A GENERALIZED HYBRID FUZZY-BAYESIAN METHODOLOGY FOR MODELING COMPLEX UNCERTAINTY

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ABSTRACT OF THE DISSERTATION

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Due to its well understood nature and its ability to model many phenomena in the physical world extremely well, probability theory is the method of choice for dealing with uncertainty in many science and engineering disciplines. However, as a tool for building representative models of complex real world systems, probability theory has a rather recent history which starts with the introduction of Bayesian Networks (BN).

Broadly construed, the BN model of a system is the compact representation of a joint probability distribution of the variables comprising the system. Many complex real-world systems are naturally represented by hybrid models which contain both discrete and continuous variables. However, when it comes to modeling uncertainty and to performing probabilistic inferencing about hybrid systems, what BNs have to offer is quite limited. Although exact inferencing in BNs composed only of discrete variables is well understood, no exact inferencing algorithms exist for general hybrid BNs.

In this thesis we concentrate on the problem of inferencing in Hybrid Bayesian Networks (HBNs). Our focus, hence our contributions are three-fold: theoretical, algorithmic and practical. From a theoretical point of view, we provide a novel framework to implement a hybrid methodology that complements probability theory with Fuzzy Sets to perform exact inferencing with general Hybrid Bayesian Networks that is composed of both discrete and continuous variables with no graph-structural restrictions to model uncertainty in complex systems. From an algorithmic perspective, we provide a suite of inferencing algorithms for general Hybrid Bayesian Networks. The suite includes two new inferencing algorithms for the two different types of Fuzzy-Bayesian Networks introduced in this study. Finally, from a practical perspective, we apply our framework, methodology, and techniques to the task of assessing system safety risk due to the introduction of emergent Unmanned Aircraft Systems (UASs) into the National Airspace System (NAS).

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Chapter 1

Introduction

Modeling complex systems is a very broad area of research where, more often than not, a multi-disciplinary approach is needed to achieve a meaningful representation of the subject matter. The analytical methods employed along the process remain as much art as science, especially, if the subject matter is safety and risk analysis of a complex system.

One aspect that particularly increases the complexity of modeling many real-world systems is the fact that they naturally include both discrete and continuous variables. We can further argue that because of the hybrid nature of real-world systems, many of them can best be modeled as hybrid stochastic processes, i.e., stochastic processes that contain both discrete and continuous variables. Due to their hybrid nature, such stochastic processes can be used in a wide variety of problem domains, such as fault diagnostics of complex machinery, pattern recognition, and risk analysis of complex systems. Although the problem domains are different, the task asked of the model is to perform probabilistic inference, such as to determine the probability of system failure given the malfunction of certain components of the machinery, to calculate the probability that a certain word is pronounced given the readings by the microphone, or to determine the likelihood that a mishap occurs given a set of precursors.

Within this context, in order to perform these tasks, an intelligent agent should be able to perform reasoning under uncertainty. As the most complex of intelligent agents, humans certainly can perform a complex reasoning task given little or no information regarding the situation they are in. The ultimate goal of a designer of an intelligent system is to mimic the human reasoning process under uncertainty and enhance it with the help of the infallible memory and unrivaled computational skills of computers.

The method of choice by the engineering and academic communities to deal with uncertainty in real-world applications is probability theory. Probability theory is a well-established area of study with an extensive historical background of successfully understanding randomness in natural phenomena. However, its application as a tool to model uncertainty in complex real-world systems is quite recent. In particular, its use as a modeling tool started with *Bayesian Networks (BNs)* in the late eighties following the introduction of the concept by Pearl [1]. In a nutshell, Bayesian Networks are *directed acyclic graphs (DAGs)* representing a probability distribution over this graphical structure. The DAG of a Bayesian Network is composed of *nodes* representing the variables in the domain of interest and *directed links* representing the conditional relations among the variables. Furthermore, each node is denoted by a *conditional probability distribution (CPD)* imposed by its parentage.

As popular tools for modeling uncertainty, Bayesian Networks are used in a variety of complex problem domains, such as troubleshooting for MS Windows, junk-email filtering, medical diagnosis, and safety risk assessment in aviation.

There are two aspects of using Bayesian Networks to model uncertainty in complex systems. First is the representation of the problem domain and second is the inferencing within the resulting graphical structure. As one might expect, the majority of the research on Bayesian Networks focused on solving the inferencing problem. The research on the inferencing aspect can be further divided into two sub-categories: inferencing in discrete only Bayesian Networks and inferencing for Hybrid Bayesian Networks, which include both discrete and continuous variables.

The problem of inferencing in discrete Bayesian Networks is fairly well understood and the overwhelming majority of existing studies either are based solely or focus mainly on discrete BNs. After the introduction of Bayesian Networks by Pearl, Lauritzen and Spiegelhalter proposed an exact inferencing algorithm for discrete BNs [2]. By exact inferencing, we mean that the inferencing algorithm results in exact answers to the probabilistic query given the graphical structure and CPDs of the BN. By now we have a few exact inferencing algorithms for discrete BNs and furthermore, we have a good understanding of the computational complexity of exact inferencing and how it relates to the graphical structure of the

BN. Particularly, the existing exact inferencing algorithms can be very efficient for small discrete BNs.

Notwithstanding the accumulated knowledge on exact inferencing and its wide acceptance on various problem domains, the discrete BNs are not always adequate, since many real-world systems are not entirely composed of discrete variables. For example, consider the complex problem of assessing the safety risk associated with operating unmanned aircraft systems (UAS) in the airspace over a populated area. A Bayesian Network model of the system may include *flight-hours, altitude, speed,* and *fuel on board*, as model variables, none of which could easily be represented by discretization without sacrificing some of the representative power of the network. However, when employing BNs, crude discretization of continuous variables is commonly used to perform exact inferencing on the system model.

We understand the need for discretization of continuous variables especially in BNs where the lack of hard data forces the analysts to resort to expert judgment to quantify the model. It is quite hard, if not impossible to generate continuous conditional distributions when the distributions are required to be constructed by subject matter expert input only. However, we further argue that using simple discretization of a problem domain to be able to perform exact inferencing is equivalent to approximate reasoning and in most cases, lead to unreliable results. Consider the variable *airspeed*, which is inherently a continuous entity. Now, for the sake of computational simplicity and exact inferencing, the analyst may choose to treat it as a discrete variable with three mutually exclusive states: *slow, medium*, and *fast*. Further assume that the crisp boundary between the states *slow* and *medium* is defined by $\leq$ 80 knots and $>$ 80 knots and we observe a reading from the sensors on board the UAS that it is cruising at 85 knots. According to our predetermined mutually exclusive three-state discretization scheme, we were observing a *medium* airspeed and perform the exact inferencing accordingly. However, one could argue that even though, the states *slow* and *medium* are different, the actual observation about the airspeed is so close to the crisp boundary separating the two states that any inferencing using this discretization scheme is fundamentally flawed to produce meaningful results.

Hybrid Bayesian Networks (HBNs), which include both continuous and discrete variables, are a generalization on discrete only Bayesian Nets. HBNs are inherently more suitable for modeling complex systems, such as visual target tracking where the variables defining location of the target and its speed are inherently continuous and speech recognition where the the bits and pieces of processed audio signals are often continuous. In this study we shall motivate our research using the problem domain of safety risk assessment in complex systems such as Unmanned Aircraft Systems.

HBNs as the generalization of discrete BNs have their own shortcomings that arise when we would like to perform exact inferencing on a general HBN. Exact inferencing on general HBNs imposes restrictions on the network structure of the HBN. The state of the art exact inferencing algorithm for HBNs, the *Lauritzen algorithm*, requires that the network satisfies the constraint that *no continuous variables have discrete children* [4]. As one would expect, this restriction places quite a burden on the *generalization* claim of the HBNs. We propose an approach which, using Fuzzy Set theory, builds on the Lauritzen algorithm to generate a *hybrid* exact inferencing algorithm for *general* HBNs.

Fuzzy Set theory, introduced by Zadeh in the late sixties [17], proposes a framework to deal with a poorly defined concept in a coherent and structured way. Examples of poorly defined concepts suitable for the application of Fuzzy logic are semantic variables, such as *heavy workload, inadequate training, fast, slow, tall, short*, etc. Within the context of our current research, Fuzzy Sets present two important application domains worthwhile for further exploration: First, Fuzzy sets provide a complete set of tools to partition continuous domains into overlapping membership regions, which result in a much more realistic discretization of the continuous domain in question and second, uncertainty regarding any empirical observation can be represented as a Fuzzy measure.

Previously, we stated that Bayesian Networks are tools to model uncertainty in the form of a probability distribution imposed by a directed acyclic graph representing the domain of interest. Hence, BNs only address the uncertainty in the form of randomness about a problem domain. However, as we elaborate in Section 3.3, uncertainty in a typical real-world application has three dimensions: *vagueness, ambiguity,* and *randomness* [29] and BNs, being solidly anchored to probability theory, only address one of its dimensions, namely randomness. For instance, consider that there is ambiguity regarding the observed evidence associated with some variable in a given Bayesian Network. We believe that Fuzzy

Set theory offers a comprehensive structure to introduce another dimension of uncertainty, namely ambiguity, to the existing framework of the classical Bayesian Networks.

1.1 Research Objective

In this thesis, we study the problem of inferencing in general Hybrid Bayesian Networks, within the context of uncertainty analysis in real-world complex systems. In particular, considering the hybrid nature of real-world systems, which include continuous and discrete variables and the lack of practical algorithmic solutions for general BNs to perform probabilistic reasoning about hybrid systems, we concentrate on the problem of exact inferencing in general HBNs with no conditional restrictions between continuous and discrete variables.

In this study, our focus, hence our contributions are three-fold: theoretical, algorithmic and practical.

From a theoretical point of view, we provide a novel framework to implement a hybrid methodology that complements probability theory with Fuzzy Sets to perform exact inferencing with general Hybrid Bayesian Networks that is composed of both discrete and continuous variables with no graph-structural restrictions to model uncertainty in complex systems.

From an algorithmic perspective, we provide a suite of inferencing algorithms for general Hybrid Bayesian Networks. The suite includes two new inferencing algorithms for the two different types of Fuzzy-Bayesian Networks introduced in this study.

Finally, from a practical perspective, we apply our framework, methodology, and techniques to the task of assessing system safety risk due to the introduction of emergent Unmanned Aircraft Systems (UASs) into the National Airspace System (NAS).

1.2 Outline

As a thesis dissertation this text is composed of five chapters: *Introduction, Bayesian Networks and Fuzzy Sets, Fuzzy-Bayesian Networks*, *Application of Research Methodology*, and *Conclusion and Future Work*.

Chapter 2 titled as *Bayesian Networks and Fuzzy Sets* could also be considered as a literature survey, where, following a brief background on basic concepts, we elaborate on selected important advanced research papers and how they relate to our proposed framework. In Chapter 3 we, first, develop a new formulation for Fuzzy probability followed by a new Fuzzy-Bayes formula. Next, we introduce the concept of Fuzzy evidence and develop a methodology to update probability distributions given Fuzzy evidence. Consequently, in Chapter 3, we introduce Fuzzy-Bayesian networks (FBNs) and propose two different types of conversions from a general Hybrid Bayesian Network to a FBN and present two different inferencing algorithms for *Type-I* and *Type-II* FBNs. In Chapter 4, we introduce a complex real-world problem domain: Unmanned Aircraft Systems (UAS). We discuss various issues that arise when trying to model system safety risk associated with such a complex problem domain as a Hybrid Bayesian Network. We present a novel methodology based on a hazard-source taxonomy and a regulatory based framework to model the UAS domain risk. We then apply the Fuzzy-Bayesian framework presented earlier in the thesis to assess the system safety risk due to the introduction of UAS into the National Airspace System. Finally, the last chapter contains concluding remarks including contributions to the field of study and some open research questions.

Chapter 2

Bayesian Networks and Fuzzy Sets

In this chapter, we present background material on Bayesian Networks and Fuzzy Sets to the extent that closely relates to our current research interests to provide some foundation for the subsequent advanced results presented in later chapters.

2.1 Bayesian Networks

This section contains background material on Bayesian Networks, which are, in simple terms, a compact way of representing joint probability distributions imposed by a Directed Acyclic Graph (DAG) network structure among a set of variables [1]. First we define Bayesian Networks and then provide an overview of some fundamental inference algorithms. We will introduce the Bayesian Networks using the discrete only case and then mainly focus on Hybrid Bayesian Nets, which are composed on both discrete and continuous variables.

2.1.1 Notation

First, we need to define the notation to be used throughout this chapter. There are three types of random variables that need to be identified to properly define a Fuzzy Bayesian Networks (FBN). These are, namely, discrete random variables, continuous random variables, and Fuzzy random variables. Traditionally, literature on Bayesian probability denotes discrete random variables by upper case letters from the beginning of the alphabet (e.g., A, B, A_i, B_{ij},...) and the values that these random variables take are denoted by corresponding lower case letters (e.g., a, b, a_i, b_{ij},...). The sets of these variables along with the frames that they are defined on are denoted by bold-face letters (e.g., $\mathbf{A}, \mathbf{B_i}, \mathbf{a_i}, \mathbf{b_{ij}}$,...). The notation used for continuous variables on the other hand, is similar with the only difference being that the letters used are selected from the end of the alphabet (e.g., $X, \mathbf{Y_i}, \mathbf{x}, \mathbf{y_{ij}}$).

While discussing Hybrid Bayesian Networks we denote the set of continuous variables by $\mathbf{\Delta}$, whereas $\mathbf{\Gamma}$ refers to the set of discrete variables in the network.

Fuzzy random variables, on the other hand, are denoted by using a hat over the corresponding random variable notation (e.g., $\hat{X}$, $\hat{\mathbf{Y}}$,...).

We will use the notation $P(X)$ to denote the *probability distribution* for variable X. The probability distribution $P(\cdot)$ refers to a *mass function* for a discrete variable and a *density function* for a continuous variable.

2.1.2 Representation of a Bayesian Network

Very concisely, we can define a Bayesian Network as the compact representation of a joint probability distribution given a set of variables $\mathbf{V}$. A Bayesian Network is defined by the following two components:

- A directed acyclic graph $\mathcal{G}$ connecting each variables $V_i \in \mathbf{V}$ into a network structure.
- A collection of *conditional probability distributions (CPDs)*, where each node (i.e., variable) V_i in the graph $\mathcal{G}$ is denoted by a conditional distribution given its parent nodes $Par(V_i)$.

An important feature of Bayesian Networks, which makes inferencing possible, is the fact that *given its parents every node is conditionally independent of the nodes which are not among its descendants.* In other words, a Bayesian Network represents the joint probability distribution over its set of variables in terms of *conditional independencies.* Formally, variable V_i only depends on its parents $Par(V_i)$ and the CPD of variable V_i parametrizes this dependency.

The local Markov property of the Bayesian Network provides a formal definition for the joint probability distribution it represents. Via the *Chain Rule* we can define this joint distribution for a Bayesian Network with n variables as follows:

$$P(V_1, \ldots, V_n) = \prod_{i=1}^{n} P(V_i \mid Par(V_i)) \tag{2.1}$$

At this stage a brief note on the computational complexity in Bayesian Networks provides some early insight on inferencing. In discrete Bayesian Networks the CPD of a variable V_i

represents a multivariate discrete distribution that parametrizes all possible combinations of the discrete states of V_i's parents. In general, the number of terms we have to consider might be exponential in the size of the network in question. We further elaborate on this issue when we discuss inferencing in Bayesian Networks.

2.1.3 Exact Inferencing in Discrete BNs

By exact inferencing we mean that, given a query, an intelligent system produces exact results. In the case of Bayesian Networks, given a set of query variables $\mathbf{Q}$ and some evidence $\mathbf{E} = \mathbf{e}$, $\forall \mathbf{E} \subseteq \mathbf{V}$, an exact inferencing algorithm should produce exact numerical results for the probability distribution $P(\mathbf{Q}|\mathbf{E} = \mathbf{e})$. The good news is that there exist well understood exact inferencing algorithms for discrete Bayesian Networks. However, the bad news is that even for a moderate size Bayesian Network, the problem of exact inferencing is NP-hard [31], [32]. Still, in many cases we can take advantage of the structure of the BN to perform probabilistic inferencing efficiently. Next we briefly discuss two such methods.

Variable Elimination

At the crux of the probabilistic inferencing in a Bayesian Network lies the summing up of the relevant terms, each of which can be computed using the chain rule in equation (2.1). Consider the discrete Bayesian Network in Figure 2.1. As an example, we want to determine the marginal distribution for variable E. That is,

$$\begin{aligned} P(E) &= \sum_{a,b,c,d} P(b,a,c,d,E) \\ &= \sum_{a,b,c,d} P(b)P(a)P(d|a,b)P(c|a)P(E|d,c) \end{aligned} \tag{2.2}$$

We can think of equation (2.2) as composed of factors (i.e., functions) involving the variables a, b, c, d, and the variable E. In this context we can write the factor $f(E, d, c)$

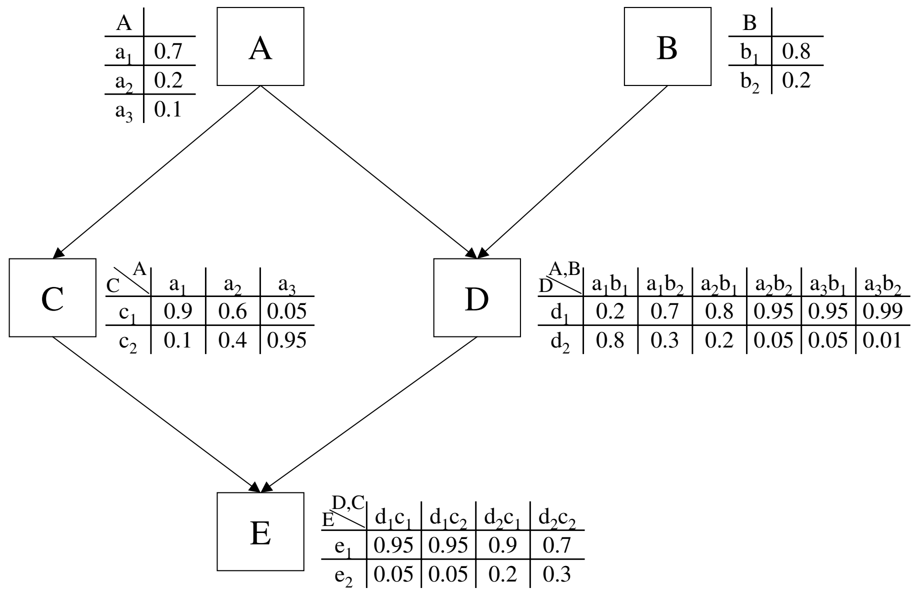


Figure 2.1: Example discrete Bayesian Network with CPDs

representing the term $P(E|d,c)$ as follows:

$$f_E(E,d,c) = \begin{array}{c|c|c|c} E & d & c & \\ \hline e_1 & d_1 & c_1 & 0.95 \\ e_1 & d_1 & c_2 & 0.95 \\ e_1 & d_2 & c_1 & 0.9 \\ e_1 & d_2 & c_2 & 0.7 \\ e_2 & d_1 & c_1 & 0.05 \\ e_2 & d_1 & c_2 & 0.05 \\ e_2 & d_2 & c_1 & 0.1 \\ e_2 & d_2 & c_2 & 0.3 \end{array} \tag{2.3}$$

Now, to compute the marginal probability $P(E)$ of equation (2.2) one needs to compute the following two operations:

- Multiplying the factors.
- Perform summations over the variables making up the factors.

The straightforward application of these two operations results in a factor over all the variables in the Bayesian Network. Note that the size of such a factor would be exponential in the number of variables and in larger Bayesian Networks its use would be clearly infeasible for all practical computational purposes.

However, if we reorganize the equation (2.2) by pushing the summations inside, such that

$$P(E) = \sum_{d,c} P(E|d,c) \sum_{a} P(a)P(c|a) \sum_{b} P(b)P(d|a,b) \tag{2.4}$$

we obtain a much more tractable form which allows practical inferencing in moderate sized Bayesian Networks.

To compute this new arrangement let us start with the terms of the innermost summation: $P(b)$ and $P(d|a,b)$. The corresponding factors are:

$$f_b(b) = \begin{array}{c|c} b & \\ \hline b_1 & 0.8 \\ b_2 & 0.2 \end{array} \quad , \quad f_d(d,a,b) = \begin{array}{c|c|c|c} d & a & b & \\ \hline d_1 & a_1 & b_1 & 0.2 \\ d_1 & a_1 & b_2 & 0.7 \\ d_1 & a_2 & b_1 & 0.8 \\ d_1 & a_2 & b_2 & 0.95 \\ d_1 & a_3 & b_1 & 0.95 \\ d_1 & a_3 & b_2 & 0.99 \\ d_2 & a_1 & b_1 & 0.8 \\ d_2 & a_1 & b_2 & 0.3 \\ d_2 & a_2 & b_1 & 0.2 \\ d_2 & a_2 & b_2 & 0.05 \\ d_2 & a_3 & b_1 & 0.05 \\ d_2 & a_3 & b_2 & 0.01 \end{array} \tag{2.5}$$

The multiplication of the factors in (2.5) results in the following factor

$$f_1(d,a,b) = \begin{array}{c|c|c|c} d & a & b & \\ \hline d_1 & a_1 & b_1 & 0.16 \\ d_1 & a_1 & b_2 & 0.14 \\ d_1 & a_2 & b_1 & 0.64 \\ d_1 & a_2 & b_2 & 0.19 \\ d_1 & a_3 & b_1 & 0.76 \\ d_1 & a_3 & b_2 & 0.198 \\ d_2 & a_1 & b_1 & 0.64 \\ d_2 & a_1 & b_2 & 0.06 \\ d_2 & a_2 & b_1 & 0.16 \\ d_2 & a_2 & b_2 & 0.01 \\ d_2 & a_3 & b_1 & 0.04 \\ d_2 & a_3 & b_2 & 0.002 \end{array} \tag{2.6}$$

Now we will evaluate the factor representing the summation $\sum_b P(b)P(d|a,b)$ which is equivalent to the summation $\sum_b f_1(d,a,b)$. The resulting factor is:

$$f_2(d,a) = \begin{array}{c|c|c} d & a & \\ \hline d_1 & a_1 & 0.3 \\ d_1 & a_2 & 0.83 \\ d_1 & a_3 & 0.958 \\ d_2 & a_1 & 0.7 \\ d_2 & a_2 & 0.17 \\ d_2 & a_3 & 0.042 \end{array} \tag{2.7}$$

With the construction of factor $f_2(d,a)$ we can rewrite the equation (2.4) as follows:

$$P(E) = \sum_{d,c} P(E|d,c) \sum_a P(a)P(c|a)f_2(d,a) \tag{2.8}$$

The important observation is that we eliminated variable b, which will not appear in any of our factors from now on. If we continue in this fashion first, we convert the terms $P(a)$ and $P(c|a)$ into their respective factors and then multiply them together with the factor $f_2(d,a)$

to obtain the factor $f_3(c,d,a)$, which is now the only factor in the innermost summation:

$$P(E) = \sum_{d,c} P(E|d,c) \sum_{a} f_3(c,d,a) \tag{2.9}$$

When we perform the summation over variable a to obtain the factor $f_4(c,d)$, thereby eliminating the variable a from our considerations:

$$P(E) = \sum_{d,c} P(E|d,c) f_4(c,d) \tag{2.10}$$

The next step is to convert the CPD $P(E|d,c)$ to its corresponding factor $f_E(E,d,c)$ and multiply it with $f_4(c,d)$.

$$\begin{aligned} P(E) &= \sum_{d,c} f_E(E,d,c) f_4(c,d) \\ &= \sum_{d,c} f_5(E,d,c) \end{aligned} \tag{2.11}$$

Consecutively, we sum out the variables d, c from $f_5(E,d,c)$ to obtain the factor $f_6(E)$:

$$f_6(E) = \begin{array}{c|l} E & \\ \hline e_1 & 0.9103 \\ e_2 & 0.0897 \end{array} \tag{2.12}$$

Note that the individual terms in (2.12) sum up to unity which confirms that the resulting factor is in fact a proper distribution representing the marginal probability distribution for variable E.

In the steps above where we perform successive variable eliminations the largest factor that we needed to deal with has only 3 variables and 12 entries as compared to the original equation (2.2) (i.e., the chain rule for the example) and its factor which has 5 variables and 48 entries. The efficiency of the variable elimination algorithm becomes more apparent when the Bayesian Network in question becomes larger.

The size of the factors generated during the process determines the complexity of the variable elimination algorithm. The key observation is that this depends on the order of elimination we choose. For example, we could start with eliminating the variable d which would lead to a factor of four variables with 24 entries, which is much larger than we had to deal with in the example above. Intuitively, we can argue that a simple greedy heuristic would be enough to determine the right order of elimination. This certainly helps [6], [8]. However the problem of determining the optimal elimination order, which minimizes the size of the maximum factor is NP-hard [9]. Additionally, in some moderate sized Bayesian Networks even the optimum elimination order results in factors that are exponential in the size of the networks. For such cases we cannot perform exact inferencing. Regardless of this, the variable elimination algorithm and the *junction tree algorithm* based on it provide the best known method for exact probabilistic inferencing in many non-trivial sized discrete Bayesian Networks.

Variable elimination can easily be adopted to handle the inclusion of an observation about one or more variables to update the inferencing results. In Bayesian Network jargon, this type of observation is often referred to as *evidence*. Handling evidence can be performed simply by representing the observed variables by indicator functions. More formally, consider that the variable X is observed to be in state x_1. We can represent this observation by the indicator function $\mathbf{I}_{x_1}(X)$, which has the entry 1 for the observed state x_1 and 0 for the rest of the states that the variable X may take. Now consider, we observe that the variable C in our example is in state c_1. Then, for example, the corresponding marginal probability distribution $P(E)$ when $C = c_1$ is observed can be computed by:

$$\begin{aligned} P(E, C = c_1) &= \sum_{a,b,c,d} P(a, b, c, d, E) \cdot \mathbf{I}_{c_1}(c) \\ &= \sum_{a,b,c,d} P(b)P(a)P(d|a, b)P(c|a)P(E|d, c) \cdot \mathbf{I}_{c_1}(c) \end{aligned} \tag{2.13}$$

where

$$\mathbf{I}_{c_1}(C) = \begin{cases} 1 & \text{if} \quad C = c_1 \\ 0 & \text{Otherwise} \end{cases}$$

We thus eliminate all the terms in equation (2.13) that are not consistent with the evidence. The actual factor representing the evidence that will be used in the variable elimination algorithm is

$$f_{Evidence}(C) = \begin{array}{c|c} C & \\ \hline c_1 & 1 \\ c_2 & 0 \end{array}$$

which at the end of the computations leads to the marginal distribution for the variable E:

$$f_6(E) = \begin{array}{c|l} E & \\ \hline e_1 & 0.216 \\ e_2 & 0.029 \end{array}$$

However note that the factor $f_6(E)$ is not a proper probability distribution since it does not add up to unity but rather to 0.245. What it actually represents is the probability $P(C = c_1)$. So, if we normalize $f_6(E)$ by dividing it by 0.245, we obtain the actual probability distribution $P(E, C = c_1)$ that we would like to determine:

$$P(E, C = c_1) = \begin{array}{c|l} E & \\ \hline e_1 & 0.882 \\ e_2 & 0.118 \end{array}$$

Junction Trees

The Junction Tree algorithm [2, 7, 8] is basically a more structured way of performing variable elimination on Bayesian Networks. The junction tree algorithm, also known as the *clique tree*, the *cluster tree*, and the *joint tree* algorithm has two main advantages over basic variable elimination:

- To compute more than one marginal distributions, the variable elimination algorithm needs to be run multiple times. On the other hand, using the junction tree algorithm, one can compute multiple marginals performing no more than twice as many operations as needed by the variable elimination for computing only one marginal [10].
- The clustering of variables in the junction tree algorithm leads to much efficient implementation of the variable implementation.

A junction tree can be considered as a graphical structure imposed by the variable elimination algorithm. Each node in the junction tree represents an operation of multiplying the factors together and is related to the variables comprising the factors. Each link connecting adjacent *cliques* (i.e., the nodes) in the junction tree indicates the result of a summation over some variables (we call this resulting set of variables the *separators*). A separator is composed of variables that two adjacent nodes share.

The junction tree induced by the variable elimination steps that we have taken in the preceding example is provided in Figure 2.2.

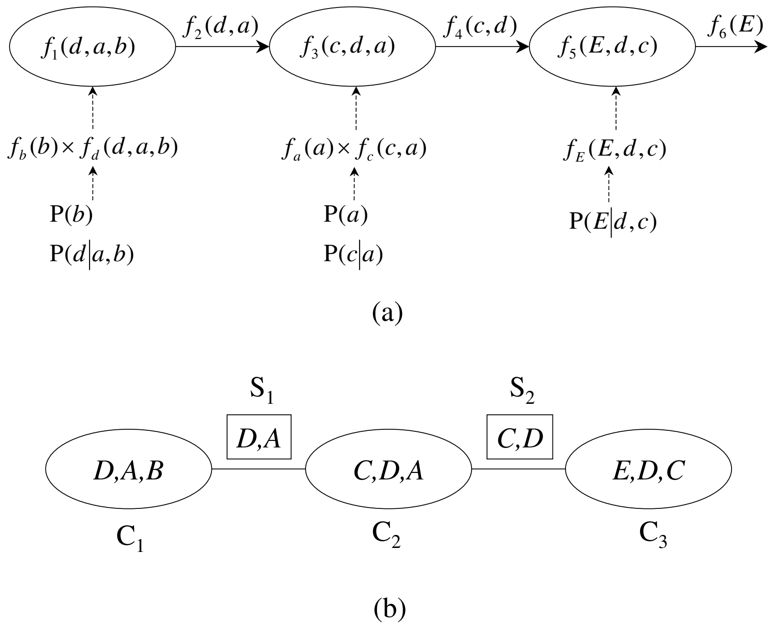


Figure 2.2: (a) The steps of the variable elimination algorithm applied to our example. (b) The junction tree induced by the variable elimination

Now we can formally define a junction tree.

Definition 1 *Consider the Bayesian Network $\mathcal{G}_{BN}$ with the random variable set $\mathbf{X}$. The junction tree over the $\mathcal{G}_{BN}$ is an undirected acyclic graph $\mathcal{T}$ composed of the vertex set $\mathbf{V}$ and the edge set $\mathbf{E}$. Each vertex (i.e. node) $C_i \in \mathbf{V}$ is associated with some set of random variables $\mathbf{X}_i \subseteq \mathbf{X}$. This association generates a clustering of variables in each vertex C_i and is called a clique. Each edge $S_i \in \mathbf{E}$ is also associated with an nonempty set variables $\mathbf{Y}_i = C_i \cap C_{i+1}$ that two adjacent cliques C_i and C_{i+1} share. The set of variables $\mathbf{Y}_i$ is called separators. We associate factors $\Phi(C_i)$ and $\Phi(S_i)$ to each clique C_i and separator S_i respectively. These factors are called potentials. Two properties further define a junction tree:*

- *For every CPD over variable set $\mathbf{Z}$, $\forall \mathbf{Z} \subseteq \mathbf{X}$ in $\mathcal{G}_{BN}$ there exist a clique C_i such that*

$\mathbf{Z} \subseteq \mathbf{X}_i$.

- *The running intersection property: Consider two cliques C_i and C_j, such that variable X is in the set of variables they share, $X \in \mathbf{X}_i \cap \mathbf{X}_j$. Then all cliques situated topographically between C_i and C_j contain the variable X.*

The following algorithm describes the process of constructing a junction tree from a Bayesian Network.

The Junction Tree Clustering Algorithm:

1. **Moralize:** Replace all directed connections with undirected edges (i.e., remove the arrows). Marry all parent nodes without a direct connection between them by an edge (hence, the term *moralization*).

2. **Triangulate:** Add arcs to generate cycles so that no cycle is longer than three arcs.

3. **Identify maximal cliques:** Identify maximal cliques in the triangulated graph.

4. **Form the tree structure:** Connect the maximal cliques by undirected arcs to form a tree graph (i.e. construct an undirected acyclic graph).

5. **Identify the separators:** For each arc on the tree identify the variables that are shared by the two cliques connected by that particular arc.

Different triangulations as the result of step 2 lead to different clustering of variables, i.e., to different cliques. Although the problem of finding the optimal triangulation is NP-hard, we can use a simple heuristic based on greedy elimination [6]. In step 3, while connecting the cliques, we should do this in such a way that the *running intersection property* is preserved. The *maximal spanning tree* algorithm can be used for this purpose [33].

Figure 2.3 illustrates the necessary steps to construct a junction tree for a sample discrete Bayesian Network.

Lauritzen and Spiegelhalter introduced and refined the junction tree algorithm as an exact probabilistic inferencing algorithm for a discrete Bayesian Network [2]. Their approach has been simply the most popular methodology used in software-based tools for BN inferencing that relies on an *message passing* algorithm.

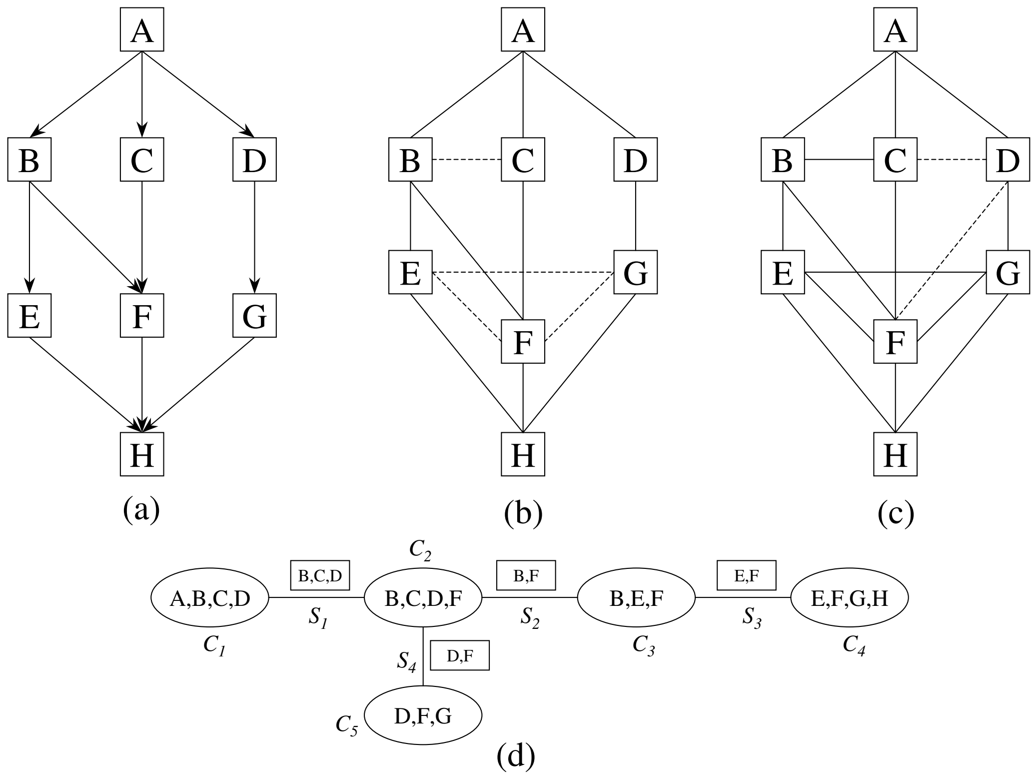


Figure 2.3: (a) A discrete Bayesian Network. (b) The moralized graph. (c) A triangulated graph. (d) A clique tree.

The message passing algorithm for a junction tree start with incorporating the CPDs into the clique, thereby initializing their potentials. This is done first, by converting the CPDs into factors as we demonstrated previously in our example for variable elimination. Second, we choose, for each factor, a clique C_i that contains all the variables in the factor. Next, for clique C_i we initiate its potential by multiplying together the factors that we associated with C_i in the previous step. In the case where no factor is associated with clique C_i we set all the terms in its potential to 1.

The variable elimination in junction trees is performed by passing messages along the tree starting with the branches and moving toward the root clique (in our example of Figure 2.3.d, C_2 is the root clique). In order for a clique to pass along a message to the next clique on the way to the root, it first needs to collect the messages coming from the branches traveling toward the root.

Table 2.1 shows the steps in the junction tree algorithm to compute the distribution for clique C_5 in Figure 2.3.d.

Table 2.1: Message passing along the junction tree to calculate the marginal distribution for clique C_5.

Step	From Clique	To Clique	Operation
0			Initialize The Potentials
1	C_1	C_2	$\sum_A \Phi(C_1) \rightarrow f_{12} \quad ; \quad f_{12} \cdot \Phi(C_2) \rightarrow \Phi(C_2)$
2	C_4	C_3	$\sum_{GH} \Phi(C_4) \rightarrow f_{43} \quad ; \quad f_{43} \cdot \Phi(C_3) \rightarrow \Phi(C_3)$
3	C_3	C_2	$\sum_E \Phi(C_3) \rightarrow f_{32} \quad ; \quad f_{32} \cdot \Phi(C_2) \rightarrow \Phi(C_2)$
4	C_2	C_5	$\sum_{BC} \Phi(C_2) \rightarrow f_{25} \quad ; \quad f_{25} \cdot \Phi(C_5) \rightarrow \Phi(C_5)$

The order of message passing indicated in Table 2.1 is one possible order. We could have started with step 2 but we could not have started with step 3 since we had to wait for the message coming from C_4.

We can use the results of Table 2.1 to accelerate the process while calculating the marginal distribution for other cliques. For example to calculate the distribution for clique C_2 we can reuse the information provided in steps 1, 2, and 3. However we cannot use the information provided in step 4 as is, because it already is multiplied by factor f_{25}. However, the good news is that we can update the message at the end of the step 4 (i.e., the potential $\Phi(C_5)$) by simply dividing it by f_{35} if only we have retained it along the process. We can consider this as resending a message backwards along a path in the junction tree. To achieve this we need to save the message sent toward the root clique C_2. This is done by storing the messages passed along the edges connecting adjacent cliques and the potentials of the separators associated with each edge is the medium to store the messages during each run of the message passing algorithm.

Finally we need to perform a calibration on the results for them to reflect marginal probability distributions for individual cliques [10].

Handling evidence in junction trees is performed in a similar way as in variable elimination. The only difference is instead of using the indicator function we convert the indicator function into a factor for the pertaining evidence variable and multiply it with the potentials of the cliques containing the evidence variable.

When the size of the discrete Bayesian Network becomes larger the junction tree algorithm becomes intractable and we need to resort to approximate methods to perform probabilistic inferencing. Sampling techniques, such as *importance sampling* and *Markov*

Chain, *Monte Carlo*, and *Gibbs sampling* are most popular approximate inferencing methods for discrete Bayesian Networks. However, in our study, we do not include the approximate reasoning in BNs.

2.1.4 General Bayesian Networks

So far our focus has been on Bayesian Networks composed of discrete random variables only and we referred to such BNs simply as discrete Bayesian Networks. In this section we briefly discuss some of the important literature on mixed (on sometimes referred to as *general* or *hybrid*) Bayesian Networks that incorporate both continuous and discrete random variables.

In general Bayesian Networks, each node could either be discrete or continuous, a real-valued scalar or vector. Exact inferencing algorithms exist when we represent the joint probability distribution over all the nodes as Conditional Gaussian (CG) [3, 4], i.e., for a given combination $\mathbf{i}$ for instantiated states of the discrete variables $\mathbf{\Delta}$, the distribution over the continuous nodes $\mathbf{\Gamma}$ has the following form:

$$f(\mathbf{\Gamma}|\mathbf{\Delta} = \mathbf{i}) = \mathcal{N}(\mathbf{\Gamma}; \boldsymbol{\mu}(\boldsymbol{i}), \boldsymbol{\Sigma}(\boldsymbol{i})) \tag{2.14}$$

where, $\mathcal{N}()$ represents a multivariate Gaussian distribution such as a multivariate Normal density function, $\boldsymbol{\mu}(\mathbf{i})$ a vector-valued entity, and $\mathbf{\Sigma}(\mathbf{i})$ a matrix-valued entity.

One should immediately note that the linear Gaussian model, i.e., the form in equation (2.14), for which exact inferencing methodologies exist excludes the cases where discrete nodes have a parent-set including continuous variables. This particular shortcoming provides the motivation for our proposed methodology that we introduce in Chapter 3.

[34] provides a good review on exact inference methodologies for the general Bayesian Networks represented by this linear Gaussian Model. For the remainder of this section we focus on the Lauritzen algorithm, which is the state-of-the-art algorithm for exact probabilistic inferencing in Bayesian Networks with discrete and continuous variables.

A computational method for exact local computations of mean and variances in Bayesian Networks modeled by Conditional Gaussian distributions has been developed by Lauritzen [4]. The model behind the computations of the Lauritzen's algorithm relies on the assumption that the conditional distribution of a continuous variable given a discrete variable is a multivariate Gaussian (i.e., the form in equation (2.14)).

Quickly reviewing, in the *univariate* case the normal distribution is defined by two variables the mean μ and the variance σ^2. Then for random variable X the density function has the form:

$$P(X) = \mathcal{N}(X; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

For the *multivariate* case the normal distribution is defined by parameters the mean *vector* and the covariance *matrix*. Formally, for a set of random variables $\mathbf{X}$, the multivariate Normal distribution is defined by

$$P(\mathbf{X}) = \mathcal{N}(\mathbf{X}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right) \tag{2.15}$$

Where, for a set of n random variables, $\boldsymbol{\mu}$ is a vector of size n and $\boldsymbol{\Sigma}$ is a matrix of size $n \times n$ and $|\boldsymbol{\Sigma}|$ is the cardinality of matrix $\boldsymbol{\Sigma}$.

A Bayesian Network can be written in the form of a multivariate Gaussian. Consider a joint normal distribution over $\{\mathbf{X}, \mathbf{Y}\}$, where $\mathbf{X} \in \mathbb{R}^n$ and $\mathbf{Y} \in \mathbb{R}^m$, we can use the following form to define it:

$$P(\mathbf{X}, \mathbf{Y}) = \mathcal{N}\left(\begin{pmatrix}\boldsymbol{\mu}_{\mathbf{X}} \\ \boldsymbol{\mu}_{\mathbf{Y}}\end{pmatrix}, \begin{bmatrix}\Sigma_{\mathbf{XX}} & \Sigma_{\mathbf{XY}} \\ \Sigma_{\mathbf{YX}} & \Sigma_{\mathbf{YY}}\end{bmatrix}\right) \tag{2.16}$$

where $\boldsymbol{\mu}_{\mathbf{X}} \in \mathbb{R}^n$ and $\boldsymbol{\mu}_{\mathbf{Y}} \in \mathbb{R}^m$ and the matrices $\Sigma_{\mathbf{XX}}$, $\Sigma_{\mathbf{XY}}$, $\Sigma_{\mathbf{YX}}$, and $\Sigma_{\mathbf{YY}}$ are of size $n \times n$, $n \times m$, $n \times m$, and $m \times m$, respectively. Next, we will use the following theorem and its corollary [5, 35, 10, 34] without a formal introduction or further detail to define the conditional distribution $P(\mathbf{Y}|\mathbf{X})$.

Theorem 2.1.1 *Let the joint distribution of variables* $\mathbf{X}$ *and* $\mathbf{Y}$ *be given by equation (2.16). Then, the conditional distribution* $P(\mathbf{Y}|\mathbf{X})$ *is a normal distribution*

$$\mathcal{N}(\mathbf{Y}; \boldsymbol{\mu}, \Sigma)$$

where,

$$\boldsymbol{\mu} = \boldsymbol{\mu}_{\mathbf{Y}} + \Sigma_{\mathbf{YX}}\Sigma_{\mathbf{XX}}^{-1}(\mathbf{x} - \boldsymbol{\mu}_{\mathbf{Y}})$$
$$\Sigma = \Sigma_{\mathbf{YY}} - \Sigma_{\mathbf{YX}}\Sigma_{\mathbf{XX}}^{-1}\Sigma_{\mathbf{XY}}$$

Corollary 2.1.2 *Consequently, when we let $|\mathbf{Y}| = 1$ we can write the conditional distribution $P(Y|\mathbf{X})$ as a normal distribution $\mathcal{N}(Y; \beta_0 + \beta^T\mathbf{X}, \sigma^2)$, where*

$$\beta_0 = \boldsymbol{\mu}_Y - \Sigma_{Y\mathbf{X}}\Sigma_{\mathbf{XX}}^{-1}\boldsymbol{\mu}_{\mathbf{X}}$$
$$\beta = \Sigma_{Y\mathbf{X}}\Sigma_{\mathbf{XX}}^{-1}$$
$$\sigma^2 = \Sigma_{YY} - \Sigma_{Y\mathbf{X}}\Sigma_{\mathbf{XX}}^{-1}\Sigma_{\mathbf{X}Y}$$

Now we can use Theorem 2.1.1 and its corollary to convert a multivariate Gaussian into a Bayesian Network. Consider an ordered series of n variables $X_1, X_2, \ldots, X_n$. Then, using Corollary 2.1.2 we can determine the conditional distribution of variable X_i given the variables up until and including the $(i-1)^{\text{th}}$ variable in the series, as follows:

$$P(X_i|X_1, \ldots, X_{i-1}) = \mathcal{N}\left(X_i; \beta_{i,0} + \sum_{j=1}^{i-1} \beta_{i,j}X_j, \sigma_i^2\right) \tag{2.17}$$

Equation 2.17 represents the simple Bayesian Network in Figure 2.4, where, for every $\beta_{i,j} \neq 0$, there exists a directed link from X_j to X_i such that $1 \leq j < i$.

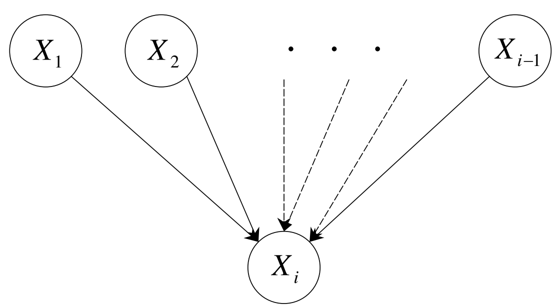


Figure 2.4: The Bayesian Network representing equation (2.17).

The conditional probability distribution (CPD) represented by equation (2.17) is called a *linear CPD*. A Bayesian Network where all the CPDs are linear is called a *linear Gaussian* [34]. Note that the above argument implies that every multivariate Gaussian can be written as a linear Gaussian, vice-versa is true, too.

To perform probabilistic inferencing on linear Gaussians we can use the variable elimination algorithm and hence the junction tree algorithm. However, since we are performing inferencing over continuous variables whose parents are a combination of discrete and continuous nodes we cannot simply represent the factors simply by tables as we did in sections

(2.1.3) and (2.1.3) for discrete only Bayesian Networks. Since the linear CPDs are conditional distributions and not Gaussians, we cannot represent the factors as Gaussians too.

Canonical characteristics (also know as *Canonical forms*) are introduced to deal with this problem [4, 35]. Within the context of representing linear Gaussians, canonical characteristics represent a function of the form $e^{Q(x)}$, where $Q(x)$ is a quadratic function. We can represent every Gaussian as canonical characteristics. Formally, we define

$$\mathcal{C}(\mathbf{X}; K, \mathbf{h}, g) = \exp\left(-\frac{1}{2}\mathbf{X}^T K \mathbf{X} + \mathbf{h}^T \mathbf{X} + g\right) \tag{2.18}$$

Then,

$$\mathcal{N}(\boldsymbol{\mu}, \Sigma) = \mathcal{C}(K, \mathbf{h}, g) \tag{2.19}$$

by rewriting and arranging equation (2.15) we get:

$$\begin{aligned} K &= \Sigma^{-1} \\ \mathbf{h} &= \Sigma^{-1}\boldsymbol{\mu} \\ g &= -\frac{1}{2}\boldsymbol{\mu}^T \Sigma^{-1}\boldsymbol{\mu} - log\left((2\pi)^{n/2}|\Sigma|^{1/2}\right) \end{aligned}$$

Being more general forms then Gaussians, we can perform algebraic operations on canonical characteristics, such as multiplication and divisions, perform marginalization over variables and enter evidence. Hence we can use them to perform message passing in a junction tree. The various operations on canonical characteristics are defined as follows:

- *Initialization:* A canonical form $\mathcal{C}(K, \mathbf{h}, g)$ can be initializes by setting $K = 0$, $\mathbf{h} = 0$, $g = 0$.

- *Multiplication*: Using equation (2.18) we get

$$C(K_1, \mathbf{h}_1, g_1) \cdot C(K_2, \mathbf{h}_2, g_2) = C(K_1 + K_2, \mathbf{h}_1 + \mathbf{h}_2, g_1 + g_2) \tag{2.20}$$

- *Division*: Division is analogous to multiplication

$$\frac{C(K_1, \mathbf{h}_1, g_1)}{C(K_2, \mathbf{h}_2, g_2)} = C(K_1 - K_2, \mathbf{h}_1 - \mathbf{h}_2, g_1 - g_2) \tag{2.21}$$

- *Extension*: Canonical forms can be extended by increasing the dimensions of K and $\mathbf{h}$ and setting the extra entries to zeros.

- *Marginalization*: Let $C(\mathbf{X}, \mathbf{Y}; K, \mathbf{h}, g)$ be a canonical form defined over continuous variable sets $\{\mathbf{X}, \mathbf{Y}\}$ such that,

$$K = \begin{bmatrix} K_{\mathbf{XX}} & K_{\mathbf{XY}} \\ K_{\mathbf{YX}} & K_{\mathbf{YY}} \end{bmatrix} \quad ; \quad h = \begin{pmatrix} h_{\mathbf{X}} \\ h_{\mathbf{Y}} \end{pmatrix} \tag{2.22}$$

  The marginalization of the variables $\mathbf{Y}$ is defined as $\int C(\mathbf{X}, \mathbf{Y}; K, \mathbf{h}, g)$. Note that the result is a function over $\mathbf{X}$. This integral is finite iff $K_{\mathbf{YY}}$ is positive definite, i.e., it is the inverse of a legal covariance matrix, in which case the result is a canonical form $C(\mathbf{X}; K', \mathbf{h}', g')$ given by [4] such that,

$$\begin{aligned} K' &= K_{\mathbf{XX}} - K_{\mathbf{XY}} K_{\mathbf{YY}}^{-1} K_{\mathbf{YX}} \\ h' &= \mathbf{h}_{\mathbf{X}} - K_{\mathbf{XY}} K_{\mathbf{YY}}^{-1} \mathbf{h}_{\mathbf{Y}} \\ g' &= g + \frac{1}{2} \left(|\mathbf{Y}| \log(2\pi) - \log |K_{\mathbf{YY}}| + \mathbf{h}_{\mathbf{Y}}^T K_{\mathbf{YY}} \mathbf{h}_{\mathbf{Y}} \right) \end{aligned} \tag{2.23}$$

- *Evidence Instantiation*: It is possible to enter evidence into a canonical form, i.e., set the values of some of its variables. The result is a canonical form over the unobserved variables. Consider the canonical form $C(\mathbf{X}, \mathbf{Y}; K, \mathbf{h}, g)$ given by equation (2.22), then instantiating $\mathbf{Y} = \mathbf{y}$ results in the canonical form $C(\mathbf{X}; K', \mathbf{h}', g')$ given by [35], such that

$$\begin{aligned} K' &= K_{\mathbf{XX}} \\ h' &= \mathbf{h}_{\mathbf{X}} - K_{\mathbf{XY}} \boldsymbol{y} \\ g' &= g + \mathbf{h}_{\boldsymbol{y}}^T - \frac{1}{2} K_{\mathbf{YY}} \boldsymbol{y} \end{aligned} \tag{2.24}$$

Representing the distributions of cliques by canonical forms provide a powerful tool to perform the basic operations needed for message passing algorithms on a junction tree. However there are two major shortcomings of the canonical characteristics:

- Since the covariance matrix is not invertible, we cannot use them to represent deterministic linear relations, such as $A = B + 3C$.

- They are computationally unstable resulting in numerical errors.

Lauritzen, who originally proposed the canonical characteristics to perform exact inference on mixed Bayesian Networks [4], improved on the original scheme, by introducing conditional forms [5].

Definition 2 *Consider the variable set* $\mathbf{V}$*, such that* $\mathbf{V} = \mathbf{X} \cup \mathbf{Y}$ *and* $\mathbf{X} \cap \mathbf{Y} = \emptyset$*, where* $|\mathbf{X}| = n$ *and* $|\mathbf{Y}| = m$*. Then, the conditional distribution* $P(\mathbf{X}|\mathbf{Y})$ *can be represented by the following form:*

$$P(\mathbf{X}|\mathbf{Y}) = w \cdot \mathcal{N}(\mathbf{X}; a + B\mathbf{Y}, C)$$

where a *is a vector of size* n *and* B *and* C *are matrices of size* $n \times m$ *and* $n \times n$*, respectively.*

Conditional forms do not demonstrate the shortcomings of the canonical characteristics, and since all operations to perform the junction tree algorithm can be performed utilizing the conditional forms, we can adopt an improved version of it to perform exact inferencing on mixed Bayesian Networks [5]. However, using conditional forms makes the operations more complex, thereby increasing the complexity of the inferencing algorithm.

We defer the discussion on the details of the Lauritzen's junction tree algorithm with conditional forms for general Hybrid Bayesian Networks to Chapter 3 where we develop and introduce our hybrid inferencing methodology for general HBNs. In particular, as part of our proposed framework, in Section 3.4.3 we introduce a variant of Lauritzen's algorithm to perform inferencing on the *Type-II* Fuzzy-Bayesian transformation of general HBNs.

2.2 Fuzzy Sets

In this section, we do not provide a review of the rather extensive topic of Fuzzy Sets. Instead, we focus on the particular research on Fuzzy Sets as they relate to Hybrid Bayesian Networks. Within this context, we briefly review some important papers that provide inspiration to our research. To facilitate the ensuing discussion we start with a concise introduction of the ideas at the crux of Fuzzy Set theory.

In classic set theory a certain element can either belong to a set or not, such as in an optimization problem a certain solution can either be feasible or not, which can be represented in mathematical terms by using an indicator function. The nature of membership to classical sets requires precision and precision assumes that the model parameters represent exactly either our perception of the phenomenon modeled or the real features of the actual system. More importantly, precision implies that our model of the real-world system does not contain any ambiguities. Therefore, an observation about a certain model parameter can only assume the 0-or-1 assignment of an indicator function over its defined domain to determine whether or not it belongs to a collection of mutually exclusive states defining the model parameter. This crisp, deterministic, and precise world view underlines a whole body of work for formal modeling and reasoning about real-world systems.

A more realistic point of view would admit that the real world is more complex and uncertain. Traditionally, uncertainty in the real world is addressed primarily by probability theory. However, as we elaborate in more detail in section 3.3, randomness is only one of the many sources of uncertainty in real world applications. Another major source of uncertainty is *ambiguity*. The question of ambiguity is directly related to the notion of *set membership*. However, this time the membership is represented by a continuous function that can assume any real number on the closed interval [0,1] instead of 0 or 1 only. Fuzzy Set theory proposed by Lotfi Zadeh [17] makes use of this more generally defined idea of membership to formally model the ambiguity in real world systems. The new idea, here, is that the notion of set membership is the key to decision making when faced with uncertainty in general. In this respect, Fuzzy Set theory could be interpreted as a generalization of the classic Set theory.

Our past research experience on modeling the safety risk in the civil aviation domain indicates that the uncertainty that needs to be quantified originates from two major sources: randomness associated with domain variables and ambiguity associated with their states. Within this context, a Fuzzy-Bayesian hybrid approach provides the most appropriate tools to tackle the problem of modeling the uncertainty associated with a real-world complex system.

Although the question of *Fuzzy-Bayesian inference* has not been a recent one [36], the majority of related research has been performed in the last decade [37, 38, 39, 40, 41, 42, 43]. We should underline the fact that almost all of these studies focus on the development of a Bayesian calculus on Fuzzy data and none of them provides a study for inferencing with

Bayesian Networks in a Fuzzy domain as we provide in this study.

A second generation of papers focus on a hybrid inferencing methodology that combines the Fuzzy Set theory and Bayesian Networks, which is rather close to the heart of our current research. In a 1998 conference paper, Pan and McMichael elaborate on the idea of using Fuzzy transformation of continuous variables in a hybrid Bayesian Network at a thought-experiment level without providing a formal methodology and much detail [16]. Then in 2000, as the follow-up of the original idea, Pan and Liu propose a general formalism for inferencing with general Hybrid Bayesian Networks [30]. However, their proposed methodology excludes any Fuzzy Set theoretic formalism and relies heavily on Conditional Gaussian and Conditional Gaussian regression models to facilitate the inferencing in Hybrid Bayesian Networks, which, in its core, is not that much different than the original Lauritzen algorithm.

Another study, which is more true-to-the-idea, by Baldwin and DiTomaso emerges in 2003 [23]. They actually propose the fuzzification of continuous variables in a hybrid BN to facilitate inferencing. Although the laid-out fuzzification scheme is well structured and complete, the proposed inferencing algorithm relies heavily on minimum entropy criteria which results in increased computational complexity and instability.

More recently Cobb et al. propose the *mixture of truncated exponentials* model as a general solution to the problem of specifying conditional distributions for continuous variables in BNs as an alternative to discretization [44]. Although the proposed methodology is not necessarily an hybrid implementation of Fuzzy and BN concepts, it provides valuable insight on alternative ways for modeling conditional distributions within the context of Bayesian Networks. About the same time, Heng and Qin propose a general formalism for hybrid BNs that utilizes *partial least-squares* along with Fuzzy sets [45]. In another important paper published quite recently in 2008, DiTomaso and Baldwin [23], improving on their earlier publication on the topic, introduce a more complete approach to theory development for dealing with continuous variables in hybrid BNs using Fuzzy Sets [21]. Finally, in their recent publication, Eleye-Datubo et al., implement a Fuzzy-Bayesian approach based on an *induced mass assignment paradigm* to model the risk associated with marine and offshore safety [46].

The literature that we briefly reviewed above indicates that the emergence of the idea of utilizing Fuzzy Sets to model conditional distributions of continuous variables to facilitate inferencing in hybrid BNs as a prominent research topic is quite recent. Furthermore, the clustering of some quality papers within the last couple of years underlines the fact that this particular problem domain is currently gaining prominence within the scientific community.

Chapter 3

Fuzzy-Bayesian Networks

In this text, Hybrid Bayesian Networks (HBN) refer to Bayesian Networks that include both discrete and continuous random variables (RV) within the same network structure. A Fuzzy-Bayesian Network (FBN) formalism is introduced to overcome the complexity and tractability issues of the existing exact inferencing algorithms with an additional emphasis on improving the representative power of general Bayesian Networks (BN) for real-world systems. In this chapter, we focus on developing an analytical approach complete with its inferencing formalism for a generic FBN.

3.1 Hybrid Bayesian Networks

In a very basic sense, general Bayesian Networks (BNs), also called Hybrid Bayesian Networks, are graphical structures, which enable us to model an uncertain domain and reason about it. Formally, a BN is a directed acyclic graph representing the joint probability distribution of a given set of random variables that include both discrete and continuous variables.

Based on a well-established probability theory, the general formalism for Bayesian Networks was first introduced by Pearl [1]. Subsequently, Lauritzen and Spiegelhalter [2] developed a complete inferencing methodology for BNs with discrete random variable only. Although the developed inferencing technique was exact, the modeling ability of discrete-only BNs was quite limited for representing real-life complex systems which quite often include continuous variables. Consequently, Lauritzen proposed an exact method for *mixed* (i.e., hybrid) BNs by making use of *Conditional Gaussians (CG)*. Today, CG based models represent the most popular class of hybrid models. However, they have two important restrictions. First, they can only model linear relations between continuous variables. Second, they do not allow discrete nodes to have continuous parents. These issues and their importance are discussed in the following sections.

Lauritzen's approach [4, 5], from an algorithmic point of view, still represents the state of art. It is based on the *junction tree algorithm* [2, 7, 8], originally developed for discrete-only BNs. Contrary to the common perception that the extension of the *junction tree algorithm* to the mixed networks is a straightforward implementation, it has been shown that in may cases the Lauritzen algorithm is intractable even for simple network structures [10]. More on this issue later.

On the other hand, various approximate algorithms were also introduced for HBNs. The most commonly used framework is based on stochastic sampling. Although, stochastic sampling applies to every class of HBNs (not only CGs), it may take a long time to converge to a reliable answer, and therefore stochastic sampling based approximate algorithms are not suitable for real-time applications.

Within this context, in the next section we provide a novel inference algorithm for HBNs. This new framework takes a different view at the whole problem from the vantage point of an analyst whose goal is to assess the safety risk associated with a complex system, for which randomness is only one of the sources of uncertainty. The proposed framework introduces vagueness to the problem in an attempt to bridge the gap between probability and possibility, which enables the application of general HBNs practical for reasoning about complex real-life systems.

3.2 Probability of a Fuzzy Event

A very heated debate has been underway since the introduction of Fuzzy logic and possibility theory about their relevance within the scientific community dominated by a world view which believes that probabilistic methods are necessary and sufficient to understand uncertainty in real-world complex systems [11], [12].

We believe that possibility theory based on Fuzzy logic and probability theory are *complementary rather than competitive* [12]. They simply address different sources of uncertainty. Probability deals with randomness, whereas Fuzzy Sets helps us to understand the vagueness. In this section we introduce the concept of a Fuzzy random variable and propose a framework to define a conditional probability formulation for various combinations of Fuzzy and crisp events.

Consider an event which is not crisply defined. Determining whether or not some observed value of crisp variable X belongs to a certain partition defined by a Fuzzy partitioning scheme along its frame $\mathbf{X}$, $\forall X \in \mathbf{X}$ can be considered such an event. In other words, this event is only defined by a possibility distribution (i.e., a Fuzzy Set). Now, let us further assume that, although the event itself is not crisp, we want to determine the probability of its occurrence. In a sense, what we are facing here is a pure stochastic problem, however, with a major twist that the event itself is vaguely defined (i.e., defined by a Fuzzy Set). There are two fundamentally different approaches one can adopt to address this question [13]. The probability of a fuzzy event should be a scalar (i.e., a crisp number or *measure*) [14] or it can be defined by a fuzzy set itself [15]. The latter point of view necessitates the adoption of an inference mechanism based solely on Fuzzy logic, which from a purely algorithmic point of view is considered *suboptimal* in the following three important aspects [16]:

- The formalism for Fuzzy inferencing set forth so far cannot be considered as *complete*.
- It is yet to be shown that experimental evidence supports the core components of Fuzzy inferencing (e.g., **MIN** and **MAX** operations).
- Fuzzy inferencing does not take into account any probabilistic information provided about the problem, thereby resulting in a one dimensional analysis of uncertainty, which is exactly what the probabilistic methods suffer from.

On the other hand, when the probability of a fuzzy event is assumed to be a scalar measure, the inference mechanism employed could capitalize the well established inferencing algorithms of probabilistic reasoning methods such as a Bayesian Network. Therefore, in our pursuit to capture various types of uncertainty that exist in real-world situations when modeling complex systems, we adopt the former point of view to develop our proposed framework.

Now we introduce the concept of a *Fuzzy random variable.* Let X be a continuous variable defined on closed interval $[a, b]$, i.e.,

$$\mathbf{X} = \{x : a \leq x \leq b\} \tag{3.1}$$

Consider the exhaustive collection of individual elements (singletons) x (i.e., the actual value that variable X takes), which make up a universe of information (discourse) $\mathbf{X}$. For the purposes of Fuzzy information, the crisp value x represents the variable X, which is defined on frame $\mathbf{X}$.

Now consider the fuzzy set $\hat{\mathbf{X}}$ defined on $\mathbf{X}$, to be

$$\hat{\mathbf{X}} = \{x, \hat{\mathbf{X}}(x) | x \in \mathbf{X}\} \tag{3.2}$$

where $\hat{\mathbf{X}}(x)$ is the membership function mapping the crisp $x, x \in \mathbf{X}$ on to the closed interval $[0, 1]$, which defines its membership to the Fuzzy Set $\hat{\mathbf{X}}$.

Now, let $\hat{U}$ be the fuzzified counterpart of crisp variable X. Also, assume that $\hat{U}$ can only take discrete states from the set $\hat{\mathbf{U}}$, which can also be considered as the frame of fuzzy variable $\hat{U}$

$$\hat{\mathbf{U}} = \{\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_m\} \tag{3.3}$$

where subscript m is the number of Fuzzy states that $\hat{U}$ can take and $\hat{X}_i$ is one of these Fuzzy states defined on $\mathbf{X}$ (i.e., frame of X), such that

$$\hat{X}_i = \{x, \hat{X}_i(x) | x \in \mathbf{X}\} \qquad 0 \leq \hat{X}_i(x) \leq 1. \tag{3.4}$$

$\hat{X}_i(x)$ in equation (3.4) denotes the degree of membership of X belonging to the fuzzy state $\hat{X}_i$. In Fuzzy literature, this function is called *membership function* and traditionally denoted by $\mu(x)$. However, to facilitate a better understanding we adopted $\hat{X}_i(x)$ to denote the membership function defining the Fuzzy Set $\hat{X}_i$, which, when the traditional notation is used, would be equivalent to

$$\hat{X}_i(x) = \mu_{\hat{X}_i}(x). \tag{3.5}$$

Before moving to the next stage in developing an expression for the probability of a Fuzzy event, let us review some key steps detailed above:

- First, consider a continuous variable X, which is defined on frame $\mathbf{X}$ (see equation (3.1)).
- Then, consider a Fuzzy Set $\hat{X}$ identified by membership function $\hat{X}(x)$ and defined on the same frame $\mathbf{X}$ (see equation (3.2)).
- Finally, define a *discrete* Fuzzy variable $\hat{U}$, which can only assume states $\hat{X}_i$ from set $\hat{\mathbf{U}} = \{\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_m\}$ where each fuzzy state $\hat{X}_i$ is defined by membership function $\hat{X}_i(x)$ (see equation (3.4)).

Next, we will introduce a new and different interpretation of a Fuzzy membership function which will in turn enable us to define the conditional probability of a fuzzy state $\hat{X}_i$ given a crisp value x of X. First, let us formally define a membership function in Fuzzy Set theory.

The representation of an object within a universe is key to any reasoning framework about that universe. In Set theory (without making a distinction between crisp and fuzzy sets), this representation is provided by the notion of *membership* to the sets defined on the universe. In classical set theory (i.e., in crisp sets), on which probability theory, as a whole, is founded, *sets* contain objects *that satisfy precise properties of membership*. In other words, an object (or an event) could either be a member of a set or not. Such a membership can be represented by an indicator function. Whereas, the membership to a Fuzzy Set is approximate. Hence, in Fuzzy Set theory, sets may contain objects *that satisfy imprecise properties of membership*.

This notion of approximate membership as compared to precise membership introduces a much more refined representation of real-world complex systems, where much of the complexity arises from nuanced interpretation of state spaces for their multitude of variables. Identifying *tall* versus *short* people among a certain population is the classic example supporting this point of view. Moving beyond the notion of binary membership makes it possible to accommodate the idea of various *degrees of membership*.

This idea was first introduced by Zadeh [17]. He defined the membership of an object to a set as a continuous function defined on the closed interval $[0, 1]$ of *Real* numbers. Since the end points 0 and 1 of the interval are included, this type of membership incorporates the notion of *no* and *full* membership of classical set theory, thereby making it a more general representation of the *membership* idea.

Let $\mathbf{X}$ be a set of observations X_i whose individual values are denoted by x_i. Now consider an event $\mathbf{A}$. Using the binary logic of membership in classic sets, the observations could only be categorized as event $\mathbf{A}$ or not. The membership function $\mathcal{X}_A(x)$ in equation (3.6) formalizes this simple example.

$$\mathcal{X}_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases} \tag{3.6}$$

Now, consider another membership function $\mu_{\hat{A}}(x)$ defined on the same set of observations $\mathbf{X}$. However, this time for each observation, $\mu_i(x)$ returns a real number between 0 and 1 (including 0 and 1) as the degree of membership to event $\mathbf{A}$, thereby generating the Fuzzy Set $\hat{\mathbf{A}}$. In classical set theory, sets (i.e., crisp sets) represent a collections of real objects or events and can be represented by a unique binary membership function such as the one in equation (3.6). However, in a set-theoretic sense, this unique binary membership function is not equivalent to a collection of real objects or events. Hence, Fuzzy Sets are not collections of objects or events. *Fuzzy Sets are always and only functions, which map a universe of objects on to the unit interval [0,1]* and every continuous function that satisfies this property is also a Fuzzy Set [18].

3.2.1 Fuzzy Bayes Formula

Within this context, we define the conditional probability of a fuzzy state $\hat{X}_i$ defined by equation (3.4) given a certain value x of the crisp variable X as the degree of the membership of x to $\hat{X}_i$.

$$P(\hat{X}_i|X = x) = \hat{X}_i(x) \qquad \forall x \in \mathbf{X} \tag{3.7}$$

In order for the conditional probability in the equation (3.7) to be considered as a viable probability an important constraint should also be introduced. We define the collection of all fuzzy events (i.e., fuzzy states) describing fuzzy information as an *orthogonal* Fuzzy information system $\hat{\mathbf{U}} = \{\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_m\}$ where by orthogonal we mean that the sum of the membership values for each Fuzzy state $\hat{X}_i$, for every observed value x, $\forall x \in \mathbf{X}$, equals unity [19]. That is,

$$\sum_{i=1}^{m} \hat{X}_i(x) = 1 \qquad \forall x \in \mathbf{X} \tag{3.8}$$

hence,

$$\sum_{i=1}^{m} P(\hat{X}_i|X = x) = 1 \qquad \forall x \in \mathbf{X} \tag{3.9}$$

Now, we can calculate the marginal probability of fuzzy state $\hat{X}_i$ using basic probability calculus. The joint probability distribution of fuzzy state and $\hat{X}_i$ crisp variable X can be written as

$$P(\hat{X}_i, X) = P(\hat{X}_i|X) \times P(X) \tag{3.10}$$

In order to obtain the marginal probability of $\hat{X}_i$ we simply sum the joint distribution over variable X. For a crisp discrete variable X, that is,

$$\begin{aligned} P(\hat{X}_i) &= \sum_{X \in \mathbf{X}} P(\hat{X}_i|X)P(X) \\ &= \sum_{x \in \mathbf{X}} \hat{X}_i(x)P(x) \end{aligned} \tag{3.11}$$

For a crisp continuous variable X, the marginal probability becomes

$$P(\hat{X}_i) = \int_{\mathbf{X}} \hat{X}_i(x)\, P(x)\, \mathrm{d}x \tag{3.12}$$

The results in equations (3.11) and (3.12) are equivalent to Zadeh's derivations for *probability measures of Fuzzy events* [20], thereby verifying the veracity of our approach when determining the probability of a fuzzy event.

Note that these results require, along with the membership function, the knowledge of the marginal distribution of the original crisp variable to determine the marginal distribution of the fuzzy event defined on the frame of the original variable. Since our ultimate goal is to generate a Fuzzy-Bayesian framework for reasoning about complex system, one can easily foresee a potential issue here.

To elaborate, in a purely Bayesian Network, the inferencing is based on conditional relations among the variables. Hence, it can only be performed when there exist a complete information set about these conditional relations. Throughout the process, the marginal distributions of individual variables are mostly unknown and in fact, one of the major objectives is to obtain marginal distributions for the variables when evidence is introduced to the model. At the crux of our proposed framework lays the partial or complete mapping (i.e., transformation) of the probabilistic information of a given problem domain on to a Fuzzy information universe, thereby introducing a more complete representation of uncertainty. Consecutively, we perform the reasoning tasks using Bayesian probabilistic axioms to avoid the shortcoming of a purely Fuzzy reasoning algorithm. Therefore, one may reach the conclusion that, at the start of the process, all we obtain is, if at all, a set of conditional probability distributions (CPDs) defining a typical *crisp* BN. Even though this may be partially true, one should also bear in mind that in practice the overwhelming majority of CPDs are determined not by data collection but through knowledge elicitation from subject matter experts (SMEs). Hence, they are nothing more than *approximations* even though the inferencing method may be *exact*. The same approach can be adopted when dealing with marginals, that is, depending on the availability of data, they can be constructed deterministically or approximated by SMEs.

We should also emphasize that our proposed framework does not require the existence of a Bayesian Network as a prerequisite. Although, it provides a novel approach to exact inferencing for general Bayesian Networks (i.e., general directed acyclic graph structure including both continuous and discrete variable with no restrictions), it can be used as a stand alone uncertainty modeling framework to reason about complex systems.

We will revisit this discussion on the usage of marginals in equations (3.11) and (3.12) again in a later section where the proof of concept is demonstrated by a simple example. Next, we build upon the results introduced so far to propose a *Bayes* rule which incorporates Fuzzy information. This rule will then be employed to develop a novel inferencing algorithm for general Hybrid Bayesian Networks.

Consider two Fuzzy variables $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ defined on the frames $\mathbf{X}$ and $\mathbf{Y}$ of two crisp variables X and Y, respectively. That is,

$$\begin{aligned} X &= \{x| \quad x \in \mathbf{X}\} \\ \hat{\mathbf{X}} &= \{x, \hat{\mathbf{X}}(x)| \quad x \in \mathbf{X}\} \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} Y &= \{y| \quad y \in \mathbf{Y}\} \\ \hat{\mathbf{Y}} &= \{y, \hat{\mathbf{Y}}(y)| \quad y \in \mathbf{Y}\} \end{aligned} \tag{3.14}$$

Assume that, fuzzy variables $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ are conditionally independent given X and Y. Hence, their joint conditional probability can be determined as follows,

$$\begin{aligned} P(\hat{\mathbf{X}}, \hat{\mathbf{Y}}|X = x, Y = y) &= P(\hat{\mathbf{X}}|X = x, Y = y) \cdot P(\hat{\mathbf{Y}}|X = x, Y = y) \\ &= P(\hat{\mathbf{X}}|X = x) \cdot P(\hat{\mathbf{Y}}|Y = y) \\ &= P(\hat{\mathbf{X}}|x) \cdot P(\hat{\mathbf{Y}}|y) \\ &= \hat{\mathbf{X}}(x) \cdot \hat{\mathbf{Y}}(y) \end{aligned} \tag{3.15}$$

Assuming crisp variables X and Y are continuous and the joint probability $P(x, y)$ is integrable on their respective domains $\mathbf{X}$ an $\mathbf{Y}$, the joint probability of $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ is defined by the following (see equation(3.12)),

$$\begin{aligned} P(\hat{\mathbf{X}}, \hat{\mathbf{Y}}) &= \int_{\mathbf{X}} \int_{\mathbf{Y}} P(\hat{\mathbf{X}}, \hat{\mathbf{Y}}|x, y) \cdot P(x, y) \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_{\mathbf{X}} \int_{\mathbf{Y}} \hat{\mathbf{X}}(x) \cdot \hat{\mathbf{Y}}(y) \cdot P(x, y) \, \mathrm{d}x \, \mathrm{d}y \end{aligned} \tag{3.16}$$

Using $P(\hat{\mathbf{X}}, \hat{\mathbf{Y}}) = P(\hat{\mathbf{X}}|\hat{\mathbf{Y}}) \cdot P(\hat{\mathbf{Y}})$ along with the equation above we can now develop an expression explicitly for the conditional probability of Fuzzy event $\hat{\mathbf{X}}$ given another Fuzzy event $\hat{\mathbf{Y}}$.

$$\begin{aligned} P(\hat{\mathbf{X}}|\hat{\mathbf{Y}}) &= \frac{\int_{\mathbf{X}} \int_{\mathbf{Y}} \hat{\mathbf{X}}(x) \cdot \hat{\mathbf{Y}}(y) \cdot P(x, y) \, \mathrm{d}x \, \mathrm{d}y}{\int_{\mathbf{Y}} \hat{\mathbf{Y}}(y) \cdot P(y) \, \mathrm{d}y} \\ &= \frac{\int_{\mathbf{X}} \int_{\mathbf{Y}} \hat{\mathbf{X}}(x) \cdot \hat{\mathbf{Y}}(y) \cdot P(x|y) \cdot P(y) \, \mathrm{d}x \, \mathrm{d}y}{\int_{\mathbf{Y}} \hat{\mathbf{Y}}(y) \cdot P(y) \, \mathrm{d}y} \end{aligned} \tag{3.17}$$

If we were dealing with two Fuzzy variables defined on frames of two crisp discrete variables then the equation (3.17) would be the following.

$$P(\hat{\mathbf{X}}|\hat{\mathbf{Y}}) = \frac{\sum_{\mathbf{X}} \sum_{\mathbf{Y}} \hat{\mathbf{X}}(x) \cdot \hat{\mathbf{Y}}(y) \cdot P(x|y) \cdot P(y)}{\sum_{\mathbf{Y}} \hat{\mathbf{Y}}(y) \cdot P(y)} \tag{3.18}$$

There are two more cases possible for a pair of Fuzzy variables. First, the conditional probability of a Fuzzy variable $\hat{\mathbf{X}}$ whose frame is crisp continuous $\mathbf{X}$ given another Fuzzy variable $\hat{\mathbf{Y}}$ with a crisp discrete frame $\mathbf{Y}$. That is,

$$P(\hat{\mathbf{X}}|\hat{\mathbf{Y}}) = \frac{\int\limits_{\mathbf{X}} \sum\limits_{\mathbf{Y}} \hat{\mathbf{X}}(x) \cdot \hat{\mathbf{Y}}(y) \cdot P(x|y) \cdot P(y) \; \mathrm{d}x}{\sum\limits_{\mathbf{Y}} \hat{\mathbf{Y}}(y) \cdot P(y)} \tag{3.19}$$

Second, the conditional probability of a Fuzzy variable $\hat{\mathbf{X}}$ whose frame is crisp discrete $\mathbf{X}$ given another Fuzzy variable $\hat{\mathbf{Y}}$ with a crisp continuous frame $\mathbf{Y}$. That is,

$$P(\hat{\mathbf{X}}|\hat{\mathbf{Y}}) = \frac{\sum\limits_{\mathbf{X}} \int\limits_{\mathbf{Y}} \hat{\mathbf{X}}(x) \cdot \hat{\mathbf{Y}}(y) \cdot P(x|y) \cdot P(y) \; \mathrm{d}y}{\int\limits_{\mathbf{Y}} \hat{\mathbf{Y}}(y) \cdot P(y) \; \mathrm{d}y} \tag{3.20}$$

With the equations (3.17), (3.18), (3.19), and (3.20), we provide new explicit formulations for all possible cases of the conditional probability $P(\hat{\mathbf{X}}|\hat{\mathbf{Y}})$ of a Fuzzy event given another Fuzzy event.

This Fuzzy / Fuzzy Bayes formulation can also be extended to Fuzzy / Crisp and Crisp / Fuzzy variable pairs and can be used as the basis of a inferencing framework about a *directed acyclic graphical* (DAG) structure such as Bayesian Networks.

Throughout this text Bayesian Networks refer to *crisp* Bayesian Networks and unless identified specifically as *Fuzzy*, all variables, discrete and continuous, refer to crisp events/objects.

For discrete Bayesian Networks, exact general inferencing algorithms exist and are widely implemented to reason about complex systems. On the other hand, for general Hybrid Bayesian Networks including both discrete and continuous variables, the exact inferencing algorithm based on Lauritzen's state-of-art junction tree algorithm require that discrete nodes cannot have continuous parents. However, in our proposed framework we address this restriction. Without going into much detail about the proposed algorithm prematurely, it may be the right place to identify the main types of *parent-child* pair nodes (i.e., variables) in Fuzzy-Bayesian Networks. These pairs of variables are illustrated in Figure 3.1.

Fuzzy discrete nodes in Figure 3.1 denote the Fuzzy counterpart of a continuous variable discretized by Fuzzy transformation. In other words, Fuzzy discrete variables are based on

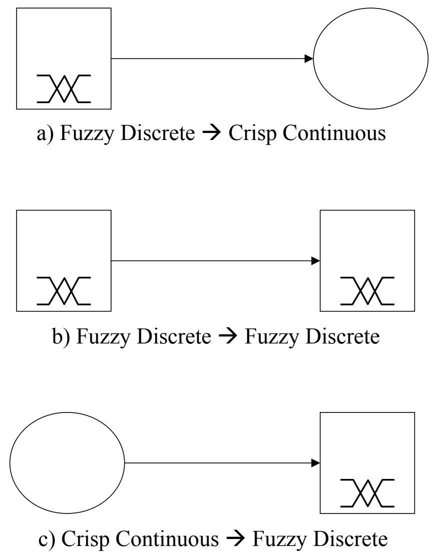


Figure 3.1: *Fuzzy-Crisp* variable pairs in Fuzzy-Bayesian Networks

the frames of crisp continuous variables. We will revisit these node pairs in the section 3.4 where we introduce the fuzzy transformation and graph restructuring ideas at the crux of our proposed methodology. However at this stage it would suffice to mention that Bayes formulas for these four pairs are basically variations of equation (3.17).

3.3 Fuzzy Evidence and Fuzzy Updating

There are two aspects of Bayesian Networks that are still subject to improvement and therefore research: how to handle continuous variables and how to deal with uncertain information as evidence? In the previous section we introduced a new Fuzzy Bayes formulation which will be the basis for our Fuzzy-Bayesian framework to deal with continuous variables in a general Hybrid Bayesian Network. In this section we introduce the idea of *Fuzzy evidence* to address the second aspect, namely the issue of uncertain evidence in Bayesian Networks.

Bayesian Networks are tools to represent a domain and its uncertainty. The representation is done by modeling the topology of the domain by an acyclic directed graph and the interactions among the domain variables by conditional probability distributions. The uncertainty modeled by a BN is represented basically by probability distributions, and hence is in the form of *randomness*. However, the uncertainty inherent in a real-world complex system is not only comprised of randomness nor is it solely in that form.

There are three types of uncertainty:

- Vagueness.
- Ambiguity.
- Randomness.

Uncertainty, such as the uncertainty about the outcome of some experiment such as a coin-toss, is in the form of *randomness*. Randomness is typically modeled using probability theory.

Now, consider a parameter in a model of a real-world system. There may be cases for which the nominal value of this parameter is known to lie within a given interval. There is uncertainty (i.e., *ambiguity*) about any nominal value chosen from that interval for that parameter.

Fuzzy Sets deal with the type of uncertainty that arises when the boundaries of a class of objects are not sharply defined (i.e., *vagueness*). Consider being *young* or *rich*, membership in such classes is a matter of degree.

By including novel concepts such as *Fuzzy random variables*, and *probability of a Fuzzy event*, we intend to include another dimension to the definition of uncertainty (i.e., randomness) that the BNs traditionally represent. This new approach to uncertainty analysis complements classical Bayesian inferencing which only addresses one of the dimensions of uncertainty, i.e., randomness, with a second aspect of uncertainty, vagueness. However, applying a Fuzzy transformation to select continuous variables in the network, which have been described above in detail, is only one of the aspects of a multi-prong approach constituting our proposed Fuzzy-Bayesian framework.

3.3.1 Fuzzy Evidence

We present the concept of *Fuzzy evidence* as the next stage in incorporating *vague* or *ambiguous* information in to a Bayesian Network. Fuzzy evidence is a type of uncertain evidence that occurs when uncertainty is presented as a fuzzy set rather than a delta or indicator function where the evidence (i.e., observation) could only indicate that the variable (i.e, event) observed is in one of the predetermined states. Hence the fuzzy evidence maps all states of the evidence to the closed continuous interval [0,1]. On the far side of the scale lays the *hard evidence*, sometimes referred to as *specific evidence* where the state in which the observation is definite and therefore an indicator function could be used to represent it.

Definition 3 *Fuzzy evidence:*

Consider variable X defined on the frame $\mathbf{X}$. Fuzzy evidence $\hat{E}_X$ is a function mapping the frame $\mathbf{X}$ to the continuous real interval [0,1] such that,

$$\sum_{x \in \mathbf{X}} \hat{E}_X(x) = 1, \qquad \forall x \in \mathbf{X}, \quad 0 \leq \hat{E}_X(x) \leq 1 \tag{3.21}$$

Suppose that we identify a continuous variable *flightcrew experience* defined on the continuous frame of the *logged flight hours of the flightcrew.* Furthermore, assume that we created a Fuzzy variable as the counterpart of the continuous variable by using orthogonal Fuzzy Sets *low, medium, and high.* For example, an observation could be interpreted as the soft evidence $E(low) = 0.6$, $E(medium) = 0.4$, and $E(high) = 0$.

In classic Bayesian Network applications, a seemingly similar type of evidence, known as *virtual* or *likelihood* evidence is used when the new information is in the form of a probability distribution [21], [22]. We should emphasize the fact that likelihood evidence, being a distribution as opposed to an indicator function shows similar characteristics as Fuzzy evidence. However, besides the updating algorithm being different, there are fundamental differences between the two concepts. Likelihood evidence represents a subjective statement that can be improved by later observations. Therefore, a continuous finding cannot be treated as likelihood evidence. Fuzzy evidence, due to its uncertain nature cannot be improved with additional observations and is suitable to be used with continuous variables since it is basically a generalization of specific or hard evidence. When instantiated, for example, the Fuzzy evidence about a continuous variable whose domain is discretized into three sates by fuzzification could assume the values 0.8, 0.2, and 0, whereas a crisp discrete variable with the same states would have the values 1, 0, and 0 as hard evidence.

Both concepts, Fuzzy and likelihood evidence have their own advantages and could be used in conjunction to produce a more realistic treatment of evidence in Bayesian Networks.

3.3.2 Fuzzy Updating

In this section we present a new approximate solution for updating a joint probability distribution when fuzzy evidence about the state of a variable is introduced. By Fuzzy updating, what we mean is the process of updating the joint distribution of a given set of variables when Fuzzy evidence is introduced.

The process of updating a joint distribution by introducing evidence can also be considered as updating a distribution given another distribution since the Fuzzy evidence we introduce, in this case, is in the form of a distribution. In other words, the question is updating a distribution when another is given as a constraint. However, the problem of updating a distribution having another probability distribution as a constraint does not have a closed form solution in classical probability theory. An approximate solution can be proposed utilizing the *relative entropy* concept of the information theory [21, 23].

Consider a prior distribution p which is subjected to constraint set $\mathbf{C}$. One can choose to update the prior distribution p with a posterior distribution q satisfying the constraint set $\mathbf{C}$ and corresponding to the least prejudiced one (i.e., with minimum relative entropy, MRE) with respect to p [24].

Conceptually, relative entropy can be considered a measure of the difference between two probability distributions. Also known as the Kullback-Leibler distance, it determines the distance or divergence between a base or *true* probability distribution and a secondary or *updated* distribution [25]. For a prior distribution P of a discrete variable X and a candidate posterior distribution q satisfying constraint $\mathbf{C}$, the Kullback-Leibler (K-L) distance is expressed as follows,

$$d_{KL}(P, Q) = \sum_{x_i \in X} P(x_i) \log_2 \frac{P(x_i)}{Q(x_i)} \tag{3.22}$$

where x_i denotes a state of the discrete variable X and the logarithm takes the base value 2 when the information measured is in *bits* and e when meaured in *nats* (natural units).

In a Bayes theoretic setup, the K-L distance between the prior and the estimated posterior can be considered as the projection of the prior onto the set of all possible distributions satisfying the data. Therefore, it can be used as a measure for the information gain caused by the introduction of new information to the system.

Equation (3.22) is a convex function which is non-negative and equals to zero when $P(x_i) = Q(x_i)$. The MRE criteria is then to determine the posterior distribution Q'_j with the minimum K-L distance to distribution P among the set of distributions $\mathbf{Q}, \forall Q_j \in \mathbf{Q}$ satisfying the constraint set $\mathbf{C}$. That is,

$$d_{KL}(P, Q'_j) = \min_{Q_j \in \mathbf{C}} d_{KL}(P, Q_j) \tag{3.23}$$

A solution to equation (3.23) exists [26]. Furthermore, as mentioned earlier, since by definition the K-L distance is a convex function the solution that exists is unique.

Consider a set of variables $\mathbf{V} = \{V_1, V_2, \ldots, V_m\}$ and their joint probability distribution $P(\mathbf{V})$ and let $P'(\mathbf{V})$ be the updated joint distribution subject to some constraint set. Then, we need to determine an updated joint distribution $P'(\mathbf{V})$, such that its K-L distance from the prior joint distribution $P(\mathbf{V})$ is minimized. That is,

$$\min_{P(\mathbf{V})} \sum_{V_i \in \mathbf{V}} P'(V_i) \log_2 \frac{P'(V_i)}{P(V_i)} \tag{3.24}$$

Now, suppose that there has been a resent observation about one of the variables V_i of set $\mathbf{V}$ and due to uncertainty associated with the observation we decided that it needs to be introduced to the system as a Fuzzy evidence. In other words, we need to update the joint probability $P(\mathbf{V})$ given the a marginal probability distribution $P(V_i)$ (recall that Fuzzy evidence transforms an observation about a continuous variable by an orthogonal collection of Fuzzy Sets and the resulting information is in the form of a distribution defined on discrete states created by the fuzzification process). Within this context, minimizing the K-L distance (i.e., relative entropy) between the updated joint distribution $P'(\mathbf{V})$ and the prior distribution $P(\mathbf{V})$ can be considered as equivalent to Jeffrey's rule of conditioning.

Jeffrey's rule of conditioning [27] is a generalized form of Bayesian *conditionalization*, where a prior distribution is updated by an evidence in the form of a probability distribution and using the same notation as in minimum entropy criteria can be formulated as follows:

$$\begin{aligned} P'(\mathbf{V}) &= \sum_j P(\mathbf{V}|V_i = v_{ij})E(V_i = v_{ij}) \\ &= \sum_j P(\mathbf{V}|v_{ij})E(v_{ij}) \end{aligned} \tag{3.25}$$

where j denotes the states of evidence variable V_i, hence v_{ij} is the j^{th} state of i^{th} variable and the summation is performed on all the states of the evidence variable V_i.

Jeffrey's rule of conditioning can also be understood as a special case of Dempster's rule of combination. It is shown that assuming additive belief functions, Jeffrey's rule follows from Dempster's rule [28].

Since Jeffrey's rule of conditioning accepts distributions as evidence, we can adopt it to propose an approximate solution for updating a joint distribution given Fuzzy evidence. Next, with an example we demonstrate the applicability of the proposed concept.

Let $\mathbf{V} = \{A, B, C\}$ be a set of discrete random variables, ,and as earlier, let $P(\mathbf{V})$ and $P'(\mathbf{V})$ denote the prior and posterior joint distributions. Then, $P(\mathbf{V}) = P(ABC)$. Now suppose that there is new information about variable A, which can assume states from its state set $\{a_1, \ldots, a_n\}$. Consider,

$$\begin{aligned} P'(\mathbf{V}) &= P(ABC|A) \\ &= \frac{P(ABC)}{P(A)} \end{aligned}$$

Then

$$P(ABC|A = a_i) = \begin{cases} \dfrac{P(a_iBC)}{P(a_i)} & \text{when } A = a_i \\ 0 & \text{when } A \neq a_i \end{cases} \tag{3.26}$$

Suppose that the evidence about A bears some uncertainty and we decided to treat it as Fuzzy evidence $\hat{E}_A(a_i)$ according to equation (3.21):

$$\sum_{i=1}^{n} \hat{E}_A(a_i) = 1, \qquad \forall a_i \in \mathbf{A}, \quad 0 \leq \hat{E}_A(a_i) \leq 1 \tag{3.27}$$

Then, applying equation (3.25), Jeffrey's rule of conditioning, the updated joint distribution $P'(ABC)$ is

$$P'(ABC) = \sum_{i}^{n} P(ABC|a_i)\hat{E}_A(a_i), \qquad i = 1, \ldots, n. \tag{3.28}$$

Representing equation (3.28) as a vector would facilitate a better understanding with regard to the composition of an updated distribution treated by a Fuzzy evidence. Note that the variables A, B, and C are denoted as sets, i.e., $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. Whereas, set pairs such as $\mathbf{BC}$ represent all possible pairwise combinations of the two variables' states. Consequently, each complete set of these combinations is multiplied by a_i, the i^{th} state of the evidence variable A.

$$P'(\mathbf{ABC}) = \begin{pmatrix} P(a_1\mathbf{BC}|a_1)\hat{E}_A(a_1)+ & 0 & + & \cdots & +0 \\ 0 \quad + & P(a_2\mathbf{BC}|a_2)\hat{E}_A(a_2) & + & \cdots & +0 \\ 0 \quad + & 0 & & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 \quad + & 0 & & \cdots & +P(a_n\mathbf{BC}|a_n)\hat{E}_A(a_n) \end{pmatrix}$$

Now, suppose that all three variables are binary, i.e., $\mathbf{A} = \{a_1, a_2\}$, $\mathbf{B} = \{b_1, b_2\}$, and $\mathbf{C} = \{c_1, c_2\}$. Then,

$$P'(\mathbf{ABC}) = P(\mathbf{ABC}|a_1)E(a_1) + P(\mathbf{ABC}|a_2)E(a_2) \tag{3.29}$$

The individual terms of the updated joint distribution, i.e., equation (3.29) can be written as follows.

$$\begin{array}{ccccc} P'(\mathbf{ABC}) & = & P(\mathbf{ABC}|a_1)E(a_1) & + & P(\mathbf{ABC}|a_1)E(a_1) \\ \hline a_1b_1c_1 & & P(a_1b_1c_1|a_1)E(a_1) & + & 0 \\ a_1b_1c_2 & & P(a_1b_1c_2|a_1)E(a_1) & + & 0 \\ a_1b_2c_1 & & P(a_1b_2c_1|a_1)E(a_1) & + & 0 \\ a_1b_2c_2 & & P(a_1b_2c_2|a_1)E(a_1) & + & 0 \\ a_2b_1c_1 & = & 0 & + & P(a_2b_1c_1|a_1)E(a_2) \\ a_2b_1c_2 & & 0 & + & P(a_2b_1c_2|a_1)E(a_2) \\ a_2b_2c_1 & & 0 & + & P(a_2b_2c_1|a_1)E(a_2) \\ a_2b_2c_2 & & 0 & + & P(a_2b_2c_2|a_1)E(a_2) \end{array} \tag{3.30}$$

Using $P(a_1b_jc_k|a_1) = \dfrac{P(a_1b_jc_k)}{P(a_i)}$, where $j = 1, 2$, and $k = 1, 2$, we obtain

$$
\begin{array}{lcccc}
P'(\mathbf{ABC}) & = & P(\mathbf{ABC}|a_1)E(a_1) & + & P(\mathbf{ABC}|a_1)E(a_1) \\
\hline
a_1b_1c_1 & & \frac{P(a_1b_1c_1)}{P(a_1)}E(a_1) & + & 0 \\
a_1b_1c_2 & & \frac{P(a_1b_1c_2)}{P(a_1)}E(a_1) & + & 0 \\
a_1b_2c_1 & & \frac{P(a_1b_2c_1)}{P(a_1)}E(a_1) & + & 0 \\
a_1b_2c_2 & & \frac{P(a_1b_2c_2)}{P(a_1)}E(a_1) & + & 0 \\
a_2b_1c_1 & = & 0 & + & \frac{P(a_2b_1c_1)}{P(a_1)}E(a_2) \\
a_2b_1c_2 & & 0 & + & \frac{P(a_2b_1c_2)}{P(a_1)}E(a_2) \\
a_2b_2c_1 & & 0 & + & \frac{P(a_2b_2c_1)}{P(a_1)}E(a_2) \\
a_2b_2c_2 & & 0 & + & \frac{P(a_2b_2c_2)}{P(a_1)}E(a_2)
\end{array}
\tag{3.31}
$$

Next, we provide a numerical example to illustrate the fuzzy updating process for a prior joint distribution P(ABC) given the Fuzzy evidence above.

Consider the joint probability distribution of Table (3.1).

Table 3.1: Prior joint distribution $P(ABC)$

$P(ABC)$	b_1c_1	b_1c_2	b_2c_1	b_2c_2
a_1	0.26	0.13	0.05	0.16
a_2	0.15	0.18	0.01	0.06

Suppose the uncertainty about the observation with regard to binary variable A is quantified as

$$\hat{E}_A(a_1) = 0.3$$

$$\hat{E}_A(a_2) = 0.7$$

Note that their summation equals to unity, thereby satisfying the orthogonality condition. Given the Fuzzy evidence $\hat{E}_A$, the updated joint distribution as the result of the Fuzzy updating scheme described by the equation set (3.31) is provided in Table (3.2). At this

Table 3.2: Updated joint distribution $P'(ABC)$ given $\hat{E}_A$

$P'(ABC)$	b_1c_1	b_1c_2	b_2c_1	b_2c_2
a_1	0.130	0.065	0.025	0.080
a_2	0.263	0.315	0.021	0.105

stage, we need to check the updated marginal distribution $P'(A)$ of variable A and verify that the updated marginal and the evidence are equal. Summing the joint distribution over A provides the marginal for the variable A. That is,

$$P'(A) = \sum_{i=1}^{2} P(a_i BC)$$

$$P'(A) = \begin{cases} 0.3 & \text{if} \quad A = a_1 \\ 0.7 & \text{if} \quad A = a_2 \end{cases}$$

Hence

$$P'(A) = \hat{E}_A(A) \tag{3.32}$$

This is indicative of the fact that the Fuzzy updating process given only one evidence variable results in a reliable updated joint distribution. Next, we will review the results of the updating process given multiple Fuzzy evidence.

Consider, now, in addition to the observations about variable A there has also been some uncertain observation about the variable B defined by Fuzzy evidence:

$$\hat{E}_B(b_1) = 0.85$$
$$\hat{E}_B(b_2) = 0.15$$

By applying the Fuzzy updating process consecutively, so that first the Fuzzy evidence $\hat{E}_A$ and then the Fuzzy evidence $\hat{E}_B$ is introduced to the joint distribution, we obtain the results for the updated joint distribution provided in Table (3.3).

Table 3.3: Updated joint distribution $P'(ABC)$ given $\hat{E}_A$ and $\hat{E}_B$

$P'(ABC)$	b_1c_1	b_1c_2	b_2c_1	b_2c_2
a_1	0.143	0.072	0.016	0.052
a_2	0.289	0.347	0.014	0.068

The resulting updated marginal distributions for variables A an B are:

$$P'(A) = \begin{cases} 0.28 & \text{if} \quad A = a_1 \\ 0.72 & \text{if} \quad A = a_2, \end{cases} \qquad P'(B) = \begin{cases} 0.85 & \text{if} \quad B = b_1 \\ 0.15 & \text{if} \quad B = b_2 \end{cases}$$

From which we see that the observation about variable B remains unchanged, as it should be, after the Fuzzy updating process, whereas the updated marginal of variable A

differs from the given evidence distribution. This discrepancy is the result of the consecutive application of the multiple updating processes due to the introduction of multiple fuzzy evidence variables. We can see this from the fact that the updated marginal $P'(B)$ reflects the same distribution for fuzzy evidence variable B, since it is the last one that updates the joint distribution. Therefore the impact of fuzzy evidence A has been affected and the updated marginal $P'(A)$ could not be equal to the evidence. In an information theoretic sense, this proposition indicates that, in the universe which also includes the updated joint distribution of Table (3.3), there exists better candidates with less relative entropy, i.e., in terms of the K-L distance, closer to the prior joint distribution, satisfying the constraint set of two given fuzzy evidences. When multiple Fuzzy evidence exits, to reach a better result, one can employ an iterative process by simply applying the same fuzzy updating methodology, such that the last updated joint and marginal distributions now become the original (i.e., prior) distributions and the same set Fuzzy evidence is applied until the updated marginals converges to the given distributions of the evidence variables. A similar approach has also been introduced independently by [21]. A single iteration of the Fuzzy updating process to our example results in unchanged marginals for both evidence variables A and B. The results of this single step iteration is provided in Table (3.4).

Table 3.4: Iterative updating of $P'(ABC)$ given $\hat{E}_A$ and $\hat{E}_B$

Iteration 0						
$P'(ABC)$	b_1c_1	b_1c_2	b_2c_1	b_2c_2	$P'(A)$	$P'(B)$
a_1	0.143	0.072	0.016	0.052	$P'(a_1) = 0.28$	$P'(b_1) = 0.85$
a_2	0.289	0.347	0.014	0.068	$P'(a_2) = 0.72$	$P'(b_2) = 0.15$
Iteration 1						
$P'(ABC)$	b_1c_1	b_1c_2	b_2c_1	b_2c_2	$P'(A)$	$P'(B)$
a_1	0.152	0.076	0.017	0.054	$P'(a_1) = 0.3$	$P'(b_1) = 0.85$
a_2	0.283	0.339	0.013	0.066	$P'(a_2) = 0.7$	$P'(b_2) = 0.15$

Note that at the end of the first iteration we have already reached convergence. That is,

$$P'(A) = \hat{E}_A(A) \quad \text{and} \quad P'(B) = \hat{E}_B(B)$$

The major issue with this iterative process is that it increases the complexity of the inherently already NP-hard Bayesian inferencing algorithm.

When applied as an inferencing method to update a joint probability distribution in the light of evidence, the MRE criteria amounts to an iterative process. Due to the increased complexity, the whole Fuzzy updating algorithm becomes too expensive for Bayesian Networks, when multiple evidence variable are introduced consecutively one at a time.

Within this context, we also present an alternative to the iterative updating, which involves simultaneously updating the joint distribution when multiple uncertain evidence is introduced.

Being a variant of the Fuzzy updating process, simultaneous Fuzzy updating uses the same equation set (3.31) based on Jeffrey's rule of conditioning. However, it relies on the assumption that when multiple evidence of uncertain nature are introduced, their uncertainty distributions are conditionally independent. Hence, we can introduce a joint distribution of multiple Fuzzy evidence and define it by the product of individual marginal uncertainty distributions of multiple evidence. That is, given the fuzzy evidence

$$\hat{\mathbf{E}}_A^T = \{e_{a_1}, e_{a_2}\} \quad \text{and} \quad \hat{\mathbf{E}}_B^T = \{e_{b_1}, e_{b_2}\}$$

we obtain the joint fuzzy evidence

$$\hat{\mathbf{E}}_{AB}^T = \{(e_{a_1} \cdot e_{b_1}), (e_{a_1} \cdot e_{b_2}), e_{a_2} \cdot e_{b_1}), (e_{a_2} \cdot e_{b_2})\}. \tag{3.33}$$

When equation (3.33) is applied to our numerical example the resulting joint Fuzzy evidence is

$$\hat{\mathbf{E}}_{AB}^T = \{0.255,\ 0.045,\ 0.595,\ 0.105\}. \tag{3.34}$$

Using the joint evidence $\hat{\mathbf{E}}_{AB}^T$ we obtain the the following updated joint probability distribution $P'(ABC)$:

Table 3.5: $P'(ABC)$given the joint Fuzzy evidence $\hat{\mathbf{E}}_{AB}^T$

$P'(ABC)$	b_1c_1	b_1c_2	b_2c_1	b_2c_2	$P'(A)$	$P'(B)$
a_1	0.170	0.085	0.011	0.034	$P'(a_1) = 0.3$	$P'(b_1) = 0.85$
a_2	0.270	0.325	0.018	0.088	$P'(a_2) = 0.7$	$P'(b_2) = 0.15$

Table (3.5) shows that without having to resort to an iterative process, we obtain a satisfactory result for the updated joint probability distribution $P'(ABC)$, which results in the correct marginal distributions $P'(A)$ and $P'(B)$ for evidence variables.

We can, therefore, conclude that, when multiple uncertain evidence is present, simultaneous fuzzy updating should be preferred over consecutive fuzzy updating especially when dealing with moderate to large sized Bayesian Networks for which the inherent complexity is already known to be high.

3.4 Fuzzy-Bayesian Networks

Historically, uncertainty about real world systems has been modeled by probabilistic tools which are crisp, deterministic and precise in character. However, as discussed in earlier sections, different forms of uncertainty exist and probability theory only addresses the randomness aspect of it. Fuzzy Set theory provides a means for representing uncertainty due to vagueness such as the uncertainty in natural language.

However, uncertainty due to vagueness (i.e., fuzziness), in fact, exists not only in human cognition and languages, but also in most systems modeled by Bayesian Networks. Consider variables such as *temperature, age* or *speed*, which are inherently continuous but represented as discrete when included in a Bayesian Network. For such variables an implicit mapping is involved whenever an observation (i.e., evidence) about them needs to be introduced to the model. Axioms of probability dictate that once such a mapping is performed on a continuous variable the resulting discrete states must cover the whole domain of the original variable and be mutually exclusive, so that every single observation falls into one and only one state and no two states co-exist at the same space and time. For the purposes of approximation and in cases without a pressing need for accuracy, such a quantification may be justifiable. However, not every continuous variable behaves sensibly under discretization. Consider *temperature* defined on the frame $[0, 40]C^o$ and we decided to use a three state discretization scheme *cold, warm*, and *hot* corresponding to the intervals $[0, 10)C^o$, $[10, 25)C^o$, $[25, 40]C^o$, respectively. A reading of $24.9C^o$ from the thermometer would fall under discrete state *cold*. Whereas, $25C^0$ would be labeled as *warm*. We believe that there is no meaningful way of determining a crisp boundary between these states. Hence, using classical sets with crisp boundaries when discretizing a continuous domain may generate some unpredictable results for Bayesian Networks

Right at this point, the concept of *fuzziness* becomes very interesting. In fact, Fuzzy Set theory and its implementation of degrees of membership idea to sets provides a structured way to improve on classical discretization techniques. Nevertheless, the distinction between Fuzzy Set theory and probability theory should be made clear. *Fuzziness describes the ambiguity of an event, whereas randomness describes the uncertainty in the occurrence of the event* [29]. Within this context, we see promise in combining the two concepts to complement each other, so that various limitations of classical Bayesian Networks will be overcome by the resulting *hybrid* methodology.

Let us now formally define general Hybrid Bayesian Networks already introduced in Section (3.1) and provide some key notation used throughout the text, then we proceed to define Fuzzy-Bayesian Networks and consequently present a proposed inferencing methodology for it.

Definition 4 *General Hybrid Bayesian Networks:*

Hybrid Bayesian Networks (HBNs) are directed acyclic graphical structures that allow us to represent a domain of interest in terms of a joint probability distribution over a set of variables $\mathbf{V}$, including a set of discrete variables $\boldsymbol{\Delta}$, such that $\boldsymbol{\Delta} \subseteq \mathbf{V}$, and a set of continuous variables $\boldsymbol{\Gamma}$, such that $\boldsymbol{\Gamma} \subseteq \mathbf{V}$. Hence, there are two key elements defining a HBN

- A network structure in the form of a directed, acyclic graph $\mathcal{G}$ where
  - each node corresponds to a variable $\Delta_k \in \boldsymbol{\Delta}$ or $\Gamma_l \in \boldsymbol{\Gamma}$ for $k = \{1, \ldots, m\}$, $l = \{1, \ldots, n\}$ and
  - the directed vertices connecting the nodes are denoted by a set of links $\mathbf{L}$.
- A set of conditional probability distributions $\mathbf{P}$, one for each node (i.e., variable) in the graph $\mathcal{G}$.

Consequently, in a concise way, we can represent a general Hybrid Bayesian Network as follows:

$$\begin{aligned} \mathcal{G}_{\text{HBN}} &= [\mathbf{V}, \mathbf{L}, \mathbf{P}] \\ &= [\boldsymbol{\Delta}, \boldsymbol{\Gamma}, \mathbf{L}, \mathbf{P}] \end{aligned} \tag{3.35}$$

The set of conditional probability distributions $\mathbf{P}$ in the above representation requires further elaboration to identify the nuances of various different forms it entails in a general HBN. Regardless of its descriptors, such as conditional, joint, or marginal, a probability distribution of a random variable V_i is denoted by $P(V_i)$. It refers to a mass function or a density function for a discrete or a continuous variable, respectively. Within this context, we can denote the set $\mathbf{P}$ by

$$\mathbf{P} = \bigcup_i P_i, \quad i = \{1, \ldots, m + n\} \tag{3.36}$$

$$P_i = P\{V_i | \boldsymbol{\Delta}_{\text{Par}}(V_i),\ \boldsymbol{\Gamma}_{\text{Par}}(V_i)\} \tag{3.37}$$

Where P_i is the CPD for the i^{th} variable V_i, and $\boldsymbol{\Delta}_{\text{Par}}(V_i)$, and $\boldsymbol{\Gamma}_{\text{Par}}$ are the sets of discrete and continuous parents of variable V_i respectively.

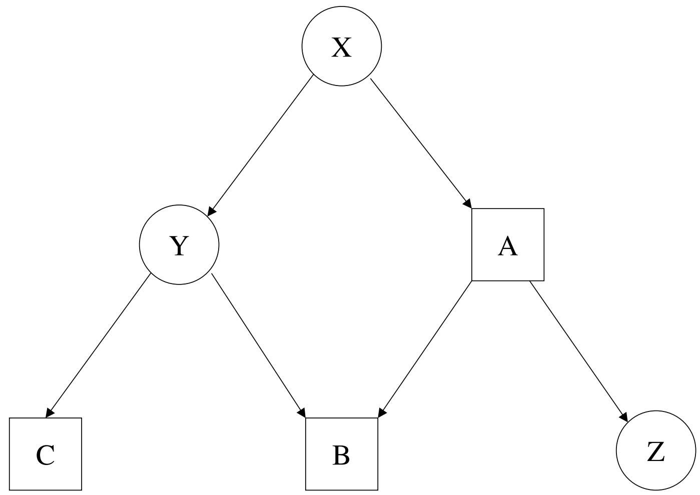


Figure 3.2: A *general* Hybrid Bayesian Network

An abstraction for a general HBN in the form of a graphical structure is provided in Figure (3.2).

Applying the notation introduced in Definition 4, we can identify the following sets of discrete and continuous variables.

$$\mathbf{\Delta} = \{A, B, C\}$$

$$\mathbf{\Gamma} = \{X, Y, Z\}$$

Additionally, note that, in Figure 3.2, discrete variables are depicted by square shaped nodes, whereas continuous variables are depicted by circle or ellipse shaped nodes, which is the convention commonly used in the literature to portray an HBN.

Recalling our earlier discussions on the Lauritzen's algorithm, although it is the state of art *exact* inferencing algorithm for mixed Bayesian Networks, it should be noted that it can only be applied to a limited subset of HBNs where no discrete variable is allowed to have continuous parent variables. Whereas, we should underline the fact that our definition and the notation thereof do not exclude that possibility and therefore is labeled as *general* HBNs.

Furthermore, we argue that the restricted usage of continuous variables in HBNs, in fact, diminishes their ability of modeling real-world complex systems in a much better and accurate way. One can argue that compromises in representation are needed to accomplish an exact reasoning about the problem domain. However, we strongly believe that, especially when less-than-adequately defined applications, especially when risk analysis of complex system are concerned, being able to build a representative model of the problem domain without any algorithm-imposed structural limitations, outweighs the benefits of using an exact algorithm on a model that represents the problem domain poorly. Nevertheless, it should be also be noted that a general representation for the probability distribution of a discrete variable given mixed parents (i.e., continuous and discrete) does not exist. Therefore approximation is needed in this type of case.

Fuzzy-Bayesian Networks emerge as powerful tools that combine the representation power of Fuzzy Set theory over poorly defined problem domains with the algorithmic strength of Bayesian Networks. Next, we propose two types of Fuzzy-Bayesian Networks. Each network represents a certain level of approximation of the problem domain of mixed nature. Progressing from the first type to the second type FBN, in fact, corresponds to an increase in accuracy toward exact inferencing, however, with the additional cost of lost generality and higher computational complexity. As a prerequisite for these two types of FBNs, parts of the network are required first to be transformed to Fuzzy domain from its original crisp environment. Approximations of a similar structural nature for general HBNs has been proposed by [30], however, without much detailed analysis and formal structure for evidence updating (i.e., inferencing). As a major improvement we propose a prerequisite step of *Crisp* to *Fuzzy* transformation, where selected parts of the network are converted to a set of Fuzzy variables and a corresponding set of Fuzzy conditional probability distribution. We also propose an exact inferencing methodology, which can treat uncertain observations as Fuzzy evidences and incorporate them into the two proposed FBN inferencing algorithms using a Fuzzy updating scheme.

Let us first start with the crisp to fuzzy transformation of a general HBN and the making of a general FBN, then we introduce two types of FBNs as approximations of general HBNs with varying degrees of accuracy and computational complexity.

3.4.1 General Fuzzy-Bayesian Networks

Given a general HBN, a general Fuzzy-Bayesian Network can be constructed by transforming all continuous variables and associated conditional probability distributions into the Fuzzy Set domain. In order to perform such a transformation the Fuzzy Sets and corresponding membership functions based on the frames of all continuous variables in the HBN need to be given or constructed first.

Recall that the joint probability distribution induced by a Bayesian Network representing n random variables $V_1, \ldots, V_n$ is defined by the *Chain Rule*:

$$P(V_1, \ldots, V_n) = \prod_{i=1}^{n} P(V_i \mid Par(V_i)) \tag{3.38}$$

Then, for a general HBN composed of n discrete variables $\Delta_1, \ldots, \Delta_n$, $\forall \Delta_i \in \boldsymbol{\Delta}$, $i = \{1, \ldots, n\}$ and m continuous variables $\Gamma_1, \ldots, \Gamma_m$, $\forall \Gamma_j \in \boldsymbol{\Gamma}$, $j = \{1, \ldots, m\}$ the joint distribution induced by the $\mathcal{G}_{\text{HBN}}$ is

$$P(\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_m) = \prod_{i=1}^{n} P(\Delta_i \mid Par(\Delta_i)) \prod_{j=1}^{m} P(\Gamma_j \mid Par(\Gamma_j)) \tag{3.39}$$

where $Par(\Delta_i)$ and $Par(\Gamma_j)$ refer to the parents of discrete and continuous variables, respectively.

Consider the general Hybrid Bayesian Network $\mathcal{G}_{\text{HBN}}$ depicted in Figure 3.2. Then the joint distribution induced by the network is

$$P(\mathbf{V}) = P(A|X)P(B|A,Y)P(C|Y)P(X)P(Y|X)P(Z|A) \tag{3.40}$$

where $\mathbf{V} = A, B, C, X, Y, Z$. We can see from equation (3.40) that the joint probability distribution is defined by six individual probability distribution, marginal and joint over clusters of random variables making up the $\mathcal{G}_{\text{HBN}}$. We can identify five types of distributions according to the the random variables comprising the following parent-child clusters:

i. Discrete given continuous: $P(A|X)$ and $P(C|Y)$

ii. Discrete given continuous and discrete: $P(B|A,Y)$

iii. Continuous: $P(X)$

iv. Continuous given continuous: $P(Y|X)$

v. Continuous given discrete: $P(Z|A)$

Throughout this chapter we introduced a succession of novel ideas dealing with the transformation of the crisp domain of a classical Bayesian Net to the hybrid domain of a Fuzzy-Bayesian Net. The concept of Fuzzy Bayes formula of section (3.2.1) is particularly important and lies at the crux of the Fuzzy Bayes transformation.

We construct the Fuzzy counterparts of the probability distributions defining the joint distribution of equation (3.40). The resulting fuzzy conditionals are variants of equations (3.17), (3.18), (3.19), and (3.20), therefore the detailed derivations are not provided here. Note that only continuous variables are subjected to the Fuzzy transformation. The discrete variables A,B, and C do not require fuzzification. However, when uncertain observations about them are used as evidence to update the joint distribution (i.e., inferencing) we employ the fuzzy updating methodology of section (3.3).

Consider the fuzzy variables $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$, and $\hat{\mathbf{Z}}$ defined on the frames $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ of the crisp variables X, Y, and Z respectively. That is,

$$\begin{aligned} \hat{\mathbf{X}} &= \{\hat{X}_1, \ldots, \hat{X}_m\} \\ \hat{\mathbf{Y}} &= \{\hat{Y}_1, \ldots, \hat{Y}_n\} \\ \hat{\mathbf{Z}} &= \{\hat{Z}_1, \ldots, \hat{Z}_r\} \end{aligned} \tag{3.41}$$

where

$$\begin{aligned} \hat{X}_i &= \{x,\ \hat{X}_i(x)|\ x \in \mathbf{X}\}, \qquad i = \{1, \ldots, m\} \\ \hat{Y}_j &= \{y,\ \hat{Y}_j(y)|\ y \in \mathbf{Y}\}, \qquad j = \{1, \ldots, n\} \\ \hat{Z}_k &= \{z,\ \hat{Z}_k(z)|\ z \in \mathbf{Z}\}, \qquad k = \{1, \ldots, r\} \end{aligned} \tag{3.42}$$

where x, y, and z are the actual values that variables X, Y, and Z take, respectively. Note that the Fuzzy random variables above in equation (3.41) are defined by bold uppercase letters contrary to our convention that random variables are denoted by uppercase letters only. When a continuous variable is discretized by Fuzzy transformation, in effect, a set of Fuzzy states is created. These Fuzzy states are themselves Fuzzy Sets defined individually and separately on the frame of the original continuous variable. In order to satisfy the completeness axiom of probability theory for a given value of the original variable its degree of membership to these Fuzzy Sets should add up to one (i.e., they have to be orthogonal). For example, in equations (3.41), and (3.42) m, n and r denote the number of Fuzzy states that the Fuzzy variables $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$, and $\hat{\mathbf{Z}}$ are consist of respectively.

Now, further assume that the discrete variables A, B, and C are composed of α, β, and ω number of states, respectively. That is,

$$\begin{aligned} A &= \{a_1, \ldots, a_f\}, \qquad & f &= \{1, \ldots, \alpha\} \\ B &= \{b_1, \ldots, b_g\}, \qquad & g &= \{1, \ldots, \beta\} \\ C &= \{c_1, \ldots, c_h\}, \qquad & h &= \{1, \ldots, \omega\} \end{aligned} \tag{3.43}$$

Then, using the *factor* $f(\ldots|\ldots)$ notation introduced in Chapter 2, the distributions representing the Fuzzy-Bayesian Net counterpart of the original HBN can be written as follows,

i. Crisp Discrete given Fuzzy Continuous: $P(A|X)$ and $P(C|Y)$

For $P(A|X) \overset{fuzzify}{\rightarrow} P(A|\hat{\mathbf{X}})$:

$$f(A, \hat{\mathbf{X}}) = \begin{array}{c|c|c} A & \hat{\mathbf{X}} & \\ \hline a_1 & \hat{X}_1 & P(a_1|\hat{X}_1) \\ a_1 & \hat{X}_2 & P(a_1|\hat{X}_2) \\ \vdots & \vdots & \vdots \\ a_f & \hat{X}_i & P(a_f|\hat{X}_i) \\ \vdots & \vdots & \vdots \\ a_{\alpha-1} & \hat{X}_{m-1} & P(a_\alpha|\hat{X}_{m-1}) \\ a_\alpha & \hat{X}_m & P(a_\alpha|\hat{X}_m) \end{array}$$

where

$$P(a_f|\hat{X}_i) = \frac{\displaystyle\int_{\mathbf{X}} \hat{X}_i(x)\, P(a_f|x)\, P(x)\, \mathrm{d}x}{\displaystyle\int_{\mathbf{X}} \hat{X}_i(x)\, P(x)\, \mathrm{d}x} \tag{3.44}$$

where $\hat{X}_i$ stands for the i^{th} Fuzzy state of the Fuzzy variable $\hat{\mathbf{X}}$ and a_f for the f^{th} state of the crisp discrete variable A; $\hat{X}_i(x)$ denotes the fuzzy membership function for the Fuzzy state $\hat{X}_i$. Note also that $P(x)$ is a probability density function, whereas $P(a_f|x)$ and $P(a_f|\hat{X}_i)$ are probability mass functions.

For $P(C|Y) \stackrel{fuzzify}{\rightarrow} P(C|\hat{\mathbf{Y}})$:

$$f(C, \hat{\mathbf{Y}}) = \begin{array}{c|c|c} C & \hat{\mathbf{Y}} & \\ \hline c_1 & \hat{Y}_1 & P(c_1|\hat{Y}_1) \\ c_1 & \hat{Y}_2 & P(c_1|\hat{Y}_2) \\ \vdots & \vdots & \vdots \\ c_h & \hat{Y}_j & P(c_h|\hat{Y}_j) \\ \vdots & \vdots & \vdots \\ c_{\omega-1} & \hat{Y}_{n-1} & P(c_\omega|\hat{Y}_{n-1}) \\ c_\omega & \hat{Y}_n & P(c_\omega|\hat{Y}_n) \end{array}$$

where

$$P(c_h|\hat{Y}_j) = \frac{\int\limits_{\mathbf{Y}} \hat{Y}_j(y)\, P(c_h|y)\, P(y)\, \mathrm{d}y}{\int\limits_{\mathbf{Y}} \hat{Y}_j(y)\, P(y)\, \mathrm{d}y} \tag{3.45}$$

again, where $\hat{Y}_j$ stands for the j^{th} Fuzzy state of the Fuzzy variable $\hat{\mathbf{Y}}$ and c_h for the h^{th} state of the crisp discrete variable C. $\hat{Y}_j(y)$ denotes the Fuzzy membership function for the Fuzzy state $\hat{Y}_j$.

ii. Discrete given continuous and discrete: $P(B|A, Y)$

For $P(B|A, Y) \stackrel{fuzzify}{\rightarrow} P(B|A, \hat{\mathbf{Y}})$:

$$f(B, A, \hat{\mathbf{Y}}) = \begin{array}{c|c|c|c} B & A & \hat{\mathbf{Y}} & \\ \hline b_1 & a_1 & \hat{Y}_1 & P(b_1|a_1, \hat{Y}_1) \\ b_1 & a_1 & \hat{Y}_2 & P(b_1|a_1, \hat{Y}_2) \\ \vdots & \vdots & \vdots & \vdots \\ b_g & a_f & \hat{Y}_j & P(b_g|a_f, \hat{Y}_j) \\ \vdots & \vdots & \vdots & \vdots \\ b_{\beta-1} & a_\alpha & \hat{Y}_{n-1} & P(b_\beta|a_\alpha, \hat{Y}_{n-1}) \\ b_\beta & a_\alpha & \hat{Y}_n & P(b_\beta|a_\alpha, \hat{Y}_n) \end{array}$$

where

$$P(b_g|a_f, \hat{Y}_j) = \frac{\int\limits_{\mathbf{Y}} \hat{Y}_j(y)\, P(b_g|a_f, x)\, P(y)\, \mathrm{d}y}{\int\limits_{\mathbf{Y}} \hat{Y}_j(y)\, P(y)\, \mathrm{d}y} \tag{3.46}$$

where b_g stands for the g^{th} state of the crisp discrete variable B.

iii. Continuous: $P(X)$

For $P(X) \overset{fuzzify}{\rightarrow} P(\hat{\mathbf{X}})$:

$$f(\hat{\mathbf{X}}) = \begin{array}{c|c} \hat{\mathbf{X}} & \\ \hline \hat{X}_1 & P(a_1|\hat{X}_1) \\ \vdots & \vdots \\ \hat{X}_i & P(a_f|\hat{X}_i) \\ \vdots & \vdots \\ \hat{X}_m & P(a_\alpha|\hat{X}_m) \end{array}$$

where

$$P(\hat{X}_i) = \int\limits_{\mathbf{X}} \hat{X}_i(x)\, P(x)\, \mathrm{d}x \tag{3.47}$$

iv. Continuous given continuous: $P(Y|X)$

For $P(Y|X) \overset{fuzzify}{\rightarrow} P(\hat{\mathbf{Y}}|\hat{\mathbf{X}})$:

$$f(\hat{\mathbf{Y}}, \hat{\mathbf{X}}) = \begin{array}{c|c|c} \hat{Y}_1 & \hat{\mathbf{X}} & \\ \hline \hat{Y}_1 & \hat{X}_1 & P(\hat{Y}_1|\hat{X}_1) \\ \hat{Y}_2 & \hat{X}_2 & P(\hat{Y}_2|\hat{X}_2) \\ \vdots & \vdots & \vdots \\ \hat{Y}_j & \hat{X}_i & P(\hat{Y}_2|\hat{X}_i) \\ \vdots & \vdots & \vdots \\ \hat{Y}_{n-1} & \hat{X}_{m-1} & P(\hat{Y}_{n-1}|\hat{X}_{m-1}) \\ \hat{Y}_n & \hat{X}_m & P(\hat{Y}_n|\hat{X}_m) \end{array}$$

where

$$P(\hat{Y}_j|\hat{X}_i) = \frac{\int_{\mathbf{Y}}\int_{\mathbf{X}} \hat{Y}_j(y)\,\hat{X}_i(x)\,P(y|x)\,P(x)\,\mathrm{d}y\,\mathrm{d}x}{\int_{\mathbf{X}} \hat{X}_i(x)\,P(x)\,\mathrm{d}x} \tag{3.48}$$

v. Continuous given discrete: $P(Z|A)$

For $P(Z|A) \stackrel{fuzzify}{\rightarrow} P(\hat{\mathbf{Z}}|A)$:

$$f(\hat{\mathbf{Z}}|A) = \begin{array}{c|c|c} \hat{\mathbf{Z}} & A & \\ \hline \hat{Z}_1 & a_1 & P(\hat{Z}_1|a_1) \\ \hat{Z}_1 & a_1 & P(\hat{Z}_1|a_1) \\ \vdots & \vdots & \vdots \\ \hat{Z}_k & a_f & P(\hat{Z}_k|a_f) \\ \vdots & \vdots & \vdots \\ \hat{Z}_{r-1} & a_{\alpha-1} & P(\hat{Z}_{r-1}|a_{\alpha-1}) \\ \hat{Z}_r & a_\alpha & P(\hat{Z}_r|a_\alpha) \end{array}$$

where

$$P(\hat{Z}_k|a_f) = \int_{\mathbf{Z}} \hat{Z}_k(z)\,P(z|a_f)\,\mathrm{d}z \tag{3.49}$$

and $\hat{Z}_k(z)$ is the fuzzy membership function for Fuzzy state $\hat{Z}_k$ of Fuzzy variable $\hat{\mathbf{Z}}$.

Now we can formally define Fuzzy-Bayesian Networks.

Definition 5 *Fuzzy-Bayesian Networks:*

Consider a general HBN defined by equation (3.35), such that

$$\begin{aligned} \mathcal{G}_{\mathrm{HBN}} &= [\mathbf{V}, \mathbf{L}, \mathbf{P}] \\ &= [\boldsymbol{\Delta}, \boldsymbol{\Gamma}, \mathbf{L}, \mathbf{P}] \end{aligned}$$

where $\boldsymbol{\Delta}$ and $\boldsymbol{\Gamma}$ are the sets of discrete and continuous variables, respectively, $\mathbf{L}$ is the set of directed vertices connecting the variables and making up the directed acyclic graph $\mathcal{G}_{\mathrm{HBN}}$, and finally $\mathbf{P}$ stands for the set of conditional distributions imposed by $\mathcal{G}_{\mathrm{HBN}}$. Then, the Fuzzy-Bayesian Net $\mathcal{G}_{\mathrm{FBN}}$ based on the hybrid Bayesian Net is defined by

$$\mathcal{G}_{\mathrm{FBN}} = [\boldsymbol{\Delta}, \hat{\boldsymbol{\Gamma}}, \mathbf{L}, \hat{\mathbf{P}}] \tag{3.50}$$

where $\hat{\mathbf{\Gamma}}$ represents the set of Fuzzy counterparts of the continuous variables constituting set $\mathbf{\Gamma}$ and $\hat{\mathbf{P}}$ stands for the set of conditional distributions imposed by the FBN, such that

$$\hat{\mathbf{P}} = \hat{\mathbf{P}}_{\mathbf{\Delta}} \cup \hat{\mathbf{P}}_{\hat{\mathbf{\Gamma}}} \tag{3.51}$$

where

$$\begin{aligned} \hat{\mathbf{P}}_{\mathbf{\Delta}} &= P(\Delta|\mathrm{Par}_{\hat{\mathbf{\Delta}}}(\Delta), \mathrm{Par}_{\hat{\mathbf{\Gamma}}}(\Delta)), \qquad \Delta \in \mathbf{\Delta} \\ \hat{\mathbf{P}}_{\hat{\mathbf{\Gamma}}} &= P(\hat{\Gamma}|\mathrm{Par}_{\hat{\mathbf{\Delta}}}(\hat{\Gamma}), \mathrm{Par}_{\hat{\mathbf{\Gamma}}}(\hat{\Gamma})), \qquad \hat{\Gamma} \in \hat{\mathbf{\Gamma}} \end{aligned} \tag{3.52}$$

where $\hat{\mathbf{P}}_{\mathbf{\Delta}}$ and $\hat{\mathbf{P}}_{\hat{\mathbf{\Gamma}}}$ represent the transformed conditional probability distributions of crisp discrete and Fuzzy variables, respectively. Note that, regardless crisp discrete or Fuzzy continuous, all parents in a general Fuzzy-Bayesian Network are actually discrete.

In our example, equations (3.44) through (3.49) constitute the set $\hat{\mathbf{P}}$. In fact, they provide a means to transform the Hybrid Bayesian Net given in Figure 3.2 into a Fuzzy-Bayesian Net. The graphical depiction of the resulting FBN, $\mathcal{G}_{\mathrm{FBN}}$ is provided in Figure 3.3.

The conditional distributions imposed by the $\mathcal{G}_{\mathrm{FBN}}$ are all now converted to *multinomials*. Hence, exact inferencing algorithms such as *variable elimination* or *junction tree algorithm* can now be applied to these multinomials to determine the joint probability distribution represented by the $\mathcal{G}_{\mathrm{FBN}}$.

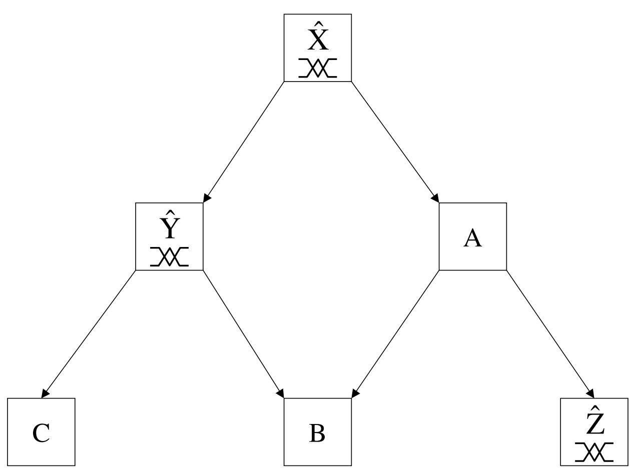


Figure 3.3: The Fuzzy-Bayesian Network $\mathcal{G}_{\mathrm{FBN}}$ derived from the $\mathcal{G}_{\mathrm{HBN}}$ of Figure 3.2

It is the complete transfer of all the crisp continuous variables to Fuzzy discrete variables that makes exact inferencing in the resulting $\mathcal{G}_{\text{FBN}}$ possible. However, two compromises were made along the way. First, the Fuzzy membership functions partitioning each continuous domain are usually build by subject matter expert input and therefore are only approximations, in addition to that, we also need the marginal distributions of the continuous variables undergoing fuzzy transformation. In most cases these marginals do not exit and we can only start the analysis with some initial estimates. We believe it is worthwhile to explore the applicability of an iterative algorithm to improve the accuracy of the initial Fuzzy membership function and marginal distributions of the continuous variables subject to fuzzification.

The resulting FBN is also suitable for Fuzzy updating introduced in section (3.3.2) when uncertain evidence in introduced to the net. We develop an adaptation of the junction tree algorithm for this purpose.

Next, we introduce two types of Fuzzy transformations. Each transformation uses the Fuzzy transformation approach detailed above as its initialization step to build the basic $\mathcal{G}_{\text{FBN}}$ in their respective algorithms. After the basic $\mathcal{G}_{\text{FBN}}$ is generated and its conditional distributions are populated by their fuzzy counterparts, each type augments this initial construct to achieve a better accuracy in representation.

Definition 6 *Fuzzy-Bayesian Network Type-I:*

Given a general Fuzzy-Bayesian net $\mathcal{G}_{\text{FBN}}$ as defined by equation (3.50), we define the Type-I Fuzzy-Bayesian net FBN as $\mathcal{G}_{\text{FBN}^I}$, such that,

$$\mathcal{G}_{\text{FBN}^I} = [\mathbf{\Delta}, \mathbf{\Gamma}, \hat{\mathbf{\Gamma}}, \mathbf{L}^I, \hat{\mathbf{P}}^I] \tag{3.53}$$

Note that in *Type-I* Fuzzy-Bayesian Network, we include the original continuous variables $\mathbf{\Gamma}$ in the acyclic graphical structure of the FBN along with their fuzzified counterparts $\hat{\mathbf{\Gamma}}$.

The directed link structure of *Type-I* FBN is denoted by $\mathbf{L}^I$. This new acyclic graphical structure is different from the original such that, for each added original continuous variable Γ, a directed link from the fuzzy counterpart $\hat{\Gamma}$ to the original continuous variable Γ is added to the original link set $\mathbf{L}$. That is,

$$\mathbf{L}^I = \mathbf{L} \cup \mathbf{L}_{\hat{\Gamma} \to \Gamma} \tag{3.54}$$

The set of conditional probability distributions $\hat{\mathbf{P}}^I$ is defined by

$$\hat{\mathbf{P}}^I = \hat{\mathbf{P}}_{\boldsymbol{\Delta}} \cup \hat{\mathbf{P}}_{\hat{\boldsymbol{\Gamma}}} \cup \hat{\mathbf{P}}_{\boldsymbol{\Gamma}} \tag{3.55}$$

where

$$\begin{aligned}
\hat{\mathbf{P}}_{\boldsymbol{\Delta}} &= P(\Delta | \mathrm{Par}_{\boldsymbol{\Delta}}(\Delta), \mathrm{Par}_{\hat{\boldsymbol{\Gamma}}}(\Delta)), \qquad \Delta \in \boldsymbol{\Delta} \\
\hat{\mathbf{P}}_{\hat{\boldsymbol{\Gamma}}} &= P(\hat{\Gamma} | \mathrm{Par}_{\boldsymbol{\Delta}}(\hat{\Gamma}), \mathrm{Par}_{\hat{\boldsymbol{\Gamma}}}(\hat{\Gamma})), \qquad \hat{\Gamma} \in \hat{\boldsymbol{\Gamma}} \\
\hat{\mathbf{P}}_{\boldsymbol{\Gamma}} &= P(\Gamma | \hat{\Gamma}), \qquad \Gamma \in \boldsymbol{\Gamma}
\end{aligned} \tag{3.56}$$

$\hat{\mathbf{P}}_{\boldsymbol{\Delta}}$ and $\hat{\mathbf{P}}_{\hat{\boldsymbol{\Gamma}}}$ are, as in equation set (3.52), multinomials and they represent the transformed conditional probability distributions of crisp discrete and Fuzzy variables, respectively. $\hat{\mathbf{P}}_{\boldsymbol{\Gamma}}$, on the other hand, denotes the conditional distribution of the original continuous variable Γ given its counterpart fuzzy variable $\hat{\Gamma}$ is approximated by a Conditional Gaussian (CG) introduced in Section 2.1.4 and defined by equation (2.14).

When applied to the $\hat{\Gamma} \to \Gamma$ pair within the context of the *Type-I* FBN, the conditional distribution of Γ given $\hat{\Gamma}$, $\hat{\mathbf{P}}_{\boldsymbol{\Gamma}} = P(\Gamma|\hat{\Gamma})$ in equation (3.56) is defined by

$$P(\Gamma = \gamma | \hat{\Gamma} = \hat{\gamma}) = \frac{1}{\sigma_{\hat{\gamma}}\sqrt{2\pi}} \exp\left(-\frac{(\gamma - \mu_{\hat{\gamma}})^2}{2\sigma_{\hat{\gamma}}^2} \right) \tag{3.57}$$

where γ and $\hat{\gamma}$ stands for the values of the variables Γ and $\hat{\Gamma}$, respectively. Note that for each fuzzy state $\hat{\gamma}$ of $\hat{\Gamma}$, there exist a pair of parameters $\{\mu_{\hat{\gamma}}, \sigma_{\hat{\gamma}}\}$.

With the introduction of equation (3.57), we are now able to define all the CPDs in a *Type-I* FBN formally and properly, hence, we can argue that *Type-I* FBNs are suitable approximations of general HBNs. The graphical depiction of the *Type-I* FBN $\mathcal{G}_{\mathrm{FBN}^I}$ is provided in Figure 3.4.

Definition 7 *Fuzzy-Bayesian Network Type-II*

The idea behind the *Type-I* FBN is to develop a methodology that converts all continuous variables in a generic HBN to fuzzy discrete counterpart variables, so that an exact inferencing algorithm such as Lauritzen's junction tree algorithm for Conditional Gaussians can subsequently be applied to the resulting acyclic graphical structure.

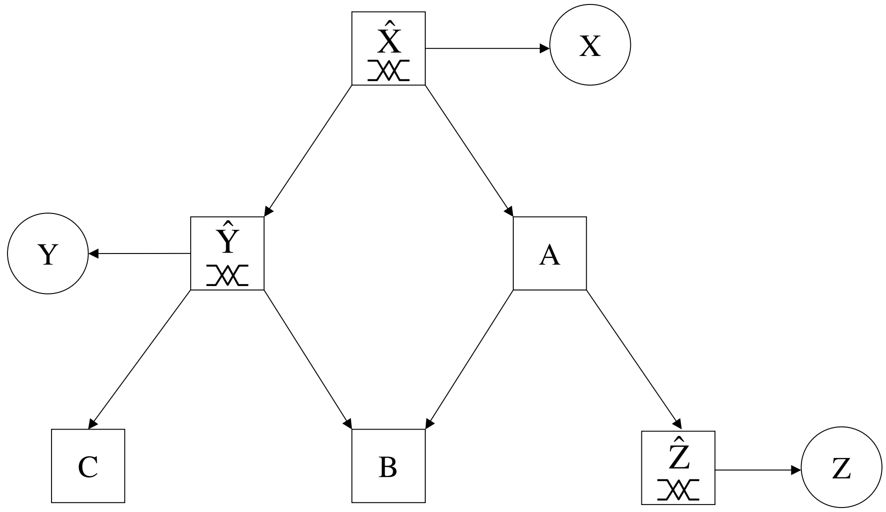


Figure 3.4: *Type-I* FBN $\mathcal{G}_{\mathrm{FBN}^I}$ derived from the $\mathcal{G}_{\mathrm{HBN}}$ of Figure 3.2

However, one can deduce from the definition of CG (see equation (3.123)) that not all the continuous variables in the HBN need to be fuzzified. We can also use an adopted version of the Junction tree algorithm for CGs, by applying the transformation defined for *Type-I* FBN only to *those set of continuous variables whose set of descendants include discrete variables.*

Formally, given a general Fuzzy-Bayesian Net $\mathcal{G}_{\mathrm{FBN}}$ as defined by equation (3.50), we define the *Type-II* Fuzzy-Bayesian net FBN as $\mathcal{G}_{\mathrm{FBN}^{II}}$, such that,

$$\mathcal{G}_{\mathrm{FBN}^{II}} = [\mathbf{\Delta}, \hat{\mathbf{\Omega}}, \mathbf{\Omega}, \mathbf{\Psi}, \mathbf{L}^{II}, \hat{\mathbf{P}}^{II}] \tag{3.58}$$

where $\mathbf{\Omega}$ denotes the set of continuous variables whose set of descendant nodes include discrete variables. $\mathbf{\Psi}$ is the set of the *remaining* continuous variables in the original set $\mathbf{\Gamma}$, such that $\mathbf{\Psi} = \mathbf{\Gamma}/\mathbf{\Omega}$. Whereas, $\hat{\mathbf{\Omega}}$ denotes the set of Fuzzy counterparts of the continuous variables in the set $\mathbf{\Omega}$. The set of directed links connecting the variables is denoted by $\mathbf{L}^{II}$, such that

$$\mathbf{L}^{II} = \mathbf{L} \cup \mathbf{L}_{\hat{\Omega}\to\Omega} \tag{3.59}$$

where $\mathbf{L}_{\hat{\Omega}\to\Omega}$ represents the set of directed links from a Fuzzy variable $\hat{\Omega}$ to its original continuous counterpart Ω. $\hat{\mathbf{P}}^{II}$ in the equation (3.58) denotes the set of CPD imposed by the $\mathcal{G}_{\mathrm{FBN}^{II}}$, such that

$$\hat{\mathbf{P}}^{II} = \hat{\mathbf{P}}_{\boldsymbol{\Delta}} \cup \hat{\mathbf{P}}_{\hat{\boldsymbol{\Omega}}} \cup \hat{\mathbf{P}}_{\boldsymbol{\Omega}} \cup \hat{\mathbf{P}}_{\boldsymbol{\Psi}} \tag{3.60}$$

where

$$\begin{aligned}
\hat{\mathbf{P}}_{\boldsymbol{\Delta}} &= P(\Delta|\mathrm{Par}_{\boldsymbol{\Delta}}(\Delta), \mathrm{Par}_{\hat{\boldsymbol{\Gamma}}}(\Delta)), \qquad \Delta \in \boldsymbol{\Delta} \\
\hat{\mathbf{P}}_{\hat{\boldsymbol{\Omega}}} &= P(\hat{\Omega}|\mathrm{Par}_{\boldsymbol{\Delta}}(\hat{\Omega}), \mathrm{Par}_{\hat{\boldsymbol{\Gamma}}}(\hat{\Omega})), \qquad \hat{\Omega} \in \hat{\boldsymbol{\Omega}} \\
\hat{\mathbf{P}}_{\boldsymbol{\Omega}} &= P(\Omega|\hat{\Omega}), \qquad \Omega \in \boldsymbol{\Omega} \\
\hat{\mathbf{P}}_{\boldsymbol{\Psi}} &= P\left(\Psi|\mathrm{Par}_{\boldsymbol{\Delta}}(\Psi), \mathrm{Par}_{\boldsymbol{\Gamma}}(\Psi)\right), \qquad \Psi \in \boldsymbol{\Psi}
\end{aligned} \tag{3.61}$$

In the equation set above, $\hat{\mathbf{P}}_{\boldsymbol{\Delta}}$ and $\hat{\mathbf{P}}_{\hat{\boldsymbol{\Omega}}}$ are multinomial (i.e., discrete) distributions. Whereas $\hat{\mathbf{P}}_{\boldsymbol{\Omega}}$, as in *Type-I* FBN, is defined by the following CLG:

$$P(\Omega = \omega|\hat{\Omega} = \hat{\omega}) = \frac{1}{\sigma_{\hat{\omega}}\sqrt{2\pi}} \exp\left(-\frac{(\omega - \mu_{\hat{\omega}})^2}{2\sigma_{\hat{\omega}}^2}\right) \tag{3.62}$$

The last conditional probability distribution in equation set (3.61) is represented by a *conditional Gaussian regression* that has been used by Lauritzen to define the conditional distribution of a continuous variable whose set of parent variables include both discrete and continuous variables.

$$P\left(\Psi = \psi|\mathrm{Par}_{\delta}(\Psi), \mathrm{Par}_{\gamma}(\Psi)\right) = \frac{1}{\sigma_{\delta}\sqrt{2\pi}} \exp\left(-\frac{(\psi - \mu_{\delta} - \alpha_{\delta\gamma})^2}{2\sigma_{\delta}^2}\right) \tag{3.63}$$

where $\mathrm{Par}_{\delta}(\Psi)$ and $\mathrm{Par}_{\gamma}(\Psi)$ represent two vectors of values for the discrete and continuous parent variables of the continuous variable Ψ, respectively. The regression parameters σ_{δ}, μ_{δ}, and $\alpha_{\delta\gamma}$ are also vectors for a given combination of discrete and continuous parents of continuous variable Ψ, δ for the discrete, γ for the continuous parents.

As compared to *Type-I*, *Type-II* FBNs brings about a much accurate approximation of the *original* general HBN, however with the additional cost of increased complexity of the exact inferencing algorithm. The *Type-II* FBN transformation of our example general HBN are depicted in Figure 3.5.

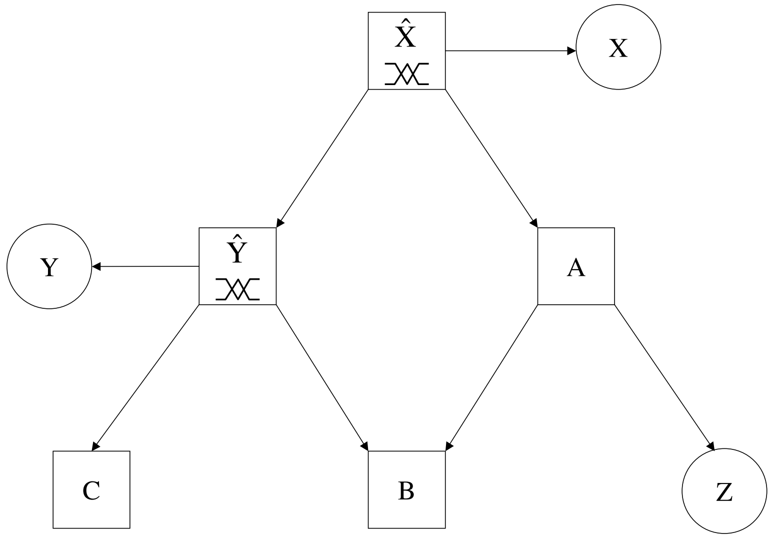


Figure 3.5: *Type-II* FBN $\mathcal{G}_{\mathrm{FBN}^{II}}$ derived from the $\mathcal{G}_{\mathrm{HBN}}$ of Figure 3.2

3.4.2 Exact inferencing in *Type-I* FBN

Inferencing in Bayesian Networks, in general, focuses on determining a joint probability distribution of variables included in the network. After a closed form solution for the joint distribution of variables is formulated, one can use it to determine the marginal probability distributions for various combinations of query variables.

In this section we propose an new exact inferencing methodology for *Type-I* FBN for hard evidence propagation.

For a *Type-I* FBN defined by Definition 6, let the joint distribution of its variables is denoted by

$$P(\boldsymbol{\Delta}, \boldsymbol{\Gamma}, \hat{\boldsymbol{\Gamma}}) \tag{3.64}$$

Using the Chain Rule introduced earlier in equations (3.38) and (3.39) we can define the joint probability distribution $P(\boldsymbol{\Delta}, \boldsymbol{\Gamma}, \hat{\boldsymbol{\Gamma}})$ formally as,

$$P(\boldsymbol{\Delta}, \boldsymbol{\Gamma}, \hat{\boldsymbol{\Gamma}}) = \prod_{\Delta \in \boldsymbol{\Delta}} P(\Delta | \mathrm{Par}_{\boldsymbol{\Delta}}(\Delta)) \prod_{\Gamma \in \boldsymbol{\Gamma}} P(\Gamma | \hat{\Gamma}) \prod_{\hat{\Gamma} \in \hat{\boldsymbol{\Gamma}}} P(\hat{\Gamma} | \mathrm{Par}_{\hat{\boldsymbol{\Gamma}}}(\hat{\Gamma})) \tag{3.65}$$

Note that the second and third terms in equation (3.65) are composed of discrete variables even though they are of crisp and fuzzy nature, respectively. Therefore, if we group them together we can create a net cluster within the original where all variables are discrete and hence the set of conditional distributions imposed by the cluster is multinomial. More formally, let us define this cluster by

$$\mathbf{\Theta} = \mathbf{\Delta} \cup \hat{\mathbf{\Gamma}} \tag{3.66}$$

hence the joint probability distribution of this discrete cluster $\mathbf{\Theta}$ is

$$\begin{aligned} P(\mathbf{\Theta}) &= P(\mathbf{\Delta}, \hat{\mathbf{\Gamma}}) \\ &= \prod_{\Theta \in \mathbf{\Theta}} P(\Theta | \mathrm{Par}_{\mathbf{\Delta}}(\Theta)) \\ &= \prod_{\Delta \in \mathbf{\Delta}} P(\Delta | \mathrm{Par}_{\mathbf{\Delta}}(\Delta)) \prod_{\hat{\Gamma} \in \hat{\mathbf{\Gamma}}} P(\hat{\Gamma} | \mathrm{Par}_{\hat{\mathbf{\Gamma}}}(\hat{\Gamma})) \end{aligned} \tag{3.67}$$

Note that the variables in cluster $\mathbf{\Theta}$ and their joint distribution constitute all by themselves a classical (i.e., discrete) Bayesian Network. Let us denote this sub-Bayesian net by BN_{sub}.

Now, consider the second term in equation (3.65), which basically denotes the conditional distributions of original continuous variables given their fuzzified counterpart discrete variable. That is,

$$P(\mathbf{\Gamma} | \hat{\mathbf{\Gamma}}) = \prod_{\Gamma \in \mathbf{\Gamma}} P(\Gamma | \hat{\Gamma}) \tag{3.68}$$

Note that all original continuous variables $\mathbf{\Gamma}$ are terminal nodes in the acyclic graphical structure $\mathcal{G}_{\mathrm{FBN}^I}$ of the *Type-I* FBN. Furthermore, when considered in conjunction with BN_{sub}, a *star* like topological structure emerges. This topology is depicted in Figure 3.6.

Next, we adopt the junction tree algorithm that we discussed in Chapter 2 and demonstrate its application to this topology. A junction tree of an acyclic graphical structure is composed of *cliques* as its nodes and *separators* as the edges connecting them. We use the classical notation introduced by Lauritzen to define the junction tree $\mathcal{T}$, such that $\mathcal{T} = (\mathbf{C}, \mathbf{S})$ where $\mathbf{C}$ denote the sets of cliques,

$$\mathbf{C} = \{C_1, \ldots, C_n\}, \quad C_k \in \mathbf{\Theta} \text{ and } k = \{1, 2, \ldots, m\} \tag{3.69}$$

and $\mathbf{S}$ represents set of the separators, the set of variables separating two adjacent cliques C_i and C_j such that

$$S_{ij} = C_i \cap C_j, \quad \forall\, C_i \neq C_j, \quad C_i \text{ and } C_j \in \mathbf{C} \tag{3.70}$$

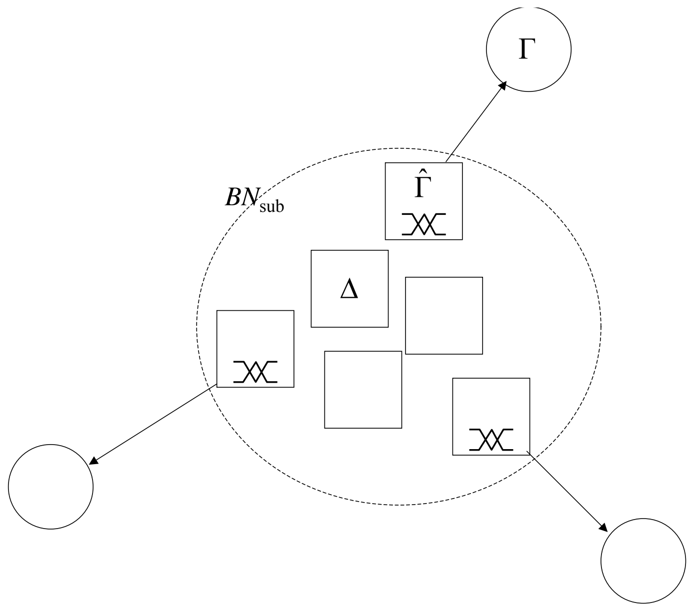


Figure 3.6: BN_{sub} and the peripheral Fuzzy discrete to continuous node pairs

For a *Type-I* FBN, consider a pair of peripheral variables, the original continuous variable Γ and its counterpart Fuzzy discrete variable $\hat{\Gamma}$ as in Figure 3.6. We can say that in the *junction tree* structure $\mathcal{T}$ composed of the clique set $\boldsymbol{\Theta}$, there exist at least one clique $C_{\hat{\Gamma}}$ that contains the Fuzzy discrete variable $\hat{\Gamma}$. Now, suppose that we create a special clique composed only from these two variables $\hat{\Gamma}$ and Γ and denote it by $\hat{C}_{\hat{\Gamma}} = \{\hat{\Gamma}, \Gamma\}$. Note that the newly created clique $\hat{C}_{\hat{\Gamma}}$ and clique $C_{\hat{\Gamma}}$ share a single variable between their respective variable sets: the Fuzzy discrete variable $\hat{\Gamma}$. Hence we can add a new separator, denoted by $\hat{S}_{\hat{\Gamma}}$, connecting/separating $\hat{C}_{\hat{\Gamma}}$ and $C_{\hat{\Gamma}}$, such that

$$\begin{aligned} \hat{S}_{\hat{\Gamma}} &= \hat{C}_{\hat{\Gamma}} \cap C_{\hat{\Gamma}} \\ &= \{\hat{\Gamma}\} \end{aligned} \tag{3.71}$$

thereby creating a improved version of the original junction tree structure $\mathcal{T}$. We shall denote this new tree structure by $\mathcal{T}_{\text{FBN}}$. Formally,

$$\mathcal{T}_{\text{FBN}} = [\mathbf{C}, \hat{\mathbf{C}}, \mathbf{S}, \hat{\mathbf{S}}] \tag{3.72}$$

where the sets $\hat{\mathbf{C}}$ and $\hat{\mathbf{S}}$ denote the collections of cliques $\hat{C}_{\hat{\Gamma}}$ and separators $\hat{S}_{\hat{\Gamma}}$ for each $\hat{\Gamma}$ in the *Type-I* FBN.

In Chapter 2, we have already seen that, the easiest and most straightforward way to perform exact inferencing on Bayesian Networks is to use the chain rule and to sum up the relevant terms each of which can be computed using the chain rule. However, we also know that in any Bayesian Network, the complexity of calculation due to the chain rule based inferencing is exponential in the number of variables making up the network. The variable elimination algorithm discussed earlier in Chapter 2 is a substantial improvement on this. A junction tree can be viewed as a graph induced by the variable elimination algorithm. In junction trees each clique C is associated with a factor $\Phi(C)$ and each separator S with a factor $\Phi(S)$. These factors are called potentials.

Potentials are representations of joint probability distributions of variables making up the cliques or separators. To initialize the junction tree algorithm, they are assigned to be a product of the CPDs associated with them. During the inferencing process these potentials are being kept updated by messages passed though the tree structure.

Junction trees, just like classical Bayes nets, represent a joint probability distribution, which is the product of the potentials of cliques $\Phi(C)$ divided by the product of the potentials of separators $\Phi(S)$. Formally, for a junction tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$ representing the set of variables $\mathbf{X}$, the joint probability distribution $P(\mathbf{X})$ is given by

$$P(\mathbf{X}) = \frac{\prod_{C_i \in \mathbf{V}} \Phi(C_i)}{\prod_{S_j \in \mathbf{E}} \Phi(S_j)} \tag{3.73}$$

where $\mathbf{V}$ and $\mathbf{E}$ represent the set of nodes and the edges in $\mathcal{T}$, respectively. Hence, we can represent the joint probability distribution for the junction tree of a *Type-I* FBN by the following expression:

$$\Phi(\mathbf{C}, \hat{\mathbf{C}}, \mathbf{S}, \hat{\mathbf{S}}) \equiv \Phi(\boldsymbol{\Delta}, \boldsymbol{\Gamma}, \hat{\boldsymbol{\Gamma}}) = \frac{\prod_{C \in \mathbf{C}} \Phi(C)}{\prod_{S \in \mathbf{S}} \Phi(S)} \times \frac{\prod_{\hat{C}_{\hat{\Gamma}} \in \hat{\mathbf{C}}} \Phi(\hat{C}_{\hat{\Gamma}})}{\prod_{\hat{S}_{\hat{\Gamma}} \in \hat{\mathbf{S}}} \Phi(\hat{S}_{\hat{\Gamma}})} \tag{3.74}$$

Within this context we can use a message passing algorithm to perform exact inferencing on a *Type-I* FBN. In a junction tree $\mathcal{T}$, a message is passed form clique C_1 to C_2 via the message passing algorithm depicted in Table 3.6.

However, we need to make a distinction between the message passing among the variables within the BN_{sub} and message passing between the peripheral pairs Γ and $\hat{\Gamma}$. Recall that BN_{sub} of Figure 3.6 is composed of cliques which include only discrete variables, originally discrete variables Δ and Fuzzy variables $\hat{\Gamma}$. Any message passing among these discrete only cliques C can be performed by a standard junction tree algorithm. Whereas, the updating

Table 3.6: Algorithm for message passing from C_1 to C_2 through separator S

Step	Action	Remarks
1	Let $f_1 = \sum_{C_1 \setminus S} \Phi(C_1)$	Sum out the variables in $C_1 \setminus S$ over $\Phi(C_1)$
2	Let $f_2 = \frac{f_1}{\Phi(C_1)}$	Divide by the last message sent
3	Let $\Phi(S) = f_1$	Store the message
4	Let $\Phi(C_2) = \Phi(C_2) \cdot f_1$	Multiply the message by $\Phi(C_2)$

between the BN_{sub} and its periphery, namely the updating between discrete clique $C_{\hat{\Gamma}}$ and mixed clique $\hat{C}_{\hat{\Gamma}} = \{\Gamma, \hat{\Gamma}\}$ through separator $\hat{S}_{\hat{\Gamma}}$ needs to be handled differently. This type of updating has two aspects:

- First, message passing from $C_{\hat{\Gamma}}$ to $\hat{C}_{\hat{\Gamma}}$.
- Second, message passing from $\hat{C}_{\hat{\Gamma}}$ to $C_{\hat{\Gamma}}$.

Let us now introduce our proposed methodology for these two forms of message passing by considering evidence propagation in both situations. Assume that an evidence $\mathbf{E}$ is introduced to the junction tree of the *Type-I* FBN. Let $\Phi'(K)$ denotes the updated potential $\Phi'(K) = \Phi(K|\mathbf{E})$ of cluster K given evidence $\mathbf{E}$. K may be a clique or separator.

Message passing from $C_{\hat{\Gamma}}$ to $\hat{C}_{\hat{\Gamma}}$: From $C_{\hat{\Gamma}}$ to $\hat{C}_{\hat{\Gamma}}$ the evidence is carried by separator $\hat{S}_{\hat{\Gamma}}$, hence its potential $\Phi(\hat{S}_{\hat{\Gamma}})$ needs to be updated so that $\hat{S}_{\hat{\Gamma}}$ can pass the message to the peripheral clique $\hat{C}_{\hat{\Gamma}}$. From equation (3.70) we know that $\hat{S}_{\hat{\Gamma}}$ contains only $\hat{\Gamma}$ and it updated potential is

$$\Phi'(\hat{S}_{\hat{\Gamma}}) = \Phi'(\hat{\Gamma}) = \sum_{C_{\hat{\Gamma}} \setminus \{\hat{\Gamma}\}} \Phi'(C_{\hat{\Gamma}}) \tag{3.75}$$

Then,

$$\text{Let } f_1 = \Phi'(\hat{\Gamma}) \tag{3.76}$$

The summation in equation (3.75) is performed over the variables in the clique $C_{\hat{\Gamma}}$ except $\hat{\Gamma}$, in effect, marginalizing the Fuzzy variable $\hat{\Gamma}$ out from the clique's potential $\Phi(C_{\hat{\Gamma}})$. Equation (3.76) completes the first step in Table (3.6).

Next, we update $\Phi(\hat{C}_{\hat{\Gamma}})$:

$$\text{Let } f_2 = \frac{f_1}{\Phi(\hat{S}_{\hat{\Gamma}})} = \frac{\Phi'(\hat{\Gamma})}{\Phi(\hat{S}_{\hat{\Gamma}})} = \frac{\Phi'(\hat{\Gamma})}{\Phi(\hat{\Gamma})} \tag{3.77}$$

since $\Phi(\hat{S}_{\hat{\Gamma}}) = \Phi(\hat{\Gamma})$ by similar reasoning as in equation (3.75). Now the message is represented by f_2. Next, we multiply the current potential $\Phi(\hat{C}_{\hat{\Gamma}})$ of clique $\hat{C}_{\hat{\Gamma}} = \{\hat{\Gamma}, \Gamma\}$ by the message to get the updated potential $\Phi'(\hat{C}_{\hat{\Gamma}})$. That is,

$$\Phi'(\hat{C}_{\hat{\Gamma}}) = \Phi(\hat{C}_{\hat{\Gamma}}) \cdot f_2 = \Phi(\hat{C}_{\hat{\Gamma}}) \cdot \frac{\Phi'(\hat{\Gamma})}{\Phi(\hat{\Gamma})} \tag{3.78}$$

The term $\Phi(\hat{C}_{\hat{\Gamma}})$ in equation (3.78) is by definition a nonnegative entity proportional to the probability representing the conditional relationship among the variables within the clique $\hat{C}_{\hat{\Gamma}}$. Since clique $\hat{C}_{\hat{\Gamma}}$ consists of two variables continuous Γ and its counter part Fuzzy discrete $\hat{\Gamma}$ and since, in the network structure $\mathcal{G}_{\mathrm{FBN}^I}$ of *Type-I* FBN, the conditional relationship between these two is always $P(\Gamma|\hat{\Gamma})$, the potential $\Phi(\hat{C}_{\hat{\Gamma}})$ is proportional to $P(\Gamma|\hat{\Gamma})$. Recall that, according to our methodology, we define $P(\Gamma|\hat{\Gamma})$ by equation (3.57),

$$P(\Gamma = \gamma|\hat{\Gamma} = \hat{\gamma}) = \frac{1}{\sigma_{\hat{\gamma}}\sqrt{2\pi}} \exp\left(-\frac{(\gamma - \mu_{\hat{\gamma}})^2}{2\sigma_{\hat{\gamma}}^2}\right) \tag{3.79}$$

Then, since $\Phi(\hat{C}_{\hat{\Gamma}}) \propto P(\Gamma|\hat{\Gamma})$, the potential $\Phi(\hat{C}_{\hat{\Gamma}})$ can correctly be represented by

$$\Phi(\hat{C}_{\hat{\Gamma}}) = \Phi(\hat{\Gamma} = \hat{\gamma}, \Gamma = \gamma) = \frac{\Phi(\hat{\gamma})}{\sigma_{\hat{\gamma}}\sqrt{2\pi}} \exp\left(-\frac{(\gamma - \mu_{\hat{\gamma}})^2}{2\sigma_{\hat{\gamma}}^2}\right) \tag{3.80}$$

Using this representation of $\Phi(\hat{C}_{\hat{\Gamma}})$ along with the equation (3.78) gives us the updated potential $\Phi'(\hat{C}_{\hat{\Gamma}})$. That is,

$$\Phi'(\hat{C}_{\hat{\Gamma}}) = \Phi'(\hat{\Gamma} = \hat{\gamma}, \Gamma = \gamma) = \frac{\Phi'(\hat{\gamma})}{\sigma_{\hat{\gamma}}\sqrt{2\pi}} \exp\left(-\frac{(\gamma - \mu_{\hat{\gamma}})^2}{2\sigma_{\hat{\gamma}}^2}\right) \tag{3.81}$$

To compute the updated marginal potential $\Phi(\Gamma = \gamma)$ of the continuous variable Γ we need to sum out its counter part fuzzy discrete $\hat{\Gamma}$ over its discrete frame $\hat{\boldsymbol{\Gamma}}$ from equation (3.81), such that

$$\Phi'(\Gamma = \gamma) = \sum_{\gamma \in \hat{\boldsymbol{\Gamma}}} \frac{\Phi'(\hat{\gamma})}{\sigma_{\hat{\gamma}}\sqrt{2\pi}} \exp\left(-\frac{(\gamma - \mu_{\hat{\gamma}})^2}{2\sigma_{\hat{\gamma}}^2}\right) \tag{3.82}$$

Message passing from $\hat{C}_{\hat{\Gamma}}$ to $C_{\hat{\Gamma}}$: Using similar steps as above, the updated potential $\Phi(C_{\hat{\Gamma}})$ of clique $C_{\hat{\Gamma}}$ when a message passed form clique $\hat{C}_{\hat{\Gamma}}$ to clique $C_{\hat{\Gamma}}$ through separator $\hat{S}_{\hat{\Gamma}}$, can be computed as follows. Again, summing out the current potential $\Phi(\hat{C}_{\hat{\Gamma}})$ over continuous variable Γ (note that $\hat{C}_{\hat{\Gamma}} \backslash \hat{S}_{\hat{\Gamma}} = \{\hat{\Gamma}, \Gamma\} \backslash \hat{\Gamma} = \{\Gamma\}$) we get the updated potential $\Phi'(\hat{S}_{\hat{\Gamma}})$ for the separator:

$$\begin{aligned}
\Phi'(\hat{S}_{\hat{\Gamma}}) &= \int\limits_{\mathbf{\Gamma}} \Phi(\hat{C}_{\hat{\Gamma}}) \mathrm{d}\Gamma \\
&= \int\limits_{\mathbf{\Gamma}} \Phi(\hat{\Gamma}, \Gamma) \mathrm{d}\Gamma \qquad \begin{array}{l} \forall \Gamma = \{\gamma | \gamma \in \mathbf{\Gamma}\} \quad \text{and} \\ \forall \hat{\Gamma} = \{\hat{\gamma} | \hat{\gamma} \in \hat{\mathbf{\Gamma}}\} \end{array} \\
&= \int\limits_{\mathbf{\Gamma}} \Phi(\hat{\gamma}, \gamma) \mathrm{d}\gamma \\
&= \Phi'(\hat{\Gamma} = \hat{\gamma})
\end{aligned} \tag{3.83}$$

hence

$$\Phi'(\hat{S}_{\hat{\Gamma}}) = \Phi'(\hat{\Gamma} = \hat{\gamma}) = \int\limits_{\gamma \in \Gamma} \frac{\Phi(\hat{\gamma})}{\sigma_{\hat{\gamma}} \sqrt{2\pi}} \exp\left(-\frac{(\gamma - \mu_{\hat{\gamma}})^2}{2\sigma_{\hat{\gamma}}^2} \right) \mathrm{d}\gamma \tag{3.84}$$

then dividing $\Phi'(\hat{S}_{\hat{\Gamma}})$ by the last message sent (i.e., $\Phi(\hat{S}_{\hat{\Gamma}}) = \Phi(\hat{\Gamma} = \hat{\gamma}) = \Phi(\hat{\gamma})$) and multiplying the message by the current potential $\Phi(C_{\hat{\Gamma}})$ we get its updated potential:

$$\Phi'(C_{\hat{\Gamma}}) = \int\limits_{\gamma \in \Gamma} \frac{\Phi(C_{\hat{\Gamma}})}{\sigma_{\hat{\gamma}} \sqrt{2\pi}} \exp\left(-\frac{(\gamma - \mu_{\hat{\gamma}})^2}{2\sigma_{\hat{\gamma}}^2} \right) \mathrm{d}\gamma \tag{3.85}$$

Numerical Example

Next, we demonstrate the message passing process with a numerical example. Consider the Hybrid Bayesian Network in Figure 3.7

The conditional probability distributions (CPDs) associated with the individual nodes (i.e., variables) are given as follows:

CPDs for the nodes A and B:

$$P(A) = \begin{cases} 0.7 & \text{for} \quad A = a_1 \\ 0.3 & \text{for} \quad A = a_2 \end{cases}, \quad P(B|A) = \begin{array}{c|cc} B|A & a_1 & a_2 \\ \hline b_1 & 0.2 & 0.45 \\ b_2 & 0.8 & 0.55 \end{array}$$

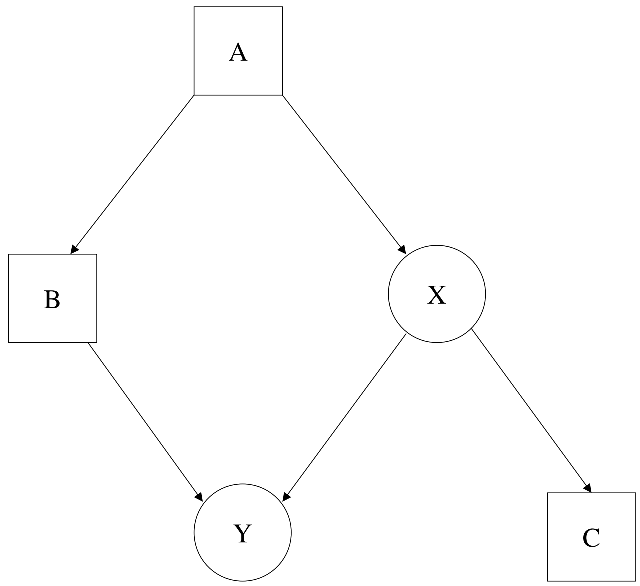


Figure 3.7: The Hybrid Bayesian Network used in the numerical example

CPD for the node X:

$$P(X|A) = \begin{cases} \begin{rcases} 1.75 & \text{for} \quad 0 \leq x < 0.5 \\ 0.25 & \text{for} \quad 0.5 \leq x \leq 1 \end{rcases} & \text{for} \quad A = a_1 \\ \\ \begin{rcases} 0.40 & \text{for} \quad 0 \leq x < 0.5 \\ 1.60 & \text{for} \quad 0.5 \leq x \leq 1 \end{rcases} & \text{for} \quad A = a_2 \end{cases}$$

Note that the CPDs of the nodes A and B are discrete distributions, whereas the CPD of node X is a mixture of uniform distributions for each discrete state of variable A, hence is a continuous distribution.

CPD for the node Y:

$$P(Y|X,B) = \begin{cases} \mathcal{N}(Y; X, 1) & \text{for} \quad B = b_1 \\ \mathcal{N}(Y; 1, 0.5) & \text{for} \quad B = b_2 \end{cases}$$

CPD for the node C:

$$P(C|X) = \begin{cases} P(c_1|X) = & \begin{cases} 0.353 & \text{for} \quad 0 \leq x < 0.2 \\ 0.667 & \text{for} \quad 0.2 \leq x < 0.8 \\ 0.5 & \text{for} \quad 0.8 \leq x \leq 1 \end{cases} \\ \\ P(c_2|X) = & \begin{cases} 0.647 & \text{for} \quad 0 \leq x < 0.2 \\ 0.333 & \text{for} \quad 0.2 \leq x < 0.8 \\ 0.5 & \text{for} \quad 0.8 \leq x \leq 1 \end{cases} \end{cases}$$

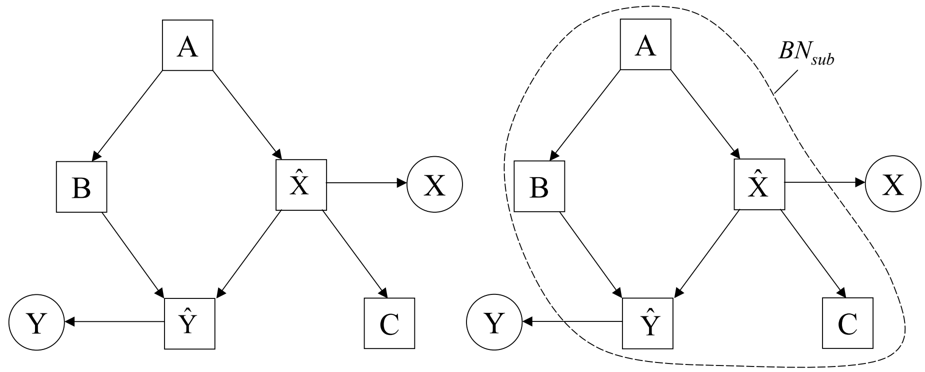
Note that the CPDs for the nodes Y and C are continuous distributions, too. In particular, the CPD of node Y is a Conditional Gaussian (CG) which, for each discrete state of variable B, the conditional is defined by a normal distribution $\mathcal{N}(V; \mu, \sigma)$, where V, μ, and σ denote the random variable, mean, and standard deviation of the distribution, respectively. On the other hand, the CPD of node C is a mixture of uniform distributions for each state of variable C.

Given the topology in Figure 3.7 and the nature of the CPDs associated with its nodes we conclude that the existing methodologies do not help us to perform exact inferencing on this hybrid Bayesian Network. We can point out two reasons:

- Discrete node C has a continuous node (i.e., X) as its parent.
- The CPDs for nodes X and Y are defined by continuous distributions other than Conditional Gaussians.

Once the inapplicability of the traditional inferencing methodologies for our example network is established we can justify the use of our proposed methodology for general HBNs. First, we convert the current topology of the net into a *Type-I* FBN. The result of this conversion is provided in Figure 3.8.

As explained earlier in this section, to transform the given HBN into the *Type-I* FNB in Figure 3.8.a, we replace the continuous nodes X and Y with their Fuzzy discrete counterpart $\hat{X}$ and $\hat{Y}$, and add the original nodes along with a directed link from the fuzzy to the

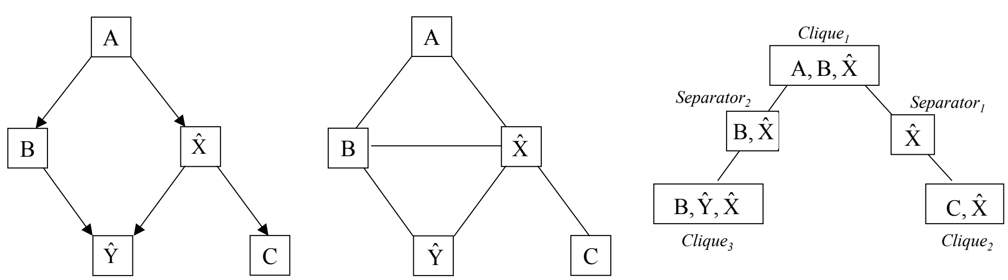


a) The *Type-I* FNB b) BN_{sub} in the *Type-I* FNB

Figure 3.8: The *Type-I* FBN corresponding to the HBN of Figure 3.7

original continuous variable. Consequently, we are now able to identify a BN_{sub} in Figure 3.8.b which is composed only of discrete variables, either crisp discrete or Fuzzy discrete.

Next, we build the junction tree associated with the BN_{sub}. This is done first moralizing (i.e., connecting every unconnected parent and removing the link directions) and second, triangulating the BN_{sub}. Consequently, the cliques are identified in the normalized and triangulated net and laid out on a tree structure where each clique is connected via a separator composed of shared variables between two neighboring cliques. This process and the resulting junction tree (JT) for the BN_{sub} is provided in Figure 3.9.



a) The BN_{sub} b) The moralized and triangulated BN_{sub} c) The *JT* for the BN_{sub}

Figure 3.9: Construction of the JT of the BN_{sub}

We can now construct the junction tree of the *Type-I* FBN using the JT of the BN_{sub} in Figure 3.9.c as its core. This is accomplished by adding two cliques composed of the

$\{\hat{X}, X\}$ and $\{\hat{Y}, Y\}$ pairs with the corresponding separators. The resulting JT is depicted in Figure 3.10 along with the notation we introduced during the formalization of the *Type-I* FBN.

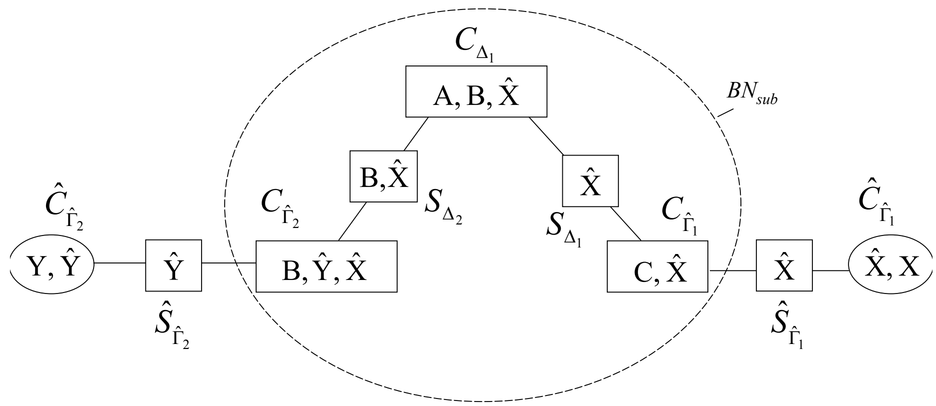


Figure 3.10: The *Junction Tree* of the *Type-I* FBN in Figure 3.8.a

According to our proposed methodology, in order to perform the exact inferencing algorithm on the JT of the Type-I FBN in Figure 3.10, we need to define the Fuzzy variables $\{\hat{X}, \hat{Y}\}$ formally by introducing the Fuzzy membership functions that will be used to convert their continuous frames into discrete states determined by the corresponding Fuzzy Sets. Consequently, we fuzzify the given CPDs of the original HBN in Figure 3.7 that include the Fuzzy variables $\{\hat{X}, \hat{Y}\}$. These fuzzified CPDs along with the others in the original HBN will, in turn, be used to determine the potentials Φ for each clique C and separator S of the JT in Figure 3.10. These potentials are then employed to pass messages along the desired branches of the junction tree to perform exact inferencing for our *Type-I* FBN.

We demonstrate the proposed inferencing methodology by passing a message from clique C_{Δ_1} to the peripheral clique $\hat{C}_{\hat{\Gamma}_1}$ as shown in Figure 3.11.

First, we need to define the Fuzzy counterpart $\hat{X}$ of continuous variables X. Let $\hat{X}$ be a binary Fuzzy variable whose states are defined on frame $\mathbf{X}$ by the Fuzzy Sets $\hat{X}_1(x)$ and

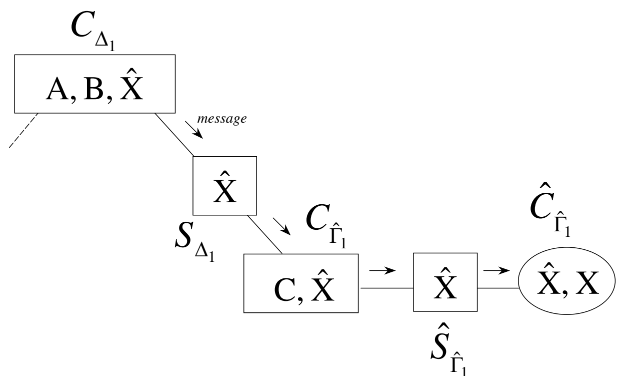


Figure 3.11: A message is passed from C_{Δ_1} to $\hat{C}_{\hat{\Gamma}_1}$

$\hat{X}_2(x)$. That is,

$$\hat{X}_1(x) = \begin{cases} 1 & \text{for} \quad 0 \leq x < 0.3 \\ \dfrac{0.7 - x}{0.7 - 0.3} & \text{for} \quad 0.3 \leq x < 0.7 \\ 0 & \text{for} \quad 0.7 \leq x < 1 \end{cases}$$

$$\hat{X}_2(x) = \begin{cases} 0 & \text{for} \quad 0 \leq x < 0.3 \\ \dfrac{x - 0.3}{0.7 - 0.3} & \text{for} \quad 0.3 \leq x < 0.7 \\ 1 & \text{for} \quad 0.7 \leq x < 1 \end{cases} \tag{3.86}$$

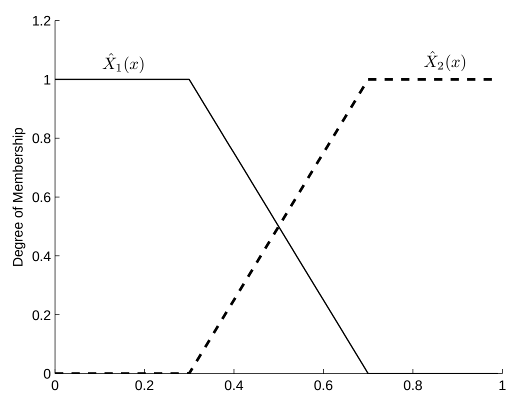


Figure 3.12: Membership functions for Fuzzy states $\hat{X}_1$ and $\hat{X}_2$

Now we need to determine the potentials for cliques C_{Δ_1}, $C_{\hat{\Gamma}_1}$, and $\hat{C}_{\hat{\Gamma}_1}$. These potentials are dependent on the following four factors: $f(\hat{X}, A)$, $f(C, \hat{X})$, $f(X, \hat{X})$, and $f(A)$, which in turn relies on the CPDs of the four node $\{A, C, X, \hat{X}\}$ in the *Type-I* FBN of Figure 3.8.a.

Hence, we compute these four CPDs next, in the process making the necessary crisp to fuzzy transformations according to the methodology we propose in Section (3.4.1).

$P(X|A) \overset{fuzzify}{\rightarrow} P(\hat{X}|A):$

$$f_1(\hat{X}, A) = \begin{array}{c|c|c} \hat{X} & A & \\ \hline \hat{X}_1 & a_1 & P(\hat{X}_1|a_1) \\ \hat{X}_1 & a_2 & P(\hat{X}_1|a_2) \\ \hat{X}_2 & a_1 & P(\hat{X}_2|a_1) \\ \hat{X}_2 & a_2 & P(\hat{X}_2|a_2) \end{array}$$

using equation (3.49) we can write the individual probabilities making up the factor $f_1(\hat{X}, A)$ as follows:

$$P(\hat{X}_i|a_j) = \int_{\mathbf{X}} \hat{X}_i(x)\, P(x|a_j)\, \mathrm{d}x; \qquad i = \{1, 2\},\ j = \{1, 2\} \tag{3.87}$$

where the frame $\mathbf{X}$ is defined on the closed interval $[0, 1]$. Hence by the equation (3.87) and using the given CPD for $P(X|A)$ and membership function $\hat{X}_1(x)$:

$$\begin{aligned} P(\hat{X}_1|a_1) &= \int_0^1 \hat{X}_1(x)\, P(x|a_1)\, \mathrm{d}x \\ &= \int_0^{0.3} 1 \times 1.75\ \mathrm{d}x + \int_{0.3}^{0.5} \left(\frac{0.7 - x}{0.7 - 0.3}\right) \times 1.75\ \mathrm{d}x + \int_{0.5}^{0.7} \left(\frac{0.7 - x}{0.7 - 0.3}\right) \times 0.25\ \mathrm{d}x \\ &= 0.8 \end{aligned}$$

$$\begin{aligned} P(\hat{X}_2|a_1) &= \int_0^1 \hat{X}_2(x)\, P(x|a_1)\, \mathrm{d}x \\ &= \int_{0.3}^{0.5} \left(\frac{x - 0.3}{0.7 - 0.3}\right) \times 1.75\ \mathrm{d}x + \int_{0.5}^{0.7} \left(\frac{x - 0.3}{0.7 - 0.3}\right) \times 0.25\ \mathrm{d}x + \int_{0.7}^{1} 1 \times 0.25\ \mathrm{d}x \\ &= 0.2 \end{aligned}$$

$$\begin{aligned} P(\hat{X}_1|a_2) &= \int_0^1 \hat{X}_1(x)\, P(x|a_2)\, \mathrm{d}x \\ &= \int_0^{0.3} 1 \times 0.4\ \mathrm{d}x + \int_{0.3}^{0.5} \left(\frac{0.7 - x}{0.7 - 0.3}\right) \times 0.4\ \mathrm{d}x + \int_{0.5}^{0.7} \left(\frac{0.7 - x}{0.7 - 0.3}\right) \times 1.6\ \mathrm{d}x \\ &= 0.26 \end{aligned}$$

$$
\begin{aligned}
P(\hat{X}_2|a_2) &= \int_0^1 \hat{X}_2(x)\, P(x|a_2)\, \mathrm{d}x \\
&= \int_{0.3}^{0.5} \left(\frac{x-0.3}{0.7-0.3}\right) \times 0.4\, \mathrm{d}x + \int_{0.5}^{0.7} \left(\frac{x-0.3}{0.7-0.3}\right) \times 1.6\, \mathrm{d}x + \int_{0.7}^{1} 1 \times 1.6\, \mathrm{d}x \\
&= 0.74
\end{aligned}
$$

and the corresponding factor is:

$$
f_1(\hat{X}, A) = \begin{array}{c|c|l} \hat{X} & A & \\ \hline \hat{X}_1 & a_1 & 0.8 \\ \hat{X}_1 & a_2 & 0.26 \\ \hat{X}_2 & a_1 & 0.2 \\ \hat{X}_2 & a_2 & 0.74 \end{array} \tag{3.88}
$$

Second, we compute the factor $f(C|\hat{X})$:

$P(C|X) \overset{fuzzify}{\rightarrow} P(C|\hat{X})$:

$$
f_2(C, \hat{X}) = \begin{array}{c|c|l} C & \hat{X} & \\ \hline c_1 & \hat{X}_1 & P(c_1|\hat{X}_1) \\ c_1 & \hat{X}_2 & P(c_1|\hat{X}_2) \\ c_2 & \hat{X}_1 & P(c_2|\hat{X}_1) \\ c_2 & \hat{X}_2 & P(c_2|\hat{X}_2) \end{array}
$$

using equation (3.44) we can write the individual probabilities making up the factor $f(C, \hat{X})$ as follows:

$$
P(c_i|\hat{X}_j) = \frac{\int_{\mathbf{X}} \hat{X}_j(x)\, P(c_i|x)\, P(x)\, \mathrm{d}x}{\int_{\mathbf{X}} \hat{X}_j(x)\, P(x)\, \mathrm{d}x}; \qquad i = \{1, 2\},\ j = \{1, 2\} \tag{3.89}
$$

As before the discrete variable C is assumed to be binary. Hence by the equation (3.89) and using the given CPD for $P(C|X)$ and membership function $\hat{X}_1(x)$, the individual components of the factor can be computed as follows:

$$P(c_1|\hat{X}_1) = \frac{\int_0^{0.2} 1 \cdot 0.353 \cdot 1.7\,\mathrm{d}x + \int_{0.2}^{0.3} 1 \cdot 0.667 \cdot 0.9\,\mathrm{d}x + \int_{0.3}^{0.7} \left(\frac{0.7-x}{0.7-0.3}\right) \cdot 0.667 \cdot 0.9\,\mathrm{d}x}{\int_0^{0.2} 1 \cdot 1.7\,\mathrm{d}x + \int_{0.2}^{0.3} 1 \cdot 0.9\,\mathrm{d}x + \int_{0.3}^{0.7} \left(\frac{0.7-x}{0.7-0.3}\right) \cdot 0.9\,\mathrm{d}x}$$

$$= 0.49198$$

$$P(c_2|\hat{X}_1) = \frac{\int_0^{0.2} 1 \cdot 0.647 \cdot 1.7\,\mathrm{d}x + \int_{0.2}^{0.3} 1 \cdot 0.333 \cdot 0.9\,\mathrm{d}x + \int_{0.3}^{0.7} \left(\frac{0.7-x}{0.7-0.3}\right) \cdot 0.333 \cdot 0.9\,\mathrm{d}x}{\int_0^{0.2} 1 \cdot 1.7\,\mathrm{d}x + \int_{0.2}^{0.3} 1 \cdot 0.9\,\mathrm{d}x + \int_{0.3}^{0.7} \left(\frac{0.7-x}{0.7-0.3}\right) \cdot 0.9\,\mathrm{d}x}$$

$$= 0.50802$$

$$P(c_1|\hat{X}_2) = \frac{\int_{0.3}^{0.7} \left(\frac{x-0.3}{0.7-0.3}\right) \cdot 0.667 \cdot 0.9\,\mathrm{d}x + \int_{0.7}^{0.8} 1 \cdot 0.667 \cdot 0.9\,\mathrm{d}x + \int_{0.8}^{1} 1 \cdot 0.5 \cdot 0.6\,\mathrm{d}x}{\int_{0.3}^{0.7} \left(\frac{x-0.3}{0.7-0.3}\right) \cdot 0.9\,\mathrm{d}x + \int_{0.7}^{0.8} 1 \cdot 0.9\,\mathrm{d}x + \int_{0.8}^{1} 1 \cdot 0.6\,\mathrm{d}x}$$

$$= 0.61561$$

$$P(c_2|\hat{X}_2) = \frac{\int_{0.3}^{0.7} \left(\frac{x-0.3}{0.7-0.3}\right) \cdot 0.333 \cdot 0.9\,\mathrm{d}x + \int_{0.7}^{0.8} 1 \cdot 0.333 \cdot 0.9\,\mathrm{d}x + \int_{0.8}^{1} 1 \cdot 0.5 \cdot 0.6\,\mathrm{d}x}{\int_{0.3}^{0.7} \left(\frac{x-0.3}{0.7-0.3}\right) \cdot 0.9\,\mathrm{d}x + \int_{0.7}^{0.8} 1 \cdot 0.9\,\mathrm{d}x + \int_{0.8}^{1} 1 \cdot 0.6\,\mathrm{d}x}$$

$$= 0.38439$$

Then the corresponding factor is;

$$f_2(C, \hat{X}) = \begin{array}{c|c|l} C & \hat{X} & \\ \hline c_1 & \hat{X}_1 & 0.49198 \\ c_1 & \hat{X}_2 & 0.61561 \\ c_2 & \hat{X}_1 & 0.50802 \\ c_2 & \hat{X}_2 & 0.38439 \end{array} \tag{3.90}$$

Next, we define the factor $f(X|\hat{X})$ using the equations (3.57) and (3.79). Consider;

$$P(X|\hat{X}) = \begin{cases} P(x|\hat{X}_1) = \frac{1}{\sigma_{\hat{X}_1}\sqrt{2\pi}} \left(-\frac{\left(x-\mu_{\hat{X}_1}\right)^2}{2\sigma^2_{\hat{X}_1}}\right) & \text{for} \quad \hat{X} = \hat{X}_1 \\ P(x|\hat{X}_2) = \frac{1}{\sigma_{\hat{X}_2}\sqrt{2\pi}} \left(-\frac{\left(x-\mu_{\hat{X}_2}\right)^2}{2\sigma^2_{\hat{X}_2}}\right) & \text{for} \quad \hat{X} = \hat{X}_2 \end{cases}$$

$$\mu_{\hat{X}} = \{0.35, 0.65\} \quad \text{and} \quad \sigma_{\hat{X}} = \{0.07, 0.07\}$$

Hence the factor for the clique $\hat{C}_{\hat{\Gamma}_1}$ in the JT of *Type-I* FBN is;

$$f_3(X, \hat{X}) = \begin{array}{c|c|l} X & \hat{X} & \\ \hline \text{x} & \hat{X}_1 & P(x|\hat{X}_1) = \mathcal{N}(X; 0.35, 0.07) \\ & \hat{X}_2 & P(x|\hat{X}_2) = \mathcal{N}(X; 0.65, 0.07) \end{array} \tag{3.91}$$

Finally, for the factor $f(A)$ we write $f(A) = P(A)$, hence using the given distribution for node A we have

$$f_4(A) = \begin{array}{c|l} A & \\ \hline a_1 & 0.7 \\ a_2 & 0.3 \end{array} \tag{3.92}$$

Summarizing, the initialization step involves incorporating the CPDs into the junction tree of *Type-I* FBN. To accomplish this, first we transfered the crisp CPDs into fuzzy CPDs for selected variables of the *Type-I* FBN to determine the factors (3.88), (3.90), (3.91), and (3.92) needed to compute the potentials of the cliques and separators along the path in Figure 3.11 that we have chosen to demonstrate our proposed methodology for message passing in a *Type-I* FBN. Then, for each such factor we choose a clique that contains all the variables in the factor. We then initialize the potential of each clique to be the product of the CPDs associated with it.

Suppose a message is passed from clique C_{Δ_1} to clique $\hat{C}_{\hat{\Gamma}_1}$ along the path shown in Figure 3.11. Next, we will initiate the potentials for cliques C_{Δ_1}, $C_{\hat{\Gamma}_1}$, and $\hat{C}_{\hat{\Gamma}_1}$.

Potential $\Phi(C_{\Delta_1})$ of clique $C_{\Delta_1} = \{A, B, \hat{X}\}$ is the product of factors $f_4(A)$, $f_5(B, A)$ and $f_1(\hat{X}, A)$. To multiply these factors we need to create a factor $f_6(A, B, \hat{X})$ over the union variables of these three original factors. Now consider a table entry in f_6 corresponding to some assignment $\{a_i, b_j, \hat{X}_k\}$. In each of the original factors $f_4,\ f_5,$ and f_1 there is precisely one entry consistent with $\{a_i, b_j, \hat{X}_k\}$. The table entry in f_6 is defined as the product of the three table entries in $f_4,\ f_5,$ and f_1. The resulting factor is the potential for clique C_{Δ_1} which is provided below:

$$\Phi(C_{\Delta_1}) = f_6(A, B, \hat{X}) = \begin{array}{c|c|c|l} A & B & \hat{X} & \\ \hline a_1 & b_1 & \hat{X}_1 & 0.7 \times 0.2 \times 0.8 = 0.112 \\ a_1 & b_1 & \hat{X}_2 & 0.7 \times 0.2 \times 0.2 = 0.028 \\ a_1 & b_2 & \hat{X}_1 & 0.7 \times 0.8 \times 0.8 = 0.448 \\ a_1 & b_2 & \hat{X}_2 & 0.7 \times 0.8 \times 0.2 = 0.112 \\ a_2 & b_1 & \hat{X}_1 & 0.3 \times 0.45 \times 0.26 = 0.0351 \\ a_2 & b_1 & \hat{X}_2 & 0.3 \times 0.45 \times 0.74 = 0.0999 \\ a_2 & b_2 & \hat{X}_1 & 0.3 \times 0.55 \times 0.26 = 0.0429 \\ a_2 & b_2 & \hat{X}_2 & 0.3 \times 0.55 \times 0.74 = 0.1221 \end{array} \tag{3.93}$$

The potential $\Phi(C_{\hat{\Gamma}_1})$ is composed only of factor $f_2(\hat{X}, C)$, hence from (3.90) we get,

$$\Phi(C_{\hat{\Gamma}_1}) = \Phi(C, \hat{X}) = \begin{array}{c|c|l} C & \hat{X} & \\ \hline c_1 & \hat{X}_1 & 0.49198 \\ c_1 & \hat{X}_2 & 0.61561 \\ c_2 & \hat{X}_1 & 0.50802 \\ c_2 & \hat{X}_2 & 0.38439 \end{array} \tag{3.94}$$

Finally, he potential $\Phi(\hat{C}_{\hat{\Gamma}_1})$ equals to $f_3(X, \hat{X})$ of equation (3.91). That is,

$$\Phi(\hat{C}_{\hat{\Gamma}_1}) = \Phi(X, \hat{X}) = \begin{array}{c|c|l} X & \hat{X} & \\ \hline \text{x} & \hat{X}_1 & \mathcal{N}(X; 0.35, 0.07) \\ & \hat{X}_2 & \mathcal{N}(X; 0.65, 0.07) \end{array} \tag{3.95}$$

Now, since all the relevant information is in the particular section of the junction tree that we are interested in we can proceed to passing a message from clique C_{Δ_1} to $\Phi(C_{\hat{\Gamma}_1})$ through separator S_{Δ_1}.

First, we update the potential of the separator S_{Δ_1}. Let $\Phi'(S_{\Delta_1})$ be that updated potential. To calculate the updated potential we perform a summation over the variables A, B which are the variables making up the clique C_{Δ_1} minus the variables in the separator S_{Δ_1}. Hence,

$$\Phi'(S_{\Delta_1}) = \sum_{C_{\Delta_1} \setminus S_{\Delta_1}} \Phi(C_{\Delta_1}) = \sum_{A,B} f_6(A, B, \hat{X}) \tag{3.96}$$

The resulting updated potential is

$$\Phi'(S_{\Delta_1}) = \begin{array}{c|l} \hat{X} & \\ \hline \hat{X}_1 & 0.638 \\ \hat{X}_2 & 0.362 \end{array} \tag{3.97}$$

Since this is the first time a message is passed through the separator S_{Δ_1} we do not need to divide it by an earlier message stored in the S_{Δ_1} to normalize the message currently being passed, hence we skip step 2 in Table 3.6. Nevertheless, we are going to store this message by letting,

$$\Phi(S_{\Delta_1}) \leftarrow \Phi'(S_{\Delta_1})$$

Next we update the clique $C_{\hat{\Gamma}_1}$ with the message carried by the separator S_{Δ_1}. That is,

$$\Phi'(C_{\hat{\Gamma}_1}) = \Phi'(S_{\Delta_1}) \times \Phi(C_{\hat{\Gamma}_1})$$

Multiplying potentials (3.94) and (3.97) according to the method outlined for the multiplication of factors we get the updated potential $\Phi'(C_{\hat{\Gamma}_1})$ for clique $C_{\hat{\Gamma}_1}$.

$$\Phi'(C_{\hat{\Gamma}_1}) = \begin{array}{c|c|l} C & \hat{X} & \\ \hline c_1 & \hat{X}_1 & 0.49198 \times 0.638 = 0.31388 \\ c_1 & \hat{X}_2 & 0.61561 \times 0.362 = 0.22285 \\ c_2 & \hat{X}_1 & 0.50802 \times 0.638 = 0.32412 \\ c_2 & \hat{X}_2 & 0.38439 \times 0.362 = 0.13915 \end{array} \tag{3.98}$$

We assign the updated potential as the current potential for the clique $C_{\hat{\Gamma}_1}$ to be used for future message passing if needed:

$$\Phi(C_{\hat{\Gamma}_1}) \leftarrow \Phi'(C_{\hat{\Gamma}_1})$$

Now we are going to pass this evolving message through separator $\hat{S}_{\hat{\Gamma}_1}$ to the peripheral clique $\hat{C}_{\hat{\Gamma}_1}$. Again, we update the potential of $\hat{S}_{\hat{\Gamma}_1}$, first:

$$\Phi'(\hat{S}_{\hat{\Gamma}_1}) = \sum_{C_{\hat{\Gamma}_1} \setminus \hat{S}_{\hat{\Gamma}_1}} \Phi(C_{\hat{\Gamma}_1}) = \sum_{C} f_2(C, \hat{X}) \tag{3.99}$$

Hence

$$\Phi'(\hat{S}_{\hat{\Gamma}_1}) = \begin{array}{c|l} \hat{X} & \\ \hline \hat{X}_1 & 0.638 \\ \hat{X}_2 & 0.362 \end{array} \tag{3.100}$$

Note that since the separator $\hat{S}_{\hat{\Gamma}_1}$ is composed of the same variable, i.e., $\hat{X}$, as the previous separator S_{Δ_1} that the message passed through and since this is the fist time that a message is processed by $\hat{S}_{\hat{\Gamma}_1}$, the potentials of both separator turned out to be equal.

Next we store the message for future use:

$$\Phi(S_{\hat{\Gamma}_1}) \leftarrow \Phi'(S_{\hat{\Gamma}_1})$$

and update the potential of clique $\hat{C}_{\hat{\Gamma}_1}$:

$$\Phi'(\hat{C}_{\hat{\Gamma}_1}) = \Phi'(\hat{S}_{\hat{\Gamma}_1}) \times \Phi(\hat{C}_{\hat{\Gamma}_1})$$

The resulting updated potential for the peripheral clique $\hat{C}_{\hat{\Gamma}_1} = \{X, \hat{X}\}$ is:

$$\Phi'(\hat{C}_{\hat{\Gamma}_1}) = \Phi'(X, \hat{X}) = \begin{array}{c|c|l} X & \hat{X} & \\ \hline \text{x} & \hat{X}_1 & 0.638 \cdot \mathcal{N}(X; 0.35, 0.07) \\ & \hat{X}_2 & 0.362 \cdot \mathcal{N}(X; 0.65, 0.07) \end{array} \tag{3.101}$$

and we store the updated potential for future messages:

$$\Phi(\hat{C}_{\hat{\Gamma}_1}) \leftarrow \Phi'(\hat{C}_{\hat{\Gamma}_1})$$

The updated potential $\Phi'(X, \hat{X})$ in (3.101) represents the updated joint distribution $P(X, \hat{X})$ after the message is absorbed by the clique $\hat{C}_{\hat{\Gamma}_1} = \{X, \hat{X}\}$. Hence we can sum out the Fuzzy variable $\hat{X}$ to get a marginal distribution of the original continuous variable X which turns out to be a mixture of two Gaussians:

$$P(X) = 0.638 \cdot \mathcal{N}(X; 0.35, 0.07) + 0.362 \cdot \mathcal{N}(X; 0.65, 0.07).$$

The marginal distribution of the continuous variable X, which is a mixture of two Gaussians as the result of this inferencing process is depicted in Fig 3.13.

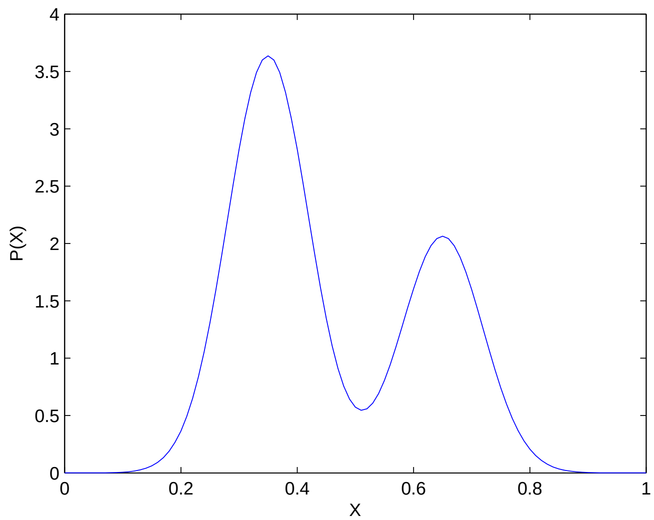


Figure 3.13: Marginal probability density function of continuous variable X

3.4.3 Exact inferencing in *Type-II* FBN

In this section we provide a formal methodology to perform inferencing about *Type-II* FNB representation a HBN. As usual, we are interested in the problem of probabilistic inferencing and once again we take the approach offered by the junction tree algorithm. We shall develop a version of the junction tree algorithm which is suited for *Type-II* FBNs, proposed in [4] and often called Lauritzen's algorithm.

Consider an HBN defined by equation (3.35) representing the joint probability distribution $P(\mathbf{\Delta},\, \mathbf{\Gamma})$. A *Type-II* FBN, $\mathcal{G}_{FBN^{II}}$, can be constructed according to Definition 7, which provides a general approximation for the HBN, if the conditional dependencies between adjacent continuous variables can be represented by a Conditional Gaussian Regression (CGR) model defined by equation (3.57), such that

$$P(\mathbf{\Delta},\, \mathbf{\Gamma}) \equiv P(\mathbf{\Delta}, \hat{\mathbf{\Omega}}, \mathbf{\Omega}, \mathbf{\Psi}) \tag{3.102}$$

As it is the case for *Type-I* inferencing, it is trivial to show that the transformation to the *Type-II* topography does not change the decomposability feature of the original HBN. Thus, using the Chain Rule of equations (3.38) and (3.39) we can define the joint probability distribution representing the *Type-II* FBN, formally, as follows:

$$\begin{aligned}
P(\mathbf{\Delta}, \hat{\mathbf{\Omega}}, \mathbf{\Omega}, \mathbf{\Psi}) = & \prod_{\Delta \in \mathbf{\Delta}} P(\Delta|\mathrm{Par}_{\mathbf{\Delta}}(\Delta)) \times \prod_{\hat{\Omega} \in \hat{\mathbf{\Omega}}} P(\hat{\Omega}|\mathrm{Par}_{\hat{\mathbf{\Omega}}}(\hat{\Omega}) \times \prod_{\Omega \in \mathbf{\Omega}} P(\Omega|\hat{\Omega}) \\
& \times \prod_{\Psi \in \mathbf{\Psi}} P(\Psi|\mathrm{Par}_{\mathbf{\Delta}}(\Psi), \mathrm{Par}_{\mathbf{\Gamma}}(\Psi)) \\
= & P(\mathbf{\Delta}, \hat{\mathbf{\Omega}}) P(\mathbf{\Omega}|\hat{\mathbf{\Omega}}) P(\mathbf{\Psi}|\mathbf{\Delta}) \\
= & P(\mathbf{\Delta}, \hat{\mathbf{\Omega}}, \mathbf{\Psi}) P(\mathbf{\Omega}|\hat{\mathbf{\Omega}})
\end{aligned} \tag{3.103}$$

Note that in equation (3.103) $P(\mathbf{\Delta}, \hat{\mathbf{\Omega}}, \mathbf{\Psi})$ represents the joint probability distribution for the discrete variable sets $\mathbf{\Delta}, \hat{\mathbf{\Omega}}$ and the continuous variable set $\mathbf{\Psi}$. Based on Definition 4, sets $\mathbf{\Delta}, \hat{\mathbf{\Omega}},$ and $\mathbf{\Psi}$ constitute a Hybrid Bayesian Network, which is a subgraph within the structure of the original HBN, $\mathcal{G}_{HBN}$, that the *Type-II* FBN is based on. We denote this subgraph HBN as HBN_{sub}. Then, given the Hybrid Bayesian Network HBN_{sub}, the joint probability distribution representing its topology is given by

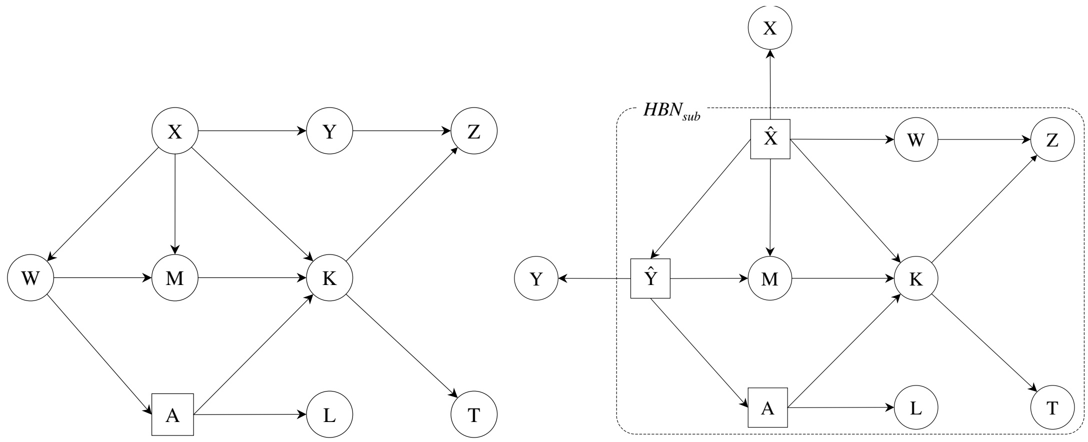
$$\begin{aligned}
P(\mathbf{\Delta}, \hat{\mathbf{\Omega}}, \mathbf{\Psi}) = & \prod_{\Delta \in \mathbf{\Delta}} P(\Delta|\mathrm{Par}_{\mathbf{\Delta}}(\Delta) \prod_{\hat{\Omega} \in \hat{\mathbf{\Omega}}} P(\hat{\Omega}|\mathrm{Par}_{\hat{\mathbf{\Omega}}}(\hat{\Omega}) \prod_{\Psi \in \mathbf{\Psi}} P(\Psi|\mathrm{Par}_{\mathbf{\Delta}}(\Psi), \mathrm{Par}_{\mathbf{\Gamma}}(\Psi)) \\
= & P(\mathbf{\Delta}, \hat{\mathbf{\Omega}}) P(\mathbf{\Psi}|\mathbf{\Delta})
\end{aligned} \tag{3.104}$$

Note also that the second term in equation (3.103), $P(\mathbf{\Omega}|\hat{\mathbf{\Omega}})$ defined by a Fuzzy-discrete and crisp-continuous variable pair represents a peripheral cluster of such variable pairs and indicates a star-shaped tree structures whose rood node is HBN_{sub}. In fact, the resulting topology is simply

$$P(\mathbf{\Omega}|\hat{\mathbf{\Omega}}) = \prod_{\Omega \in \mathbf{\Omega}} P(\Omega|\hat{\Omega}) \ . \tag{3.105}$$

The notion of HBN_{sub} is illustrated in Figure 3.14b.

Consider that a Junction Tree is constructed based on HBN_{sub}. Then, hybrid cliques $\hat{C}_{\hat{\Omega}} = \{\hat{\Omega}, \Omega\}$ can be used to attach original continuous variables $\mathbf{\Omega}$ through discrete (i.e, fuzzy-discrete) separators $\hat{S}_{\hat{\Omega}} = \{\hat{\Omega}\}$ to this Junction Tree. The evidence propagation between a clique $C_{\hat{\Omega}}$, that contains the fuzzy-discrete counterpart variable $\hat{\Omega}$ and a hybrid clique $\hat{C}_{\hat{\Omega}}$ through a separator $\hat{S}_{\hat{\Omega}}$ can be performed using a similar mechanism developed for *Type-I* FBNs and presented in Section 3.4.2. This observation, in fact, simplifies our job



(a) A General HBN (b) The *Type-II* FBN

Figure 3.14: The *Type-II* FBN as the result of transforming an example HBN with variable sets $\mathbf{\Delta} = \{A\}$ and $\mathbf{\Gamma} = \{K, L, M, T, X, W, Y, Z\}$. Note also that HBN_{sub} notion is illustrated in (b).

extensively. Therefore, we can only concentrate developing a message passing methodology within the topology of HBN_{sub}, composed of variables $\mathbf{\Delta}, \hat{\mathbf{\Omega}}, \mathbf{\Psi}$.

A junction tree $\mathcal{T}_{HBN_{sub}}$ can be constructed using the junction tree clustering algorithm introduced in Section 2.1.3 based on the topology of HBN_{sub}, such that

$$\mathcal{T}_{HBN_{sub}} = [\mathbf{C}_d, \mathbf{C}_h, \mathbf{C}_c, \mathbf{S}_d, \mathbf{S}_h, \mathbf{S}_c] \tag{3.106}$$

where $\mathbf{C}_d$ is a set of discrete cliques of purely discrete variables, $\mathbf{C}_h$ represents a set of hybrid cliques of discrete and continuous variables, and $\mathbf{C}_c$ is a set of continuous cliques of purely continuous variables. Similarly, $\mathbf{S}_d, \mathbf{S}_h,$ and $\mathbf{S}_c$ are sets of discrete, hybrid, and continuous separators, respectively. Recalling that the message passing in a junction tree is performed along the separators that connect two adjacent cliques, let us take a closer look to the connections within $\mathcal{T}_{HBN_{sub}}$. Let $C - S - C'$ denote such a connection where adjacent cliques C and C' are connected through separator S. Then, we can represent all the connections in $\mathcal{T}_{HBN_{sub}}$ by the following seven classes:

$$C_d - S_d - C'_d \tag{3.107}$$

$$C_d - S_d - C'_h \tag{3.108}$$

$$C_h - S_d - C'_h \tag{3.109}$$

$$C_h - S_h - C'_h \tag{3.110}$$

$$C_h - S_c - C'_h \tag{3.111}$$

$$C_h - S_d - C'_c \tag{3.112}$$

$$C_c - S_c - C'_c \tag{3.113}$$

where $C_d \neq C'_d,\ C_d, C'_d \in \mathbf{C}_d$, $C_h \neq C'_h,\ C_h, C'_h \in \mathbf{C}_h$, $C_c \neq C'_c,\ C_c, C'_c \in \mathbf{C}_c$, and $S_d \in \mathbf{S}_d$, $S_h \in \mathbf{S}_h$, $S_c \in \mathbf{S}_c$.

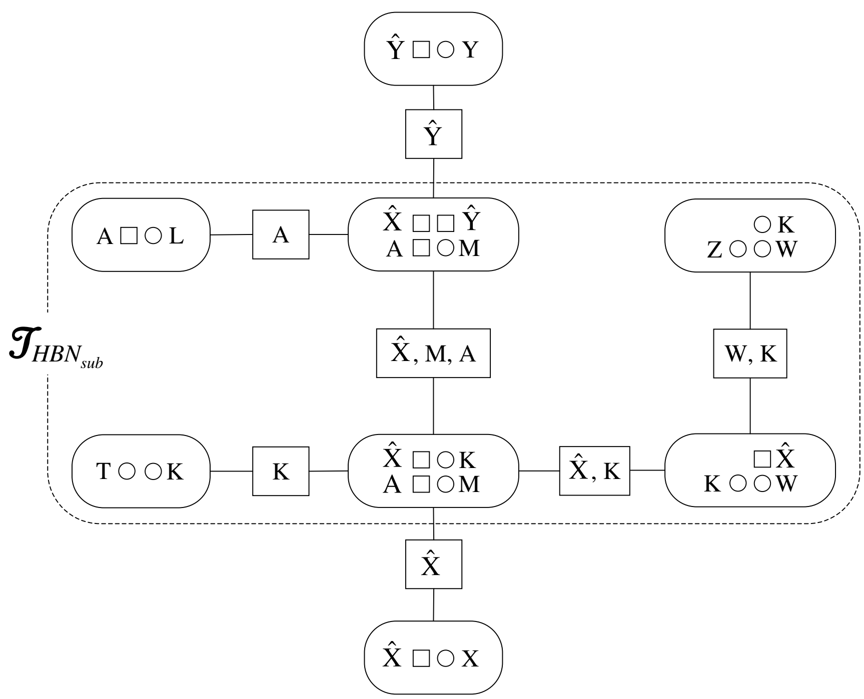


Figure 3.15: The junction tree associated with *Type-II* FBN of Figure 3.14b. Note that junction tree $\mathcal{T}_{HBN_{sub}}$ corresponding to HBN_{sub} is also denoted here.

Considering the fact that these seven classes given by equations (3.107) to (3.113) are an exhaustive list of connections in $\mathcal{T}_{HBN_{sub}}$, we can make the following important observations regarding the topology of a hybrid junction tree defined by equation (3.106):

- A discrete clique C_d and a continuous clique C_c are not adjacent and along the path that connect them there must be a hybrid clique C_h.

- Hence, in HBN_{sub}, consistent with the topology of the *Type-II* FBN, in term of ordering, discrete variables precede continuous variables, i.e., discrete variables do not appear as descendants of continuous variables.

Using this ordering, A hybrid junction tree $\mathcal{T}_{HBN_{sub}}$ can be constructed such that,

i. A hybrid clique in the tree becomes the root of $\mathcal{T}_{HBN_{sub}}$.

ii. If there a continuous cliques on any branch of $\mathcal{T}_{HBN_{sub}}$, then these continuous cliques must be farthest away from the root clique. Furthermore, there must be at least one hybrid cluster separating these continuous cliques and the discrete cliques.

Lauritzen called such a root clique as a *strong root* [4]. Before providing a formal definition of a strong root, let us elaborate on the how it relates to inferencing in junction trees.

At the crux of exact inferencing in a Hybrid Bayesian Network, as in HBN_{sub}, where continuous variables do not have discrete variables among their descendants, lies the decomposition of a suitable modified decomposable network into partly independent components formed by the cliques of the HBN. Such a decomposition, however, also needs to take into account the asymmetry of conditional dependencies between continuous and discrete variables as defined by Conditional Gaussians, in particular, by equations (3.57) and (3.63), for *Type-I* and *Type-II* FBNs respectively. The reader may refer to Leimer [88] for a detailed theoretical study as well as proofs. Here, we provide the formal definitions that will help us state the fundamental theorem that ensures message passing within hybrid junction trees representing *Type-II* FBNs, in general and within junction tree $\mathcal{T}_{HBN_{sub}}$, in particular. We start with the notion of decomposition:

Definition 8 *Let $\mathcal{G}$ be an undirected hybrid graph with a vertex set $\mathbf{V}$ and $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be three disjoint subsets of $\mathbf{V}$. The triplet $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is said to form a decomposition of $\mathcal{G}$ if $\mathbf{V} = \mathbf{A} \cup \mathbf{B} \cup \mathbf{C}$ and the following three conditions hold:*

i. *C seperates A form B.*

ii. *C is a complete subset of V.*

iii. $\mathbf{C} \subseteq \mathbf{\Delta} \vee \mathbf{B} \subseteq \mathbf{\Gamma}$

If the conditions outlined in Definiton 8 are satisfied then we say that $\mathcal{G}$ is *decomposed* into $\mathcal{G}_{\mathbf{A}\cup\mathbf{C}}$ and $\mathcal{G}_{\mathbf{B}\cup\mathbf{C}}$. If only the conditions (i) and (ii) are satisfied then we say that the triplet ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$) constitutes a *weak decomposition*. Figure 3.16 illustrates the concept of strong and weak decomposition and the *forbidden connection* in a decomposable graph.

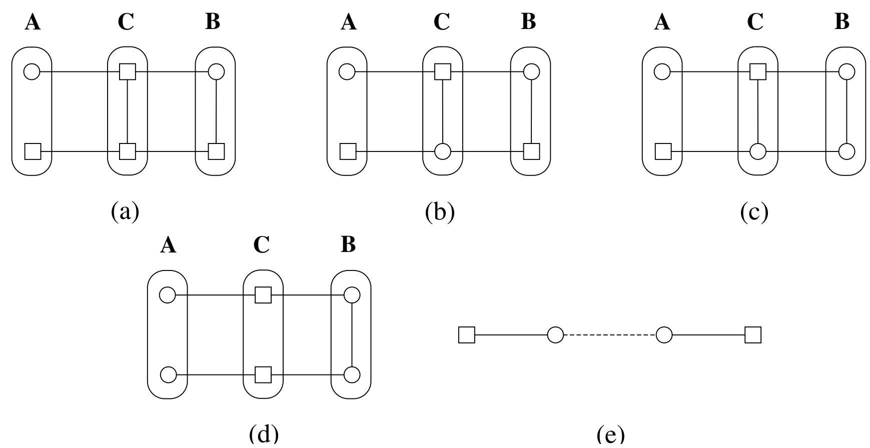


Figure 3.16: Illustration of Decomposability: For sets $\mathbf{A}, \mathbf{B}, \mathbf{C}$, where discrete and continuous nodes are depicted by squares and circles respectively, we have a strong decomposition with $\mathbf{C} \subseteq \mathbf{\Delta}$ in (a) and with $\mathbf{B} \subset \mathbf{\Gamma}$ in (b). (c) illustrates a weak decomposition where none of these two conditions are satisfied. (d) does not denote a decomposition since $\mathbf{C}$ is not complete. Finally, in (e) we see a path that is not permissible in a decomposable graph.

A graph that can be successfully decomposed into its cliques is called a *decomposable* graph. Definition 9 formalizes the concept.

Definition 9 *An undirected, hybrid graph $\mathcal{G}$ is decomposable if it is complete of if there exist a composition $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, such that $\mathbf{A}$ and $\mathbf{B}$ are nonempty sets, into subgraphs $\mathcal{G}_{\mathbf{A}\cup\mathbf{C}}$ and $\mathcal{G}_{\mathbf{B}\cup\mathbf{C}}$.*

Being triangulated and not having any path of particular type are the two key feature of decomposable graphs. Recalling our discussions on construction of junction trees, a junction tree is formed based a triangulated graph. Therefore the cliques in a junction tree can be arranged to constitute a decomposition of the hybrid graph it represents.

Now, we are ready to define the strong root formally and provide the fundamental theorem which assures that message passing in *Type-II* FBNs tractable.

Definition 10 *A clique R in a clique tree T is a strong root if every two adjacent cliques C1 and C2 such that C1 is closer to R than C2 satisfy*

$$(C_1 \cap C_2) \subseteq \mathbf{\Delta} \vee (C_2 \setminus C_1) \subseteq \mathbf{\Gamma} \tag{3.114}$$

If we consider the statement above in terms of decomposition, we see that the condition stated in 3.114 is equivalent to a strong decomposition $\mathcal{G}_{C_1 \cup C_2}$ formed by the triplet $(C_1 \setminus C_2,\ C_2 \setminus C_1,\ C_1 \cap C_2)$. In other words, as we stated in item ii while discussing the construction of of a hybrid junction tree, if a separator between two adjacent cliques is not purely discrete then the clique furthest away from the root has only continuous vertices. In the junction tree illustrated in Figure 3.15 cliques $\{\hat{X}, M, A, \hat{Y}\}$ and $\{A, L\}$ are possible strong roots.

The reason we use strongly rooted trees is that it ensures that message passing is not only well defined, but also leads to the exact results. In general, this is not the case when using trees that are not strongly rooted.

Theorem 3.4.1 *Having a strong root is a sufficient, but not a necessary condition for the operations of the message passing algorithm to be well defined in hybrid clique trees [88].*

So far, we have motivated strongly rooted trees as a way to make sure that all the operations required to perform message passing on $\mathcal{T}_{HBN_{sub}}$ are well defined. However, it is possible to find clique trees which are not strongly rooted but in which message passing is still possible [10].

Leimer [88] (Statement iii' of Theorem $2'$) ensures that *the cliques of a decomposable Type-II FBN can be organized in a junction tree with at least one strong root.* Therefore, in the following discussion, we assume that a hybrid junction tree $\mathcal{T}_{HBN_{sub}}$ representing the *Type-II* FBN with at least one strong root has been already constructed. Figure 3.15 illustrates such a junction tree for a sample *Type-II* FBN.

Now, we can focus on message passing for the seven classes of possible connections between neighboring cliques in a $\mathcal{T}_{HBN_{sub}}$ defined by equations (3.107) to (3.113). A closer inspection of these connections shows that class (3.107) is a discrete-to-discrete connection that is typical to junction trees of discrete only BNs and hence, associated message passing can be perform by a standard junction tree algorithm such as Lauritzen's JT algorithm

[89]. In terms of directional symmetry, the connection classes (3.108), (3.110), (3.111), and (3.113) demonstrate symmetrical connections, whereas classes (3.112) and (3.112) are asymmetrical and therefore the associated bi-directional message passing should be treated differently. However we can treat the connection class (3.110) with all hybrid elements as a meta class which encompasses all other connection classes. Therefore, we only need to demonstrate our message passing scheme for the connection class (3.110). Message passing in all other classes is performed in a similar way, though to increase computational efficiency it implementation may be handled differently.

Consider two neighboring hybrid cliques $\mathbf{A}_h$ and $\mathbf{B}_h$ connected by a hybrid separator $\mathbf{S}_h$, such that

$$\mathbf{A}_h = \mathbf{\Delta}_A \cup \mathbf{\Gamma}_A \tag{3.115}$$

$$\mathbf{B}_h = \mathbf{\Delta}_B \cup \mathbf{\Gamma}_B \tag{3.116}$$

$$\mathbf{S}_h = (\mathbf{\Delta}_A \cap \mathbf{\Delta}_B) \cup (\mathbf{\Gamma}_A \cap \mathbf{\Gamma}_B) \tag{3.117}$$

where $\mathbf{\Delta}_X$ and $\mathbf{\Gamma}_X$ denote the sets of discrete and continuous variables in the clique $\mathbf{X}$ respectively.

Consider a set of evidence $\mathbf{E}$ is introduced to the *Type-II* FBN and distributed along the associated junction tree and let $\mathbf{\Phi}'(\mathbf{A}_h)$ indicates the updated potential of clique $\mathbf{A}_h$ from it original potential $\mathbf{\Phi}(\mathbf{A}_h)$ due to the introduction of evidence $\mathbf{E}$. Formally,

$$\mathbf{\Phi}'(\mathbf{A}_h) = \mathbf{\Phi}(\mathbf{A}_h|\mathbf{E}) \tag{3.118}$$

Next we propagate the updated evidence from clique $\mathbf{A}_h$ to clique $\mathbf{B}_h$ through separator $\mathbf{S}_h$. First step to perform this propagation is to update the potential of $\mathbf{S}_h$ as follows

$$\mathbf{\Phi}'(\mathbf{S}_h) = \int\limits_{-\infty}^{\infty} \ldots \Gamma_{A_h} \int\limits_{-\infty}^{\infty} \left(\sum \ldots \Delta_{A_h} \sum \mathbf{\Phi}(\mathbf{\Delta}_{A_h}, \mathbf{\Gamma}_{A_h})\right) \prod_{\Gamma_{A_h}} d\Gamma_{A_h} \tag{3.119}$$

where $\Delta_{A_h} \in \mathbf{\Delta}_{A_h}$ and $\Gamma_{A_h} \in \mathbf{\Gamma}_{A_h}$ enumerate the elements of discrete and continuous variables in clique $\mathbf{A}_h$, respectively.

Then, the updated potential $\mathbf{\Phi}'(\mathbf{B}_h)$ of clique $\mathbf{B}_h$ is determined by

$$\mathbf{\Phi}'(\mathbf{B}_h) = \mathbf{\Phi}(\mathbf{\Delta}_{B_h}, \mathbf{\Gamma}_{B_h}) \frac{\mathbf{\Phi}'(\mathbf{\Delta}_{S_h}, \mathbf{\Gamma}_{S_h})}{\mathbf{\Phi}(\mathbf{\Delta}_{S_h}, \mathbf{\Gamma}_{S_h})} \tag{3.120}$$

Apparently, to develop a explicit tractable formalism for message passing as defined in equations (3.119) and (3.120), we need to have a model to define the potentials of a hybrid clique and separator. This model should also provide closed form solutions for basic local operations in a hybrid junction tree including the instantiation, frame extensions, marginalization, multiplication, and divisions of associated clique potentials.

The Conditional Gaussian Regression (CGR) model is the generalized form of the Conditional Gaussian model that we introduced while defining the *Type-I* FBNs in Definition 6 and developing the associated inferencing algorithm in Section 3.4.2.

We shall implement a Conditional Gaussian Regression model to define the conditional distribution of continuous variable W given a hybrid set of parents inside the HBN_{sub}.

In particular, the conditional distribution of W given continuous parent set $\mathbf{Z}$ an discrete parent set $\mathbf{I}$ is given by

$$P(W|\mathbf{I}=i,\mathbf{Z}=z)=\mathcal{N}\left(\alpha(i)+\beta(i)z,\gamma(i)\right) \tag{3.121}$$

Note that for each instantiation of continuous and discrete parents i an z respectively, $\gamma(i)$ is a nonnegative real number, $\alpha(i)$ is a real number, and $\beta(i)$ is a vector of the same size as the the cardinality of $\mathbf{Z}$. In case $\gamma(i)=0$, this distribution indicates a linear deterministic dependence between $\mathbf{W}$ and $\mathbf{Z}$.

Formally, Conditional Gaussian Regression (CGR), also know as Conditional Linear Gaussian (CLG) [5], is a Hybrid Bayesian Network, where

- Discrete nodes cannot have continuous parents, hence their CPDs are discrete and,
- CPD of any continuous variable is a linear CPD given any combinations of its discrete parents. Such that, if a continuous variable W has the following discrete and continuous sets of parents

$$\begin{aligned}\boldsymbol{\Delta}_{\mathrm{Par}}(W)&=\{\Delta_1,\ldots,\Delta_k\}, \qquad \boldsymbol{\Delta}_{\mathrm{Par}}(W)\in\boldsymbol{\Delta}\\ \boldsymbol{\Gamma}_{\mathrm{Par}}(W)&=\{\Gamma_1,\ldots,\Gamma_l\}, \qquad \boldsymbol{\Gamma}_{\mathrm{Par}}(W)\in\boldsymbol{\Gamma}\end{aligned} \tag{3.122}$$

The CPD of W is defined by

$$P(W|\boldsymbol{\delta},\boldsymbol{\gamma})=\mathcal{N}\left(W:\ \beta_{\boldsymbol{\delta},0}+\sum_{i=1}^{l}\beta_{\boldsymbol{\delta},i}\gamma_i,\ \sigma_{\delta}^2\right) \tag{3.123}$$

> Note that in equation (3.123) for every assigned value set δ; $\quad \delta \in \mathbf{\Delta}_{\mathrm{Par}}(W)$, we have a set $\{\beta_{\delta,0}, \ldots, \beta_{\delta,k}\}$ and σ_{δ}^{2}.

Note that if all the discrete variables are given, then the CPDs of the continuous variables are all linear CPDs (see equation equation (2.17)). Thus given any assignment of the discrete variables a CGR model is reduced to a LG and therefore represents a normal distribution. It follows that the joint distribution represented by a CGR is a mixture of Normal distribution where every mixture component corresponds to an instantiation of the discrete variables.

The key decision in adapting the clique tree algorithm into CGR model that we want to use is how to represent the clique potentials and the separators and in particular how to represent the functions over continuous variables.

We already showed that the joint probability distribution represented by CGR is a mixture of Gaussians, thus the potential representing the joint distribution of a hybrid clique C_h is also a mixture of Normal distributions. Consider a hybrid clique (or separator) with the set of continuous variables $\mathbf{\Gamma}_h$ and the set of discrete variables $\mathbf{\Delta}_h$. Then, the marginal distribution over a subset of variables $\{\mathbf{\Delta}_h, \mathbf{\Gamma}_h\}$ can be represented as a mixture of Gaussians over $\mathbf{\Gamma}_h$ for every assignment of $\mathbf{\Delta}_h$. The probability $P(\mathbf{\Delta}_h = \delta_h)$ is determined by adding the probabilities of the individual Gaussians in the mixture for an instantiation of $\mathbf{\Delta}_h = \delta_h$. If $\mathbf{\Delta}_h$ represents all the discrete variables then for every instantiation of $\mathbf{\Delta}_h = \delta_h$ we get a single Gaussian rather than a mixture. For the purposes of message passing to and from a hybrid clique, this leads to a natural way of representing an intermediate function or potential over $\{\mathbf{\Delta}_h, \mathbf{\Gamma}_h\}$ as a table with one entry for every possible assignment to $\mathbf{\Delta}$. Each entry contains a mixture of Gaussians over the continuous variables.

It becomes clearer that in order to perform basic operations of message passing about the junction tree $\mathcal{T}_{HBN_{sub}}$ of the *Type-II* FBN by using CGR models to represent the potentials of hybrid cliques we need to deal with a mixture of Gaussians. Therefore, a more detailed look at a mixture of Gaussians is in order.

A mixture of Gaussians over the a set of variables $\mathbf{X}$ can be represented as a set of pairs $\{w_i, \mathcal{N}(\mathbf{X}; \boldsymbol{\mu}_i, \Sigma_i)\}$ where w_i is the weight of the i^{th} mixture component. If $\Sigma_i w_i = 1$ we say that the mixture is *normalized* then it represents a probability density: with probability w_i

$\mathbf{X}$ has the normal distribution $\mathcal{N}(\mathbf{X}; \boldsymbol{\mu}_i, \Sigma_i)$. Alternatively, we can also represent a mixture using *canonical forms* as defined by equation (2.18). In this case the weight w_i becomes a part of the canonical form by incorporating it into the parameter g_i we add $\log w_i$ to g_i.

However, the use of mixtures of Gaussians within the context of message passing in *Type-II* FBNs has two serious problems:

- The size of the functions representing cliques in a *Type-II* junction tree is not fixed, For example, a function over one continuous variable can be as simple as one weight and univariate Gaussian and as complex as a mixture with as many as domain variables.

- Some of the operations used in the clique tree algorithm are not defined. In particular, dividing two mixtures of Gaussians does not result in a mixture of Gaussians and does not have a closed form. Furthermore, marginalization is not defined for mixtures of functions of cliques.

We now present an alternative approach to the mixture representation which can successfully be used to perform basic operations of message passing such as marginalization. In this alternative representation we still represent a potential or function over $\{\boldsymbol{\Delta}_h, \boldsymbol{\Gamma}_h\}$ as a table, which has one entry for every assignment of $\boldsymbol{\Delta}_h = \delta_h$, however, now every entry contains just one canonical form rather than a mixture. Let us refer this new data table or structure as *canonical factors* to facilitate the narrative. We just stated in previous paragraphs that the marginal distribution over continuous variables can be a mixture of Gaussians. However, it is obvious that the newly defined conditional factors do not except mixtures except for the case where all the relevant discrete variables are included in their domain and thus the mixture is represented by one Gaussian. To overcome this obstacle we shall approximate the mixtures that cannot be represented just one Gaussian. This approximation is performed by *collapsing* the original Gaussians of the mixture to obtain just one Gaussian as the representation of the actual mixture density. The notion of collapsing is illustrated for an example mixture of two Gaussians in Figure 3.17.

Now the question is how to perform the collapsing operation. Recall that the mean vector $\boldsymbol{\mu}$ an the covariance matrix Σ are the two parameters that define Gaussians. Then our objective while approximating a mixture should be is to come up with a Gaussian which

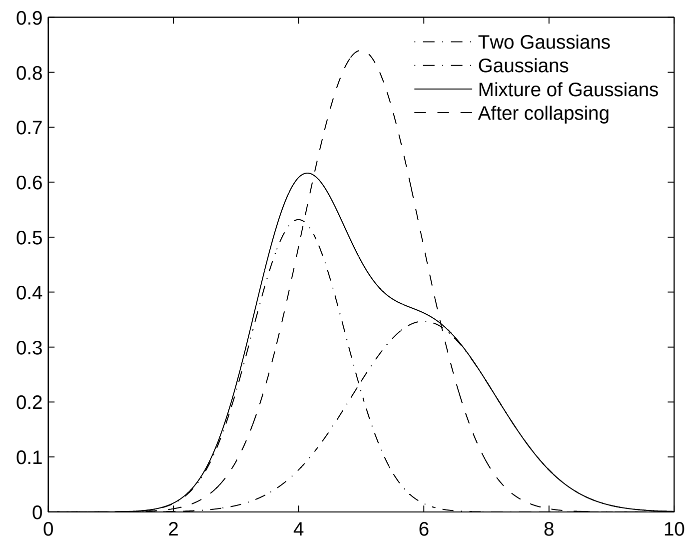


Figure 3.17: Collapsing a mixture of two Gaussians

has the same $\boldsymbol{\mu}$ and Σ as the original mixture. The following theorem [81] formalizes the notion of collapsing a mixture of Gaussians.

Theorem 3.4.2 *Let Q be the density function of a normalized mixture of n Gaussians $(\omega_i, \mathcal{N}(\mathbf{X}; \boldsymbol{\mu}_i, \Sigma_i))$. Let $P = \mathcal{N}(\mathbf{X}; \boldsymbol{\mu}, \Sigma)$ be a normal distribution defined as:*

$$\boldsymbol{\mu} = \sum_i \omega_i \boldsymbol{\mu}_i$$

$$\Sigma = \sum_i \omega_i \Sigma_i + \sum_i \omega_i (\boldsymbol{\mu}_i - \boldsymbol{\mu})(\boldsymbol{\mu}_i - \boldsymbol{\mu})^T$$

Then P has the same first two moments (i.e., means and covariances) of Q. Furthermore, P minimizes the KL-divergence between Q and any other normal distribution, i.e. for any other normal distribution $P' = \mathcal{N}(\boldsymbol{\mu}', \Sigma')$ we have $d_{KL}(P, Q) \leq d_{KL}(P', Q)$, where the equality holds iff $P \equiv P'$.

Having chosen a representation for the potentials and separators we need to verify that we can perform the various operations required by message passing algorithms. With the important exception of marginalization which we shall soon discuss all these operations are a natural generalization of the operations defined on discrete factors introduced in Section 2.1.3 and the operations defined for canonical form outlined in Section 2.1.4:

- *Extension*: We can extend the domain of the canonical factor by adding discrete variables (just like in discrete factors) and by adding continuous variables to every canonical form (as defined in Section 2.1.4).
- *Multiplication and Division:* We multiply / divide canonical factors by multiplying / dividing the relevant entries (just like in discrete factors) as defined in Equation 2.20 and Equation 2.21.
- *Instantiation:* We instantiate a discrete set of observation $\mathbf{d}$ by setting the entries which are not consistent with $\mathbf{d}$ to zero. We instantiate a set of continuous observations $\mathbf{x}$ by simply instantiating every canonical form with $\mathbf{d}$.

Regarding the marginalization operation, we treat the continuous and discrete cases differently. We can marginalize continuous variables by marginalizing each of the associated canonical forms. Marginalization discrete variables are slightly more complicated and requires combining multiple canonical forms into one. In particular, assume that we have a canonical factor representing the potential of clique $\{\mathbf{X}, \mathbf{A}, \mathbf{B}\}$, where $\mathbf{A}, \mathbf{B} \subseteq \mathbf{\Delta}$ and $\mathbf{X} \subseteq \mathbf{\Gamma}$,and we want to marginalize the canonical factor over $\mathbf{X}, \mathbf{A}$. To perform the marginalization, for every value of $\mathbf{A} = \mathbf{a}$ we need to identify entries that are consistent with $\mathbf{a}$ and combine them into one. The operation of combining a few canonical forms is equivalent to the collapsing operation. However, the collapsing operation was defined for a mixture of Gaussians and not for a mixture of general canonical forms. Indeed, combining canonical forms into one is well defined if the canonical forms can be represented as Gaussians. Recalling the fact that collapsing operation is essentially an approximation, marginalization that involves collapsing Gaussians is called *weak marginalization* as opposed to *strong marginalization* that does not involve any approximations. The following definition and theorem formalizes this discussion.

Definition 11 *Marginalization of canonical factors can be weak or strong. Weak marginalization involves approximation such as collapsing of mixtures. Whereas, strong marginalization does not involve approximation. Strong marginalization can be applied to the following cases:*

- *Only continuous variables are to be marginalized.*
- *The canonical factor in question is defined over discrete variables only.*
- *All the canonical forms are identical.*

If none of these conditions hold then the marginalization is weak.

It is clear that, considering the composition of a hybrid clique, we can conclude that the marginalization operation over a hybrid clique is a weak marginalization. Thus, the collapsing approach is required to perform the marginalization.

Theorem 3.4.3 *The weak marginalization results in a closed form solution and hence is well defined iff all the canonical forms of the potential can be represented by Gaussians. The weak marginalization operation is performed using Theorem 3.4.2. Weak marginalization involves collapsing and the resulting marginals are an approximation of the true marginal distribution, preserving the mean and the variances (i.e., the first two moments).*

Not being able to combine general canonical forms imposes restriction while determining elimination order during message passing in a junction tree. In Theorem 3.4.1, we already indicated that such an ordering that ensures message passing exist in a junction tree with a strong root (see Definition 10). In a junction tree with strong roots, it is possible to find a clique C_R among the strong roots such that when a message is send towards C_R there will be no need for weak marginalization. Not having to perform weak marginalization means that when sending a message towards the root C_R either there are no continuous variables on the separators or there was no need to sum out any discrete variables along the path.

When the junction tree $\mathcal{T}_{HBN_{sub}}$ is strongly rooted it is possible to calibrate it using message passing. First we send a message towards the root. As indicated above, this run is possible since it does not involve any weak marginalization. Then, we send a message originating from the root clique C_R along the branches of the tree. This second run requires us to perform weak marginalization, but it will be always possible. The reason is that in this second run we always send messages from cliques that already received and processed all their messages and represent a probability function. Therefore, the canonical forms for each clique represented by Gaussians and can be collapsed. After running the calibration algorithm each potential in the tree contains the correct joint distribution over its variables, which can be used to determine marginals distributions for the individual variables making up the clique.

```
Calibration of a Hybrid Junction Tree with Strong Roots
1. Pick a strong root C_R
2. Initialize canonical factors in the potentials
3. Multiply potentials by the CPDs
4. Enter evidence
    5. Discrete: in any potential
    6. Continuous: in all potentials; remove observed
       variables from the junction tree
7. Send messages towards root clique C_R using the
   algorithm on Table 3.6
8. Send message from the root towards the branches using
   the algorithm on Table 3.6
```

Figure 3.18: Calibration Algorithm for Strongly Rooted Hybrid Junction Trees

As in purely discrete junction trees, it is possible to introduce evidence into hybrid junction trees such as $\mathcal{T}_{HBN_{sub}}$. Discrete evidence can be introduced in $\mathcal{T}_{HBN_{sub}}$ by multiplying one of the potentials by the relevant indicator function (as in the case of discrete junction trees such as *Type-I* case). Continuous evidence can be introduced by instantiating every relevant potential. This process of calibration is presented as a calibration algorithm which also includes evidence instantiation is shown in Figure 3.18. Note that evidence can also be introduced after calibrating the junction tree, in which case the continuous evidence also needs to be instantiated in the separators.

This concludes our discussion on exact inferencing for *Type-II* FBNs. Starting the next chapter, we shall develop a real-world application of the Fuzzy-Bayesian hybrid methodology outlined here as well as present the results of the *Type-I* and *Type-II* inferencing to the HBN representing the domain model of the application.

Chapter 4

Application of Research Methodology

In this chapter we focus on the application of the hybrid Fuzzy-Bayesian framework introduced in the preceding chapters of this thesis. In particular, our application concentrates on a real-world problem domain that would immensely benefit from the inclusion of various complex variables of a hybrid nature to the domain model while performing uncertainty analysis about the system. Unmanned Aircraft Systems (UASs) are selected as our domain of interest. In the succeeding sections, first we provide background information on the UAS as an emerging technology, how it relates to our research, and the objectives of our application. Then, we introduce a novel methodology for hazard taxonomy development with particular emphasis on aviation related hazard-source identification and present the Hazard Classification and Analysis System (HCAS) for the UAS. Consequently, we present a new regulation based framework for modeling domain safety risk in complex aviation systems as a hybrid Bayesian Network and apply it to the UAS domain. Finally, on the resulting hybrid BN we apply our hybrid Fuzzy-Bayesian methodology introduced in the preceding chapters and conclude with presenting and analyzing the results.

4.1 Application Domain: Unmanned Aircraft Systems - A System Safety Analysis

Concisely, an unmanned aircraft (UA) is an aircraft that does not have a human pilot on board. The Unmanned Aircraft Program Office (UAPO) of the Federal Aviation Authority (FAA) formally defines a UAS as follows.

> "A UAS is the UA and all of the associated support equipment, control station, data links, telemetry, communications and navigation equipment, etc., necessary to operate the unmanned aircraft.

> The UA is the flying portion of the system, flown by a pilot via a ground control system, or autonomously through use of an on-board computer, communication links and any additional equipment that is necessary for the UA to operate safely. The FAA issues an experimental airworthiness certificate for the entire system, not just the flying portion of the system." [47]

Other commonly used terms for UA and UAS are Drones, Unmanned Aerial Vehicle (UAV), Remotely Operated Aircraft (ROA), and Remotely Piloted Vehicles (RPV).

As a most cutting edge field, military acquisitions constitute the bulk of the UAS research and implementations. Since WWII, the UAS concept has been periodically visited only to be abandoned until the next technological advance brings it a new shot of life. This situation changed in 1991 during the Gulf war with the effective use of a low-tech, short-range UAS called Pioneer [48].

Figure 4.1: A Pioneer UAV is catapulted from a launching rail set up atop an M-814 5-ton cargo truck during a test in support of Operation Desert Shield in 1991 [49].

After the first Gulf war, UASs have steadily gained acceptance by militaries around the world as well as in US. Most of the world's major aerospace companies, numerous research institutes and universities, hundreds of component manufacturers and small companies involved heavily in competing for this rapidly expending market are moving ahead at different rates on designing and building a broad range of UAS types. However, there are major regulatory and technological barriers to overcome for wider acceptance. Particularly, the issue of how UASs will be integrated in the National Airspace System (NAS) is considered by regulatory agencies and industry alike as one such major obstacle [50].

At the crux of the integration issue lies the safety concerns, which require an analytical look at the NAS, as a whole, which is increasingly becoming a complex array of military, commercial, and private aircraft. This increased system complexity necessitates the application of systematic safety risk analysis methods to understand and eliminate where possible, reduce, and mitigate risk factors, especially when a new technology or mode of operation such as UAS is introduced to the system.

As the NAS becomes increasingly more complex and constrained, the associated hazard and safety risk modeling must also mature in sophistication. Thus, there is a need for advanced studies focusing on risk-based system safety analysis of emergent UAS operations. This chapter presents a novel regulatory-based integrated approach to system safety and risk analysis of the UAS operations and their interaction with the current NAS and the future Next Generation (NextGen) Airspace.

However, before introducing the components of our methodology for the application, it is beneficial to briefly discuss the concept of system safety analysis and elaborate on the need for such an analysis within the context of emerging UAS operations.

System safety analysis emerged as a separate and unique discipline in the early 1960s with the advent of NASA's Apollo missions and the development of the U.S. Air Force Ballistic Missiles [51]. The essential generalized approach of systems analysis is depicted in Fig 4.2.

The step-by-step approach typically involves describing or characterizing the system under study, establishing boundary conditions, and delving into analysis to support some type of modeling. Models may take a variety of forms, such as mathematical programs, simulations, probability or fuzzy set applications, event trees, fault trees, etc. Emerging from the analysis and modeling phases may be a synthesis of ideas or issues. Frequently, the next step after modeling involves prototype development followed by verification and validation of any proposed action plans.

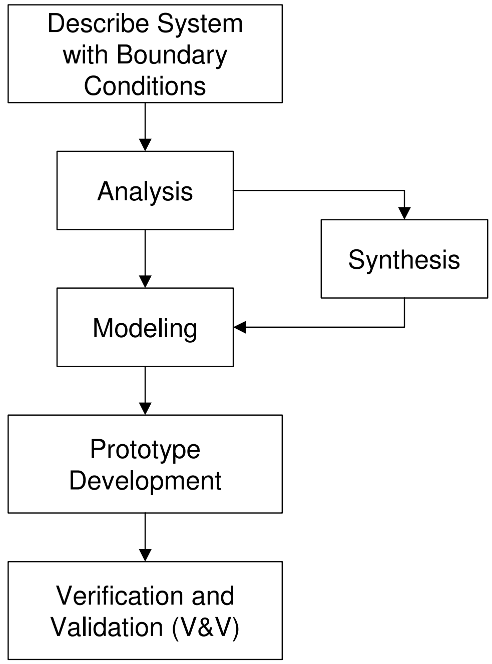


Figure 4.2: A Generalized Systems Approach [52].

Consistent with the basic systems analysis approach, a generalized system safety approach may be developed as shown in Fig 4.3. In this sense, we now have an application of the generalized systems approach focused on the issue of safety. The realization of the systems approach to system safety involves some additional steps, such as clearly defining objectives and system metrics, hazard identification, classification and analysis, safety risk modeling and analysis, and safety risk assessment and prioritization. To these prioritized risks, mitigations are developed and applied to reduce the likelihood or severity of the identified hazards. Once action plans are developed and then applied, validation of their effect is needed to assess their effectiveness in controlling or managing the risk. Feedback from the validation step may necessitate that the system or process be modified which in turn leads back to a new hazard identification, analysis and classification step.

Safe integration of UASs into the NAS presents significant challenges to all stakeholders in the aviation community. Although the main thrust of this emerging technology originates from entrepreneurs, both civilian and military, the burden of the safe integration arguably lies on the shoulders of regulatory agencies such as FAA. The question of safety associated with this integration arises principally due to the unknowns of potential hazards

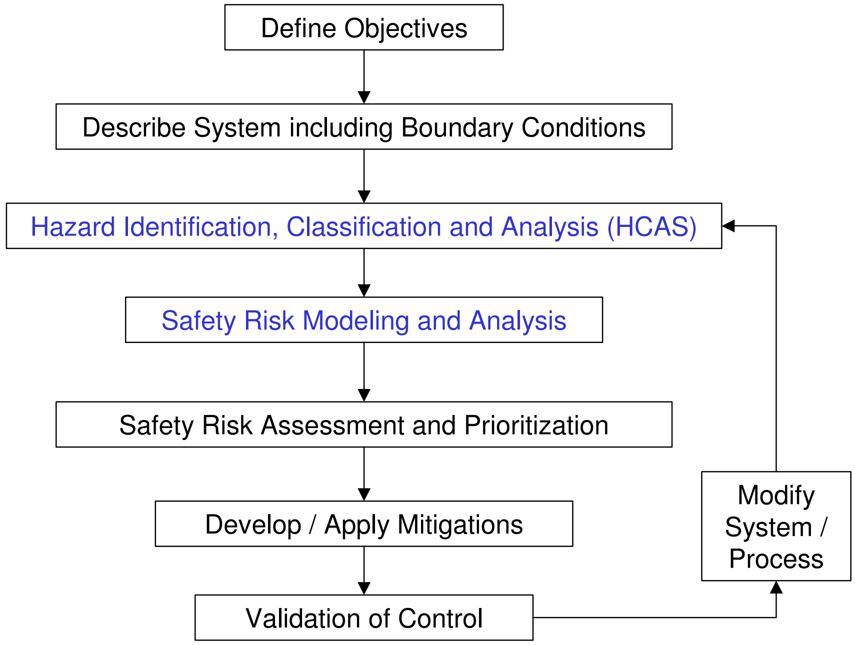


Figure 4.3: A Generalized System Safety Approach [52].

and associated risks while operating in the NAS and interacting with existing NAS users. It should also be noted that UAS operation, which represents a new class are governed by the existing aviation regulations [54].

In the past, when incidents or accidents occurred, a forensic approach was undertaken in the hazard analysis phase of the system safety approach. Heinrich suggested the *domino* theory of accidents [55]. Five dominoes- social environment and ancestry, undesirable traits (e.g., recklessness, violent temper, lack of knowledge, etc.), unsafe acts or behaviors, accident, and injury- formed the basis of the domino effect techniques. His idea was that accidents are a sequence of events in a predetermined proceed/follow relationship, like a row of falling dominoes. This view changes the focus of accident investigations toward the events involved, rather than the conditions surrounding the accident environment. The objective is for analysts and investigators to understand the accident phenomenon on the basis of the chain of events that had occurred. His theory was that if a set of *unsafe conditions set* up a row of vulnerable dominoes, an *unsafe act* would start them toppling. However, should a domino in the sequence be removed, no injury or loss will be incurred. Under this concept, the investigator looks for information that will help reconstruct the chain of events that constituted the accident. The National Transportation Safety Board (NTSB) uses a variant of the sequence of events approach in their analysis of aircraft accidents. While such a forensic approach has merit and has been important and useful to system safety analysts in the past, it is very scenario driven and quite dependent on the contextual factors involved in the specific incident or accident. As such, the safety recommendations emerging from such a forensic analysis may be quite aircraft type or airport specific. As shown in Fig 4.3, the system safety approach involves an identify-analyze-control method of safety as opposed to a *fly-fix-fly* approach [51]. One key hypothesis of our application is that the system safety approach is better suited to safety analysis of new classes of aircraft where data is sparse and operations are limited. In particular, UAS represent a new class of aircraft that is emerging in the current NAS and that will most likely be an integral component of the U.S. Next Generation (NextGen) Air Transport System and the Single European Sky (SESAR) [56].

A second hypothesis of our application is that safety hazards may be derived top-down as opposed to bottom-up. Rather than collecting hazard information from a case-by-case or scenario approach, especially for novel aircraft systems where such data is usually not available, the conjecture is that *the Title 14 Code of Federal Regulations* or *CFRs* may be used to derive hazards as well as their underlying causal factors by utilizing a systems analysis approach.

Thus, our application on the UAS integration into the NAS proposes a regulatory-based integrated approach to system safety and risk analysis of the UAS operations and their interaction with the current NAS and the future NextGen Airspace. Three distinct yet closely related areas of analysis comprise the main thrust of the proposed approach: taxonomy development, causal factor identification, and modeling complex uncertainty.

The next sections describe the elements of our approach in more detail.

4.2 Development of a System-level Taxonomy for Categorization of UAS Hazards

UAS having being successfully employed in the last decade by various military applications are, inevitably, making their way into the civilian world. This new frontier in civil aviation adds another dimension to the ever-increasing complexity of the current NAS in the United States. The future inclusion of private and commercial operations of the UAS into the NAS, unavoidably, raises safety concerns. As the NAS becomes increasingly more complex and constrained, the associated hazard and safety risk modeling must also mature in sophistication. Thus, there is a need for advanced studies focusing on risk-based system safety analysis of emergent UAS operations.

One of the first steps in the proposed UAS system safety analysis is hazard identification and analysis. To that end, a new hazard taxonomy was developed. This taxonomy, termed the Hazard Classification and Analysis System (HCAS) identifies four main hazard system sources: Airmen, UAS, Operations, and Environment. The basic framework of the proposed taxonomy is based on the FAA regulatory perspective (i.e., Title 14, Code of Federal Regulations (14 CFR) chapters on Aircraft, Airmen, Certification/Airworthiness, Flight Operations, etc.). Such an approach uniquely distinguishes the HCAS taxonomy from all other UAS hazard analyses being performed by the Department of Defense (DoD), the RTCA-Special Committee (SC) 203 [5,6], etc.

Safety analysis has a fundamental role to play in the identification of hazard source potentials, the understanding of the underlying causal factors, the likelihood assessment of these factors, the severity evaluation of the potential consequence(s) of mishaps, and the prioritization of mitigations.

A sound system-level safety analysis relies heavily on properly identifying the key components of the area of interest. In particular, the identification of potential hazard sources and sub-sources within the systemic structure of the problem domain should be considered as a fundamentally important step in system safety analysis. Furthermore, since semantics play a crucial role while defining the domain variables, a systematic taxonomy that balances fidelity and generalization provides a solid foundation for a meaningful and relevant system safety analysis. Within this context, we present the HCAS taxonomy specifically designed and developed to identify and categorize individual system-level hazard sources for the UAS operations.

4.2.1 Hazard Classification and Analysis System (HCAS)

HCAS categorizes the UAS hazards consistent with the 14 CFR Sub-chapters, thereby establishing the taxonomy on the FAA regulatory framework. The advantage of the proposed approach is to allow direct association of hazards identified with regulatory requirements or vice versa. The system not only provides the FAA as well as the UAS community the tools to determine safety and regulatory implications of UAS operating in the NAS, but also falls in directly under the FAA Safety Management System (SMS) Doctrine. Particularly, as described in [58], the taxonomy was uniquely developed but was inspired by the research of Hayhurst et al. [59] and the RTCA Special Committee 203 [50].

A set of hypothesized UAS mishap scenarios provided by the FAA are employed to verify and test the robustness of the taxonomy. These hypothesized scenarios were not detailed or specific operational scenarios but were rather more akin to *thought experiments* of possible UAS mishaps due to their inclusion in the NAS. In that sense, these hypothesized scenarios are generated scenarios representing possible hazardous situations in UAS domain. Raheja and Allocco [57] term them "scenario themes" and propose a semi-formal methodology for the development and categorization of such scenarios. Broadly construed, scenario development and characterization include the following: scenario description, initial contributors, subsequent contributors, life-cycle phase, possible effect, system state and exposure, and recommendations, precautions and controls. In a *conversation* or *dialogue* with subject matter experts, knowledge about accidents or possible accidents is elicited using *scenario themes* that are short, concise statements that describe the primary and main contributory hazards. In the case of the scenario development for the UAS, 208 hazard scenario themes were identified in multiple sessions with experts [57]. Such *representative* scenario themes are presented in Table 4.1. Scenario statements typically provide text as to how or why potential accidents may occur. The accident life-cycle from design, certification, flight standards, operations, maintenance and training, etc. should be considered. [58, 61, 62, 63],

Table 4.1: Representative UAS Scenario Themes [60].

List of Hazard Scenarios	
Seq.#	**Hazard Scenario/ Description/ Discussion**
1	The transitional planning for UAS NAS integration is Less Than Adequate (LTA) and the planning does not allow for system design, development time, and maturity. Situation results in increased accident risks.
2	Assumptions concerning system reliability and availability maturity are LTA. There are transitional risks associated with UAS design, development time, system reliability, reliability growth, availability and maturity. Current situation results in vehicle not meeting expected NAS-level availability and reliability requirements associated with Catastrophic and Hazardous risks, consequently there may be increased accident risk.
3	Current DoD and contractor system safety and system reliability analyses and related data are not accessible to the FAA. As a result there are inappropriate assumptions made concerning knowledge of system safety and system reliability that may be increased risk.
4	Due to physical design limitations vehicles may not meet NAS-level availability and reliability requirements associated with Catastrophic and Hazardous risks, consequently there may be increased accident risk.

and [64] also use induction from scenarios to develop risk models of commercial aircraft accidents for the assessment of a portfolio of new aeronautical products. In the situation of emergent aeronautical operations where actual accident or incident data is sparse, inductive reasoning from hypothesized scenarios is a plausible alternative.

De Jong et al. [53] present an approach to pushing the boundary between imaginable and unimaginable hazards that keeps the performance of the hazard identification process separate from the hazard analysis and hazard mitigation processes, so the idea of developing *UAS scenario themes* is consistent with the de Jong method for hazard identification. These hypothesized UAS scenarios supported *analytic generalization* and were primarily used to develop concise terminology of system and sub-system hazard sources. The semantics of the hazards were aligned in a general way with the wording of the main CFR chapters and also vetted with industry subject matter experts.

At the crux of the HCAS taxonomy lie two closely related yet distinct concepts: *hazards* and *hazard sources*. Based on Leveson's definition of hazard [65] we adopted our own definitions for both concepts within the context of this application:

Hazard: A hazard is a state or set of conditions of a system that, together with other conditions in the environment of the system, may lead to an accident (loss event).

Hazard Source: Hazard source are primarily components of the UAS domain; hence a state or set of conditions of these components may lead to hazardous potential for the domain itself. Each hazard source category corresponds to a key component of the domain of interest. Thus, these components do not represent neither individual hazards nor categories of hazards of the UAS domain.

HCAS is a continuously evolving taxonomy. The current version of the taxonomy has been developed in multiple phases as the product of numerous knowledge elicitation sessions with subject matter experts spanning a time period of two years. The succeeding sections illustrate this evolutionary development process.

HCAS version 1.0

The idea behind the HCAS development effort is to provide a structured framework to identify and classify both system and sub-system hazard sources for UAS operations. Based on the above hazard and hazard source definitions, in this early version of HCAS, three systems-level hazard sources are identified as *Aircraft, Control Station*, and *UAS-NAS Interconnectivity*. These system-level hazard sources form the three main HCAS *cubes* depicted in Fig 4.4.

It is acknowledged that these three primary hazard sources are operative in a context, so "Environment" is included as the backdrop. Formally we can define these three system level hazard sources as follows;

UAS System Level Hazard Sources:

Aircraft: Design and Human Factors Issues associated with the UA and technology (includes hardware & software) onboard.

Control Station: Design and Human Factors Issues associated with the facilities, functions, equipment, and staff necessary to control and maintain the UA.

UAS: Design and Human Factors Issues associated with the NAS interconnectivity for the UA operations.

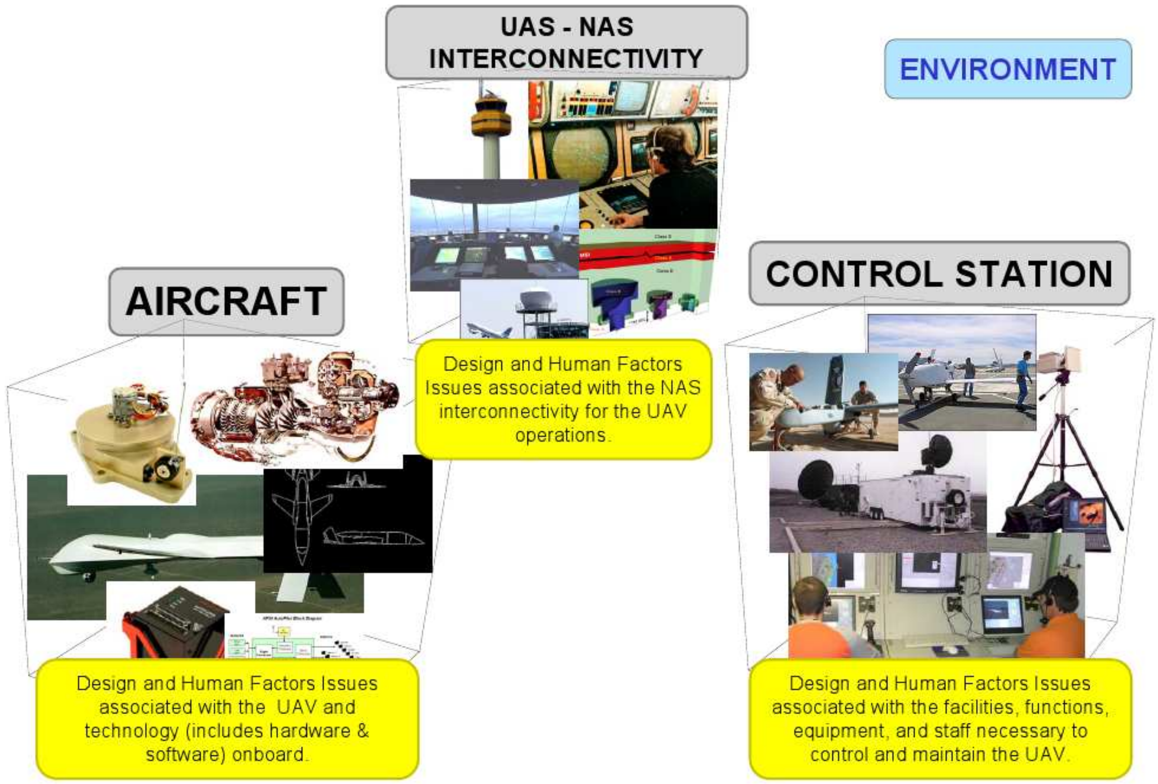


Figure 4.4: Three system-level hazard sources for UAS operations.

For each of the system-level hazard sources, subsystem elements are also identified. For example, for the system hazard source of "Aircraft", the subsystem hazard sources of *aerodynamics, airframe, payload, propulsion, avionics hardware and software, sensors/antennas, control and communication link, onboard emergency recovery, verification and validation,* and *human factors* are included. The resulting HCAS version 1.0 is presented in Fig 4.5. Note also that the interactions among the three system hazard sources are depicted in Fig 4.5. The notion of a "hazard source" is consistent with "hazardous element" of Ericsons "hazard triangle" [66] and also recognizes that a hazard needs a trigger or initiator to move it from a dormant to an active state, thus focusing on the hazards potential to do harm.

HCAS version 1.0 presented in [67, 60] was verified from an analysis of 208 hypothesized UAS scenarios discussed above as well as some real UAS mishaps. Once possible hazards for a given scenario set are categorized, an implicit prioritization of the hazards may be obtained by recomputing frequency counts as percentages. Such an approach provides a possible structured means to systematically weigh the hazards [60].

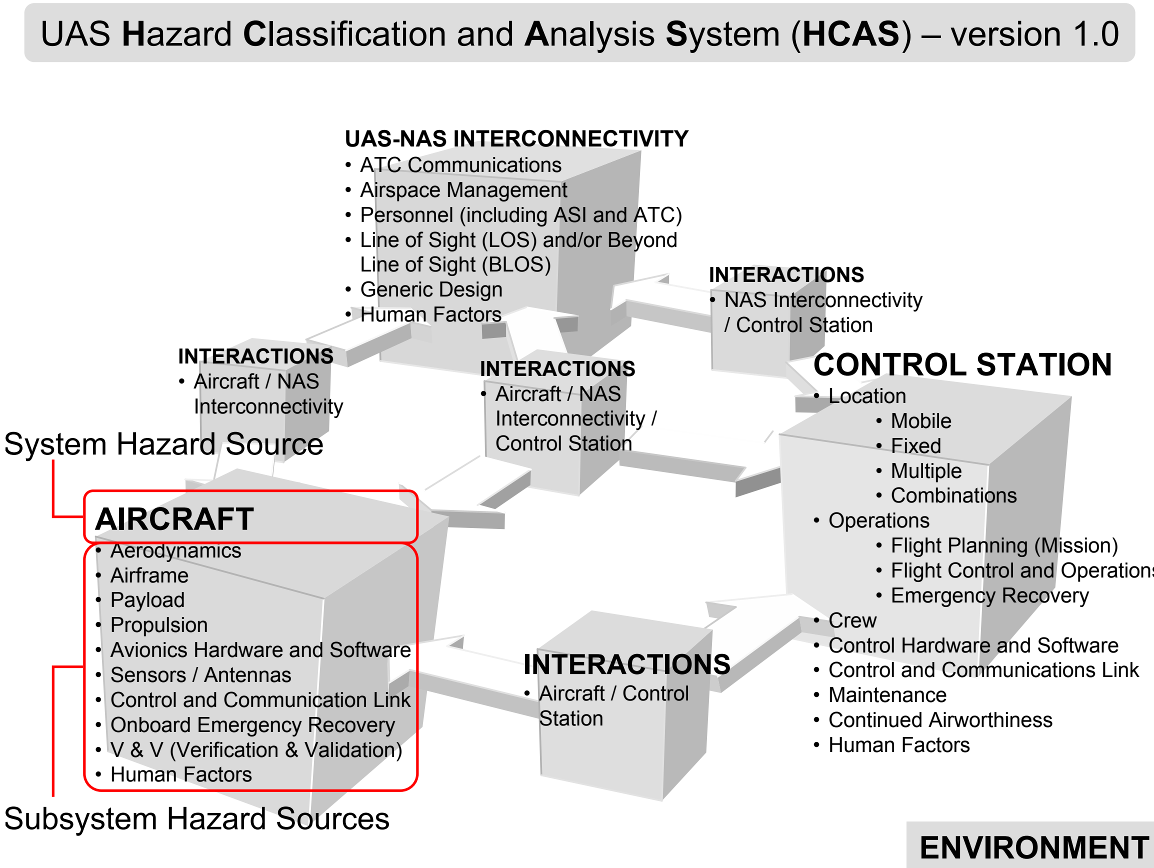


Figure 4.5: System and subsystem level hazard sources in HCAS version 1.0

HCAS version 2.0

In subsequent discussions with the FAA's UAS Program Office, there was a recommendation that the proposed Rutgers UAS hazard framework needs to consider, to some extent, alignment with the FAAs regulatory perspective dealing with Title 14, Code of Federal Regulations (14 CFR) chapters on Aircraft, Airmen, Certification / Airworthiness, Flight Operations, etc.. To incorporate the input of subject matter experts, an attempt to move HCAS version 1.0 closer to alignment with the current FAA aviation regulations was developed and is presented in Fig 4.6. This alignment is also consistent, to some extent, with the FAA UAS operations guidance document [54].

Note that in HCAS version 2.0, the system source of Control Station is now subsumed by a system hazard source termed as UAS. Also, the system source of Aircraft from HCAS

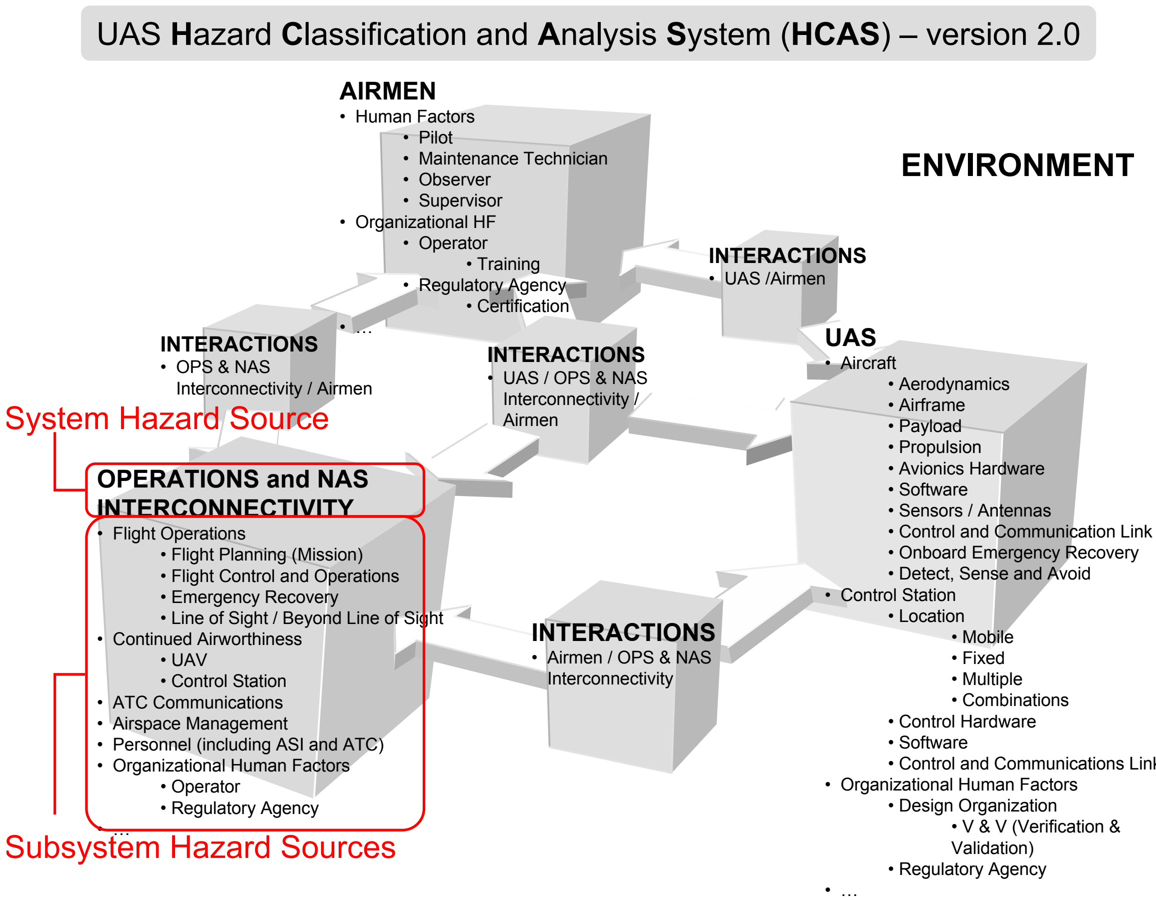


Figure 4.6: System and subsystem level hazard sources in HCAS version 2.0

version 1.0 is subsumed under UAS in version 2.0. Airmen is extracted out from version 1.0 into a separate system hazard source in version 2.0. A new system hazard source termed Operations and NAS Interconnectivity is created in version 2.0. The Environment still serves as a backdrop in HCAS version 2.0. Elements of human factors now appear in the three main HCAS system hazard sources. An excellent overview of UAS human factors issues is presented in [69]. The HCAS version 2.0 cube model is more aligned, to some extent, with the FAAs 14 CFR chapters. [59] provides excellent reflections on the potential hazards of the integration of UAS into the NAS and the implications for regulations. [70, 71, 72, 73, 74, 75], and [76] provide additional insights on safety issues associated with UAS operations.

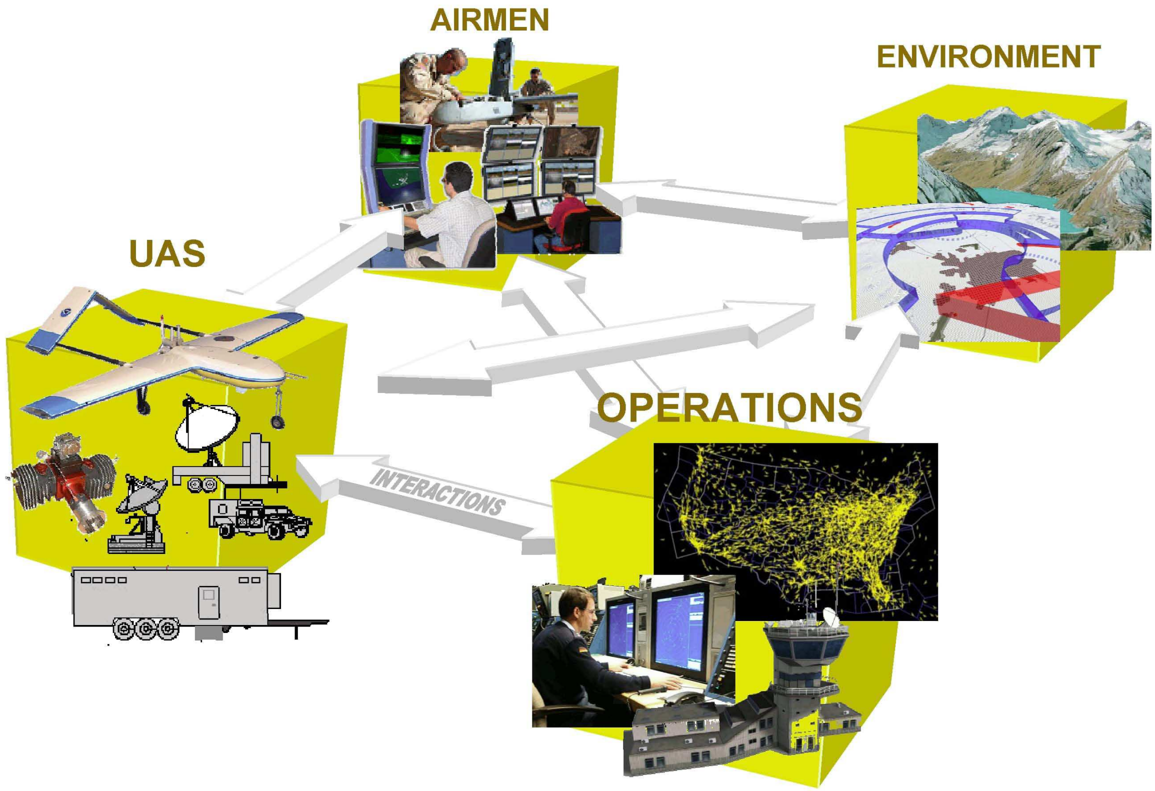


Figure 4.7: Four UAS Systems as the foundation of the current HCAS taxonomy

HCAS version 3.5

Subsequent reviews and critiques of the HCAS version 2.0 by industry subject matter experts indicated a need to incorporate further changes to have the version 2.0 become more aligned with FAA regulations and UAS guidance material.

Fundamentally different than the earlier versions that are built on three main system-level UAS components (see Figure 4.4), the most current version of the HCAS taxonomy is founded on four UAS system. This new foundation is illustrated in Figure 4.7. In particular, HCAS version 3.5 now identifies four primary hazard sources, namely *UAS, Airmen, Operations,* and *Environment.* Some of the significant changes include embedding the Control Station system source under the original Aircraft system source renamed as UAS. A fourth cube termed as Environment was added to the revised version and numerous sub-system hazard sources added. A detailed review of the HCAS taxonomy by industry subject matter experts improved the taxonomy by moving it to be more closely aligned with the existing FAA 14 CFR chapters. The expertise of the subject matter experts proved quite

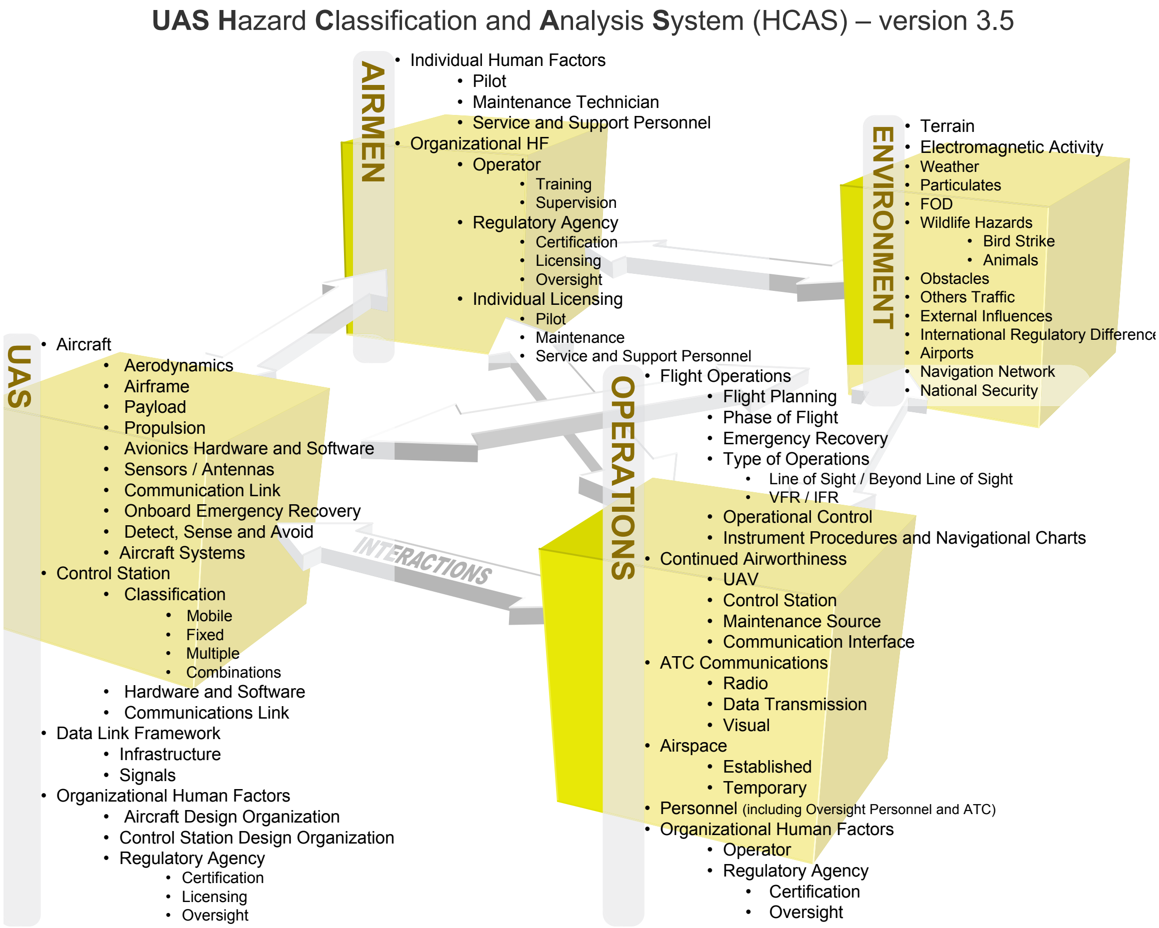


Figure 4.8: System and subsystem level UAS hazard sources - HCAS version 3.5

valuable in providing additional detail for the Environment hazard system source. Summary papers on HCAS versions 1, 2 and 3 are presented in [60] and in [68]. HCAS version 3.5 depicted in Figure 4.8 and a numbered outline form of the HCAS version 3.5 is also provided in Figure 4.9.

There is one particular aspect of our approach to taxonomy development that needs to be underlined. Our approach, while originally scenario-based, evolved into a more regulatory-based perspective during the course of the development. In a sense, this focus shift was natural considering the fact that, right from the start, our goal was to develop a generalized taxonomy for system-level UAS hazards that would have applicability across a broad spectrum of FAA regulations. This aspect of our approach uniquely distinguishes the HCAS taxonomy from all other UAS hazard analyses being performed by the Department of Defense (DoD), the RTCA-Special Committee (SC) 203, etc.

1. UAS (Systems Level)

1.1. Aircraft (Subsystems Level)
- 1.1.1. Aerodynamics
- 1.1.2. Airframe
- 1.1.3. Payload
- 1.1.4. Propulsion
- 1.1.5. Avionics Hardware and Software
- 1.1.6. Sensors / Antennas
- 1.1.7. Communication Link
- 1.1.8. Onboard Emergency Recovery
- 1.1.9. Detect, Sense and Avoid
- 1.1.10. Aircraft Systems

1.2. Control Station
- 1.2.1. Classification
  - 1.2.1.1. Mobile
  - 1.2.1.2. Fixed
  - 1.2.1.3. Multiple
  - 1.2.1.4. Combinations
- 1.2.2. Hardware and Software
- 1.2.3. Communications Link

1.3. Data Link Framework
- 1.3.1. Infrastructure
- 1.3.2. Signals

1.4. Organizational Human Factors
- 1.4.1. Aircraft Design Organization
- 1.4.2. Control Station Design Organization
- 1.4.3. Regulatory Agency
  - 1.4.3.1. Certification (includes authorizations and special approvals, operations specifications, designations, etc.)
  - 1.4.3.2. Licensing
  - 1.4.3.3. Oversight

2. AIRMEN

2.1. Individual Human Factors (HF)
- 2.1.1. Pilot
- 2.1.2. Maintenance Technician
- 2.1.3. Service and Support Personnel

2.2. Organizational HF
- 2.2.1. Operator
  - 2.2.1.1. Training
  - 2.2.1.2. Supervision
- 2.2.2. Regulatory Agency
  - 2.2.2.1. Certification
  - 2.2.2.2. Licensing
  - 2.2.2.3. Oversight
- 2.2.3. Individual Licensing
  - 2.2.3.1. Pilot
  - 2.2.3.2. Maintenance
  - 2.2.3.3. Service and Support Personnel

3. OPERATIONS

3.1. Flight Operations
- 3.1.1. Flight Planning
- 3.1.2. Phases of Flight (include pre and post flight operations by the ground support personnel, see and avoid, right of way-conflict resolution)
- 3.1.3. Emergency Procedures
- 3.1.4. Type of Operations
  - 3.1.4.1. Line of Sight / Beyond Line of Sight
  - 3.1.4.2. VFR / IFR
- 3.1.5. Operational Control
- 3.1.6. Instrument Procedures and Navigation Charts

3.2. Continued Airworthiness
- 3.2.1. UAV
- 3.2.2. Control Station
- 3.2.3. Maintenance Source (Facility and Individual)
- 3.2.4. Communication Interface (Data Link)

3.3. ATC Communications
- 3.3.1. Radio
- 3.3.2. Data Transmission
- 3.3.3. Visual

3.4. Airspace
- 3.4.1. Established
- 3.4.2. Temporary

3.5. Personnel (including Oversight Personnel and ATC)

3.6. Organizational Human Factors
- 3.6.1. Operator
- 3.6.2. Regulatory Agency
  - 3.6.2.1. Certification
  - 3.6.2.2. Licensing
  - 3.6.2.3. Oversight

4. ENVIRONMENT

4.1. Terrain

4.2. Electromagnetic Activity

4.3. Weather

4.4. Particulates (including Volcanic Ash)

4.5. FOD

4.6. Wild Life Hazards
- 4.6.1. Bird Strike
- 4.6.2. Animals

4.7. Obstacles

4.8. Other Traffic

4.9. External Influences (Social, Political)

4.10. International Regulatory Differences

4.11. Airports

4.12. Navigation Network (ground and space based infrastructure and signals)

4.13. National Security

5. INTERACTIONS

5.1. UAS / Environment

5.2. Operations / Environment (includes NAS interconnectivity)

5.3. Airmen / Environment

5.4. UAS / Operations / Environment

5.5. UAS / Airmen / Environment

5.6. Operations / Airmen / Environment

5.7. UAS / Operations / Airmen / Environment

Figure 4.9: HCAS version 3.5 - Outline

The HCAS taxonomy may also be used to construct influence/causal factor diagrams representing hypothetical or notional UAS safety risk scenarios. The use of modifiers placed on the HCAS taxonomy elements, such as “inappropriate”, “inadequate”, etc. may be used to create such an influence diagram. These influence diagrams may then be used to study the interactions among various causal factors associated with the hazards. Conceptually, HCAS represents a hierarchical structure for UAS hazard sources. In particular, at the very top, there are system-level hazard sources, which, in lower levels, are decomposed into their subsystem-level hazard sources. Since civil UAS operations are relatively new and emergent, databases of mishaps are not readily available. This idea is further explored in the next section to develop a methodology for modeling safety risk of the UAS domain as a hybrid Bayesian Network.

4.3 A Regulatory-Based Safety Risk Modeling Approach

The HCAS taxonomy provides a systematic approach to identify hazards associated with UAS operations in the NAS. However, hazards are not causal factors, which are the essential building blocks of influence diagrams (i.e., Bayesian Nets) representing various risk scenarios of the UAS operations. Thus, the decomposition of hazards into their constituent causal factors is another important step in the development of a comprehensive scheme for UAS safety risk modeling. Underlying causes of the hazards, such as failure modes, operator and software errors, design flaws, etc., need to be identified in order to eventually determine the mishap risk and the hazard mitigations. However, HCAS is not a taxonomy of causal factors. Although the resulting taxonomy for the UAS hazard sources is intended to be generic and inclusive, it represents an inductive reasoning approach with particular emphasis on a given set of UAS hazard scenarios. Hence, to determine a taxonomy of UAS causal factors, which are, strictly speaking, hierarchically at a lower level than hazard sources, we chose to employ deductive reasoning and based our analysis on the current FAA regulations for commercial civil aviation. Knowledge elicitation sessions with subject matter experts are heavily utilized throughout this process. Subsequently, individual causal factors are mapped to the taxonomy of UAS hazard sources resulting in a seamless analysis that is generic enough to cover most possible UAS operational scenarios yet provides the necessary level of fidelity to map their prominent features into a database.

At the crux of our regulatory-based approach lie the following assumptions:

- UAS integration will impact the entire NAS because of the wide-ranges of UAS size, weight, performance characteristics, airspace access, and unique operation issues.
- There are no sufficient data and proven methods to perform UAS safety analysis with the traditional event-driven approach.
- The regulations provide the essential safety net for the NAS safety.

- There exist a set of causal factors, which can be identified, associated with each relevant regulatory section.
- With proper descriptions of causal factors, the interdependencies (linkages) among themselves can be demonstrated.
- These linkages form the basis to analyze UAS safety risk by applying probabilistic reasoning methodologies such as BNs through the Hazard Classification and Analysis System (HCAS) model.

Within the context of these assumption, we derive the individual causal factors from existing regulations governing all current aviation-related operations in the NAS. The role of the HCAS taxonomy introduced in the prior section is to provide structure to this regulatory based casual factor identification process.

The current structure of the Federal Aviation Regulations (FARs) in the US represents a hierarchy. The FARs as part of Title 14 of the Code of Federal Regulations (CFR) are organized into *Subchapters*. Each subchapter is then organized into *Parts*. Each part deals with a specific type of aviation activity. For example, *14 CFR Part 121* contains rules and requirements for Domestic, Flag, and Supplemental Operations of US registered aircraft. Individual FAR Parts are further divided sequentially into *Subparts*, *Section*, and *Subsections*. The derivation process for the causal factors closely mimics this hierarchical structure. In particular, causal factors are extracted from within the context of a FAR Part keeping possible applicability for UAS operations into consideration. Consequently, each causal factor is categorized under a subsystem level hazard source defined by the HCAS taxonomy, thereby establishing a viable connection between regulations and hazard sources. Figure 4.10 is a notional diagram depicting the connection between regulations and hazard sources (i.e., HCAS element) through causal factors.

This regulatory-based process has two main objectives; first, for each FAR part, to identify, describe, and define the causal factors; second, to determine interactions and connections among these causal factors. These connections constitute the foundation upon which the Bayesian Networks representing causal dependencies within the context of a UAS risk or hazard source will be constructed.

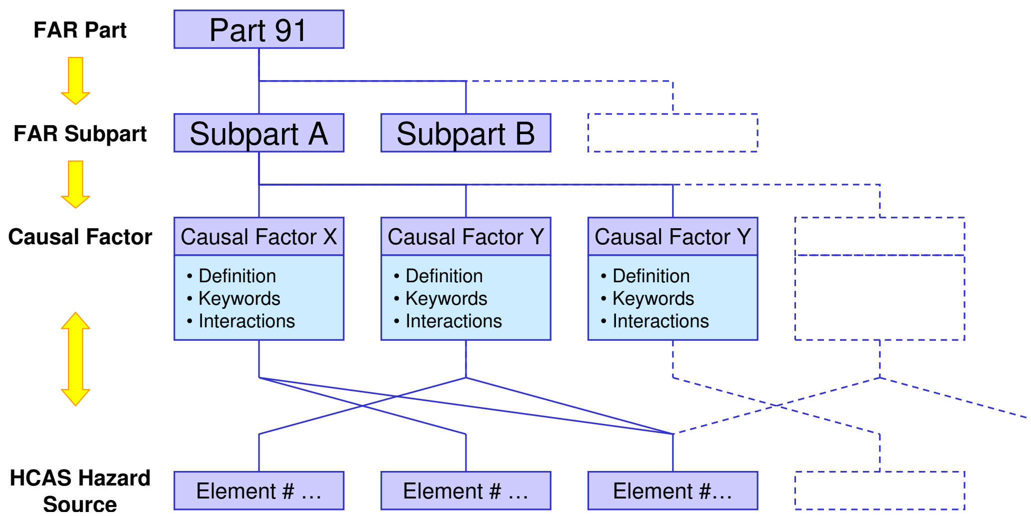


Figure 4.10: Causal Factors are the Link between FARs and HCAS Taxonomy

Since individual causal factors are identified and defined solely based on FARs, their derivation, as a creative process relies heavily on knowledge elicitation sessions with subject matter experts and will need some vetting within the aviation community.

4.3.1 Regulatory-based Causal Factor Framework (RCFF)

Based on the ideas outlined in the preceding section, we introduce a Regulatory-based Causal Factor Framework (RCFF) to study the potential safety impacts of introducing emerging UAS operations into the well-established National Airspace System (NAS). Formally, RCFF is a systematic process for the creation of causal factors that are derived from the regulations to functions to hazards to causal factors [67, 90]. It provides a qualitative means of identifying and assessing hazards controlled by existing regulations. The RCFF is a novel system safety process for analyzing hazards and associated causal factors due to introducing new technology into NAS. Introducing these new technologies to the NAS not only has the potential of impacting the entire system (NAS), but also leads to greater uncertainties of their safety impacts due to the very limited knowledge with no actual operational data in the NAS. Safety risk analyses, essentially events-driven and largely built upon past experience, and vast amount of actual operational data, may not provide adequate technical information for risk controls. The proposed RCFF approach is attempting to overcome some of these uncertainties by utilizing the existing regulations, which provide the minimum safety standards, as a measure to assess whether all potential risk areas are addressed while using the event-driven approach.

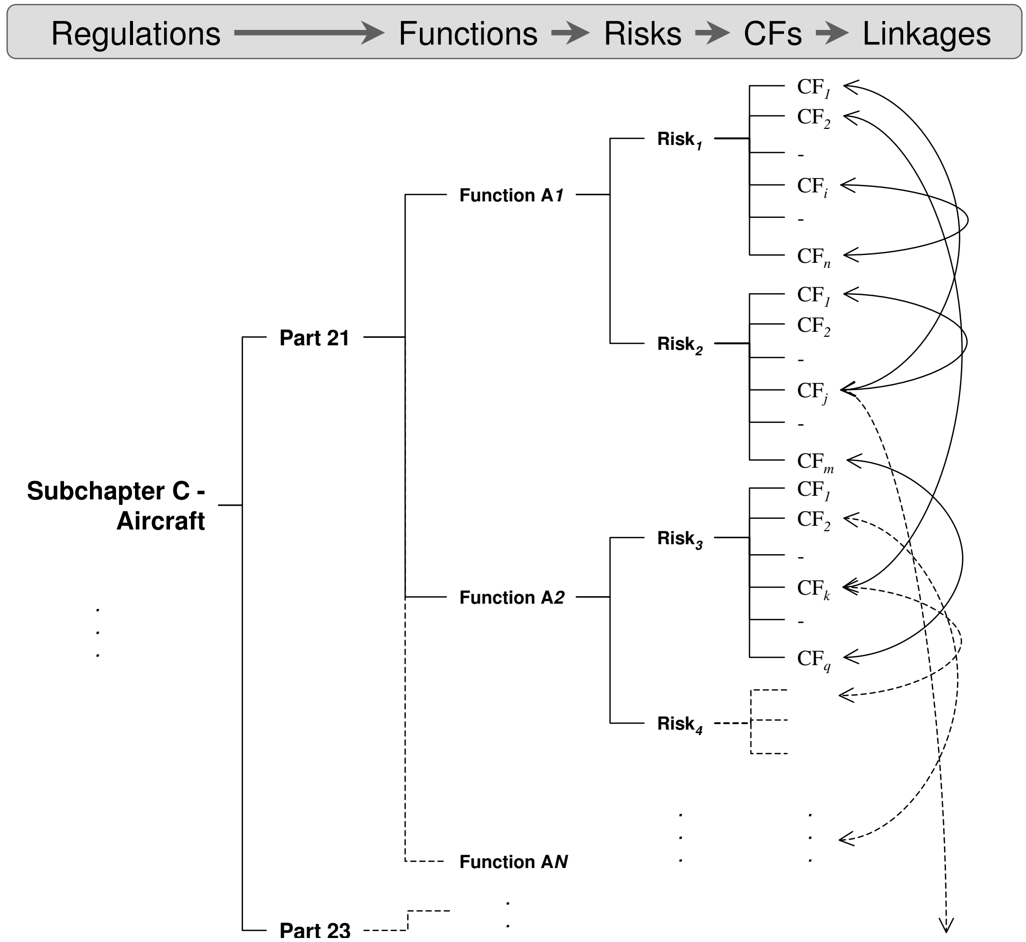


Figure 4.11: A sample conceptual RCFF hierarchy for 14 CFR Subchapter C-Aircraft (CFs: Causal Factors).

Conceptually, our proposed approach to identify causal factors based on existing regulatory structure represents a hierarchical framework. At the very top, covering the whole NAS, Federal Aviation Regulations (FARs) provide the minimum requirement for safe operations. Within the context of FAR Subchapters, functional models provide fidelity to conceptualize the risk associated with the proposed UAS operations. Consequently, groups of causal factors are identified to outline the underpinnings of each UAS related risk. However, unlike conventional hierarchical methodologies such as Fault Trees, the proposed framework, illustrated in Figure 4.11, also emphasizes the interactions and connectivity among various components and compartments comprising the whole domain.

4.4 UAS Domain Safety Risk Model (DSRM)

In particular, the process of building a UAS domain safety risk model using the RCFF methodology starts with one of the 14 CFR Subchapters and related FAR parts (i.e., regulations). For each FAR part, generic functions of operational activities are identified. These functions provide context to identify safety risk related to UAS operations and create a domain where the subsequent causal modeling effort becomes conceptually relevant to current UAS safety concerns. The idea of operational functions are successfully employed by [77] and [78] to develop system engineering models for 14 CFR Part 121 air carrier operational and Part 137 oversight activities, respectively.

In particular, the Air Carrier Operations System Model (ACOSM) of [77] concentrates on the following key air carrier operations processes:

- Operational management.
- Air transportation.
- Aircraft maintenance.
- Personnel training.
- Operational resource provisions.

For example, the ACOSM identifies "perform air carrier operations" as the main context activity (function), which describes the system itself. Here, "perform air carrier operations" is understood as a set of activities directly related to the movement of aircraft with passengers and/or cargo from the departing airport to the destination airport, conducted by air carriers operated under 14 CFR Part 121. The context activity, "perform air carrier operations", is then decomposed into the following five subactivities:

- Manage air carrier operations.
- Perform air transportation.
- Perform aircraft maintenance.
- Perform personnel training.

- Provide air carrier operation resources.

Note that the subactivities follow a simple pattern that utilizes three fundamental functions, namely, *manage, perform* and *provide resources*. This triad is used in the following sections to provide context for our UAS domain safety risk model.

Determining operational functions is a relatively simple process compared to identifying individual causal factors and determining their interactions. The latter, which is essentially a creative process, requires a fairly good understanding of regulations, hence a detailed study of FARs. Thus, heavy involvement of subject matter experts is necessary to identify causal factor and their connections.

Continuing with the development of the methodology, at this stage of the model development, we identify a set of causal factors, for each functional domain and determine possible connections among these causal factors. These connections are *undirected*, i.e., the causal factor pairs are not *ordered* since the interactions do not imply any causality at this point. Nevertheless, the resulting structure exhibits the fundamental attributes of a *graph*, where causal factors are *nodes* or *vertices* and the connections/interactions are *edges*. Figure 4.12 illustrates this concept.

In order to construct a *Directed Acyclic Graph (DAG)*, we need to determine a *direction of causality* for each connected causal factor pair. Consequently, the resulting *directed graph* needs to be revised to identify and eliminate the connections causing *cycling*. While eliminating these connections, as a rule of thumb, the disruption of the main causal path originating from the most latent organizational factors and leading to the more pronounced individual factors should be avoided. Note also that there may be more than one such causal paths depending on the context of the functional domain. Again, throughout this process subject matter experts are extensively utilized.

The next step in building the UAS domain risk model is to identify the risks associated with each operational function governed by the particular FAR part that we focus on at that time and determine the nodes (i.e, causal factors) in the DAG with most immediate and strongest casual impact on these risks. In this sense, the risk nodes serve as the *terminating node* or *sink* of the DAG.

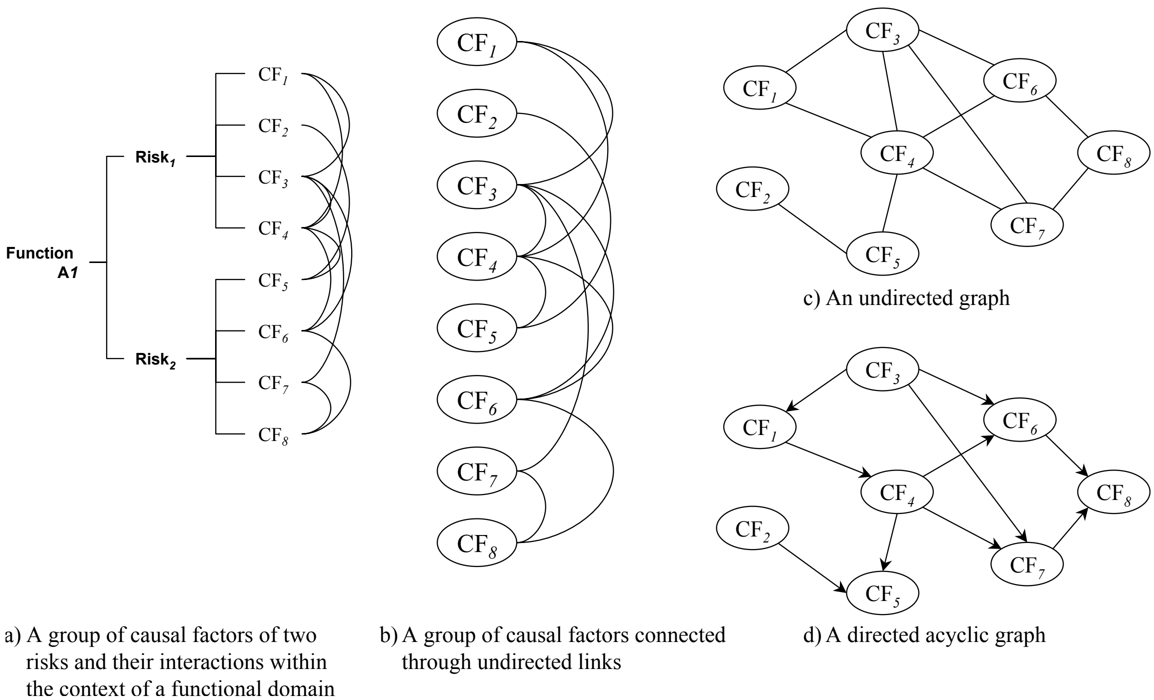


Figure 4.12: Transitional steps from a functional domain of the RCFF to a directed acyclic graph.

We restricted our preceding discussions mainly to hazards and hazard sources rather than risks. Furthermore, we introduced the HCAS taxonomy in section 4.2.1, which represents a systematic approach to identification and categorization of *UAS hazard sources*, but not *UAS risks*. However, conceptually, the distinction between hazard and risk is smaller than one might think. Broadly construed, risk is a concept that denotes the precise probability of specific eventualities. Qualitatively, risk is proportional to both the expected losses which may be caused by an event and to the probability of this event, i.e.;

$$Risk = (\text{Probability of event occurring}) \times (\text{Impact of event occurring}) \tag{4.1}$$

A hazard, on the other hand, is a situation which poses a level of threat to life, health, property or environment. Most hazards are dormant or potential, with only a theoretical risk of harm. Thus, we can reformulate equation (4.1) as equation (4.2) below to achieve a representation of risk as a function of hazard, i.e.;

$$Risk = \begin{pmatrix} \text{Likelihood of the hazard} \\ \text{turning into an incident} \end{pmatrix} \times \begin{pmatrix} \text{Severity of the incident} \\ \text{if it were to occur} \end{pmatrix} \tag{4.2}$$

Equation (4.2) indicates a relationship between risk and hazard, where the potential hazard is transformed into a kinetic situation (i.e., risk) via a likelihood and a severity. Therefore, in situation where severity is assumed to be constant, we can safely argue that risk is proportional to the probability of the hazard potential. This reasoning is further extended by dropping the probability and stating that "risk is proportional to hazard". For the purposes of the UAS domain safety risk modeling methodology we replace risk with hazard and identify hazards using the HCAS taxonomy.

A conceptual illustration of the UAS domain safety risk model based on the RCFF methodology outlined above is provided in Fig 4.13.

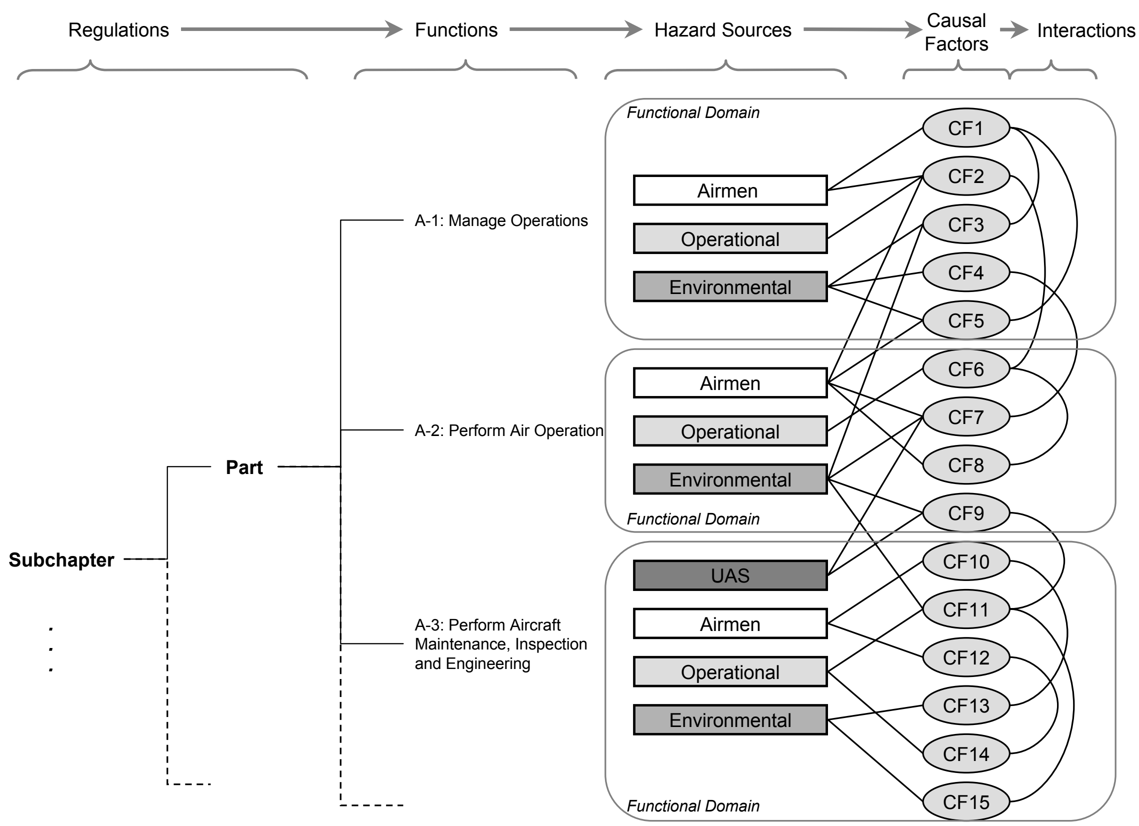


Figure 4.13: A conceptual illustration of the UAS domain safety risk model

4.4.1 14 CFR Part 91 UAS Domain Safety Risk Model

In this section we apply the the regulatory-based causal factor framework (RCFF) methodology to 14 CFR Part 91 "General Operating and Flight Rules" to develop a UAS Domain Safety Risk Model (DSRM) introduced in the preceding sections.

14 CFR Part 91 of Subchapter F-*Air Traffic and General Operating Rules* is organized into 12 Subpart, listed in Table 4.2. Broadly construed, Part 91 is a set of regulations that define the operation of small non-commercial aircraft within the US, however, many other countries defer to these rules. These rules set conditions, such as weather, under which the aircraft may operate, flight operations, equipment, maintenance and alterations, among others.

Table 4.2: List of 14 CFR Part 91 Subparts

Subpart #	**Subpart Title**
A	General
B	Flight Rules
C	Equipment, Instrument, and Certificate Requirements
D	Special Flight Operations
E	Maintenance, Preventive Maintenance, and Alterations
F	Large and Turbine-Powered Multiengine Airplanes and Fractional Ownership Program Aircraft
G	Additional Equipment and Operating Requirements for Large and Transport Category Aircraft
H	Foreign Aircraft Operations and Operations of U.S.-Registered Civil Aircraft Outside of the US; and Rules Governing Persons On Board Such Aircraft
I	Operating Noise Limits
J	Waivers
K	Fractional Ownership Operations
L	Continued Airworthiness and Safety Improvements

Although Part 91 covers a large spectrum, subparts B and C constitute its core by outlining fundamental requirements for flight operations, equipment and certification. Thus, as suggested by subject matter experts, our application focuses on these two subparts and individual causal factors are identified accordingly.

The modeling process start with the identification of individual causal factors of the hazards that Part 91 controls. As an essentially creative process, subject matter experts, during focused sessions under our moderation, derive individual causal factors based on their understanding of the regulatory text and on their expertise on the problem domain. The causal factors for subparts B and C as the result of such knowledge elicitation sessions are given in Tables 4.3 and 4.4, respectively.

Note that the set of causal factors identified in Tables 4.3 and 4.4 demonstrates a high level of resolution in terms of the detail that can be achieved using FAR regulations as the sole data source according to the RCFF methodology. For the purposes of the application presented here, we only concentrated to a functional domain representing Part 91 operations, however for a larger functional domain spanning on multiple FAR Parts, this level of detail will result in oversized hybrid Bayesian Networks that are unpractical to populate and propagate. Therefore, it is suggested that when larger domains are concerned the level of

Table 4.3: Causal Factors associated with Part 91 Subpart B - Flight Rules

	Causal Factor	HCAS Element#
1	Inadequate Preflight Planning	3.1.1, 3.1.2, 3.1.5
2	Inadequate Preflight Information	3.1.1, 3.1.2, 3.1.5
3	Crewmember Not at Station	2.1.1, 3.1.2, 3.3
4	Occupants Not Secured	2.1.1
5	Occupants Not Informed of Use of Restraining Systems	2.1.1
6	Proximity to Other Aircraft	1.1.9, 2.1.1, 4.8
7	Right of Way Rules Not Followed	2.1.1, 1.1.9, 4.8
8	Failure to See and Avoid	2.1.1, 4.8
9	Failure to Comply with Airspace Speed Limits	2.1.1, 3.3, 3.4
10	Failure to Comply with Minimum Safe Altitudes in Congested and Non-Congested Areas	4.1, 4.7
11	Inaccurate Altimeter Setting	2.1.1, 1.1.10, 1.1.6
12	Failure to Comply with ATC Clearances and Instructions	2.1.1, 3.3
13	Failure to Comply with ATC Light Signals	2.1.1, 3.3.3
14	Failure to Follow Requirements in Designated Airspace	2.1.1
15	Not Following Flight Restrictions	2.1.1
16	Not Complying with Fuel Requirements	3.1.1, 3.1.2, 3.1.4.2
17	Incomplete VFR Flight Plan Information	2.1.1, 3.1.2
18	Not Complying with VFR or Special VFR Weather Minimums	2.1.1, 3.1.1, 3.1.2, 3.1.4.2, 4.3
21	Not Complying With VFR/IFR Cruising Altitude Requirements	2.1.1, 3.1.1, 3.1.2, 3.1.4.2
22	Not Complying with Minimum IFR Altitude Requirements	2.1.1, 3.1.6, 3.1.4, 3.1.1
23	Flying with VOR Equipment that Does Not Meet the Check Requirements for IFR Operations	2.1.1
24	Operating IFR Without an ATC Clearance and Flight Plan in Controlled Airspace	2.1.1, 3.1.4.
25	Failure to Use Published Instrument Approach Procedures	2.1.1, 3.1.1, 3.1.4.2
26	Pilot Conducts Cat II or Cat III Operations Without Complying With Proper Training, Authorization or Procedure Requirements (manual)	2.1.1, 3.1.4.2, 2.2.2.2
27	Failure to Follow Procedure for Transitioning From the Instrument to the Visual Portion of an Instrument Approach Using Normal Maneuvers	2.1.1, 3.1.1, 3.1.4.2
28	Not Complying with IFR Takeoff Procedures or Minimums	2.1.1, 3.1.4.2, 4.3, 3.1.6
29	Flying in RVSM Airspace Without Complying to RVSM Requirements	2.1.1, 1.1.5, 1.1.10, 1.4.3.2, 2.2.1.1, 2.2.2.2, 3.6.2.2
30	Not Complying with IFR Course Requirements	2.1.1
31	Failure to Maintain Communications with ATC While Flying Under IFR	2.1.1
32	Failure to Comply with Loss of Communication Procedures	2.1.1, 3.1.3, 1.1.5
33	Failure of an Aircraft Flying IFR to Notify ATC of Malfunction of Certain Required Equipment	2.1.1, 3.1.4.2, 1.1.5, 1.1.6, 1.1.10, 3.2.1

detail employed for the causal factors identification process should be kept under control to achieve safety risk models that are practically and computationally viable given the resource available.

Table 4.4: Causal Factors associated with Part 91 Subpart C - Equipment, Instrument, and Certificate Requirements

	Causal Factor	HCAS Element#
1	Fuel tank installed not in accordance per FAA regulatory requirements	1.1.2, 1.1.10
2	Lack of required FAA aircraft certifications	3.2.1, 3.6.2.1
3	Inoperative or missing equipment required for the type of operation	1.1, 3.1.4, 3.2
4	Inoperative or missing emergency locator transmitter(ELT)	1.1.5, 1.1.6
5	Position or anti-collision lights inoperative or not turned on	1.1.10, 1.1.9, 4.8
6	Passengers not provided with sufficient supplemental oxygen	1.1.10, 3.1.1, 3.1.2, 3.1.3
7	Lack of accurate altitude information	1.1.5, 1.1.6, 3.2.1, 3.3.2
8	Lack of or inoperative altitude alert system	1.1.10, 1.2.2
9	Lack of approved Traffic alert and Collision Avoidance System (TCAS)	1.1.10, 1.1.9, 3.2.1, 4.8
10	Inoperative or missing Terrain Awareness and Warning System (TAWS)	1.1.10, 1.1.9, 3.2.1, 4.1, 4.7

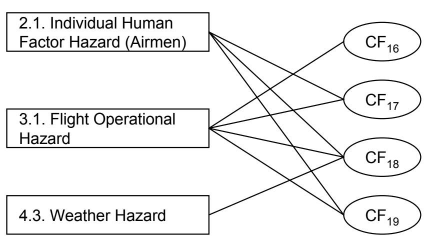
The *HCAS element#* columns in Tables 4.3 and 4.4 indicate the set of specific HCAS taxonomy items related to each causal factor.

The functional activities controlled by Subparts B and C is given in Table 4.5. These functions provide the contextual domain where the UAS domain safety risk model is developed.

Table 4.5: Functional Activities of Part 91 Subpart B and C

Regulation	**Functional Activity**
Subpart B - Flight Rules	A1.1 Perform Flight Operations A1.2 Perform Visual Flight Operations A1.3 Perform Instrument Flight Operations
Subpart C - Equipment, Instrument, and Certificate Requirements	A2.1 Manage Equipment, Instrument, and Certificate Requirements

Consequently, for each functional domain identified in Table 4.5 we determine a set of hazard sources that constitute the sink nodes in the Bayesian Network we are about to construct. HCAS taxonomy provides structure during this step of the process and the information in the *HCAS element #* columns of Tables 4.3 and 4.4 determine the set of hazard sources pertinent to each functional activity. Then, we connect each causal factor to the set of hazard sources identified in the HCAS element column with an undirected edge. For example, in Table 4.3, HCAS elements 2.1.1 and 3.1.2 are identified as the hazard sources for causal factor # 17 "Incomplete VFR Flight Plan Information". While identifying hazards we choose to stay at the subsystem level according to the HCAS taxonomy and identify elements "2.1. Individual Human Factor Hazard (Airmen)" and "3.1. Flight Operational Hazard" as the hazards associated with causal factor # 17 "Incomplete VFR Flight Plan Information". This step is illustrated in Figure 4.14 for functional activity "Perform Visual Flight Operations" which is comprised of four causal factors: CF# 16, CF#17, CF#18, and CF#19.



Causal Factor	Definition	HCAS #
Not Complying with Fuel Requirements	FAR define minimum fuel requirements for various flight operations. Failure to comply fuel requirements VFR/IFR	3.1.1, 3.1.2, 3.1.4.2
Incomplete VFR Flight Plan Information	VFR flight plans are optional method for pilots to facilitate search and rescue efforts.	2.1.1, 3.1.2
Not Complying with VFR or Special VFR Weather Minimums	FARs establish what VFR means in different airspace categories for visibility and distance from cloud requirements.	2.1.1, 3.1.1, 3.1.2, 3.1.4.2, 4.3,
Not Complying With VFR/IFR Cruising Altitude Requirements.	FARs establish VFR/IFR cruising altitudes based on magnetic courses to separate traffic.	2.1.1, 3.1.1, 3.1.2, 3.1.4.2,

Figure 4.14: Causal Factors and Hazards for the Functional Activity "Perform Visual Flight Operations"

We repeat this process for each casual factor under each functional domain. Next, for all causal factors, we determine their interactions including those that connect the causal factors of different functional domains. Consequently, we arrive at an undirected graph structure where on one side of the structure hazard sources accept connections from causal factors and on the other side causal factors are connected among themselves.

This undirected graphical structure is called the initial domain safety risk model (DSRM). The initial DSRM based on the 14 CFR Part 91 Subparts B and C constructed according the the regulatory-based causal factor framework is provided in Fig 4.15 and Fig 4.16. Note that even though the DSRM for Part B and Part C are presented in separate figures, there are links or edges that connect both DSRMs, thereby creating a larger more complex domain model.

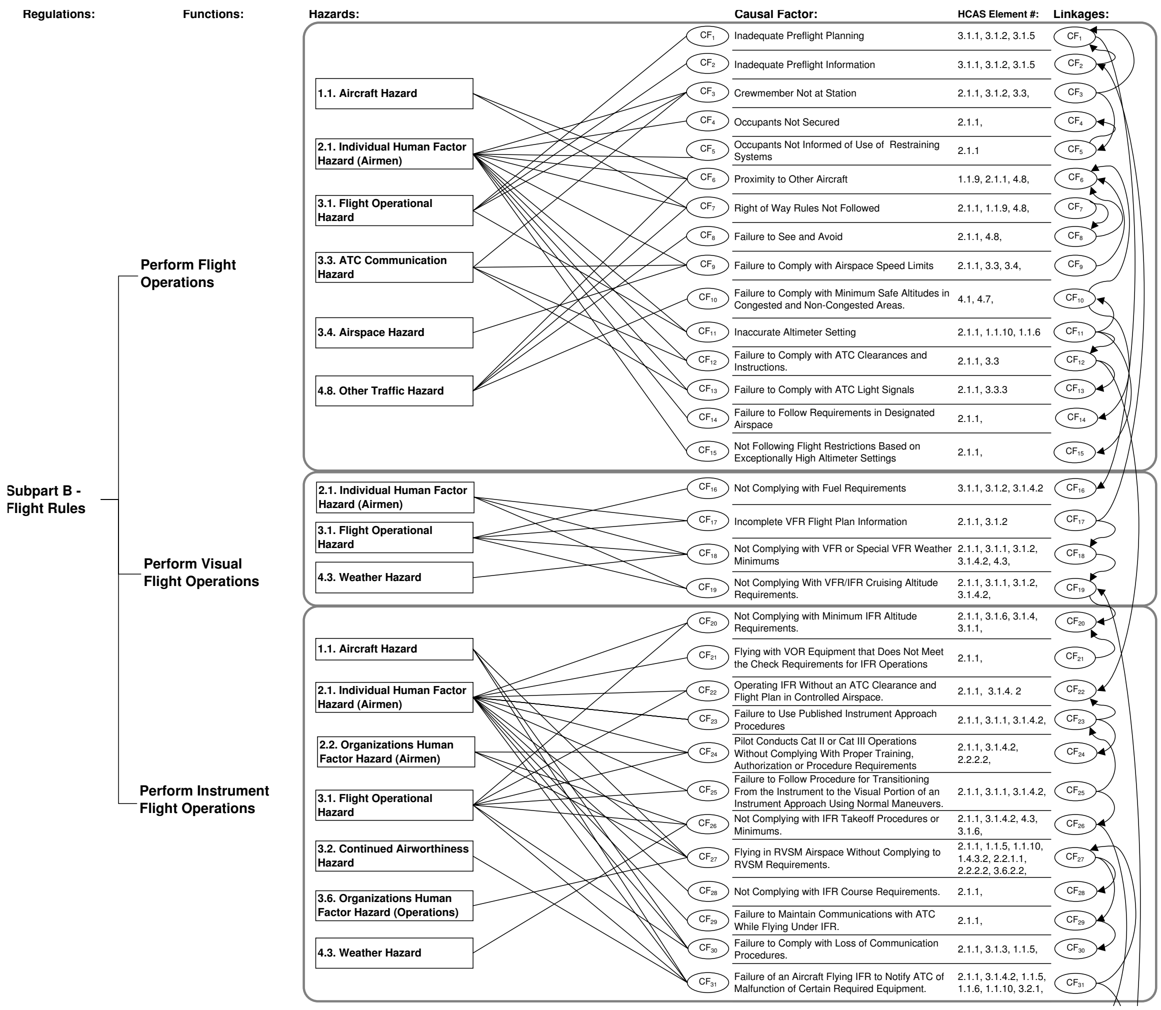


Figure 4.15: The initial DSRM (14 CFR Part 91 Subparts B)

In the initial domain safety risk model in Figures 4.15 and 4.16, there are four distinct functional domains, 11 hazards and 40 causal factors. The causal factors, also listed in Table 4.3 are derived based on the regulation sections that constitute the subparts with the help of subject matter experts. For example, Causal Factor #1 *Inadequate Preflight Planning* based on Section 91.103 “Preflight Action” is described by SMEs as follows;

> “Pilot fails to follow all necessary steps or makes errors (e.g. calculations, decisions, etc) during preflight planning based on the available preflight information.”

For the same causal factor, SMEs also identified possible associations to HCAS elements 3.1.1 “Flight Planning”, 3.1.2 “Phases of Flight”, and 3.1.5 “Operational Control”. Hence

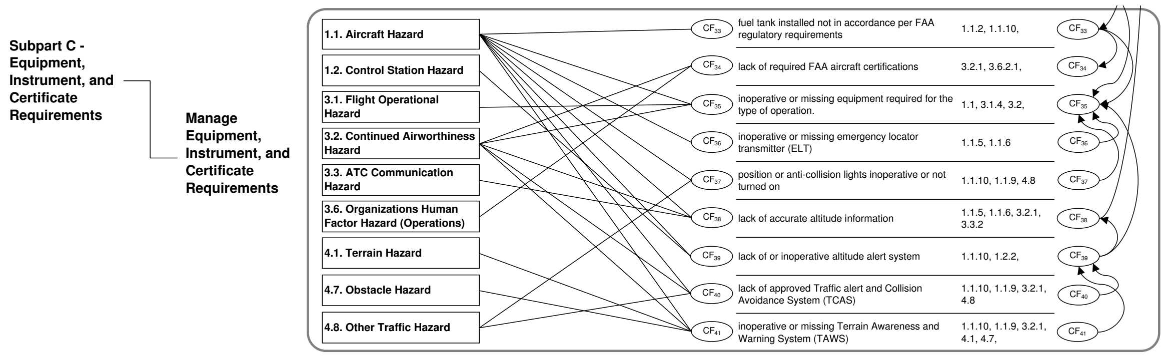


Figure 4.16: Part of the initial DSRM (14 CFR Part 91 Subparts C)

Causal Factor #1 is linked to 3.1 "Flight Operational Hazard" which represents a higher level grouping of these three HCAS elements.

Finally, taking the context of functional domains into consideration, SMEs identify a set of preliminary links between causal factors. At this stage of the process, these links do not imply any causality. The causality of the connections is established as part of the next step where this undirected graph is transfered to a general hybrid Bayesian Network representing the UAS domain safety risk model. This transformation is the subject of the next section.

The process of constructing an initial undirected DSRM can be summarized as follows;

- Identify functional activities, i.e., functional domains, based on the regulatory framework around which the model is to be constructed;
- Identify and describe Causal Factors using the regulatory text;
- Determine associated hazard sources, i.e., HCAS elements for each causal factor;
- Determine hazard groups based on the HCAS elements for each functional domain;
- Connect each causal factor to its hazard group by undirected edges;
- Determine possible prominent interactions between causal factors and depict these interactions by undirected edges.

Starting from the next section, we develop a Hybrid Bayesian Network representing the UAS DSRM and apply our Fuzzy-Bayesian methodology introduced in Chapter 3 to perform probabilistic inferencing about the resulting Fuzzy Bayesian Network (FBN).

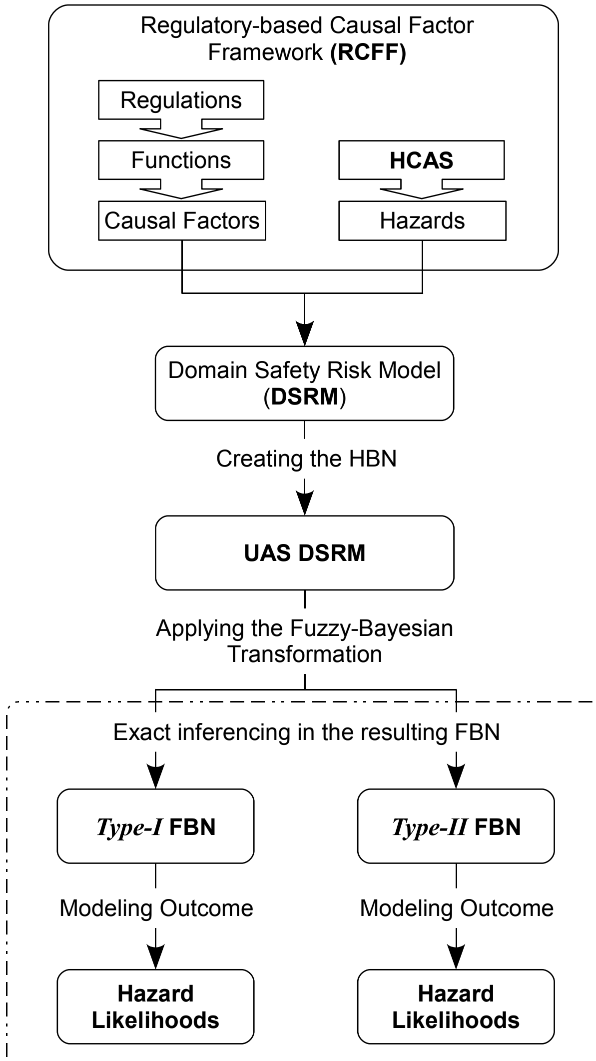


Figure 4.17: The flowchart representing the progression of concepts and ideas introduced so far within the context of this application.

Figure 4.17 provides a flowchart that outlines this rather lengthy process, which start with the identification of causal factors and hazards according to the RCCF and ends with the computation of hazard likelihood as the modeling results of the Type-I and Type-II FBNs using the algorithms developed in Sections 3.4.2 and 3.4.3.

4.4.2 The Hybrid Bayesian Network

In this section we focus on constructing and refining the network structure of the UAS domain safety risk model (DSRM) introduced in the preceding section.

The raw model presented in two parts in Figures 4.15 and 4.16 includes 50 nodes comprised of 37 causal factors and 13 hazard elements. Even, without taking the linkages between the pairs of nodes into consideration, this preliminary topography of the model constitutes a relatively large network as a BN. However, as discussed above, in order for us to apply our Fuzzy-Bayesian methodology outlined in Chapter 3 to the UAS DSRM, we need to process this preliminary model so that it ultimately becomes a proper Hybrid Bayesian Network about which probabilistic/possibilistic reasoning can be performed.

The process of construction and refinement of the raw model starts with the identifying the the causal interactions between the pairs of causal factors and the linkages between the causal factors and hazard elements. In fact, the actual topology of the UAS DSRM is constructed by these linkages provided that the linkages indicate a casual direction underlying causal dependencies as a Bayesian Network. As outlined in section 4.4.1 these links are identified by using the definitions of the causal factors as determined by the subject matter experts based on their interpretation of FAR Part 91. However, to bring structure to this process and facilitate the repeatability of the methodology, we employed a pattern matching approach for the identification of the links between two causal factors. In particular, along with a definition for each causal factor we also identified a set of keywords/phrases to complement the definition. These sets of keywords/phrases provide context to the definitions and emphasize prominent attributes of the causal factors.

	Causal Factor	Definition	Keywords	HCAS #
CF_{16}	Not Complying with Fuel Requirements	FAR define minimum fuel requirements for various flight operations. Failure to comply fuel requirements VFR/IFR	Fuel requirements, VFR, IFR, day, night, Fuel reserve	3.1.1, 3.1.2, 3.1.4.2
CF_{17}	Incomplete VFR Flight Plan Information	VFR flight plans are optional method for pilots to facilitate search and rescue efforts.	VFR flight plan,	2.1.1, 3.1.2
CF_{18}	Not Complying with VFR or Special VFR Weather Minimums	FARs establish what VFR means in different airspace categories for visibility and distance from cloud requirements.	VFR weather minimums, airspace, distance from clouds, visibility, (class B, C, D E, G), special VFR minimums	2.1.1, 3.1.1, 3.1.2, 3.1.4.2, 4.3,
CF_{19}	Not Complying With VFR/IFR Cruising Altitude Requirements.	FARs establish VFR/IFR cruising altitudes based on magnetic courses to separate traffic.	VFR cruising altitude, IFR cruising altitude,	2.1.1, 3.1.1, 3.1.2, 3.1.4.2,

Figure 4.18: Definitions and Keywords/phrases for the causal factors comprising the Functional Activity "Perform Visual Flight Operations"

A sample set of keywords/phrases are provided in Figure 4.18 for the causal factors comprising the functional activity "Perform Visual Flight Operations" of Figure 4.14.

We implement a two step approach while identifying the causal dependencies between two causal factors. The initial step of this approach involves the identification of possible connections among the causal factors. At this stage these connections only imply interactions, thus the undirected edge contains no information regarding the conditional dependency between the two causal factors it connects. Nonetheless, these undirected edges provide the foundation upon which the causal structure of the hybrid Bayesian Network is to be constructed. In order to determine an initial set of undirected edges we use the set of keywords identified by SME for each causal factor. Employing the concepts of pattern matching and classification for text mining, for each causal factor we simply look for the keywords/phrases shared with other causal factors. If we identify one or more keywords/phrases shared by two causal factors we connect them by an undirected edge. During this process, to determine meaningful matches between causal factors, the priority is given to searching common or similar phrases. If for a causal factor we fail to identify a keyphrase shared with any other causal factor, individual key words become the common patterns to look for to establish a preliminary connection between two causal factors. Practice shows that these initial set of undirected edges need to be reviewed by subject matter experts to identify and remove the connections that cannot be justified within the context of their definitions. At this stage, the SMEs also look for connections that might be overlooked by the pattern matching process.

Casual Factors	Definitions	Keywords/Phrases	HCAS #	
Not Complying With VFR/IFR Cruising Altitude Requirements.	FARs establish VFR/IFR cruising altitudes based on magnetic courses to separate traffic.	VFR cruising altitude, IFR cruising altitude,	2.1.1, 3.1.1, 3.1.2, 3.1.4.2,	CF_{19}
Failure to Comply with Minimum Safe Altitudes in Congested and Non-Congested Areas.	To avoid a collision with obstacles, people, vessels, on the ground and on the water, Minimum safe altitudes are established for congested and non-congested areas.	congested, obstacles, altitude, persons, vessels, distance, non-congested, hazards to person or property on the ground,	4.1, 4.7,	CF_{10}
Not Complying with Minimum IFR Altitude Requirements.	Regulations specify additional minimum altitudes for IFR operations except for takeoff and landing. These relate to charted minimum altitudes and radio navigational aid reception ranges.	IFR altitude minimum, minimum obstruction clearance altitude (MOCA), minimum en route altitude (MEA), minimum crossing altitude(MCA), minimum reception altitude (MRA),	2.1.1, 3.1.6, 3.1.4, 3.1.1,	CF_{20}
lack of accurate altitude information	Lack of data correspondence between automatically reported pressure altitude data and the pilot's altitude reference.	altitude information, transponder, altitude encoding, calibration, altimeter	1.1.5, 1.1.6, 3.2.1, 3.3.2	CF_{38}

Figure 4.19: CF19 is connected to CF10, CF20 and CF38

As an example of application of this process to the UAS domain safety risk model, the undirected edges that connects the Causal Factor # 19 "Not Complying With VFR/IFR

Cruising Altitude Requirements" (CF19) to three other causal factors, namely CF10, CF20, and C38 are provided in Figure 4.19.

Note that although the keyword "altitude" is common in all four causal factors depicted in Figure 4.19 the undirected edges connecting them are ultimately verified and determined by SMEs especially taking the context of their definitions into consideration.

The undirected network as the outcome of the first stage of the process to identify the causal dependencies among the causal factors of the UAS DSRM is provided in Figure 4.20. Note that Figure 4.20 does not include the nodes representing the hazard elements.

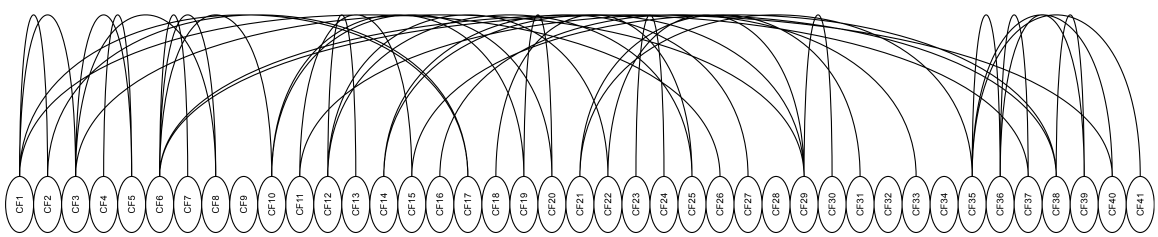


Figure 4.20: The casual factors of the UAS Domain Risk Model connected through undirected links

Note also that in Figure 4.20, the raw network consisted of a set of causal factors and connections that do not imply any causality is in the same form as in Figure 4.12.b. Recalling our discussion in Section 4.4 about the process of the transformation from a functional domain of the RCFF to a directed acyclic graph, our next step is to rearrange these causal factors into a proper graph topology where we can proceed with the second step of our approach. An undirected graph as the result of such an rearrangement is provided in Figure 4.21.

The second step of our approach in the process of constructing the hybrid Bayesian Network representing the UAS domain safety risk model (DSRM) is to identify the conditional dependencies among the causal factors. In a Bayesian Network these conditional dependencies are depicted by directed links between causal factors.

We utilize the undirected graph of Figure 4.21 when identifying causal/conditional interactions within the network structure of the UAS DSRM model.

Throughout this process, once again, subject matter experts determine the direction of the edges. This direction not only depicts a conditional probability distribution over two

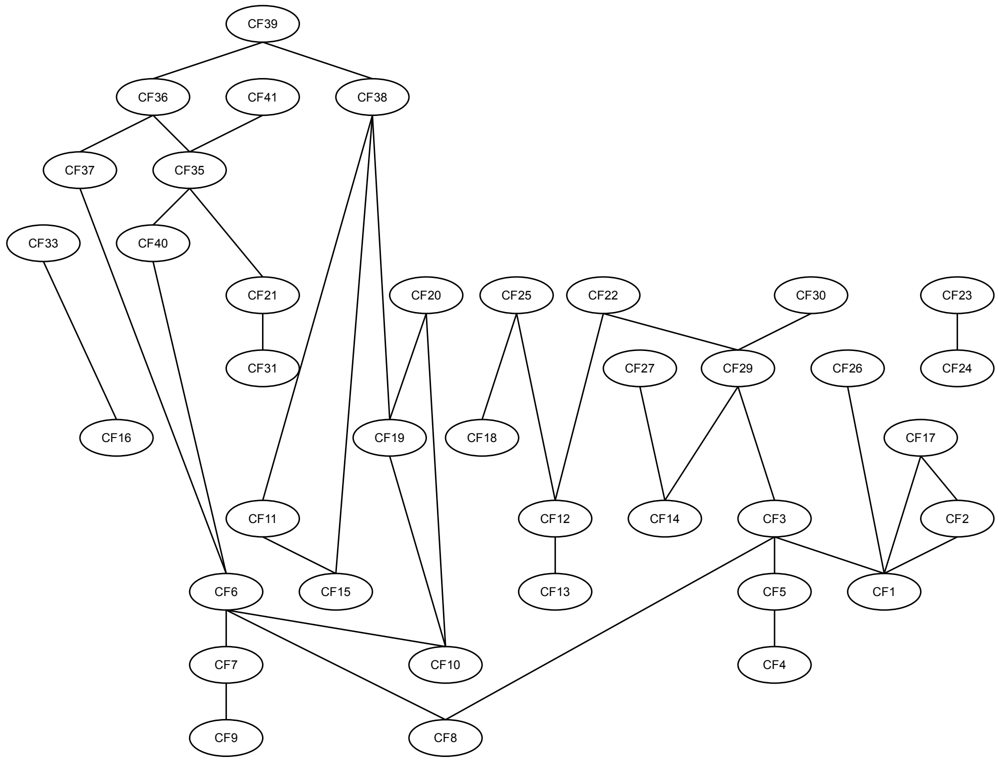


Figure 4.21: An undirected graph of the casual factors of the UAS Domain Risk Model

random variables but also illustrates a causal interaction between two factors contributing to various functional domain hazards. To identify such a dependency SMEs mainly utilize the definitions of causal factors and the context that the four functional activities listed in Table 4.5 provide to the overall model domain.

Note that the graph in Figure 4.21 is not fully connected due to the fact that the causal factor pairs "CF33-CF16" and "CF23-CF24" are not connected to the rest of the graph. As an immediate consequence of this, unless new edges are added to make the graph fully connected we cannot construct a Bayesian Net by simply converting the undirected edges to directed ones. However, note also that Figure 4.21 does not include the hazard nodes without which our model is not complete. With the inclusion of the hazard nodes we see that the graph becomes connected.

Once the hazard nodes are included to the undirected graph and new edges between the hazards and causal factors are added to the network structure, we determine the possible directions for each edge, specifically, taking the "acyclicity" requirement of BNs into consideration.

The Figure 4.22 provides the final directed acyclic graph (DAG) depicting the UAS DSRM.

There are a few features in the final DAG of the UAS model that need further discussion. While determining the direction of conditional dependencies among the variables of the model, we eliminated some linkages identified during the keyword/phrase matching process outlined above. There are two reasons to perform such a "clean up" in the model. First, the final graph needs to be acyclic. Second, whenever there is a direct connection between two casual factors any secondary conditional dependency through other causal factors only complicates the topology and ultimately the propagation over the final BN.

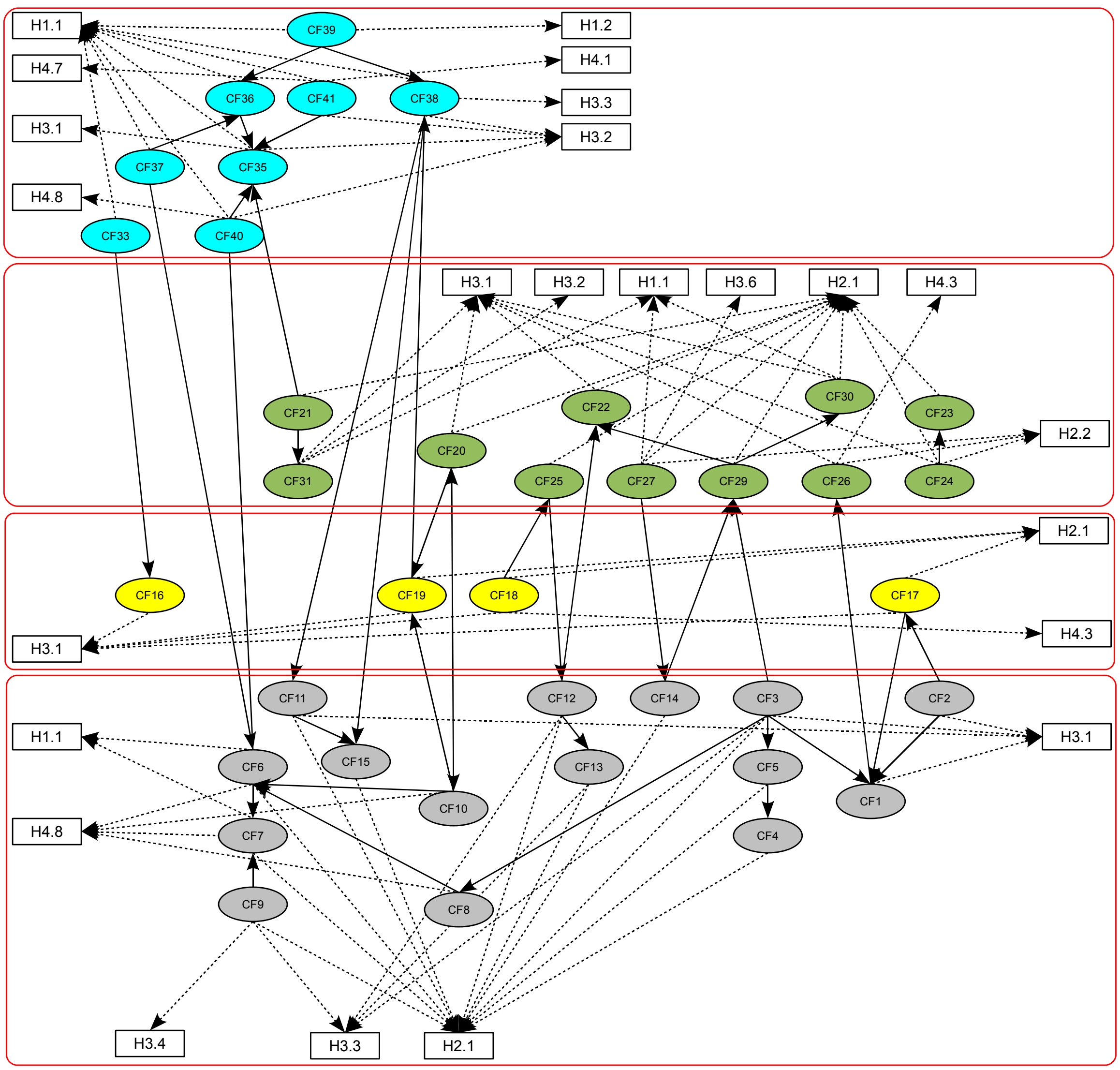


Figure 4.22: A DAG of the UAS Domain Risk Model

In Figure 4.22, the connections leading to the Hazard nodes are shown as dashed lines whereas the edges between two causal factors are drawn as solid lines. Although these two connections are depicted differently, strictly speaking, they are the same in nature and represent a conditional dependency between two nodes/variables that they link in the resulting BN. Also note that the causal factors and hazards are depicted by ellipse and rectangle shaped nodes, respectively. We have chosen to use different styles when depicting the edges and nodes only to enhance the legibility of the network topology and to assist readers understanding of the conditional interactions in a rather crowded model.

Finally, Figure 4.22 divides the model into four distinct functional domains as outlined in Table 4.5 and the causal factors and hazards are arranged in the model topology accordingly. Within this context, some hazard nodes that are conditionally dependent of (i.e., connected to) the causal factors from multiple functional domains are repeated in these domains. However, the important observation here is that although the boundaries of the functional domains are emphasized by the illustration of the model, there are edges crossing over these boundaries and connecting the four distinct domains into one coherent model representing the system safety risk about one larger domain. This representation of the system safety risk is in line with the thinking and philosophy of the Regulatory-based Causal Factor Framework (RCFF) and Domain Safety Risk Modeling (DSRM) approach outlined in Sections 4.3 and 4.4, respectively.

Model Variables

In this section we review the types of variables included in the UAS DSRM depicted in Figure 4.22. In particular we focus on continuous variable in the model.

A closer look to the variables -both causal factors and hazards of the UAS DRSRM reveals that the model contains variables that can be quantitatively expressed through observations. Consider the causal factor "proximity to other aircraft" (CF6), which can be observed and measured in *feet* on a continuous scale, thus it should be represented by a continuous variable to capture a better approximation within the context of a Bayesian Network model such as our UAS DSRM.

Recognizing the continuous nature of the causal factor CF6 may be rather straightforward as compared to some other casual factors, such as causal factor "failure to comply with airspace speed limits" (CF9), among others. A closer inspection of the causal factor, however, reveals that the *speed* of the aircraft is the variable underlying the causal factor "failure to comply with airspace speed limits" (CF9). Hence "airspeed" as the continuous variable replaces CF6 in the final Hybrid Bayesian Network (HBN) representation of the UAS DSRM. This replacement takes place only in the name and definition of the causal factor and we continue to denote it by "CF6".

Analogously, the causal factor "Not complying with fuel requirements" CF16 is closely related to the "fuel on board" of the aircraft and therefore a quantitative variable provides a better approximation of the real word variable that CF6 represents.

A review of the UAS DSRM is conducted to identify possible quantifiable variables underlying the descriptions/definitions of the causal factors constituting the model. The quantifiable causal factors identified through this process are then replaced by associated continuous variables in the final model.

The set of continuous variables representing the causal factors that are quantifiable in the final hybrid Bayesian Network of the UAS Domain Safety Risk Model is listed in Table 4.6.

Additionally, we consider the hazards identified in the UAS DSRM as continuous variables. In fact, by defining the hazard nodes as continuous entities we are able to asses them quantitatively on a predetermined continuous scale. We believe that the quantitative depiction of individual hazards as the model outcome is a substantial improvement on the qualitative depiction of hazards that the aviation community is accustomed to.

The Final HBN

All other nodes except the causal factors identified as quantifiable in Table 4.6 and the hazards are considered as qualitative, thus as discrete variables. Using the notation introduced in Chapter 3 and in Figure 3.2, namely depicting the continuous variables with ellipses and the discrete variables with rectangles, the Figure 4.23 illustrates the final Hybrid Bayesian Network of the UAS Domain Safety Risk Model. On this HBN we apply in the following

Table 4.6: The Set of Continuous Variables Representing the Causal Factors in the HBN of the UAS DSRM

#	Causal Factor	Continuous Variable	
6	Proximity to Other Aircraft	Proximity to Other Aircraft	CF_6
9	Failure to Comply with Airspace Speed Limits	Airspeed	CF_9
10	Failure to Comply with Minimum Safe Altitudes in Congested and Non-Congested Areas.	Altitude	CF_{10}
12	Failure to Comply with ATC Clearances and Instructions.	Proximity to Obstacle	CF_{12}
16	Not Complying with Fuel Requirements	Fuel on Board	CF_{16}
18	Not Complying with VFR or Special VFR Weather Minimums	Visibility (VFR-Cruse)	CF_{18}
19	Not Complying With VFR/IFR Cruising Altitude Requirements.	Crusing Altitude	CF_{19}
20	Not Complying with Minimum IFR Altitude Requirements.	Minimum IFR Altitude	CF_{20}
26	Not Complying with IFR Takeoff Procedures or Minimums.	Visibility (IFR-Takeoff)	CF_{26}
27	Flying in RVSM Airspace Without Complying to RVSM Requirements.	Vertical Seperation	CF_{27}
28	Not Complying with IFR Course Requirements.	Diversion from IFR course	CF_{28}

section the Fuzzy-Bayesian methodology and the algorithms introduced and outlined in detail in Chapter 3.

In Figure 4.23, “CFXX” stands for “Causal Factor XX” and HX.X stands for Hazard Element X.X from within the HCAS Taxonomy. Note also that the Hybrid Bayesian Network depicted above preserves the original domain model of Figures 4.15 and 4.16 which identify four functional models, namely “A1.1 Perform Flight Operations”, “A1.2 Perform Visual Flight Operations”, “A1.3 Perform Instrumental Flight Operations”, and “A2.1 Manage Equipment, Instrument, and Certificate Requirement” to cover the aviation operations governed by 14 CFR Part 91 Subparts B and C.

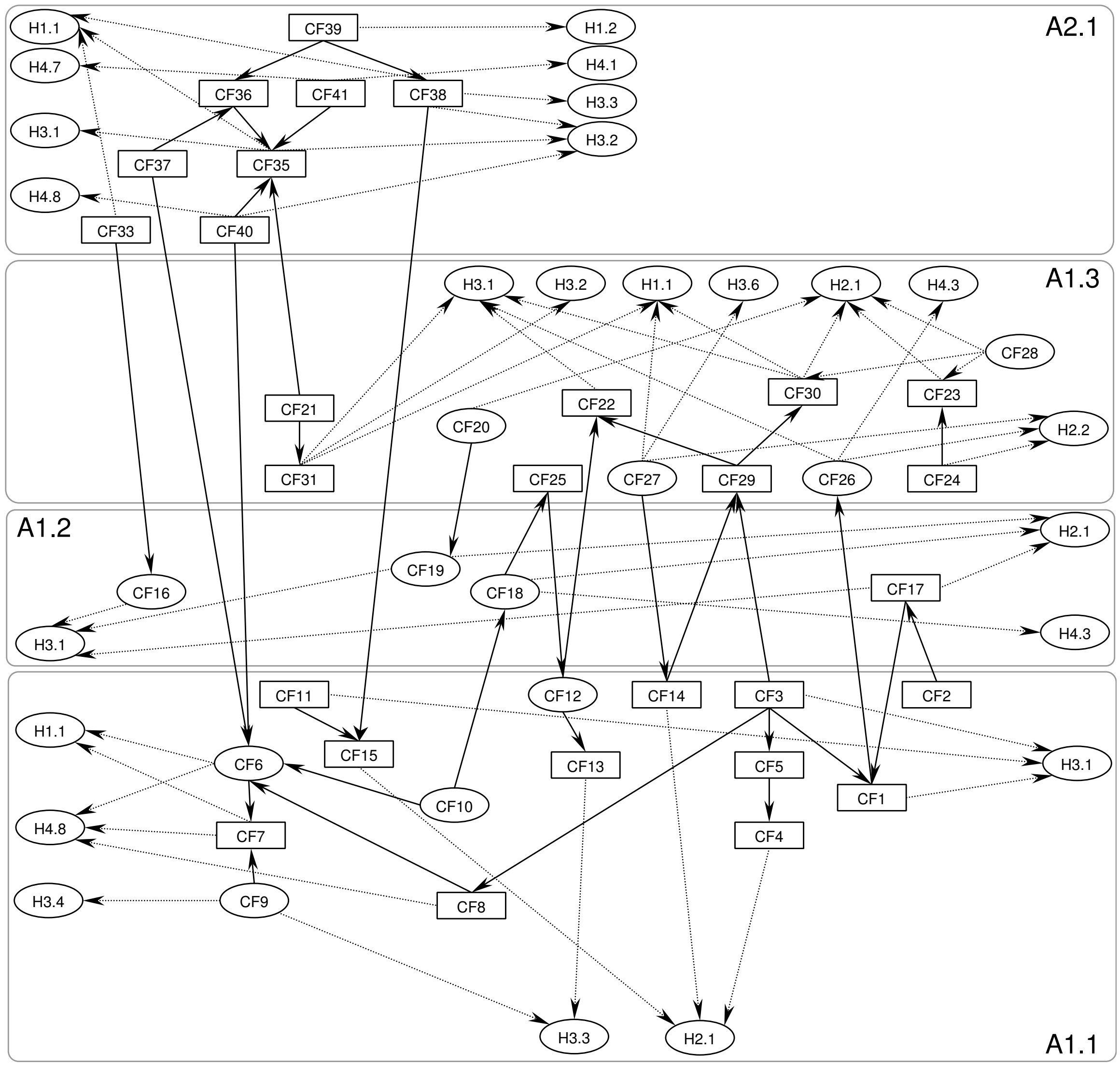


Figure 4.23: The Final Hybrid Bayesian Network Representing the UAS Domain Risk Model.

Populating the Final HBN

A Bayesian Network can only be considered complete when the conditional distributions of variables - conditional probability tables (CPTs) for discrete variables and conditional probability distributions (CPDs) for continuous variables - imposed by the network topology is defined and the BN is fully populated.

During the process of populating the model one can utilize multiple sources depending on the domain of interest and the availability of data to define and construct the CPTs and the CPDs representing the conditional dependencies between a variable and its parents.

Clemens [79] describes a process that outlines a hierarchy of data sources and their usability. This process depicted in Figure 4.24 indicates an order of preference of data sources for performing risk analysis about complex engineering systems.

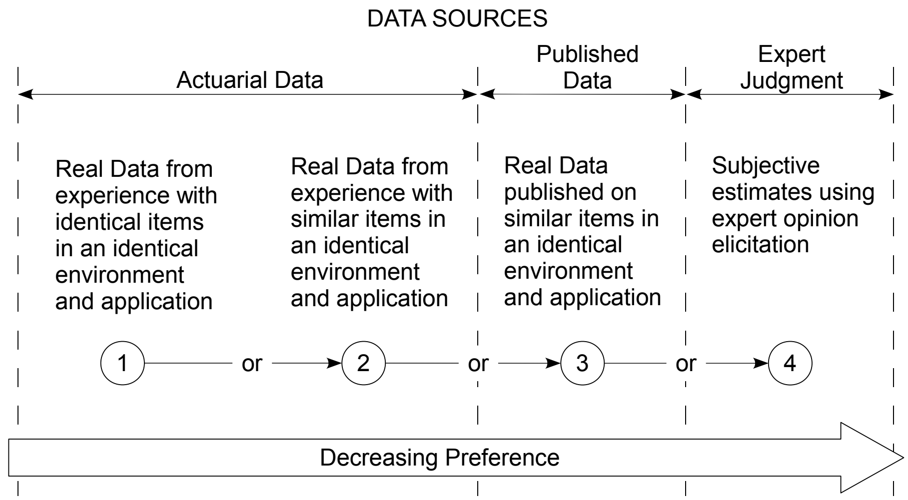


Figure 4.24: Data Sources for Performing Risk Analysis (Adapted from Clemens [79])

According to this hierarchy, if the data required to perform risk analysis is provided by preexisting data for the same identical items or components of the system, this preexisting data should be used. Such a perfect case is rarely encountered in real world situations, especially in new technology applications. The next best thing is the preexisting data, however, this time for similar items or components of the systems. If neither of these scenarios is available, published data on similar systems can be used. Finally, if neither preexisting nor published data does not exist, expert knowledge provides a valuable data source to perform the required analysis.

Although we can define an order of preference among various data sources, a practical approach with broader applicability would be to combine different data sources depending their availability and the requirements dictated by the risk analysis methodology employed. For example, Bayesian Networks can combine both subjective and objective information with successful result to perform risk analysis of complex systems [80]. In fact, the HBN in Figure 4.23 representing the UAS DSRM would be a good fit for such an hybrid approach which utilizes quantitative and qualitative data about the UAS domain. In particular, the continuous variables of the model would benefit form hard data as the result of actual observations of the system. Whereas, conditional probability tables of the discrete variables can be successfully populated by qualitative judgments of the subject matter experts during knowledge elicitation sessions. Ayyub [80] provides an extensive discussion on expert-opinion elicitation.

However, as we discussed in detail in section 4.1, the cutting-edge nature of our domain of interest, makes it quite difficult to find preexisting hard data applicable to our problem domain. Furthermore, recalling the systems level approach that we took while developing our modeling methodology and ultimately building the final UAS DSRM, due to the unique nature of the model variables derived from the aviation regulations, the data requirements of our UAS domain safety risk model are best supported by subjective knowledge based on the experience of the SMEs on the problem domain. Consider the discrete variables CF1 "Inadequate Preflight Planning", even for the conventional operations of commercial aviation, quantifying the conditional probability table of CF1 given other causal factors using preexisting hard data presents quite a difficult challenge. The lack of a common taxonomy for the existing accident/incident databases is arguable the major source for this challenge. [64] provides an extensive discussion on existing aviation accident/incident databases. Considering the shortcomings of the existing aviation databases, for a safety risk model of the emergent UAS operations, the knowledge elicited form SMEs provides a viable data source. Thus, in addition to the restrictions imposed by the UAS problem domain, as the natural extension of our system safety risk modeling methodology, expert-opinion emerges as the main data source while populating the UAS DSRM in Figure 4.23.

However, the application presented here within the context of this thesis is mainly for the purposes of illustrating the applicability of our research methodology, namely a set of new propagation algorithms for *Type I* and *Type II* Fuzzy-Bayesian Networks, to a real-world problem domain. Therefore, the final application model is populated using synthetic data and the algorithms are propagated and results are generated accordingly.

The data used to populate the UAS DSRM are in the form of unique conditional probability distributions for each discrete and continuous variables making up the HBN. Next, we present the conditional probability distributions for selected model variables.

A closer look to the HBN of Figure 4.23 indicates that the final UAS Domain Safety Risk Model includes the following six types of conditional interactions between the child and parent variables constituting the network.

i. Discrete given discrete, such as $CF1 \leftarrow (CF3,\ CF17)$

ii. Discrete given continuous, such as $CF7 \leftarrow CF6$

iii. Discrete given discrete and continuous, such as $CF22 \leftarrow (CF29,\ CF12)$

iv. Continuous given discrete, such as $CF26 \leftarrow CF1$

v. Continuous given continuous, such as $CF18 \leftarrow CF10$

vi. Continuous given continuous and discrete, such as $CF6 \leftarrow (CF10,\ CF8)$

There are 63 nodes in the HBN model of the UAS DSRM in Figure 4.23. Due to the relatively larger size of the network, it is not feasible to list all the synthetic data used to populate the HBN here in this thesis. Instead, for each of the six conditional dependencies that repeats throughout the hybrid network, a representative parent-child combination is identified. A conditional probability distribution associated with the child-node defines each parent-child combination. Next, we provide the synthetic data used to populate the conditional probability distributions (CPDs) of these six representative nodes from within the UAS DSRM.

i. **Discrete Given Discrete: CPD of node CF1** "*Inadequate Preflight Planing*"$=$ $P(CF1|CF3,\ CF17)$.

$CF3:$ *Crewmember Not At Station*

$CF17:$ *Incomplete VFR Flight Plan Information*

$CF1,\ CF3,\ CF17 \in \mathbf{\Delta}$

Unless otherwise indicated, all discrete causal factors are binary variables, which can only assume "true" or "false".

		CF3, CF17			
$P(CF1 \mid CF3, CF17)=$		true,false	true,false	true,false	true,false
CF1	true	0.85	0.70	0.65	0.15
	false	0.15	0.30	0.35	0.85

ii. **Discrete Given Continuous: CPD of node CF7** "*Right of way rules are not followed*"$= P(CF7 \mid CF6)$:

$CF6$: *Proximity to other aircraft*

$CF7 \in \boldsymbol{\Delta}, \quad CF6 \in \boldsymbol{\Gamma}$

$CF6$ proximity to other aircraft denoted by "d" is measured in feet, such that $CF6$: $D = \{d : 0 \leq d \leq 10000\ ft\}$; then the CPD of CF7 is given by

$$P(CF7 \mid CF6)= \begin{cases} P(CF7{=}true \mid CF6)= & \begin{cases} 0.0025 \times e^{-.005d} & \text{for} \quad 0 \leq d < 10000ft \\ 0 & \text{otherwise} \end{cases} \\ \\ P(CF7{=}false \mid CF6)= & \begin{cases} 0.0025 \times e^{-.005(2000-d)} & \text{for} \quad 0 \leq d < 10000ft \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

iii. **Discrete Given Discrete and Continuous: CPD of node CF22** "*Operating IFR without an ATC clearance and flight plan in controlled airspace*"
$= P(CF22 \mid CF29, CF12)$:

$CF29$: *Failure to maintain communications with ATC while flying under IFR*

$CF12$: *Proximity to obstacle* (see Table 4.6)

$CF22, CF29 \in \boldsymbol{\Delta}, \quad CF12 \in \boldsymbol{\Gamma}$

$CF12$ proximity to obstacle denoted by "h" is measured in feet, such that

$CF12 : H = \{h : 0 \leq h \leq 2000\ ft\}$; then the CFD of CF22 is given by

$$P(CF22|CF29{,}CF12) = \begin{cases} P(true|true,\, CF12) = \begin{cases} \frac{0.005}{4} \cdot e^{-.005h} & \text{for} \quad 0 \leq h \leq 2000ft \\ 0 & \text{otherwise} \end{cases} \\ P(false|false,\, CF12) = \begin{cases} \frac{0.005}{8} \cdot e^{-.005(2000-h)} & \text{for} \quad 0 \leq h \leq 2000ft \\ 0 & \text{otherwise} \end{cases} \\ P(true|false,\, CF12) = \begin{cases} \frac{0.005}{8} \cdot e^{-.005h} & \text{for} \quad 0 \leq h \leq 2000ft \\ 0 & \text{otherwise} \end{cases} \\ P(false|false,\, CF12) = \begin{cases} \frac{0.005}{4} \cdot e^{-.005(2000-h)} & \text{for} \quad 0 \leq h \leq 2000ft \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

iv. **Continuous Given Discrete: CPD of node CF26** "*Not Complying with IFR Takeoff Procedures or Minimums*"$= P(CF26|CF1)$.

Causal factor $CF26$ is defined by *visibility (IFR-Takeoff)* as a continuous variable (see Table 4.6), which is measured in *feet*, such that

$CF26 : W = \{w : w \in \mathbb{R}^{+}\}$; and

$CF26 \in \mathbf{\Gamma}, \quad CF1 \in \mathbf{\Delta}$.

Then the CFP for node $CF26$ is defined by the following expression:

$$P(CF26|CF1) = \begin{cases} 0.005 \cdot e^{-.005 \cdot w} & \text{for} \quad CF1 = true \\ \mathcal{N}(W;\ 2000, 500^2) & \text{for} \quad CF1 = false \end{cases} \tag{4.3}$$

v. **Continuous Given Continuous: CPD of node CF18** "*Not complying with VFR or Special VFR Weather Minimums*"$= P(CF18|CF10)$.

$CF10$: *Failure to comply with Minimum Safety Altitude in congested and non-congested areas*

Causal factors $CF18$ and $CF10$ are represented by the continuous variables "Visibility (VFR-Cruse)" and "Altitude", respectable (see Table 4.6). The continuous variables "Altitude" and "Visibility" denoted by "l" and "v are measured in feet, such that

$CF10 : L = \{l : 0 \leq l\}$

$CF18 : V = \{v : 0 \leq v\}$ and

$CF10, CF18 \in \mathbf{\Gamma}$.

Then, the conditional probability distribution $P(CF18|CF10)$ as follows:

$$P(CF18|CF10) = \mathcal{N}(V; L - 5000, 500^2) \tag{4.4}$$

Equation (4.4), which is essentially a bivariate Normal distribution, indicates a linear CPD outlined in Section 2.1.4 and, particularly, in equation (2.17).

vi. **Continuous Given Discrete and Continuous: CPD of node CF6** "*Proximity to Other Aircraft*"$= P(CF6|CF10, CF8)$.

$CF8$: *Failure to See and Avoid*, which is a binary discrete variable, which can only assume "true" or "false" as its value.

$CF6, CF10 \in \mathbf{\Gamma}, \quad CF8 \in \mathbf{\Delta}$.

Then the CPD of the node CF6 is given by the following expression:

$$P(CF6|CF10, CF8) = \begin{cases} \mathcal{N}(D; L, 1000^2) & \text{for} \quad CF8 = true \\ \mathcal{N}(D; 4000, 1000^2) & \text{for} \quad CF8 = false \end{cases} \tag{4.5}$$

where,

$CF6 : D = \{d : 0 \leq d \leq 10000\, ft\}$

$CF10 : L = \{l : 0 \leq l\}$

The CPD data provided here in this section is used to populate the six unique types of conditional dependencies between the six selected child nodes and their parent variables. The CPDs of the remaining 44 nodes are populated by the data that, in terms of type and form, fits in one of the types listed above and are not provided here to preserve the focus of this Chapter's narrative.

4.5 Application of the Fuzzy Bayesian Methodology to the UAS DSRM

The types and forms of the data that populates the final HBN of the UAS DSRM are introduced in the preceding section. Given the existing topology of the hybrid Bayesian Net of Figure 4.23, a review of this data concludes that current popular propagation algorithms could not be used to perform exact inferencing about the UAS DSRM because of the following two reasons:

- The network includes discrete variables, which have continuous parents.
- The conditional dependencies between some continuous variables and their parents are defined by distributions other than Conditional Gaussians.

Hence, we can justify the use of the Fuzzy Bayesian Methodology developed in this thesis to perform probabilistic reasoning on the UAS DSRM and calculate marginal distributions of various hazard identified as the result of the modeling process.

Within this context, using the propagation algorithms developed and introduced in Chapter 3, we first provide the results of the *Type I* and *Type II* FBN Fuzzy-Bayesian transformation and then present the marginal probability distributions of the hazards nodes, which are calculated as the outcome of the UAS Domain Safety Risk Model.

4.5.1 The *Type I* FBN and Modeling Results

As introduced in Section 3.4.1 and discussed in detail in Definition 6, the directed acyclic graph (DAG) of a Hybrid Bayesian Network $\mathcal{G}_{\text{HBN}}$ defined by equation (3.35), is transformed into a *Type I* Fuzzy Bayesian Network $\mathcal{G}_{\text{FBN}^I}$, first, by replacing all its continuous variables $\mathbf{\Gamma}$ with their Fuzzy counterparts $\hat{\mathbf{\Gamma}}$ and then, by adding the original crisp-continuous variables to the network along with a directed link originating from the Fuzzy counterparts to the original continuous variables. All original continuous variables are thereby conditionally dependent to only their Fuzzy counterparts. The resulting network which is also a DAG is defined by equation (3.53).

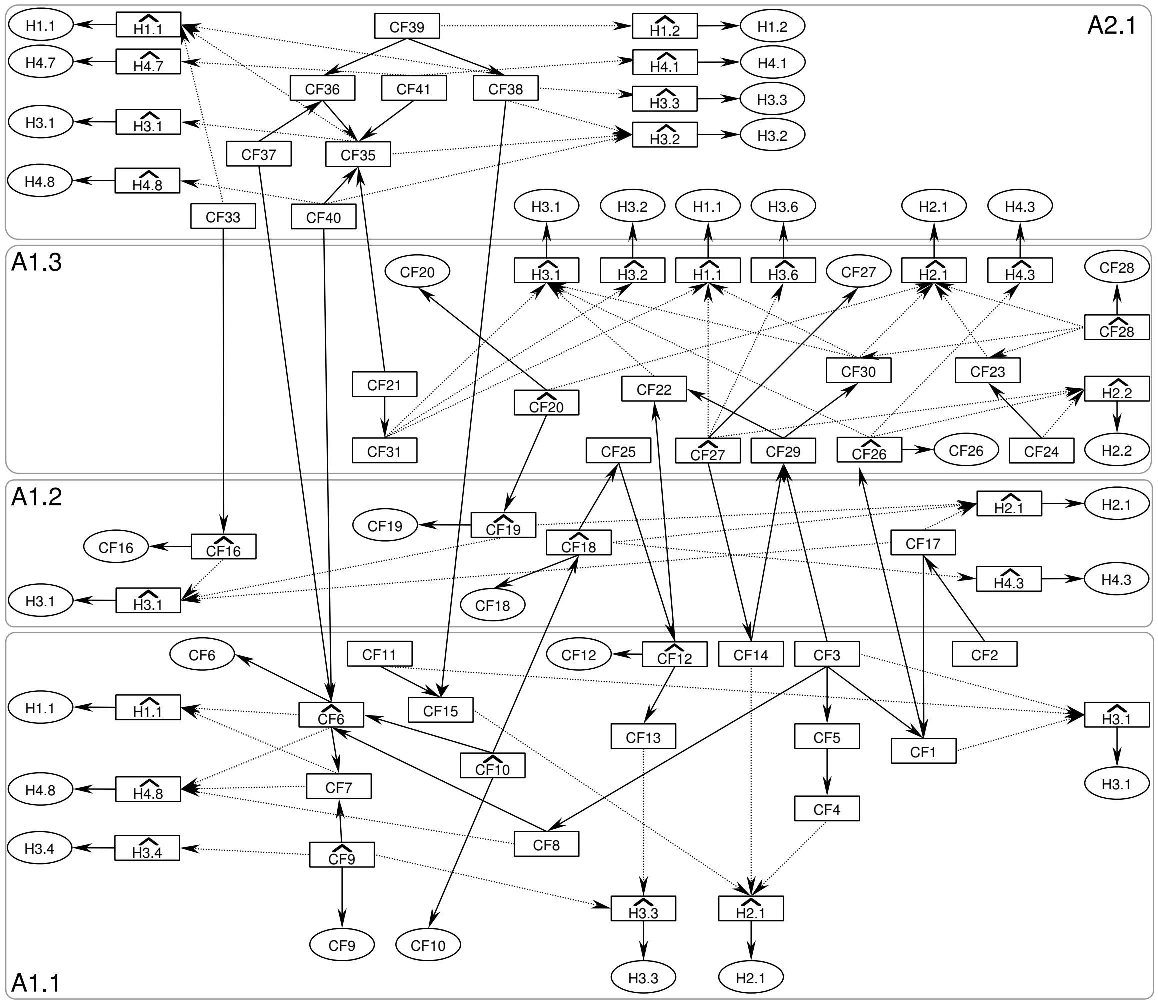


Figure 4.25: *Type-I* FBN $\mathcal{G}_{\mathrm{FBN}^I}$ of the UAS Domain Risk Model as the result of the transformation.

Accordingly, the conditional variables of the UAS DSRM identified in Table (4.6) constitute the Set $\mathbf{\Gamma}$,

$$\mathbf{\Gamma} = \{CF_6,\, CF_9,\, CF_{10},\, CF_{12},\, CF_{16},\, CF_{18},\, CF_{19},\, CF_{20},\, CF_{26},\, CF_{27},\, CF_{28}\}.$$

The corresponding Fuzzy-discrete counterparts are given by set $\hat{\mathbf{\Gamma}}$,

$$\hat{\mathbf{\Gamma}} = \{\widehat{CF_6},\, \widehat{CF_9},\, \widehat{CF_{10}},\, \widehat{CF_{12}},\, \widehat{CF_{16}},\, \widehat{CF_{18}},\, \widehat{CF_{19}},\, \widehat{CF_{20}},\, \widehat{CF_{26}},\, \widehat{CF_{27}},\, \widehat{CF_{28}}\}.$$

The *Type-I* FBN, $\mathcal{G}_{\mathrm{FBN}^I}$, as the result of this transformation is depicted in Figure 4.25.

Note that in Figure 4.25, as the result of transforming UAS DSRM of Figure 4.23 to a *Type-I* Fuzzy Bayesian Network, $\widehat{CF_{xx}}$ and $\widehat{H_{x.x}}$ represent the counterpart Fuzzy-discrete variables of the original continuous variables CF_{xx} and $H_{x.x}$, respectively.

Furthermore, the new Fuzzy-discrete variables are depicted by rectangle nodes in accordance with the convention used throughout this thesis. The node cluster which is comprised of a crisp continuous node and its Fuzzy-discrete counterpart is illustrated in Figure 4.26.

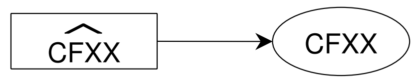


Figure 4.26: Sample cluster of continuous variable CF_{xx} and its Fuzzy counterpart $\widehat{CF_{xx}}$

Additionally, the Fuzzy variable $\widehat{CF_{xx}}$ replaces original continuous variable CF_{xx} and acquires all its original connections as parent or child variable in the *Type-I* FBN transformation of the UAS DSRM.

In order to perform the inferencing algorithm about the UAS DSRM in Figure 4.25 as outlined in Section 3.4.2, we need to complete the transformation of the crisp domain of the original model into the Fuzzy domain of the *Type-I* FBN. To complete the transformation we need two types of additional information regarding the newly created Fuzzy variables and their conditional dependency towards their original continuous counterparts. In particular we need to define the following:

- The fuzzy states, which the continuous variables will be fuzzified into and associated membership functions.
- The conditional probability distributions for each continuous variable given their newly created fuzzy counterparts.

These two information are required to finalize populating the *Type-I* FBN in Figure 4.25 and to perform the propagation algorithm developed in Section 3.4.2 and illustrated by Numerical Example 3.4.2.

For the purposes of this application, Fuzzy-discrete counterparts of all continuous variables included in the UAS DSRM are defined using the following generic formulation: Consider continuous variable CF_i, such that

$$CF_i : X = \{x : x_L \leq x \leq x_U\}, \qquad x \in \mathbb{R}^+ \tag{4.6}$$

for $i = \{6, 9, 10, 12, 16, 18, 19, 20, 26, 27, 28\}$, where x_L, x_U are lower and upper bounds of the domain upon which the continuous variable $CF_i : X$ is defined, respectively. Then, based on the continuous frame of CF_i the Fuzzy-discrete counterpart variable $\widehat{CF_i}$ is defined as having three Fuzzy states, such that

$$\widehat{CF_i} = \{\widehat{CF_{i_1}}, \widehat{CF_{i_2}}, \widehat{CF_{i_3}}\} \tag{4.7}$$

along with the following associated membership functions:

$$\widehat{CF_{i_1}}(x) = \begin{cases} 1 \\ \dfrac{x_2 - x}{x_2 - x_1} \\ 0 \\ 0 \\ 0 \end{cases} \quad \widehat{CF_{i_2}}(x) = \begin{cases} 0 \\ \dfrac{x - x_1}{x_2 - x_1} \\ 1 \\ \dfrac{x_4 - x}{x_4 - x_3} \\ 0 \end{cases} \quad \widehat{CF_{i_3}}(x) = \begin{cases} 0 & x_L \leq x < x_1 \\ 0 & x_1 \leq x < x_2 \\ 0 & x_2 \leq x < x_3 \\ \dfrac{x - x_3}{x_4 - x_3} & x_3 \leq x < x_4 \\ 1 & x_4 \leq x < x_U \end{cases}$$

where $x_L,\ x_1,\ x_2,\ x_3,\ x_4,\ x_U \in \mathbb{R}^+$ are considered as the characteristics that define the transition regions between adjacent Fuzzy states and the shape of their membership functions. Note also that while defining the Fuzzy states we use a generic *trapezoidal* membership function.

Table 4.7: The Fuzzy States and associated membership function characteristics for the Continuous Variables of *Type-I* FBN

CF_i	$\widehat{CF_i} = \{\widehat{CF_{i_1}}, \widehat{CF_{i_2}}, \widehat{CF_{i_3}}\}$	x_L	x_1	x_2	x_3	x_4	$x_U(unit)$
CF_6	$\widehat{CF_6}$ = {near, moderate, far}	0	2000	3000	5000	6000	$10000ft$
CF_9	$\widehat{CF_9}$ = {slow, medium, fast}	0	40	60	120	140	$\infty\ kts$
CF_{10}	$\widehat{CF_{10}}$ = {low, medium, high}	0	1500	2500	4500	5500	$0ft$
CF_{12}	$\widehat{CF_{12}}$ = {near, moderate, far}	0	250	500	1000	1500	$2000ft$
CF_{16}	$\widehat{CF_{16}}$ = {low, medium, full}	0	3	6	15	20	$30lt$
CF_{18}	$\widehat{CF_{18}}$ = {poor, moderate, clear}	0	250	750	1500	2000	$\infty\ ft$
CF_{19}	$\widehat{CF_{19}}$ = {low, medium, high}	0	1500	2500	7500	10000	$20000ft$
CF_{20}	$\widehat{CF_{20}}$ = {low, medium, high}	0	1000	1250	2500	3000	$\infty\ ft$
CF_{26}	$\widehat{CF_{26}}$ = {poor, moderate, clear}	0	100	300	500	750	$1000ft$
CF_{27}	$\widehat{CF_{27}}$ = {narrow, moderate, wide}	0	500	750	1500	2000	$\infty\ ft$
CF_{28}	$\widehat{CF_{28}}$ = {small, medium, large}	0	1500	3000	5000	7500	$\infty\ ft$

For all continuous variables identified during the structuring process of the UAS DSRM and consequently listed in Table 4.6, the counterpart Fuzzy-discrete variables along with the associated Fuzzy states and the characteristics of their membership functions are provided in Table 4.7.

On the other hand, the conditional probability distribution for the original continuous variable CF_i given its Fuzzy-discrete counterpart $\widehat{CF_i}$, defined by equations (4.6) and (4.7), respective, is given by equation (4.8).

$$P(CF_i|\widehat{CF_i}) = \begin{array}{c|c|l} CF_i & \widehat{CF_i} & \\ \hline & \widehat{CF}_{i_1} & \mathcal{N}(X;\mu_{i_1},\sigma_{i_1}) \\ \text{x} & \widehat{CF}_{i_2} & \mathcal{N}(X;\mu_{i_2},\sigma_{i_2}) \\ & \widehat{CF}_{i_3} & \mathcal{N}(X;\mu_{i_3},\sigma_{i_3}) \end{array} \tag{4.8}$$

$$\text{for } i = \{6, 9, 10, 12, 16, 18, 19, 20, 26, 27, 28\},$$

where, $\mathcal{N}(X;\mu_{i_j},\sigma_{i_j})$ denotes a Normal distribution with mean μ_{i_j} and variance σ_{i_j} for $j = \{1, 2, 3\}$ over the continuous variable $CF_i : X$. Note also that x represents any value that $CF_i : X$ may assume on its continuous frame.

Table 4.8: Mean μ_{i_j} and Variance σ_{i_j} Values for $P(CF_i|\widehat{CF}_i)$ defined by equation (4.8)

$(CF_i\|\widehat{CF_i})$	(μ_{i_1},σ_{i_1})	(μ_{i_2},σ_{i_2})	(μ_{i_3},σ_{i_3})
$(CF_6\|\widehat{CF_6})$	$(2000, 1000)$	$(800, 100)$	$(5000, 1000)$
$(CF_9\|\widehat{CF_9})$	$(50, 10)$	$(120, 15)$	$(200, 15)$
$(CF_{10}\|\widehat{CF_{10}})$	$(1250, 150)$	$(2500, 200)$	$(500, 75)$
$(CF_{12}\|\widehat{CF_{12}})$	$(150, 30)$	$(750, 100)$	$(1500, 225)$
$(CF_{16}\|\widehat{CF_{16}})$	$(5, 1.2)$	$(12, 2)$	$(2, 0.75)$
$(CF_{18}\|\widehat{CF_{18}})$	$(300, 75)$	$(750, 60)$	$(1000, 100)$
$(CF_{19}\|\widehat{CF_{19}})$	$(1300, 225)$	$(5000, 300)$	$(500, 65)$
$(CF_{20}\|\widehat{CF_{20}})$	$(275, 50)$	$(850, 65)$	$(2500, 125)$
$(CF_{26}\|\widehat{CF_{26}})$	$(250, 15)$	$(500, 30)$	$(900, 45)$
$(CF_{27}\|\widehat{CF_{27}})$	$(875, 90)$	$(55, 10)$	$(600, 85)$
$(CF_{28}\|\widehat{CF_{28}})$	$(1000, 250)$	$(1500, 300)$	$(3000, 425)$

The actual mean and variance values of the Normal distributions that are used to define the conditional distributions of the continuous variables identified in Table 4.6 given their

Fuzzy counterparts are provided in Table 4.8.

Using the information in Tables 4.7 and 4.8 along with the formulations introduced above, we can extract the actual data used to populate the continuous Causal Factor nodes of UAS DSRM.

As an example, the data for the continuous node CF_6 is provided below. For all the Causal Factors in Table 4.6, the required data to perform the *Type-I* FBN exact inferencing algorithm can be written in a similar fashion.

$\mathbf{CF_6}$: Proximity to Other Aircraft, defined by

$$CF_6 : D = \{d : 0 \leq d \leq 10000\, ft\}$$

Fuzzy Counterpart Variable: $\widehat{CF_6}$, with Fuzzy states $\widehat{CF_{6_1}}$:"near", $\widehat{CF_{6_2}}$:"moderate", and $\widehat{CF_{6_3}}$:"far", such that $\widehat{CF_6} = \{\widehat{CF_{6_1}}, \widehat{CF_{6_2}}, \widehat{CF_{6_3}}\}$ with membership functions

$$\widehat{CF_{6_1}}(d) = \begin{cases} 1 \\ \frac{3000-d}{1000} \\ 0 \\ 0 \\ 0 \end{cases} \qquad \widehat{CF_{6_2}}(d) = \begin{cases} 0 \\ \frac{d-2000}{1000} \\ 1 \\ \frac{6000-d}{1000} \\ 0 \end{cases} \qquad \widehat{CF_{6_3}}(d) = \begin{cases} 0 & 0 \leq d < 2000 \\ 0 & 2000 \leq d < 3000 \\ 0 & 3000 \leq d < 5000 \\ \frac{d-5000}{1000} & 5000 \leq d < 6000 \\ 1 & 6000 \leq d < 10000 \end{cases}$$

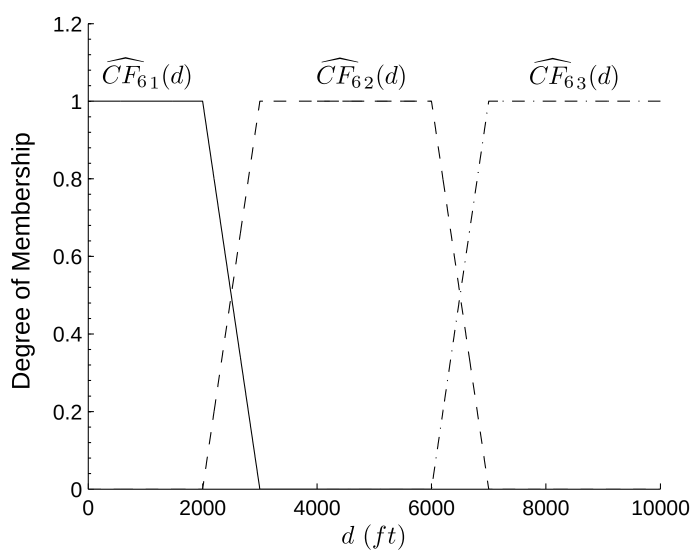


Figure 4.27: Membership functions for Fuzzy states $\widehat{CF_{6_1}}$, $\widehat{CF_{6_2}}$, and $\widehat{CF_{6_3}}$

And the conditional dependency between the continuous variable and its fuzzy counterpart is given by the CPD depicted in equation (4.9).

$$P(CF_6|\widehat{CF}_6) = \begin{array}{c|c|c} CF_6 & \widehat{CF}_6 & \\ \hline & \widehat{CF}_{6_1} & \mathcal{N}(D;\mu_{6_1},\sigma_{i_1}) \\ d & \widehat{CF}_{6_2} & \mathcal{N}(D;\mu_{i_2},\sigma_{i_2}) \\ & \widehat{CF}_{6_3} & \mathcal{N}(D;\mu_{i_3},\sigma_{i_3}) \end{array} \tag{4.9}$$

At this stage, the only remaining information needed to perform our exact inferencing methodology about the *Type-I* FBN of the UAS DSRM is the data associated with the Hazard nodes.

As can be seen in Figure 4.25, we had already identified the Hazard nodes in our model as continuous variables. In particular, we define all Hazard nodes on the same continuous fame and use the same Fuzzy states and membership functions to identify their Fuzzy-discrete counterpart. Furthermore, the conditional dependency between the Hazard and its Fuzzy counterpart is represented by the same CPD. This approach not only helps to control the complexity of the FBN, but also provides a consistent way to quantify hazard associated with the model.

Formally, the continuous hazard nodes are populated according to the following formulation. Consider continuous variables $H_{i.j}$, such that

$$H_{i.j} : H = \{h : 0 \leq h \leq 10\}, \qquad h \in \mathbb{R}^+ \tag{4.10}$$

for $(i.j) = \{(1.1), (1.2), (2.1), (2.2), (3.1), (3.2), (3.3), (3.4), (3.6), (4.1), (4.3), (4.7), (4.8)\}$,

Then, based on the continuous frame of $H_{i.j}$ the Fuzzy-discrete counterpart variable $\widehat{H}_{i.j}$ is defined as having three Fuzzy states, "$\widehat{H}_{i.j_1} : low$", "$\widehat{H}_{i.j_2} : moderate$", and "$\widehat{H}_{i.j_3} : high$", such that

$$\widehat{H}_{i.j} = \{\widehat{H}_{i.j_1}, \widehat{H}_{i.j_2}, \widehat{H}_{i.j_3}\} \tag{4.11}$$

along with the following corresponding membership functions:

$$\widehat{H}_{i.j_1}(h) = \begin{cases} 1 \\ \dfrac{4-h}{2} \\ 0 \\ 0 \\ 0 \end{cases} \quad \widehat{H}_{i.j_2}(h) = \begin{cases} 0 \\ \dfrac{h-2}{2} \\ 1 \\ \dfrac{8-h}{2} \\ 0 \end{cases} \quad \widehat{H}_{i.j_3}(h) = \begin{cases} 0 & 0 \leq h < 2 \\ 0 & 2 \leq h < 4 \\ 0 & 4 \leq h < 6 \\ \dfrac{h-6}{2} & 6 \leq h < 8 \\ 1 & 8 \leq h < 10 \end{cases}$$

Finally, the conditional probability distribution of the continuous Hazard node $H_{i.j}$ given its Fuzzy counterpart $\widehat{H}_{i.j}$ is given by equation (4.12).

$$P(H_{i.j}|\widehat{H}_{i.j}) = \begin{array}{c|c|l} H_{i.j} & \widehat{H}_{i.j} & \\ \hline & \widehat{H}_{i.j_1} & \mathcal{N}(H; \mu_{i_1}, \sigma_{i_1}) \\ h & \widehat{H}_{i.j_2} & \mathcal{N}(H; \mu_{i_2}, \sigma_{i_2}) \\ & \widehat{H}_{i.j_3} & \mathcal{N}(H; \mu_{i_3}, \sigma_{i_3}) \end{array} \tag{4.12}$$

So far we provided the information required to populate the *Type-I* FBN transformation of the UAS DSRM and to perform the probabilistic reasoning about the network applying the inferencing algorithm developed in Section 3.4.2 .

Next, we present the results of our exact inferencing algorithm for *Type-I* FBN applied on the UAS DSRM. Reiterating the concept illustrated in Figure 4.17, the marginal probability distributions of the 13 hazards identified in the final HBN of the UAS DSRM of Figure 4.23 are the outcome of the application of our research methodology. Therefore, we concentrate solely on hazard nodes while determining the results of applying the Type-I inferencing algorithm to the FBN in Figure 4.25. As the results of *Type-I* FBN application, we present two sets of marginal distributions for the Hazard nodes:

- The marginal distributions after the junction three associated with the network is initiated, i.e., a message is passed, *for the first time*, through the JT. This set of marginal distributions determine a baseline probability distribution for the hazard variables.

- The marginal distributions of hazards when evidence is introduced to the model. These marginals could then be compared to the baseline distributions of the same hazards to reveal the impact of the evidence as an increase or decrease in the hazard likelihoods.

In the latter case, a set observations regarding some selected continuous and discrete variables constitutes evidence, thereby outlining a scenario about the problem domain modeled by the Bayesian Network. Therefore, For the purposes of this application, a scenario is simply a collection of variables with known values. Within the same context, the scenario depicted in Table 4.9 is used as evidence to generate associated marginals.

Table 4.9: The Set Causal Factors and their values used as the Synthetic Scenario

CF_i	Definition		Value
CF_1	Inadequate Preflight Training	$\Leftarrow$	$True$
CF_9	Airspeed	$\Leftarrow$	150 kts
CF_{10}	Altitude	$\Leftarrow$	$3000 ft$
CF_{14}	Failure to Follow Requirements in Designated Airspace	$\Leftarrow$	$True$
CF_{16}	Remaining Fuel On Board	$\Leftarrow$	40%
CF_{17}	Incomplete VFR Flight Plan Information	$\Leftarrow$	$True$
CF_{30}	Failure to Comply with Loss of Communication Procedures	$\Leftarrow$	$True$

For the *Type-I* UAS DSRM, the baseline distributions and the distributions after the evidence is introduced are provided in Figure 4.29 and 4.30.

Matlab is extensively used while performing inferencing about the UAS DSRM and generating the results presented in Figures 4.28 and 4.29 and it is quite infeasible to repeat the calculations that we already outlined in detail with a numerical example in Section 3.4.2. Therefore, the construction of the Junction Tree associated with the *Type-I* FBN of the UAS DSRM, and the steps of the application of the *Type-I* inferencing algorithm are not presented here.

The results of the *Type-I* inferencing on UAS DSRM are presented in Figures 4.29 and 4.30. To elaborate on the information that these plots provide, the results for Hazard $H_{1.1}$ given in Figure 4.28 are discussed next as an example.

In Figure 4.28 the dotted line represents the baseline marginal distribution for Hazard

$H_{1.1}$ "*Aircraft Design Related Hazards*", whereas the solid line represents the marginal probability distribution of the hazard after the evidence associated with the scenario outlined in Table 4.9 is introduced to the model.

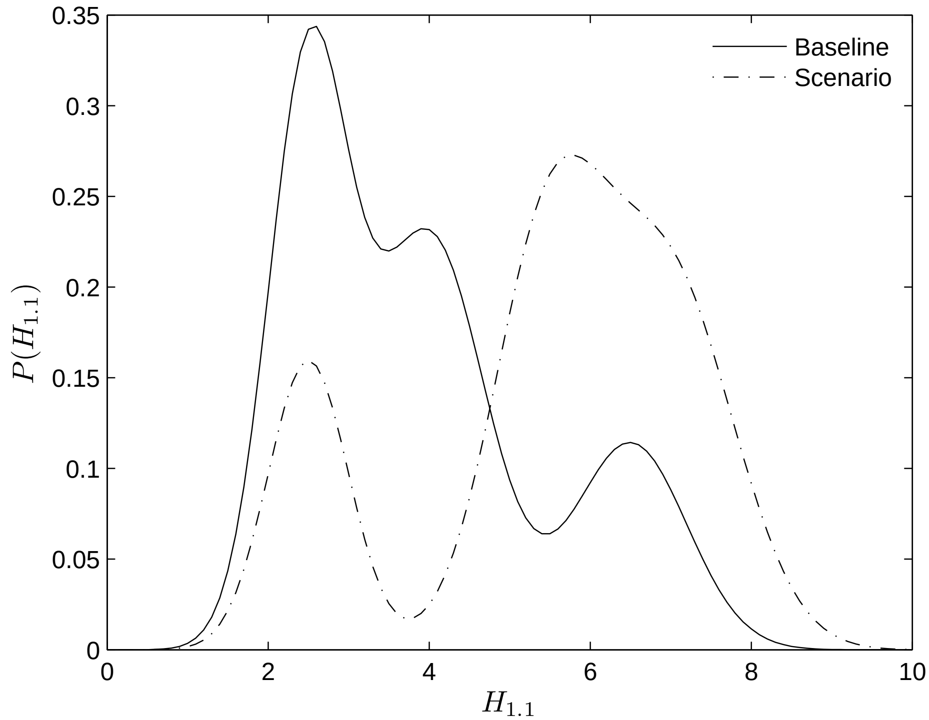


Figure 4.28: Marginal Probability Density Functions of Hazard $H_{1.1}$ "*Aircraft Design Related Hazards*" when the model is initiated with the initial CPDs (i.e., *Baseline Case*) and when the evidence as a scenario is introduced to the model (i.e., *Scenario Case*).

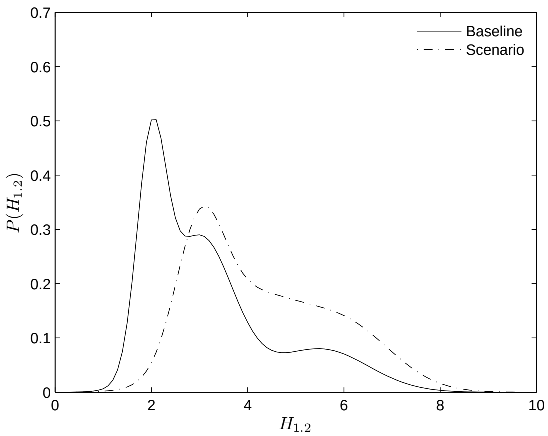
Juxtaposing the baseline and scenario plots of the marginal distributions make it possible to visualize the relative change in the likelihood of individual hazards. For example, in Figure 4.28, one can observe a shift towards a higher hazard value in the probability density after the evidence is introduced to the network as compared to the baseline density for the same hazard. Thus, we can deduce that the scenario outlined in Table has a negative impact on likelihood of occurrence of the individual hazard element $H_{1.1}$ *Aircraft Design Related Hazards.*

Furthermore, one can also apply fuzzy transformation on these continuous probability density functions to determine the membership values associated with the states of Fuzzy-discrete counterpart hazards. For example, using the membership functions of equation

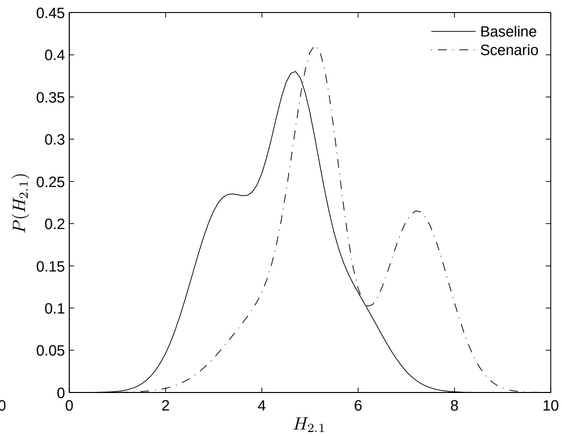
(4.11) we determine the Fuzzy hazard $\widehat{H}_{1.1}$ for the baseline and scenario cases as follows:

$$\widehat{H}_{1.1_{Baseline}} = \{Low = 0.51,\ Moderate = 0.32,\ High = 0.17\}$$
$$\widehat{H}_{1.1_{Scenario}} = \{Low = 0.23,\ Moderate = 0.34,\ High = 0.43\}.$$

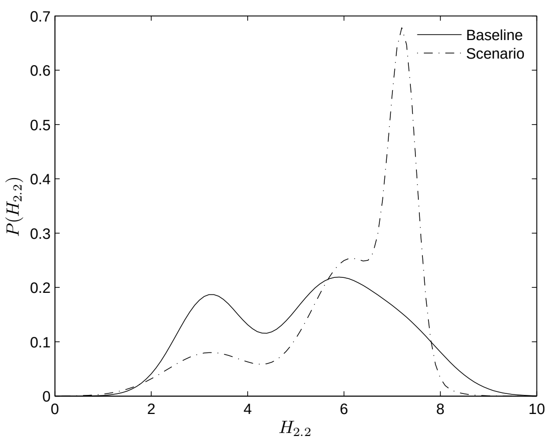
These Fuzzy hazard values also verify the shift towards a higher value as the result of the scenario introduced to the model.



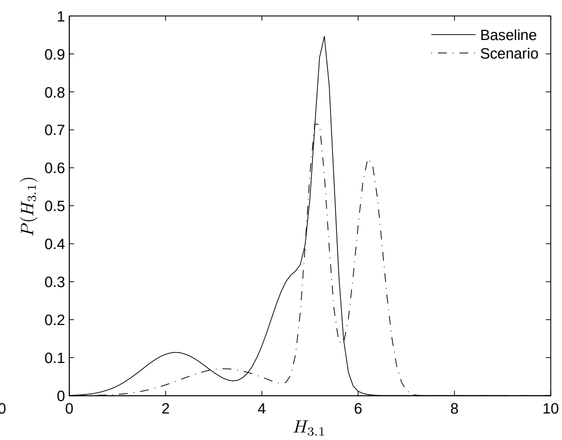
(a) Marginal $p.d.f$ of $H_{1.2}$ Control Station



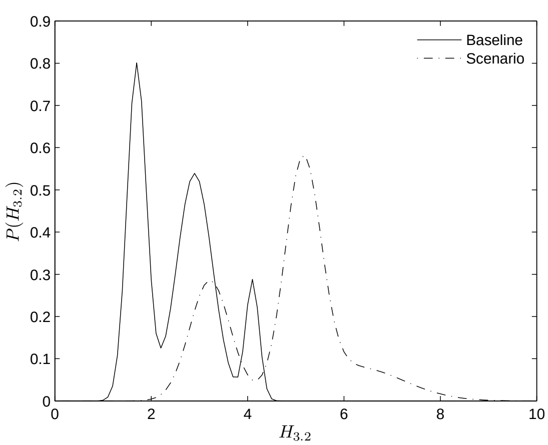
(b) $H_{2.1}$: Individual HF



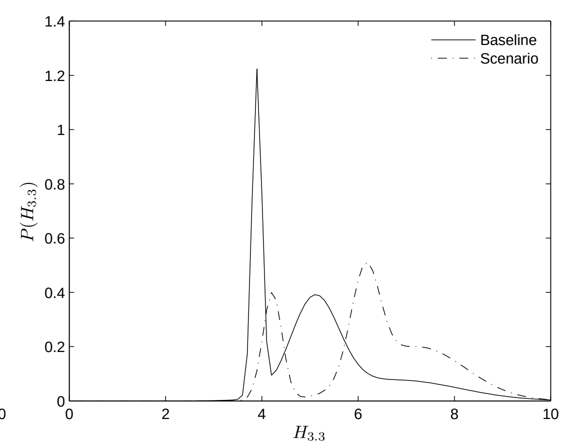
(c) $H_{2.2}$: Operational HF



(d) $H_{3.1}$: Flight Operations

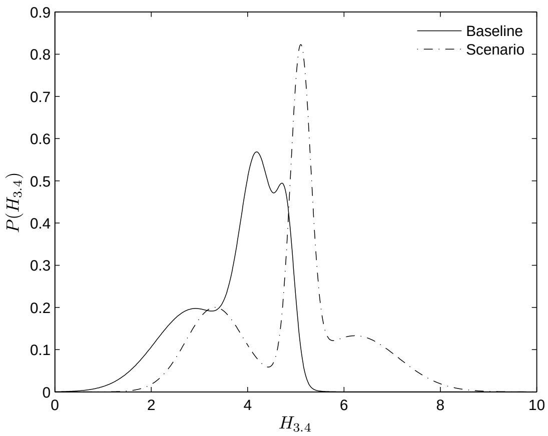


(e) $H_{3.2}$:Continued Airworthiness

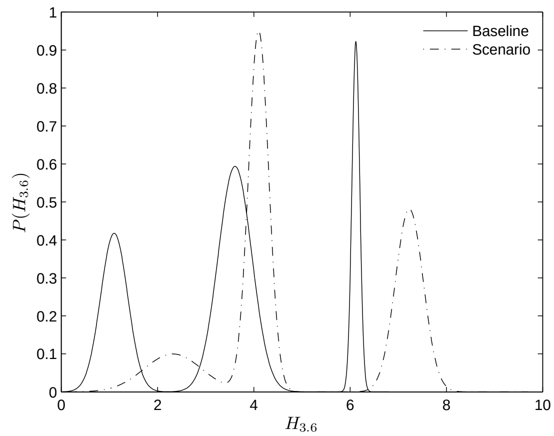


(f) $H_{3.3}$: ATC Communications

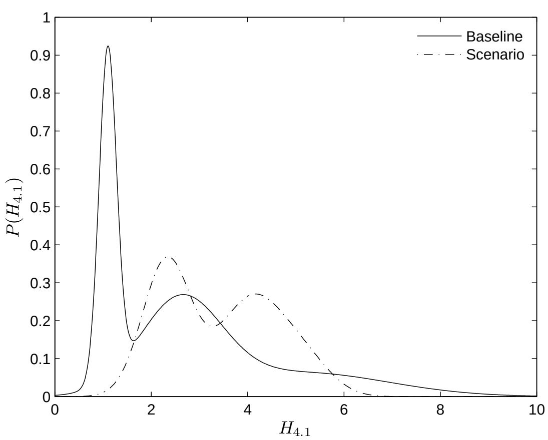
Figure 4.29: Marginal Probability Density Functions ($p.d.f$) of *Type-I* UAS DSRM Hazards - Part 1. (Note that the vertical axes are not on different scale)



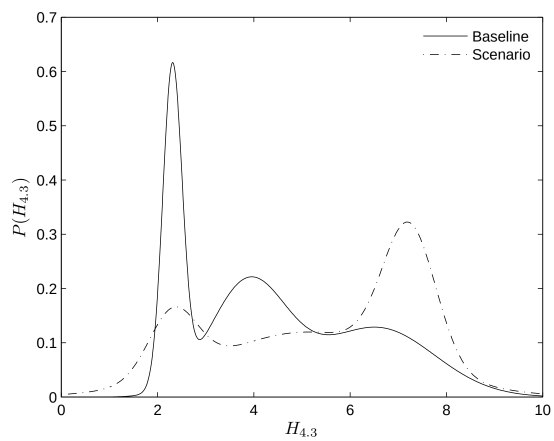
(a) $H_{3.4}$: Airspace



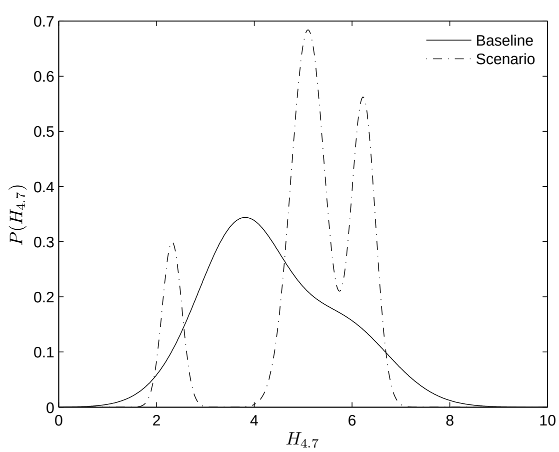
(b) $H_{3.6}$: Organizational HF



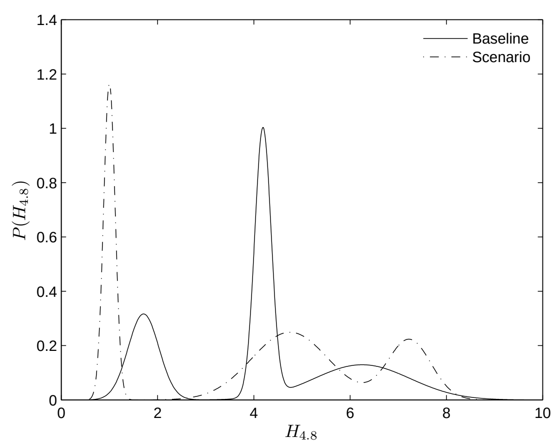
(c) $H_{4.1}$: Terrain



(d) $H_{4.3}$: Weather



(e) $H_{4.7}$: Obstacles



(f) $H_{4.8}$: Other Traffic

Figure 4.30: Marginal Probability Density Functions ($p.d.f$) of *Type-I* UAS DSRM Hazards - Part 2. (Note that the vertical axes are not on different scale)

4.5.2 The *Type-II* FBN and Modeling Results

Recalling the Fuzzy-Bayesian framework outlined in Chapter 3, we introduced two different forms of transformation of a general HBN into a FBN, namely *Type-I* and *Type-II* FBNs, and developed two exact inferencing algorithms to perform reasoning about the resulting hybrid networks.

In this section, we focus on the *Type-II* FBN transformation of the UAS DSRM depicted in Figure 4.23 and present the results of the *Type-II* inferencing in a similar fashion as in the preceding section.

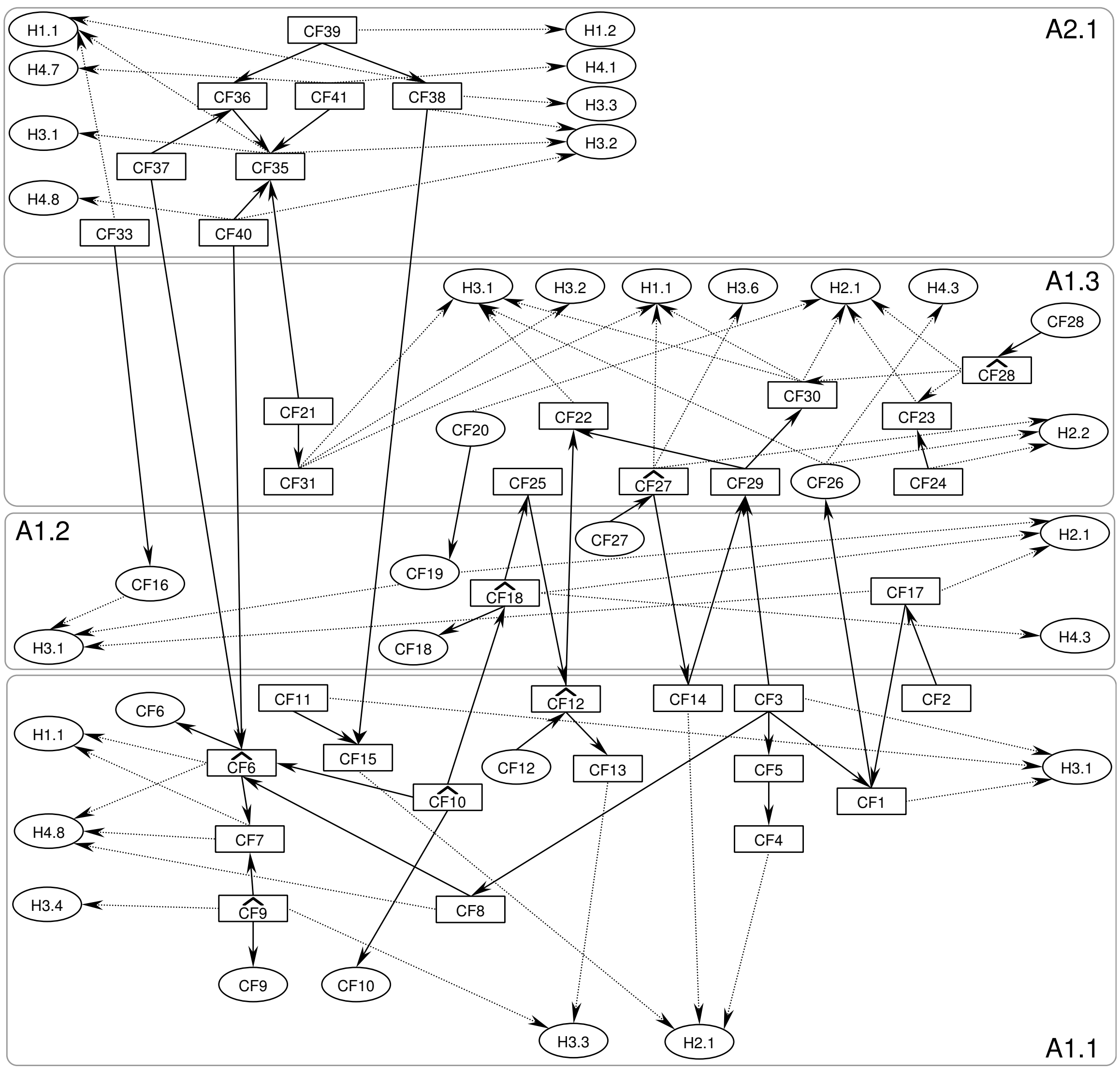


Figure 4.31: *Type-II* FBN $\mathcal{G}_{\mathrm{FBN}^{II}}$ of the UAS Domain Risk Model as the result of the transformation.

As outlined in Definition 7, a Hybrid Bayesian Network $\mathcal{G}_{\text{HBN}}$ defined by equation (3.35), is transformed into a *Type-II* FBN $\mathcal{G}_{\text{FBN}^{II}}$, by applying the fuzzification transformation as defined for Type-I transformation, but only to those continuous variables whose descendants in the original HBN include discrete variables. The *Type-II* FBN $\mathcal{G}_{\text{FBN}^{II}}$ as the results of Type-II transformation of the UAS DSRM is provided in Figure 4.31.

Note that in the original HBN of Figure 4.23, the continuous variables whose descendants include discrete nodes constitute Set $\mathbf{\Gamma}_2$, $\mathbf{\Gamma}_2 \subset \mathbf{\Gamma}$, such that

$$\mathbf{\Gamma}_2 = \{CF_6,\ CF_9,\ CF_{10},\ CF_{12},\ CF_{18},\ CF_{27},\ CF_{28}\}.$$

Additionally, since all hazard nodes are terminating nodes, i.e., sinks, they have parents, but no descendants. Therefore, the Fuzzification transformation is only applied to the continuous variables in $\mathbf{\Gamma}_2$ to obtain the counterpart Fuzzy variables which constitute Set $\widehat{\mathbf{\Gamma}}_2$, such that

$$\widehat{\mathbf{\Gamma}}_2 = \{\widehat{CF}_6,\ \widehat{CF}_9,\ \widehat{CF}_{10},\ \widehat{CF}_{12},\ \widehat{CF}_{18},\ \widehat{CF}_{27},\ \widehat{CF}_{28}\}.$$

Although the topology of *Type-I* and *Type-II* FBNs are quite different, the conditional dependencies between child and parent nodes demonstrate identical attributes. In particular, the Fuzzy counterpart variables in Set $\mathbf{\Gamma}_2$ are generated by equation (4.7) using the same Fuzzy states and membership functions identified in Table 4.7 for the *Type-I* UAS DSRM. Furthermore, the CPD of continuous variable CF_j given Fuzzy variable $\widehat{CF}_j$, where $CF_j \in \mathbf{\Gamma}_2$ and $\widehat{CF}_j \in \widehat{\mathbf{\Gamma}}_2$, is identified by equation (4.8) using the same mean and variance values in Figure 4.8 identified for the *Type-I* FBN.

Finally, the CPDs for the remaining nodes including the Fuzzy variables in Set $\hat{\mathbf{\Gamma}}_2$ and continuous Hazard nodes, needed to finalize populating the *Type-II* FBN in Figure 4.31, follow one of the same six types of conditional dependencies identified while populating the original HBN in Section 4.4.2. Therefore, essentially, the same set of data with minor modifications is used to populate the *Type-II* FBN of the UAS DSRM and we are not going to provide this data here. However, the reader may refer to Sections 4.4.2 and 4.5.1 for the details of specific types and forms of the data used.

Similar to the *Type-I* case, as the outcome of the *Type-II* UAS DSRM, we concentrate on the marginal densities of individual hazards elements identified in the original risk model.

These marginal densities, determined by applying the *Type-II* inferencing algorithm to the *Type-II* FBN are provided in Figures 4.32, 4.33, and 4.34.

The details of the *Type-II* propagation are not provided here due to the same reasoning as for the *Type-I* case, however, the reader may refer to Section 3.4.3 for a detailed discussion on and application of the exact inferencing algorithm for *Type-II* FBNs.

The results for the *Type-II* UAS DSRM entails two marginal densities presented in a similar fashion as for the *Type-I* FBN: a marginal density for the hazard associated with the Baseline *Type-II* model and a marginal density determined after evidence is introduced to the network. The scenario outlined in Table 4.9 is used as the evidence.

Figure 4.32 illustrates the marginal densities for hazard $H_{1.1}$ "*Aircraft Design Related Hazards*" computed by the *Type-II* propagation for the baseline case and for the scenario. Note that there is an increase in the medium hazard value as well as in the variance of hazard density after the evidence is introduced to the model.

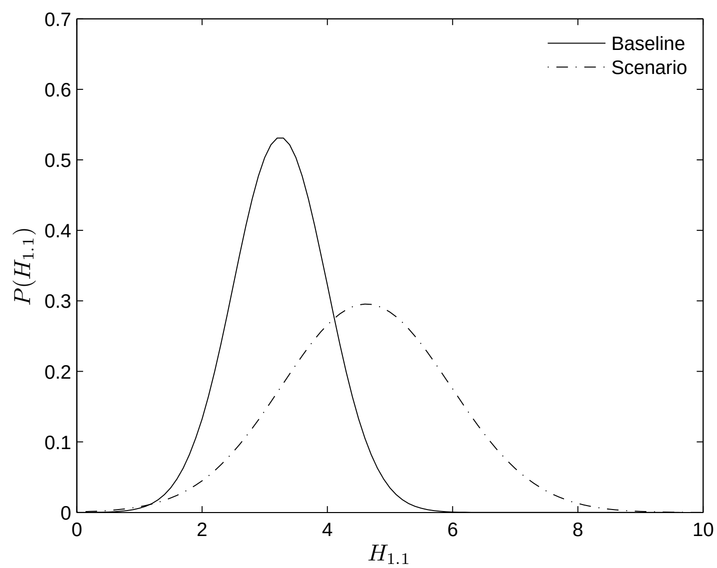
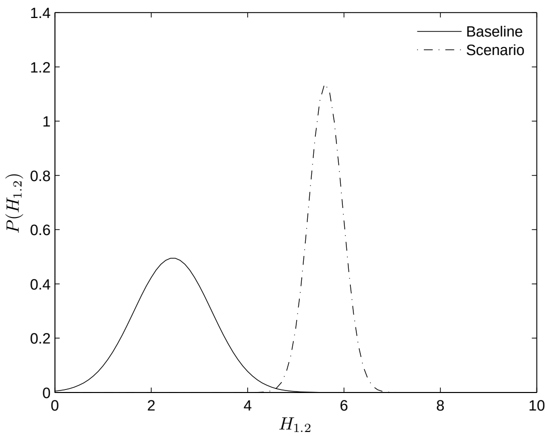
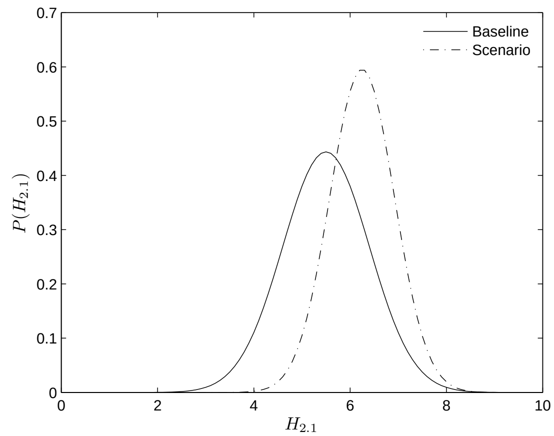


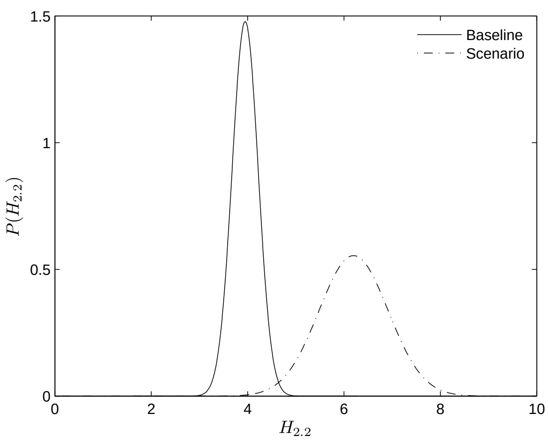
Figure 4.32: Marginal Probability Density Functions of Hazard $H_{1.1}$ "*Aircraft Design Related Hazards*" of the *Type-II FBN* when the model is initiated with the initial CPDs (i.e., *Baseline Case*) and when the evidence as a scenario is introduced to the model (i.e., *Scenario Case*).



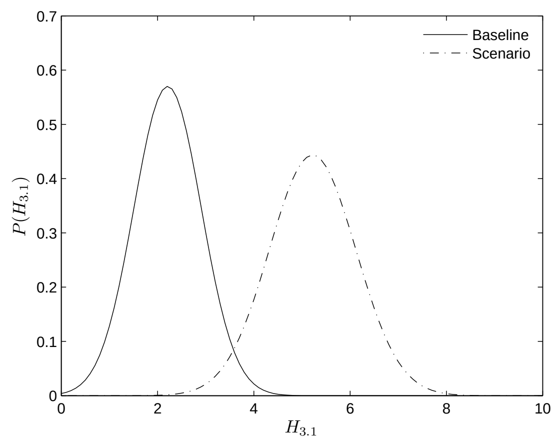
(a) Marginal $p.d.f$ of $H_{1.2}$ Control Station



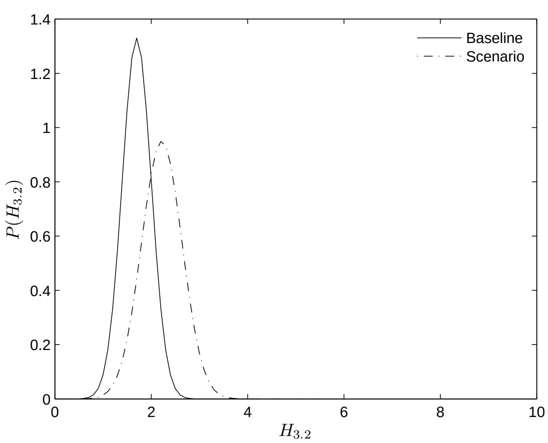
(b) $H_{2.1}$: Individual HF



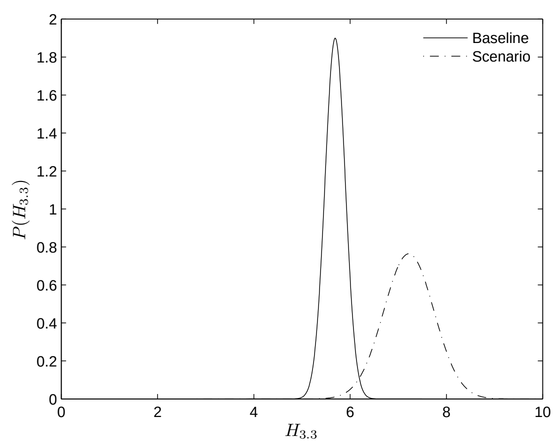
(c) $H_{2.2}$: Operational HF



(d) $H_{3.1}$: Flight Operations

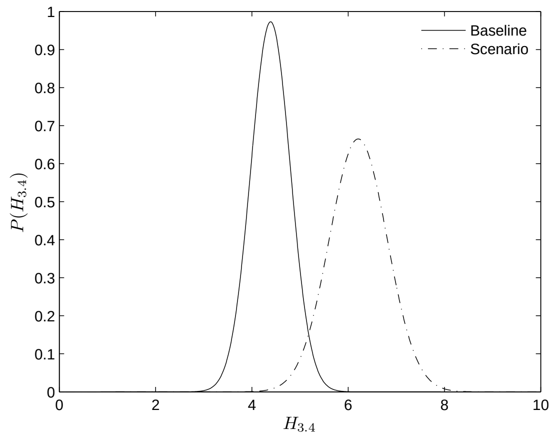


(e) $H_{3.2}$:Continued Airworthiness

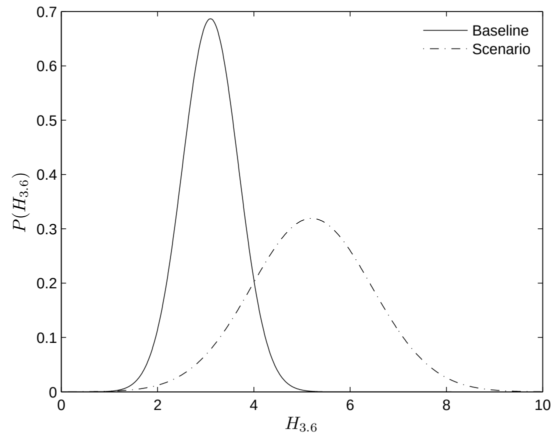


(f) $H_{3.3}$: ATC Communications

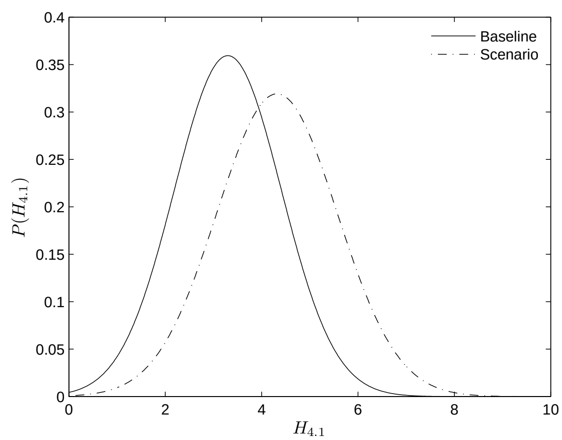
Figure 4.33: Marginal Probability Density Functions ($p.d.f$) of *Type-II* UAS DSRM Hazards - Part 1. (Note that the vertical axes are not on different scale)



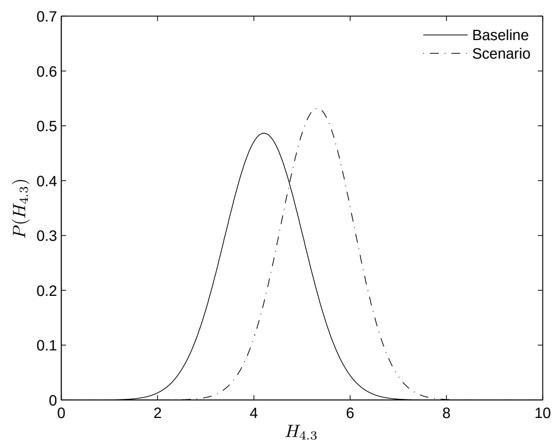
(a) $H_{3.4}$: Airspace



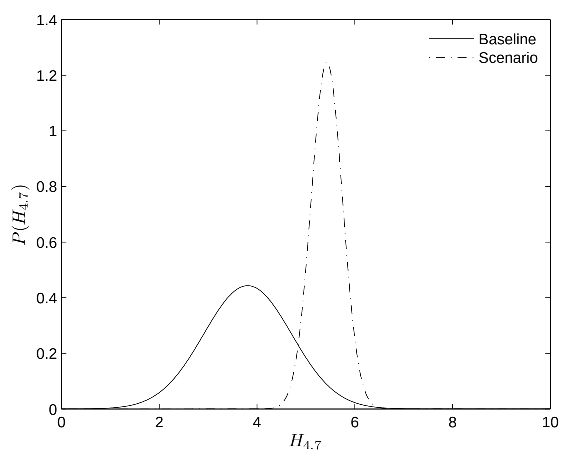
(b) $H_{3.6}$: Organizational HF



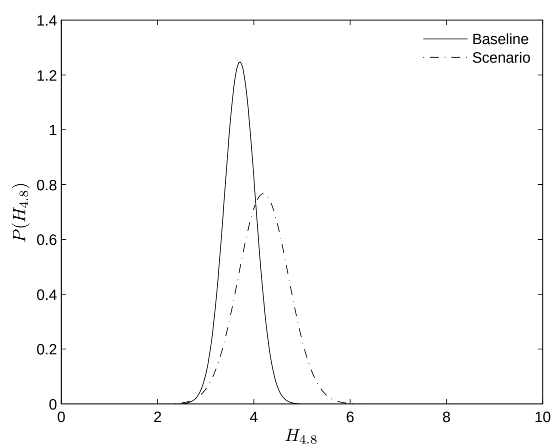
(c) $H_{4.1}$: Terrain



(d) $H_{4.3}$: Weather



(e) $H_{4.7}$: Obstacles



(f) $H_{4.8}$: Other Traffic

Figure 4.34: Marginal Probability Density Functions (*p.d.f*) of *Type-II* UAS DSRM Hazards - Part 2. (Note that the vertical axes are not on different scale)

4.5.3 A Discussion on the Results

Even a cursory look to compare the results of *Type-II* propagation depicted in Figures 4.32, 4.33, and 4.34 and the results of the *Type-I* propagation provided in Figures 4.28, 4.29, and 4.30 indicates a fundamental difference between the marginal densities of the hazards. Particularly, the marginal densities of individual hazards elements emerge as ordinary Gaussians (i.e., Normal distributions) in the *Type-II* UAS DSRM, whereas, the marginals for the *Type-I* model are a mixture of three Gaussians.

When performing inferencing on the *Type-I* UAS DSRM, the original continuous variables hazard $H_{i.j} \in \mathbf{\Gamma}$ and causal factor $CF_i \in \mathbf{\Gamma}$ defined by equations (4.11) and (4.6), respectively, occur only in special cliques $\hat{C}_{\hat{\Gamma}} = \{\hat{\Gamma}, \Gamma\}$, where the original continuous variables are paired with their Fuzzy-Discrete counterparts $\widehat{H}_{i.j} \in \hat{\mathbf{\Gamma}}$ and $\widehat{C}_i \in \hat{\mathbf{\Gamma}}$, such that

$$\widehat{C}_{\widehat{H}_{i.j}} = \{\widehat{H}_{i.j}, H_{i.j}\} \text{ and } \widehat{C}_{\widehat{CF}_i} = \{\widehat{CF}_i, CF_i\}. \tag{4.13}$$

As discussed in detail in Section 3.4.2, these special cliques in the Junction Tree of a *Type-I* FBN emerge due the location of the original continuous variables at the periphery of the network after the *Type-I* transformation. In particular, the topography of a *Type-I* FBN can be rearranged into a *star* shape such that all discrete variables, including the newly created Fuzzy-discrete ones, constitute the core of the star, whereas the continuous variables are located only at the tip of its arms (see Figure 3.6).

Once data is entered into the *Type-I* FBN and the associated junction tree is initiated by passing a message back and forth, for each clique, including the ones in equation (4.13), a probability distribution representing the joint distribution of the variables that the clique is composed of appears. The marginal distributions of the individual variables making up the clique are determined using this joint distribution.

According to the Conditional Gaussian (CG) model at the crux of our *Type-I* inferencing algorithm, the CPD of a continuous variable is a linear CPD given any combination of discrete parents (see equation (3.123)) and if all the discrete variables are given then the CPDs of the continuous variables are all linear CPDs. Thus, given any assignment of the discrete variables a CG is reduced to a Linear Gaussian (LG) and therefore represents a Normal distribution. It follows that the CPD of a continuous variable given an instantiation of its Fuzzy counter part (i.e., given one of its Fuzzy states) is a Normal distribution defined by equation (3.123). According to the *Type-I* algorithm, this CPD of the continuous variable dictates the form of the final joint distribution of the continuous variable and its Fuzzy-counterpart in the special peripheral cliques, defined by equation (4.13), after the junction tree is initiated or evidence is introduced. Thus, the marginal distribution of the continuous variable X is calculated by summing out the Fuzzy-discrete counterpart $\widehat{X}$ over the joint CLG as indicated by equation (3.101) in the numerical example in Section 3.4.2.

As illustrated in Figure 3.13, this marginal density is in the form of a mixture of Gaussians and the number of Normal distributions in the mixture is dictated by the number of Fuzzy states used to fuzzify the original continuous variable X and to create its counterpart Fuzzy-discrete variable $\hat{X}$.

Likewise, the marginal distribution for a hazard node $H_{i.j}$ defined by equation (4.10) is calculated by summing out the Fuzzy-discrete counterpart variable $\widehat{H}_{i.j}$ given by equation (4.11) over the joint distribution $f(\widehat{C}_{\widehat{H}_{i.j}})$ of clique $\widehat{C}_{\widehat{H}_{i.j}}$ after the network is initiated or evidence is entered, such that

$$f(\widehat{C}_{\widehat{H}_{i.j}}) = f(H_{i.j}, \widehat{H}_{i.j}) = \begin{array}{c|l} H_{i.j}, \widehat{H}_{i.j} & \\ \hline h, \widehat{H}_{i.j_1} & w_1 \cdot \mathcal{N}(H; \mu_{i_1}, \sigma_{i_1}) \\ h, \widehat{H}_{i.j_2} & w_2 \cdot \mathcal{N}(H; \mu_{i_2}, \sigma_{i_2}) \\ h, \widehat{H}_{i.j_3} & w_3 \cdot \mathcal{N}(H; \mu_{i_3}, \sigma_{i_3}) \end{array} \tag{4.14}$$

and

$$\begin{aligned} f(H_{i.j}) &= \sum_{\widehat{H}_{i.j}} f(\widehat{C}_{\widehat{H}_{i.j}}) \\ &= w_1 \cdot \mathcal{N}(H; \mu_{i_1}, \sigma_{i_1}) + w_2 \cdot \mathcal{N}(H; \mu_{i_2}, \sigma_{i_2}) + w_3 \cdot \mathcal{N}(H; \mu_{i_3}, \sigma_{i_3}) \end{aligned} \tag{4.15}$$

Since, the same three Fuzzy states given by equation (4.11) are used to create the Fuzzy-discrete counterparts of the original hazard variables, their marginals, defined by equation (4.15) illustrated by Figures 4.28, 4.29, and 4.30 are mixtures of three Gaussians. This is the only place throughout the *Type-I* propagation where we have to deal with a mixture of Gaussians. Thus we do not have to perform basic operations of message-passing such as extension, multiplication or marginalization with mixtures of Gaussians.

On the other hand, this is not the case for the *Type-II* transformation. The junction tree of the FBN after the *Type-II* transformation can also be arranged into a star-shape, where, similar to the *Type-I* case, we use a CG model to represent the hybrid cliques which appear at the tip of the arms of the star (see Figures 3.14b and 3.15). However, the core of the star denoted by HBN_{sub} in Figure 3.14b, contrary to the *Type-I* case, is itself a Hybrid Bayesian Network, where continuous variables do not have discrete descendants. To represent the conditional distributions of these continuous variables given their hybrid parents in the *Type-II* FBN we use a CGR model. As we showed in Section 3.4.3 while developing the *Type-II* inferencing algorithm, the joint probability distribution represented by CGR is a mixture of Gaussians, thus the potential representing the joint distribution of a hybrid clique in the junction tree $\mathcal{T}_{HBN_{sub}}$ is also a mixture of Normal distributions. However, the use of mixtures for message passing in $\mathcal{T}_{HBN_{sub}}$ of the *Type-II* FBN is problematic and as presented in Section 3.4.3, we use an alternative approach to the mixture representation. Our approach involves collapsing the original Gaussians of the mixture to obtain just one Gaussian as the representation of the actual mixture density. Consequently, for these continuous variables whose conditional distributions are modeled by a CGR model, the *Type-II* inferencing outlined in Section 3.4.3 results in *univariate Normal* distributions as their marginal densities. Since, according to the *Type-II* transformation scheme, none of the hazard nodes do require a Fuzzy transformation, their conditional distributions as continuous variables given hybrid parentage are modeled by CGR. Thus, the hazard nodes appear inside the core junction tree $\mathcal{T}_{HBN_{sub}}$ and the joint distributions of the hybrid cliques within which they appear are given by mixtures of Gaussians. In order to perform inferencing in the *Type-II* FBN of the UAS DSRM, we approximate these mixtures by collapsing them into a single Normal distribution. Thus, *Type-II* propagation on the UAS DSRM give rise to a set of univariate Gaussians as the hazard marginal probability density functions provided in Figures 4.32, 4.33, and 4.34 as the results of the *Type-II* UAS DSRM.

Above, we provide an explanation for the basic form difference between the marginal densities of the results for the *Type-I* and *Type-II* inferencing about the UAS DSRM. Next, we elaborate on how to compare their results quantitatively.

An intuitive way of comparing multiple probability distributions is to compare associated mean and variances, however, provided that the distributions are of the same kind. Thus, the results of the *Type-I* and *Type-II* inferencing of the UAS DSRM give rise to densities that need some further manipulation for an easy quantitative comparison. This manipulation entails *collapsing* the mixture of Gaussians to obtain a single univariate Normal distribution representing the original mixture. Before going into detail, however, let us elaborate on mixtures on Gaussians.

As we have already outlined in Section 3.4.3 while developing the *Type-II* inferencing mechanism, a mixture of Gaussians over the variables $\mathbf{X}$ can be represented formally as a set of pairs $(\omega_i, \mathcal{N}(\mathbf{X}; \boldsymbol{\mu}_i, \Sigma_i))$ where ω_i is the weight of the i-th mixture component. If $\sum_i \omega_i = 1$ we say that the mixture is *normalized* in which case it represents a probability density function: with probability ω_i, $\mathbf{X}$ has the normal distribution $\mathcal{N}(\mathbf{X}; \boldsymbol{\mu}_i, \Sigma_i)$.

We approximate a mixture with a Gaussian which has the same mean vector and covariance matrix as the entire mixture. The Theorem 3.4.2 provided in Section 3.4.3 presents a formalism for collapsing a mixture of Gaussians. If applied to the results of the *Type-I* inferencing, Theorem 3.4.2 provides a tool to collapse the marginal densities presented in

Table 4.10: Mean and variance values for the marginal probability densities of the individual hazard nodes $H_{i.j}$ of the UAS DSRM as the result of the *Type-I* and *Type-II* propagation.

	μ				σ			
	Baseline		Scenario		Baseline		Scenario	
	Type-I	Type-II	Type-I	Type-II	Type-I	Type-II	Type-I	Type-II
$H_{1.1}$	3.92	3.25	5.51	4.62	0.62	0.75	0.70	1.35
$H_{1.2}$	3.24	2.45	4.36	5.62	0.64	0.81	0.85	.35
$H_{2.1}$	4.33	5.52	5.66	6.25	0.62	0.91	0.64	0.67
$H_{2.2}$	3.50	3.95	5.83	6.20	0.41	0.27	0.93	0.72
$H_{3.1}$	4.43	2.21	5.32	5.22	0.42	0.72	0.35	0.93
$H_{3.2}$	2.54	1.70	4.79	2.23	0.28	0.30	0.51	0.42
$H_{3.3}$	5.16	5.69	6.30	7.22	0.57	0.21	0.68	0.52
$H_{3.4}$	3.77	4.39	4.90	6.21	0.51	0.41	0.53	0.61
$H_{3.6}$	3.34	3.16	4.93	5.20	0.28	0.58	0.30	1.25
$H_{4.1}$	3.25	3.34	3.41	4.32	1.03	1.11	0.59	1.37
$H_{4.3}$	4.84	4.21	5.94	5.47	0.86	0.82	1.47	0.75
$H_{4.7}$	4.63	3.81	5.59	5.43	0.99	0.93	0.33	0.32
$H_{4.8}$	4.23	3.76	4.53	4.21	0.48	0.32	0.55	0.57

Figures 4.28, 4.29, and 4.30 to compute mean and variance values representing each mixture.

In Table 4.10, the mean and variance values after collapsing the marginal probability density functions of the hazard nodes are provided for both *Type-I* and *Type-II* FBNs of the UAS DSRM for the purposes of comparison.

4.6 A Discussion on Validation

This section provide a discussion regarding the question of validation within the context of our methodology for UAS safety risk modeling outlined throughout this Chapter.

Broadly speaking, any attempt to model a complex system or a real phenomenon results in an approximation of reality with varying degrees of veracity. Thus, a study to compare and contrast the results of the model and the reality is generally considered as important aspect of the modeling methodology. UAS DSRM, for which inferencing results are presented in the preceding Sections is the product of the Regulatory-based Causal Factor Framework (RCFF) developed to model safety risk associated with aviation operations.

In essence, the RCFF and the UAS DSRM are decision support tools and, as a general practice, decision support tools are validated by comparing the results of test runs against preexisting data or expert judgment. Although the role and importance of validating decision support systems are well documented, in the early literature on decision support systems [82, 83], there does not exist a definition shared by the majority for their validation, verification, and evaluation. In a review paper on decision support systems validation, Balci and Sargent indicates that over 16 terms are used somewhat interchangeably [84]. A more recent attempt by Gonzales and Barr [85] tries to clarify the meaning of the terms validation and verification and provides an extensive review on their implementation in the field of contemporary decision support system development.

Based on their experiences of developing decision support systems and using validation methods in computer-based modeling, O'Keefe et al. mention the following major questions concerning validation [86]: what to validate, what to validate against, and what to validate with.

Regarding to what to validate, one can validate any intermediate results, the final result (often called conclusion), the reasoning of the system, or any combination of the three [83]. Having said that, it may be difficult, even impossible, to classify the results of a decision support system as "right" or "wrong". For such situations, we can use the experts to classify the level of accuracy of the solution presented by the system into several categories [87].

Decision support systems can be validated against known results as well as against expert knowledge [64]. The best case scenario would be the existence of many documented previous cases or scenarios, covering the whole spectrum of the problem domain. However, typically, only a limited number of test cases are available. The concept of validation should not be considered as a binary variable in which decision support systems are absolutely valid or absolutely invalid. Since the decision support systems are representations or abstractions of reality pertinent to a problem domain, we cannot expect perfect performance. The performance level acceptable to the users of the system is called acceptable performance range and can be determined during the phases of model development [86].

Every decision support system is designed for a specific purpose or application and its veracity should only be assessed in terms of that purpose with regard to a predetermined problem domain. In other words, an decision support system can be valid for one input domain and completely irrelevant for another.

The literature on decision support systems defines two types of validation: qualitative and quantitative validation. O'Keefe mentions seven widely accepted quantitative validation techniques: face validation, predictive validation, Turing tests, field tests, subsystem validation, sensitivity analysis, and visual interaction [86]. For example, as a popular technique, in face validation, system developers, intended users of the system, and experts on the problem domain subjectively compare systems performance against human expert performance.

On the other hand, qualitative validation entails subjective comparisons of performance. This does not imply that such approaches are informal, we can find highly formal qualitative validation methods [87] in literature. For an introductory example of qualitative validation on a CBR system for modeling aviation accidents, the reader may refer to [64], where the performance of the CBR system is subjectively evaluated by human experts.

If the results of the decision support system can be quantified, then we can employ quantitative (i.e., statistical) validation techniques. Where appropriate, qualitative and quantitative methods can also be combined.

Within the framework of these general concepts about validating decision support systems as outlined above, we can now elaborate on the question of validation on the UAS DSRM and, particularly, on the RCFF.

In Sections 4.1 and 4.2 we discuss the emergent nature of UAS and mention that, as far as the information on UAS related causal factors and hazards are concerned, data on UAS operations is practically nonexistent. Coupled with our usage of synthetic data to populate the UAS DSRM, this lack of historical data renders the applicability of quantitative validation techniques such as face validation impossible. What about qualitative techniques? Provided that the data used is not synthetic, a semi-formal qualitative validation approach could have been applied where the results of the UAS DRSM are evaluated subjectively by experts, who has not involved with the development of the model. However the results of this application are based pure on synthetic data and developed solely for the purposes of demonstrating the applicability of our Fuzzy-Bayesian hybrid inferencing methodology to a real-world problem domain. Therefore unless populated by real data, including both actuarial data and expert judgments, any further discussion on validation is premature. However, we can present a concise argument on the repeatability of the regulatory-based modeling results presented as the UAS DSRM. A closer look to the backgrounds of the SMEs whose knowledge has been primarily utilized to construct the UAS domain safety risk model indicates a rather wide coverage in terms of expertise in and understanding of the problem domain. We believe this diversity of experience resulted in a UAS DSRM model with sufficient representative power so that the modeling results should be considered repeatable provided that a similarly diverse group of SMEs is tasked with the developed modeling framework. The background summaries for the SMEs whom we have worked with in this thesis are provided in Appendix A without exposing their actual identities.

From our experience with much smaller discrete only BN models of aviation accidents [58, 61, 62, 63, 64], we can foresee that populating the UAS DSRM with real data will require numerous knowledge elicitation sessions as well as an extensive effort to collect field data on UAS accidents/incidents. Such a study, which requires time, resources, and most importantly access to UAS operations, is considered as a possible avenue of future work to improve upon this study.

We conclude our discussion on validation by mentioning the fact that the HCAS taxonomy and the RCFF presented in Sections 4.2 and 4.3 have been vetted by aviation experts with system safety research backgrounds as well as by the larger academic community through various FAA program review meetings and conference proceedings [68, 67, 60, 52, 91, 90].

Chapter 5

Conclusions and Future Work

5.1 Summary

Discrete Bayesian Networks provide a formal framework for representing a joint probability distribution given a set of discrete random variables. Exact inferencing solutions for discrete Bayesian Networks, such as the well-known junction tree algorithm by Lauritzen, exist and are well understood. Contrary to the discrete-only case, a general solution for exact inferencing about general Hybrid Bayesian Networks, where continuous and discrete variables may appear anywhere within the network topology, has yet to be developed. Unfortunately, the general HBNs, as the generalization of the BN concept, provide a better modeling representation for the vast majority of real-world applications. Thus, there exist a real demand among the practitioners of uncertainty analysis and modeling of real-world applications for a general operational solution for the representation of and inferencing about general HBNs.

In this thesis, we focus on the larger problem of inferencing in general Hybrid Bayesian Networks. In particular, we have tried to achieve two main objectives:

- Develop a complete formal theoretical framework for exact inferencing on general HBNs using a Fuzzy-Bayesian approach and,
- Demonstrate applicability of the resulting hybrid methodology to perform uncertainty analysis for a real-world complex system.

In Chapter 2, we started our discussions with presenting an overview of discrete Bayesian Networks and providing a detailed review of exact inferencing schemes such as variable elimination and junction tree algorithms as they apply to discrete-only networks. Discrete Bayesian Networks, as a fully-developed and well-understood research area, provide a practical, yet powerful approximation for modeling conditional dependencies among system variables and for performing uncertainty analysis of real-world systems. However, their discrete-only structure constitute a major limitation to how close the final network can approximate the conditional interactions related to the often-continuous nature of real-world phenomenon by using a discrete only representation. General Hybrid Bayesian Networks as a generalization of the discrete-only networks provide a solution to overcome this limitation of representation. We extended our discussions in Chapter 2 by reviewing a constrained form of general Hybrid Bayesian Networks as well as by providing an overview of the theoretical foundation which enables inferencing about this constrained form possible. We showed that a multivariate Gaussian can be converted to a Bayesian Network and a constrained form of general Hybrid Bayesian Networks can be represented by linear Gaussians if all conditional dependencies are modeled by linear CPDs. Some of the insights for discrete Bayesian Networks can also be used for these hybrid networks but there are some important differences, which dictate a particular ordering inside their network topology. In particular, we saw that the junction tree algorithm can be extended for exact inferencing about HBNs, provided that the joint distribution of the continuous variables given an instantiation of the discrete variables is assumed to be multivariate Gaussian. Within this context, we outlined the Lauritzen algorithm for these constrained hybrid networks, where conditional distribution of continuous variables given a set of discrete parents is modeled by a CG distribution. Due to this modeling assumption of the Lauritzen algorithm, which is the current state of the art probabilistic inferencing scheme for the HBNs, there is a strict topological limitation such that continuous variables cannot be parents of discrete variables. This asymmetry between continuous and discrete variables limits the applicability of HBNs in general practice and motivates our research.

We concluded Chapter 2 with a general discussion on Fuzzy Sets as they relate to our research objectives. We elaborated on our past experience on risk analysis of commercial civil aviation operations and on modeling aviation accident scenarios using discrete Bayesian Networks and argued that the uncertainty associated with complex systems has two main sources: randomness associated with domain variables and ambiguity associated with their observed states. Thus, we suggested that a hybrid approach which complements Bayesian probability theory with Fuzzy Sets, which are, in essence, a generalization of the classical

Set theory, provides a more complete and realistic representation for modeling complex uncertainty related to real-world applications.

We started Chapter 3 by laying down the theoretical foundations of a novel formal methodology for exact inferencing in general Hybrid Bayesian Networks. Our methodology takes a hybrid approach to the problem of inferencing and utilizes Fuzzy sets to represent the continuous variables and related conditional interactions dictated by the hybrid network. Within this context, we presented the notion of probability of a Fuzzy event and introduced the concept of a Fuzzy random variable. In particular, the probability of a Fuzzy event is a purely stochastic problem where the event itself is vaguely defined i.e., represented by Fuzzy sets. There are two possible approaches one can adopt: The probability of a Fuzzy event is a scalar (i.e., a crisp real number or a measure) or it can be represented by a Fuzzy set. We saw that the latter option necessitates the adoption of an inferencing mechanism based solely on Fuzzy logic, which from an algorithmic point of view is considered to be suboptimal. Whereas, if the probability of a Fuzzy event is assumed to be a scalar measure then the inferencing mechanism could capitalize the well established algorithms developed for probabilistic reasoning methods such as Bayesian Networks. Based on this analysis, we introduced a novel Fuzzy Bayes formulation and outlined a formalism to determine the conditional probability due to the interactions of Fuzzy and crisp variables.

There are two aspects of Bayesian Networks that are still subject to improvement and therefore research: how to represent continuous variables in a general HBN setting and how to deal with uncertain information as evidence? Our Fuzzy Bayes formulation provides a mechanism to represent continuous variables and associated conditional dependencies in a general HBN setting. In Chapter 3 we also presented a formalism for handling uncertain evidence. In particular, we introduced the notion of Fuzzy evidence to incorporate vague or ambiguous information into a Bayesian Network. We defined Fuzzy evidence as a type of uncertain evidence, where observations are presented as Fuzzy sets rather than delta or indicator functions, which place the observed variable in one of the mutually exclusive states. Thus, Fuzzy evidence maps the observation to a set of predetermined Fuzzy states defined on the closed interval $[0, 1]$. We discussed alternative representations of uncertain evidence in current practiced and showed that, as opposed to virtual or likelihood evidence, where uncertainty is presented as a probability distribution, Fuzzy evidence is suitable to be used in conjunction with continuous variables. Consequently, we presented a formal methodology, a new approximate solution, for updating joint probability distributions when Fuzzy evidence about the distribution variables is introduced. We utilized the relative entropy concept of the information theory when formalizing our Fuzzy updating methodology. In particular, we outlined an updating scheme for the prior distribution, where the posterior (or updated) distribution satisfying a constraint set (i.e., Fuzzy evidence) has the minimum relative entropy with respect to the prior distribution. We demonstrated the applicability of our solution for updating with single and multiple uncertain evidences with a detailed numerical example and concluded that, when multiple uncertain evidence is present, simultaneous updating should be preferred over consecutive updating especially when dealing with moderate to large size Bayesian Networks for which the inherent complexity is already known to be high.

After developing the necessary Fuzzy-Bayesian theoretical background, for the remainder of Chapter 3, we focused on formalizing our hybrid methodology for inferencing in general HBNs. We started with introducing general Fuzzy-Bayesian Networks, which, given a general HBN, can be constructed by transforming all continuous variables and associated conditional probability distributions into the Fuzzy domain. We provided explicit formulations to perform these transformations, which require the Fuzzy sets and corresponding membership functions defined on the frames of all continuous variables in the HBN to be given or constructed first. In the resulting FBN, all originally continuous variables are now replaced by their counterpart Fuzzy-discrete variables whose states correspond to the Fuzzy states identified for the original continuous variable for the purposes of this transformation. Furthermore, after the transformation, all conditional distributions in the FBN can be represented by discrete multinomial distributions. It follows that exact inferencing algorithms such as variable elimination or junction tree algorithm for discrete BNs can be applied to perform probabilistic reasoning about general FBNs.

Although, with general FBNs we achieved practical exact inferencing for general HBNs, since only the Fuzzy-discrete transformations, not the original continuous variables, are present in general FBNs, there is still room for improvement to reach a better approximation of the original hybrid network. Therefore, we introduced two new forms of Fuzzy transformation for general HBNs, namely *Type-I* and *Type-II* FBNs, with increased sophistication in their representation of the original HBN and complexity in inferencing. a *Type-I* FBN is created in two consecutive steps. First steps involve, as in a general FBN transformation, the replacement of all continuous variables in the original general HBN with their Fuzzy-discrete counterparts. In the second step, the original continuous variables are added and connected to the network with a directed link originating from their respective Fuzzy-discrete counterpart. The resulting form represents an HBN where continuous variables have only discrete parents, for which we showed that exact inferencing solutions exist. We used a CG model to represent the conditional distributions of the original continuous variables given their Fuzzy-discrete counterparts. We showed that, while building a junction tree from *Type-I* FBNs, this pair of variables, namely, the original continuous variables and their Fuzzy-discrete transformations, forms a hybrid clique. We demonstrated that the junction tree can be manipulated so that these hybrid cliques are placed at the periphery of the tree, thereby giving rise to a star-shaped formation. At the core of this formation lies a junction tree comprised solely of discrete variables. The message passing in this sub-tree can be performed using existing algorithms. For the message passing between the discrete-only core and the peripheral hybrid cliques, we developed a formal propagation mechanism and demonstrated its application with a numerical example.

As the second form of transformation for general HBNs, we introduced *Type-II* FBNs which involve a finer approximation when representing the original hybrid network as compared to the *Type-I* transformation and hence, present a greater computational challenge. To construct a *Type-II* FBN, the same fuzzy-transformation, as defined for the *Type-I* case, is applied however this time only to those continuous variables whose descendants in the original HBN include discrete variables. For the transformed part of the resulting hybrid network, similar to *Type-I* FBNs, we use a CG to model the conditional distributions of the original continuous variables given their Fuzzy-discrete counterparts. Whereas, for the conditional distribution of the remaining continuous variables in the *Type-II* FNB we use a CGR model. We showed that the junction tree of a *Type-II* FBN can also be rearranged into a star-shaped form, where the hybrid cliques composed only of original continuous variables and their Fuzzy counterparts appear at the periphery of a hybrid core. Furthermore, the continuous variables inside this hybrid core now, do not have discrete descendants.

Message passing between the hybrid core and peripheral hybrid cliques can be handled with the propagation mechanism similar to the one that we developed for *Type-I* FBNs. Whereas the message passing within the hybrid core should be treated separately. We presented a detailed analysis of the types of cliques and their connections making up the core. We saw that the joint probability distribution represented by CGR are a mixture of Gaussians, thus the potential representing the joint distribution of a hybrid clique inside the core is also a mixture of Normal distributions. However the use of mixtures of Gaussians within the context message passing in *Type-II* FBNs is problematic because first, the size of the functions representing cliques in a *Type-II* junction tree is not fixed and second, some operations used in the clique tree algorithm are not defined for mixtures. Therefore we presented an alternative approach to the mixture representation which involves approximation of the mixture by just one Gaussian. The approximation is performed by a collapsing operation. We demonstrated that the collapsed mixture can successfully be used to perform basic operations of message passing. We showed that an exact inferencing solution exits for the hybrid core if associated variables are arranged into cliques to form a junction tree with strong roots. We outlined the decomposability of graphs and strong root concepts and presented a framework to perform message passing in *Type-II* FBNs and concluded Chapter 3 by presenting a calibration algorithm for strongly rooted hybrid junction trees. A comparative analysis of *Type-I* and *Type-II* FBNs are presented in Table 5.1.

In Chapter 4, we applied the research methodology developed in Chapter 3 to a real-world problem domain. Risk analysis of unmanned aircraft systems is the problem domain that we have chosen for the application of our research methodology. As mentioned in Chapter 4, the application component of this research is funded by the UAS research program of the FAA Research and Technology Development Office. Ultimately, our goal with this application was to demonstrate that general HBNs provide a suitable modeling tool-set to perform risk and uncertainty analysis of real-world systems, in general and of emergent UAS operations, in particular. In Chapter 4 we presented the components of a novel approach to

Table 5.1: Comparison of inferencing algorithms for *Type-I* and *Type-II* FBNs

	***Type-I* FBNs**	***Type-II* FBNs**
Representation of general HBNs	• All continuous variables are fuzzified to create Fuzzy-discrete counterparts. • CPDs of continuous variables given Fuzzy-discrete counterparts are assumed to be CG.	• A better approximation of a general HBN. • Only the continuous variables with discrete descendants are fuzzified. • CPDs of continuous variables given Fuzzy-discrete counterpart are assumed to be CG. • CPDs of the remaining continuous variables including the ones with hybrid parentage are modeled by a CGR model. • The joint distributions of the hybrid cliques presented by CGRs are mixtures of Gaussians. • To perform message passing in the *Type-II* junction tree mixtures are approximated by single Gaussians.
Inferencing	• Exact	• Approximate
Computational Complexity	• Comparable to discrete BNs	• High compared to *Type-I* FBNs

system safety analysis for emerging aviation systems/operations, for which current analysis techniques that rely heavily on preexisting data are shown to be inappropriate. As the first component of our approach, we introduced a novel methodology for hazard taxonomy development with particular emphasis on aviation related hazard-source identification and presented the Hazard Classification and Analysis System (HCAS) for the UAS. As the second component, for the identification of causal factors and their possible interactions leading to hazard sources, we presented a new regulation based framework, which employs deductive reasoning and relies on detailed analysis of the current FAA regulations for commercial civil aviation. The third and final component involves the construction of a general HBN representing a preliminary safety risk model for UAS operations limited to the regulatory domain that the regulations selected for this application cover. These components are the result of a multi-year development effort and throughout this long and arduous process, the inputs of a diverse panel of subject matter experts with extensive aviation background are heavily utilized not only for model construction but also for concept development. The meeting and knowledge elicitation sessions conducted as part of this intensive development effort are listed in Table 5.2.

After the UAS Domain Safety Risk Model is constructed as a general HBN, we applied

Table 5.2: The schedule of meetings that have been performed within the context of UAS system safety research.

Schedule of Meetings

Date	Place	# of SMEs	Hours
6/25/09	Telecon	5	1
5/7 - 5/8/09	Rutgers	6	$1\frac{1}{2}$ days
5/12/09	Telecon	5	2
5/30/09	IERC	Approx. 30	Conference Session
4/7/09	Telecon	5	2
3/9/09	Telecon	7	2
2/3 - 2/4/09	Rutgers	10	$1\frac{1}{2}$ days
2/9/09	FAATC	2	2
2/19/09	Telecon	5	2
1/28/09	Telecon	5	2
11/18 - 11/19/09	Colorado	Approx. 40	UAS Program Review
10/7 - 10/8/08	FAATC	5	$1\frac{1}{2}$ days
10/29/08	Telecon	5	2
9/5/08	FAATC	3	1 day
9/17/08	ICAS2008, Alaska	Approx. 40	Conference Session
7/23/08	Rutgers	5	1 day
6/10/08	AUVSI UAS Workshop	Approx. 80	1 day
10/23 - 10/25/07	FAATC	Approx. 50	UAS Program Review
9/19 - 9/07	FAATC	3	$1\frac{1}{2}$ days
8/15/07	ISSC, Baltimore	Approx. 60	Conference Session
8/27/07	FAA HQ	5	$\frac{1}{2}$ day

the Fuzzy-Bayesian framework developed in Chapter 3 to perform inferencing on the UAS DSRM. We used synthetic date to populate the *Type-I* and *Type-II* FBN transformations of the UAS DSRM. As the results of applying respective inferencing algorithms, for *Type-I* and *Type-II* of UAS DSRM, we presented the marginal probability distribution of hazard elements for the baseline case and for a scenario where a collection of synthetic evidence is introduced to the model. Finally, we concluded Chapter 4 with a discussion on the validation of the UAS DSRM model.

5.2 Contributions

In this thesis we concentrate on the problem of inferencing in general Hybrid Bayesian Networks. In particular, we try to understand and tackle the issues that exact inferencing in general HBNs faces. In this context, our contributions to the larger research domain of representation and inferencing in general HBNs are three-fold: theoretical, algorithmic and practical. Specifically, our major contributions can be outlined as follows:

- From a theoretical point of view, we complement classical probability theory with Fuzzy set theory to develop a hybrid formalism to understand and model complex uncertainty associated with real-world systems. To that end, we provide a novel framework to implement a hybrid Fuzzy-Bayesian methodology to perform exact inferencing in general HBNs where continuous and discrete variables may appear anywhere within the network topology.

- From an algorithmic perspective, we provide a suite of inferencing algorithms for general Hybrid Bayesian Networks. In particular, we introduce two transformations for general HBNs to create *Type-I* and *Type-II* Fuzzy-Bayesian Networks and present formal representation techniques and separate inferencing mechanisms for *Type-I* and *Type-II* FBNs.

- Finally, from a practical perspective, we apply our framework, methodology, and techniques to the task of assessing system safety risk due to the introduction of emergent Unmanned Aircraft Systems (UASs) into the National Airspace System (NAS).

We also believe that, as a major contribution to the research domain of system safety analysis, the HCAS taxonomy and the regulatory-based framework developed to identify hazards and causal factors and to model their interactions as a general HBN provides a novel methodology and a valuable tool-set for practitioners to understand, analyze and model system safety risk associated with emerging aviation technologies and operations, such as UASs.

Supporting the major contributions, we can list the following as our minor contributions to research field of complex uncertainty analysis:

- Fuzzy-Bayes Formula: We introduce a new Fuzzy-Bayes formulation to define the conditional probability of a Fuzzy event given another Fuzzy event, which can also be extended to Fuzzy/Crisp and Crisp/Fuzzy variable pairs.

- Fuzzy updating: We present a new approximate solution for updating a joint probability distribution of a set of variables when fuzzy evidence about the state of one or more variable is introduced.

5.3 Limitations and Future Directions

It is our hope that this thesis provides useful tools to overcome the challenges that inferencing with general HBNs present and demonstrates the power of Fuzzy Bayesian Networks while modeling complex uncertainty in real-world applications. Even though our discussions throughout this thesis addressed important questions that arise with FBNs, obviously there is still room for improvement. Here, we discuss the limitations of the developed FBN formalism and review some exiting research directions that build on top of the work presented in this thesis.

In this thesis, while developing our Fuzzy-Bayesian formalism and presenting the fuzzifications needed for HBN-to-FBN transformations, the inferencing algorithms and the results of message propagation, we based our analysis exclusively on Fuzzy sets defined by trapezoidal membership functions. The extension of the Fuzzy-Bayesian framework outlined in this thesis by using Fuzzy sets with general memberships functions will surely prove to be computationally challenging, yet it presents an interesting area for future research.

It is the complete transfer of selected crisp continuous variables to Fuzzy discrete variables that makes exact inferencing in *Type-I* and *Type-II* FBNs possible. However, two compromises were made along the way. First, the Fuzzy membership functions partitioning continuous domains are usually build by using subject matter expert knowledge and therefore are only approximations. Additionally, we also need the marginal distributions for continuous variables undergoing fuzzy transformation. In most cases these marginals do not exits and we can only start the analysis with some initial estimates. We believe it is worthwhile to explore the applicability of an iterative algorithm to improve the accuracy of the initial Fuzzy membership functions and marginal distributions of the continuous variables subject to fuzzification.

Our inferencing mechanism for FBNs relies on two fundamentally important assumptions about the conditional distributions of continuous variables given their parents. In particular we assume that the conditional distributions of continuous variables given discrete parents are modeled by Conditional Gaussians, whereas the conditional distribution of a continuous variable given a hybrid parentage is modeled by a CG regression model. However, in some case, depending on the nature of system domain, these models may not be appropriate to capture and represent the conditional dependencies between system variables.

The application component of this thesis involves the development of the UAS domain safety risk model as a general HBN, the transformation of the resulting hybrid network to *Type-I* and *Type-II* FBNs and ultimately, the application of the developed inferencing algorithms. Throughout this process, as we outlined in Chapter 4, we used synthetic data to populated the *Type-I* and *Type-II* FBN transformations of the UAS DSRM. Although populating a complex hybrid network such as UAS DSRM will obviously require substantial resources as well as special access to operational data, we believe that it present an interesting future research effort to improve on the work in this thesis.

Finally, we presented a discussion on validation, but did not provide a validation study on the UAS DSRM, because of the reasons outlined in Section 4.6. Therefore, once real data is collected for the hybrid model, the development of a formal methodology to validating the UAS DSRM presents an interesting future research direction, which, we believe, will greatly contribute the larger domain of safety and risk modeling of aviation systems.

5.4 Conclusions

It is our hope that this thesis presents a convincing formal argument on the usefulness of Fuzzy Bayesian Networks in understanding and modeling complex uncertainty associated with real-world applications. FBNs provide a more realistic explicit representation of uncertainty by complementing randomness with ambiguity and combine it with enough expressive power to model discrete as well as continuous phenomena in real-world applications. Although we concentrate on the system safety and risk modeling of complex systems as the application of FBNs, there are various problem domains, such as fault diagnosis, pattern matching and recognition, and decision support tools among others, for which FBNs should be considered as perfect fit. In particular, we believe that HBN-based efforts dealing with uncertainty modeling of complex systems that inherently involve conditional dependencies among a hybrid set of domain variables will greatly benefit from the implementation of the Fuzzy-Bayesian framework presented in this thesis.

In this thesis we have answered some fundamental questions on inferencing in HBNs using a hybrid Fuzzy-Bayesian approach. Perhaps even more importantly, we provided a practical application of the hybrid approach by performing uncertainty analysis of a real-world complex system. In particular, we presented a preliminary system safety risk model for emerging UAS operations. We believe that this is a strong indication of FBNs practical relevance going beyond intellectual exercise. We hope that this will serve as motivation for other researchers to further explore the potential of FBNs as well as general HBNs in various other real-world applications.

Appendix A

SME Backgrounds

SME - 1:

SME-1 has over 40 years of aviation experience in a career heavily involved in regulatory control and oversight. He has been involved with and served as the Senior Executive responsible for the oversight of Federal Aviation Regulation implementation and certification of several major airlines, repair facilities, and airmen. He served as advisor to the Associate Administrator for Certification and Regulations pertaining to all areas of aircraft maintenance. SME-1 also served as a Manager in the Aircraft Maintenance Division of the FAA responsible for developing international and domestic airworthiness, rulemaking, and regulatory control. Additional, he was the Assistant Division Manager of Flight Standards, responsible for fourteen Flight Standards District Offices and the regional Operations Branch Manager.

SME - 2:

SME-2 has over 40 years of aviation experience. He served in a variety of aviation safety positions while employed with Federal Aviation Administration for over 25 years. These positions include principal maintenance inspector for a major US Airline at the district office level and assignments with the Safety Analysis and Management Branch in the regional office. SME-2 also served as team leader on special projects such as the initial approval of ETOPS operations. Most recently, SME-2, as Manager of the Air Transportation Branch, was selected for a variety of management positions at the FAA Washington Headquarters. He provided the leadership during the development, implementation, and evaluation of the National Aviation Safety Inspection Program (NASIP). He coordinated inspection activities with the DOD Air Carrier Analysis and Survey Office. He was team leader of many internal regional Flight Standards evaluations. Working in partnership with the aviation industry, he published a wide variety of national policy materials, which included modified regulatory requirements for public use aircraft. He participated with the Air Transportation Association Maintenance Operations Committee and served as the FAA Alternate Executive Director for the Aviation Rulemaking Advisory Committee (ARAC) Air Carrier and General Aviation Maintenance Issues Group.

SME - 3:

SME-3 has over 32 years of aviation experience in a career heavily involved in regulatory control, management and oversight. His most recent assignment was Manager of the Flight Standards Service International Program Staff. During this assignment he was responsible for the management of the International Aviation Safety Assessment program, international harmonization and the newly implemented air carrier code international code-share review program. While in this position SME-3 had program oversight and conducted numerous International Aviation Safety Assessments (IASA) of the civil aviation authorities and represented the FAA in consultations to foreign authorities. He also served as the Assistant Manager of the National Field Office for which he provided management oversight of the National Aviation Safety Inspection Program, FAA inspector training, and safety analysis of daily air carrier operations. He has supported FAA research as a subject matter expert in the development of the FAA Air Carrier Operations Systems Model, and safety risk and hazard assessment programs.

SME - 4:

SME-4 has been a member on numerous FAA regulatory or guidance development teams and has performed numerous safety audits of foreign and domestic airlines and repair stations. He has provided remote and on-site compliance audits and technical support to numerous Federal Aviation Regulations (FAR) Part 145 foreign repair stations in Japan and Taiwan regarding Federal Aviation Administration (FAA) certification requirements. He has assistance for a transfer of several Lockheed L-1011 aircraft from a foreign airline to a domestic airline, including on-site records inspections. SME-4 drafted a rule and related advisory material for the FAA on aging airplane safety. He has developed a generic flight operations manual for use by foreign operators and developed an Advisory Circular (AC) on flight in icing conditions (AC 91-74). He has also worked on FAA research projects in the areas of Continuing Analysis and Surveillance System (CASS) and the accomplished background studies in support of research and advisory circular development for FAA repair station Approved Training Programs.

SME - 5:

SME-5 has 50 years of aviation experience in a career heavily involved in regulatory control and oversight and industry operational experience. His most recent assignment was Vice President of Safety for for a major US airline where he served as the senior executive responsible for the safety management and quality assurance of a scheduled 14 CFR part 121 airline until his retirement in 2004. Prior to his industry position SME-3 retired from the Federal Aviation Administration (FAA)in 1996 where he held a variety of positions with increasing responsibilities, He was a principal operations safety inspector, regional operation specialist and Flight Standards District Office manager in Baton Rough, LA and Dallas TX. He has served as a FAA Principal Operations Inspector for a variety of Part 121 and 135 carriers and supplemental air cargo operators during his career in the FAA and has been team leader and/or participated on numerous regional and national safety evaluation teams. These positions pertained to all areas of FAA's oversight, certification and surveillance of the air transportation system. Prior to joining the Federal Aviation Administration SME-5 retired from the US Army as a Warrant Officer flying a variety of military fixed and rotary wing aircraft.

SME - 6:

SME-6 has a managerial position in the Unmanned Aircraft Systems (UAS) Research program at the United States Federal Aviation Administration (FAA). He has managed FAA research initiatives in a broad range of technical areas including airport pavement technology, airport planning & design, airframe structures, flight control systems, and currently unmanned aircraft systems. He served as the program manager for the FAA Airworthiness Assurance Center of Excellence program, a consortium of 28 universities across the

United States of America with numerous industry partners. In addition, SME-6 had the opportunities to be detailed at Engineering Division of FAA Aircraft Certification Services, FAA International Aviation Office, and the National Transportation Safety Board (NTSB). SME-6 also initiated and managed several FAA joint research projects with international entities including the Chinese Civil Aviation Authority (CAA), the Netherlands CAA, and International Air Transport Association (IATA). He holds a Ph.D. in engineering from Columbia University and worked as a post-doctor research fellow at Princeton University. Thereafter, he worked as a senior research engineer in an engineering consulting firm to provide Systems Engineering and Technical Assistance (SETA) to the United State Air Force Research Laboratories (AFRL) before joining the FAA.

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