FACTORIZATION OF ISOMETRIES OF HYPERBOLIC

4-SPACE AND A DISCRETENESS CONDITION

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A dissertation submitted to the Graduate School-Newark

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Graduate Program in Mathematical Sciences

written under the direction of

Professor Jane P. Gilman

and approved by

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Newark, New Jersey

May, 2009

Abstract

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Discreteness Condition

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Gilman's NSDC condition is a sufficient condition for the discreteness of a two generator subgroup of $PSL(2, \mathbb{C})$. We address the question of the extension of this condition to subgroups of isometries of hyperbolic 4-space. While making this new construction, namely the NSDS condition, we are led to ask whether every orientation preserving isometry of hyperbolic 4-space can be factored into the product of two half-turns. We use some techniques developed by Wilker to first, define a half-turn suitably in dimension 4 and then answer the former question. It turns out that defining a half-turn in this way in any dimension n enables us to generalize some of Gilman's theorems to dimension $n \geq 4$. We also give an exposition on part of Wilker's work and give new proofs for some of his results.

To my parents, my sister and to Neha.

Acknowledgements:

A large number of people have had an influence on the writing and completion of this thesis. Let me begin by giving my sincere and heartfelt thanks to my advisor Jane Gilman, who not only introduced me to the wonderful world of hyperbolic geometry, but was constantly encouraging and ever patient with my questions and she continues to be a role model and mentor. I would like to thank Mark Feighn for his insights and questions and for his availability and willingness to field my numerous questions. I am indebted to Lee Mosher for his critical analysis of my work, his pertinent comments and his help and encouragement through some difficult times. A special note of thanks to Ara Basmajian for leading me to think in new directions during the course of some greatly enjoyable conversations. I thank Jacob Sturm for helping me to organize my thoughts and present mathematics in a structured manner. I am grateful to Robert Sczech and John Loftin for their unwavering support and encouragement through some difficult times. To William Keigher, I owe a debt of gratitude for helping me to grow as a teacher of mathematics. Susan Tel deserves a special word of thanks for making my stay at the department so enjoyable and for always being available and welcoming. I am thankful to my fellow graduate students Gregory Fein, Andrew Silverio and Vidur Malik for some engaging conversations and insightful questions.

I owe a great debt of gratitude to my father Chander Puri, who opened up the world of mathematics to me and showed me the possibilities that lay within. His untiring support through difficult periods has been invaluable and will always be cherished. To my mother Sukesh Puri, I am greatly indebted as it was she who made possible not only an academic, but a well rounded education for me and for her love and support over the last six years. My sister Kajal Puri Joshi has been a great friend and a constant source of positive energy and I owe her more than she knows. Finally, to my partner Neha Gondal, who has been a pillar of strength and support through good times and bad, mere words fail me in the expression of my gratitude. It is to my family that I dedicate this work.

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1 Introduction

There are several methods which enable us to test whether a given subgroup of isometries of three-dimensional hyperbolic space is discrete. Gilman [5] described one such method, a sufficient condition for discreteness of a group generated by two isometries of three-dimensional hyperbolic space, $\mathbb{H}^3$, namely the NSDC condition, along with a means to construct discrete subgroups of the group of isometries of $\mathbb{H}^3$.

Our goal here is to extend this construction to hyperbolic space of dimension four. One of the main questions that arises here is whether an orientation preserving isometry of hyperbolic 4-space can be written as a composition of half-turns. While this is true for hyperbolic 3-space, the answer is not known in higher dimensions. In dimension four, the action of the Möbius group can be viewed explicitly on the boundary of hyperbolic space ($\hat{\mathbb{E}}^3$). Wilker [11] proves that every orientation preserving Möbius transformation acting on $\hat{\mathbb{E}}^3$ can be factored as the product of two suitably defined half-turns and we give a new proof for all but one case of this theorem. We also explain why this new definition of half-turn is the proper and natural generalization for our purposes. In recent work, Basmajian and Maskit [2] have explored a similar question in dimension n although their definition of half-turn is broader than ours. With our initial goal in mind, we use some of Wilker's [11] techniques, elaborate on and motivate some of his definitions and constructions as we proceed.

2   Organization

The organization of this work is as follows. In sections 3, 4 and 5 we review the necessary background. This is standard material and is drawn from [3, 9]. Sections 6 and 7 constitute the main results of this thesis. In section 6, we analyze and describe Wilker's techniques [11] and give some new proofs and motivation for his results. We give a new proof for the first five cases of the half-turn theorem, namely Theorem 6.53. We begin section 7 with the half-turn theorem restated for hyperbolic 4-space, Corollary 7.1. We then proceed to state and prove the discreteness theorem 7.14 and conclude with section 7.1, wherein we define the associated LM-supergroup to a given two generator group and we describe the extended discreteness condition via Theorem 7.16: that is if the associated supergroup has the NSDS property, then the given two generator group is discrete.

3   Definitions and Preliminaries

We refer to Ratcliffe [9] for the definitions in this section. Consider the n-dimensional vector space $\mathbb{R}^n$. A vector in $\mathbb{R}^n$ is an ordered n-tuple $x = (x_1, ..., x_n)$ of real numbers. Let x and y be vectors in $\mathbb{R}^n$.

Definition 3.1. Let

$$d_E(x, y) = |x - y| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Then d_E is a metric on $\mathbb{R}^n$ called the *Euclidean metric*. The metric space

consisting of $\mathbb{R}^n$ together with the metric d_E is called *Euclidean n-space*.

Henceforth, we shall use $\mathbb{E}^n$ to denote the metric space $(\mathbb{R}^n, d_E)$.

Definition 3.2. An $m-plane$ of $\mathbb{E}^n$ is a coset $a + V$ of an m-dimensional vector subspace V of $\mathbb{R}^n$. A *straight line* of $\mathbb{E}^n$ is a 1-plane of $\mathbb{E}^n$.

Definition 3.3. The *(n-1)-sphere* of $\mathbb{E}^n$ of *radius r centered at* $a = (a_1, ..., a_n)$ is defined to be the set

$$\left\{ x = (x_1, ..., x_n) \in \mathbb{E}^n : d_E(x, a) = \sqrt{\sum_{i=1}^{n} (x_i - a_i)^2} = r \right\}.$$

Every $(m-1)$-sphere, $m < n$, in $\mathbb{E}^n$ is the intersection of an $(n-1)$-sphere in $\mathbb{E}^n$ with an m-plane of $\mathbb{E}^n$. A *circle* is a 1-sphere of $\mathbb{E}^n$.

Identify $\mathbb{E}^{n-1}$ with $\mathbb{E}^{n-1} \times \{0\}$ in $\mathbb{E}^n$. Therefore, $\mathbb{E}^{n-1}$ will be the plane in $\mathbb{E}^n$ exactly consisting of points of the form $(x_1, ..., x_{n-1}, 0)$, where $x_i \in \mathbb{R}$.

Let $\mathbb{U}^n = \{(x_1, ..., x_n) \in \mathbb{E}^n : x_n > 0\}$.

Definition 3.4. The *hyperbolic metric* d_H on $\mathbb{U}^n$ is given by

$$\cosh d_H(x, y) = 1 + \frac{|x - y|^2}{2x_n y_n}.$$

The metric space consisting of $\mathbb{U}^n$ together with the metric d_H is called the *upper half-space model* of hyperbolic n-space.

Henceforth, we shall use $\mathbb{H}^n$ to denote the metric space $(\mathbb{U}^n, d_H)$.

Definition 3.5. A subset P of $\mathbb{H}^n$ is a *hyperbolic m-plane* of $\mathbb{H}^n$ if and only if P is the intersection of $\mathbb{H}^n$ with either an m-plane of $\mathbb{E}^n$ orthogonal to $\mathbb{E}^{n-1}$ or with an m-sphere of $\mathbb{E}^n$ orthogonal to $\mathbb{E}^{n-1}$.

A subset L of $\mathbb{H}^n$ is a *hyperbolic line* of $\mathbb{H}^n$ if and only if L is the intersection of $\mathbb{H}^n$ with either a straight line of $\mathbb{E}^n$ orthogonal to $\mathbb{E}^{n-1}$ or with a circle of $\mathbb{E}^n$ orthogonal to $\mathbb{E}^{n-1}$.

Now, let $\mathbb{B}^n = \{x \in \mathbb{E}^n : |x| < 1\}$.

Definition 3.6. The *Poincaré metric* d_B on $\mathbb{B}^n$ is given by

$$\cosh d_B(x, y) = 1 + \frac{2\,|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}.$$

The metric space consisting of $\mathbb{B}^n$ together with the metric d_B is called the *conformal ball model* of hyperbolic n-space.

Henceforth, we shall use B^n to denote the metric space $(\mathbb{B}^n, d_B)$.

Definition 3.7. A subset P of B^n is said to be a *hyperbolic m-plane* of B^n if and only if P is the intersection of B^n with an m-dimensional vector subspace of $\mathbb{E}^n$ or with an m-sphere of $\mathbb{E}^n$ orthogonal to B^n.

A *hyperbolic line* of B^n is a 1-plane of B^n.

The definitions of line, plane, distance etc. will hence become clear from the space under consideration.

4 Reflections, Stereographic Projection and Möbius Transformations

The material in this section is standard and may be found in Beardon [3] and Ratcliffe [9].

4.1 Reflections

Let a be a unit vector in $\mathbb{E}^n$ and $t \in \mathbb{R}$.

Definition 4.1. Then, the *hyperplane* of $\mathbb{E}^n$ with unit normal vector a and passing through the point ta is given by

$$P(a, t) = \{x \in \mathbb{E}^n : a \cdot x = t\}.$$

Every hyperplane of $\mathbb{E}^n$ is of this form, and every hyperplane as exactly two representations $P(a, t)$ and $P(-a, -t)$.

Definition 4.2. The *reflection* ρ in the plane $P(a, t)$ is given by the explicit formula

$$\rho(x) = x + 2(t - a \cdot x)a.$$

Let a be a point of $\mathbb{E}^n$ and $r \in \mathbb{R}^+$.

Definition 4.3. The *sphere* of $\mathbb{E}^n$ of *radius r centered at a* is defined to be the set

$$S(a, r) = \{x \in \mathbb{E}^n : |x - a| = r\}.$$

Definition 4.4. The *reflection* (or *inversion*) σ of $\mathbb{E}^n$ in the sphere $S(a,r)$ is given by the explicit formula

$$\sigma(x) = a + \left(\frac{r}{|x-a|} \right)^2 (x-a).$$

Theorem 4.5. *[9] If σ is the reflection of $\mathbb{E}^n$ in the sphere $S(a,r)$, then*

(1) $\sigma(x) = x \Leftrightarrow x \in S(a,r)$

(2) $\sigma^2(x) = x \ \forall \ x \neq a$; and

(3) for each $x, y \neq a$,

$$|\sigma(x) - \sigma(y)| = \frac{r^2 \, |x-y|}{|x-a| \, |y-a|}.$$

Remark 4.6. Every reflection of $\mathbb{E}^n$ in a hyperplane or sphere is conformal and reverses orientation.

4.2   Stereographic Projection and Möbius Transformations

Identify $\mathbb{E}^n$ with $\mathbb{E}^n \times \{0\}$ in $\mathbb{E}^{n+1}$.

Definition 4.7. The *stereographic projection* π of $\mathbb{E}^n$ onto $S^n - \{e_{n+1}\}$ is defined by projecting x in $\mathbb{E}^n$ towards (or away from) e_{n+1} until it meets the sphere S^n in the unique point $\pi(x)$ other than e_{n+1}. Calculation yields the explicit formula

$$\pi(x) = \left(\frac{2x_1}{1+|x|^2}, ..., \frac{2x_n}{1+|x|^2}, \frac{|x|^2 - 1}{|x|^2 + 1} \right).$$

Remark 4.8. The map π is a bijection of $\mathbb{E}^n$ onto $S^n - \{e_{n+1}\}$ and after some more calculation we see that

$$\pi^{-1}(y) = \left(\frac{y_1}{1 - y_{n+1}}, ..., \frac{y_n}{1 - y_{n+1}} \right).$$

Let ∞ be a point not in $\mathbb{E}^{n+1}$ and define $\hat{\mathbb{E}}^n = \mathbb{E}^n \cup \{\infty\}$. Now, we may extend π to a bijection $\hat{\pi} : \hat{\mathbb{E}}^n \to S^n$ by setting $\hat{\pi}(\infty) = e_{n+1}$, and we define a metric d on $\hat{\mathbb{E}}^n$ by the formula

$$d(x, y) = |\hat{\pi}(x) - \hat{\pi}(y)|.$$

The metric d is called the *chordal metric* on $\hat{\mathbb{E}}^n$. The metric space $\hat{\mathbb{E}}^n$ is compact and is obtained from $\mathbb{E}^n$ by adjoining one point at infinity. For this reason, $\hat{\mathbb{E}}^n$ is called the *one-point compactification* of $\mathbb{E}^n$.

Let $P(a, t)$ be a hyperplane of $\mathbb{E}^n$. Let

$$\hat{P}(a, t) = P(a, t) \cup \{\infty\}.$$

We note that the subspace $\hat{P}(a, t)$ of $\hat{\mathbb{E}}^n$ is homeomorphic to S^{n-1}. If ρ is the reflection of $\mathbb{E}^n$ in $P(a, t)$, then we define the extension $\hat{\rho} : \hat{\mathbb{E}}^n \to \hat{\mathbb{E}}^n$ of ρ by setting $\hat{\rho}(\infty) = \infty$. The map $\hat{\rho}$ is called the *reflection* of $\hat{\mathbb{E}}^n$ in the extended hyperplane $\hat{P}(a, t)$. Note that every reflection of $\hat{\mathbb{E}}^n$ in an extended hyperplane is a homeomorphism.

Let σ be the reflection of $\mathbb{E}^n$ in the sphere $S(a, r)$. We now extend σ to a map $\hat{\sigma} : \hat{\mathbb{E}}^n \to \hat{\mathbb{E}}^n$ by setting $\hat{\sigma}(a) = \infty$ and $\hat{\sigma}(\infty) = a$. The map $\hat{\sigma}$ is called the

reflection of $\hat{\mathbb{E}}^n$ in the sphere $S(a, r)$. Note that every reflection of $\hat{\mathbb{E}}^n$ in a sphere of $\mathbb{E}^n$ is a homeomorphism.

Definition 4.9. A *sphere* of $\hat{\mathbb{E}}^n$ is defined to be either a Euclidean sphere $S(a, r)$ or an extended plane $\hat{P}(a, t) = P(a, t) \cup \{\infty\}$. $\hat{P}(a, t)$ is topologically a sphere.

Definition 4.10. A *Möbius transformation* of $\hat{\mathbb{E}}^n$ is a finite composition of reflections of $\hat{\mathbb{E}}^n$ in spheres. The set of all Möbius transformations of $\hat{\mathbb{E}}^n$ is denoted $\mathcal{M}_n$ and forms a group under composition.

Definition 4.11. A group G *acts on a set X* if and only if there is a function from $G \times X$ to X, written $(g, x) \mapsto gx$, such that for all $g, h \in G$ and $x \in X$, we have

(1) $1 \cdot x = x$ and

(2) $g(hx) = (gh)x$.

A function from $G \times X$ to X satisfying conditions (1) and (2) is called an action of G on X.

Example 4.12. If X is a metric space, then the group $I(X)$ of isometries of X acts on X by $\phi x = \phi(x)$. We note that the group of isometries of $\mathbb{H}^n$, denoted $I(\mathbb{H}^n)$, and the group $\mathcal{M}_{n-1}$ are isomorphic. Hence, $\mathcal{M}_{n-1}$ acts on $\mathbb{H}^n$.

5 Poincaré Extension

We review the Poincaré extension and note that this is a critical tool in our analysis of factorization into half-turns. Once again, we refer to Beardon [3] and Ratcliffe [9] for the construction below.

We recall the identification of $\mathbb{E}^{n-1}$ with $\mathbb{E}^{n-1} \times \{0\}$ in $\mathbb{E}^n$, whereby a point $x \in \mathbb{E}^{n-1}$ corresponds to the point $\tilde{x} = (x, 0)$ of $\mathbb{E}^n$. Let ϕ be a Möbius transformation of $\hat{\mathbb{E}}^{n-1}$. We may extend ϕ to a Möbius transformation of $\hat{\mathbb{E}}^n$ as follows. If ϕ is the reflection of $\hat{\mathbb{E}}^{n-1}$ in $\hat{P}(a, t)$, then $\tilde{\phi}$ is the reflection of $\hat{\mathbb{E}}^n$ in $\hat{P}(\tilde{a}, t)$. If ϕ is the reflection of $\hat{\mathbb{E}}^{n-1}$ in $S(a, r)$, then $\tilde{\phi}$ is the reflection of $\hat{\mathbb{E}}^n$ in $S(\tilde{a}, r)$. In both these cases

$$\tilde{\phi}(x, 0) = (\phi(x), 0) \ \forall \ x \in \mathbb{E}^{n-1}.$$

Thus, $\tilde{\phi}$ extends ϕ. In particular, $\tilde{\phi}$ leaves $\hat{\mathbb{E}}^{n-1}$ invariant. It also leaves invariant *upper half-space*

$$\mathbb{U}^n = \{(x_1, ..., x_n) \in \mathbb{E}^n : x_n > 0\}.$$

Now, we assume that ϕ is an arbitrary Möbius transformation of $\hat{\mathbb{E}}^{n-1}$. Then ϕ is a composition of reflections, say $\phi = \sigma_1 \ldots \sigma_m$. Let $\tilde{\phi} = \tilde{\sigma}_1 \ldots \tilde{\sigma}_m$. Then $\tilde{\phi}$ extends ϕ and leaves $\mathbb{U}^n$ invariant. It may be shown that $\tilde{\phi}$ depends only on ϕ and not on the decomposition $\phi = \sigma_1 \ldots \sigma_m$. The map $\tilde{\phi}$ is called the *Poincaré extension* of ϕ.

5.1 Classification of Möbius Transformations

Let ϕ be a Möbius transformation of $\hat{\mathbb{E}}^n$ (resp. S^n). By the Brouwer fixed point theorem [7], ϕ fixes a point in $\overline{\mathbb{U}^{n+1}}$ (resp. $\overline{\mathbb{B}^{n+1}}$). Then, ϕ is said to be

1. *elliptic* if it fixes a point of $\mathbb{U}^{n+1}$ (resp. $\mathbb{B}^{n+1}$);

2. *parabolic* if it fixes no point of $\mathbb{U}^{n+1}$ (resp. $\mathbb{B}^{n+1}$) and fixes a unique point of $\hat{\mathbb{E}}^n$ (resp. S^n);

3. *loxodromic* otherwise.

We note that if ϕ fixes more than two points of $\hat{\mathbb{E}}^n$ (resp. S^n), then it must fix a point in $\mathbb{U}^{n+1}$ (resp. $\mathbb{B}^{n+1}$) and is therefore, elliptic. Thus, a loxodromic transformation fixes exactly two points of $\hat{\mathbb{E}}^n$ (resp. S^n).

6 Exposition on Wilker's Inversive Geometry

In his paper titled *Inversive Geometry* [11], Wilker takes a seemingly different approach to Möbius transformations when compared with the one above. We will describe his approach and motivate and elaborate on some of his constructions and definitions by relating them to the standard approach described thus far.

6.1 The Möbius Group revisited

Recall that $S^n = \{x \in \mathbb{E}^{n+1} : |x| = 1\}$ is the unit n-sphere in Euclidean $(n+1)$-space.
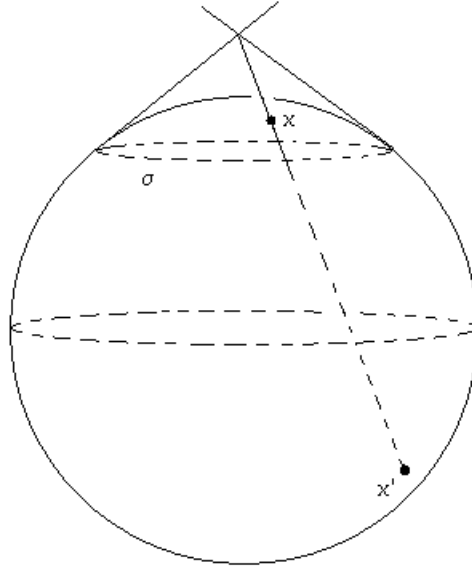
Figure 1: **Inversion in a** 1**-sphere in** S^2

Definition 6.1. An $(n-1)$-*sphere,* σ, *lying on* S^n is defined as the intersection of S^n with an n-plane in $\mathbb{E}^{n+1}$ (provided the plane is not tangent to or disjoint from S^n).

Definition 6.2. $\bar{\sigma} : S^n \to S^n$ is called *inversion* in the $(n-1)$-sphere σ and is defined as follows. If σ is an equatorial or great $(n-1)$-sphere, then $\bar{\sigma}$ is the restriction to S^n of reflection in the Euclidean n-plane which intersects S^n in σ. If σ is not an equatorial $(n-1)$-sphere, then there exists a point $x_\sigma \in \mathbb{E}^{n+1}$, all of whose tangent lines to S^n touch S^n exactly in σ. In this case, $\bar{\sigma}$ interchanges the points x and x' where the secant xx' passes through x_σ.

Remark 6.3. *(Unification)* By referring to Sections 4.2 and 5, we may give these two cases a unified definition. Note that $S^n \subset \hat{\mathbb{E}}^{n+1}$. In the first case,

where σ is an equatorial $(n-1)$-sphere, notice that $\bar{\sigma}$ is the restriction to S^n of inversion in n-sphere Σ which is orthogonal to S^n and intersects S^n in σ. Similarly, when σ is not an equatorial $(n-1)$-sphere, then $\bar{\sigma}$ is the restriction of inversion in the n-sphere Σ which is orthogonal to S^n and intersects S^n in σ. We note that Σ must pass through ∞ in the first case and cannot pass through ∞ in the second case.

Remark 6.4. The inversion $\bar{\sigma}$ is a bijection and, moreover, is an involution.

Definition 6.5. Let $\mathcal{M}_n$ denote the set of all bijections on S^n which can be expressed as a product of inversions in $(n-1)$-spheres as defined above. $\mathcal{M}_n$ forms a group called the n-dimensional *Möbius group* and its elements are called *Möbius transformations*.

Remark 6.6. We make an observation about stereographic projection.

Let Σ be an n-sphere in $\mathbb{E}^{n+1}$ and let x_0 be a point on Σ. Let $\mathbb{E}_0^{n'}$ be the Euclidean n-plane tangent to Σ at x_0. Now, choose any Euclidean n-plane not through x_0 which is parallel to $\mathbb{E}_0^{n'}$ and call it $\mathbb{E}_0^n$. We will now define *stereographic projection* from $\mathbb{E}_0^n$ to Σ as follows. The set of all Euclidean lines through x_0 and not in $\mathbb{E}_0^{n'}$ establishes a 1-1 correspondence between $\Sigma - \{x_0\}$ and $\mathbb{E}_0^n$. The image of each point x of $\mathbb{E}_0^n$ is that point $\pi(x)$ on Σ, where the line joining x_0 and x intersects Σ. Let ∞ be a point not in $\mathbb{E}^{n+1}$ and define $\hat{\mathbb{E}}_0^n = \mathbb{E}_0^n \cup \{\infty\}$. Then, we may extend the above defined *stereographic projection* π to a bijection $\hat{\pi} : \hat{\mathbb{E}}_0^n \rightarrow \Sigma$ by setting $\hat{\pi}(\infty) = x_0$.

We have now the concept of an *extended Euclidean plane* or an *n-sphere through ∞*, $\hat{\mathbb{E}}_0^n = \mathbb{E}_0^n \cup \{\infty\}$ where $\mathbb{E}_0^n$ is a Euclidean n-plane in $\mathbb{E}^{n+1}$.

If S^n is the unit n-sphere in $\mathbb{E}^{n+1}$ and $\mathbb{E}_0^n$ is the plane $\mathbb{E}^n \times \{0\}$ with unit normal $e_{n+1} = (0, \ldots 0, 1)$, then we get the explicit representation given above in section 4.2. The point of this remark is that stereographic projection can be carried out by taking any point on any give n-sphere to ∞.

6.2   Caps, Clusters and the Lorentz Group

We begin this section with a discussion of the Lorentz group for which we refer to Ratcliffe [9]. This standard discussion is then related to Möbius transformations on S^n and to Wilker's definition of n-caps.

Definition 6.7. Let x and y be vectors in $\mathbb{R}^n$. The *Lorentzian inner product* of x and y is defined to be the real number

$$x \circ y = x_1 y_1 + \ldots + x_{n-1} y_{n-1} - x_n y_n.$$

The inner product space consisting of the vector space $\mathbb{R}^n$ together with the Lorentzian inner product is called *Lorentzian n-space* and is denoted $\mathbb{R}^{n-1,1}$. The *Lorentzian norm* of x is defined to be the complex number $\|x\| = (x \circ x)^{\frac{1}{2}}$, where $(x \circ x)^{\frac{1}{2}}$ is the principal branch of the square root function.

The set $\{x \in \mathbb{R}^n : \|x\| = 0\}$ is a hypercone $\mathfrak{C}^{n-1}$ in $\mathbb{R}^n$ called *the light cone* of $\mathbb{R}^n$. If $\|x\| = 0$, then x is said to be *light-like*. A light-like vector x is said to be positive (resp. negative) if $x_n > 0$ (resp. $x_n < 0$).

If $\|x\| > 0$, then x is said to be *space-like*. The set of all space-like vector is said to be the *exterior* of the light cone.

If $\|x\|$ is positive imaginary, then x is said to be *time-like.* The set of all time-like vectors is said to be the *interior* of the light cone. A time-like vector is said to be *positive* (resp. *negative*) if $x_n > 0$ (resp. $x_n < 0$).

Definition 6.8. A function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a *Lorentz transformation* if and only if

$$\phi(x) \circ \phi(y) = x \circ y \ \ \forall \ x, y \in \mathbb{R}^n.$$

A real $n \times n$ matrix A is said to be *Lorentzian* if and only if the associated linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$, defined by $A(x) = Ax$, is Lorentzian.

The set of all Lorentzian $n \times n$ matrices together with matrix multiplication forms a group $O(n-1, 1)$ called the *Lorentz group* of $n \times n$ matrices. The group $O(n-1, 1)$ is naturally isomorphic to the group of Lorentz transformations of $\mathbb{R}^n$. A Lorentzian matrix A is said to be *positive* (resp. *negative*) if it transforms positive time-like vectors into positive (resp. negative) time-like vectors. A Lorentzian matrix is either positive or negative.

Let $PO(n-1, 1)$ denote the set of all positive matrices in $O(n-1, 1)$. Then $PO(n-1, 1)$ is a subgroup of index two in $O(n-1, 1)$ and is called the *positive Lorentz group.*

Definition 6.9. Two vectors $x, y \in \mathbb{R}^n$ are *Lorentz orthogonal* if and only if $x \circ y = 0$.

Definition 6.10. Let V be a vector subspace of $\mathbb{R}^n$. Then V is said to be

(1) *time-like* if and only if V has a time-like vector,

(2) *space-like* if and only if every non-zero vector of V is space-like, or

(3) *light-like* otherwise.

Definition 6.11. The *Lorentzian complement* of a vector subspace V of $\mathbb{R}^n$ is defined as the set

$$V^L = \{x \in \mathbb{R}^n : x \circ y = 0 \ \forall \ y \in V\}.$$

We note that $(V^L)^L = V$ and that V is time-like if and only if V^L is space-like.

Theorem 6.12. *[9] [Page 59]*

Let x, y be positive (negative) time-like vectors in $\mathbb{R}^n$. Then $x \circ y \leq \|x\|\|y\|$ with equality if and only if x and y are linearly dependent.

If x and y are positive (negative) time-like vectors in $\mathbb{R}^n$, then by theorem 6.12, there is a unique nonnegative real number $\eta(x, y)$ such that

$$x \circ y = \|x\|\|y\| \cosh(\eta(x, y)).$$

$\eta(x, y)$ is called the *Lorentzian time-like angle* between x and y. Note that $\eta(x, y) = 0$ if and only if x and y are positive scalar multiples of each other.

Let $F^n = \{x \in \mathbb{R}^{n+1} : \|x\|^2 = -1\}$. The set F^n is a hyperboloid of two sheets in $\mathbb{R}^{n+1}$. The set of all x in F^n such that $x_1 > 0$ (resp. $x_1 < 0$) is called the *positive* (resp. *negative*) sheet of F^n.

The *hyperboloid model* H^n of hyperbolic n-space is defined to be the positive sheet of F^n. The hyperbolic distance between x and y in F^n is given by

$$d_H(x, y) = \eta(x, y).$$

where $\eta(x, y)$ is the Lorentzian time-like angle between x and y.

Definition 6.13. A *hyperbolic m-plane* of H^n is the intersection of H^n with an $(m + 1)$-dimensional time-like vector subspace of $\mathbb{R}^{n+1}$.

Note that a *hyperbolic line* is a hyperbolic 1-plane of H^n and that a *hyperplane* of H^n is a hyperbolic $(n - 1)$-plane of H^n.

Remark 6.14. If x is a space-like vector in $\mathbb{R}^{n+2}$, then the Lorentzian complement of the subspace $\langle x \rangle$ spanned by x is an $(n + 1)$-dimensional time-like vector subspace, P, of $\mathbb{R}^{n+1,1}$. We say that x is a *Lorentz normal vector* to P. If $\|x\| = 1$, then it is a *unit Lorentz normal vector* to P. We note that $\mathbb{P} = P \cap H^n$ is a hyperplane of H^n called the hyperplane of H^n Lorentz orthogonal to x.

We now make a definition and an observation to help motivate some of Wilker's [11] construction.

Definition 6.15. The *Lorentz reflection* in the $(n + 1)$-dimensional time-like vector subspace P of $\mathbb{R}^{n+1,1}$ with unit Lorentz normal x is a map $\rho : \mathbb{R}^{n+1,1} \to \mathbb{R}^{n+1,1}$ such that

1. $\rho(y) = y \Leftrightarrow y \in P$,

2. $\rho(y) = y + tx$ and

3. $\|\rho(y)\| = \|y\|$.

Remark 6.16. In Definition 6.15, we note that $P = \{u \in \mathbb{R}^{n+2} : u \circ x = 0\}$. If ρ is the Lorentz reflection in P, then for each $y \in \mathbb{R}^{n+1,1}$,

$$(y + tx) \circ (y + tx) = y \circ y$$

$$\Rightarrow y \circ y + t(x \circ y) + t(y \circ x) + t^2 = y \circ y$$

$$\Rightarrow t^2 + 2t(x \circ y) = 0 \Rightarrow t = 0, -2(x \circ y).$$

If $y \notin P$, then $t \neq 0$ (as $\rho(y) \neq y$). If $y \in P$, then $t = 0 = -2(x \circ y)$.

Thus, we have that the Lorentz reflection in P is given by the formula

$$\rho(y) = y - 2(x \circ y)x.$$

We now turn our attention to Wilker's discussion and relate his construction to the Lorentz group. We define an n-cap on S^n. By subsequently lifting to $(n+2)$ coordinates, this equips us with a means to parameterize Lorentz orthogonal hyperplanes in H^{n+1}.

Definition 6.17. The *closed n-cap* C on S^n with center $c \in S^n$ and angular radius $\theta, 0 < \theta < \pi$ is defined as

$$C = \{x \in S^n : x \cdot c \geq \cos \theta\}.$$

Remark 6.18. *Geometric meaning:* [Figure 2]

Note that $x \cdot c = |x||c| \cos \phi$ where ϕ is the angle between the vectors x and c $(0 < \phi < \pi)$. Therefore, $x \cdot c = \cos \phi$. Now, as $\cos \theta$ is decreasing for $0 < \theta < \pi$, we have $\phi \leq \theta \Rightarrow \cos \phi \geq \cos \theta$. Hence, points in the cap are those vectors $x \in S^n$ which make angle less than or equal to θ with the vector $c \in S^n$.

Also, there is a unique point x_σ from which all tangents to S^n meet S^n in exactly the $(n-1)$-sphere σ. Observe that an easy calculation tells us that $x_\sigma = (\sec \theta)c$, that is x_σ is the vector with magnitude $\sec \theta$ and unit vector c
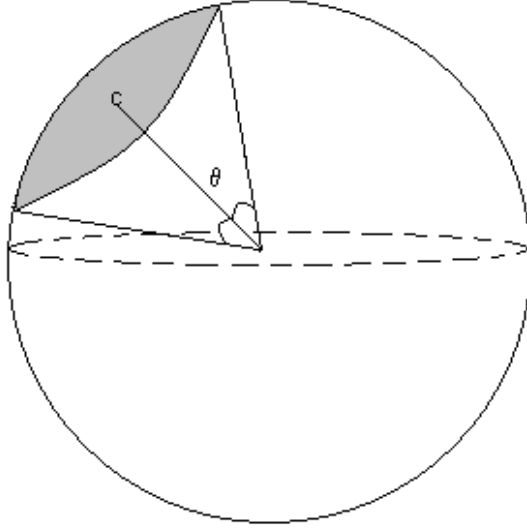
Figure 2: **A 2-cap in S^2 with center c and angular radius θ.**

where c is the center of the n-cap bounded by σ.

The condition for $x \in S^n$ to belong to the cap C can be rewritten as

$$x \cdot c - 1 \cdot \cos \theta \geq 0.$$

Since $\sin \theta > 0$,

$$x \cdot \frac{c}{\sin \theta} - 1 \cdot \frac{\cos \theta}{\sin \theta} \geq 0.$$

Let us define the *lift* of the point x by the $(n+2)$-vector

$$\mathbf{x} = (x, 1),$$

and the *lift* of the cap by the $(n+2)$-vector

$$\mathbf{c} = ((\csc\theta)c, \cot\theta).$$

Consider the Lorentzian inner product in $\mathbb{R}^{n+2}$. Then, a vector $\mathbf{c}$ is the lift of a cap if and only if $\mathbf{c} \circ \mathbf{c} = 1$ and a vector $\mathbf{x}$ is the lift of a point if and only if $\mathbf{x} \circ \mathbf{x} = 0$ and $x_{n+2} = 1$. Also, point $\mathbf{x}$ belongs to the cap $\mathbf{c}$ if and only if $\mathbf{x} \circ \mathbf{c} \geq 0$.

Note that we may allow the coordinates of a point to be positive homogeneous. That is for $\lambda > 0$, $\mathbf{x}$ and $\lambda\mathbf{x}$ are lifts of the same point $x \in S^n$. We can do this since multiplying by a positive constant neither changes the sign of the inner product, nor does it change the property of being a point (if $\lambda > 0$, then $\mathbf{x} \circ \mathbf{x} = 0 \Leftrightarrow (\lambda\mathbf{x}) \circ (\lambda\mathbf{x}) = 0$). Therefore, we may make the modification that $\mathbf{x}$ is the lift of a point if and only if $\mathbf{x} \circ \mathbf{x} = 0$ and $\mathbf{x} = \lambda(x, 1)$ for some $\lambda > 0$. Clearly, one must fix the sign of λ since the condition for $\mathbf{x}$ to belong to cap $\mathbf{c}$ must remain invariant.

Definition 6.19. The *complementary cap* of a cap C is the cap C' with center $-c$ (the antipode of c) and angular radius $\pi - \theta$. Thus, in (n+2)-coordinates, the lift of C' is $\mathbf{c}' = (\csc(\pi - \theta)(-c), \cot(\pi - \theta)) = ((-\csc\theta)c, -\cot\theta) = -\mathbf{c}$.

Remark 6.20. Note that a point x belongs to the common boundary $(n-1)$-sphere of C and C' if and only if $\mathbf{x} \circ \mathbf{c} = 0$, where $\mathbf{x}$ and $\mathbf{c}$ are the lifts of x and C, respectively.

Theorem 6.21. *Inversion of Lorentz space $\mathbb{R}^{n+1,1}$ in the lift of the common*

boundary of C and C' is given by the linear transformation

$$\rho(\mathbf{u}) = \mathbf{u} - 2(\mathbf{c} \circ \mathbf{u})\mathbf{c}.$$

Proof. Note that

$$\rho(\mathbf{u}) \circ \rho(\mathbf{v}) = [\mathbf{u} - 2(\mathbf{c} \circ \mathbf{u})\mathbf{c}] \circ [\mathbf{v} - 2((\mathbf{c} \circ \mathbf{v})\mathbf{c}]$$

$$= \mathbf{u} \circ \mathbf{v} - 2(\mathbf{c} \circ \mathbf{u})(\mathbf{c} \circ \mathbf{v}) - 2(\mathbf{c} \circ \mathbf{v})(\mathbf{c} \circ \mathbf{u}) + 4(\mathbf{c} \circ \mathbf{u})(\mathbf{c} \circ \mathbf{v})(\mathbf{c} \circ \mathbf{c})$$

$$= \mathbf{u} \circ \mathbf{v}.$$

Thus, ρ preserves the bilinear form and hence maps caps to caps. Consider two cases now:

<u>Case 1</u>: $c_{n+2} = 0$. Then, $\mathbf{c} = (c, 0)$ because $\cot \theta = 0$ implies that $\theta = \pi/2$ and that $\csc \theta = 1$. Furthermore, the cap C is bounded by an equatorial $(n-1)$-sphere and c is the unit normal to that Euclidean n-plane, reflection in which induces inversion in the common boundary of C and $-C$. Now, let $\mathbf{x} = (x, 1)$ be the lift of a point on S^n. Now,

$$\rho(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x} \circ \mathbf{c})\mathbf{c}$$

$$\Rightarrow \rho(x, 1) = (x, 1) - 2[(x, 1) \circ (c, 0)](c, 0)$$

$$= (x, 1) - 2(x \cdot c)(c, 0) = (x - 2(x \cdot c)c, 1).$$

Therefore, we see that every point x on S^n is taken to point $x' = x - 2(x \cdot c)c$

which is nothing but the definition of reflection in an n-flat through the origin with unit normal c.

<u>Case 2</u>: $c_{n+2} \neq 0$. Then, $\mathbf{c} = ((\csc\theta)c, \cot\theta)$ and let $x_\sigma = (\sec\theta)c \in \mathbb{E}^{n+1}$. Note that x_σ is that very point outside S^n, all of whose tangents to S^n touch S^n along the common boundary of C and $-C$. Also, this is point of concurrence of all line segments joining each point of S^n to its image under inversion in the common boundary of C and $-C$. We also notice that this common boundary is the set $\{x \in S^n : x \cdot c = \cos\theta\}$. Inversion herein is the restriction to S^n of inversion in the n-sphere in $\mathbb{E}^{n+1}$ centered at $x_\sigma = (\sec\theta)c$ with radius $\tan\theta$. Using these observations, we compute the image of point x under inversion in σ as follows:

$$\bar{\sigma}(x) = x_\sigma + \frac{\tan^2\theta}{|x - x_\sigma|^2}(x - x_\sigma)$$

$$= \lambda x + (1 - \lambda)x_\sigma,$$

where $\lambda = \frac{\tan^2\theta}{|x-x_\sigma|^2}$. Therefore, in $(n+2)$ coordinates, we have,

$$\rho(x, 1) = (x, 1) - 2[((\csc\theta)c, \cot\theta) \circ (x, 1)]((\csc\theta)c, \cot\theta).$$

After some manipulation, we get

$$\rho(x, 1) = \lambda^{-1}(\lambda x + (1 - \lambda)x_\sigma, 1).$$

As $\rho(\mathbf{x}) \circ \rho(\mathbf{x}) = \mathbf{x} \circ \mathbf{x} = 0$, we have that $\rho(\mathbf{x})$ is the lift of a point and this point on S^n lies on the line joining x_σ and x. Thus, it is just the required image of x under inversion in the boundary of C and C'.

$\square$

Remark 6.22. This theorem yields a beautiful geometric picture. The cap C in S^n lifts to a space-like vector $\mathbf{c}$ of $\mathbb{R}^{n+1,1}$ such that $\|\mathbf{c}\| = 1$. We note that the subspace $P_{\mathbf{c}} = \{\mathbf{w} \in \mathbb{R}^{n+1,1} : \mathbf{w} \circ \mathbf{c} = 0\}$ is an $(n+1)$-dimensional subspace of $\mathbb{R}^{n+2}$ that intersects the light-cone in the set $B_{\mathbf{c}} = \{\mathbf{x} \in P_{\mathbf{c}} : \|\mathbf{x}\| = 0\}$. $B_{\mathbf{c}}$ consists exactly of the lifts of those points $x \in S^n$ that constitute the boundary of the cap C.

Remark 6.23. We notice that a Möbius transformation of S^n is a product of reflections in $(n-1)$-spheres on S^n i.e. a product of inversions in the boundaries of n-caps on S^n. These extend via the Poincaré extension to inversions in $\mathbb{R}^{n+1}$ that preserve the unit ball $\mathbb{B}^{n+1}$. In fact, these are just reflections in hyperplanes of hyperbolic space B^{n+1} and any isometry of B^{n+1} may be written as a product of reflections in these hyperplanes.

Now, the inversion in the boundary of an n-cap C is given by Lorentz reflection in the $(n + 1)$-dimensional subspace $P_{\mathbf{c}}(= \langle \mathbf{c} \rangle^L)$. The intersection of each such subspace with the positive sheet of F^{n+1} is a hyperplane of hyperbolic space in the hyperboloid model H^{n+1}. Thus, when restricted to H^{n+1}, each of these transformations restricts to reflection in a hyperplane of H^{n+1} and is an isometry of H^{n+1}. Furthermore, any isometry of H^{n+1} may be written as a product of such reflections.

Corollary 6.24. *The n-dimensional Möbius group $\mathcal{M}_n$ is isomorphic to a subgroup of the $(n+2)$-dimensional positive Lorentz group.*

Proof. Each inversion in an $(n-1)$-sphere of S^n is given by a Lorentz reflection

in the $(n+1)$-dimensional subspace $\langle \mathbf{c} \rangle^L$. This reflection is given by $\rho_{\mathbf{c}}(\mathbf{u}) = \mathbf{u} - 2(\mathbf{u} \circ \mathbf{c})\mathbf{c}$, where $\mathbf{c}$ is the lift of the cap $C \subset S^n$ bounded by σ.

Now, $\mathcal{M}_n$ is generated by all such inversions in S^n. By definition, the map $\bar{\sigma} \mapsto \rho_{\mathbf{c}}$ is a homomorphism. Therefore, the group generated by all Lorentz reflections is isomorphic to $\mathcal{M}_n$. Moreover, it is clear from the proof of the theorem that any such Lorentz reflection is a positive Lorentz transformation. Thus, the group generated by Lorentz reflections, LR, is a subgroup of $PO(n+1, 1)$.

$\square$

Lemma 6.25. *It is possible to have a set of $n+2$ n-caps on the n-sphere such that any two of them are externally tangent.*

Proof. Consider $\hat{\mathbb{E}}^n$. Let C_{n+1} and C_{n+2} be any two $(n-1)$-spheres tangent at ∞ such that the Euclidean distance between them in $\mathbb{E}^n$ is $2r$. Now, consider the unique $(n-1)$-sphere through ∞, D, whose Euclidean distance in $\mathbb{E}^n$ from each one of C_{n+1} and C_{n+2} is r. Choose any regular $(n-1)$-simplex $\{x_1, \ldots, x_n\}$ in D such that $d_E(x_i, x_j) = 2r$. Then, let C_i be the $(n-1)$-sphere centered at x_i with radius r. Notice that the set $\{C_1, \ldots, C_{n+2}\}$ has the property that any two spheres (or planes) are tangent. Moving back to S^n by taking ∞ to any convenient point on S^n, we have the desired result.

$\square$

Definition 6.26. An ordered set of $n+2$ n-caps, any two of which are externally tangent, is called a *cluster*.

Remark 6.27. If $\mathbf{c} = ((\csc\theta)c, \cot\theta)$ and $\mathbf{d} = ((\csc\psi)d, \cot\psi)$ represent be

the respective lifts of externally tangent n-caps, then $c \cdot d = \cos(\theta + \psi)$ and $\mathbf{c} \circ \mathbf{d} = -1$. The caps in a cluster $\mathbf{Cl} = (\mathbf{c_1}, \ldots \mathbf{c_{n+2}})$ therefore satisfy

$$c_{ij} = \mathbf{c_i} \circ \mathbf{c_j} = \begin{cases} +1 & \text{if } i = j \\ -1 & \text{if } i \neq j \end{cases}$$

The cluster matrix $M = (c_{ij}) = 2I - J$, where I is the $(n+2) \times (n+2)$ identity matrix and J is the $(n+2) \times (n+2)$ matrix of 1's. We note that $J^2 = (n+2)J$ and that $\frac{1}{2}[I - (1/n)J] = M^{-1}$. Thus, M is nonsingular and in turn the vectors in $\mathbf{Cl}$ are linearly independent and form a basis for $\mathbb{E}^{n+2}$.

Remark 6.28. We claim that an element of $PO(n+1,1)$ is determined by its action on a given cluster $\mathbf{Cl}$. To see this, first we observe that a positive Lorentz transformation takes clusters to clusters. Also, we notice that the vectors in a cluster form a basis for $\mathbb{R}^{n+2}$. Thus, given clusters $\mathbf{Cl}$ and $\mathbf{Cl'}$, there is a unique linear transformation that takes $\mathbf{Cl}$ to $\mathbf{Cl'}$. Moreover, this transformation must be Lorentz since it must preserve the Lorentzian inner product and it must be positive since it maps points to points.

6.3   Euclidean Isometries and Similarities

We conduct a survey of Euclidean isometries and Euclidean similarities in this section as these are central to the study of the Möbius group, a fact that will be proved later. The definitions in this section may be found in [11].

Definition 6.29. An *isometry* is a mapping $f : \mathbb{E}^n \to \mathbb{E}^n$ such that $|f(x) - f(y)| = |x - y|$.

We shall see that any isometry can be written as a product of at most $n + 1$ reflections, and so the isometries form a subgroup of $\mathcal{M}_n$.

Definition 6.30. A *similarity* with scale factor $k > 0$ is a mapping $g : \mathbb{E}^n \to \mathbb{E}^n$ such that $|g(x) - g(y)| = k\,|x - y|$.

The similarity g can be written $g = fd$ where f is an isometry and d is the *dilation* $x \mapsto kx$. Moreover, d can be written as the product of two inversions $p : x \mapsto (k_1/|x|)^2 x$ and $q : x \mapsto (k_2/|x|)^2 x$ such that $\frac{k_1}{k_2} = \sqrt{k}$. Thus $g \in \mathcal{M}_n$.

Computation:

$$pq(x) = p(q(x)) = p\left(\left(\frac{k_2}{|x|}\right)^2 x\right) = p\left(\frac{k_2{}^2 x}{|x|^2}\right) = \frac{k_1{}^2}{\left|\frac{k_2{}^2 x}{|x|^2}\right|^2} \cdot \frac{k_2{}^2 x}{|x|^2}$$

$$= k_1{}^2 k_2{}^2 x \div \frac{k_2{}^4 |x|^2 |x|^2}{|x|^4} = \frac{k_1{}^2}{k_2{}^2} x = kx.$$

We shall see that similarities constitute the subgroup of $\mathcal{M}_n$ which stabilizes ∞.

Definition 6.31. Two subsets $S = \{x_i\}$ $(i \in I)$ and $S' = \{x'_i\}$ $(i \in I)$ of $\mathbb{E}^n$ are said to be similar (resp. congruent) if there is a constant $k > 0$ (resp. $(k = 1)$) such that $|x_i - x_j| = k|x'_i - x'_j|$ for all $i, j \in I$.

Lemma 6.32. *If $S = \{x_i\}$ $(i = 1, ..., m)$ and $S' = \{x'_i\}$ $(i = 1, ..., m)$ are congruent m-point subsets of $\mathbb{E}^n$, then there is an isometry $f : \mathbb{E}^n \to \mathbb{E}^n$ such that $f(S) = S'$ and f can be written as the product of at most m reflections.*

Proof. We proceed by induction on the number of points k, in S and S'. If $k = 1$, then $S = \{x\}$ and $S' = \{x'\}$ and there is an isometry f_1, namely

reflection in the $(n-1)$-plane midway between x and x' such that $f_1(x) = x'$. We assume that the lemma is true if S and S' each have k points. Now, let S and S' have $k+1$ points each. Therefore, $S = \{x_1, \ldots, x_{k+1}\}$ and $S' = \{x'_1, \ldots, x'_{k+1}\}$. Then, consider the sets $S_k = \{x_1, \ldots, x_k\}$ and $S'_k = \{x'_1, \ldots, x'_k\}$. By the induction hypothesis, there exists an isometry $f_k : \mathbb{E}^n \to \mathbb{E}^n$ such that for each $i = 1, \ldots, k$, $f_k(x_i) = x'_i$ and $f_k = \eta_1 \ldots \eta_r$ where each η_i is reflection in an $(n-1)$-plane and $r \leq k$. If $f_k(x_{k+1}) = x'_{k+1}$, then we are done. If not, then let us note that

$$\left| x'_i - x'_{k+1} \right| = |x_i - x_{k+1}| = |f_k(x_i) - f_k(x_{k+1})| = |x'_i - f_k(x_{k+1})|.$$

Furthermore, the locus of all points equidistant from x'_{k+1} and $f_k(x_{k+1})$ is an $(n-1)$-plane P, that contains the points x_i, $(i = 1, \ldots, k)$. Let η_{r+1} be the reflection in P. Then η_{r+1} fixes each point x_i, $(i = 1, \ldots, k)$. Define the map $f_{k+1} = f_k \eta_{r+1}$. Then, for each $i = 1, \ldots, k+1$, we have $f_{k+1}(x_i) = x'_i$ and that $f_{k+1} = \eta_1 \ldots \eta_{r+1}$ where $r + 1 \leq k+1$.

$\square$

Definition 6.33. An *n-simplex* $S = \{x_i\}$ $(i = 1, \ldots, n+1)$ is a set of $n+1$ points of $\mathbb{E}^n$ which do not lie on an $(n-1)$-plane.

Note that any set congruent to an n-simplex is also an n-simplex and that isometries map n-simplices to congruent n-simplices.

Lemma 6.34. *An isometry* $f : \mathbb{E}^n \to \mathbb{E}^n$ *is determined by its effect on an arbitrary n-simplex. Moreover, it can be written as the product of at most $n+1$ reflections.*

The lemma says that there cannot be two distinct isometries whose respective restrictions to the n-simplex are equal.

Proof. Let S be an n-simplex and f and g be two isometries such that $f(x_i) = g(x_i) = x_i' \ \forall \ x_i \in S$. We will show that $f(x) = g(x) \ \forall \ x \in \mathbb{E}^n$. If not, then there is a point $c \in \mathbb{E}^n$ such that $f(c) \neq g(c)$. Then for each $i = 1, \ldots, n + 1$,

$$|f(c) - x_i'| = |f(c) - f(x_i)| = |c - x_i| = |g(c) - g(x_i)| = |g(c) - x_i'|.$$

The set of points that are equidistant from each of $f(c)$ and $g(c)$ form an $(n-1)$-plane P of $\mathbb{E}^n$ and by our observation above, $f(S) = g(S) \subset P$. This contradicts the fact that $f(S) = g(S)$ is an n-simplex.

Thus, f is determined by its action on an n-simplex and since an n-simplex has $n + 1$ points, we may use Lemma 6.32 to show that f may be written as the product of at most $n + 1$ reflections.

$\square$

Corollary 6.35. *The group of isometries of $\mathbb{E}^n$ is uniquely transitive on any class of mutually congruent n-simplices. In other words, given two congruent n-simplices S and S', there is a unique isometry f such that $f(S) = S'$.*

Lemma 6.36. *The similarities of $\mathbb{E}^n$ form a group $\mathcal{S}$. The mapping $\Phi : \mathcal{S} \to \mathbb{R}^+$ such that $\Phi(g) = k$ taking every similarity g to its scale factor k is a homomorphism whose kernel is the isometries. The similarities are uniquely transitive on any class of mutually similar n-simplices.*

Proof. Clearly, $\mathcal{S}$ is a group since any similarity is the composition of an isometry and a dilation. Also, it is quite clear that the kernel of Φ is the isometries as 1 is the identity element of $\mathbb{R}^+$ under the operation of multiplication. This leaves the question of unique transitivity.

Let S and S' be similar n-simplices. Then, there is a dilation d such that $d(S)$ and S' are congruent. Then, by corollary 6.35 above, we have a unique isometry f such that $f(d(S)) = S'$ and then clearly $g = fd$ is a similarity taking S to S'. Now, g is unique because if there is another similarity h such that $h(S) = S'$, then $g^{-1}h(S) = S$. Since $g^{-1}h$ is a similarity that fixes an n-simplex, it is in fact an isometry that fixes an n-simplex. Thus, $g^{-1}h$ is the identity or $g = h$.

$\square$

Lemma 6.37. *A proper similarity (one where the scale factor $k \neq 1$) can be written as the commuting product of an isometry with a fixed point and a dilation with the same fixed point.*

Proof. Let g be a proper similarity. As $k \neq 1$ we have that either g or g^{-1} is a contraction mapping and by the Contraction mapping theorem [8], it has a unique fixed point x_0 (note that g and g^{-1} share this fixed point). Let $d(x) = k(x - x_0) + x_0$ be the dilation with the same fixed point and scale factor. Then $gd^{-1} = f$ is an isometry with fixed point x_0. Then $g = fd$ and also $fd = df$ by factoring f and d into their respective inversions (in flats through x_0 and in spheres centered at x_0 respectively) and noticing that these particular inversions commute.

$\square$

6.4   The Complete Description of the Möbius Group

Lemma 6.38. *Let* **Cl** $= (C_1, \ldots C_{n+2})$ *and* **Cl**$' = (C'_1, \ldots C'_{n+2})$ *be any two clusters on the n-sphere S^n. Then there is a sequence of at most $n + 2$ inversions whose product maps* **Cl** *to* **Cl**$'$.

Proof. Let us consider the situation in $\hat{\mathbb{E}}^n$ with ∞ at the point of contact of C'_{n+1} and C'_{n+2}. So, C'_{n+1} and C'_{n+2} are parallel *half-spaces* and all the other C'_is are congruent n-balls sandwiched between these two half-spaces. The centers of $C'_1, \ldots, C'_n$ are vertices of a regular $(n-1)$-simplex, say $(x'_1, \ldots, x'_n)$ and these points lie in the $(n-1)$-flat that is midway between the boundaries of C'_{n+1} and C'_{n+2}. We complete this $(n-1)$-simplex to an n-simplex $S' = (x'_1, \ldots, x'_{n+1})$ by adding the point of contact of C'_1 and C'_{n+1}.

Let x_0 be the point of contact of C_{n+1} and C_{n+2}. Two cases arise:

Case(i) $x_0 \neq \infty$: Then invert in an $(n-1)$-sphere σ centered at x_0 and control its radius such that the distance between the boundaries of the images $\bar{\sigma}(C_{n+1})$ and $\bar{\sigma}(C_{n+2})$ (which are parallel half-spaces) is equal to that between the boundaries of C'_{n+1} and C'_{n+2}. Then, introduce an n-simplex $S = (x_1, \ldots, x_{n+1})$ just as before. S is clearly congruent to S'. By our earlier work, the isometry that takes S to S' will also take $\bar{\sigma}(\mathcal{C})$ to $\mathcal{C}'$. This isometry costs at most $n + 1$ inversions. Therefore, including $\bar{\sigma}$, we have at most $n + 2$ inversions.

Case (ii) $x_0 = \infty$: We already have that C_{n+1} and C_{n+2} are two parallel half spaces and so again define an n-simplex S which will at worst be similar to S'.

Then, we can dilate S through a transformation which takes C_1 to C_1'. This dilation will take up two inversions but will also make S congruent to S' and will make sure that they have at least one point in common. Now, S can be taken to S' through an isometry that costs at most n inversions because one point is already in common. Again, the total is $n + 2$ inversions.

$\square$

Theorem 6.39. $\mathcal{M}_n$ *is isomorphic to the $n + 2$ dimensional positive Lorentz group, $PO(n+1, 1)$. $\mathcal{M}_n$ is uniquely transitive on clusters and its most general element can be written as the product of at most $n + 2$ inversions.*

Proof. As $\mathcal{M}_n$ is transitive on clusters, so is the subgroup LR, generated by Lorentz reflections. But since $PO(n + 1, 1)$ is uniquely transitive on clusters, we have that $LR = PO(n + 1, 1)$ and that $\mathcal{M}_n \cong LR \cong PO(n + 1, 1)$.

$\square$

Corollary 6.40. *When $\mathcal{M}_n$ acts on $\hat{\mathbb{E}}^n$, its most general element is either a similarity of $\mathbb{E}^n$ or the product of an inversion and an isometry of $\mathbb{E}^n$.*

Proof. Consider a cluster $\mathbf{Cl}'$ with ∞ as the point of contact of C_{n+1}' and C_{n+2}'. Now, any element of $\mathcal{M}_n$ is determined by its action on a cluster. So, let $\mathbf{Cl} = h^{-1}(\mathbf{Cl}')$. Thus, $h(\mathbf{Cl}) = \mathbf{Cl}'$ and following the notation and argument of lemma 6.38, if $x_0 = \infty$, then h is a similarity and if $x_0 \neq \infty$ then h is the product of an inversion and an isometry.

$\square$

Corollary 6.41. *An element of $\mathcal{M}_n$ which fixes a point of S^n can be considered as a Euclidean similarity.*

Proof. If h fixes $x_0 \in S^n$, then consider the action of $\mathcal{M}_n$ on $\hat{\mathbb{E}}^n$ by taking x_0 to ∞. Then h acts as a similarity on $\hat{\mathbb{E}}^n$.

$\square$

Corollary 6.42. *An element of $\mathcal{M}_n$ which acts without fixed points on S^n is conjugate to an isometry of S^n, the extension of whose action to E^{n+1} is an orthogonal transformation.*

Proof. A map $h \in \mathcal{M}_n$ which acts on S^n can be extended via the Poincaré extension to a map $\tilde{h} \in \mathcal{M}_{n+1}$ which acts on $\mathbb{E}^{n+1} \cup \{\infty\}$ and maps the ball $|x| \leq 1$ continuously onto itself. If h has no fixed point, then $\tilde{h}$ has no fixed point on Σ but by the Brouwer fixed point theorem [7], $\tilde{h}$ must have a fixed point x_0 with $|x_0| < 1$.

Let Σ_0 be the n-sphere orthogonal to S^n such that $\bar{\Sigma}_0(x_0) = 0$. Then $\tilde{h}' = \bar{\Sigma}_0 \tilde{h} \bar{\Sigma}_0$ fixes 0 and S^n and hence fixes $\bar{S}^n(0) = \infty$. It follows that $\tilde{h}'$ is a Euclidean isometry which can be factored as the product of reflections in Euclidean n-planes through 0.

Finally, restrict back down to S^n and we get $h' = \bar{\sigma}_0 h \bar{\sigma}_0$ where $\sigma_0 = \sigma_0 \cap S^n$. Then h' factors as the product of reflections in equatorial $(n-1)$ spheres of S^n and is hence an isometry of S^n.

$\square$

6.5 The Product of two Inversions

In this section, we state an important theorem and corollary proved by Wilker [11] which will be used in our analysis in section 6.6.

Definition 6.43. If α and β are $(n-1)$-spheres on S^n, then we define the pencil perpendicular to α and β, $(\alpha, \beta)^{\perp}$ as the set of all circles (1-spheres) in S^n that are orthogonal to both α and β.

Definition 6.44. If α and β are $(n-1)$-spheres on S^n, then we define the pencil of α and β, (α, β) as the set of all $(n-1)$-spheres in S^n that are orthogonal to each circle in $(\alpha, \beta)^{\perp}$.

Theorem 6.45. *Let α and β be $(n-1)$-spheres on S^n. Then the transformation $h = \bar{\alpha}\bar{\beta}$ lies in one of the following conjugacy classes:*

(i) Translation if α and β are tangent,

(ii) Rotation by an angle 2θ if α and β intersect at an angle θ,

(iii) Dilation δ if α and β are disjoint and separated by inversive distance δ.

Corollary 6.46. *[11]*

Let α, β and γ be three $(n-1)$-spheres in a pencil on Σ. Then there is a fourth $(n-1)$-sphere δ in the pencil such that $\bar{\alpha}\bar{\beta}\bar{\gamma} = \bar{\delta}$.

6.6 Classification of Möbius Transformations of $S^n, (n = 2, 3)$

Recall that an arbitrary element of $\mathcal{M}_n$ may be written as the product of at most $n + 2$ inversions. If it has a fixed point, it is conjugate to a Euclidean similarity and if not, then it is conjugate to a spherical isometry. Thus, we proceed by listing all possible Euclidean isometries, Euclidean similarities and

finally, spherical isometries. Although we exhaust all possibilities, we are particularly interested in the orientation preserving cases (products of an even number of reflections).

Remark 6.47. In the second case, the transformation is necessarily elliptic as a hyperbolic isometry in dimension $n+1$ because $\mathbb{S}^n$ or Σ is the boundary of $\mathbb{B}^{n+1}$. Now, by the Brouwer fixed point theorem [7], the Poincaré extension of such an element must have a fixed point in the interior of the ball $\mathbb{B}^{n+1}$.

Remark 6.48. Additionally, spherical isometries can be factored as the product of reflections in at most $n+1$ equatorial $(n-1)$-spheres. This is because in the proof of Corollary 6.42, we move up to $\mathbb{E}^{n+1}$ via Poincaré extension. There, we have that the extension is a Euclidean isometry with fixed point 0. Therefore, one of the points in the $(n+1)$-simplex we use to determine this isometry is already on the mirror. That reduces the required reflections from $n+2$ to $n+1$ in number.

Let us begin by analyzing all possible Möbius transformations of S^2:

Case 1) *Euclidean isometries:* These are products of reflections in at most three straight lines in $\mathbb{E}^2$.

Reflection in one line yields just ordinary reflection.

Reflection in two lines throws up two distinct possibilities. If the lines intersect at angle θ, then the resultant transformation is a rotation of angle 2θ about the point of intersection. This extends to an *elliptic* transformation as its

Poincaré extension will fix a hyperbolic line in $\mathbb{H}^3$. If the lines are parallel and the Euclidean distance between them is d, then the resultant transformation is a Euclidean translation of translation length $2d$. This extends to a *parabolic* transformation as it fixes no point in $\mathbb{H}^3$ and only one point (∞ in this case) on the boundary of $\mathbb{H}^3 = \hat{\mathbb{E}}^2$.

If the transformation requires three reflections, then the lines can neither be concurrent nor parallel. Let the three lines be l_1, l_2 and l_3 respectively. Now the transformation is $g = \bar{l}_1\bar{l}_2\bar{l}_3$. Note that $\bar{l}_1\bar{l}_2 = \bar{l}'_1\bar{l}'_2$ where $l'_2 \perp l_3$ and that $\bar{l}'_2\bar{l}_3 = \bar{l}''_2\bar{l}'_3$ where $l'_3 \perp l'_1$. Therefore, $g = \bar{l}'_1\bar{l}''_2\bar{l}'_3$ is a *glide-reflection* of $\mathbb{E}^2$ (the product of a translation and a reflection).

Case 2) *Euclidean Similarities:* A similarity is the commuting product of an isometry with a fixed point and a dilation with the same fixed point. The only isometries with a fixed point (in $\mathbb{E}^2$) are the ordinary reflection and the rotation. Thus, there are three types of similarities.

The first is dilation, which extends to a *pure hyperbolic* transformation acting on $\mathbb{H}^3$ since it fixes two points on the boundary of $\mathbb{H}^3$ (0 and ∞ in this case) and no other points in $\hat{\mathbb{E}}^2$ or in $\mathbb{H}^3$.

The second is the product of a dilation and a reflection.

The third is the product of a dilation and a rotation with the same fixed point. This third type of transformation extends to a *pure loxodromic* transformation of $\mathbb{H}^3$ as it fixes only two points on the boundary $\hat{\mathbb{E}}^2$ and no others. Note that

the difference between the *pure hyperbolic* and *pure loxodromic* transformations is that although each of them fix two points of $\hat{\mathbb{E}}^2$ and no others, the pure loxodromic transformation also has a non-trivial rotational element.

Case 3) *Spherical Isometries:* If the transformation does not have a fixed point in $\hat{\mathbb{E}}^2$, it is conjugate to a spherical isometry of $S^2 \subset \mathbb{E}^3$ This transformation is the product of at most three inversions in great circles on S^2. The products of reflection in one or two circles must fix a point on S^2, thus eliminating these from consideration. The product of three inversions may be rearranged as in the case of the glide-reflection, hence yielding the product of a rotation (as two great circles must intersect twice) and a reflection.

We now turn our attention to the Möbius transformations of S^3:

Case 1) *Euclidean Isometries:* These are the products of reflections in at most four 2-planes.

Again, reflection in one 2-plane is ordinary reflection.

The product of reflection in two 2-planes yields the same cases as in $\mathbb{E}^2$, namely, the elliptic case i.e. rotation about a line in $\mathbb{E}^3$ that is fixed pointwise by the transformation and the parabolic case, which we now rename *pure parabolic*, that is translation that does not fix any point in $\mathbb{E}^3$. Here, we will refer to the elliptic transformation as *elliptic of type I*.

If the transformation requires three reflections in 2-planes, then they can nei-
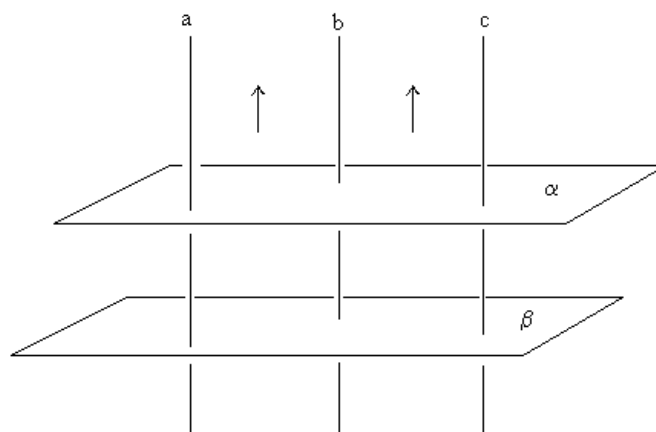
Figure 3: **A Pure Parabolic transformation in $\mathbb{E}^3$ as the product of reflection in two Euclidean 2-planes.**
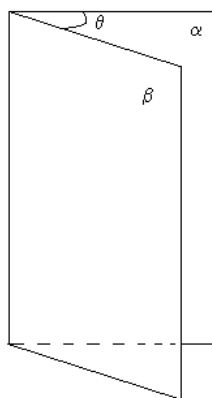

Figure 4: **An Elliptic element of type I in $\mathbb{E}^3$ as the product of reflection in two intersecting Euclidean 2-planes.**

ther intersect in a line nor be parallel. Now, we have two possibilities. If the line of intersection of two of the planes is parallel to the third plane, we may rearrange the transformation as in $\mathbb{E}^2$, thus yielding *glide-reflection*. If the third plane intersects this line of intersection of the first two planes in a point, we have the product of rotation about a line and a reflection.

If the transformation g actually requires four reflections, then it cannot have a fixed point. This is clear form the proof of the fact that every isometry can be written as the product of $n+1$ reflections in $(n-1)$-planes. If an isometry has a fixed point, then this point is already in the desired position and hence, the required number of reflections reduces from $n+1$ to just n. Thus, g does not have a fixed point in $\mathbb{E}^3$. Now, choose a point o and a translation t such that $gt(o) = o$. Now, since gt is orientation preserving with a fixed point in $\mathbb{E}^3$, it must be a rotation, say $gt = r$. Thus, $g = rt^{-1}$ where t^{-1} has two translational components at most, t_1 which is translation in a direction perpendicular to the axis of rotation of r and t_2 which is translation along the axis of rotation of r. Thus, $g = rt_1t_2$. Now, note that $rt_1 = r_1$ is just rotation about an axis which is parallel to the axis of r. Thus, $g = r_1t_2$, the product of rotation about a line and translation along that line. We notice that this transformation extends to a parabolic transformation acting on $\mathbb{H}^4$ as it fixes just one point in $\hat{\mathbb{E}}^3$ and no other. But we note that this transformation has a non-trivial rotational element and we will thus call it *screw-parabolic*.

Case 2) *Euclidean Similarities:* Again, the isometries with a fixed point are reflection, rotation and the product of rotation and reflection. Thus, there are
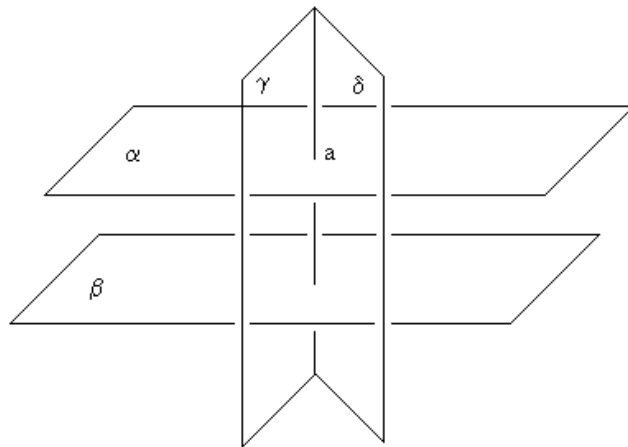
Figure 5: **A Screw Parabolic transformation in $\mathbb{E}^3$ as the product of reflection in four Euclidean 2-planes.**

four classes of similarities.

The first is dilation, which extends to a *pure hyperbolic* transformation acing on $\mathbb{H}^4$ since it fixes two points on the boundary of $\mathbb{H}^4$ (0 and ∞ in this case) and no other points in $\hat{\mathbb{E}}^3$ or in $\mathbb{H}^4$.

The second is the product of a dilation and a reflection.

The third is the product of a dilation and a rotation whose axis passes through the fixed points of the dilation. This third type of transformation extends to a *pure loxodromic* transformation of $\mathbb{H}^4$ as it fixes only two points on the boundary $\hat{\mathbb{E}}^3$ (0 and ∞ in this case) and no others. Again, this pure loxodromic transformation has a non-trivial rotational element.

The fourth is the product of a dilation, a rotation and a reflection.
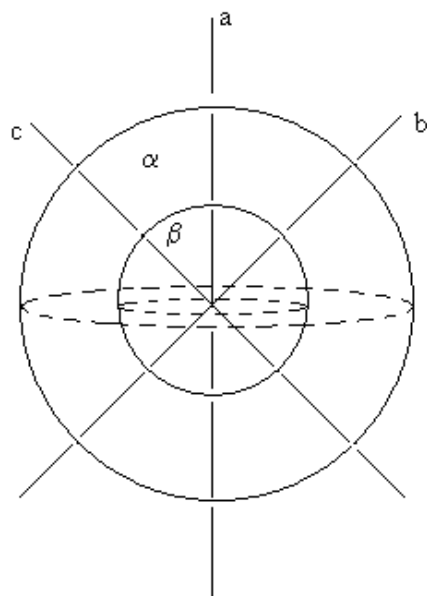
Figure 6: **A Pure Hyperbolic transformation in $\mathbb{E}^3$ as the product of reflection in two concentric 2-spheres.**
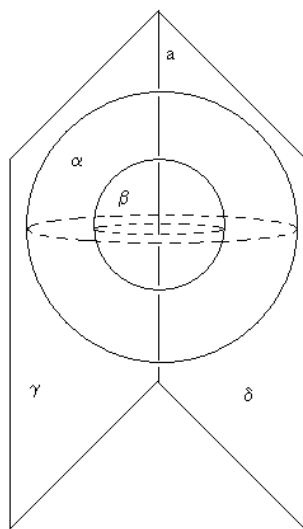

Figure 7: **A Pure Loxodromic transformation in $\mathbb{E}^3$ as the product of reflection in two concentric 2-spheres and two Euclidean 2-planes through the two fixed points.**

Case 3) *Spherical Isometries:* These are the products of inversion in at most four equatorial 2-spheres on $S^3 \subset \mathbb{E}^4$. Now, products of inversion in one or two equatorial 2-spheres clearly have fixed points on S^3. In fact, by lemma 6.49, so does the product of inversion in three equatorial 2-spheres.

Lemma 6.49. *Two equatorial $(n-1)$-spheres on the surface of $\mathbb{S}^n$ must intersect in an $(n-2)$-sphere. Moreover, m equatorial $(n-1)$-spheres on the surface of $\mathbb{S}^n$ intersect in a k-sphere where $k \geq n - m$.*

Proof. An equatorial $(n-1)$-sphere on $\mathbb{S}^n$ arises as the intersection of a Euclidean n-plane through the origin with $\mathbb{S}^n$. Two such Euclidean n-planes must intersect in a Euclidean $(n-1)$-plane through the origin. This Euclidean $(n-1)$-plane must intersect $\mathbb{S}^n$ in a $(n-2)$-sphere which turns out to be the intersection of the equatorial $(n-1)$-spheres. Now, if we consider the intersection of three equatorial $(n-1)$-spheres, note that the Euclidean n-planes whose intersections with $\mathbb{S}^n$ give rise to them must have a Euclidean l-plane through the origin as their intersection where $l \geq n - 2$. Thus, the intersection of this l-plane with $\mathbb{S}^n$ must be an $(l-1)$-sphere which constitutes the intersection of the three equatorial $(n-1)$-spheres. One can use an inductive argument continuing in this way to show that the intersection of m equatorial $(n-1)$-spheres on the surface of $\mathbb{S}^n$ is a k-sphere for $k \geq n - m$. $\square$

Now, if this transformation does not fix a point on S^3, then it must be the product of inversion in four equatorial 2-spheres, $\sigma_1, \sigma_2, \sigma_3$ and σ_4 such that $\cap_{i=1}^4 \sigma_i = \phi$. Move one of the points $\sigma_1 \cap \sigma_2 \cap \sigma_3$ to ∞ via stereographic

projection. Then, in $\hat{\mathbb{E}}^3$, we have three Euclidean 2-planes, say α, β and γ such that they intersect in just one point in $\mathbb{E}^3$ and a 2-sphere δ that intersects each of α, β and γ in a circle but again the intersection of all four is empty. Thus, this transformation is the composition of two rotations, one about the circle $\alpha \cap \beta$ through ∞ and one about the circle $\gamma \cap \delta$, not through ∞. We notice that this transformation extends to an elliptic transformation acting on $\mathbb{H}^4$ as it must fix a point of $\mathbb{H}^4$ by the Brouwer fixed point theorem [7]. We shall refer to this transformation as *elliptic of type II*.

We are now ready to define a half-turn in S^3.

Definition 6.50. A *half turn* about a circle C in S^3 is the composition of reflection in two orthogonal 2-spheres, S_C and T_C, such that $S_C \cap T_C = C$.

Remark 6.51. We notice that there are two types of elliptic element in $\mathcal{M}_3$ and remark that our definition of half-turn cannot be restated as in dimension three, i.e. we cannot say simply that a half-turn is an elliptic element of order two. For example, the antipodal map in $\mathbb{E}^4$ is an elliptic element in the ball model hyperbolic four-space B^4. It is of order two but cannot be written as the product of two reflections (it is the product of four reflections and no less). However, we may rephrase our definition as follows.

Definition 6.52. A *half turn* of $\mathbb{H}^4$ is an elliptic type I element of order two. This isometry fixes a hyperbolic 2-plane in $\mathbb{H}^4$ pointwise.

Theorem 6.53. *Every orientation-preserving transformation in $\mathcal{M}_3$ can be written as the product of half-turns about two circles.*

Proof. Let g represent the transformation under consideration. We proceed with a case by case analysis:

Case 1) g *is pure parabolic:* If the transformation g is parabolic and not screw-parabolic, then we may assume that the fixed point is ∞. Then $g = \bar{\alpha}'\bar{\beta}'$ where α' and β' are 2-spheres tangent at ∞. Let c be any circle that is tangent to α' at ∞ and let α be the 2-sphere through c tangent to α' at ∞. Then the half-turn about c may be written as $\bar{\alpha}\bar{\gamma}$ where γ is the sphere through c which is orthogonal to α. Also, $g = \bar{\alpha}\bar{\beta}$ where β is a 2-sphere tangent to α at ∞. Then, let $b = \gamma \cap \beta$. The half-turn about b is $\bar{\gamma}\bar{\beta}$. Therefore, the product of these half-turns is $\bar{\alpha}\bar{\gamma}\bar{\gamma}\bar{\beta} = \bar{\alpha}\bar{\beta} = g$.

Case 2) g *is screw-parabolic:* Once again, let us consider the case where the fixed point is at ∞. Notice that there is a unique circle a through ∞ that is left invariant by g. Then $g = \bar{\alpha}'\bar{\beta}'\bar{\gamma}'\bar{\delta}'$ where α and β are 2-spheres orthogonal to a and are tangent to each other at ∞ whereas γ and δ are (distinct) 2-spheres through a. Let c be any circle through ∞ which is orthogonal to a. Then, the half-turn about c may be written as $\bar{\alpha}\bar{\gamma}$ where α is the 2-sphere through c orthogonal to a and γ is a 2-sphere through a (and is clearly orthogonal to α). Then, we observe that the rotational element of g may be written as $\bar{\gamma}\bar{\delta}$ where δ is another 2-sphere through a whereas the translational element of g may be written as $\bar{\alpha}\bar{\beta}$ where $\beta(\neq \alpha)$ is a 2-sphere orthogonal to a and tangent to α at ∞. Let $b = \beta \cap \delta$. Then, the half-turn about b is $\bar{\beta}\bar{\delta}$ and also $\bar{\alpha}'\bar{\beta}' = \bar{\alpha}\bar{\beta}$ and $\bar{\gamma}'\bar{\delta}' = \bar{\gamma}\bar{\delta}$. Therefore, the product of half-turns about b and c is $\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta} = g$.
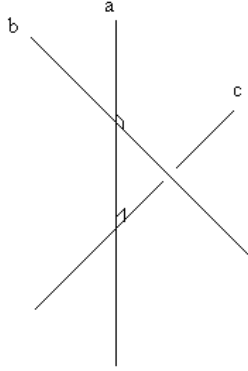
Figure 8: **A screw parabolic transformation as the product of half turns about two circles through the fixed point and orthogonal to axis a.**

Case 3) *g is pure hyperbolic:* We may assume that one of the fixed points are ∞ and 0. Then, $g = \bar{\alpha}'\bar{\beta}'$ where α' and β' are concentric spheres (not through ∞) centered at 0. Let c be any circle centered at 0 (and not through ∞). Then the half-turn about c can be written as $\bar{\alpha}\bar{\gamma}$ where α is the 2-sphere through c centered at 0 and γ is a 2 sphere through c, ∞ and 0 and orthogonal to α. Now, $g = \bar{\alpha}\bar{\beta}$ where β is a 2-sphere also centered at 0. Let $b = \beta \cap \gamma$. Then, the half-turn about b is $\bar{\gamma}\bar{\beta}$. Thus, the product of the two half-turns about b and c is $\bar{\alpha}\bar{\gamma}\bar{\gamma}\bar{\beta} = \bar{\alpha}\bar{\beta} = g$.

Case 4) *g is pure loxodromic:* As in Case 3), assume that the fixed points are at ∞ and 0. We note that in this case (as in the screw-parabolic case), there is a unique circle a through 0 and ∞, such that a is left invariant by g. Then, $g = \bar{\alpha}'\bar{\beta}'\bar{\gamma}'\bar{\delta}'$ where α' and β' are concentric 2-spheres centered at 0 whereas
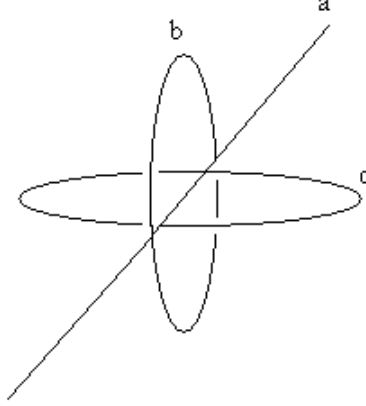
Figure 9: **A loxodromic as the product of half turns about two concentric circles through the fixed points and orthogonal to axis a.**

γ' and δ' are (distinct) 2-spheres through a and are orthogonal to each of α' and β'. Let c be any circle (not through ∞) centered at 0 and intersecting a twice at right angles. Then, the half-turn about c may be written as $\bar{\alpha}\bar{\gamma}$ where α is the 2-sphere through c centered at 0 and γ is the 2-sphere through c, ∞ and 0 (and consequently through a) and orthogonal to α. Now, note that the rotational element of g may be written as $\bar{\gamma}\bar{\delta}$ where δ is another 2-sphere through a while the dilation element of g may be written as $\bar{\alpha}\bar{\beta}$ where β is another 2-sphere (not through ∞) centered at 0 and orthogonal to a. Then, let $b = \beta \cap \delta$. The half-turn about b is then $\bar{\beta}\bar{\delta}$ and we have $\bar{\alpha}'\bar{\gamma}' = \bar{\alpha}\bar{\gamma}$ and $\bar{\beta}'\bar{\gamma}' = \bar{\beta}\bar{\gamma}$. Thus, the product of half-turns about b and c is $\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta} = g$.

Case 5) *g is elliptic of type I:* We may assume that one of the fixed points is ∞. Here, there is a unique circle a through ∞ that is fixed by g pointwise. Now,
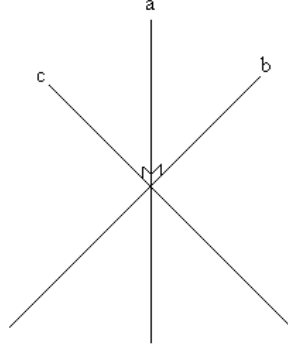
Figure 10: **An elliptic type I element as the product of half-turns in circles orthogonal to** a

$g = \bar{\alpha}'\bar{\beta}'$ where α' and β' are (distinct) 2-spheres such that $\alpha' \cap \beta' = a$. Let c be a circle through ∞ which intersects a orthogonally. Then the half-turn about c may be written as $\bar{\alpha}\bar{\gamma}$ where α is the 2-sphere determined by a and c and γ is the sphere through c orthogonal to α. We note that $g = \bar{\alpha}\bar{\beta}$ where β is a 2-sphere through a orthogonal to γ. Let $b = \beta \cap \gamma$. Then then half-turn about b may be written as $\bar{\gamma}\bar{\beta}$. Therefore, the product of the half-turns about c and b is $\bar{\alpha}\bar{\gamma}\bar{\gamma}\bar{\beta} = \bar{\alpha}\bar{\beta} = g$.

Case 6) *g is elliptic of type II:* We cannot improve upon Wilker's proof of this case and refer to [11] for the proof. We note that in this case, g may be written as the product of half-turns about circles b and c, where b and c are interlocked.

$\square$

7 A Discreteness Condition for Subgroups of Isometries of $\mathbb{H}^4$

Using the machinery developed thus far, we extend some of the results given by Gilman [5] to hyperbolic four-space, $\mathbb{H}^4$. We also note that some of the theorems and definitions go through for dimension $n \geq 3$. We begin with a corollary of Theorem 6.53.

Corollary 7.1. *Every orientation preserving isometry of $\mathbb{H}^4$ can be written as the product of half-turns about two hyperbolic 2-planes.*

Proof. We know that the transformations of $\mathcal{M}_3$ extend via the Poincaré extension to the isometries of $\mathbb{H}^4$. The property of preserving or reversing orientation is also preserved. Thus, every half turn must extend to an isometry of $\mathbb{H}^4$ and we have that every isometry of $\mathbb{H}^4$ may be written as the product of two half turns in $\mathbb{H}^4$.

$\square$

Definition 7.2. A *topological group* is a group G that is also a topological space such that the multiplication $(g, h) \mapsto gh$ and inversion $g \mapsto g^{-1}$ in G are continuous functions.

Remark 7.3. The topology of S^n determines a natural topology on $\mathcal{M}_n$, the metric topology defined by the metric

$$d_S(\phi, \psi) = \sup_{x \in S^n} |\phi(x) - \psi(x)|.$$

The group $\mathcal{M}_n$, with this topology is a topological group [9].

Definition 7.4. A *discrete group* is a topological group G all of whose points are open.

Definition 7.5. A subgroup G of $\mathcal{M}_n$ is *elementary* if and only if G has a finite orbit in the closure $\overline{\overline{\mathbb{H}}}^n$.

Definition 7.6. The *end-sphere* of an m-plane in the upper half-space model of $\mathbb{H}^n$ is the set of points where the plane intersects the boundary $\hat{\mathbb{E}}^{n-1}$. The end-sphere is also known as the *horizon* of the m-plane. If $m = 2$, then this set of points is called the *end-circle*. We will refer to the m-plane as the *cap* of its end-sphere.

Remark 7.7. The end sphere of an m-plane in $\mathbb{H}^n$ is either an $(m-1)$-sphere in $\hat{\mathbb{E}}^{n-1}$ or an extended $(m-1)$-plane in $\hat{\mathbb{E}}^{n-1}$.

Definition 7.8. The *half-turn* H_T about an $(n-2)$-plane T of $\mathbb{H}^n$ is defined as the composition of reflections in two orthogonal $(n-1)$-planes whose intersection is T. We refer to T as the *axial plane* of H_T.

Remark 7.9. If C is a circle in $\hat{\mathbb{E}}^3$, let P_C be the 2-plane in $\mathbb{H}^4$ whose horizon is C. Then C is the end-circle of P_C. Let H_{P_C} be the half-turn about the 2-plane P_C in $\mathbb{H}^4$. The hyperbolic plane P_C is the *axial plane* of the half-turn and we call the circle C *the end-circle of the half-turn.*

Theorem 7.10. *A half-turn fixes every hyperplane whose horizon passes through its end-sphere.*

Proof. Let S_C be any $(n-2)$-sphere in $\hat{\mathbb{E}}^{n-1}$ that contains the end-sphere C. Let T_C be the $(n-2)$-plane in $\mathbb{H}^n$ whose horizon in $\hat{\mathbb{E}}^{n-1}$ is C. Let P_{S_C} be

the hyperplane ($(n-1)$-plane) in $\mathbb{H}^n$ whose horizon is S_C. We will denote the half-turn about T_C as H_{T_C}. We must show that $H_{T_C}(P_{S_C}) = P_{S_C}$.

Clearly, H_{T_C} fixes T_C pointwise. Also, T_C lies on P_{S_C}. Let $x \in P_{S_C}, x \notin T_C$. Then let ℓ_x be the line through x which is perpendicular to T_C. Let $p_x = l_x \cap T_C$. Note that $H_{T_C}(\ell_x) = \ell_x$. Now, $H_{T_C}(x)$ is the point on ℓ_x which is the same distance from p_x as x. Since ℓ_x lies on P_{S_C}, $H_{T_C}(x) \in P_{S_C}$ and $H_{T_C}(P_{S_C}) = P_{S_C}$.

$\square$

Definition 7.11. Let $n \geq 3, n \in \mathbb{N}$. Three $(n-3)$-spheres C_A, C_B and C_N have the *non-separating disjoint sphere property*, (the *NSDS property*) if there are three disjoint $(n-2)$-spheres S_{C_A}, S_{C_B} and S_{C_N} in $\hat{\mathbb{E}}^{n-1}$ containing C_A, C_B and C_N respectively such that no one sphere separates the other two.

Definition 7.12. Let $n, m \geq 3, n, m \in \mathbb{N}$. Then the $(n-3)$-spheres $C_{A_1}, C_{A_2}, \ldots C_{A_m}$ in $\hat{\mathbb{E}}^{n-1}$ have the *non-separating disjoint sphere property* (the *NSDS property*), if there exist m disjoint $(n-2)$ spheres $S_{A_1}, S_{A_2}, \ldots S_{A_m}$ in $\hat{\mathbb{E}}^{n-1}$ such that for each $i = 1, \ldots m$, C_{A_i} is contained in S_{A_i} and no one of the $(n-2)$-spheres separates any others.

Definition 7.13. In the definition of NSDS, if we allow the possibility that two or more of the $(n-2)$-spheres are tangent instead of disjoint, the $(n-3)$-spheres have the *non-separating disjoint or tangent sphere property*, (the *NSD/TS property*).

Theorem 7.14. *Let $T_1, T_2, \ldots T_m$ ($m \geq 3$) be the $(n-2)$-planes in $\mathbb{H}^n$ whose horizons on $\hat{\mathbb{E}}^{n-1}$ are $C_1, C_2 \ldots C_m$ respectively. If $C_1, C_2, \ldots C_m$ have the*

NSD/TS property, then the group generated by the half turns

$$\langle H_{T_1}, H_{T_2}, \ldots H_{T_m} \rangle$$

is discrete.

Proof. Let $S_{C_1}, S_{C_2}, \ldots S_{C_m}$ be the $(n-2)$-spheres in $\hat{\mathbb{E}}^{n-1}$ that contain $C_1, C_2 \ldots C_m$ respectively. Then we may apply the Poincaré Polyhedron Theorem [9] to the region bounded by $P_1, P_2 \ldots P_m$, the hyperplanes in $\mathbb{H}^n$ whose horizons are $S_{C_1}, S_{C_2}, \ldots S_{C_m}$ respectively with side pairing transformations $H_{T_1}, H_{T_2}, \ldots H_{T_m}$, respectively. This shows discreteness as the half-turns fix the corresponding hyperplanes by Theorem 7.10.

$\square$

7.1 The Construction in $\mathbb{H}^4$

We make a construction analogous to Gilman's [5] construction of NSDC groups. Note that in $\mathbb{H}^3$, given two isometries with disjoint axes, there exists a common perpendicular hyperbolic line to each of the axes in $\mathbb{H}^3$. This is a unique fact about $\mathbb{H}^3$ that distinguishes it from $\mathbb{H}^4$. The construction in $\mathbb{H}^4$ is as follows.

Let A and B be isometries of $\mathbb{H}^4$ and let $G = \langle A, B \rangle$. We may assume that neither A nor B is elliptic (Remark 7.17). Then, $A = H_{L_A} \cdot H_{M_A}$ where H_{L_A} and H_{M_A} are half-turns about the 2-planes L_A and M_A. Similarly, $B = H_{L_B} \cdot H_{M_B}$. Then, we make the following definition.

Definition 7.15. Let $4G_{LM} = \langle H_{L_A}, H_{M_A}, H_{L_B}, H_{M_B} \rangle$. Then $G = \langle A, B \rangle$ is a subgroup of $4G_{LM}$ and we call $4G_{LM}$ the *associated LM-supergroup* of G. There are many choices of four generator supergroups associated to a given group G and therefore, these are indexed by the choice of axial planes for the associated half-turns.

Theorem 7.16. *Let the end-circles of L_A, M_A, L_B and M_B be l_A, m_a, l_B and m_B respectively. If l_A, m_A, l_B and m_B have the NSDS property, then the group $4G_{LM}$ is discrete. Moreover, G is a subgroup of $4G$ and the discreteness of $4G$ yields the discreteness of G. We call G and $4G_{LM}$ NSDS groups.*

Proof. The result is clear from Theorem 7.14.

$\square$

Remark 7.17. If either A or B is an elliptic isometry of $\mathbb{H}^4$, then $4G$ cannot have the NSDS or the NSD/TS property. Without loss of generality, let A be elliptic. Then A is elliptic type I or elliptic type II. By Theorem 6.53, the restriction of A to $\hat{\mathbb{E}}^3$ can be factored into the product of half-turns about two circles which either touch twice or are interlocked. In either case, there do not exist 2-spheres in $\hat{\mathbb{E}}^3$ that are non-separating and disjoint.

Remark 7.18. In order to distinguish the case $n = 4$ from $n = 3$, we note that in $\mathbb{H}^3$, if the two transformations A and B share a fixed point, then the group $G = \langle A, B \rangle$ is not discrete. However, in $\mathbb{H}^4$, we may have that the transformations A and B share a fixed point and yet the group G is NSDS as illustrated below.

Example 7.19. Let A and B be two screw-parabolic isometries of $\mathbb{H}^4$ with a shared fixed point at ∞. Then, in their action on the boundary of $\mathbb{H}^4$, there are unique circles ax_A and ax_B through ∞ in $\hat{\mathbb{E}}^3$ that are left invariant by A and B respectively. Let n denote the common normal to a and b in $\hat{\mathbb{E}}^3$. Then, by our analysis in section 6.6, there exist circles l_A and l_B through ∞ and orthogonal to ax_A and ax_B respectively, such that $A = H_N \cdot H_{L_A}$ and $B = H_{L_B} \cdot H_N$, where L_A, L_B and N are the caps of l_A, l_B and n in $\mathbb{H}^4$ respectively. If l_a, l_B and n have the NSD/TS property, then the group $3G = \langle H_{L_A}, H_N, H_{L_B} \rangle$ is discrete and consequently, G is discrete.

We note here that in order for G to be NSDS, the circles ax_A and ax_B must lie on a 2-sphere in $\hat{\mathbb{E}}^3$. In other words, their caps in $\mathbb{H}^4$ must lie on a hyperplane of $\mathbb{H}^4$. Another point of distinction is that G is elementary as every transformation in the group fixes ∞, whereas NSDC groups acting on $\mathbb{H}^3$ are non-elementary [5].

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Curriculum Vitae

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