VISUALIZING SOLUTIONS OF THE CIRCULAR RESTRICTED THREE-BODY PROBLEM

by

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The stability of a satellite near the Lagrange points is studied in a Circular Restricted Three-Body Problem (CR3BP). The Runge Kutta method is used to trace out the orbital path of the satellite over a period of time. Various initial positions near the Lagrange points and velocities are used to produce various paths the satellite can take. The primary paths focused on are horseshoe paths. Horseshoe orbits are shown to be sometimes stable and sometimes chaotic.
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1 Introduction

The Italian-French mathematician Joseph-Louis Lagrange was interested in finding a way to compute orbits for an arbitrary number of celestial bodies. One body in a universe is not difficult and just sits still. In a universe with two bodies, both bodies will orbit in ellipses around their common center of mass. Now consider a universe consisting of three bodies and the mathematics explodes. Three bodies of similar mass and of roughly equal distance from each other will usually skip about chaotically until one of the masses gets hurled into a void or until a collision happens.

The problem of how a third, small mass orbits around two orbiting large masses is well known as the Restricted Three Body Problem (RTBP). The third body is considered to have zero mass. That means the third body exerts no gravitational force on the two large bodies, although the large bodies exert a force on it. This is a classical problem in dynamical systems that is still investigated by scientists today. A special case of this is the Circular Restricted Three Body Problem (CR3BP). Here the two large masses revolve about their common center of mass in circles. This case can model the trajectory of an object (e.g. spacecraft or satellite) in the gravitational potential of two huge orbiting bodies, such as the Earth and the moon or the Sun and Jupiter.

Because the mathematical calculations are very complicated involving more than two bodies, Lagrange wanted to find a simpler way. There is a situation where you would have to calculate every gravitational interaction between every object at every point along its trajectory. So, Lagrange drew up a simple conclusion:

*The trajectory of an object is determined by finding a path that minimizes the action over time*[1].

This way of thinking allowed Lagrange to re-formulate the classical Newtonian mechanics and derive a new system of calculations. In his system, he adopted a *rotating frame of reference*, where the two large masses hold fixed positions. A rotating frame of reference is a special case of a non-inertial reference frame in which the coordinate system is rotating relative to an inertial reference frame. All non-inertial reference frames exhibit fictitious forces, therefore the centrifugal force and the Coriolis force have to be figured in. His continued work led him to hypothesize how a third body of negligible mass would orbit two larger bodies which are almost in circular orbit.

Lagrange’s studies of this problem in 1772, eventually lead to the discovery of five special points.
These points are referred to as Lagrange points, libration points or L-points. They have been labeled $L_1$, $L_2$, $L_3$, $L_4$ and $L_5$. In the CR3BP, the Lagrange points are positions in space where the third body will remain at rest in the rotating coordinate system, because the gravitational and centrifugal forces balance each other. Some example systems where the five Lagrangian points exist are in the Earth-Moon and Sun-Jupiter systems.

In this paper, we will take a look at the stability of the Lagrange points in the Sun-Jupiter system. Hence, our large focus will be on the boundaries of stability near $L_4$ and $L_5$, varying the $x$-$y$ positions and velocities of the orbiting body. We will be tracing the orbital path an object will take, if it is not exactly on one of these two points. The Sun, Jupiter and the object are all unequal masses, where the Sun and Jupiter are very massive in size to the object. Our set of initial conditions will also lead to probable solutions. But before we look any further we will find out why these two points are more special than the others and some other studies that concentrate on these two points.

The points $L_4$ and $L_5$ rest at the third corners of the two equilateral triangles in the plane of orbit whose common base is the line between the centers of the two large masses. Because of this arrangement, $L_4$ and $L_5$ are sometimes called triangular Lagrange points. $L_4$ and $L_5$ are resistant to perturbations unlike the other Lagrange points. This stability have led to the collection of asteroids around these points.

This collection of asteroids are referred to as Trojan asteroids. Trojan asteroids are a large group of objects that share the orbit of the planet Jupiter around the Sun, but remain 60 degrees behind or ahead of Jupiter in its orbit. Although the term Trojan asteroid originally referred only to the Jupiter Trojans, we know that there are other small bodies that have similar relationships with other massive bodies. For example, Neptune and Mars have Trojans, and so do some of the satellites of Saturn.

In any case, Trojan asteroids are of importance to us, because it is believed that some of these objects take an orbital path called a horseshoe orbit. In the inertial coordinate coordinate system, an asteroid in a Sun-Jupiter horseshoe orbit, moves along a circular orbit alternately catching up to Jupiter and falling behind. But in the rotating coordinate system, the orbit of the asteroid is shaped like a horseshoe, or an elongated kidney bean going back and forth between the $L_4$ and $L_5$. 
positions. Vanderbei discusses horseshoe orbits in his studies on Saturn’s rings.

Contrary to the accepted theory, Vanderbei argues that Saturn’s rings and inner moons are in much more stable orbits by examining in detail horseshoe orbits[2]. He uses the Saturn-Janus-Epimetheus system as an example. He states that orbits of the moons Janus and Epimetheus have almost identical radii about 151000 km from the center of Saturn and their mass ratios are large enough to satisfy the arrangement of the Lagrange points $L_4$ and $L_5$[2]. Further, he goes on to say the following:

*The moon orbits are nearly circular and one body is approximately 50 km closer to Saturn than the other, with the closer body having a slightly shorter period than the other. But in an $L_4-L_5$ configuration, the body (e.g. Epimetheus) that is closer to Saturn accelerates forward to catch up with the other body (e.g. Janus). Therefore, this acceleration brings the trailing body to a higher, slower orbit and the other body decelerates to a lower, faster orbit. Over a period of time, the roles of the moon would switch again this time the former body ending up in a higher orbit and the latter in lower orbit.[2]*

This indefinite switching of orbital roles are referred to as a horseshoe orbit.

Llibre and Ollé also study horseshoe [periodic] orbits in the CR3BP. Their investigation also centers on the same, Saturn and its two co-orbital satellites. In this CR3BP, they set Saturn and Janus to primary masses having a value of $\mu$ and $1 - \mu$, while Epimetheus has an infinitesimal mass[3, p. 1088]. From this, Llibre and Ollé generate horseshoe [periodic] orbits using various values of the mass parameter $\mu$, ranging from zero to very small values[3, p. 1088]. Thus, Llibre and Ollé do numerical studies on the motion of co-orbital satellites Janus and Epimetheus to compute stable horseshoe [periodic] orbits that fit this real orbit.
2 Lagrange Points

2.1 Formulating the Restricted Three Body Problem

Before we proceed any further, let’s take a look at the approach for finding the Lagrange points. Using Newton’s laws of motion and Newton’s law of gravity, we will be able to seek solutions to the equations of motion in which the three bodies will be at a constant separation. Considering the two primary masses \( M_1 \) and \( M_2 \) and their positions \( \vec{r}_1 \) and \( \vec{r}_2 \), respectively, we have the following equation

\[
\vec{F} = -\frac{GM_1 m}{|\vec{r} - \vec{r}_1|^3} (\vec{r} - \vec{r}_1) - \frac{GM_2 m}{|\vec{r} - \vec{r}_2|^3} (\vec{r} - \vec{r}_2)
\]  

representing the sum of the total force exerted on the third mass \( m \) at position \( \vec{r} \)[4, p. 1]. From this we have defined the mathematical formulation of the CR3BP.

To simplify this method, we will establish the standard non-dimensional convention as a representation of the CR3BP. Our first decision is to define the distance between the two primary masses \( M_1 \) and \( M_2 \) as the following

\[
|r_2 - r_1| = 1
\]  

since the third body’s mass is too small to be significant, our choice to have the sum of the primary masses

\[
M_1 + M_2 = 1
\]  

would be a non-complicated and practical. We can then set the smaller primary mass to \( \mu \). This establishes mass system as

\[
\begin{align*}
M_1 &= \mu \\
M_2 &= 1 - \mu
\end{align*}
\]  

where \( \mu \) represents the gravitational parameter. The gravitational parameter \( \mu \) can also be stated
as the mass ratio

$$\mu = \frac{M_1}{M_1 + M_2}$$  \hspace{1cm} (5)

Our next choice involves choosing in this frame, a co-rotating frame of reference. Since Lagrange points are stationary, this is the best way of finding the solutions. We will then have the two large masses sustain fixed positions and a frame of reference where the origin is at the center of mass. By applying Kepler’s third law

$$\omega = \frac{G(M_1 + M_2)}{r_{12}^3}$$  \hspace{1cm} (6)

we have defined the two large masses’ angular velocity $\omega$. In Equation (6), $r_{12}$ is distance from $M_1$ to $M_2$. With this equation you can apply it to the RTBP. Thus the radius vector sums

$$\vec{r} - \vec{r}_1$$  
$$\vec{r} - \vec{r}_2$$

from $M_1$ to $m$ and $M_2$ to $m$, respectively, can be restated as the magnitudes

$$r_1^2 = (x - 1 + \mu)^2 + y^2$$  \hspace{1cm} (7)
$$r_2^2 = (x + \mu)^2 + y^2$$

In System (7), $r_1$ and $r_2$ represents the distance $M_1$ and $M_2$ each from $m$, respectively.

By introducing a non-inertial frame of reference, we have to now to account for its drawback by appending pseudo-forces to the equations of motion. The first one we will look at is the centrifugal force. If you are at rest in a rotating system in the inertial frame, then $v = \omega r$, where $v$ is the true velocity. We already know that

$$a = \frac{v^2}{r} \text{ (uniform circular motion)}$$  \hspace{1cm} (8)

and thus

$$a = \frac{(\omega r)^2}{r} = \omega^2 r$$
The centrifugal force is found to be

\[ \vec{a}_{\text{centrifugal}} = \omega^2 \vec{r} \]

Now switching over to the second force, we will allow \( \Delta \vec{v} \) to represent the velocity in the rotating system. By substituting \( v + \Delta v \) into Equation (8), we end up with the following

\[ a = \frac{(v + \Delta v)^2}{r} \]
\[ = \frac{(\omega r + \Delta v)^2}{r} \]
\[ = \omega^2 r^2 + 2\omega r \Delta v + \Delta v^2 \]
\[ = \omega r + 2\omega \Delta v \]

where we ignore the tiny term \( \frac{\Delta v^2}{r} \). The centrifugal force is \( \omega^2 r \) and Coriolis force is \( 2\omega \Delta v \).

Finally we arrive at a new representation of the CR3BP equations of motion. By applying Equations (2 - 10) to Equation (1), the outcome will be

\[ \ddot{x} - 2\omega \dot{y} - \omega^2 x = -G \frac{M_1}{r_1^3} (x - 1 + \mu) - G \frac{M_2}{r_2^3} (x + \mu) \]
\[ \ddot{y} - 2\omega \dot{x} - \omega^2 y = -G \frac{M_1}{r_1^3} y - G \frac{M_2}{r_2^3} y \]

The velocities at the \( x \) and \( y \) positions are represented by \( \dot{x} \) and \( \dot{y} \). And the accelerations at the \( x \) and \( y \) positions are \( \ddot{x} \) and \( \ddot{y} \). Furthermore, we can simplify Equation (1) by assuming the gravitational constant is one \( (G = 1) \). Next, the rotational velocity is set to one \( (\omega = 1) \) from Kepler’s third law (i.e. Equation 6). Once the assumptions are incorporated into Equation (10), we will have the following system

\[ \ddot{x} - 2\dot{y} - x = -\frac{\mu(x - 1 + \mu)}{r_1^3} - \frac{(1 - \mu)(x + \mu)}{r_2^3} \]
\[ \ddot{y} - 2\dot{x} - y = -\frac{\mu y}{r_1^3} - \frac{(1 - \mu)y}{r_2^3} \]

The system above consists of second-order, nonlinear, coupled ordinary differential equations mathematically representing the CR3BP. The system has been normalized and the length, time and mass are all in terms of unit measurements.
Now it is time to look into a geometrical representation of a three-body in figure 1\(^1\) taken from the Project Geryon web pages of the Colorado Center for Astrodynamics Research (CCAR)[5]. The origin is the center of mass (i.e. barycentre) of the two large planetary bodies (i.e. \(P_1\) and \(P_2\)). Mass \(M_1\) \((P_1)\) is at a distance of \(\mu\) from the origin and mass \(M_2\) \((P_2)\) is at a distance of \(1 - \mu\).

### 2.2 Establishing Stability

So far we have established the mathematical formula of the CR3BP. We can use this formula to examine the stability of the Lagrange points to get a better understanding of the CR3BP. Mathematically it can be shown that a satellite or moon having a velocity of zero and positioned exactly on one of these points would remain there. Our main focus is to investigate a very small mass near a Lagrange point.

First, let’s assume the object is located close to a Lagrange point. Secondly its velocity at this point is zero. Therefore the location of this object can be stated as

\[
\begin{align*}
x &\approx x_e + \delta x \\
y &\approx y_e + \delta y
\end{align*}
\]

(12)

where the subscript \(e\) represents the equilibrium location for the object or a Lagrange point. We can see the stability of the \(\delta x\) and \(\delta y\) by looking at their growth. A stable equilibrium point will produce a \(\delta x\) and \(\delta y\) that stay bounded. In the case where the equilibrium point is quasi-stable or unstable, \(\delta x\) and \(\delta y\) will grow immensely. In the final step, Equation (12) is substituted into the equations of motion and rearranged into the following state-space form

\[
\begin{align*}
\dot{x}_1 &= \dot{x} = x_2 \\
\dot{y}_1 &= \dot{y} = y_2 \\
\dot{x}_2 &= 2\dot{y}_1 + x_1 + \frac{\mu}{r_1^3}(x_1 - 1 + \mu) + \frac{1 - \mu}{r_2^3}(x_1 + \mu) \\
\dot{y}_2 &= -2\dot{x}_1 + y_1 \left(1 + \frac{\mu}{r_1^3} - \frac{1 - \mu}{r_2^3}\right)
\end{align*}
\]

representing the two dimensional CR3BP.

\(^1\)see supplementary file figure1.PNG
3 Runge Kutta Method

In able to determine the stability of the Lagrange points, we must find the solutions to the CR3BP equations. Since the system consists of two second order differential equations, a preferred method would be one that bypass the calculation and evaluation of the derivatives. The Runge-Kutta algorithm is a good choice for this approximation, while still giving a high-order truncation error[6, p. 254]. This algorithm is known to work accurately and well with a wide range of problems.

The Runge-Kutta formula (RK4) that we will be applying is the fourth order. This means there are four "f substitutions" involved[7, p. 233].

Let’s take a look at the single variable problem

\[ \dot{x} = f(x, t) \]

with an initial condition \( x(0) = x_0 \). The value \( x_n \) will represent the variable \( x \) at time \( t_n \). The method approximates \( x_{n+1} \) at time \( t_n + h \) using \( x_n \) and \( t_n \). It involves weighted averages of numerical values of \( f(t, x) \) at several times within the interval \((t_n, t_n + h)\). The formula for the algorithm is as follows

\[ x_{n+1} = x_n + \frac{h}{6} (k_1 + 2k_2 + 2k_3 + k_4) \]

where

\[ k_1 = hf(x_n, t_n) \]
\[ k_2 = hf(x_n + \frac{1}{2} k_1, t_n + \frac{h}{2}) \]
\[ k_3 = hf(x_n + \frac{1}{2} k_2, t_n + \frac{h}{2}) \]
\[ k_4 = hf(x_n + k_3, t_n + h) \].

But since we have four variables \( x_1, x_2, y_1 \) and \( y_2 \) representing the position and velocity
respectively, we need to first convert the differential equations into the following first order system

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, y_1, x_2, y_2, t) \\
\dot{y}_1 &= f_2(x_1, y_1, x_2, y_2, t) \\
\dot{x}_2 &= f_3(x_1, y_1, x_2, y_2, t) \\
\dot{y}_2 &= f_4(x_1, y_1, x_2, y_2, t).
\end{align*}
\]

For this case, \(x_{1,n}, x_{2,n}, y_{1,n}\) and \(y_{2,n}\) are the values at \(t_n\). This results into the formula

\[
\begin{align*}
x_{1,n+1} &= x_{1,n} + \frac{h}{6}(k_{11} + 2k_{21} + 2k_{31} + k_{41}) \\
x_{2,n+1} &= x_{2,n} + \frac{h}{6}(k_{12} + 2k_{22} + 2k_{32} + k_{42}) \\
y_{1,n+1} &= y_{1,n} + \frac{h}{6}(k_{13} + 2k_{23} + 2k_{33} + k_{43}) \\
y_{2,n+1} &= y_{2,n} + \frac{h}{6}(k_{14} + 2k_{24} + 2k_{34} + k_{44})
\end{align*}
\]
where

\[k_{11} = h f_1(x_{1,n}, x_{2,n}, y_{1,n}, y_{2,n}, t_n)\]
\[k_{12} = h f_2(x_{1,n}, x_{2,n}, y_{1,n}, y_{2,n}, t_n)\]
\[k_{13} = h f_3(x_{1,n}, x_{2,n}, y_{1,n}, y_{2,n}, t_n)\]
\[k_{14} = h f_4(x_{1,n}, x_{2,n}, y_{1,n}, y_{2,n}, t_n)\]
\[k_{21} = h f_1(x_{1,n} + \frac{1}{2}k_{11}, x_{2,n} + \frac{1}{2}k_{12}, y_{1,n} + \frac{1}{2}k_{13}, y_{2,n} + \frac{1}{2}k_{14}, t_n + \frac{h}{2})\]
\[k_{22} = h f_2(x_{1,n} + \frac{1}{2}k_{11}, x_{2,n} + \frac{1}{2}k_{12}, y_{1,n} + \frac{1}{2}k_{13}, y_{2,n} + \frac{1}{2}k_{14}, t_n + \frac{h}{2})\]
\[k_{23} = h f_3(x_{1,n} + \frac{1}{2}k_{11}, x_{2,n} + \frac{1}{2}k_{12}, y_{1,n} + \frac{1}{2}k_{13}, y_{2,n} + \frac{1}{2}k_{14}, t_n + \frac{h}{2})\]
\[k_{24} = h f_4(x_{1,n} + \frac{1}{2}k_{11}, x_{2,n} + \frac{1}{2}k_{12}, y_{1,n} + \frac{1}{2}k_{13}, y_{2,n} + \frac{1}{2}k_{14}, t_n + \frac{h}{2})\]
\[k_{31} = h f_1(x_{1,n} + \frac{1}{2}k_{21}, x_{2,n} + \frac{1}{2}k_{22}, y_{1,n} + \frac{1}{2}k_{23}, y_{2,n} + \frac{1}{2}k_{24}, t_n + \frac{h}{2})\]
\[k_{32} = h f_2(x_{1,n} + \frac{1}{2}k_{21}, x_{2,n} + \frac{1}{2}k_{22}, y_{1,n} + \frac{1}{2}k_{23}, y_{2,n} + \frac{1}{2}k_{24}, t_n + \frac{h}{2})\]
\[k_{33} = h f_3(x_{1,n} + \frac{1}{2}k_{21}, x_{2,n} + \frac{1}{2}k_{22}, y_{1,n} + \frac{1}{2}k_{23}, y_{2,n} + \frac{1}{2}k_{24}, t_n + \frac{h}{2})\]
\[k_{34} = h f_4(x_{1,n} + \frac{1}{2}k_{21}, x_{2,n} + \frac{1}{2}k_{22}, y_{1,n} + \frac{1}{2}k_{23}, y_{2,n} + \frac{1}{2}k_{24}, t_n + \frac{h}{2})\]
\[k_{41} = h f_1(x_{1,n} + \frac{1}{2}k_{31}, x_{2,n} + \frac{1}{2}k_{32}, y_{1,n} + \frac{1}{2}k_{33}, y_{2,n} + \frac{1}{2}k_{34}, t_n + h)\]
\[k_{42} = h f_2(x_{1,n} + \frac{1}{2}k_{31}, x_{2,n} + \frac{1}{2}k_{32}, y_{1,n} + \frac{1}{2}k_{33}, y_{2,n} + \frac{1}{2}k_{34}, t_n + h)\]
\[k_{43} = h f_3(x_{1,n} + \frac{1}{2}k_{31}, x_{2,n} + \frac{1}{2}k_{32}, y_{1,n} + \frac{1}{2}k_{33}, y_{2,n} + \frac{1}{2}k_{34}, t_n + h)\]
\[k_{44} = h f_4(x_{1,n} + \frac{1}{2}k_{31}, x_{2,n} + \frac{1}{2}k_{32}, y_{1,n} + \frac{1}{2}k_{33}, y_{2,n} + \frac{1}{2}k_{34}, t_n + h).\]
4 Solution for the Lagrange Points

Since we now have the CR3BP equations and RK4 as our method of choice, our next step now is to find the Lagrange points. The masses are assumed to be at rest in the rotating frame in able to solve for the five points. Thus, the velocity and acceleration components are zero. The Equation (11) will then simplify to

\[ x = \mu (x - 1 + \mu) + \frac{(1 - \mu)(x + \mu)}{r_1^3} \\
\[ y = \frac{(1 - \mu)y}{r_2^3} + \mu y \frac{y}{r_2^3} \]

where the five equilibrium points can be found. Figure 2 displays CCAR’s geometric model of the five points position in the CR3BP[5].

4.1 Analyzing the Stability of the Collinear Solutions

If \( y \) is set to zero in Equation (13), then the three collinear points (\( L_1, L_2, L_3 \)) can be found from the following equation

\[ x - \frac{\mu(x - 1 + \mu)}{|x - 1 + \mu|^3} - \frac{(1 - \mu)(x + \mu)}{|x + \mu|^3} = 0 \] (14)

along \( x \)-axis. Using any numerical method you will find the three real \( x \) roots of Equation (14) for \( 0 \leq \mu \leq 1 \). Thus we will be able to study the stability of these three points.

First, let’s consider the model we are going to study. In a CR3BP, the small primary body Jupiter will have a mass of \( \mu = .001 \). The Sun will be the large body with a mass of \( 1 - \mu \). Both Jupiter and the Sun are lying on the \( x \)-axis with a coordinate of .999 for Jupiter and \(-.001 \) for the Sun. The infinitesimal body can be a satellite or asteroid. We can then plug this value of \( \mu \) into Equation (14) to solve for the three stable locations for the zero mass body.

For each observation, we will actually begin with the already computed collinear point. This point will be set as the initial position. We would then set the initial velocities (\( \dot{x}, \dot{y} \)) to zero. We will plug these values into the Runge Kutta formula and iterate this method for \( n \) number of times.

\[ \text{see supplementary file figure2.png} \]
(a) The orbital path of the object in the time of 4.0

(b) The orbital path of the object in the time of 100.0

(c) The orbital path of the object in the time of 500

Figure 3: The evolution of orbital path of the object after drifting from $L_1$ when the initial $v_x$ and $v_y$ are 0.
<table>
<thead>
<tr>
<th>Time (t)</th>
<th>Position</th>
<th>Velocity</th>
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Table 1: Tabular representation of Figure 3a
For accuracy, it is sufficient for our stepsize to be set to .1 (i.e. \( h = .1 \)). The output of each iteration of the method will allow us to trace the path of an object over a period of time \( t \). Our initial time will be zero and the time step will be equal to the stepsize. We can now move forward to examine our first point.

We will take a look first at the Lagrange point \( L_1 \). This point lies somewhere between the Jupiter and the Sun. In this Sun-Jupiter system, it can be shown by any numerical means that the location of this point is estimated at \( x = .9313 \). If we take a look at Figure 3a, we can see the object slowly drifting from \( L_1 \). At the twenty-fifth iteration (i.e. \( t = 2.5 \)), we can see a significant increase from the previous position of the object (see Table 1). The delta values of the \( x-y \) position increase for the period of time given in Figure 3a. In that same figure we can see up to forty iterations (i.e \( t = 4.0 \)).

Also in the time given time, the velocity in the \( x \) direction increases, while the velocity in the \(-y\) direction increases up 3.0 time units. After 3.0 time units, the velocity in the \( y \) direction tends to fluctuate in the positive and negative. These results alone show us the instability of the \( L_1 \) point.

Despite instability at the \( L_1 \) point, an object starting near there could still end up in a stable orbit around the Sun for a period of time. If we look at Figure 3b, we can see the path of object up to 100 units of time. This path shows the object orbiting around the Sun (i.e. \( x = -.001 \)) and orbiting near Jupiter (i.e. \( x = .999 \)). The velocity has also appeared to be fluctuating in the
(a) The object’s first path from the time of 0 to 74.7

(b) The object’s second path from the time of 74.8 (position: \(x = 1.02057316, y = .00614222\)) to 140.0

(c) The object spiraling out of the Solar System after the time of 140.0 (position: \(1.00466454, y = -.00975784\))

Figure 5: The three orbital paths of the object drifting from \(L_1\) when the initial \(v_x\) and \(v_y\) are .1.
Figure 6: The three orbital paths taken when the object drifts from $L_2$ with initial $v_x$ and $v_y$ are 0.0.
negative and positive in both the $x$ and $y$ directions. Even at a point on the path further in time (i.e. $t = 500^3$) shows the object relatively on the same orbital path (see Figure 3c).

Now let us see what the results are if we change the initial velocities. In our first case, we will set velocity in the $x$ direction to .1 and the velocity in the $y$ direction will remain at zero. If we take a look at the Figure 4, we will see that the object drifts into a chaotic orbit and gets kicked out of the Solar System where it goes into a spiral (i.e. Archimedean spiral). In an inertial reference frame, the object leaves the Solar System on a hyperbolic path, asymptotically approaching a straight line at a constant velocity. In a rotating frame, the straight line becomes a spiral of Archimedes. The second case, Figure 5 illustrates the object going into two orbits and then spiraling out of the Solar System. If we closely examine the graph, we will see the orbital change of the object near Jupiter (i.e. $x = .999$). The velocity in the $x$ and $y$ directions for this case are both equal to .1. We even have a case where the object immediately begins to spiral out of the Solar System as it moves away from $L_1$ (see Figure A.1) in Appendix A.1. In this example, the velocity in the $x$ direction is zero and the velocity in the $y$ direction is .1. In all cases we can see fluctuations in the velocity as the object is getting kicked out of the Solar System (see Table A.1 in Appendix A.1). The absolute delta velocities ($|\Delta v_x|, |\Delta v_y|$) show the various changes in velocities between each point over a period time.

Our next observation will be the Lagrange point $L_2$. It lies on a line defined by the Sun and Jupiter, beyond Jupiter. The numerical estimation for this point is $x = 1.0699$. Looking at Figure 6a, we can not clearly see that it goes into three orbits before it exits the Solar System. Now let’s pay close attention to the second and third orbits. If we isolate these two paths, we will see that the object is in a horseshoe orbit in Figures 6b and 6c. A horseshoe orbit is of interest to us because, the object is in a stable orbit that makes a visit near the points $L_3$, $L_4$ and $L_5$. Once again near $L_2$ the object goes into another orbit, but this time it is leaving the Solar System.\(^4\)

Changing the initial velocities again, but this time for $L_2$, will also gives use unique results. For Figure 7, there are three orbits in each. Once again it may not be obvious to see the three orbits.\(^5\) The object in both graphs exhibit very similar behavior. The first orbit in each graph is further from the center $(0,0)$ than the second orbit. Just like in the $L_1$ plots, the object changed its orbit again near Jupiter. Then the object goes back into an outer orbit very similar to the first. But for

\(^3\)This will be the maximum set for all plots due to constraints of computer resources used.

\(^4\)No more graphs will be depicted since the previous two are self-explanatory for an object spiraling

\(^5\)see Appendix A.2 for breakdown of each orbit
(a) The object’s path from the time of 0.0 to 500 with initial $v_y$ is .1
(b) The object’s path from the time of 0.0 to 368.7 with initial $v_y$ is -.1

Figure 7: The three orbital paths taken when the object drifts from $L_2$ with initial $v_x$ is .1.

the object in Figure 7b at the end of the third orbit, near Jupiter sends the object on a spiraling path out of the Solar System. The initial velocity in the $x$-direction is .1 for both graphs, while the initial velocity in the $y$-direction is .1 and −.1, respectively. Another interesting graph is one where the object seems to be in a chaotic orbit as it drifts away from Jupiter. After a time of 4.4 at coordinate $(1.06189797, −0.02070420)$, the object eventually goes into a stable orbit in Figure 8b. At the time of 54.5 at coordinate $(1.09161248, −0.01489693)$ the object goes into another orbit in Figure 8c. As you can see both changes take place near Jupiter. The velocity in the $x$-direction is 0 and in the $y$-direction is −.1 for this case.

The point $L_3$ is the final collinear point under analysis. This point lies on the line defined by the Sun and Jupiter, beyond the Sun. Its estimated location is at $x = −1.0004$. If the object was placed near this location, we would have another occurrence of the object drifting away from the equilibrium point (i.e. $L_3$) into a horseshoe orbit. Taking a look at Figure 9b, it appears the object is on a smooth horseshoe path. But Figure 9a displays the object moving in a wave-like motion. The object still maintains a horseshoe orbit up to a time of 500.0.

For other variations of initial velocities, we have also produced graphs which include horseshoe orbits. For Figure 10b, where the velocity in the $x$-direction has changed to .1 and in the $y$-direction it has remained the same, we can see when the object has entered a second orbit that is in a shape of
(a) The object’s first path taken from the time of 0.0 to 4.4

(b) The object’s second path from the time of 4.5 to 54.4

(c) The object’s third path from the time of 54.5 to 500

Figure 8: The three paths an object takes as it drifts from $L_2$ when the initial $v_x$ is 0.0 and $v_y$ is −.1.
Figure 9: The object drifts from $L_3$ into a horseshoe orbit with initial $v_x$ is 0.0 and $v_y$ is 0.0.

The object’s path for the first 5.0 units of time.

In Figure 11a, the object is in a horseshoe orbit when the velocity in the $x$-direction has changed to $-1$. When it moves near Jupiter in Figure 11b, it starts to orbit around the planet. After a time of 273.5, it stops orbiting Jupiter and begins to go into a spiral out of the Solar System. We have another example where the object goes into three different orbits (see Figure 12). This is one case where the object does not change course near Jupiter. The second orbit is a horseshoe. The velocity in the $y$-direction being $-1$ and the $x$-direction being the same.
Figure 10: The three paths the object takes drifting from $L_3$ when the initial $v_x$ is .1 and $v_y$ is 0.0.
Figure 11: The two orbital paths taken when the object drifts from $L_3$ with initial $v_x = -1$. 

(a) The object’s horseshoe path from the time of 0.0 to 267.0

(b) The object’s path from the time of 267.1 to 273.5
Figure 12: The three paths the object takes drifting from $L_3$ when the initial $v_x$ is 0.0 and $v_y$ is $-1$. 
4.2 Analyzing the Stability of the Triangular Solutions

We have now arrive at the main focus of our paper, the triangular Lagrange points $L_4$ and $L_5$. These equilateral triangular points can be found when $r_1$ and $r_2$ are 1. From Equation (11), we will discover that the locations of $L_4$ and $L_5$ are coordinates $(\mu + \frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(\mu + \frac{1}{2}, -\frac{\sqrt{3}}{2})$, respectively. The two points represent the third corners of the two equilateral triangles in the plane of orbit whose common base is the line between the centers of the two large masses. In this scenario, the point is ahead of ($L_4$), or behind ($L_5$) the smaller mass with regard to its orbit around the larger mass.

Now let us explain the reason why $L_4$ and $L_5$ are in balance. First the distances from each point to the two large masses are equal and the gravitational forces from the two large bodies are in the same ratio as the masses of the two bodies. Thus, the resultant force acts through the barycentre of the system. A barycentre is the center of gravity where two or more celestial bodies orbit each other. Also the geometry of the triangle ensures that the resultant acceleration is to the distance from the barycentre in the same ratio as for the two large bodies. Since the barycentre is both the center of mass and the center if rotation of the system, the resultant is all that is required to keep a body at the orbit of the Lagrange point in equilibrium with the rest of the system.

As we a move along in our study of the triangular points, we will assume the same initial conditions mentioned earlier. Our investigation and analysis is solely on $L_4$. Since $L_5$ is a $y$-point reflection of $L_4$, every result obtained will be applicable and true also to $L_5$.

In our observation of $L_4$, Figure 13 has a graph of point. But there is not only one point at this location but a collection of points up to time of 500. Here you see the object at an orbital equilibrium on the triangular point. The delta values for the velocities are also zero. The reason for the stability at this point is due to the Coriolis force.

We will further investigate into the $L_4$ by using various initial velocities. Our primary interest will be the graphs that include horseshoe orbits. In our first case where $v_x = .1$ and $v_y = .1$, the object starts off in a horseshoe orbit in Figure 14a. It maintains this path up to a time of 289.1, before transitioning into another orbit near Jupiter in Figure 14b. In Figure 15 we can see the object in its four orbits, the first one a horseshoe. The fourth path is a mini-orbit that is occurring near Jupiter just before it exits the Solar System. The initial velocities are $v_x = 0$ and $v_y = .1$ for this case.
Figure 13: The object stable on the $L_4$ point.

(a) The object’s horseshoe path from the time of 0.0 to 289.1
(b) The object’s path from the time of 289.2 to 500.0

Figure 14: The two orbital paths taken when the object drifts from $L_4$ with initial $v_x$ is .1 and $v_y$ is .1.
Our last investigation leads to a repositioning of the object near $L_4$ and initial velocities remaining the same. The next two cases produces horseshoe orbits that are unique to the others we have previously studied. The horseshoe orbits are chaotic. Figure 16 comprises of the breakdown of the first example. The object is originally placed $-0.027$ in the $x$-direction from $L_4$. The object first moves away further from $L_4$ and then eventually orbits around $L_5$. As it is making its way back to the $L_4$ the object reverses its path back to $L_5$ as it is near the $x$-axis. This happens on two occasions (see Figures 16b - 16d). On the third try going back to $L_4$ the object crosses over the $x$-axis in Figure 16e. Another example of a chaotic horseshoe orbit exist when the object is $-0.0153$ in the $y$-direction away from $L_4$ (see Appendix A.3). The difference in this example is the object reverses back to $L_5$ on the third occasion and does not cross the $x$-axis by the time of 500.
Figure 15: The four orbital paths taken when the object drifts from $L_4$ when the initial $v_x$ is 0.0 and $v_y$ is −.1.

(a) The object in a horseshoe orbit from the time of 0 to 134.5

(b) The object’s second orbit from the time of 134.6 to 303.1

(c) The object’s third orbit from the time of 303.2 to 371.2

(d) The object in a mini-orbit from time of 371.3 to 376.3 in transition to leaving the Solar System
(a) The object in a horseshoe orbit from the time of 0.0 to 500

(b) The object’s path from the time of 0 to 175.5

(c) The object’s path from the time of 175.6 to 312.5

(d) The object’s path from the time of 312.5 to 433.5

(e) The object’s path from the time of 433.6 to 500 crossing back over the x-axis

Figure 16: The orbital path taken when the object is placed −.027 in the x-direction from $L_4$. 

5 Conclusion

The solutions of the three body problem are much more varied than those of the two body problem. Even if we restrict our attention to orbits which start near one of the Lagrange points, the mathematics does indeed explode. Often a solution will seem to be stable for a while, but then a significant change in the orbit occurs near one of the Lagrange points. This is especially likely to happen near $L_1$ and $L_2$, because these points are near Jupiter, and the gravitational pull of Jupiter has an effect on the object’s orbit. But if the body is moving very slowly, there can also be a drastic change near $L_3$, from a horseshoe orbit to a half-horseshoe and back again. A body nearly at rest near $L_3$ is delicately balanced, and even the tiny pull of Jupiter from the other side of the Sun can tip the orbit to one side or the other.

By using Maple programming (see Appendix B), we were able to execute the RK4 method. The program produced a Cartesian graph of points. Each point represented a location of the object or satellite at specific time. Thus, we are able to visualize the object’s path drifting from (near) a specific Lagrange point.
Appendices
Appendix A

A.1 More $L_1$

Figure A.1: The object spiraling when the initial $v_x$ is 0.0 and $v_y$ is .1.
| Time ($t$) | $|\Delta v_x|$ | $|\Delta v_y|$ |
|------------|-------------|-------------|
| 0.1        | 0.01986535  | 0.00371961  |
| 0.2        | 0.01900747  | 0.01105292  |
| 0.3        | 0.01743831  | 0.01814801  |
| 0.4        | 0.01543449  | 0.02510977  |
| 0.5        | 0.01344434  | 0.03251123  |
| 0.6        | 0.01242471  | 0.04205547  |
| 0.7        | 0.01585754  | 0.05975139  |
| 0.8        | 0.06055506  | 0.13393346  |
| 0.9        | 1.61529404  | 0.15151497  |
| 1.0        | 0.01232626  | 0.35714787  |
| 1.1        | 0.04333637  | 0.35506800  |
| 1.2        | 0.08370598  | 0.34886927  |
| 1.3        | 0.12714049  | 0.33796077  |
| 1.4        | 0.17150805  | 0.32190481  |
| 1.5        | 0.21537819  | 0.30047514  |
| 1.6        | 0.25762687  | 0.27362867  |
| 1.7        | 0.29730120  | 0.24147754  |
| 1.8        | 0.33562525  | 0.20426650  |
| 1.9        | 0.36566199  | 0.16235387  |
| 2.0        | 0.39293149  | 0.11620385  |
| 2.1        | 0.41478194  | 0.06636039  |
| 2.2        | 0.43070527  | 0.01344927  |
| 2.3        | 0.44027806  | 0.04183985  |
| 2.4        | 0.44316564  | 0.09876316  |
| 2.5        | 0.43912607  | 0.15653387  |
| 2.6        | 0.42801347  | 0.21433359  |
| 2.7        | 0.40978060  | 0.27132394  |
| 2.8        | 0.38448032  | 0.32658222  |
| 2.9        | 0.35226609  | 0.37943181  |
| 3.0        | 0.31339109  | 0.42900884  |
| 3.1        | 0.26820626  | 0.47437996  |
| 3.2        | 0.21715706  | 0.51486744  |
| 3.3        | 0.16077900  | 0.54974918  |
| 3.4        | 0.09969198  | 0.57837045  |
| 3.5        | 0.03459354  | 0.60014564  |
| 3.6        | 0.03374898  | 0.61456713  |
| 3.7        | 0.10450732  | 0.62121337  |
| 3.8        | 0.17680195  | 0.61975593  |
| 3.9        | 0.24971259  | 0.60966540  |
| 4.0        | 0.32228936  | 0.59171619  |
| 4.1        | 0.39356441  | 0.56490004  |
| 4.2        | 0.46256404  | 0.52987822  |
| 4.3        | 0.52832097  | 0.48658242  |
| 4.4        | 0.58988673  | 0.43541424  |
| 4.5        | 0.64634404  | 0.37679327  |

Table A.1: Delta velocity values for the object moving in Figure A.1
A.2  More $L_2$ graphs

(a) The object’s first path taken from the time of 0.0 to 325.0
(b) The object’s second path from the time of 325.1 to 426.1
(c) The object’s third path from the time of 426.2 to 500

Figure A.2: Three separate plots for the three orbits of the object drifting from $L_2$ with the initial $v_x$ is .1 and $v_y$ is .1
(a) The object’s first path taken from the time of 0.0 to 125.9
(b) The object’s second path from the time of 126.0 to 283.4
(c) The object’s third path from the time of 283.5 to 368.7

Figure A.3: Three separate plots for the three orbits of the object drifting from $L_2$ with the initial $v_x$ is .1 and $v_y$ is $-1$. 
A.3 More $L_4$ graphs

(a) The object's path from the time of 0.0 to 194.6

(b) The object's path from the time of 194.7.0 to 320.7

(c) The object's path from the time of 320.8 to 440.8

(d) The object's path from the time of 440.9 to 500

Figure A.4: Four separate plots for the breakdown of the chaotic horseshoe orbit of the object $-0.0153$ in the $y$-direction from $L_4$. 
Appendix B

Maple Code

> restart:with(plots):with(plottools):Digits:=10:
> RK4:=proc(x0,y0,vx0,vy0,t0)

> description "4-variable order 4 runge-kutta method":

> f[1]:=(x1,x2,y1,y2,t)->x2:

> f[3]:=(x1,x2,y1,y2,t)->y2:

> f[2]:=(x1,x2,y1,y2,t)->-.999*(x1+.001)/sqrt(((x1+.001)^2+y1^2)^3)

> -.001*(x1-.999)/sqrt(((x1-.999)^2+y1^2)^3)+x1+2*y2:

> f[4]:=(x1,x2,y1,y2,t)->-.999*y1/sqrt(((x1+.001)^2+y1^2)^3)

> -.001*y1/sqrt(((x1-.999)^2+y1^2)^3)+y1-2*x2:

> F:=(x1,x2,y1,y2,t,n)->h*f[n](x1,x2,y1,y2,t):

> x1[0]:=x0:

> x2[0]:=vx0:

> y1[0]:=y0:

> y2[0]:=vy0:

> t[0]:=t0:

> h:=.1:
n:=5000:

pt[0]:=[x1[0],y1[0]]:

printf("t[0]= %f:\n",t[0]);

printf("x1[0]= %2.8f\t\ty1[0]= %2.8f\n",x1[0],y1[0]);

printf("x2[0]= %2.8f\t\ty2[0]= %2.8f\n\n\n",x2[0],y2[0]);

for p from 0 to (n-1) do

for i from 1 to 4 do

G[1,i]:=F(x1[p],x2[p],y1[p],y2[p],t[p],i);

end do:

for j from 1 to 4 do

G[2,j]:=F(x1[p]+.5*G[1,1],x2[p]+.5*G[1,2],y1[p]+.5*G[1,3],y2[p]+.5*G[1,4],t[p]+.5*h,j);

end do:

for k from 1 to 4 do

G[3,k]:=F(x1[p]+.5*G[2,1],x2[p]+.5*G[2,2],y1[p]+.5*G[2,3],y2[p]+.5*G[2,4],t[p]+.5*h,k);

end do:
> end do:

> for m from 1 to 4 do

> t[p]+h,m);

> end do:

> x1[p+1]:=x1[p]+(G[1,1]+2.0*G[2,1]+2.0*G[3,1]+G[4,1])/6.0;

> x2[p+1]:=x2[p]+(G[1,2]+2.0*G[2,2]+2.0*G[3,2]+G[4,2])/6.0;

> y1[p+1]:=y1[p]+(G[1,3]+2.0*G[2,3]+2.0*G[3,3]+G[4,3])/6.0;


> t[p+1]:=t[p]+h;

> printf("t[%d]= %f:\n",p+1,t[p+1]);

> printf("x1[%d]= %2.8f\tx1y1[%d]= %2.8f\n",p+1,x1[p+1],p+1,y1[p+1]);

> printf("x2[%d]= %2.8f\tx2y2[%d]= %2.8f\n\n",p+1,x2[p+1],p+1,y2[p+1]);

> pt[p+1]:=[x1[p+1],y1[p+1]];

> end do:

> L4plot:= pointplot([seq(pt[j],j=0..n)], color=blue,symbol=point,
> symbolsize=14):
> display(L4plot, insequence=true):

> end proc;

> RK4(.499, sqrt(3.)/2., 0., 0., 0., 0.);
Bibliography


