VERTEX OPERATOR ALGEBRAS AND INTEGRABLE SYSTEMS

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A thesis submitted to the

Graduate School—New Brunswick

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree of

Master of Science

Graduate Program in Physics and Astronomy

Written under the direction of

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and approved by

New Brunswick, New Jersey

January, 2009

ABSTRACT OF THE THESIS

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The goal of this thesis is to explicitly construct vertex operator algebras and their representations from classical integrable systems. We first construct a module for the corresponding affine Lie algebra of level 0 from the dual space of the space of functions on the solutions space of an integrable system, by applying the formal uniformization theorem of Barron, Huang and Lepowsky. Then we show that this module is in fact a module for the corresponding vertex operator algebra. We hope that our construction of modules for vertex operator algebras associated to affine Lie algebras will lead us to a better understanding of integrable systems in terms of the representation theory of vertex operator algebras.

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Chapter 1

Introduction

Integrable systems arise in many areas of sciences and also in mathematics itself. Classicial integrable systems were solved sucessfully using the method of inverse scattering methods by Zakharov, Shabat, Ablowitz, Kaup, Newell, and Segur (often referred as the *ZS-AKNS* method; see [BC] for the rigorous foundation). The KdV equation, the KP equation, the nonlinear Schrödinger equation and the Sine-Gordon equation are important examples of classical integrable systems. These equations have a number of common properties including the existence of Lax pairs, the reality property, and a construction of solutions using loop group actions (dressing actions; see [TU1] for a systematic treatment). The theory for *ZS-AKNS* method is an approach to classical integrable systems before the actions of infinite-dimensional Lie algebras were discovered.

The best way to study the loop actions on the spaces of solutions of classical integrable systems is perhaps in term of Lax pairs. Classical integrable systems are in fact obtained as consistency conditions for the Lax pairs. The spaces of solutions of integral systems are in fact certain quotient spaces of the spaces of solutions of the corresponding Lax pair equations. Given a solution E of the Lax pair equations of a classical integrable system, loop groups act on E and gives a new solution of the Lax pair equation and thus gives a new solution of the integrable system.

The goal of this thesis is to explicitly construct vertex operator algebras and their representations from classical integrable systems. We first construct a module for the corresponding affine Lie algebra of level 0 from the dual space of the space of functions on the solutions space of an integrable system, by applying the formal uniformization theorem of Barron, Huang and Lepowsky [BHL]. Then we show that this module is in fact a module for the corresponding vertex operator algebra. We hope that our construction of modules for vertex operator algebras associated to affine Lie algebras will lead us to a better understanding of integrable systems in terms of the representation theory of vertex operator algebras.

Chapter 2

A brief review of classical integral systems and the dressing action

In this section, we review the basic definitions, constructions and examples in the theory of classical integrable systems. The material in this section in this section is from [P], [TU1], [TU2].

2.1 G-hierarchy in integrable system

Let $\langle . \rangle$ be a non-degenerate, **ad**-invariant bilinear form on $\mathfrak{g}$, where $\mathfrak{g}$ is a semi-simple, complex Lie algebra. Fix $a \in \mathfrak{g}$ such that the centralizer $\mathfrak{g}_a$ of a in $\mathfrak{g}$ is a maximal abelian subalgebra $\mathfrak{A}$ of $\mathfrak{g}$. Let $\mathfrak{g}_a^\perp = \{\xi \in \mathfrak{g} | \langle \xi, \mathfrak{g}_a \rangle = 0\}$ and let $P = S(\mathbb{R}, \mathfrak{g}_a^\perp)$ be schwartz class maps from $\mathbb{R}$ into $\mathfrak{g}_a^\perp$. The elements of P are "potentials." It can be shown that given $b \in \mathfrak{A}$, there exists a sequence $\{Q_{b,j}(u) \mid j \in \mathbb{Z}_+\}$ of polynomial differential operators from P to $C^\infty(\mathbb{R}, \mathfrak{g}_a^\perp)$ [TU1]. determined uniquely by the recursive formula

$$(Q_{b,j}(u))_x + [u, Q_{b,j}(u)] = [Q_{b,j+1}(u), a], Q_{b,0} = b, Q_{a,1}(u) = u \tag{2.1.1}$$

The differential equation

$$u_t = (Q_{b,j})_x + [u, Q_{b,j}(u)] \tag{2.1.2}$$

is called the (b, j)-*flow*. The hierarchy of these (b, j)-flows is called the G-*hierarchy*.

The recursive formula (2.1.1) implies that u is a solution of the (b, j)-flow (2.1.2) if and only if the connection

$$\theta_\lambda = (a\lambda + u)dx + (b\lambda^j + Q_{b,1}\lambda^{j-1} + ... + Q_{b,j}(u))dt \tag{2.1.3}$$

is flat for all $\lambda \in \mathbb{C}$.

Remark 2.1.1. Note that $\mathbf{ad}(a)$ annihilates $\mathfrak{g}_a$ and leaves $\mathfrak{g}_a^\perp$ invariant.

Remark 2.1.2. For $A = a\lambda + u$, $B = b\lambda^j + Q_{b,1}\lambda^{j-1} + ... + Q_{b,j}(u)$, u being a solution of the (b,j)-flow (2.1.2) is also equivalent to any one of the follwing: (1) $[\frac{\partial}{\partial x} + A, \frac{\partial}{\partial t} + B] = 0$.
(2) $B_x - A_t + [A, B] = 0$.
(3) The system

$$\begin{cases} E_x = EA, \\ E_t = EB, \\ E(0,0,\lambda) = I, \end{cases}$$

has a solution. (E is called trivialization of θ_λ.)

2.2 $\mathcal{U}$-hierarchy

Let σ be an anti-linear involution of the Lie algebra $\mathfrak{g}$, and $\mathcal{U}$ the space of fixed points of σ in $\mathfrak{g}$. We call $\mathcal{U}$ the *real form* of $\mathfrak{g}$. If a, $b \in \mathcal{U}$, it can be shown that $Q_{b,j}(u) \in \mathcal{U}$. Thus we have:

1. By the recursive formula (2.1.1), the (b,j)-flow in the $\mathcal{U}$-hierarchy leaves $S(\mathbb{R}, \mathfrak{g}_a^\perp \cap \mathcal{U})$ invariant.

2. The Lax pair defined by (2.1.3) is a $\mathfrak{g}$-valued 1-form satisfying the $\mathcal{U}$-reality condition

$$\sigma(\theta_{\bar{\lambda}}) = \theta_\lambda \tag{2.2.4}$$

   Note that $\mathfrak{g}_a^\perp \cap \mathcal{U} = \mathcal{U}_a^\perp$.

The hierarchy of the restrictions of the (b,j)-flows in the G-hierarchy to $S(\mathbb{R}, \mathcal{U}_a^\perp)$ is called the *$\mathcal{U}$-hierarchy.*

Remark 2.2.1. We shall use the notation $P = S(\mathbb{R}, \mathcal{T}_a^\perp)$, where

$$\mathcal{T}_a^\perp = \begin{cases} \mathfrak{g}_a^\perp, & \text{for } G\text{-hierarchy}, \\ \mathcal{U}_a^\perp, & \text{for } \mathcal{U}\text{-hierarchy}. \end{cases}$$

Note that $\mathcal{T}_a$ is the maximal abelian subalgebra $\mathfrak{A}$ of $\mathfrak{g}$ or $\mathcal{U}$ in the case of G-hierarchy or $\mathcal{U}$-hierarchy, respectively.

The following theorem proved using the inverse scattering method enables us to solve the Cauchy problem for the (b,j)-flows:

Theorem 2.2.2. *There is an open dense subset P_o of $P = S(\mathbb{R}, \mathcal{T}_a^{\perp})$ such that if $u_0 \in P_o$, then the Cauchy problem for the (b,j)-flow*

$$\begin{cases} u_t = (Q_{b,j})_x + [u, Q_{b,j}(u)], \\ u(x,0) = u_0(x), \end{cases}$$

has a unique solution u. Moreover, $u(x,t)$ is defined for all $(x,t) \in \mathbb{R}^2$ and $u(.,t) \in S(\mathbb{R}, \mathcal{T}_a^{\perp})$

2.3 Example: $SL(2,\mathbb{C})$-hierarchy

Let G be the Lie group $SL(2,\mathbb{C})$,

$$a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and $b = a$. Then $\mathfrak{sl}(2,\mathbb{C})_a = \mathfrak{A} = \mathbb{C}a$ and

$$\mathfrak{sl}(2,\mathbb{C})_a^{\perp} = \left\{ \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \,\middle|\, q, r \in \mathbb{C} \right\}.$$

So

$$Q_{a,1}(u) = u = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$$

To get the (a,j)-flows for $j > 1$, we proceed as follows: Write

$$Q_{a,j}(u) = T_{b,j}(u) + P_{b,j}(u) \in \mathfrak{sl}(2,\mathbb{C})_a + \mathfrak{sl}(2,\mathbb{C})_a^{\perp}$$

where $T_{b,j}(u)$ and $P_{b,j}(u)$ are in fact the diagonal and off diagonal parts of $Q_{a,j}(u)$, respectively. By Remark 2.1.1, $\mathbf{ad}(a)$ annihilates $T_{b,j}(u)$ and is an isomorphisom on $P_{b,j}(u)$. Indeed, for any

$$\begin{pmatrix} 0 & w \\ v & 0 \end{pmatrix} \in P_{b,j}(u),$$

$$\mathbf{ad}(a)\begin{pmatrix}0 & w\\ v & 0\end{pmatrix} = 2i\begin{pmatrix}0 & w\\ -v & 0\end{pmatrix}.$$

So

$$\mathbf{ad}(a)^{-1}\begin{pmatrix}0 & w\\ v & 0\end{pmatrix} = \frac{i}{2}\begin{pmatrix}0 & -w\\ v & 0\end{pmatrix} \tag{2.3.5}$$

We also have

$$[a, Q_{b,j+1}(u)] = \mathbf{ad}(a)P_{b,j+1}(u)$$

Then by the recursive formula (2.1.1), we get

$$P_{b,j+1}(u) = -\mathbf{ad}(a)^{-1}((P_{b,j})_x + \pi_1([u, Q_{b,j}])), \tag{2.3.6}$$

$$(T_{b,j+1})_x = -\pi_0([u, P_{b,j}]), \tag{2.3.7}$$

where π_0 and π_1 are projections from $\mathfrak{sl}(2,\mathbb{C})$ onto $\mathfrak{sl}(2,\mathbb{C})_a$ and $\mathfrak{sl}(2,\mathbb{C})_a^{\perp}$, respectively. Using (2.3.5), (2.3.6), (2.3.7), we find

$$Q_{a,2} = \begin{pmatrix} i\frac{qr}{2} & i\frac{q_x}{2}\\ -i\frac{r_x}{2} & -i\frac{qr}{2}\end{pmatrix}$$

and

$$Q_{a,3} = \begin{pmatrix} \frac{1}{4}(qr_x - q_x r) & -\frac{1}{4}(q_{xx} - 2q^2 r)\\ -\frac{1}{4}(r_{xx} - 2qr^2) & -\frac{1}{4}(qr_x - q_x r)\end{pmatrix}.$$

So the first three flows in the $SL(2)$-hierarchy are

$$\begin{aligned}
&q_t = q_x,\ r_t = r_x,\\
&q_t = \frac{i}{2}(q_{xx} - 2q^2 r),\ r_t = -\frac{i}{2}(r_{xx} - 2qr^2),\\
&q_t = -\frac{1}{4}(q_{xxx} - 6qrq_x),\ r_t = -\frac{1}{4}(r_{xxx} - 6qrr_x)
\end{aligned}$$

2.4 Example: $\mathfrak{su}(2)$-hierarchy

Let τ be the involution of $\mathfrak{sl}(2,\mathbb{C})$ defined by $\tau(\xi) = -\bar{\xi}^t$. Then the space $\mathcal{U}$ of the fixed points of τ is $\mathfrak{su}(2,\mathbb{C})$. Let

$$a = \begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix} \in \mathcal{U}.$$

Then $\mathfrak{g}_a = \mathfrak{A} = \mathbb{C}a$ and

$$\mathfrak{g}_a^{\perp} \cap \mathcal{U} = \left\{ \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \Bigm| q \in \mathbb{C} \right\}$$

So the $\mathfrak{su}(2)$-hierarchy is the restriction of the $SL(2,\mathbb{C})$-hierarchy to the subspace given by $r = -\bar{q}$. The first three flows in the $\mathfrak{su}(2)$-hierarchy are

$$\begin{aligned} q_t &= q_x, \\ q_t &= \frac{i}{2}(q_{xx} + 2|q^2|q), \\ q_t &= -\frac{1}{4}(q_{xxx} + 6|q|^2 q). \end{aligned}$$

2.5 Factorizations and Dressing actions

Let G be a Lie group and G^+, G^- be subgroups of G. Assume that the maps

$$G^+ \times G^- \longrightarrow G : (g_+, g_-) \longrightarrow g_+ g_-,$$

$$G^- \times G^+ \longrightarrow G : (g_-, g_+) \longrightarrow g_- g_+$$

given by the multiplication are bijections. Then we have the following unique factorization: Given any $g \in G$, there exist unique g_+, $h_+ \in G_+$ and g_-, $h_- \in G_-$ such that

$$g = g_+ g_- = h_- h_+$$

The dressing action of G_+ on G_- is defined as follows: For $g_\pm \in G_\pm$, factor $g_+ g_-$ as $\tilde{g}_- \tilde{g}_+$ with $\tilde{g}_\pm \in G_\pm$. Then the dressing action of g_+ on g_- is given by $g_+ * g_- = \tilde{g_-}$. The dressing action of G_- on G_+ can be defined similarly.

2.6 Factorizations of loop groups

Let G be a complex, semi-simple Lie group and σ an involution of G. Given an open subset $\mathcal{O}$ of $S^2 = \mathbb{C} \cup \infty$, let $\mathrm{Hol}(\mathcal{O}, G)$ be the group of all holomorphic maps $\mathcal{O}$ to G with multiplication defined by $(fg)(\lambda) = f(\lambda)g(\lambda)$ for $f, g \in \mathrm{Hol}(\mathcal{O}, G)$. For $\epsilon > 0$, let

$$\mathcal{O}_\infty^\epsilon = \left\{ \lambda \in S^2 \mid |\lambda| > \frac{1}{\epsilon} \right\}, \tag{2.6.8}$$

$$\mathcal{O}_0^\epsilon = \{ \lambda \in \mathbb{C} \mid |\lambda| < \epsilon \} \tag{2.6.9}$$

Let e be the identity of G. We consider the following groups:

$$\Lambda(G) = \mathrm{Hol}(\mathbb{C} \cap \mathcal{O}_\infty^{\frac{1}{\epsilon}}, G), \tag{2.6.10}$$

$$\Lambda_+(G) = \mathrm{Hol}(\mathbb{C}, G), \tag{2.6.11}$$

$$\Lambda_-(G) = \{f \in \mathrm{Hol}(\mathcal{O}_\infty^{\frac{1}{\epsilon}}, G) \mid f(\infty) = e\}, \tag{2.6.12}$$

$$\Lambda^\sigma(G) = \{f \in \Lambda(G) \mid \sigma(f(\bar{\lambda})) = f(\lambda)\}, \tag{2.6.13}$$

$$\Lambda_+^\sigma(G) = \Lambda_+(G) \cap \Lambda^\sigma(G), \tag{2.6.14}$$

$$\Lambda_-^\sigma(G) = \Lambda_-(G) \cap \Lambda^\sigma(G). \tag{2.6.15}$$

Then we have the following theorem:

Theorem 2.6.1 (Birkhoff Factorization Theorem [TU1])**.** *The maps*

$$\Lambda_+^\sigma(G) \times \Lambda_-^\sigma(G) \longrightarrow \Lambda^\sigma(G)$$

and

$$\Lambda_-^\sigma(G) \times \Lambda_+^\sigma(G) \longrightarrow \Lambda^\sigma(G)$$

obtained from the multiplication map are injective and the images are open and dense. In particular, there exists an open dense subset $\Lambda_o^\sigma(G)$ of $\Lambda^\sigma(G)$ such that given $g \in \Lambda_o^\sigma(G)$, g can be factored uniquely as $g = g_+g_- = h_-h_+$ with g_+, $h_+ \in \Lambda_+^\sigma(G)$ and g_-, $h_- \in \Lambda_-^\sigma(G)$

2.7 The dressing action on $\mathcal{U}$-hierarchy

The action of $\Lambda_-^\sigma(G)$

Let u be a solution of the (b, j)-th flow. Then by Remark 2.1.2, $E(x,t)(\lambda) = E(x,t,\lambda)$ is holomorphic in $\lambda \in \mathbb{C}$, i.e. $E(x,t) \in \Lambda_+(G)$ for all (x,t). Since the corresponding θ_λ satisfies the $\mathcal{U}$-reality condition (2.2.4), $E(x,t)$ satisfies

$$\sigma(E(x,t)(\bar{\lambda})) = E(x,t)(\lambda).$$

So $E(x,t) \in \Lambda_+^\sigma(G)$. We need the following theorem:

Theorem 2.7.1 ([TU1]). *Let $g \in \Lambda_-^\sigma(G)$, u a solution of the (b,j)-th flow in the $\mathcal{U}$-hierarchy, and E the trivialization of θ_λ as in Remark 2.1.2. Then there is an open subset $\mathcal{O}$ of (0,0) in $\mathbb{R}^2$ such that for all $(x,t) \in \mathcal{O}$ there exist $\tilde{E}(x,t) \in \Lambda_+^\sigma(G)$ and $\tilde{g}(x,t) \in \Lambda_-^\sigma(G)$ such that*
(i) $gE(x,t) = \tilde{E}(x,t)\tilde{g}(x,t)$.
(ii) $(\tilde{E}^{-1}(x,t)\tilde{E}_x(x,t))(\lambda)$ is of the form $a\lambda + \tilde{u}(x,t)$ and $\tilde{u} = u + [a, \tilde{g}_1]$, where $\tilde{g}_1$ is the coefficient of λ^{-1} in the expansion of $\tilde{g}^{-1}$ as

$$\tilde{g}^{-1}(x,t)(\lambda) = I + \tilde{g}_1(x,t)\lambda^{-1} + \tilde{g}_2(x,t)\lambda^{-2} + ...$$

(iii) $\tilde{u}$ is a solution of the (b,j)-flow in the $\mathcal{U}$-hierarchy.

This theorem gives a method of finding new solutions of the $\mathcal{U}$-hierarchy from known solutions. Given a solution $u \in C_0^\infty(\mathcal{O}, \mathcal{T}_a^\perp)$, we can solve for $E(x,t) \in \Lambda_+^\sigma(G)$, and then use the dressing action of g on $E(x,t)$ to produce $\tilde{E} \in \Lambda_+^\sigma(G)$ and obtain a new solution $\tilde{u}$ of the (b,j)- flow in the $\mathcal{U}$-hierarchy.

Remark 2.7.2. By (i), we have

$$g * E = \tilde{E}.$$

We shall denote $\tilde{u}$ by $g * u = \tilde{u}$. So $*$ is an action of $\Lambda_-^\sigma(G)$ on the space of solutions of the $\mathcal{U}$-hierarchy.

Chapter 3

Actions of central extensions of the full loop group

The material in this section is from [CH]. We already have an action $*$ of $\Lambda_-^\sigma(G)$ on the space of (local) solutions of the integrable system. It is natural to ask whether the action can be extended to an action of the full loop group. In this section, we extend this action to actions of a subgroup of the full loop group containing enough elements in $\Lambda_+^\sigma(G)$.

From the Birkhoff factorization theorem (Theorem 2.6.1), we know that there is an open dense subset $\Lambda_o^\sigma(G)$ of $\Lambda^\sigma(G)$ such that any element in $\Lambda_o^\sigma(G)$ can be decomposed uiniquely as g_+g_- for $g_+ \in \Lambda_+^\sigma(G)$ and $g_- \in \Lambda_-^\sigma(G)$. and as h_-h_+ for $h_- \in \Lambda_-^\sigma(G)$ and $h_+ \in \Lambda_+^\sigma(G)$.

Given $g_- \in \Lambda_-^\sigma(G)$, we define an action, also denoted as $*$, of an open dense subset of $\Lambda_+^\sigma(G)$ on g_- as follows: Consider the map $\ell_{g_-} : \Lambda_+^\sigma(G) \to \Lambda^\sigma(G)$ defined by $\ell(g_+) = g_+g_-$ for $g_+ \in \Lambda_+^\sigma(G)$. Since ℓ_{g_-} is a continuous map, the inverse image $\Lambda_o^\sigma(G)_+$ of the open dense subset $\Lambda_o^\sigma(G)$ must be an open dense subset of $\Lambda_+^\sigma(G)$. For any $g_+ \in \Lambda_o^\sigma(G)_+$, $g_+g_- \in \Lambda_o^\sigma(G)$. So there exist unique $h_- \in \Lambda_-^\sigma(G)$, $h_+ \in \Lambda_+^\sigma(G)$ such that $g_+g_- = h_-h_+$. We define $g_+ * g_- = h_-$.

We now define an action, also denoted as $*$, of $\Lambda_o^\sigma(G)$ on $\Lambda_-^\sigma(G)$ as follows: Given any $g \in \Lambda_o^\sigma(G)$, there exit unique $h_- \in\in \Lambda_-^\sigma(G)$, $h_+ \in \Lambda_+^\sigma(G)$ such that $g = h_-h_+$. For any $g_- \in \Lambda_-^\sigma(G)$, we define $g * g_- = h_-(h_+ * g_-)$.

Proposition 3.0.3. *This indeed defines an action of $O^\sigma(G)$ on $\Lambda_-^\sigma(G)$.*

Proof. The action of $\Lambda_o^\sigma(G)_+$ on $\Lambda_-^\sigma(G)$ just defined is nothing but the dressing action.

Let $g^{(1)}, g^{(2)} \in \Lambda_o^\sigma(G)$. Then there exist $h_-^{(1)}, h_-^{(2)} \in \Lambda_-^\sigma(G)$, $h_+^{(1)}, h_+^{(2)} \in \Lambda_+^\sigma(G)$ such that $g^{(i)} = h_-^{(i)} h_+^{(i)}$ for $i = 1, 2$. Let $g_- \in \Lambda_-^\sigma(G)$. By Theorem 2.6.1,

$$\begin{aligned} h_+^{(2)} g_- &= h_-^{(20)} h_+^{(20)}, \\ h_+^{(1)} h_-^{(2)} &= h_-^{(12)} h_+^{(12)}, \\ h_+^{(12)} h_-^{(20)} &= h_-^{(122)} h_+^{(122)}. \end{aligned}$$

Then we have

$$\begin{aligned} h_+^{(12)} h_+^{(2)} g_- &= h_+^{(12)} h_-^{(20)} h_+^{(20)} \\ &= h_-^{(122)} h_+^{(122)} h_+^{(20)} \end{aligned}$$

and

$$\begin{aligned} h_+^{(1)} h_-^{(2)} h_-^{(2)} &= h_-^{(12)} h_+^{(12)} h_-^{(20)} \\ &= h_-^{(12)} h_-^{(122)} h_+^{(122)}. \end{aligned}$$

Using all these formulas, we obtain

$$\begin{aligned} (g^{(1)} g^{(2)}) * g_- &= (h_-^{(1)} h_+^{(1)} h_-^{(2)} h_+^{(2)}) * g_- \\ &= (h_-^{(1)} h_-^{(12)} h_+^{(12)} h_+^{(2)}) * g_- \\ &= (h_-^{(1)} h_-^{(12)})((h_+^{(12)} h_+^{(2)}) * g_-) \\ &= h_-^{(1)} h_-^{(12)} h_-^{(122)} \\ &= h_-^{(1)} (h_-^{(12)} h_-^{(122)}) \\ &= h_-^{(1)} (h_+^{(1)} * (h_-^{(2)} h_-^{(20)})) \\ &= (h_-^{(1)} h_+^{(1)}) * (h_-^{(2)} h_-^{(20)}) \\ &= (h_-^{(1)} h_+^{(1)}) * (h_-^{(2)} (h_+^{(2)} * g_-)) \\ &= (h_-^{(1)} h_+^{(1)}) * ((h_-^{(2)} h_+^{(2)}) * g_-) \\ &= g^{(1)} * (g^{(2)} * g_-), \end{aligned}$$

proving that we do have an action of the group $\Lambda_o^\sigma(G)$. □

We now consider a central extension $\widetilde{L}(G)$ of the loop group $L(G)$ of central charge c. This central extention gives central extensions $\widetilde{\Lambda}(G)$ and $\widetilde{\Lambda}^\sigma(G)$ of the groups $\Lambda(G)$

and $\Lambda^\sigma(G)$, respectively. In particular, we have a homomorphism from $\widetilde{\Lambda}^\sigma(G)$ to $\Lambda^\sigma(G)$ whose kernel is the center of $\widetilde{\Lambda}^\sigma(G)$. Moreover, $\Lambda_-(G)$, $\Lambda_+(G)$, $\Lambda^\sigma_-(G)$ and $\Lambda^\sigma_+(G)$ can actually be embedded as subgroups $\widetilde{\Lambda}_-(G)$, $\widetilde{\Lambda}_+(G)$, $\widetilde{\Lambda}^\sigma_-(G)$ and $\widetilde{\Lambda}^\sigma_+(G)$ of $\Lambda(G)$ and $\Lambda^\sigma(G)$, respectively. From Proposition 3.0.3, we obtain:

Corollary 3.0.4. *The composition of the homomorphism from $\widetilde{\Lambda}^\sigma(G)$ to $\Lambda^\sigma(G)$ and the action of $\Lambda^\sigma_o(G)$ on $\Lambda^\sigma_-(G)$ gives an action of the preimage $\widetilde{\Lambda}^\sigma_o(G)$ of $\Lambda^\sigma_o(G)$ in $\widetilde{\Lambda}^\sigma(G)$ on $\Lambda^\sigma_-(G)$.*

Let u_0 be a solution and $\mathcal{S}_{u_0}$ the orbit containing u_0 under the action of $\Lambda^\sigma_-(G)$. Let $(\Lambda^\sigma_-(G))_{u_0}$ be the isotropy group at u_0 of the action of $\Lambda^\sigma_-(G)$ on the solution space. Then $\mathcal{S}_{u_0}$ can be identified canonically with $\Lambda^\sigma_-(G)/(\Lambda^\sigma_-(G))_{u_0}$. Let $\rho : \Lambda^\sigma_-(G)/(\Lambda^\sigma_-(G))_{u_0} \to \mathcal{S}_{u_0}$ be the inverse of this canonical identification and $\pi : \Lambda^\sigma_-(G) \to \Lambda^\sigma_-(G)/(\Lambda^\sigma_-(G))_{u_0}$ be the projection. By Corollary 3.0.4, $\widetilde{\Lambda}^\sigma_o(G)$ acts on $\Lambda^\sigma_-(G)$. In particular, we obtain the following:

Theorem 3.0.5. *Given any element $u \in \mathcal{S}_{u_0}$, choose an element h_- of $\Lambda^\sigma_-(G)$ such that $\rho(\pi(h_-)) = u$. Then for any $\tilde{g} \in \widetilde{O}^\sigma(G)$, $\rho(\pi(\tilde{g} * h_-)) \in \mathcal{S}_{u_0}$. In particular, $\rho(\pi(\tilde{g} * h_-))$ is a solution of the $\mathcal{U}$-hierarchy.*

Remark 3.0.6. Note that the solution $\rho(\pi(\tilde{g} * h_-))$ depends not only on u and $\tilde{g}$ but also on h_-. Thus this does not give an action of $\widetilde{O}^\sigma(G)$ on $\mathcal{S}_{u_0}$.

We have the following result:

Proposition 3.0.7. *For any solution u_0 of the integrable system, the isotropy subgroup $(\Lambda^\sigma_-(G))_{u_0}$ contains the intersection $\tilde{T}^\sigma_-(G) = \tilde{\Lambda}^\sigma_-(G) \cap \tilde{\Lambda}^\sigma(T)$ of the central extension of the group $\Lambda^\sigma_-(G)$ and the central extension of the loop group of a maximal abelian subgroup T of G.*

Proof. This follows easily from the construction of the action of $\Lambda^\sigma_-(G)$ on the solution space in [TU1] and [TU2]. $\square$

Chapter 4

A module for the affine Lie algebra $\hat{\mathcal{U}}$ and for the corresponding vertex operator algebra

In this section, we use the method developed in [H] and the action obtained in the preceding section to construct a lower truncated module for the affine Lie algebra $\hat{\mathcal{U}}$ and the corresponding vertex operator algebra. The material in this section is also from [CH]. For details, the reader is referred to [CH].

We take a basis $h^{(a)}$, $a = 1, \ldots, \dim\mathcal{U}$ of the Lie algebra $\mathcal{U}$. Any element of $\Lambda^{\sigma}_{-}(G)$ near the identity can be written using the exponential map as

$$\exp\left(\sum_{i\in -\mathbb{Z}_+}\sum_{a=1}^{\dim\mathcal{U}} D_i^{(a)} h^{(a)} \otimes t^i\right),$$

where $D_i^{(a)} \in \mathbb{C}$. Let $X_i^{(a)} = h^{(a)} \otimes t^i$, then we have

$$[X_i^{(b)}, X_j^{(c}] = [h^{(b)}, h^{(c)}] \otimes t^{i+j}$$

$i \in \mathbb{Z},\ a = 1, \ldots, \dim\mathcal{U}$.

Let $D(\mathcal{S}_{u_0})$ be the space of functions on $\mathcal{S}_{u_0}$ such that for $F \in D(\mathcal{S}_{u_0})$,

$$F\left(\left(\exp\left(\sum_{i\in -\mathbb{Z}_+}\sum_{a=1}^{\dim\mathcal{U}} D_i^{(a)} X_i^{(a)}\right)\right) * u_0\right)$$

is a polynomial in $D_i^{(a)}$.

Let W_{u_0} be the subspace of $(D(\mathcal{S}_{u_0}))^*$ spanned by elements of the form

$$F \mapsto \frac{\partial}{\partial A_{i_1}^{(a_1)}} \cdots \frac{\partial}{\partial A_{i_n}^{(a_m)}} F\left(\left(\exp\left(\sum_{j\in -\mathbb{Z}_+}\sum_{b=1}^{\dim\mathcal{U}} A_j^{(b)} X_j^{(b)}\right)\right) * u_0\right)\Bigg|_{A_j^{(b)}=0}$$

for $a_1, \cdots, a_m = 1, \ldots, \dim\,\mathcal{U}$ and $i_1, \ldots, i_n \in -\mathbb{Z}_+$. Define the weight to be $-(a_1 i_1 + \cdots + a_m i_n)$, then $W(u_0)$ is $\mathbb{N}$-graded. We now give a structure of a module for the affine Lie algebra of level 0.

For $i \in \mathbb{Z}$, $a = 1, \ldots, \dim \mathcal{U}$ and $F \in D(\mathcal{S}_{u_0})$, let $\mathcal{T}^{(a)}(i)F \in D(\mathcal{S}_{u_0})$ be defined by

$$(\mathcal{T}^{(a)}(i)F)(u) = \frac{\partial}{\partial A_{\pm i}^{(a)}} F\left(\left(\exp\left(\sum_{j \in \mathrm{sgn}(i)\mathbb{Z}_+} \sum_{b=1}^{\dim \mathcal{U}} A_j^{(b)} X_j^{(b)}\right)\right) * u\right)\Bigg|_{A_i^{(b)}=0} \tag{4.0.1}$$

for $u \in \mathcal{S}_{u_0}$, where $\mathrm{sgn}(i)$ is 1, 0 or -1 if i is positive, 0 or negative. Given $f \in (D(\mathcal{S}_{u_0}))^*$, we define $T^{(a)}(i)f \in (D(\mathcal{S}_{u_0}))^*$ for $i \in \mathbb{Z}$ and $a = 1, \ldots, \dim \mathcal{U}$ by

$$(T^{(a)}(i)f)(F) = f(\mathcal{T}^{(a)}(i)F)$$

for $F \in D(\mathcal{S}_{u_0})$.

Theorem 4.0.8. *For $i \in \mathbb{Z}$ and $a = 1, \ldots, \dim \mathcal{U}$, we have $T^{(a)}(i)W_{u_0} \subset W_{u_0}$. Moreover, $X_i^{(a)} \mapsto T^{(a)}(i)$ give a representation the affine Lie algebra of $\mathcal{U}$ of level 0 on W_{u_0}.*

Sketch of the proof The proof is similar to the proof in [H] for the representation of the Virasoro algebra, except that instead of the results in Chapters 3 and 4 in [H], we have to use the Birkhoff factorization theorem (Theorem 2.6.1) and the formal uniformization theorem of Barron, Huang and Lepowsky in [BHL] in the case of affine Lie algebras.

The first part of the theorem can be proved using the same method as in citeH. For the seond part, let f_{ab}^c for $a, b, c = 1, \ldots, \dim \mathcal{U}$ be the structure constants of the Lie algebra $\mathcal{U}$ in the basis $h^{(a)}$, $a = 1, \ldots, \dim \mathcal{U}$. Then we want to prove

$$[T^{(a)}(i), T^{(b)}(j)] = \sum_{c=1}^{\dim \mathcal{U}} f_{ab}^c T^{(c)}(i+j).$$

In the case that $i, j \in \mathbb{N}$, this can be proved easily by using the definition and the Campbell-Baker-Hausdorff formula. In the cases of $i \in \mathbb{Z}_+$, $j \in -\mathbb{Z}_+$, or $i \in -\mathbb{Z}_+$, $j \in \mathbb{Z}_+$ or $i = 0$, $j \in -\mathbb{Z}_+$ or $i \in -\mathbb{Z}_+$, $j = 0$, the Campbell-Baker-Hausdorff formula cannot be used. Instead, we have to use a formula proved in [BHL]. We discuss only the case $i \in \mathbb{Z}_+$, $j \in -\mathbb{Z}_+$ and $i + j \neq 0$; the other cases are similar. From Application

4.2 in [BHL], we have

$$\begin{aligned}
&\exp\left(\sum_{i\in\mathbb{Z}_+}\sum_{a=1}^{\dim\mathcal{U}} A_i^{(a)}X_i^{(a)}\right)\exp\left(\sum_{i\in-\mathbb{Z}_+}\sum_{a=1}^{\dim\mathcal{U}} B_i^{(a)}X_i^{(a)}\right)\\
&\quad=\exp\left(\sum_{i\in-\mathbb{Z}_+}\sum_{a=1}^{\dim\mathcal{U}} C_i^{(a)}(A,B)X_i^{(a)}\right)\exp\left(\sum_{i\in\mathbb{Z}_+}\sum_{a=1}^{\dim\mathcal{U}} D_i^{(a)}(A,B)X_i^{(a)}\right)\cdot\\
&\qquad\cdot\exp\left(\sum_{a=1}^{\dim\mathcal{U}} E_0^{(a)}(A,B)X_0^{(a)}\right),
\end{aligned}$$

where $C_i^{(a)}(A,B)$, $D_i^{(a)}(A,B)$ and $E_0^{(a)}(A,B)$ are series in the variables $A_i^{(a)}$ and $B_i^{(a)}$ whose first few terms are given explicitly by

$$\begin{aligned}
C_i^{(a)}(A,B) &= \sum_{i\in\mathbb{Z}_+}\sum_{a=1}^{\dim\mathcal{U}} B_i^{(a)}X_i^{(a)}\\
&\quad+\sum_{\substack{i\in Z_+,k\in\mathbb{Z}_+\\ i+j<0}}\sum_{c=1}^{\dim\mathcal{U}} f_{bc}^a A_i^{(b)}B_j^{(c)}X_{i+j}^{(a)}+P^-(A,B),\\
D_i^{(a)}(A,B) &= \sum_{i\in\mathbb{Z}_+}\sum_{a=1}^{\dim\mathcal{U}} A_i^{(a)}X_i^{(a)}\\
&\quad+\sum_{\substack{i\in Z_+,k\in\mathbb{Z}_+\\ i+j>0}}\sum_{b,c=1}^{\dim\mathcal{U}} f_{bc}^a A_i^{(b)}B_j^{(c)}X_{i+j}^{(a)}+P^+(A,B),\\
E_0^{(a)}(A,B) &= \sum_{b,c=1}^{\dim\mathcal{U}}\sum_{i\in\mathbb{Z}_+} f_{bc}^a A_i^{(b)}B_{-i}^{(c)}X_0^{(a)}+P^0(A,B)
\end{aligned}$$

where $P^-(A,B)$, $P^+(A,B)$ and $P^0(A,B)$ contain only terms of total order three or more in the $A_i^{(a)}$'s and $B_i^{(a)}$'s, with order at least one in the $A_i^{(a)}$'s and at least one in the $B_i^{(a)}$'s.

Using this formula and the definition of $T^{(a)}(i)$, we can prove the bracket formula

$$[T^{(a)}(i),T^{(b)}(j)]=\sum_{c=1}^{\dim\mathcal{U}} f_{ab}^c T^{(c)}(i+j).$$

For more details, see [CH]. □

Using the theory of vertex operator algebras, we obtain from Theorem 4.0.8 the following result:

Theorem 4.0.9. *The W_{u_0} for the affine Lie algebra $\hat{\mathcal{U}}$ has a natural structure of a $M(0,0)$-module $M(0,0)$, where $M(0,0)$ is the vertex operator algebra whose underlying* *vector space is the generalized Verma module for the affine Lie algebra* $\hat{\mathcal{U}}$ *with level* 0 *and lowest weight* 0.

Proof. It is clear that W_{u_0} as an $\hat{\mathcal{U}}$-module is lower-truncated. From the theory of vertex operator algebras, any lower-truncated $\hat{\mathcal{U}}$-module of level 0 must be an $M(0,0)$-module. In particular, W_{u_0} is an $M(0,0)$-module. $\square$

References

[BHL] Barron, K., Huang Y.-Z., and Lepowsky, J., *Factorization of formal Exponentials and uniformization*, J. of Algebra, **228**(2000), 551-579

[BC] Beals, R., Coifman, R.R.,*Scattering and inverse scattering for first order systems*, Commun. Pure Appl. Math., **37**(1984),39-90

[B] Billig, Y., *Sine-Gordon equation and representations of affine algebra* $\widehat{sl_2}$. Journal of Functional Analysis, **192** (2002), 295-318

[B2] Billig, Y., *Vertex operator representations for toroidal Lie algebras* J. Math. Phys., **39** (1998), p. 3844-3864

[CH] Chen, S.-J., Huang Y.-Z, *Vertex operator and classical integrable system*, in preparation

[FB] Frenkel, E., Ben-Zvi, D., *Vertex Algebras and algebraic curves, second Ed.* Mathematical surveys and monograph, **88**(2004)

[H] Huang Y.-Z, *Two-Dimensional Conformal Geometry and Vertex Operator Algebras, Progress in Math.*, **148**, Birkhäuser, Boston, 1998

[K] Kac, V.G.: *Infinite dimensional Lie Algebras*. 3rd ed., Cambridge University Press 1990

[P] Palais, Richard S.,*The Symmetries of Solitons*, Bulletin. Amer. Math. Soc.,New Series **34**, No. 4, 339-403 (1997) [ISSN 0273-0979]

[TU1] Terng, C.L., Uhlenbeck, K., *Bäcklund transformations and loop group actions*, Comm. Pure Appl.Math., **53** (2000), 1-75

[TU2] Terng, C.L., Uhlenbeck, K.,*Poisson actions and scattering theory for integrable systems*, Surveys in Differential Geometry: Integrable Systems, A supplement to J. Differential Geometry, **4** (1999), 315-402