D_v and ker D_R have the same dimension, it is enough to produce a pair of injective linear maps, $\Phi : \ker\ D_v \longrightarrow \ker\ D_R$ and $\Psi : \ker\ D_R \longrightarrow \ker\ D_v$. Write f^R for the pregluing map.

Claim 1: $\Phi := (1 - Q_R D_R) df^R : \ker\ D_v \longrightarrow \ker\ D_R$ is injective.

By hypothesis $\ker D_v$ is one-dimensional, so we know that an explicit basis is $\{\partial_s v\}$. It suffices therefore to show that for sufficiently large R, there is a constant $c > 0$ such that

$$\|\partial_s v\|_{1,p} \leq c\|(1 - Q_R D_R) df^R(\partial_s v)\|_{1,p}. \tag{6.8}$$

We prove the estimate by proving two separate inequalities

$$\begin{aligned} \|df^R(\partial_s v)\|_{1,p,R} &\geq c_1\|\partial_s v\|_{1,p} &(6.9)\\ \|D_R df^R(\partial_s v)\|_{0,p,R} &\leq \epsilon(R)\|\partial_s v\|_{1,p} &(6.10)\end{aligned}$$

where $c_1 > 0$ and $\epsilon(R) \to 0$ as $R \to \infty$. Together these imply (6.8), since the uniform bound on the right inverse

$$\|Q_R\xi\|_{1,p} \leq C\|\xi\|_{0,p}$$

holds for R sufficiently large, and so

$$\begin{aligned} \|(1 - Q_R D_R) df^R(\partial_s v)\|_{1,p} &\geq \|df^R(\partial_s v)\|_{1,p} - \|Q_R D_R df^R(\partial_s v)\|_{1,p}\\ &\geq c_1\|\partial_s v\|_{1,p} - C\|D_R df^R(\partial_s v)\|_{0,p}\\ &\geq c_1\|\partial_s v\|_{1,p} - C\epsilon(R)\|\partial_s v\|_{1,p}\\ &\geq c^{-1}\|\partial_s v\|_{1,p} &(6.11)\end{aligned}$$

for some $c > 0$ for sufficiently small $\epsilon(R)$.

To prove (6.9), we write $v_\lambda(s,t) := v(s+\lambda, t)$. With this notation, $\partial_\lambda v_\lambda\big|_{\lambda=0} = \partial_s v$. Thus,

$$df^R(\partial_s v)(s,t) \quad = \quad \frac{d}{d\lambda}\Big|_{\lambda=0} v_\lambda \#_R u_1,$$

which by construction is supported only on the region $s \geq 3R/2$ on the striplike end. The pre-gluing map on this region is

$$f^R(v_\lambda, u_1) = v_\lambda \#_R u_1(s,t) = \begin{cases} \exp_{\underline{y}(t)}(\beta(-s+\frac{3R}{2})\eta_\lambda(s-2R,t)), & s \in [\frac{3R}{2}, \frac{3R}{2}+1] \\ v_\lambda(s-2R,t), & s \geq \frac{3R}{2}+1. \end{cases}$$

and we need to take the derivative with respect to λ. On the region $s \geq 3R/2+1$ we have that

$$\begin{aligned} df^R(\partial_s v)(s,t) &= \frac{d}{d\lambda}\Big|_{\lambda=0} v_\lambda(s-2R,t) \\ &= (\partial_s v)(s-2R,t). \end{aligned}$$

Note that

$$\begin{aligned} \|df^R(\partial_s v)(s,t)\|_{1,p} &\geq \|df^R(\partial_s v)(s,t)\|_{1,p;[3R/2+1,\infty)} \\ &= \|(\partial_s v)(s-2R,t)\|_{1,p;[3R/2+1,\infty)} \\ &= \|\partial_s v\|_{1,p;[-R/2+1,\infty)}. \end{aligned}$$

Because of the exponential convergence of $\partial_s v$, there is a $c_1 > 0$ such that for all sufficiently large R,

$$\|\partial_s v\|_{1,p;[-R/2+1,\infty)} \geq c_1 \|\partial_s v\|_{1,p}$$

which proves (6.9).

To prove (6.10), observe first that by construction, $D_R df^R(\partial_s v)$ is supported only on the interval $s \in [3R/2, 3R/2+1]$. It follows that the L^p norm of $D_R df^R(\partial_s v)$ is controlled by the $W^{1,p}$ norm of $df^R(\partial_s v)$ on that interval. Introduce the abbreviated notation $\beta_R(s) := \beta(-s+3R/2)$ and $\eta_R(s,t) := \eta(s-2R,t)$. On the interval $s \in [3R/2, 3R/2+1]$,

$$\begin{aligned} df^R(\partial_s v)(s,t) &= \frac{d}{d\lambda}\Big|_{\lambda=0} \exp_{\underline{y}(t)}(\beta_R \eta_\lambda(s-2R,t)) \\ &= \frac{d}{d\lambda}\Big|_{\lambda=0} \exp_{\underline{y}(t)}(\beta_R(s)\eta_R(s+\lambda,t)) \\ &= d\exp_{y(t)}(\beta_R\eta_R)\frac{d}{d\lambda}\Big|_{\lambda=0}(\beta_R\eta_R(s+\lambda,t)) \\ &= \beta_R d\exp_{y(t)}(\beta_R\eta_R)(\partial_s\eta)(s-2R,t). \end{aligned}$$

The identity

$$\exp_{y(t)} \eta(s-2R+\lambda,t) = v(s-2R+\lambda),$$

implies that

$$d\exp_{y(t)}(\eta(s-2R,t))(\partial_s\eta)(s-2R,t) \quad = \quad \partial_s v(s-2R).$$

The linear operator $d\exp_y(\eta) : T_yM \to T_{\exp_y\eta}M$ is the identity for $\eta = 0$, so is invertible for small η. In particular for sufficiently large R, the exponential convergence of trajectories means that $\eta(s-2R,t)$ is uniformly small for $s \in [3R/2, 3R/2+1]$. Therefore on this interval, we can write

$$df^R(\partial_s v)(s,t) \quad = \quad \beta_R(s)d\exp_{y(t)}(\beta_R\eta_R)[d\exp_{y(t)}(\eta(s-2R,t))]^{-1}\partial_s v(s-2R).$$

Thus there is a constant $c_2 \geq 0$ such that

$$\begin{aligned}
\|df^R(\partial_s v)(s,t)\|_{1,p;[3R/2,3R/2+1]} &\leq c_2\|\partial_s v(s-2R,t)\|_{1,p;[3R/2,3R/2+1]} \\
&= c_2\|\partial_s v(s,t)\|_{1,p;[-R/2,-R/2+1]} \\
&\leq \epsilon(R)\|\partial_s v(s,t)\|_{1,p}
\end{aligned}$$

where the last inequality and the term $\epsilon(R)$ reflects the fact that the ratio

$$\|\partial_s v\|_{1,p;[-R/2,-R/2+1]}/\|\partial_s v\|_{1,p}$$

goes to 0 as $R \to \infty$. This proves (6.10), hence we have proved Claim 1.

Now let $\beta : \mathbb{R} \to [0,1]$ be a smooth cut-off function such that $\beta(s) = 0$ for $s \leq -1/4$ and $\beta(s) = 1$ for $s \geq 1/4$, and $0 \leq \dot\beta \leq 3$. Define a shifted and rescaled cut-off function $\beta_R(s) := \beta((s-R)/R)$. Then β_R has the properties that $\beta_R = 0$ for $s \leq 3R/4$ and $\beta_R = 1$ for $s \geq 5R/4$, and $0 \leq \dot\beta_R = \dot\beta/R \leq 3/R$.

Claim 2: $\Psi := (1-Q_{u_1^R}D_{u_1^R})(1-\beta_R)\times(1-Q_{v^R}D_{v^R})\beta_R : \ker\ D_R \longrightarrow \ker D_{u_1^R} \oplus \ker D_{v^R}$ is injective.

Let $\xi \in \ker\ D_R$. Then $\xi = (1-\beta_R)\xi + \beta_R\xi$. On the support of $(1-\beta_R)\xi$, the linearized operators D_R and $D_{u_1^R}$ coincide, so

$$\begin{aligned}
D_{u_1^R}(1-\beta_R)\xi &= D_R(1-\beta_R)\xi \\
&= -\dot\beta_R\xi + (1-\beta_R)D_R\xi \\
&= -\dot\beta_R\xi.
\end{aligned}$$

Hence

$$\|D_{u_1^R}(1-\beta_R)\xi\|_{0,p} \leq 3/R\|\xi\|_{0,p}.$$

Similarly on the support of $\beta_R\xi$, the linearized operators D_R and D_{v^R} coincide, so

$$\begin{aligned} D_{v^R}\beta_R\xi &= D_R\beta_R\xi \\ &= \dot{\beta_R}\xi + \beta_R D_R\xi \\ &= \dot{\beta_R}\xi, \end{aligned}$$

hence

$$\|D_{v^R}\beta_R\xi\|_{0,p} \leq 3/R\|\xi\|_{0,p}.$$

Let c_1 and c_2 be uniform bounds for the right inverses $Q_{u_1^R}$ and Q_{v^R} respectively. Then we have:

$$(1-Q_{u_1^R}D_{u_1^R})(1-\beta_R)\times(1-Q_{v^R}D_{v^R})(\xi) = [(1-\beta_R)\xi + Q_{u_1^R}(\dot{\beta_R}\xi), \beta_R\xi - Q_{v_R}(\dot{\beta_R}\xi)]$$

Combine the identity $(1-\beta_R)\xi + \beta_R\xi = \xi$ with the estimates

$$\begin{aligned} \|Q_{u_1^R}(\dot{\beta_R}\xi)\|_{1,p} &\leq 3c_1/R\|\xi\|_{0,p} \\ \|Q_{v^R}(\dot{\beta_R}\xi)\|_{1,p} &\leq 3c_2/R\|\xi\|_{0,p} \end{aligned}$$

to get that for R sufficiently large, there is a constant $c \geq 0$ such that

$$\|(1-Q_{u_1^R}D_{u_1^R})(1-\beta_R)\times(1-Q_{v^R}D_{v^R})(\xi)\|_{1,p} \geq c\|\xi\|_{0,p}.$$

This proves Claim 2, completing the proof of Lemma 6.7.3. $\square$

6.8 Surjectivity of the gluing map

The final step is to prove the surjectivity of the gluing maps near the broken tuples. We begin by defining a neighborhood $U_\epsilon \subset \mathcal{B}^{d,1}$ that is to be associated to a broken tuple of each type and a sufficiently small $\epsilon > 0$. Our goal will be to show that for sufficiently small ϵ and sufficiently large R, the gluing map associated to the given tuple surjects onto $\mathcal{M}_{d,1}(\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_d)^1 \cap U_\epsilon$. We will prove surjectivity separately for the different types of gluing constructions.

Type 1

Proposition 6.8.1. *Let* $(r_1, \underline{u}_1) \in \mathcal{M}_{d-e+1,1}(\underline{x}_0, \ldots, \underline{x}_i, \underline{y}, \underline{x}_{i+e+1}, \ldots, \underline{x}_d)$ *and* $(r_2, \underline{u}_2) \in \mathcal{M}_e(\underline{y}, \underline{x}_{i+1}, \ldots, \underline{x}_{i+e})$ *be regular, and let* U_ϵ *be a neighborhood as defined above. Given* $\delta > 0$*, there is an* $\epsilon > 0$ *such that the following holds. If* $(r, \underline{u}) \in U_\epsilon \cap \mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_d)^1$*, then there is a pre-glued curve* $(r_R, \underline{u}_R)$ *and a* $(\rho, \underline{\xi}) \in T_{r_R}\mathcal{R}^{d,1} \times \Omega^0(\mathcal{S}_{r_R}, \underline{u}_R^* TM)$ *such that* $\exp_{r_R} \rho = r$*,* $\exp_{\underline{u}_R} \xi = \underline{u}$*,* $|\rho| + \|\underline{\xi}\|_{1,p} < \delta$*, and* $(\rho, \xi) \in \operatorname{im} Q_R$*.*

Proof. We will prove it by contradiction. Suppose there were a $\delta > 0$, and sequences $\epsilon_\nu \to 0$, $\delta_\nu \leq \epsilon_\nu \to 0$, $R_\nu = -\log(\delta_\nu) \to \infty$, and $(r_\nu, \underline{u}_\nu) \in \mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_d)^1$, and $\widetilde{r}_{1,\nu} \to r_1$, $\widetilde{r}_{2,\nu} \to r_2$ such that $r_\nu = \widetilde{r}_{1,\nu} \#_{\delta_\nu} \widetilde{r}_{2,\nu}$, with the following properties as $\nu \to \infty$: $E(u_\nu) \to E(u_1) + E(u_2)$, and writing (r^{R_ν}, u^{R_ν}) for the preglued curve constructed with gluing length R_ν,

$$\inf\{|\rho| + \|\xi\|_{1,p} \big| (r_\nu, u_\nu) = (\exp_{r^{R_\nu}} \rho, \exp_{u^{R_\nu}} \xi)\} \geq \delta. \tag{6.12}$$

Our goal is to contradict 6.12.

Writing $\rho = \rho_g + \rho_1 + \rho_2$ where ρ_g is the component of ρ in the direction of the gluing parameter, it follows from the choice of gluing length R_ν that $\rho_g = 0$. For sufficiently small ϵ_ν, the condition $\exp_{r^{R_\nu}} \rho = r_\nu$ determines ρ uniquely, so let us call it ρ_ν. The convergence of $\widetilde{r}_{i,\nu}$ to r_i for $i = 1, 2$ implies that $|\rho_\nu| \to 0$.

By assumption, $\underline{u}_\nu \to \underline{u}_1$ uniformly on compact subsets of $\mathcal{S}_{r_1}$ and $\underline{u}_\nu \to \underline{u}_2$ uniformly on compact subsets of $\mathcal{S}_{r_2}$; moreover since the maps are pseudoholomorphic, convergence on these compact subsets is uniform in all derivatives. The preglued maps $\underline{u}^{R_\nu}$ have the same convergence properties, so for large ν there is a unique section $\underline{\xi}_\nu \in \Omega^0(\mathcal{S}_{r^{R_\nu}}, (\underline{u}^{R_\nu})^* TM)$ such that $\exp_{\underline{u}^{R_\nu}} \underline{\xi}_\nu = \underline{u}_\nu$. So it is enough to show that $\|\underline{\xi}_\nu\|_{1,p} < \delta$ for sufficiently large ν, contradicting (6.12). Equivalently, we will show that the L^p norms of $\underline{\xi}_\nu, \nabla_s \underline{\xi}_\nu$ and $\nabla_t \underline{\xi}_\nu$ can be made arbitrarily small by taking ν sufficiently large.

It follows from the uniform convergence in all derivatives on compact subsets that on such subsets of $\mathcal{S}_{r_1} \cup \mathcal{S}_{r_2}$, the L^p norms of $\underline{\xi}_\nu, \nabla_s \underline{\xi}_\nu$ and $\nabla_t \underline{\xi}_\nu$ all go to zero as $\nu \to \infty$. We can choose these compact subsets to be such that their complement is on the striplike ends and neck. Hence, without loss of generality, it suffices to prove that the L^p norms of $\underline{\xi}_\nu, \nabla_s\underline{\xi}_\nu$ and $\nabla_t\underline{\xi}_\nu$ converge to zero along the striplike ends and neck of the preglued surfaces. The exponential convergence of $\underline{u}_\nu$ as well as $\underline{u}_{R_\nu}$ along the striplike ends means that the L^p norms of ξ_ν and its first derivatives can be made arbitrarily small too; so the essential thing to prove is that the L^p norm of $\underline{\xi}_\nu$ along the neck of the preglued surface can be made arbitrarily small with sufficiently large ν.

The neck consists of two finite strips of length R_ν identified along an end to form a single strip,

$$[-R_\nu, R_\nu] \times [0,1] \cong [0, R_\nu] \times [0,1] \cup [0, R_\nu] \times [0,1]/\sim,$$

where $\sim$ is the identification of $(R_\nu, 1-t)$ of the first strip with (R_ν, t) of the second, for $t \in [0,1]$.

Let $\epsilon_0 > 0$ be given. Fix $R > 0$ large enough that $\lim_{\nu\to\infty} E(\underline{u}_\nu; [R, R_\nu] \times [0,1] \cup [R, R_\nu] \times [0,1]/\sim) < \epsilon_0$. Without loss of generality we can assume that $\epsilon_0 > 0$ is small enough that $|\partial_s \underline{u}_\nu|$ satisfies, by Proposition B.0.11,

$$|\partial_s \underline{u}_\nu| \leq ce^{-\kappa^2 s}$$

for all ν and for all $s \in [R, R_\nu]$, for some $c, \kappa > 0$.

Since $\underline{u}_\nu$ satisfies Floer's inhomogeneous pseudoholomorphic equation (2.2) on this strip, this implies the inequality

$$|\partial_t \underline{u}_\nu(s,t) - X_{H_t}(\underline{u}_\nu(s,t))| \leq ce^{-\kappa^2 s}. \tag{6.13}$$

Let ϕ_t be the flow of the Hamiltonian vector field X_{H_t}, and consider the function $\widetilde{\underline{u}_\nu} := \phi_{1-t}(\underline{u}_\nu(s,t))$. Then

$$\partial_t \widetilde{\underline{u}_\nu} = (\phi_{1-t})_*(\partial_t \underline{u}_\nu - X_{H_t}(\underline{u}_\nu))$$

and (6.13) leads to an estimate

$$\begin{aligned} \operatorname{dist}(\widetilde{\underline{u}_\nu}(s,1) - \widetilde{\underline{u}_\nu}(s,0)) &\leq \int_0^1 |\partial_t \widetilde{\underline{u}_\nu}| \, dt \\ &\leq \widetilde{c}e^{-\kappa^2 s}. \end{aligned}$$

Since $\widetilde{\underline{u}_\nu}(s,1) \in L_1$ and $\widetilde{\underline{u}}_\nu(s,0) \in \phi_1(L_0)$, this means that both are very close to an intersection point $p \in \phi_1(L_0) \cap L_1$. The assumption of transversality of their intersection implies that there is a constant $a > 0$ such that

$$\begin{aligned}
\operatorname{dist}(\widetilde{u}_\nu(s,0),p) &\leq a \operatorname{dist}(\widetilde{u}_\nu(s,0),L_1) \\
&\leq a \operatorname{dist}(\widetilde{u}_\nu(s,0),\widetilde{u}_\nu(s,1)) \\
&\leq a\, \widetilde{c}\, e^{-\kappa^2 s}.
\end{aligned}$$

Now for every other t,

$$\begin{aligned}
\operatorname{dist}(\widetilde{u}_\nu(s,t),p) &\leq \operatorname{dist}(\widetilde{u}_\nu(s,t),\widetilde{u}_\nu(s,0)) + \operatorname{dist}(\widetilde{u}_\nu(s,0),p) \\
&\leq \widetilde{c}e^{-\kappa^2 s}(t) + a\, \widetilde{c}\, e^{-\kappa^2 s} \\
&\leq b\, e^{-\kappa^2 s}.
\end{aligned}$$

In terms of the original function u_ν, and writing $\phi_{1-t}x(t) = p$, this estimate translates back into an estimate

$$\operatorname{dist}(u_\nu(s,t),x(t)) \leq \widetilde{b}e^{-\kappa^2 s}.$$

By construction, the preglued curves u^{R_ν} satisfy a similar inequality, and so

$$\begin{aligned}
\operatorname{dist}(u_\nu(s,t),u^{R_\nu}(s,t)) &\leq \operatorname{dist}(u_\nu(s,t),x(t)) + \operatorname{dist}(x(t),u^{R_\nu}(s,t)) \\
&\leq C\, e^{-\kappa^2 s}.
\end{aligned}$$

This implies

$$|\xi_\nu(s,t)| \leq C'\, e^{-\kappa^2 s}.$$

Taking the L^p norm on a strip $[R,R_\nu] \times [0,1]$ gives

$$\begin{aligned}
\int_R^{R_\nu} \int_0^1 |\xi_\nu|^p\, ds\, dt &\leq C' \int_R^{R_\nu} e^{-p\kappa^2 s}\, ds \\
&= \frac{C'}{p\kappa^2}(e^{-p\kappa^2 R} - e^{-p\kappa^2 R_\nu}),
\end{aligned}$$

and this can be made arbitrarily small by choosing R large enough. By symmetry the same estimate holds for the strip $[R,R_\nu] \times [0,1]$ on the other side of the neck. Thus, the L^p norm of ξ on the neck can be made arbitrarily small as $\nu \to \infty$.

Now we consider the L^p norms of $\nabla_s\xi_\nu$ and $\nabla_t\xi_\nu$. First note that since u_ν converges uniformly in all its derivatives on compact subsets of $\mathcal{S}_{r_1}$ and $\mathcal{S}_{r_2}$ to the limits u_1 and u_2, we see that on such compact subsets we have uniform estimates for $|\nabla_s\xi_\nu| \to 0$ and $|\nabla_t\xi| \to 0$. On the striplike ends of $\mathcal{S}_{r_\nu}$, the exponential convergence of u_ν and u^{R_ν} to the same limits mean that the L^p norms here can be made arbitrarily small. Therefore what we need to show is that the L^p norms of $\nabla_s\xi_\nu$ and $\nabla_t\xi_\nu$ on the neck, which varies in length with ν, can be made arbitrarily small with large ν.

Write $\exp : TM \to M$, and consider $d\exp : T(TM) \to TM$. At a fixed point $(p,\xi) \in TM$ we can take a tangent vector $(\zeta,\eta), \zeta \in T_pM, \eta \in T_pM$, and write

$$d\exp_{(p,\xi)}(\zeta,\eta) = D_1\exp_{(p,\xi)}(\zeta) + D_2\exp_{(p,\xi)}(\eta)$$

where $D_1\exp_{p,\xi}$ corresponds to varying the basepoint p while keeping all else fixed, and $D_2\exp_{p,\xi}$ corresponds to fixing the basepoint p and varying the tangent vector ξ. In particular $D_1\exp_{p,0}$ and $D_2\exp_{(p,0)}$ are the identity, so for small values of ξ they are invertible. So

$$\begin{aligned}\partial_s u_\nu &= \partial_s \exp_u^{R_\nu}\xi_\nu\\ &= (D_1\exp)_{(u^{R_\nu},\xi_\nu)}\partial_s u^{R_\nu} + (D_2\exp)_{(u^{R_\nu},\xi_\nu)}(\nabla_s\xi_\nu)\end{aligned}$$

which implies

$$(D_2\exp)^{-1}_{(u^{R_\nu},\xi_\nu)}(\partial_s u_\nu - (D_1\exp)_{(u^{R_\nu},\xi_\nu)}\partial_s u^{R_\nu}) = \nabla_s\xi_\nu. \tag{6.14}$$

Similarly,

$$(D_2\exp)^{-1}_{(u^{R_\nu},\xi_\nu)}(\partial_t u_\nu - (D_1\exp)_{(u^{R_\nu},\xi_\nu)}\partial_t u^{R_\nu}) = \nabla_t\xi_\nu. \tag{6.15}$$

First we analyze (6.14). The operators $(D_2\exp)^{-1}_{(u^{R_\nu},\xi_\nu)}$ and $(D_1\exp)_{(u^{R_\nu},\xi_\nu)}$ can be uniformly bounded for all $p \in M, \xi \in T_pM$ with $|\xi| < \delta$, so by Proposition B.0.11 we get on each strip $[R, R_\nu] \times [0,1]$ on either side of the neck,

$$\begin{aligned}|\nabla_s\xi_\nu| &\leq c_1|\partial_s u_\nu| + c_2|\partial_s u^{R_\nu}|\\ &\leq Ae^{-\kappa^2 s}\end{aligned}$$

for some constant $A > 0$, and therefore

$$\int_R^{R_\nu}\int_0^1 |\nabla_s\xi_\nu|^p\ ds\ dt \ \leq\ \frac{A}{p\kappa^2}(e^{-p\kappa^2 R} - e^{-p\kappa^2 R_\nu})$$

which can be made arbitrarily small by choosing R large enough. The same estimate holds by symmetry on the other side of the neck. From this we conclude that $\|\nabla_s \xi_\nu\|_{L^p}$ can be made arbitrarily small for large ν.

Now we analyze (6.15). We can write

$$\begin{aligned} \partial_t u_\nu &= J_t(u_\nu)(\partial_s u_\nu) + X_{H_t}(u_\nu) \\ \partial_t u^{R_\nu} &= J_t(u^{R_\nu})(\partial_s u^{R_\nu}) + X_{H_t}(u^{R_\nu}) + E_\nu(s,t) \end{aligned}$$

where $E_\nu(s,t)$ is an error term that is supported only on the compact interval $s \in [R_\nu/2, R_\nu/2+1]$ of each of the two strips making up the neck, with $|E_\nu(s,t)| \le \delta_\nu \to 0$ as $\nu \to \infty$. Together with (6.15) this yields a pointwise estimate

$$\begin{aligned} |\nabla_t \xi_\nu| &= |(D_2 \exp)^{-1}_{(u^{R_\nu},\xi_\nu)}(\partial_t u_\nu - (D_1 \exp)_{(u^{R_\nu},\xi_\nu)} \partial_t u^{R_\nu})| \\ &\le c|\partial_t u_\nu - (D_1 \exp)_{(u^{R_\nu},\xi_\nu)} \partial_t u^{R_\nu}| \\ &\le c_1|\partial_s u_\nu| + c_2|\partial_s u^{R_\nu}| \\ &\quad + c_3|X_{H_t}(u_\nu) - (D_1 \exp)_{(u^{R_\nu},\xi_\nu)} X_{H_t}(u^{R_\nu})| \\ &\quad + c_4|E_\nu(s,t)| \\ &\le c_5(|\partial_s u_\nu| + |\partial_s u^{R_\nu}| + \mathrm{dist}(u_\nu, u^{R_\nu}) + |E_\nu(s,t)|). \end{aligned}$$

From this, applying the estimates for $|\partial_s u_\nu|, |\partial_s u^{R_\nu}|$ and $\mathrm{dist}(u_\nu, u^{R_\nu})$ and $|E_\nu(s,t)|$ we get

$$\begin{aligned} \int_R^{R_\nu} \int_0^1 |\nabla_t \xi_\nu|^p \, ds \, dt &\le c_6 \int_R^{R_\nu} \int_0^1 \big(|\partial_s u_\nu|^p + |\partial_s u^{R_\nu}|^p \\ &\quad + \mathrm{dist}(u_\nu, u^{R_\nu})^p + |E_\nu(s,t)|^p\big) \, ds \, dt \\ &\le c_7 \int_R^{R_\nu} e^{-\kappa^2 s} \, ds + \int_{R_\nu/2-1}^{R_\nu/2} \delta_\nu^p \, ds \\ &\le c_8(e^{-\kappa^2 R} - e^{-\kappa^2 R_\nu}) + c_7 \delta_\nu^p, \end{aligned}$$

and it is clear that this can be made arbitrarily small by taking R large enough.

This provides a contradiction to (6.12). Hence, given $\delta > 0$, there is an $\epsilon > 0$ such that whenever $(r, \underline{u}) \in \mathcal{M}_{d,1}(\underline{x}_0, \dots, \underline{x}_d) \cap U_\epsilon$, there is a gluing length R such that $r = \exp_{r_R} \rho$, $\underline{u} = \exp_{\underline{u}_R} \xi$, with $|\rho| + \|\xi\|_{1,p} \le \delta$, and with $\rho_g = 0$. By (6.6), this implies that $(\rho, \xi) \in \mathrm{im}\, Q_R$. $\square$

Surjectivity for Type 2

Proposition 6.8.2. *Let* $(r_0, \underline{u}_0), (r_1, \underline{u}_1), \ldots, (r_k, \underline{u}_k)$ *be regular, and define* U_ϵ *as above. Given* $\delta > 0$*, there is an* $\epsilon > 0$ *such that the following holds. If* $(r, \underline{u}) \in U_\epsilon \cap \mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_d)^1$*, then there is a pre-glued curve* $(r_R, \underline{u}_R)$ *and a* $(\rho, \underline{\xi}) \in T_{r_R}\mathcal{R}^{d,1} \times \Omega^0(\mathcal{S}_{r_R}, \underline{u}_R^* TM)$ *such that* $\exp_{r_R} \rho = r$*,* $\exp_{\underline{u}_R} \xi = \underline{u}$*,* $|\rho| + \|\underline{\xi}\|_{1,p} < \delta$*, and* $(\rho, \xi) \in \operatorname{im} Q_R$*.*

Proof. The proof is so similar to the proof for Type 1 that, rather than repeat all the calculations, we will only outline the argument. As before we observe that for sufficiently small $\epsilon > 0$, if $\operatorname{dist}_{\mathcal{R}}(r, r_0 \#_0 \{r_1, \ldots, r_k\}) < \epsilon$, the local charts near the boundary provide a unique way of writing $r = \tilde{r}_0 \#_{\delta_r} \{\tilde{r}_1, \ldots, \tilde{r}_k\}$ with $0 \le \delta_r \le \epsilon$ and with each $\tilde{r}_i$ in an ϵ-neighborhood of r_i. So now suppose that there were a $\delta > 0$ and sequences $\epsilon_\nu \to 0$, $(r_\nu, \underline{u}_\nu) \in \mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_d)^1$, and $0 < \delta_\nu \le \epsilon_\nu$, $R_\nu = -\log \delta_\nu \to \infty$ such that $r_\nu = \tilde{r}_{0,\nu} \#_{\delta_\nu} \{\tilde{r}_{1,\nu}, \ldots, \tilde{r}_{k,\nu}\}$, and $\operatorname{dist}_{\underline{M}}(\underline{u}(z), \underline{u}_i(z)) \le \epsilon_\nu$ on all compact subsets of $\mathcal{S}_{r_i}$, but

$$\inf\{|\rho| + \|\xi\|_{1,p} \big| r_\nu = \exp_{R_\nu} \rho, \underline{u}_\nu = \exp_{\underline{u}_{R_\nu}} \xi\} \ge \delta. \tag{6.16}$$

Then the convergence would be uniform in all derivatives on those compact subsets, and we could choose the compact subsets to be large enough that their complements comprise the striplike ends and the k necks of the glued surfaces $\mathcal{S}_{r_0 \#_{\delta_\nu}\{r_1, \ldots, r_k\}} = \mathcal{S}_{r_{R_\nu}}$. However on these striplike ends and these necks, the energy of $\underline{u}_\nu$ must approach 0, and the same exponential decay arguments would imply that $\underline{u}_\nu = \exp_{\underline{u}_{R_\nu}} \xi_\nu$ for some $\xi_\nu \in \Omega^0(\mathcal{S}_{r_{R_\nu}}, \underline{u}_{R_\nu}^* T\underline{M})$ with $\|\xi_\nu\|_{1,p} \to 0$ as $\nu \to \infty$. Since $|\rho_\nu| \to 0$ also, we would get a contradiction to (6.16). Hence given $\delta > 0$ we could find an $\epsilon > 0$ such that $(r, \underline{u}) \in \mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_d)^1 \cap U_\epsilon$ could be written as $r = \exp_{r_R} \rho, \underline{u} = \exp_{\underline{u}_R} \xi$ for some preglued curve $(r_R, \underline{u}_R)$, with $|\rho| + \|\xi\|_{1,p} \le \delta$. Moreover, this ρ is such that $\rho_g = 0$, so it follows from (6.7) that $(\rho, \xi) \in \operatorname{im} Q_R$. □

Surjectivity for Type 3

Proposition 6.8.3. *Let* $(r_0, \underline{u}_0)$ *and* $\underline{v}$ *be regular. Given* $\delta > 0$*, there is an* $\epsilon > 0$ *such that the following holds. If* $(r, \underline{u}) \in U_\epsilon \cap \mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_d)^1$*, then* $(r, \underline{u})$ *is in the image of the gluing map.*

Proof. We will prove this in a slightly different way from the previous cases. We will argue that it suffices to prove the following:

Claim 1: *Given $R_1 >> 0$, and $\delta > 0$, there is an $\epsilon > 0$ such that the following holds. If $(r, \underline{u}) \in U_\epsilon \cap \mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_d)^1$, then there is an $R \geq R_1$, and a preglued curve $(r_R, \underline{u}_R)$ and a $(\rho, \underline{\xi}) \in T_{r_R}\mathcal{R}^{d,1} \times \Omega^0(\mathcal{S}_{r_R}, \underline{u}_R^* T\underline{M})$ such that $\exp_{r_R} \rho = r, \exp_{\underline{u}_R} \xi = \underline{u}$, and $|\rho| + \|\underline{\xi}\|_{1,p} < \delta$.*

To see how Claim 1 implies Proposition 6.8.3, the argument is as follows. From Section 6.7 we know that the image of the gluing map is contained in the one dimensional component of the moduli space of pseudoholomorphic quilted disks, $\mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_d)^1$. By the continuity of g, the image of $[R_0, \infty)$ is a connected component of this one-dimensional manifold. The implicit function theorem also tells us that in a local trivialization about a preglued curve $(r_R, \underline{u}_R)$, we get a local chart for $\mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_d)^1$. This chart contains $g(R)$, so the piece of the manifold $\mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_d)^1$ covered by the chart intersects the image of the gluing map. So we want to show that if R is sufficiently big, and δ is sufficiently small, then the piece of the one-manifold determined by the local chart about $(r_0, \underline{u}_R)$ is contained in the image of the gluing map.

The preglued curves $\underline{u}_R$ are defined on the same domain $\mathcal{S}_{r_0}$, but by construction any two of them will differ by a translation in the s direction sufficiently far along the striplike end Z_i. The magnitude of the distance between these translations depends on distances between points in $\underline{v}$, and the size of the difference in gluing lengths. Since $\underline{v}$ is non-constant we can chooose $\delta > 0$ small enough that for any R_1, there will eventually be an $R' > R_1$ such that the preglued curves $\underline{u}_R$ for $R \geq R'$ can not be written $\underline{u}_R = \exp_{\underline{u}_{R_1}} \xi$ with $\|\xi\|_{1,p} \leq \delta$. In particular, for $R \geq R'$ the preglued curves $(r_0, \underline{u}_R)$ are not in a δ-neighborhood of the local trivialization about $(r_0, \underline{u}_{R_1})$, and similarly $(r_0, \underline{u}_{R_1})$ is not in a δ-neighborhood of the local trivialization about $(r_0, \underline{u}_R)$. So considering the gluing map $g : [R_0, \infty) \to \mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_d)^1$, by making $\delta > 0$ smaller if necessary we can assume that the hypotheses of the implicit function theorem are satisfied. Then we can fix an $R_1 > R_0$ such that the respective δ-neighborhoods of $(r_0, \underline{u}_{R_1})$ and $(r_0, \underline{u}_{R_0})$

are disjoint. If we suppose, as in Claim 1, that $(r, \underline{u}) \in \mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_d)^1$ is such that $r = \exp_{r_0} \rho, \underline{u} = \exp_{\underline{u}_R} \xi$ for some $R \geq R_1$, and $|\rho| + \|\xi\|_{1,p} < \delta$, then $(r, \underline{u})$ is in the local chart around $(r_0, \underline{u}_R)$. But since $R \geq R_1$ we see that $g(R_0)$ is not in this chart, and we can choose an $R_2 >> R$ large enough that $g(R_2)$ is also not in that chart, but by the connectedness of the image of the gluing map this means that the whole chart is contained in the image of the gluing map.

Proof of Claim 1 For the sake of contradiction suppose that the assertion were false. Then there would be some $\delta > 0$, and sequences $\epsilon_\nu \to 0$, $R_\nu = -\log \epsilon_\nu \to \infty$, $\tau_\nu \geq 2R_\nu \to \infty$, and $(r_\nu, \underline{u}_\nu) \in \mathcal{M}_{d,1}(\underline{x}_0, \ldots, \underline{x}_d)^1$ such that

- $\mathrm{dist}_{\mathcal{R}}(r_\nu, r_0) < \epsilon_\nu$,
- $|E(\underline{u}_0) + E(\underline{v}) - E(\underline{u}_\nu)| < \epsilon_\nu$,
- $\mathrm{dist}_{\underline{M}}(\underline{u}_\nu(z), \underline{u}_0(z)) < \epsilon_\nu$ for all $z \in \mathcal{S}_{r_0}^{R_\nu}$, and
- $\mathrm{dist}(\underline{u}_\nu(s + \tau_\nu, t), \underline{v}_(s, t)) < \epsilon_\nu$ for all $s \in [-R_\nu, R_\nu]$,

and yet for every ν,

$$\inf\{|\rho| + \|\xi\|_{1,p} \big| r_\nu = \exp_{r_0}(\rho), \underline{u}_\nu = \exp_{\underline{u}_R}(\xi)\} \geq \delta. \tag{6.17}$$

For large ν, the condition $\exp_{r_0} \rho = r_\nu$ uniquely determines $\rho =: \rho_\nu$, and the convergence implies that $|\rho_\nu| \to 0$. So the quantity $|\rho|$ becomes insignificant in (6.17). We will arrive at a contradiction by showing that the norms $\|\underline{\xi}\|_{1,p}$ in (6.17) must also go to 0 for large ν. The assumptions show that $\underline{u}_\nu$ converges to $\underline{u}_0$ uniformly on compact subsets of $\mathcal{S}_{r_0}$, and since both are pseudoholomorphic the convergence is uniform in all derivatives. On the striplike end Z_i, $\underline{u}_\nu(s + \tau_n, t)$ converges uniformly on compact subsets of $\mathbb{R} \times [0, 1]$ to $\underline{v}(s, t)$; and since they are pseudoholomorphic curves the convergence is uniform in all derivatives. Moreover, the preglued curves $(r_0, \underline{u}_{\tau_\nu})$ converge in the same way. For each ν the energy of $\underline{u}_\nu$ restricted to the subsets $[R(\epsilon_\nu), \tau_n - R(\epsilon_\nu)] \times [0, 1]$ and $s \geq \tau_n + R(\epsilon_\nu)$ of the striplike end Z_i goes to zero. Thus, the proof reduces to the same calculations as done for Type 1. That is, the uniform estimates of convergence on those compact subsets of $\mathcal{S}_{r_0}$, combined with exponential decay estimates based on the vanishing energy of the strips in the complement of those compact subsets, show that for sufficiently large ν there is a unique section $\xi_\nu \in \Omega^0(\mathcal{S}_{r_0}, \underline{u}_{\tau_\nu})$ for which $\exp_{\underline{u}_{\tau_\nu}} \xi_\nu = \underline{u}_\nu$, and $\|\xi_\nu\|_{1,p} \to 0$, contradicting 6.17. $\square$

Appendix A

$W^{1,p}$ embeddings.

Here we collect some relevant $W^{1,p}$ embedding statements, following Adams [1]. They are needed to show that for each quilted surface S constructed in Chapter 4, there exists a constant c_p such that

$$\|f\|_{L^\infty(S)} \leq c_p(S)\|f\|_{W^{1,p}(S)}$$

and that for the families constructed in Chapter 4, there is a uniform bound $c_p(S) \leq c_0$.

Theorem A.0.4. *Let $S \subset \mathbb{R}^2$ be a compact Lipschitz domain. Let $u \in C^\infty(S)$. Then there is a constant c, depending only on p, such that*

$$\sup_S |u(s,t)| \leq c\|u\|_{W^{1,p}_{std}}.$$

Here the $W^{1,p}_{std}$ norm refers to the standard volume form $ds \wedge dt$ on $\mathbb{R}^2$.

For a general volume form dvol_S, we have the following consequence of Theorem A.0.4.

Corollary A.0.5. *Let $S \subset \mathbb{R}^2$ be a compact Lipschitz domain, and $\mathrm{dvol}_S = f(s,t)ds\wedge dt$ a volume form on S. Let $u \in C^\infty(S)$. Then there is a constant $c = c(p)$ (in fact it is the same constant as in Theorem A.0.4) such that*

$$\sup_S |u(s,t)| \leq \frac{c}{(f_{min})^{1/p}}\|u\|_{W^{1,p}(S)}$$

for all $z \in S$, where $\|\cdot\|_{W^{1,p}(S)}$ denotes the $W^{1,p}$ norm defined by the volume form dvol_S, as in (5.6), and $f_{min} = \min_S f(s,t)$.

Proof. Since $f(s,t)ds \wedge dt$ is a volume form, $f(s,t) > 0$ for all $(s,t) \in S$. Moreover, S is compact so f achieves a minimum $f_{min} > 0$. Hence, using the constant c of Theorem

A.0.4,

$$
\begin{aligned}
\sup_{(s,t)\in S} |u(s,t)| &\leq c\|u\|_{W^{1,p}} \\
&= c\left(\iint\limits_S |u|^p + |du|^p ds\wedge dt\right)^{1/p} \\
&= c\left(\iint\limits_S |u|^p + |du|^p \frac{f(s,t)}{f(s,t)} ds\wedge dt\right)^{1/p} \\
&\leq c\left(\iint\limits_S |u|^p + |du|^p \frac{f(s,t)}{f_{min}} ds\wedge dt\right)^{1/p} \\
&= \frac{c}{f_{min}}\left(\iint\limits_S |u|^p + |du|^p f(s,t) ds\wedge dt\right)^{1/p} \\
&= \frac{c}{f_{min}}\|u\|_{W^{1,p}(S)}.
\end{aligned}
$$

□

A similar result holds for unbounded domains $\mathbb{R}^2$ whose geometry satisfies a *cone condition*:

Definition A domain $\Omega \subset \mathbb{R}^2$ satisfies the *cone condition* if there is a finite cone $C = C(r_c, \theta_c)$ such that each $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C.

Theorem A.0.6. *Let $\Omega \subset \mathbb{R}^2$. Suppose that Ω satisfies the cone condition for some finite cone $C = C(r_c, \theta_c)$, and let $p > 2$. Then there is a constant $c = c(r_c, \theta_c, p) > 0$ such that for every $f \in C^\infty(S) \cap W^{1,p}(\Omega)$, and every $x \in \Omega$,*

$$|f(x)| \leq c\|f\|_{W^{1,p}}.$$

The theorem relies on the following Lemma.

Lemma A.0.7.

$$\frac{1}{\mathrm{vol}(C)} \iint\limits_{C_x(r_c,\theta_c)} |u(y)-u(x)|\mathrm{dvol}_y \leq \frac{1}{\theta_c} \iint\limits_{C_x(r_c,\theta_c)} \frac{|Du(y)|}{|x-y|}\ \mathrm{dvol}_y$$

where $\mathrm{vol}(C) = \frac{1}{2}\theta_c r_c^2$ *is the area of the cone, and* dvol_y *is the standard volume form on* $\mathbb{R}^2$*, with the subscript* y *to indicate integration with respect to* $y \in C_x(r_c, \theta_c)$*.*

Proof. Without loss of generality we can chose polar coordinates on $\mathbb{R}^2$ such that

$$C_x(r_c, \theta_c) = \{x + re^{i\theta} | 0 \le r \le r_c,\ 0 \le \theta \le \theta_c\}.$$

Then we have, for any $0 \le \theta \le \theta_c$, and $0 \le s \le r_c$,

$$\begin{aligned} |u(x + se^{i\theta}) - u(x)| &= \left| \int_0^s \frac{d}{dt}(u(x + te^{i\theta}))\ dt \right| \\ &= \left| \int_0^s \frac{d}{dt}(u(x + te^{i\theta}))\ dt \right| \\ &\le \int_0^s |Du(x + te^{i\theta})|\ dt. \end{aligned}$$

Integrating both sides with respect to θ in the cone C gives

$$\begin{aligned} \int_{\theta=0}^{\theta_c} |u(x + se^{i\theta}) - u(x)| d\theta &\le \int_{\theta=0}^{\theta_c} \int_{t=0}^{s} |Du(x + te^{i\theta})|\ dt\ d\theta \\ &= \int_{t=0}^{s} \int_{\theta=0}^{\theta_c} \frac{|Du(x + te^{i\theta})|}{t}\ t\ dt\ d\theta \\ (\text{putting } y = x + te^{i\theta}) \quad &= \iint_{C_x(s,\theta_c)} \frac{|Du(y)|}{|x - y|}\ \mathrm{dvol}_y \\ &\le \iint_{C_x(r_c,\theta_c)} \frac{|Du(y)|}{|x - y|}\ \mathrm{dvol}_y. \end{aligned}$$

Multiplying both sides by s and integrating over $0 \le s \le r_c$ gives the inequality

$$\begin{aligned} \iint_{C_x(r_c,\theta_c)} |u(y) - u(x)|\ \mathrm{dvol}_y &\le \left(\int_{s=0}^{r} s\ ds \right) \iint_{C_x(r_c,\theta_c)} \frac{|Du(y)|}{|x - y|}\ \mathrm{dvol}_y \\ &= \frac{r_c^2}{2} \iint_{C_x(r_c,\theta_c)} \frac{|Du(y)|}{|x - y|}\ \mathrm{dvol}_y. \end{aligned}$$

Dividing both sides by $\mathrm{vol}(C) = \frac{1}{2}\theta_c r^2$ gives

$$\frac{1}{\mathrm{vol}(C)} \iint_{C_x(r_c,\theta_c)} |u(y) - u(x)| \mathrm{dvol}_y \le \frac{1}{\theta_c} \iint_{C_x(r_c,\theta_c)} \frac{|Du(y)|}{|x - y|}\ \mathrm{dvol}_y.$$

□

Proof of Theorem. Fix $x \in \Omega$; we now write C_x for the cone. Then

$$
\begin{aligned}
|u(x)| &= \frac{1}{\mathrm{vol}(C)} \iint_{C_x} |u(x)| \mathrm{dvol}_y \\
&\leq \frac{1}{\mathrm{vol}(C)} \iint_{C_x} |u(y) - u(x)| \mathrm{dvol}_y + \frac{1}{\mathrm{vol}(C)} \iint_{C_x} |u(y)| \mathrm{dvol}_y \\
\text{by Lemma} &\leq \frac{1}{\theta_c} \iint_{C_x} \frac{|Du(y)|}{|x-y|} \mathrm{dvol}_y + \frac{1}{\mathrm{vol}(C)} \iint_{C_x} |u(y)| \mathrm{dvol}_y \\
\text{Hölder's inequality} &\leq \frac{1}{\theta_c} \left(\iint_{C_x} |Du(y)|^p \mathrm{dvol}_y \right)^{1/p} \left(\iint_{C_x} \frac{1}{|x-y|^q} \mathrm{dvol}_y \right)^{1/q} \\
&\quad + \frac{1}{\mathrm{vol}(C)} \left(\iint_{C_x} |u(y)|^p \mathrm{dvol}_y \right)^{1/p} \left(\iint_{C_x} \mathrm{dvol}_y \right)^{1/q} \\
&\leq \frac{1}{\theta_c} \|u\|_{W^{1,p}(\Omega)} \left(\int_0^{\theta_c} d\theta \int_{t=0}^{r_c} t^{1-q} \, dt \right)^{1/q} + \mathrm{vol}(C)^{1/q-1} \|u\|_{W^{1,p}(\Omega)} \\
&= (\theta_c^{1/q-1} \left(\frac{r_c^{2-q}}{2-q} \right)^{1/q} + \mathrm{vol}(C)^{1/q-1}) \|u\|_{W^{1,p}(\Omega)}
\end{aligned}
$$

where $q = p/(p-1) = 1 + 1/(p-1) < 2$ since $p > 2$. □

Theorem A.0.8. *Let $S = S_1 \cup \ldots \cup S_l$ be a surface defined by a union of open sets S_i, such that each S_i is one of the following types:*

1. *The closure of S_i is diffeomorphic to a compact Lipschitz domain $\widetilde{S}_i \subset \mathbb{R}^2$.*
2. *The closure of S_i is diffeomorphic to a domain $\widetilde{S}_i \subset \mathbb{R}^2$ that satisfies the cone condition, for some cone $C_i = (r_i, \theta_i)$,* and *the volume form on S restricted to S_i is the pull-back of the standard volume form on R^2.*

Then there is a constant c, depending on p, the cones in the cone condition, and the volume form dvol_S restricted to the S_i's of type (a), such that

$$\sup_S |u| \leq c \|u\|_{W^{1,p}(S)}$$

for all $u \in C^\infty(S) \cap W^{1,p}(S)$.

Proof. Let $\{\rho_i\}$ be a partition of unity subordinate to the cover $S_1 \cup \ldots \cup S_l$ of S. Then $u = \sum_{i=1}^{l} \rho_i u$, and each $\rho_i u \in C^\infty(S_i) \cap W^{1,p}(S_i)$. By Corollary A.0.5 and Theorem A.0.6, there is a constant c_i which depends on p and, in the case of the subsets of type (a), the volume form dvol_S restricted to those components, such that

$$\sup_S |\rho_i u| = \sup_{S_i} |\rho_i u| \leq c_i \|\rho_i u\|_{W^{1,p}(S_i)} = c_i \|\rho_i u\|_{W^{1,p}(S)} \leq c_i \|u\|_{W^{1,p}(S)}.$$

Hence,

$$\begin{aligned}
\sup_S |u| &= \sup_S |\sum_{i=1}^{l} \rho_i u| \\
&\leq \sup_S \sum_{i=1}^{l} |\rho_i u| \\
&\leq \sum_{i=1}^{l} \sup_S |\rho_i u| \\
&\leq \sum_{i=1}^{l} c_i \|u\|_{W^{1,p}(S)} \\
&= \left(\sum_{i=1}^{l} c_i \right) \|u\|_{W^{1,p}(S)} \\
&= c \|u\|_{W^{1,p}(S)}.
\end{aligned}$$

□

Appendix B

Exponential decay

Results on exponential decay for Floer trajectories with small energy follow from two things: a convexity estimate for the energy density of a trajectories in a sufficiently small neighborhood of a generalized intersection point, and a mean-value inequality that converts L^2 energy density estimates to pointwise estimates. The results we collect here are based on the convexity estimates in [13], and mean-value inequality in [18].

Without loss of generality, we only need to consider solutions to

$$\partial_s u + J_t \partial_t u = 0, \tag{B.1}$$
$$u(s,0) \subset L_0, u(s,1) \subset L_1,$$

where L_0 and L_1 are transversely intersecting Lagrangians. This is because a solution of the inhomogeneous equation,

$$\partial_s u + J_t(\partial_t u - X_{H_t}(u)) = 0 \tag{B.2}$$
$$u(s,0) \subset L_0, u(s,1) \subset L_1,$$

can be translated into a solution of type (B.1) by setting $\tilde{u}(s,t) = \phi_{1-t}(u(s,t))$, where ϕ_t is the time t flow of the Hamiltonian vector field X_{H_t}, and $\tilde{J}_t := (\phi_{1-t}^{-1})^* J_t$, which satisfies

$$\partial_s \tilde{u} + \tilde{J}_t \partial_t \tilde{u} = 0,$$
$$\tilde{u}(s,0) \subset \phi_1(L_0), \tilde{u}(s,1) \subset \phi_0(L_1) = L_1,$$

and by assumption the Hamiltonian perturbation is such that $\phi_1(L_0)$ intersects L_1 transversely.

Consider a solution $u : I \times [0,1] \to M$ of (B.1) where $I = [-T,T], [T,\infty)$ or $(-\infty,-T]$ for some $T > 0$. We assume that I is fixed. The *energy density* of u on this strip is defined to be

$$e(s,t) := \omega(\partial_s u(s,t), J_t \partial_s u(s,t)) = \|\partial_s u(s,t)\|^2_{J_t} = \|\partial_t u(s,t)\|^2_{J_t} \tag{B.3}$$

By [18, Lemma A.1], the energy density satisfies inequalities

$$\begin{aligned} \triangle e &\leq ae^2 \\ \frac{\partial e}{\partial \nu}\Big|_{\mathbb{H}} &\leq be^{3/2}, \end{aligned}$$

where $a \geq 0$ and $b \geq 0$ depend only on the data M, ω, J and Lagrangians L_0, L_1. Then by Theorem 1.3 of [18], we can choose δ to be small enough that there is a mean-value inequality

$$|e(s,t)| \leq \frac{C}{r^2} \int_{D_r(s,t)} e(s,t) ds\ dt \tag{B.4}$$

where $D_r(s,t) \subset I \times [0,1]$ is the partial disk of radius $r > 0$ centered at (s,t). Fix $r = \frac{1}{2}$, so that each partial disk is at most the intersection of a disk with a half-plane. (B.4) implies

$$\begin{aligned} |e(s,t)| &\leq C' \int_{D_r(s,t)} e(s,t) ds\ dt \\ &\leq C' E(u) \end{aligned}$$

so if $\delta > 0$ is chosen to be small enough, we have a uniform bound

$$\|\partial_t u\|^2_{J_t} = \|\partial_s u\|^2_{J_t} \leq C'\delta, \tag{B.5}$$

at least for points (s,t) where s is a distance at least $1/2$ from the boundary of the interval I. In particular, for a fixed value of s, the path $\gamma_s : [0,1] \to \underline{M}$ defined by $\gamma_s(t) := u(s,t)$ satisfies

$$\begin{aligned} \mathrm{dist}_M(\gamma_s(0), \gamma_s(1)) &\leq C_1 \int_0^1 \|\dot{\gamma_s}\|_{J_t}\ dt \\ &\leq C_1 (\int_0^1 \|\partial_t u\|^2_{J_t}\ dt)^{1/2} \\ &\leq C_1 (C'\delta)^{1/2}. \end{aligned} \tag{B.6}$$

Thus if $\delta > 0$ is small, each path γ_s lies entirely in a small neighborhood of an intersection point of L_0 and L_1. By the transversality of the intersection $L_0 \cap L_1$, and the compactness of M, intersection points are isolated in M so if δ is small enough, all paths γ_s are close to the *same* intersection point $p \in L_0 \cap L_1$ for all $s \in I_{1/2}$ where by $I_{1/2}$ we mean either $[-T+1/2, T-1/2], [T+1/2, \infty)$ or $(-\infty, -T-1/2]$.

Define $f : I_{1/2} \to \mathbb{R}$ by

$$f(s) = \frac{1}{2}\int_0^1 \|\partial_t u\|_{J_t}^2 \, dt. \tag{B.7}$$

It follows from [13] that given an intersection point $p \in L_0 \cap L_1$, there is a neighborhood U of p such that the the function $f(s)$ satisfies a convexity estimate

$$\ddot{f}(s) \geq \kappa^2 f(s) \tag{B.8}$$

for some $\kappa > 0$. Therefore, choose δ small enough that all paths γ_s are contained in such a neighborhood. Combining this convexity estimate with the mean value inequality again, we can prove the following standard exponential decay results for strips.

Proposition B.0.9. *Let $u : [0, \infty) \times [0, 1] \longrightarrow M$ be a solution of (B.1), such that $E(u) < \infty$. Then there exist constants $\kappa > 0$ and $A > 0$ such that*

$$|\partial_s u(s,t)| \leq A \; e^{-\kappa s}. \tag{B.9}$$

Proof. We can safely ignore a compact subset $[0, T] \times [0, 1]$ of the strip, which can be bounded by some fixed constant. So choosing sufficiently large T we can assume without loss of generality assume that the strip is of the form $[T, \infty) \times [0, 1]$, the energy of u restricted to this strip is less than δ for which the convexity estimate (B.8) holds. Note that the fact that $E(u) < \infty$ implies that $\partial_s u \to 0$ as $s \to \infty$, so in particular $f(s) = \|\partial_s u\|_{J_t}^2 \to 0$ as $s \to \infty$. Then for $s \geq T$, we have that

$$\ddot{f}(s) \geq \kappa^2 f(s).$$

This convexity estimate on $f(s)$ implies (explained, for instance, in [13]) an inequality $f(s) \leq ce^{-\kappa s}$ for some $c > 0$, i.e.,

$$f(s) = \int_0^1 \|\partial_s u(s,t)\|_{J_t}^2 \, dt \leq ce^{-\kappa s}.$$

By the mean-value inequality (B.4) applied to disks of radius 1/2 we get

$$\begin{aligned}
\|\partial_s u\|_{J_t}^2 &\leq C\int_{D_{1/2}(s,t)} \|\partial_s u\|_{J_t}^2 \; ds\; dt \\
&\leq C\int_{s-1/2}^{s+1/2} f(s)\; ds \\
&\leq Cce^{-\kappa(s-1/2)} \\
&= Ae^{-\kappa s}.
\end{aligned}$$

□

Proposition B.0.10. *There is a* $\delta > 0$ *so that the following holds. For any solution* $v : [-\rho,\rho]\times[0,1]\to M$ *of (B.1) with* $E(v) < \delta$*, there is a* $\kappa > 0$ *such that*

$$E(v;[-\rho+T,\rho-T]\times[0,1]) \leq e^{-\kappa T}E(v) \tag{B.10}$$

for all $1 \leq T \leq \rho/2$.

Proof. Take δ to small enough that $E(v) < \delta$ implies the *a priori* estimate (B.5) for all $-\rho+1 \leq s \leq \rho-1$, as well as the convexity estimate (B.8). Let us write

$$E(T) := E(v;[-\rho+T,\rho-T]\times[0,1]).$$

Then in terms of f, we have

$$E(T) = \int_{-\rho+T}^{\rho-T} f(s)ds$$

is a monotone decreasing function of T. Taking the derivative with respect to T,

$$E'(T) \;=\; -f(\rho-T)-f(-\rho+T).$$

So

$$\begin{aligned}
E''(T) &= \dot f(\rho-T)-\dot f(-\rho+T) \\
&= \int_{-\rho+T}^{\rho-T}\ddot f(s)ds \\
&\geq \triangle^2\int_{-\rho+T}^{\rho-T} f(s)ds \\
&= \kappa^2E(T).
\end{aligned}$$

Thus $e^{-\kappa T}(E'(T)+\kappa E(T))$ is monotone increasing. An inequality $E'(T)+\kappa E(T)>0$ would imply that

$$e^{-\kappa T}(E'(T)+\kappa E(T))>\alpha>0,$$

so that

$$\begin{aligned} E'(T)+\kappa E(T) &> e^{\kappa T}\alpha \\ \implies (e^{\kappa T}E(T))' &> e^{2\kappa T}\alpha \end{aligned}$$

which would imply that $E(T)$ grows exponentially, which is impossible since by construction $E(T)$ decreases with T. Hence

$$E'(T)+\kappa E(T)\le 0,$$

so $e^{\kappa T}E(T)$ is monotone decreasing, and

$$\begin{aligned} e^{\kappa T}E(T) &\le E(0)=E(v) \\ \implies E(T) &\le e^{-\kappa T}E(v). \end{aligned}$$

□

Applying the mean-value inequality we get the following corollary, which describes the behavior of long pseudo-holomorphic strips with small energy.

Corollary B.0.11. *There is a $\delta>0$ so that the following holds. For any solution $v:[-\rho,\rho]\times[0,1]\to M$ of (B.1) with $E(v)<\delta$, there is a $\kappa>0$ and $A>0$ which depend only on M,ω,J_t, L_0 and L_1, such that*

$$\|\partial_s v\|^2_{J_t}\le A\delta e^{-\kappa|s|} \tag{B.11}$$

for all $s\in[-\rho+1,\rho-1]$.

Proof. This is just an application of the mean-value inequality to the previous lemma.

By (B.4) using $r = 1/2$ we have, for each $s \in [-\rho + 1/2, \rho - 1/2]$,

$$\begin{aligned}
\|\partial_s u(s,t)\|^2_{J_t} &\leq 4C \int_{D_{1/2}(s,t)} \|\partial_s u\|^2_{J_t}\ ds\ dt \\
&\leq 4C \int_{-|s|+1/2}^{|s|-1/2} \int_0^1 \|\partial_s u\|^2_{J_t}\ ds\ dt \\
&= 4CE(|s|-1/2) \\
&\leq 4CE(v)e^{-\kappa(|s|-1/2)} \\
&\leq 4C\delta e^{\kappa/2} e^{-\kappa|s|} \\
&=: A\delta e^{-\kappa|s|}.
\end{aligned}$$

□

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