AN INTRODUCTION TO THE CURVATURE OF SURFACES

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ABSTRACT OF THE THESIS

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Curvature is fundamental to the study of differential geometry. It describes different geometrical and topological properties of a surface in $\mathbb{R}^3$. Two types of curvature are discussed in this paper: intrinsic and extrinsic. Numerous examples are given which motivate definitions, properties and theorems concerning curvature.

1 Introduction

For surfaces in $\mathbb{R}^3$, there are several different ways to measure curvature. Some curvature, like normal curvature, has the property such that it depends on how we embed the surface in $\mathbb{R}^3$. Normal curvature is extrinsic; that is, it could not be measured by being on the surface. On the other hand, another measurement of curvature, namely Gauss curvature, does not depend on how we embed the surface in $\mathbb{R}^3$. Gauss curvature is intrinsic; that is, it can be measured from on the surface.

In order to engage in a discussion about curvature of surfaces, we must introduce some important concepts such as regular surfaces, the tangent plane, the first and second fundamental form, and the Gauss Map. Sections 2,3 and 4 introduce these preliminaries, however, their importance should not be understated as they lay the groundwork for more subtle and advanced topics in differential geometry. For example, the first fundamental form plays a very special role in the calculation of curvature. It also provides a way of calculating angles between vectors and distance at a point p on a regular surface.

Once these preliminary concepts are introduced, it is then possible to define the different types of curvature and how to calculate them. Of particular importance is Gauss's Theorema Egregium, which is discussed in section 4. Gauss was able to prove that Gauss curvature K is an intrinsic quantity even though it can be calculated in terms of extrinsic quantities. Section 4 offers several examples to go over the geometry of points on a regular surface for different values of Gauss curvature K. Section 5 offers a potpourri of some geometrical and topological facts concerning curvature, including mean curvature, and a counterexample to show that the converse of Theorema Egregium is false. Section 6 offers some closing remarks about the importance of curvature in differential geometry, which is central to the study of the differential geometry of surfaces.

2 Surfaces and Tangent Planes in $\mathbb{R}^3$

In differential geometry, we require that our surfaces are smooth so that they provide the means to do calculus. To do this, we must map points of an open set in $\mathbb{R}^2$ to points in $\mathbb{R}^3$, namely, our regular surface S. However, since we require that S be smooth, we must map in such a way that the mapping is differentiable. The mapping must also be continuous, injective, with continuous inverse, and the differential of the mapping must also be injective. The result is called a regular surface. Intuitively, this means that we are deforming pieces of a plane (the open set in $\mathbb{R}^2$) in such a way that what we end up with is something that does not have sharp edges or self intersections. More formally, we state the definition the following definition.

Definition 1. *We call $S \subset \mathbb{R}^3$ a regular surface if for each $\mathbf{V} \subset \mathbb{R}^3$ there is a map $\mathbf{x}$:$\mathbf{U} \to \mathbf{V} \cap \mathbf{S}$, where $\mathbf{U} \subset \mathbb{R}^2$ is an open set, such that the following are satisfied*

1. *The map $\mathbf{x}(\mathbf{u}, \mathbf{v}) = (x(u, v), y(u, v), z(u, v))$ has continuous partial derivatives.*
2. *$\mathbf{x}$ is a homeomorphism. That is, $\mathbf{x}$ is a one-to-one correspondence between $\mathbf{U}$ and $\mathbf{V} \cap \mathbf{S}$ and this correspondence is continuous. Additionally, the inverse of $\mathbf{x}$ is continuous.*
3. *The differential $d\mathbf{x}_p$ is one to one for each p in $\mathbf{U}$.*

The map $\mathbf{x}$ is called a parameterization. Property (1) is essential for us to be able to do calculus. Property (2) guarantees that there won't be any self intersections. Property (3) guarantees that at each point p in S there is a tangent plane (see section 2.4 of [1] for a proof of this). Note that it may take more than one parameterization $\mathbf{x}$

to cover all of S. For example, if we consider the unit sphere in cartesian coordinates $\mathbf{S^2} = \{(x, y, z) \in R^3 | x^2 + y^2 + z^2 = 1\}$, it would take six parameterizations to cover all of $\mathbf{S^2}$:

$$\begin{aligned}
\mathbf{x}_1 &= (x, y, +\sqrt{1-(x^2+y^2)}) \\
\mathbf{x}_2 &= (x, y, -\sqrt{1-(x^2+y^2)}) \\
\mathbf{x}_3 &= (x, +\sqrt{1-(x^2+z^2)}, z) \\
\mathbf{x}_4 &= (x, -\sqrt{1-(x^2+z^2)}, z) \\
\mathbf{x}_5 &= (\sqrt{1-(y^2+z^2)}, y, z) \\
\mathbf{x}_6 &= (-\sqrt{1-(y^2+z^2)}, y, z)
\end{aligned}$$

Also note that the parameterization for a regular surface need not be unique; that is, in our particular example, the fact that the sphere is a regular surface does not depend on the choice of parameterization [1]. This can be to our advantage as some parameterizations lend themselves to simpler calculations. Consider the following example.

Example 1. *Let* $\mathbf{x}(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ *where* $0 < \theta < \pi$ *and* $0 < \phi < 2\pi$. *Notice that* $\mathbf{x}(u, v)$ *is the standard representation of the unit sphere in spherical coordinates. We can verify that the unit sphere is in fact a regular surface by verifying that properties 1-3 from Definition (1) hold. Property 1 is easily satisfied as* $\sin\theta\cos\phi, \sin\theta\sin\phi$ *and* $\cos\theta$ *are continuously differentiable functions. For property 2, first notice that our parameterization leaves out a semicircle (including the north and south poles)* $C = \{(x, y, c) | y = 0, x \geq 0\}$. *Since* $z = \cos\theta$, *then* $\theta = \cos^{-1} z$ *is* *uniquely determined for a given z. Now given*

$$\begin{aligned} x &= \sin\theta\cos\phi \\ y &= \sin\theta\sin\phi \end{aligned}$$

defined on $S^2 - C$, we can rewrite the above equations as

$$\begin{aligned} \frac{x}{\sin\theta} &= \cos\phi \\ \frac{y}{\sin\theta} &= \sin\phi. \end{aligned}$$

That is, we've found ϕ uniquely in terms of θ and so $\mathbf{x}(\theta,\phi)$ is one-to-one which means $\mathbf{x}^{-1}$ exists. The fact that $\mathbf{x}(\theta,\phi)$ is differentiable implies that it is also continuous and so $\mathbf{x}(\theta,\phi)$ is a homeomorphism. For property 3, notice that the differential is

$$\mathbf{dx_p} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{pmatrix}.$$

Calculating the cross product of $\mathbf{x}_\theta$ and $\mathbf{x}_\phi$,

$$\begin{aligned} \mathbf{x}_\theta \times \mathbf{x}_\phi &= \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{pmatrix} \\ &= \vec{i}(\sin^2\theta\cos\phi) - \vec{j}(\sin^2\theta\sin\phi) + \vec{k}(\sin\theta\cos\phi), \qquad (1) \end{aligned}$$

we note that the length of (1) is

$$= |\vec{i}(\sin^2\theta\cos\phi) - \vec{j}(\sin^2\theta\sin\phi) + \vec{k}(\sin\theta\cos\phi)|$$

$$\begin{aligned} &= \sqrt{\sin^4\theta\cos^2\phi + \sin^4\theta\sin^2\phi + \sin^2\theta\cos^2\theta} \\ &= \sqrt{\sin^4\theta + \sin^2\theta\cos^2\theta} \\ &= \sqrt{\sin^2\theta} = \sin\theta \end{aligned}$$

which is nonzero when $0 < \pi < \theta$. *As such, the differential is one to one and property 3 is satisfied. Since properties 1-3 are satisfied, the sphere is a regular surface.*

It would be much simpler if we did not have to verify properties 1-3 in the definition of a regular surface each time we wanted to check if a given set is a regular surface. Fortunately, there are easier ways. Suppose that your parameterization $\mathbf{x(u,v)} = (u, v, z(u, v))$ where $z(u, v)$ is differentiable with respect to u and v. That is, $\mathbf{x}$ is the graph of a differentiable function. Then $\mathbf{x}$ is a regular surface. To demonstrate this, first note that $\mathbf{x_u} = (1, 0, z_u)$ and $\mathbf{x_v} = (0, 1, z_v)$. Since z is differentiable with respect to u and v, condition 1 for a regular surface is satisfied. Note that if we have points $u_1, v_1, u_2, v_2 \in \mathbf{U}$ where $u_1 \neq u_2$ and $v_1 \neq v_2$, then the image under $\mathbf{x}$ produces two unique points $(u_1, v_1, z(u_1, v_1)), (u_2, v_2, z(u_2, v_2))$ and so $\mathbf{x}$ is one-to-one. Since $\mathbf{x}$ is injective, $\mathbf{x^{-1}}$ exists. It is clearly continuous as each component of $\mathbf{x}(u, v)$ is a continuous function. We note that $\mathbf{x^{-1}}$ is simply the projection of the point $(u, v, z(u, v))$ to the point (u, v) and thus is a continuous function. Condition 2 is now satisfied ($\mathbf{x}$ is a homeomorphism). Finally, note that for some point $p \in \mathbf{U}$, the columns of the differential

$$\mathbf{dx_p} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial z(u,v)}{\partial u} & \frac{\partial z(u,v)}{\partial v} \end{pmatrix}$$

are linearly independent, which means that we have satisfied condition 3. Therefore, the graph of a differentiable function $f(x, y)$ is a regular surface.

Example 2. *Consider the set* $S = \{(x, y, z) \in \mathbf{R^3}$ *where* $z = x^2 - y^2\}$*. Note that we can write* $S = \{(u, v, f(u, v)\}$*, a hyperbolic paraboloid, where* $f(u, v) = u^2 - v^2$ *is a differentiable function. Using the fact that we wrote* S *as the graph of a differentiable function, then* S *is a regular surface.*

Other examples of regular surfaces are provided by the level sets of a differentiable function. Recall that a level set of a differentiable function $f : \mathbf{R^n} \to \mathbf{R}$ corresponding to some real number k is the set $\{(x_1, x_2, ..., x_n)$ where $f(x_1, x_2, ..., x_n) = k\}$. The sphere provides a familiar example of a level set of the function $f(x, y, z) = x^2 + y^2 + z^2 = k$ (in the case of the unit sphere, $k = 1$). To make the notion precise, consider the following definition.

Definition 2. *Suppose we have a differentiable mapping* $F : U \subset \mathbf{R^n} \to \mathbf{R^m}$ *where* U *is an open set. We call a point* $p \in U$ *a critical point of the mapping* F *if* $\mathbf{dx_p}$ *is not an onto mapping. Similarly, we call the point* $F(p)$ *a critical value of* F*. Points in the image of* F *that are not critical values are called regular values.*

Now that we know what a regular value is, we can relate level sets to regular surfaces with the following theorem.

Theorem 1. *Suppose* k *is a regular value of a differentiable function* $F : \mathbf{R^3} \to \mathbf{R}$*. Let* $S = \{(x, y, z) where f(x, y, z) = k\}$*. Then* $F^{-1}(k)$ *is a regular surface.*

For a proof of Theorem 1, refer to [1].

Example 3. *Consider the differentiable function* $F(x, y, z) = x^2 + y^2 - z^2$*. The differential* $\mathbf{dx}$ *will not be a matrix but rather just the gradient of* F*, and so* $\mathbf{dx} = (2x, 2y, -2z)$*. Now note that* $\mathbf{dx}$ *fails to be surjective when* $x = y = z =$ *0; that is,*

$\mathbf{dx} = 0$ *at the point* $(0,0,0)$*. In fact, it is the only critical value of* F*. It is safe to say that 1 would then be a regular value of* F*. By Theorem* 1*, we have that the level surface* $F(x,y,z) = x^2 + y^2 - z^2 = 1$*, which is a hyperboloid, is a regular surface.*

Now that we've defined a regular surface, we introduce the definition of the tangent plane.

Definition 3. *We say* v *is a tangent vector at a point* $p \in S$ *if there is a differentiable curve* $\alpha : (-\epsilon, \epsilon) \to S$ *with* $\alpha(0) = p$ *and* $\alpha'(0) = v$*. The set of all tangent vectors at a point* $p \in S$ *is called the tangent space or tangent plane, denoted by* T_pS*.*

Property 3 in the definition of a regular surface is important because it allows us to assign a set of tangent vectors for each point $p \in S$. In fact, given a regular surface S parameterized by $\mathbf{x} : U \subset \mathbf{R^2} \to S$, the differential $\mathbf{dx_q}$ gives a set of linearly independent tangent vectors at a point $q \in U$. Our parameterization $\mathbf{x}$ determines a basis $\{\frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v}\}$ that we call the tangent space of S at p, or T_pS.

Given a level surface of a differentiable function, $f(x,y,z) = 0$, where 0 is a regular value of f, we can write down the tangent plane at a point (x_0, y_0, z_0) with the equation

$$f_x(x_0,y_0,z_0)(x - x_0) + f_y(x_0,y_0,z_0)(y - y_0) + f_z(x_0,y_0,z_0)(z - z_0) = 0 \qquad (2)$$

Given a point $p = (x_0, y_0, z_0) \in S$, define a curve $\alpha(x(t), y(t), z(t))$ where $x(t), y(t)$ and $z(t)$ are differentiable functions of t and $\alpha(0) = p$. Now

$$f(\alpha(t)) = f(x(t), y(t), z(t)) = 0 \qquad (3)$$

and by differentiating both sides of (3), we have

$$
\begin{aligned}
f'(x(0), y(0), z(0)) &= \\
f_x \cdot x'(0) + f_y \cdot y'(0) + f_z \cdot z'(0) &= \\
(f_x, f_y, f_z) \cdot (x'(0), y'(0), z'(0)) &= \\
\nabla f \cdot (x'(0), y'(0), z'(0)) &= 0 \qquad (4)
\end{aligned}
$$

Equation (4) tells us that ∇f, the gradient of f, is perpendicular to the vector tangent to the curve $\alpha(0) = p$, namely, $(x'(0), y'(0), z'(0))$. This implies that ∇f is the normal to the surface at the point $p = (x_0, y_0, z_0)$. Since the plane goes through (x_0, y_0, z_0), the equation of the plane T_pS is simply (2).

3 The First Fundamental Form

The notion of a tangent plane plays an important role in differential geometry: it allows one to assign a metric, or a way of measuring distance, to the regular surface. This metric is called the First Fundamental Form. Let $w \in T_p(S)$ at a point $p \in S$. As in the previous section, we know that there is a differentiable curve, $\alpha(t) = \mathbf{x}(u(t), v(t))$ such that $\alpha(0) = p$ and using the inner product of R^3 and the chain rule to compute the length of the tangent vector $\alpha'(0)$ at p,

$$\begin{aligned}
\|\alpha'(0)\|^2 &= \langle \alpha'(0), \alpha'(0) \rangle \\
&= \langle \mathbf{x_u} u' + \mathbf{x_v} v', \mathbf{x_u} u' + \mathbf{x_v} v' \rangle \\
&= \langle \mathbf{x_u} u', \mathbf{x_u} u' \rangle + \langle \mathbf{x_u} u', \mathbf{x_v} v' \rangle + \langle \mathbf{x_u} u', \mathbf{x_v} v' \rangle + \langle \mathbf{x_v} v', \mathbf{x_v} v' \rangle \\
&= \langle \mathbf{x_u}, \mathbf{x_u} \rangle (u')^2 + 2 \langle \mathbf{x_u}, \mathbf{x_v} \rangle (u'v') + \langle \mathbf{x_v}, \mathbf{x_v} \rangle (v')^2.
\end{aligned}$$

We define the first fundamental form for a regular surface S at a point p, $\mathbf{I_p}$, as

$$E du^2 + F du dv + G dv^2$$

where the coefficients

$$\begin{aligned}
E &= \langle \mathbf{x_u}, \mathbf{x_u} \rangle \\
F &= \langle \mathbf{x_u}, \mathbf{x_v} \rangle \\
G &= \langle \mathbf{x_v}, \mathbf{x_v} \rangle
\end{aligned}$$

are called the coefficients of the first fundamental form. An important thing to note about the first fundamental form is that it is an intrinsic quantity of S. That is, we can refer to measurements such as length without ever leaving the surface. Put another way, that means we need not refer to the space that S lies in, which is $\mathbf{R^3}$ (i.e it does not matter how we embed S in $\mathbf{R^3}$).

Example 1. *Consider the cylinder with radius 1 parameterized by* $\mathbf{x}(u,v) = (\cos u, \sin u, v)$ *where* $0 < u < 2\pi$ *and* $v \in \mathbf{R}$. *To find the coefficients of the first fundamental form, first we compute* $\mathbf{x_u}$ *and* $\mathbf{x_v}$

$$\begin{aligned}
\mathbf{x_u} &= (-\sin u, \cos u, 0) \\
\mathbf{x_v} &= (0, 0, 1)
\end{aligned}$$

and then find E, F, *and* G

$$\begin{aligned}
E &= \langle \mathbf{x_u}, \mathbf{x_u} \rangle = \langle (-\sin u, \cos u, 0), (-\sin u, \cos u, 0) \rangle \\
&= \sin^2 u + \cos^2 u = 1 \\
F &= \langle \mathbf{x_u}, \mathbf{x_v} \rangle = \langle (-\sin u, \cos u, 0), (0, 0, 1) \rangle = 0 \\
G &= \langle \mathbf{x_v}, \mathbf{x_v} \rangle = \langle (0, 0, 1), (0, 0, 1) \rangle = 1
\end{aligned}$$

Therefore, the first fundamental form for the cylinder of radius 1 is

$$\mathbf{I_p} = du^2 + dv^2.$$

As mentioned before, the first fundamental form provides a natural way to measure distance. To illustrate this, take the cylinder from the previous example and consider a helix on the cylinder that is parameterized by a curve $\alpha(t)$ where

$$\alpha(t) = (\cos t, \sin t, t).$$

From standard results of calculus, we know that we could find the length of the curve α, denoted by $s(t)$, from 0 to 2π by first differentiating α with respect to the parameter t, then finding the length of α' by taking the inner product, and finally integrating the result from 0 to 2π. More precisely, we have the formula,

$$L = s(t) = \int_0^{2\pi} |\alpha'(t)|\, dt$$

For a good derivation of the formula for arc length of a curve, see any standard calculus text, such as [4] or [2]. Computing arc length for the cylinder, we have

$$\begin{aligned} \alpha(t) &= (\cos t, \sin t, t) \\ \alpha'(t) &= (-\sin t, \cos t, 1) \\ |\alpha'(t)| &= \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2} \end{aligned}$$

and so

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{2}\, dt \\ &= 2\sqrt{2}\pi. \end{aligned} \tag{5}$$

Note that we could also use the definition of the first fundamental form to calculate arc length as it provides a metric for doing so. For the cylinder with parameterization given in Example 1, observe that

$$\begin{aligned} \frac{du}{dt} &= 1 \\ \frac{dv}{dt} &= 1 \end{aligned}$$

and so using the definition of $\mathbf{I_p}$, this is the same as if we had calculated

$$\begin{aligned} L &= \int_0^{2\pi} (\mathbf{I_p}(\alpha'(t))^{1/2}\,dt \\ &= \int_0^{2\pi} (E(u')^2 + 2Fu'v' + G(v')^2)^{1/2}\,dt \\ &= \int_0^{2\pi} \sqrt{2}\,dt \end{aligned} \tag{6}$$

where equation (6) will yield the same result as equation (5).

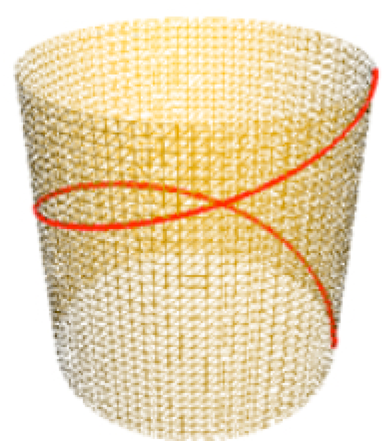


Figure 1: A Helix superimposed on a Cylinder. Image produced with Mac OS X Grapher

We introduce our next concept with another example:

Example 2. *Two orthonormal vectors* $v_1, v_2 \in \mathbf{R^3}$ *parameterize a plane in* $\mathbf{R^3}$ *passing through a point* $p_0 = (x_0, y_0, z_0)$ *via the parameterization* $\mathbf{x}(x, y) = p_0 + xv_1 + yv_2$.

Computing E, F, G, we have

$$\begin{aligned} E &= \langle \mathbf{x_x}, \mathbf{x_x} \rangle = \langle v_1, v_1 \rangle = 1 \\ F &= \langle \mathbf{x_x}, \mathbf{x_y} \rangle = \langle v_1, v_2 \rangle = 0 \\ G &= \langle \mathbf{x_y}, \mathbf{x_y} \rangle = \langle v_2, v_2 \rangle = 1 \end{aligned}$$

as the dot product of orthonormal vectors v_i and v_j is 1 if $i = j$ and 0 if $i \neq j$. Therefore, the first fundamental form for the plane is

$$\mathbf{I_p} = dx^2 + dy^2.$$

Note that from examples 1 and 2, the first fundamental form for the plane is the same as that of the cylinder. The reason is that the first fundamental form is an intrinsic geometric property. That is, our notion of measuring distance is the same whether the plane is flat or it is rolled up into a cylinder. To make this notion precise, we require two definitions.

Definition 4. *Given two regular surfaces S_1, S_2, we say that S_1 is diffeomorphic to S_2 if there is a differentiable map $\psi : S_1 \to S_2$ with differentiable inverse $\psi^{-1} : S_2 \to S_1$. We call this kind of map a diffeomorphism.*

The concept of diffeomorphism is similar to the concept of isomorphism. That is, two surfaces that are diffeomorphic to each other are essentially equivalent to each other with respect to differentiability.

Definition 5. *Let $\psi : S_1 \to S_2$ be a diffeomorphism. We say that ψ is an isometry if it preserves the inner product. That is, for all $w_1, w_2 \in T_p(S_1)$,*

$$\langle w_1, w_2 \rangle = \langle d\psi(w_1), d\psi(w_2) \rangle.$$

Since ψ preserves the metric, it follows that the coefficients of the first fundamental forms between the two surfaces would be equal. However, we must take care to note that this concept is a local one. For example, the cylinder and the plane are not globally isometric to each other. This is due to the fact that there isnt a global homeomorphism between the two ob jects. To see this, note that any curve in the plane can be continuously deformed to a point, yet, we can nd a curve on the cylinder (slice the cylinder perpendicularly with a plane) that cannot be deformed to a point. This indicates that topologically speaking, the plane and cylinder are different. However, as we saw, the plane and cylinder do share the same rst fundamental form for coordinate neighborhoods and so they are locally isometric.

4 Curvature of a surface in $\mathbb{R}^3$

Let S be a surface in $\mathbb{R}^3$. For each point $p \in S$, we can write down the formula for the unit normal as

$$\mathbf{N} = \frac{\mathbf{x_x} \times \mathbf{x_y}}{\mid \mathbf{x_x} \times \mathbf{x_y} \mid}.$$

$\mathbf{N}$ has a more special role in differential geometry than merely being the unit normal at a point p. Note that $\mathbf{N}$ takes it's values in the unit sphere $\mathbf{S^2}$.

Definition 6. *Let S be a regular surface with a differentiable field of unit normal vectors N. We call the map $\mathbf{N} : S \to \mathbf{S^2}$ the Gauss map of S.*

If you consider a curve α on a regular surface S, notice that at each point $p \in S$, the unit normal $\mathbf{N}$ will map to a point on the unit sphere; that is, as you travel along α, $\mathbf{N}$ will sweep out points on $\mathbf{S^2}$. In differential geometry, what we're really interested in is the differential of the Gauss map at a point p as it gives us information on how a regular surface S curves near the point p. There are two properties of the differential of the Gauss map that we'd like to exploit.

Lemma 1. *For each point $p \in S$, the differential $d\mathbf{N}$ is self adjoint or symmetric.*

For a proof of this lemma, refer to page 140 of [1]. The fact that dN is linear and symmetric is important. In fact, we can write down a matrix for dN, which will be explained later in this paper. But first a little linear algebra. Recall from linear algebra that a quadratic form on $\mathbf{R^n}$ is a function $Q(\mathbf{x}) = \mathbf{x^T}A\mathbf{x}$ where $\mathbf{x}$ is a vector in $\mathbf{R^n}$ and A is a symmetric $n \times n$ matrix. Conversely, given any symmetric matrix A, we can associate a quadratic form Q using the same formula. For more properties of quadratic forms, the reader may refer to [3] , [7] or any text on linear algebra. For a more concrete introduction, consider the following example.

Example 1. *Let* $A = \begin{pmatrix} 2 & 7 \\ 7 & 2 \end{pmatrix}$ *be a symmetric matrix and* $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ *be any vector in* $\mathbf{R^2}$*. Then the quadratic form* Q *associated with the matrix* A *is*

$$\begin{aligned} Q(\mathbf{x}) &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 2 & 7 \\ 7 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 2x_1 + 7x_2 \\ 7x_1 + 2x_2 \end{pmatrix} \\ &= x_1(2x_1 + 7x_2) + x_2(7x_1 + 2x_2) \\ &= 2x_1^2 + 7x_1x_2 + 7x_1x_2 + 2x_2^2 \\ &= 2x_1^2 + 14x_1x_2 + 2x_2^2 \end{aligned}$$

Note that Q has a cross term x_1x_2. This can actually be removed via a change of variables that orthogonally diagonalizes our matrix A. To do so, first note that the characteristic equation of our matrix A is

$$(2-\lambda)(2-\lambda) - 49 = \lambda^2 - 4\lambda - 45 \tag{7}$$

and so A has eigenvalues $\lambda = 9, -5$. The corresponding normalized eigenvectors are

$$\begin{aligned} \mathbf{v_1} &= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \\ \mathbf{v_2} &= \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}. \end{aligned}$$

Note that the vectors are also orthonormal, as they should be since A is a symmetric matrix. This is known as the Spectral Theorem (for the precise statement of the theorem, see [7]). Since our vectors are orthonormal, they provide a basis for $\mathbf{R^2}$. We can then rewrite A as $PDP^{-1} = PDP^{\mathbf{T}}$, where

$$\begin{aligned} P &= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\ D &= \begin{pmatrix} 9 & 0 \\ 0 & -5 \end{pmatrix} \end{aligned}$$

Now we make the change of variables $\mathbf{x} = P\mathbf{y}$ where P is our matrix defined above and $\mathbf{y}$ is a new vector in $\mathbf{R^2}$. Making the change of variables for Q,

$$\begin{aligned} 2x_1^2 + 14x_1x_2 + 2x_2^2 &= \mathbf{x^T}A\mathbf{x} \\ &= (P\mathbf{y})^T A(P\mathbf{y}) \\ &= \mathbf{y}^T P^T AP\mathbf{y} \\ &= \mathbf{y}^T D\mathbf{y} \\ &= 9y_1^2 - 5y_2^2. \end{aligned}$$

The above example prompts the following theorem known as the Principle Axes Theorem.

Theorem 2. *Let A be an $n \times n$ symmetric matrix. Then there exists an orthogonal change of variables, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into a quadratic form $\mathbf{y}^T A\mathbf{y}$ such that the new quadratic form does not have a cross term.*

The proof of the above theorem can be found in [3], page 453. Continuing on, suppose we constrain the length of our vector $\mathbf{y}$ to have length equal to 1. Note that

$$-5y_2^2 \leq 9y_2^2$$

and so

$$\begin{aligned} Q(\mathbf{y}) &= 9y_1^2 - 5y_2^2 \\ &\leq 9y_1^2 + 9y_2^2 \\ &= 9(y_1^2 + y_2^2) \\ &= 9. \end{aligned}$$

The last equation says that 9 is the maximum value of Q. Similarly,

$$-5y_1^2 \leq 9y_1^2$$

and so

$$\begin{aligned} Q(\mathbf{y}) &= 9y_1^2 - 5y_2^2 \\ &\geq -5y_1^2 - 5y_2^2 \\ &= -5(y_1^2 + y_2^2) \\ &= -5, \end{aligned}$$

which says that -5 is the minimum value of Q. Notice that -5 $= m =$min$\{\mathbf{x}^T A\mathbf{x}\}$ and $9 = m =$max$\{\mathbf{x}^T A\mathbf{x}\}$ also correspond to the minimum and maximum eigenvalues of our symmetric matrix A. To summarize, we state the following theorem (refer to [3] for a proof).

Theorem 3. *Let A be an $n \times n$ symmetric matrix and define m and M as above. Then M is the greatest eigenvalue of A and m is the least eigenvalue of A. The value of $\mathbf{x}^T A\mathbf{x}$ is M when $\mathbf{x}$ is a unit eigenvector $\mathbf{v_1}$ corresponding to M and the value of $\mathbf{x}^T A\mathbf{x}$ is m when $\mathbf{x}$ is a unit eigenvector $\mathbf{v_2}$ corresponding to m.*

The columns of P, which form an orthonormal basis are called the principle directions associated to the quadratic form Q. Theorems 2 and 3 tell us that the value of our quadratic form Q is smallest in the principal direction that corresponds to the smallest eigenvalue and greatest in the principal direction that corresponds to the greatest eigenvalue.

Returning back to differential geometry, we now know that we can associate a quadratic form to our symmetric linear map dN. This prompts the following definition:

Definition 7. *Let S be a regular surface with a differentiable field of unit normal vectors N and let v be a vector in $T_p(S)$. We associate a quadratic form, $\mathbf{II_p}$, to the Gauss Map, dN, defined by $\mathbf{II_p} = -v^T dN(v)v = -\langle dN(v), v\rangle$ called the second fundamental form of S at a point p.*

However, the second fundamental form $\mathbf{II_p}$ is more than just a quadratic form. To give it a geometric meaning, suppose we have a differentiable curve parameterized by arc length, $\alpha(s)$, on a regular surface S with $\alpha(0) = p \in S$. Restrict our unit normal N to the curve $\alpha(s)$. That is, consider $(N \circ \alpha)(s) = N(\alpha(s)) = N(s)$. If we differentiate α at p, then α' is a unit tangent vector at the point p. As a result, $N(s)$ is perpendicular to α' as $N(s)$ is normal to the surface S and α' is tangent to S (at a point p). To be more precise, we have $\langle N(s), \alpha'(s)\rangle = 0$. Note that since α is parameterized by arc length,

$$\frac{d}{dt}N(s) = \frac{d}{dt}N\frac{ds}{dt} = N'$$

and so writing out $\mathbf{II_p}$ at p in terms of α, we have

$$\mathbf{II_p}(\alpha'(0)) = -\langle dN(\alpha'(0)), \alpha'(0)\rangle = -\langle N'(0), \alpha'(0)\rangle \tag{8}$$

We can simplify (8) by differentiating the expression $\langle N(s), \alpha'(s)\rangle = 0$. We have

$$\begin{aligned} 0 &= \frac{d}{ds}\langle N(s), \alpha'(s)\rangle = \langle N'(s), \alpha'(s)\rangle + \langle N(s), \alpha''(s)\rangle && (9) \\ \Leftrightarrow &\ -\langle N'(s), \alpha'(s)\rangle = \langle N(s), \alpha''(s)\rangle && (10) \end{aligned}$$

and substituting the last equation back into (8), we obtain

$$-\langle N'(0), \alpha'(0)\rangle = \langle N(0), \alpha''(0)\rangle. \tag{11}$$

However, note that in the last expression, we can use the Frenet formula (see [1]) $\alpha'' = kn$ where n is the normal vector to the curve α and k is the curvature. So we have

$$\langle N(0), \alpha''(0)\rangle = \langle N(0), kn(0)\rangle$$

The last expression says that $\mathbf{II_p}$ is the projection of $\alpha'' = kn$ onto the unit normal N. We call this value the normal curvature $k_n(p)$. That is, the second fundamental form $\mathbf{II_p}$ for a unit vector v in $T_p(S)$ is equal to the normal curvature of a regular curve α passing through p and whose tangent is v. Furthermore, recall from Theorems 2 and 3 that if $\{v_1, v_2\}$ is an orthonormal basis then for each $p \in S$ our linear symmetric matrix dN has principal directions and minimal and maximal eigenvalues that correspond to the principal directions. By convention, we have $dN(v_1) = -k_1 v_1$, $dN(v_2) = -k_2 v_2$ and call k_1, k_2 the maximum and minimum curvature of $\mathbf{II_p}$ in the direction of the eigenvectors v_1, v_2, respectively. Computing $k_n(p)$ is not very difficult if you are working in an orthonormal basis. So suppose that the tangent space $T_p(S)$ for a regular surface S is given by an orthonormal basis $\{v_1, v_2\}$. Then we can express any vector $v \in T_p(S)$ as a linear combination of v_1 and v_2. In fact, if we let θ be the angle from v_1 to v, we can express v as

$$v = v_1 \cos\theta + v_2 \sin\theta.$$

With this in mind, let's express $k_n(p)$ in terms of v. We write

$$\begin{aligned} k_n(p) &= -\langle dN(v), v\rangle \\ &= -\langle dN(v_1 \cos\theta + v_2 \sin\theta), v_1 \cos\theta + v_2 \sin\theta\rangle \end{aligned}$$

and since dN is linear the above becomes

$$-\langle dN(v_1 \cos\theta) + dN(v_2 \sin\theta), v_1 \cos\theta + v_2 \sin\theta\rangle . \tag{12}$$

Now using the fact that $dN(v_1) = -k_1 v_1, dN(v_2) = -k_2 v_2$ and that $\{v_1, v_2\}$ is an orthonormal basis, equation (12) becomes

$$\begin{aligned} &= -\langle -k_1(v_1 \cos\theta) - k_2(v_2 \sin\theta), v_1 \cos\theta + v_2 \sin\theta\rangle \\ &= \langle k_1(v_1 \cos\theta) + k_2(v_2 \sin\theta), v_1 \cos\theta + v_2 \sin\theta\rangle \\ &= k_1 v_1 \cdot v_1 \cos^2\theta + 2k_1 v_1 \cdot v_2 \sin\theta \cos\theta + k_2 v_2 \cdot v_2 \sin^2\theta \end{aligned}$$

$$= \quad k_1 \cos^2\theta + k_2 \sin^2\theta = k_n(p).$$

A nice little fact about normal curvature is the following:

Lemma 2. *The sum of the normal curvatures for any pair of orthogonal directions at a point $p \in S$ is constant.*

Proof. Let $\{v_1, v_2\}$ be an orthonormal basis for $T_p(S)$ and choose a vector $v \in T_p(S)$. Then as we saw earlier, we can express v as $v = v_1 \cos\theta + v_2 \sin\theta$ and the normal curvature, $k_n(p)$ in the direction of θ is just $k_1 \cos^2\theta + k_2 \sin^2\theta$. Now, if θ was our given direction, a direction orthogonal to θ would be just $(\theta + \frac{\pi}{2})$. So we can choose a vector $\hat{v} = v_1 \cos(\theta + \frac{\pi}{2}) + v_2 \sin(\theta + \frac{\pi}{2})$ and compute the normal curvature at $\hat{v}$. Hence we compute the normal curvature at $\hat{v}$,

$$\hat{k}_n = k_1 \cos^2(\theta + \frac{\pi}{2}) + k_2 \sin^2(\theta + \frac{\pi}{2}) = k_1 \sin^2\theta + k_2 \cos^2\theta$$

where the last step is justified as sin and cos are orthogonal functions. Now

$$\begin{aligned} k_n + \hat{k}_n &= k_1 \cos^2\theta + k_2 \sin^2\theta + k_1 \sin^2\theta + k_2 \cos^2\theta \\ &= k_1(\sin^2\theta + \cos^2\theta) + k_2(\cos^2\theta + \sin^2\theta) \\ &= k_1 + k_2. \end{aligned}$$

Note that the expression $k_n + \hat{k}_n$ is not dependent on θ; therefore, we have the sum of the normal curvatures for any pair of orthogonal directions at a point p is constant. □

It is convenient to introduce three more definitions related to normal curvature. Suppose we have a curve α such that for each point p on α, the tangent vector α' is a principal direction. We then say α is a *line of curvature*. Additionally, suppose that $k_n = 0$ at a point p in some direction $v \in T_p(S)$. Then we say v is an *asymptotic direction*. Furthermore, we call α an *asymptotic curve* if for each point p on α, the tangent vector is an asymptotic direction.

All of the calculations above were done assuming that $T_p(S)$ was given by an orthonormal basis $\{v_1, v_2\}$. However, when we have a regular surface S parameterized by $\mathbf{x}(u, v)$, $T_p(S)$ has a natural basis given by $\{\mathbf{x_u}, \mathbf{x_v}\}$ which isn't necessarily orthonormal (or even orthogonal). So we need a way to express dN as a matrix in terms of the basis $\{\mathbf{x_u}, \mathbf{x_v}\}$. To start, let $\alpha(t) = \mathbf{x}(u(t), v(t))$ be a curve on S such that $\alpha(0) = p$. Then the tangent vector, $\alpha' = \mathbf{x_u}u' + \mathbf{x_v}v'$ is in $T_p(S)$ and

$$dN(\alpha') = N'(u(t), v(t)) = N_u u' + N_v v'.$$

Earlier, we noted that N_u and N_v were in $T_p(S)$. Hence we can express them as linear combinations of $\mathbf{x_u}$ and $\mathbf{x_v}$

$$N_u = a\mathbf{x_u} + b\mathbf{x_v} \tag{13}$$

$$N_v = c\mathbf{x_u} + d\mathbf{x_v} \tag{14}$$

and then rewrite dN as

$$\begin{aligned} dN(\alpha') &= (a\mathbf{x_u} + b\mathbf{x_v})u' + (c\mathbf{x_u} + d\mathbf{x_v})v' \\ &= (au' + cv')\mathbf{x_u} + (bu' + dv')\mathbf{x_v} \\ \Leftrightarrow \quad dN &= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}. \end{aligned}$$

In order to find a, b, c, d, we need to first calculate $\mathbf{II_p}(\alpha')$

$$\begin{aligned} &= -\langle dN(\alpha'), \alpha'\rangle \\ &= -\langle N_u u' + N_v v', \mathbf{x_u} u' + \mathbf{x_v} v'\rangle \\ &= -(\langle N_u, \mathbf{x_u}\rangle (u')^2 + \langle N_u, \mathbf{x_v}\rangle (u'v') + \langle N_v, \mathbf{x_u}\rangle (u'v') \\ &\quad + \langle N_v, \mathbf{x_v}\rangle (v')^2). \end{aligned} \tag{15}$$

We can further simplify equation (15) by noting the following. Since N is perpendicular to $\mathbf{x_u}$ and $\mathbf{x_v}$,

$$\begin{aligned} 0 = \frac{d}{dv}\langle N, \mathbf{x_u}\rangle &= \langle N_v, \mathbf{x_u}\rangle + \langle N, \mathbf{x_{uv}}\rangle \\ &\Leftrightarrow \langle N_v, \mathbf{x_u}\rangle = -\langle N, \mathbf{x_{uv}}\rangle \\ 0 = \frac{d}{du}\langle N, \mathbf{x_v}\rangle &= \langle N_u, \mathbf{x_v}\rangle + \langle N, \mathbf{x_{uv}}\rangle \\ &\Leftrightarrow \langle N_u, \mathbf{x_v}\rangle = -\langle N, \mathbf{x_{uv}}\rangle \\ &\Rightarrow \langle N_u, \mathbf{x_v}\rangle = \langle N_v, \mathbf{x_u}\rangle. \end{aligned}$$

Using the last equation, we can rewrite (15) as

$$\begin{aligned} -(\langle N_u, \mathbf{x_u}\rangle (u')^2 + 2\langle N_u, \mathbf{x_v}\rangle (u'v') + \langle N_v, \mathbf{x_v}\rangle (v')^2) \\ = e(u')^2 + fu'v' + g(v')^2 \end{aligned}$$

where

$$\begin{aligned} e &= -\langle N_u, \mathbf{x_u}\rangle \\ f &= -\langle N_u, \mathbf{x_v}\rangle = -\langle N_v, \mathbf{x_u}\rangle \\ g &= -\langle N_v, \mathbf{x_v}\rangle \end{aligned}$$

are called the coefficients of the second fundamental form. Now given (13) and (14),

$$\begin{aligned} -e &= \langle a\mathbf{x_u} + b\mathbf{x_v}, \mathbf{x_u}\rangle = a\langle \mathbf{x_u}, \mathbf{x_u}\rangle + b\langle \mathbf{x_u}, \mathbf{x_v}\rangle \\ &= aE + bF \\ -g &= \langle c\mathbf{x_u} + d\mathbf{x_v}, \mathbf{x_v}\rangle = c\langle \mathbf{x_u}, \mathbf{x_v}\rangle + d\langle \mathbf{x_v}, \mathbf{x_v}\rangle \\ &= cF + dG \end{aligned}$$

and

$$\begin{aligned} -f &= \langle c\mathbf{x_u} + d\mathbf{x_v}, \mathbf{x_u}\rangle = c\langle \mathbf{x_u}, \mathbf{x_u}\rangle + d\langle \mathbf{x_u}, \mathbf{x_v}\rangle \\ &= cE + dF \\ -f &= \langle a\mathbf{x_u} + b\mathbf{x_v}, \mathbf{x_v}\rangle = a\langle \mathbf{x_u}, \mathbf{x_v}\rangle + b\langle \mathbf{x_v}, \mathbf{x_v}\rangle \\ &= aF + bG \end{aligned}$$

where E, F, G were the coefficients of the first fundamental form. What we have done is written a matrix whose coefficients are the coefficients second fundamental form as the product of our matrix dN and a matrix whose coefficients are the coefficients of the first fundamental form. That is, from the equations above, we have

$$\begin{pmatrix} -e & -f \\ -f & -g \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \tag{16}$$

We can finally solve for dN by computing the inverse of

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

and multiplying on the right of (16) by the inverse. That is,

$$
\begin{aligned}
dN &= \frac{1}{EG-F^2}\begin{pmatrix} -e & -f \\ -f & -g \end{pmatrix}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \\
&= \frac{1}{EG-F^2}\begin{pmatrix} -eG+fF & eF-fE \\ -fG+gf & fF-gE \end{pmatrix}
\end{aligned}
$$

Now, if we calculate the determinate of dN, we have

$$
\begin{aligned}
det(dN) &= \frac{1}{(EG-F^2)^2}((-eG+fF)(fF-gE)-(-fG+gf)(eF-fE)) \\
&= \frac{1}{(EG-F^2)^2}(-eGfF+eGgE+f^2F^2-gFgE+fGeF-f^2GE \\
&\quad -geF^2+gFfE) \\
&= \frac{1}{(EG-F^2)^2}((efGE-f^2GE)+f^2F^2-fFgE+fFgE-egF^2) \\
&= \frac{1}{(EG-F^2)^2}(EG(eg-f^2)-F^2(eg-f^2)) \\
&= \frac{1}{(EG-F^2)^2}((EG-F^2)(eg-f^2) \\
&= \frac{eg-f^2}{(EG-F^2)} \\
&= K
\end{aligned}
$$

and we call K the Gaussian curvature of S at a point p. Additionally, define

$$H = \frac{1}{2}\text{trace}(dN) = \frac{1}{2}\frac{eG-2fF+gE}{EG-F^2}$$

to be the mean curvature of S at a point p. Here, K and H were defined in terms of the first and second fundamental forms; that is, in terms of the basis $\{\mathbf{x_u}, \mathbf{x_v}\}$. However, if we're working with an orthonormal basis for $T_p(S)$, then $K = k_1k_2$ and

$H = \frac{1}{2}(k_1 + k_2)$ as the determinant and trace for our symmetric matrix dN is just the product and sum of our eigenvalues of dN, respectively. Furthermore, we can classify a point p on a surface S by examining the value of K.

Definition 8. *We say a point p on S is*

1. *Elliptic if $K > 0$*
2. *Hyberbolic if $K < 0$*
3. *Parabolic if $K = 0$ but the matrix $dN \neq 0$*
4. *Planar if the matrix $dN = 0$*

That is, K tells us something about the local geometry of the surface.

Example 2. *Let $\mathbf{x}(\theta, \phi) = (a \sin\theta \cos\phi, a \sin\theta \sin\phi, a \cos\theta)$ where $0 < \theta < \pi$, $0 < \phi < 2\pi$, and $a > 0$ be a parameterization for a sphere with radius a. Computing the first order partial derivatives for $\mathbf{x}(\theta, \phi)$,*

$$\begin{aligned}
\mathbf{x}_\theta &= (a \cos\theta \cos\phi, a \cos\theta \sin\phi, -a \sin\theta) \\
\mathbf{x}_\phi &= (-a \sin\theta \sin\phi, a \sin\theta \cos\phi, 0)
\end{aligned}$$

we can then find the coefficients of the first fundamental form:

$$\begin{aligned}
E &= \langle \mathbf{x}_\theta, \mathbf{x}_\theta \rangle = a^2 \cos^2\theta \cos^2\phi + a^2 \cos^2\theta \sin^2\phi + a^2 \sin^2 \\
&= a^2(cos^2\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta) = a^2 \\
F &= \langle \mathbf{x}_\theta, \mathbf{x}_\phi \rangle = -a^2 \cos\theta \cos\phi \sin\theta \sin\phi + a^2 \cos\theta \cos\phi \sin\theta \sin\phi + 0 \\
&= 0 \\
G &= \langle \mathbf{x}_\phi, \mathbf{x}_\phi \rangle = a^2 \sin^2\theta \sin^2\phi + a^2 \sin^2\theta \cos^2\phi + 0 \\
&= a^2 \sin^2\theta(\cos^2\phi + \sin^2\phi) = a^2 \sin^2\theta.
\end{aligned}$$

Computing the second order partial derivatives, we obtain

$$\begin{aligned}
\mathbf{x}_{\theta\theta} &= (-a\sin\theta\cos\phi, -a\sin\theta\sin\phi, -a\cos\theta) \\
\mathbf{x}_{\theta\phi} &= (-a\cos\theta\sin\phi, a\cos\theta\cos\phi, 0) \\
\mathbf{x}_{\phi\phi} &= (-a\sin\theta\cos\phi, -a\sin\theta\sin\phi, 0)
\end{aligned}$$

To calculate the unit normal to the surface, N, we first calculate the vector cross product of $\mathbf{x}_\theta$ and $\mathbf{x}_\phi$:

$$\begin{aligned}
\mathbf{x}_\theta \times \mathbf{x}_\phi &= \begin{pmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ a\cos\theta\cos\phi & a\cos\theta\sin\phi & -a\sin\theta \\ -a\sin\theta\sin\phi & a\sin\theta\cos\phi & 0 \end{pmatrix} \\
&= (a^2\sin^2\theta\cos\phi, a^2\sin^2\theta\sin\phi, a^2\sin\theta\cos\theta)
\end{aligned}$$

Normalizing the result,

$$\begin{aligned}
\|\mathbf{x}_\theta \times \mathbf{x}_\phi\| &= \sqrt{a^4(\sin^4\theta\cos^2\phi + \sin^4\theta\sin^2\phi + \sin^2\theta\cos^2\theta} \\
&= \sqrt{a^4(\sin^4\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta\cos^2\theta)} \\
&= \sqrt{a^4(\sin^4\theta + \sin^2\theta\cos^2\theta)} \\
&= \sqrt{a^4(\sin^2\theta(\sin^2\theta + \cos^2\theta)} \\
&= \sqrt{a^4\sin^2\theta} = a^2\sin\theta
\end{aligned}$$

and so

$$N = \frac{\mathbf{x}_\theta \times \mathbf{x}_\phi}{\|\mathbf{x}_\theta \times \mathbf{x}_\phi\|} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Now we find the coefficients of the second fundamental form:

$$\begin{aligned} e &= \langle N, \mathbf{x}_{\theta\theta}\rangle = -a(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta) = -a \\ f &= \langle N, \mathbf{x}_{\theta\phi}\rangle = -a\sin\theta\cos\phi\cos\theta\sin\phi + a\sin\theta\cos\phi\cos\theta\sin\phi = 0 \\ g &= \langle N, \mathbf{x}_{\phi\phi}\rangle = -a\sin^2\theta\cos^2\phi - a\sin^2\theta\sin^2\phi = -a\sin^2\theta. \end{aligned}$$

Since we have the coefficients of the first and second fundamental forms, we can easily write down the Gaussian curvature, which is

$$K = \frac{eg - f^2}{EG - F^2} = \frac{a^2\sin^2\theta}{a^4\sin^2\theta} = \frac{1}{a^2}.$$

Notice that the sphere has constant Gaussian curvature; each point on the surface is an elliptic point. This means that both principal curvatures k_1, k_2 *have the same sign and that any pair of curves passing through a point* p *on the sphere have their respective normal vectors pointing toward the same side of the tangent plane. Additionally, the mean curvature is*

$$H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2} = \frac{1}{2}\frac{-a(a^2\sin^2\theta) + a^2(-a\sin^2\theta)}{a^4\sin^2\theta} = \frac{-1}{a}$$

and is also constant over the entire surface. Also note that as the radius a *increases,* K *decreases; the sphere appears more plane like as* a *gets larger.*

It is clear that the plane is a trivial example of a surface that has a point p such that p is a planar points. However, the next example shows a surface that has exactly one planar point.

Example 3. *The graph of* $z = x^3 - 3y^2x$ *is a regular surface known as the monkey saddle. We can parameterize the surface with* $\mathbf{x}(u, v) = (u, v, u^3 - 3v^2u)$*. Computing*

Figure 2: The Monkey Saddle. Produced with Mac OS X Grapher

the first order partial derivatives for $\mathbf{x}(u,v)$,

$$\begin{aligned}\mathbf{x_u} &= (1,0,3u^2-3v^2)\\ \mathbf{x_v} &= (0,1,-6uv)\end{aligned}$$

we can then find the coefficients of the first fundamental form:

$$\begin{aligned}E &= \langle \mathbf{x_u},\mathbf{x_u}\rangle = 1+(3u^2-3v^2)^2 = 1+9(u^2-v^2)^2\\ F &= \langle \mathbf{x_u},\mathbf{x_v}\rangle = -6uv(3u^2-3v^2) = -18uv(u^2-v^2)\\ G &= \langle \mathbf{x_v},\mathbf{x_v}\rangle = 1+(-6uv)^2 = 1+36u^2v^2\end{aligned}$$

Computing the second order partial derivatives, we obtain

$$\begin{aligned}\mathbf{x_{uu}} &= (0,0,-6u)\\ \mathbf{x_{uv}} &= (0,0,-6v)\\ \mathbf{x_{vv}} &= (0,0,-6u)\end{aligned}$$

To calculate the unit normal to the surface, N, we first calculate the vector cross product of $\mathbf{x_u}$ and $\mathbf{x_v}$:

$$\begin{aligned} \mathbf{x_u} \times \mathbf{x_v} &= \begin{pmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 0 & 3u^2 - 3v^2 \\ 0 & 1 & -6uv \end{pmatrix} \\ &= (3v^2 - 3u^2, 6uv, 1) \end{aligned}$$

Normalizing the result,

$$\begin{aligned} \|\mathbf{x_u} \times \mathbf{x_v}\| &= \sqrt{1 + 36(uv)^2 + 9(v^2 - u^2)^2} \\ &= \sqrt{1 + 9(u^2 + v^2)} \end{aligned}$$

and so

$$N = \frac{\mathbf{x_u} \times \mathbf{x_v}}{\|\mathbf{x_u} \times \mathbf{x_v}\|} = \frac{(3v^2 - 3u^2, 6uv, 1)}{\sqrt{1 + 9(u^2 + v^2)}}.$$

Now we find the coefficients of the second fundamental form:

$$\begin{aligned} e &= \langle N, \mathbf{x_{uu}} \rangle = \frac{-6u}{\sqrt{1 + 9(u^2 + v^2)}} \\ f &= \langle N, \mathbf{x_{uv}} \rangle = \frac{-6v}{\sqrt{1 + 9(u^2 + v^2)}} \\ g &= \langle N, \mathbf{x_{vv}} \rangle = \frac{-6u}{\sqrt{1 + 9(u^2 + v^2)}}. \end{aligned}$$

We can now write down the Gaussian curvature

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-36(u^2 + v^2)}{(1 + 9(u^2 + v^2))^2}$$

Note that the point $(0, 0, 0)$ on the monkey saddle is the image of the point $(0, 0)$ and *that $K = 0$. Not only is the Gaussian curvature 0, but given that $e = f = g = 0$ at $(0,0)$, then*

$$\begin{aligned} dN &= \frac{1}{EG - F^2} \begin{pmatrix} -e & -f \\ -f & -g \end{pmatrix} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \\ &= \frac{1}{EG - F^2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \\ &= 0 \end{aligned}$$

and so the point $(0,0)$ is a planar point. Additionally, for any other point (u,v) where u, v are not both equal to 0, each point is classified as a hyperbolic point as K will be less than 0. Because k_1, k_2 must have opposite sign at a hyperbolic point p, curves passing through p can have their normal vectors point towards either side of the tangent plane.

When a regular surface is given by the graph of a differentiable function f, calculations tend to be easier. We can write down a formula for K in terms of our differentiable function f. That is, we don't have to compute e, f, g, E, F, G if we know the surface is given as a graph of a differentiable function. As with the monkey saddle example, parameterize the surface with $\mathbf{x}(u,v) = (u, v, f(u,v))$. The coefficients of the first fundamental form are given by

$$\begin{aligned} E &= \langle \mathbf{x_u}, \mathbf{x_u} \rangle = \langle (1,0,f_u), (1,0,f_u) \rangle = 1 + f_u^2 \\ F &= \langle \mathbf{x_u}, \mathbf{x_v} \rangle = \langle (1,0,f_u), (0,1,f_v) \rangle = f_u f_v \\ G &= \langle \mathbf{x_v}, \mathbf{x_v} \rangle = \langle (0,1,f_v), (0,1,f_v) \rangle = 1 + f_u^2. \end{aligned}$$

The vector cross product of partial derivatives $\mathbf{x_u}$ and $\mathbf{x_v}$ is

$$\begin{aligned}\mathbf{x_u} \times \mathbf{x_v} &= \begin{pmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{pmatrix} \\ &= (-f_u, -f_v, 1)\end{aligned}$$

and so the unit normal is just

$$N = \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}}.$$

The coefficients of the second fundamental form are then

$$\begin{aligned} e &= \langle N, \mathbf{x_{uu}} \rangle = \left\langle \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}}, (0, 0, f_{uu}) \right\rangle = \frac{f_{uu}}{\sqrt{1 + f_u^2 + f_v^2}} \\ f &= \langle N, \mathbf{x_{uv}} \rangle = \left\langle \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}}, (0, 0, f_{uv}) \right\rangle = \frac{f_{uv}}{\sqrt{1 + f_u^2 + f_v^2}} \\ g &= \langle N, \mathbf{x_{vv}} \rangle = \left\langle \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}}, (0, 0, f_{vv}) \right\rangle = \frac{f_{vv}}{\sqrt{1 + f_u^2 + f_v^2}}. \end{aligned}$$

We now have all we need to write down a formula for Gaussian curvature:

$$K = \frac{eg - f^2}{EG - F^2} = \frac{f_{uu} f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2}.$$

In section (2), we demonstrated that the graph of a differentiable function $f(x, y)$ is a regular surface and it turns out that the converse is also true (see page 63 of [1]). Equally as important, we can write the second fundamental form in as the Hessian of $f(x, y)$. For a thorough treatment of regular surfaces as a graph of a differentiable function, see [6].

There is a very important link between Gaussian Curvature and the First Fundamental Form. The formula for the Gaussian curvature involves the first and second fundamental forms, or, when working in an orthonormal basis, the principal curvatures. We discussed earlier that the first fundamental form is an intrinsic quantity, however, the second fundamental form is not. Before we state a very important theorem, consider the following two examples.

Example 4. *Let* $\mathbf{x}(u,v) = (\cosh u \cos v, \cosh u \sin v, u)$ *be a parameterization for the catenoid, which is the surface of revolution of the catenary. Computing the first order partial derivatives*

$$\begin{aligned}
\mathbf{x_u}(u,v) &= (\sinh u \cos v, \sinh u \sin v, 1) \\
\mathbf{x_v}(u,v) &= (-\cosh u \sin v, \cosh u \cos v, 0)
\end{aligned}$$

we can then find the coefficients of the first fundamental form

$$\begin{aligned}
E &= \langle \mathbf{x_u}, \mathbf{x_u} \rangle = \sinh^2 u \cos^2 v + \sinh^2 u \sin^2 v + 1 \\
&= \sinh^2 u + 1 = \cosh^2 u \\
F &= \langle \mathbf{x_u}, \mathbf{x_v} \rangle \\
&= -\sinh u \cosh u \cos v \sin v + \sinh u \cosh u \cos v \sin v = 0 \\
G &= \langle \mathbf{x_v}, \mathbf{x_v} \rangle = \cosh^2 u \sin^2 v + \cosh^2 u \cos^2 v \\
&= \cosh^2 u.
\end{aligned}$$

Computing the second order partial derivatives, we obtain

$$\mathbf{x_{uu}}(u,v) = (\cosh u \cos v, \cosh u \sin v, 0)$$

$$\begin{aligned}
\mathbf{x_{uv}}(u,v) &= (-\sinh u \sin v, \sinh u \cos v, 0) \\
\mathbf{x_{vv}}(u,v) &= (-\cosh u \cos v, -\cosh u \sin v, 0)
\end{aligned}$$

To calculate the unit normal to the surface, N, we first calculate the vector cross product of $\mathbf{x_u}$ and $\mathbf{x_v}$:

$$\begin{aligned}
\mathbf{x_u} \times \mathbf{x_v} &= \begin{pmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \sinh u \cos v & \sinh u \sin v & 1 \\ -\cosh u \sin v & \cosh u \sin v & 0 \end{pmatrix} \\
&= (\cosh u \cos v, \cosh u \sin v, \sinh u \cosh u)
\end{aligned}$$

Normalizing the result, we have

$$\begin{aligned}
\|\mathbf{x_u} \times \mathbf{x_v}\| &= \sqrt{\cosh^2 u \cos^2 v + \cosh^2 u \sin^2 v + \sinh^2 u \cosh^2 u} \\
&= \sqrt{\cosh^2 u + \sinh^2 u \cosh^2 u} \\
&= \sqrt{\cosh^2 u (1 + \sinh^2 u)} \\
&= \cosh^2 u
\end{aligned}$$

and so

$$N = \frac{\mathbf{x_u} \times \mathbf{x_v}}{\|\mathbf{x_u} \times \mathbf{x_v}\|} = (\frac{\cos v}{\cosh u}, \frac{\sin v}{\cosh u}, \tanh u).$$

We can now find the coefficients of the second fundamental form

$$\begin{aligned}
e &= \langle N, \mathbf{x_{uu}} \rangle = \cos^2 v + \sin^2 v = 1 \\
f &= \langle N, \mathbf{x_{uv}} \rangle = -\tanh u \cos v \sin v + \tanh u \sin v \cos v = 0 \\
g &= \langle N, \mathbf{x_{vv}} \rangle = -\cos^2 v - \sin^2 v = -1.
\end{aligned}$$

Since we have the coefficients of the first and second fundamental forms, we can easily write down the Gaussian curvature, which is

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-1}{\cosh^4 u}.$$

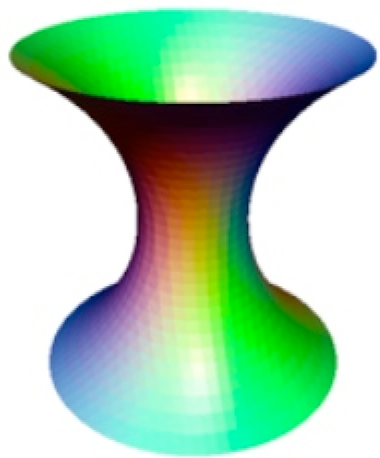


Figure 3: The Catenoid. Produced with Mac OS X Grapher

Example 5. *Let* $\sigma(u, v) = (u \cos v, u \sin v, v)$ *be a parameterization for the helicoid. Now make the change of parameters* $\hat{u} = \sinh u, \hat{v} = v$ *and we get a new parameterization* $\hat{\sigma}(u, v) = (\sinh u \cos v, \sinh u \sin v, v)$. *Computing the first order partial derivatives of* $\hat{\sigma}$

$$\begin{aligned}
\hat{\sigma}_{\mathbf{u}}(u, v) &= (\cosh u \cos v, \cosh u \sin v, 0) \\
\hat{\sigma}_{\mathbf{v}}(u, v) &= (-\sinh u \cos v, \sinh u \cos v, 1),
\end{aligned}$$

we can then find the coefficients of the first fundamental form

$$\hat{E} = \langle \hat{\sigma}_{\mathbf{u}}, \hat{\sigma}_{\mathbf{u}} \rangle = \cosh^2 u \cos^2 v + \cosh^2 u \sin^2 v$$

$$
\begin{aligned}
&= \cosh^2 u(\cos^2 v + \sin^v) = \cosh^2 u \\
\hat{F} &= \langle \hat{\sigma}_{\mathbf{u}}, \hat{\sigma}_{\mathbf{v}} \rangle \\
&= -\cosh u \sinh u \cos v \sin v + \cosh u \sinh u \cos v \sin v = 0 \\
\hat{G} &= \langle \hat{\sigma}_{\mathbf{v}}, \hat{\sigma}_{\mathbf{v}} \rangle = \sinh^2 u \sin^2 v + \sinh^2 u \cos^2 v + 1 \\
&= \sinh^2(\sin^2 v + \cos^2 v) + 1 = \sinh^2 + 1 = \cosh^2 u.
\end{aligned}
$$

From the above calculations, we see that $E = \hat{E}, F = \hat{F}, G = \hat{G}$ *where* $E, F,$ *and* G *were the coefficients of the first fundamental form for the catenoid. That is, the catenoid and helicoid are locally isometric. To carry this example further, we compute the second order partial derivatives of* $\hat{\sigma}$

$$
\begin{aligned}
\hat{\sigma}_{\mathbf{uu}}(u, v) &= (\sinh u \cos v, \sinh u \sin v, 0) \\
\hat{\sigma}_{\mathbf{uv}}(u, v) &= (-\cosh u \sin v, \cosh u \cos v, 0) \\
\hat{\sigma}_{\mathbf{vv}}(u, v) &= (-\sinh u \cos v, -\sinh u \sin v, 0).
\end{aligned}
$$

To calculate the unit normal to the surface, $\hat{N}$*, we first calculate the vector cross product of* $\hat{\sigma}_{\mathbf{u}}$ *and* $\hat{\sigma}_{\mathbf{v}}$*:*

$$
\begin{aligned}
\hat{\sigma}_{\mathbf{u}} \times \hat{\sigma}_{\mathbf{v}} &= \begin{pmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \cosh u \cos v & \cosh u \sin v & 0 \\ -\sinh u \cos v & \sinh u \cos v & 1 \end{pmatrix} \\
&= (\cosh u \sin v, -\cosh u \cos v, \sinh u \cosh u)
\end{aligned}
$$

Normalizing the result, we obtain

$$
\|\hat{\sigma}_{\mathbf{u}} \times \hat{\sigma}_{\mathbf{v}}\| = \sqrt{\cosh^2 u \sin^2 v + \cosh^2 u \cos^2 v + \sinh^2 u \cosh^2 u}
$$

$$
\begin{aligned}
&= \sqrt{\cosh^2 u + \sinh^2 u \cosh^2 u} \\
&= \sqrt{\cosh^2 u (1 + \sinh^2 u)} \\
&= \cosh^2 u
\end{aligned}
$$

and so

$$\hat{N} = \frac{\hat{\sigma}_{\mathbf{u}} \times \hat{\sigma}_{\mathbf{v}}}{\|\hat{\sigma}_{\mathbf{u}} \times \hat{\sigma}_{\mathbf{v}}\|} = (\frac{\sin v}{\cosh u}, \frac{-\cos v}{\cosh u}, \tanh u).$$

We can now find the coefficients of the second fundamental form

$$
\begin{aligned}
\hat{e} &= \left\langle \hat{N}, \hat{\sigma}_{\mathbf{uu}} \right\rangle = \tanh u \cos v \sin v - \tanh u \cos v \sin v = 0 \\
\hat{f} &= \left\langle \hat{N}, \hat{\sigma}_{\mathbf{uv}} \right\rangle = -\sin^2 v - \cos^2 v = -(\sin^2 v + \cos^2 v) = -1 \\
\hat{g} &= \left\langle \hat{N}, \hat{\sigma}_{\mathbf{vv}} \right\rangle = -\tanh u \cos v \sin v + \tanh u \cos v \sin v = 0.
\end{aligned}
$$

Since we have the coefficients of the first and second fundamental forms, we can easily write down the Gaussian curvature for the helicoid, which is

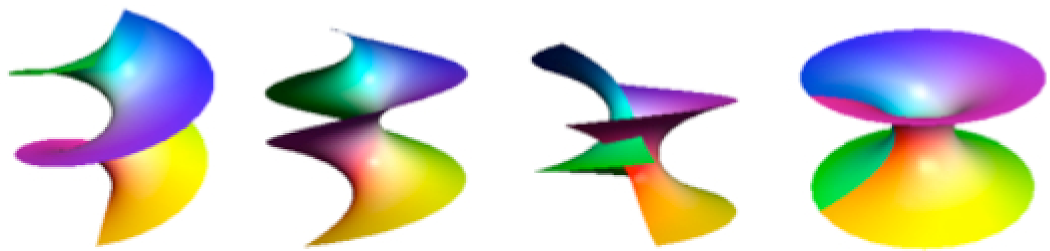
$$\hat{K} = \frac{\hat{e}\hat{g} - \hat{f}^2}{\hat{E}\hat{G} - \hat{F}^2} = \frac{-1}{\cosh^4 u}.$$

Below is a sequence of figures illustrating the continuous deformation of the helicoid into the catenoid given by the parameterization

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (\cos\theta)(\sinh v)(\sin u) \;+\; (\sin\theta)(\cosh v)(\cos u) \\ (-\cos\theta)(\sinh v)(\cos\ u) \;+\; (\sin\theta)(\cosh v)(\sin u) \\ (u)\cos\theta + (v)\sin\theta \end{bmatrix}
$$

where $-\pi < u < \pi, -\infty < v < \infty, 0 < \theta < \pi$ *[Weinstein].*

As discussed before Examples 4 and 5, the formula for Gaussian curvature depends



(a) $\theta = 0$ (b) $\theta = 0.5$ (c) $\theta = 1.5$ (d) $\theta = \frac{\pi}{2}$

Figure 4: Various stages of the continuous helicoid to catenoid deformation. Images produced with Mac OS X Grapher

on both the first and second fundamental forms, where the second fundamental form is not an intrinsic quantity. As such, it is surprising that $K = \hat{K}$. The two previous examples suggest perhaps that Gaussian curvature itself is an intrinsic quantity, and it turns out that this suggestion would be true. More precisely, we state Gauss's Theorema Egregium:

Theorem 4. *The Gaussian curvature K of a surface is invariant by local isometries [1].*

That is, one regular surface S_1 can be mapped isometrically into another regular surface S_2 only if the Gaussian curvature for each point $p \in S_1$ is equal to the Gaussian curvature for the corresponding point in S_2. Gauss proved his theorem by writing K in terms of the first fundamental form, which indicates that K is an intrinsic quantity and is thus invariant by local isometries (isometries preserve distance). In order to do so, we need to introduce the Christoffel symbols, $\Gamma^k_{ij}, i, j, k = 1, 2$, which are quantities that can be expressed in terms of the derivatives of the coefficients of the first fundamental form. First, Gauss defined a set of equations, called the Gauss Equations, which expressed the derivatives of $\mathbf{x_u}, \mathbf{x_v}$, and N (where $\mathbf{x}$ is a parameterization for a regular surface) in the basis $\{\mathbf{x_u}, \mathbf{x_v}, N\}$:

$$
\begin{aligned}
\mathbf{x_{uu}} &= \Gamma_{11}^1 \mathbf{x_u} + \Gamma_{11}^2 \mathbf{x_v} + eN && (17)\\
\mathbf{x_{uv}} &= \Gamma_{12}^1 \mathbf{x_u} + \Gamma_{12}^2 \mathbf{x_v} + fN && (18)\\
\mathbf{x_{vu}} &= \Gamma_{21}^1 \mathbf{x_u} + \Gamma_{21}^2 \mathbf{x_v} + fN && (19)\\
\mathbf{x_{vv}} &= \Gamma_{22}^1 \mathbf{x_u} + \Gamma_{22}^2 \mathbf{x_v} + gN && (20)\\
N_u &= a\mathbf{x_u} + b\mathbf{x_v} && (21)\\
N_v &= c\mathbf{x_u} + d\mathbf{x_v}. && (22)
\end{aligned}
$$

We've seen the last two expressions when we wrote down the matrix for the Gauss Map. We can write down an explicit formula for Γ_{ij}^k, but in $\mathbf{R^3}$, it is easier to express these symbols via a system of equations. To derive such a system, we take the dot product of equations 17 through 22 with $\mathbf{x_u}$ and $\mathbf{x_v}$:

$$
\begin{aligned}
\langle \mathbf{x_{uu}}, \mathbf{x_u} \rangle &= \left\langle \Gamma_{11}^1 \mathbf{x_u}, \mathbf{x_u} \right\rangle + \left\langle \Gamma_{11}^2 \mathbf{x_v}, \mathbf{x_u} \right\rangle + \langle eN, \mathbf{x_u} \rangle \\
&= \Gamma_{11}^1 E + \Gamma_{11}^2 F = \frac{1}{2} E_u \\
\langle \mathbf{x_{uu}}, \mathbf{x_v} \rangle &= \left\langle \Gamma_{11}^1 \mathbf{x_u}, \mathbf{x_v} \right\rangle + \left\langle \Gamma_{11}^2 \mathbf{x_v}, \mathbf{x_v} \right\rangle + \langle eN, \mathbf{x_v} \rangle \\
&= \Gamma_{11}^1 E + \Gamma_{11}^2 G = F_u - \frac{1}{2} E_v \\
\langle \mathbf{x_{uv}}, \mathbf{x_u} \rangle &= \left\langle \Gamma_{12}^1 \mathbf{x_u}, \mathbf{x_u} \right\rangle + \left\langle \Gamma_{12}^2 \mathbf{x_v}, \mathbf{x_u} \right\rangle + \langle fN, \mathbf{x_u} \rangle \\
&= \Gamma_{12}^1 E + \Gamma_{12}^2 F = \frac{1}{2} E_v \\
\langle \mathbf{x_{uv}}, \mathbf{x_v} \rangle &= \left\langle \Gamma_{12}^1 \mathbf{x_u}, \mathbf{x_v} \right\rangle + \left\langle \Gamma_{12}^2 \mathbf{x_v}, \mathbf{x_v} \right\rangle + \langle fN, \mathbf{x_v} \rangle \\
&= \Gamma_{12}^1 F + \Gamma_{12}^2 G = \frac{1}{2} G_u \\
\langle \mathbf{x_{vv}}, \mathbf{x_u} \rangle &= \left\langle \Gamma_{22}^1 \mathbf{x_u}, \mathbf{x_u} \right\rangle + \left\langle \Gamma_{22}^2 \mathbf{x_v}, \mathbf{x_u} \right\rangle + \langle gN, \mathbf{x_u} \rangle \\
&= \Gamma_{22}^1 E + \Gamma_{22}^2 F = F_v - \frac{1}{2} G_u
\end{aligned}
$$

$$\begin{aligned}\langle \mathbf{x_{vv}}, \mathbf{x_v}\rangle &= \left\langle \Gamma^1_{22}\mathbf{x_u}, \mathbf{x_v}\right\rangle + \left\langle \Gamma^2_{22}\mathbf{x_v}, \mathbf{x_v}\right\rangle + \langle gN, \mathbf{x_v}\rangle \\ &= \Gamma^1_{22}F + \Gamma^2_{22}G = \frac{1}{2}G_v\end{aligned}$$

Differentiating $\mathbf{x_{uu}}$ and $\mathbf{x_{uv}}$ by v and u, respectively, we have

$$\begin{aligned}(\mathbf{x_{uu}})_\mathbf{v} &= \Gamma^1_{11}\mathbf{x_{uv}} + \Gamma^2_{11}\mathbf{x_{vv}} + eN_v + (\Gamma^1_{11})_v\mathbf{x_u} + (\Gamma^2_{11})_v\mathbf{x_v} \\ &\quad + e_vN \qquad (23) \\ (\mathbf{x_{uv}})_\mathbf{u} &= \Gamma^1_{12}\mathbf{x_{uu}} + \Gamma^2_{12}\mathbf{x_{vu}} + fN_u + (\Gamma^1_{12})_u\mathbf{x_u} + (\Gamma^2_{12})_u\mathbf{x_v} \\ &\quad + e_uN. \qquad (24)\end{aligned}$$

Substituting the Gauss equations back into the above, we obtain

$$\begin{aligned}\Gamma^1_{11}\mathbf{x_{uv}} &= \Gamma^1_{11}(\Gamma^1_{12}\mathbf{x_u} + \Gamma^2_{12}\mathbf{x_v} + fN) && (25) \\ &= \Gamma^1_{11}\Gamma^1_{12}\mathbf{x_u} + \Gamma^1_{11}\Gamma^2_{12}\mathbf{x_v} + f\Gamma^1_{11}N && (26) \\ \Gamma^2_{11}\mathbf{x_{vv}} &= \Gamma^2_{11}(\Gamma^1_{22}\mathbf{x_u} + \Gamma^2_{22}\mathbf{x_v} + gN) && (27) \\ &= \Gamma^2_{11}\Gamma^1_{22}\mathbf{x_u} + \Gamma^2_{11}\Gamma^2_{22}\mathbf{x_v} + g\Gamma^2_{11}N && (28) \\ \Gamma^1_{12}\mathbf{x_{uu}} &= \Gamma^1_{12}(\Gamma^1_{11}\mathbf{x_u} + \Gamma^2_{11}\mathbf{x_v} + eN) && (29) \\ &= \Gamma^1_{12}\Gamma^1_{11}\mathbf{x_u} + \Gamma^1_{12}\Gamma^2_{11}\mathbf{x_v} + e\Gamma^1_{12}N && (30) \\ \Gamma^2_{12}\mathbf{x_{vu}} &= \Gamma^2_{12}(\Gamma^1_{12}\mathbf{x_u} + \Gamma^2_{11}\mathbf{x_v} + fN) && (31) \\ &= \Gamma^2_{12}\Gamma^1_{12}\mathbf{x_u} + \Gamma^2_{12}\Gamma^2_{12}\mathbf{x_v} + f\Gamma^2_{12}N. && (32)\end{aligned}$$

Since mixed partial derivatives commute, we set equation (23) equal to equation (24). Furthermore, if we use the above values to substitute back into (23) and (24), we obtain the equality

$$\begin{aligned} & \Gamma^1_{11}(\Gamma^1_{12}\mathbf{x_u} + \Gamma^2_{12}\mathbf{x_v} + fN) + \Gamma^2_{11}(\Gamma^1_{22}\mathbf{x_u} + \Gamma^2_{22}\mathbf{x_v} + gN) + eN_v + (\Gamma^1_{11})_v\mathbf{x_v} + \\ & e_vN \\ = \; & \Gamma^1_{12}(\Gamma^1_{11}\mathbf{x_u} + \boldsymbol{\Gamma^2_{11}}\mathbf{x_v} + eN) + \Gamma^2_{12}(\Gamma^1_{12}\mathbf{x_u} + \Gamma^2_{11}\mathbf{x_v} + fN) + fN_u + (\Gamma^1_{12})_u\mathbf{x_u} + \\ & (\Gamma^2_{12})_u\mathbf{x_v} + e_uN \end{aligned} \tag{33}$$

Using the fact that $\{\mathbf{x_u}, \mathbf{x_v}, N\}$ are linearly independent vectors, the above equality implies that the coefficients of $\mathbf{x_v}$ from both sides of the above equations are equal. Furthermore, equations (21) and (21) allow us to write N_u, N_v in terms of the coefficients a, b, c, d and $\mathbf{x_u}, \mathbf{x_v}$ where a, b, c, d were entries of the matrix that represented the Gauss map dN. So with the above, we have

$$\begin{gathered} \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} + (\Gamma^2_{11})_v + ed = \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} + (\Gamma^2_{12})_u + fb \\ \Leftrightarrow \\ \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} + (\Gamma^2_{12})_u - \Gamma^1_{11}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22} - (\Gamma^2_{11})_v = ed - fb \\ = \tfrac{1}{EG-F^2}(e(fF - gE) - f(eF - fE)) = \tfrac{1}{EG-F^2}(-E(eg - f^2) \\ = -EK \qquad . \end{gathered}$$

The last equation shows that we can write K in terms of the coefficients of the first fundamental form and their derivatives, thus showing that K is an intrinsic quantity and thus invariant under isometries. This is indeed surprising since the formula for K can also be expressed as

$$K = \frac{eg - f^2}{EG - F^2}$$

$$K = k_1 k_2$$

where the principle curvatures k_1, k_2 and the coefficients of the second fundamental form e, f, g are all extrinsic quantities. This is the reason why Gauss's theorem is appropriately named Theorema Egregium, which in English translates to "remarkable theorem."

5 Some Important Facts About Curvature

For a more intuitive flavor of K and Theorema Egregium, consider the plane and the cylinder. In section two, we spoke about how the cylinder and plane were locally isometric to each other. If one considers a piece of paper to be the plane, then our isometry simply rolls the paper into a cylinder (note this is a local isometry). This mapping preserves distances and angles (i.e the metric); an isometry would not stretch, crumple, or shrink the plane into the cylinder. So at a given point on the cylinder, the Gaussian curvature K will equal 0, which is the same as for the plane, and is guaranteed by Gauss's Theorema Egregium. This is very different from attempting to wrap a ball with a piece of paper; it cannot be done without distorting the paper. In doing so, we cannot possibly find a way to conform the paper to the ball isometrically. More precisely, Theorema Egregium guarantees that this is true since each point in a plane has $K = 0$ while given a sphere of radius r, each point of the sphere has $K = 1/r^2$. Therefore the plane cannot be mapped isometrically into the sphere.

While the last section showed that the Gaussian Curvature K of a regular surface is an intrinsic quantity, the converse of Theorema Egregium does not hold. That is, given two regular surfaces S_1 and S_2 with Gaussian Curvature K_1 and K_2, respectively, and a function $f : S_1 \to S_2$ such that $K_1(f(p)) = K_{2(p)}$ for all $p \in S_1$, then it does not follow that f is an isometry. To illustrate the point, consider the following regular surfaces parameterized by

$$\begin{aligned}
\mathbf{x}(u, v) &= (u \cos v, u \sin v, \log u) \\
\hat{\mathbf{x}}(u, v) &= (u \cos v, u \sin v, v)
\end{aligned}$$

Computing the partial derivatives of $\mathbf{x}(u, v)$,

$$\begin{aligned} \mathbf{x}_u &= (\cos v, \sin v, 1/u) \\ \mathbf{x}_v &= (-u \sin v, u \cos v, 0) \end{aligned}$$

we can then compute the coefficients of the first fundamental form

$$\begin{aligned} E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 1 + 1/u^2 \\ F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0 \\ G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = u^2. \end{aligned}$$

Since we have E, F, G, we can find the unit normal

$$N = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \frac{1}{\sqrt{1+u^2}}(-\cos v, \sin v, u)$$

and the coefficients of the second fundamental form

$$\begin{aligned} e &= \langle N, \mathbf{x}_{uu} \rangle = \frac{-1}{u\sqrt{1+u^2}} \\ f &= \langle N, \mathbf{x}_{uv} \rangle = 0 \\ g &= \langle N, \mathbf{x}_{vv} \rangle = \frac{u}{\sqrt{1+u^2}}. \end{aligned}$$

From the coefficients of the first and second fundamental forms, we compute

$$K_1 = \frac{eg - f^2}{EG - F^2} = (\frac{-1}{1+u^2})(\frac{1}{1+u^2}) = \frac{1}{(1+u^2)^2}.$$

In the same fashion, we compute the partial derivatives of $\hat{\mathbf{x}}(u, v)$,

$$\begin{aligned}\hat{\mathbf{x}}_u &= (\cos v, \sin v, 0) \\ \hat{\mathbf{x}}_v &= (-u \sin v, u \cos v, 1)\end{aligned}$$

The coefficients of the first fundamental form are

$$\begin{aligned}\hat{E} &= \langle \hat{\mathbf{x}}_u, \hat{\mathbf{x}}_u \rangle = 1 \\ \hat{F} &= \langle \hat{\mathbf{x}}_u, \hat{\mathbf{x}}_v \rangle = 0 \\ \hat{G} &= \langle \hat{\mathbf{x}}_v, \hat{\mathbf{x}}_v \rangle = u^2 + 1.\end{aligned}$$

Since we have $\hat{E}, \hat{F}, \hat{G}$, we can find the unit normal

$$\hat{N} = \frac{\hat{\mathbf{x}}_u \times \hat{\mathbf{x}}_v}{\|\hat{\mathbf{x}}_u \times \hat{\mathbf{x}}_v\|} = \frac{1}{\sqrt{1+u^2}}(\sin v, -\cos v, u)$$

and the coefficients of the second fundamental form

$$\begin{aligned}\hat{e} &= \langle N, \hat{\mathbf{x}}_{uu} \rangle = 0 \\ \hat{f} &= \langle N, \hat{\mathbf{x}}_{uv} \rangle = \frac{-1}{\sqrt{1+u^2}} \\ \hat{g} &= \langle N, \hat{\mathbf{x}}_{vv} \rangle = 0.\end{aligned}$$

From the coefficients of the first and second fundamental forms, we compute

$$K_2 = \frac{eg - f^2}{EG - F^2} = (\frac{1}{1+u^2})(\frac{-1}{1+u^2}) = \frac{-1}{(1+u^2)^2}.$$

So we see that while S_1 and S_2 have the same Gaussian Curvature, the coefficients of the first fundamental form for S_1 and S_2 are not equal. Therefore they cannot be isometric to each other.

Up to this point, we've concentrated mainly on the Gaussian Curvature K. However, in the previous section, we noted another type of curvature, namely the mean curvature given by the formula

$$H = \frac{1}{2}\text{trace}(dN) = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}$$

where dN is the matrix for the Gauss Map. Unlike the Gaussian curvature K, H is not intrinsic. As a simple example, consider the plane and cylinder (assume the cylinder has radius 1 for simplicity). We already know that $E = \hat{E}, G = \hat{G}, F = \hat{F}$ where E, G, F are coefficients of the first fundamental form for the plane and $\hat{E}, \hat{F}, \hat{G}$ are the coefficients of the first fundamental form for the cylinder. Additionally, we know that the plane and cylinder are isometric and have equal Gaussian curvature. Computing the mean curvature for the plane is simple: in the plane, if we intersect the plane with another, we get a straight line. In fact, any intersection yields a straight line. Since a straight line has 0 curvature, the principle curvatures $k_1 = k_2 = 0$. In addition to the formula above, we can also compute $H = \frac{(k_1+k_2)}{2}$ which is just 0 for the plane. Similarly, when we cut the cylinder with a plane perpendicularly, we're left with a circle of radius 1, which has curvature equal to 1. Continuing to slice the cylinder, the normal sections will vary from the circle to a straight line parallel to the cylinder, which has 0 curvature; the maximum and minimum principle curvatures are 1 and 0, respectively. So $H = 1/2$ for the cylinder, which is not equal to the mean curvature for the plane, even though the two surfaces are isometric. That is, H depends on the way in which a surface is embedded in $\mathbb{R}^n$ and hence is extrinsic.

In section 3, we defined points of a regular surface S based on the value of K. That is, a point could be called elliptic, hyperbolic, parabolic or planar depending on the value of K (and in the case of planar, if the matrix $dN = 0$). On a similar note, when $H = 0$ for all points $p \in S$, we say that S is a minimal surface. The word "minimal" is used to describe such surfaces because surfaces with mean curvature 0 minimize surface area. For surfaces of revolution, this is a classical variational problem. That is, given some curve $y = f(x)$ on the interval $[a, b]$, which function f produces the least surface area when revolved around the $y-$axis [5]? When a surface is given by the graph of a function, we can produce a partial differential equation to solve which would produce an f that minimizes surface area. So let $\mathbf{x}(x, y) = (x, y, f(x, y))$ be a parameterization for a regular surface given by the graph of a function f. In section 4 we wrote down all we need to compute H. We had calculated the following:

$$\begin{aligned}
\mathbf{x}_x &= (1, 0, f_x) \\
\mathbf{x}_y &= (0, 1, f_y) \\
\mathbf{x}_{xx} &= (0, 0, f_{xx}) \\
\mathbf{x}_{xy} &= (0, 0, f_{xy}) \\
\mathbf{x}_{yy} &= (0, 0, f_{yy}) \\
E &= \langle \mathbf{x}_x, \mathbf{x}_x \rangle = 1 + f_x^2 \\
F &= \langle \mathbf{x}_x, \mathbf{x}_y \rangle = f_x f_y \\
G &= \langle \mathbf{x}_y, \mathbf{x}_y \rangle = 1 + f_y^2 \\
e &= \langle N, \mathbf{x}_{xx} \rangle = \frac{f_{xx}}{\sqrt{f_x^2 + f_y^2 + 1}} \\
f &= \langle N, \mathbf{x}_{xy} \rangle = \frac{f_{xy}}{\sqrt{f_x^2 + f_y^2 + 1}} \\
g &= \langle N, \mathbf{x}_{yy} \rangle = \frac{f_{yy}}{\sqrt{f_x^2 + f_y^2 + 1}}
\end{aligned}$$

With the above we can write down the mean curvature for the surface parameterized by the graph of a function f as

$$H = \frac{f_{xx}(1+f_x^2) - 2f_{xy}f_xf_y + f_{yy}(1+f_y^2)}{2(\sqrt{f_x^2+f_y^2+1})(1+f_y^2+f_x^2+f_x^2f_y^2)}.$$

When we set $H = 0$, the following equation

$$0 = f_{xx}(1+f_x^2) - 2f_{xy}f_xf_y + f_{yy}(1+f_y^2) \tag{34}$$

is known as the minimal surface equation and can be solved to produce an f such that the surface given by the graph of f is a minimal surface. For information on how to solve (34), consult a text on partial differential equations, such as [8]. There are numerous examples of minimal surfaces, however, other than the plane, the earliest non-trivial examples of minimal surfaces were the helicoid and catenoid, discovered by Meusnier in 1776 (see [1] page 205). Interestingly, the catenoid is the only surface of revolution which is minimal (see [1] page 202). An example of a recently discovered minimal surface is the genus one helicoid pictured below.

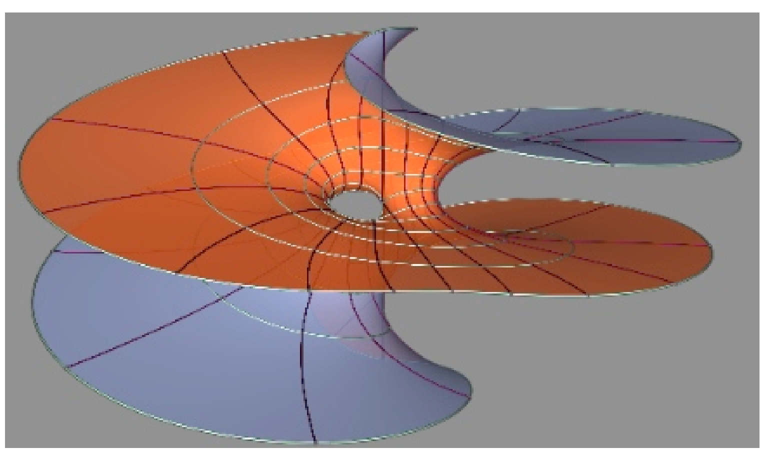


Figure 5: Genus 1 Helicoid discovered by Hoffman, Karcher and Wei [9]. Reproduced under Creative Commons License 3.0

The mean curvature, and in particular, when $H = 0$, can give us some insight into the geometry and topology of a regular surface. Recalling that given a vector $v \in T_p(S)$ is an asymptotic direction if the normal curvature, k_n is 0, then consider the following lemma.

Lemma 1. *If the mean curvature is zero at a non-planar point p, then p has two orthogonal asymptotic directions.*

Proof. If p is a non-planar point, then our principle curvatures k_1, k_2 are non zero at p. Let $\{e_1, e_2\}$ be an orthonormal basis of $T_p(S)$ and let $v \in T_p(S)$. Since our basis for the tangent space is orthonormal, we may write $v = e_1 \cos\theta + e_2 \sin\theta$. Now v is an asymptotic direction if

$$k_n = \langle dN(v), v\rangle = k_1 \cos^2\theta + k_2 \sin^2\theta = 0.$$

Since $H = 0$ at p, then $k_1 = -k_2$ and our above equation can be written as

$$k_1(\cos^2\theta - \sin^2\theta = 0$$
$$\Rightarrow \cos^2\theta = \sin^2\theta$$
$$\Rightarrow \theta = \tfrac{\pi}{4}, \tfrac{3\pi}{4}.$$

Plugging in θ, we can write our two asymptotic directions as

$$\begin{aligned} v_1 &= \frac{\sqrt{2}}{2}e_1 + \frac{\sqrt{2}}{2}e_2 \\ v_2 &= -\frac{\sqrt{2}}{2}e_1 + \frac{\sqrt{2}}{2}e_2 \end{aligned}$$

and observe that

$$\langle v_1, v_2\rangle = -\frac{1}{2}e_1e_1 + \frac{1}{2}e_1e_2 - \frac{1}{2}e_1e_2 + \frac{1}{2}e_2e_2$$

$$= -\frac{1}{2} + \frac{1}{2} = 0$$

which shows that v_1, v_2 are two orthogonal asymptotic directions. $\square$

As far as topology is concerned, recall that a compact surface is one that is closed and bounded. Examples of compact surfaces include the sphere and torus. Note how the plane and catenoid, both minimal surfaces, are not compact. This prompts the following lemma.

Lemma 2. *There are no compact (closed and bounded) minimal surfaces.*

Proof. Let's assume that there is a compact minimal surface S. Since S is closed and bounded, we can enclose S in a sphere $\widehat{S}$ centered at a point O such that there is some point p in S that touches $\widehat{S}$. Let $\alpha(s)$ be any curve on S parameterized by arc length such that $\alpha(0) = p$. Since $\alpha(s)$ is a vector valued function, note that $|\alpha(s)|$ is equal to the distance of a point on the curve to the origin O and observe that $\alpha(0) = p$ is a local maximum. Let $h(s) = |\alpha(s)|^2$. Since $\alpha(0)$ is a local maximum, $h'(0) = 0$ and $h''(0) \leq 0$. Computing $h'(s)$ and $h''(s)$,

$$\begin{aligned} h'(s) &= \frac{d}{ds}(|\alpha(s)|^2) = 2\langle \alpha(s), \alpha'(s)\rangle \\ h''(s) &= \frac{d}{ds}(2\langle \alpha(s), \alpha'(s)\rangle) = 2\langle \alpha'(s), \alpha'(s)\rangle + \langle \alpha(s), \alpha''(s)\rangle \end{aligned} \tag{35}$$

Since $h'(0) = 0 = 2\langle \alpha(s), \alpha'(s)\rangle$, the vector $\alpha(0) = \vec{OP}$ is orthogonal to the tangent vector $\alpha'(0)$ where $\vec{OP}$ is the vector from the origin O to the point P. Since $\alpha'(0)$ is a vector in $T_p(S)$, it follows that $\vec{OP}$ is normal to S. If we plug in $s = 0$ in equation (35), we have

$$h''(0) = 2|\alpha'(0)|^2 + 2\left\langle \vec{OP}, \alpha''(0)\right\rangle \tag{36}$$

and using the Frenet Relation $\alpha'' = k\mathbf{n}$ (see [1]) where k is the curvature of α and $\mathbf{n}$ is the unit normal, equation (36) becomes

$$h''(0) = 2|\alpha'(0)|^2 + 2\left\langle \vec{OP}, k\mathbf{n}(0) \right\rangle \leq 0. \tag{37}$$

Since $|\alpha'(0)| = 1$,

$$\begin{aligned} h''(0) &= 2|\alpha'(0)|^2 + 2\left\langle \vec{OP}, k\mathbf{n}(0) \right\rangle \\ &= 2 + 2\left\langle \vec{OP}, k\mathbf{n}(0) \right\rangle \\ &= 2 + 2\left\langle \frac{\vec{OP}}{|\vec{OP}|}, k\mathbf{n}(0) \right\rangle |\vec{OP}| \\ &= 2 + 2\left\langle \mathbf{N}, \mathbf{n} \right\rangle k|\vec{OP}| \\ &= 2 + 2k_n(p)|\vec{OP}| \leq 0 \end{aligned} \tag{38}$$

where N is the unit normal to S and $k_n(p)$ is the normal curvature. Using equation (38) and the definition of normal curvature, we have

$$1 + \langle dN(v), v \rangle |\vec{OP}| \leq 0$$
$$\Rightarrow \langle dN(v), v \rangle \leq \tfrac{-1}{|\vec{OP}|}.$$

From the above equation, if v_1, v_2 are principle directions in $T_p(S)$, then

$$\langle dN(v_1), v_1 \rangle = k_1 \leq \frac{-1}{|\vec{OP}|}$$
$$\langle dN(v_2), v_2 \rangle = k_2 \leq \frac{-1}{|\vec{OP}|}$$

where $k_1 < 0, k_2 < 0$ are the principle curvatures. If both principle curvatures are less than 0, then $k_1 \neq -k_2$, however, this is what would be required for S to be a minimal surface. Therefore, this contradicts the assumption that S is a minimal surface and so we conclude that there are no compact minimal surfaces. $\square$

Note that since $k_1 < 0, k_2 < 0$, $K = k_1 k_2 > 0$. Thus the above calculations also show that if a regular surface S is closed and bounded, than it has at least one elliptic point.

6 Conclusion

Curvature is fundamental to the study of differential geometry. In this paper, two types of curvature for surfaces were discussed: intrinsic (Gaussian) and extrinsic (normal, principle and mean). It is rather remarkable that even though we can express Gaussian curvature K in terms of the first and second fundamental forms, where the second fundamental form is not an intrinsic quantity, K is still intrinsic. Additionally, if we do the calculations with an orthonormal basis, we can express K as the product of principle curvatures k_1 and k_2, where the principle curvatures are the maximum and minimum normal curvatures (obtained from intersecting the surface with a plane) and are extrinsic quantities. Yet K is still intrinsic. We also saw how the value of K at a point p describes the geometry near p. For example, when $K > 0$ at a point p, we find that the normal vectors for any curve going through p all point towards the same side of the tangent plane at p. When $K < 0$, the normal vectors can point towards either side of the tangent plane. Additionally, we saw how curvature plays a role in the topology of regular surfaces (i.e compact surfaces have at least one elliptic point). A far more complicated example of this which is beyond the scope of this paper is Hilbert's theorem, which states that a complete surface S with constant negative curvature cannot be isometrically immersed in $\mathbb{R}^3$ [1].

There are still numerous topics in differential geometry, such as geodesics and the exponential map that are beyond the scope of this paper. Additionally, the concepts that were discussed in this paper also extend to n dimensional space and to abstract manifolds, which is the branch of mathematics called Riemannian Geometry. In Riemannian Geometry, the Riemannian curvature tensor R_{ijkl} is analogous to the matrix dN that was discussed. That is, from it, we can extract quantities such as sectional curvature, which is the n dimensional analogue of Gauss Curvature. Also, the ideas presented in this paper extend to physics in the area of study called general relativity. For example, Einstein considered space and time to be a 4 dimensional manifold endowed with a metric. With this, he was able to describe gravity mathematically by quantifying how much space time curves in the presence of mass (although this is a rather simplistic explanation). All of these concepts are explained in terms of curvature of manifolds, and as such, curvature represents the most important and essential ingredient in the subject.

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