ESSAYS ON BAYESIAN ANALYSIS OF FINANCIAL ECONOMICS

by

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This dissertation consists of three essays with each essay forming a chapter. The regression models in these three chapters are different but share the same feature: the error terms of the models all follow ARMA-GARCH error processes generated either from normal or exponential power distributions.

In the first chapter I present a spot asset pricing model that is known as the CKLS model. Two CKLS models are compared. In one model the ARMA-GARCH error process is generated by the exponential power distribution while in the other model the error process is generated by the normal distribution. Using monthly U.S. federal funds rate I estimate the parameters of the CKLS models. From the predictive densities I obtain the distributions of the mean squared errors of forecast (MSEF) and the predictive deviance information criterion (PDIC). In addition I use the Bayes factor and the deviance information criterion (DIC). Markov Chain Monte Carlo (MCMC) algorithms, which are stochastic numerical integration methods, are used. I find that in general the CKLS model with the error term generated by the exponential power distribution is chosen over the model with the normal error term.
In the second chapter I first compare two MCMC algorithms: random walk draw and non-random walk draw for a Markov switching regression model. Two Markov switching models are compared: one with the variance of the normal distribution generated by the state space variable and the other with the constant variance. The realized volatilities of MMM Company are used to estimate and compare the models. The mean squared errors (MSE) and mean squared errors of forecast (MSEF) are used as the model selection criteria. I find that the model with the constant variance is chosen over the model with the state space variance by the MSE but the latter is chosen over the former by the MSEF.

In the third chapter I estimate a bivariate copula model. Each of the two regressions is generated by the exponential power distribution. I use monthly data on SP500 and FTSE100. Results show that the correlation parameter for SP500 and FTSE100 is .6893.
Acknowledgement

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1 Introduction

Better modeling and forecasting macroeconomic variables and financial time series such as interest rates, stock index returns and volatility are important issues in both finance and economics. In my dissertation, I focus on model designing, model comparison and model selection. My objective is to apply econometrics to financial economics. Bayesian inference is used to analyze time series in economics. The three chapters in my dissertation are as follows.

In the 1st chapter, I create a CKLS-ARMA-GARCH-EPD model for short-term interest rate. This model adds some new features to the CKLS model suggested by Chan, Karolyi, Longstaff and Sanders (1992). Firstly, the diffusion parameter follows a GARCH process to capture the high degree of volatility persistence (see Engle (1982) and Bollerslev (1987)). Secondly, exponential power distribution (EPD) (see Subbotin (1923)) is assumed for the error term to better model the fat tail property of interest rate.

Markov Chain Monte Carlo algorithm (MCMC) is used to estimate this model. Then, the predictability of this model is compared with other model. Bayes Factor (see Gelfand and Dey (1994)), Deviance Information Criterion (DIC) (see Spiegelhalter(2002)), Predictive Deviance Information Criterion (PDIC) and Mean Squared Error of Forecast are generated within the MCMC algorithms and used as model selection criteria.

The DIC shows CKLS-ARMA-GARCH-EPD has better in-sample fit. The long-run equilibrium value for short-term interest is 1.53% calculated
from the CKLS-ARMA-GARCH-EPD. And out-of-sample forecast comparison shows the model with better in-sample fit is not necessary the one with better forecastability. Rolling estimation scheme shows CKLS-ARMA-GARCH-EPD predicts long-run better.

One of the future extensions for chapter 1 is to add some macro variables of Bernake et al. (2005)) into CKLS-ARMA-GARCH-EPD model and then compare its forecastability with other models. Another extension could add a regime switch parameter in CKLS-ARMA-GARCH-EPD model (see Dueker et al. (2007)). One can also add 2 or 3 latent factors in this model (see Sorwar(2005)). CKLS-ARMA-GARCH-EPD model can be used to price derivatives. Or we can perform some non-nested model comparison. Last but not the least, we can add jump components into this model.

In the 2nd chapter, we study Markov switching models by Bayesian inference. Markov switching model (hereafter denoted as MSW ) was introduced to economics by Hamilton (1989) . We first compare two MCMC algorithms: random walk draw and non-random walk draw for a Markov switching model with state variable in the variance(i.e., MSW-ARMA-GARCH-NORMAL-\(S_t\)-in-variance, see Das and Yoo(2004)). Then, we compare this Markov switching model with other model (i.e., MSW-ARMA-GARCH-NORMAL-\(S_t\)-in-mean, see Yoo(2006)) by MSE and MSEF for the realized volatility of MMM stock prices. For realized volatility, one can refer to Andersen and Bollerslev (1997).

Results show algorithm of non-random walk draw is better. And MSW-
ARMA-GARCH-NORMAL-$S_t$-in-mean has better in-sample fit, but MSW-ARMA-GARCH-NORMAL-$S_t$-in-variance has better out-of-sample forecastability. Future extensions for chapter 2 could be follows. First, we can compare Markov switching model with other nonlinear models such as TARMA. Secondly, we can consider other model selection criteria. Thirdly, we can also extend previous Markov switching models by assuming other error distributions such as the exponential power distribution. Last but not least, we can design a new model which includes two regime switching variables: one regime switching variable in the mean component, another regime switching variable in the variance component.

In chapter 3, we design a Gaussian copula model with ARMA-GARCH-EPD error terms (i.e., Copula-ARMA-GARCH-EPD). Copulas are convenient tools to construct multivariate joint distributions from margins. In financial risk assessment and actuarial analysis, dependence modeling with copula functions is widely used. Li (2000) introduces Gaussian copula model into pricing Collateralized Debt Obligations (CDOs). We design Markov Chain Monte Carlo (MCMC) algorithms to estimate the Copula-ARMA-GARCH-EPD model. Monthly data on SP500 and FTSE100 are analyzed.

Results show the MCMC algorithm is convergent. The correlation parameter $\rho$ for SP500 and FTSE100 is .6893. One of the future extensions for chapter 3 is to compare Copula-ARMA-GARCH-EPD model with other nonlinear models. Another extension could add a regime switch parameter in Copula-ARMA-GARCH-EPD model (see Dueker et al. (2007)). Or we
can add some macro variables of Bernake et al. (2005)) into Copula-ARMA-GARCH-EPD model and then compare its forecastibility with other models. Last but not the least, one may use Copula-ARMA-GARCH-EPD to price CDOs.
2 CKLS models for US short-term interest rate

2.1 Motivation

A short term interest rate is one of the most important economic indicators. Better modeling and forecasting the short term interest rate are crucial for risk management, derivatives pricing and monetary policy.

Since the mid 1970’s, asset prices have been modeled by diffusion processes. In 1977, Vasicek(1977) applied an Ornstein–Uhlenbeck process to model the short term interest rate. Since the Vasicek model can generate negative interest rates, Cox, Ingersoll and Ross(1985) suggested a square root process. Chan, Karolyi, Longstaff and Sanders(1992) proposed a more flexible model now known as the CKLS model. The CKLS model nests a class of asset pricing models as shown in Table 1.

In CKLS formula, \( r_t \) is the asset price at time \( t \) or interest rate level. \( W_t \)

<table>
<thead>
<tr>
<th>Model</th>
<th>Differential Equation</th>
<th>( \beta = 0 )</th>
<th>( \gamma = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Merton(1973)</td>
<td>( dr_t = \alpha dt + \sigma dW_t )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Black and Scholes(1973)</td>
<td>( dr_t = \beta r_t dt + \sigma r_t dW_t )</td>
<td>( \alpha = 0 )</td>
<td>( \gamma = 1 )</td>
</tr>
<tr>
<td>Cox (1975)</td>
<td>( dr_t = \beta r_t dt + \sigma r_t^2 dW_t )</td>
<td>( \alpha = 0 )</td>
<td></td>
</tr>
<tr>
<td>Vasicek(1977)</td>
<td>( dr_t = (\alpha + \beta r_t)dt + \sigma dW_t )</td>
<td>( \gamma = 0 )</td>
<td></td>
</tr>
<tr>
<td>Dothan(1978)</td>
<td>( dr_t = \sigma r_t dW_t )</td>
<td>( \alpha = 0 )</td>
<td>( \beta = 0 )</td>
</tr>
<tr>
<td>Brennan and Schwartz(1980)</td>
<td>( dr_t = (\alpha + \beta r_t)dt + \sigma r_t dW_t )</td>
<td>( \gamma = 1 )</td>
<td></td>
</tr>
<tr>
<td>Cox et al.(1980)</td>
<td>( dr_t = \sigma r_t^{1.5} dW_t )</td>
<td>( \alpha = 0 )</td>
<td>( \beta = 0 )</td>
</tr>
<tr>
<td>Cox et al.(1985)</td>
<td>( dr_t = (\alpha + \beta r_t)dt + \sigma r_t^{1/2} dW_t )</td>
<td>( \gamma = 0.5 )</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Asset Pricing Models Nested in the CKLS Model
is a Wiener Process (or Brownian motion), $\alpha$, $\beta$, $\sigma$ and $\gamma$ are parameters. $\beta$ measures the speed of mean reversion. $r_t$ mean-reverts towards the level $\alpha/\beta$. $\sigma$ is the diffusion parameter. $\gamma$ is the constant elasticity of variance with respect to interest rate level.


There are two limitations of the CKLS model. One is the diffusion parameter\(^1 \sigma\) of the CKLS model is a constant, which may not capture the extremely high degrees of volatility in the asset prices. The other limitation is the CKLS model discretized by Euler-Maruyama scheme assumes normal distribution for the error term. The discrete time specification of the CKLS

\(^1\text{For discrete-time specification of the CKLS model, } \sigma \text{ is also the parameter in variance if variance exists or is finite.}\)
To overcome the limitations of CKLS model and better model the high degrees of volatility persistence, we estimate the CKLS model with an ARMA-GARCH error term that follows an exponential power distribution (denoted as CKLS-ARMA-GARCH-EPD or Model 1). The reasons we use GARCH for conditional variance are 1) in the literature of finance and economics, the ARCH model of Engle(1982) and the GARCH model of Bollerslev(1986) have been used widely to model the high degrees of volatility persistence. 2) Brenner et al.(1996) and Koedijk et al.(1997) show the CKLS-GARCH model performs much better statistically than either the CKLS or the GARCH models with white noise errors.

Even though normal distribution assumption has been widely used in many models, there are non-normal distributions used in literature. For example, Mandelbrot(1963) and Fama(1963) used the stable Paretian family of distribution. Bollerslev(1987) extended the GARCH model to allow for conditionally t-distributed errors. Liu and Brorsen(1995) assumed a stable distribution for the error term of a GARCH model with application to foreign currency returns. Nelson(1991) and Bali and Wu(2006) used the generalized

\[ r_t - r_{t-1} = \alpha + \beta r_t + \sigma r_t^\gamma \varepsilon_t, \varepsilon_t \sim N(0, 1) \]  

To discretize a continuous-time model, we can use Euler-Maruyama scheme, Milstein scheme or other more sophisticated approximation techniques. We choose Euler-Maruyama scheme for simplicity.
error distribution (or exponential power distribution, i.e., EPD\(^3\)) for the error term\(^4\). We employ the exponential power distribution (EPD) for the error in our model. In addition, we specify that the error term follows an ARMA-GARCH model.


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\(^3\)EPD is also known as the generalized error distribution. The EPD is proposed by Subbotin (1923). For more information about EPD, please see Box and Tiao (1973), and Tsurumi and Shimizu (2008).

in the GMM estimates of the speed of mean reversion $\beta$ in CKLS model. Qian et al. (2005) showed the Bayesian and MLE perform better than GMM.

The second is the maximum likelihood method (MLE) including quasi MLE. Aït-Sahalia (2002) used a closed-form approximation for the likelihood function and then used maximum likelihood method to estimate the discretely sampled diffusion. For more information about MLE, see Marsh and Rosenfeld (1982), Nelson (1991), Brenner et al. (1996), Koedijk et al. (1997), Aït-Sahalia (1999, 2002), Bali (2000), Das (2002), Jagannathan et al. (2003), Demirtas (2006), Bali and Wu (2006), Dueker et al. (2007), and Christiansen (2008). The disadvantage of MLE is this method is restrictive because few short term interest rate models have known or simple likelihood functions.

The third is the Bayesian method, i.e. Markov Chain Monte Carlo method (MCMC), see Elerian et al. (2001), Eraker (2001, 2004), Jones (2003), Hong (2005), Goldman (2005), Sorwar (2005), Griffin and Steel (2006), Sanford et al. (2006), Li et al. (2006) and Qian et al. (2005). The advantages of MCMC are that it can estimate the posterior distributions of the parameters easily; it can estimate the unobserved volatility; it can be used to estimate models with complicate posterior distributions and it can also estimate the predictive density for the forecast value.

In this paper, we use Bayesian method (MCMC). The in-sample fit of this new model is analyzed by comparing with CKLS-ARMA-GARCH-NORMAL. Bayes Factor (see Gelfand and Dey (1994)) and Deviance Information Criterion (DIC, see Spiegelhalter (2002)) are used as in-sample fit criteria. The
long run equilibrium of short-term interest rate is calculated.

Lastly, we also test one hypothesis: if one model has better in-sample fit, will it also be the model with better forecastability? Predictive Deviance Information Criterion (PDIC) and the distributions of Mean Squared Error of Forecast (MSEF) are generated within the Markov Chain Monte Carlo algorithms and used as out-of-sample model selection criteria.

Simulation results show that the MCMC algorithms used to estimate and forecast CKLS-ARMA-GARCH-EPD and CKLS-ARMA-GARCH-NORMAL are efficient since the filtered Fluctuation test (FT) or the filtered Kolmogorov Smirnov test (KS) (see Ploberger, Kramer and Kontrus(1989), Smirnov(1939) and Goldman et.al. (2007)) indicates convergence.

The results of in-sample model comparison are as follows. DIC shows CKLS-ARMA-GARCH-EPD model fits in-sample data better since its DIC value is smaller. However, Bayes Factor has decisive evidence supporting Model 2. DIC uses effective number of parameters instead of the true number of parameters to compare models. So, we prefer to use the DIC criterion and pick CKLS-ARMA-GARCH-EPD model as the one with better in-sample fit. And the long-run equilibrium short-term interest rate is 1.53%, which is calculated using the posterior means of parameter estimates from CKLS-ARMA-GARCH-EPD model.

The results of forecast show a model with better in-sample fit does not necessary have better out-of-sample forecastibility. One-period ahead PDIC shows CKLS-ARMA-GARCH-NORMAL has better out-of-sample forecast
since its PDIC value is smaller. If we use one-period ahead Mean Squared Error of Forecast (MSEF), CKLS-ARMA-GARCH-EPD is better. Since MSEF criterion comes from sample theory and PDIC criterion comes from Bayesian theory, we prefer the results of PDIC.

To test the robustness of previous results, 3-period, 6-period and 9-period ahead forecast values (fixed estimation scheme) are also calculated. The results of one-period ahead PDIC and MSEF are robust. For PDIC, CKLS-ARMA-GARCH-NORMAL model always has better forecastibility since its PDICs are smaller. For MSEF, CKLS-ARMA-GARCH-EPD model has better forecastibility since its MSEFs are smaller. When out-of-sample period increases from 1 to 9, the MSEF of CKLS-ARMA-GARCH-NORMAL increases greatly.

To check the robustness of the results for PDIC from previous fixed estimation scheme, we use rolling estimation scheme to derive PDIC for 10-period, 15-period, 20-period and half-sample ahead data. The results of rolling estimation scheme show CKLS-ARMA-GARCH-EPD model has better forecastibility. The PDIC for CKLS-ARMA-GARCH-EPD model becomes smaller as the forecast period increases from 10 to half-sample.

This paper is organized as follows. Section 2 introduces the models and methodology. The effectiveness of the MCMC algorithms is shown in section 3. Section 4 shows the empirical results for model estimation and predictability. Conclusions are given in section 5.
2.2 CKLS-ARMA-GARCH models and MCMC algorithms

We specify the CKLS-ARMA-GARCH model as:

\[ r_t = a + br_{t-1} + r_{t-1}^c u_t \]  \hfill (2)

where

\[ u_t = \sum_{j=1}^{p} \phi_j u_{t-j} + \sum_{j=1}^{q} \theta_j e_{t-j} + e_t \]  \hfill (3)

\[ e_t = \sigma_t \varepsilon_t \]  \hfill (4)

\[ \sigma_t^2 = \alpha_0 + \sum_{i=1}^{r} \alpha_i e_{t-i}^2 + \sum_{i=1}^{s} \beta_i \sigma_{t-i}^2 \]  \hfill (5)

\[ \alpha_0 > 0, \alpha_i \geq 0, i = 1, \ldots, r. \quad \beta_i \geq 0, i = 1, \ldots, s. \]  \hfill (6)

\[ 1 \geq \sum_{i=1}^{\max(r,s)} (\alpha_i + \beta_i) \]  \hfill (7)

\( r_t \) is spot rate. The error term \( \varepsilon_t \) follows the exponential power distribution (EPD) with variance normalized to be unity. The probability density function of \( \varepsilon_t \) is given by

\[ f(\varepsilon_t|Y, X) = \frac{1}{\lambda 2^{1+1/\alpha} \Gamma(1 + 1/\alpha)} \exp\left\{ -\frac{1}{2} |\varepsilon_t|^{\alpha} \right\} \]  \hfill (8)

\[ \lambda = \sqrt{\frac{2^{-2/\alpha} \Gamma(1/\alpha)}{\Gamma(3/\alpha)}} \]  \hfill (9)
\(\lambda\) is the constant to make the variance of \(\varepsilon_t\) unity. The EPD distribution has been used in many studies (see Box and Tiao(1973), Nelson(1991) and Tsurumi and Shimizu(2008), among others). Let us call equation 2 with the EPD error term as Model 1 or CKLS-ARMA-GARCH-EPD model.

Majority of the CKLS models in the literature employ the normal error term: \(\varepsilon_t \sim N(0, 1)\). Let us label the model with the normal error term Model 2 or CKLS-ARMA-GARCH-NORMAL model. We shall compare the performance of Model 1 and Model 2 to see which model fits in-sample data better and which one performs better judged by forecast performance.

### 2.2.1 Densities of CKLS-ARMA-GARCH models

We use the posterior density as our target density in the MCMC algorithms to draw the parameter estimates. After we estimate these models, we compare these models using in-sample selection criteria such as Bayes Factor and Deviance Information Criterion. Also, we calculate the forecast values using the posterior means of parameters. Out-of-sample comparisons are made by Predictive Deviance Information Criterion (PDIC) and Mean Squared Error of Forecast (MSEF).

Assume \(\Theta = \{a, b, c, \{\phi_i\}_{i=1}^p, \{\theta_i\}_{i=1}^q, \{\alpha_i\}_{i=0}^r, \{\beta_i\}_{i=1}^s, \alpha\}\) for Model 1, the posterior distribution for \(Y\) is:

\[
p(\Theta | Y, X) \propto \prod_{t=1}^{T} \frac{r_{t-1} \sigma_{t}^{-1}}{\lambda^{2^{1+1/\alpha}} \Gamma(1 + 1/\alpha)} \exp \left\{ -\frac{1}{2} \frac{\varepsilon_t^\alpha}{\lambda^\alpha} \right\} p(\Theta) \tag{10}
\]
where

\[ \varepsilon_t = \frac{e_t}{\sigma_t} \]  \hspace{1cm} (11)  

\[ e_t = \frac{r_t - (a + br_{t-1}) - \sum_{j=1}^{p} \phi_j u_{t-j} - \sum_{j=1}^{q} \theta_j e_{t-j}}{r_{t-1}^{c}} \]  \hspace{1cm} (12)  

\[ \lambda = \sqrt{\frac{2(-2/\alpha)\Gamma(1/\alpha)}{\Gamma(3/\alpha)}} \]  \hspace{1cm} (13)  

And

\[ p(\Theta) = p\{a, b, c, \{\phi_i\}_{i=1, \ldots, p}, \{\theta_i\}_{i=1, \ldots, q}, \{\alpha_i\}_{i=0, \ldots, r}, \{\beta_i\}_{i=1, \ldots, s}, \alpha\} \]  \hspace{1cm} (14)  

\[ = p(a)p(b)p(c)\prod_{i=1}^{p} p(\phi_i)\prod_{i=1}^{q} p(\theta_i)\prod_{i=0}^{r} p(\alpha_i)\prod_{i=1}^{s} p(\beta_i)p(\alpha) \]

\[ = N_a(0, .1)N_0(0, .1)N_c(0, .1)\prod_{i=1}^{p} N_{\phi_i}(0, .1)\prod_{i=1}^{q} N_{\theta_i}(0, .1) \]

\[ \times \prod_{i=0}^{r} N_{\alpha_i}(0, .1)\prod_{i=1}^{s} N_{\beta_i}(0, .1)N_a(0, .1) \]

is the prior. We assume priors for each parameters are independent. For model 2, we set parameter \( \alpha = 2 \).

2.2.2 MCMC algorithms

In order to implement this model, for parameter \((\alpha_i, \beta_i)\), we follow the approximation of Nakatsuma(1998) to get proposal density:

\[ \varepsilon_i^2 = \alpha_0 + \sum_{i=1}^{t} (\alpha_i + \beta_i) \varepsilon_{t-i}^2 - \sum_{i=1}^{s} \beta_i w_{t-i} + w_t, \ w_t \sim N(0, 2\sigma_t^2) \]  \hspace{1cm} (15)
Where $w_t = \varepsilon_t^2 - \sigma_t^2$, $l = \max(r, s)$, $\alpha_i = 0$ for $i > r$, $\beta_i = 0$ for $i > s$, $\varepsilon_i^2 = 0$ and $w_t = 0$ for $t \leq 0$.

Nakatsuma(1998) used nonlinear least square estimation to draw parameters in the MA and GARCH processes. Qian et al.(2005) used random walk draws for all parameters. We follows Qian et al.(2005)’s modifications.

Different from Nakatsuma(1998), we need to draw both parameter $c$ in the CKLS model and the parameter $\alpha$ in EPD. Different from Qian et al.(2005), we use Efficient Jump Algorithms to draw parameter $\{\theta_{i}\}_{i=1}^{q}$ and $\{\beta_{i}\}_{i=1}^{s}$, and we need to draw parameter $\alpha$ in EPD.

Nakatsuma(1998) and Qian et al.(2005) focus on parameter estimation. However, we not only estimate parameters in $\Theta$ but also calculate the predictive values $\tilde{Y}$ using the posterior means of parameters. In addition, we focus on model selection.

In our MCMC algorithm, we also compare following two algorithms for drawing parameter $\alpha$:

1) Efficient Jump Algorithm.

2) Modified Efficient Jump Algorithm with Inverse Gaussian as the proposal density.
2.3 Simulation results

Simulation data we used are:

\[ a = .2, \ b = .4, \ c = .5 \]
\[ \phi_1 = .4, \ \theta_1 = .4 \]
\[ \alpha_0 = .4, \ \alpha_1 = .7, \ \beta_1 = .1 \]
\[ \alpha = 1 \]

The sample size is \( T=1000 \). We run 18000 MCMC iterations. And we discard the first 3000 draws and keep every 10th draws. All acceptance rates are higher than 0.28.

We check the convergence of the draws by plotting the draws, by the filtered Fluctuation test or filtered Kolmogorov-Smirnov test (see Ploberger, Kramer and Kontrus(1989), Smirnov(1939) and Goldman et.al.(2007)). We judge convergence by 3 criteria: 1) eye rolling convergence. i.e., the draws plotted in graphs are convergent. 2) P-value of filtered Fluctuation test is no less than 5% significance level. 3) or P-value of filtered Kolmogorov-Smirnov test is no less than 5% significance level. If any one of these 3 criteria is satisfied, we conclude the draws are convergent.

Simulation results show the draws from our MCMC algorithms are convergent. The plots of the draws indicate convergence\(^5\). And the convergence tests also indicate convergence(see Table 2 ). In Table 2, for each parameter,Graphs will be available upon request.
either the P-value of filtered Fluctuation test or filtered Kolmogorov-Smirnov test is no less than 5% significance level.

The summary statistics of the MCMC draws are given in Table 3. We can see the posterior means of parameters drawn from our MCMC algorithm are very close to the true values.

The $\alpha$ draws from model 1 using simulated data are plotted in Figure 1, 2, 3 and 4. We can see the draws of $\alpha$ are convergent and the mean of $\alpha$ is around 1, which is the true simulation parameter for Laplace distribution. The results from Efficient Jump Algorithm (see Figure 3 and 4) are similar to those from the Modified Efficient Jump with Inverse Gaussian as proposal density (see Figure 1 and 2).

### 2.4 Application to US federal funds rate

<table>
<thead>
<tr>
<th>Parameters</th>
<th>P-value of FT</th>
<th>P-value of KST</th>
</tr>
</thead>
<tbody>
<tr>
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<td></td>
<td></td>
</tr>
<tr>
<td>$a$</td>
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<td>.84</td>
</tr>
<tr>
<td>$b$</td>
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<td>0</td>
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</tr>
<tr>
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<td>.87</td>
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<td>$b$</td>
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<td>.84</td>
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<tr>
<td>$c$</td>
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<td>.54</td>
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Table 2: P-values of Filtered Fluctuation Tests and Filtered Komogorov-Smirnov Tests
<table>
<thead>
<tr>
<th>True Value</th>
<th>Mean</th>
<th>St.Dev.</th>
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<td></td>
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<td>$b$</td>
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<tr>
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<td>.4</td>
<td>.2182</td>
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<td>$\theta_1$</td>
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<td>.4889</td>
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<tr>
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<td>.5042</td>
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<td>$\beta_1$</td>
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<tr>
<td>$c$</td>
<td>.5</td>
<td>.4020</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>1</td>
<td>.8843</td>
</tr>
<tr>
<td>Model 2: CKLS-ARMA-GARCH-NORMAL</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a$</td>
<td>.2</td>
<td>.2288</td>
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<td>$b$</td>
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<td>.1</td>
<td>.1934</td>
</tr>
<tr>
<td>$c$</td>
<td>.5</td>
<td>.4448</td>
</tr>
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</table>

Table 3: Posterior Summaries
Figure 1: MCMC Draws of Parameter $\alpha$ Using Algorithm of Modified Efficient Jump with Inverse Gaussian as Proposal Density

Figure 2: PDF of MCMC Draws of $\alpha$ Using Algorithm of Modified Efficient Jump with Inverse Gaussian as Proposal Density
Figure 3: MCMC Draw of Parameter $\alpha$ Using Algorithm of Efficient Jump

Figure 4: PDF of MCMC Draws of $\alpha$ Using Algorithm of Efficient Jump
Eraker et al. (2003), Fan (2003), and Qian et al. (2008). Quarterly data are used in Dueker et al. (2007).

2.4.1 Descriptive statistics


The descriptive statistics are listed in Table 4. The kurtosis is 5.2, which is greater than 3. And the skewness is 1.22, which is skewed to the right.

We also plot the unconditional density of the Federal funds rate in Figure 5. Compared to standard normal density, the unconditional density of the Federal funds rate is leptokurtic.

2.4.2 Parameter estimates and long-run equilibrium interest rate

The posterior means and standard deviations of the parameters are given in Table 5 for all models. The density for \( \alpha \) is plotted in Figure 6. We can see the posterior mean of \( \alpha \) is approximately .8424, far from Normal case \( \alpha = 2.\)

\(^6\)Chan et al. (1992) use monthly US Treasury bill yield between June, 1964 and December, 1989 (306 observations) in their paper to estimate the CKLS model by GMM.
### Table 4: Descriptive Statistics for Federal Funds Rate

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Mean</td>
<td>6.30</td>
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<tr>
<td>Min</td>
<td>0.98</td>
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<td>Max</td>
<td>19.10</td>
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<tr>
<td>Std.Dev.</td>
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<tr>
<td>Skewness</td>
<td>1.22</td>
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<tr>
<td>Kurtosis</td>
<td>5.20</td>
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</table>

Figure 5: Unconditional Density of Federal Funds Rate
<table>
<thead>
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<th>Model 1: CKLS-ARMA-GARCH-EPD</th>
<th>Mean</th>
<th>St. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>2.8913</td>
<td>.15</td>
</tr>
<tr>
<td>$b$</td>
<td>-.8882</td>
<td>1.15</td>
</tr>
<tr>
<td>$\phi_1$</td>
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<tr>
<td>$\beta_1$</td>
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<tr>
<td>$c$</td>
<td>.6788</td>
<td>.34</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>.8424</td>
<td>.23</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Model 2: CKLS-ARMA-GARCH-NORMAL</th>
<th>Mean</th>
<th>St. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
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</tr>
<tr>
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<td>-.5145</td>
<td>1.15</td>
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<td>$\phi_1$</td>
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<tr>
<td>$c$</td>
<td>.6178</td>
<td>.27</td>
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</table>

Table 5: Posterior Means and Standard Deviations of the Parameters
Figure 6: PDF of $\alpha$ Draws Using Interest Rate Data and Efficient Jump Algorithm
And the long-run equilibrium short-term interest rate is 1.53%, which is calculated using the posterior means of parameters $a$ and $b$ by following formula:

$$
\frac{a}{-(b - 1)} = \frac{2.8913}{-(.8882 - 1)} = 1.53
$$

In the paper of CKLS(1992), the long-run equilibrium short-term interest rate is 6.89%, which is calculated using the data in Table III of their paper (page 1218 of CKLS(1992)). We think the difference is caused by different models, different estimation techniques, and different data sets. Chan et al.(1992) use monthly US Treasury bill yield between June, 1964 and December, 1989 (306 observations) in their paper to estimate the CKLS model by GMM. But we use monthly US effective federal funds rate to estimate the CKLS-ARMA-GARCH-EPD model by MCMC.

2.4.3 In-sample-fit model selection criteria and results

To compare CKLS-ARMA-GARCH-EPD with CKLS-ARMA-GARCH-NORMAL, we first want to see which model has better in-sample fit. The model selection criteria are the Bayes Factor (see Gelfand and Dey (1994)) and the Deviance Information Criterion (see Spiegelhalter(2002)).

For Bayes Factor, the formula can be simplified as follows (see Gelfand
and Dey (1994), equation 21 on page 510):

\[
\log P_{SBF} \approx \log \lambda_n + \frac{p_2 - p_1}{2} \\
= \log \frac{L(\hat{\theta}_1; Y, M_1)\pi_1(\hat{\theta}_1)}{L(\hat{\theta}_2; Y, M_2)\pi_2(\hat{\theta}_2)} + \frac{p_2 - p_1}{2}
\]  

(16)

We get Bayes Factor by following implementation:

\[
\log \hat{P}_{SBF} = \log \frac{1}{n_{rep}} \sum_{i=1}^{n_{rep}} L(\hat{\theta}_1^{(i)}; Y, M_1)\pi_1(\hat{\theta}_1^{(i)}) + \frac{p_2 - p_1}{2} \\
= \log \frac{1}{n_{rep}} \sum_{i=1}^{n_{rep}} L(\hat{\theta}_2^{(i)}; Y, M_2)\pi_2(\hat{\theta}_2^{(i)}) + \frac{p_2 - p_1}{2}
\]  

(17)

where \( \hat{L}(\hat{\theta}_1^{(i)}; Y, M_1) \) is the estimated likelihood function at \( ith \) draw for all parameter \( \theta_1^{(i)} \) in Model 1. \( \hat{L}(\hat{\theta}_2^{(i)}; Y, M_2) \) is the estimated likelihood function at the \( ith \) draw for all parameter \( \theta_2^{(i)} \) in Model 2. \( \pi_1(\cdot) \) and \( \pi_2(\cdot) \) are priors. \( p_i \) is the number of parameters to be estimated in model \( i \). \( n_{rep} \) is the number of accepted draws.

The Deviance Information Criterion (DIC) is suggested by Spiegelhalter (2002). The formula of DIC is as follows (see Spiegelhalter (2002), page 603, equation 36, 37):

\[
DIC \equiv D(\bar{\theta}) + 2P_D = \bar{D}(\bar{\theta}) - P_D
\]  

(18)
We calculate DIC by following implementation:\footnote{For more information about DIC implementation, one can also refer to Chen(2008).}:

\[
\hat{\text{DIC}} \approx \hat{D}(\hat{\theta}) - \hat{P}_D
\]

\[
\hat{D}(\hat{\theta}) \approx -2 \times \frac{1}{n_{\text{rep}}} \sum_{i=1}^{n_{\text{rep}}} \log L(y|\theta^{(i)})
\]

\[
\hat{P}_D \approx \hat{D}(\hat{\theta}) - \hat{D}(\bar{\theta})
\]

Where \( \bar{\theta} \) is the posterior mean of parameters. \( Y = \{r_t\}_{t=1}^T \), and \( X \) are in-sample data. \( P_D \) is the effective number of parameters. \( D(\theta) \) is the deviance information. \( L(.) \) is the likelihood function:

\[
L(\Theta|Y, X) = f(Y|\Theta, X)
\]

\[
\propto \prod_{t=1}^{T} \frac{r_{t-1}^{-c} \sigma_t^{-1}}{\lambda^{2^{1+1/\alpha} \Gamma(1+1/\alpha)}} \exp \left\{ -\frac{1}{2} \frac{\varepsilon_t^{\alpha}}{\lambda^2} \right\}
\]

\[
\varepsilon_t = \frac{e_t}{\sigma_t}
\]

\[
e_t = \frac{r_t - (a + br_{t-1})}{r_{t-1}} - \sum_{j=1}^{p} \phi_j u_{t-j} - \sum_{j=1}^{q} \theta_j e_{t-j}
\]

\[
\lambda = \sqrt{\frac{2(-2/\alpha) \Gamma(1/\alpha)}{\Gamma(3/\alpha)}}
\]

The values of in-sample model selection criteria are listed in Table 6. DIC criterion shows CKLS-ARMA-GARCH-EPD (i.e., Model 1) fits in-sample data better since its DIC value is smaller. But Bayes Factor shows Model 2 is decisively better than Model 1. DIC uses effective number of parameters.
instead of the true number of parameters \( p_i \) to compare models. So, we prefer to use DIC criterion and pick CKLS-ARMA-GARCH-EPD model as the model with better in-sample fit.

### 2.4.4 Out-of-sample forecast model selection criteria and results

To see which model has better out-of-sample forecastability, we use the Predictive Deviance Information Criterion (PDIC) and Mean Squared Errors of Forecast (MSEF).

We calculate the Predictive Deviance Information Criterion (PDIC) by the same formula of DIC using forecast values \( \hat{y} \). The formula of PDIC is as follows

\[
\hat{D}_{\text{IC}} \approx \hat{D}(\theta) - \hat{P}_D
\]

\[
\hat{D}(\theta) \approx -2 \times \frac{1}{n_{\text{rep}}} \sum_{i=1}^{n_{\text{rep}}} \log L(\hat{y}|\theta^{(i)})
\]

\[
\hat{P}_D \approx \hat{D}(\theta) - \hat{D}(\theta)
\]

Where \( \theta \) is the posterior mean of parameters. \( \hat{y} = \{\hat{y}_t\}_{t=1}^{M} \) is the forecast values. \( \hat{x} \) is out-of-sample data. \( \hat{\sigma} \) is the forecast values for conditional standard deviation. \( P_D \) is the effective number of parameters. \( D(\theta) \) is the
deviance information. $L(.)$ is the likelihood function:

$$L(\Theta|\hat{Y}, \hat{X}) = f(\hat{Y}|\Theta, \hat{X})$$

$$\propto \prod_{t=1}^{M} \frac{\hat{r}_{t-1}^{1-1/\alpha} \Gamma(1+1/\alpha)}{\lambda^{1+1/\alpha}} \exp \left\{ -\frac{1}{2} \left| \frac{\hat{\varepsilon}_t}{\hat{\lambda}} \right|^\alpha \right\}$$

$\hat{\varepsilon}_t = \frac{\hat{\varepsilon}_t}{\hat{\sigma}_t}$

$\lambda = \sqrt{\frac{2(-2/\alpha) \Gamma(1/\alpha)}{\Gamma(3/\alpha)}}$

We also calculate Mean Squared Errors of Forecast (MSEF):

$$MSEF = \frac{1}{m} \sum_{i=1}^{m} (\hat{y}_{n+i}^{(j)} - y_{n+i})^2$$

where $m$ is the number of forecast period beyond the end of the sample period; $\hat{y}_{n+i}^{(j)}$ is the forecast value for period $n+i$ generated by Model $j$, $j = 1, 2$. $y_{n+i}$ is the actually realized value in period $n+i$. In sample theory we have one $\hat{y}_{n+i}^{(j)}$. In Bayesian analysis we can generate $\hat{y}_{n+i}^{(j)}$ and obtain the distribution of the MSEF given the actually realized $y_{n+i}$'s. For one period ahead forecast (i.e., $m = 1$), the MSEF becomes the squared errors of forecast (SEF):

$$SEF = (\hat{y}_{n+1}^{(j)} - y_{n+1})^2$$

1-period ahead PDIC shows CKLS-ARMA-GARCH-NORMAL (i.e., Model
2) has better out-of-sample predictive power since its PDIC value is smaller (see Table 7). 1-period ahead MSEF (i.e. SEF) shows CKLS-ARMA-GARCH-EPD (i.e., Model 1) is better since its MSEF is smaller. The densities and cumulative densities of the 1-period ahead MSEF (i.e., SEF)'s are given in Figure 7, 8, 9 and 10. We see that the cumulative density of the SEF’s for Model 1 clearly dominates that for Model 2. That is, CDF of the SEF for model 1 goes to 1 more quickly than that of model 2.

Figure 7: PDF of SEF (Model 1: CKLS-ARMA-GARCH-EPD)
Figure 8: PDF of SEF (Model 2: CKLS-ARMA-GARCH-NORMAL)
Figure 9: CDF of SEF (Model 1: CKLS-ARMA-GARCH-EPD)
Figure 10: CDF of SEF (Model 2: CKLS-ARMA-GARCH-NORMAL)
2.4.4.1 Check robustness (fixed estimation scheme) To test the robustness of previous results for 1-period ahead forecast, 3-period, 6-period and 9-period ahead values of PDIC and MSEF are also calculated (see Table 7). Table 7 shows the results of 1-period ahead PDIC and MSEF are robust. That is, PDIC shows CKLS-ARMA-GARCH-NORMAL (i.e., Model 2) always has better forecastability since its PDICs are smaller. And MSEF shows CKLS-ARMA-GARCH-EPD (i.e., Model 1) always has better forecastability since its MSEFs are smaller. The MSEF of CKLS-ARMA-GARCH-NORMAL increases greatly when out-of-sample period increases. Since MSEF comes from sample theory and PDIC comes from Bayesian theory, we prefer the results coming from PDIC. The results of forecast show the model with better in-sample fit is not necessary the one with better forecastability.

2.4.4.2 Check robustness (rolling estimation scheme) To test the robustness of PDIC results from fixed estimation scheme, we use rolling estimation scheme to derive PDIC for 10-period, 15-period, 20-period and half-sample ahead data. The results are listed in Table 8. The results of rolling estimation scheme show CKLS-ARMA-GARCH-EPD model has better forecastability. From Table 8, we can see the PDIC for CKLS-ARMA-GARCH-EPD model becomes smaller as the forecast period increases from 10 to half-sample. For the half sample case, we plot the PDICs of both models in Figure 11. The graph in Figure 11 shows the PDICs of CKLS-ARMA-GARCH-EPD
are always smaller than those of CKLS-ARMA-GARCH-NORMAL, and the difference between them is bigger when out-of-sample period increases.

![Diagram showing a comparison of half-sample PDICs between CKLS-ARMA-GARCH-EPD and CKLS-ARMA-GARCH-NORMAL.](image)

Figure 11: Half-sample PDICs Comparison

### 2.5 Chapter conclusions

In this paper, we firstly propose a short-term interest rate model. The model is the CKLS-ARMA-GARCH model with error terms distributed as exponential power distribution (EPD). We denote this model as CKLS-ARMA-GARCH-EPD (or Model 1).
### Table 6: In-sample Fit (Jan. 63 - Dec. 07)

<table>
<thead>
<tr>
<th></th>
<th>PDIC</th>
<th>MSEF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Model 1</td>
<td>Model 2</td>
</tr>
<tr>
<td>1-period ahead</td>
<td>13.07</td>
<td>9.78</td>
</tr>
<tr>
<td>3-period ahead</td>
<td>21.98</td>
<td>17.54</td>
</tr>
<tr>
<td>6-period ahead</td>
<td>24.11</td>
<td>19.93</td>
</tr>
<tr>
<td>9-period ahead</td>
<td>20.31</td>
<td>16.16</td>
</tr>
</tbody>
</table>

Note: Model 1 is CKLS-ARMA-GARCH-EPD. Model 2 is CKLS-ARMA-GARCH-NORMAL.

### Table 7: Multi-period ahead Out-of-Sample Model Selection Using the Estimates from Period Jan. 63 - Dec. 07 (Fixed Estimation Scheme)

<table>
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<tr>
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<th>PDIC</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Model 1</td>
</tr>
<tr>
<td>10-period ahead</td>
<td>-</td>
</tr>
<tr>
<td>15-period ahead</td>
<td>Smaller</td>
</tr>
<tr>
<td>20-period ahead</td>
<td>Smaller except the first 10 periods</td>
</tr>
<tr>
<td>Half sample ahead</td>
<td>Smaller</td>
</tr>
</tbody>
</table>

Note: Model 1 is CKLS-ARMA-GARCH-EPD. Model 2 is CKLS-ARMA-GARCH-NORMAL.

### Table 8: Multi-period ahead Out-of-Sample Model Selection (Rolling Estimation Scheme)
Bayesian MCMC algorithms are developed to estimate this model. Then, both in-sample fit and out-of-sample forecastability of this model are compared with CKLS-ARMA-GARCH-NORMAL (or Model 2). Bayes Factor (see Gelfand and Dey (1994)) and Deviance Information Criterion (DIC, see Spiegelhalter (2002)) are used as in-sample fit criteria. Predictive Deviance Information Criterion (PDIC) and Mean Squared Error of Forecast (MSEF) are used as criteria for out-of-sample forecast. Empirical data of US Federal funds rates are analyzed.

Simulation results show that the MCMC algorithms used to estimate Model 1 and 2 are efficient since the filtered Fluctuation test (FT) and filtered Kolmogorov Smirnov test (KS) (see Ploberger, Kramer and Kontrus (1989), Smirnov (1939) and Goldman et al. (2007)) indicate convergence.

We figure out the long-run equilibrium short-term interest rate is 1.53% calculated by the posterior means of CKLS-ARMA-GARCH-EPD. In the paper of CKLS (1992), their long-run equilibrium short-term interest rate is 6.89%. We think the difference is caused by different models, different estimation techniques and different data sets.

The results of in-sample model selection are as follows. DIC shows CKLS-ARMA-GARCH-EPD fits in-sample data better since its DIC value is smaller. However, Bayes Factor has decisive evidence supporting Model 2. DIC uses effective number of parameters instead of the true number of parameters to compare models. So, we prefer the result of DIC criterion and pick CKLS-ARMA-GARCH-EPD model as the one with better in-sample fit.
By fixed estimation scheme, both 1-period and multi-period ahead PDICs show CKLS-ARMA-GARCH-NORMAL model has better out-of-sample forecastibility since its PDIC values are smaller. However, if we use criterion of Mean Squared Error of Forecast (MSEF), then CKLS-ARMA-GARCH-EPD is better. Since MSEF comes from sample theory and PDIC comes from Bayesian theory, we prefer the results coming from PDIC.

Results from both in-sample fit and out-of-sample forecast via fixed estimation scheme show a model with better in-sample fit does not necessary have better out-of-sample forecastibility. Results from out-of-sample forecast via rolling estimation scheme show CKLS-ARMA-GARCH-EPD model can predict long run better since its PDICs are smaller as forecast period increases.

Future extensions include but not limit to follows. We can add macro variables of Bernake et al.(2005) into the CKLS-ARMA-GARCH-EPD model (see Bernake et al.(2005) for the Factor-Augmented (FAVAR) model). One can also introduce a regime switching variable in CKLS-ARMA-GARCH-EPD model (see Dueker et al.(2007)). 2 or 3 latent factors can also be added to CKLS-ARMA-GARCH-EPD model (see Andersen and Lund (1997), Jones (2003), Bali (2000) and Sorwar (2005)). Or one can use our model to price derivatives. Last but not the least, one can check the robustness of our model using international data.
3 Markov switching models for realized volatility

3.1 Motivation

Many financial time series display asymmetric behavior. For example, large negative returns appear more frequently than large positive returns and sudden dramatic political events have big effects on financial time series. In order to capture such asymmetric behavior, nonlinear time series models are developed\(^8\).

A lot of attempts try to model this kind of nonlinearity explicitly. One approach is to define different regimes, i.e., state of the world, and allow the dynamic behavior of economic variable to depend on the regimes (see Priestly (1980, 1988)). That means certain properties of the time series such as its mean, variance or autocorrelation are different in different regimes. In this case, we call this time series state-dependent or regime-switching.

Regime-switching process can be deterministic or stochastic (see Figure 12). One example of deterministic regime process is the seasonal effect. Stochastic regime can be found in LeBaron (1992), Kräger and Kugler (1993)

---

\(^8\)Various statistical tests have been used to detect the nonlinearity in stock prices and exchange rates (see Hinich and Patterson (1985), Scheinkmann and LeBaron (1989), Hsieh (1989,1991), Crato and de Lima (1994), and Brooks (1996)).
and Chappell et al. (1996).

Stochastic regime-switching models can be divided into two categories (see Figure 12). The first category assumes that the regimes can be characterized or determined by an observable variable, i.e., the regimes that have occurred in the past and present are known with certainty, and can be calculated by statistical tools. Examples are Threshold Autoregressive model (TAR)\(^9\), Self-Exciting TAR (SETAR) model with threshold value determined by the time series itself, or Smooth Transition AR (STAR) model including Logistic STAR (LSTAR)\(^{10}\).

The second category assumes that the regimes are unobserved or hidden, and they are determined by an underlying unobservable stochastic process, which we denote as \(s_t\). We can not be sure that a particular regime will occur at a particular point of time, but can only assign probabilities to the occurrence of the different regimes. When the property of \(s_t\) in a model is specified to be a first order Markov-process, i.e. the current regime only depends on the regime one period ahead, then this kind of model is called Markov switching model (MSW).

Markov switching model (hereafter denoted as MSW) was introduced to economics by Hamilton (1989). Since then, many extensions of Markov switching model are developed\(^{11}\). For example, Cai (1994), and Hamilton and

---

\(^{10}\)See Bacon and Watts (1971), Chan and Tong (1986), Granger and Terävirta (1993), and Teräsvirta (1994, 1998))

At the beginning of development of Markov switching models, most of the researches are used maximum likelihood estimation (MLE) to estimate the parameters in the models\textsuperscript{12}.

However, as the Markov switching models are extended to be more and more complicated, MLE becomes computationally unfeasible. For example, Rachev and Fabozzi (2006).

when the MA\((q)\) error terms are added into the models, then the likelihood function depends on all previous \(q\) periods of the past history of the state variables, which makes the joint process become a non-Markovian process and is hard to be estimated by MLE. In addition, for MSW-GARCH, if there are only 2 states with \(N\) sample size, then there will be \(2^N\) cases of the likelihood function to be consider.

There are two ways to deal with previous problem of MLE. One is to use reparameterization to recover a Markovian frame work from the non-Markovian models (see Jones(1987) and Barnett et.al.(1996)). Another is to use Markov Chain Monte Carlo (MCMC) to deal with the non-Markovian structure directly \(^{13}\) (see Billio et.al.(1999), Kaufmann et.al.(2000), Das and Yoo(2004), Yoo(2006), and Henneke, Rachev and Fabozzi(2006)). MCMC is a very powerful tool for the numerical computation of integrals, which can be used easily to estimate complicated models.

Markov switching models have been widely applied to GDP, T-bill rates, exchange rates and stock prices\(^{14}\), but there is little application to the realized volatility, a new measure of volatility first introduced by Andersen and Bollerslev (1997). There are a lot of researches done to model and forecast the realized volatility \(^{15}\) because of the importance of volatility in finance. How-

\(^{13}\)For more discussion of MCMC, please refer to Robert and Casella (1999).


ever, most researches use linear models, GARCH, RiskMetrics, FIEGARCH or TARMA.

The first contribution of this paper is to design a random walk draw algorithm for a Markov switching model (MSW-ARMA-GARCH-$s_t$-in-variance). And then the estimates from this algorithm are compared with those of non-random walk draw (see Das and Yoo(2004)).

The 2nd contribution of this paper is to compare two Markov switching models by MSE and MSEF. We try to test one hypothesis: is the model with $s_t$ in variance better than the one with $s_t$ in mean? A Markov-switching model with state variable in variance (i.e., MSW-ARMA-GARCH-$s_t$-in-variance) is estimated by MCMC algorithm of Das and Yoo(2004). And another Markov-switching model with state variable in mean (i.e., MSW-ARMA-GARCH-$s_t$-in-mean) is estimated by MCMC algorithm of Yoo(2006). Metropolis Hasting algorithm and Gibbs samplers are used. In order to generate latent variable $s_t$, single-move method is used.

The 3rd contribution is to study the realized volatility using Markov switching models. Even though there is few research on realized volatility using Markov-switching models, these MSW models are different from ours. For example, Maheu and McCurdy(2002) found that a duration dependent Markov-switching ARMA model is better than other linear models for the realized volatility of the DM/$ exchange rate. Different from us, their model

\footnote{For single-move method, see Carlin, Polson, and Stoffer (1992), and Albert and Chib (1993). For multi-move, see Carter and Kohn (1994),and Kim and Nelson (1999).}
This paper is organized as follows. Section 2 provides details about the Markov switching ARMA-GARCH models. Bayesian inference and the MCMC algorithm for the estimation are given in section 3 and 4. Section 5 explains the effectiveness of the MCMC algorithm using numerical examples. In section 6, the estimation results from the realized volatility of MMM company are presented. Conclusions are given in section 7.

3.2 Markov switching models

The Markov switching models with ARMA-GARCH errors are as follows:

3.2.1 Model 1: MSW-ARMA-GARCH-S_t-in-variance

\[ y_t = x_t \gamma + u_t, \] (27)

\[ u_t = \sum_{i=1}^{p} \phi_i u_{t-i} + e_t + \sum_{i=1}^{q} \theta_i e_{t-i}, \quad e_t \sim N(0, \sigma_t^2), \] (28)

\[ \sigma_t^2 = \alpha_0 + s_t \mu + \sum_{i=1}^{r} \alpha_i e_{t-i}^2 + \sum_{i=1}^{s} \beta_i \sigma_{t-i}^2, \] (29)

\[ \alpha_0 > 0, \alpha_0 + \mu > 0, \alpha_i \geq 0, i = 1, 2, \ldots r; \beta_i \geq 0, i = 1, 2, \ldots s, \]

\[ 1 \geq \sum_{i=1}^{\max(r,s)} (\alpha_i + \beta_i) \] (30)

For model 1, the Markov switching variable \( s_t \) is in conditional variance (i.e., GARCH term, equation 29).
3.2.2 Model 2: MSW-ARMA-GARCH-S, in-mean

\[ y_t = x_t \gamma_1 + s_t \mu + u_t, \]  
\[ u_t = \sum_{i=1}^{p} \phi_i u_{t-i} + e_t + \sum_{i=1}^{q} \theta_i e_{t-i}, \quad e_t \sim N(0, \sigma^2_t), \]  
\[ \sigma^2_t = \alpha_0 + \sum_{i=1}^{r} \alpha_i e^2_{t-i} + \sum_{i=1}^{s} \beta_i \sigma^2_{t-i}, \]  
\[ \alpha_0 > 0, \alpha_i \geq 0, i = 1, 2, \ldots r, \beta_i \geq 0, i = 1, 2, \ldots s, \]
\[ \max(r,s) \sum_{i=1}^{r,s} (\alpha_i + \beta_i) \leq 1. \]  

For model 2, the Markov switching variable \( s_t \) is in the mean component of dependent variable (equation 31).

For both models, we assume the state variable \( s_t \) is a two-state, first order Markov switching process. The hidden transition probabilities for this Markov switching process are:

\[ Pr[s_t = 0|s_{t-1} = 0] = p_{00}, Pr[s_t = 1|s_{t-1} = 0] = p_{01}. \]  
\[ Pr[s_t = 0|s_{t-1} = 1] = p_{10}, Pr[s_t = 1|s_{t-1} = 1] = p_{11}. \]  

We also impose constraints (30) and (34) on the coefficients of GARCH to guarantee that the conditional variance is positive.

For notation simplicity, we use following parameter definitions for model
\[ \phi = [\phi_1, ... \phi_p]', \quad \theta = [\theta_1, ... \theta_q]', \quad \alpha = [\alpha_0, \mu, \alpha_1, ... \alpha_r]', \quad \beta = [\beta_1, ..., \beta_q]', \]
\[ \delta = [\gamma, \phi, \theta, \alpha, \beta, p_{00}, p_{11}]', \quad S = [s_1, ..., s_T]', \quad Y = [y_1, ..., y_T]' \]

For model 2, in addition to previous definitions, we define two more notations: \( \alpha = [\alpha_0, \alpha_1, ... \alpha_r]' \), \( \gamma = [\gamma_1, \mu]' \).

### 3.3 Bayesian inference

For Bayesian inference\(^\text{17}\), we need to derive the posterior distribution of all parameters and the state variables \( S = [s_1, ..., s_T]' \) conditional on the observed data \( Y \). According to Bayes’ rule, the posterior distribution of model 1 or model 2 is:

\[
p(\delta, S|Y) = p(\delta, S, Y)/p(Y) \quad \text{(property of conditional density)} \quad (36)
\]
\[
= p(\delta, S)p(Y|\delta, S)/p(Y) \quad \text{(property of conditional density)} \quad (37)
\]
\[
\propto p(\delta, S)p(Y|\delta, S) \quad \text{(}p(Y)\text{ is known)} \quad (38)
\]
\[
\propto p(\delta)p(S|\delta)p(Y|\delta, S) \quad \text{(property of conditional density)} \quad (39)
\]

The first term of equation (39) is the prior distribution for all parameters.

\(^\text{17}\)In the classical framework, inference on Markov-switching models first estimates the models’ unknown parameters, then makes inferences on the unobserved Markov switching variables \( s_t, t = 1, ..., T \). In the Bayesian analysis, both the parameters and the Markov-switching variable \( s_t, t = 1, ..., T \) are treated as random variables.
This prior $p(\delta)$ is assumed to be:

$$p(\delta) = p(\gamma, \phi, \theta, \alpha, \beta, p_{00}, p_{11})$$

$$= p(\gamma)p(\phi)p(\theta)p(\alpha)p(\beta)p(p_{00})p(p_{11})$$

$$= N(\mu_\gamma, \Sigma_\gamma) * N(\mu_\phi, \Sigma_\phi) * N(\mu_\theta, \Sigma_\theta)$$

$$* N(\mu_\alpha, \Sigma_\alpha) * N(\mu_\beta, \Sigma_\beta)$$

$$* Beta(u_{00}, u_{01}) * Beta(u_{11}, u_{10})$$

The equal sign in (41) comes from the assumption of independence, i.e., we assume the prior distributions of parameters are independent. $N(.)$ is the normal density function, and $Beta(.)$ is the beta density function.

The second term of equation (39) is:

$$p(S|\delta) = p(S|\gamma, \phi, \theta, \alpha, \beta, p_{00}, p_{11})$$

$$= p(S|p_{00}, p_{11}) \quad \text{Since the independence assumption.}$$

$$= \prod_{t=1}^{T} p(s_{t+1}|s_t, p_{00}, p_{11}) \quad \text{Since Markov property.}$$

$$= p_{00}^{n_{00}}(1 - p_{00})^{n_{01}} p_{11}^{n_{11}} (1 - p_{11})^{n_{10}}$$

where $n_{ij}$ is the number of the transitions from state i to state j.
The third term in equation (39) is:

\[
p(Y|\delta, S) = \prod_{t=1}^{T-1} p(y_t|y_{t-1}, \delta, S)
\]

\[
= \prod_{t=1}^{T-1} \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left[-\frac{c_t^2}{2\sigma_t^2}\right]
\]

\[
= T_1 \prod_{t=1}^{T-1} \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left[-\frac{c_t^2}{2\sigma_t^2}\right]
\]

### 3.4 MCMC implementation

Markov Chain Monte Carlo (MCMC) algorithm is used to estimate the parameters. MCMC is one way of the numerical integrations based on the Clifford-Hammersley theorem. This theorem shows that a joint distribution can be characterized by its complete conditional distributions. When applied in Bayesian, that means the posterior distributions, \( p(\delta, S|Y) \), can be characterized by its complete conditional distributions: \( p(\delta|S, Y) \) and \( p(S|\delta, Y) \).

In the MCMC algorithm, both Gibbs Samplers (GS) and Metropolis Hastings (MH) are used. Gibbs Samplers are used to draw \( p_{00}, p_{11} \) since its complete conditional distribution is assumed to be beta distribution, which is known. Metropolis Hastings (MH) algorithms are used to draw other parameters such as \( \gamma, \phi, \theta, \alpha, \) and \( \beta \), since the complete conditional distributions of these parameters are unknown.
3.4.1 Algorithm using non-random walk draw

We draw parameters such as $\gamma, \phi, \theta, \alpha$, and $\beta$ using non-random walk draw (see Appendix i), the probability to accept a proposal value is

$$\rho = \min \left\{ \frac{p(\tilde{\delta}|Y, S) * f(\delta^{(j-1)})}{p(\delta^{(j-1)}|Y, S) * f(\tilde{\delta})}, 1 \right\}$$

(45)

Where $f(.)$ is the proposal distribution function. $\delta^{(j-1)}$ is the $j-1$ draw. $\tilde{\delta}$ is a generated value for $\delta$. We use the method of Nakatsuma(2000) to choose the proposal distribution $f(.)$\(^{18}\).

3.4.2 Algorithm using random walk draw

We also use random walk draw in the MCMC algorithm to draw parameters $\gamma, \phi, \theta, \alpha$, and $\beta$ for model 1 (see Appendix ii). In the random walk draw, the probability to accept a proposal value is $\rho = \min \left\{ \frac{p(\tilde{\delta}|Y, S)}{p(\delta^{(j-1)}|Y, S)}, 1 \right\}$, since the proposal distribution function $f(.)$ in equation 45 is symmetric, i.e., $f(\delta^{(j-1)}, \tilde{\delta}) = f(\tilde{\delta}, \delta^{(j-1)})$.

Since $S = [s_1, ..., s_T]$ are latent variables, data augmentation are used. That is, the parameter $\delta$ is augmented with the states $S$. Single move method is used to generate $S$ since the process is non-Markovian and multi-move method is infeasible.

Parameters in $\delta$ are divided into 3 groups: $\delta_1 = (p_{00}, p_{11}), \quad \delta_2 = (\gamma, \phi, \theta), \quad \delta_3 =$

\(^{18}\)Almost any distribution can be chosen as the proposal distribution. However, for the speed of the convergence, it is crucial to select a proper proposal distribution. See Robert and Casella(1999).
Then, the MCMC algorithm is:

1. Draw $s_t$ from $p(s_t|S_{-t}, Y, \delta)$ for all $t = 1, 2, ... T$ by single-move GS method.

2. Draw $\delta_1$ from $p(\delta_1|S, Y, \delta_2, \delta_3) = p(\delta_1|S)$, the Beta distribution, by GS.

3. Draw $\delta_2$ from $p(\delta_2|S, Y, \delta_3, \delta_1) = p(\delta_2|S, Y, \delta_3)$ by MH algorithm.

4. Draw $\delta_3$ from $p(\delta_3|S, Y, \delta_2, \delta_1) = p(\delta_3|S, Y, \delta_2)$ by MH algorithm.

We use equations (27), (28) for model 1 (or equations (31), (32) for model 2) to draw proposal density for $\delta_2 = (\gamma, \phi, \theta)$. The corresponding likelihood function is:

$$p(Y|S, \delta_2, \delta_3) = \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi} \sigma_t^2} \exp\left[-\frac{e_t^2}{2\sigma_t^2}\right] \tag{46}$$

For $\delta_3 = (\alpha, \beta)$, we follow the approximation of Nakatsuma (2000) to get proposal densities:

$$e_t^2 = \alpha_0 + \sum_{i=1}^{l} (\alpha_i + \beta_i) e_{t-i}^2 + w_t - \sum_{i=1}^{s} \beta_i w_{t-j}, \quad w_t \sim N(0, 2\sigma_t^2), \tag{47}$$

where: $l = max\{r, s\}$, $\alpha_i = 0$ for $i > r$, $\beta_i = 0$ for $i > s$,

$$e_t^2 = 0, \quad w_t = 0 \quad \text{for} \quad t \leq 0$$
The corresponding likelihood function is:

\[
p(e^2|Y, S, \delta_2, \delta_3) = \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi(2\sigma_t^2)}} \exp\left[-\frac{w_t^2}{2(2\sigma_t^4)}\right]
\]  

(48)

where \( e^2 = [e_1^2, ..., e_T^2]' \).

For more detailed of MCMC implementation, please refer to the Appendix i 
& ii.

### 3.5 Simulation results

In this section, we use simulation examples to show the effectiveness of the MCMC algorithms. We estimate model 1 and model 2 while setting \( p = 1, q = 1, r = 1, s = 1 \), i.e., two Markov switching models with ARMA(1,1) -GARCH(1,1) error term.

We first design an algorithm using random walk draws to estimate model 1 (see Appendix ii). The results\(^{19}\) are listed in Table 9. The sample size is 1000, the draws discarded are 1000 and every 5th draw is kept. It takes about 6 hours to run the program. In Table 9, all parameters are close to true value except \( \mu, \alpha_1 \) and \( p_{11} \). The parameter \( \mu \) has an opposite sign compared with the true parameter. The mean of \( \alpha_1 \) is double of the true parameter. For \( p_{11} \), the value is just a little bit higher than 0.5, which is far away from true value 0.98.

\(^{19}\)We try different parameters and sample size to see the results.
Then, we estimate model 1 using non-random walk draw with the proposal densities listed in Appendix i. The results for model 1 are in Table 10. Sample size is 500, 100 draws are discarded and every 1 draw of 500 is used. It takes 44 minutes to get the result. In Table 10, every posterior mean is closed to its true value. Compared with Table 9, the results are obvious better than that. So we choose this algorithm for model 1 to estimate the realized volatility in next section\(^{20}\).

The simulation results for model 2 are in Table 11\(^{21}\). Sample size is 200, draws discarded are 8000 and every 2 draws from 3000 are used. It takes one and a half hour to get the result. In Table 11, every posterior mean is closed to its true value.

In Figure 13 and 14, we can see that the estimated state variable \(\hat{s}_t\) from the simulations are very similar to the true state variable \(s_t\). So, the MCMC algorithm in Appendix i estimates both the parameters and the states very well.

\(^{20}\)Note: MSW-ARMA-GARCH-\(S_t\)-in-variance is estimated by MCMC algorithm of Das and Yoo(2004).

\(^{21}\)Note: MSW-ARMA-GARCH-\(S_t\)-in-mean is estimated by MCMC algorithm of Yoo(2006).
<table>
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<th></th>
<th>True</th>
<th>Mean</th>
<th>St.Dev.</th>
</tr>
</thead>
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Table 9: Simulation Results for Model 1 (Random Walk Draw)

<table>
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<td>$\alpha_1 + \beta_1$</td>
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Table 10: Simulation Results for Model 1 (Non-Random Walk Draw)
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<th>St.Dev.</th>
</tr>
</thead>
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Table 11: Simulation Results for Model 2 (Non-random Walk Draw)

Figure 13: S(t) and Estimated S(t) for Model 1 (Note: The bold line is the estimated value.)
3.6 Application to realized volatility

In this section, we apply the algorithms for model 1 and model 2 to the realized volatility of MMM company. The data are 5-minute returns from TAQ database. The sample period is from January 4, 1993 to December 31, 2004. We use 600 observations. The first 500 data are used for estimation and in-sample fit evaluation. And the last 100 observations are used for out-of-sample forecast evaluation.

The results are listed in Table 12. From these estimates, it is hard to tell which model is better. So we calculate in-sample Mean Squared Errors (MSE) and Mean Squared Error of Forecast (MSEF) for out-of sample fore-
cast evaluation (see Table 13). Model 2 has better in-sample fit since its MSE value is smaller. But model 1 has better out-of-sample forecastability since it has lower MSEF.

3.7 Chapter conclusions

Markov switching models have been widely applied to GDP, T-bill rates, exchange rates and stock prices\(^{22}\). In this paper, we design a random walk algorithm for a Markov switching model with regime switching in variance (i.e., MSW-ARMA-GARCH-\(S_t\)-in-variance). Then, the estimates from this algorithm are compared with those of Das and Yoo(2004).

Also, we compare two Markov switching models by MSE and MSEF. A Markov-switching model with state variable in variance (i.e., MSW-ARMA-GARCH-\(S_t\)-in-variance) is estimated by MCMC algorithm of Das and Yoo(2004). And another Markov-switching model with state variable in mean (i.e., MSW-ARMA-GARCH-\(S_t\)-in-mean) is estimated by MCMC algorithm of Yoo(2006). Metropolis Hasting algorithm and Gibbs Samplers are used. In order to generate latent variable \(s_t\), single-move method\(^{23}\) is used.

Lastly, we use previous two Markov switching models to analyze the realized volatility of MMM. The realized volatility is a new measure of volatil-


<table>
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<tr>
<th>Parameter</th>
<th>Model 1 Mean</th>
<th>St.Dev.</th>
<th>Model 2 Mean</th>
<th>St.Dev.</th>
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<tr>
<td>$P_{11}$</td>
<td>.897</td>
<td>.086</td>
<td>.244</td>
<td>.196</td>
</tr>
<tr>
<td>$\alpha_1 + \beta_1$</td>
<td>.444</td>
<td>.000</td>
<td>1.177</td>
<td>.067</td>
</tr>
</tbody>
</table>

Note: Model 1: MSW-ARMA-GARCH-S$_t$-in-variance  
Model 2: MSW-ARMA-GARCH-S$_t$-in-mean

Table 12: Estimation Results for the Realized Volatility of MMM Company

<table>
<thead>
<tr>
<th>Models</th>
<th>MSE</th>
<th>MSEF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model 1: MSW-ARMA-GARCH-S$_t$-in-variance</td>
<td>.0019</td>
<td>.0006</td>
</tr>
<tr>
<td>Model 2: MSW-ARMA-GARCH-S$_t$-in-mean</td>
<td>.0009</td>
<td>.9678</td>
</tr>
</tbody>
</table>

Table 13: MSE and MSEF for the Realized Volatility of MMM Company
ity, first introduced by Andersen and Bollerslev (1997). There are a lot of researches being done to model and forecast the realized volatility. Most of these researches are analyzed the realized volatility by linear models, GARCH, RiskMetrics, FIEGARCH or TARMA models. However, using MSW-ARMA-GARCH models to model the realized volatility is new.

The conclusions of this paper are as follows. Algorithm of random walk draw for model 1 is not better than the algorithm of non-random walk draw in Das and Yoo (2004). According to MSE, we find out model 2 has better in-sample fit. However, MSEF shows model 1 has better forecastibility.

Future extension could be follows. First, we can compare Markov switching model with other nonlinear models such as TARMA. Secondly, we can consider other model selection criteria. Thirdly, we can also extend previous Markov switching models by assuming other error distributions such as the exponential power distribution. Last but not least, we can design a new model which includes two regime switching variables: one regime switching variable in the mean component, another regime switching variable in the variance component.
4 Bayesian analysis of SP500 and FTSE100 using a Gaussian Copula model

4.1 Motivation

In this paper, we first design a Gaussian copula model with ARMA-GARCH-EPD error terms (i.e., Copula-ARMA-GARCH-EPD). Then, we design Markov Chain Monte Carlo (MCMC) algorithms to estimate this model. Monthly data on SP500 and FTSE100 are analyzed.

Copulas (see Sklar(1959)) are convenient tools to construct multivariate joint distributions from margins. In financial risk assessment and actuarial analysis, dependence modeling with copula functions is widely used. Li (2000) introduces Gaussian copula model into pricing Collateralized Debt Obligations (CDOs).

Researches on copulas have been fast growing for the following reasons: (1) As shown in Blyth (1996), Shaw (1997), and Embrechts et.al. (1999), copulas can handle nonlinear dependence measures such as Kendall’s $\tau$, Spearman’s $\rho$, and Gini indice $\gamma$ that may be more appropriate for measuring dependence among data.\(^{24}\) (2) The empirical distributions of many financial time series data are skewed and leptokurtic. This implies that models with normal error distribution may be inadequate. Copulas allow us to build multivariate distributions from leptokurtic and/or skewed marginal distributions,

\(^{24}\)Additional references are Karolyi and Stultz (1996), Forbes and Rigobon (1999), Straetmans (2000), Login and Solnik (2001), and Ang and Bekaert (2002).
and thus we can find dependence among data.\textsuperscript{25}

Papers applying copulas are many and the application of copulas spreads across various fields. For example, in health economics, Zimmer and Trivedi (2006) applied a trivariate copula to analyze family health care demand. Romeo \textit{et. al.} (2006) used bivariate copula models to study multivariate survival data. Li \textit{et. al.} (2006) used a copula variance-components method for genetic studies.\textsuperscript{26} In finance, Meucci (2006) used copula-opinion pooling methodology to extend the Black-Litterman model for portfolio management. Arakelian and Dellaportas (2006) studied a threshold model using copulas. Patton (2006) used copulas to model five Asian stock indices to measure financial contagion.\textsuperscript{27}

Bayesian and maximum likelihood procedures have been used to estimate copula models. For example, Xu (2004) designed an MCMC algorithm to estimate a mixed copula model. Chen \textit{et. al.} (2004) developed two goodness-of-fit tests for copula models. Chen \textit{et. al.} (2005) proposed a sieve maximum likelihood estimation (MLE) to efficiently estimate copula models. An efficient Bayesian approach for estimating a Gaussian copula model was presented by Pitt, Chan and Kohn (2006).

In this paper we combine the Gaussian copula with ARMA-GARCH-EPD

\textsuperscript{25}See Mandelbrot (1963), Fama (1965), Praetz (1972), Blattberg and Gonedes (1974).


marginal distributions. Results show the MCMC algorithm is convergent by the filtered Fluctuation test and Kolmogorov-Smirnov test in Goldman et al. (2007). Using monthly data on SP500 and FTSE100, we estimate the Gaussian copula model with ARMA-GARCH-EPD margins.

The organization of the paper is as follows. Section 2 discusses the model: Gaussian copula with ARMA-GARCH-EPD margins. Section 3 presents the posterior density for Copula-ARMA-GARCH-EPD. MCMC algorithm and a numerical example are showed in section 4. Section 5 illustrates an empirical analysis of SP500 and FTSE100. Concluding remarks are given in section 6.

4.2 Copula-ARMA-GARCH-EPD model

The Copula-ARMA-GARCH-EPD is a multivariate regression model:

\[ Y = Xb + u \]  

where \( Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_M \end{bmatrix}, X = \begin{bmatrix} X_1 & 0 & \ldots & 0 \\ 0 & X_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & X_M \end{bmatrix}, b = \begin{bmatrix} b_1 \\ \vdots \\ b_M \end{bmatrix}, u = \begin{bmatrix} u_1 \\ \vdots \\ u_M \end{bmatrix} \).

Or,

\[ Y_i = X_i b_i + u_i, \; i = 1, \ldots, M. \]
where

\[ u_{it} = \sum_{j=1}^{p} \phi_{ij} u_{it-j} + \sum_{j=1}^{q} \theta_{ij} e_{it-j} + e_{it} \quad (51) \]

\[ e_{it} = \sigma_{it} \varepsilon_{it}, \; \varepsilon_{it} \sim EPD(1, \; \zeta_i) \quad (52) \]

\[ \sigma_{it}^2 = \alpha_{i0} + \sum_{j=1}^{r} \alpha_{ij} e_{it-j}^2 + \sum_{j=1}^{s} \beta_{ij} \sigma_{it-j}^2 \quad (53) \]

\[ \alpha_{i0} > 0, \; \alpha_{ij} \geq 0, \; j = 1, ..., r, \; \beta_{ij} \geq 0, \; j = 1, ..., s. \quad (54) \]

\[ 1 \geq \sum_{j=1}^{\max(r,s)} (\alpha_j + \beta_j) \quad (55) \]

\[ Y_i = (y_{i1}, ..., y_{Ti})' \text{ is a } T \times 1 \text{ vector, } X_i = (x_{i1}', ..., x_{Ti}')' \text{ is a } T \times k_i \text{ matrix} \]

\[ b_i \text{ is a } k_i \times 1 \text{ vector of regression coefficients, and } u_i \text{ is a } T \times 1 \text{ vector of error terms.} \]

The sample size is \( T \) and \( M \) is the number of equations.

And \( f_i(\varepsilon_{it}) \) is the PDF of exponential power distribution (EPD)

\[ f_i(\varepsilon_{it}) = \frac{1}{2^{1+1/\zeta_i} \sigma_{it} \Gamma(1+1/\zeta_i)} \exp \left\{ -\frac{1}{2} \frac{\varepsilon_{it}^2}{\sigma_{it}^2} \right\} \quad (56) \]

\[ 0 < \sigma_{it} < \infty, \; 0 < \zeta_i < \infty, \; i = 1, ..., M. \]

\( EPD(\sigma_{it}, \zeta_i) \) becomes the normal distribution when \( \zeta_i = 2 : \)

\[ f_i(\varepsilon_{it}) = \frac{1}{\sqrt{2\pi \sigma_{it}}} \exp \left\{ -\frac{1}{2} \frac{\varepsilon_{it}^2}{\sigma_{it}^2} \right\}, \; 0 < \sigma_{it} < \infty, \; i = 1, ..., M. \quad (57) \]

\[ ^{28}\text{For density of EPD, please refer to Tsurumi and Shimizu(2006).} \]
4.3 Posterior density

4.3.1 Joint density through copula

Let $X_1, \ldots, X_M$ be the random variables. Their joint density through copula\footnote{For definition of copula, please see Sklar’s Theorem on page 18 of Nelsen (2005).} is given by

$$h(X_1, \ldots, X_M) = c(F_1(x_1), \ldots, F_M(x_M); \rho) \prod_{i=1}^{M} f_i(x_i) \quad (58)$$

where $c(F_1(x_1), \ldots, F_M(x_M); \rho)$ is the density of copula; $F_i(x_i)$ is the cumulative function and $F_i(x_i) = \int_{-\infty}^{x_i} f_i(X_i) dX_i$; $f_i(X_i)$ is the marginal density of random variable $X_i$; $x_i$ is the realized value for random variable $X_i$, $i = 1, \ldots, M$.

If all the marginal distributions $f_i(X_i)$’s are normal, then the joint density in equation 58 is the multivariate normal PDF. If $X_i$ is not normal, then the joint density $h(X_1, \ldots, X_M)$ is not multivariate normal even if we use the Gaussian copula (see Appendix iv). We can construct a non-normal joint distribution from margins, and then we can estimate the correlation matrix $\rho$. 
4.3.2 Gaussian copula density

The density of Gaussian copula\(^{30}\) is given by

\[
c(F_1(e_1), \ldots, F_M(e_M); \rho) = |\rho|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} A'( \rho^{-1} - I) A \right\}
\]  \hspace{1cm} (59)

where

\[
A = (a_1, \ldots, a_M)'
\]  \hspace{1cm} (60)

\[
a_i = \Phi^{-1}(F_i(e_i))
\]  \hspace{1cm} (61)

\[
F_i(e_i) = \int_{-\infty}^{e_i} f_i(\varepsilon_i) d\varepsilon_i
\]  \hspace{1cm} (62)

And \(f_i(\varepsilon_i)\) is the PDF of \(\varepsilon_i\). \(F_i(e_i)\) is the CDF of \(\varepsilon_i\). \(\rho\) is a positive definite matrix with diagonal elements being unity. \(\Phi(.)\) is the standardized normal CDF. And \(I\) is the identity matrix.

4.3.3 Posterior density for Copula-ARMA-GARCH-EPD

We define \(\Theta_i \equiv \{b_i, \{\phi_{ij}\}_{j=1\ldots p}, \{\theta_{ij}\}_{j=1\ldots q}, \{\alpha_{ij}\}_{j=0\ldots r}, \{\beta_{ij}\}_{j=1\ldots s}, \zeta_i\} \) and \(\Theta \equiv (\Theta_1, \ldots, \Theta_M, \rho)\). The posterior density of \(\Theta\) for Copula-ARMA-GARCH-EPD is

\[
p(\Theta|Y, X) = \prod_{t=1}^{T} c(F_1(e_{1t}) \ldots F_M(e_{Mt}); \rho) f_1(\varepsilon_{1t}) \ldots f_M(\varepsilon_{Mt}) * p(\Theta)
\]

\(^{30}\)For density of copula, please refer to page 8 of Bouyé (2000).
where

\[
c(F_1(e_{1t}) \ldots F_M(e_{ Mt}); \rho) = |\rho|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} A_t'(\rho^{-1} - I) A_t \right\}
\]  \hspace{1cm} (63)

\[
A_t = (a_{1t}, \ldots, a_{Mt})'
\]  \hspace{1cm} (64)

\[
a_{it} = \Phi^{-1}(F_{it}(e_{it}))
\]  \hspace{1cm} (65)

\[
F_{it}(e_{it}) = \int_{-\infty}^{e_{it}} f_i(\varepsilon_{it}) d\varepsilon_{it}
\]  \hspace{1cm} (66)

\[
f_i(\varepsilon_{it}) = \frac{1}{2^{1+1/\zeta_i} \sigma_{it} \Gamma(1 + 1/\zeta_i)} \exp \left\{ -\frac{1}{2} |\varepsilon_{it}|^{1/\zeta_i} \right\}
\]  \hspace{1cm} (67)

\[
0 < \sigma_{it} < \infty, \; 0 < \zeta_i < \infty, \; i = 1, \ldots, M.
\]  \hspace{1cm} (68)

\[
\varepsilon_{it} = e_{it}/\sigma_{it}
\]  \hspace{1cm} (69)

\[
\sigma_{it}^2 = \alpha_{i0} + \sum_{j=1}^{r} \alpha_{ij} \varepsilon_{it-j}^2 + \sum_{j=1}^{s} \beta_{ij} \sigma_{it-j}^2
\]  \hspace{1cm} (70)

\[
e_{it} = u_{it} - \sum_{j=1}^{p} \phi_{ij} u_{it-j} - \sum_{j=1}^{q} \theta_{ij} \varepsilon_{it-j}
\]  \hspace{1cm} (71)

\[
u_{it} = Y_{it} - X_{it} b_{it}, \; i = 1, \ldots, M. \; \; t = 1, \ldots, T.
\]  \hspace{1cm} (72)

\(p(\Theta)\) is the prior density for \(\Theta\). We use prior: \(p(\Theta) = 1\).

### 4.4 MCMC algorithms and simulation results

We estimate the parameters \(\theta\) of Copula-ARMA-GARCH-EPD by the MCMC algorithms that are explained in the Appendix iii. The sample size is 300. We run 18000 MCMC iterations and discard the first 3000 draws. Every 10th draw is kept. All acceptance rates are higher than .31. It takes 16 hours to get the results.
We check the convergence of the draws by plotting the draws and by the filtered Fluctuation test and Kolmogorov-Smirnov test that are given in Goldman et al. (2007). The plots of the draws indicate convergence (see Figure 15 and 16). And the convergence tests also indicate convergence since the p-value of filtered Fluctuation test or filtered Kolmogorov-Smirnov test is no less than 5% significance level\textsuperscript{31} (see Table 14).

The summary statistics of the MCMC draws are given in Table 15. The

\textsuperscript{31}We judge convergence by 3 criteria: 1) eye rolling convergence, i.e., the draws plotted in a graph are convergent, 2) P-value of Fluctuation test is no less than 5% significance level, 3) or P-value of Kolmogrov-Smirnov test is no less than 5% significance level. If one of these 3 criteria is satisfied, we conclude the draws are convergent.
Figure 15: MCMC Draws for Parameters (in the order of Table 14)

Figure 16: PDFs of Accepted MCMC Draws (in the order of Table 14)
<table>
<thead>
<tr>
<th>Parameters</th>
<th>True</th>
<th>Mean</th>
<th>St.Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_1$</td>
<td>3</td>
<td>3.19</td>
<td>.27</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>.7</td>
<td>.66</td>
<td>.05</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>.5</td>
<td>.55</td>
<td>.08</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>.3</td>
<td>.36</td>
<td>.14</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>.3</td>
<td>.27</td>
<td>.08</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>.6</td>
<td>.56</td>
<td>.10</td>
</tr>
<tr>
<td>$\zeta_1$</td>
<td>2</td>
<td>1.85</td>
<td>.20</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>.8</td>
<td>1.06</td>
<td>.29</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>.65</td>
<td>.67</td>
<td>.06</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>.6</td>
<td>.52</td>
<td>.08</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>.2</td>
<td>.24</td>
<td>.11</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>.2</td>
<td>.17</td>
<td>.07</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>.7</td>
<td>.69</td>
<td>.10</td>
</tr>
<tr>
<td>$\zeta_2$</td>
<td>2</td>
<td>2.16</td>
<td>.21</td>
</tr>
<tr>
<td>$\rho$</td>
<td>.5</td>
<td>.44</td>
<td>.06</td>
</tr>
</tbody>
</table>

Table 15: Posterior Summaries

Summary statistics are posterior means and standard deviations. We see that the posterior means are close to the true values.

### 4.5 Empirical application: SP500 and FTSE100

With increasing globalization of the financial markets around the world, we would expect that the returns of the financial instruments in different markets are correlated. If we incorporate such correlation into the prediction of the financial returns in different markets, we may improve the predictability of the returns. Let us take up SP500 and FTSE100 that are two leading stock indicators in U.S. and in Europe, respectively.
As are the cases of many financial return data, the returns on SP500 and on FTSE100 are leptokurtic with robust tails, and thus a bivariate normal distribution may not be appropriate to model the returns.

There have been many empirical studies on the stock indices such as SP500 and FTSE100. For example, Amendola and Stort (2006) studied asymmetries in the conditional mean and variance of the returns on SP500, FTSE100 and NASDAQ, one stock index at a time. Using the EPD distributions, Liu, Wu, and Lee (2004) obtained the maximum likelihood estimates of VaR models of Dow Jones Industrial Average (DJIA), Taiwan Stock Exchange index (TAIEX) and FTSE100, one stock index at a time. Also, there are studies on the correlation structure of stock indices. For example, using Skewed Student-t distribution and quasi-maximum likelihood estimation, Jondeau and Rockinger (2006) estimated a Copula-GARCH model for the four major stock indices in the US, UK, Germany, and France (SP500, FTSE100, DAX and CAC-40). Different from their study, we use a Copula-ARMA-GARCH-EPD model.

We use the monthly data from January 1981 to October 2006 with a total of 319 observations. The data are taken from Global Financial Data. We use these observations to estimate the parameters of Copula-ARMA-GARCH-EPD. The returns are calculated as the monthly rates of change.

The descriptive statistics of these data are shown in Table 16. The means

\[\text{For more research on SP 500 and/or FTSE100, see Taylor(2004), Bhardwaj and Swanson (2005), Savva et.al. (2005), Bao et. al. (2006), Hu(2006), and Li et. al. (2006).} \]
### Table 16: Descriptive Statistics for FTSE100 and SP500

<table>
<thead>
<tr>
<th></th>
<th>FTSE100</th>
<th>SP500</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0085</td>
<td>0.0086</td>
</tr>
<tr>
<td>St.Dev.</td>
<td>0.0462</td>
<td>0.0431</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.9050</td>
<td>-0.5895</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>6.6729</td>
<td>5.4560</td>
</tr>
<tr>
<td>Min</td>
<td>0.0121</td>
<td>0.0102</td>
</tr>
<tr>
<td>Max</td>
<td>0.1443</td>
<td>0.1318</td>
</tr>
<tr>
<td>Correlation</td>
<td></td>
<td>0.7058</td>
</tr>
</tbody>
</table>

The returns on SP500 and FTSE100 are slightly skewed to the left and they are leptokurtic indicating that the tails of the distributions are fatter than those of the normal distribution. The correlation is about 0.7058.

The posterior means and standard deviations of the parameters are given in Table 17. We observe that the posterior mean of the correlation parameter is around 0.6893.

### 4.6 Chapter Conclusions

A copula is a convenient vehicle to obtain a joint distribution from marginal distributions. In this chapter, we first design a Gaussian copula model with ARMA-GARCH-EPD error terms (i.e., Copula-ARMA-GARCH-EPD). And then, we design Markov Chain Monte Carlo (MCMC) algorithms to estimate this model. Monthly data on SP500 and FTSE100 are analyzed. Results show MCMC algorithm is convergent. The correlation parameter $\rho$ for SP500
<table>
<thead>
<tr>
<th>Parameters</th>
<th>Mean</th>
<th>St.Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_1$</td>
<td>-.0231</td>
<td>.0000</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>-.9853</td>
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<tr>
<td>$\theta_1$</td>
<td>.9893</td>
<td>.0109</td>
</tr>
<tr>
<td>$\varpi_1$</td>
<td>.0024</td>
<td>.0005</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>.0527</td>
<td>.0416</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>.1734</td>
<td>.1297</td>
</tr>
<tr>
<td>$\zeta_1$</td>
<td>2.2455</td>
<td>.1312</td>
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<tr>
<td>$\gamma_2$</td>
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<td>.0032</td>
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<td>$\phi_2$</td>
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<td>.0484</td>
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<tr>
<td>$\theta_2$</td>
<td>-.9205</td>
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<tr>
<td>$\varpi_2$</td>
<td>.0016</td>
<td>.0003</td>
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<tr>
<td>$\alpha_2$</td>
<td>.0893</td>
<td>.0635</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>.1276</td>
<td>.0938</td>
</tr>
<tr>
<td>$\zeta_2$</td>
<td>1.6710</td>
<td>.0675</td>
</tr>
<tr>
<td>$\rho$</td>
<td>.6893</td>
<td>.0422</td>
</tr>
</tbody>
</table>

Table 17: Posterior Means and Standard Deviations of the Parameters

and FTSE100 is .6893.

One of the future extensions is to compare Copula-ARMA-GARCH-EPD
model with other nonlinear models. Another extension could add a regime
switch parameter in Copula-ARMA-GARCH-EPD model (see Dueker et al.
(2007)). Or we can add some macro variables of Bernake et al.(2005)) into
Copula-ARMA-GARCH-EPD model and then compare its forecastibility
with other models. Last but not the least, one may use Copula-ARMA-
GARCH-EPD to price Collateralized Debt Obligations (CDOs).
5 Appendices

Appendix i shows the MCMC algorithm with non-random walk draw for model 1 (MSW-ARMA-GARCH-$S_t$-in-variance). In Appendix ii, the MCMC algorithm with random walk draw for model 1 is given. Appendix iii shows the MCMC algorithm for Copula-ARMA-GARCH-EPD. Appendix iv presents the simulation results for Copula-EPD and Copula-NORMAL.

5.1 Appendix i: MCMC algorithm using non-random walk draw

**Generate $S$**

$S$ is generated by single move method. The distribution of $p(s_t|Y, S_{-t}, \delta)$ can be derived as follows\(^{33}\):

$$
p(s_t|Y, S_{-t}, \delta) \propto p(s_{t+1}|s_t, \delta_1)p(s_t|s_{t-1}, \delta_1)$$

\[\times p(y_{t|t-1}, S, \delta)\cdots p(y_{T|T-1}, S, \delta)\]

$$\propto p(s_{t+1}|s_t, \delta_1)p(s_t|s_{t-1}, \delta_1)$$

\[\times \prod_{s=t}^{T}[(\sigma_s^2)^{-1/2}\exp[-\frac{C_s^2}{2\sigma_s^2}]]\]

After we calculate the $p(s_t = 1|Y, S_{-t}, \delta)$, we compare it with a random number from the uniform distribution. If this probability is greater than the

---

\(^{33}\)For more details of derivation for equation (73), please refer to the proof of the Lemma on page 11 of Yoo 2006.
random number, then we set $s_t = 1$, otherwise 0.

**Generate** $\delta_1 = (p_{00}, p_{11})$

The posterior conditional distribution of $p_{00}$ is:

$$p(p_{00}|Y, S, \delta_{-p_{00}}) \propto p(p_{00}) \times p(S|\delta_1)$$

$$\propto p_{00}^{n_{00}-1}(1 - p_{00})^{n_{01}-1} p_{00}^{n_{00}}(1 - p_{00})^{n_{01}}$$

That is, we generate $p_{00}$ by Beta distribution, i.e., $p_{00}|S \sim Beta(u_{00} + n_{00}, u_{01} + n_{01})$. Same reasoning, we generate $p_{11}|S$ from Beta$(u_{11} + n_{11}, u_{10} + n_{10})$.

**Generate** $\gamma$

By proper transformation\textsuperscript{34}, we use following proposal distribution to draw $\gamma$:

$$\gamma|Y, X, \Sigma, \delta_{-\gamma} \sim N(\hat{\mu}_\gamma, \hat{\Sigma}_\gamma),$$

Where:

$$\hat{\mu}_\gamma = \hat{\Sigma}_\gamma (X'_\gamma \Sigma^{-1} Y_\gamma + \Sigma^{-1} \mu_\gamma), \quad \hat{\Sigma}_\gamma = (X'_\gamma \Sigma^{-1} X_\gamma + \Sigma^{-1})^{-1},$$

$$\Sigma = diag\{\sigma_1^2, \ldots, \sigma_T^2\}, \quad Y_\gamma = [y_1^*, \ldots y_n^*]' , \quad X_\gamma = [x_1^*, \ldots x_n^*]' ,$$

$$y_t^* = y_t - \sum_{i=1}^{p} \phi_i y_{t-i} - \sum_{i=1}^{q} \theta_i y_{t-i}^*, \quad x_t^* = x_t - \sum_{i=1}^{p} \phi_i x_{t-i} - \sum_{i=1}^{q} \theta_i x_{t-i}^*,$$

$$e_t \equiv y_t^* - x_t^* \gamma, \quad y_t^* = 0, \quad x_t^* = 0 \quad \text{for all} \quad t \leq 0.$$  

\textsuperscript{34}See Chib and Greenberg(1994), Yoo (2006), and Das and Yoo (2005).
Generate $\phi$

By proper transformation, we draw $\phi$ using the following proposal distribution:

$$
\phi|Y, X, \Sigma, \delta_{-\phi} \sim N(\hat{\mu}_\phi; \hat{\Sigma}_\phi),
$$

(76)

Where:

$$
\hat{\mu}_\phi = \hat{\Sigma}_\phi (X'_{\phi} \Sigma^{-1} Y_{\phi} + \Sigma_{\phi}^{-1} \mu_{\phi}), \quad \hat{\Sigma}_\phi = (X'_{\phi} \Sigma^{-1} X_{\phi} + \Sigma_{\phi}^{-1})^{-1},
$$

$$
\Sigma = diag\{\sigma_{1}^2, ..., \sigma_{T}^2\}, \quad Y_{\phi} = [\bar{y}_1, ..., \bar{y}_n]', \quad X_{\phi} = [\bar{x}_1, ..., \bar{x}_n]'.
$$

$$
\bar{y}_t = y_t - x_t \gamma - \sum_{i=1}^{q} \theta_i \bar{y}_{t-i}, \quad \bar{x}_t = [\bar{y}_{t-1}, ..., \bar{y}_{t-p}]
$$

$$
e_t = 0, y_t = 0, \bar{y}_t = 0, \text{for } t \leq 0.
$$

Generate $\theta$

We use the following proposal density to draw $\theta$.

$$
\theta|Y, X, \Sigma, \delta_{-\theta} \sim N(\hat{\mu}_\theta; \hat{\Sigma}_\theta),
$$

(77)

where:

$$
\hat{\mu}_\theta = \hat{\Sigma}_\theta (X'_{\theta} \Sigma^{-1} Y_{\theta} + \Sigma_{\theta}^{-1} \mu_{\theta}), \quad \hat{\Sigma}_\theta = (X'_{\theta} \Sigma^{-1} X_{\theta} + \Sigma_{\theta}^{-1})^{-1},
$$

$$
\Sigma = diag\{\sigma_{1}^2, ..., \sigma_{T}^2\}, \quad Y_{\theta} = [\nabla_1 \theta^* - e_1(\theta^*), ..., \nabla_n \theta^* - e_n(\theta^*)]',
$$

$$
X_{\theta} = [\nabla'_1, ..., \nabla'_n]', \quad \nabla_t = [\nabla_{1t}, ..., \nabla_{qt}],
$$

$$
\nabla_{it} = -e_{t-i}(\theta^*) - \sum_{j=1}^{q} \theta_i^* \nabla_{jt-i}, \quad i = 1, ..., q, \quad \theta^* = arg \min_{\theta} \sum_{t=1}^{n} [e_t(\theta)]^2 / \sigma_t^2.
$$

Generate $\alpha$
Same transformation method used for AR coefficient $\phi$, we can rewrite (47). The proposal density used to draw $\alpha$ is:

$$
\alpha|Y, X, \Sigma, \delta_{-\alpha} \sim N(\hat{\mu}_{\alpha}, \hat{\Sigma}_{\alpha}),
$$

(78)

where:

$$
\hat{\mu}_{\alpha} = \hat{\Sigma}_{\alpha}(X'_\alpha \wedge^{-1} Y_\alpha + \Sigma^{-1}_\alpha \mu_\alpha), \quad \hat{\Sigma}_{\alpha} = (X'_\alpha \wedge^{-1} X_\alpha + \Sigma^{-1}_\alpha)^{-1},
$$

$$
\wedge = \text{diag}\{2\sigma^4_1, ..., 2\sigma^4_T\}, \quad Y_\alpha = [e^2_1, ..., e^2_n]', \quad X_\alpha = [\xi', ..., \xi'],
$$

$$
\xi_t = [\eta_t, \bar{e}^2_{t-1}, \ldots, \bar{e}^2_{t-s}], \quad \eta_t = 1 + \sum_{i=1}^n \beta_i \hat{\eta}_{t-i}, \quad \bar{e}^2_t = e^2_t + \sum_{i=1}^n \beta_i \bar{e}^2_{t-i}.
$$

**Generate $\beta$**

Similar transformation method as MA coefficient $\theta$, the proposal density to draw $\beta$ is:

$$
\beta|Y, X, \Sigma, \delta_{-\beta} \sim N(\hat{\mu}_{\beta}, \hat{\Sigma}_{\beta}),
$$

(79)

where:

$$
\hat{\mu}_{\beta} = \hat{\Sigma}_{\beta}(X'_\beta \wedge^{-1} Y_\beta + \Sigma^{-1}_\beta \mu_\beta), \quad \hat{\Sigma}_{\beta} = (X'_\beta \wedge^{-1} X_\beta + \Sigma^{-1}_\beta)^{-1},
$$

$$
\wedge = \text{diag}\{2\sigma^4_1, ..., 2\sigma^4_T\}, \quad Y_\beta = [w_1(\beta^*), \hat{\nu}_1 \beta^*, ..., w_n(\beta^*) + \hat{\nu}_n \beta^*]',
$$

$$
X_\beta = [\hat{\nu}'_1, ..., \hat{\nu}'_n]', \quad \hat{\nu}_t = [\hat{\nu}_{1t}, ..., \hat{\nu}_{qt}],
$$

$$
\hat{\nu}_{it} = -\bar{e}^2_{t-i} + w_{t-i}(\beta^*) + \sum_{j=1}^s \beta^*_j \hat{\nu}_{i,t-j}, \quad i = 1, ..., s, \beta^* = \arg \min_\beta \sum_{t=1}^T [w_t(\beta)]^2 / 2\sigma^4_t.
5.2 Appendix ii: MCMC algorithm using random walk draw

Appendix ii gives the proposal density for random walk draw\(^ {35} \). For \( S \) and \( p_{00}, p_{11} \), they are the same as that in Appendix i.

**Generate \( \gamma \)**

We use following proposal distribution to draw \( \gamma \):

\[
\gamma^{(j)}|Y, X, \Sigma, \delta_\gamma \sim N(\gamma^{(j-1)}, \hat{\Sigma}_{\gamma^{(j-1)}}),
\]

where:

\[
\hat{\Sigma}_{\gamma^{(j-1)}} = (X_{\gamma^{(j-1)}} \Sigma^{-1}_{(j-1)} X_{\gamma^{(j-1)}} + \Sigma^{-1}_{\gamma})^{-1},
\]

\[
\Sigma_{(j-1)} = \text{diag}\{\sigma_1^2, \ldots, \sigma_T^2\},
\]

\( Y_\gamma = [y_1^*, \ldots y_n^*]' \), \( X_\gamma = [x_1^*, \ldots x_n^*]' \).

\[
y_t^* = y_t - \sum_{i=1}^{p} \phi_i y_{t-i} - \sum_{i=1}^{q} \theta_i y_{t-i}, \quad x_t^* = x_t - \sum_{i=1}^{p} \phi_i x_{t-i} - \sum_{i=1}^{q} \theta_i x_{t-i}^*
\]

\( e_t \equiv y_t^* - x_t^* \gamma, y_t^* = 0, x_t^* = 0 \) for all \( t \leq 0 \).

**Generate \( \phi \)**

\(^ {35} \)See Qian, Ashizawa and Tsurumi(2004)
We draw $\phi$ using following proposal distribution:

$$\phi^{(j)} | Y, X, \Sigma, \delta, \phi \sim N(\phi^{(j-1)}, \hat{\Sigma}_{\phi^{(j-1)}}),$$  \hspace{1cm} (81)

where:

$$\hat{\Sigma}_{\phi^{(j-1)}} = (X_{\phi^{(j-1)}}^\prime \Sigma_{\phi^{(j-1)}}^{-1} X_{\phi^{(j-1)}} + \Sigma_{\phi}^{-1})^{-1},$$

$$\Sigma_{(j-1)} = diag\{\sigma_1^2, ..., \sigma_T^2\},$$

$$Y_\phi = [\tilde{y}_1, ..., \tilde{y}_n]', X_\phi = [\tilde{x}_1, ..., \tilde{x}_n]',$$

$$\tilde{y}_t = y_t - x_t\gamma - \sum_{i=1}^{q} \theta_i \tilde{y}_{t-i}, \quad \tilde{x}_t = [\tilde{y}_{t-1}, ..., \tilde{y}_{t-p}]$$

$$e_t = 0, y_t = 0, \tilde{y}_t = 0, \text{for } t \leq 0.$$

**Generate $\theta$**

The proposal density used to draw $\theta$ is:

$$\theta^{(j)} | Y, X, \Sigma, \delta, \theta \sim N(\theta^{(j-1)}, \hat{\Sigma}_{\theta^{(j-1)}}),$$  \hspace{1cm} (82)

where:

$$\hat{\Sigma}_{\theta^{(j-1)}} = (X_{\theta^{(j-1)}}^\prime \Sigma_{\theta^{(j-1)}}^{-1} X_{\theta^{(j-1)}} + \Sigma_{\theta}^{-1})^{-1},$$

$$\Sigma_{(j-1)} = diag\{\sigma_1^2, ..., \sigma_T^2\},$$

$$X_\theta = [\tilde{x}_1, ..., \tilde{x}_n], \quad \tilde{x} = [\tilde{y}_{t-1}, ..., \tilde{y}_{t-q}],$$

$$\tilde{y}_t = y_t - x_t\gamma + \sum_{i=1}^{p} \phi_i (y_{t-i} - x_{t-i}\gamma) - \sum_{i=1}^{q} \theta_{i} \tilde{y}_{t-i}$$

$$y_t = 0, \tilde{y}_t = 0, \text{for } t \leq 0.$$
Generate $\alpha$

The proposal density used to draw $\alpha$ is:

$$\alpha^{(j)}|Y, X, \Sigma, \delta_{-\alpha} \sim N(\alpha^{(j-1)}, \hat{\Sigma}_{\alpha^{(j-1)}}),$$

(83)

where:

$$\hat{\Sigma}_{\alpha^{(j-1)}} = (X_{\alpha^{(j-1)}} \wedge_{(j-1)}^{-1} X_{\alpha^{(j-1)}} + \Sigma_{\alpha}^{-1})^{-1},$$

$$\wedge_{(j-1)} = diag\{2\sigma_1^4, \ldots, 2\sigma_T^4\},$$

$$X_{\alpha} = [\eta_1, \ldots, \eta_n], \quad \eta_t = 1 + \sum_{i=1}^n \beta_i \eta_{t-i}.$$ 

Generate $\beta$

We use following proposal density to draw $\beta$.

$$\beta^{(j)}|Y, X, \Sigma, \delta_{-\beta} \sim N(\beta^{(j-1)}, \hat{\Sigma}_{\beta^{(j-1)}}),$$

(84)

where:

$$\hat{\Sigma}_{\beta^{(j-1)}} = (X_{\beta^{(j-1)}} \wedge_{(j-1)}^{-1} X_{\beta^{(j-1)}} + \Sigma_{\beta}^{-1})^{-1},$$

$$\wedge_{(j-1)} = diag\{2\sigma_1^4, \ldots, 2\sigma_T^4\},$$

$$X_{\beta} = [x_1^T, \ldots, x_n^T], \quad x_t^T = -[y_{t-1}^T, \ldots, y_{t-s}^T],$$

$$y_t^T = e_t^2 - \alpha_0 - \sum_{i=1}^l (\alpha_i + \beta_i)e_{t-i}^2 + \sum_{i=1}^s \beta_i y_{t-i}^T,$$

$$y_t^T = 0, \text{ for } t \leq 0.$$
5.3 Appendix iii: MCMC algorithm for Copula-ARMA-GARCH-EPD

To draw parameter $\Theta = (\Theta_1, ..., \Theta_M, \rho)$ in Copula-ARMA-GARCH-EPD model, we use following posterior distribution:

$$h(\Theta|Y, X) = p(\Theta) \prod_{i=1}^{n+m} c(F_1(e_{i1})...F_M(e_{iM}); \rho) f_1(\varepsilon_{i1})...f_M(\varepsilon_{iM})$$

Parameters in $\Theta = (\Theta_1, ..., \Theta_M, \rho)$ are $\Theta_i = \{b_i; \{\phi_{ij}\}_{j=1..p}; \{\theta_{ij}\}_{j=1..q};$

$\{\alpha_{ij}\}_{j=0..r}; \{\beta_{ij}\}_{j=1..s}; \zeta_i\}, i = 1, ..., M$. We divide parameters $\Theta = (\Theta_1, ..., \Theta_M, \rho)$ into $M + 1$ blocks: $\Theta_i$ ($i = 1, ..., M$) and $\rho$.

Then, for each parameter $\Theta_i$, the MCMC algorithm is36:

1. Draw $b_i$ from independent multivariate Normal by the MH algorithm using the method of random walk draw.

2. Draw each parameter in $\{\phi_{ij}\}_{j=1..p}$ by MH algorithm using random walk draw.

3. Draw each parameter in $\{\theta_{ij}\}_{j=1..q}$ by MH algorithm using random walk draw.

4. Draw each parameter in $\{\alpha_{ij}\}_{j=0..r}$ by MH algorithm using random walk draw.

36For more detailed information, one can refer to the algorithm for ARMA-GARCH error term in Nakatsuma(1998) and Qian et al.(2005).
5. Draw each parameter in \( \{\beta_{ij}\}_{j=1}^s \) by Efficient Jump algorithm.

6. Draw each parameter in \( \zeta_i \) by Efficient Jump algorithm.

For parameter \( \rho \), the algorithm is:

Draw covariance matrix \( \Sigma = \begin{bmatrix}
\sigma_{11}, \ldots, \sigma_{1M} \\
\vdots \\
\sigma_{M1}, \ldots, \sigma_{MM}
\end{bmatrix} \)

from Inverted Wishart by Gibbs Sampling and calculate \( \delta_4 = \rho = D^{-1} \Sigma D^{-1} \), where

\[
D = \begin{bmatrix}
\sqrt{\sigma_{11}}, \ldots, 0 \\
\vdots \\
0, \ldots, \sqrt{\sigma_{MM}}
\end{bmatrix}.
\]
5.4 Appendix iv: Simulation results for Copula-EPD and Copula-NORMAL

Setting $M = 2$ we run following simulation:

\[
Y_1 = X_1\beta_1 + \varepsilon_1
\]
\[
Y_2 = X_2\beta_2 + \varepsilon_2
\]
\[
\varepsilon_1 \sim EPD(\sigma_1, \alpha_1), \quad \varepsilon_2 \sim EPD(\sigma_2, \alpha_2).
\]
\[
T = 200
\]
\[
X_i = [1, \ldots, 1]'', \quad i = 1, 2.
\]
\[
\beta_1 = .8, \quad \beta_2 = .7, \quad \sigma_1 = \sigma_2 = 1, \quad \alpha_1 = \alpha_2 = 1
\]
\[
\rho = \begin{bmatrix}
1 & .2 \\
.2 & 1
\end{bmatrix}
\]

The surface graph of the joint density of Gaussian copula with EPD\textsuperscript{37} margins is presented in Figure 17. This density is different from the density of the multivariate EPD distribution in Figure 1 of Gòmez et.al. (1998). In Gòmez et.al. (1998), the multivariate EPD distribution has only one shape

\textsuperscript{37}For density of EPD, please refer to Tsurumi and Shimizu(2006). $EPD(\sigma_i, \alpha_i)$ becomes the normal distribution when $\alpha_i = 2$. If $\varepsilon_i$ is distributed as $EPD(\sigma_i, \alpha_i)$, then the PDF of $f_i(\varepsilon_i)$ is:

\[
f_i(\varepsilon_i) = \frac{1}{2^{1+1/\alpha_i}\sigma_i \Gamma(1+1/\alpha_i)} \exp\left\{-\frac{1}{2} \left| \frac{\varepsilon_i}{\sigma_i} \right|^{\alpha_i} \right\}
\]

\[
0 < \sigma_i < \infty, \quad 0 < \alpha_i < \infty, \quad i = 1, \ldots, M.
\]
parameter $\alpha$, whereas in the Gaussian copula with EPD margins the shape parameter can be different for each marginal distribution.

To verify that the joint density of Gaussian copula with normal margins is multivariate normal, we generated data by setting $\alpha_1 = \alpha_2 = 2$ while keeping the other parameter settings as the same as before. The surface of the joint density of the Gaussian copula with normal margins is presented in Figure 18.

In conclusion, simulation results show that the joint density is far from normal even if we use Gaussian copula with error term $\varepsilon_i$ ($i = 1, 2$) drawn
Figure 18: Density of Gaussian Copula with Normal Margins
from exponential power distribution (i.e., Copula-EPD). If the $\varepsilon_i$ ($i = 1, 2$) are drawn from normal distributions (i.e., Copula-NORMAL) then Gaussian copula yields the bivariate normal distribution.
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Curriculum Vitae

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