ESSAYS IN MECHANISM DESIGN

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ABSTRACT OF THE DISSERTATION

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This dissertation consists of three essays in the theory of mechanism design under incomplete information. In the first essay, we analyze an implementation problem in which monetary transfers are feasible, valuations are interdependent and the set of available choices lies in a product space of lattices. This framework is general enough to subsume many interesting examples, including allocation problems with multiple objects. We identify a class of social choice rules which can be implemented in ex post equilibrium. We identify conditions under which ex post efficient social choice rules are implementable using monotone selection theory. The key conditions are extensions of the single crossing property and supermodularity. These conditions can be replaced with more tractable conditions in multiobject allocation problems with either two objects or two agents. I also show that the payments which implement monotone social decision rules coincide with the payments of (1) the classical Vickrey-Clarke-Groves mechanism with private values, and (2) the generalized Vickrey auction introduced by Ausubel [1999] in multiunit allocation problems.

The second essay generalizes the analysis of optimal (revenue maximizing) mechanism design for the seller of a single object introduced by Myerson [1981]. We consider a problem in which the seller has several heterogeneous objects and buyers' valuations depend on each other's private information. We analyze two nonnested environments in which incentive constraints can be replaced with more tractable monotonicity conditions. We establish conditions under which these monotonicity conditions can be ignored, and show that several earlier analyses of the optimal mechanism design problem can be unified and generalized. In particular, problems with two complementary goods in Levin [1997] and multiunit auction problems in Maskin and Riley [1989] and Branco [1996] are special cases.

The third essay considers the problem of selling internet advertising slots to advertisers. Under suitable conditions, we solve for the payments imposed by an optimal mechanism

and show that it can be decentralized via prices using a linear assignment approach. At every configuration of private information, optimal mechanism can be interpreted as a menu consisting of a price for every slot.

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Dedication

To my first teachers, Şükrüye and Osman Ülkü

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Chapter 1

Introduction

1.1 Mechanism Design Under Incomplete Information

The theory of mechanism design analyzes resource allocation under incomplete information. The main objective of the theory is to identify, among all possible allocation schemes, the ones that are optimal for different parties involved in the process. The analysis depends heavily on the revelation principle which implies that attention can be restricted, without loss of generality, to those allocation schemes which (1) ask agents to report their private information, or their types, $t = (t_1, ..., t_n)$, (2) choose an allocation $\varphi(t) = (\varphi_1(t), ..., \varphi_n(t))$ and payments $x(t) = (x_1(t), ..., x_n(t))$ depending on the reported types and (3) induce truthful reporting. Such allocation schemes are direct revelation mechanisms of the form (φ, x), and they are the primary focus of this dissertation. That agents should report their information truthfully imposes certain incentive constraints on the choice of a mechanism. These incentive constraints are incentive compatibility, the requirement that it is some kind of an equilibrium in the game induced by the mechanism for agents to report their types truthfully, and individual rationality, the requirement that at this equilibrium, each agent gets at least the payoff he would get from his outside option. Thus, a typical mechanism design problem is one of finding a mechanism which is optimal for a given party (or for a group of people) among the class of mechanisms which satisfy these incentive constraints.

Several different versions of the mechanism design problem can be studied, depending on the details of the model at hand. A model needs to specify, among other things, the exact nature of the allocation problem and the private information possessed by economic agents, whether agents' valuations are private or interdependent, and what kind of equilibrium strategies agents are required to use. To be specific, let us consider the classical example of allocating, or selling, an object, a private good, to one of several buyers $i = 1, ..., n$ who have private information regarding their valuations. An allocation is a choice of a buyer to give the object to, or a randomization among the buyers. A natural and tractable choice is to let each agent's private information, or type, be a random variable t_i taking values on the real line, which determines his valuation fully or partially. The random variables $t_1, ..., t_n$ could be independent or correlated, they could be drawn from the same distribution, or from different distributions. If t_i completely determines each agent i's valuation, we can let $v_i(t_i) = t_i$ be i's valuation function. In this case valuations are said to be private. Valuations are interdependent if the collective type vector $(t_1, ..., t_n)$ determines the valuation $v_i(t_1, ..., t_n)$ of each agent i. Finally, we can impose various versions of incentive constraints on the mechanisms. It is commonplace in models with private information, to require turthful revelation to be either a Bayes-Nash equilibrium or a dominant strategy equilibrium of the revelation game induced by the mechanism. In models with interdependent values, the equilibrium concept used is usually Bayes-Nash or ex post Nash.

Myerson [1981] is the classical paper that studies the seller-optimal mechanism design problem in the single object environment with private values, independently distributed types and Bayesian incentive constraints. Under certain special conditions, Myerson [1981] shows that standard auction procedures are optimal for the seller. The main contribution of Myerson [1981] is a reformulation of the optimal mechanism design problem in which incentive constraints are replaced with a more tractable monotonicity constraint, and payments received by the seller are replaced by "virtual" valuations of the buyers. Virtual valuations are modifications of buyers' valuations, which take into consideration the distribution of their types. For a class of "regular" problems the monotonicity constraints in the reformulation can safely be ignored and the seller should give the object to the buyer with the highest nonnegative virtual valuation and keep the object if all virtual valuations are negative. This allocation rule, coupled with a payment scheme identified by Myerson [1981] solves the optimal mechanism design problem.

Implementation theory studies the related but different problem of attaining a given allocation rule, φ, as part of a mechanism (φ, x) which satisfies incentive constraints. In other words, given an allocation rule φ, implementation theory analyzes whether one can find payments x such that (φ, x) becomes feasible in mechanism design problems. Such payments x are said to implement the allocation rule φ. Two questions of interest in implementation theory are whether certain desirable allocation rules are implementable, and whether a class of implementable allocation rules can be identified . The Vickrey-

Clarke-Groves (VCG) mechanism (Vickrey [1961], Clarke [1971] and Groves [1973]) gives a positive answer to the first question when valuations are private. Any allocation rule which maximizes the sum of agents' valuations at every type configuration is implemented by VCG payments.

1.2 Ex Post Implementation over Lattices

Models in which implementation questions are addressed start out with a set of social choices, or allocations, available to a planner whose decision has to be based on private information of economic agents. The set of choices typically consists of allocations of a given supply of a resource or resources. Consider a combinatorial allocation problem in which members of a set of objects Ω are to be allocated to members of a set of agents N. The important characteristic of such a setting is that if agent i receives object a, then agent $j \neq i$ may receive any object $b \neq a$ but not a. In many problems these restrictions are either too stringent or too loose. If each object is a membership to a club, for example, and if a club may have sufficiently many members, then both i and j may receive a. On the other hand in problems with topological structures, if i receives a and j is not "close" to i, then j can not be given objects which are "close" to a.

The first essay of this dissertation analyzes a model in which such considerations can be addressed. In particular, we model the set of choices available to the planner to be a subset of a product of lattices. In the combinatorial allocation problem we mentioned above, the set of choices is $\{(A_1, ..., A_n) : A_i \subseteq \Omega \text{ and } A_i \cap A_j \neq \emptyset \text{ if } i \neq j\}$ and is a subset of $(2^{\Omega})^n$. If we want to study a combinatorial model in which agents are not riva in using the objects, as in the club membership example, then the set of choices is exactly $(2^{\Omega})^n$. Our model can be used to analyze both problems. A multiunit allocation problem in which the set of allocations is $\{(q_1, ..., q_n) \in \Re^n_+ : \sum q_i \leq Q\}$ is another special case of our environment.

Other important features of the model we analyze are interdependence of valuations and imposition of ex post incentive constraints. In such a setting we introduce a class of mechanisms, the Monotone-Implementation mechanisms, denoted (φ, x^{φ}) which are ex post incentive feasible (i.e., ex post incentive compatible and ex post individually rational) if φ satisfies an individual monotonicity property. In other words, the class of allocation rules which satisfy the individual monotonicity property are ex post implementable. We furthermore show that the payments x^{φ} coincide with (1) the VCG payments if φ is individually monotone and ex post efficient and if values are private, and (2) the generalized Vickrey auction payments (Ausubel [1999]) in multiunit allocation problems with interdependent values. We also identify conditions which guarantee that ex post efficient allocation rules are individually monotone using supermodularity theory. In combinatorial allocation problems complementarity between the objects plays a key role.

1.3 The Optimal Combinatorial Mechanism Design Problem

In many problems of interest in economics, an uninformed party has to allocate several objects among privately informed agents. Such problems are multiobject, or combinatorial, mechanism design problems. In our analysis of the optimal combinatorial mechanism design problem in the second essay, we extend Myerson's (1981) single object analysis in a number of directions. We formulate the problem with interdependent values and ex post incentive constraints. Then, we reformulate this problem using virtual valuations and monotonicity constraints. We define the class of regular problems as those in which monotonicity constraints can be ignored and establish conditions that guarantee regularity in two nonnested environments. These are the conditions that guarantee that an allocation rule maximizes the sum of virtual valuations at every type vector also satisfies monotonicity constraints. The conditions are analogs of the conditions identified in the first essay, which guarantee that ex post efficient allocation rules are monotone.

Special cases of our results appear in Branco [1996], Levin [1997], Monteiro [2002] and Ledyard [2007]. All of these papers treat combinatorial mechanism design problems in a variety of environments. Branco [1996] studies a model with interdependent values, multiple identical objects and decreasing marginal utilities. Monteiro [2002] analyzes a private values model with identical objects but without the decreasing marginal utilities assumption of Branco, allowing for synergies or complementarities between objects. Levin [1997] analyzes a problem with full complementarity and solves the mechanism design problem for two complementary objects and private values. Ledyard [2007] analyzes a combinatorial problem with several nonidentical objects and with private values, however with a special valuation structure: each agent has a positive valuation for exactly one specific subset of the grand set of objects. We show that these models are special cases of our model and that our approach unifies their treatment of the optimal mechanism design problem.

1.4 An Application: Internet Advertising with Unit Demand

In the third essay, we formulate an optimal mechanism design in which agents demand only one object. This formulation is geared towards the internet advertising application. Internet search engines sell to potential advertisers advertisement spots displayed following a keyword search. Each advertisement spot is a different object, as different locations on the screen have varying degrees of success in attracting users to visit the displayed sponsor. Hence the internet advertising problem is a combinatorial mechanism design problem and methods of the second essay can be applied in it s analysis. Furthermore special features of the internet advertising problem make it possible to identify the optimal mechanism, which is like an auction, under convenient assumptions and decentralize it using type contingent prices. In particular, we show that the payments of the optimal mechanism solve the dual of the linear assignment problem in which spots are allocated to advertisers in decreasing order of their virtual valuations.

Chapter 2

Ex Post Implementation Over Lattices

2.1 Introduction

Consider a setting in which a social decision is going to be made based on information privately held by a group of individuals. Suppose that these individuals are asked to report what they know to an outside party, who will then base the social decision on the collected information. If the social decision will affect their well being, these individuals will take into consideration the possibility of changing the social decision in their favor by misreporting. The theory of implementation is concerned with the identification of those social choice rules, and the environments in which they operate, which will give holders of private information the incentive to report it truthfully. In particular the theory investigates the effects of monetary transfers on the possibility of creating the incentives to report truthful information, whenever such transfers are possible. If for a given decision rule such transfers exist, the rule is said to be implementable. To be precise, a social decision rule coupled with a payment scheme, a mechanism in short, induces a game of incomplete information played by the members of the society, the agents. In this game, each agent reports his private information, or his type, in ignorance of what others report. Implementation is in Bayes-Nash strategies (ex post Nash strategies, dominant strategies) if it is a Bayes-Nash (ex post Nash, dominant strategy) equilibrium for agents to report their true types.

Work on implementation theory has considered mainly two related but different questions: (A) whether a given -desirable- social choice rule (for example a total welfare maximizing, i.e., efficient social choice rule) is implementable, and (B) whether a class of implementable social choice rules can be identified. An answer to the question (B) helps answer question (A) as well, since once a class of implementable social choice rules is identified, then conditions may be analyzed under which a desirable rule belongs to this class.

These two questions are usually studied in models that differ on a variety of dimensions including: (1) whether values are private or interdependent, (2) the kind of equilibrium strategies agents are required to use, (3) the nature of private information, and (4) the specification of the set of possible choices. The research frontier is currently expanding as models are getting more general and assumptions are being dropped in these dimensions.

This chapter answers question (B) in a fairly general model in terms of dimensions (1), (2) and, in particular, in dimension (4). We also indirectly provide an answer to question (A) using the class of social choice rules we identify. In terms of the dimensions of difference we referred to above, our model possesses the following features:

(1) Values are interdependent.

(2) Agents use ex post Nash equilibrium strategies. Ex post Nash equilibrium allows us to ignore assumptions regarding the distribution of private information, since agents do not need to have a prior to play these strategies. Truthful revelation of private information is an ex post Nash equilibrium if, at every type vector, it is a best response for an agent to report truthfully when he believes that everyone else is doing exactly the same.

(3) Private information is one dimensional and can take values in an interval on the real line. The one dimensionality restriction is almost necessary as recent work by Jehiel and Moldovanu (2001) shows that efficient ex post implementation is in general not possible when private information is multidimensional.

(4) Agents' valuations are defined over lattices, and social decisions belong to a partially ordered set. For example, agents' valuations can be defined over the power set of a given set, and social choices could be allocations of the members of this set between agents.

An answer to question (A) has been given by Vickrey (1961), Clarke (1971) and Groves (1973) in their analyses of what is now called the Vickrey-Clarke-Groves (VCG) mechanism. The VCG mechanism is capable of implementing efficient social choice rules, i.e., social choice rules which maximize the total welfare in the society at every configuration of private information, in dominant strategies. This means that, faced with the VCG payments, an individual prefers to report his information truthfully regardless of how he expects others will behave. This strong result crucially depends on the assumption of private values. If values are interdependent, then the VCG mechanism does not have any desirable incentive properties.

Several recent papers study extensions of the VCG mechanism to interdependent value settings. Ausubel (1999), analyzes a typical "auction problem" where multiple identical objects are to be allocated to several agents with interdependent values. A social choice rule in this problem maps agents' valuations to allocations. Ausubel shows that generalized VCG payments can be constructed which implement the efficient allocation rule in ex post Nash strategies. In related auction problems Dasgupta and Maskin (2000) and Perry and Reny (2001) develop auction mechanisms which have minimal informational requirements on the auctioneer. These auctions induce truthful bidding as ex post Nash equilibrium and implement an efficient allocation of objects. Our work is closest to but more general than Ausubel (1999). We analyze a general implementation problem of which resource allocation problems with several homogenous or heterogeneous objects may be considered special cases.

An answer to question (B) has been provided by Crémer and McLean (1985), who find that in an interdependent values model, "monotone" decision rules are implementable in ex post Nash strategies. Crémer and McLean (1985) also show that in problems where a fixed supply of a divisible object will be allocated between agents with interdependent values, efficient allocation rules are "monotone" under certain conditions. These conditions include a single crossing property on valuations and an interpersonal valuation comparison condition, which roughly states that the information possessed by an agent has a greater effect on his payoff, than on other agents' payoffs. Analogous conditions appear in many other papers in the literature, and they will certainly play a key role in our model. Moreover, under a certain full rank condition on the distribution of private information, Crémer and McLean (1985) show that it is possible to extract full surplus from the agents in the allocation problem they analyze. It is important to note that in the Crémer-McLean model private information of each agent belongs to a finite and completely ordered set, for example to a finite subset of the positive reals. The generalization of the Crémer-McLean transfer scheme and full surplus extraction result to a model where private information can take values in an interval is an open question in the literature.

The chapter proceeds as follows. We introduce the model in the next section. In Section 1.3 we introduce the Monotone-Implementation mechanism and identify a class of social choice rules it implements. This class consists of rules which satisfy a monotonicity property. In Section 1.4 we show that the Monotone-Implementation mechanism is a generalization of the VCG mechanism by showing that the two mechanisms coincide in private values models under certain mild conditions. In Section 1.5 we identify conditions under which efficient social choice rules are monotone in the sense of Section 1.3. These conditions include supermodularity of valuations. In Section 1.6 we discuss how these conditions relate to the conditions studied in the literature and examples in which some of them can be dropped. Section 1.7 relates our monotone implementation mechanism to the generalized Vickrey auction analyzed by Ausubel (1999) and Section 1.8 concludes.

2.2 The Model

We will analyze an extension of the classical mechanism design problem in which agents' valuations are defined over a lattice and we begin with some notation. Let $N = \{1, ..., n\}$ be the set of agents. For each i, let L_i be a lattice with associated partial order $\precsim_i$ and join and meet operators $\vee$ and $\wedge$.[1] Let $\prec_i$ be the induced strict order on L_i and let $L_N = \times_{i\in N} L_i$ and $L_{-i} = \times_{j\neq i} L_j$. We will denote elements of L_N typically by q and write $q = (q_i, q_{-i})$ where q_{-i} lists all coordinates of q corresponding to the members of $N\backslash\{i\}$.

We will let the space of social outcomes be a subset C of L_N. For some of the results, we will need to impose restrictions on C, for example we will need C to be a lattice.

Agents have private information in the form of one dimensional types. Agent i's private information, or his type, is $t_i \in T_i = [a_i, b_i]$. We will let $T = \times_{i\in N} T_i$, $T_{-i} = \times_{j\neq i} T_j$ denote members of T and T_{-i} by t and t_{-i} and write $t = (t_i, t_{-i})$. We allow for informational externalities, but no allocation externalities and assume quasilinearity in money. The valuation function for agent i is a map $v_i : L_i \times T \to \Re$. In particular, agent i's payoff depends on (1) the i^{th} component of a social outcome q and (2) the collective private information vector. If the social outcome is $q = (q_i, q_{-i})$, the type vector is $t = (t_i, t_{-i})$, and he pays x_i, agent i's payoff is $v_i(q_i, t_i, t_{-i}) - x_i$. We maintain the following technical assumptions on valuations v_i.

Assumption 1 For every i, t_{-i} and q_i, $v_i(q_i, \cdot, t_{-i})$ is nondecreasing and differentiable on $[a_i, b_i]$.

[1] We write $\vee$ and $\wedge$ instead of the more precise $\vee_i$ and $\wedge_i$ to lighten the notation and no confusion should result.

Assumption 2 For every i and t_{-i}, $v_i(\cdot,\cdot,t_{-i})$ satisfies nondecreasing differences (NDD), i.e., for every $t_i' < t_i$ and $q_i' \prec_i q_i$

$$v_i(q_i, t_i', t_{-i}) - v_i(q_i', t_i', t_{-i}) \leq v_i(q_i, t_i, t_{-i}) - v_i(q_i', t_i, t_{-i}).$$

Assumption 1 implies, by a result in Koliha (2006), that $v_i(q_i,\cdot,t_{-i})$ is absolutely continuous and enables us to use the Fundamental Theorem of Calculus. We will denote the derivative of $v_i(q_i,\cdot,t_{-i})$ by $\partial v_i(q_i,\cdot,t_{-i})$. Note that in the presence of Assumption 1, Assumption 2 implies $q_i' \prec_i q_i \Rightarrow \partial v_i(q_i', t_i, t_{-i}) \leq \partial v_i(q_i, t_i, t_{-i})$ for every i.

2.3 Ex-Post Implementation via Monotone Implementation Mechanisms

A (revelation) mechanism f is a pair (φ, x) consisting of a social choice rule $\varphi : T \to C$ and a payment rule $x : T \to \Re^n$. We will write $\varphi(t) = (\varphi_1(t), ..., \varphi_n(t))$ and $x(t) = (x_1(t), ..., x_n(t))$. We will be interested in the following ex post conditions satisfied by social choice rules and mechanisms.

A social choice rule φ is ex post outcome efficient if for every t,

$$\varphi(t) \in \arg \max_{(q_1,\ldots,q_n)\in C} \sum_{i\in N} v_i(q_i, t).$$

A mechanism (φ, x) satisfies ex post Nash incentive compatibility (XIC) if for every i, t_i and t_{-i},

$$t_i \in \arg \max_{t_i' \in [a_i, b_i]} v_i(\varphi_i(t_i', t_{-i}), t_i, t_{-i}) - x_i(t_i', t_{-i}),$$

ex post individual rationality (XIR) if for every i and t,

$$v_i(\varphi_i(t), t) - x_i(t) \geq 0,$$

and feasibility if for every t,

$$\sum_{i\in N} x_i(t) \geq 0 \text{ for every } t.$$

A mechanism that satisfies XIC, XIR and feasibility is said to be ex post incentive feasible. A social choice rule φ is ex post (Nash) implementable if there exists x such that the mechanism (φ, x) is ex post incentive feasible.

First we will investigate which social choice rules are ex post implementable. Throughout the paper we will restrict attention to social choice rules for which $t_i \mapsto \partial v_i(\varphi_i(t_i, t_{-i}), t_i, t_{-i})$ is Lebesgue integrable for every i and t_{-i}. The following class of mechanisms will play a key role.

Definition 1 Let φ be a social choice rule. A monotone implementation (MI) mechanism is a pair (φ, x^φ) satisfying, for every i and $t = (t_i, t_{-i})$,

$$x_i^\varphi(t) := v_i(\varphi_i(t), t) - \int_{a_i}^{t_i} \partial v_i(\varphi_i(z, t_{-i}), z, t_{-i}) dz. \tag{1}$$

The ex post incentive feasibility of an MI mechanism depends on the following property of its social choice rule.

Definition 2 A social choice rule φ is individually monotone if for every i and t_{-i}

$$t_i' < t_i \Rightarrow \varphi_i(t_i', t_{-i}) \preccurlyeq_i \varphi_i(t_i, t_{-i}).$$

Our first result shows that all individually monotone social choice rules are ex post implementable.

Proposition 1 If Assumptions 1 and 2 hold and the social choice rule φ is individually monotone, then the MI mechanism (φ, x^φ) is ex post incentive feasible.

Proof. Fix i, t_{-i} and $t_i' < t_i$ and an individually monotone social choice rule φ. We have

$$\begin{aligned}
& x_i^\varphi(t_i, t_{-i}) - x_i^\varphi(t_i', t_{-i}) \\
= \; & v_i(\varphi_i(t_i, t_{-i}), t_i, t_{-i}) - v_i(\varphi_i(t_i', t_{-i}), t_i', t_{-i}) - \int_{t_i'}^{t_i} \partial v_i(\varphi_i(z, t_{-i}), z, t_{-i}) dz \\
\leq \; & v_i(\varphi_i(t_i, t_{-i}), t_i, t_{-i}) - v_i(\varphi_i(t_i', t_{-i}), t_i', t_{-i}) - \int_{t_i'}^{t_i} \partial v_i(\varphi_i(t_i', t_{-i}), z, t_{-i}) dz \\
= \; & v_i(\varphi_i(t_i, t_{-i}), t_i, t_{-i}) - v_i(\varphi_i(t_i', t_{-i}), t_i, t_{-i})
\end{aligned}$$

where the inequality follows from individual monotonicity of φ and Assumption 2. Similarly if $t_i < t_i'$. This shows that the MI mechanism is ex post incentive compatible. That

$0 \leq x_i^{\varphi}(t) \leq v_i(\varphi_i(t), t)$ follows directly from the definition and hypotheses and therefore (φ, x^{φ}) is ex post incentive feasible. ■

2.4 Private Values

In order to highlight the relationship between MI mechanisms and VCG mechanisms, we specialize the model to the case of private values by making the following assumption.

Assumption 3 For each $i, q_i, t_i, t'_{-i} \neq t_{-i}$, $v_i(q_i, t_i, t_{-i}) = v_i(q_i, t_i, t'_{-i})$.

In this section we will abuse notation and write $v_i(q_i, t_i)$. We will also need the following notation:

$$
\begin{aligned}
C_i &= \{q_i \in L_i : (q_i, q_{-i}) \in C \text{ for some } q_{-i} \in L_{-i}\} \\
C_{-i} &= \{q_{-i} \in L_{-i} : (q_i, q_{-i}) \in C \text{ for some } q_i \in L_i\} \\
C_{-i}(q_i) &= \{q_{-i} \in L_{-i} : (q_i, q_{-i}) \in C\}
\end{aligned}
$$

Obviously $\cup_{q_i} C_{-i}(q_i) = C_{-i}$.

Definition 3 A Vickrey-Clarke-Groves (VCG) mechanism is a pair (φ, y^{φ}) where φ is ex post outcome efficient and for every i and $t = (t_i, t_{-i})$

$$y_i^{\varphi}(t) = \max_{q_{-i} \in C_{-i}} \sum_{j \neq i} v_j(q_j, t_j) - \max_{q_{-i} \in C_{-i}(\varphi_i(t))} \sum_{j \neq i} v_j(q_j, t_j). \tag{2}$$

It is easy to verify that VCG mechanisms are ex post incentive efficient. A naive extension of VCG mechanisms to interdependent value environments is obtained by modifying payments into

$$y_i^{\varphi}(t) = \max_{q_{-i} \in C_{-i}} \sum_{j \neq i} v_j(q_j, t) - \max_{q_{-i} \in C_{-i}(\varphi_i(t))} \sum_{j \neq i} v_j(q_j, t).$$

However this extension is not ex post incentive compatible. We now argue that the MI mechanism is a suitable extension of VCG mechanisms to interdependent value environments by showing that the two mechanisms coincide under private values. We will need to make the following assumptions.

Assumption 4 For every i and t_i, $v_i(\inf L_i, t_i) = 0$.

Assumption 5 The set C is finite and it contains $(\inf L_i)_{i\in N}$. Furthermore, for every i, $C_{-i}(\inf L_i) = C_{-i}$.

Note that if the map $q_i \mapsto C_{-i}(q_i)$ is nonincreasing, that is if $q_i' \preceq q_i \Rightarrow C_{-i}(q_i) \subseteq C_{-i}(q_i')$, then the condition $C_{-i}(\inf L_i) = C_{-i}$ in Assumption 5 would follow.

Proposition 2 If Assumptions 1-5 hold and if φ is an ex post outcome efficient and individually monotone social choice rule, then $x^\varphi = y^\varphi$, i.e., agents' payments are the same under the MI mechanism and the VCG mechanism.

Proof. Fix an ex post outcome efficient and individually monotone social choice rule φ, an agent i and a type vector t. Let $\Phi_i(t) := \{\varphi_i(z, t_{-i}) : z \leq t_i\}$. Note that $\Phi_i(t)$ has to contain finitely many elements because of the finiteness condition in Assumption 5 and is a completely ordered subset of L_i since φ is individually monotone. Write $\Phi_i(t) = \{q_i^1, ..., q_i^m\}$ where $m = |\Phi_i(t)|$ and $q_i^k \lesssim_i q_i^{k+1}$. Divide $[a_i, t_i]$ into subsets $\Gamma_i^1, ..., \Gamma_i^m$ such that $\varphi_i(z, t_{-i}) = q_i^k$ if and only if $z \in \Gamma_i^k$. Note that Γ_i^k is nonempty and connected, and that $[a_i, t_i] = \cup_{k=1}^m \Gamma_i^k$. Consequently $\inf \Gamma_i^k = \sup \Gamma_i^{k-1}$. Let $\gamma_i^k = \inf \Gamma_i^k$. Fix k. For every $z \in \Gamma_i^k$ we must have

$$\begin{aligned} v_i(q_i^k, z) + \max_{q_{-i}\in C_{-i}(q_i^k)} \sum_{j\neq i} v_j(q_j, t_j) &= v_i(\varphi_i(z, t_{-i}), z) + \sum_{i\in N} v_j(\varphi_j(z, t_{-i}), t_j) \\ &\geq v_i(q_i^{k-1}, z) + \max_{q_{-i}\in C_{-i}(q_i^{k-1})} \sum_{j\neq i} v_j(q_j, t_j) \end{aligned}$$

where the inequality follows from ex post efficiency. Similarly, for every $z' \in \Gamma_i^{k-1}$ we must have

$$v_i(q_i^{k-1}, z') + \max_{q_{-i} \in C_{-i}(q_i^{k-1})} \sum_{j \neq i} v_j(q_j, t_j) \geq v_i(q_i^k, z') + \max_{q_{-i} \in C_{-i}(q_i^k)} \sum_{j \neq i} v_j(q_j, t_j).$$

Using continuity of valuations in types and taking limits as $z \to \gamma_i^k$ and $z' \to \gamma_i^k$ we get

$$v_i(q_i^k, \gamma_i^k) - v_i(q_i^{k-1}, \gamma_i^k) = \max_{q_{-i} \in C_{-i}(q_i^{k-1})} \sum_{j \neq i} v_j(q_j, t_j) - \max_{q_{-i} \in C_{-i}(q_i^k)} \sum_{j \neq i} v_j(q_j, t_j) \quad (3)$$

Now we can write, by letting $\gamma_i^{m+1} = t_i$ and $\gamma_i^0 = \inf L_i$

$$\begin{aligned} x_i^{\varphi}(t) &= v_i(\varphi_i(t), t_i) - \int_{a_i}^{t_i} \partial v_i(\varphi_i(z, t_{-i}), z) dz \\ &= v_i(q_i^m, t_i) - \sum_{k=1}^{m} \int_{\gamma_i^k}^{\gamma_i^{k+1}} \partial v_i(q_i^k, z) dz \\ &= \sum_{k=1}^{m} [v_i(q_i^k, \gamma_i^k) - v_i(q_i^{k-1}, \gamma_i^k)] \\ &= \sum_{k=1}^{m} \left[\max_{q_{-i} \in C_{-i}(q_i^{k-1})} \sum_{j \neq i} v_j(q_j, t_j) - \max_{q_{-i} \in C_{-i}(q_i^k)} \sum_{j \neq i} v_j(q_j, t_j) \right] \\ &= \max_{q_{-i} \in C_{-i}(\inf L_i)} \sum_{j \neq i} v_j(q_j, t_j) - \max_{q_{-i} \in C_{-i}(q_i^k)} \sum_{j \neq i} v_j(q_j, t_j) \\ &= y_i^{\varphi}(t). \end{aligned}$$

The first equality is by definition. The second is by the construction of sets Γ_i^k. The third equality follows from Assumption 4 and the fourth from Equation (3). The fifth follows from a telescoping argument and the last equality from Assumption 5. ■

Assumption 5 holds for several important problems. For example let Ω be a finite set, $L_i = 2^{\Omega}$, and

$$C = \{(A_1, ..., A_n) \colon \cup_i A_i \subseteq \Omega \text{ and } A_i \text{ are disjoint}\}.$$

2.5 Implementing Ex Post Efficient Social Choice Rules

If we are interested in implementing an ex post outcome efficient social choice rule, then we must make additional assumptions that guarantee that such a social choice rule is individually monotone. To that end, we need the following notation: for every i, t and q_i, let $v^*_{-i}(q_i, t)$ be the maximal feasible total valuation of agents other than i if the i^{th} component of the social outcome is to be q_i, that is,

$$v^*_{-i}(q_i, t) = \max_{q_{-i} \in C_{-i}(q_i)} \sum_{j \neq i} v_j(q_i, t)$$

Assumption 6 Valuations satisfy the extended strict single crossing property, i.e., for every $i, t = (t_i, t_{-i}), t'_i < t_i$ and $q'_i \prec q_i$,

$$v_i(q_i, t'_i, t_{-i}) - v_i(q'_i, t'_i, t_{-i}) \geq v^*_{-i}(q'_i, t'_i, t_{-i}) - v^*_{-i}(q_i, t'_i, t_{-i}) \\ \Rightarrow v_i(q_i, t) - v_i(q'_i, t) > v^*_{-i}(q'_i, t) - v^*_{-i}(q_i, t).$$

Assumption 7 For every i and t, $v_i(\cdot, t)$ is supermodular, i.e., for every q'_i, $q_i \in L_i$,

$$v_i(q'_i, t) + v_i(q_i, t) \leq v_i(q_i \vee_i q'_i, t) + v_i(q_i \wedge q'_i, t).$$

Assumption 8 $C \subseteq L_N$ is a lattice.

Topkis (1998, Lemma 2.2.3) shows that if C is a lattice, then so are C_{-i} and $C_{-i}(q_i)$ for every q_i.

Proposition 3 If Assumptions 1, 2, 6–8 hold and if φ is an ex post outcome efficient social choice rule, then the MI mechanism (φ, x^φ) is ex post incentive feasible.

In order to prove Proposition 3, we need the following two technical results.

Lemma 1 (Monotone Selection Theorem) Let K be a lattice with partial order $\precsim$, $h : K \times [a, b] \to \Re$ and $l(z) \in \arg\max_{l \in K} h(l, z)$ for every z. Suppose that (1) h satisfies the strict single crossing property, i.e., for every $l' \prec l$ and $z' < z$, if $h(l', z') \leq h(l, z')$, then $h(l', z) < h(l, z)$, and that (2) $h(\cdot, z)$ is supermodular for every z. Then $z' < z$ implies $l(z') \precsim l(z)$.

Proof. See Topkis (1998), Theorem 2.8.4. ■

Lemma 2 (Preservation of supermodularity) For each $i \in N$, the function

$$q_i \mapsto v^*_{-i}(q_i, t) = \max_{q_{-i} \in C_{-i}(q_i)} \sum_{j \neq i} v_j(q_j, t)$$

is supermodular on the lattice C_{-i}.

Proof. See Topkis(1998), Theorem 2.7.6 or Corollary 2.7.2. ■

Proof of Proposition 3. It suffices to show that any ex post outcome efficient social choice rule satisfies individual monotonicity. If φ is ex post outcome efficient, then for every i and $t = (t_i, t_{-i})$

$$\varphi_i(t) \in \arg\max_{q_i \in C_i} \left[v_i(q_i, t) + v^*_{-i}(q_i, t) \right].$$

Note that $(q_i, t_i) \mapsto v_i(q_i, t) + v^*_{-i}(q_i, t)$ satisfies the strict single crossing property by Assumption 3. By Lemma 2 $q_i \mapsto v^*_{-i}(q_i, t)$ is supermodular. Since the sum of supermodular functions is also supermodular, $q_i \mapsto v_i(q_i, t) + v^*_{-i}(q_i, t)$ is supermodular and we deduce from Lemma 1 that $t'_i < t_i \Rightarrow \varphi_i(t'_i, t_{-i}) \precsim_i \varphi_i(t_i, t_{-i})$. Thus φ is individually monotone and Proposition 1 implies that (φ, x^φ) is ex post incentive feasible. ■

Assumption 8 is only used in order to utilize the conclusion of Lemma 2. Consequently, we can replace Assumption 8 with the conclusion of Lemma 2.

Assumption 8′ For each $i \in N$, C_i is a lattice and the function

$$q_i \mapsto v^*_{-i}(q_i, t) = \max_{q_{-i} \in C_{-i}(q_i)} \sum_{j \neq i} v_j(q_j, t)$$

is supermodular on C_i.

Now the following slightly more general result holds:

Proposition 4 If Assumptions 1, 2, 6, 7 and 8′ hold and if φ is an ex post outcome efficient social choice rule, then the MI mechanism (φ, x^φ) is ex post incentive feasible.

2.6 Discussion and Examples

Assumptions 1, 2 and 7 are quite common and, indeed, are satisfied by almost every example that appears in the literature dealing with implementation and mechanism design. However, Assumptions 6 and 8 (and 8') require further comment. Informally, Assumption 6 requires that an increase in the type of agent i must have a "bigger" effect on i's marginal valuation than on the marginal valuations of other agents. Such an assumption seems critical to proving individual monotonicity of ex post outcome efficient mechanisms. Similar assumptions designed for the same purpose have appeared in, e.g., Cremer and McLean (1985), Ausubel (1999), and Perry and Reny (1999).

Assumption 5 is satisfied in many but not all cases of interest.

Example 1 Consider the standard example in which L_i is a sublattice of $\mathbb{R}^m_+$ with $0 \in L_i$ and $\precsim_i$ is the usual partial order on $\mathbb{R}^m_+$, i.e., $x \precsim y$ if and only if $x_k \leqq y_k$ for each $k \in \{1, .., m\}$. Let $Q \in \mathbb{R}^m_{++}$ and let

$$C := \{(q_1, ..., q_n) \in L_N : \sum_{i \in N} q_i \precsim Q\}.$$

Then C is not a generally a sublattice of L_N and Assumption 8 will not be satisfied. In certain special cases, however, Assumption $8'$ will be satisfied. If $m = 1$, for example, then $q_i \mapsto v^*_{-i}(q_i, t)$ is function of a real variable and is trivially supermodular. Letting L_i be the set of positive integers and $Q = m > 0$, we obtain the classic model in which m identical indivisible objects are to be allocated to n agents. If $n = 2$, then $q_2 \mapsto v^*_1(q_2, t)$ is supermodular on L_2 and $q_1 \mapsto v^*_2(q_1, t)$ submodular on L_1. To see this, suppose that $q'_1, q''_1 \in C_1 = \{q_1 \in L_1 | q_1 \precsim Q\}$. Then

$$\begin{aligned} C_2(q'_1) &= \{q_2 \in L_2 : q_2 \precsim Q - q'_1\} \text{ and} \\ C_2(q''_1) &= \{q_2 \in L_2 : q_2 \precsim Q - q''_1\}. \end{aligned}$$

Let

$$q'_2 \in \arg\max_{q_2 \in C_2(q'_1)} v_2(q_2, t) \text{ and } q''_2 \in \arg\max_{q_2 \in C_2(q''_1)} v_2(q_2, t)$$

Since

$$q_2' \precsim Q - q_1' \text{ and } q_2'' \precsim Q - q_1''$$

it follows that

$$\begin{aligned} q_2' \vee q_2'' &\precsim (Q - q_1') \vee (Q - q_1'') = Q - (q_1' \vee q_1'') \text{ and} \\ q_2' \wedge q_2'' &\precsim (Q - q_1') \wedge (Q - q_1'') = Q - (q_1' \wedge q_1''). \end{aligned}$$

Since $q_2' \vee q_2'' \in L_2$ and $q_2' \wedge q_2'' \in L_2$, we conclude that

$$\begin{aligned} v_2^*(q_1', t) + v_2^*(q_1'', t) &= v_2(q_2', t) + v_2(q_2'', t) \\ &\leq v_2(q_2' \vee q_2'', t) + v_2(q_2' \wedge q_2'', t) \\ &\leq v_2^*(q_1' \vee q_1'', t) + v_2^*(q_1' \wedge q_1'', t) \end{aligned}$$

where the first inequality follows from the supermodularity of $v_2(\cdot, t)$. Similarly for $v_1^*(\cdot, t)$.

Example 2: Consider the special case of Example 1 in which n is a positive integer, $m = 2, Q = (1, 1)$ and $L_i = L = \{(0,0), (0,1), (1,0), (1,1)\}$. Furthermore, suppose that $v_i((0,0), t) = 0$ for all i and t. This case corresponds to the allocation of two indivisible objects to n agents and was considered by Levin (1997). In this very simple case, $C_i = L$ for each i and $q_i \mapsto v_{-i}^*(q_i, t)$ is supermodular on L if and only if

$$v_{-i}^*((0,1), t) + v_{-i}^*((1,0), t) \leq v_{-i}^*((0,0), t) + v_{-i}^*((1,1), t) = v_{-i}^*((0,0), t).$$

To prove this inequality, suppose that $v_j((1,0), t) = v_{-i}^*((0,1), t)$ and $v_k((0,1), t) = v_{-i}^*((1,0), t)$. If $j = k$, then supermodularity of $v_k(\cdot, t)$ implies that

$$\begin{aligned} v_{-i}^*((0,1), t) + v_{-i}^*((1,0), t) &= v_j((1,0), t) + v_j((0,1), t) \\ &\leq v_j((1,1), t) + v_j((0,0), t) \\ &= v_{-i}^*((0,0), t). \end{aligned}$$

If $j \neq k$, then $q_j = (1,0), q_k = (0,1)$ and $q_q = (0,0)$ for all other $q \neq i$ is feasible for the problem

$$\max_{q_{-i} \in C_{-i}(0,0)} \sum_{j \neq i} v_j(q_i, t)$$

from which we deduce that

$$v^*_{-i}((0,1),t) + v^*_{-i}((1,0),t) = v_j((1,0),t) + v_k((0,1),t) \leq v^*_{-i}((0,0),t).$$

The argument used in Example 2 is specific to the case of two indivisible objects and breaks down with more that two objects. Unfortunately, the function $v^*_{-i}(\cdot,t)$ will not generally be supermodular on C_i when C is not a lattice, not even in the special "budget constrained" model of Examples 1 and 2 when n and m are both greater than 2. On the other hand, the many applications involve valuation functions whose special structure does not require that Assumptions 8 or 8' hold and these are treated in the next chapter.

2.7 Allocation of an Indivisible Object

Consider a multinit allocation problem in which a Q units of of an indivisible object will be allocated between the agents. In this application we have:

$$\begin{aligned}
L_i &= \mathbb{Z}_+ \\
C &= \{(q_1, ..., q_n) \in \mathbb{Z}^n_+ : \sum_{i \in N} q_i \leq Q\} \\
C_i &= [0, Q] \\
C_{-i} &= \{q_{-i} \in \mathbb{Z}^{n-1}_+ : \sum_{j \neq i} q_i \leq Q\} \\
C_{-i}(q_i) &= \{q_{-i} \in \mathbb{Z}^{n-1}_+ : \sum_{j \neq i} q_i \leq Q - q_i\}
\end{aligned}$$

where $\mathbb{Z}_+$ is the set of nonnegative integers. Suppose that there exist maps $w_i : L_i \times T \to \Re_+$ such that

$$v_i(q_i, t) = \sum_{k=1}^{q_i} w_i(k, t).$$

The number $w_i(k,t)$ is the marginal valuation of agent i for the kth unit, if the type vector is t. Suppose that $w_i(0,t) = 0$.

In this environment, an efficient social choice rule must be monotone if Assumption 6 is satisfied. Note that Assumptions 7 and $8'$ do not impose any restriction as $v_i(\cdot, t)$ is defined on a completely ordered set and is therefore trivially supermodular. Even though Assumption 8 is definitely not satisfied by the set of social choices, Proposition 4 implies that under Assumptions 1, 2 and 6, the MI mechanism implements efficient allocation rules.

Furthermore the payments in Equation (1) which characterizes the MI mechanism may be given an interpretation in terms of the marginal values. Fix an individually monotone allocation rule φ, an agent i, and a type vector t. Define $\gamma_i^k(t_{-i}) = \inf\{y : \varphi_i(y, t_{-i}) = k\}$ for $k = 1, ..., \varphi_i(t)$ and let $\gamma^{k+1} = t_i$. In words, $\gamma_i^k(t_{-i})$ is agent i's critical type for the kth unit. We have

$$
\begin{aligned}
x_i^{\varphi}(t) &= \sum_{k=1}^{\varphi_i(t)} w_i(k,t) - \int_{a_i}^{t_i} \sum_{k=1}^{\varphi_i(y,t_{-i})} \partial w_i(k,y,t_{-i}) dy \\
&= \sum_{k=1}^{\varphi_i(t)} w_i(k,t) - \sum_{l=1}^{\varphi_i(t)} \int_{\gamma_i^l(t_{-i})}^{\gamma_i^{l+1}(t_{-i})} \sum_{k=1}^{l} \partial w_i(k,y,t_{-i}) dy \\
&= \sum_{k=1}^{\varphi_i(t)} w_i(k, \gamma_i^k(t_{-i}), t_{-i}).
\end{aligned}
$$

Thus, in a multiunit model with an indivisible object MI mechanism asks each agent to pay the sum of marginal valuations for each unit that he obtains, where these marginals are evaluated at his critical type for that unit. Analogous results have been obtained in Ausubel (1999) in his analyziz of the generalization of the Vickrey auction to an interdependent value environment.

2.8 Conclusion

In this chapter we developed an approach to the ex post Nash implementation problem when the social choice set is a subset of a product lattice. We identify a class of implementable social decision rules which satisfy the individual monotonicity property. The Monotone-Implementation mechanism we introduce coincides with the VCG mechanism on a class of social choice problems with private values and it coincides with the generalization of the Vickrey multiunit auction to interdependent value environments by Ausubel (1999). We analyze conditions under which efficient allocation rules are individually monotonic. An extended single crossing property plays a key role. Our methods further require either (a) the social choice set be a lattice and a supermodularity condition hold, or (b) the social choice set be a partially ordered set, not necessarily a lattice and a larger set of supermodularity conditions hold.

A typical example in which our methods apply is the combinatorial allocation problem which we analyze in depth in the next chapter. In particular, just like in any other constrained allocation problem, the social choice set in the combinatorial allocation problem is not a lattice and therefore several supermodularity restrictions must be imposed for our methods to work.

There are many problems where supermodularity conditions do not play a role, for example when the valuations are defined over completely ordered sets. Further work is necessary to understand how the Monotone-Implementation mechanism functions in these environments.

Chapter 3

Optimal Combinatorial Mechanism Design

3.1 Introduction

In many problems of interest in economics, an uninformed party has to allocate several objects among privately informed agents. Such is the nature of the problem faced by an internet search engine in selling advertisement spots displayed after a keyword search, by the FCC in selling radiospectrum licenses and by the FAA in selling airport arrival and departure gates. An important common feature in these problems is that the objects offered for sale are heterogeneous and they may be related in a complex way. Different advertisement spots will not generally attract the same number of users. An arrival gate and a departure gate in suitable locations at suitable times may be complements while two arrival gates at the same airport at the same time are substitutes. A wireless communication company may view radiospectrum licenses for two neighboring locations as complements and licenses for two distant locations as substitutes.

The main purpose of this chapter is to analyze the combinatorial mechanism design problem in some generality. The literature on mechanism design with independently distributed private information has established many celebrated results. In the seminal paper of this literature, Myerson [1981] considers an environment in which a principal interacts with several privately informed agents in order to allocate a single object and, in return, collect payments. The revelation principle implies that the mechanism can be chosen from among those which collect valuation reports from the agents and then determine an allocation and payments. Myerson characterizes the incentive constraints via "monotonicity" and "envelope" conditions. For each agent i and each of his types t_i, let $Q_i(t_i)$ be the expected probability of winning the object and let $U_i(t_i)$ be the expected payoff from reporting truthfully. Loosely speaking, monotonicity requires Q_i to be nondecreasing and the envelope condition requires that $Q_i(t_i) = U_i'(t_i)$. Myerson then reformulates the principal's revenue maximization problem as one of maximizing the expected sum of "virtual" valuations of the agents subject to his monotonicity constraint, where an agent's virtual valuation for the object is his actual valuation less the reciprocal of the hazard rate of the distribution of his valuation. Next, he asks when the constraints in the reformulated problem will not be binding and shows that under a regularity condition, a solution to the mechanism design problem can be obtained by focusing on the simpler problem of maximizing the expected sum of virtual valuations without the incentive constraints.

In this chapter, we extend Myerson's analysis in a number of directions. First, ours is a multi-object (combinatorial) setting in which objects could be complements or substitutes. Second, we formulate the optimal mechanism design problem with interdependent values and ex post incentive constraints. Agents have interdependent values if their valuations for sets of objects depend on each other's information. Incentive constraints are ex post if agents' behavior constitute an ex post Nash equilibrium in the game induced by the mechanism. Ex post incentive compatibility implies that agents don't need to know the distribution of other agents' private information.

Some important features of our setting are as follows. We work with several objects but one dimensional type spaces. Each agent i is equipped with a valuation function v_i which associates a real number with each type vector $t = (t_1, ..., t_n)$ and each set of objects A in a grand set Ω. Thus each t generates a vector of valuations $(v_i(A,t))_{A\subseteq\Omega}$. Our assumptions on v_i make it possible to analyze complements, substitutes and more hybrid valuation structures. We assume that v_i is common knowledge so that the principal need only elicit one dimensional type reports from the agents in order to calculate their valuations for all subsets of Ω. Our approach allows us to focus on the multidimensionality associated with allocating sets, in the absence of the well known problems of incentive characterization in models with multidimensional private information. We will identify conditions under which the former kind of multidimensionality is analytically tractable.

Besides Myerson [1981], the papers that are most closely related to this chapter are those of Branco [1996], Levin [1997], Monteiro [2002] and Ledyard [2007]. All of these papers treat combinatorial mechanism design problems in a variety of environments. Myerson [1981] solves the mechanism design problem for a single object and with a restricted form of interdependence of valuations. Branco [1996] studies a model with interdependent values, multiple identical objects and decreasing marginal utilities. Monteiro [2002] analyzes a private values model with identical objects but without the decreasing marginal utilities assumption of Branco, allowing for synergies or complementarities between objects. Levin [1997] analyzes a problem with full complementarity and solves the mechanism design problem for two complementary objects and private values.[1] Ledyard [2007] analyzes a combinatorial problem with several nonidentical objects and with private values, however with a special valuation structure: each agent has a positive valuation for exactly one specific subset of the grand set of objects. We show that these models are special cases of our model and that our approach unifies their treatment of the optimal mechanism design problem. In related work, Maskin and Riley [1989] and Ausubel and Cramton [1999] analyze the mechanism design problem when the principal has a continuum of identical objects with private and interdependent values, respectively. Our approach can be extended to include these models as special cases as well and we discuss this extension in Section 5.

The plan of the chapter is as follows. In Section 2.2 we introduce the environment. In Section 2.3, we characterize ex post incentive compatibility using monotonicity and envelope conditions in a way that extends Myerson's analysis. Our monotonicity condition accommodates multiple nonidentical objects and nonlinear valuations. In Section 2.4 we analyze the optimal mechanism design problem and derive a reformulation of it. The solution to this reformulation also solves the original problem when coupled with the right payments. We identify conditions that guarantee regularity for different classes of valuations. First, we develop a supermodularity based analysis of the sufficiency conditions making use of the theory of monotone selection in maximization problems over lattices. Next, we analyze problems in which preferences over sets can be represented by valuation functions over real numbers, or more generally over any completely ordered set. In these problems supermodularity conditions do not impose any restriction. Many examples of combinatorial problems studied in the literature fall in this category, including some interesting problems with submodular valuations which violate supermodularity conditions of the first approach. In Section 2.5 we discuss extensions to various related models, including those with looser feasibility requirements. Section 2.6 concludes.

[1] Levin uses a direct approach tailored for the two object scenario instead of a Myersonesque reformulation.

3.2 The Model

We consider a mechanism design problem in which (possibly a strict subset of) a finite set Ω of indivisible objects will be allocated by an uninformed principal among privately informed agents in return for monetary transfers. All actors are risk-neutral. Let $N = \{1, ..., n\}$ be the set of agents. The space of outcomes is $C \times \Re^n$ where

$$C = \{(A_1, ..., A_n) : \cup_i A_i \subseteq \Omega \text{ and } A_i \cap A_j = \emptyset \text{ if } i \neq j\} \tag{1}$$

is the set of lists of n pairwise disjoint subsets of Ω. The set A_i in the list $(A_1, ..., A_n)$ identifies the objects allocated to agent i. Note that a list $(A_1, ..., A_n) \in C$ need not cover Ω, i.e., some members of Ω may remain unallocated to any agent. The requirement that the sets A_i and A_j be disjoint for different agents i and j ensures that no single object is allocated to multiple agents. Note that in general C is not a lattice when it is ordered by the componentwise extension of the set order $\subseteq$. In the special case when $n = 1$, then $C = 2^\Omega$ is a lattice.

Agents have private information in the form of one dimensional types. We will assume that the private information is independently distributed across agents. The type of agent i is a random variable $\tilde{t}_i$ with a positive density f_i and associated distribution F_i on a support $T_i = [a_i, b_i]$. We denote by t_i a typical element of T_i. We define random vectors $\tilde{t} = (\tilde{t}_1, ..., \tilde{t}_n)$, $\tilde{t}_{-i} = (\tilde{t}_1, ..., \tilde{t}_{i-1}, \tilde{t}_{i+1}, ..., \tilde{t}_n)$, write $\tilde{t} = (\tilde{t}_i, \tilde{t}_{-i})$ and denote the typical realizations of these random vectors by t and t_{-i}. We let f and f_{-i} be the joint densities for $\tilde{t}$ and $\tilde{t}_{-i}$, with associated distributions F and F_{-i}. We denote by $\mathbb{E}_i, \mathbb{E}_{-i}$ and $\mathbb{E}$, the expectations computed with respect to F_i, F_{-i} and F.

We allow for informational externalities but there are no externalities pertaining to the allocation of objects. The payoff of agent i depends on the set of objects he receives, the size of his payment, and the realized collective private information vector. Given an outcome $(A_1, ..., A_n, x_1, ..., x_n) \in C \times \Re^n$, and a type vector t, i's payoff is $v_i(A_i, t) - x_i$ where $v_i : 2^\Omega \times T \to \Re$ is his valuation function. We maintain the following assumptions on valuations throughout the paper.

Assumption 1 For each i, t_{-i} and A, $v_i(A, \cdot, t_{-i})$ is differentiable (right differentiable at a_i and left differentiable at b_i) and nondecreasing.[2]

Assumption 2 For each i, t_i and A, $v_i(A, t_i, \cdot)$ is (Lebesgue) integrable.

These assumptions place minimal restrictions on how valuations depend on the collective type vector. In particular, we make no curvature assumption regarding the way in which v_i depends on agent i's type (cf. Maskin and Riley [1984 and 1989], Levin [1997], Krishna and Maenner [2001], Figueroa and Skreta [2007]). In addition, different authors have specified various assumptions regarding the way in which v_i depends on sets of objects. In particular, objects may be complements as in Levin [1997], or substitutes as in Branco [1996] and Monteiro [2002]. Agents may be "single-minded," in the sense of having positive valuation for only one specific set of objects as in Ledyard [2006]. In general, of course, an agent may view some objects as complements and others as substitutes. One agent may view some objects as complements, while another agent may view the same objects as substitutes, and different types of an agent may have different valuation structures. The assumptions that we will make below will include the aforementioned models as special cases.

The principal attaches no value to the objects and his payoff is simply the sum of payments $\sum x_i$. At the cost of additional notation, all our results directly extend to a setting in which the principal has positive valuations for various sets of objects, as long as these valuations do not depend on agents' private information. This is, in fact, a point of departure from Myerson [1981] who assumes a very restricted form of interdependence of valuations. In Myerson's model, the information held by i affects the valuation of each $j \neq i$ as well as the valuation of the principal through what Myerson calls revision effects. The exact form of his results depends critically on the symmetry and the linearity of revision effects. Although we allow virtually any form of interdependence between the agents, we rule out the possibility that an agent's information has any effect on the principal's valuation. We will discuss the generalization of Myerson's full fledged model with revision effects to multiple objects in Section 6.

For future reference let us record some technical definitions. In order to save space, we

[2] We will denote the derivative of v_i with respect to t_i by $\partial v_i(A, \cdot, t_{-i})$.

will make use of the difference operator which we denote by Δ. In particular, for any real valued function h,

$$\Delta_{\alpha,\beta}[h] := h(\alpha) - h(\beta).$$

Suppose that L is a lattice ordered with the weak order $\preceq$ and the induced strong order $\prec$ and K is an interval in $\Re$. Denote generic elements of L by l and l', those of K by k and k' and the meet and join operations on members of L by $\wedge$ and $\vee$ respectively. Two lattices of interest are $L = 2^{\Omega}$ and $L = \Re^m$.

We will be interested in variants of three kinds of properties which a map $\phi : L \times K \to \Re$ may satisfy.

Definition 1 (Monotone differences) The map ϕ satisfies nondecreasing differences (NDD) if $\Delta_{k,k'}[\Delta_{l,l'}[\phi(\cdot,\cdot)]] \geq 0$ for every $l' \preceq l$ and every $k' \leq k$, and strictly increasing differences (SID) if $\Delta_{k,k'}[\Delta_{l,l'}[\phi(\cdot,\cdot)]] > 0$ for every $l' \prec l$ and every $k' < k$.

Definition 2 (Single crossing) The map ϕ satisfies the single crossing property (SCP) if $\Delta_{l,l'}[\phi(\cdot, k')] > (\geq)0$ implies $\Delta_{l,l'}[\phi(\cdot, k)] > (\geq)0$ for every $l' \prec l$ and $k' < k$, and the strict single crossing property (SSCP) if $\Delta_{l,l'}[\phi(\cdot, k')] \geq 0$ implies $\Delta_{l,l'}[\phi(\cdot, k)] > 0$ for every $l' \prec l$ and $k' < k$.

Definition 3 (Supermodularity) For any $k \in K$, the map $\phi(\cdot, k)$ is supermodular if $\phi(l, k)+\phi(l', k) \leq \phi(l \vee l', k)+\phi(l \wedge l', k)$ for every l and l', strictly supermodular if $\phi(l, k)+\phi(l', k) < \phi(l \vee l', k) + \phi(l \wedge l', k)$ for every unordered pair l and l', and pseudo-supermodular if $\max\{\phi(l, k), \phi(l', k)\} \geq (>)\phi(l \wedge l', k)$ implies $\phi(l \vee l', k) \geq (>) \min\{\phi(l, k), \phi(l', k)\}$ for every l, l' and k. The map ϕ is supermodular if $\phi(l, k) + \phi(l', k') \leq \phi(l \vee l', \max\{k, k'\}) + \phi(l \wedge l', \min\{k, k'\})$ for every (l, k) and (l', k'), and strictly supermodular if $\phi(l, k) + \phi(l', k') < \phi(l \vee l', \max\{k, k'\}) + \phi(l \wedge l', \min\{k, k'\})$ for every (l, k) and (l', k') such that l and l' are unordered.

It is fairly easy to prove that NDD implies SCP, SID implies SSCP and supermodularity implies pseudo-supermodularity. Moreover ϕ is supermodular (strictly supermodular) if and only if ϕ satisfies NDD (SID) and $\phi(\cdot, k)$ is supermodular (strictly supermodular) for every

k. For a thorough treatment, see Milgrom and Shannon [1994], Topkis [1998] and also Agliardi [2000] who introduces the concept of pseudo-supermodularity.

3.3 Mechanisms

In this section we will define mechanisms and discuss the concept of feasibility that we will adopt and its ramifications. The revelation principle tells us that, regardless of his objective, the principal need only consider direct mechanisms which ask agents to report their types, induce truthful reporting, and determine allocation and payments depending on the reported types. We will be interested in deterministic mechanisms that induce truthful reporting as an ex post Nash equilibrium.

A (direct and deterministic) mechanism consists of an allocation rule $S : T \to C$ and a payment rule $x : T \to \Re^n$ and is denoted (S, x). We will write $S(t) = (S_1(t), ..., S_n(t))$ and $x(t) = (x_1(t), ..., x_n(t))$. Given a mechanism (S, x), the ex post payoff to agent i when the type vector is $t = (t_i, t_{-i})$ and all agents report truthfully is

$$V_i(t|S, x) = v_i(S_i(t), t) - x_i(t).$$

Whenever convenient, we will suppress the dependence of the ex post payoff on the underlying mechanism and simply write $V_i(t)$. A mechanism (S, x) satisfies *ex post* Nash incentive compatibility (XIC) if $V_i(t) \geq v_i(S_i(t_i', t_{-i}), t) - x_i(t_i', t_{-i})$ for every $i, t = (t_i, t_{-i})$ and $t_i' \neq t_i$, and ex post individual rationality (XIR) if $V_i(t) \geq 0$ for every i and t. A mechanism that satisfies both XIC and XIR is said to be ex post incentive feasible. We will denote by $\mathcal{F}$ the set of ex post incentive feasible mechanisms. An allocation rule S is ex post Nash implementable if there is a payment rule x such that the mechanism $(S, x) \in \mathcal{F}$.

3.3.1 Characterizing Incentives

We begin by characterizing the class of mechanisms that satisfy XIC using, as is standard in the literature, "monotonicity" and "envelope" conditions that are appropriate for our combinatorial setting. These conditions, in their various versions, are fundamental workhorses of mechanism design theory.

Definition 4 An allocation rule S satisfies monotonicity *(*M) if for every $i, t = (t_i, t_{-i}), t'_i \neq t_i$,

$$v_i(S_i(t), t) \geq v_i(S_i(t), t'_i, t_{-i}) + \int_{t'_i}^{t_i} \partial v_i(S_i(y, t_{-i}), y, t_{-i}) dy. \tag{M}$$

We leave it to the reader to verify that in the standard single unit environment with private values, where $|\Omega| = 1$ and $v_i(\Omega, t) = t_i$, M is equivalent to the following condition: the agent to whom the object is allocated does not change when only that agent's type goes up. This property is easily recognized to be the ex post version of the monotonicity condition in Myerson [1981]. Toward the end of this section we will identify two important environments in which M is implied by more useful and appealing conditions.

Definition 5 A mechanism (S, x) satisfies the envelope condition (E) if for every $i, t = (t_i, t_{-i}), t'_i \neq t_i$,

$$V_i(t) = V_i(t'_i, t_{-i}) + \int_{t'_i}^{t_i} \partial v_i(S_i(y, t_{-i}), y, t_{-i}) dy. \tag{E}$$

Using results in Milgrom and Segal [2002] and Koliha [2006], the following result characterizes XIC. In particular, it shows that XIC implies E under two standard assumptions: monotonicity and differentiability of v_i in t_i.

Lemma 1 A mechanism (S, x) satisfies XIC if and only if (S, x) satisfies condition E and S satisfies condition M.

Proof. $(\Rightarrow)$ Suppose that (S, x) satisfies XIC. Since $v_i(A, \cdot, t_{-i})$ is differentiable and non-decreasing, $\partial v_i(A, \cdot, t_{-i})$ is Lebesgue integrable and Proposition 1 in Koliha [2006] applies, rendering $v_i(A, \cdot, t_{-i})$ absolutely continuous. Now condition E follows from Theorem 2 in Milgrom and Segal [2002]. To see that S satisfies condition M, note that for every $i, t = (t_i, t_{-i})$ and $t'_i \neq t_i$

$$\begin{aligned}
\int_{t'_i}^{t_i} \partial v_i(S_i(y, t_{-i}), y, t_{-i}) dy &= V_i(t) - V_i(t'_i, t_{-i}) \\
&= v_i(S_i(t), t) - x_i(t) \\
&\quad -[v_i(S_i(t'_i, t_{-i}), t'_i, t_{-i}) - x_i(t'_i, t_{-i})] \\
&\leq v_i(S_i(t), t) - x_i(t) - [v_i(S_i(t), t'_i, t_{-i}) - x_i(t)] \\
&= v_i(S_i(t), t) - v_i(S_i(t), t'_i, t_{-i})
\end{aligned}$$

where the inequality follows from XIC.

($\Leftarrow$) Suppose that S satisfies condition M and (S, x) satisfies condition E. For every $i, t = (t_i, t_{-i})$ and $t_i' \neq t_i$

$$\begin{aligned} V_i(t) - V_i(t_i', t_{-i}) &= \int_{t_i'}^{t_i} \partial v_i(S_i(y, t_{-i}), y, t_{-i}) dy \\ &\leq v_i(S_i(t), t) - v_i(S_i(t), t_i', t_{-i}) \end{aligned}$$

from which XIC follows. ■

Lemma 1, in particular the exact form of condition M, depends critically on the technical assumption regarding the way in which agents' valuations depend on their own types. The literature does not seem to have an agreed upon way to model this particular relationship. In Maskin and Riley [1984, 1989] and in Levin [1997] valuations are concave in agents' own types whereas in Krishna and Maenner [2001] and Figueroa and Skreta [2007], they are convex. Krishna [2002] shows that in the special case of linear valuations, as in Myerson [1981], a "subgradient condition" can be used to characterize incentive compatibility. For example, consider the case in which $v_i(A, t) = \mu_i(A)t_i$ for every i, A and t where $\mu_i : 2^\Omega \to R_+$. Then a mechanism (S, x) satisfies XIC if and only if for every i, t_i and t_{-i}, $\mu_i(S_i(t_i, t_{-i}))$ is a subgradient of $V_i(\cdot, t_{-i})$ at t_i. Unfortunately examples can be constructed showing that this result does not work in general, even when valuations are convex in types. This is also apparent from a close reading of Rochet [1987]. In general, condition M is indispensable in the characterization of XIC.

We will finish this section by recording three corollaries of Lemma 1. Corollary 1 obtains a revenue equivalence theorem for combinatorial mechanism design problems. Corollary 2 characterizes ex post Nash implementable allocation rules. Corollary 3 characterizes ex post incentive feasible mechanisms.

Corollary 1 All mechanisms which satisfy XIC, which have the same allocation rule and which leave the lowest type of each agent with the same ex post payoff generate the same ex post revenue.

Proof. If a mechanism satisfies XIC, then by Lemma 1 it also satisfies E implying that

$$V_i(t) = V_i(a_i, t_{-i}) + \int_{a_i}^{t_i} \partial v_i(S_i(y, t_{-i}), y, t_{-i}) dy$$

for each i and t. Consequently, an agent's payment depends only on the allocation rule and the payoff received by his lowest type. ■

Corollary 2 An allocation rule S is ex post Nash implementable if and only if S satisfies condition M.

Proof. The only if part trivially follows from the definition of ex post Nash implementability and Lemma 1. Suppose that S satisfies condition M and choose x such that for every i and t_{-i}, $x_i(a_i, t_{-i}) \leq v_i(S_i(a_i, t_{-i}), a_i, t_{-i})$ and if $t_i > a_i$ then

$$\begin{aligned} x_i(t_i, t_{-i}) &= v_i(S_i(t_i, t_{-i}), t_i, t_{-i}) - \int_{a_i}^{t_i} \partial v_i(S_i(y, t_{-i}), y, t_{-i}) dy \\ &\quad - [v_i(S_i(a_i, t_{-i}), a_i, t_{-i}) - x_i(a_i, t_{-i})] \end{aligned} \qquad (2)$$

This choice of x implies condition E. Thus (S, x) must satisfy XIC. Since $\partial v_i(A, t_i, t_{-i}) \geq 0$ for every i, A and $t = (t_i, t_{-i})$, this choice of x also implies XIR. We conculde that $(S, x) \in \mathcal{F}$. ■

Corollary 3 A mechanism $(S, x) \in \mathcal{F}$ if and only if (S, x) satisfies condition E, S satisfies condition M, and for every i and t_{-i}, $V_i(a_i, t_{-i}|S, x) \geq 0$.

Proof. If $(S, x) \in \mathcal{F}$, then Conditions E and M follow from Lemma 1 and XIR implies that $V_i(a_i, t_{-i}|S, x) \geq 0$. If $S \in \mathcal{M}$ and (S, x) satisfies Condition E, then XIC follows from Lemma 1. Using condition E, we can write

$$V_i(t|S, x) = V_i(a_i, t_{-i}|S, x) + \int_{a_i}^{t_i} \partial v_i(S_i(y, t_{-i}), y, t_{-i}) dy$$

which is nonnegative since $V_i(a_i, t_{-i}|S, x) \geq 0$ and v_i is increasing in t_i. Hence (S, x) satisfies XIR as well. ■

3.3.2 Restricted Environments

There are two important and nonnested environments in which condition M is implied by simpler and more appealing conditions. Below we identify these environments and the corresponding conditions guaranteeing condition M.

Environment A For each i and t_{-i}, $v_i(\cdot,\cdot,t_{-i}) : 2^\Omega \times T_i \to \Re_+$ satisfies NDD.

In our model NDD implies a complementarity relationship between an agent's private information and the objects. It can be shown that v_i satisfies NDD in (A, t_i) if and only if for every $t_{-i}, t'_i < t_i, \omega \in \Omega$, and $A \subseteq \Omega \backslash \{\omega\}$,

$$v_i(A \cup \{\omega\}, t'_i, t_{-i}) - v_i(A, t'_i, t_{-i}) \leq v_i(A \cup \{\omega\}, t_i, t_{-i}) - v_i(A, t_i, t_{-i}),$$

that is, the marginal value of attaining another object (when the agent already has the set A and when the collective type vector for the remaining agents is t_{-i}) is higher for higher types. This does not mean that the objects are complements. In fact valuations may satisfy NDD even if all agents think that the objects in Ω are perfect substitutes. Note that an agent i treats the objects as complements (substitutes) for every collective type realization if for each t, $v_i(\cdot, t)$ is supermodular (submodular). A map is submodular it its negative is supermodular.

We will call a map $\sigma : [a, b] \to 2^\Omega$ weakly expanding if $a \leq y' < y \leq b$ implies $\sigma(y') \subseteq \sigma(y)$. We will skip the proof of the following result.

Lemma 2 In Environment A, an allocation rule S satisfies condition M if $S_i(\cdot, t_{-i})$ is weakly expanding for every i and t_{-i}.

Environment B For each i, there exist maps $\mu_i : 2^\Omega \to \Re$ and $w_i : \mu_i(2^\Omega) \times T \to \Re_+$ such that

1. for every A and t, $v_i(A, t) = w_i(\mu_i(A), t)$,

2. for every t_{-i}, $w_i(\cdot,\cdot,t_{-i}) : \mu_i(2^\Omega) \times T_i \to \Re_+$ satisfies NDD,

where $\mu_i(2^\Omega) = \{\mu_i(A) : A \in 2^\Omega\}$.

To use the terminology introduced by Mookherjee and Reichelstein [1992], in Environment B, valuations satisfy the "one-dimensional condensation property", that is, they depend on scalars associated with sets, rather than the sets themselves. An important example is the case of perfect substitutes where μ_i is the counting measure.

Lemma 3 In Environment B, an allocation rule S satisfies condition M if $\mu_i(S_i(\cdot, t_{-i}))$ is nondecreasing on $[a_i, b_i]$ for every i and t_{-i}.

Note that the sufficient conditions for condition M obtained in Lemmas 2 and 3 are usually nonnested. In Environment B, we may have an allocation rule S with nondecreasing $\mu_i(S_i(\cdot, t_{-i}))$, without $S_i(\cdot, t_{-i})$ being weakly expanding. To see this, consider a single agent problem with $[a_1, b_1] = [0, 1], \Omega = \{\alpha, \beta\}$, $v(A, t) = \mu(A)t$, $\mu(\emptyset) = \mu(\{\alpha\}) = 0$ and $\mu(\{\beta\}) = \mu(\{\alpha, \beta\}) = 1$. Consequently, $w : \{0, 1\} \times T \to \Re$ is defined as $w(z, t) = zt$ and satisfies NDD. If $S(t) = \{\alpha\}$ when $0 \leq t < \frac{1}{2}$ and $S(t) = \{\beta\}$ when $\frac{1}{2} \leq t \leq 1$, then $t \mapsto \mu(S(t))$ is nondecreasing but $t \mapsto S(t)$ is not weakly expanding.

3.4 Optimal Mechanism Design

In this section we will analyze the optimal mechanism design problem:

$$\max_{(S,x)\in\mathcal{F}} \mathbb{E} \sum_{i\in N} x_i(\tilde{t}) \tag{OMD}$$

In order to reformulate this problem in a way so that the choice of the allocation rule can be separated from the choice of payments, we need to define virtual valuations. Agent i's virtual valuation is a map $u_i : 2^\Omega \times T \to \Re$ defined by

$$u_i(A, t) = v_i(A, t) - \partial v_i(A, t) \frac{1 - F_i(t_i)}{f_i(t_i)}.$$

A well-known result in mechanism design theory allows us to replace payments with virtual valuations in this problem. We record this result next.

Lemma 4 If (S, x) satisfies Condition E, then

$$\mathbb{E}\sum_{i\in N} x_i(\tilde{t}) = \mathbb{E}[\sum_{i\in N} u_i(S_i(\tilde{t}), \tilde{t}) - \sum_{i\in N} V_i(a_i, \tilde{t}_{-i})]. \tag{3}$$

Proof. By a simple integration by parts argument we obtain for any i and t_{-i}

$$\mathbb{E}_i \int_{a_i}^{\tilde{t}_i} \partial v_i(S_i(y, t_{-i}), y, t_{-i}) dy = \mathbb{E}_i \partial v_i(S_i(\tilde{t}_i, t_{-i}), \tilde{t}_i, t_{-i}) \lambda_i(\tilde{t}_i) \tag{4}$$

where $\lambda_i(t_i) = (1 - F_i(t_i))/f_i(t_i)$. Using Condition E, we can write, for any i and t_i

$$\begin{aligned}\mathbb{E}_{-i} x_i(t_i, \tilde{t}_{-i}) &= \mathbb{E}_{-i}[v_i(S_i(t_i, \tilde{t}_{-i}), t_i, \tilde{t}_{-i}) - V_i(a_i, \tilde{t}_{-i})] \\ &\quad - \mathbb{E}_{-i} \int_{a_i}^{t_i} \partial v_i(S_i(y, \tilde{t}_{-i}), y, \tilde{t}_{-i}) dy.\end{aligned}$$

Computing expectations with respect to t_i we obtain

$$\begin{aligned}\mathbb{E} x_i(\tilde{t}) &= \mathbb{E} v_i(S_i(\tilde{t}), \tilde{t}) - \mathbb{E}_{-i} V_i(a_i, \tilde{t}_{-i}) \\ &\quad - \mathbb{E}_{-i}\mathbb{E}_i \int_{a_i}^{\tilde{t}_i} \partial v_i(S_i(y, \tilde{t}_{-i}), y, \tilde{t}_{-i}) dy.\end{aligned}$$

Using Equation 4 and summing over i, finishes the proof. ■

Following Myerson [1981], the next result reformulates OMD using Lemma 4. The reformulation separates the choice of the allocation rule from the choice of the payment rule and this will play a key role in the ensuing analysis. We will denote by $\mathcal{M}$ the class of allocation rules satisfying condition M.

Proposition 1 If S^* solves the reformulated problem

$$\max_{S\in\mathcal{M}} \mathbb{E}\sum_{i\in N} u_i(S_i(\tilde{t}), \tilde{t}) \tag{R}$$

and if

$$x_i^*(t) = v_i(S_i^*(t), t) - \int_{a_i}^{t_i} \partial v_i(S_i^*(y, t_{-i}), y, t_{-i}) dy \tag{5}$$

for every i and t, then the mechanism (S^*, x^*) solves OMD.

Proof. The allocation rule S^* satisfies Condition M and the choice of x^* implies that the mechanism (S^*, x^*) satisfies Condition E. Therefore, by Lemma 1, (S^*, x^*) satisfies XIC. The choice of x^* also indicates that (S^*, x^*) satisfies XIR. Note that for every i and t_{-i}, $V_i(a_i, t_{-i}|S^*, x^*) = 0$ since $x_i^*(a_i, t_{-i}) = v_i(S_i^*(a_i, t_{-i}), a_i, t_{-i})$. Thus (S^*, x^*) is feasible in OMD. For any other ex post incentive feasible mechanism (S, x) we have,

$$\begin{aligned}
\mathbb{E}\sum_{i\in N} x_i(\tilde{t}) &= \mathbb{E}[\sum_{i\in N} u_i(S_i(\tilde{t}), \tilde{t}) - \sum_{i\in N} V_i(a_i, \tilde{t}_{-i}|S, x)] \\
&\leq \mathbb{E}\sum_{i\in N} u_i(S_i(\tilde{t}), \tilde{t}) \\
&\leq \mathbb{E}\sum_{i\in N} u_i(S_i^*(\tilde{t}), \tilde{t}) \\
&= \mathbb{E}\sum_{i\in N} x_i^*(\tilde{t})
\end{aligned}$$

where the first equality follows from Lemma 4, the first inequality follows because (S, x) must satisfy XIR, the second inequality is by hypotheses and the second equality follows from Lemma 4 and the observation that $V_i(a_i, t_{-i}|S^*, x^*) = 0$ for every i and t_{-i}. ■

Two remarks on Proposition 1 are in order.

Remark 1 Fix

$$S^* \in \arg\max_{S\in\mathcal{M}} \mathbb{E}\sum_{i\in N} u_i(S_i(\tilde{t}), \tilde{t})$$

and consider the set $X(S^*) = \{x : (S^*, x) \in F\}$. This set is nonempty since $(S^*, x^*) \in F$ if x^* is as defined in Equation 5. If $\xi \in X(S^*)$, then for every i and t, we have

$$\begin{aligned}
x_i^*(t) &= v_i(S_i^*(t), t) - \int_{a_i}^{t_i} \partial v_i(S_i^*(y, t_{-i}), y, t_{-i})dy \\
&\geq v_i(S_i^*(t), t) - \int_{a_i}^{t_i} \partial v_i(S_i^*(y, t_{-i}), y, t_{-i})dy \\
&\quad -[v_i(S_i^*(a_i, t_{-i}), a_i, t_{-i}) - \xi_i(a_i, t_{-i})] \\
&= \xi_i(t)
\end{aligned}$$

where the first equality is by definition. the inequality follows from the fact that (S^*, ξ) must satisfy XIR, and the second equality follows because (S^*, ξ) must satisfy Condition E.

Thus x^* achieves the highest possible revenue for the mechanism designer within the class of mechanisms $\{(S^*, x) : x \in X(S^*)\}$.

Remark 2 The implication in Proposition 1 can be reversed. Suppose that the mechanism (S^*, x^*) solves OMD. Then the following must hold: S^* solves Problem R and payments satisfy Equation 2 with $x_i^*(a_i, t_{-i}) \leq v_i(S_i^*(a_i, t_{-i}), a_i, t_{-i})$ for every i *and* t_{-i}.

Proposition 1 is useful in separating the choice of the allocation rule from the choice of the payments. In order to solve OMD, the allocation rule can be chosen to solve Problem R and payments can be derived by using this allocation rule in Equation 5. But solving Problem R may still be formidable as we don't know much about the structure of the constraint set $\mathcal{M}$ in general. Regularity addresses exactly this issue.

Definition 6 The optimal mechanism design problem is regular if whenever S is such that $S(t)$ solves

$$\max_{(A_1,\ldots,A_n)\in C} \sum_{i\in N} u_i(A_i, t) \qquad (\mathrm{OP}_n)$$

for every t, then S satisfies Condition M.

If the optimal mechanism design problem is regular, the mechanism designer need only choose an allocation such that $S(t)$ solves the optimal partitioning problem with n agents (OP_n) and then choose payments as in Proposition 1. The resulting mechanism will solve OMD.

We move on to establishing sufficient conditions for regularity in Environments A and B.

3.4.1 Regularity with Supermodularity: Environment A

In this subsection we will restrict attention to Environment A in which each v_i satisfies NDD in (A, t_i). It is instructive to start with the single agent problem. When there is only one agent Problem OP_1 becomes $\max_{A\in 2^\Omega} u(A, t)$, where, importantly, the constraint set 2^Ω is a lattice. The theory of monotone selection of optimizers (Topkis [1998], Milgrom and Shannon [1994] and Agliardi [2000]) can now be used to show that an optimal selection

$t \mapsto S^*(t)$ is monotonic in the sense of Condition M if (1) $u(\cdot, t)$ is pseudo-supermodular for every t, (2) u satisfies the strict single crossing property, and (3) $S^*(t') \subseteq S^*(t)$ whenever $t' < t$. Thus sufficient conditions for regularity can be obtained by referring to established results of the literature when there is a single agent.

However this simplicity rests on two important features of the single agent problem. First, when there is a single agent, $C = 2^\Omega$ is a lattice and therefore $S(t') \cup S(t)$ and $S(t') \cap S(t)$ are elements of C. This ceases to be the case in general, as the constraint set in the multiagent problem OP_n, defined in Equation 1, is not a lattice when it is ordered with the componentwise extension of the standard set order $\subseteq$. Second, the complications arising from interdependence of valuations are absent in the single agent problem. In multiagent problems t_i has an effect on u_j. Loosely speaking, we must make sure that t_i has a larger effect on the set received by i than on the set received by $j \neq i$.

We will now identify conditions under which monotone comparative static results can be obtained in $\text{OP}_{n.}$ We need some more notation. Define, for each i, A and t

$$u^*_{N\backslash i}(A_i, t) = \max\{\sum_{j \neq i} u_j(A'_j, t) : (A_i, A'_{-i}) \in C\}$$

where A'_{-i} lists $\{A'_j : j \neq i\}$. In words, $u^*_{N\backslash i}(A_i, t)$ is the largest sum of virtual valuations of all other agents conditional on i getting set A_i when the type vector is t.

Definition 7 Virtual valuations satisfy the extended strict single crossing property (E-SSCP) if for every $i, t = (t_i, t_{-i}), t'_i < t_i$ and $A'_i \subset A_i$

$$u_i(A_i, t'_i, t_{-i}) - u_i(A'_i, t'_i, t_{-i}) \geq u^*_{N\backslash i}(A'_i, t'_i, t_{-i}) - u^*_{N\backslash i}(A_i, t'_i, t_{-i})$$
$$\Rightarrow u_i(A_i, t) - u_i(A'_i, t) > u^*_{N\backslash i}(A'_i, t) - u^*_{N\backslash i}(A_i, t)$$

To explain E-SSCP, fix t_{-i} and sets A_i and A'_i with $A'_i \subset A_i$. Now consider two "type dependent" plans. In the first plan, the set A_i is assigned to agent i and the remaining objects in $\Omega\backslash A_i$ are allocated to the agents in $N\backslash i$ in an optimal fashion. In the second plan, the set A'_i is assigned to agent i and the remaining objects in $\Omega\backslash A'_i$ are allocated to the agents in $N\backslash i$ in an optimal fashion. If the first plan dominates the second when agent

i is of type t'_i, i.e., if

$$u_i(A_i, t'_i, t_{-i}) + u^*_{N\backslash i}(A_i, t'_i, t_{-i}) \geq u_i(A'_i, t'_i, t_{-i}) + u^*_{N\backslash i}(A'_i, t'_i, t_{-i}),$$

then E-SSCP requires that the first plan strictly dominate the second when agent i is of type $t_i > t'_i$. If valuations are private, then the right hand sides of the displayed inequalities in the definition of E-SSCP are 0. In this case virtual valuations satisfy E-SSCP if and only if they satisfy SSCP. In this sense, E-SSCP is an extension of SSCP.

Definition 8 Virtual valuations satisfy the extended supermodularity property (E-SUPM) if for every i and t,

1. $u_i(\cdot, t) : 2^{\Omega} \to \Re$ is supermodular, and
2. $u^*_{N\backslash i}(\cdot, t) : 2^{\Omega} \to \Re$ is supermodular.

If the virtual valuations satisfy E-SUPM, then Problem OP_n can be reformulated in a way that the constraint set becomes a lattice and the objective function becomes supermodular over the constraint set. If Condition E-SSCP is also satisfied, then the objective function in the reformulated problem satisfies SSCP. We record this finding in the next result.

Proposition 2 In Environment A, the optimal mechanism design problem is regular if virtual valuations satisfy Conditions E-SSCP and E-SUPM.

Proof. Pick a selection $S(t) = (S_1(t), ..., S_n(t))$ that solves OP_n for every t. For every i, $S_i(t)$ must also solve

$$\max_{A_i \in 2^{\Omega}} [u_i(A_i, t) + u^*_{N\backslash i}(A_i, t)]..$$

Note that the constraint set in this problem is a lattice. Condition E-SSCP implies that the objective function satisfies SSCP over $2^{\Omega} \times T_i$ and Condition E-SUPM implies that it is supermodular on 2^{Ω}. By Theorem 2.8.4 in Topkis (1998) we conclude that $S_i(\cdot, t_{-i})$ is weakly expanding. By Lemma 2, this implies that $S(\cdot)$ satisfies Condition M and we conclude that OMD is regular. ■

In an interesting paper Levin [1997] considers the optimal mechanism design problem with two complementary goods and private values. His results are extended by Proposition 2 to any number of objects and interdependent values. In his analysis, Levin uses a direct argument tailored to the two object case. Without invoking the machinery of supermodular optimization, he employs an exhaustive analysis of all possible cases to prove that the optimal allocation has the expansion property of our Lemma 2. The assumptions that Levin makes in his two object model coincide exactly with the assumptions of Corollary 2 when specialized to a two object problem. Even though a direct extension of Levin's analysis to more than two objects can be discouragingly cumbersome, our analysis shows that his result works for arbitrary number of complementary objects.

To see how Levin's model can be incorporated in our model, let $\Omega = \{\omega_1, \omega_2\}$, and for each agent i let there exist nonnegative and differentiable functions $v_{i1}, v_{i2}, \varkappa_i$ such that:

$$v'_{i1}(t_i), v'_{i2}(t_i) > 0, \;\; \varkappa'_i(t_i) \geq 0$$

$$v''_{i1}(t_i), v''_{i2}(t_i), \varkappa''_i(t_i) \leq 0$$

$$v_i(S, t_i) = \begin{cases} 0 & \text{if } S = \emptyset \\ v_{i1}(t_i) & \text{if } S = \{\omega_1\} \\ v_{i2}(t_i) & \text{if } S = \{\omega_2\} \\ v_{i1}(t_i) + v_{i2}(t_i) + \varkappa_i(t_i) & \text{if } S = \{\omega_1, \omega_2\} \end{cases}$$

Now assume that the hazard rates of type distributions are nondecreasing (Levin's Assumption 2) so that v_i and u_i satisfy SID. This implies, together with private values that the E-SSCP condition is satisfied. Finally assume that $\varkappa_i(t_i) - \frac{1-F_i(t_i)}{f_i(t_i)} \varkappa'_i(t_i) \geq 0$ (Levin's Assumption 3) so that $u_i(\cdot, t_i)$ is supermodular. Then, as discussed in Example 2 of the previous chapter, the second requirement in the E-SUPM condition is also satisfied. Now Proposition 2 applies and we conclude that the mechanism design problem is regular.

3.4.2 Regularity without Supermodularity: Environment B

In this subsection we restrict attention to Environment B in which a one dimensional condensation property holds together with a NDD condition: for every i there exist maps $\mu_i : 2^\Omega \to \Re$ and $\hat{v}_i : \mu_i(2^\Omega) \times T \to \Re_+$ such that $v_i(A,t) = \hat{v}_i(\mu_i(A), t)$ and $\hat{v}_i(\cdot, \cdot, t_{-i})$ has NDD. In order to motivate the analysis, consider the following example with private values, multiple identical units and decreasing marginal valuations. Agents are only interested in the number of units they obtain and their valuations take the form $v_i(A,t) = \mu(|A|)t_i$ where μ is strictly concave on $\{1, ..., |\Omega|\}$. Suppose that t_i is distributed uniformly over $[0,1]$ so that $\frac{1-F_i(t_i)}{f_i(t_i)} = 1 - t_i$. Proposition 2 can not be used to determine the regularity of OMD in this environment, since the maps $\{u_i(\cdot, t_i)\}_{t_i \in [0,1]}$ are not supermodular: for every $i, t_i > \frac{1}{2}$ and $\omega_1, \omega_2 \in \Omega$,

$$\begin{aligned} u_i(\{\omega_1, \omega_2\}, t) - u_i(\{\omega_1\}, t) &= [\mu(2) - \mu(1)][2t_i - 1] \\ &< [\mu(1) - \mu(0)][2t_i - 1] \\ &= u_i(\{\omega_2\}, t) - u_i(\emptyset, t). \end{aligned}$$

Multiunit environments with decreasing marginal valuations are one of the important mechanism design environments to study, and Environment B is the right environment to analyze them. Define

$$\begin{aligned} \hat{u}_i(z,t) &= \hat{v}_i(z,t) - \partial \hat{v}_i(z,t) \frac{1 - F_i(t_i)}{f_i(t_i)} \\ C_\mu &= \{(\mu_1(A_1), ..., \mu_n(A_n)) : \cup_i A_i \subseteq \Omega \text{ and } A_i \text{ are disjoint}\}, \text{ and} \\ \hat{u}^*_{N \backslash i}(z_i, t) &= \max\{\sum_{j \neq i} \hat{u}_j(a_j, t) : (a_1, ..., a_{i-1}, z_i, a_{i+1}, ..., a_n) \in C_\mu\} \end{aligned}$$

Adapting Definition 7, we will say that the virtual valuations $\hat{u}_i$, $i = 1, ..., n$, satisfy the E-SSCP if for every $i, t_{-i}, z'_i < z_i$ and $t'_i < t_i$

$$\begin{aligned} \hat{u}_i(z_i, t'_i, t_{-i}) - \hat{u}_i(z'_i, t'_i, t_{-i}) &\geq \hat{u}^*_{N\backslash i}(z'_i, t'_i, t_{-i}) - \hat{u}^*_{N\backslash i}(z_i, t'_i, t_{-i}) \\ &\Rightarrow \hat{u}_i(z_i, t) - \hat{u}_i(z'_i, t) > \hat{u}^*_{N\backslash i}(z'_i, t) - \hat{u}^*_{N\backslash i}(z_i, t). \end{aligned}$$

The conditions that guarantee regularity do not involve supermodularity in Environment B.

Proposition 3 In Environment B, the optimal mechanism design problem is regular if the virtual valuations satisfy the E-SSCP.

Proof. If $\{(S_1(t), ..., S_n(t)) : t \in T\}$ is a selection of maximizers in Problem OP_n, then for every i and t, $\mu_i(S_i(t))$ solves

$$\max_{z_i \in \{\mu_i(A):A \subseteq \Omega\}} [\hat{u}_i(z_i, t) + \hat{u}^*_{-i}(z_i, t)].$$

E-SSCP implies that $(z_i, t_i) \mapsto [\hat{u}_i(z_i, t_i, t_{-i}) + \hat{u}^*_{-i}(z_i, t_i, t_{-i})]$ satisfies SSCP implying that $t_i \mapsto \mu_i(S_i(t_i, t_{-i}))$ is nondecreasing for every i and t_{-i}. In Environment B, this is sufficient for Condition M. ■

Note that $z_i \mapsto \hat{u}_i(z_i, t) + \hat{u}^*_{-i}(z_i, t)$ is defined on a completely ordered set and supermodularity does not impose any restriction on it. Therefore we don't have a supermodularity condition in Proposition 3. A second important aspect of this result is the nature of the strong monotonicity conditions obtained. Suppose that $(S_i(t))_{i \in N}$ is a selection from the solutions of Problem OP_n. In the proof of Proposition 2, we show that $t'_i < t_i \Rightarrow S_i(t'_i, t_{-i}) \subseteq S_i(t'_i, t_{-i})$ whereas in the proof of Proposition 3, we show that $t_i \mapsto \mu_i(S_i(t_i, t_{-i}))$ is a nondecreasing function. These two strong monotonicity conditions are, in general, nonnested.

As in Environment A, more tractable sufficient conditions for regularity can be obtained in Environment B in the special case of private values. In particular we obtain the following corollary to Proposition 4, whose proof we skip.

Corollary 4 In Environment B and with private values, the optimal mechanism design problem is regular if for each i, $(z_i, t_i) \mapsto \hat{u}_i(z_i, t_i)$ satisfies SSCP.

Two interesting problems in which Proposition 3 and Corollary 4 apply are the optimal mechanism design problem with multiple identical units, analyzed by Branco [1996] and

Monteiro [2002] and the optimal mechanism design problem with single-minded agents. We briefly analyze how these problems can be mapped into our model next.

3.4.2.1 Identical Objects

In problems with identical objects, agents are only interested in the cardinality of the set of objects that they receive. Hence, the one-dimensional condensation property is satisfied with $\mu_i(A) = |A|$, i.e., for every A and t,

$$v_i(A,t) = w_i(|A|\,,t)$$

where $w_i : \{0,1,..,m\} \times T \to \Re_+$ and $m = |\Omega|$. In order, to apply Proposition 3, we need to identify conditions under which w_i satisfies NDD and E-SSCP. To this end define, abusing notation slightly, the maps $w_{ik} : T \to \Re$ and $u_{ik} : T \to \Re$ by

$$\begin{aligned} w_{i0}(t) &= w_i(0,t), \\ u_{i0}(t) &= u_i(0,t) \\ w_{ik}(t) &= w_i(k,t) - w_i(k-1,t) \text{ if } k = 1,...,m, \text{ and} \\ u_{ik}(t) &= u_i(k,t) - u_i(k-1,t) \text{ if } k = 1,...,m. \end{aligned}$$

so that $v_i(A,t) = \sum_{k=1}^{|A|} w_{ik}(t)$ and $u_i(A,t) = \sum_{k=1}^{|A|} u_{ik}(t)$.

Proposition 4 Suppose that for each k, i and t,

1. $w_{ik}(\cdot, t_{-i})$ is nondecreasing,

2. $u_{ik}(t) \geq \widehat{u}_{ik+1}(t)$,

3. $u_{ik}(\cdot, t_{-i})$ is strictly increasing.

4. $u_{ik}(t_i', t_{-i}) > u_{jk'}(t_i', t_{-i})$ whenever $u_{ik}(t_i, t_{-i}) \geq u_{jk'}(t_i, t_{-i})$ and $t_i' > t_i$.

Then the associated optimal mechanism design problem is regular.

Proof. For every t, let $u^{(1)}(t) \geq u^{(2)} \cdots \geq u^{(m)}(t)$ be the first m highest elemenst of $\{u_{ik} : i = 1, ..., n \text{ and } k = 1, ...m\}$ and define the set of "winning" marginal virtual valuations as

$$W(t) = \{u^{(1)}(t), ..., u^{(m)}(t)\} \cap \{u_{ik}(t) : u_{ik}(t) \geq 0\}.$$

Let $W_i(t) = W(t) \cap \{u_{ik}(t) : k = 1, ..., m\}$ be the set of agent i's winning bids. For this identical units problem, $\mu(A) = |A|$ and therefore

$$C_\mu = \{(a_1, ..., a_n) : \sum a_i \leq m \text{ and } a_i \text{ is a nonnegative integer}\}.$$

The optimal partitioning problem becomes

$$\max_{(A_1,...,A_n)\in C} \sum_{i=1}^{n} \sum_{k=1}^{|A_i|} u_{ik}(t)$$

and if $S(t)$ is a solution to this problem, then $(|S_1(t)|, ..., |S_n(t)|)$ solves

$$\max_{(a_1,...,a_n)\in C_\mu} \sum_{i=1}^{n} \sum_{k=1}^{a_i} u_{ik}(t)$$

implying that $|S_i(t)| = |W_i(t)|$. Fix an agent j and types $t'_j < t_j$ and t_{-j}. Suppose that $S(t'_j, t_{-j})$ and $S(t_j, t_{-j})$ solve the OP_n problem at the corresponding type vectors. Let $|S_j(t'_j, t_{-j})| = k' > 0$. Then $u_{jk'}(t'_j, t_{-j}) \in W(t'_j, t_{-j})$, i.e., $\{u_{j1}(t'_j, t_{-j}), ..., u_{jk'}(t'_j, t_{-j})\}$ is a subset of $W(t'_j, t_{-j})$ by assumption 2 of the proposition. Towards a contradiction suppose that $u_{jk'}(t_j, t_{-j}) \notin W(t_j, t_{-j})$. By assumption 3, $u_{jk'}(t_j, t_{-j}) > u_{jk'}(t'_j, t_{-j})$ and therefore $u_{jk'}(t_j, t_{-j}) > 0$. Then there must be some i and k such that $u_{ik}(t'_j, t_{-j}) \notin W(t'_j, t_{-j})$ but $u_{ik}(t_j, t_{-j}) \in W(t_j, t_{-j})$. In particular this implies that $u_{ik}(t'_j, t_{-j}) \leq u_{jk'}(t'_j, t_{-j})$ but $u_{ik}(t_j, t_{-j}) \geq u_{jk'}(t_j, t_{-j})$ violating assumption 4. So it must be the case that $u_{jk'}(t_j, t_{-j}) \in W(t_j, t_{-j})$ and $|S_j(t'_j, t_{-j})| \leq |S_j(t_j, t_{-j})|$. Since by assumption 1 $(k, t_i) \mapsto w_i(k, t_i, t_{-i})$ satisfies NDD for every i and t_{-i}, the conditions for Environmen B are satisfied, this proves that S satisfies Condition M and the OMD is regular. ■

3.4.2.2 Single-minded Agents

Ledyard [2006] considers a private values model in which for each i, there exists a special set of objects A_i^* such that $v_i(A, t_i) = t_i$ if $A_i^* \subseteq A$ and zero otherwise. Letting $\mu_i(A) = 1$ if $A_i^* \subseteq A$ and zero otherwise, and letting $v_i(A, t_i) = \mu_i(A)t_i$ we can conclude, by Corollary 4, that the mechanism design problem with such single-minded agents is regular if the hazard rates are nondecreasing.

3.5 Extensions

In this section we will discuss four directions in which our model can be extended without substantially changing the analysis. We first consider a specific form of interdependent valuations whereby agents' private information has an effect on each other's as well as the mechanism designer's valuations. Next we consider fluid models in which there is a continuous supply of one or many objects which are perfectly divisible. Then we remark on the effects of weakening the notion of incentive feasibility that we adapt and allowing for stochastic mechanisms. Finally we consider more general mechanism design problems in which the mechanism designer maximizes the expected sum of a weighted average of all parties involved. Optimal mechanism design is a special case of this more general formulation, in which all weight is placed on the welfare of the mechanism designer.

3.5.1 Myerson's revision effects

The main difference between Myerson's original formulation of the mechanism design problem and the present model lies in the principal's valuation structure. In Myerson's model an agent's type affects the valuations all other agents and the principal linearly and in exactly the same way, a feature which Myerson calls *revision effects*. In our model the precise statement of revision effects is as follows.

Definition 9 Valuations exhibit revision effects if for each i there exist maps $e_i : 2^\Omega \times T_i \to \Re$ and $w_i : 2^\Omega \times T_i \to \Re$ such that

1. $e_i(\cdot, t_i)$ is additive, i.e., if $A \cap A' = \emptyset$, then $e_i(A \cup A', t_i) = e_i(A, t_i) + e_i(A', t_i)$, *2.* $v_i(A,t) = w_i(A,t_i) + \sum_{j\neq i} e_j(A,t_j)$, and

3. the principal's valuation is given by $v_0(A,t) = \sum_{j\in N} e_j(A,t_j)$.

If valuations exhibit revision effects, then the principal's valuation is also a function of the collective type vector and the optimal mechanism design problem becomes:

$$\max_{(S,x)\in\mathcal{F}} \ \mathbb{E}\sum_i [x_i(\tilde{t}) - v_0(S_i(\tilde{t}),\tilde{t})]$$

As a result, simple changes in the arguments show that the reformulated problem becomes:

$$\max_{S\in\mathcal{M}} \ \mathbb{E}\sum_i \sigma_i(S_i(\tilde{t}),\tilde{t}_i)$$

where $\sigma_i : 2^\Omega \times T_i \to \Re$ is defined by

$$\sigma_i(A,t_i) = w_i(A_i,t_i) - e_i(A_i,t_i) - \partial w_i(A_i,t_i)\frac{1-F_i(t_i)}{f_i(t_i)}.$$

Quite remarkably, σ_i does not depend on t_{-i} even though agents have interdependent valuations. As a result we get the following proposition whose proof we skip.

Proposition 5 Suppose that valuations satisfy revision effects and define σ_i as above. The mechanism design problem is regular if for each i, w_i has NDD, σ_i has SSCP and $\sigma_i(\cdot,t_i)$ is supermodular.

3.5.2 Fluid Models

In important work, Maskin and Riley [1989] and Ausubel and Cramton [1999] analyze the multiunit optimal mechanism design problem in a slightly different environment than ours. They analyze a problem in which the object is divisible with a fixed supply of q_0 units and valuations take the form $v_i : [0,q_0] \times T \to \Re$. In particular they hypothesize that $v_i(q,t) = \int_0^q p_i(y,t)dy$ for some demand function $p : \Re_+ \times T \to \Re_+$. In Maskin and Riley [1989] values are private and the demand function is symmetric across agents. Since the object is perfectly divisible in these models, we will call them *fluid* models.

The techniques of supermodular optimization can be employed in fluid models which are more general then the ones analyzed by Maskin and Riley [1989] and Ausubel and Cramton [1999], as we outline next.

Suppose there are m fluid objects and the supply constraints are given by q_0^k, $k = 1, ..., m$. A feasible allocation is a vector $q = (q_1, ..., q_n)$ where $q_i = (q_i^k)_{k=1}^m$ and $\sum_i q_i^k \leq q_0^k$. Let $Q = [0, q_0^1] \times \cdots \times [0, q_0^m]$. Note that Q is a lattice ordered with the partial order $\leq$ given by $\bar{q}_i \leq q_i$ if $\bar{q}_i^k \leq q_i^k$ for every k. Suppose that valuations take the form $v_i : Q \times T \to \Re$ where $v_i(q_i, \cdot, t_{-i})$ is differentiable and increasing and define $u_i(q_i, t) = v_i(q_i, t) - \partial v_i(q_i, t)\frac{1-F_i(t_i)}{f_i(t_i)}$. Now appropriately modifying the proof of Proposition 2, we can prove that if each v_i satisfies NDD on $Q \times T_i$, if virtual valuations satisfy the appropriate modifications of Conditions E-SSCP and E-SUP then the mechanism design problem is regular and maximizing for every type profile the sum of virtual valuations is a legitimate way of solving for the revenue-maximizing mechanism. In Maskin and Riley [1989] and Ausubel and Cramton [1999], $m = 1$ and, not surprisingly, the conditions they identify for regularity do not require supermodularity.

3.5.3 Interim Incentives and Stochastic Mechanisms

The notion of feasibility that we adopt in this paper has two restrictions: mechanisms must be (1) ex post incentive feasible, (2) deterministic. In this subsection we will argue that these two restrictions are, in some sense, without loss of generality.

By a stochastic mechanism, we mean a mechanism that determines probabilities of different allocations based on type reports. Hence a deterministic mechanism (S, x) is a stochastic mechanism that puts probability one on the allocation $S(t)$ at every collective type report t. Thus, a stochastic mechanism is a pair (q, x) where $x : T \to \Re^n$ determines payments and $q : C \times T \to [0, 1]$ is such that $q(\cdot, t)$ is a probability distribution over C for every t. In particular, for every t, q satisfies:

$$\begin{aligned} q(S,t) &\geq 0 \\ \sum_{S \in C} q(S,t) &= 1. \end{aligned}$$

Note that we require $\sum_S q(S,t) = 1$ since C is defined such that an allocation $S \in C$, need not cover Ω. Given (q,x) the ex post payoff of agent i is

$$V_i(t_i', t_{-i}|q,x) = \sum_{S\in C} q(S, t_i', t_{-i}) v_i(S_i, t) - x_i(t_i', t_{-i})$$

if he reports t_i' and other agents report t_{-i}. Note that i's payoff only depends on the ith component of the allocation $S(t)$.

Bayesian (interim) incentive feasibility is weaker than ex post incentive feasibility. A mechanism is Bayesian incentive feasible if incentive constraints hold when they are evaluated at the interim stage, after agents learn their own type, but in ignorance of each other's types. Let F_B be the class of interim incentive feasible deterministic mechanisms and $\mathring{F}$ and $\mathring{F}_B$ be the corresponding classes of ex post and interim incentive feasible stochastic mechanisms. To be precise,

$$(S,x) \in F_B \Leftrightarrow \forall i, t_i \begin{cases} t_i \in \arg\max_{t_i' \in T_i} \mathbb{E}V_i(t_i', \tilde{t}_{-i}|S,x) \\ \mathbb{E}V_i(t_i, \tilde{t}_{-i}|S,x) \geq 0 \end{cases}$$

$$(q,x) \in \mathring{F} \Leftrightarrow \forall i, t_i, t_{-i} \begin{cases} t_i \in \arg\max_{t_i' \in T_i} V_i(t_i', t_{-i}|q,x) \\ V_i(t_i, t_{-i}|q,x) \geq 0 \end{cases}$$

$$(q,x) \in \mathring{F}_B \Leftrightarrow \forall i, t_i \begin{cases} t_i \in \arg\max_{t_i' \in T_i} \mathbb{E}V_i(t_i', \tilde{t}_{-i}|q,x) \\ \mathbb{E}V_i(t_i, \tilde{t}_{-i}|q,x) \geq 0 \end{cases}$$

The optimal mechanism design problem can be formulated in four different ways.

$$\max_{(S,x)\in F} \mathbb{E}\sum_{i\in N} x_i(\tilde{t}) \tag{OMD}$$

$$\max_{(S,x)\in F_B} \mathbb{E}\sum_{i\in N} x_i(\tilde{t}) \tag{OMD$'$}$$

$$\max_{(q,x)\in \mathring{F}} \mathbb{E}\sum_{i\in N} x_i(\tilde{t}) \tag{OMD$''$}$$

$$\max_{(q,x)\in \mathring{F}_B} \mathbb{E}\sum_{i\in N} x_i(\tilde{t}) \tag{OMD$'''$}$$

The constraint set is the largest in OMD$'''$ and smallest in OMD. In Section 2.4 we analyzed OMD. Myerson [1981], Branco [1996] and Levin [1997], among others, analyze OMD$'''$ in different environments.

It can be shown, by changing the definition of regularity appropriately for each problem, that

1. Identical conditions imply regularity in all four problems.

2. Under these sufficient conditions,

   (a) there are deterministic solutions to OMD$''$ and OMD$'''$, and

   (b) there are ex post incentive feasible solutions to OMD$'$ and OMD$'''$.

We will illustrate these assertions for OMD$'''$ only, but in two environments, first, in the single object environment of Myerson, and next, in our combinatorial environment. It is an open question whether conditions can be identified which guarantee regularity in the Bayesian optimal mechanism design problem, for example in OMD$'''$, which lead to a solution that is not ex post incentive feasible.

3.5.3.1 The Environment in Myerson (1981)

Let $\Omega = \{\omega\}$. Abusing notation, replace the allocation rule by a map $q : T \to \Re^n$ where $q(t) = (q_1(t), ..., q_n(t))$ is such that $q_i(t) \geq 0$ and $\sum_i q_i(t) \leq 1$. The number $q_i(t)$ is the probability that the object will be given to agent i. Let $v_i(\{\omega\}, t) = t_i$ and $v_i(\emptyset, t_i) = 0$.

Myerson's monotonicity condition is:

$$\text{for every } i,\ t_i \mapsto Q_i(t_i) := \mathbb{E}q_i(t_i, \tilde{t}_{-i}) \text{ is nondecreasing}$$

The virtual valuation of an agent is given by $c_i(t_i) = t_i - \lambda_i(t_i)$. It can be shown that an expected revenue maximizing mechanism can be constructed by choosing an allocation rule q which solves

$$\max \mathbb{E} \sum_{i \in N} q_i(\tilde{t}) c_i(\tilde{t}_i) \text{ subject to } q \text{ satisfies Myerson monotonicity.}$$

The unconstrained version of this reformulation can be solved by solving the following problem parametrized by t:

$$\max_{(q_1(t),\ldots,q_n(t))} \sum_{i \in N} q_i(t) c_i(t_i).$$

This is exactly the single object and stochastic analog of our OP_n. Ignoring ties, the solution is given by setting $q_i(t) = 1$ if $c_i(t_i) = \max\{c_j(t_j) : j \in N\}$ and if $c_i(t_i) > 0$. Suppose that q^* is constructed such that $q^*(t)$ solves this problem at every t. It does not necessarily follow that q^* should be Myerson monotonic and therefore that it should solve the reformulation. However, if c_i is strictly increasing, then this is indeed the case. If i would win the object at a lower type, then he definitely wins it at a higher type. Indeed, Myerson's regularity condition is precisely that c_i should be increasing, which follows if λ_i is nonincreasing.

It is important to note, however that if c_i is increasing for every i, then $q_i^*(\cdot, t_{-i})$ is nondecreasing for every i and t_{-i}. This result is stronger than Myerson monotonicity which characterizes Bayesian incentive compatibility. In fact, when coupled with the payments identified by Myerson, q^* is ex post incentive compatible.

Under the sufficient conditions for regularity in OMD‴, we conclude that the optimal mechanism is deterministic and ex post incentive compatible. It can also be shown that this mechanism is ex post individually rational and therefore ex post incentive feasible. Thus, with a single object the solutions of OMD and OMD‴ coincide. It is not clear whether conditions can be identified in Myerson's model under which a Bayesian incentive feasible optimal mechanism can be obtained, which is not ex post incentive feasible.

3.5.3.2 The Multiobject Environment

The observation that optimal mechanism design with Bayesian feasibility conditions leads to optimal mechanisms which are ex post incentive feasible does not depend on the features of the Myerson problem which we generalize in this paper. In our multiobject model with interdependent types and valuations nonlinear in types, the same conclusion follows.

Consider problem OMD$'''$ in the multiobject setting. Since F is a strict subset of $\mathring{F}_B$, it may be expected that the value of this problem should exceed the value of OMD. However, just like in Myerson (1981), the conditions we have to impose to analyze the problem constrained by $\mathring{F}_B$ imply that the optimal mechanism be in F. We proceed with an outline of the arguments.

An allocation function q satisfies the monotonicity condition M_B if for each i, t_i and t_i'

$$\mathbb{E} \sum_{S \in C} q(S; t_i', \tilde{t}_{-i}) \left[v_i(S_i, t_i, \tilde{t}_{-i}) - v_i(S_i, t_i', \tilde{t}_{-i}) \right] \leq \mathbb{E} \int_{t_i'}^{t_i} \sum_{S \in C} q(S; y, \tilde{t}_{-i}) \partial v_i(S_i, y, \tilde{t}_{-i}) dy.$$

Note that Condition M is exactly the ex post and deterministic version of M_B. Let $\mathcal{M}_B$ be the class of allocation rules satisfying M_B. It is straightforward to check that in the single object case with private values M_B coincides with the monotonicity condition used by Myerson [1981], which we introduced in Section 2.5.3.1. Condition M_B reduces in Myerson's model to:

$$\mathbb{E} q_i(t_i', \tilde{t}_{-i})(t_i - t_i') \leq \mathbb{E} \int_{t_i'}^{t_i} q_i(y, \tilde{t}_{-i}) dy.$$

To see that M_B implies Myerson's condtion, observe that for any i, t_i' and t_i, we must have

$$\begin{aligned} \mathbb{E} q_i(t_i', \tilde{t}_{-i})(t_i - t_i') &\leq \mathbb{E} \int_{t_i'}^{t_i} q_i(y, \tilde{t}_{-i}) dy \\ &= -\mathbb{E} \int_{t_i'}^{t_i} q_i(y, \tilde{t}_{-i}) dy \\ &\leq -\mathbb{E} q_i(t_i, \tilde{t}_{-i})(t_i' - t_i) \\ &= \mathbb{E} q_i(t_i, \tilde{t}_{-i})(t_i - t_i'). \end{aligned}$$

Now, if $t_i' < t_i$, it follows that $\mathbb{E}q_i(t_i', \tilde{t}_{-i}) \leq \mathbb{E}q_i(t_i, \tilde{t}_{-i})$ and Myerson's condition holds. To see that Myerson's condition implies M_B, fix $i, t_i' < t_i$. We have

$$\begin{aligned}
\mathbb{E}\int_{t_i'}^{t_i} q_i(y, \tilde{t}_{-i})dy &= \int_{t_i'}^{t_i} \mathbb{E}q_i(y, \tilde{t}_{-i})dy \\
&= \int_{t_i'}^{t_i} Q_i(y)dy \\
&\geq \int_{t_i'}^{t_i} Q_i(t_i')dy \\
&= Q_i(t_i')(t_i - t_i').
\end{aligned}$$

Similarly if $t_i' > t_i$ and M_B follows. Thus M_B is an extension of Myerson's monotonicity condition and it will serve precisely the same purpose Condition M serves in Section 4, i.e., Condition M_B will constrain a reformulation of OMD$'''$.

Consider the following reformulation of OMD$'''$.

$$\max_{q \in \mathcal{M}_B} \mathbb{E} \sum_{i \in N} \sum_{S \in C} q(S, \tilde{t}) u_i(S_i, \tilde{t}) \tag{R$'''$}$$

Proposition 6 If q solves R$'''$ and payments are defined by

$$x_i(t) = \sum_{S \in C} q(S, t) v_i(S_i, t) - \int_{a_i}^{t_i} \sum_{S \in C} q(S, y, t_{-i}) \partial v_i(S_i, y, t_{-i}) dy$$

for each i and $t = (t_i, t_{-i})$, then the mechanism (q, x) solves OMD$'''$.

We will omit the proof which proceeds in a fashion similar to the arguments of Section 2.4. The critical observation is that Bayesian incentive compatibility is equivalent to M_B plus an envelope condition. A solution to the reformulated problem must satisfy M_B. The envelope condition follows by choice of payments. Therefore the mechanism proposed by the Proposition Bayesian incentive compatible, and in fact Bayesian incentive feasible. Using independence of types, an exact analog of Lemma 4 indicates that maximizing the expected sum of payments is equivalent to maximizing the expected sum of virtual valuations while leaving all agents with zero expected surplus at their lowest types. Since the mechanism of the Proposition has these properties, the result follows.

Thus, the principal need only solve R‴ and determine payments as in the Proposition 6 to solve OMD‴. Note that payments in Propositions 1 and 6 coincide. There is no easy way to solve R‴, as we do not know much about the constraint set $\mathcal{M}_B$. To solve the unconstrained version of R‴, i.e, the problem

$$\max_{q:C\times T\to\Re} \mathbb{E}\sum_{i\in N}\sum_{S\in C} q(S,\tilde{t})u_i(S_i,\tilde{t})$$

one can solve

$$\max_{q(\cdot,t):C\to\Re}\sum_{i\in N}\sum_{S\in C} q(S,t)u_i(S_i,t)$$

at every t. But the objective of this maximization problem is a convex combination of the numbers $\{\sum_i u_i(S_i,t) : (S_1,...,S_n) \in C\}$ and the choice variables are the weights. So, in order to solve the unconstrained version of R‴, it suffices to solve the optimal partitioning problem

$$\max_{S\in C}\sum_{i\in N} u_i(S_i,t)$$

for each $t \in T$. Note that this is precisely the same partitioning problem OP_n which appeared in Section 2.4. If $S(t)$ denotes a solution to OP_n, then defining $q^*(S(t),t) = 1$ for all t yields an optimal solution to the unconstrained version R‴. Of course, q^* need not be a solution to R‴. Regularity addresses exactly this issue.

Definition 10 The optimal mechanism design problem OMD‴ is regular if, whenever (i) $S(t)$ is a solution to OP_n and (ii) $q^*(S(t),t) = 1$ for each t, then q^* satisfies M_B.

Note that M_B reduces to Condtion M if q is deterministic. Thus the conditions under which the allocation rule constructed by solving the OP_n are precisely the same conditions that guarantee regularity in Section 2.4. Under these conditions the optimal mechanism is (1) nonstochastic, (2) satisfies M and (3) satisfies the envelope condition of Section 2.3 by choice of payments in Proposition 6. This implies that the resulting mechanism is ex post incentive feasible.

3.5.4 Other Objective Functions

The optimal mechanism design problem is a special case of a more general mechanism design problem where the objective is to maximize the expectation of a weighted sum of both the principal's and the agents' welfare. Let $\beta_0, \beta_1, ..., \beta_n$ be nonnegative numbers adding up to 1. The general formulation of the mechanism design problem would be:

$$\max_{(S,x)\in\mathcal{F}} \mathbb{E}[\sum_{i\in N} \beta_0 x_i(\tilde{t}) + \sum_{i\in N} \beta_i v_i(S_i(\tilde{t}), \tilde{t})]$$

A solution to this problem will be an ex post incentive feasible mechanism which is not "ex ante dominated" by any other ex post incentive feasible mechanism in the sense of Holmstrom and Myerson [1983].[3] Now the optimal mechanism design problem is the special case in which $\beta_0 = 1$.

Using Lemma 5 we can replace payments with virtual valuations and the problem becomes

$$\max_{(S,x)\in\mathcal{F}} \mathbb{E} \sum_i \varphi_i(S_i(\tilde{t}), \tilde{t})$$

where $\varphi_i(S_i(\tilde{t}), \tilde{t}) = \beta_0 x_i(\tilde{t}) + \beta_i v_i(S_i(\tilde{t}), \tilde{t})$. Now regularity can be obtained in Environments A and B by placing analogous restrictions on the maps φ_i.

3.6 Conclusion

In this paper, we consider mechanism design problems with multiple possibly nonidentical objects and interdependent values, assuming that agents' preferences are parametrized by one dimensional types. We identify conditions and assumptions under which the analysis of the linear, single-object problem of Myerson [1981] extends to a fully nonlinear combinatorial problem with a general interdependent valuation structure. We show that ex post incentive constraints can be characterized with only minimal assumptions on agents' valuations. We define regularity as the condition under which the incentive constraints are not binding and identify sufficiency conditions for it. Our model is rich enough to incorporate problems with

[3] The nature of the domination relation between mechanisms can be altered by letting agents' welfare weights depend on their own types (interim domination) or by letting all weights depend on the collective type vector. It can be shown that if a mechanism is not ex ante dominated by any other feasible mechanism, then it is not interim (or ex post) dominated by any feasible mechanism either.

complements, substitutes and hybrid problems in which objects are complements for some types and substitutes for others. In our analysis of the efficient mechanism design problem we identify conditions under which the Vickrey-Clarke-Groves mechanism can be extended to interdependent value environments while preserving its ex post incentive compatibility properties. In our analysis of the optimal mechanism design problem we show that our approach unifies and generalizes earlier work on the problem in more restricted settings.

A very useful aspect of the notion of regularity used by Myerson [1981] is that its converse also lends itself to a tractable analysis. In the linear model of Myerson, regularity is equivalent to the condition that the hazard rates of type distributions are nondecreasing. Sufficient conditions for regularity in our model are conditions on valuation functions as well as type distributions. Hence the interesting question about the analysis of irregular problems may be far too complicated in a nonlinear model.

The methods we employ apply to more general settings as long as the lattice structure is preserved. As an example, consider a single agent mechanism design problem in which the outcome space is a lattice L_1 and the valuation of the agent is a map $v : L_1 \times [a, b] \to \Re$ given by $v(q, t) = g(w(q), t)$ where $w : L_1 \to L_2$ is isotone, L_2 is a lattice and $g : L_2 \times T \to \Re$. If $L_1 = L_2 = 2^{\Omega}$ for some finite set Ω and if w is the identity map, we specialize to Environment A. If $L_1 = 2^{\Omega}$, $L_2 = \Re$ and w is a set function we specialize to Environment B.

Chapter 4

Mechanism Design with Unit-Demand: An Application to Internet Advertising

4.1 Introduction

Consider an internet search engine selling to potential advertisers various advertisement spots displayed on the computer screen following a keyword search. Each advertisement spot is a different object, as different locations on the screen have varying degrees of success in attracting users to visit the displayed sponsor. The search engine must, in ignorance of the valuations of the sponsors, find a way to sell the spots in the most profitable way. Internet advertising is a source of substantial income for internet giants like Google and Yahoo!, and it is an attractive new environment of analysis for economists.

The internet advertising environment fits nicely into the framework of the previous chapters and methods of mechanism design can be used to analyze the expected revenue maximization problem of the seller. An important feature of the internet advertising setup is that it is reasonable to model private information as one dimensional. The private information of an advertiser is the expected amount of money spent by the user who clicks on his advertisement. If the expected expenditure of the user is independent of the spot in which the advertisement is displayed, then private information is just a number. Another important feature of this setup is that the objects at hand are physically ranked from top to bottom. Thus, it is reasonable to assume that advertisers' valuations are also ranked as such, i.e., that each advertiser likes the top spot the best, the second spot next, and so on. This common ranking feature simplifies the problem quite a bit.

The analyses of internet advertising have always assumed, implicitly or explicity, that each advertiser demands at most one spot. This means that if a sponsor is allocated a set of spots, his valuation is exactly what it would have been had he been allocated the top spot in that set. This assumption is made for a very good reason. Under the assumption of unit demand, together with other assumptions which we discuss below, mechanisms which can be considered extensions of standard auctions can be made revenue maximizing by choice of a reserve price. In this chapter our goal is to analyze mechanism design problems with unit demand. As in Chapter 3, the level of generality of our analysis will allow for asymmetric agents and interdependent valuations. We will argue that the unit demand model can not be analyzed as a special case of Environment A or Environment B, which we studied in Chapter 3. The unit demand problem usually lacks the supermodularity features which are needed in Environment A and it lacks the type-independent scalarization feature necessary in Environment B. However the unit demand problem associated with internet advertising is a special case of Environment B, and we analyze the optimal mechanism design problem as a special case of the methods of Chapter 3.

Although the assumptions we make in analyzing internet advertising are a compromise from generality, these assumptions are standard in the literature. In particular, this structure allows us to identify the revenue maximizing mechanism. It also makes it possible for us to interpret the payments in the optimal mechanism as the solution to the dual of a linear program in which spots are assigned to advertisers in a way to maximize revenue. This implies that the optimal mechanism can be decentralized via prices.

Although quite new, the literature on internet advertising already has two strands. In one strand are papers which analyze the equilibrium behavior in the auctions that are or could be used to sell the spots to the advertisers. For example Varian (2006) and Edelman et al. (2007) analyze a particular auction, called the generalized second price auction, and show that it has a symmetric equilibrium in which higher types submit higher bids. In this auction each advertiser submits a bid and bidders are ranked in decreasing order of their bids. Advertisers then submit the search engine a fixed payment every time their ad is clicked on. In the generalized second price auction the per click payment of the kth highest bidder is the next highest bid, or zero if he is the lowest bidder. Feng et al. (2006) analyze what could be called a discriminatory auction in which the kth highest bidder pays the search engine his own bid per click. One can imagine other standard auction formats in this setting, which allocate the spots in the same way, but determine payments differently. In a Vickrey auction, an advertiser's payment would coincide with the externality he imposes on the others by his presence. In a uniform price auction, each advertiser would pay the highest losing bid per click. It can be shown that these four auctions are revenue equivalent under suitable conditions and they are also equivalent to any other auction which possesses an equilibrium at which bidders use increasing bid functions and which leaves the lowest types with zero expected surplus. Furthermore, in our model the standard auctions are optimal with an appropriately chosen reserve price. Hence, there is a sense in which the basic results of auction theory extend to the internet advertising environment. Note, however, that the standard auctions we considered assign each advertiser at most one spot. As such, in the absence of the unit demand assumption, they are not optimal with or without appropriate reserve prices.

A second strand of the literature analyzes the internet advertising literature from a mechanism design point of view. Iyengar and Jumar (2006) and Feng (2007) find the optimal mechanism within the class of mechanisms which allocate each advertiser at most one spot. This chapter complements these two papers. We show, first, that the regularity conditions in the optimal mechanism design problem can be studied using the analysis of the previous chapter and we derive the optimal mechanism. We then show that the payments in the optimal mechanism can be given the interpretation of prices by analyzing the internet advertising problem in a linear assignment framework.

4.2 The Unit Demand Model

Let $\Omega = \{\omega_1, ..., \omega_m\}$ be a finite set of objects and let $N = \{1, ..., n\}$ be a finite set of agents. Each agent i has a type $\tilde{t}_i$ which is a random variable taking takes values in $[a_i, b_i]$. We assume that types are independently drawn from continuous and strictly increasing distributions F_i, $i \in N$, whose inverse hazard rates are λ_i. We will denote the expectation operator by $\mathbb{E}$ and realizations of $\tilde{t}_i$ by t_i and write $\tilde{t} = (\tilde{t}_i, \tilde{t}_{-i})$ and $t = (t_i, t_{-i})$. Agents have interdependent valuations and unit demand. To be precise, for each i there exists a scalar valued map $g_i : \Omega \times T \to \Re_+$ such that the valuation of i for every $A \subseteq \Omega$ at every type vector $t = (t_i, t_{-i})$ is given by the unit demand valuation

$$v_i(A, t) = \max_{\omega \in A} g_i(\omega, t).$$

In Chapter 3, we presented two nonnested environments in which regularity conditions can be obtained that allow us to solve the optimal mechanism design problem. However, the valuation functions as defined above do not fit into either environment. Since $v_i(\cdot, t)$ is actually submodular, we cannot apply the results developed for environment A except in very special cases. Furthermore, there may not exist a set function $A \mapsto \mu_i(A)$ and a function w_i such that $v_i(A,t) = w_i(\mu_i(A), t)$ so the results in environment B may not be applicable either. These comments apply to the general unit demand valuation function defined above and it turns out that the unit demand valuations associated with internet click auctions have a special feature that allows us to apply the results developed for environment B.

Let $\Omega = \{1, ..., m\}$ be the set of ad positions that will be displayed by an internet search engine after a keyword search ranked from top to bottom. Each position $k \in \Omega$ is associated with a number α_k interpreted as the number of user clicks on the ad displayed at that position. Suppose that the positions $1, .., m$ are ranked according to "clicks per unit time" so that $\alpha_1 > \cdots > \alpha_m$. Let $N = \{1, ..., n\}$ be the set of potential advertisers. In the click auction, the payoff to advertiser i who is assigned position k when the type profile is t is defined as $g_i(\alpha_k, t)$ so that

$$v_i(S,t) = \max_{k \in S} g_i(\alpha_k, t).$$

In the case of click auctions, it is assumed that $k < j$ implies $g_i(\alpha_k, t) > g_i(\alpha_j, t)$. Now define $\mu(S) = \min\{k : k \in S\}$. Then

$$v_i(S,t) = \max_{k \in S}\{g_i(\alpha_k, t)\} = \; g_i(\alpha_{\mu(S)}, t)$$

Defining

$$w_i(k,t) = g_i(\alpha_k, t)$$

it follows that

$$v_i(S,t) = \; g_i(\alpha_{\mu(S)}, t) = w_i(\mu(S), t)$$

and condition 1 in the definition of environment B is satisfied. If we make the additional assumption that

$$w_i(k,t) - w_i(k+1,t) \geq w_i(k,t_{-i},t_i') - w_i(k+1,t_{-i},t_i')$$

whenever $t_i > t_i'$ and $1 \leq k \leq m-1$, then w_i satisfies NDD and condition 2 in the definition of Environment B is satisfied.

We will use our methods to derive the optimal mechanism for the data of a symmetric private values auction model proposed by Feng et al. Adopting their notation, let $w_i(k,t_{-i},t_i) = V_k(t_i)$ where $x \mapsto V(x)$ is twice differentiable, nondecreasing and concave for each k and suppose that $F_i = F, a_i = a$ and $b_i = b$ for all i. Furthermore, it is assumed that NDD is satisfied: if $x > x'$ and $1 \leq k \leq m-1$, then

$$V_k(x) - V_{k+1}(x) \geq V_k(x') - V_{k+1}(x').$$

As a consequence of NDD and the differentiability of $V_k(\cdot)$, it follows that

$$V_k'(x) \geq V_{k+1}'(x) \text{ for each } x \in [a,b].$$

The common inverse hazard rate will be denoted λ and it is assumed that λ is nonincreasing. Consequently, using the notation of Chapter 3, the problem OP_n becomes

$$\max_{(S_1,..,S_n)} \sum_{i \in N} \left[V_{\mu(S_i)}(t_i) - V_{\mu(S_i)}'(t_i)\lambda(t_i) \right].$$

For each k, consider the function

$$x \mapsto u_k(x) = V_k(x) - V_k'(x)\lambda(x)$$

Since

$$u_k'(x) = V_k'(x) - V_k''(x)\lambda(x) - V_k'(x)\lambda'(x) \geq 0$$

it follows that $x \mapsto u_k(x)$ is nondecreasing for each k. Now suppose that the type profile t

has the property that $t_1 > \cdots > t_n$. Then $u_k(t_1) > \cdots > u_k(t_n)$ and to solve the problem

$$\max_{(S_1,..,S_n)} u_{\mu(S_1)}(t_1) + \ldots + u_{\mu(S_n)}(t_n)$$

we need only use the following procedure. First, define the m numbers $\sigma_1(t), .., \sigma_m(t)$ as follows:

$$\begin{aligned} \sigma_1(t) &\in \arg\max_{k\in\Omega} u_k(t_1) \\ \sigma_2(t) &\in \arg\max_{k\in\Omega\setminus\{\sigma_1(t)\}} u_k(t_2) \\ &\vdots \\ \sigma_m(t) &\in \arg\max_{k\in\Omega\setminus\{\sigma_1(t),..,\sigma_{m-1}(t)\}} u_k(t_m) \end{aligned}$$

Assume that $\{k : u_{\sigma_k(t)}(t_k) \geq 0\} \neq \varnothing$ and define

$$i^* = \max\{k : u_{\sigma_k(t)}(t_k) \geq 0\}.$$

An optimal solution to OP_n is then defined by

$$\begin{aligned} S_i^*(t) &= \{\sigma_i(t)\} \text{ if } 1 \leq i \leq i^* \\ &= \emptyset \text{ if } i > i^*(t). \end{aligned}$$

Note that to use Corollary 4 to conclude that this solution corresponds to an optimal mechanism, we need to make sure that the maps $(k, t_i) \mapsto u_k(t_i)$ satisfy SSCP.

Note also that $t_i > t_j$ does not imply that agent i is assigned a higher advertising position than agent j in an optimal solution to OP_n. In order to obtain a solution in which $t_i > t_j$ implies that agent i is assigned a higher advertising position than agent j, it is sufficient that the virtual valuations satisfy the following additional monotonicity condition: for each x and $2 \leq k \leq m$,

$$u_k(x) \geq 0 \Rightarrow u_{k-1}(x) \geq u_k(x).$$

In this case, $t_1 > \cdots > t_n$ implies that $\sigma_i(t) = i$ if $1 \leq i \leq i^*$ where $i^* = \max\{k : u_k(t_k) \geq$

$0\}$. In the next section, we interpret the optimal solution and associated payments when $w_i(k, t_i) = \alpha_k t_i$.

4.3 An Optimal Mechanism for Internet Advertising with Linear Valuations

The optimal solution computed in the previous section has a particularly appealing interpretation when $w_i(k, t_i) = \alpha_k t_i$. In this case,

$$u_k(x) = \alpha_k \left[x - \lambda(x)\right] \equiv \alpha_k c(x)$$

Pick a vector of types t and assume that $t_1 \geq \cdots \geq t_n$. Note that c is strictly increasing since λ is nonincreasing. Then

$$\{k : u_k(t_k) \geq 0\} = \{k : c(t_k) \geq 0\} = \{k : t_k \geq c^{-1}(0)\}$$

Suppose that $\{k : t_k \geq c^{-1}(0)\}$ is nonempty and let $i^* = m$ or $i^* = \max\{i : t_i \geq c^{-1}(0)\}\}$ whichever number is lower.

Proposition 1 Under the assumptions above the mechanism (S^*, x^*) defined for every i and t by

$$\begin{aligned} S_i^*(t) &= \begin{cases} \{i\} & \text{if } i \leq i^* \\ \emptyset & \text{if } i > i^* \end{cases} \\ x_i^*(t) &= \begin{cases} \sum_{s=i}^{i^*-i} (\alpha_s - \alpha_{s+1}) t_{s+1} + \alpha_{i^*} c^{-1}(0) & \text{if } i < i^* \\ \alpha_{i^*} c^{-1}(0) & \text{if } i = i^* \end{cases} \end{aligned}$$

solves the optimal mechanism design problem for internet advertising.

Proof. Fix t and assume without loss of generality that t_i weakly decreases in i. Then $c(t_1) > \cdots > c(t_n)$ and since the problem is regular by assumption, the optimal allocation rule S^* solves

$$\max_{(A_1,\ldots,A_n) \in C} \sum_{i \in N} \max\{\alpha_k : k \in A_i\} c(t_i).$$

This implies that the allocation rule in the statement proposition is optimal. Suppose that $i^* = \max\{i : t_i \geq c^{-1}(0)\} \leq m$ so that there are at least as many spots as there are advertisers whose types exceed $c^{-1}(0)$. The expression for optimal mechanism payments given in Equation 1 becomes, for every $i \in \{1, ..., i^* - 1\}$,

$$\begin{aligned}
x_i(t) &= v_i(S_i(t), t_i) - \int_a^{t_i} \partial v_i(S_i(y, t_{-i}), y) dy \\
&= \alpha_i t_i - \sum_{s=i}^{i^*-i} \int_{t_{s+1}}^{ts} \alpha_s dy - \int_{c^{-1}(0)}^{t_i^*} \alpha_{i^*} dy \\
&= \alpha_i t_i - \sum_{s=i}^{i^*-i} \alpha_s (t_s - t_{s+1}) - \alpha_{i^*}(t_{i^*} - c^{-1}(0)) \\
&= \sum_{s=i}^{i^*-i} (\alpha_s - \alpha_{s+1}) t_{s+1} + \alpha_{i^*} c^{-1}(0)
\end{aligned}$$

which finishes the proof. ■

Note that at the optimal mechanism, agents who do not receive a position do not pay. Note also that there is a strictly positive probability that some positions will remain empty. In particular if there is no agent i such that $t_i \geq c^{-1}(0)$, then no position is allocated.

If $i < i^*$, then $x_i(t)$ depends on the payments of all advertisers placed in lower positions. Simple calculations show that

$$\alpha_i c^{-1}(0) \leq x_i(t) \leq \alpha_i t_{i+1}.$$

If advertisers bid their true types, then the upper bound for the optimal payment is the payment collected by the generalized second price auction and the lower bound is the payment collected by a mechanism which imposes a uniform price of $c^{-1}(0)$ per click. It is known however that these mechanisms do not induce truthful bidding. However they have equilibria with increasing bid functions. We can therefore conclude, using standard revenue equivalence arguments, that they would earn the search engine exactly the same revenue as the optimal mechanism if they are complemented by the reserve price $c^{-1}(0)$.

A special case of the internet advertisement auction in which positions are perfect substitutes can be analyzed by letting $\alpha_k = \alpha$ for each k. This is exactly the symmetric version of the multiunit problem analyzed by Branco (1996). Under the assumption of unit demand, the optimal mechanism assigns a position to each agent whose type exceeds $c^{-1}(0)$ and collects $\alpha c^{-1}(0)$ from him. Note that with $m = 1$ we recover the single object mechanism design problem.

4.4 A Linear Assignment Approach

The internet advertising problem is similar to a linear assignment problem. This similarity, within the context of the generalized second price auction, is discussed in both Edelman et al. (2007) and Varian (2006). The optimal mechanism design problem for internet advertisement could also be approached as a linear assignment problem. Let us introduce the following notation. Let $z_{ik} = 1$ if spot k is assigned to agent i and 0 otherwise. Let $z_{0k} = 1$ if k is not assigned to any agent and 0 otherwise. In effect, we are treating the search engine as agent 0. Assume, once again, that $t_1 \geq \cdots \geq t_n$ and recall that $c(\cdot)$ is increasing. Assume that for some i, $t_i \geq c^{-1}(0)$. A version of the linear assignment problem whose dual has an interesting and straightforward decentralization interpretation is as follows.

$$\max_{\{x_{ik}\}} \sum_{i=1}^{n} \sum_{k=1}^{m} \alpha_k t_i z_{ik} + \sum_{k=1}^{m} \alpha_k c^{-1}(0) z_{0k}$$

$$\text{subject to} \begin{cases} \sum_{i=0}^{n} z_{ik} = 1 \text{ for each } k \\ \sum_{k=1}^{m} z_{ik} \leq 1 \text{ for each } i \neq 0 \\ \sum_{k=0}^{n} z_{0k} \leq m \\ z_{ik} \leq 1 \text{ for each } i \text{ and } k \\ z_{ik} \geq 0 \text{ for each } i \text{ and } k. \end{cases}$$

The last summation term in the objective accounts for the fact that optimally, the principal would not like to assign a position to an agent whose type is less than $c^{-1}(0)$. Some of the constraints in this problem are clearly redundant. If $z_{ik} \leq 1$ for each i and k, then $\sum_k z_{0k} \leq m$ for each k. Moreover if $z_{ik} \geq 0$ for each i and k and $\sum_{i=0}^{n} z_{ik} = 1$ for each k, then $z_{ik} \leq 1$ for each i and k. So the problem becomes:

$$\max_{\{x_{ik}\}} \sum_{i=1}^{n} \sum_{k=1}^{m} \alpha_k t_i z_{ik} + \sum_{k=1}^{m} \alpha_k c^{-1}(0) z_{0k}$$

$$\text{subject to} \begin{cases} \sum_{i=0}^{n} z_{ik} = 1 \text{ for each } k \\ \sum_{k=1}^{m} z_{ik} \leq 1 \text{ for each } i \neq 0 \\ z_{ik} \geq 0 \text{ for each } i \text{ and } k. \end{cases}$$

Once again, let $i^* = \min\{m, \max\{i : t_i \geq c^{-1}(0)\}\}$. The solution to the primal entails assigning the first i^* positions to the first i^* advertisers and leaving any remaining positions open. Hence a solution to the primal is given by $z_{ii} = 1$ if $i \leq i^*$, and $z_{0k} = 1$ otherwise. The optimal value of the primal is $\sum_{i=1}^{i^*} \alpha_i t_i + \sum_{i=i^*+1}^{m} \alpha_i c^{-1}(0)$.

The dual problem is

$$\min_{\{q_k\},\{p_i\}} \sum_{k=1}^{m} q_k + \sum_{i=1}^{m} p_i$$

$$\text{subject to} \begin{cases} q_k + p_i \geq \alpha_k t_i \text{ for each } i \text{ and } k \\ p_i \geq 0 \text{ for each } i. \end{cases}$$

Using the theory of duality we can find a solution to the dual.

Proposition 2 The dual of the internet advertisement problem is solved by:

$$q_k = \begin{cases} x_k(t) \text{ if } k \leq i^* \\ \alpha_k c^{-1}(0) \text{ otherwise} \end{cases}$$

$$p_i = \begin{cases} \alpha_i t_i - q_i \;\; \textit{if } i \leq i^* \\ 0 \text{ otherwise} \end{cases}$$

This solution is facilitated by the informed guess that the payments in the optimal mechanism, when presented as prices to the agents, will lead to self-selection and solve the dual: $q_k = x_k(t)$ may be given the interpretation of the price of position k. In effect, the search engine presents the advertisers with a menu $\{(k, q_k)\}_{k=1,\ldots,m}$ and each $i = 1, \ldots, i^*$ optimally picks item i from the menu. The rest of the agents do not find it optimal to purchase an item. The conditions $p_i \geq 0$ may be viewed as ex post individual rationality and the conditions $p_i \geq \alpha_k t_i - q_k$ may be viewed as ex post incentive compatibility conditions.

Proof of Proposition 2. Note that dual program achieves the optimal value of the primal at the proposed solution. So all that needs to be checked is feasibility. We have $p_i \geq 0$ since if $i \leq i^*$, then

$$\begin{aligned} p_i &= \alpha_i t_i - x_i(t) \\ &= \sum_{s=i}^{i^*-1} \alpha_s(t_s - t_{s+1}) + \alpha_{i^*}(t_{i^*} - c^{-1}(0)) \\ &\geq 0. \end{aligned}$$

In order to check $q_k + p_i \geq \alpha_k t_i$ for each i and k we need to focus on several cases. Suppose that $k = i$. If $i \leq i^*$ then $p_i + q_i = \alpha_i t_i$. If $i > i^*$, then $p_i + q_i = \alpha_i c^{-1}(0) \geq \alpha_i t_i$. If $k \leq i^*$ then

$$\begin{aligned} x_i(t) - x_k(t) &= \sum_{s=i}^{k-1}(\alpha_s - \alpha_{s+1})t_{s+1} \\ &\leq \sum_{s=i}^{k-1}(\alpha_s - \alpha_{s+1})t_i \\ &= (\alpha_i - \alpha_k)t_i \end{aligned}$$

which in turn implies

$$\begin{aligned} p_i &= \alpha_i t_i - x_i(t) \\ &\geq \alpha_k t_i - x_k(t). \end{aligned}$$

If $i < i^* < k$, we have

$$\begin{aligned} p_i &= \alpha_i t_i - x_i(t) \\ &= \sum_{s=i}^{i^*-1} \alpha_s (t_s - t_{s+1}) + \alpha_{i^*}(t_{i^*} - c^{-1}(0)) \\ &\geq \sum_{s=i}^{i^*-1} \alpha_{i^*} (t_s - t_{s+1}) + \alpha_{i^*}(t_{i^*} - c^{-1}(0)) \\ &= \alpha_{i^*}(t_i - c^{-1}(0)) \\ &\geq \alpha_k (t_i - c^{-1}(0)). \\ &= \alpha_k t_i - q_k. \end{aligned}$$

If $i = i^*$, then

$$\begin{aligned} p_{i^*} &= \alpha_{i^*} t_{i^*} - x_{i^*}(t) \\ &= \alpha_{i^*}(t_{i^*} - c^{-1}(0)) \\ &\geq \alpha_k (t_{i^*} - c^{-1}(0)) \\ &= \alpha_k t_i - q_k. \end{aligned}$$

If $i^* < i$, then

$$\begin{aligned} p_i &= 0 \\ &\geq \alpha_k t_i - \alpha_k c^{-1}(0) \\ &= \alpha_k t_i - q_k. \end{aligned}$$

Now suppose that $k < i$. If $i \leq i^*$, then

$$\begin{aligned} x_k(t) - x_i(t) &= \sum_{s=k}^{i-1} (\alpha_s - \alpha_{s+1}) t_{s+1} \\ &\geq \sum_{s=k}^{i-1} (\alpha_s - \alpha_{s+1}) t_i \\ &= (\alpha_k - \alpha_i) t_i \end{aligned}$$

which implies

$$\begin{aligned} p_i &= \alpha_i t_i - x_i(t) \\ &\geq \alpha_k t_i - x_k(t) \\ &= \alpha_k t_i - q_k. \end{aligned}$$

If $k < i^* < i$, then

$$\begin{aligned} p_i &= 0 \\ &\geq \alpha_k c^{-1}(0) - x_k(t) \\ &\geq \alpha_k t_i - q_k. \end{aligned}$$

If $i^* \leq k$, then

$$\begin{aligned} p_i &= 0 \\ &\geq \alpha_k t_i - \alpha_k c^{-1}(0) \\ &= \alpha_k t_i - q_k. \end{aligned}$$

This finishes the proof. ■

4.5 Conclusion

We analyze a multiobject mechanism design problem with agents who have unit demands at a fairly general level. The main benefit of assuming unit demand is that a mechanism can be assumed, without loss of generality, to give each agent at most one object. The internet advertising problem provides an interesting and tractable example. In particular, the assumptions of linearity of valuations in types and private values imply that the standard monotone hazard rate condition is sufficient for regularity. The assumption of unit demand makes it possible to solve for the combinatorial optimization problem in which, for each type vector, advertisement spots are allocated between advertisers in a way to maximize the sum of virtual valuations. If the monotone hazard rate condition holds, then the optimal mechanism identifies a critical type at which the virtual valuation is zero and ranks the advertisers in the order of their types provided that their types exceed the critical type. The payment of each advertiser who gets a spot can easily be calculated. These payments form a solution to the dual of the linear assignment problem in which spots are allocated to advertisers in a way to maximize total welfare provided that their type exceeds the critical type.

Even though the generalization of the internet advertising problem to valuations beyond the unit demand assumption is interesting, it lacks tractability as the combinatorial optimization problem OP_n becomes more involved. However, the regularity conditions for such problems can easily be obtained and computational methods can be used to solve the problem. In particular, as long as the valuations are linear in types and isotone in the set of objects received, the monotone hazard rate condition will continue to be sufficient for regularity.

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