SINGULAR HARMONIC MAPS INTO HYPERBOLIC SPACES AND APPLICATIONS TO GENERAL RELATIVITY

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ABSTRACT OF THE DISSERTATION

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Harmonic maps with singular boundary behavior from a Euclidean domain into hyperbolic spaces arise naturally in the study of axially symmetric and stationary spacetimes in general relativity. In particular, the study of multi-black-hole configurations and the force between co-axially rotating black holes requires, as a first step, an analysis on the boundary regularity of the "next order term" of those harmonic maps. We carry out this analysis by considering those harmonic maps as solutions to some homogeneous divergence systems of partial differential equations with singular coefficients. We then apply our result to study the regularity of axially symmetric and stationary electrovac spacetimes, which extends previous works by Weinstein [22], [23] and by Li and Tian [10], [11], [12]. This dissertation is based on a preprint of the author [16].

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Chapter 1

Introduction

A spacetime in general relativity is a 4-manifold M equipped with a Lorentzian metric g which satisfies the Einstein field equations,

$$R_{ab} - \frac{1}{2} R\, g_{ab} = 2\, T_{ab}.$$

Here R_{ab} and R denote respectively the Ricci and scalar curvature of M, and T_{ab} is the energy-momentum tensor which represents matter. It is of interest to study the regularity of spacetimes in equilibrium consisting of more than one body. In [1], Bach and Weyl showed that any axially symmetric and static vacuum spacetime outside two bodies possesses a conical singularity along the axis connecting the bodies. Later, Bunting and Massood-ul-Alam [3] cleverly used the positive mass theorem of Schoen and Yau to show that there does not exist any regular static "multiple-body vacuum spacetimes". For non-static spacetimes, much less is known. Regardless of regularity, Weinstein [22],[23],[27] used the harmonic map reduction, known as the Ernst-Geroch reduction, to construct a family of multi-black-hole axially symmetric and stationary vacuum/electrovac solutions, of which a regular one must be a member. In the vacuum case, Li and Tian [11], [12] and Weinstein [22], [23], [24] independently proved some regularity results for the reduced harmonic maps and then used them to show that within the solutions constructed by Weinstein, there is a continuum of irregular solutions. In this paper we bridge the methods used by Li and Tian and by Weinstein to extend the above regularity results to the axisymmetric stationary electrovac case.

To describe the reduction used by Weinstein, we first introduce the notion of singular harmonic maps (see [25]). Let Γ be a subset of the z-axis in $\mathbb{R}^3$ obtained by removing some bounded line segments. Let h be the Newtonian potential created by a charge distribution of strictly positive density along Γ. Note that h is a harmonic function on

$\mathbb{R}^3 \setminus \Gamma$ and h 'behaves like' some negative multiple of $\log \rho$ near an interior point of Γ where ρ is the distance to the z-axis. Let $\mathbb{H}$ be either the real or the complex hyperbolic plane. For $\mathbb{H}_{\mathbb{R}}$, we use the standard half-plane model $\{(X, Y) : X > 0\}$ with the metric

$$ds^2 = X^{-2}(dX^2 + dY^2).$$

For $\mathbb{H}_{\mathbb{C}}$, we model it by $\mathbb{R}^4 = \{(u, v, \chi, \psi)\}$ with the metric

$$ds^2 = du^2 + e^{4u}(dv - \psi\, d\chi + \chi\, d\psi)^2 + e^{2u}(d\chi^2 + d\psi^2).$$

(See Appendix A for a derivation of this line element from the standard disk model of $\mathbb{H}_{\mathbb{C}}$.) Then given a geodesic ζ in $\mathbb{H}$, $\zeta \circ h$ is a harmonic map from $\mathbb{R}^3 \setminus \Gamma$ into $\mathbb{H}$. Moreover, as $x \to \Gamma$, $\zeta \circ h(x)$ approaches the ideal point $\zeta(+\infty) \in \partial\mathbb{H}$. Recall that a map $\Phi : \Omega \subset \mathbb{R}^3 \to \mathbb{H}$ is harmonic if it satisfies in local coordinates the equations

$$\Delta\Phi^\alpha + \Gamma^\alpha_{\beta\gamma}(\Phi) D\Phi^\beta \cdot D\Phi^\gamma = 0$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of $\mathbb{H}$.

Definition 1.1 *Let Γ be a subset of the z-axis in $\mathbb{R}^3$ obtained by removing n bounded line segments. To each component Γ_j of Γ, $1 \le j \le n+1$, we associate an ideal point p_j $\in \partial H$. Pick any normalized geodesic ζ_j so that $\zeta_j(+\infty) = p_j$. Let h be the Newtonian potential created by a line charge distribution of strictly positive density on Γ. We say that a map $\Phi : \mathbb{R}^3 \setminus \Gamma \to \mathbb{H}$ is a singular harmonic map controlled by the distribution h and the ideal points p_j if Φ is harmonic and near each component Γ_j the hyperbolic distance between Φ and $\zeta_j \circ h$ is bounded. Sometimes, we simply say that Φ is a singular harmonic map controlled by h.*

The Ernst-Geroch reduction formulation states that every axially symmetric, stationary exterior solution of the Einstein vacuum equation can be conveniently put in the form (see [22] or [8], for example)

$$ds^2 = -\frac{\rho^2}{X}\, dt^2 + X(d\varphi - p\, dt)^2 + \frac{1}{X}\, e^{2\mu}(d\rho^2 + dz^2)$$

where ρ is the distance to the z-axis, φ is the cylindrical angle around the z-axis, and X, p and μ are functions on $\mathbb{R}^3 \setminus \Gamma$ which are determined by the following four conditions.

(i) X, p and μ are independent of the angle variable φ.

(ii) X is the first component of some axially symmetric singular harmonic map (X, Y) from $\mathbb{R}^3 \backslash \Gamma$ into the real hyperbolic plane $\mathbb{H}_{\mathbb{R}}$ which is controlled by the Newtonian potential created by a uniform line charge distribution h of unit density and some ideal points on $\partial\mathbb{H}_{\mathbb{R}}$. In particular, (X, Y) satisfies the harmonic map equations

$$\Delta X = \frac{|DX|^2 - |DY|^2}{X}, \tag{1.1}$$

$$\Delta Y = \frac{2DX \cdot DY}{X}. \tag{1.2}$$

(iii) p satisfies

$$dp = -\frac{\rho}{X^2} Y_z \, d\rho + \frac{\rho}{X^2} Y_\rho \, dz. \tag{1.3}$$

(iv) μ satisfies

$$d\mu = \frac{\rho}{4X^2}(X_\rho^2 - X_z^2 + Y_\rho^2 - Y_z^2) \, d\rho + \frac{\rho}{2X^2}(X_\rho X_z + Y_\rho Y_z) \, dz. \tag{1.4}$$

Notice that the harmonic map equations (1.1) and (1.2) give the integrability conditions needed to integrate (1.3) and (1.4).

Using this reduction and a variational approach, Weinstein proved the existence of a family of (possibly singular) spacetimes which can be interpreted as equilibrium configurations of asymptotically flat co-axially rotating vacuum black holes ([22], [23]). Moreover, he showed that this family is uniquely parametrized by $3n-1$ parameters (n is the number of black holes) which are interpreted as the masses and angular momenta of the black holes, and the distances between them. This result implies in particular Robinson's theorem on the uniqueness of the Kerr solutions [18], [19].

After a first look at the reduction, one might have the impression that given any solution to the singular harmonic map equations (with a right rate of singularity, of course), one could easily cook up a solution to the Einstein vacuum equations. This is in fact not the case, which was probably first observed by Bach and Weyl. As one tries to construct a spacetime out of a singular harmonic map, one might possibly introduce a conical singularity along Γ. A necessary and sufficient condition for regularity is

$$\lim_{\rho \to 0} (\mu - \log X + \log \rho) = 0 \text{ along } \Gamma. \tag{1.5}$$

Bach and Weyl showed in [1] that this limit is nonzero in the static setting, i.e. $Y \equiv 0$. Their method does not seem welcoming of the general setting since it relies on the explicit form of the solution. Even though there has been some progress in getting the explicit form of solutions for multiple-body spacetimes, e.g. the famous double Kerr solutions of Kramer and Neugebauer [9], the dependence of the validity or invalidity of (1.5) on the parameters seems still unclear in general.

Concerning the equation (1.5) itself, some regularity structure across the symmetry axis Γ of the harmonic map (X, Y) is required in order to make sense of the limit on the left hand side. Note that because of the singularity of h, (X, Y) necessarily approaches $\partial\mathbb{H}_{\mathbb{R}}$ near Γ. Therefore, the equations (1.1)-(1.2) are not satisfied across Γ, and so one cannot apply directly well-known results on the regularity of harmonic maps into negatively curved targets to obtain the required regularity for X and Y near Γ. This regularity issue was settled independently by Weinstein and by Li and Tian. Weinstein showed that, about any interior point of Γ, any singular harmonic maps corresponding to the spacetimes he constructed in [22] and [23] can be "decomposed" into an explicit singular part and a C^∞-regular part. Independently, using a different approach, Li and Tian [11], [12] proved a weaker version: $C^{k,\alpha}$ regularity for the regular part, which suffices to justify the limit on the left hand side of (1.5). On the other hand, their result remains valid even if the harmonic map (X, Y) is not axisymmetric or the singular control h is the potential of a uniform charge distribution of arbitrary positive density along Γ. To describe their results more precisely, we define the following weighted spaces.

Definition 1.2 *Let Ω be a domain in $\mathbb{R}^n$ and Σ a subset of Ω. Let w be a positive measurable function in Ω. We denote by $L^p(\Omega, w)$ $(1 \le p \le \infty)$ the space of p-integrable functions with respect to the measure $w(x)\,dx$, equipped with its standard Banach space structure, i.e. with the norm*

$$\|g\|_{L^p(\Omega,w)} = \left\{\int_\Omega |g|^p \, w \, dx\right\}^{1/p}.$$

The spaces $W^{1,p}_{\Sigma}(\Omega, w)$ and $W^{1,p}_{0,\Sigma}(\Omega, w)$ are respectively the completions of $C^\infty_c(\bar{\Omega}\setminus\Sigma)$

and $C_c^\infty(\Omega \setminus \Sigma)$ *with respect to the norm*

$$\|g\|_{W^{1,p}_\Sigma(\Omega,w)} = \left\{ \int_\Omega \left[|g|^p + |Dg|^p w \right] dx \right\}^{1/p}.$$

When $p = 2$, *we also write* $H^1_\Sigma(\Omega, w)$ *and* $H^1_{0,\Sigma}(\Omega, w)$ *for* $W^{1,2}_\Sigma(\Omega, w)$ *and* $W^{1,2}_{0,\Sigma}(\Omega, w)$. *Note that these two spaces are Hilbert spaces. Also, if* w *is bounded from below by a strictly positive number, they are naturally embedded into the familiar Sobolev spaces* $H^1(\Omega)$ *and* $H^1_0(\Omega)$, *respectively.*

Li and Tian [11], [12] and Weinstein [22], [23] proved the following.

Theorem A (Y.Y. Li - G. Tian; G. Weinstein) *Let* Γ *be the* z*-axis in* $\mathbb{R}^3$ *with some line segments removed and* ρ *the distance function to the* z*-axis. Let* h *be the potential of a uniform line charge distribution of density* $\gamma > 0$ *along* Γ. *If* (X, Y) *is a singular harmonic maps from* $\mathbb{R}^3 \setminus \Gamma$ *into* $\mathbb{H}_\mathbb{R}$ *controlled by* h *and some ideal points on* $\partial\mathbb{H}_\mathbb{R}$ *with* $\log X + h \in H^1(\mathbb{R}^3)$, $Y \in H^1(\mathbb{R}^3, e^{2h}))$, *then* $\log X + h$ *and* Y *are* $C^{k,\alpha}$ *across the interior of* Γ *for any integer* k *and* $\alpha \in (0, 1)$ *with* $k + \alpha < 4\gamma$ *and* C^∞ *in* $\mathbb{R}^3 \setminus \Gamma$. *Also, near any compact subset of the interior of any component of* Γ, Y *has the asymptotic expansion*

$$Y = C + O(\rho^{k+\alpha})$$

where C *is a constant (see [11],[12]).*

Moreover, if (X, Y) *is axially symmetric about the* z*-axis and* $\gamma = 1$, $(\log X + h, Y)$ *is everywhere* C^∞ *except possibly at the endpoints of* Γ *(see [22],[23]).*

Remark 1.1 *In the above theorem, we can allow* $k + \alpha = 4\gamma$. *See Corollary 3.1 and Remark 3.2.*

After settling the regularity of the reduced harmonic maps, Li and Tian [11], [12], [10] and Weinstein [24] went forward to study if there is a conical singularity in an axially symmetric stationary vacuum spacetime. As mentioned above, such conical singularity exists if and only if equation (1.5) is violated. Using Theorem A, the two works showed that the limit on the left hand side of equation (1.5) exists and depends continuously on the masses, angular momenta and distances between two bodies. More importantly, they proved that there are several continuous sets of parameters which give rise to spacetimes violating (1.5). Unfortunately, their results do not reveal whether there is any set of parameter that realizes (1.5) except those corresponding to the Kerr spacetimes, i.e. one-body spacetimes.

Later, in an attempt to shed more insight into the problem, Weinstein started analyzing a generalization of the problem for Einstein-Maxwell equations ([25], [26], [27]). It should be noted that there exists regular axially symmetric stationary charged spacetimes which have more than one black hole: the Papapetrou-Majumdar solutions [17], [13], [7]. However, these have degenerate event horizons. It is not known if there are any regular charged spacetimes having non-degenerate event horizons and more than one body.

Similar to the case of vacuum, by the Ernst-Geroch formulation, one can write the metric of any axially symmetric stationary charged spacetime in the form (see [27], [8])

$$ds^2 = -\rho^2 e^{2u}\,dt^2 + e^{-2u}(d\varphi - p\,dt)^2 + e^{2\mu+2u}(d\rho^2 + dz^2)$$

where ρ is again the distance to the z-axis, φ is the cylindrical angle around the z-axis and u, p and μ are determined by

(i) u, p and μ are independent of the angle variable φ;

(ii) u is the first component of some axially symmetric singular harmonic map (u, v, χ, ψ) from $\mathbb{R}^3 \setminus \Gamma$ into the complex hyperbolic plane $\mathbb{H}_{\mathbb{C}}$, i.e. it satisfies

$$\Delta u - 2\,e^{4u}\,|Dv - \psi\,D\chi + \chi\,D\psi|^2 - e^{2u}(|D\chi|^2 + |D\psi|^2) = 0, \tag{1.6}$$

$$\operatorname{div}\left[e^{4u}(Dv - \psi\,D\chi + \chi\,D\psi)\right] = 0, \tag{1.7}$$

$$\operatorname{div}(e^{2u}D\chi) - 2e^{4u}D\chi\cdot(Dv - \psi\,D\chi + \chi\,D\psi) = 0, \tag{1.8}$$

$$\operatorname{div}(e^{2u}D\psi) + 2e^{4u}D\psi\cdot(Dv - \psi\,D\chi + \chi\,D\psi) = 0; \tag{1.9}$$

moreover, it is singular near Γ and the singular rate is controlled by the Newtonian potential creaated by a uniform line charge distribution $\frac{h}{2}$ of density $\frac{1}{2}$ and some ideal points on $\partial\mathbb{H}_{\mathbb{C}}$;

(iii) p satisfies

$$dp = -\rho\, e^{4u}(v_z - \psi\, \chi_z + \chi\, \psi_z)d\rho + \rho\, e^{4u}(v_\rho - \psi\, \chi_\rho + \chi\, \psi_\rho)\, dz; \tag{1.10}$$

(iv) and μ satisfies

$$\begin{aligned} d\mu = \rho[(u_\rho^2 - u_z^2) + \frac{e^{4u}}{4}((v_\rho - \psi\, \chi_\rho + \chi\, \psi_\rho)^2 - (v_z - \psi\, \chi_z + \chi\, \psi_z)^2) \\ + e^{2u}(\chi_\rho^2 - \chi_z^2 + \psi_\rho^2 - \psi_z^2)]d\rho \\ + 2\rho[u_\rho\, u_z + \frac{e^{4u}}{4}(v_\rho - \psi\, \chi_\rho + \chi\, \psi_\rho)(v_z - \psi\, \chi_z + \chi\, \psi_z) \\ + e^{2u}(\chi_\rho\, \chi_z + \psi_\rho\, \psi_z)]dz. \end{aligned} \tag{1.11}$$

Again, the harmonic map equations (1.6)-(1.9) are the integrability conditions for (1.10) and (1.11). Moreover, in the absence of charge, i.e. $\chi = \psi = 0$, the system (1.6)-(1.9) reduces to the system (1.1)-(1.2) via the transformation $X = e^{-2u}$ and $Y = 2v$.

The existence problem for the asymptotically flat case was settled by Weinstein himself. Physically speaking, this solution represents (possibly singular) asymptotically flat co-axially rotating electro-vacuum, or charged, black holes in gravitational equilibrium. Moreover, he also showed that this family of solutions is parametrized by $4n-1$ parameters, n being the number of black holes, which can be interpreted as the masses, angular momenta and charges of the black holes, and the distances between them. Especially, when $n = 1$, he recovered the uniqueness of the Kerr-Newman solutions which was proved independently by Mazur and Bunting ([14], [15], [2]).

Here we would like to address the regularity of the harmonic maps obtained by performing the Ernst-Geroch reduction to the solutions constructed by Weinstein. In this work, we prove:

Theorem B *Let Γ be the z-axis in $\mathbb{R}^3$ with some bounded line segments removed and let ρ be the distance function to the z-axis. Let h be the potential of a uniform charged distribution of density $\gamma > 0$. If (u, v, χ, ψ) is a singular harmonic map from $\mathbb{R}^3 \setminus \Gamma$ into the complex hyperbolic plane $\mathbb{H}_{\mathbb{C}}$ controlled by $\frac{h}{2}$ and some ideal points on $\partial\mathbb{H}_{\mathbb{C}}$ and if $u - \frac{h}{2} \in H^1(\mathbb{R}^3)$, $v \in H^1(\mathbb{R}^3, e^{2h})$, and χ, $\psi \in H^1(\mathbb{R}^3, e^h)$, then $u - \frac{h}{2}$, v χ, $\psi \in C^{k,\alpha}$*

across the interior of Γ *for any* $k+\alpha < 2\gamma$. *In addition, near any compact subset of the interior of any component of* Γ, v, χ *and* ψ *admit the asymptotic expansion*

$$\begin{aligned} v &= C_1 - C_2\,\chi + C_1\psi + O(\rho^{2k+2\alpha}), \\ \chi &= C_2 + O(\rho^{k+\alpha}), \\ \psi &= C_2 + O(\rho^{k+\alpha}), \end{aligned}$$

where C_1, C_2 *and* C_3 *are constants.*

Moreover, if (u, v, χ, ψ) *is axially symmetric about* Γ *and* $\gamma = 1$, $u - h$, v, χ, ψ *are everywhere* C^∞ *except possibly at the endpoints of* Γ.

Remark 1.2 *By letting* $u = -\frac{1}{2}\log X$, $v = \frac{1}{2}Y$, *and* $\chi = \psi = 0$ *in Theorem B, we recover Theorem A.*

Remark 1.3 *The conclusions of Theorem B can be extended to singular harmonic maps with values in any real, complex or quaternionic hyperbolic spaces, i.e. symmetric spaces of rank one. See Remark 2.1.*

As an immediate consequence of Theorem B, we also prove:

Theorem C *The metric components of the co-axially rotating, stationary, multiple-black-hole, charged spacetimes constructed by Weinstein in [27] are* C^∞ *outside the event horizons.*

Again, we emphasize that this theorem does <u>not</u> rule out the possibility of having conical singularity along the symmetry axis. For the metric to be regular across the axis, one needs

$$\lim_{\rho\to 0}(\mu + 2u + \log\rho) = 0 \text{ along } \Gamma. \tag{1.12}$$

In view of Theorem B, one can verify easily that the limit on the left hand side exists. It is then possible to carry out an analysis similar to that in [11], [12] and [24] to prove nonexistence of regular spacetimes corresponding to certain class of parameters. However, we will not pursue this direction in the present work.

The rest of this dissertation is organized as follows. In Chapter 2, we carry out some preliminary analysis on the harmonic map equations (1.6)-(1.9) which allows us to consider it as a special case of a broader class of singular quasilinear elliptic systems. In Chapter 3 we consider the case when the model problem is a single linear equation. In Chapter 4, we return to the study of the model problem in the general setting. The proofs of Theorems B and C are carried out in Chapter 5. Finally, for completeness, we include in Appendix A a quick review on hyperbolic spaces and in appendix B a brief study the weighted Sobolev and Lebesgue spaces $W^{1,p}_{\Sigma}(\Omega, w)$ and $L^p(\Omega, w)$ defined in Definition 1.2.

Chapter 2

The model PDE problem

In $\mathbb{R}^3$, let Γ be the z-axis with some line segments removed. Let h be the Newtonian potential created by a uniform line charge distribution of density $\gamma > 0$ along Γ. We would like to understand the regularity of a singular harmonic map (u, v, χ, ψ) from $\mathbb{R}^3 \setminus \Gamma$ into $\mathbb{H}_{\mathbb{C}}$ controlled by h. Recall that (u, v, χ, ψ) satisfies (1.6)-(1.9),

$$\begin{aligned}
\Delta u - 2\, e^{4u}\, |Dv - \psi\, D\chi + \chi\, D\psi|^2 - e^{2u}(|D\chi|^2 + |D\psi|^2) &= 0,\\
\mathrm{div}\, \left[e^{4u}(Dv - \psi\, D\chi + \chi\, D\psi)\right] &= 0,\\
\mathrm{div}\, (e^{2u} D\chi) - 2e^{4u} D\chi \cdot (Dv - \psi\, D\chi + \chi\, D\psi) &= 0,\\
\mathrm{div}\, (e^{2u} D\psi) + 2e^{4u} D\psi \cdot (Dv - \psi\, D\chi + \chi\, D\psi) &= 0.
\end{aligned}$$

Since $\mathbb{H}_{\mathbb{C}}$ has negative sectional curvature, standard harmonic map theory implies that (u, v, χ, ψ) is smooth away from the singularity set Γ. Thus we only need to study its behavior near Γ. Also, as we are only interested in the interior of Γ, it suffices to restrict our attention to a subset Ω of $\mathbb{R}^3$ such that $\Sigma = \Omega \cap \Gamma$ has only one component whose endpoints lie on $\partial\Omega$. Also, we only need to pick one controlling ideal point $p \in \partial\mathbb{H}_{\mathbb{C}}$. Using an isometry of $\mathbb{H}_{\mathbb{C}}$, we can assume without loss of generality that p is the $+\infty$ point sitting on the u-axis of $\mathbb{H}_{\mathbb{C}}$. We pick the geodesic going to p to be

$$t \mapsto \zeta(t) = (t, 0, 0, 0) \in \mathbb{H}_{\mathbb{C}}.$$

As noted earlier, $\zeta(\frac{h}{2})$ is a singular harmonic map from $\Omega \setminus \Sigma$ into $\mathbb{H}_{\mathbb{C}}$. Also, as $\frac{h}{2}$ and p control (u, v, χ, ψ), we must have

$$d_{\mathbb{H}_{\mathbb{C}}}\Big((\tfrac{1}{2}h, 0, 0, 0), (u, v, \chi, \psi)\Big) < C < \infty \text{ in } \Omega.$$

After some calculation, this turns out to be equivalent to

$$\big|u - \frac{h}{2}\big| + e^{2h}\,|v| + e^{h}(|\chi| + |\psi|) < C < \infty \text{ in } \Omega.$$

We have shown:

Lemma 2.1 *Let Ω be a subset of $\mathbb{R}^3$ and Σ be a curve in Ω. Let h be the Newtonian potential created by a uniform line charge distribution of density $\gamma > 0$ along Σ. Then (u, v, χ, ψ) is a singular harmonic map from $\Omega \setminus \Sigma$ into the complex hyperbolic plane $\mathbb{H}_{\mathbb{C}}$ controlled by $\frac{h}{2}$ and the ideal point $(+\infty, v_0, \chi_0, \psi_0) \in \partial\mathbb{H}_{\mathbb{C}}$ if and only if it satisfies the harmonic map equations* (1.6)-(1.9) *and satisfies the estimate*

$$\big|u - \frac{h}{2}\big| + e^{2h}\,|v - v_0 - \psi_0\,\chi + \chi_0\,\psi| + e^h(|\chi - \chi_0| + |\psi - \psi_0|) < C < \infty \text{ in } \Omega. \quad (2.1)$$

This a priori estimate shows that the harmonic map equations (1.6)-(1.9) is a singular semilinear elliptic system. As h 'behaves like' $-2\gamma \log \rho$ near Σ, the singularity types are negative powers of the distance function to a line.

Next, we observe that our target manifold is $\mathbb{H}_{\mathbb{C}}$, a symmetric space. By Noether's theorem, the harmonic map equations (1.6)-(1.9) induce several conservation laws, each for an isometry of $\mathbb{H}_{\mathbb{C}}$. In fact, there are even enough symmetries to turn (1.6)-(1.9) into divergence form:

$$\operatorname{div}\Big[Du - 2e^{4u}v\,Dv + (2e^{4u}\,v\,\psi - e^{2u}\,\chi)D\chi - (2e^{4u}\,v\,\chi + e^{4u}\,\psi)D\psi\Big] = 0, \quad (2.2)$$

$$\operatorname{div}\Big[e^{4u}(Dv - \psi\,D\chi + \chi\,D\psi)\Big] = 0, \quad (2.3)$$

$$\operatorname{div}\Big[- 2e^{4u}\,\psi\,Dv + (e^{2u} + 2e^{4u}|\psi|^2)D\chi - 2e^{4u}\,\chi\,\psi\,D\psi\Big] = 0, \quad (2.4)$$

$$\operatorname{div}\Big[2e^{4u}\,\chi\,Dv - 2e^{4u}\,\chi\,\psi\,D\chi + (e^{2u} + 2e^{4u}|\chi|^2)D\psi\Big] = 0. \quad (2.5)$$

(In the vacuum case, this divergence structure is more apparent (cf. [21], Chapter 7).)

Write $u^1 = u - h/2$, $u^2 = v$, $u^3 = \chi$, $u^4 = \psi$, and note that h is harmonic, the above system can be rewritten in the form

$$\operatorname{div}\,(c_{\alpha\beta}(x,u)\,Du^\beta) = 0, \qquad 1 \le \alpha \le 4$$

where $c_{\alpha\beta}$ is a 4×4 matrix of coefficients given by

$$\begin{bmatrix} 1 & -2e^{4u}v & 2e^{4u}v\,\psi - e^{2u}\chi & -2e^{4u}v\,\chi + e^{2u}\psi \\ 0 & e^{4u} & -e^{4u}\psi & e^{4u}\chi \\ 0 & -2e^{4u}\psi & e^{2u} + 2e^{4u}|\psi|^2 & -2e^{4u}\chi\,\psi \\ 0 & 2e^{4u}\chi & -2e^{4u}\chi\,\psi & e^{2u} + 2e^{4u}|\chi|^2 \end{bmatrix}.$$

The drawback of this way of writing the harmonic map equations is that the coefficients $c_{\alpha\beta}$ might not satisfy the Legendre-Hadamard condition. Indeed, for $\xi = (\xi^1 = a, \xi^2 = 1, \xi^3 = 0, \xi^4 = 0) \in \mathbb{R}^4$,

$$c_{\alpha\beta}\xi^\alpha\xi^\beta = a^2 - 2e^{4u}v\, a + e^{4u}.$$

It is readily seen that when $e^{2u}v > 1$, the right hand side changes sign as a varies in $\mathbb{R}$. Nevertheless, when the a priori estimate (2.1) holds, the Legendre-Hadamard condition can be recovered (for a different but equivalent set of coefficients). To this end we rewrite (2.2)-(2.5) in the form

$$\mathrm{div}\,(a_{\alpha\beta}(x,u)\, Du^\beta) = 0, \qquad 1 \le \alpha \le 4$$

where $a_{\alpha\beta}$ is a 4×4 matrix of coefficients given by

$$\begin{bmatrix} 1 & -2e^{4u}v & 2e^{4u}v\,\psi - e^{2u}\chi & -2e^{4u}v\,\chi + e^{2u}\psi \\ 0 & 2le^{4u} & -2le^{4u}\psi & 2le^{4u}\chi \\ 0 & -2le^{4u}\psi & le^{2u} + 2le^{4u}|\psi|^2 & -2le^{4u}\chi\,\psi \\ 0 & 2le^{4u}\chi & -2le^{4u}\chi\,\psi & le^{2u} + 2le^{4u}|\chi|^2 \end{bmatrix}$$

and l is some constant to be determined shortly. Then, for $\xi \in \mathbb{R}^4$,

$$\begin{aligned} a_{\alpha\beta}(x,u)\,\xi^\alpha\,\xi^\beta = {} & |\xi^1|^2 + 2le^{4u}|\xi^2 - \psi\xi^3 + \chi\xi^4|^2 + le^{2u}(|\xi^3|^2 + |\xi^4|^2) \\ & - [2e^{4u}v(\xi^2 - \psi\xi^3 + \chi\xi^4) - e^{2u}\psi\xi^3 + e^{2u}\chi\xi^4]\xi^1 \end{aligned}$$

In view of (2.1), we can always pick l sufficiently large and positive λ, Λ such that

$$\begin{aligned} \lambda[|\xi^1|^2 + e^{4u}|\xi^2|^2 + e^{2u}(|\xi^3|^2 + |\xi^4|^2)] & \le a_{\alpha\beta}(x,u)\,\xi^\alpha\,\xi^\beta \\ & \le \Lambda[|\xi^1|^2 + e^{4u}|\xi^2|^2 + e^{2u}(|\xi^3|^2 + |\xi^4|^2)] \text{ for any } \xi \in \mathbb{R}^4. \end{aligned}$$

We are thus led to study the following problem:

Problem 2.1 *Let Ω be a domain in $\mathbb{R}^n$ and Σ be a $(n-k)$-submanifold of Ω, $k \ge 2$. Consider the system*

$$\mathrm{div}\,(a_{\alpha\beta}(x,u)\, Du^\beta) = 0 \ \text{in}\ \Omega \setminus \Sigma, \qquad 1 \le \alpha \le m, \tag{2.6}$$

where $a_{\alpha\beta} : (\Omega \setminus \Sigma) \times \mathbb{R}^m \to \mathbb{R}$ *are given smooth functions and* $u : \Omega \to \mathbb{R}^m$ *is the unknown.*

Let $d = d_\Sigma$ *denote the distance function to* Σ*. Assume that there exist constants* $k(1), k(2), \ldots, k(m) \geq 0$ *such that for any given* $R > 0$*, there exist* $\lambda = \lambda(R) > 0$ *and* $\Lambda = \Lambda(R)$ *such that*

$$\lambda \sum_{\alpha=1}^{m} d(x)^{-2k(\alpha)} |\xi^\alpha|^2 \leq a_{\alpha\beta}(x, u)\, \xi^\alpha\, \xi^\beta \leq \Lambda \sum_{\alpha=1}^{m} d(x)^{-2k(\alpha)} |\xi^\alpha|^2 \tag{2.7}$$

for any $x \in \Omega \setminus \Sigma$*,* $\xi \in \mathbb{R}^m$ *and* $u \in \mathbb{R}^m$ *satisfying* $d(x)^{-k(\alpha)}|u^\alpha| \leq R$*,* $1 \leq \alpha \leq m$*.*

Assume that u *solves* (2.6) *in some appropriate sense. Determine how regular* u *is!*

Remark 2.1 *Analogous to the case of harmonic maps into the complex hyperbolic plane, regularity for harmonic maps with prescribed singularity into any real, complex, or quaternionic hyperbolic space in the sense described in [25] can always be recast as a part of Problem 1. Consequently, the result in Theorem B extends parallelly to those cases.*

We now describe what we mean by a solution to (2.6). The singularity of the coefficients $a_{\alpha\beta}$ and the aforementioned existence result established by Weinstein makes it reasonable to assume that $u^\alpha \in H^1_\Sigma(\Omega, d^{-2k(\alpha)})$. In addition, in a compactly supported open subset of $\Omega \setminus \Sigma$, $a_{\alpha\beta}$ behaves nicely and so (2.6) should hold in the usual weak sense, i.e.

$$\int_\Omega a_{\alpha\beta}(x, u)\, Du^\alpha\, D\xi^\beta\, dx = 0, \qquad \xi \in C_c^\infty(\Omega \setminus \Sigma, \mathbb{R}^m).$$

This suggests the following definition.

Definition 2.1 *A measurable function* $u : \Omega \to \mathbb{R}^m$ *with* $u^\alpha \in H^1_\Sigma(\Omega, d^{-2k(\alpha)})$ *is said to be a weak solution of* (2.6) *if*

$$\int_\Omega a_{\alpha\beta}(x, u)\, Du^\alpha\, D\xi^\beta\, dx = 0$$

for any $\xi^\alpha \in H^1_{0,\Sigma}(\Omega, d^{-2k(\alpha)})$*.*

Chapter 3

The case of a single linear equation

In this chapter, we consider a special case of Problem 1 where the unknown is a scalar map and the coefficient is independent of the unknown. Recall that Ω is an open subset of $\mathbb{R}^n$ and Σ is a $(n-k)$-dimensional submanifold of Ω with $k \geq 2$. Let d be the distance function to Σ and w be a weight that satisfies

$$\lambda\, d(x)^{-\gamma} \leq w(x) \leq \Lambda\, d(x)^{-\gamma},$$

for some $\gamma > 0$, $0 < \lambda \leq \Lambda < \infty$. We are interested in the regularity of weak solutions of

$$\operatorname{div}\big(w(x)Du\big) = 0 \text{ in } \Omega \setminus \Sigma. \tag{3.1}$$

Recall that by weak solution we mean a function $u \in H^1_\Sigma(\Omega, d^{-\gamma})$ such that

$$\int_\Omega w(x)\, Du\, D\xi\, dx = 0 \text{ for any } \xi \in H^1_{0,\Sigma}(\Omega, d^{-\gamma}).$$

This problem has been studied extensively in the literature. When $\gamma < k-2$, w belongs to the Muckenhoupt class A_2 and the result of Fabes, Kenig and Serapioni [4] implies that u is Hölder continuous across Σ. Unfortunately, in our application to the problem from general relativity, w might get too singular in a way that it is not even integrable across Σ, and so their work does not apply directly. Moreover, for other purposes, we are also interested in whether w vanishes along Σ and how fast it decays there. A result in this direction was given by Li and Tian in [11] when Σ has codimension 2. We generalize their work to the case where Σ is a general submanifold of any codimension $k \geq 2$ and sharpen the decay estimate to its optimal form.

Theorem 3.1 *Assume that $w \in C^2(\Omega \setminus \Sigma)$ and*

$$\lambda\, d^{-\gamma} \leq w \leq \Lambda\, d^{-\gamma}$$

for some $\gamma \geq 0$ *and* $0 < \lambda \leq \Lambda < \infty$. *Moreover, assume that* $|D(d^\gamma\, w)| = o(d^{-1})$ *in a neighborhood of* Σ.

Then any $u \in H^1_\Sigma(\Omega, w)$ *which solves*

$$\mathrm{div}\,(w\,Du) = 0$$

in $\Omega \setminus \Sigma$ *is locally Hölder continuous in* Ω *and enjoys*

$$\sup_\omega d^{-(\gamma-k+2)^+}\,|u| \leq C\,\|d^{-\frac{\gamma}{2}}u\|_{L^2(\Omega)}$$

for any ω *compactly supported in* Ω. *The constant* C *depends only on* Λ, λ *and the modulus of continuity of* $d|D(d^\gamma\, w)|$.

Remark 3.1 *To see that the decay estimate is optimal, consider the case where* Σ *is an* $n-k$ *hyperplane and* $w = d^{-\gamma}$.

When $\gamma > k-2$, (3.1) *has a special solution* $u = d^{\gamma-k+2}$ *which belongs to* $H^1_\Sigma(\Omega, d^{-\gamma})$. *This solution vanishes along* Σ *and exhibits Hölder continuity along* Σ.

When $\gamma < k-2$, *the above special solution does not belong to* $H^1_\Sigma(\Omega, d^{-\gamma})$. *Moreover, constant functions are solutions which are in* $H^1_\Sigma(\Omega, d^{-\gamma})$ *and, of course, they need not vanish along* Σ. *Nevertheless, as mentioned above,* w *is in the Muckenhoupt class* A_2 *and so* u *is Hölder continuous across* Σ.

Before proving Theorem 3.1, let us give an application which improves Theorem A.

Corollary 3.1 *Under the hypotheses of Theorem A,* $(\log X + h, Y)$ *is of class* $C^{k,\alpha}$ *where* k *is the biggest integer smaller than* 4γ *and* $\alpha = 4\gamma - k$. *Moreover, near any compact subset of the interior of any component of* Γ, Y *admits the expansion*

$$Y = C + O(\rho^{4\gamma})$$

for some constant C.

Proof. Focus our attention to a small neighborhood Ω of an interior point of Γ, we can assume that $Y \in H^1(\Omega, d^{-4\gamma})$ and $Y|_\Sigma = 0$ where $\Sigma = \Gamma \cap \Omega$ (see Lemma 2.1). This implies that $Y \in H^1_\Sigma(\Omega, d^{-4\gamma})$, Rewrite (1.2) as

$$\operatorname{div}(X^{-2}\, DY) = 0$$

and apply Theorem 3.1, we get the required decay for Y and then its $C^{k,\alpha}$ regularity. To get the regularity for X, we rewrite (1.1) as

$$\Delta(\log X + h) = -\frac{|DY|^2}{X^2}$$

and apply a simple estimate for the Poisson equation. □

Remark 3.2 *To see that the $C^{k,\alpha}$ regularity in Corollary 3.1 is optimal, consider for example the case where Γ is the whole z-axis, $h = -2\gamma \log \rho$, $\gamma > 0$ and*

$$X = \frac{\rho^{2\gamma}}{\rho^{4\gamma}+1},$$
$$Y = \frac{\rho^{4\gamma}}{\rho^{4\gamma}+1}.$$

This example also shows that the $C^{k,\alpha}$ regularity in Theorem B is almost optimal as a harmonic map into the real hyperbolic plane is a special harmonic map into the complex hyperbolic plane.

We will prove Theorem 3.1 through a sequence of lemmas. Also, to avoid technicality, we will assume that Σ is a $(n-k)$-hyperplane and $w = d^{-\gamma}$. In fact, if $\bar{w} = d^{\gamma} w$ is not constant, the proofs below require minor changes.

Also, we will frequently use the following inequality (see Appendix B) without explicitly mentioning,

$$\int_{\omega} d^{-\gamma-2}\, |f|^2\, dx \le C \int_{\Omega} [|f|^2 + d^{-\gamma}\, |Df|^2]\, dx$$

for any $f \in H^1_{\Sigma}(\Omega, d^{-\gamma})$ and ω compactly supported in Ω.

Lemma 3.1 *Theorem 3.1 holds for $0 \le \gamma < 2(k-2)$.*

Proof. For $0 \le \gamma \le k-2$, the assertion is a consequence of the aforementioned result of Fabes et al. [4]. Assume that $k-2 < \gamma < 2(k-2)$. Set $v = d^{-\gamma+k-2} u$ and $\tilde{w} = d^{-2(k-2)+\gamma}$. Then $v \in H^1(\Omega, \tilde{w})$ and satisfies in $\Omega \setminus \Sigma$ the equation

$$\operatorname{div}(\tilde{w}\, Dv) = 0.$$

Since $\gamma < 2(k-2)$, $\tilde{w}$ belongs to the Muckenhoupt class A_2 and the above equation is satisfied across Ω. Again, the result of Fabes et al. in [4] applies showing that v is locally bounded in Ω and for any ω compactly supported in Ω,

$$\sup_{\omega} |v| \le C \, \| d^{\frac{\gamma}{2}-k+2} v \|_{L^2(\Omega)} \le C \, \| d^{-\frac{\gamma}{2}} v \|_{L^2(\Omega)}.$$

$\square$

Lemma 3.2 *Assume that $\gamma \ge 2(k-2)$ and u is as in Theorem 3.1. Then u satisfies*

$$\sup_{\omega} d^{-\frac{\gamma}{2}} |u| \le C \, \| d^{-\frac{\gamma}{2}} u \|_{L^2(\Omega)}$$

for any ω compactly supported in Ω.

Proof. Write $v = d^{-\frac{\gamma}{2}} u$. Then $v \in H^1(\Omega) \cap L^2(\Omega, d^{-2})$ and satisfies in $\Omega \setminus \Sigma$ the equation

$$\Delta v - c\, v = 0$$

where

$$c = \frac{\gamma}{2} (\frac{\gamma}{2} - k + 2) \, \frac{1}{d^2}.$$

Note that c is positive. We therefore can apply De Giorgi or Moser techniques to show that v is locally bounded and

$$\sup_{\omega} |v| \le C \, \| v \|_{L^2(\Omega)}.$$

The lemma is proved. $\square$

Lemma 3.3 *Assume that $u \in H^1(\Omega) \cap C^0(\bar{\Omega})$ solves the following inhomogeneous equation in $\Omega \setminus \Sigma$*

$$\mathrm{div}\, (w(x) D u) = f, \tag{3.2}$$

where f is smooth away from Σ and $|f| \le C\, d^{-\gamma-2+\mu}$ in a neighborhood of Σ for some $0 < \mu < \lambda - k + 2$. Assume in addition that u vanishes along Σ. Then for any $\lambda \le \mu$, $d^{-\lambda} u$ is locally bounded and for any ω compactly supported in Ω,

$$\sup_{\omega} d^{-\lambda} |u| \le C \sup_{\Omega} |u|.$$

Proof. Fix some ω compactly supported in Ω. Fix some $0 < h < 1$. Assume for the moment that $d^{-c}\,u$ is bounded. We will show that if $c + h \le \mu$ then

$$\sup_{\omega} d^{-c-h}\,|u| \le C\,\sup_{\omega} d^{-c}\,|u|.$$

This obviously implies our result.

Fix $x_0 \in \Omega$. Consider the function

$$\theta(x) = d^{c+h}(x) + d^{c}(x)\,|x - x_0|^2.$$

We compute

$$\begin{aligned} D\theta &= (c+h)\,d^{c+h-1}\,Dd + c\,d^{c-1}\,|x-x_0|^2\,Dd + 2\,d^c\,(x-x_0),\\ \mathrm{div}\,(d^{-\gamma}\,D\theta) &= (c+h)(c+h-\gamma+k-2)\,d^{-\gamma+c+h-2}\\ &\quad + c(c-\gamma+k-2)\,d^{-\gamma+c-2}\,|x-x_0|^2\\ &\quad + 2(2c-\gamma)\,d^{-\gamma+c-1}\,Dd\cdot(x-x_0) + 2n\,d^{-\gamma+c}. \end{aligned}$$

Since $h < 1$, this implies that

$$\mathrm{div}\,(d^{-\gamma}\,D\theta) \le C(c+h)(c+h-\gamma+k-2)\,d^{-\gamma+c+h-2} \le -C\,|f|.$$

in some neighborhood U of Σ. Take another neighborhood V of Σ such that $\bar{V} \subset U$. Set $\delta = \mathrm{dist}\,(V, \partial U) > 0$. For $x_0 \in V$ with $|x - x_0| > \delta$, we have

$$|u| \le \|d^{-c}u\|_{L^\infty(U)} d^c \le \|d^{-c}u\|_{L^\infty(U)}\,\delta^{-2}\,\theta$$

Also, θ vanishes on Σ. Hence, by the maximum principle,

$$|u| \le \|d^{-c}u\|_{L^\infty(U)}\,\delta^{-2}\,\theta \text{ in } B_\delta(x_0).$$

In particular, for $x = x_0$, we have $|u(x_0)| \le \|d^{-c}u\|_{L^\infty(U)}\,\delta^{-2}\,d^{c+h}(x_0)$. Since x_0 is arbitrary, this proves the lemma. $\square$

Lemma 3.4 *Let $v \in H^1(\Omega) \cap L^2(\Omega, d^{-2})$ solve the following equation in $\Omega \setminus \Sigma$:*

$$\Delta v - p\,\frac{Dd}{d}\cdot Dv - q\frac{1}{d^2}v = 0 \tag{3.3}$$

where p and q are two real numbers satisfying

- *$q \geq 0$,*
- *$(p-k+2)^2 + 4q - p^2 > 0$,*
- *and $p - k + 2 \leq \frac{2q}{k-2}$ or $\frac{4q^2}{(k-2)^2} - \frac{4q(p-k+2)}{k-2} + p^2 - 4q < 0$.*

Then $v \in L^\infty_{loc}(\Omega)$ and for any ω compactly supported in Ω,

$$\sup_\omega |v| \leq C \, \|v\|_{L^2(\Omega)}.$$

The constant C depends continuously on p and q.

Remark 3.3 *The classical De Giorgi - Nash - Moser estimate does not apply here as $d^{-1} \notin L^{1,n-1+\epsilon}$ and $d^{-2} \notin L^{1,n-2+\epsilon}$ for any $\epsilon > 0$.*

Proof. We can assume that Ω is the unit ball B_1. We will show that v is bounded in $B_{1/2}$. The idea is to adapt De Giorgi's proof exploiting the fact that q is negative.

Let $A(k,\rho) = \{x \in B_\rho : u(x) > k\}$ and

$$\Phi(k,\rho) = \int_{A(k,\rho)} |u-k|^2 \, dx.$$

Let η be a standard cut-off function supported in B_ρ. Observe that we can take $\eta^2 (u-k)^+$ as a test function for (3.3). This yields

$$\begin{aligned}\int_{A(k,\rho)} \eta^2 \, |Du|^2 \, dx \leq \int_{A(k,\rho)} \{&\epsilon \, \eta^2 \, |Du|^2 + C_\epsilon \, |u-k|^2 \, |D\eta|^2 - p \, \eta^2 \, \frac{Dd}{d} \, Du \, (u-k)^+ \\ &- q \, \eta^2 \, \frac{1}{d^2} \, |u-k|^2 \} dx. \end{aligned} \tag{3.4}$$

On the other hand, by integrating by parts, we see that

$$\begin{aligned}\pm \int_{A(k,\rho)} \eta^2 \, \frac{Dd}{d} \, Du \, (u-k)^+ \, dx &= \pm \frac{1}{2} \int_{A(k,\rho)} \eta^2 \, \frac{Dd}{d} \, D(|u-k|^2) \, dx \\ &= \mp \frac{1}{2} \int_{A(k,\rho)} [\eta^2 \, \mathrm{div}\,(\frac{Dd}{d}) + 2 \, \eta \, D\eta \, \frac{Dd}{d}] \, |u-k|^2 \, dx \\ &\leq -(\pm \frac{k-2}{2} + \epsilon') \int_{A(k,\rho)} \eta^2 \, \frac{1}{d^2} \, |u-k|^2 \, dx \\ &\qquad + C_{\epsilon'} \int_{A(k,\rho)} |u-k|^2 \, |D\eta|^2 \, dx.\end{aligned}$$

Substituting the above estimate into (3.4), we arrive at

$$\int_{A(k,\rho)} \eta^2 \, |Du|^2 \, dx \leq \int_{A(k,\rho)} \{\epsilon \, \eta^2 \, |Du|^2 + C \, |u-k|^2 \, |D\eta|^2 - (p-b) \, \eta^2 \, \frac{Dd}{d} \, Du \, (u-k)^+ \\ - (q - \frac{(k-2)b}{2} - \epsilon') \, \eta^2 \, \frac{1}{d^2} \, |u-k|^2\} dx$$

where b is any real number. If we requires that $b < \frac{2q}{k-2}$, this implies

$$\int_{A(k,\rho)} \eta^2 \, |Du|^2 \, dx \leq \int_{A(k,\rho)} \{\alpha \, \eta^2 \, |Du|^2 + C \, |u-k|^2 \, |D\eta|^2\} dx$$

where

$$\alpha = \frac{(p-b)^2}{4(q - \frac{(k-2)b}{2} - \epsilon')} + \epsilon.$$

Our hypotheses on p and q are exactly to allow us to find some $b < \frac{2q}{k-2}$ so that $\alpha < 1$. We have therefore shown that

$$\int_{A(k,\rho)} \eta^2 \, |Du|^2 \, dx \leq C \int_{A(k,\rho)} |u-k|^2 \, |D\eta|^2 \, dx. \tag{3.5}$$

Using this estimate, we can run through De Giorgi's proof for local boundedness and obtain

$$\sup_{B_{1/2}} u \leq C \, \|u^+\|_{L^2(B_1)}.$$

The lemma follows. $\square$

Proof of Theorem 3.1. The case $0 \leq \gamma < 2(k-2)$ is taken care by Lemma 3.1. We hence assume that $\gamma \geq 2(k-2)$.

By Lemma 3.2, $d^{-\frac{\gamma}{2}} u$ is locally bounded. Thus, by elliptic theory, u is continuous in Ω and vanishes along Σ. Lemma 3.3 hence shows that $d^{-\lambda} u$ is locally bounded for any $\lambda < \gamma - k + 2$. By elliptic theory, we infer that $d^{-\lambda+1}|Du|$ is locally bounded for the same λ's.

Take a sequence $\lambda_m < \gamma - k + 2$ converging to $\gamma - k + 2$. Set $v_m = d^{-\lambda_m} u$. Since $\lambda_m < \gamma - k + 2$, the above estimates on the sizes of u and Du show that $v_m \in H^1(\Omega) \cap L^2(\Omega, d^{-2})$. Moreover, in $\Omega \setminus \Sigma$, v_m satisfies

$$\Delta v_m - p_m \, \frac{Dd}{d} \cdot Dv_m - q_m \, \frac{1}{d^2} \, v_m = 0$$

where $p_m = \gamma - 2\lambda_m$ and $q_m = \lambda_m(\gamma - k + 2 - \lambda_m) > 0$. An application of Lemma 3.4 shows that, for any ω compactly supported in Ω,

$$\sup_{\omega} d^{-\lambda_m}|u| = \sup_{\omega} |v_m| \le C_m(\omega)\,\|v_m\|_{L^2(\Omega)} = C_m(\omega)\,\|d^{-\lambda_m}u\|_{L^2(\Omega)}.$$

More importantly, as $\lambda_m \to \gamma - k + 2$, the constant C_m does not blow up as

$$(p_m - k + 2)^2 + 4q_m - p_m^2 \to (\gamma - k + 2)^2 - (\gamma - 2k + 4)^2 > 0$$

and

$$p_m - k + 2 - \frac{2q_m}{k-2} \to -\gamma + k - 2 < 0.$$

As a consequence, if $\omega \subset\subset \omega' \subset\subset \Omega$,

$$\sup_{\omega} d^{-\gamma+k-2}|u| \le C\,\|d^{-\gamma+k-2}u\|_{L^2(\omega')} \le C\,\sup_{\omega'} d^{-\gamma+k-2+\epsilon}|u|$$
$$\le C\,\sup_{\omega'} d^{-\frac{\gamma}{2}}|u| \le C\,\|d^{-\frac{\gamma}{2}}u\|_{L^2(\Omega)}.$$

In any case, the theorem is ascertained. $\square$

Chapter 4

Hölder regularity

We next switch our attention back to the system (2.6), i.e.

$$\operatorname{div}\left(a_{\alpha\beta}(x,u)\,Du^{\beta}\right)=0,\qquad 1\le\alpha\le m.$$

Recall that $u^{\alpha}\in H^1_{\Sigma}(\Omega,d^{-2k(\alpha)})$ is a weak solution of (2.6) if

$$\int_{\Omega} a_{\alpha\beta}(x,u)\,Du^{\alpha}\,D\xi^{\beta}\,dx=0,\qquad \forall\ \xi^{\alpha}\in H^1_{0,\Sigma}(\Omega,d^{-2k(\alpha)}).$$

Throughout the chapter, the coefficients $a_{\alpha\beta}$ are assumed to satisfy the ellipticity condition (2.7), i.e. for any $R>0$, there exists $\lambda=\lambda(R)>0$ and $\Lambda=\Lambda(R)<\infty$ such that

$$\lambda\sum_{\alpha=1}^{m} d(x)^{-2k(\alpha)}|\xi^{\alpha}|^2\le a_{\alpha\beta}(x,u)\,\xi^{\alpha}\,\xi^{\beta}\le\Lambda\sum_{\alpha=1}^{m} d(x)^{-2k(\alpha)}|\xi^{\alpha}|^2$$

for any $x\in\Omega\setminus\Sigma$ and $u\in\mathbb{R}^m$ verifying $d(x)^{-k(\alpha)}u^{\alpha}\le R$. Here $k(1)$, $k(2)$, …, $k(m)$ are non-negative numbers. Let τ be the smallest integer such that $k(\tau)>0$. (If there is no such index, set $\tau=m+1$.)

In general, without any additional assumption on the coefficients $a_{\alpha\beta}$ or the unknowns u^{α}, one does not expect u^{α} to be regular, or even Hölder continuous, across Σ. For example, when the coefficients $a_{\alpha\beta}$ satisfies the classical ellipticity, i.e. $k(\alpha)=0$, it is well-known that u satisfies (2.6) in Ω in the weak sense and so is regular at any point $x\in\Sigma$ at which

$$\liminf_{\delta\to 0}\frac{1}{\delta^{n-2}}\int_{B_{\delta}(x)}|Du|^2\,dx=0.$$

(See [5], Chapter 4, for example.) If $a^{ij}_{\alpha\beta}$ do not depend on u and are smooth across Σ, this set is empty and so u is regular in Ω. Nonetheless, if $a^{ij}_{\alpha\beta}$ is allowed to depend on u, the points where regularity fails might constitute a non-vacuous subset of Σ (see [5], Chapter 2).

We show that the above phenomenon also persists for the problem we are considering. For any $B_\delta(x) \subset \Omega$, we define

$$E_\delta(x) = \frac{1}{\delta^{n-2}} \int_{B_\delta(x)} \sum_{\alpha=1}^{m} d(z)^{-2k(\alpha)} |Du^\alpha(z)|^2 \, dz. \tag{4.1}$$

Also, define

$$b_{\alpha\beta}(x, \tilde{u}) = d(x)^{-[k(\alpha)+k(\beta)]} \, a(x, d(x)^{-k(\gamma)} \tilde{u}^\gamma), \qquad x \in \Omega \setminus \Sigma, \tilde{u} \in \mathbb{R}^m.$$

By (2.7), the functions $b^{i,j}_{\alpha\beta}(x, \tilde{u})$ remain bounded as long as $\{\tilde{u}^\alpha\}_{\alpha=1}^m$ stays bounded.

Theorem 4.1 *Let Ω be a domain in $\mathbb{R}^n$ and let Σ be a $(n-k)$-submanifold of Ω, $k \geq 2$. Assume that $a_{\alpha\beta}$ are smooth functions on $(\Omega \setminus \Sigma) \times \mathbb{R}^m$ which satisfy the ellipticity condition (2.7) with $k(\alpha)$ being either zero or bigger than $k-2$. Let τ be the smallest index so that $k(\tau) > k-2$. (If all $k(\alpha)$ vanish, we set $\tau = m+1$.) Assume that*

$$b_{\alpha\beta}(x, \tilde{u}^1, \ldots, \tilde{u}^{\tau-1}, 0) = 0 \text{ unless } \alpha = \beta. \tag{4.2}$$

Let $u^\alpha \in H^1_\Sigma(\Omega, d^{-2k(\alpha)})$ be a weak solution of (2.6).

If $u^\alpha \in L^\infty(\Omega, d^{-k(\alpha)})$, then there exists $\varepsilon_0 > 0$ such that if $E_\delta(x_0) \leq \varepsilon_0$ for some $0 < \delta < \operatorname{dist}(x_0, \partial\Omega)/4$, then

$$E_\sigma(x) \leq C \, \sigma^{2\lambda}, \qquad x \in B_{\delta/2}(x_0), \sigma < \delta/4,$$

for some $\lambda = \lambda(k(\alpha), \|u^\alpha\|_{L^\infty(\Omega, d^{-k(\alpha)})}) > 0$ and $C = C(k(\alpha), \|u^\alpha\|_{L^\infty(\Omega, d^{-k(\alpha)})}) \geq 0$. In particular, u is Hölder continuous in $B_{\delta/4}(x_0)$ and for $\alpha \geq \tau$, $d^{-k(\alpha)-\lambda}|u^\alpha|$ is bounded in $B_{\delta/4}(x_0)$.

Remark 4.1 *The requirement that $k(\tau) > k-2$ is probably merely technical. It would be interesting to remove this extra hypothesis. However, in doing so, one must be careful with the assumption that $d^{-k(\alpha)}u^\alpha$ is bounded. For, in case of a single linear equation, if $k(\alpha) < k-2$, there are examples that u^α may not vanish along Σ as fast as $d^{k(\alpha)}$ (see Remark 3.1).*

In the context of harmonic maps, an ϵ-regularity result is usually established using some monotonicity formula. For example, this is the case in the work of Li and Tian

[11]. It seems that their proof does not apply in our context. However, as demonstrated in Chapter 2, our harmonic map problem has an equivalent homogeneous divergence form. This special structure makes it apparent to obtain a Caccioppoli inequality which we will state shortly. When restricted to the vacuum case in [11], this Caccioppoli inequality implies the monotonicity formula therein.

Observe that, as long as we stay away from Σ, (2.6) is an elliptic system for u with bounded smooth coefficients, so classical elliptic theory tells us that a Caccioppoli inequality holds there. However, as we move towards Σ, the coefficients of these equations behave badly signifying some possible deterioration in the structure of the Caccioppoli inequality. This is indeed true as stated in the following result.

Proposition 4.1 (Caccioppoli inequality) *Let $u^\alpha \in H^1_\Sigma(\Omega, d^{-2k(\alpha)}) \cap L^\infty(\Omega, d^{-k(\alpha)})$ be a weak solution of* (2.6)*. Let x_0 be a point in Ω and δ_0 the distance from x_0 to $\partial\Omega$. There exists $C = C(\Lambda, \lambda) > 0$ such that for $\delta \le \delta_0/2$ and $b \in \mathbb{R}^m$ whose last $m - \tau + 1$ components vanish when $B_{2\delta}(x_0) \cap \Sigma \neq \emptyset$,*

$$\int_{B_{\delta/2}(x_0)} \sum_{\alpha=1}^m d^{-2k(\alpha)} |Du^\alpha|^2 \, dz \le \frac{C}{\delta^2} \int_{B_\delta(x_0)} \sum_{\alpha=1}^m d^{-2k(\alpha)} \, |u^\alpha - b^\alpha|^2 \, dz.$$

Proof. Fix $\varepsilon > 0$ and $\delta < \delta_0$. Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a cut-off function satisfying $\eta(z) = 0$ if $|z - x_0| \ge \delta$ and $\eta(z) = 1$ if $|z - x_0| \le \delta/2$, and $|D\eta| \le 4\delta^{-1}$.

Observe that our constraints on b imply that $u^\alpha - b^\alpha \in H^1_\Sigma(\Omega, d^{-2k(\alpha)})$. Thus, by Corollary B.1 in Appendix B, we can take $\xi = \eta^2(u - b)$ as a test function for (2.6). Using this special choice of ξ and recalling (2.7) yields

$$\begin{aligned}\int_\Omega \eta^2 \sum_{\alpha=1}^m d^{-2k(\alpha)} \, |Du^\alpha|^2 \, dz &\le C \int_\Omega \eta^2 a_{\alpha\beta} \, Du^\alpha \, Du^\beta \, dz \\ &= -C \int_\Omega \eta a_{\alpha\beta} \, Du^\beta \, D\eta \, (u^\alpha - b^\alpha) \, dz \\ &\le C \int_\Omega \sum_{\alpha=1}^m \left[\eta^2 \, d^{-2k(\alpha)} \, |Du^\alpha|^2 + |D\eta|^2 \, d^{-2k(\alpha)} \, |u^\alpha - b^\alpha|^2 \right] dz.\end{aligned}$$

This implies the inequality in question. □

We next study the limiting system when doing a blow-up analysis and then prove that smallness in energy density implies regularity. We first recall a well known result (see [5], Chapter 4).

Lemma 4.1 *Let $a^{ij}_{\alpha\beta(h)}$ be a sequence of measurable functions satisfying*

$$\lambda|p|^2 \le a^{ij}_{\alpha\beta(h)}(x,v)p^\alpha_i p^\beta_j \le \Lambda|p|^2, \qquad (x,v,p) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{mn}$$

for some $0 < \lambda \le \Lambda < \infty$ and converging in $L^2(B_1(0))$ to $a^{ij}_{\alpha\beta}$ verifying the same ellipticity bound. Let $f_{(h)}$ be a sequence in $L^2(B_1(0);\mathbb{R}^m)$ converging weakly in $L^2(B_1(0))$ to f. Let $u_{(h)}$ be a sequence in $H^1_{loc}(B_1(0);\mathbb{R}^m) \cap L^2(B_1(0))$ such that

$$\frac{\partial}{\partial x^i}\big(a^{ij}_{\alpha\beta(h)}\frac{\partial u^\beta_{(h)}}{\partial x^j}\big) = f^\alpha_{(h)}$$

in the weak sense in $B_1(0)$.

If $u_{(h)}$ converges weakly in $L^2(B_1(0);\mathbb{R}^m)$ to u then $u \in H^1_{loc}(B_1(0);\mathbb{R}^m)$ and

- *for any $\rho < 1$, $u_{(h)}$ converges strongly to u in $L^2(B_\rho(0);\mathbb{R}^m)$ and $Du_{(h)}$ converges weakly to Du in $L^2(B_\rho(0);\mathbb{R}^m)$,*

- *and more importantly, u satisfies in the weak sense in $B_1(0)$ the system*

$$\frac{\partial}{\partial x^i}\big(a^{ij}_{\alpha\beta}\frac{\partial u^\beta}{\partial x^j}\big) = f.$$

Lemma 4.2 *Let $u^\alpha \in H^1_\Sigma(\Omega, d^{-2k(\alpha)}) \cap L^\infty(\Omega, d^{-k(\alpha)})$ be a weak solution to (2.6). Let $B_{(h)} = B_{\delta_{(h)}}(x_{(h)})$ be a sequence of balls in Ω such that the ratios $\frac{d(x_{(h)})}{\delta_{(h)}}$ are all less than some fixed κ. Let $\Sigma_{(h)}$ be the image of Σ under the map $x \mapsto (x - x_{(h)})/\delta_{(h)}$ and $d_{(h)}$ the distance function to $\Sigma_{(h)}$. Assume that $\Sigma_{(h)}$ stabilizes to some Σ_*, i.e. $d_{(h)}$ converges uniformly away from Σ_* to d_*, the distance function to Σ_*.*

Set $\varepsilon_{(h)} = E_{\delta_{(h)}}(x_{(h)})^{1/2}$. Define

$$u^\alpha_{(h)}(x) = \frac{1}{\varepsilon_{(h)}\,\delta^{k(\alpha)}_{(h)}}[u^\alpha(x_{(h)} + \delta_{(h)}\,x) - \bar{u}^\alpha_{(h)}], \qquad x \in B_1(0),$$

where

$$\bar{u}^\alpha_{(h)} = \begin{cases} \text{the average of } u^\alpha \text{ over } B_{(h)} \text{ if } k(\alpha) = 0, \\ 0 \text{ otherwise.} \end{cases}$$

If $\varepsilon_{(h)} \to 0$, there exists $u^\alpha_* \in H^1_{k(\alpha)\bar{v}^0_*}(B_{1/2}(0))$ such that, up to extracting a subsequence,

- *$u^\alpha_{(h)}$ converges weakly in $H^1(B_{1/2}(0))$ and strongly in $L^2(B_{1/2}(0))$ to u^α_*,*

- $d_{(h)}^{-k(\alpha)}\, u_{(h)}^{\alpha}$ *converges strongly to* $d_*^{-k(\alpha)}\, u_*^{\alpha}$ *in* $L^2(B_{1/2}(0))$ *and* $d_{(h)}^{-k(\alpha)}\, Du_{(h)}^{\alpha}$ *converges weakly to* $d_*^{-k(\alpha)}\, Du_*^{\alpha}$ *in* $L^2(B_{1/2}(0))$.

Moreover, u_* *satisfies*

$$\operatorname{div}\left(d_*^{-[k(\alpha)+k(\beta)]}\, b_{\alpha\beta *}\, Du^{\beta}\right) = 0$$

in the weak sense in $B_{1/2}(0) \setminus \Sigma_*$, *where* $b_{\alpha\beta *}$ *are constant and satisfy*

$$\nu_1 \sum_{\alpha=1}^{m} d_*^{-2k(\alpha)}\, |p^{\alpha}|^2 \leq d_*^{-[k(\alpha)+k(\beta)]}\, b_{\alpha\beta *}\, p^{\alpha}\, p^{\beta} \leq \nu_2 \sum_{\alpha=1}^{m} d_*^{-2k(\alpha)}\, |p^{\alpha}|^2,$$

and the ratio ν_2/ν_1 *does not depend on the sequence of balls* $B_{\delta_{(h)}}(x_{(h)})$. *Additionally,* $b_{\alpha\beta *}$ *can be written in the form*

$$b_{\alpha\beta *} = b_{\alpha\beta}(x_*, \tilde{u}_*^1, \ldots, \tilde{u}_*^{\tau-1}, 0)$$

for some accumulation point x_* *of the sequence* $x_{(h)}$.

Proof. Let's assume for the moment that the above convergences have been established. Passing to a subsequence, we can assume that $x_h \to x_*$, $d_{(h)}^{-k(\alpha)}\, u_{(h)}^{\alpha} \to d_*^{-k(\alpha)}\, u_*^{\alpha}$ a.e. Also, as

$$\delta_{(h)}^{-2k(\alpha)} |\bar{u}_{(h)}^{\alpha}|^2 \leq \frac{C}{\delta_{(h)}^{n}} \int_{B_{(h)}} d^{-2k(\alpha)} |u^{\alpha}|^2\, dz \leq C,$$

we can assume that $\delta_{(h)}^{-k(\alpha)}\, \bar{u}_{(h)}^{\alpha} \to \tilde{u}_*^{\alpha}$. It follows that

$$\tilde{u}^{\alpha}(x_{(h)} + \delta_{(h)} x) \;=\; d_{(h)}^{-k(\alpha)}(x)\, \delta_{(h)}^{-k(\alpha)} [\varepsilon_{(h)}\, \delta_{(h)}^{k(\alpha)}\, u_{(h)}^{\alpha}(x) + \bar{u}_{(h)}^{\alpha}] \;\to\; \tilde{u}_*^{\alpha}\, d_*(x)^{-k(\alpha)} \;\; \textit{a.e.},$$

Therefore, if we define

$$\begin{aligned} a_{\alpha\beta(h)}(x) &= \delta_{(h)}^{-[k(\alpha)+k(\beta)]}\, a_{\alpha\beta}\Big(x_{(h)} + \delta_{(h)} x, u(x_{(h)} + \delta_{(h)} x)\Big) \\ &= d_{(h)}^{-[k(\alpha)+k(\beta)]}\, b_{\alpha\beta}\Big(x_{(h)} + \delta_{(h)} x, \tilde{u}(x_{(h)} + \delta_{(h)} x)\Big). \end{aligned}$$

then

$$a_{\alpha\beta(h)}(x) \to d_*(x)^{-[k(\alpha)+k(\beta)]}\, b_{\alpha\beta}\Big(x_*, \tilde{u}_*^{\alpha}\, d_*(x)^{-k(\alpha)}\Big) \;\textit{a.e.}$$

By the Lebesgue dominated convergence theorem, this can be regarded as convergence in $L^2_{loc}(B_1(0) \setminus \Sigma_*)$. An application of Lemma 4.1 yields the second half of the Lemma.

It remains to check the convergences. We have

$$\int_{B_1(0)} \sum_{\alpha=1}^{m} d_{(h)}^{-2k(\alpha)} \, |Du_{(h)}^{\alpha}|^2 \, dx = \frac{1}{\delta_{(h)}^{n-2} \, \varepsilon_{(h)}^2} \int_{B_{(h)}} \sum_{\alpha=1}^{m} d^{-2k(\alpha)} \, |Du^{\alpha}|^2 \, dx = 1. \tag{4.3}$$

Here we have used the assumption that $E_{\delta_{(h)}}(x_{(h)}) = \varepsilon_{(h)}^2$. In particular, this implies that the L^2 norm of $Du_{(h)}^{\alpha}$ over $B_1(0)$ is uniformly bounded for $\alpha < \tau$. The standard Poincaré inequality then implies that the H^1 norm of $u_{(h)}^{\alpha}$ over $B_1(0)$ is also uniformly bounded for $\alpha < \tau$. For $\alpha \geq \tau$, we use Corollary B.1 in Appendix B to show that the L^2 norm and so the H_{Σ}^1 norm of $u_{(h)}^{\alpha}$ over $B_{3/4}(0)$ is uniformly bounded. One can then modify the proof of Proposition B.2 in Appendix B to get the required convergences. The details work out in exactly the same manner and hence are omitted. □

Lemma 4.3 *Let $u^{\alpha} \in H_{\Sigma}^1(\Omega, d^{-2k(\alpha)}) \cap L^{\infty}(\Omega, d^{-k(\alpha)})$ be a weak solution to* (2.6)*. Let $B_{(h)} = B_{\delta_{(h)}}(x_{(h)})$ be a sequence of balls in Ω such that the ratios $\frac{d(x_{(h)})}{\delta_{(h)}}$ are all at least some fixed κ.*

Set $\varepsilon_{(h)} = E_{\delta_{(h)}}(x_{(h)})^{1/2}$ and define

$$u_{(h)}^{\alpha}(x) = \frac{1}{\varepsilon_{(h)} \, d(x_{(h)})^{k(\alpha)}} [u^{\alpha}(x_{(h)} + \delta_{(h)} \, x) - \bar{u}_{(h)}^{\alpha}], \qquad x \in B_1(0),$$

where $\bar{u}_{(h)}^{\alpha}$ is the average of u^{α} over $B'_{(h)} = B_{\kappa\delta_{(h)}/2}(x_{(h)})$.

If $\varepsilon_{(h)} \to 0$, there exists $u_* \in H^1(B_{\kappa/2}(0))$ such that, up to extracting a subsequence, $u_{(h)}$ converges weakly in $H^1(B_{\kappa/2}(0))$ and strongly in $L^2(B_{\kappa/2}(0))$ to u_*. Moreover, u_* satisfies

$$\operatorname{div}\left(a_{\alpha\beta*}(x) \, Du^{\beta}\right) = 0$$

in the weak sense in $B_1(0) \setminus \Sigma_*$, where $a_{\alpha\beta*}$ are smooth and satisfy

$$\nu_1 \sum_{\alpha=1}^{m} |p^{\alpha}|^2 \leq a_{\alpha\beta*} \, p^{\alpha} \, p^{\beta} \leq \nu_2 \sum_{\alpha=1}^{m} |p^{\alpha}|^2,$$

and the ratio ν_2/ν_1 does not depend on the sequence of balls $B_{\delta_{(h)}}(x_{(h)})$ and does not exceed $C \, \kappa^{-2 \max k(\alpha)}$.

Proof. The proof of this lemma is similar to but easier than that of Lemma 4.2 and is omitted. □

Having a Caccioppoli inequality at hand and some understanding of the limiting system, we are ready to go forward with our proof of Theorem 4.1, i.e. smallness in density $E_\delta(x)$ implies regularity. Recall we require here that any limiting system decouples into m independent equations by asking (4.2), namely

$$b_{\alpha\beta}(x, \tilde{u}^1, \ldots, \tilde{u}^{\tau-1}, 0) = 0 \text{ unless } \alpha = \beta.$$

Lemma 4.4 *Let $u^\alpha \in H^1_\Sigma(\Omega, d^{-2k(\alpha)}) \cap L^\infty(\Omega, d^{-k(\alpha)})$ be a weak solution to* (2.6)*. Assume in addition that* (4.2) *holds and all the $k(\alpha)$ are either zero or bigger than $k-2$. Let x_0 be a point in Ω and $\delta_0 = \operatorname{dist}(x_0, \partial\Omega)$. There exists ε_0 and λ_0 such that for any $x \in B_{\delta_0/2}(x_0)$ and $\delta \in (0, \delta_0/4)$ satisfying $E_\delta(x) \leq \varepsilon_0^2$ there holds $E_{\lambda_0\,\delta}(x) \leq E_\delta(x)/2$.*

Proof. Arguing indirectly, we assume that the conclusion fails. Fix some λ_0 and κ for the moment. They will be determined by a set of constraints which will be formulated in the sequel. We can find sequences $\varepsilon_{(h)} \to 0$, $x_{(h)} \in B_{\delta_0/2}(x_0)$, and $\delta_{(h)} \in (0, \delta_0/4)$ such that $E_{\delta_{(h)}}(x_{(h)}) = \varepsilon_{(h)}^2$ but $E_{\lambda_0\,\delta_{(h)}}(x_{(h)}) > \varepsilon_{(h)}^2/2$. In addition, we can assume that one of the following two cases occurs: All the ratios $\frac{d(x_{(h)})}{\delta_{(h)}}$ are less than κ or all are not.

<u>Case 1</u>: $\frac{d(x_{(h)})}{\delta_{(h)}} < \kappa$ for all h.

Define $d_{(h)}$, $u^\alpha_{(h)}$, d_*, Σ_*, u^α_* and $b_{\alpha\beta *}$ as in Lemma 4.2.

In the ball $B_{1/2}(0)$, u^α_* satisfies

$$\operatorname{div}(d^{-2k(\alpha)}\, Du^\alpha) = 0.$$

Hence, by Theorem 3.1,

$$|u^\alpha_*(x) - u^\alpha_*(0)| \leq C|x| \qquad \alpha < \tau$$

and

$$|u^\alpha_*(x)| \leq C\, d(x)^{2k(\alpha)-k+2} \qquad \alpha \geq \tau$$

for some C independent of the balls $B_{\delta_{(h)}}(x_{(h)})$.

Therefore, by the Caccioppoli inequality,

$$
\begin{aligned}
\frac{E_{\lambda_0\,\delta_{(h)}}(x_{(h)})}{\varepsilon_{(h)}^2} &\le \frac{C}{\varepsilon_{(h)}^2(\lambda_0\delta_{(h)})^n}\int_{B_{2\lambda_0\delta_{(h)}}(x_{(h)})}\Big\{\sum_{\alpha=1}^{\tau-1}|u^\alpha-\bar u^\alpha_{(h)}-\varepsilon_{(h)}\,m_{(h)}^{k(\alpha)}\,u_*^\alpha(0)|^2 \\
&\qquad\qquad + \sum_{\alpha=\tau}^{m} d^{-2k(\alpha)}\,|u^\alpha|^2\Big\}dz \\
&\le \frac{C}{\lambda_0^n}\int_{B_{2\lambda_0}(0)}\Big\{\sum_{\alpha=1}^{\tau-1}|u^\alpha_{(h)}-u_*^\alpha(0)|^2+\sum_{\alpha=\tau}^{m} d_{(h)}^{-2k(\alpha)}\,|u^\alpha_{(h)}|^2\Big\}dz \\
&\le \frac{C}{\lambda_0^n}\int_{B_{2\lambda_0}(0)}\sum_{\alpha=1}^{m}|d_{(h)}^{-k(\alpha)}\,u^\alpha_{(h)}-d_*^{-k(\alpha)}\,u^\alpha_*|^2\,dz \\
&\qquad + \frac{C}{\lambda_0^n}\int_{B_{2\lambda_0}(0)}\Big\{|z|^2+\sum_{\alpha=\tau}^{m} d_*^{2k(\alpha)-2k+4}\Big\}dz \\
&\le \frac{C}{\lambda_0^n}\int_{B_{2\lambda_0}(0)}\sum_{\alpha=1}^{m}|d_{(h)}^{-k(\alpha)}\,u^\alpha_{(h)}-d_*^{-k(\alpha)}\,u^\alpha_*|^2\,dz \\
&\qquad + C_1\,\lambda_0^2+C_2\,(\kappa+2\lambda_0)^{2k(\tau)-2k+4}.
\end{aligned}
$$

Hence if

$$
C_1\,\lambda_0^2+C_2\,(\kappa+2\lambda_0)^{2k(\tau)-2k+4}\le\frac{1}{8}, \tag{4.4}
$$

we infer that $\frac{\varepsilon_{(h)}^2}{2}<E_{\lambda_0\,\delta_{(h)}}(x_{(h)})\le\frac{\varepsilon^2}{4}$ for h large enough, a contradiction.

Case 2: $\frac{d(x_{(h)})}{\delta_{(h)}}\ge\kappa$ for all h.

Let $u^\alpha_{(h)}$, u^α_* and $a_{\alpha\beta *}$ as in Lemma 4.3.

In $B_{\kappa/2}(0)$, u_* satisfies

$$
\operatorname{div}\left(a_{\alpha\beta *}(x)Du^\beta\right)=0
$$

where the $a_{\alpha\beta *}$ are smooth and satisfy

$$
\nu_1\sum_{\alpha=1}^{m}|p^\alpha|^2\le a_{\alpha\beta *}\,p^\alpha\,p^\beta\le\nu_2\sum_{\alpha=1}^{m}|p^\alpha|^2,
$$

with $\nu_2/\nu_1\le C\,\kappa^{-2\max k(\alpha)}$. Also, the L^2 norm of u_* in $B_{\kappa/2}(0)$ is universally bounded. Therefore

$$
\sum_{\alpha=1}^{m}|u_*^\alpha(x)-u_*^\alpha(0)|\le C\,\kappa^{-\frac{n+A}{2}}|x|,\quad x\in B_{\kappa/4}(0)
$$

for some $A>0$. Like case 1, the constant C is independent of the balls $B_{\delta_{(h)}}(x_{(h)})$.

We again invoke the Caccioppoli inequality and obtain

$$\begin{aligned}
&\frac{E_{\lambda_0\,\delta_{(h)}}(x_{(h)})}{\varepsilon_{(h)}^2}\\
&\quad\le \frac{C}{\varepsilon_{(h)}^2(\lambda_0\delta_{(h)})^n}\int_{B_{2\lambda_0\delta_{(h)}}(x_{(h)})}\sum_{\alpha=1}^m d^{-2k(\alpha)}\,|u_\alpha-\bar u_{(h)}^\alpha-\varepsilon_{(h)}\,d(x_{(h)})^{k(\alpha)}\,u_*^\alpha(0)|^2\,dz\\
&\quad\le \frac{C}{\lambda_0^n}\int_{B_{2\lambda_0}(0)}\sum_{\alpha=1}^m |u_{(h)}^\alpha-u_*^\alpha(0)|^2\,dz\\
&\quad\le \frac{C}{\lambda_0^n}\int_{B_{2\lambda_0}(0)}\sum_{\alpha=1}^m |u_{(h)}^\alpha-u_*^\alpha|^2\,dz+\frac{C}{\kappa^{n+A}\lambda_0^n}\int_{B_{2\lambda_0}(0)}|z|^2\,dz\\
&\quad\le \frac{C}{\lambda_0^n}\int_{B_{2\lambda_0}(0)}\sum_{\alpha=1}^m |u_{(h)}^\alpha-u_*^\alpha|^2\,dz+C_3\,\kappa^{-n-A}\,\lambda_0^2.
\end{aligned}$$

Similar to case 1, if we can choose λ_0 and κ such that

$$C_3\,\kappa^{-n-A}\,\lambda_0^2\le\frac{1}{8}, \tag{4.5}$$

this will lead us to an absurdity.

To complete the proof, we need to furnish (4.4) and (4.5). This is always doable by first picking λ_0 small enough such that $\lambda_0\le\frac{1}{4\sqrt{C_1}}$ and $(8C_3\,\lambda_0^2)^{\frac{1}{n+A}}\le(\frac{1}{16C_2})^{\frac{1}{B}}-2\lambda_0$, and then picking κ in between the last two figures. Here $B=2k(\tau)-2k+4>0$. $\square$

Proof of Theorem 4.1. Let $u^\alpha\in H^1_\Sigma(\Omega,d^{-2k(\alpha)})\cap L^\infty(\Omega,d^{-k(\alpha)})$ be a weak solution of (2.6). Let ε_0 be as in Lemma 4.4. Assume that $E_\delta(x_0)\le\varepsilon_0$ for some $0<\delta<\operatorname{dist}(x_0,\partial\Omega)/4$.

It follows from Lemma 4.4 and standard iteration techniques (cf. [6]) that there exists positive constants C and λ depending only on $k(\alpha)$ and the $L^\infty(\Omega,d^{-k(\alpha)})$-norm of u^α such that

$$E_\sigma(x)=\frac{1}{\sigma^{n-2}}\int_{B_\sigma(x)}\sum_{\alpha=1}^m d^{-2k(\alpha)}\,|Du^\alpha|^2\,dz\le C\,\sigma^{2\lambda},\qquad x\in B_{\delta/2}(x_0),\sigma<\delta/4.$$

It follows from Morrey's lemma that u is Hölder continuous in $B_{\delta/2}(x_0)$.

Moreover, if $k(\alpha)>0$, the above estimate implies a rate of vanishing of u^α near Σ. To see this, observe that if $\sigma<d(x)/2$, then

$$\int_{B_\sigma(x)}|Du^\alpha|^2\,dz\le C\,d(x)^{2k(\alpha)}\,\sigma^{n-2+2\lambda},\qquad 1\le\alpha\le m,$$

which implies

$$\operatorname*{osc}_{B_{\sigma/2}(x)} u^{\alpha} \leq C\, d(x)^{k(\alpha)}\, \sigma^{\lambda}.$$

(See Theorem 7.19 in [6], for example.) Since u^{α} vanishes along Σ when $k(\alpha) > 0$, this implies that $d^{k(\alpha)-\lambda} u^{\alpha}$ is bounded in $B_{\delta/4}(x_0)$. The proof is complete. $\square$

Chapter 5

Applications

We now turn to the proof of Theorem B and C. We first consider a special case of Problem 1 in which the smallness assumption in Theorem 4.1 is fulfilled. The condition we impose additionally on the system (2.6) is a common feature that harmonic map problems into hyperbolic spaces share. This was first used by Li and Tian in [11] in the case of real hyperbolic spaces. We doubt that this phenomenon has some connection to the underlying geometry of the target manifolds, but we have very limited evidence.

Proposition 5.1 *Suppose all the assumptions of Theorem 4.1 hold. Assume in addition that* $\tau = 2$. *If*

$$a_{\beta 0} = \delta_{\beta 0} \text{ and } a_{0\beta} = -l(\alpha)\, a_{\alpha\beta}\, u^\alpha, \qquad 2 \le \beta \le m, \tag{5.1}$$

for some $l(1) = 0$, $l(2), \ldots, l(m) > 0$, *then the* u^α *are Hölder continuous across* Σ. *Moreover, for* $\alpha > 1$, $|u^\alpha| \le C\, d^{k(\alpha)+\lambda}$ *for some* $\lambda > 0$.

First observe that, due to (5.1), the first equation in the system (2.6) can be rewritten as

$$\Delta u^1 = \sum_{2 \le \alpha, \beta \le m} l(\alpha)\, a_{\alpha\beta}\, Du^\alpha\, Du^\beta. \tag{5.2}$$

Note that the right hand side, which we will abbreviate as $F(x, u, Du)$, is always nonnegative by ellipticity.

Lemma 5.1 *Let* $u^\alpha \in H^1_\Sigma(\Omega, d^{-2k(\alpha)}) \cap L^\infty(\Omega, d^{-k(\alpha)})$ *be a weak solution to* (2.6). *Let* x_0 *be a point in* Ω *and* $\delta_0 = \operatorname{dist}(x_0, \partial\Omega)$. *Then there exists* K *such that if* $\delta < \delta_0/4$ *and* $x \in B_{\delta_0/2}(x_0)$,

$$\int_{B_\delta(x)} |x - z|^{-n+2} \sum_{\alpha=1}^{m} d^{-2k(\alpha)}\, |Du^\alpha|^2\, dz \le K,$$

and so

$$E_\delta(x) = \frac{1}{\delta^{n-2}} \int_{B_\delta(x)} \sum_{\alpha=1}^{m} d^{-2k(\alpha)} \, |Du^\alpha|^2 \, dz \le K.$$

Proof. Note that u^1 satisfies equation (5.2) in the sense of distributions throughout Ω.

For ε sufficiently small, let $\eta_\varepsilon : [0,\infty) \to \mathbb{R}$ be a cut-off function satisfying

$$\eta_\varepsilon(t) = \begin{cases} 0 & \text{if } t \le \varepsilon \text{ or } t \ge 2\delta, \\ 1 & \text{if } 2\varepsilon \le t \le \delta, \end{cases}$$

and $|\eta_\varepsilon'|$, $|\eta_\varepsilon''| \le C$ in $[\delta, 2\delta]$, and $|\eta_\varepsilon'| \le C\varepsilon^{-1}$, $|\eta_\varepsilon''| \le C\varepsilon^{-2}$ in $[\varepsilon, 2\varepsilon]$.

Let R be a positive number such that $|u^1| \le R$. Let $G_x(z) = |x-z|^{-n+2}$ and set $G_x^\varepsilon(z) = \eta_\varepsilon(|x-z|) G_x(z)$. Inserting $G_x^\varepsilon(z)(u^1 + R + 1)$ as a test function into equation (5.2) and note that F is nonnegative, we get

$$\begin{aligned}
\int_{B_{2\delta}(x)} G_x^\varepsilon(z)\, F(z,u,Du)\, dz &\le \int_{B_{2\delta}(x)} F(z,u,Du)\, G_x^\varepsilon(z)\, (u^1 + R + 1)\, dz \\
&= -\int_{B_{2\delta}(x)} Du^1\, D[G_x^\varepsilon(z)\, (u^1 + R + 1)]\, dz \\
&= -\int_{B_{2\delta}(x)} \left[G_x^\varepsilon(z) |Du^1|^2 + (u^1 + R + 1)\, Du^1\, DG_x^\varepsilon \right] dz.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{B_\delta(x)\setminus B_{2\varepsilon}(x)} G_x(z) \sum_{\alpha=1}^{m} d^{-2k(\alpha)}\, |Du^\alpha|^2\, dz \\
&\quad \le C \int_{B_\delta(x)\setminus B_{2\varepsilon}(x)} G_x(z) \left[F(z,u,Du) + |Du^1|^2 \right] dz \\
&\quad \le -C \int_{B_{2\delta}(x)} (u^1 + R + 1)\, Du^1\, DG_x^\varepsilon\, dz. \\
&\quad = C \int_{B_{2\delta}(x)} (u^1 + R + 1)^2\, \Delta[\eta_\varepsilon(|x-z|)\, G_x(z)]\, dz \\
&\quad \le C \int_{B_{2\delta}(x)} \left[|\eta_\varepsilon''(|x-z|)|\, |x-z|^{-n+2} + |\eta_\varepsilon'(|x-z|)|\, |x-z|^{-n+1} \right] dz \\
&\quad \le K.
\end{aligned}$$

The conclusion then follows from Lebesgue's monotone convergence theorem. □

Proposition 5.2 (Smallness of density) *Let* $u^\alpha \in H^1_\Sigma(\Omega, d^{-2k(\alpha)}) \cap L^\infty(\Omega, d^{-k(\alpha)})$ *be a weak solution to* (2.6). *Let* x_0 *be a point in* M *and* $\delta_0 = \operatorname{dist}(x_0, \partial M)$. *Let* K *be* *the constant obtained in Lemma 5.1. For given $\varepsilon > 0$, $\delta < \delta_0/4$, and $x \in B_{\delta_0/2}(x_0)$, there exists $\sigma = \sigma(\varepsilon, \delta, x)$ between $e^{-2K(n-2)/\varepsilon}\delta$ and δ such that*

$$E_\sigma(x) \le \varepsilon.$$

Proof. Fix $\varepsilon > 0$. For $\delta' = e^{-2K(n-2)/\varepsilon}\delta$, we have

$$\begin{aligned}
\int_{\delta'}^{\delta} \frac{E_\sigma(x)}{\sigma} d\sigma &= \int_{\delta'}^{\delta} \sigma^{-n+1} \int_{B_\sigma(x)} \sum_{\alpha=1}^{m} d^{-2k(\alpha)} \, |Du^\alpha|^2 \, dz \, d\sigma \\
&= (-n+2)\sigma^{-n+2} \int_{B_\sigma(x)} \sum_{\alpha=1}^{m} d^{-2k(\alpha)} \, |Du^\alpha|^2 \, dz \, \Bigg|_{\delta'}^{\delta} \\
&\qquad + (n-2) \int_{\delta'}^{\delta} \sigma^{-n+2} \int_{\partial B_\sigma(x)} \sum_{\alpha=1}^{m} d^{-2k(\alpha)} \, |Du^\alpha|^2 \, dS(z) \, d\sigma \\
&= (-n+2)\sigma^{-n+2} \int_{B_\sigma(x)} \sum_{\alpha=1}^{m} d^{-2k(\alpha)} \, |Du^\alpha|^2 \, dz \, \Bigg|_{\delta'}^{\delta} \\
&\qquad + (n-2) \int_{B_\delta(x) \setminus B_{\delta'}(x)} |x-z|^{-n+2} \sum_{\alpha=1}^{m} d^{-2k(\alpha)} \, |Du^\alpha|^2 \, dz
\end{aligned}$$

Hence, by Lemma 5.1

$$\int_{\delta'}^{\delta} \frac{E_\sigma(x)}{\sigma} d\sigma \le 2K(n-2).$$

The assertion follows immediately from an argument by contradiction. □

Proof of Proposition 5.1. By Proposition 5.2,

$$\liminf_{\sigma \to 0} E_\sigma(x) = 0 \qquad \forall \, x \in \Omega.$$

Theorem 4.1 then applies yielding the assertion. □

Finally, we consider Theorem B. Recall that u, v, χ and ψ satisfy (1.6)-(1.9), i.e.

$$\begin{aligned}
\Delta u - 2\, e^{4u} \, |Dv - \psi \, D\chi + \chi \, D\psi|^2 - e^{2u}(|D\chi|^2 + |D\psi|^2) &= 0, \\
\operatorname{div}\left[e^{4u}(Dv - \psi \, D\chi + \chi \, D\psi) \right] &= 0, \\
\operatorname{div}\,(e^{2u} D\chi) - 2e^{4u} D\chi \cdot (Dv - \psi \, D\chi + \chi \, D\psi) &= 0, \\
\operatorname{div}\,(e^{2u} D\psi) + 2e^{4u} D\psi \cdot (Dv - \psi \, D\chi + \chi \, D\psi) &= 0.
\end{aligned}$$

We will use various equations derivable from (1.6)-(1.9). The first set gives the formulas for the Laplacians of u, χ and ψ.

$$\Delta(u - \frac{h}{2}) = \Delta u = 2e^{4u}|Dv - \psi\, D\chi + \chi\, D\psi|^2 + e^{2u}(|D\chi|^2 + |D\psi|^2), \tag{5.3}$$

$$\Delta\chi = -2Du \cdot D\chi + 2e^{2u} D\psi \cdot (Dv - \psi\, D\chi + \chi\, D\psi), \tag{5.4}$$

$$\Delta\psi = -2Du \cdot D\psi - 2e^{2u} D\chi \cdot (Dv - \psi\, D\chi + \chi\, D\psi), \tag{5.5}$$

The second makes way to apply Lemma 3.3.

$$\operatorname{div}(e^{4u} Dv) = e^{4u} Du \cdot (\psi\, D\chi - \chi\, D\psi) + e^{4u}(\psi\, \Delta\chi - \chi\, \Delta\psi), \tag{5.6}$$

$$\operatorname{div}(e^{2u} D\chi) = 2e^{4u} D\psi \cdot (Dv - \psi\, D\chi + \chi\, D\psi), \tag{5.7}$$

$$\operatorname{div}(e^{2u} D\psi) = -2e^{4u} D\chi \cdot (Dv - \psi\, D\chi + \chi\, D\psi). \tag{5.8}$$

The proof consists of two parts. In the first part, we prove the general case where γ is arbitrary. In the second part, we consider the physical case where $\gamma = 1$ and (u, v, χ, ψ) is axially symmetric around the z-axis.

For the first part, we apply Theorem 4.1 and Corollary 5.1 to obtain Hölder continuity. Unlike proving Corollary 3.1, we cannot apply Theorem 3.1 due to the coupling of the unknowns. Instead, we repeatedly apply Lemma 3.3, a primitive of Theorem 3.1, to (5.6)-(5.8) to get $C^{k,\alpha}$ regularity.

For the second part, we follow the reversed Ernst-Geroch reduction scheme to prove smoothness under axial symmetry. This was used by Weinstein in his work on the vacuum case. His idea is the following. The system (1.1)-(1.2) was obtained by applying the Ernst-Geroch formulation for axisymmetric stationary vacuum solutions, which was done in the order of the axial Killing vector field being applied first and the stationary Killing vector field second. This choice of order made it more convenient to establish existence and uniqueness but regularity. If one applies the reduction scheme in the reversed order, i.e. to the stationary Killing vector field first and then to the axial Killing vector field, one obtains a harmonic map problem from $\mathbb{R}^3$ into the real hyperbolic plane in the usual sense. Similarly, if one applies the reversed Ernst-Geroch reduction scheme to the Einstein-Maxwell equations, the system acquired is, though no longer a harmonic map problem, still elliptic. To finish one needs to prove some initial regularity for the acquired system. In the vacuum case, this can be done much simpler by a clever use of the De Giorgi-Nash-Moser regularity result for scalar elliptic equation (see [22]). It seems unlikely that this technique applies in the electrovac case. It is precisely at this point that we need to use the $C^{k,\alpha}$ regularity obtained in the general case to go forward.

Proof of Theorem B. Note that since the complex hyperbolic plane is a manifold of negative sectional curvature, (u, v, χ, ψ) is C^∞ in $\mathbb{R}^3 \setminus \Sigma$. Thus we only need to consider the regularity question around Γ. Without loss of generality, we assume that the origin is an interior point of Γ, $h = -2\gamma \log \rho$ and that the controlling ideal point on $\partial \mathbb{H}_{\mathbb{C}}$ is $(+\infty, 0, 0, 0)$ as in Chapter 2.

Step 1: $C^{k,\alpha}$ regularity in the general case.

Let $\bar{u} = u - h$. Then, as shown in Chapter 2, $(\bar{u}, v, \chi, \psi)$ satisfies the hypotheses of Theorem 4.1 and Proposition 5.1. Therefore, $(\bar{u}, v, \chi, \psi)$ is locally Hölder continuous in Ω and there is some $\lambda > 0$ so that

$$\frac{1}{\delta^{n-2}} \int_{B_\delta} \Big[|D\bar{u}|^2 + \rho^{-4\gamma} |Dv|^2 + \rho^{-2\gamma} (|D\chi|^2 + |D\psi|^2) \Big] \leq C\, \delta^\lambda.$$

It follows that $|D\bar{u}| = O(\rho^{-1+\lambda})$, $|Dv| = O(\rho^{-1+2\gamma+2\lambda})$, and $|D\chi| + |D\psi| = O(\rho^{-1+\gamma+\lambda})$. Thus, by (5.3), $|\Delta u| = O(\rho^{-2+2\lambda})$.

The above estimates show that $|D\psi|\, |Dv - \psi\, D\chi + \chi\, D\psi| = O(\rho^{-2+3\gamma+2\lambda})$. Thus, if $\lambda < \frac{\gamma}{2}$, an application of Lemma 3.3 to (5.7) shows that $|\chi| = O(\rho^{\gamma+2\lambda})$. Similarly, $|\psi| = O(\rho^{\gamma+2\lambda})$. Applying basic gradient estimates for Poisson equations to (5.7) and (5.8), we deduce that $|D\chi| + |D\psi| = O(\rho^{-1+\gamma+2\lambda})$. Using equations (5.4) and (5.5) we infer that $|\Delta\chi| + |\Delta\psi| = O(\rho^{-2+\gamma+2\lambda})$. Hence the right hand side of (5.6) is at most $O(\rho^{-2-2\gamma+4\lambda})$. Lemma 3.3 applies again, but to (5.6), yielding $|v| = O(\rho^{2\gamma+4\lambda})$. Gradient estimates for Poisson equations then show that $|Dv| = O(\rho^{-1+2\gamma+4\lambda})$.

Repeating the argument in the previous paragraph we see that $|v| = O(\rho^{2\gamma+2\lambda})$ and $|\chi| + |\psi| = O(\rho^{\gamma+\lambda})$ for any $\lambda < \gamma$. The regularity result in the general case follows.

Step 2: Smoothness when (u, v, χ, ψ) is symmetric about Γ and $\gamma = 1$.

It suffices to show that (u, v, χ, ψ) is C^∞ in $B_\delta(0)$ for some $\delta > 0$ small enough.

Step 2(a): Construction of new functions.

We first exploit the symmetry to construct new functions defined in a neighborhood of the origin. Let D_2 and div_2 denote the gradient and divergence operators in the half plane $\{(\rho, z) \in \mathbb{R}^2 | \rho > 0\}$. Then, in terms of cylindrical coordinates, (1.6)-(1.9) are equivalent to

$$\mathrm{div}_2\,(\rho\, D_2 u) - 2\, e^{4u}\, \rho |D_2 v - \psi\, D_2\chi + \chi\, D_2\psi|^2 - e^{2u}\, \rho(|D_2\chi|^2 + |D_2\psi|^2) = 0, \tag{5.9}$$

$$\mathrm{div}_2\left[e^{4u}\,\rho\,(D_2 v - \psi\, D_2\chi + \chi\, D_2\psi)\right] = 0, \tag{5.10}$$

$$\mathrm{div}_2\,(e^{2u}\,\rho\, D_2\chi) - 2e^{4u}\,\rho\, D_2\psi\cdot(D_2 v - \psi\, D_2\chi + \chi\, D_2\psi) = 0, \tag{5.11}$$

$$\mathrm{div}_2\,(e^{2u}\,\rho\, D_2\psi) + 2e^{4u}\,\rho\, D_2\chi\cdot(D_2 v - \psi\, D_2\chi + \chi\, D_2\psi) = 0. \tag{5.12}$$

Set $\omega = (\omega_\rho, \omega_z) = D_2 v - \psi\, D_2\chi + \chi\, D_2\psi$. Using (5.10)-(5.12), we define p, $\tilde{\chi}$ and $\tilde{\psi}$ by

$$dp = -2e^{4u}\,\rho\,\omega_z\, d\rho + 2e^{4u}\,\rho\,\omega_\rho\, dz, \tag{5.13}$$

$$d\tilde{\chi} = -(e^{2u}\,\rho\,\psi_z + p\,\chi_\rho)d\rho + (e^{2u}\,\rho\,\psi_\rho - p\,\chi_z)dz, \tag{5.14}$$

$$d\tilde{\psi} = (e^{2u}\,\rho\,\chi_z - p\,\psi_\rho)d\rho - (e^{2u}\,\rho\,\chi_\rho + p\,\psi_z)dz. \tag{5.15}$$

Note that they are well-defined up to a constant. Also, by the $C^{k,\alpha}$ regularity result from the general case, p, $\tilde{\chi}$ and $\tilde{\psi}$ are locally bounded.

Set

$$\begin{aligned}\tilde{\omega} = (\tilde{\omega}_\rho, \tilde{\omega}_z) &= \frac{1}{2}\Big((e^{-4u}\,\rho^{-1}\, p^2 + \rho)p_z - 4\rho\, p\, u_z, -(e^{-4u}\,\rho^{-1}\, p^2 + \rho)p_\rho + 4\rho\, p\, u_\rho + 2p\Big)\\ &= \frac{1}{2}\Big(2p^2\,\omega_\rho + \rho\, p_z - 4\rho\, p\, u_z, 2p^2\,\omega_z - \rho\, p_\rho + 4\rho\, p\, u_\rho + 2p\Big).\end{aligned}$$

A straightforward calculation using (5.9), (5.13), (5.14) and (5.15) shows that

$$\begin{aligned}2(\tilde{\omega}_{\rho,z} - \tilde{\omega}_{z,\rho}) &= 4p(p_z\,\omega_\rho - p_\rho\,\omega_z) - 4p\,[(\rho\, u_z)_z + (\rho\, u_\rho)_\rho]\\ &\qquad + 2p^2(\omega_{\rho,z} - \omega_{z,\rho}) + \rho^2[(\rho^{-1}p_z)_z + (\rho^{-1}p_\rho)_\rho] - 4\rho(p_z\, u_z + p_\rho\, u_\rho)\\ &= -4e^{2u}\,\rho\, p(|D_2\chi|^2 + |D_2\psi|^2) + 4(e^{4u}\,\rho^2 + p^2)(\psi_\rho\,\chi_z - \chi_\rho\,\psi_z)\\ &= 4(\tilde{\psi}_\rho\,\tilde{\chi}_z - \tilde{\chi}_\rho\tilde{\psi}_z).\end{aligned} \tag{5.16}$$

Thus there exists $\tilde{v}$ such that

$$d\tilde{v} = (\tilde{\omega}_\rho + \tilde{\psi}\,\tilde{\chi}_\rho - \tilde{\chi}\,\tilde{\psi}_\rho)d\rho + (\tilde{\omega}_z + \tilde{\psi}\,\tilde{\chi}_z - \tilde{\chi}\,\tilde{\psi}_z)dz. \tag{5.17}$$

Like $\tilde\chi$ and $\tilde\psi$, $\tilde v$ is defined up to a constant but remains locally bounded.

Finally, define $\tilde u$ by

$$\tilde u = -\frac{1}{2}\log(e^{2u}\,\rho^2 - e^{-2u}\,p^2) = -\frac{1}{2}\log(e^{2\bar u} - e^{-2\bar u}\,\rho^2\,p^2). \tag{5.18}$$

Recall that $\bar u = u - h = u + \gamma\log\rho$. Since $\bar u$ and p are locally bounded, $\tilde u$ is bounded in $B_{2\delta}(0)$ for δ small enough.

Step 2(b): Equations governing $(\tilde u, \tilde v, \tilde\chi, \tilde\psi)$.

By construction, $(\tilde u, \tilde v, \tilde\chi, \tilde\psi)$ is smooth in $B_\delta(0)\setminus\Gamma$. We claim that it satisfies the following equations in $B_\delta(0)\setminus\Gamma$:

$$\Delta\tilde u - 2\,e^{4\tilde u}|D\tilde v - \tilde\psi\,D\tilde\chi + \tilde\chi\,D\tilde\psi|^2 + e^{2\tilde u}(|D\tilde\chi|^2 + |D\tilde\psi|^2) = 0, \tag{5.19}$$

$$\operatorname{div}\left[e^{4\tilde u}\,(D\tilde v - \tilde\psi\,D\tilde\chi + \tilde\chi\,D\tilde\psi)\right] = 0, \tag{5.20}$$

$$\operatorname{div}\,(e^{2\tilde u}\,D\tilde\chi) + 2e^{4\tilde u}\,D\tilde\psi\cdot(D\tilde v - \tilde\psi\,D\tilde\chi + \tilde\chi\,D\tilde\psi) = 0, \tag{5.21}$$

$$\operatorname{div}\,(e^{2\tilde u}\,D\tilde\psi) - 2e^{4\tilde u}\,D\tilde\chi\cdot(D\tilde v - \tilde\psi\,D\tilde\chi + \tilde\chi\,D\tilde\psi) = 0. \tag{5.22}$$

In cylindrical coordinates, these equations take the form

$$\operatorname{div}_2\,(\rho\,D_2\tilde u) - 2\,e^{4\tilde u}\,\rho\,|\tilde\omega|^2 + e^{2\tilde u}\,\rho(|D_2\tilde\chi|^2 + |D_2\tilde\psi|^2) = 0, \tag{5.23}$$

$$\operatorname{div}_2\,(e^{4\tilde u}\,\rho\,\tilde\omega) = 0, \tag{5.24}$$

$$\operatorname{div}_2\,(e^{2\tilde u}\,\rho\,D_2\tilde\chi) + 2e^{4\tilde u}\,\rho\,D_2\tilde\psi\cdot\tilde\omega = 0, \tag{5.25}$$

$$\operatorname{div}_2\,(e^{2\tilde u}\,\rho\,D_2\tilde\psi) - 2e^{4\tilde u}\,\rho\,D_2\tilde\chi\cdot\tilde\omega = 0. \tag{5.26}$$

Define

$$\tilde p = \frac{e^{-2u}p}{e^{2u}\,\rho^2 - e^{-2u}\,p^2} = \frac{e^{-2u}p}{e^{-2\tilde u}}. \tag{5.27}$$

Then

$$d\tilde p = -2e^{4\tilde u}\,\rho\,\tilde\omega_z\,d\rho + 2e^{4\tilde u}\,\rho\,\tilde\omega_\rho\,dz, \tag{5.28}$$

which shows that $(\tilde u, \tilde v, \tilde\chi, \tilde\psi)$ satisfies (5.24).

By (5.14), (5.15) and (5.27),

$$d\chi = -(e^{2\tilde u}\,\rho\,\tilde\psi_z - \tilde p\,\tilde\chi_\rho)d\rho + (e^{2\tilde u}\,\rho\,\tilde\psi_\rho + \tilde p\,\tilde\chi_z)dz. \tag{5.29}$$

This implies that

$$-(e^{2\tilde{u}}\,\rho\,\tilde{\psi}_z + \tilde{p}\,\tilde{\chi}_\rho)_z = (e^{2\tilde{u}}\,\rho\,\tilde{\psi}_\rho - \tilde{p}\,\tilde{\chi}_z)_\rho.$$

In view of (5.28), (5.26) follows. Similarly, (5.25) holds due to

$$d\psi = (e^{2\tilde{u}}\,\rho\,\tilde{\chi}_z + \tilde{p}\,\tilde{\psi}_\rho)d\rho - (e^{2\tilde{u}}\,\rho\,\tilde{\chi}_\rho - \tilde{p}\,\tilde{\psi}_z)dz. \tag{5.30}$$

We next verify (5.23). In the interior of $\{\tilde{p} = 0\}$, p and $\tilde{\omega}$ vanish, and $\tilde{u} = -\bar{u}$, so (5.23) follows immediately from (5.9), (5.14) and (5.15). Thus, by continuity, it suffices to consider the region where $\tilde{p} \neq 0$. We note that

$$p = \frac{e^{-2\tilde{u}}\,\tilde{p}}{e^{-2u}} = \frac{e^{-2\tilde{u}}\,\tilde{p}}{e^{2\tilde{u}}\rho^2 - e^{-2\tilde{u}}\,\tilde{p}^2}.$$

In view of (5.13), this implies that

$$\omega = \frac{1}{2}\Big((e^{-4\tilde{u}}\,\rho^{-1}\,\tilde{p}^2 + \rho)\tilde{p}_z - 4\rho\,\tilde{p}\,\tilde{u}_z, -(e^{-4\tilde{u}}\,\rho^{-1}\,\tilde{p}^2 + \rho)\tilde{p}_\rho + 4\rho\,\tilde{p}\,\tilde{u}_\rho + 2\tilde{p}\Big),$$

and so, by (5.16) and (5.27),

$$\omega_{\rho,z} - \omega_{z,\rho} = 4e^{4\tilde{u}}\,\rho\,\tilde{p}\,|\tilde{\omega}|^2 - 2\tilde{p}\,[(\rho\,\tilde{u}_z)_z + (\rho\,\tilde{u}_\rho)_\rho] + 2(e^{4\tilde{u}}\,\rho^2 + \tilde{p}^2)(\tilde{\psi}_\rho\,\tilde{\chi}_z - \tilde{\chi}_\rho\,\tilde{\psi}_z).$$

On the other hand, by (5.29) and (5.30)

$$\begin{aligned}\omega_{\rho,z} - \omega_{z,\rho} &= 2(\psi_\rho\,\chi_z - \chi_\rho\,\psi_z)\\ &= 2e^{2\tilde{u}}\,\rho\,\tilde{p}(|D_2\tilde{\chi}|^2 + |D_2\tilde{\psi}|^2) + 2(e^{4\tilde{u}}\,\rho^2 + \tilde{p}^2)(\tilde{\psi}_\rho\,\tilde{\chi}_z - \tilde{\chi}_\rho\,\tilde{\psi}_z).\end{aligned}$$

The above relations imply that

$$4e^{4\tilde{u}}\,\rho\,\tilde{p}\,|\tilde{\omega}|^2 - 2\tilde{p}\,[(\rho\,\tilde{u}_z)_z + (\rho\,\tilde{u}_\rho)_\rho] = 2e^{2\tilde{u}}\,\rho\,\tilde{p}(|D_2\tilde{\chi}|^2 + |D_2\tilde{\psi}|^2).$$

Since $\tilde{p}$ is nonzero, (5.23) follows.

Step 2(c): Smoothness of $(\tilde{u}, \tilde{v}, \tilde{\chi}, \tilde{\psi})$.

We will use the following lemma, which is easy to prove.

Lemma 5.2 *Assume that f is a regular function in $B_1(0) \setminus \Gamma$ and that $|f| = O(\rho^{-1+\alpha})$ for some $\alpha \in (0, 1)$. If Nf be the Newtonian potential of f with respect to $B_1(0)$, i.e.*

$$Nf(x) = \int_{B_1(0)} \Phi(x - y) f(y)\,dy$$

where Φ is the fundamental solution of the Laplace equation, then Nf is $C^{1,\alpha}$ in $B_1(0)$.

As a consequence, if $u \in L^1(B_1(0))$ is a weak solution of

$$\Delta u = f$$

in $B_1(0)$, then u is $C^{1,\alpha}$ in $B_1(0)$.

We have shown that (5.19)-(5.22) hold in $B_\delta(0) \setminus \Gamma$. Moreover, by the regularity result for the general case, $\tilde{u}$, $\tilde{v}$, $\tilde{\chi}$ and $\tilde{\psi}$ belong to $H^1(B_\delta(0)) \cap C^{0,\alpha}(B_1(0))$ for any $\alpha \in (0,1)$. Hence, since Γ is of codimension 2, they satisfy (5.19)-(5.22) in $B_\delta(0)$ in the weak sense.

On the other hand, using the regularity result for the general case, we can show that $\tilde{u}$, $\tilde{v}$, $\tilde{\chi}$ and $\tilde{\psi}$ are $C^{0,\alpha}$ in $B_\delta(0)$ for any $\alpha \in (0,1)$. Moreover,

$$|D\tilde{v}| + |D\tilde{\chi}| + |D\tilde{\psi}| = O(\rho^{\alpha-1}).$$

Applying Lemma 5.2 to (5.19), (5.21), (5.22) and then (5.20), we can show that $\tilde{u}$, $\tilde{v}$, $\tilde{\chi}$ and $\tilde{\psi}$ are $C^{1,\alpha}$. This allows us to bootstrap in between (5.19)-(5.22) to obtain smoothness for $\tilde{u}$, $\tilde{v}$, $\tilde{\chi}$ and $\tilde{\psi}$ in $B_\delta(0)$.

<u>Step 2(d):</u> Smoothness of $(\bar{u}, v, \chi, \psi)$.

The smoothness of χ and ψ follows from (5.29) and (5.30). By (5.28), $\tilde{p}$ and so $e^{-2u}p = e^{-2\tilde{u}}\,\tilde{p}$ are smooth. That of e^{-2u} and of $\bar{u}$ follows from the identity

$$\rho^2\, e^{-2\bar{u}} = e^{-2u} = e^{2\tilde{u}}\,\rho^2 - e^{-2\tilde{u}}\,\tilde{p}^2.$$

Next, since $\rho^2\, p$ is smooth and $p \in C^{0,\alpha}$, p is smooth too. The smoothness of v follows from (5.13). The proof of the theorem is complete. $\square$

Proof of Theorem C. The theorem follows immediately from Theorem B and the results in [27].

Appendix A

Hyperbolic spaces

In this appendix, we present some relevant facts about the complex hyperbolic plane $\mathbb{H}_{\mathbb{C}} \cong U(1,2)/(U(1)\times U(2))$.

In the disk model, the hyperbolic plane $\mathbb{H}_{\mathbb{C}}$ is modeled by the disk $D = \{\zeta = (\zeta^1, \zeta^2) \in \mathbb{C}^n : |\zeta|^2 < 1\}$ with the metric element

$$ds^2 = \Big\{\frac{\delta_{ij}}{1-|\zeta|^2} + \frac{\zeta^i\,\bar{\zeta}^j}{(1-|\zeta|^2)^2}\Big\}\, d\bar{\zeta}^i\; d\zeta^j = \frac{|d\zeta|^2}{1-|\zeta|^2} + \frac{|\zeta\cdot d\zeta|^2}{(1-|\zeta|^2)^2}.$$

The action of a matrix $A = (a_{ij}) \in U(1,2)$ on D is given by

$$A\cdot\zeta = \Big(\frac{a_{21}+a_{22}\zeta_1+a_{23}\zeta_2}{a_{11}+a_{12}\zeta_1+a_{13}\zeta_2}, \frac{a_{31}+a_{32}\zeta_1+a_{33}\zeta_2}{a_{11}+a_{12}\zeta_1+a_{13}\zeta_2}\Big).$$

The isotropy group at the origin is $U(1)\times U(2)$.

The geodesics at 0 are precisely the $\mathbb{R}$-lines through 0. All other geodesics can be obtained from those by some motion in $U(1,2)$. This implies that

$$\cosh\,\mathrm{dist}(\zeta,\hat{\zeta}) = \frac{|1-\zeta\cdot\hat{\zeta}|}{\sqrt{1-|\zeta|^2}\sqrt{1-|\hat{\zeta}|^2}}.$$

We now derive the (u, v, χ, ψ) parametrization of $\mathbb{H}_{\mathbb{C}}$ from the disk model. See [25] for a geometric interpretation.

Define

$$\begin{aligned}
u &= \log\frac{|1+\zeta_1|}{\sqrt{1-|\zeta|^2}},\\
v &= \frac{1}{2}\,\mathrm{Im}\,w_1,\\
\chi &= \mathrm{Re}\,w_2\\
\psi &= \mathrm{Im}\,w_2,
\end{aligned}$$

where

$$w_1 = \frac{1-\zeta_1}{1+\zeta_1},$$
$$w_2 = \frac{\zeta_2}{1+\zeta_1}.$$

Note that

$$e^{-2u} = \frac{1-|\zeta|^2}{|1+\zeta_1|^2} = \mathrm{Re}\, w_1 - |w_2|^2.$$

Hence, for a given (u, v, χ, ψ), we can recover ζ by

$$\zeta_1 = \frac{1-w_1}{1+w_1},$$
$$\zeta_2 = \frac{2w_2}{1+w_1},$$

and

$$w_1 = e^{-2u} + \chi^2 + \psi^2 + 2\mathbf{i}\, v,$$
$$w_2 = \chi + \mathbf{i}\, \psi.$$

By a direct calculation, we see that the line element in terms of (u, v, χ, ψ) is

$$ds^2 = du^2 + e^{4u}(dv - \psi\, d\chi + \chi\, d\psi)^2 + e^{2u}(d\chi^2 + d\psi^2).$$

Appendix B

The space $\mathbf{W^{1,p}_{\Sigma}(\Omega, w)}$

We will study the space $W^{1,p}_{\Sigma}(\Omega, w)$ defined in the introduction. Most of the results here are probably known to readers. However we show them for completeness.

Throughout this appendix, we will frequently make the following assumptions on Ω, Σ and w.

(C1) Σ is a $(n-k)$-dimensional submanifold of $\mathbb{R}^n$ (possibly with or without a boundary) and Ω is a bounded domain in $\mathbb{R}^n$.

(C2) φ is a positive C^2 function in a punctured neighborhood of Σ such that $\varphi(x) \to 0$ as $x \to \Sigma$.

(C3) w is a positive measurable function defined on Σ such that $w = w(\varphi)$ in the domain of φ.

(C4) w is bounded on any compact subset of $\bar{\Omega} \setminus \Sigma$.

(C5) w is bounded from below by a positive number on any compact subset of $\bar{\Omega} \setminus \Sigma$.

Lemma B.1 *Let $1 < p < \infty$. Assume (C1)-(C3). Let α and β be functions on $(0, \infty)$ which are locally bounded measurable functions on compact subsets of $(0, \infty)$ and satisfy*

$$\alpha(t) \geq \sup_{\varphi(x)=t} \frac{\Delta\varphi(x)}{|D\varphi(x)|^2},$$

$$\beta(t) \geq \sup_{\varphi(x)=t} |D\varphi(x)|^{-(p-2)/(p-1)}.$$

Let A be an anti-derivative of α and define

$$\tilde{w}_\mu(t) = e^{p\,\mu\,A(t)}\, w^{-1/(p-1)}(t) \left\{ \int_0^t \beta(s)\, w^{-1/(p-1)}(s)\, e^{\mu\,A(s)}\, ds \right\}^{-p}$$

and $\tilde{w}_\mu(x) = \tilde{w}_\mu(\varphi(x))$.

(i) If $\beta\, w^{-1/(p-1)} e^{-A/(p-1)}$ *is locally integrable near* 0*, then there exists* $\delta > 0$ *such that* $d(\Omega^c, \Sigma) < \delta$ *implies*

$$\int_\Omega |f|^p\, \tilde{w}_{-1/(p-1)}\, dx \le C \int_\Omega |Df|^p\, w\, dx \qquad f \in W^{1,p}_{0,\Sigma}(\Omega, w).$$

The constant C *depends only on* p*.*

(ii) If, in addition, α *is positive and the level surfaces of* φ *are regular hypersurfaces, then there exists* $\delta > 0$ *such that* $d(\Omega^c, \Sigma) < \delta$ *implies*

$$\int_\Omega |f|^p\, \tilde{w}_\mu\, dx \le C \int_\Omega |Df|^p\, w\, dx \qquad f \in W^{1,p}_{0,\Sigma}(\Omega, w),$$

for any μ *for which* $\beta\, w^{-1/(p-1)}\, e^{\mu A}$ *is locally integrable near* 0*. The constant* C *depends only on* p *and* μ*.*

Proof. It suffices to consider non-negative $f \in C_c^\infty(\Omega \setminus \Sigma)$.

(i) Let $B_t = \{d(x, \Sigma) \ge t\}$. For t sufficiently small, f vanishes outside of B_t. Thus, if $\vec{F}$ is a C^1 vector-valued function on $\Omega \setminus \Sigma$ then

$$0 = \int_{B_t} \operatorname{div}(f^p\ \vec{F})\, dx = \int_{B_t} [f^p \operatorname{div}(\vec{F}) + p\, f^{p-1}\, Df \cdot \vec{F}]\, dx.$$

Hence, if $\operatorname{div}(\vec{F})$ is positive,

$$\int_{B_t} f^p \operatorname{div}(\vec{F})\, dx \le C \int_{B_t} |\nabla f|^p\, \frac{|\ \vec{F}\ |^p}{(\operatorname{div}(\vec{F}))^{p-1}}\, dx. \tag{B.1}$$

Define

$$F(t) = -e^{-A(t)} \left\{ \int_0^t \beta(s)\, w^{-1/(p-1)}(s)\, e^{-A(s)/(p-1)}\, ds \right\}^{-p+1} < 0,$$

so that

$$F' + \alpha\, F = (p-1)\, \beta\, w^{-1/(p-1)}\, |F|^{p/(p-1)}.$$

Hence, if $\vec{F}\,(x) = F(\varphi(x)) D\varphi(x)$ then

$$\begin{aligned}
\operatorname{div}(\vec{F}) &= |D\varphi|^2 \big(F' + \frac{\Delta\varphi}{|D\varphi|^2}\, F\big) \ge |D\varphi|^2 (F' + \alpha\, F) \\
&= (p-1)\, |D\varphi|^2\, \beta\, w^{-1/(p-1)}\, |F|^{p/(p-1)} \\
&\ge (p-1)\, w^{-1/(p-1)}\, |F|^{p/(p-1)}\, |D\varphi|^{p/(p-1)} \\
&= (p-1)\, w^{-1/(p-1)} |\ \vec{F}\ |^{p/(p-1)} = (p-1) \tilde{w}_{-1/(p-1)}.
\end{aligned}$$

Also, this implies

$$\frac{|\overrightarrow{F}|^p}{(\operatorname{div}(\overrightarrow{F}))^{p-1}} \le (p-1)^{p-1} w.$$

(i) then follows from (B.1).

(ii) We localize the proof of (i). Since α is positive, A is strictly increasing. Let t_j be an increasing sequence of positive real numbers such that $0 \le A(t_{j+1}) - A(t_j) \le 1$. Let $S_j = \{d(x,\Sigma) = t_j\}$ and $A_j = \{t_{j+1} \le d(x,\Sigma) \le t_j\}$. Similar to (i), any vector-valued function $\overrightarrow{F}$ which is differentiable in A_j such that $\operatorname{div}(\overrightarrow{F}) > 0$ gives rise to

$$-D_j + \frac{1}{2}\int_{A_j} f^p \operatorname{div}(\overrightarrow{F})\, dx \le C \int_{A_j} |\nabla f|^p \frac{|\overrightarrow{F}|^p}{(\operatorname{div}(\overrightarrow{F}))^{p-1}}\, dx. \tag{B.2}$$

where

$$D_j = \int_{S_j} f^p\ \overrightarrow{F} \cdot \nu_j\, d\sigma - \int_{S_{j+1}} f^p\ \overrightarrow{F} \cdot \nu_{j+1}\, d\sigma.$$

Here ν_j denotes the normal vector to S_j in the direction of increasing distance.

For $t_j \le t \le t_{j+1}$, define

$$F_j(t) = -e^{-A(t)} \left\{ \int_{t_j}^t \beta(s)\, w^{-1/(p-1)}(s)\, e^{-A(s)/(p-1)}\, ds + K_j \right\}^{-p+1} < 0,$$

where K_j is a constant to be specified later. Set $\overrightarrow{F_j}(x) = F_j(\varphi(x))\, D\varphi(x)$. Again,

$$F_j'(t) + \alpha\, F_j(t) = (p-1)\beta\, w^{-1/(p-1)}(t)\, |F_j(t)|^{p/(p-1)},$$

which implies $\operatorname{div}(\overrightarrow{F_j}) \ge C\, w^{-1/(p-1)} |\overrightarrow{F_j}|^{p/(p-1)}$ and $\frac{|\overrightarrow{F_j}|^p}{(\operatorname{div}(\overrightarrow{F_j}))^{p-1}} \le Cw$.

On the other hand, recalling the definition of F_j and the sequence $\{t_j\}$, we have

$$\begin{aligned}
|F_j|^{p/(p-1)}(t) &= e^{-p\,A(t)/(p-1)} \left\{ \int_{t_j}^t \beta(s) w^{-1/(p-1)}(s)\, e^{-A(s)/(p-1)}\, ds + K_j \right\}^{-p} \\
&= \left\{ \int_{t_j}^t \beta(s) w^{-1/(p-1)}(s)\, e^{(A(t)-A(s))/(p-1)}\, ds + e^{A(t)/(p-1)} K_j \right\}^{-p} \\
&\ge C \left\{ \int_{t_j}^t \beta(s) w^{-1/(p-1)}(s)\, e^{-\mu(A(t)-A(s))}\, ds + e^{A(t)/(p-1)} K_j \right\}^{-p} \\
&= C e^{p\,\mu\, A(t)} \left\{ \int_{t_j}^t \beta(s) w^{-1/(p-1)}(s)\, e^{\mu\, A(s)}\, ds + e^{(\mu+1/(p-1))A(t)} K_j \right\}^{-p} \\
&\ge C e^{p\,\mu\, A(t)} \left\{ \int_{t_j}^t \beta(s) w^{-1/(p-1)}(s)\, e^{\mu\, A(s)}\, ds + e^{(\mu+1/(p-1))A(t_j)} K_j \right\}^{-p}.
\end{aligned}$$

Hence, by setting

$$K_j = e^{-(\mu+1/(p-1))A(t_j)} \int_0^{t_j} \beta(s)\, w^{-1/(p-1)}(s)\, e^{\mu\, A(s)}\, ds$$

we arrive at

$$w^{-1/(p-1)}|F_j|^{p/(p-1)} \geq C\tilde{w}_\mu(t).$$

The above estimates allow us to rewrite (B.2) as

$$-D_j + C_1 \int_{A_j} f^p\, \tilde{w}\, dx \leq C_2 \int_{A_j} |Df|^2\, w\, dx$$

where C_1 and C_2 is independent of j. Summing over j and recall that $f \in C_c^\infty(\Omega \setminus \Sigma)$, we get the assertion.

$\square$

Remark B.1 *(i) It happens in many cases that* $w = o(\tilde{w}_\mu)$.

(ii) For $\varphi(x) = d(x,\Sigma)$, *we can take* $\alpha(t) = k/t$ *and* $\beta(t) = 1$. *Here* k *is the codimension of* Σ. *In that case,*

$$\tilde{w}_\mu(t) = t^{p\,k\,\mu}\, w^{-1/(p-1)}(t) \left\{ \int_0^t w^{-1/(p-1)}(s)\, s^{k\,\mu}\, ds \right\}^{-p}.$$

In particular, if $w(x) = d(x,\Sigma)^{-\gamma}$, *we can take* $\tilde{w}(x) = d(x,\Sigma)^{-\gamma-p}$.

(iii) The proof is actually valid if we only assume that $f \in W^{1,p}_\Sigma(\Omega, w)$ *and* $f = 0$ *at points on* $\partial\Omega$ *where the normal vector is not perpendicular to* $D\varphi$.

Proposition B.1 *Let* $1 < p < \infty$. *Assume (C1)-(C4). Let* δ *and* μ *be as in Lemma B.1. Define*

$$\tilde{w}(x) = \begin{cases} \tilde{w}_\mu(x) & \text{if } d(x,\Sigma) < \delta, \\ 1 & \text{otherwise.} \end{cases}$$

(i) For any $f \in W^{1,p}_{0,\Sigma}(\Omega, w)$, $f \in L^p(\Omega, \tilde{w})$ *and*

$$\int_\Omega |f|^p \tilde{w}\, dx \leq C \int_\Omega [|f|^p + |Df|^p\, w]\, dx.$$

(ii) For any $f \in W^{1,p}_\Sigma(\Omega, w)$, $f \in L^p_{loc}(\Omega, \tilde{w})$ *and*

$$\int_\omega |f|^p \, \tilde{w} \, dx \leq C \int_\Omega [|f|^p + |Df|^p \, w] \, dx$$

for any ω *compactly supported in* Ω.

Proof. Part (i) follows immediately from Lemma B.1 and the classical Poincare inequality.

For part (ii), it suffices to consider ω for which there is some r such that each connected component of $\omega \cap \{d(x, \Sigma) \leq r\}$ is bounded by the hypersurface $\{d(x, \Sigma) = r\}$ and two arbitrary hypersurfaces tangential to $D\varphi$. Also, we can assume that $f \in C^\infty_c(\bar{\Omega} \setminus \Sigma)$. We will show that

$$\int_\omega |f|^p \, \tilde{w} \, dx \leq C \int_\omega [|f|^p + |Df|^p \, w] \, dx \tag{B.3}$$

Let η be a standard cut-off function which vanishes in $\{d(x, \Sigma) > r\}$ and is identically 1 in $\{d(x, \Sigma) < r/2\}$. Split $f = f_1 + f_2 = f\eta + f(1 - \eta)$. Then f_1 vanishes on $\{x \in \partial\omega | d(x, \Sigma) = r\}$. The remainder of $\partial\omega$ is tangential to $D\varphi$. Thus by the remark following Lemma B.1,

$$\begin{aligned}
\int_\omega |f_1|^p \, \tilde{w} \, dx &\leq C \int_\omega |D(f\eta)|^p \, w \, dx \\
&\leq C \int_{\{r/2 < d(x,\Sigma) < r\}} |f|^p \, w \, dx + C \int_\omega |Df|^p \, w \, dx \\
&\leq C(r) \int_\omega |f|^p \, dx + C \int_\omega |Df|^p \, w \, dx.
\end{aligned}$$

For f_2, we compute

$$\int_\omega |f_2|^p \tilde{w} \, dx = \int_{\{r/2 < d(x,\Sigma) < r\}} |f_2|^p \, \tilde{w} \, dx \leq C(r) \int_\omega |f|^p \, dx.$$

The result follows from Minkowski inequality. □

Proposition B.2 *Let* $1 < p < \infty$. *Assume (C1)-(C5). Define* $\tilde{w}$ *as in Proposition B.1. Let* $\bar{w}$ *be a positive measurable weight in* Ω *such that* $\bar{w} = o(\tilde{w})$ *in a neighborhood of* Σ *and* $\bar{w}$ *is bounded on compact subsets of* $\bar{\Omega} \setminus \Sigma$. *Then* $W^{1,p}_\Sigma(\Omega, w)$ *is compactly embedded into* $L^p_{loc}(\Omega, \bar{w})$.

Proof. Let f_n be a bounded sequence in $W^{1,p}_\Sigma(\Omega)$ and ω compactly supported in Ω. By Proposition B.1 and our assumption on $\bar{w}$, $f_n \in L^p(\Omega, \bar{w})$. We will show that, up to extracting a subsequence, f_n converges in $L^p(\omega, \bar{w})$. Indeed, by (C5), we can assume that f_n converges to f pointwise and in $L^p_{loc}(\bar{\Omega} \setminus \Sigma)$. We claim that f_n converges to f in $L^p(\omega, \bar{w})$. To this end it suffices to show that

$$\lim_{n\to\infty} \int_\omega |f_n|^p \, \bar{w} \, dx = \int_\omega |f|^p \, \bar{w} \, dx.$$

Fix a $\varepsilon > 0$. We observe that, $|f_n|^p \, \bar{w}$ converges to $|f|^p \, \bar{w}$ in $L^1(\omega \cap \{d(x,\Sigma) \geq t\})$ for any t, so

$$\lim_{n\to\infty} \int_{\omega\cap\{d(x,\Sigma)\geq t\}} |f_n|^p \, \bar{w} \, dx = \int_{\omega\cap\{d(x,\Sigma)\geq t\}} |f|^p \, \bar{w} \, dx.$$

On the other hand, by Proposition B.1,

$$\begin{aligned} \int_{\omega\cap\{d(x,\Sigma)\leq t\}} |f_n|^p \, \bar{w} \, dx &\leq \sup_{0\leq s\leq t} \frac{\bar{w}(s)}{\tilde{w}(s)} \int_{\omega\cap\{d(x,\Sigma)\leq t\}} |f_n|^p \, \tilde{w} \, dx \\ &\leq C(\omega) \sup_{0\leq s\leq t} \frac{\bar{w}(s)}{\tilde{w}(s)} \int_\Omega [|f_n|^p + |Df_n|^p \, w] \, dx \leq \varepsilon \end{aligned}$$

for t sufficiently large. By Fatou's lemma, this implies

$$\int_{\omega\cap\{d(x,\Sigma)\leq t\}} |f|^p \, \bar{w} \, dx \leq \varepsilon.$$

Combining the above estimates, we infer that

$$\limsup_{n\to\infty} \left| \int_\omega |f_n|^p \, \bar{w} \, dx - \int_\omega |f|^p \, \bar{w} \, dx \right| \leq 3\varepsilon.$$

Letting $\varepsilon \to 0$, the assertion follows. $\square$

Remark B.2 *Using the inequality* (B.3) *instead of Proposition B.1, we can show that* $W^{1,p}_\Sigma(\Omega, w)$ *embeds compactly into* $L^p(\Omega, \bar{w})$ *whenever there is some* r *such that each component of* $\Omega \cap \{d(x,\Sigma) \leq r\}$ *is a set bounded by* $\{d(x,\Sigma) = r\}$ *and two hypersurfaces tangential to* $D\varphi$.

Proposition B.3 *Let* $1 < p < \infty$. *Assume (C1)-(C5) and that* Ω *intersects* Σ. *Define* $\tilde{w}$ *as in Proposition B.1. Let* $\bar{w}$ *be a positive measurable weight in* Ω *such that* $\bar{w} = o(\tilde{w})$ *in a neighborhood of* Σ *and* $\bar{w}$ *is bounded on compact subsets of* $\bar{\Omega} \setminus \Sigma$. *Assume in* *addition that $\bar{w}$ is bounded from below. If $\tilde{w}$ is nowhere locally integrable along Σ, then for any ω compactly supported in Ω,*

$$\int_{\omega} |f|^p \, \bar{w} \, dx \leq C \int_{\Omega} |Df|^p \, w \, dx.$$

Proof. It suffices to show that

$$\int_{\omega} |f|^p \, \bar{w} \, dx \leq C \int_{\omega} |Df|^p \, w \, dx$$

for any ω compactly supported in Ω such that, for some r, $\omega \cap \{d(x, \Sigma) \leq r\}$ is a set bounded by $\{d(x, \Sigma) = r\}$ and two hypersurfaces tangential to $D\varphi$.

Arguing indirectly, we assume that there exists f_h such that

$$1 = \int_{\omega} |f_h|^p \, \bar{w} \, dx \geq h \int_{\omega} |Df|^p \, w.$$

Since $\bar{w}$ is bounded from, f_h is uniformly bounded in $L^p(\omega)$, and so in $W^{1,p}_{\Sigma}(\omega, w)$ (as $h \to \infty$). Hence, by Remark B.2, we can assume that there exists $f \in W^{1,p}_{\Sigma}(\omega, w)$ such that f_h converges weakly in $W^{1,p}_{\Sigma}(\omega, w)$ and strongly in $L^p(\omega, \bar{w})$ to f. This results in

$$\int_{\omega} |Df|^p \, w \, dx \leq \liminf_{h \to \infty} \int_{\omega} |Df_h|^p \, w \, dx = 0,$$

which show that f is constant in ω. But as $|f|^p \, \tilde{w}$ is locally integrable in ω (cf. Proposition B.1) and $\tilde{w}$ is nowhere locally integrable long Σ, f must vanish identically. This contradicts

$$\int_{\omega} |f|^p \, \bar{w} \, dx = \lim_{h \to \infty} \int_{\omega} |f_h|^p \, \bar{w} \, dx = 1.$$

The proof is complete. □

Finally, we restate the above results for the case $w = d(x, \Sigma)^{-\gamma}$.

Corollary B.1 *(i) For $f \in W^{1,p}_{0,\Sigma}(\Omega, d^{-\gamma})$,*

$$\int_{\Omega} |f|^p \, d^{-\gamma - p}, dx \leq C \int_{\Omega} |Df|^p \, d^{-\gamma} \, dx.$$

(ii) For $f \in W^{1,p}_{\Sigma}(\Omega, d^{-\gamma})$ and ω compactly supported in Ω,

$$\int_{\omega} |f|^p \, d^{-\gamma - p} \, dx \leq C \int_{\Omega} [|f|^p + |Df|^p \, d^{-\gamma}] \, dx.$$

If, in addition, Ω intersects Σ and $\gamma > k-2$, then

$$\int_\omega |f|^p \, d^{-\gamma'} \, dx \leq C \int_\Omega |Df|^p \, d^{-\gamma} \, dx,$$

where $0 \leq \gamma' < \gamma + p$. If Ω and ω are fixed concentric balls and Σ is straight, the constant C depends only on p, γ, γ' and the distance from the center to Σ. The bigger the distance, the bigger the constant.

(iii) The embedding $W^{1,p}_\Sigma(\Omega, d^{-\gamma}) \hookrightarrow L^p_{loc}(\Omega, d^{-\gamma'})$ is compact for any $0 \leq \gamma' < \gamma + p$.

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