LATTICE SUBGROUPS OF KAC-MOODY GROUPS

BY ILA LEIGH COBBS

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ABSTRACT OF THE DISSERTATION

Lattice Subgroups of Kac-Moody Groups

by Ila Leigh Cobbs

Dissertation Director: Lisa Carbone

We utilize graphs of groups and the corresponding covering theory to study lattices in type-∞ Kac-Moody groups over a finite field $\mathbb{F}_q$, including results for both cocompact and nonuniform lattices. For every prime power q we give a sufficient condition for the rank 2 Kac-Moody group G to contain a cocompact lattice with quotient a simplex, and we show that this condition is satisfied when $q = 2^s$. Under further restrictions, we show that there is an infinite descending chain of cocompact lattices, and we demonstrate such a chain for $q = 2$. Moreover we characterize the quotient graphs of groups for each lattice. Our method involves extending coverings of edge-indexed graphs to covering morphisms of graphs of groups. We also show how this gives rise to other infinite families of cocompact lattices in G.

When $q = 2$ we are also able to embed the infinite descending chain in the rank 3 Kac-Moody group as a chain of lattices in the subgroup generated by all non-maximal standard parabolic subgroups. In addition we embed a non-discrete subgroup in the rank 3 Kac-Moody group whose quotient is a simplex.

We next give graphs of groups descriptions for known nonuniform lattices of Nagao-type. For the nonuniform lattices $SL_2(\mathbb{F}_q[t])$ and $PGL_2(\mathbb{F}_q[t])$ we use the theory of ramified coverings to construct the graphs of groups for their congruence subgroups. We also examine the same construction employed by Morgenstern, identifying and repairing

an error in his work. All graphs of groups for non-uniform lattices constructed here satisfy the structure theorem for graphs of groups.

Acknowledgements

This dissertation would not be complete but for the contributions, mathematical and otherwise, of a number of people. First this work contains co-authored material from two papers. Chapters 2-5 are taken from [CC] and written in cooperation with my advisor Lisa Carbone. This material also includes contributions by Gabriel Rosenberg (Section 2.3.1) and Anne Thomas (Section 4.1). The results in chapters 8 and 9 are taken from [CCM] and include those obtained in cooperation with Lisa Carbone and Scott Murray. In particular, the computer generated examples in Chapter 8 are due to Murray. All proofs presented in this dissertation are my original work, unless explicitly stated otherwise. I thank all the contributors to these papers for their insights and for the pleasure of working with them.

Still others have offered crucial insights. I have enjoyed a number of helpful conversations with Lisa Carbone and Scott Murray and along with them, Delaram Kharobei and Gunther Cornelissen. Frédéric Paulin explained his construction and Bertrand Rémy clarified the details of his paper with Ronan ([RR]). Gunther Cornelissen provided several useful references including the works of Gekeler-Nonnengardt, Rust, and Max Gebhardt. Max Gebhardt provided details of his unpublished computations which corroborate some of the results in Chapter 9. Dimitri Leemans assisted with the theory and computation for coset graphs.

I cannot say enough about those who have stuck with me throughout my time at Rutgers. Lisa, thank you for all your time, patience, and sound advice. Dr. Greenfield has been an amazing source of humor and support, both mathematically and pedagogically. My fellow graduate students (especially the roommates) have provided encouragement, commiseration, and inspiration. Sara, your friendship means more than you know. I am grateful to all those at CUMC for providing a wonderful church home.

To my family, especially my wonderful Mom, I love you and thank you for always being there. Finally I offer praise to God for all the opportunities and for His love.

Dedication

I would like to dedicate this work to all my students, past and future. You are my inspiration.

And to my grandfather, Albert Cobbs, who likes to boast that he planted the seeds for my enjoyment of mathematics. You have given me so much and your life sets an example that I will always strive to follow.

Table of Contents

List of Figures

Chapter 1

Introduction

This thesis examines and in many cases constructs lattice subgroups of Kac-Moody groups by constructing first their graphs of groups. The theory of graphs of groups has its roots in the work of Serre [S]. He shows that when a group acts on a tree, it is possible to recover a presentation for the group, along with additional properties, from its quotient graph of groups. The group is constructed as a fundamental group of graphs of groups. This notion is analogous to its counterpart in algebraic topology and indeed, the two notions of fundamental group are equivalent when taken with respect to a group which acts freely on a tree.

Bass ([B]) and later Bass-Kulkarni ([BK]) expand on the theory by developing the notion of covering morphisms for graphs of groups. Their work gives a constructive method for realizing subgroups (of a group acting on a tree) since such a covering morphism induces an embedding of the corresponding fundamental groups.

In [BL], the theory of graphs of groups is applied extensively to the study of tree lattices. Here we mean a lattice to be a discrete group which carries a finite invariant measure. Bass and Lubotzky show that when a group acts on a tree, its lattice subgroups may be characterized as those which act with finite stabilizers and have either finite quotient graphs (cocompact lattices) or infinite quotient graphs but finite covolume (nonuniform lattices).

The work in [BL] is partly motivated by presenting a theory for general tree lattices in analogy with the theory for lattices in semisimple, non-archimedean Lie groups of rank one, which act on their Bruhat-Tits tree. The Lie group case is also treated in works such as [L1, L2]. Subsequent works by Carbone and Carbone-Garland use the methods of Bass and Lubotzky to study lattices in Kac-Moody groups over finite fields.

Kac-Moody groups arise from infinite-dimensional Kac-Moody Lie algebras. They carry a BN-pair structure and act on their Tits buildings, which are trees in the case of rank 2 Kac-Moody groups.

Infinite families of both cocompact and nonuniform lattices in Kac-Moody groups have been constructed in [CG] and [RR]. What this work offers is an application of the graphs of groups theory to provide a systematic construction and study of lattices. In chapter 3 we supply sufficient conditions for the existence of cocompact lattices with quotient a simplex and show that these conditions are satisfied when the base field has characteristic 2. We also give a further condition, which holds over the field of two elements, that gives rise to a lattice corresponding to a 'tripod' graph of groups. Chapter 4 again applies coverings of graphs of groups in order to embed the 'tripod' lattice into a subgroup of a rank 3 Kac-Moody group. Also in this chapter we apply the analogous theory for coverings of triangle of groups [GP] (or more generally the complex of groups theory developed by Haefliger [H]) in order to embed a non-discrete subgroup in the rank 3 Kac-Moody group which has quotient a simplex.

In [RR], Rémy and Ronan showed that Bourdon's cocompact lattices $\Gamma_{r,q+1}$, $r \geq 5$, $q \geq 3$ [Bo1, Bo2], can be embedded into the closure of *right-angled* Kac-Moody groups in the automorphism groups of their buildings, $I_{r,q+1}$ for q a prime power. The Kac-Moody groups we consider here are examples of right-angled Kac-Moody groups, however there is no overlap with the setting of [RR]. The analog of 'r' in our results, that is, the type of fundamental polygon for the Weyl group, is 2 or 3, while the results of [RR] and [Bo1], [Bo2] hold only for $r \geq 5$. It is intriguing however that the lattices of [RR] (Section 5C) appear to be quotients of the lattices we construct in the rank 3 Kac-Moody group.

The 'tripod' lattice exhibited in ranks 2 and 3 over the field of $q = 2$ elements likely occurs over larger fields in rank 2. However for Kac-Moody groups of rank 3, the tree inscribed in the Tits building is homogeneous only when $q = 2$. Since our results depend on this homogeneity, we do not expect our methods in rank 3 to extend easily to other values of q, though a generalization may be possible.

The sufficient condition which establishes the 'tripod' lattice in Chapters 3 and 4 also yields infinite descending chains of cocompact lattices. These are the first examples of infinite descending chains of lattices in Kac-Moody groups. We give the construction of two classes of descending chains in Chapter 5. A key tool in these constructions is the method of 'open fannings'. This method was first used in [C2] to construct the first non-uniform lattices in $Aut(X)$ for any locally finite tree X which is the universal cover of a finite connected graph.

In Chapter 6 we give a third class of infinite descending chains, this time using only covering theory. We classify the graphs of groups which give rise to cocompact lattices in certain cases by providing necessary and sufficient conditions for the existence of covering morphisms in these cases. This classification provides yet further examples of infinite families of cocompact lattices in rank 2 Kac-Moody groups. As a final example, we show in Section 6.3 that any free group may be embedded as a cocompact lattice in a rank 2 Kac-Moody group.

Since the lattices obtained over the field of two elements are free products of finite groups, it follows that they are residually finite ([Ma]) and hence contain an abundance of normal subgroups of finite index. The existence of descending chains of lattices is therefore not surprising, though we do not use residual finiteness. Instead our strategy of extending covering morphisms of edge-indexed graphs to covering morphisms of graphs of groups provides a new tool for constructing descending chains of lattices in locally compact groups.

As mentioned previously, cocompact lattices such as those constructed in Chapters 3, 5 and 6 are characterized by finite graphs of finite groups. Nonuniform lattices have infinite graphs of finite groups and are more complex. One interesting open question is whether or not nonuniform lattices satisfy the Structure Theorem, stating that any lattice has a quotient graph consisting of a finite 'core' graph with finitely many infinite rays (cusps) attached. The structure theorem holds in the Lie group case, as shown in various cases by both Lubotzky and Raghunathan ([BL]).

What is known in the Kac-Moody group setting is that the Structure Theorem holds for affine Kac-Moody groups. Affine Kac-Moody groups arise from Lie algebras associated to Cartan matrices of affine type, while the larger class of hyperbolic Kac-Moody groups arise from Cartan matrices of hyperbolic type. All known examples of non-uniform lattices of hyperbolic Kac-Moody groups satisfy the Structure Theorem, but the question remains open. This motivates the work of Chapters 7, 8 and 9, where we construct graphs of groups for well-known nonuniform lattices of affine Kac-Moody groups which are likely to have natural analogues in the hyperbolic setting. This lays the ground work for potentially using graphs of groups to help answer the Structure Theorem question for hyperbolic Kac-Moody groups.

For rank 2 affine Kac-Moody groups we have convenient descriptions in terms of the matrix group $SL_2(\mathbb{F}_q((t^{-1})))$. The nonuniform lattices we study in Chapters 7, 8 and 9 are nonuniform lattices of this matrix group. Chapter 7 gives graphs of groups for Nagao-type lattices, such as $SL_2(\mathbb{F}_q[t])$, with quotient an infinite ray. We then combine these to give a graph of groups for one of the generating BN-pair groups, also a nonuniform lattice but with quotient a bi-infinite ray. Chapter 8 adopts the method of ramified coverings to give the graphs of groups for congruence subgroups of the Nagao-type lattice $SL_2(\mathbb{F}_q[t])$. In particular we give the size of the core graph together with the number of cusps.

The work done in Chapter 8 was originally inspired by a work of Morgenstern [M]. His method was to construct the quotient graph for a congruence subgroup as a ramified covering of the quotient graph for $PGL_2(\mathbb{F}_q[t])$. This idea is consistent with the theory of branched topological coverings, and in Morgenstern's setting, coincides with a method suggested by Drinfeld in his theory of modular curves over function fields ([D]). Similar constructions of fundamental domains of lattices for congruence subgroups were constructed by Gekeler and Nonnengardt ([GN2]) and Rust ([Ru]) using essentially the same method.

We remark that the method of constructing a fundamental domain for a congruence subgroup as a ramified covering is unclear in the settings of Morgenstern, Gekeler-Nonnengardt and Rust. However, we have been able to verify that a correctly constructed ramified covering gives rise to a covering morphism of graphs of groups in the sense of Bass' covering theory for graphs of groups ([B]). Thus the ramified covering should coincide with the quotient graph. In general the structural properties of the quotient graphs obtained as ramified coverings are difficult to determine and detailed drawings of these graphs are non-trivial to obtain.

One important property of quotient graphs is that they are always connected. The works of [GN] and [Ru] do not verify this property for their constructions as ramified coverings. Morgenstern's work yields examples which we will show are disconnected and hence cannot be quotient graphs. This contradicts some of the results in [M], and indicates an error in the ramified covering. Chapter 9 discusses the sources of this error and gives a corrected ramified covering for the congruence subgroups of $PGL_2[t]$. We show that these quotient graphs are in fact isomorphic to their counterparts in $SL_2[t]$ constructed in Chapter 8.

Chapter 2

Preliminaries

Let G be a completion of Tits' Kac-Moody group functor over a finite field $\mathbb{F}_q$. Then G is locally compact and totally disconnected ([CG], [RR]). Completions of the Tits functor have been described by Carbone and Garland ([CG]) and Rémy and Ronan ([RR]). If G is the Kac-Moody group of a generalized Cartan matrix A, then we call G *affine* if A is positive semi-definite but not positive definite. If A is neither positive definite nor positive semi-definite, but every proper indecomposable submatrix is either positive definite or positive semi-definite, we say that G has *hyperbolic type.* If every proper indecomposable submatrix of A is positive definite, we say that G has *compact hyperbolic type.* Thus if A has a proper indecomposable affine submatrix, we say that G has *noncompact hyperbolic type.*

This work considers only the following class of Kac-Moody groups G over a finite field $\mathbb{F}_q$. We suppose that G is complete using the Rémy-Ronan completion, that G is either of affine or hyperbolic type, and that G has 'type ∞', that is, the Weyl group W is a free product of $\mathbb{Z}/2\mathbb{Z}$'s. This coincides with the class of affine or hyperbolic Kac-Moody groups corresponding to generalized Cartan matrices $A = (A_{ij})_{i,j\in I}$ where all m_{ij} equal ∞ for $i \neq j$. In particular, this includes all rank 2 Kac-Moody groups, whose generalized Cartan matrices form the infinite family

$$A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}, \ ab \in \mathbb{Z}_{\geq 4},$$

which is of affine type if $ab = 4$ and of (compact) hyperbolic type if $ab > 4$.

If A is of affine type then there are two possible generalized Cartan matrices, namely

$$A_1^{(1)} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \; A_2^{(2)} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$

If $rank(G) = 3$, we may apply the classification of symmetrizable hyperbolic Dynkin diagrams ([Sa]) to deduce that G is of noncompact hyperbolic type and that the generalized Cartan matrix of G is one of the following:

$$A = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}, \; A = \begin{pmatrix} 2 & -1 & -2 \\ -4 & 2 & -1 \\ -2 & -4 & 2 \end{pmatrix}.$$

In both cases $W \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ and the fundamental chamber for W is an ideal triangle in the hyperbolic plane. Our interest in the Kac-Moody groups in this class comes in part from the fact that the corrected automorphic forms of the corresponding generalized Kac-Moody algebras play an important role in high-energy physics ([GN] and [HM]). If G has type ∞ and $rank(G) > 3$ then G no longer has hyperbolic type. We will not say more about this case here.

2.1 Locally compact Kac-Moody groups

Though there is no obvious infinite dimensional generalization of finite dimensional Lie groups, Tits associated a group functor G_A on the category of commutative rings, such that for any symmetrizable generalized Cartan matrix A and any ring R there exists a group $G_A(R)$ ([Ti2], [Ti4]). Tits defined not one group, but rather *minimal* and *maximal* groups. The value of the Tits functor G_A over a field k is called a *minimal Kac-Moody group.* The *maximal* or *complete Kac-Moody group* is defined relative to a completion of the Kac-Moody algebra and contains $G_A(k)$ as a dense subgroup.

2.1.1 BN-pair and Tits building

Tits' Kac-Moody group functor may be described by certain group theoretic data, called a *Tits system* or (B, N)*-pair.* This data carries a great deal of information about the group and its subgroups, and in particular determines a simplicial complex, a *Tits building* X on which the group acts faithfully and cocompactly. A (B,N)-pair can be associated to a Kac-Moody group on either the minimal or complete level. Here we describe briefly the (B,N)-pair associated to a completion of Tits' functor over a finite field.

Let A be an $l \times l$ symmetrizable generalized Cartan matrix. Let $G = G_A(\mathbb{F}_q)$ be a completion of Tits' functor associated to A and the finite field $\mathbb{F}_q$. The existence of such a completion was noted by Tits ([Ti4]). Explicit completions have been constructed using distinct methods by Carbone and Garland ([CG]) and by Remy and Ronan ([RR]) (subsection 2.1.3 below). A complete Kac-Moody group G over a finite field is locally compact, totally disconnected and the Tits building X is locally finite. In this subsection we give a brief description of the Tits system for G and its corresponding Tits building.

A completion G of Tits' functor over the finite field $\mathbb{F}_q$ has subgroups $B^{\pm} \subseteq G$, $N \subseteq G$, and Weyl group $W = N/H$, where $H = N \cap B^{\pm}$ is a normal subgroup of N. We have $B^{\pm} = HU^{\pm}$ where U^+ is generated by all positive real root groups , U^- is generated by all negative real root groups, B^+ is compact, in fact a profinite neighborhood of the identity in G, and B^- is discrete. Then (B^+,N) and (B^-,N) are BN-pairs, and

$$G \quad = \quad B^+NB^- \quad = \quad B^-NB^+.$$

It follows that

$$G \quad = \quad \sqcup_{w\in W}\ B^{\pm}wB^{\pm}.$$

Let S be the standard generating set for the Weyl group W consisting of simple root reflections. Let $U \subsetneq S$. The *standard parabolic subgroups* are

$$P_U \quad = \quad \sqcup_{w\in\langle U\rangle}\ B^{\pm}wB^{\pm}.$$

A *parabolic* subgroup is any subgroup containing a conjugate of $B^{\pm}$. The Tits building of G is a simplicial complex X of dimension $dim(X) = |S| - 1$. In fact we associate a building $X^{\pm}$ to each BN-pair (B^+,N) and (B^-,N). The buildings X^+ and X^- are isomorphic as chamber complexes and have constant thickness $q+1$ (see [DJ], Appendix KMT).

The vertices of X are given by cosets of the maximal parabolic subgroups in G. The incidence relation is described as follows. The $r+1$ vertices $P_1, \ldots, P_{r+1}$ span an r-simplex if and only if the intersection $P_1 \cap \cdots \cap P_{r+1}$ is parabolic, that is, contains a conjugate of $B^{\pm}$. If the root system is infinite, the Weyl group W is infinite, so by the Solomon-Tits theorem, X is contractible. The group G acts by left multiplication on cosets.

2.1.2 Kac-Moody groups and Tits' presentation

In this subsection we define minimal Kac-Moody groups over arbitrary fields by generators and relations, following Tits ([Ti2]).

Let W be the Weyl group of a symmetrizable Kac-Moody algebra $\mathfrak{g}$. We introduce an auxiliary group $W^* \subseteq \mathrm{Aut}(\mathfrak{g})$, generated by elements $\{w_i^*\}_{i \in I}$, where

$$w_i^* \;=\; \exp(\mathrm{ad}\, e_i)\exp(-\,\mathrm{ad}\, f_i)\exp(\mathrm{ad}\, e_i) \;=\; \exp(-\,\mathrm{ad}\, f_i)\exp(\mathrm{ad}\, e_i)\exp(-\,\mathrm{ad}\, f_i).$$

The group W^* is a central extension of W, that is, there is a surjective homomorphism $\epsilon : W^* \to W$ which sends w_i^* to w_i for all i. We define certain elements of $\mathfrak{g}$, denoted $\{e_\alpha\}_{\alpha \in \Phi}$. Given $\alpha \in \Phi$, write α in the form $w\alpha_j$ for some $j \in I$ and $w \in W$, choose $w^* \in W^*$ which maps onto w, and set $e_\alpha = w^* e_{\alpha_j}$. It is clear from [Ti2, (3.3.2)] that e_α belongs to the root space g^α, e_α is uniquely determined up to sign, and for all $i \in I$, $w_i^* e_\alpha = \eta_{\alpha,i} e_{w_i \alpha}$ for some constants $\eta_{\alpha,i} \in \{\pm 1\}$. These constants $\{\eta_{\alpha,i}\}$ will appear in the definition of Kac-Moody groups.

Let A be a symmetrizable generalized Cartan matrix. Let k denote an arbitrary field. The group $G = G_A(k)$ defined below is called the *incomplete simply-connected Kac-Moody group* corresponding to A. The presentation of G is 'almost canonical' except for the choice of elements $\{e_\alpha\}$ which determine the constants $\{\eta_{\alpha,i}\}$.

By definition, $G_A(k)$ is generated by a set of symbols denoted $\{\chi_\alpha(u) \mid \alpha \in \Phi, u \in k\}$ satisfying relations (R1)-(R7) below. Let $i, j \in I,\ u, v \in k$ and let $\alpha, \beta \in \Phi$.

(R1) $\chi_\alpha(u+v) = \chi_\alpha(u)\chi_\alpha(v)$;

(R2) Let (α, β) be a *prenilpotent pair,* that is, there exist $w,\ w' \in W$ such that

$$w\alpha,\ w\beta \in \Phi^+ \text{ and } w'\alpha,\ w'\beta \in \Phi^-.$$

Then

$$[\chi_\alpha(u), \chi_\beta(v)] = \prod_{m,n=1}^{\infty} \chi_{m\alpha+n\beta}(C_{mn\alpha\beta} u^m v^n)$$

where the product on the right hand side is taken over all real roots of the form $m\alpha+n\beta$, $m, n \geq 1$, in some fixed order, and $C_{mn\alpha\beta}$ are integers independent of k (but depending on the order).

For each $i \in I$ and $u \in k^*$ set

$\chi_{\pm i}(u) = \chi_{\pm\alpha_i}(u)$,

$\widetilde{w}_i(u) = \chi_i(u)\chi_{-i}(-u^{-1})\chi_i(u)$,

$\widetilde{w}_i = \widetilde{w}_i(1)$ and $h_i(u) = \widetilde{w}_i(u)\widetilde{w}_i^{-1}$.

The remaining relations are

(R3) $\widetilde{w}_i\chi_\alpha(v)\widetilde{w}_i^{-1} = \chi_{w_i\alpha}(\eta_{\alpha,i}u)$,

(R4) $h_i(u)\chi_\alpha(v)h_i(u)^{-1} = \chi_\alpha(vu^{-\langle\alpha,\alpha_i^\vee\rangle})$ for $u \in k^*$,

(R5) $\widetilde{w}_i h_j(u)\widetilde{w}_i^{-1} = h_j(u)h_i(u^{-a_{ji}})$,

(R6) $h_i(uv) = h_i(u)h_i(v)$ for $u, v \in k^*$, and

(R7) $[h_i(u), h_j(v)] = 1$ for $u, v \in k^*$.

An immediate consequence of relations (R3) is that $G_A(k)$ is generated by $\{\chi_{\pm i}(u)\}$.

Remark. In [KP, Proposition 2.3], it is shown that a pair (α, β) is prenilpotent if and only if $\alpha \neq -\beta$ and $|(\mathbb{Z}_{>0}\alpha + \mathbb{Z}_{>0}\beta) \cap \Phi| < \infty$. Thus the product on the right-hand side of (R2) is finite.

The above presentation can be viewed as an analog of the Steinberg presentation for classical groups with $\chi_\alpha(u)$ playing the role of $\exp(ue_\alpha)$. In [CG], the authors give a representation-theoretic interpretation of Kac-Moody groups which makes the above analogy precise.

Next we introduce several subgroups of $G = G_A(k)$:

1. *Root subgroups* U_α. For each $\alpha \in \Phi$ let $U_\alpha = \{\chi_\alpha(u) \mid u \in k\}$. By relations (R1), each U_α is isomorphic to the additive group of k.

2. *The 'extended Weyl group'* $\widetilde{W}$. Let $\widetilde{W}$ be the subgroup of G generated by elements $\{\widetilde{w}_i\}_{i\in I}$. One can show that $\widetilde{W}$ is isomorphic to the group W^* introduced before, so there is a surjective homomorphism $\epsilon : \widetilde{W} \to W$ such that $\epsilon(\widetilde{w}_i) = w_i$ for $i \in I$. Given $\widetilde{w} \in \widetilde{W}$ and $w \in W$, we will say that $\widetilde{w}$ is a representative of w if $\epsilon(\widetilde{w}) = w$. It will be convenient to identify (non-canonically) W with a subset (not a subgroup) of $\widetilde{W}$ which contains exactly one representative of every element of W. By abuse of notation, the set of those representatives will also be denoted by W. It follows from relations (R3) that $wU_\alpha w^{-1} = U_{w\alpha}$ for any $\alpha \in \Phi$ and $w \in W$.

3. *'Unipotent' subgroups.* Let $U^+ = \langle U_\alpha \mid \alpha \in \Phi^+\rangle$, and $U^- = \langle U_\alpha \mid \alpha \in \Phi^-\rangle$.

4. *'Torus' ('diagonal' subgroup).* Let $H = \langle\{h_i(u) \mid i \in I, u \in k\}\rangle$. One can show that relations (R6)-(R7) are defining relations for H, so H is isomorphic to the direct sum of l copies of k^*.

5. *'Borel' subgroups.* Let $B^+ = \langle U^+, H\rangle$ and $B^- = \langle U^-, H\rangle$. By relations (R4), H normalizes both U^+ and U^-, so we have $B^+ = HU^+ = U^+H$ and $B^- = HU^- = U^-H$.

6. *'Normalizer.'* Let N be the subgroup generated by $\widetilde{W}$ and H. Since $\widetilde{W}$ normalizes H, we have $N = \widetilde{W}H$. It is also easy to see that $N/H \cong W$.

Tits proved that (B^+, N) and (B^-, N) are BN-pairs of G [Ti2]. In fact, G admits the stronger structure of a twin BN-pair, though we shall not use this additional structure. From now on, we write B for B^+ and U for U^+. From Tits [Ti1, Proposition 5] we have that the group U is generated by the elements $\{\chi_\alpha(u) \mid \alpha \in \Phi^+, u \in k\}$ subject to relations (R1) and (R2) above.

2.1.3 Complete Kac-Moody groups

Distinct completions of Tits' minimal group have been given in the papers of Carbone and Garland [CG] and Rémy and Ronan [RR]. We shall primarily use the Rémy-Ronan completion, which we review in this subsection.

Let X be the building associated with the positive BN-pair (B, N), and consider the action of G on X. We define a topology on G by taking a subbase of neighborhoods of the identity to consist of stabilizers of vertices of X. We shall call this topology the *building topology.* The completion of G in its building topology will be referred to as the *Rémy-Ronan completion* and denoted by $\widehat{G}$. Let Z be the kernel of the natural map $G \to \widehat{G}$ (or, equivalently, the kernel of the action of G on X). Using results of Kac and Peterson [KP], Rémy and Ronan [RR, 1.B] showed that Z is a subgroup of H and hence is finite. Furthermore, Z coincides with the center of G.

Now let $\widehat{B}$ (resp. $\widehat{U}$) be the closure of B (respectively U) in $\widehat{G}$. The natural images of N and H in $\widehat{G}$ are discrete, and therefore we will denote them by the same symbols (without hats).

The following theorem is a collection of results from [Re] and [RR]:

Theorem 2.1. *Let $\widehat{G}$, $\widehat{B}$ and N be as above. Then:*

(a) The pair $(\widehat{B}, N)$ is a BN-pair of $\widehat{G}$. Moreover, if $\widehat{X}^+$ is the associated building, there exists a $\widehat{G}$-equivariant isomorphism between X^+ and $\widehat{X}^+$. In particular, the Coxeter group associated to $(\widehat{B}, N)$ is isomorphic to $W = W(A)$.

(b) The group $\widehat{B}$ is an open profinite subgroup of $\widehat{G}$. Furthermore, $\widehat{U}$ is an open pro-p subgroup of $\widehat{B}$.

2.1.4 Kac-Moody Groups of Type ∞

Let G be a Kac-Moody group and suppose that G has type ∞. Then the associated Weyl group $W = N/H$, as described using the BN-pair, is a free product of copies of $\mathbb{Z}/2\mathbb{Z}$. In particular, let G be a Kac-Moody group of type ∞ over a finite field $\mathbb{F}_q$. Then the Weyl group has the form

$$W \;=\; \langle w_i \;\mid\; i = 1, ..., n \rangle \;\cong\; *_{i=1,...,n}\mathbb{Z}/2\mathbb{Z}\;,$$

where $n = rank(G)$. If G is affine or hyperbolic and has type ∞, then G has rank 2 or rank 3.

Rank 2

Let G be rank 2 Kac-Moody group over a finite field $\mathbb{F}_q$. Then

$$W \;=\; \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}\;.$$

The Tits building X is the $(q+1)$-regular tree X_{q+1}. The vertex set VX is given by the set of cosets of the maximal standard parabolic subgroups

$$P_1 \;:=\; B \sqcup Bw_1B\;, \text{ and}$$

$$P_2 \;:=\; B \sqcup Bw_2B\;.$$

The oriented edge set is given by

$$EX = G/B \sqcup \overline{G/B}\;,$$

where $\overline{G/B}$ denotes the edges of opposite orientation.

We have X a *homogeneous*,bipartite tree of degree

$$[P_1 : B] = [P_2 : B] = q + 1.$$

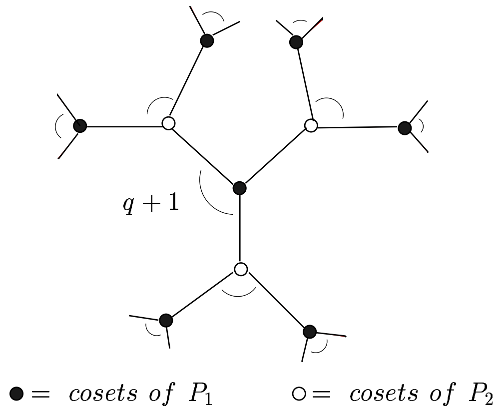

Figure 2.1: Bruhat-Tits tree for rank 2 Kac-Moody group over $\mathbb{F}_q$

Since G acts with two orbits on vertices and a single orbit on edges, we can recover, by application of the fundamental theorem of Bass-Serre [S], a presentation for G as an amalgamated free product

$$G \;=\; P_1 *_B P_2 \;.$$

Rank 3

Let G be a rank 3 Kac-Moody group of type ∞ over a finite field $\mathbb{F}_q$. Then

$$W \quad = \quad \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}\;.$$

The Tits building X is the hyperbolic plane tessellated by triangles, together with $q-1$ triangles glued along each edge of a triangle in the plane.

The vertex set (vertices of the triangles) VX is given by the set of cosets of the maximal standard parabolic subgroups

$$P_{1,2} \quad := \quad B \bigsqcup_{w\in<w_1,w_2>} BwB \;, \text{ and}$$

$$P_{2,3} \quad := \quad B \bigsqcup_{w\in<w_2,w_3>} BwB \;, \text{ and}$$

$$P_{1,3} \quad := \quad B \bigsqcup_{w\in<w_1,w_3>} BwB \;.$$

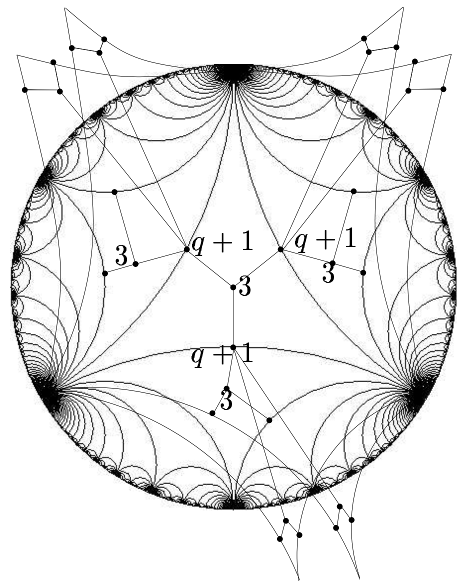

Figure 2.2: Tits building for rank 3 Kac-Moody group with $(3, q+1)$-biregular inscribed tree

The edges correspond to cosets of

$$Q_1 \quad := \quad B \sqcup Bw_1B \text{ , and}$$

$$Q_2 \quad := \quad B \sqcup Bw_2B \text{ , and}$$

$$Q_3 \quad := \quad B \sqcup Bw_3B \text{ .}$$

(We use Q_i to differentiate from the maximal parabolic subgroups P_i in rank 2.)

The triangular faces correspond to cosets of B.

We can embed a $(q+1, 3)$-bi-regular tree in X by taking the a maximal tree $\mathcal{X}$ of the barycentric subdivision, as shown. The action of G on X induces an action on $\mathcal{X}$.

For a Kac-Moody group of type ∞ over a field of two elements, we will use the action of G on its building (or embedded tree) to exhibit (lattice) subgroups by constructing their quotient graphs of groups. This technique is explained in the next section.

2.2 Lattices and covering theory

Let G be a locally compact group acting on a set X with compact open stabilizers, and let μ be a (left) Haar measure on G. Let $\Gamma \leq G$ be a discrete subgroup with quotient $p : G \longrightarrow \Gamma\backslash G$. We call Γ a *G-lattice* if $\mu(\Gamma\backslash G) < \infty$, and a *uniform* or *cocompact* G-lattice if $\Gamma\backslash G$ is compact. When G is unimodular, $\mu(G_x)$ is constant on G-orbits, so we can define:

$$Vol(G\backslash\backslash X) = \mu(G\backslash\backslash X) := \sum_{x \in V(G\backslash X)} \frac{1}{\mu(G_x)}.$$

Theorem 2.2. ([BL], (1.6)) *Suppose that a group G acts on a set X with compact open stabilizers. For a discrete subgroup $\Gamma \leq G$, the following conditions are equivalent:*

(a) $Vol(\Gamma\backslash\backslash X) < \infty$.

(b) Γ is a G-lattice (hence G is unimodular), and $\mu(G\backslash\backslash X) < \infty$.

In this case:

$$Vol(\Gamma\backslash\backslash X) = \mu(\Gamma\backslash G) \cdot \mu(G\backslash\backslash X).\square$$

Let $\Gamma \leq G$ be discrete. Then the diagram of natural projections

$$\begin{array}{ccc} & X & \\ \swarrow^{p_\Gamma} & & \searrow^{p_G} \\ \Gamma\backslash X & \xrightarrow{p} & G\backslash X \end{array}$$

commutes. Assume that $Vol(\Gamma\backslash\backslash X) < \infty$. Then Γ is a G-lattice. To determine if Γ is uniform or non-uniform in G, we use the following:

Lemma 2.3. ([BL], (1.5.8)) *Let $x \in VX$. The following conditions are equivalent:*

(a) Γ is a uniform G-lattice.

(b) Some fiber $p^{-1}(p_G(x)) \cong \Gamma\backslash G/G_x$ is finite.

(c) Every fiber of p is finite.

Now let G be a Kac-Moody group of noncompact hyperbolic type. Let X be the Tits building of G. Then $G\backslash X$ is not compact. Suppose that G contains a cocompact G-lattice Γ. By the Lemma above, this implies that $\Gamma\backslash G/G_x$ is finite for any $x \in VX$, that is, $\Gamma\backslash G/gP_ig^{-1}$ is finite for any $g \in G$, where P_i is a maximal parabolic subgroup of G. Even though $G\backslash X$ is not compact, the number of orbits of G on X is finite.

2.2.1 Edge-indexed graphs

Let A be a connected locally finite graph, with sets VA of vertices and EA of oriented edges. The initial and terminal vertices of $e \in EA$ are denoted by $\partial_0 e$ and $\partial_1 e$ respectively. The map $e \mapsto \overline{e}$ is orientation reversal, with $\partial_{1-j}\overline{e} = \partial_j e$ for $j = 0, 1$.

A *graph of groups* $\mathbb{A} = (A, \mathcal{A}_v, \mathcal{A}_e, \alpha_e)$ over a connected graph A consists of an assignment of vertex groups $\mathcal{A}_v$ for each $v \in VA$ and edge groups $\mathcal{A}_e = \mathcal{A}_{\overline{e}}$ for each $e \in EA$, together with monomorphisms $\alpha_e : \mathcal{A}_e \to \mathcal{A}_{\partial_0 e}$ for each $e \in EA$. We refer the reader to [B] for the definitions of the *fundamental group* $\pi_1(\mathbb{A}, a_0)$ and *universal covering tree* $X = \widetilde{(A, a_0)}$ of a graph of groups $\mathbb{A} = (A, \mathcal{A}_v, \mathcal{A}_e, \alpha_e)$, with respect to a basepoint $a_0 \in VA$.

An *edge-indexed graph* (A, i) consists of an underlying graph A together with an assignment of a positive integer $i(e) \in \mathbb{Z}_{>0}$ to each edge $e \in EA$. Let $\mathbb{A} = (A, \mathcal{A})$ be a graph of groups. Then $\mathbb{A}$ naturally gives rise to an edge-indexed graph $I(\mathbb{A}) = (A, i)$, with for each $e \in EA$, the map $i : EA \to \mathbb{Z}_{>0}$ given by $i(e) = [\mathcal{A}_{\partial_0 e} : \alpha_e \mathcal{A}_e]$, which we assume to be finite.

Given an edge-indexed graph (A, i), a graph of groups $\mathbb{A}$ such that $I(\mathbb{A}) = (A, i)$ is called a *grouping* of (A, i). We call $\mathbb{A}$ a *finite grouping* if the vertex groups $\mathcal{A}_v$ are all finite, and a *faithful grouping* if $\mathbb{A}$ is a faithful graph of groups, that is if the fundamental group $\pi_1(\mathbb{A}, a_0)$ acts faithfully on the universal covering tree $X = \widetilde{(\mathbb{A}, a_0)}$.

Example: Let G be a Kac-Moody group over a finite field $\mathbb{F}_q$. The action of G on $X = X_{q+1}$ gives rise to a quotient graph of groups, whose vertices and edges are the G-orbits of vertices in edges in X.

$$\begin{tikzcd} P_1 \arrow[r, dash, "B"] & P_2 \end{tikzcd}$$

Figure 2.3: graph of groups for a rank 2 Kac-Moody group

The Kac-Moody group G is the fundamental group of this graph of groups, and the Tits building X is the universal covering tree. The corresponding edge-indexed graph is a pair of edges $\{e, \overline{e}\}$ with $i(e) = i(\overline{e}) = [P_i : B] = q + 1$.

We now describe a method for constructing lattices in $\mathrm{Aut}(X)$ which follows naturally from the fundamental theory of Bass–Serre (see [B, S]), and was first suggested in [BK]. We begin with an edge-indexed graph (A, i). Then (A, i) determines a universal covering tree $X = \widetilde{(A, i, a_0)}$ up to isomorphism. Let $\mathbb{A}$ be a finite grouping of (A, i). Then there is a homomorphism

$$\pi_1(\mathbb{A}, a_0) \to \mathrm{Aut}(X).$$

This map is a monomorphism if and only if $\mathbb{A}$ is faithful, in which case we may identify $\pi_1(\mathbb{A}, a_0)$ with its image in $\mathrm{Aut}(X)$. Since $\mathbb{A}$ is a finite grouping, this image is discrete. By the discussion in Section 2.2.1 above, the image of $\pi_1(\mathbb{A}, a_0)$ is a lattice in $\mathrm{Aut}(X)$ if and only if $\mathbb{A}$ is a faithful graph of finite groups of finite volume.

2.2.2 Coverings

We have described in Section 2.2.1 above how to construct lattices in $\mathrm{Aut}(X)$ as fundamental groups Γ of graphs of groups. In order to determine if such a Γ embeds into a subgroup $G < \mathrm{Aut}(X)$ (e.g. a Kac-Moody group) we will use covering morphisms of graphs of groups.

Definition 2.3. *Let $\mathbb{A} = (A, \mathcal{A}_v, \mathcal{A}_e, \alpha_e)$ and $\mathbb{A}' = (A', \mathcal{A}'_v, \mathcal{A}'_e, \alpha'_e)$ be graphs of groups. A* covering morphism *$\Phi = (\varphi, (\delta)) : \mathbb{A} \to \mathbb{A}'$ consists of:*

1. *a graph morphism $\varphi : A \to A'$;*
2. *monomorphisms*

$$\varphi_a : \mathcal{A}_a \to \mathcal{A}'_{\varphi(a)} \quad (a \in A), \qquad \varphi_e = \varphi_{\bar{e}} : \mathcal{A}_e \to \mathcal{A}'_{\varphi(e)} \quad (e \in EA);$$

3. *For each $e \in EA$ with $a = \partial_0 e$ an element $\delta_e \in \mathcal{A}'_{\varphi(a)}$ such that the following two conditions hold:*

(a) the following diagram commutes:

$$\begin{tikzcd} \mathcal{A}_e \arrow[r, "\alpha_e"] \arrow[d, "\varphi_e"] & \mathcal{A}_a \arrow[d, "\varphi_a"] \\ \mathcal{A}'_{\varphi(e)} \arrow[r, "\mathrm{ad}(\delta_e)\circ\alpha'_{\varphi(e)}"] & \mathcal{A}'_{\varphi(a)} \end{tikzcd}$$

where $\mathrm{ad}(x)(s) = xsx^{-1}$.

(b) For $f \in EA'$, $a' = \partial_0 f$ *and* $a \in \varphi^{-1}(a')$, *the map*

$$\Phi_{a/f} : \left(\coprod_{e \in \varphi_{(a)}^{-1}(f)} \mathcal{A}_a/\alpha_e\mathcal{A}_e \right) \longrightarrow \mathcal{A}'_{\varphi(a)}/\alpha'_f\mathcal{A}'_f$$

defined by

$$\Phi_{a/f}([s]_e) = [\varphi_a(s)\delta_e]_f$$

is bijective (where $s \in \mathcal{A}_a$, $[s]_e$ *is the class of* s *in* $\mathcal{A}_a/\alpha_e\mathcal{A}_e$, *and* $\varphi_{(a)}^{-1}(f) = \{e \in EA \mid \varphi(e) = f, \partial_0(e) = a\}$*).*

Covering morphisms of graphs of groups were originally defined by Bass (Definitions 2.1 and 2.6 of [B]). By Proposition 2.7 of [B], a covering morphism $\Phi : \mathbb{A} \to \mathbb{A}'$ induces a monomorphism of fundamental groups $\Phi_{a_0} : \pi_1(\mathbb{A}, a_0) \to \pi_1(\mathbb{A}', a_0')$ and an isomorphism of universal covers $\widetilde{\Phi} : \widetilde{(\mathbb{A}, a_0)} \to \widetilde{(\mathbb{A}', a_0')}$ (where $a_0 \in VA$ and $a_0' = \varphi(a_0)$ for $\varphi : A \to A'$ the underlying graph morphism of the covering Φ).

We will embed lattices in a Kac-Moody group G by constructing coverings of the corresponding graphs of groups. The following lemma describes characteristics of the lattice derived from the corresponding graph of groups. The lemma will be useful, in particular for the embedding of free groups as cocompact lattices in rank 2 Kac-Moody groups (see Section 6.3).

Lemma 2.4. *Let* G *be a rank 2 locally compact Kac-Moody group over* $\mathbb{F}_q$. *Let* $\mathbb{A} = (A,\ \mathcal{A}_v,\ \mathcal{A}_e,\ \alpha_e)$ *be any graph of groups. Let* $\mathbb{G} = G\backslash\backslash X$ *be the graph of groups for* G *with respect to its action on* $X = X_{q+1}$. *Suppose there is a covering morphism* $\Phi : \mathbb{A} \to \mathbb{G}$. *Then*

1. $\Gamma = \pi_1(\mathbb{A})$ *is a subgroup of* $G = \pi_1(\mathbb{G})$.

2. If $\mathbb{A}$ *is a graph of finite groups, then* $\Gamma = \pi_1(\mathbb{A})$ *is a discrete subgroup* G.

3. If $\mathbb{A}$ *is a finite graph of finite groups, then* $\Gamma = \pi_1(\mathbb{A})$ *is a cocompact lattice in* G.

4. If $\mathbb{A}$ *is an infinite graph of finite groups, with* $Vol(\mathbb{A}) := \sum_{v \in VA} \frac{1}{|\mathcal{A}_v|} < \infty$, *then* $\Gamma = \pi_1(\mathbb{A})$ *is a nonuniform lattice in* G.

Proof. (of Lemma 2.4)

(1) is a restatement of Bass's equivalence between a covering morphism of graphs of groups and an embedding of fundamental groups of graphs of groups ([B], Proposition 2.7).

For (2), let G_0 be the faithful quotient of G on X. By (1), Γ is a subgroup of G, and hence acts on X. Since $\mathbb{A}$ is a graph of finite groups, the action has finite vertex stabilizers, and thus Γ is discrete in $Aut(X)$. Moreover, G_0 is a closed subgroup of $Aut(X)$ and the quotient topology on G_0 coincides with the quotient topology on $Aut(X)$ ([CG], Section 9). Therefore Γ is discrete in $Aut(X)$ if and only if Γ is discrete in G.

For (3), since A is a finite graph, the map $A = \pi_1(\mathbb{A})\backslash X \longrightarrow G\backslash X$ has finite fibers. Since $\pi_1(\mathbb{A})$ is discrete in G it follows from ([BL], 1.5) that $\pi_1(\mathbb{A})$ is cocompact in G.

For (4), by ([BL], 1.6) $Vol(\mathbb{A}) < \infty$ if and only if $\mu(\Gamma\backslash G) < \infty$ and

$$\mu(\mathbb{G}) := \frac{1}{\mu(P_1)} + \frac{1}{\mu(P_2)} < \infty ,$$

where μ is the Haar measure on G. Since Γ is discrete in G and $\mu(\Gamma\backslash G) < \infty$ and A is an infinite graph, it follows that Γ is nonuniform in G.

$\square$

2.3.1 Extending Coverings of Edge-indexed Graphs to Covering Morphisms of Graphs of Groups

In the last section we described a technique for embedding fundamental groups of graphs of groups by constructing covering morphisms of graphs of groups. This technique requires us to find an infinite family of elements δ_e that produce corresponding commutative diagrams and bijections on cosets.

In some cases it is possible to construct a covering morphism of graphs of groups by first constructing a simpler *covering of edge-indexed graphs* (see [BL]),

$$p : (B, j) \longrightarrow (A, i).$$

Here $p : B \longrightarrow A$ is a graph morphism such that for all $e \in EA$, $\partial_0(e) = a$, and $b \in p^{-1}(a)$, we have

$$i(e) = \sum_{f \in p_{(b)}^{-1}(e)} j(f),$$

where $p_{(b)} : E_0^B(b) \longrightarrow E_0^A(a)$ is the local map on stars $E_0^B(b) = \{f \in EB \mid \partial_0(e) = b\}$ and $E_0^A(a)\{e \in EA \mid \partial_0(e) = a\}$ of vertices $b \in VB$ and $a \in VA$. If $b \in VB$ and $p(b) = a \in VA$, then we can identify

$$\widetilde{(A, i, a)} = X = \widetilde{(B, j, b)}$$

so that the diagram of natural projections

$$\begin{tikzcd} & X \arrow[dl, "p_B"'] \arrow[dr, "p_A"] & \\ B \arrow[rr, "p"] & & A \end{tikzcd}$$

commutes.

Let $\varphi : (A, i) \to (A', i')$ be a covering of edge-indexed graphs. In this section we consider the following natural question:

Question 2.4. *Are there faithful finite groupings $\mathbb{A}$ and $\mathbb{A}'$ of (A,i) and (A',i') respectively such that φ extends to a covering morphism $\Phi : \mathbb{A} \longrightarrow \mathbb{A}'$?*

A positive answer to Question 2.4 would give rise to a pair $\Gamma \leq \Gamma'$ of discrete subgroups of $\mathrm{Aut}(X)$, where $X = \widetilde{(A,i)} = \widetilde{(A',i')}$ is the universal covering tree, and $\Gamma = \pi_1(\mathbb{A}, a)$ and $\Gamma' = \pi_1(\mathbb{A}', a')$ are the respective fundamental groups, with basepoints $a \in VA$ and $a' = \varphi(a)$.

Theorem 2.5 gives a sufficient condition for Question 2.4 to have a positive answer in the case that $\mathbb{A}$ and $\mathbb{A}'$ are abelian groupings. In this case, the definition of a covering morphism is simplified as follows.

Let $\Phi : \mathbb{A} \to \mathbb{A}'$ be a covering morphism, as in Definition 2.3, with monomorphisms

$$\varphi_a : \mathcal{A}_a \to \mathcal{A}'_{\varphi(a)} \quad (a \in A), \qquad \varphi_e = \varphi_{\bar{e}} : \mathcal{A}_e \to \mathcal{A}'_{\varphi(e)} \quad (e \in EA),$$

and conjugating elements $\delta_e \in \mathcal{A}'_{\varphi(a)}$ (where $\partial_0 e = a$).

Consider the case where each action $\mathrm{ad}(\delta_e)$ is trivial. This must occur in particular when the groups $\mathcal{A}'_{\varphi(a)}$ are abelian. Since the maps φ_a and φ_e are monomorphisms, we may identify the groups $\mathcal{A}_a$ and $\mathcal{A}_e$ with their images in $\mathcal{A}'_{\varphi(a)}$ and $\mathcal{A}'_{\varphi(e)}$ respectively. Condition (3a) of Definition 2.3 then becomes

$$\text{(3a}'\text{)} \qquad \alpha_e = \alpha'_{\varphi(e)}|_{\mathcal{A}_e}.$$

We have the following.

Theorem 2.5. *([CC]) Let $\varphi : (A,i) \to (A',i')$ be a covering of edge-indexed graphs. Let $\mathbb{A}$ and $\mathbb{A}'$ be finite abelian groupings of (A,i) and (A',i') respectively. Suppose further that*

1. *For all $a \in VA$ and $e \in EA$, we have $\mathcal{A}_a \leq \mathcal{A}'_{\varphi(a)}$ and $\mathcal{A}_e \leq \mathcal{A}'_{\varphi(e)}$.*

2. *For all $e \in EA$, we have $\alpha_e = \alpha'_{\varphi(e)}|_{\mathcal{A}_e}$.*

3. For all $a \in VA$ and $f \in EA'$ such that $\partial_0 f = \varphi(a)$, and for all $e \in \varphi_{(a)}^{-1}(f)$, we have $\mathcal{A}_a \cap \alpha'_f \mathcal{A}'_f = \alpha_e \mathcal{A}_e$.

Then φ extends to a covering morphism $\Phi = (\varphi, (\delta)) : \mathbb{A} \longrightarrow \mathbb{A}'$.

This theorem is due to Rosenberg and its statement and proof may be found in [CC].

2.5 Nagao Lattice

The Remy-Ronan complete Kac-Moody group $\widehat{G}$ described in Section 2.1.3 has a nice matrix description when the group has affine type. In particular the rank 2 affine Kac-Moody group associated to the Cartan matrix $A_1^{(1)}$ is isomorphic to $PSL_2(\mathbb{F}_q((t^{-1})))$. In this case, nonuniform lattices of $G = SL_2(\mathbb{F}_q((t^{-1})))$ yield nonuniform lattices in a rank 2 affine Kac-Moody group. We give the BN-pair description for G in Chapter 7 and for now remark that its Tits building is also the $q+1$-regular infinite tree $X = X_{q+1}$.

This work will focus on known nonuniform lattices of Nagao type. A nonuniform lattice in $G = SL_2(\mathbb{F}_q((t^{-1})))$ will be of Nagao type if it corresponds to a graph of groups of the form

$$\overset{\Gamma_0^+}{\bullet}\overset{\Gamma_0}{\rule{3em}{0.4pt}}\overset{\Gamma_1}{\bullet}\overset{\Gamma_1}{\rule{6em}{0.4pt}}\overset{\Gamma_2}{\bullet}\overset{\Gamma_2}{\rule{6em}{0.4pt}}\overset{\Gamma_3}{\bullet}\rule{3em}{0.4pt}\cdots$$

Figure 2.4: graph of groups for a Nagao-type lattice in $SL_2(\mathbb{F}_q((t^{-1})))$

where the edge monomorphisms are inclusion maps with $[\Gamma_0^+ : \Gamma_0] = q + 1$ and for $i \geq 0$, $[\Gamma_{i+1} : \Gamma_i] = q$. Thus the graph of groups is an infinite ray whose edge groups form an ascending chain of groups. A more general description of a 'Nagao ray' and corresponding groupings may be found in ([BL], Chapter 10).

One well known nonuniform lattice of Nagao type is $\Gamma = \mathrm{SL}_2(\mathbb{F}_q[t]) \leq G = \mathrm{SL}_2(\mathbb{F}_q((t^{-1})))$. Restricting the action of G on X to Γ, the quotient graph $X_1 = \Gamma \backslash X$ is the infinite ray of vertices

$$\Lambda_0 \to \Lambda_1 \to \Lambda_2 \to \cdots,$$

and the corresponding graph of groups is described in the following result of Serre.

Proposition 2.6 (Proposition 3, p. 87, [S]). *Let* $\Gamma_0 = \mathrm{SL}_2(\mathbb{F}_q)$ *and for* $n \geq 1$*,*

$$\Gamma_n = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \, a \in \mathbb{F}_q^{\times}, b \in \mathbb{F}_q[t], \deg(b) \leq n \right\}$$

(a) The vertices Λ_n *are pairwise inequivalent mod* Γ*.*

(b) Γ_n *is the stabilizer of* Λ_n *in* Γ*.*

(c) Γ_0 *acts transitively on the set of edges with origin* Λ_0*.*

(d) For $n \geq 1$*,* Γ_n *leaves the edges* $\Lambda_n \Lambda_{n+1}$ *and* $\Lambda_{n+1}\Lambda_n$ *[we only give the first kind of edge above] fixed, and acts transitively on the set of edges with origin* Λ_n *which are distinct from* $\Lambda_n\Lambda_{n+1}$*, and on the set of edges with terminus* Λ_n *which are distinct from* $\Lambda_{n+1}\Lambda_n$*.*

Chapter 3

Existence of cocompact lattices in rank 2 Kac-Moody groups

In this chapter and in chapter 5 we will prove the following.

Theorem 3.1. *Let A be a rank 2 affine or hyperbolic generalized Cartan matrix. Let G be a locally compact Kac-Moody group associated to A over a finite field $\mathbb{F}_q$. Let X be the Tits building of G, the homogeneous tree $X = X_{q+1}$.*

(1) For every prime power q, there is a finite group $M_q(x)$ acting transitively on the edges in the star of a vertex $x \in VX$.

(2) Let $M_q = M_q(x_1)$ and $\widetilde{M_q} = \widetilde{M_q}(x_2)$ denote the groups of (1) corresponding to adjacent vertices x_1 and x_2 in the Tits building X for G. If

$$Stab^X_{M_q}(x_2) = M_q \cap \widetilde{M_q} = Stab^X_{\widetilde{M_q}}(x_1)$$

then $\Gamma \cong M_q *_{M_q \cap \widetilde{M_q}} \widetilde{M_q}$ is a cocompact lattice in G with quotient a simplex.

(3) If further M_q and $\widetilde{M_q}$ are abelian, there is an infinite descending chain of cocompact lattices $...\Gamma_3 \leq \Gamma_2 \leq \Gamma_1 \leq \Gamma$.

(4) When $q = 2^s$, the condition in (2) is satisfied, with $M_q \cap \widetilde{M_q} = \{1\}$. When $q = 2$, the condition in (3) is also satisfied, as $M_q \cong \mathbb{Z}/3\mathbb{Z} \cong \widetilde{M_q}$.

Our method is constructive. The lattices of Theorem 3.1 are tree lattices for the homogeneous tree $X = X_{q+1}$. Thus the theories of Bass-Serre theory for constructing lattices via their actions on trees are accessible for the Kac-Moody groups of Theorem 3.1 ([B], [BL], [L2], [S]). Part (1) of Theorem 3.1 is adapted from Lemma 3.5 of [L2] to the setting of rank 2 affine or hyperbolic Kac-Moody groups using the Levi decomposition of the parabolic subgroups due to [RR].

Lemma 3.2. ([L2], 3.5) *For every prime power q, $SL_2(\mathbb{F}_q)$ contains a subgroup M_q acting transitively on the projective line $\mathbb{P}^1(\mathbb{F}_q)$.*

◦ *For q =11, 19, 29, A_5 embeds in $PSL_2(\mathbb{F}_q)$, so take M_q to be the preimage of A_5 in $SL_2(\mathbb{F}_q)$. Then $|M_q| = 120$.*

◦ *For q =7, S_4 embeds in $PSL_2(\mathbb{F}_q)$, so take M_q to be the preimage of S_4 in $SL_2(\mathbb{F}_q)$. Then $|M_q| = 48$.*

◦ *For $q \neq 2^s$, 5, 7, 11, 19, 29, M_q is the normalizer of a non-split Cartan subgroup. Then $|M_q| = 2(q+1)$.*

◦ *For $q = 2^s$ M_q is a non-split Cartan subgroup. Then $|M_q| = q+1$.*

◦ *For q =5, A_4 embeds in $PSL_2(\mathbb{F}_q)$, so take M_q to be the preimage of A_4 in $SL_2(\mathbb{F}_q)$. Then $|M_q| = 24$.*

We extend this lemma to the setting of all Kac-Moody groups with a rank 2 generalized Cartan matrix over $\mathbb{F}_q$ by an embedding of $SL_2(\mathbb{F}_q)$. Rémy and Ronan define the *Levi factor* of a standard parabolic subgroup of type i, $i = 1, 2$, as

$$L_i \;=\; (SL_2(\mathbb{F}_q) \times (\mathbb{F}_q^{\times}))_i$$

where the subscript i indicates that L_i is generated by $H \cong \mathbb{F}_q^{\times} \times \mathbb{F}_q^{\times}$ and the root groups $U_{\pm\alpha_i}$. Rémy and Ronan show that standard parabolic subgroup of type i is the semidirect product

$$P_i \;=\; L_i \ltimes U^i,$$

where U^i is a pro-p group and is the normal closure of the group generated by all positive root groups except U_{α_i}. We refer the reader to [RR] for a definition of the action of the group L_i on U^i. We remark that the Levi factors of parabolic subgroups are isomorphic for rank 2 affine and hyperbolic Kac-Moody groups over $\mathbb{F}_q$.

By a slight abuse of notation, we let M_q denote the image of the group M_q of ([L2], Lemma 3.5) in the Levi factor L_1 and we let $\widetilde{M_q}$ denote the image of M_q in L_2. We have the following.

Theorem 3.3. *Let G be a topological Kac-Moody group, with any rank 2 generalized Cartan matrix over $\mathbb{F}_q$, in the Rémy-Ronan completion. Let $X = X_{q+1}$ be the Bruhat-Tits tree for G. Suppose that M_q and $\widetilde{M_q}$ of (i), (ii), and (iii) below satisfy $Stab_{M_q}(x_2) = M_q \cap \widetilde{M_q} = Stab_{\widetilde{M_q}}(x_1)$ for adjacent vertices $x_1, x_2 \in VX$. Then G contains a cocompact lattice subgroup $\Gamma = M_q *_{M_q \cap \widetilde{M_q}} \widetilde{M_q}$, where*

(i) If $q = 2^s$, M_q is the image of the non-split Cartan subgroup of $SL_2(\mathbb{F}_q)$ of order $q+1$ in L_1, and $\widetilde{M_q}$ is its image in L_2.

(ii) If $q \neq 2^s, 5, 7, 11, 19, 29$, M_q is the image of the normalizer of a non-split Cartan subgroup of $SL_2(\mathbb{F}_q)$ of order $2(q+1)$in L_1, and $\widetilde{M_q}$ is its image in L_2.

(iii) If $q = 11, 19, 29$, M_q is the preimage of A_5 in L_1, and $\widetilde{M_q}$ is its preimage in L_2.

(iv) If $q = 7$, M_q is the preimage of S_4 in L_1, and $\widetilde{M_q}$ is its preimage in L_2.

(v) If $q = 5$, M_q is the preimage of A_4 in L_1, and $\widetilde{M_q}$ is its preimage in L_2.

Recall that a rank 2 Kac-Moody group G over a finite field $\mathbb{F}_q$ acts on the tree $X = X_{q+1}$ with quotient a simplex. The quotient graph of groups is given below.

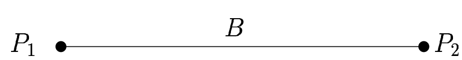

Figure 3.1: graph of groups for a rank 2 Kac-Moody group (also Figure 2.3)

As discussed in section 2.1.4, B is the minimal parabolic subgroup, and the Weyl group is given by

$$W = < w_1, w_2 > \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}.$$

The vertex groups are the standard parabolic subgroups

$$P_1 = B \sqcup Bw_1B \ , \ P_2 = B \sqcup Bw_2B \ .$$

G has the corresponding amalgamated product decomposition $G = P_1 *_B P_2$. We restate the following lemma of Lubotzky to give a sufficient condition on the action of an amalgamated product of the form $\Gamma = \mathcal{A}_1 *_{\mathcal{A}_1\cap\mathcal{A}_2} \mathcal{A}_2$ yielding Γ as a cocompact subgroup of G. We then use this lemma to construct a cocompact lattice in a rank 2 Kac-Moody group over a field of characteristic 2.

Lemma 3.4 ([L2], Lemma 3.1). *Let G be a rank 2 Kac-Moody group over the field $\mathbb{F}_q$ and let $X = X_{q+1}$ be the Bruhat–Tits tree of G. Let x_1, x_2 be adjacent vertices of X and let $\mathcal{A}_1$ and $\mathcal{A}_2$ be finite subgroups of G such that*

(1) $\mathcal{A}_i$ fixes x_i for $i = 1, 2$, and $Stab_{\mathcal{A}_i}(x_{3-i}) = \mathcal{A}_1 \cap \mathcal{A}_2$.

(2) For $i = 1, 2$ $\mathcal{A}_i$ acts transitively on the $q+1$ neighbors of x_i.

Then the group $\Gamma = \mathcal{A}_1 *_{\mathcal{A}_1\cap\mathcal{A}_2} \mathcal{A}_2$ is a cocompact lattice in G .

We can give an alternate proof of this lemma, demonstrating the utility of covering theory for graphs of groups.

Proof. We note that $\Gamma = \pi_1(\mathbb{A})$ where $\mathbb{A}$ is the graph of groups:

$$A_1 \quad \longleftarrow \quad A_1 \cap A_2 \quad \longrightarrow \quad A_2$$

Since $\mathbb{A}$ is a finite graph of finite groups, it follows that Γ is a cocompact lattice in $\mathrm{Aut}(X)$. Thus to prove Γ is a lattice in G, it suffices to construct a covering morphism

$$\Phi = (\varphi, (\delta)) : \mathbb{A} \to \mathbb{G},$$

where $\mathbb{G}$ is the graph of groups for G:

$$P_1 \quad \longleftarrow \quad B \quad \longrightarrow \quad P_2.$$

To do this, let (y_1, y_2) be the edge in X such that $P_i = Stab_G(y_i)$ for $i = 1, 2$. Note that G acts transitively on the edges of X. Thus there is an element $g \in G$ such that g sends the edge (x_1, x_2) to (y_1, y_2). Then for any $\gamma \in A_i$, $g\gamma g^{-1}(y_i) = g\gamma(x_i) = g(x_i) = y_i$, whence $\mathrm{ad}(g) : A_i \to P_i$ by $\gamma \to g\gamma g^{-1}$ is a monomorphism. Similarly $\mathrm{ad}(g) : A_1 \cap A_2 \to B$ is also a monomorphism. Thus the following diagram commutes, where each horizontal arrow designates the inclusion map.

$$\begin{tikzcd}
A_1 \cap A_2 \arrow[r] \arrow[d, "\mathrm{ad}(g)"] & A_i \arrow[d, "\mathrm{ad}(g)"] \\
B \arrow[r] & P_2
\end{tikzcd}$$

To show that these maps do indeed yield a covering morphism, it remains to check that

$$\varphi_{i/b} : A_i/A_1 \cap A_2 \to P_i/B$$

$$\gamma(A_1 \cap A_2) \mapsto g\gamma g^{-1}B \text{ for } i = 1, 2$$

are bijections.

We know that each A_i acts transitively on the $q+1$ neighbors of x_i. That is, the projection $p : X \to \Gamma\backslash\backslash X$ sends all the neighbors of x_1 (including x_2) to the vertex corresponding to $A_2 = \Gamma_{x_2}$. Similarly, all the neighbors of x_2 map to the vertex corresponding to A_1. So the index of the edge (and its reverse) corresponding to $A_1 \cap A_2$ is

$$[A_1 : A_1 \cap A_2] = [A_2 : A_1 \cap A_2] = q + 1.$$

But we also have $[P_i : B] = q + 1$ for $i = 1, 2$, and therefore we have the same number of cosets. Thus is suffices to show that $\varphi_{i/b}$ is injective.

Let $\gamma \in A_1$. If $g\gamma g^{-1}B = B$, then

$$g\gamma g^{-1} \in B \implies g\gamma g^{-1}(y_1, y_2) = (y_1, y_2) \implies \gamma(x_1, x_2) = g^{-1}(y_1, y_2) = (x_1, x_2)$$

and hence $\gamma \in A_1 \cap A_2$. Thus $\gamma(A_1 \cap A_2)$ is the trivial coset, and the map is injective.

$\square$

Recall that the subgroups M_q in Lemma 3.2 ([L2], Lemma 3.5) act transitively on the projective line. Moreover the group P_i has Levi factor $L_i = (SL_2(\mathbb{F}_q) \times (\mathbb{F}_q^\times))_i$ where the subscript i indicates that L_i is generated by $H \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$ and the root groups $U_{\pm\alpha_i}$. By a slight abuse of notation, we let M_q denote the image of the group M_q of Lemma 3.2 in the Levi factor L_1 and we let $\widetilde{M_q}$ denote the image of M_q in L_2. It is then clear that M_q fixes x_1 and $\widetilde{M_q}$ fixes x_2. Since M_q and $\widetilde{M_q}$ each act transitively on $\mathbb{P}_{q+1}$, it follows from the construction of X that M_q (resp. $\widetilde{M_q}$) acts transitively on the $q+1$ neighbors of x_1 (resp. neighbors of x_2). Then Lemma 3.5 follows directly from Lemma 3.4.

Lemma 3.5. *Let G be a rank 2 Kac-Moody group over $\mathbb{F}_q$. Let x_1 and x_2 be adjacent vertices in the Tits building $X = X_{q+1}$ such that $P_i = Stab(x_i)$ for $i = 1, 2$. Let M_q and $\widetilde{M_q}$ be subgroups of P_1 and P_2 as described above. If $Stab_{M_q}(x_2) = M_q \cap \widetilde{M_q} = Stab_{\widetilde{M_q}}(x_1)$, then*

$$M_q *_{M_q \cap \widetilde{M_q}} \widetilde{M_q}$$

is a cocompact lattice in G.

Corollary 3.6. *Let G be a rank 2 Kac-Moody group over $\mathbb{F}_q$, and suppose $q = 2^s$. Let M_q and $\widetilde{M_q}$ be as in Lemma 3.5. Then*

1. *$Stab_{M_q}(x_2) = Stab_{\widetilde{M_q}}(x_1) = \{1\}$*
2. *$M_q * \widetilde{M_q}$ is a cocompact lattice in G.*

Proof. Note that the star in X of x_1, denoted $Star_X(x_1)$ consists of the $q+1$ edges with initial vertex x_1. Moreover M_q has order $q+1$ and acts transtively on $Star_X(x_1) = \mathbb{P}^1(\mathbb{F}_q)$ by Lemma 3.2. By transitivity of M_q on $Star_X(x_1)$, the orbit of the edge (x_1, x_2) has cardinality $q+1$. By the orbit-stabilizer theorem it follows that the stabilizer of (x_1, x_2) in M_q is trivial. The group M_q fixes x_1, and hence $Stab_{M_q}(x_2) = \{1\}$. A similar argument shows that $Stab_{\widetilde{M_q}}(x_1) = \{1\}$. Thus $Stab_{M_q}(x_2) = Stab_{\widetilde{M_q}}(x_1) = M_q \cap \widetilde{M_q} = \{1\}$, and the result follows from Lemma 3.5. $\square$

A similar line of argument was used in Section 6.1 of [LW] to embed $\mathbb{Z}/(q+1)\mathbb{Z} * \mathbb{Z}/(q+1)\mathbb{Z}$ as a cocompact lattice in $SL_2(\mathbb{F}_q((t^{-1})))$ for $q = 2^s$.

When $q = 2$, we have $M_q \cong \widetilde{M_q} \cong \mathbb{Z}/3\mathbb{Z}$. The following corollary gives the cocompact lattice subgroup Γ (in rank 2) with quotient a simplex introduced in Theorem 3.1.

Corollary 3.7. *Let G be a rank 2 Kac-Moody group over $\mathbb{F}_2$. Then $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ is a cocompact lattice in G.*

Remark:

Note that the two factors of the cocompact lattice in the corollary are distinct copies of $\mathbb{Z}/3\mathbb{Z}$ with trivial intersection. In fact it is possible to show that

$$M_2 \cong < \chi_1(1)\chi_{-1}(1) > \text{ and } \widetilde{M_2} \cong < \chi_2(1)\chi_{-2}(1) >,$$

where $\chi_{\pm i}(1) \in P_i - B$. For simplicity we just write $\mathbb{Z}/3\mathbb{Z}$.

For the remainder of this section, let G be a rank 2 Kac-Moody group over a finite field F_q, and consider a cocompact lattice Γ of the form $M_q *_M \widetilde{M_q}$. The main idea is to now use covering theory to construct further cocompact lattices of G which embed in Γ. More precisely, we seek to exhibit edge-indexed graphs (B, j) for which a covering $p : (B, j) \to (A, i)$ exists, where (A, i) is the edge-indexed graph corresponding to a previously constructed cocompact lattice, such as $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. We then extend the coverings to covering morphisms $\varphi : \mathbb{B} \to \mathbb{A}$ of graphs of groups, yielding an embedding of fundamental groups $\pi_1(\mathbb{B}) \to \pi_1(\mathbb{A})$.

The following lemma carries out this strategy of extending coverings of edge-indexed graphs more generally. In particular we construct subgroups of an arbitrary amalgamated free product $\Gamma = \mathcal{A}_1 *_{\mathcal{A}_3} \mathcal{A}_2$, where $\mathcal{A}_3$ embeds in $\mathcal{A}_1$ and $\mathcal{A}_2$ as a finite-index subgroup. The construction uses Theorem 2.5 in Section 2.3.1 to extend coverings of edge-indexed graphs. This necessitates the condition that the groups $\mathcal{A}_1$ and $\mathcal{A}_2$ are abelian. From this lemma we then specialize to the Kac-Moody setting.

Lemma 3.8. *Let $\Gamma = \mathcal{A}_1 *_{\mathcal{A}_3} \mathcal{A}_2$ be a group, and suppose that $[\mathcal{A}_1 : \mathcal{A}_3] = a < \infty$ and $[\mathcal{A}_2 : \mathcal{A}_3] = b < \infty$. Let X be the locally finite tree on which Γ acts with quotient a simplex. Suppose further that $\mathcal{A}_1$ and $\mathcal{A}_2$ are abelian. Then $\Gamma_1 = \underset{k=1,\ldots,a}{\overset{*}{\mathcal{A}_3}} (\mathcal{A}_2)_k$ is a subgroup of Γ. Moreover, if $\mathcal{A}_2$ is finite then Γ_1 is a cocompact lattice in Γ.*

Proof. Let X be the (a, b) bi-regular tree on which $\Gamma = \mathcal{A}_1 *_{\mathcal{A}_3} \mathcal{A}_2$ acts with quotient a simplex. Let $(A, i) = (\Gamma \backslash X, i)$ be the resulting edge-indexed graph, and let (B, j) be the edge-indexed "a-star" shown in the figure below.

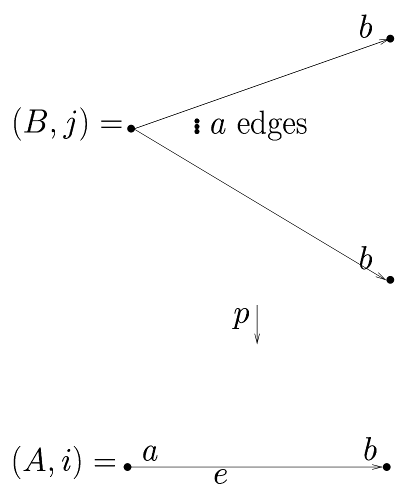

Figure 3.2: edge-indexed covering of the simplex by the a-star

The graph morphism $p : (B, j) \to (A, i)$ depicted is a covering of edge-indexed graphs. Assume without loss of generality that $\mathcal{A}_3 \leq \mathcal{A}_1$, and let $\alpha : \mathcal{A}_3 \to \mathcal{A}_2$ be a monomorphism. Let $\mathbb{A}$ be the graph of groups associated with the action of Γ on X. Give (B, j) the abelian grouping $\mathbb{B}$ as shown in the next figure.

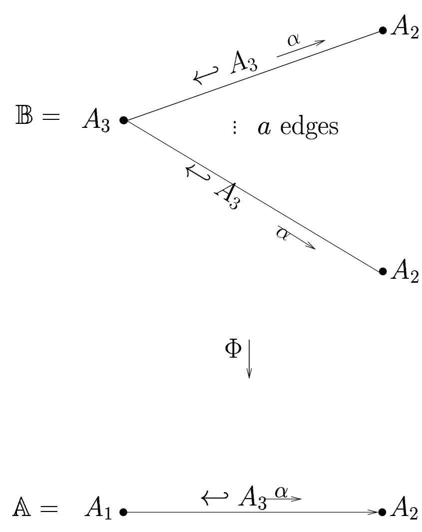

Figure 3.3: abelian grouping of the a-star, covering the simplex grouping

It is straightforward to check that the abelian groupings $\mathbb{A}$ and $\mathbb{B}$, together with the edge-indexed covering p, satisfy the sufficient conditions listed in Theorem 2.5.

Therefore p extends to a covering morphism $\varphi : \mathbb{B} \to \mathbb{A}$ of graphs of groups. By Proposition 2.7 of [B] it follows that $\pi_1(\mathbb{B}) = \underset{k=1,\ldots,a}{\overset{*}{\mathcal{A}_3}} (\mathcal{A}_2)_k$ embeds in $\pi_q(\mathbb{A}) = \Gamma$. If $\mathcal{A}_2$ is finite then $\underset{k=1,\ldots,a}{\overset{*}{\mathcal{A}_3}} (\mathcal{A}_2)_k$ is a cocompact lattice in Γ.

□

Given that the rank 2 Kac-Moody group G has a cocompact lattice whose quotient is a simplex, we now have a sufficient condition yielding a "$q+1$-star" which embeds in G as a cocompact lattice.

Corollary 3.9. *Let G be a rank 2 Kac-Moody group over $\mathbb{F}_q$, and suppose that $\mathcal{A}_1 *_{\mathcal{A}_3} \mathcal{A}_2$ is a cocompact lattice in G with quotient a simplex. Suppose further that $\mathcal{A}_1$ and $\mathcal{A}_2$ are abelian. Then*

$$\underset{k=1,\ldots,q+1}{\overset{*}{\mathcal{A}_3}} (\mathcal{A}_2)_k$$

is a cocompact lattice in G.

In particular, we have an embedding of such a $q+1$-star when $\mathbb{F}_q$ is a field of characteristic 2 if the subgroups M_q and $\widetilde{M_q}$ are abelian.

Corollary 3.10. *Let G be a rank 2 Kac-Moody group over $\mathbb{F}_q$, and suppose $q = 2^s$. Let $M_q * \widetilde{M_q}$ be the cocompact lattice given in Corollary 3.6. If M_q and $\widetilde{M_q}$ are abelian, then*

$$\underset{k=1,\ldots,q+1}{*} (\widetilde{M_q})_k$$

is a cocompact lattice in G.

In particular the sufficient condition holds for the field of 2 elements. This yields the cocompact lattice which is the second subgroup in the infinite descending chain in Theorem 3.1 (3).

Corollary 3.11. *Let G be a rank 2 Kac-Moody group over $\mathbb{F}_2$. Then*

$$\Gamma_1 \;\cong\; \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$$

is a cocompact lattice in G.

Remark:

Note that each copy of $\mathbb{Z}/3\mathbb{Z}$ in the graph of groups $\mathbb{B}$ is identical to the image subgroup $\widetilde{M_2} \cong < \chi_2(1)\chi_{-2}(1) >$ in $\mathbb{A}$ (see remark following the proof of Corollary 3.7).

There are infinitely many cocompact lattices of G which may be constructed using the technique described in this section. In particular in Section 5 we give the general construction for an infinite descending chain of subgroups of an amalgamated free product and provide two examples in the Kac-Moody setting when $\mathbb{F}_q = \mathbb{F}_2$ is a field of two elements.

Chapter 4

Rank 3 Complex of Groups

In this chapter and the next, we prove the following.

Theorem 4.1. *Let G be a locally compact rank 3 Kac-Moody group of type ∞ over a finite field $\mathbb{F}_q$. Let X be the Tits building of G. Suppose that $q = 2$.*

1. *Let Γ denote the free product $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. Then Γ is an $\mathcal{X}$-lattice for the tree $\mathcal{X} = X_3$.*
2. *The group Γ can be embedded in the subgroup $\mathcal{Q}$ of G generated by all non-maximal standard parabolic subgroups. Moreover the image Γ_1' of Γ in $\mathcal{Q}$ is a cocompact lattice in $\mathcal{Q}$.*
3. *There is non-discrete subgroup $\Lambda' \leq G$ with $\Gamma_1' \leq \Lambda'$ and $\Lambda' \backslash X$ a simplex (ideal triangle) in the Tits building.*
4. *There is an infinite descending chain $\ldots \Gamma_3' \leq \Gamma_2' \leq \Gamma_1'$ of subgroups which are cocompact lattices in $\mathcal{Q}$ and whose images in G are discrete.*

Let G be a rank 3 Kac-Moody group of type ∞ over the field $\mathbb{F}_2$, and X its Tits building. As described in section 2.1.4, X is a simplicial complex consisting of a tiling of the hyperbolic plane by ideal triangles together with glueings of a third triangle at each edge of the plane. The vertices of the complex are given by cosets of the maximal standard parabolic subgroups:

$$P_{i,i+1 mod\ 3} = \bigsqcup_{w \in \langle w_i, w_{(i+1) mod\ 3} \rangle} BwB, \ i = 1, \ldots, 3 \ .$$

The edges are given by cosets of

$$Q_i \quad = \quad \bigsqcup_{w \in \langle w_i \rangle} BwB \quad = \quad B \ \sqcup \ Bw_iB, \ i = 1, ..., 3 \ .$$

The faces are given by cosets of B. (Here we are, by abuse of notation, writing w_i for the element $\widetilde{w}_i$.)

There are $q + 1 = 3$ faces adjoining each edge. Note that

$$P_{i-1,i} \cap P_{i,i+1} = Q_i \ \text{ and } \cap Q_i \ = \ B.$$

In general each edge coset is the intersection of the corresponding vertex cosets, and each face is the intersection of the adjoining edges cosets (or vertex cosets).

As G has type ∞, the Weyl group has the form

$$W \ = \ \langle w_1, \ w_2, \ w_3 \mid w_i^2 = 1, \ (w_iw_j)^{\infty} = 1, \ i,j = 1,2,3, \ i \neq j \rangle \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$$

The fundamental chamber for W is an ideal triangle in the hyperbolic plane.

The quotient by the action of G on X is the following triangle of groups:

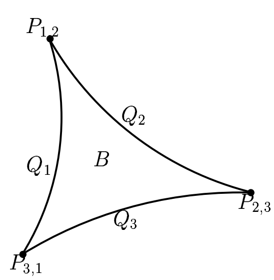

Figure 4.1: quotient triangle of groups for a rank 3 Kac-Moody group

Triangles of groups have been studied in various contexts. Haefliger [H] developed the theory for general complexes of groups, describing the underlying complexes as small categories without loops (scwols). We refer the reader to III.C of [BH] for the definitions of scwols, complexes of groups, and fundamental groups of complexes of groups. For our purposes, we note that the triangle of groups above is a (simple) complex of groups, and the Kac-Moody group G over the field $\mathbb{F}_2$ is the corresponding fundamental group.

Lin and Thomas [LT] give a very nice treatment of the coverings of complexes of groups. They also show that a covering of complexes of groups induces an embedding of the corresponding fundamental groups, as with the graph of groups theory. We may thus construct a subgroup of a rank 3 Kac-Moody group by exhibiting a triangle of groups ...

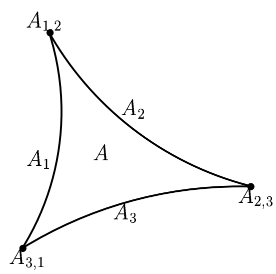

Figure 4.2: arbitrary triangle of groups

... and constructing a covering from this triangle of groups to the triangle of groups for G. Such a covering of triangles of groups may be constructed by exhibiting the following:

$$\text{monomorphisms } \varphi_{ij} : A_{ij} \to P_{ij}, \quad \varphi_i : A_i \to Q_i \ , \quad i : A \to B$$

such that these monomorphisms induce bijections

$$A_{ij}/A \to P_{ij}/B \ , \quad A_i/A \to Q_i/B \ , \quad \text{and} \quad A_{ij}/A_i \to P_{ij}/Q_i \ .$$

4.1 Existence of cocompact lattices in rank 3 type ∞ Kac-Moody groups over $\mathbb{F}_2$

Let G be a rank 3 Kac-Moody group of type ∞ over the field $\mathbb{F}_2$, and X its Tits building as described in the previous section. We now use coverings of complexes of groups to construct a non-discrete subgroup whose quotient is also an ideal triangle of groups, that is whose quotient is a simplex. This is the non-discrete subgroup Λ' in rank 3 introduced in Theorem 3.1.

Lemma 4.2. *Let G be a rank 3 Kac-Moody group of type ∞ over the field $\mathbb{F}_2$. Let Λ' be the fundamental group of the ideal triangle depicted below. Then Λ' is a non-discrete subgroup of G.*

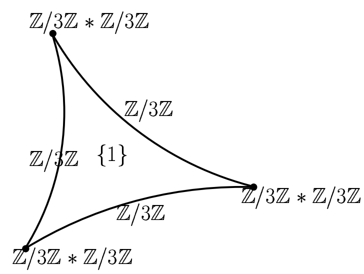

Figure 4.3: triangle of groups for Λ'

Remark: Since Λ' is a faithful complex of finite groups with universal cover X, we may identify Λ' with a discrete subgroup of $Aut(X)$.

The construction in the proof below is due to Anne Thomas.

Proof. We construct a covering of complexes of groups from the triangle of groups for Λ' to the triangle of groups for G.

The face group of Λ' is trivial so we let the local map i here be the natural inclusion of the identity element into B. For the edge groups, write $\mathbb{Z}/3\mathbb{Z} = \langle \varepsilon \rangle$ for the cyclic group of order 3. For each $i = 1, 2, 3$ we claim that there is a group monomorphism $\varphi_i : \langle \varepsilon \rangle \to Q_i$.

By definition of the elements $w_i = \widetilde{w}_i$, and using the fact that in the field $\mathbb{F}_2$ we have $1 = -1$,

$$\chi_i(1)\chi_{-i}(1) = w_i \chi_i(1)^{-1}.$$

Since $\chi_i(1) = \chi_{\alpha_i}(1) \in U_{\alpha_i} \subseteq U \subseteq B$, we have that

$$\varphi_i(\varepsilon) = w_i \chi_i(1)^{-1} \in B w_i B = Q_i - B$$

as required. In particular, $\varphi_i(\varepsilon) \neq 1$ and $< \varphi_i(\epsilon) > \cap < \varphi_j(\epsilon) > = \{1\}$ for $i \neq j$.

Next we show that $\varphi_i(\varepsilon)$ has order 3. For this, it is enough to show that $(\varphi_i(\varepsilon))^3 = 1$. We compute

$$\begin{aligned}
(\varphi_i(\varepsilon))^3 &= \chi_i(1)\chi_{-i}(1)\chi_i(1)\chi_{-i}(1)\chi_i(1)\chi_{-i}(1) \\
&= w_i\chi_{-i}(1)\chi_i(1)\chi_{-i}(1) \\
&= w_i\chi_{-i}(1)w_i^{-1}w_i\chi_i(1)w_i^{-1}w_i\chi_{-i}(1)w_i^{-1}w_i \\
&= \chi_i(1)\chi_{-i}(1)\chi_i(1)w_i \\
&= w_i^2 \\
&= 1.
\end{aligned}$$

We thus have a monomorphism

$$\varphi_i : \mathbb{Z}/3\mathbb{Z} \to Q_i$$

for each $i = 1, 2, 3$. Moreover, the elements $\{\varphi_i(1), \varphi_i(\varepsilon), \varphi_i(\varepsilon^2)\}$ form a set of coset representatives of Q_i/B since $\varphi_i(\varepsilon)$ has order 3 and does not lie in B.

For the vertex groups, since $\varphi_i(\varepsilon) \in Q_i - B$, we have that for $i = 1, 2, 3$,

$$\varphi_i(\langle\varepsilon\rangle) \cap \varphi_{i+1}(\langle\varepsilon\rangle) = \{1\}.$$

Hence we obtain an embedding

$$\langle\varepsilon\rangle * \langle\varepsilon\rangle \hookrightarrow P_{i,i+1}.$$

Moreover the image $\varphi_i(\langle\varepsilon\rangle)$ in $P_{i,i+1}$ forms a set of coset representatives for $P_{i,i+1}/Q_{i+1}$, and similarly for the image of $\varphi_{i+1}(\langle\varepsilon\rangle)$ and $P_{i,i+1}/Q_i$. Finally the image of $\langle\varepsilon\rangle * \langle\varepsilon\rangle$ in $P_{i,i+1}$ forms a set of coset representatives for $P_{i,i+1}/B$ since this image only intersects B trivially.

We have thus constructed a covering of complexes of groups from the complex of groups for Λ' to the complex of groups for G. It follows that Λ' embeds as a non-discrete subgroup in G (or in its completion). $\square$

Lemma 4.3. *Let G be a rank 3 Kac-Moody group of type ∞ over the field $\mathbb{F}_2$. Let $\mathcal{Q} \leq G$ be the subgroup generated by the non-maximal parabolic subgroups of G. The group $Z/3\mathbb{Z} * Z/3\mathbb{Z} * Z/3\mathbb{Z}$ embeds as a cocompact lattice $\Gamma_1 \leq Q$.*

4.2 Actions of cocompact lattices on ideal complexes and on their inscribed trees

A locally compact Kac-Moody group G of rank 2, and hence any lattice subgroup, comes equipped with an action on a simplicial tree, the Tits building of G.

In [C1] the author showed that all locally compact Kac-Moody groups G of rank 3 noncompact hyperbolic type over finite fields $\mathbb{F}_q$ have the Haagerup property, and she exhibited an action of G on a simplicial tree $\mathcal{X}$ where certain lattices act discretely, that is, with finite vertex stabilizers. When G has type ∞, the tree $\mathcal{X}$ is the bihomogeneous bipartite tree $\mathcal{X}_{3,q+1}$ ([C1]). When $q = 2$, $\mathcal{X}$ is the homogeneous bipartite tree denoted $\mathcal{X}_3$.

Thus we have actions of the rank 2 and rank 3 Kac-Moody groups over the field $\mathbb{F}_2$ on the trivalent tree. We show that the cocompact lattice Γ_1 in the rank 2 Kac-Moody group embeds in the rank 3 Kac-Moody group G over $\mathbb{F}_2$. In fact the edge-indexed graph quotient graph of Γ_1 on $\mathcal{X}$ is inscribed in the fundamental chamber for the Weyl group. Our methods show that Γ_1 also acts discretely on $\mathcal{X}$.

Theorem 4.4. *Let G be a rank 3 locally compact Kac-Moody group of type ∞ over a finite field $\mathbb{F}_2$. Let X be the Tits building of G, and let $\mathcal{X} = \mathcal{X}_3$ be the bihomogeneous bipartite simplicial tree inscribed in X. Let Γ_1 be the non-discrete subgroup of Corollary 3.11. Then Γ_1 is an $\mathcal{X}$-lattice. Moreover Γ_1 can be embedded in the subgroup $\mathcal{Q}$ of G generated by all non-maximal parabolic subgroups. The image Γ_1' of Γ_1 in $\mathcal{Q}$ is a cocompact lattice. Thus Γ_1 acts discretely on $\mathcal{X}$.*

Chapter 5

Infinite descending chains of cocompact lattices

In this section, we consider groups of the form $\Gamma_1 = \underset{k=1,\ldots,a}{\overset{*}{\mathcal{A}_3}} (\mathcal{A}_2)_k$, an amalgamated free product of copies of a group $\mathcal{A}_2$ over a subgroup $\mathcal{A}_3$ of finite index. The group $\Gamma_1 \cong \Gamma_1' \cong \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, which embeds in a type-∞ Kac-Moody group over $\mathbb{F}_2$ as described in Corollary 3.11 for rank 2 and Theorem 4.4 for rank 3, is an example of such an amalgamated free product.

Remark: The notation reflects the fact that we will be building on the constructions in Lemma 3.8.

We will construct infinite descending chains of subgroups of Γ_1. Specializing to the Kac-Moody setting, this construction will yield infinite descending chains of subgroups which embed as cocompact lattices in subgroups of a type ∞ Kac-Moody group over the field $\mathbb{F}_2$.

Theorem 5.1. *Let G be a locally compact Kac-Moody group of type ∞ over the field $\mathbb{F}_2$ with Weyl group $W =< w_i >$. Let $\widetilde{G}$ be the subgroup generated by the parabolic subgroups of the form $B \sqcup Bw_iB$. Then $\widetilde{G}$ contains an infinite descending chain $\ldots \Gamma_3 \leq \Gamma_2 \leq \Gamma_1$ of cocompact lattices with distinct fundamental domains, with $\Gamma_k \cong \underset{j=1,\ldots,n_k}{*} (\mathbb{Z}/3\mathbb{Z})_j$, where $n_k = 1 + 3^k - \sum_{i=0}^{k-1} 3^i$ and $Vol(\Gamma_k) = 2(3)^{k-1}$. Hence the Γ_k are pairwise non-conjugate.*

In general we can construct a infinite descending chain of subgroups in $\Gamma_1 = \underset{k=1,\ldots,a}{\overset{*}{\mathcal{A}_3}} (\mathcal{A}_2)_k$ by iterating the technique described in the proof of Lemma 3.8. We use the action of Γ_1 on an a-regular tree X. We first build an infinite sequence of coverings of edge-indexed graphs over the quotient of Γ_1 on X. We then extend this to an infinite sequence of covering morphisms of finite graphs of finite groups. The following theorem gives the sufficient conditions for these infinite sequences, and Theorem 5.1 will follow as a special case, with $\mathcal{A}_2 \cong \mathbb{Z}_3$ and $\mathcal{A}_3 \cong \{1\}$.

Theorem 5.2. *Let $\mathcal{A}_2$ be a group with subgroup $\mathcal{A}_3$ of finite index $[\mathcal{A}_2 : \mathcal{A}_3] = a < \infty$, with $a \geq 3$. Let $\Gamma_1 = \underset{\substack{\mathcal{A}_3 \\ k=1,\dots,a}}{*} (\mathcal{A}_2)_k$ be an amalgamated free product. Let X be the a-regular tree on which Γ_1 acts with quotient an a-star. Let $(A, i) = (\Gamma_1 \backslash X, i)$ denote the edge-indexed quotient graph for Γ_1 on X, and let $\mathbb{A} = \Gamma_1 \backslash\backslash X$ denote the corresponding graph of groups. Suppose that $\mathcal{A}_2$ is abelian. Then there exists an infinite sequence of coverings of finite edge-indexed graphs*

$$\ldots \longrightarrow (B_3, j_3) \longrightarrow (B_2, j_2) \longrightarrow (B_1, j_1) = (A, i)$$

and an infinite sequence of covering morphisms of graphs of groups

$$\ldots \longrightarrow \mathbb{B}_3 \longrightarrow \mathbb{B}_2 \longrightarrow \mathbb{B}_1 = \mathbb{A}$$

with $\mathbb{B}_k$ a grouping of (B_k, j_k)such that $\Gamma_{k+1} = \pi_1(\mathbb{B}_{k+1}) \leq \Gamma_k = \pi_1(\mathbb{B}_k)$ and $\Gamma_k \cong \underset{\substack{\mathcal{A}_3 \\ j=1,\dots,n_k}}{*} (\mathcal{A}_2)_j$, where $n_k = 1 + a^k - \sum_{j=1}^{k-1} a^j$. Moreover, if $|\mathcal{A}_2| = c < \infty$, then $Vol(\Gamma_k) = \frac{2a^k}{c}$. In this case the Γ_k are pairwise non-conjugate and form a descending chain of cocompact lattices in Γ_1.

To produce the infinite sequence of coverings of finite edge-indexed graphs in Theorem 5.2, we use an iteration of a method known as 'open fanning' of 'arithmetic bridges' in the edge-indexed graphs in the sequence. This method was first used to prove existence of nonuniform coverings over finite edge-indexed graphs and hence to prove existence of nonuniform lattices on uniform trees ([C2]). This method was also used by Gabrial Rosenberg ([Ro]) to exhibit infinite ascending chains of cocompact lattices with arbitrarily small covolumes in automorphism groups of locally finite trees.

In our edge-indexed graph (A, i), the 'arithmetic bridge' can be taken to be any single separating edge e with an index (ramification factor) $i(e) = [\mathcal{A}_2 : \mathcal{A}_3] = a$. The open fanning then has the schematic diagram:

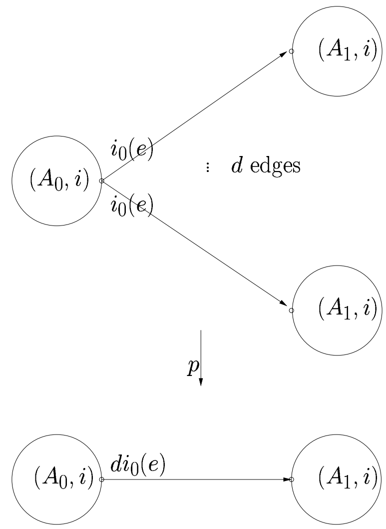

Figure 5.1: schematic of an open fanning along an edge

Proof. (of Theorem 5.2 – open fanning on a single edge)

We now use the method of open fannings on a single edge to recursively construct a sequence of edge-indexed coverings as follows:

1. Let $(B_1, i_1) = (A, i) = (\Gamma_1 \backslash X, i)$ be the edge-indexed 'a-star' and choose an edge e of B_1 with index $i_1(e) = i(e) = a$.

2. Let (B_2, j_2) be an open a-fanning on the edge e, as shown.

3. For each $k \geq 2$, choose an edge e of B_k with index $i_k(e) = a$ and let (B_{k+1}, j_{k+1}) be an open a-fanning of B_k on e.

Note that the a-star (B_1, i_1) has a edges of index a. An easy induction shows that for $k \geq 1$, B_k has

$$n_k = 1 + a^k - \sum_{j=1}^{k-1} a^j \geq a \text{ edges of index } a.$$

Thus the recursion is well-defined. Moreover we may associate an abelian grouping $\mathbb{B}_k$ to (B_k, i_k) consisting of copies of $\mathcal{A}_2$ at each initial vertex of these n_k edges and copies of $\mathcal{A}_2$ at each remaining vertex and along each edge. It is straightforward to check that for $k \geq 1$, the groupings $\mathbb{B}_k$ and $\mathbb{B}_{k+1}$, together with the edge-indexed covering

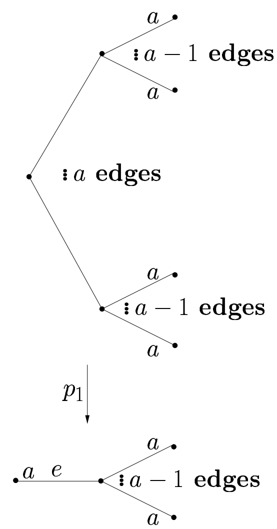

Figure 5.2: edge-indexed covering $p_1 : (B_2, j_2) \longrightarrow (B_1, j_1)$ (open fanning of the a-star)

$p_k : (B_{k+1}, i_{k+1}) \to (B_k, i_k)$, satisfy the sufficient conditions of Theorem 2.5. Therefore for $k \geq 1$, p_k extends to a covering morphism $\varphi_k : \mathbb{B}_{k+1} \to \mathbb{B}_k$ of graphs of groups. By Proposition 2.7 of [B] it follows that $\Gamma_{k+1} = \pi_1(\mathbb{B}_{k+1}) = \mathop{\ast}\limits_{\substack{\mathcal{A}_3 \\ j=1,\ldots,n_{k+1}}} (\mathcal{A}_2)_j$ embeds in $\Gamma_k = \pi_1(\mathbb{B}_k) = \mathop{\ast}\limits_{\substack{\mathcal{A}_3 \\ j=1,\ldots,n_k}} (\mathcal{A}_2)_j$ (and ultimately in $\Gamma_1 = \pi_1(\mathbb{B}_1)$). Thus these Γ_k form an infinite descending chain of subgroups. If $|\mathcal{A}_2| = c < \infty$, then another easy induction shows for $k \geq 1$,

$$Vol(\Gamma_{k+1}) \;=\; aVol(\Gamma_k) = 2\frac{a^{k+1}}{c}.$$

□

In constructing the chain of subgroups in Theorem 5.2, we use an open fanning on a single edge, that is an arithmetic bridge of size 1, at each step. Changing the size of the bridge at any step will yield further (distinct) descending chains. In this manner we can construct an infinite number of infinite descending chains of cocompact lattices. We give one more example of this method, in the Kac-Moody setting over the field of two elements, constructing a chain by fanning on a bridge of two edges at each step.

Theorem 5.3. *Let G be a locally compact Kac-Moody group of type ∞ over the field $\mathbb{F}_2$ with Weyl group $W =< w_i >$. Let $\widetilde{G}$ be the subgroup generated by the parabolic* *subgroups of the form* $B \sqcup B w_i B$*. Then* $\widetilde{G}$ *contains an infinite descending chain* $\ldots \Gamma_3 \leq \Gamma_2 \leq \Gamma_1$ *of cocompact lattices with distinct fundamental domains, with*

$$\Gamma_k \cong *_3 \mathbb{Z}/3\mathbb{Z} *_{m_k} \mathbb{Z} \ \ \textit{where}$$

$$m_k = \begin{cases} 0 & \textit{if } k = 1 \\ 2 & \textit{if } k = 2 \\ 8 \sum_{j=0}^{\frac{k-3}{2}} 3^{2j} & \textit{if } k \textit{ odd, } k \geq 3 \\ 24 \sum_{j=0}^{\frac{k-4}{2}} 3^{2j} + 2 & \textit{if } k \textit{ even, } k \geq 4, \end{cases}$$

and $Vol(\Gamma_k) = 2 \cdot 3^{k-1}$.

Proof. As before, we recursively construct a sequence of edge-indexed coverings as follows:

1. Let $(B_1, i_1) = (A, i)$ be the edge-indexed tripod (3-star). Choose two edges e_1, e_2 of B_1 with index $i_1(e_1) = i_1(e_2) = 3$.

2. Let (B_2, i_2) be an open fanning of (B_1, i_1) on the bridge $\{e_1, e_2\}$.

3. For each $k \geq 2$, choose two edges e_1, e_2 of B_k with index 3 and let (B_{k+1}, i_{k+1}) be an open fanning of B_k on $\{e_1, e_2\}$.

The general schematic for the open fanning on two edges is given above. Note that no graph in the sequence (after the initial tripod) is a tree. In particular these cycles introduce copies of $\mathbb{Z}$ in the free product decomposition of the fundamental groups. It is easy to see from the schematic that for $k \geq 1$, if B_k has m_k edges outside its spanning tree then B_{k+1} has $3m_k + 2$ edges outside its spanning tree. The first figure shows that $m_1 = 0$ and $m_2 = 2$, and an easy induction combined with the recursive relationship

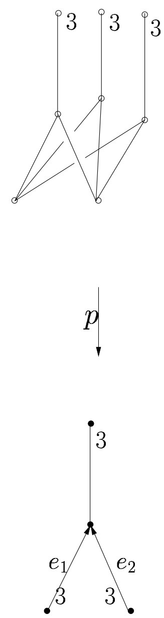

Figure 5.3: covering of edge-indexed graphs $p : (B_2, i_2) \longrightarrow (B_1, i_1)$

gives the general formula

$$m_k = \begin{cases} 0 & \text{if } k = 1 \\ 2 & \text{if } k = 2 \\ 8 \sum_{j=0}^{\frac{k-3}{2}} 3^{2j} & \text{if } k \text{ odd, } k \geq 3 \\ 24 \sum_{j=0}^{\frac{k-4}{2}} 3^{2j} + 2 & \text{if } k \text{ even, } k \geq 4, \end{cases}$$

Note also from the schematic that if B_k has three edges of index 3 (one in the center subgraph plus the two edges of the bridge), then B_{k+1} also has three edges of index 3 (one in each of copies of the central subgraph of B_k). Since the tripod has three edges of index 3, it follows that every graph in the sequence has exactly three edges of index 3. Then the corresponding groupings $\mathbb{B}_k$ consist of three copies of $\mathbb{Z}/3\mathbb{Z}$ at each initial vertex of these index 3 edges and trivial groups elsewhere. As before each $\mathbb{B}_k$ is an abelian grouping with trivial edge groups, and thus for $k \geq 2$ the edge-indexed covering $p_k : (B_k, i_k) \to (B_{k-1}, i_{k-1})$ extends to a covering morphism $\varphi_k : \mathbb{B}_k \to \mathbb{B}_{k-1}$. This gives the desired sequence of corresponding covering morphisms of graphs of groups

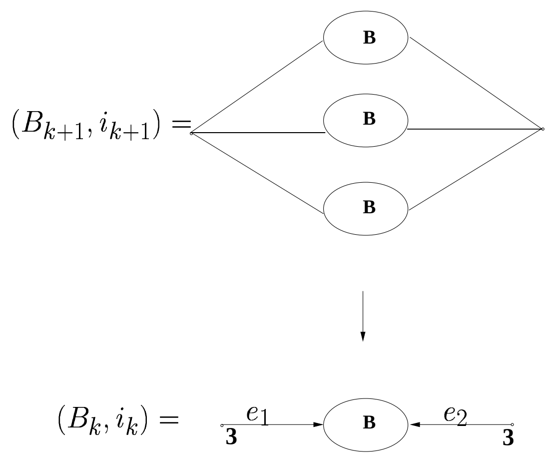

Figure 5.4: schematic of open fanning on two edges

and the embeddings of their fundamental groups

$$\Gamma_k \;=\; \pi_1(\mathbb{B}_k) \;\cong\; *_3\mathbb{Z}/3\mathbb{Z} *_{m_k} \mathbb{Z}$$

These Γ_k form another infinite descending chain of cocompact lattices.

Finally, note from the schematic that the open fanning of (B_k, j_k) produces three copies of the middle subgraph with the same grouping on these vertices. The two initial vertices of the bridge fan from degree 1 vertices to degree three vertices, changing the groups at these vertices from $\mathbb{Z}/3\mathbb{Z}$ to trivial groups. This observation yields a recursive definition of the covolume. Since $Vol(\Gamma_1) = 2$, another easy induction shows for $k \geq 2$,

$$Vol(\Gamma_k) \;=\; 3(Vol(\Gamma_{k-1}) - 2(\frac{1}{3})) \;+\; 2 \;=\; 3Vol(\Gamma_{k-1}) \;=\; 2 \cdot 3^{k-1}.$$

$\square$

Remark: Note that the two-edge open fanning yields a distinct infinite descending chain of cocompact lattices with the same covolumes as the chain constructed by a one-edge open fanning.

Chapter 6

Further Examples

6.1 Another Infinite Descending Chain

In the previous two examples, we gave infinite towers of cocompact lattices constructed using open fannings on edges. Now we give a third infinite tower, this time using the technique of double covers. Unlike the previous two examples, this tower begins with a simplex on a single edge and does not include the tripod.

Theorem 6.1. *Let G be a locally compact rank 2 Kac-Moody group over the field $\mathbb{F}_2$. Then G contains an infinite descending chain $\ldots \Gamma_2 \leq \Gamma_1 \leq \Gamma_0$ of cocompact lattices with distinct fundamental domains, with*

$$\Gamma_0 \cong \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}, \quad \textit{and for } k \geq 1, \Gamma_k \cong \mathop{*}_{j=1,\ldots,2^k} (\mathbb{Z}/3\mathbb{Z})_j \ * \ \mathbb{Z},$$

where $Vol(\Gamma_0) = \frac{2}{3}$ and for $k \geq 1$, $Vol(\Gamma_k) = \frac{2^{k+2}}{3}$. Hence the Γ_k are pairwise non-conjugate.

As in the previous chapter, we will prove the existence of this chain by providing the construction more generally as a descending chain over an appropriate amalgamated free product.

Theorem 6.2. *Let $\Gamma_0 = \mathcal{A}_1 *_{\mathcal{A}_3} \mathcal{A}_2$ be a group, and suppose that $[\mathcal{A}_1 : \mathcal{A}_3] = a < \infty$ and $[\mathcal{A}_2 : \mathcal{A}_3] = a < \infty$. Let X be the locally finite (a-regular) tree on which Γ_0 acts with quotient a simplex. Let $(A, i) = (\Gamma_0 \backslash X, i)$ denote the edge-indexed quotient graph for Γ_0 on X, and let $\mathbb{A} = \Gamma_0 \backslash\backslash X$ denote the corresponding graph of groups. Suppose further that $\mathcal{A}_1$ and $\mathcal{A}_2$ are abelian. Then there exists an infinite sequence of coverings* *of finite edge-indexed graphs*

$$\ldots \longrightarrow (B_2, j_2) \longrightarrow (B_1, j_1) \longrightarrow (B_0, j_0) = (A, i)$$

and an infinite sequence of covering morphisms of graphs of groups

$$\ldots \longrightarrow \mathbb{B}_2 \longrightarrow \mathbb{B}_1 \longrightarrow \mathbb{B}_0 = \mathbb{A}$$

with $\mathbb{B}_k$ *a grouping of* (B_k, j_k) *such that for each* $k \geq 1$, $\Gamma_k = \pi_1(\mathbb{B}_k) \leq \Gamma_{k-1} = \pi_1(\mathbb{B}_{k-1})$. *Moreover, if* $|\mathcal{A}_1| = |\mathcal{A}_2| = c < \infty$, *then* $Vol(\Gamma_0) = \frac{2}{c}$ *and for* $k \geq 1$, $Vol(\Gamma_k) = \frac{2^{k+1}(a-1)}{c}$. *In this case the* Γ_k *are pairwise non-conjugate and form a descending chain of cocompact lattices in* Γ_0. *In particular, if* $\mathcal{A}_3$ *is trivial, then there is a descending chain of cocompact lattices of the form*

$$\Gamma_k \cong \left(\mathop{*}_{j=1,\ldots,2^{k-1}(a-2)} (\mathcal{A}_1)_j \right) * \left(\mathop{*}_{j=1,\ldots,2^{k-1}(a-2)} (\mathcal{A}_2)_j \right) * \mathbb{Z}.$$

Proof. As before, we recursively construct a sequence of edge-indexed coverings as follows:

1. Let (B_0, i_0) and (B_1, i_1) be the edge-indexed graphs shown. There is a natural covering $p_0 : (B_1, i_1) \longrightarrow (B_0, i_0)$.

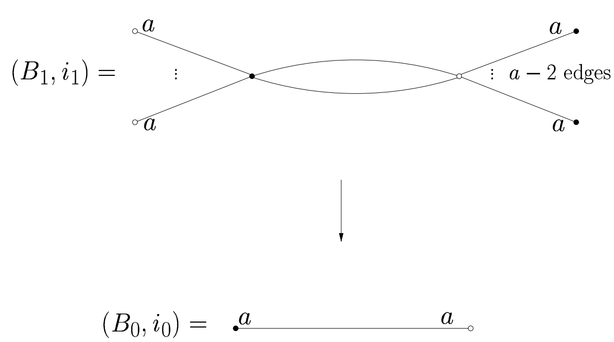

Figure 6.1: 'double' cover of single edge $p_0 : (B_1, i_1) \to (B_0, i_0)$

2. For each $k \geq 1$, let (B_{k+1}, i_{k+1}) be the 'double cover' of (B_k, i_k). That is, let B_{k+1} be a 2^{k+1}-cycle with $(a-2)$ dangling edges incident to each vertex of the cycle, let $i_{k+1}(e) = 1$ for every edge e whose initial vertex is in the cycle, and let the remaining edges have index a. (See figure below for 2nd covering.)

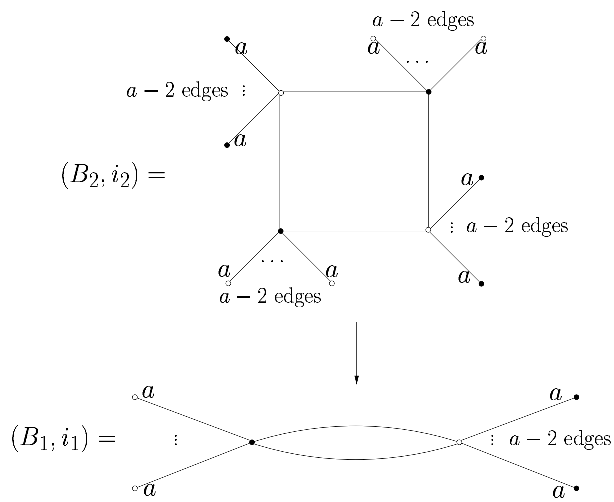

Figure 6.2: Second 'double' cover in chain; $p_1 : (B_2, i_2) \to (B_1, i_1)$

Note that for $k \geq 1$, B_k has $2^k(a-2)$ edges of index a ($(a-2)$ edges terminating at each vertex of the cycle). If we form a bipartition of the vertices of the graph into "black" and "white" vertices, then exactly half of these index a edges have initial vertices which are white (resp. black). Then we can choose corresponding groupings $\mathbb{B}_k$ as follows:

1. $2^{k-1}(a-2)$ copies of $\mathcal{A}_1$ at each black initial vertex of an edge of index a,

2. $2^{k-1}(a-2)$ copies of $\mathcal{A}_2$ at each white initial vertex of an edge of index a,

3. copies of $\mathcal{A}_3$ at all vertices of the cycle and along all edges of the graph.

By hypothesis, each $\mathbb{B}_k$ is then an abelian grouping. It is straightforward to check that for $k \geq 0$, the groupings $\mathbb{B}_k$ and $\mathbb{B}_{k+1}$, together with the edge-indexed covering $p_k : (B_{k+1}, i_{k+1}) \to (B_k, i_k)$, satisfy the sufficient conditions of Theorem 2.5. Therefore for $k \geq 0$, p_k extends to a covering morphism $\varphi_k : \mathbb{B}_{k+1} \to \mathbb{B}_k$ of graphs of groups. Also each graph B_k consists of a single 2^k-cycle and so has only one edge outside its spanning tree. This gives the desired sequence of corresponding covering morphisms of graphs of groups and the embeddings of their fundamental groups Γ_k. When $\mathcal{A}_3$ is trivial, so are all edge groups in the sequence of graphs of groups, and each Γ_k for $k \geq 1$ has a presentation as a free product of the vertex groups together with a single copy of $\mathbb{Z}$ corresponding to the cycle:

$$\Gamma_k \cong \left(\mathop{*}_{j=1,\ldots,2^{k-1}(a-2)} (\mathcal{A}_1)_j \right) * \left(\mathop{*}_{j=1,\ldots,2^{k-1}(a-2)} (\mathcal{A}_2)_j \right) * \mathbb{Z}.$$

If $|\mathcal{A}_1| = |\mathcal{A}_2| = c < \infty$, then the groups Γ_k form a descending chain of cocompact lattices. From the construction, $Vol(\Gamma_0) = \frac{2}{c}$ and $Vol(\Gamma_1) = \frac{4a}{c}$. Then each double cover doubles the covolume, giving a simple recursive relation and thus for $k \geq 1$,

$$Vol(\Gamma_{k+1}) = 2(Vol(\Gamma_k)) = \frac{2^{k+2}(a-2)}{c}.$$

$\square$

Remarks:

As in the previous chapter, there are infinitely many descending chains or infinite families that may be constructed in a similar fashion to the chain just constructed. To illustrate, we mention a few possibilites.

1. Replace the initial 2-cycle in the chain above with any even cycle of length $n = 2k$, again placing $(a-2)$ dangling edges at each vertex of the cycle. Create an abelian grouping as in Theorem 6.2 and extend the edge-indexed covering to a covering of graphs of groups over the simplex. For instance if we consider the cocompact lattice $M_2 * \widetilde{M_2}$ in the rank 2 Kac-Moody group over $\mathbb{F}_2$, this method allows us to construct, for any $k \geq 1$, subgroups of the form

$$\left(\mathop{*}_{j=1,\ldots,k} (M_2)_j \right) * \left(\mathop{*}_{j=1,\ldots,k} (\widetilde{M_2})_j \right) * \mathbb{Z}.$$

2. Increasing the number of cycles in the underlying graph would yield fundamental groups with sucessively more HNN-extenstions in the decomposition. When the

original edge group $\mathcal{A}_3$ is trivial, this corresponds to factors of free groups of successively higher order. Once again, we give an example in the rank 2 Kac-Moody group over $\mathbb{F}_2$. Modifying the graph B_2 as shown below gives rise to a natural grouping with fundamental group

$$\left(\mathop{*}_{j=1,2}(M_2)_j\right) * \left(\mathop{*}_{j=1,2}(\widetilde{M_2})_j\right) * F_2.$$

$(B_2, i_2) =$

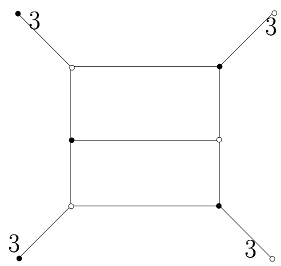

Figure 6.3: Modified 4-cycle graph with extra cycle

3. Any terminal vertex (vertex of degree 1) may be replaced with a vertex of degree a, creating $(a-1)$ new terminal vertices and increasing the number of copies of the corresponding vertex group by $a-2$. Below we illustrate such a modification made to the graph B_2, which may be equipped with a natural grouping with fundamental group

$$\left(\mathop{*}_{j=1,\ldots,3}(M_2)_j\right) * \left(\mathop{*}_{j=1,2}(\widetilde{M_2})_j\right) * \mathbb{Z}.$$

$(B_2, i_2) =$

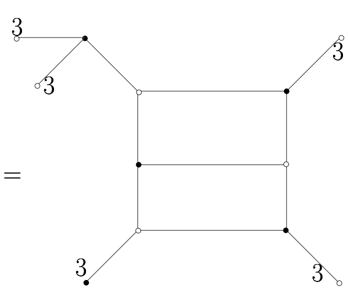

Figure 6.4: Modified 4-cycle graph with extra branch vertex

Once again, infinite descending chains may be built over any of these examples using the method of double covers or the method of open fannings.

6.2 Necessary and Sufficient Conditions for Constructing Covers

Let G be a rank 2 Kac-Moody group over $\mathbb{F}_q$ and suppose that G has a cocompact lattice of the form $\Gamma = \mathcal{A}_1 *_{\mathcal{A}_3} \mathcal{A}_2$. We have shown in Chapter 3 and Section 6.1 that it is possible to realize an infinite number of cocompact lattices in Γ (and thus in G), provided that $\mathcal{A}_1$ and $\mathcal{A}_2$ are abelian. In this section we take a more detailed look at what kind of cocompact lattice subgroups we may expect to see in characteristic 2.

Recall from Corollary 3.6 that a Kac-Moody group over a field of characteristic 2 has a cocompact lattice of the form $\Gamma \cong M_q * \widetilde{M_q}$. Since any cocompact lattice of G which embeds in Γ gives rise to a covering morphism of graphs of groups, we can deduce several characteristics for an appropriate graph of groups. We will do this in a moment, but first we remark that one trait of any graph of groups corresponding to a cocompact lattice subgroup of Γ is that each of its vertex groups is a subgroup of M_q or $\widetilde{M_q}$, and each of its edge groups is trivial. From Bass-Serre theory we can deduce a presentation for all cocompact lattices of G which embed in Γ. That is, every such cocompact lattice subgroup $H \leq \Gamma$ has the form

$$H = (*_j (M_q)_j) * \left(*_k (\widetilde{M_q})_k \right) * F_n \ ,$$

for some j and k, where n is the number of edges outside a maximal tree T of $H \backslash X_{q+1}$. This result is also a special case of Kurosh's Theorem.

Now let $\mathbb{A}'$ be the graph of groups for Γ acting on the Bruhat-Tits tree $X = X_{q+1}$. As demonstrated in Chapters 3 and 5, when M_q and $\widetilde{M_q}$ are abelian we may construct cocompact lattices in Γ (and thus in G) by constructing coverings of edge-indexed graphs. These coverings then give rise to covering morphisms of graphs of groups $\varphi : \mathbb{A} \longrightarrow \mathbb{A}'$, where $\mathbb{A}$ is the graph of groups for a cocompact lattice. We give the following necessary and sufficient conditions for the structure of such a graph of groups.

Proposition 6.3. *Let G be a rank 2 Kac-Moody group over $\mathbb{F}_q$ and suppose $q = 2^s$ for some integer $s \geq 1$. Let $\Gamma = M_q * \widetilde{M_q}$ be the cocompact lattice of G indicated by Corollary 3.6. Suppose that M_q and $\widetilde{M_q}$ are abelian. Let $\mathbb{A}$ be a graph of groups of Γ with edge-indexed graph (A, i). Suppose that $\Phi : \mathbb{A} \to \mathbb{A}'$ is a covering morphism. Then:*

1. *The graph A has a bipartition of vertices $VA = V_1 \cap V_2$ such that*
   - *(a) For all $v \in V_1$, each vertex group $\mathcal{A}_v \leq M_q$ (up to isomorphism)*
   - *(b) For all $v \in V_2$, each vertex group $\mathcal{A}_v \leq \widetilde{M_q}$ (up to isomorphism)*
2. *Every edge group $\mathcal{A}_e$ is trivial.*
3. *The vertex degree $deg(v)$ divides $q+1$ for all $v \in VA$ and $i(e) = \frac{q+1}{deg(v)}$ for all edges $e \in EA$ with initial vertex $\partial_0(e) = v$.*

Moreover, for any graph of groups $\mathbb{A}$ satisfying $(i), (ii)$, and (iii), there is a covering morphism $\Phi : \mathbb{A} \to \mathbb{A}'$.

Proof. Denote the vertices of $\mathbb{A}'$ corresponding to M_q and $\widetilde{M_q}$ as a_1 and a_2, respectively. Suppose there is covering morphism $\Phi : \mathbb{A} \to \mathbb{A}'$. Then there is a graph morphism mapping the graph A to the single edge. Hence $V_1 = \{v \in VA \mid \varphi(v) = a_1\}$ and $V_2 = \{v \in VA \mid \varphi(v) = a_2\}$ form the desired bipartition of vertices whose vertex groups embed in M_q and $\widetilde{M_q}$, respectively. Moreover each edge group of $\mathbb{A}$ embeds in the single trivial edge group of $\mathbb{A}'$. Therefore (1) and (2) hold. For any vertex $v \in VA$, each edge such that $\partial_0(e) = v$ has index $i(e) = [\mathcal{A}_v : \mathcal{A}_e] = [\mathcal{A}_v : \{1\}] = |\mathcal{A}_v|$. Since the covering morphism φ induces an underlying edge-indexed covering and since the single edge in $\mathbb{A}'$ and its reverse have index $|M_q| = |\widetilde{M_q}| = q+1$, it follows that

$$q + 1 = \sum_{\partial_0(e)=v} i(e) = deg(v)|\mathcal{A}_v|.$$

Conversely, if $\mathbb{A}$ is a graph of groups for which $(i), (ii)$, and (iii) hold, then by (i) we have a graph morphism φ mapping the underlying graph A to a single edge. By (iii), this yields a covering morphism of edge indexed graphs $p : (A, i) \to (q+1, q+1)$. By

(i) we can identify the vertex groups $\mathbb{A}$ with abelian subgroups of M_q and $\widetilde{M}_q$. Since the edge groups are all trivial we can then apply Theorem 2.5, and thus we have the desired covering morphism. $\square$

Since the hypothesis that M_q and $\widetilde{M}_q$ are abelian certainly holds when $q = 2$, let's look at some properties which arise in this case.

Remarks: It follows from the proposition that every finite graph of groups which covers $M_2 * \widetilde{M}_2$ satisfies the following.

1. In the edge-indexed graph, every vertex will have degree 1, giving the corresponding edge an index of 3, or degree 3, giving the corresponding 3 edges an index of 1.

2. Every vertex group will be $\mathbb{Z}/3\mathbb{Z}$ (up to isomorphism) or trivial.

3. All cycles must be even, and no more than three cycles will meet at any vertex.

4. The corresponding cocompact lattice has the form $*_j\mathbb{Z}/3\mathbb{Z} * F_n$ when the graph has j vertices of degree 1 and n minimal cycles.

6.3 Free Groups as Cocompact Lattices

Our final example of an infinite family of cocompact lattices shows that any free group may be embedded as a cocompact lattice in a rank 2 Kac-Moody group.

Proposition 6.4. *Let G be a rank 2 locally compact Kac-Moody group over a finite field $\mathbb{F}_q$. Let $X = X_{q+1}$ be the Tits building of G, a $(q+1)-$regular tree. Let A be any $(q+1)-$regular bipartite graph. Then*

1. *The free group $F_s = \pi_1(A)$ is a discrete subgroup of G, where s is the number of edges outside any maximal tree in A.*

2. *If A is finite, then the free group $F_s = \pi_1(A)$ is a cocompact lattice in G, where s is the number of edges outside any maximal tree in A.*

To prove this, we construct a covering morphism of graphs of groups, where

$$\begin{tikzcd} P_1 \bullet \arrow[rr, dash, "B"] & & \bullet P_2 \end{tikzcd}$$

Figure 6.5: graph of groups for rank 2 Kac-Moody group (also Figure 2.3)

is the graph of groups for G.

Proof. (of Proposition)

Let A be any $(q+1)$-regular bipartite graph and let $\mathbb{A} = (A, \{1\}, \{1\})$ be the corresponding graph of groups with trivial vertex and edge groups. Then $F_s = \pi_1(\mathbb{A})$, where s is the number of edges outside any maximal tree of A. By Lemma 2.4, it is enough to show that there is a covering morphism $\mathbb{A} \longrightarrow \mathbb{G}$. Since A is bipartite there is a graph morphism from A to the simplex (the underlying graph of $\mathbb{G}$). For each $a \in VA, e \in EA$, the vertex map φ_a and the edge map φ_e are inclusions of the trivial group. Choose $\{\delta_e\}$ to be a complete set of coset representatives of P_i/B for $i = 1, 2$. Since each edge and vertex group is trivial the diagram

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d] & \{1\} \arrow[d] \\ B \arrow[r, "\mathrm{ad}(\delta_e)"] & P_i \end{tikzcd}$$

commutes for any $e \in EA$. Moreover, if the quotient graph for G is the single edge f with vertices a_1 and a_2, then for $i = 1, 2$ the map

$$\coprod_{e \in \varphi^{-1}_{(a_i)}(f)} \{1\}_e \longrightarrow P_i/B \text{ by } \{1\}_e \mapsto \delta_e B$$

is map from $q+1$ copies of the trivial group (because A is $(q+1)$-regular) to distinct cosets of P_i/B. Thus the map is bijective, and the desired covering morphism is obtained.

$\square$

Chapter 7

Parabolic Subgroups as Nonuniform Lattices

In this chapter we transition to constructing graphs of groups for known nonuiform lattices of $G = SL_2(\mathbb{F}_q((t^{-1})))$. These give nonuniform lattices in $PSL_2(\mathbb{F}_q((t^{-1})))$, which is isomorphic to the rank 2 affine Kac-Moody group associated to the Cartan matrix $A_1^{(1)}$. Moreover the constructions of these graphs of groups strongly rely on the BN-pair structure of G. We hope to build on this technique to obtain analogues in the hyperbolic Kac-Moody groups.

The graphs of groups constructed in this chapter are for parabolic subgroups of G. We give one maximal parabolic subgroup which is a Nagao-type lattice whose fundamental domain is an infinite ray corresponding to half of the 'standard apartment'. We also give a minimal parabolic subgroup whose fundamental domain is the full standard apartment.

7.1 SL_2 Subgroups and Graphs of Groups

We first wish to recall the graph of groups presentation for $SL_2(\mathbb{F}_q((t^{-1})))$, give a detailed description of its BN-pair, and fix the notation for several subgroups used in the lattice constructions.

7.1.1 BN–pair subgroups for SL_2

Let $G = SL_2(\mathbb{F}_q((t^{-1})))$, and fix the following subgroups:

$$B = \left\{ \begin{pmatrix} a & b \\ c/t & d \end{pmatrix} \in SL_2(\mathbb{F}_q[[t^{-1}]]) \right\},$$

$$B^- = \left\{ \begin{pmatrix} a & tb \\ c & d \end{pmatrix} \in SL_2(\mathbb{F}_q[t]) \right\},$$

$$N = G \cap \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \cup G \cap \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$$

Then (G, B, N) and (G, B^-, N) are BN-pairs, in fact they form a twin BN-pair, with isomorphic Weyl groups $W \cong D_\infty$, generated by

$$w_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 & t \\ -1/t & 0 \end{pmatrix}.$$

We have

$$G = BNB^- = B^-NB$$

and $G = \sqcup_{w \in W} BwB$. It follows that the map

$$w \mapsto B^-wB$$

is a bijection

$$W \longrightarrow B^-\backslash G/B.$$

In particular, this gives us a correspondance between the Weyl group elements, which index the standard apartment (see Section 7.1.2), and the quotient by the action of B^- on the postive edges of the Bruhat-Tits tree $X = X_{q+1}$ (corresponding to G/B). Thus we can conclude that the quotient graph $B^-\backslash X$ is a bi-infinite ray. We will compute the corresponding graph of groups at the end of this chapter.

For the BN-pair (G, B^-, N) we also have a second (twin) Bruhat decomposition $G = \sqcup_{w \in W} B^-wB^-$. If we consider the Bruhat decompositions over the subgroups of W with a single generator we obtain the *standard parabolic subgroups*

$$P_1 = B \sqcup Bw_1B = SL_2(\mathbb{F}_q[[t^{-1}]])$$

$$P_2 = B \sqcup Bw_2B = \left\{ \begin{pmatrix} a & tb \\ c/t & d \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{F}_q[[t^{-1}]]) \right\}$$

$$P_1^- = B^- \sqcup B^- w_1 B^- = SL_2(\mathbb{F}_q[t])$$

$$P_2^- = B^- \sqcup B^- w_2 B^- \cong SL_2(\mathbb{F}_q[t]).$$

We have bijections

$$W^+ \longrightarrow P_i^- \backslash G/B,$$

where W^+ is 'half' of W, so that the quotient graph $P_i^- \backslash X \leftrightarrow P_i^- \backslash G/B$ is a semi-infinite ray. We will compute the graph of groups $P_1^- \backslash\backslash X$ in Section 7.2.

Set

$$U = \left\{ \begin{pmatrix} 1 & 0 \\ c/t & 1 \end{pmatrix} \in SL_2(\mathbb{F}_q[[t^{-1}]]) \right\},$$

$$U^- = \left\{ \begin{pmatrix} 1 & tb \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{F}_q[t]) \right\}.$$

7.1.2 Standard Apartment

As a group with a BN pair, $G = SL_2(\mathbb{F}_q((t^{-1})))$ acts on its Bruhat–Tits building, the $(q+1)$-regular tree $X = X_{q+1}$. The associated edge of groups has vertex groups P_1 and P_2 and edge group B, giving us the graph of groups presentation $G = P_1 *_B P_2$.

We denote the *standard apartment* of X by the bi-infinite ray $\mathcal{A}$ of vertices

$$\cdots \to x_{-3} \to x_{-2} \to x_{-1} \to x_1 \to x_2 \to \cdots \quad .$$

For $n \geq 1$ we write e_n for the oriented edge with initial vertex x_n and terminal vertex x_{n+1}, and for $n \leq -1$ we write e_n for the oriented edge with initial vertex x_n and terminal vertex x_{n-1}. We denote the edge with initial vertex x_1 and terminal vertex x_{-1} by e_0. We identify the vertex stabilized by the group P_1 with the vertex x_{-1}, and the vertex stabilized by P_2 with x_1.

One important property of the standard apartment is that it is the fundamental domain for the action of the BN-pair subgroup B on the Bruhat-Tits tree. We will give a full graph of groups description for B^- at the end of this chapter.

7.1.3 Finite Subgroups of SL_2

We now define finite subgroups which will arise in the construction of the graph of groups presentation for our lattices in this chapter. Each of these subgroups has a natural analogue in the hyperbolic case.

$$D(\mathbb{F}_q) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_q^\times \right\},$$

$$U^+(\mathbb{F}_q) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}_q \right\}, \quad U^-(\mathbb{F}_q) = \left\{ \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \mid c \in \mathbb{F}_q \right\},$$

$$U^+(t\mathbb{F}_q) = \left\{ \begin{pmatrix} 1 & tb \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}_q \right\}, \quad U^-(t\mathbb{F}_q) = \left\{ \begin{pmatrix} 1 & 0 \\ ct & 1 \end{pmatrix} \mid c \in \mathbb{F}_q \right\}.$$

We use Tits' convention and set

$$\chi_{\alpha_i}(s) = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \mid s \in \mathbb{F}_q \right\},$$

$$\chi_{-\alpha_i}(s) = \left\{ \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix} \mid s \in \mathbb{F}_q \right\}.$$

Thus we may identify U_{α_i} with $U^+(\mathbb{F}_q)$ and $U_{-\alpha_i}$ with $U^-(-\mathbb{F}_q)$ (this is consistent with [RR] and consistent with the standard Kac-Moody group relations).

7.1.4 Parabolic Subgroup Intersections

In the next two sections we give constructions of graphs of groups by considering the action of the maximal standard parabolic subgroup P_1^- on the halves of the standard apartment $\mathcal{A}$ of the Bruhat-Tits tree X. We showed in Section 7.1.1 that the quotient graph $P_1^-\backslash X$ is an infinite ray, one half of the standard apartment. We will use this structure to computer the vertex groups, an infinite ascending chain unique up to isomorphism (see). A conjugate copy of this chain will give the vertex stabilizers in P_1^1 for the second half of the standard apartment. This will allow us to compute the vertex (and edge) stabilizers in $B^- \leq P_1^-$, whose quotient graph is the entire standard apartment.

We will need the following lemma regarding intersections of parabolic subgroups:

Lemma 7.1.

$$(1) \quad P_1^- \cap P_1 = SL_2(\mathbb{F}_q) = \langle D(\mathbb{F}_q), U^-(\mathbb{F}_q), U^+(\mathbb{F}_q) \rangle$$

$$\begin{aligned}(2) \quad P_1^- \cap P_2 &= \left\{ \begin{pmatrix} a & b_0 + b_1 t \\ 0 & a^{-1} \end{pmatrix} \mid a, a^{-1} \in \mathbb{F}_q^\times,\ b_0,\ b_1 \in \mathbb{F}_q \right\} \\ &= \langle D(\mathbb{F}_q), U^+(\mathbb{F}_q), U^+(t\mathbb{F}_q) \rangle.\end{aligned}$$

Proof. (1) follows immediately from the fact that $\mathbb{F}_q[[t^{-1}]] \cap \mathbb{F}_q[t] = \mathbb{F}_q$. For (2), recall any matrix in P_2 has the form $\begin{pmatrix} \alpha(t^{-1}) & \beta_{-1}t + \beta(t^{-1}) \\ t^{-1}\gamma(t^{-1}) & \delta(t^{-1}) \end{pmatrix}$,
where $\alpha(t^{-1}), \beta(t^{-1}), \gamma(t^{-1}), \delta(t^{-1}) \in \mathbb{F}_q[[t^{-1}]]$,
and any matrix in P_1^- has the form $\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$, where $a(t), b(t), c(t), d(t) \in \mathbb{F}_q[t]$.
Two such matrices are equal (and have determinant 1) when
$\alpha(t^{-1}) = a(t) = a_0$,$\gamma(t^{-1}) = c(t) = 0$, $\delta(t^{-1}) = d(t) = a_0^{-1}$ and $b(t) = \beta_{-1}t + \beta(t^{-1}) = b_0 + b_1 t$. Since any matrix of this form is in both P_2 and P_1^-, the proof is complete. ☐

7.2 Alternate Construction of the Nagao lattice

Let $\mathcal{A}^+$ denote half of the standard apartment, with vertices

$$x_1,\ x_2,\ x_3,\ \ldots$$

corresponding to cosets

$$P_2,\ w_2P_1,\ w_2w_1P_2,\ \ldots\ .$$

Note that the stabilizers in G of these vertices are the corresponding conjugates

$$P_2,\ w_2P_1w_2^{-1},\ w_2w_1P_2w_1^{-1}w2^{-1},\ \ldots\ .$$

That is, $gwP = wP$ if and only if $g \in wPw^{-1}$. Now we consider the vertex stabilizers in P_1^-.

For $n \geq 1$, let $\Gamma_n = Stab_{P_1^-}(x_n)$. Recall from Lemma 7.1 that

$$\Gamma_1 = P_1^- \cap P_2 = \left\{ \begin{pmatrix} a & b_0 + b_1 t \\ 0 & a^{-1} \end{pmatrix} \mid a, a^{-1} \in \mathbb{F}_q^\times,\ b_0,\ b_1 \in \mathbb{F}_q \right\}.$$

The sequence Γ_n, $n \geq 1$ satisfies

(1) $\Gamma_n \subset \Gamma_{n+1}$.

(2) $[\Gamma_{n+1} : \Gamma_n] = q$.

(3) $\cup_{n\geq 1} \Gamma_n \subset Stab_G(\infty)$.

(4) $\Gamma_1 = \left\{ \begin{pmatrix} a & b_0 + b_1 t \\ 0 & a^{-1} \end{pmatrix} \mid a, a^{-1} \in \mathbb{F}_q^\times,\ b_0,\ b_1 \in \mathbb{F}_q \right\}$.

Remark: Nagao's Theorem (Theorem 7.2) is well known. We give a new proof which seems likely to generalize the statement to hyperbolic Kac-Moody groups.

Theorem 7.2. *The groups*

$$\Gamma_n = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a, a^{-1} \in \mathbb{F}_q^\times,\ b \in \mathbb{F}_q[t],\ deg(b) \leq n \right\}$$

are the unique sequence, up to isomorphism, satisfying conditions (1), (2), (3) and (4) above.

Proof: An ascending union of finite subgroups stabilizes a half apartment and hence is contained in the stabilizer of the corresponding end. In this case, the stabilizer of the positive half $\mathcal{A}^{=}$ of the standard apartment is the Borel subgroup of G [S]. The groups Γ_n are thus the unique sequence, up to isomorphism, satisfying (1), (2), (3) and (4). □

It follows that

$$P_1^- \cap w_2 P_1 w_2^{-1} = \Gamma_2$$

$$P_1^- \cap w_2 w_1 P_2 w_1^{-1} w_2^{-1} = \Gamma_3$$

$$P_1^- \cap w_2 w_1 w_2 P_1 w_2^{-1} w_1^{-1} w_2^{-1} = \Gamma_4$$

$$\vdots$$

In general

$$P_1^- \cap w P_i w^{-1} = \Gamma_n,$$

where w is the unique element of the set $\{w_2,\ w_2w_1,\ w_2w_1w_2,\ \ldots\}$ with $l(w) = n-1$, and moreover $i = 1 \Leftrightarrow l(w) \in \mathbb{O}$.

We have $\Gamma_n = D(\mathbb{F}_q)A_n$, where A_n are the groups

$$A_n = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}_q[t],\ deg(b) \leq n \right\}.$$

For $n \geq 1$, define

$$B_{x_n}^- = B^- \cap \Gamma_n = \left\{ \begin{pmatrix} a & tb \\ 0 & a^{-1} \end{pmatrix} \mid a, a^{-1} \in \mathbb{F}_q^{\times},\ b \in \mathbb{F}_q[t],\ deg(b) \leq n-1 \right\}$$

$$U_{x_n}^- = B^- \cap A_n = \left\{ \begin{pmatrix} 1 & tb \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}_q[t],\ deg(b) \leq n-1 \right\}.$$

The notation is suggestive. Indeed, since $B^- \leq P_1^-$ and $U^- \leq P_1^-$, we have

Theorem 7.3. *For $n \geq 1$,*

$$Stab_{B^-}(x_n) = B^- \;\cap\; \Gamma_n = B^-_{x_n}$$

$$Stab_{U^-}(x_n) = B^- \;\cap\; A_n = U^-_{x_n}. \;\square$$

Remark: The groups $B^-_{x_n}$ also form an ascending chain which satisfies (1), (2), and (3) above.

The ascending chain $\Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_3 \subseteq \ldots$ gives the vertex groups in the graph of groups presentation of the Nagao lattice $P_1^- = SL_2(\mathbb{F}_q[t])$. This ascending chain also gives the edge groups for the graph of groups presentation. Let

$$e_1,\; e_2,\; e_3,\; \ldots$$

denote the edges of half of the standard apartment of X, where

$$e_i = (x_i, x_{i+1}) \;\text{ for }\; i \geq 1.$$

Then, the edge groups are given by

$$Stab_{P_1^-}(e_i) = \Gamma_i \cap \Gamma_{i+1} = \Gamma_i \;.$$

Restricting these edge stabilizers to B^-, we have

$$B^-_{e_i} = Stab_{B-}(e_i) = B^- \cap \Gamma_i = B^-_{x_i} \;.$$

7.3 Lattice corresponding to the standard apartment

In this section we construct the graph of groups for the nonuniform lattice B^-, which has the standard apartment as its quotient graph. We already found the vertex and edge groups associated with the positive half of the standard apartment in the previous section. Next we consider the groups associated with the vertices and edges of the other half of the standard apartment. Again we begin by looking at the action of P_1^- on this infinte ray.

Let $\mathcal{A}^-$ denote half of the standard apartment, with vertices

$$x_{-1},\ x_{-2},\ x_{-3},\ x_{-4}\ \ldots$$

corresponding to cosets

$$P_1, w_1P_2,\ w_1w_2P_1,\ w_1w_2w_1P_2,\ \ldots\ .$$

As before, the stabilizers in G of these vertices are the corresponding conjugates

$$P_1,\ w_1P_2w_1^{-1},\ w_1w_2P_1w_2^{-1}w_1^{-1},\ \ldots\ .$$

We now consider the stabilizers in P_1^- for these vertices.

For $n \geq 1$, let $\Gamma_{-n} = Stab_{P_1^-}(x_{-n})$. Recall from 7.1 that

$$\Gamma_{-1} = P_1^- \ \cap\ P_1 \ = \ SL_2(\mathbb{F}_q).$$

A simple computation shows that

$$\Gamma_{-2} = \ P_1^- \ \cap \ w_1P_2w_1^{-1} \ = \ \left\{ \begin{pmatrix} a & 0 \\ b_0 + b_1t & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_q^\times,\ b_0, b_1 \in \mathbb{F}_q \right\}.$$

The sequence of vertex stabilizers Γ_{-n} must satisfy

(1) $\Gamma_{-n} \subset \Gamma_{-(n+1)}$, $n \geq 2$

(2) $[\Gamma_{-(n+1)} \ :\ \Gamma_{-n}] = q$, $n \geq 2$

(3) $\cup_{n\geq 2}\, \Gamma_{-n} \ \subset \ Stab_G(-\infty)$.

(4) $\Gamma_{-2} \ = \left\{ \begin{pmatrix} a & 0 \\ b_0 + b_1t & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_q^\times,\ b_0, b_1 \in \mathbb{F}_q \right\}.$

Remark We have excluded Γ_{-1} since $SL_2(\mathbb{F}_q)$ is not a subgroup of Γ_{-2}.

We then claim, by the same reasoning as before,

Theorem 7.4. *For $n \geq 2$, the groups*

$$\Gamma_{-n} \;=\; \left\{ \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_q^{\times}\,,\; c \in \mathbb{F}_q[t], deg(c) \leq n-1 \right\}$$

are the unique sequence, up to isomorphism, satisfying conditions (1), (2), (3), and (4) above. $\square$

As before, for $n \geq 2$ we define

$$B^-_{x_{-n}} \;=\; B^- \;\cap\; \Gamma_{-n} \;=\Gamma_{-n}\;, \quad \text{and}$$

$$B^-_{x_{-1}} \;=\; B^- \;\cap\; \Gamma_{-1} \;=\left\{ \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_q^{\times}\,,\; c \in \mathbb{F}_q, \right\}\;.$$

Again, since $B^- \leq P_1^-$, we have

Theorem 7.5. *For $n \geq 1$,*

$$Stab_{B^-}(x_{-n}) = B^- \;\cap\; \Gamma_{-n} = B^-_{x_{-n}}\;. \;\square$$

Remark: For $n \geq 1$ the groups $B^-_{x_{-n}}$ also form an ascending chain which satisfies (1), (2), and (3) above. In fact, it is the same ascending chain as in Theorem 7.4 with an additional group $B^-_{x_{-1}}$ at the beginning.

Using the vertex stabilizers we now find the edge stabilizers. Let

$$e_{-1},\; e_{-2},\; e_{-3},\; \ldots$$

denote the edges of the 'negative' half $\mathcal{A}^-$ of the standard apartment of X, where

$$e_{-i} = (x_{-(i+1)}, x_{-i}) \;\; \text{for} \;\; i \geq 1.$$

Then, the edge stabilizers with respect to the action of P_1^- are given by

$$Stab_{P_1^-}(e_{-1}) = \Gamma_{-2} \cap \Gamma_{-1} \;=\; \left\{ \begin{pmatrix} a & 0 \\ b_0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_q^\times,\; b_0 \in \mathbb{F}_q \right\},$$

$$\text{and for } i \geq 2, Stab_{P_1^-}(e_{-i}) = \Gamma_{-(i+1)} \cap \Gamma_{-i} = \Gamma_{-i} \;.$$

Restricting these edge stabilizers to B^-, we have

$$B^-_{e_{-i}} = Stab_{B^-}(e_{-i}) = B^- \cap Stab_{P_1^-}(e_{-i}) = B^-_{x_{-i}} \;.$$

Finally, denote the central "core" edge of the standard apartment by $e_0 = (x_1, x_{-1})$ and define

$$B^-_{e_0} \;=\; Stab_{B^-}(e_0) = B^-_{x_{-1}} \;\cap\; B^-_{x_1} = D(\mathbb{F}_q)\;.$$

Thus we have divided the fundamental domain for B^- into two infinite rays and computed the corresponding vertex and edge stabilizers by interesecting B^- with the stabilizers in P_1^-. Together the two rays combine to create the standard apartment $\mathcal{A} = B^-\backslash X$ with vertex groups $\{B^-_{x_n} \mid n \in \mathbb{Z}^\times\}$ and edge groups $\{B^-_{e_n} \mid n \in \mathbb{Z}\}$ satisfying:

1. $B^-_{e_0} = B^-_{x_{-1}} \cap B^-_{x_1} = D(\mathbb{F}_q)$,
2. $B^-_{e_n} = B^-_{x_n} \cap B^-_{x_{n+1}} = B^-_{x_n}\;,\; n \geq 1$,
3. $B^-_{e_{-n}} = B^-_{x_{-n}} \cap B^-_{x_{-(n+1)}} = B^-_{x_{-n}}\;,\; n \geq 1$,
4. $[B^-_{x_1} \;:\; B^-_{e_0}] \;=\; [B^-_{x_{-1}} \;:\; B^-_{e_0}] \;=\; q$,
5. $[B^-_{x_{n+1}} \;:\; B^-_{x_n}] \;=\; [B^-_{x_{-(n+1)}} \;:\; B^-_{x_{-n}}] \;=\; q\;,\; n \geq 1$.

That is, we have single (q, q)-indexed core edge with two infinite rays (cusps): one ray of $(1, q)$-indexed edges and another ray of $(q, 1)$-indexed edges.

Chapter 8

Fundamental domains for congruence subgroups of $\mathrm{SL}_2(\mathbb{F}_q[t])$ as ramified coverings

Suppose a group G acts on a graph X. Taking a quotient by the action of G yields a graph $G\backslash X$ whose vertices and edges correspond to the G-orbits. Initial and terminal vertices of an edge $G \cdot e$ are given by $\partial_i(G \cdot e) = G \cdot \partial_i(e)$ for $i = 0, 1$, respectively. We call $G\backslash X$ the quotient graph with respect to its action on X. We will often identify $G\backslash X$ with its image in X; this subgraph is called the *fundamental domain*.

Let Y be the fundamental domain of X with respect to the action of G. Suppose N is a normal subgroup of G. It is possible to reconstruct $N\backslash X$ as a covering of Y in the following way. Identify Y with the fundamental domain of $N\backslash X$ with respect to the action of Γ/N. Take the vertices of $N\backslash X$ to be orbits of the vertices in Y with respect to this action. Identify the orbits with cosets of the group Γ/N by the stabilizers of the vertices in Y. Treat the edges in a similar fashion. Two vertices are adjacent if the corresponding cosets of non-trivial intersection. This construction of $N\backslash X$ is called a *ramified covering* of $G\backslash X$.

In this chapter we will use ramified coverings to characterize the quotient graphs for congruence subgroups of $\Gamma = \mathrm{SL}_2(\mathbb{F}_q[t])$ in $G = \mathrm{SL}_2(\mathbb{F}_q((t^{-1})))$. The Tits building of G is the $q+1$-regular infinite tree $X = X_{q+1}$.

The quotient graph $X_1 = \Gamma\backslash X$ is the infinite ray

$$\Lambda_0 \to \Lambda_1 \to \Lambda_2 \to \cdots ,$$

and the corresponding graph of groups is described in Section 2.5.

We will use this fundamental domain for Γ to construct the fundamental domains for congruence subgroups of Γ. Since these subgroups are normal, we have an action by the quotient groups.

Lemma 8.1 (DD). *Let G be a group and T a tree. Suppose G acts on T. If N is a normal subgroup of G, then G/N acts on the connected graph $N\backslash T$. Each $Nt \in N\backslash T$ has stabilizer*

$$(G/N)_{Nt} \;=\; NG_t/N.$$

Therefore given a normal subgroup N of Γ we may describe the vertices (resp. edges) of $N\backslash X$ not only as N-orbits with respect to the action of N on X, but as elements of G/N-orbits of $\{Nv \;:\; v \in V(G\backslash X)\}$ (resp. of $\{Ne \;:\; e \in E(G\backslash X)\}$). That is we may construct $N\backslash X$ as a ramified covering of $G\backslash X$.

8.1 The action of $\Gamma/\Gamma(g)$

Let $\Gamma = \mathrm{SL}_2(\mathbb{F}_q[t])$, which is a nonuniform lattice in $G = \mathrm{SL}_2(\mathbb{F}_q((t^{-1})))$.

For $g \in \mathbb{F}_q[t]$, define the corresponding congruence subgroup

$$\Gamma(g) \;=\; \{A \in SL_2(\mathbb{F}_q[t]) \mid A \equiv I_2 \bmod g\}.$$

Since $\Gamma(g)$ is normal in Γ, the quotient graph $X_g = \Gamma(g)\backslash X$ is a ramified covering of the quotient graph $X_1 = \Gamma\backslash X$. We have projections $\pi_g : X \to X_g$ and $\pi_1 : X_g \to X_1$.

Let $N = \Gamma(g) \leq \Gamma$. As described using Bass-Serre theory, the vertices of $X_g = N\backslash X$ are given by the distinct N-orbits of vertices in X:

$$V(X_g) \;=\; \{N \cdot v \;:\; v \in V(X)\}.$$

Similarly the edges of X_g are given by

$$E(X_g) \;=\; \{N \cdot e \;:\; e \in E(X)\}.$$

Then Γ/N acts on X_g by

$$\gamma N(N \cdot x) \;=\; N \cdot \gamma x.$$

(Here x is either a vertex or an edge of X.)

We may partition $V(X)$ into the Γ-orbits of the vertices Λ_i of the infinite ray $X_1 = \Gamma\backslash X$. Or equivalently we may partition the vertices of X into coset spaces Γ/Γ_i, where Γ_i is the stabilizer of Λ_i in Γ, as given in Proposition 2.6. That is

$$V(X) = \sqcup_{i\geq 0}\Gamma/\Gamma_i.$$

Similarly

$$E(X) = \sqcup_{i\geq 0}\Gamma/(\Gamma_i \cap \Gamma_{i+1}).$$

We use this description of $\Gamma\backslash X$ to construct our ramified cover. First, the vertices and edges of $X_g = N\backslash X$ as follows.

$$\begin{aligned} V(X_g) &= \sqcup_{i\geq 0}\Gamma(g)\cdot\Gamma/\Gamma_i && \leftrightarrow \sqcup_{i\geq 0}\Gamma_i\cdot\Gamma/\Gamma(g) \\ E(X_g) &= \sqcup_{i\geq 0}\Gamma(g)\cdot\Gamma/(\Gamma_i\cap\Gamma_{i+1}) && \leftrightarrow \sqcup_{i\geq 0}(\Gamma_i\cap\Gamma_{i+1})\cdot\Gamma/\Gamma(g) \end{aligned}$$

We now define the coset spaces $L_i = \Gamma_i\cdot\Gamma/\Gamma(g)$ for $i \geq 0$, and we call L_i the set of vertices in *level* i.

8.2 Structure of fundamental domains for congruence subgroups

Let $\Gamma = \mathrm{SL}_2(\mathbb{F}[t])$, $g \in \mathbb{F}[t]$ with $n = \deg(g)$, and $\Gamma(g)$ the congruence subgroup mod g. It will be useful to have a matrix group description for $\Gamma/\Gamma(g)$. We show here that

$$\Gamma/\Gamma(g) \cong \mathrm{SL}_2(R_g)$$

where $R_g = \mathbb{F}[t]/(g)$. The argument is the same as in ([Sh]) for the classical setting $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 8.2. *The map* $\mathrm{SL}_2(\mathbb{F}[t]) \to \mathrm{SL}_2(R_g)$ *given by* $A \mapsto A \bmod (g)$ *is surjective.*

Proof. Let $\overline{A} \in \mathrm{SL}_2(R_g)$ and let $A \in M_2(\mathbb{F}[t])$ be a matrix such that $A \bmod (g) = \overline{A}$. We seek a matrix in $\mathrm{SL}_2(\mathbb{F}[t])$ which is congruent to $A \bmod g$. By the Smith normal form there exist matrices $U, V \in SL_2(\mathbb{F}[t])$ such that UAV is diagonal. Then $UAV = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$, with $ad \equiv 1 \bmod (g)$. Let $B = \left(\begin{smallmatrix} a & ad-1 \\ 1-ad & 2d-ad^2 \end{smallmatrix}\right)$. Then $\det(B) = 1$, so $B \in \mathrm{SL}_2(\mathbb{F}[t])$. Moreover $B \equiv UAV \bmod (g)$. Therefore $U^{-1}BV^{-1} \in \mathrm{SL}_2(\mathbb{F}[t])$ and $U^{-1}BV^{-1} \equiv A \bmod (g)$, as desired. $\square$

As stated previously, when $\mathbb{F} = \mathbb{F}_q$ is a finite field, Γ acts on the tree $X = X_{q+1}$. We now seek a precise description of the graph structure for $\Gamma(g)\backslash X$.

For each $i \geq 0$, denote by L_i the set of vertices in X_g which project to the vertex Λ_i in the ray. That is $L_i = \Gamma_i \cdot \Gamma/\Gamma(g)$ (see previous section). We may also identify L_i with the $\Gamma/\Gamma(g)$-orbit of $\Gamma(g)\Lambda_i$. Since any edge of $\Gamma\backslash X$ has endpoints Λ_i and Λ_{i+1} for some $i \geq 0$, it follows that any edge of X_g has endpoints in L_i and L_{i+1} for some $i \geq 0$.

We have the following formula for the number of vertices in each level:

Lemma 8.3. $|L_i| = \frac{|\mathrm{SL}_2(R_g)|}{|\Gamma_i|/|\Gamma_i \cap \Gamma(g)|}$

Proof. The classic Orbit-Stabilizer Theorem gives

$$|L_i| = |\Gamma/\Gamma(g)|/|\mathrm{Stab}_{\Gamma/\Gamma(g)}(\Lambda_i)|.$$

By Proposition 2.6, we have $\mathrm{Stab}_\Gamma(\Lambda_i) = \Gamma_i$. Then from the previous section, we have

$$\mathrm{Stab}_{\Gamma/\Gamma(g)}(\Lambda_i) = (\Gamma(g)\Gamma_i)/\Gamma(g).$$

Using this equality and the second isomorphism theorem, the stabilizer of Λ_i in $\Gamma/\Gamma(g)$ is

$$\mathrm{Stab}_{\Gamma/\Gamma(g)}(\Lambda_i) = \Gamma(g)\Gamma_i/\Gamma(g) \cong \Gamma_i/[\Gamma_i \cap \Gamma(g)]\ .$$

Thus the number of vertices in L_i is given by the number of vertices in

$$[\Gamma/\Gamma(g)]/[\Gamma(g)\Gamma_i/\Gamma(g)] \ \cong [\Gamma/\Gamma(g)]/[\Gamma_i/[\Gamma_i \cap \Gamma(g)]] \ \cong \mathrm{SL}_2(\mathbb{F}_q[t]/(g))/[\Gamma_i/[\Gamma_i \cap \Gamma(g)]]\ .$$

$\square$

The adjacency relation between two vertices of levels $L_i = \Gamma_i \cdot \Gamma/\Gamma(g)$ and $L_{i+1} =$

$\Gamma_{i+1} \cdot \Gamma/\Gamma(g)$ is as follows.

1. The neighbors of a level i vertex $\Gamma_i\sigma$ in L_{i+1} are $\Gamma_{i+1}\sigma'$, where $\sigma' \in \Gamma_i\sigma$.

2. For $i \geq 1$, the neighbors of $\Gamma_i\sigma$ in L_{i-1} are $\Gamma_{i-1}\sigma'$, where $\sigma' \in \Gamma_i\sigma$.

It is useful to phrase this in terms of coset intersections.

Let

$H = \Gamma/\Gamma(g) \cong SL_2(R_g)$, where $R_g = \mathbb{F}_q[t]/(g) \leftrightarrow \{a_0+a_1t+...+a_{n-1}t^{n-1}|a_i \in \mathbb{F}_q\}$, and

$$H_i = \Gamma_i/(\Gamma_i \cap \Gamma(g)).$$

The vertices

$$hH_i \text{ and } kH_{i+1} \text{ are connected by an edge if and only if } hH_i \cap kH_{i+1} \neq \emptyset.$$

Since X_g becomes a collection of disjoint infinite rays beginning at each vertex of level L_{n-1}, it suffices to describe the graph induced by the first n levels. It is useful to note that $\Gamma_i \cap \Gamma(g) = \{1\}$ for $i \leq n-1$, where $n = deg(g)$. Thus $H_i = \Gamma_i$ for $i \leq n-1$ and we may describe the first n levels as cosets of matrix groups $H_0, H_1, \ldots, H_{n-1}$ in $SL_2(R_g)$.

A fundamental consequence is that these adjacency relations yield a connected graph.

Proposition 8.4. *Let $X_g = \Gamma(g)\backslash X$. The subgraph induced by the vertices in $L_0, L_1, ..., L_{n-1}$ is connected if, and only if, $H = \langle H_0, H_{n-1}\rangle$.*

The following proof is due to Scott Murray.

Proof. Suppose that $H = \langle H_0, H_{n-1}\rangle$. Clearly there is a path connecting H_ia and H_ja for every $i, j = 0, \ldots, n-1$. Let H_ia and H_jb be two vertices. Write

$$a^{-1}b = h_1k_1h_2k_2 \cdots h_mk_m$$

for $h_l \in H_0$ and $k_l \in H_{n-1}$. Then we have a path from $H_i a$ to $H_{n-1}a = H_{n-1}k_m a$, to $H_0 k_m a = H_0 h_m k_m a$, to $H_{n-1}h_m a = H_{n-1}k_{m-1}h_m k_m a$, and so on, to $H_0 h_1 k_1 \cdots h_m k_m a = H_0 a$, and so to $H_j b$.

Conversely, we can write $a \in H$ as a word in elements of the groups H_i using the path from H_0 to $H_0 a$. Since $H_1 \leq H_2 \leq \cdots$, we are done. □

Proposition 8.4 may be used to show connectedness. We state the result here. The proof is due to Scott Murray and may be found in [CCM].

Theorem 8.5. *Hence the graph X_g is connected.*

Figure 8.1 gives a schematic drawing of the graph X_g, whose properties are summarized below.

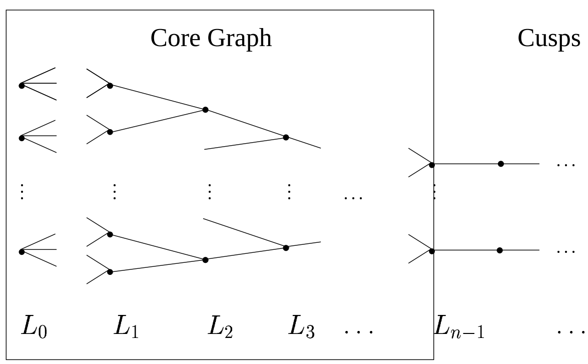

Figure 8.1: schematic drawing of quotient graph $X_g = \Gamma(g)\backslash X$

1. X_g is bipartite since it is a quotient of the bipartite graph X.
2. The vertex set is given by $V(X_g) = \sqcup L_i$, and the sets $\{v \in L_i \mid i \text{ is even }\}$, $\{v \in L_i \mid i \text{ is odd }\}$ form a bipartition of the vertex set.
3. The edges run between sequential levels, and the edges between vertices in L_i and L_{i+1} project to the edge $\Lambda_i\Lambda_{i+1}$ in X. The subgraph induced by L_0 and L_1 is a $(|H_0|/|H_0 \cap H_1|, |H_1|/|H_0 \cap H_1|) = (q+1, q)$-regular, bipartite graph. Then,

for $i = 1, ..., n-1$, each vertex in L_i has degree $q+1$ with $|H_i|/|H_i \cap H_{i-1}| = q$ edges incident to vertices in L_{i-1} and only $|H_i|/|H_i \cap H_{i+1}| = 1$ edge incident to a vertex in L_{i+1}. For $i \geq n$, each vertex in L_i has degree 2 with $|H_{n-1}|/|H_{n-1}| = 1$ edge each incident to a vertex in L_{i-1} and a vertex in L_{i+1}. Thus the graph "collapses" in a q-fold manner until it reaches the level L_{n-1}. At this point the graph becomes a collection of disjoint rays, one ray for each vertex in L_{n-1}.

4. The graph has $|L_{n-1}|$ cusps.

Moreover we can give the graph of groups, denoted $\Gamma(g)\backslash\backslash X$. We label each vertex and edge with its stabilizer under the action of $\Gamma(g)$. First we consider the $\Gamma(g)$ stabilizers of the vertices in $\Gamma\backslash X$.

$$Stab_{\Gamma(g)}(\Lambda_i) = \Gamma_i \cap \Gamma(g) =$$

$$= \begin{cases} \{1\} & \text{if } i < n \\ U_i \;=\; \left\{ \begin{pmatrix} 1 & g(t)f(t) \\ 0 & 1 \end{pmatrix} \mid f(t) \in \mathbb{F}_q[t],\ deg(f) \leq i-n \right\} & \text{if } i \geq n \end{cases}$$

Since the Γ-stabilizer of any vertex in L_i is conjugate in Γ to Γ_i, it follows that the $\Gamma(g)$-stabilizer of any vertex in L_i is conjugate in Γ to $\Gamma_i \cap \Gamma(g)$. Thus the "core" vertices are labeled with the trivial group, and the "cusp" vertex groups along each ray are of the form

$$s_j U_i s_j^{-1} \ ,$$

where $\{s_j \mid j = 1, \ldots, k = |L_{n-1}|\}$ is a set of coset representatives of $(\Gamma/\Gamma(g))/(\Gamma_i \cap \Gamma(g))$. The edge groups, that is the edge stabilizers, are intersections of the corresponding initial and terminal vertex groups. Thus the edge monomorphisms are simply inclusion maps.

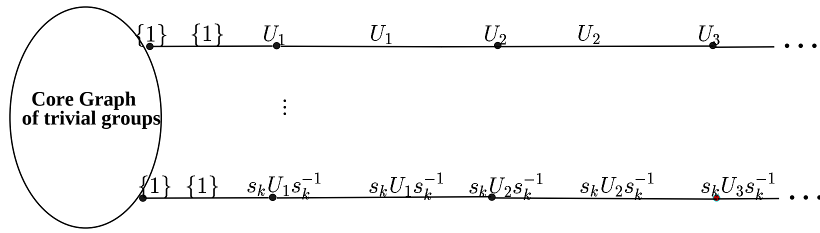

Figure 8.2: graphs of groups $\Gamma(g)\backslash\backslash X$ (all vertex and edge groups in core are trivial)

8.3 Congruence mod t^n

We will now give a more detailed description regarding the size of the graph X_g when $g = t^n$. This setting will be apt to giving detailed drawings using Magma and later to contrasting our construction with that of Morgenstern. The large size will also reveal a difficulty in obtaining detailed examples.

Let $R = \mathbb{F}_q[t]$ and for $n \geq 1$,

$$R_n = R_{t^n} = \mathbb{F}_q[t]/t^n\mathbb{F}_q[t] \cong \{a_0 + a_1t + ... + a_{n-1}t^{n-1}|a_i \in \mathbb{F}_q\},$$

$$|R_n| = q^n.$$

The invertible elements in the ring R_n are those with non-zero constant term:

$$R_n^\times \;=\; \{a_0 + a_1t + ... + a_{n-1}t^{n-1} \mid a_0 \in \mathbb{F}_q^*, a_i \in \mathbb{F}_q\},$$

$$|R_n^\times| = q^{n-1}(q-1).$$

We can give the formula in 8.3 in terms of n and q. First, it is easy to show that

$$|\Gamma_0| = (q-1)q(q+1), \quad |\Gamma_i| = (q-1)q^{i+1} \text{ for } i \geq 1, \text{ and,}$$

$$|\Gamma_i \cap \Gamma(g)| = \begin{cases} 1 & \text{if } i < n \\ q^{i-n+1} & \text{if } i \geq n \end{cases}$$

A counting argument using row reduction of matrices (due to Scott Murray) yields:

$$|SL_2(R_n)| = |R_n^{\times}|\ |R_n|\ (2|R_n|\ -\ |R_n^{\times}|) = q^{3n-2}(q-1)(q+1).$$

We can then say that the number of vertices in each level is given by:

$$|L_i| = \begin{cases} q^{3(n-1)} & \text{if } i = 0 \\ (q+1)q^{3(n-1)-i} & \text{if } 0 < i < n \\ (q+1)q^{2(n-1)} & \text{if } i \geq n \end{cases}$$

In particular the graph X_{t^n} has $(q+1)q^{2(n-1)}$ cusps.

From here it is also possible to construct the covolume for the graph of groups.

Lemma 8.6. $Vol(X_{t^n}) = \frac{2q^{3n-2}}{q-1}$.

Proof. For $i = 0, ..., n-1$, L_i contains $|L_i|$ trivial vertex groups. For every $i \geq n$, each of the $(q+1)q^{2(n-1)}$ cusps has a vertex group of size $|\Gamma_i \cap \Gamma(g)| = q^{i-n+1}$ in level i. Thus the covolume is given by

$$Vol(X_{t^n}) = \sum_{v \in VX_{t^n}} \frac{1}{\text{size of vertex group}} = \sum_{i=0}^{n-1} |L_i| + (q+1)q^{2(n-1)} \sum_{i=1}^{\infty} \frac{1}{q^i} = \frac{2q^{3n-2}}{q-1}.$$

∎

8.4 Detailed examples of fundamental domains for congruence subgroups

Let's look at some specific examples of the graph X_g for the congruence subgroups of $\Gamma = SL_2(\mathbb{F}_q[t])$.

(1) First, when $g(t) = t$, we have $|L_0| = 1$, and $|L_i| = q+1$ for $i \geq 1$. Thus the graph X_g is just a "star graph" with a central vertex and $q+1$ rays.

(2) Now, let $g(t) = t^2$. Here, $|L_0| = q^3$ and $|L_i| = (q+1)q^2$ for $i \geq 1$. The bipartite graph between the first two levels is $(q+1, q)$-regular, and the cusps are attached to this bipartite graph (no collapsing takes place). Below is the graph when $q = 2$.

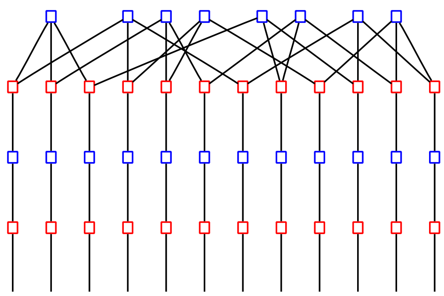

Figure 8.3: $n = 2$, $q = 2$

(3) Let $g(t) = t^3$. Here, $|L_0| = q^6$, $|L_1| = (q+1)q^5$ and $|L_i| = (q+1)q^4$ for $i \geq 2$. The bipartite graph between the first two levels is $(q+1, q)$-regular, and then the graph collapses once by a factor of q before extending onward as infinite rays. Below is the graph when $q = 2$.

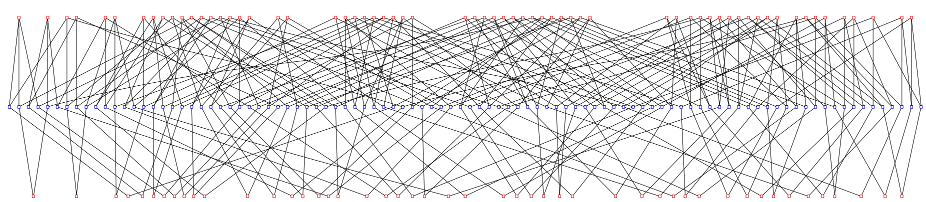

Figure 8.4: $n = 3$, $q = 2$

Here we use the computer algebra system Magma as a computational tool to construct these graphs. Due to the large size of the core graph it is impractical to draw the examples generated by Magma for larger values of q and n. However these examples do allow us to make some critical observations about the graph structure. To help draw these graphs for small values of n and q we use Magma as described below.

Let $g = t^n$ and $R_g = \mathbb{F}_q[t]/t^n$. Let $H = SL(2, R_g)$. Define the subgroups $H_0 = SL(2, \mathbb{F}_q)$ and for $1 \leq i \leq n-1$, $H_i = T + (\mathbb{F}_q + \mathbb{F}_q t + ... + \mathbb{F}_q t^i)E_{1,2}$ where T is the group of all diagonal matrices in H, and $E_{1,2}$ is an elementary matrix. We wish to construct the core part of the coset graph X_g, up to the infinite rays. The vertices correspond to all cosets of H_i in H, and there is an edge between two vertices hH_i and hH_{i+1} exactly when these cosets intersect nontrivially.

Since Magma has little functionality for matrix groups over the ring R_g, we constructed the groups as permutations on submodules of R_g^2 with a single generator (resp. elements of R_g^2). We determine if two cosets intersect by a brute-force method. This approach is only suitable for small groups.

Chapter 9

Fundamental domains for congruence subgroups of $\mathrm{PGL}_2(\mathbb{F}_q[t])$

Our methods in the previous chapter used to construct fundamental domains as ramified coverings are not new. The constructions mimic work done for congruence subgroups of $PGL_2(\mathbb{F}_q[t])$ by Morgenstern [M], and similar methods have been applied in [GN] and [R]. However, at least some of Morgenstern's constructions fail to give fundamental domains, as we explain in this chapter.

Morgenstern ([M]) claimed to construct fundamental domains of lattices for congruence subgroups of the group $\Gamma = \mathrm{PGL}_2(\mathbb{F}_q[t])$ which is a nonuniform lattice subgroup of $G = \mathrm{PGL}_2(\mathbb{F}_q((t^{-1})))$. These congruence subgroups have the form

$$\Gamma(g) = \{A \in \mathrm{PGL}_2(\mathbb{F}_q[t]) \mid A \equiv I_2 \bmod g\}$$

for some $g \in \mathbb{F}_q[t]$. He employed the method of ramified coverings described in the previous chapter. This method for producing the fundamental domain for a subgroup of a given group is consistent with the theory of branched topological coverings, and in Morgenstern's setting, coincides with a method suggested by Drinfeld in his theory of modular curves over function fields ([D]). Similar constructions of fundamental domains of lattices for congruence subgroups were constructed by Gekeler and Nonnengardt ([GN2]) and Rust ([Ru]) using essentially the same method.

Morgenstern's motivation was to provide the first known examples of linear families of bounded concentrators. These are obtained as subgraphs denoted $D_g(0-1)$ of the fundamental domains of lattices for congruence subgroups $\Gamma(g)$ for $\Gamma = \mathrm{PGL}_2(\mathbb{F}_q[t])$ ([M]).

We prove that these constructions do not yield the desired ramified coverings, and in particular yield graphs that are not connected in characteristic 2. Since the fundamental domain is the quotient graph of the action of a group on a tree, the quotient must be connected. It follows that Morgenstern's graphs cannot be quotient graphs by the action of congruence subgroups on the Bruhat-Tits tree. Moreover certain subgraphs $D_g(0-1)$ of Morgenstern's graphs which he claims to be expanders are not connected in characteristic 2.

We repair Morgenstern's method of constructing ramified coverings to give fundamental domains of congruence subgroups of $\mathrm{PGL}_2(\mathbb{F}_q[t])$ as we gave in the SL_2 setting in the previous chapter.

9.1 Morgenstern's PGL_2 graph

Let $\Gamma = PGL(\mathbb{F}_q[t])$. For a polynomial $g \in \mathbb{F}_q[t]$, let $\Gamma(g) = \{A \in \Gamma : A \equiv I \bmod g\}$. Both Γ and $\Gamma(g)$ act on the $q+1-$regular tree $X = X_{q+1}$. In Morgenstern's paper, he gives the structure for a graph X'_g as follows.

Let $degree(g) = n$ and $R_g = \mathbb{F}_q[t]/g$. Define the groups $H_0 = SL(2, \mathbb{F}_q)$ and for $1 \leq i \leq n-1$, $H_i = T + (\mathbb{F}_q + \mathbb{F}_q t + ... + \mathbb{F}_q t^i)E_{1,2}$ where T is the group of all diagonal matrices in $SL_2(\mathbb{F}_q)$, and $E_{1,2}$ is an elementary matrix, as in the previous chapter. For all $i \geq n$, let $H_i = H_{n-1}$. Finally, let $H = PGL_2(R_g) = GL_2(R_g)/Z$, where $Z = \mathbb{F}_q^\times I_2$ is the center of $GL_2(R_g)$. The vertices of Morgenstern's graph X'_g are all cosets of $\overline{H_i} = H_i Z/Z$ in $\mathrm{PGL}_2(R_g)$. There is an edge between two vertices $h\overline{H_i}$ and $h\overline{H_{i+1}}$ exactly when these cosets intersect nontrivially.

If we let $\overline{L_i} = PGL_2(R_g)/\overline{H_i}$ for each $i \geq 0$, then Morgenstern's description of the graph X'_g is nearly identical to our description of the ramified covering X_g in the previous chapter. In particular, the number of vertices in each graph is the same as $|\overline{L_i}| = |L_i| = |SL_2(R_g)|/|H_i|$. So X'_g has a core graph with the same number of vertices as for X_g and $|\overline{L_n}| = |L_n|$ infinite rays are attached to this core.

Moreover the structure of the core graphs of X_g and X'_g are similar. That is X'_g is a bipartite graph where the subgraph induced by $\overline{L_0}$ and $\overline{L_1}$ is a $(|\overline{H_0}|/|\overline{H_0} \cap$

$\overline{H_1}|, |\overline{H_1}|/|\overline{H_0} \cap \overline{H_1}|) = (q+1, q)$-regular, bipartite graph. Then, for $i = 1, ..., n-1$, each vertex in $\overline{L_i}$ has degree $q+1$ with $|H_i|/|H_i \cap H_{i-1}| = q$ edges incident to vertices in $\overline{L_{i-1}}$ and only $|\overline{H_i}|/|\overline{H_i} \cap \overline{H_{i+1}}| = 1$ edge incident to a vertex in $\overline{L_{i+1}}$. Thus the graph "collapses" in a q-fold manner until it reaches the level $\overline{L_{n-1}}$. At this point the graph becomes a collection of disjoint rays, one ray for each vertex in $\overline{L_{n-1}}$.

Despite the similarity in structure, the graphs X_g and X'_g are not always isomorphic, as we will see in the next section. This is ultimately a consequence of the fact that Morgenstern fails to produce a valid ramified covering, at least in some cases. Though he claims that $X'_g \cong \Gamma(g)\backslash X$, this is not necessarily the case.

Theorem 9.1. *Let $\Gamma = PGL(\mathbb{F}_q[t])$ and $\Gamma(g)$ be the congruence subgroup with respect to a polynomial $g \in \mathbb{F}_q[t]$. Let X_g and X'_g be the coset graphs described above. Then there exist a prime power q and a polynomial $g \in \mathbb{F}_q[t]$ such that $X'_g \not\cong \Gamma(g)\backslash X$ and $X'_g \not\cong X_g$.*

The original proof shown here is simply a counter-example obtained by brute force computations of coset intersections.

Proof. Let $g = t^2$ and note that the graph X_g is connected if and only if the subgraph induced by the vertices in L_0 and L_1 is connected. It is straightforward to check that for $q = 2$, the resulting $(3, 2)$-biregular graph on $(8, 12)$ vertices has two connected components. Since the quotient graph $\Gamma(g)\backslash X$ must be connected, as is X_g (see Theorem 8.5), the proof is complete. □

The hand-computed counter-example in the proof above is not an isolated incident. Further brute-force computations performed with Magma show that the graph X'_g is disconnected in a number of cases, including the following:

1. $q = 2$ and $g = t^n$, $2 \leq n \leq 26$;

2. $q = 4$ and $g = t^n$, $2 \leq n \leq 13$;

3. $q = 8$ and $g = t^n$, $2 \leq n \leq 7$;

4. $q = 16$ and $g = t^n$, $2 \leq n \leq 4$.

We can also point to independent results which give these counterexamples. Counterexamples for $q = 2$ and $n = 3, 4, 5, 6$ were found independently by Ortwin Scheja and Max Gebhardt. They used the computer algebra system SIMATH.

One easy observation is that all the disconnected examples occur over a field of characteristic 2. This led to the following results, whose proofs are given in our joint work [CCM] and may be attributed to Scott Murray.

Proposition 9.2. *Morgenstern's graph X'_g is connected if, and only if, every element of $R^\times$ is an $\mathbb{F}$-linear combination of elements of $R^{\times 2}$.*

Corollary 9.3. *X'_{t^n} is connected if, and only if, q is odd or $n = 1$.*

The results above contradict Morgenstern's claim that X'_g is the quotient graph for the congruence subgroup $\Gamma(g)$, as well as some of his connectedness results, which we will discuss in the next section. These contradictions, and in particular the failure of X'_g to behave as a ramified covering, ultimately stem from the incorrect identification $\Gamma/\Gamma(g) \cong PGL_2(R_g)$. The following gives the correct isomorphism.

Proposition 9.4. *Let $R = \mathbb{F}_q[t]/(g)$. For each of the groups Γ below, let $\Gamma(g) = \{ A \in \Gamma \ : \ A \equiv I_2 \bmod (g) \}$ be the congruence subgroup mod g.*

1. *If $\Gamma = \mathrm{SL}_2(\mathbb{F}_q[t])$, then $\Gamma/\Gamma(g) \cong \mathrm{SL}_2(R)$.*
2. *If $\Gamma = \mathrm{GL}_2(\mathbb{F}_q[t])$, then $\Gamma/\Gamma(g) \cong \mathrm{SL}_2(R) \rtimes (\mathbb{F}_q^\times \oplus (1))$.*
3. *If $\Gamma = \mathrm{PGL}_2(\mathbb{F}_q[t])$, then $\Gamma/\Gamma(g) \cong (\mathrm{SL}_2(R) \rtimes (\mathbb{F}_q^\times \oplus (1)))/\mathbb{F}_q^\times I_2$.*

Proof. (written with Scott Murray)

The first isomorphism (1) follows from Proposition 8.2.

For the second isomorphism, first decompose the group of invertible matrices

$$\mathrm{GL}_2(\mathbb{F}_q[t]) = \mathrm{SL}_2(\mathbb{F}_q[t]) \rtimes (\mathbb{F}_q^\times \oplus (1)).$$

Since $\Gamma(g) = \{ A \ : det(G) \in \mathbb{F}_q^\times \ A \equiv I_2 \bmod (g) \} \leq \mathrm{SL}_2(\mathbb{F}_q[t])$, it follows that

$$\Gamma/\Gamma(g) = (\mathrm{SL}_2(\mathbb{F}_q[t])/\Gamma(g)) \rtimes (\mathbb{F}_q^\times \oplus (1)) \cong SL_2(R) \rtimes (\mathbb{F}_q^\times \oplus (1)).$$

The third isomorphism follows from the second by first taking the pre-images of the projective matrices in $GL_2(\mathbb{F}_q[t])$.

$$\Gamma/\Gamma(g) \;\cong\; \mathrm{GL}_2(\mathbb{F}_q[t])/\{A \in GL_2(\mathbb{F}_q[t]) \;:\; A \equiv \lambda I_2 \bmod (g) \text{ for some } \lambda \in \mathbb{F}_q^\times\}.$$

$$= \mathrm{GL}_2(\mathbb{F}_q[t])/\mathbb{F}_q^\times\{A \in GL_2(\mathbb{F}_q[t]) \;:\; A \equiv I_2 \bmod (g)\}.$$

$$= \;(\mathrm{SL}_2(R) \rtimes (\mathbb{F}_q^\times \oplus (1))/\mathbb{F}_q^\times I_2.$$

□

9.2 Subgraphs of levels $0, 1$

Let $\Gamma = PGL(\mathbb{F}_q[t])$, $g \in \mathbb{F}_q[t]$ a polynomial of degree n, $\Gamma(g)$ a congruence subgroup, and X_g' the graph of Morgenstern described in the previous section. Morgenstern constructed X_g' as a means of providing examples of linear families of bounded concentrators. These examples were obtained as subgraphs of X_g', namely given g, let $D_g(0-1)$ be the subgraph induced by the vertices in the first two levels L_0 and L_1. However, a necessary property for a bounded concentrator is connectedness. Since we showed that X_g' is disconnected in characteristic 2, it follows that the subgraphs $D_g(0-1)$ must also be disconnected.

Corollary 9.5. *(to Proposition 9.2) If q is even, then $D_g(0-1)$ is disconnected.*

Proof. In any level L_i with $i \geq 1$, every vertex has a neighbor in L_{i-1}. Thus any vertex in level i is connected via a path to a vertex in level L_1. If $D_g(0-1)$ is connected, then any two vertices in L_1 are also connected by a path. Thus any two vertices in the graph may be connected via vertices in L_1. □

This contradicts the following claim of Morgenstern.

Proposition 9.6 (Proposition 4.2, M). *If $q \geq 4$, or $q = 3$ and $g(x)$ is irreducible of degree greater than 2, then $D_g(0-1)$ is connected.*

We stated that the full graphs X_g' are connected in odd characteristic in the previous section. What remains to be seen is whether or not the subgraphs $D_g(0-1)$ are connected in odd characteristic and if they have the claimed expansion properties. This is beyond the scope of the current work, but we do wish to give a few remarks regarding a result of Morgenstern's which underlies his results on connectedness and the expansion properties of $D_g(0-1)$.

The chief idea in the proof of Proposition 4.2 [M] is a lower bound for $N_0(S)$, the set of vertices in L_0 which are adjacent to a subset $S \subseteq L_1$ of vertices in L_1. We give Morgenstern's lemma here.

Lemma 9.7 (M, Lemma 4.1). *For every* $S \subseteq L_1$, $\frac{|N_0(S)|}{|S|} \geq \frac{q|L_1|}{(q-3)|S|+4|L_1|}$.

We tested this lower bound on one set of vertices $S \subseteq L_1$ corresponding to a connected component of our example when $q = 4,\ n = 2$. The bound failed (our size of L_1 matches the formula given by Morgenstern). We speculate that this bound will fail in general for characteristic 2. It is interesting to note that we first tried a random search of the subsets of L_1 but did not find a counterexample. It is possible that the bound holds for most subsets of L_1. We do not know what happens to this bound in odd characteristic.

When Morgenstern's fundamental domains for congruence subgroups are disconnected, all connected components of the fundamental domain are isomorphic. Moreover, there is a group acting freely by permuting the components. This follows from a general property of coset graphs: Since H acts transitively of the cosets of H_0, there must be an element of H taking any component to any other component, and so the components must be isomorphic.

9.3 Fundamental domains in PGL_2 as ramified coverings

Let $\Gamma = PGL(\mathbb{F}_q[t])$, $g \in \mathbb{F}_q[t]$ a polynomial of degree n, and $\Gamma(g)$ a congruence subgroup. We explained in Section 9.1 that the discrepancy in Morgenstern's construction as a ramified covering (and therefore a quotient graph) is due to an incorrect computation of $\Gamma/\Gamma(g)$. We offered the following correction in Proposition 9.4:

$$\Gamma/\Gamma(g) \;\cong\; (\mathrm{SL}_2(R) \;\rtimes\; (\mathbb{F}_q^\times \oplus (1)))/\mathbb{F}_q^\times I_2.$$

This correction, when combined with Morgenstern's construction, gives a valid quotient graph $\overline{X_g} = \Gamma(g)\backslash X$. Moreover, this quotient graph is isomorphic to that for the corresponding SL_2 congruence subgroup.

Theorem 9.8. *The ramified coverings X_g and $\overline{X_g}$ are isomorphic. In particular, $\overline{X_g}$ is connected.*

We thus provide new families of subgraphs which potentially have the expansion properties claimed by Morgenstern, though we have not verified this.

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Vita

Ila Leigh Cobbs

2009	Ph. D. in Mathematics, Rutgers University, New Brunswick, NJ
1999-2003	B. Sc. in Applied Mathematics from Texas A& M University, College Station, TX
1999	Graduated from Bosqueville High School, Waco, TX.
2003-2009	Teaching assistant, Department of Mathematics, Rutgers University