GAMBLING THEORY AND STOCK OPTION MODELS

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ABSTRACT OF THE DISSERTATION

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This thesis investigates problems both in gambling theory and in stock option models. In gambling theory, we study the difference between the Vardi casino and the Dubins-Savage casino. In the simple Dubins-Savage casino there is only one table in which a sub-fair gamble is available fixed odds ratio, r and the problem is to change a fortune of size f to a fortune of size 1 with maximum probability before going broke. Vardi proposed the casino where there is available a table for each odds ratio r. Since the Dubins-Savage casino can be duplicated in the Vardi casino, it is clear that the Vardi casino will provide a bigger probability to achieve the goal than the Dubins-Savage casino. A main result of the thesis is to show that the advantage of the Vardi casino is surprisingly small. This implies the surprising conclusion that it does not really help the gambler to have a variety of gambles available, and raises the question of why casinos in the real world have such a variety of gambles. In particular, the optimal probabilities of the Vardi casino and the Dubins-Savage casino with odds ratio r = 1 (red-and-black) agree to three decimal places. We further conjecture that the largest difference

between the Vardi and the Dubins-Savage optimal probabilities occurs at $f = 1/3$. The thesis also studies the two classic stochastic models involved in finance and economics, the additive Bachelier model and the multiplicative Black-scholes model. Both models have advantages and shortcomings. Chen et al [6] introduced a general class of models with decreasing-return- to-scale indexed by a parameter interpolating between the additive ($\theta = 0$) and the multiplicative ($\theta = 1$) cases. We study the American and the Russian option under the decreasing-return-to-scale models and give the optimal policy of each option for these new models. The two parts of the thesis are related through the fact that gambling is involved in each case, this despite the fact that investors often prefer to believe there is no gambling involved in their activity. Of course gamblers often believe this as well. Furthermore, among the stocks with the same negative drift, in order to maximize the probability to achieve a particular amount of fortune to survive for the gamblers problem of stocks (see [29] [30]), they need to buy those stocks with big volatilities (odds ratios).

Preface

Gambling is one of the oldest businesses in the world. Lots of people are drawn to it. They spend lots of time and money on gambling. Some become rich, but most of them lose their money. For us, it may be more important that gambling has helped develop both statistics and probability theory. To quote the pioneer Louis Bachelier [1], 'It is almost always gambling that enables one to form a fairly clear idea of a manifestation of chance; It is gambling that gave birth to the calculus of probability; it is to gambling that this calculus owes its first faltering utterances and its most recent developments; it is gambling that enables us to conceive of this calculus in the most general way; it is, therefore, gambling that one must strive to understand, but one should understand it in a philosophic sense, free from all vulgar ideas.

Stock and derivatives, in some sense, are simply a fancy version of gambling. The only difference is that gambling is discrete while stock and derivatives are continuous and gambling has a clear winning odds ratio and probability while stock and derivatives don't. Actually, the stock and derivatives market are the biggest casino in the world. Lots of people study them and build models for stock and economic fluctuations. However, today's finance crisis proved that no model can perfectly fit the real world. And failure to understand the model, especially the assumptions of the model, will lead to serious consequences, not only for individuals, but for their countries as well.

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Dedication

To my beloved son

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Chapter 1
Introduction

In the Dubins-Savage book [1] it is supposed that a gambler, with an initial fortune, f, less than 1, has to achieve fortune 1. For us, there are two casinos available: the Dubins-Savage casino and the Vardi casino. The Dubins-Savage casino has only one table while the Vardi casino has an infinity of tables. Different tables have different odds ratios, r but the same expected unit return c. To maximize the probability to achieve the goal, the gambler needs to play boldly in the Dubins-Savage casino, which is a subtle and celebrated theorem of Dubins and Savage. The optimal strategy in the Vardi casino is to play timidly. Of course, the Vardi casino provides a bigger probability to achieve the goal. However, the difference is, surprisingly, not big. Particularly, for $c = -.02$, the difference between the utility function of the Vardi casino and the Dubins-Savage casino with odds ratio $r = 1$ (red-and-black) agree to three decimals. And we further conjecture that the biggest difference happens at $f = 1/3$, where the difference is 0.007814498.

Stock and derivative markets are just a version of gambling. However, unlike the real casinos, the apprepriate models are not so clear. Of the two classic stochastic models in finance and economics, the Bachelier model and the Black-Scholes model, one is additive and one is multiplicative in the action of noise on price. Both have their own advantages and shortcomings. Ren-raw Chen, Oded Palmon and Larry Shepp [5] introduced a general class of models with "decreasing-return-to-scale", indexed by a parameter θ interpolating between the additive ($\theta = 0$) and the multiplicative ($\theta = 1$) cases. We show that among the stocks with the same negative drift, investors need to buy those with higher volatility to maximize the probability to achieve a certain amount of fortune. Thich is just like the case of the Vardi casino. We also study the American and so-called Russian option under the 'decreasing-return-to-scale' models and give the optimal policy of American option and Russian option problem for each of these new models.

1.1 Gambling theory

Mr. A owns \$100,000 to a loan shark and will be killed at dawn if the loan is not repaid in full, but he only has \$10,000. Mr. B wants to buy a TV (price \$1000), although he only has \$100. The only way that Mr. A and Mr. B can make enough money is through gambling in a casino. So basically, each gambler starts with a fortune $f < 1$ and can stake any amount of money in his possession, gaining r times the stake with probability ω and losing the stake with probability $\overline{\omega} = 1 - \omega$ ($r > 0, 0 < \omega < 1$). Gamblers want to find the optimal strategy to maximize the probability to achieve the goal, i.e. $f = 1$. Hence, as in [1], the gambler's problem is formulated as following:

A gambler begins with a fortune f_0, and as play progresses he moves successively through a sequence of fortunes $f_1, f_2, \cdots$. Let F be the set of all fortunes. A gamble is a probability measure γ on subsets of fortunes. A casino is a function Γ that associates with each f a nonempty set $\Gamma(f)$ of gambles γ among which the gambler is allowed to choose when his fortune is f. A partial history p is a finite (possible vacuous) sequence of fortunes $(f_1, \cdots, f_n)$, and a strategy is a function σ associating a gamble $\sigma(p)$ with each partial history p. Equivalently, a strategy σ is a sequence $\sigma_0, \sigma_1, \cdots$, where σ_0 is a gamble, and for each positive n, σ_n is a function that associates with each finite sequence of fortunes $(f_1, \cdots, f_n)$ a gamble $\sigma_n(f_1, \cdots, f_n)$.

The gambler's desire or aspiration is to maximize the probability of terminating play at a fortune that is satisfactory. It can be summarized by a function u from fortunes to real numbers, called the *utility function*. The worth or utility of the fortune f to the gambler is $u(f)$. Since the gambler's only objective is to cease gambling with a numerical fortune as large as f' if possible, then $u(f)$ can be taken to be 0 or 1 according as f is or is not exceeded by f'. In this paper, $f' = 1$ and $u(f) = 1\,(0)$ if $f \geq\ (\leq) 1$. According to the intended interpretation of utility, the gambler will value a policy π according to the expected value under π of the utility of its terminal fortune f_t. The gambler's objective is to make $u(\pi)$ as large as possible subject to his initial fortune f and the rules of the gambling house Γ or, if he finds that preferable, to settle for $u(f)$. Therefore, let

$U(f)$=the maximum of sup $u(\pi)$ and $u(f)$

confining the supremum to those policies π that are available in Γ at f, and call the number $U(f)$ the *utility* of Γ at f. Hence, the "gambler's problem" is to find U when Γ and u are given and to determine policies that are optimal or nearly optimal. Like other problems dealing with an "unknown future", we need some kind of martingale tools to solve the problem. Here in order to prove some function Q is the utility funcction we need to show that it is *excessive* [1] for Γ, i.e. $\gamma Q \leq Q(f)$ for all the f and γ in $\Gamma(f)$.

In the case of superfair casino, the law of large numbers will guarantee the gambler achieves his goal if he plays timidly each time. So the gambler's problem will focus on the sub-fair casino cases. There are some variations of the gambler's problem. In the well-known casino of Dubins and Savage, there is only one table n the casino and the gambler starts with initial fortune of size $0 < f < 1$, and wants to achieve fortune of size 1. He can bet any amount $0 < s \leq f$, gaining a fixed odds ratio r times the stake with probability ω or losing the stake with probability $\overline{\omega} = 1 - \omega$. The expected return on a dollar bet is

$$r * \omega - 1 * \overline{\omega} = (r+1) * \omega - 1 = c < 0.$$

Dubins and Savage [1] showed that the gambler maximizes the chance to achieve the goal if he plays "boldly": he should bet his entire fortune or just enough of it to reach the goal. Some may think that this is so obvious it does not even require proof, but there are variants which are even more "obvious" that are false. For example, suppose the gambler's entire fortune is reduced by dividing by $1+\alpha$, where α is a small positive number, after every bet. It is called a casino with presence of inflation. Now it is even more obvious that you want to minimize the number of bets by playing boldly. Although the bold strategy is indeed optimal for sub-fair primitive casinos with inflation for certain initial fortunes or odds ratios [6] [9], in general, bold play is not optimal; the gambler can get a strictly greater survival probability by making a smaller bet when the initial fortune belongs to a certain infinite set [7]. Yehuda Vardi raised the question of how-to-gamble-if-you-must in a casino where any bet is available as long as its expected value is less than or equals to some particular number c. We call this casino Vardi's casino. Vardi's casino is a typical casino in which there are a variety of bets available such as roulette or slots with long odds. He asked whether this would give a much better chance for the gambler to achieve the goal under the optimal play. Larry Shepp [11] showed that the gambler's optimal probability to achieve fortune 1 with initial fortune f is

$$V(f) = 1 - (1-f)^{(1+c)},\ 0 \leq f \leq 1.$$

It is only attainable within ϵ, and an ϵ-optimal betting strategy is to bet *small* amounts at the *odds* just enough to provide a fortune 1 if the gambler wins. For Vardi's casino with inflation, Grigorescu *et al*. [13] showed that whether the bold play is optimal also depends on the choice of parameters.

So what is the advantage of the Vardi's casino over the Dubins-Savage casino for real gambling? Say, $\omega = 0.49$ and $c = \omega * 1 + (1-\omega) * (-1) = -0.02$ in the Dubins-Savage casino. This is roughly the case for the pass bet at the craps table. One might believe that this quantity would be large for $c = -0.02$ because of the existence of casinos in which there are a wide variety of odds available, but it is not so. This thesis shows that the two casinos give nearly the same value and the difference is less than 0.008. The fact that the difference is so small makes one wonder why there are casinos with a wide variety of odds. A gambler who wants to turn a fortune of size $f < 1$ into one of size one hardly does better in the Vardi's casino than in the Dubins-Savage casino. This means that in a real casino it does not really help the gambler to have variety of gambles available. This is quite surprising.

1.2 Stock model

Playing the stock and derivatives markets, in some sense, is simply a fancy version of gambling. The only difference is that gambling is discrete while stock and derivatives are continuous and gambling has a clear winning odds ratio and probability while playing the stock and derivatives market don't. So analogous to the theorem of the Vardi casino, one would expect that one should invest in those stocks with higher volatility in order to achieve a certain fortune. However, in order to measure the odds and the expected return, we need models to formulate the stock and derivatives problems.

In 1900, mathematician Louis Bachelier at the Sorbonne said in his thesis, "Theorie de la speculation", the price of an asset underlying an option could be modeled as a Brownian motion, also known as a continuous random walk. Just as a drunken sailor is equally likely to wander left or right, Bachelier assumed that financial assets are priced so that they're equally likely to rise or fall by the same amount. This leads to the Bachelier model:

$$dX(t) = \mu dt + \sigma dW(t)$$

or equivalently

$$X(t) = x_0 + \mu t + \sigma W(t)$$

where $X(t)$ is the stock price, μ is the drift, σ is the diffusion, $W(t)$ is the standard Brownian motion and $x_0 = X(0)$ is the stock price at time 0. The Bachelier model is effective in the sense that the optimal decisions made within the model are at least qualitatively consistent with observed decisions made by real firms [2]. However, some people don't like to use it to model the stock price. In the Bachelier model, the price of the stock could take negative values with probability $e^{\frac{-2\mu x}{\sigma^2}}$, and negative value is an 'impossibility' for stocks. In early days of market modelling, it was seen as a disadvantage, and in 1965, Paul Samuelson [14] [15] proposed that price changes should instead be proportional to the level of the asset price. Under Samuelson's geometric Brownian motion, even long stream of bad luck would leave the price positive. Based on Samuelson's work, in 1973 Fischer Black, Myron Scholes and Robert Merton [4] introduced the Black-Scholes-Merton model:

$$dX(t) = X(t)(\mu dt + \sigma dW(t))$$

or equivalently,

$$X(t) = x_0 e^{\mu t - \frac{1}{2}\sigma^2 t + \sigma W(t)}$$

where $X(t)$ is the stock price, μ is the growth rate, σ is the volatility, $W(t)$ is the standard Brownian motion and $x_0 = X(0)$ is the stock price at time 0. As we can see, $X(t)$ will never go negative or hit zero in this model. There were other "returns-to-scale" reasons advanced in [4] for preferring a multiplicative model. So the Black-Scholes-Merton model soon became the fundamental model for financial engineering.

The derivatives market today is huge. At the end of June 2007, the notional value of over-the-counter derivatives was \$516 trillion, according to the Basel, Switzerland-based Bank for International Settlements, the bank for central banks. The fundamental theory of derivatives is based on the Black-Scholes-Merton model and martingale theory. However, a model is only a model. Although the Black-Scholes-Merton model is very successful and the idea that 'the price change should be proportional to the level of the asset price' does somehow reflect the reality, there are some potential dangers in using it in modeling a stock and derivative price. And failure to understand the model, especially the assumptions of the model, will lead to serious consequences. As Samuelson said, "I'm never sure of the more than 10 million people who are exposed to that, whether there are more than 10 of them who understand my nuances". And the stock market crash of 2008 seems to have justified his words.

The first danger is that, in the model, parameters like interest rate and volatility are fixed, while in reality they are not. And in the model, the price is markovian. But in reality people never treated stock prices as Markovian. They always want to predict the future by estimating the past. Today the world is facing the worst financial crisis since WW2. The Dow Jones Industrial Average index (DJIA) fell from the peak of 14,000 points to nearly 8,000 points. SP 500 and NASDAQ and all the other stock indices around the world sank as well (see figure 1.1, 1.2, 1.3). The bank system froze up. The whole world is in the shadow of recession. No doubt that there is a strong relationship between interest rate and stock market. The interest rate of the United States was very low since 1990s. The Federal funds overnight interest rate historical data is showed in figure 1.4. And when the interest rate is low, the stock price goes up. The reason is that when $r < \mu$, by the Black-Scholes-Merton model,

$$EX(t)e^{-rt} = Ex_0 e^{(\mu-r)t-\frac{\sigma^2}{2}t+\sigma W(t)} = x_0 e^{(\mu-r)t} \to \infty \text{ as } t \to \infty.$$

So everyone will buy the stock. Furthermore, as stated in chapter 3, when $\mu > r$, the value of some options like the American call option will go to ∞ under the Black-Scholes-Merton model. So low interest rate can easily blow up a bubble of prosperity and make every one rich, at least at first. But when the interest rate changed, everything changed immediately.

The second danger of the Black-Scholes-Merton model is that under the model, the stock price will never go to zero. Under the Bachelier model, $X(t)$ could reach

Figure 1.1: DJIA 2003-2008

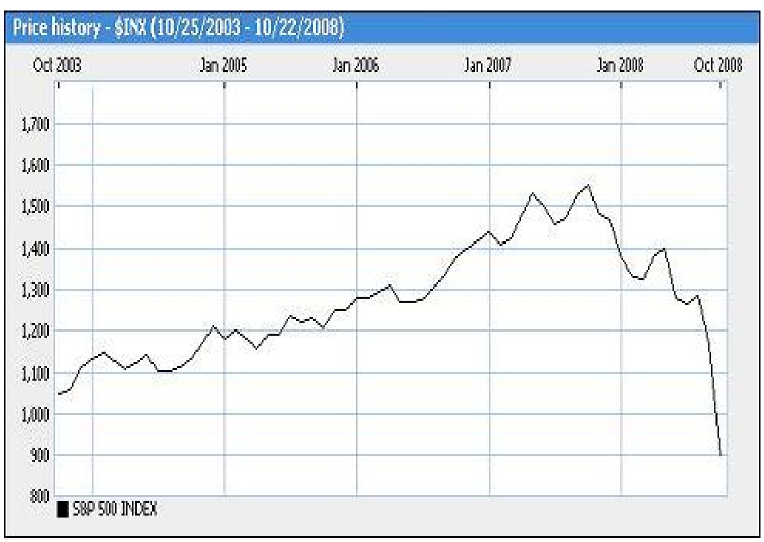

Figure 1.2: SP 500 2003-2008

Figure 1.3: NASDAQ 2003-2008

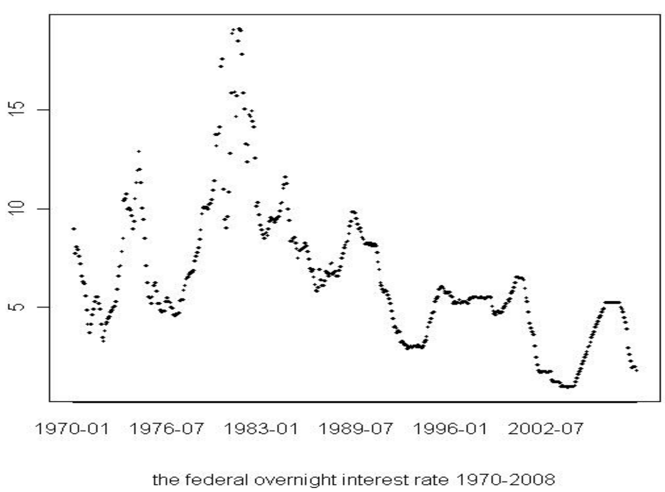

Figure 1.4: federal reserve overnight interest rate 1970-2008

Figure 1.5: Lehman brothers stock price 1998-2008

zero. It would be reasonable to simply regard reaching zero level as bankruptcy, so negative values can't occur in the Black-Scholes-Merton model which means that a company will never go bankrupt. But in reality, no company lasts forever. When the market situation is good, the Black-Scholes-Merton model seems to be OK. However, when the market situation goes bad, the stock price drops dramatically and will go to zero when the company goes bankrupt, just like the Lehman Brothers and Bear Sterns (see figure 1.5). In these cases, the Black-Scholes-Merton obviously failed and investors lost their money. Hence, investors who have observed the market closely might consider this aspect of the Black-Scholes-Merton highly unrealistic and any conclusions based on such a model dangerous.

So the 'never goes to zero' thing is not all good. Samuelson said that financial engineering is like the science that can help mankind or create atomic bombs. "Under proper regulation and with optimal transparency, it can spread risk efficiently and in that sense reduce intrinsic riskiness, But sans transparency and lacking understanding of the arithmetic of cancerous leveraging, maybe it introduces into modern finance new fragility?" What he said become true. Now the 'atomic bomb' called 'subprime mortgage' has exploded and the whole world is in serious trouble.

Because both of the models have their own advantages and shortcomings, Renraw Chen, Oded Palmon and Larry Shepp [5] introduced a general class of models with "decreasing-return-to-scale" indexed by a parameter θ interpolating between the Bachelier model ($\theta = 0$) and the Black-Scholes-Merton model ($\theta = 1$). i.e. The price of a company $X(t)$ at each time t is assumed to vary according to the stochastic differential equation

$$dX(t) = X^{\theta}(t)(\mu dt + \sigma dW(t)),\ t > 0,\ X(0) = x,$$

where the exponent $0 \leq \theta \leq 1$ is fixed, μ, known, is a measure of the rate of growth, σ, known , is volatility, r, known, is the fixed rate of interest. This class of models is somewhere between the Bachelier model and the Black-Scholes-Merton model. For $0 < \theta < 1$, the group of models are robust under the change of interest rate because $X(t)e^{-rt}$ is dominated by the e^{-rt} term.

$dX(t)e^{-rt} = e^{-rt}(-rX(t) + \mu X^{\theta}(t))dt + e^{-rt}\sigma X^{\theta}(t)dW(t)$. So when $X(t)$ becomes big, the $-rX(t) + \mu X^{\theta}(t)$ term will become negative no matter how big or small the r is. Hence, the $X(t)e^{-rt}$ will not go to ∞ even when the interest rate is small. Furthermore, for $0 < \theta < 1$, under optimal profit taking, the firm goes bankrupt w.p.1 in a finite time as in the case $\theta = 0$, the probability, $Q(x)$, of the firm reaching zero starting from x, i.e., going bankrupt in finite time, is positive and is given by :

$$Q(x) = \frac{\int_x^\infty e^{-\frac{2\mu}{\sigma^2(1-\theta)}u^{(1-\theta)}}du}{\int_0^\infty e^{-\frac{2\mu}{\sigma^2(1-\theta)}u^{(1-\theta)}}du}$$

which reduced to the started value $e^{-\frac{2\mu x}{\sigma^2}}$ when $\theta = 0$. Thus the new θ-model allows bankruptcy while incorporating the advantage of variable "returns-to-scale" present in the Black-Scholes-Merton model. Furthermore, this group of models are robust against the change of interest rate. Shepp *et al.* [5] introduced the optimal policy for profit taking under "decreasing-return-to-scale" model. In chapter 3, we show that the result of gambler's problem under "decreasing-return-to-scale" models is consistent with the gambling theory in the Vardi casino. i.e. investors need to buy those stocks with higher volatility when market is going down. We also study the American and Russian option under the "decreasing-return-to-scale" models. The American option and Russian option are two classic options widely used in finance engineering. Here we give the optimal policy and the price formula of American option and Russian option problem for each of these new models. In chapter 4, we conclude our work and discusses some related open research questions.

Chapter 2

How to gamble if you must: the difference between two casinos

A gambler plays in two different casinos: the Vardi casino and the Dubins-Savage casino. One has infinitely many tables and one has only one table available. All the tables have the same expected unit return. No doubt that the Vardi casino will provide bigger utility value(i.e. bigger probability to achieve the goal). But the difference is actually surprisingly small.

2.1 Introduction

Mr. A owns \$100,000 to a loan shark and will be killed at dawn if the loan is not repaid in full but he only has \$10,000. The only way that Mr. A can make enough money is through gambling at a casino. Mr. A starts with a fortune of size $f < 1$ and plays in a casino with tables indexed by the odds $r \geq 0$. There are two casinos available. One is the Dubins-Savage casino where there is only one table available and the gambler can bet any amount $0 < s \leq f$, gaining a fixed odds ratio r times the stake with probability ω or losing the stake with probability $\overline{\omega} = 1 - \omega$. The expected return on a dollar bet is

$$r * \omega - 1 * \overline{\omega} = (r + 1) * \omega - 1 = c < 0.$$

The other one is the Vardi casino where there are infinitely many tables and the gambler can stake any amount in his possession on any table indexed by odds $r \in \Re$ with the same expected winnings c. Obviously the Vardi Casino will provide a bigger chance to achieve the goal because it offers more choices.

So what is the advantage of having infinitely more choice of tables? What's the difference between the maximal probability to achieve the goal (reach fortune one) in two Casinos? One might believe that it would be large because of the existence of casinos in which there are a wide variety of odds available. However, we will show that the Vardi casino gives an advantage but not a very large one. For example, the two probability functions agree to 3 decimals when c = -0.02 (roughly the expected unit return for 'craps' game). And we conjecture that the biggest difference access at the fortune of size 1/3, where the difference is 0.007814498.

2.1.1 Formulation of the problem

As in [1],we formally formulate the above problem as a Dubins-Savage gambling problem in which the set of fortunes, the utility function, and the set of available gambles are, respectively, as follows:

$$F = [0, \infty),$$

$$u(f) = \begin{cases} 0, & if\ 0 \le f < 1 \\ 1, & if\ f \ge 1 \end{cases}$$

$$\Gamma(f) = \begin{cases} \omega\delta(f + rs) + \overline{\omega}\delta(f - s) : 0 \le s \le f, & if\ 0 \le f < 1 \\ \delta(f), & if\ f \ge 1 \end{cases}$$

Here, for $0 \le x < \infty, \delta(x)$ denotes the probability measure that assigns probability 1 to $\{x\}$. The reason that $\Gamma(f)$ consists only of $\delta(f)$ for $f \ge 1$ is that, when the gambler has a fortune $f \ge 1$, he has reached the goal already and need not gamble any more.

For each integer $n \ge 1$, let f_{n-1} be the gambler's fortune before the nth play (with f_0 denoting the initial fortune). A strategy $\sigma = \{s_1, s_2, \cdots\}$ is a sequence of stakes, where $0 \le s_n \le f_{n-1}$ is the gambler's stake on the nth play. Given the gambler's fortune $f_{n-1} < 1$ before the nth play and the stake s_n on the nth play, his fortune f_n (after the nth play and before the (n+1)th play) will be $f_{n-1} + rs_n$

with probability ω and $f_{n-1} - s_n$ with probability $\overline{\omega} = 1 - \omega$. The utility of the strategy σ is $U_\sigma(f) = P\{f_n \geq 1 \text{ for some } n \geq 0 | f_0 = f\}$ for $f \geq 0$. The utility of the game, i.e. the maximum probability to achieve the goal, is defined as $U(f) = sup\{U_\sigma(f)\}$, where the supremum is taken over all possible strategies.

2.1.2 The Dubins-Savage casino

In the Dubins-Savage casino, the gambler can bet any amount $0 < s \leq f$, gaining a fixed odds ratio r times the stake with probability ω or losing the stake with probability $\overline{\omega} = 1 - \omega$. The expected return on a dollar bet is

$$r * \omega - 1 * \overline{\omega} = (r+1) * \omega - 1 \leq c < 0.$$

Dubins and Savage[1] showed that the optimal strategy is to play boldly. The bold stake at the fortune f is defined by $b(f) = min\{f, (1-f)/r\}$ if $0 \leq f < 1$ and $b(f) = 0$ if $f \geq 1$. The gambler is said to use the bold strategy if he stakes the bold stake $b(f)$ whenever he has a fortune f (and stops playing as soon as he is either 'broke', i.e. his fortune equals 0, or reaches his goal). Let 'D' denote the Dubins-Savage casino and $D(f)$ the utility of the bold strategy under the Dubins-Savage casino. It is obvious that $D(0) = 0$ and $D(f) = 1$ for $f \geq 1$. For $0 < f < 1$, we have, by considering one play,

$$D(f) = \overline{\omega} D(f - b(f)) + \omega D(f + rb(f)).$$

Therefore,

$$D(f) = \begin{cases} \omega D((1+r)f), & for\ 0 \leq f \leq 1/(1+r), \\ \omega + \overline{\omega} D(f - 1/(1+r)), & for\ 1/(1+r) \leq f < 1, \\ 1, & for\ f \geq 1, \end{cases}$$

Clearly, the utility function of bold strategy satisfies (as Dubins and Savage showed)

$$D(\tfrac{1}{1+r}) = \omega$$

$$D((\tfrac{1}{1+r})^2) = \omega^2$$

$$D((\tfrac{1}{1+r}) + (\tfrac{1}{1+r}) * (\tfrac{r}{1+r})) = \omega + \omega * \overline{\omega}, \text{ etc.}$$

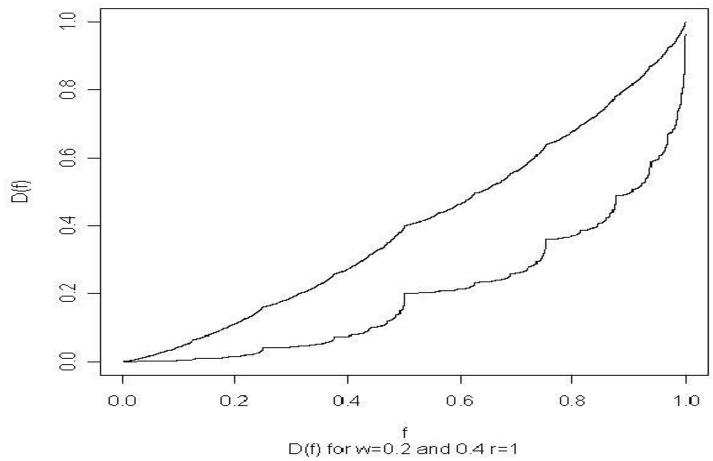

Figure 2.1: The utility function for the Dubins-Savage casino

In general, $D(f)$ is given explicitly at

$f = \sum_{i \geq 0} \frac{r^i}{(1+r)^{n_i}}, 1 \leq n_0 < n_1 < \cdots$, by

$D(f) = \sum_{i \geq 0} \omega^{n_i - i}(1-\omega)^i$.

It is a singular function. Figure 2.1 shows the utility function $D(f)$ for $c = -0.6$ and $c = -0.2$,$r = 1$ (black & red).

Particularly, for $r = 1$, $\omega = 0.49$ and $c = -0.02$ (roughly the case for the pass bet at the craps table)

$D(f) = \sum_{i \geq 0} \omega^{n_i - i}(1-\omega)^i = \sum_{i \geq 0} 0.51^i * 0.49^{n_i - i}$

at $f = \sum_{i \geq 0} \frac{1}{2^{n_i}}$, $1 \leq n_0 < n_1 < \cdots$.

2.1.3 The Vardi Casino

Yehuda Vardi asked the question of how-to-gamble-if-you-must in a casino where any bet is available as long as its expected return on a dollar bet is the same. We call this casino the Vardi casino. The Vardi casino is a typical casino in which there are a variety of bets available, as with roulette or slots with long odds. He asked whether this would give a much better chance for the gambler to achieve the goal under the optimal play. Let 'V' denote the Vardi casino and $V(f)$ denote the

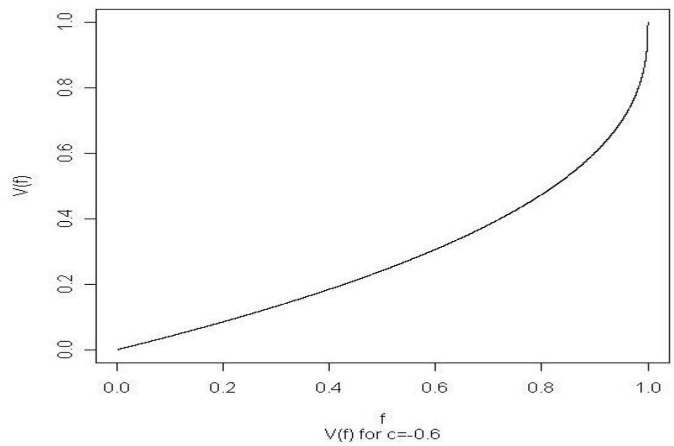

Figure 2.2: The utility function for the Vardi casino

utility function of the Vardi casino. Larry shepp [11] showed that the gambler's optimal probability to achieve fortune 1 with initial fortune f is

$V(f) = 1 - (1-f)^{(1+c)}$, $0 \leq f \leq 1$.

And it is only attainable within ϵ, and an ϵ-optimal betting strategy is to bet *small* amounts, say δ at the odds $\frac{1-f}{\delta}$.

Figure 2.2 shows the utility function of the Vardi casino at c=-0.6.

So what is the advantage of the Vardi's casino over the Dubins-Savage casino for real gamble, say craps game or $\omega = 0.49$ $r = 1$ and $c = \omega * 1 + (1-\omega) * (-1) = -0.02$ in the Dubins-Savage casino. One might believe that this quantity would be large for $c = -0.02$ because of the existence of casinos in which there are a wide variety of odds available. But it is not so. This thesis shows that the two casinos give nearly the same value and the difference is less than 0.008. The fact that the difference is so small makes one wonder why there are casinos with a wide variety of odds. A gambler who wants to turn a fortune of size $f < 1$ into one of size one hardly does better in the Vardi's casino than in the Dubins-Savage casino. This seems quite surprising.

The difference between two casinos

The limit of the Vardi casino actually can be approached by the Dubins-Savage casino while we let the odds ratio $r \to \infty$.

For any $f \in (0,1)$,

$(\frac{r}{1+r})^{n+1} \leq 1-f \leq (\frac{r}{1+r})^n$,where $n = [\frac{log(1-f)}{log(\frac{r}{1+r})}]$,

hence, $1-(\frac{r-c}{1+r})^{n+1} \leq D(f) \leq 1-(\frac{r-c}{1+r})^n$.

When $r \to \infty$, $D(f,r) \to 1-(1-f)^{1+c}$, which means that, essentially, a larger odds ratio will provide a better chance to achieve the goal.

We still need to answer the question of whether the various different casinos give very different values for the utility function or not. Thus if $D(f,r)$ is the utility function of the Dubins-Savage casino with odds r, and $V(f)$ is the utility function of the Vardi casino with the same value of c, then clearly

$V(f) \geq D(f)$,

but define $\delta(f,r,c) = V(f) - D(f,r), 0 < f < 1$ and $\theta(r,c) = sup_{0<f<1}\delta(f,r)$ which measure the improvement that the Vardi casino achieves over the Dubins casino. One might believe it would be large for $c = -0.02$ because of the existence of casinos in which there are a wide variety of odds available. But it is not so: the two casinos give nearly the same value and $\theta(1,-0.02) < 0.008$.

Theorem: For c=-0.02, $\theta(1,c) < 0.008$.

Proof: Let $f_i = i/2^n, i = 1,2,\cdots,2^n$. For $f_{i-1} < f < f_i, \delta(f) < V(f_i) - D(f_{i-1}) = V(f_i) - V(f_{i-1}) + V(f_{i-1}) - D(f_{i-1})$.

So, $\theta(1,c) \leq max_{i=1,\cdots 2^n}(V(f_i) - D(f_i)) + max_{i=1,\cdots 2^n}(V(f_i) - V(f_{i-1})) = M_{n1} + M_{n2}$.

For $n = 10$, $M_{n1} = 0.007804855$, $M_{n2} = 0.001121776$,

For $n = 14$, $M_{n1} = 0.00781388$, $M_{n2} = 7.410857e-05$,

For $n = 16$, $M_{n1} = 0.007814341$, $M_{n2} = 1.904801e-05$,

So, $\theta(1,c) < 0.008$.

Figure 2.3 shows the difference between two utility functions as $r = 1$ and

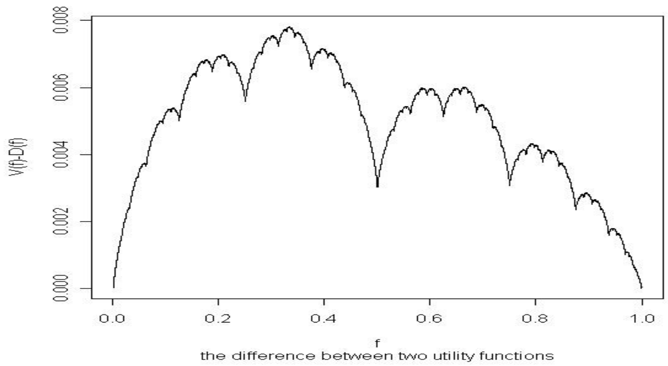

Figure 2.3: The difference between two utility functions

$c = -0.02$ in the Dubins casino. This result is quite surprising. People can't really expect too much by trying all kinds of games. What really matters is the expected unit return. Then why would casinos in the real world have such a variety of gambles, since they aren't really producing any advantage?

Corollary: For the r satisfies $D(\frac{1}{2}, r) \geq 0.49$, $\theta(r, -0.02) \leq 0.008$.

Proof: Based on the Dubins and Savage's theorem (see [1] p112-113), if $D(\frac{1}{2}, r) \geq 0.49$, then $D(f, r)$ majorzes $D(f, 1)$, i.e. $D(f, r) \geq D(f, 1)$. Hence, $\theta(r, -0.02) \leq 0.008$.

Conjecture 1. For $r = 1$, the biggest $\delta(f, r)$ would happen at $f = \frac{1}{3}$.

Conjecture 2. $\theta(r, c)$ is decreasing on r. Hence,

$sup_{0<f<1}[V(f) - inf_{r\geq 1} D(f, r)] < 0.008$.

Figure 2.4 shows the difference between two utility functions for c=-0.02 and $r = 3$ in the Dubins-Savage casino:

Figure 2.5 shows the maximum difference between two utility functions for c=-0.02 and different choices of r in Dubins-Savage casino. We can see that the maximum differnce is decreasing in the odds ratio r, which is quite reasonable since $V(f) = lim_{r\to\infty} D(f)$. (This shows that high risk does give some kind of

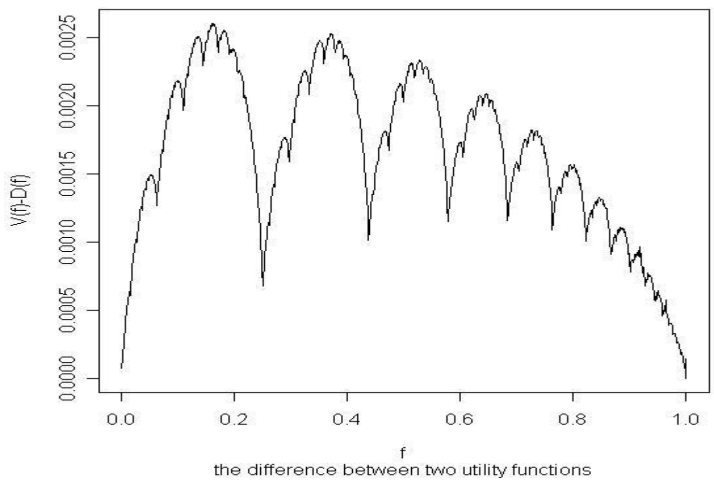

Figure 2.4: The difference between two utility functions, r=3

advantage in gambling, although it is really very small).

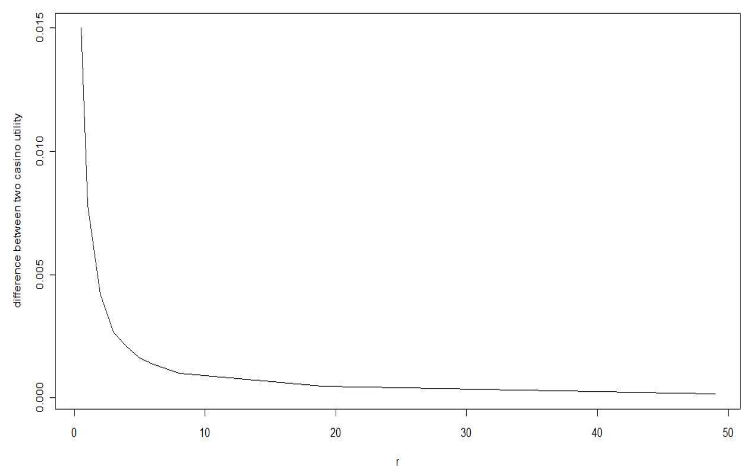

Figure 2.5: The difference between two utility functions for c=-0.02 and different r

Chapter 3

American option and Russian Option for decreasing-return-to-scale model

The Black-Scholes-Merton model and the Bachelier model each have advantages and shortcomings. Ren-raw Chen, Oded Palmon and Larry Shepp introduced a general class of models with "decreasing-return-to-scale" indexed by a parameter θ interpolating between the Bachelier model ($\theta = 0$) and the Black-Scholes-Merton ($\theta = 1$) model. Here we study the American option and Russian option under "decreasing-return-to-scale" model.

3.1 Introduction

There are two classic stochastic models in economy and finance, the Black-Scholes-Merton model and the Bachelier model. One is multiplicative and one is additive.

Suppose that the stock price (or the value) of a firm at time t is X(t). In the Black-Scholes model, $X(t)$ is assumed to vary according to the following stochastic differential equation

$$dX(t) = X(t)(\mu dt + \sigma dW(t)),\ t > 0,\ X(0) = x$$

where $X(t)$ is the stock price, μ is the drift, σ is the diffusion, $W(t)$ is the standard Brownian motion and $x_0 = X(0)$ is the stock price at time 0. Hence, $X(t) = X(0)e^{\mu t - \sigma^2/2t + \sigma W(t)}, t > 0$. It is a very useful in a model for optimal hiring and firing decisions in corporate planning. The size of the company, measured in number of personnel, naturally enters multiplicatively and this model does give effective results for the problem of when to hire and fire. And it is widely used in stock option modeling. The idea that ‘price change should be proportional to the level of the asset price’ does somehow reflect the reality. There are some other “returns-to-scale” reasons advanced in [4] for preferring the Black-Scholes-Merton model to the Bachelier model. However, as we can see, $X(t)$ will never go below or hit 0. This may be a problem when you use it to make economic decisions because all the companies will go bankrupt, eventually.

In the Bachelier model, $X(t)$ is assumed to vary according to the following stochastic differential equation:

$$dX(t) = \mu dt + \sigma dW(t),\ t > 0,\ X(0) = x$$

where $X(t)$ is the stock price, μ is the drift, σ is the diffusion, $W(t)$ is the standard Brownian motion and $x_0 = X(0)$ is the stock price at time 0. Hence, $X(t) = X(0) + \mu t + \sigma W(t),\ t > 0$. It is effective in the sense that the optimal decisions made within the model are at least qualitatively consistent with observed decisions made by real companies. It uses additive Brownian motion to model the underlying economic fluctuations, and it was noted in [2] that unrealistic results are obtained if multiplicative rather than additive Brownian motion is used instead to model the economic fluctuations.

Because both of the models have their own advantage and shortcomings, Renraw Chen, Oded Palmon and Larry Shepp introduced a general class of models with “decreasing-return-to-scale” indexed by a parameter θ interpolating between the additive ($\theta = 0$) and the multiplicative ($\theta = 1$) cases. That is, the price of a company $X(t)$ at each time t is assumed to vary according to the stochastic differential equation

$$dX(t) = X^{\theta}(t)(\mu dt + \sigma dW(t)),\ t > 0,\ X(0) = x,$$

where the exponent $0 \leq \theta \leq 1$ is fixed, μ, known, is a measure of the rate of income, σ, known , is volatility, and r, known, is the fixed rate of interest. This class of models is somewhere between the Bachelier model and the Black-Scholes model. This new θ-model allows bankruptcy while incorporating the advantage of variable "returns-to-scale" present in the Black-Scholes-Merton model. Also, it is robust against the change of parameters. Finally, it is consistent with gambling theory, i.e. you need to invest in those stocks with higher volatility when the market is going down.

Suppose a gambler is gambling in a Vardi casino of stocks. All stocks have the same negative drift μ. His initial fortune is f and he wants to make his fortune 1. In the original Vardi casino, a gambler needs to stake on the table with odds ratio as large as possible to maximize the probability to achieve the goal. In the stock market which follows the "decreasing-return-to-scale" model we also need to invest in those stocks with higher volatility (odds).

Actually, for $0 \leq \theta < 1$, the probability, $Q(x)$, of the stock price reaching a particular value x_u before 0 starting from x is positive and is given by:

$$Q(x) = \frac{\int_0^x e^{\frac{-2\mu}{\sigma^2}\frac{s^{1-\theta}}{1-\theta}} ds}{\int_0^{x_u} e^{\frac{-2\mu}{\sigma^2}\frac{s^{1-\theta}}{1-\theta}} ds}$$

It is increasing in the volatility σ, which means investor needs to buy those stocks with higher volatility. And Q is bounded above by $\frac{x}{x_u}$. Indeed, it is clear because the process $X(t)$ is a supermartingale. Hence $EX(\tau) \leq X(0) = x$ for any stopping time τ which implies that $Q(x)x_u \leq x$. (See Appendix for more details)

In [5], Ren-raw Chen, Oded Palmon and Larry Shepp gave the optimal strategy for corporate policy. Here, we study the American and the Russian option under the 'decreasing-return-to-scale' models and give the optimal policy for each of these new models.

3.2 American option

Suppose an employee of the firm is offered the choice of taking a bonus of Y dollars at time $t = 0$, or the right to an amount Z times $(X(\tau, \omega) - k)^+$ where Y, Z, k are specified numbers. The employee can execute the second choice only once at any future time τ. Suppose the employee considers that the true interest rate is r and that the second choice is really worth Z times

$V(x) = \sup_\tau V(x,\tau)$, where $V(x,\tau) = E_x(X(\tau)-k)^+e^{-r\tau}$.

where $X(t)$ is the stock price of a firm at each time t and is assumed to vary according to the stochastic differential equation

$dX(t) = X^\theta(t)(\mu dt + \sigma dW(t)),\ t > 0,\ X(0) = x,$

where the exponent $0 \leq \theta \leq 1$ is fixed, μ, known, is a measure of the rate of income, σ, known , is volatility, and $W(t)$ is the standard Brownian motion. The additive and multiplicative cases are $\theta = 0$ and $\theta = 1$ respectively.

In the Black-Scholes model, $\theta = 1$. For this θ, $V_\theta(x) = sup_\tau E_x e^{-r\tau}(X(\tau)-k)^+$, where τ is any stopping time of $X(t)$, satisfies for $\mu < r$,

$V_1(x) = Ax^{\gamma_+},\ x < \alpha;$

$V_1(x) = (x-k)^+,\ x \geq \alpha.$

where $\gamma_+ = \frac{\frac{\sigma^2}{2}-\mu+\sqrt{(\frac{\sigma^2}{2}-\mu)^2+2r\sigma^2}}{\sigma^2}$,

$\alpha = \frac{k}{1-\frac{1}{\gamma_+}}$,

$A = \frac{\alpha^{1-\gamma_+}}{\gamma_+}$.

When $\mu > r$, $V_1(x) = \infty$, since $E_x e^{\sigma W(t)+(\mu-r-\frac{\sigma^2}{2})t} = e^{(\mu-r)t} \to \infty$ as $t \to \infty$.

When $\mu = r$, there is no optimal solution. But one can achieve any close to x. Then the employee can compare the $V_1(x)$ with Y and make his optimal decision. Would the result be similar when $0 < \theta < 1$?

3.2.1 Statement of results

For $0 < \theta < 1$, the process is now defined by

$dX(t) = X^\theta(t)(\mu dt + \sigma dW(t)),\ t > 0,\ X(0) = x,$

And we need to find out $V_\theta(x;\mu,\theta,r) = sup_\tau E_x e^{-r\tau}(X(\tau)-k)^+$?

We will guess the optimal policy as in the case $\theta = 1$, namely that there is a constant $\alpha = \alpha(\theta,\mu,\sigma,r)$ so that while $X(t) < \alpha$, one does not execute the option and when $X(t) \geq \alpha$, one executes the option immediately. We will prove that if we can find a number α and a function $\widehat{V}(x), x > 0$ satisfying the five conditions:

$I.\ \widehat{V}(x) \in C^2, \widehat{V}(x) \geq (x-k)^+, x \geq 0$

$II.\ \frac{\sigma^2}{2}x^{2\theta}\widehat{V}''(x) + \mu x^\theta \widehat{V}'(x) - r\widehat{V}(x) = 0$, for $0 < x \leq \alpha$

$III.\ \frac{\sigma^2}{2}x^{2\theta}\widehat{V}''(x) + \mu x^\theta \widehat{V}'(x) - r\widehat{V}(x) \leq 0$, for $x \geq \alpha$,

$IV.\ 0 \leq \widehat{V}'(x) \leq 1$, $0 \leq x \leq a$, $\widehat{V}'(x) \equiv 1$, $x \geq \alpha$,

$V.\ \widehat{V}(0) = 0$

then $V_\theta(x) = \widehat{V}(x)$.

To prove this, let's look at the process

$Y(t) = e^{-rt}\widehat{V}(X(t), S(t))$.

$dY(t) = -re^{-rt}\widehat{V}(x)dt + e^{-rt}\widehat{V}'(x)dx + 1/2e^{-rt}\widehat{V}''(x)(dx)^2$

$= e^{-rt}(-r\widehat{V}(x) + x^\theta\mu\widehat{V}'(x) + \frac{\sigma^2}{2}x^{2\theta}\widehat{V}''(x))dt + e^{-rt}\sigma\widehat{V}'(x)dW(t)$

Based on the conditions above, the process $Y(t)$ is a super-martingale. Then we can use the fact that a non-negative supermartingale satisfies for every stopping time τ, $EY(\tau) \leq EY(0)$. i.e.

$EY(\tau) \leq Y(0) = \widehat{V}(x)$,

and since this holds for every stopping time, we also have $V_\theta(x) \leq \widehat{V}(x)$.

To prove the reverse inequality we have from the first two conditions that if one uses the right stopping time τ, which is the first hitting time of α, then equality holds throughout the last line and since $V_\theta(x) \geq V(x,\tau) = \widehat{V}(x)$, we see that $V_\theta = \widehat{V}$.

Theorem: For $\mu < r$, the optimal strategy under the θ model is to execute the option at the first time that $X(t)$ is larger than or equal to some particular level α, And the value of the option is $V(x) = Af(x), 0 \leq x \leq \alpha$

$V(x) = x - k, \alpha \leq x < \infty$

where $\alpha = (\frac{z(1-\theta)}{c})^{\frac{1}{1-\theta}}$,

z is the largest negative root of

$\frac{zM'(1+a-b,2-b,z)}{M(1+a-b,2-b,z)} = \frac{1-\theta}{c}\frac{r}{\mu}z - \mu - \frac{1-\theta}{c}\gamma\mu z.$

and $A = \frac{\alpha - k}{f(\alpha)}$.

3.2.2 Solution and proof

If we guess that the best $\tau = \tau_a$ for some α, where τ_α is the first hitting time of α, then we 'know' that $\alpha > k$ of course. We would expect that if $X(t) = x < \alpha$, then $Y(t) = \widehat{V}(X(t))e^{-rt}$ would be a local martingale in the 'continue-to-observe' region, $x < \alpha$, i.e. the process Y(t) would have no drift term.

$dY(t) = -re^{-rt}dt\widehat{V}(x) + e^{-rt}\frac{d}{dx}\widehat{V}(x)x^\theta(\mu dt + \sigma dW) + \frac{1}{2}e^{-rt}\frac{d^2}{dx^2}\widehat{V}(x)x^{2\theta}\sigma^2 dt$

and there will be no drift term if and only if for $x < \alpha$,

$0 = -r\widehat{V}(x) + \frac{d}{dx}\widehat{V}(x)x^\theta\mu + \frac{1}{2}\frac{d^2}{dx^2}\widehat{V}(x)x^{2\theta}\sigma^2$.

Suppose $f(x)$ solves the differential equation for $\widehat{V}$ then

$\frac{\sigma^2}{2}x^{2\theta}f''(x) + \mu x^\theta f'(x) - rf(x) = 0.$

let $f(x) = e^{\gamma\frac{x^{1-\theta}}{1-\theta}}h(-(2\gamma + \frac{2\mu}{\sigma^2})\frac{x^{1-\theta}}{1-\theta})$, where γ is any root of $\frac{\sigma^2}{2}\gamma^2 + \mu\gamma - r = 0$, then

$-(2\gamma + \frac{2\mu}{\sigma^2})\frac{\sigma^2}{2}\frac{x^{1-\theta}}{1-\theta}h''(-(2\gamma + \frac{2\mu}{\sigma^2})\frac{x^{1-\theta}}{1-\theta}) + ((\sigma^2\gamma + \mu)\frac{x^{1-\theta}}{1-\theta} - \frac{\theta}{1-\theta}\frac{\sigma^2}{2})h'(-(2\gamma + \frac{2\mu}{\sigma^2})\frac{x^{1-\theta}}{1-\theta}) - \frac{\sigma^2}{2}\frac{\gamma}{-(2\gamma+\frac{2\mu}{\sigma^2})}\frac{\theta}{1-\theta}h(-(2\gamma + \frac{2\mu}{\sigma^2})\frac{x^{1-\theta}}{1-\theta}) = 0$

i.e.

$h''(z) + (z - \frac{\theta}{1-\theta})h'(z) + \frac{\gamma}{2\gamma+\frac{2\mu}{\sigma^2}}\frac{\theta}{1-\theta}h(z) = 0$

This is Kummer's equation and has a general solution,

$h(z) = AM(a, b, z) + Bz^{1-b}M(1 + a - b, 2 - b, z)$, where

$a = \frac{\gamma}{c}\frac{\theta}{1-\theta}$,

$b = -\frac{\theta}{1-\theta}$,

$c = -(2\gamma + \frac{2\mu}{\sigma^2})$ and

$M(a, b, z) = 1 + \frac{az}{b} + \cdots + \frac{(a)_n z^n}{(b)_n n!} + \cdots$ is the standard hypergeometric function [3].

Since $\widehat{V}(0) = 0$, $f(x) = Axe^{\gamma\frac{x^{1-\theta}}{1-\theta}}M(1 + a - b, 2 - b, c\frac{x^{1-\theta}}{1-\theta})$.

We can now construct $\widehat{V}$ in terms of f as follows:

$\widehat{V}(x) = Af(x), 0 \leq x \leq \alpha$

$\widehat{V}(x) = x - k, \alpha \leq x < \infty$

Since $\widehat{V}'(\alpha) = 1$ and $\widehat{V}''(\alpha) = 0$, we have $\mu\alpha^\theta = r\widehat{V}(\alpha)$ and we see that

$\alpha = (\frac{z(1-\theta)}{c})^{\frac{1}{1-\theta}}$

where z is the largest negative root of

$\frac{zM'(1+a-b,2-b,z)}{M(1+a-b,2-b,z)} = \frac{1-\theta}{c}\frac{r}{\mu}z - \mu - \frac{1-\theta}{c}\gamma\mu z.$

And $A = \frac{\alpha-k}{f(\alpha)}$.

Now we still need to prove that equality holds for the stopping time τ_α. If we define $Y(t) = Y(\tau_\alpha)$ for $t > \tau_\alpha$, then Y is a uniformly integrable martingale and this means equality holds in

$EY(\tau_\alpha) = Y(0) = \widehat{V}(x).$

qed.

3.3 Russian option

The Russian option was first introduced by L.A. Shepp and A.N.Shiryaev [20] [17]. It's payoff is the maximum value of the underlying asset(stock) price attained over some interval of time. Because it depends on the value of the past, it is called 'path-dependent' option. Say, the stock price at time t is $X(t)$ and the maximum price over the time period (0,t) is $S(t) = s \vee max_{0\leq u\leq t}X(u)$ where s is the initial value of $S(t)$. One can get the payoff $S(t)$ upon executeing the option at time t. So the Russian option is some kind of 'reduced regret' option. In order to find the optimal execution time, we need to find

$V_\theta(x,s) = V_\theta(x,s,r,\mu,\sigma) = sup_\tau V_\theta(x,s,\tau) = sup_\tau E_{x,s}e^{-r\tau}S(\tau),$

where $X(t)$ follows the stochastic differential equation

$dX(t) = X^\theta(t)(\mu dt + \sigma dW(t)),\ t > 0,\ X(0) = x,$

and the supremum is taken over all stopping times τ. Note that the process S(t) alone is not Markovian, we need to study the joint process (X(t), S(t)). Hence the

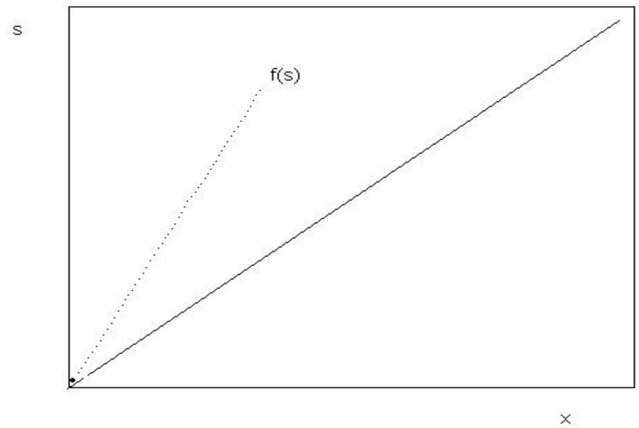

Figure 3.1: the state space of (X(t),S(t))

"state space" of the problem is the set of (x, s) with $x \leq s$, as pictured in figure 3.1, where the dotted line represents an unknown "free boundary", $x = f(s)$. We want to find this function $f(s)$ and then for $f(s) < x \leq s$ we should continue observing the fluctuations, while for $0 < x \leq f(s)$ we should stop immediately (execute the option). Once we guess the answer, we will give a rigorous proof of its correctness.

In the Black-Scholes model, $\theta = 1$, and it was shown that for this θ, when $r > max(0, \mu)$,

$$V(x,s) = \begin{cases} \frac{s}{\gamma_+ - \gamma_-}(\gamma_+(\frac{\alpha x}{s})^{\gamma_-} - \gamma_-(\frac{\alpha x}{s})^{\gamma_+}), & f(s) \leq x \leq s, \\ s, & 0 \leq x \leq f(s), \end{cases}$$

with $f(s) \equiv s/\alpha$, and where

$\alpha = (\frac{1-\frac{1}{\gamma_-}}{1-\frac{1}{\gamma_+}})^{\frac{1}{\gamma_+ - \gamma_-}}$

and $\gamma_\pm = \frac{\frac{\sigma^2}{2} - \mu \pm \sqrt{(\frac{\sigma^2}{2} - \mu)^2 + 2r\sigma^2}}{\sigma^2}$ are the solutions to $r = \mu\gamma + \frac{1}{2\sigma^2}\gamma(\gamma - 1)$. The optimal strategy is to execute the option immediately as $X(t) \leq f(S(t))$.

When $r < 0$ or $r \leq \mu$, $V(x, s, r, \mu, \sigma) = \infty$.

How about $0 < \theta < 1$?

3.3.1 Statement of result

The case, $0 < \theta < 1$, is similar to the $\theta = 1$ case. The "state space" of $(X(t), S(t))$ is separated by a 'free boundary', $x = g(s)$. We want to find this function g and then for $g(s) < x \leq s$ we should continue observing the fluctuations, while for $0 < x \leq g(s)$ we should stop immediately (execute the option). So if we can find the boundary $g(s)$ and function $\widehat{V}_\theta(x, s), x > 0$ satisfying the following conditions:

I. $\widehat{V}(x, s) \in C^2, \widehat{V}(x, s) \geq s, \widehat{V}_1(x, s) \geq 0, 0 < x \leq s, s > 0,$

II. $\frac{\sigma^2}{2} x^{2\theta} \widehat{V}_{11}(x, s) + \mu x^\theta \widehat{V}_1(x, s) - r\widehat{V}(x, s) = 0$, for $g(s) < x \leq s$

III. $\frac{\sigma^2}{2} x^{2\theta} \widehat{V}_{11}(x, s) + \mu x^\theta \widehat{V}_1(x, s) - r\widehat{V}(x, s) \leq 0$, for $0 < x \leq g(s)$,

IV. $\widehat{V}_2(s, s) \equiv 0$,

V. $\widehat{V}(x, s) = s$, for $0 < x \leq g(s)$,

then $V_\theta(x, s) = \widehat{V}_\theta(x, s)$.

To prove this, let's look at the process

$Y(t) = e^{-rt}\widehat{V}(X(t), S(t))$.

$dY(t) = -re^{-rt}\widehat{V}(x, s)dt + e^{-rt}\widehat{V}_1(x, s)dx + 1/2e^{-rt}\widehat{V}_{11}(x, s)(dx)^2 + e^{-rt}\widehat{V}_2(x, s)ds$
$= e^{-rt}(-r\widehat{V} + x^\theta \mu \widehat{V}_1 + \frac{\sigma^2}{2} x^{2\theta} \widehat{V}_{11})dt + e^{-rt}\widehat{V}_2 ds + e^{-rt}\sigma \widehat{V}_1 dW$

On account of the second, third and fourth conditions, $Y(t)$ is a super martingale. Hence for $\tau \leq \infty$, the first hitting time of zero, we have

$V_\theta(x, s) \leq Y(0) = \widehat{V}(x, s)$.

and since this holds for every τ, we also have $V_\theta(x, s) \leq \widehat{V}(x, s)$.

Now we need to prove the reverse inequality. We have to find a stopping time which realizes the value $V(x, s)$. Of couse this will be the first time t for which $X(t) \leq g(S(t))$. Then $V_\theta(x, s) \geq V_\theta(x, s, \tau) = \widehat{V}(x, s)$. So we have

Theorem: For $0 < \theta < 1$, the optimal strategy of Russian option under θ model is to execute the option at the first time that $X(t) \leq g(S(t))$ and the optimal value of the option is $V(X(t), S(t)) = \widehat{V}(X(t), S(t))$ where $g(s)$ and $\widehat{V}(x, s)$ are given by equations 3.1 and 3.2 below.

3.3.2 Solution

We guess the best τ would be the first time that $X(t)$ falls into the region $X(t) \leq g(S(t))$ for some function $x = g(s)$ and we would expect that if $X(t) > g(S(t))$, then $Y(t) = \widehat{V}(X(t))e^{-rt}$ would be a local martingale in the 'continue-to-observe' region, $x > g(s)$, i.e. the process $Y(t) = e^{-rt}\widehat{V}(X(t), S(t))$ would have no drift term.

$dY(t) = e^{-rt}(-r\widehat{V} + x^{\theta}\mu\widehat{V}_1 + \frac{\sigma^2}{2}x^{2\theta}\widehat{V}_{11})dt + e^{-rt}\widehat{V}_2 ds + e^{-rt}\sigma\widehat{V}_1 dW$

and there will be no drift term if and only if for $s \geq x \geq g(s)$,

$0 = \frac{\sigma^2}{2}x^{2\theta}\widehat{V}_{11}(x, s) + \mu x^{\theta}\widehat{V}_1(x, s) - r\widehat{V}(x, s)$

and $\widehat{V}_2(x, s) = 0$ when $x = s$.

The equation above has a general solution:

$\widehat{V}(x, s) = A(s)f_1(x) + A_2(s)f_2(x)$

where $f_1(x) = e^{\gamma\frac{x^{1-\theta}}{1-\theta}}M(a, b, c\frac{x^{1-\theta}}{1-\theta})$

$f_2(x) = xe^{\gamma\frac{x^{1-\theta}}{1-\theta}}M(1 + a - b, 2 - b, c\frac{x^{1-\theta}}{1-\theta})$

$a = \frac{\gamma}{c}\frac{\theta}{1-\theta}$,

$b = -\frac{\theta}{1-\theta}$,

$c = -(2\gamma + \frac{2\mu}{\sigma^2})$ and

$M(a, b, z) = 1 + \frac{az}{b} + \cdots + \frac{(a)_n z^n}{(b)_n n!} + \cdots$ is the standard hypergeometric function.

Hence, we have

$\widehat{V}(x, s)|_{x=g(s)} = A_1(s)f_1(g(s)) + A_2(s)f_2(g(s)) = s,$

$\widehat{V}_1(x, s)|_{x=g(s)} = A_1(s)f_1'(g(s)) + A_2(s)f_2'(g(s)) = 0$,

$\widehat{V}_2(x, s)|_{x=s} = A_1'(s)f_1(s) + A_2'(s)f_2(s) = 0,$

Hence,

$A_1(s) = \frac{sf_2'(x)}{f_1(x)f_2'(x) - f_1'(x)f_2(x)}|_{x=g(s)},$

$A_2(s) = -\frac{sf_1'(x)}{f_1(x)f_2'(x) - f_1'(x)f_2(x)}|_{x=g(s)},$

Let $f_1(x) = e^{\gamma\frac{x^{1-\theta}}{1-\theta}}y_1(x),\ f_2(x) = e^{\gamma\frac{x^{1-\theta}}{1-\theta}}y_2(x),$

Then,

$$f_1'(x) = e^{\gamma \frac{x^{1-\theta}}{1-\theta}}(\gamma x^{-\theta} y_1(x) + c x^{-\theta} y_1'(x))$$
$$f_2'(x) = e^{\gamma \frac{x^{1-\theta}}{1-\theta}}(\gamma x^{-\theta} y_2(x) + c x^{-\theta} y_2'(x))$$
$$f_1(x) f_2'(x) - f_1' f_2(x) = e^{2\gamma \frac{x^{1-\theta}}{1-\theta}} c x^{-\theta}(y_1(x) y_2'(x)) - e^{2\gamma \frac{x^{1-\theta}}{1-\theta}} c x^{-\theta}(y_1'(x) y_2(x))$$
$$= e^{2\gamma \frac{x^{1-\theta}}{1-\theta}} c x^{-\theta}(y_1(x) y_2'(x) - y_1'(x) y_2(x))$$
$$= e^{(2\gamma + c)\frac{x^{1-\theta}}{1-\theta}} c x^{-\theta}(1-b)(c\frac{x^{1-\theta}}{1-\theta})^{-b} e^{c\frac{x^{1-\theta}}{1-\theta}}$$
$$= c^{\frac{1}{1-\theta}}(1-\theta)^{\frac{-1}{1-\theta}} e^{-\frac{2\mu}{\sigma^2}\frac{x^{(1-\theta)}}{1-\theta}}.$$

Hence,

$$A_1(s) = s f_2'(x)(\tfrac{c}{1-\theta})^{\frac{1}{1-\theta}} e^{-\frac{2\mu}{\sigma^2}\frac{x^{(1-\theta)}}{1-\theta}}|_{x=g(s)} = s e^{(\gamma + \frac{2\mu}{\sigma^2})\frac{x^{(1-\theta)}}{1-\theta}} x^{-\theta}(\gamma y_2(x) + c y_2'(x))|_{x=g(s)},$$
$$A_2(s) = -s f_1'(x)(\tfrac{c}{1-\theta})^{\frac{1}{1-\theta}} e^{-\frac{2\mu}{\sigma^2}\frac{x^{(1-\theta)}}{1-\theta}}|_{x=g(s)} = s e^{(\gamma + \frac{2\mu}{\sigma^2})\frac{x^{(1-\theta)}}{1-\theta}} x^{-\theta}(\gamma y_1(x) + c y_1'(x))|_{x=g(s)},$$

$$\widehat{V}(x,s) = s e^{(\gamma + \frac{2\mu}{\sigma^2})\frac{x^{(1-\theta)}}{1-\theta}} x^{-\theta}(\gamma y_2(x) + c y_2'(x)) f_1(x) + s e^{(\gamma + \frac{2\mu}{\sigma^2})\frac{x^{(1-\theta)}}{1-\theta}} x^{-\theta}(\gamma y_1(x) + c y_1'(x)) f_2(x)|_{x=g(s)} \quad (3.1)$$

and

$$[(\gamma y_2(g(s)) + c y_2'(g(s))) + s(g^{-\theta}(s) - \theta g^{-1}(s))(\gamma y_2(g(s)) + c y_2'(g(s)))g'(s) + s c g^{-\theta}(s)$$
$$(\gamma y_2'(g(s)) + c y_2''(g(s)))g'(s)] y_1(s) - [(\gamma y_1(g(s)) + c y_1'(g(s))) + s(g^{-\theta}(s) - \theta g^{-1}(s))$$
$$(\gamma y_1(g(s)) + c y_1'(g(s)))g'(s) + s c g^{-\theta}(s)(\gamma y_1'(g(s)) + c y_1''(g(s)))g'(s)] y_2(s) = 0. \quad (3.2)$$

This equation has a unique solution. If there were two solutions, say $g_1(s)$ and $g_2(s)$, and $g_1(s) \leq g_2(s)$. Then for $g_1(s) \leq x \leq g_2(s)$, $\widehat{V}(x,s) = s$ and $\widehat{V}_1(x,s) = \widehat{V}_{11}(x,s) = 0$. But in this case, condition II fails. $\frac{\sigma^2}{2} x^{2\theta} \widehat{V}_{11}(x,s) + \mu x^{\theta} \widehat{V}_1(x,s) - r\widehat{V}(x,s) = -rs < 0$, for $g_1(s) \leq x \leq g_2(s)$. Hence, $g_1(s) = g_2(s)$, $0 < s < \infty$. i.e. the solution is unique. Unfortunately, there is no simple expression of the solution. We can only numerical approximate it. The figure 3.2 shows a numerical calculation of $g(s)$ for $r = 0.05, \mu = 0.04, \sigma = 0.2$. As long as we have $g(s)$, we can get $\widehat{V}(x,s)$ and complete the whole argument.

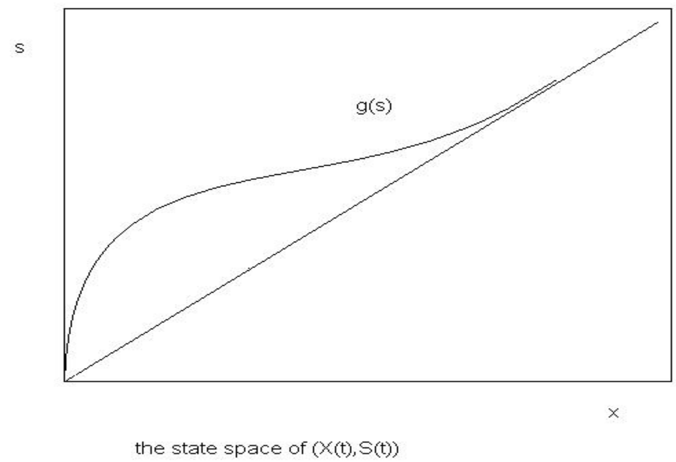

Figure 3.2: the state space of (X(t),S(t)) and the boundary g(s)

3.4 Appendix

Suppose a gambler is gambling in a Vardi casino of stocks. All stocks have the same negative drift μ. His initial fortune is f and he wants to make his fortune 1. In the original Vardi casino, a gambler needs to stake on the table with odds ratio as large as possible to maximize the probability to achieve his goal, to make his forture reach some level f'. In a stock market following the "decreasing-return-to-scale" model we also need to invest in those stocks with higher volatility (odds).

Actually, for $0 \leq \theta < 1$, the initial stock price is x and the gambler quits when the stock price hits $x_u > x$ or zero. The probability, $Q(x)$, of the stock price reaching value x_u before 0 starting from x, is positive and is given by:

$$Q(x) = \frac{\int_0^x e^{\frac{-2\mu}{\sigma^2}\frac{s^{1-\theta}}{1-\theta}} ds}{\int_0^{x_u} e^{\frac{-2\mu}{\sigma^2}\frac{s^{1-\theta}}{1-\theta}} ds}$$

Proof:

Let $Y(t) = f(x(t))$.

$$dY(t) = f'(X(t))dX(t) + \frac{1}{2}f''(X(t))(dX(t))^2$$

$$= f'(X(t))X^{\theta}(t)(\mu dt + \sigma dW(t)) + \frac{1}{2}f''(X(t))\sigma^2 X^{2\theta}(t)dt$$

Let $Y(t)$ be a martingale, i.e.

$f'(x)x^\theta\mu + \frac{\sigma^2}{2}f''(x)x^{2\theta} = 0$

We can easily get the solution

$f(x) = \int_0^x e^{\frac{-2\mu}{\sigma^2}\frac{s^{1-\theta}}{1-\theta}}ds$

Let $\tau = min(\tau_{x_u}, \tau_0)$, where τ_0 and τ_{x_u} are the first hitting time of 0 and x_u. Since $Y(t)$ is bounded for $0 \leq t \leq \tau$, we have

$EY(\tau) = Y(0) = f(x)$

But if $p = P(\tau_0 > \tau_{x_u})$ and $q = 1 - p$, then

$EY(\tau) = pf(x_u) + qf(0)$

It follows that

$p = p(\sigma) = \frac{f(x)-f(0)}{f(x_u)-f(0)} = \frac{\int_0^x e^{\frac{-2\mu}{\sigma^2}\frac{s^{1-\theta}}{1-\theta}}ds}{\int_0^{x_u} e^{\frac{-2\mu}{\sigma^2}\frac{s^{1-\theta}}{1-\theta}}ds}$

It is clear that $p(\sigma)$ is a monotone increasing function in σ, which means that higher volatility will provide a higher chance to reach the goal.

Of course we really need to prove that $\tau < \infty$ w.p. 1. To do this it seems easiest to evaluate $E\tau$; once this is finite then of course τ is finite. Let $\phi(x) = E_x\tau$. Then $\phi(0) = \phi(x_u) = 0$ are boundary conditions for ϕ, and that ϕ satisfies the local equation

$\phi(x) = dt + E\phi(x + dx)$, $0 < x < x_u$.

Using Ito calculus, this leads directly to the equation

$0 = 1 + x^\theta\mu\phi'(x) + x^{2\theta}\frac{\sigma^2}{2}\phi''(x)$.

The solution via the Green function is, as is easily verified, for any $0 < \theta < 1$,

$\phi(x) = \int_0^x g(s)h(s)ds + g(x)\int_x^{x_u} h(s)ds$

where

$g(x) = \int_0^x e^{\frac{-2\mu}{\sigma^2}\frac{s^{1-\theta}}{1-\theta}}ds$

and

$h(x) = \frac{2}{\sigma^2}x^{-2\theta}\frac{1}{g'(x)}$, where $g'(x) = \frac{-2\mu}{\sigma^2}\frac{x^{1-\theta}}{1-\theta}$.

qed.

Chapter 4

Conclusions and Future Work

Larry Shepp [11] showed the ϵ-optimal strategy for Vardi casino without inflation and Grigirescu et al. [13] showed the optimal strategy for the Vardi casino with inflation. We showed that the difference between the Dubins-Savage casino and the Vardi casino is not big. However, in real casino, you can't have an infinitely big odds ratio. So what is the optimal strategy and maximal probability to achieve the goal for a casino that has two tables with odds ratio r_1 and r_2 or talbes with odds ratio $r \in [r_1, r_2]$? This problem remains open.

For stock and derivatives, the Black-Scholes-Merton model seems to be good in the short term, since the parameters will not change too much, hence could be treated as fixed and the company won't bankrupt so fast. But in the long term, we have to consider the potential risk of bankruptcy and change of parameters, especially in the risk free interest rate r and the volatility σ. The "decreasing-return-to-scale" model is a good try. We can also try to use multi-dimensional stochastic process to model the stock price and the underlying company. i.e.

$$X(t) = (X_1(t), X_2(t))$$

where $X_1(t)$ stands for the stock price and $X_2(t)$ stands for the underlying company, and

$$dX_1(t) = \mu_1(X_1(t), X_2(t))dt + \sigma_1(X_1(t), X_2(t))dW_1(t)$$

$$dX_2(t) = \mu_2(X_1(t), X_2(t))dt + \sigma_2(X_1(t), X_2(t))dW_2(t)$$

We can choose the right $\mu_1, \mu_2, \sigma_1, \sigma_2$ to formulate the stock price. For example,

$dX_1(t) = X_1(t)(\mu_1 - \lambda \log \frac{X_1(t)}{X_2(t)})dt + \sigma_1 X_1(t) dW_1(t)$

$dX_2(t) = \mu_2(X_1(t), X_2(t))dt + \sigma_2(X_1(t), X_2(t))dW_2(t)$

The idea of this model is that the price of the stock should reflect the true value of the underlying company. As we can see, when the stock price goes to high or too low away from the true value of the company, the drift will make it return to the true value. And when the company go bankrupt, the stock price will go to zero as well. However, it is hard to describe the true value of the underlying company. If one treats the stock price as the value of the company, then $X_2(t) = X_1(t)$, and hence the model reduces to the Black-Scholes-Merton model. If we use a jump process to describe X_2, then within two adjacent jumps,

$dX_1(t) = X_1(t)(\mu_1 - \lambda \log X_1(t))dt + \sigma_1 X_1(t) dW_1(t)$

In this case,

$d \log X_1(t) = (\mu_1 - \frac{\sigma_1^2}{2} - \lambda \log X_1 + \lambda \log X_2)dt + \sigma_1 dW_1(t)$

$de^{\lambda t} \log X_1(t) = e^{\lambda t}(\lambda \log X_1(t)dt + d \log X_1(t)) = e^{\lambda t}((\mu_1 - \frac{\sigma_1^2}{2} + \log X_2(t))dt + \sigma_1 dW_1(t))$

Hence,

$\log X_1(t) = e^{-\lambda t} \log X_1(0) + \int_0^t e^{-\lambda(t-u)}(\mu_1 - \frac{\sigma_1^2}{2} + \lambda \log X_2(u))du + \int_0^t e^{\lambda u} \sigma_1 dW_1(u)$

and

$X_1(t) = X_1(0)^{e^{-\lambda t}} e^{(\mu_1 - \frac{\sigma_1^2}{2})\frac{1-e^{-\lambda t}}{\lambda} + \int_0^t \lambda e^{-\lambda(t-u)} \log X_2(u)du + \sigma_1 W^*(\frac{e^{2\lambda t}-1}{2\lambda})}$

If we further assume that the $X_2(t)$ is a constant before there comes any new information about the company, then

$X_1(t) = X_2(0)^{1-e^{-\lambda t}} X_1(0)^{e^{-\lambda t}} e^{(\mu_1 - \frac{\sigma_1^2}{2})\frac{1-e^{-\lambda t}}{\lambda} + \sigma_1 W^*(\frac{e^{2\lambda t}-1}{2\lambda})}$

If we assume that the $X_2(t)$ increases at an exponential rate, i.e. $X_2(t) = X_2(0)e^{\mu_2 t}$, before new information comes, then

$X_1(t) = X_2(0)^{1-e^{-\lambda t}} X_1(0)^{e^{-\lambda t}} e^{(\mu_1 - \frac{\sigma_1^2}{2})\frac{1-e^{-\lambda t}}{\lambda} + \frac{\mu_2}{\lambda}(\lambda t - 1 + e^{-\lambda t}) + \sigma_1 W^*(\frac{e^{2\lambda t}-1}{2\lambda})}$

$= X_2(t)(\frac{X_1(0)}{X_2(0)})^{e^{-\lambda t}} e^{(\mu_1 - \mu_2 - \frac{\sigma_1^2}{2})\frac{1-e^{-\lambda t}}{\lambda} + \sigma_1 W^*(\frac{e^{2\lambda t}-1}{2\lambda})}.$

Based on the above formula, we can get the option price under this model.

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