AUTOMATED DISCOVERY AND PROOF IN THREE COMBINATORIAL PROBLEMS

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ABSTRACT OF THE DISSERTATION

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In this Ph.D. disseration, I will go over advances I have made in three combinatorial problems. The running theme throughout these three problems is the novel use of computers to aid not only in the discovery of the theorems proved, but also in the proofs themselves. The first problem concerns the quantity $f_\Delta(n)$, defined as the size of the largest subset of $[n]$ avoiding differences in Δ. Originally motivated by the Triangle Conjecture of Schützenberger and Perrin, we again define an enumeration scheme that will find, and prove automatically, the sequence $\{f_\Delta(n)\}_{n=1}^{\infty}$ for any prescribed Δ. Although the Triangle Conjecture has long been refuted, we present an asymptotic version of it and prove it. The second problem involves the enumeration of spanning trees in grid graphs – graphs of the form $G \times P_n$ (or C_n) for arbitrary G. An enumeration scheme is developed based on the partitions of $[n]$, yielding an algorithmic method to completely solve the sequence for any G. These techniques yield a surprising consequence: sequences obtained in this manner are divisibility sequences. The final problem is the firefighter problem, a dynamic graph theory problem modeling the spread of diseases, information, etc. We will present the problem as it applies on the two-dimensional grid and prove new upper and lower bounds, found mainly through computer experimentation.

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Dedication

This thesis is dedicated to my family, and specifically to my wife Audria, who was just my girlfriend when I started graduate school. The first three years apart were rough and rocky, but my love for her grows with each passing day, and I look forward to transferring a good portion of my time spent in grad school to time spent with her.

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Chapter 1

Introduction

It is no coincidence that ENIAC, the first electronic computer built, was frequently described as a "giant" or "electronic" *brain* [32]. The evolution of computers' brains, mathematically speaking, has followed that of humans, although the timeline has been compressed considerably. In human evolution, mathematics has broadly evolved from the concept of number to the concept of computation to the concept of mathematical proof and rigor. Similarly, the evolution for the computer is the same, being that the basic element for a computer is *bit*, or binary digit. All constructs that are used by computers are built up from the bit, whether it be the `unsigned int` or `float` or the `array`. Manipulation of these and other basic computer constructs were obtained through the basic operations and extended through computer methods. Finally, in a body of work that is only about 20 years old, these concepts have been abstracted *inside* the computer to allow the ability to *prove*.

The abstraction away from number and computation in a computer has generally been of two distinct forms. First, there is the notion of *formal proof*, which seeks to harness the computational power of the computer to provide strict proofs of mathematical statements, given in a predefined axiomatic system (usually ZFC or PA). The motivation leading to formal proof is manyfold, but its usefulness is seen when one considers that all proofs read in mathematical journals or explained on the board are not actual proofs in the axiomatic framework, but digests of those that can be read, and more importantly, understood by a human. However, Kempe's flawed proof of the Four-Color Theorem is one of numerous shortcomings of the informal proof method that is widely employed today. This is one of the issues that Formal Proof seeks to remedy. In the near future, according to the belief, mathematicians will be able to include formalized proofs, with the assistance of computer programs like *Isabelle* or *HOL-Light*, with their paper submissions to journals (see [26]).

For the time being, formal proof purveyors spend most of their time finding formal proofs for popular and important theorems, such as the The Prime Number Theorem (for which the author was a contributing member of – see [3]), the Quadratic Reciprocity Theorem, and the Fundamental Theorems of Algebra and Arithmetic. It is a venture the author deems worthwhile and productive for the long-term future of mathematics, not least because he was a participant of the progress at one point, and hopes to be again in the future. This pursuit is worthwhile because it is futile to believe that computers can be separated from mathematics in and real form for any real benefit. One of the main complaints that is given about proof assistants like *Isabelle* or *HOL-Light* is that they are too hard to learn. This is valid, but one that should not scuttle the formal proof debate. Rather, as we see in other areas where computers play a leading role, the initial attempts are not so pleasing to the consumer, yet through refinements a good product is made. For example, Apple's iPod was not the first mp3 music player that was developed; rather, it fixed the shortcomings of previous music players and its success is well-deserved for that. Similarly, the superficially technical shortcomings of formal proof will be fixed. For a good review of the current state of the art of formal proof as of 2009, one only needs to look at the AMS Special Issue on Formal Proof, consisting of [23], [20], [26], and [54].

The second form of abstraction away from pure computation is not an abandonment of computation *per se*, but it is the notion that proofs are, no matter how they are written, a computational notion in their purest form. This gives rise to the notion of *symbolic computation*, which seeks to reduce proofs to statements that can be solved computationally, instead of being proved verbally. A trivial example of the notion of symbolic computation is given in the following example.

Example Prove that there is no quadratic function $f(x)$ satisfying $f(1) = -1, f(2) = 12, f(3) = 31$, and $f(4) = 55$.

The answer is as follows: a quadratic function $f(x) = Ax^2 + Bx + C$ satisfying those four conditions would imply that

$$\begin{aligned} A + B + C &= -1 \\ 4A + 2B + C &= 12 \\ 9A + 3B + C &= 31 \\ 16A + 4B + C &= 55, \end{aligned}$$

which is a system of four equations with three variables. Generally these overloaded systems have no solutions, which is what happens in this case. This example, although somewhat contrived, shows how a "proof" can be reduced down to mere computation, and the fact that a contradiction appears in the computation immediately solves the problem.

A more involved example, and a fundamental theory in the field of symbolic computation so far, is the Wilf-Zeilberger Theory [55] of hypergeometric functions and their implication that every binomial identity can be verified or refuted through an effective algorithm, which is, in its essence, a series of computations. Other examples abound but the reader should realize that neither this author, nor any other respectable mathematician in this field, is trying to do away with theory or arguing that there will be a point in the near future where theory is irrelevant and computers rule the world. On the contrary, theory will always be the driving force. The theory we create is necessary to justify the use of the computers to find out much more than can be found out without computers at all. Behind all of the use of computers in this thesis, there is always an invisible guiding hand behind it, carefully planned out by the author.

This thesis will not attempt to take a stance or argue about which style of mathematics is better, more useful, or most industrious. Mathematics, although universal at its root, is still a human endeavor for the humans that use it and apply it. It will still be shaped and influenced by the people in power, as it was during the Bourbaki Revolution that led to our current period of formal, rigorous, proof-based mathematics. This thesis attempts to straddle the divide between the two by using the computer for what it's best suited for at this stage in its development: pure computation and bookkeeping. Specifically, the main vehicle for intuition and discovery for two of the three problems discussed in this thesis is the *enumeration scheme* which, at the basic level, is a (large) system of interconnected recurrence relations.

The first problem relates to *avoiding* (or *missing*) *differences*. The main problem concerns the number $f_\Delta(n)$, which is defined to be the size of the largest subset X of $[n] = \{1, 2, \ldots, n\}$ such that $\forall x, y \in X, x \neq y \to |x - y| \notin \Delta$. A small amount of work has been previously done, mainly concerning the computation of the number

$$\mu(\Delta) = \lim_{n\to\infty} f_\Delta(n),$$

which originated from a question posed by Motzkin [33]. In this thesis we will effectively give a complete theory of the structure of the sequence $\{f_\Delta(n)\}_{n=0}^{\infty}$ using computational means arising from a complicated enumeration scheme that arises with the consideration of an extra parameter. Additionally, we will discuss three different, but equally important, variations of the number $f_\Delta(n)$ that extend the definition of what it means to avoid a difference while also considering a cyclic version of the number $f_\Delta(n)$. On the way, we will answer new questions and pave new paths toward Szemerédi's Theorem, while also giving an asymptotic version of the long-refuted Triangle Conjecture of Schützenberger and Perrin.

The second problem discussed is the counting of spanning trees of grid graphs. A *spanning tree* is a minimally connected spanning subgraph, and a *grid graph* is a graph of the form $G \times P_n$, where P_n is the *path graph* on n vertices. For any specific graph G and n, the number of spanning trees of $G \times P_n$ can be computed effectively using the Matrix-Tree Theorem of Kirchhoff. However, what is desired, and given in this thesis, is an effective way, given G, of computing the *sequence* of spanning trees of $G \times P_n$ for n from 1 to ∞. The algorithmic method that is developed in this thesis is of the Wilf-Zeilberger flavor in that only a finite amount of computation is needed to obtain all information possible about the complete sequence of integers. A consequence of this method is that it obtains a $O(n)$ algorithm to compute the number of spanning trees of $G \times P_n$; however, the "up-front" charge in this algorithm is potentially astronomically high as to be absurd. Regardless, it is a $O(n)$ algorithm. While most of the results obtained through the process developed in this thesis are not new, thanks to conversations with Richard Guy [22] these methods *easily* and *immediately* admit a combinatorial proof of a fascinating property: all of these sequences that are sequences of spanning trees in grid graphs are *divisibility sequences*, meaning sequences $\{a_n\}_{n=0}^{\infty}$ satisfying $n|m \to a_n|a_m$. Through these methods, we also discover many deep, related relationships between sequences that have not been investigated in any proper manner. These relationships that are given here, and others found by Guy, are "towards a multiplicative theory of divisibility sequences," as he (somewhat redundantly) puts it. Until now, most enumeration schemes were created as a means to find the value of one object by finding the values of many others. While that is done in this thesis, this divisibility example shows that there are important things to be discovered when analyzing the enumeration scheme as an object in its own right, for relationships like the Split-Merge Lemma (Lemma 3.9.2) are consequences that come directly from the enumeration scheme itself, and not any applications of it.

The final problem, although not as related unrelated to the first two, is the *firefighter problem*. Although there are many variations which will be discussed later on, the basic problem involves an underlying graph G and a vertex which is initially *on fire* at time $t = 0$. Then, at each time step, t is incremented, a certain number, $f(t)$, of *firefighters* is placed on vertices of G that are not on fire, and each vertex that is on fire has its fire spread to adjacent vertices that are neither on fire nor protected by a firefighter. This process continues indefinitely and necessarily stops for a finite graph G, where the main question is "what is the least number of vertices that necessarily will catch on fire?" For the case of infinite G the question is still the same, but the more relaxed question to ask is whether or not the process itself will stop. The same question that could be asked is whether or not the minimum number of vertices that will catch on fire is finite or not. The firefighter problem is a good model to use in discrete mathematics to answer questions relating to epidemiology, rumor spreading, and the transmittal of "viral" information over the internet. In this thesis we go a very long way to solving the firefighter problem in the specific case where $G = \mathbb{L}_2$, the two-dimensional infinite grid, and $f(t)$ is *not* constant, which was the norm previously. It was well-known that $f(t) = 2$ admitted a finite solution to the firefighter problem whereas $f(t) = 1$ did not. With my colleague Professor Kah Loon Ng [35], we found that having $f(t) = 1.5+\varepsilon$ (which will be carefully defined) firefighters per turn admits a finite solution to the firefighter problem, and this author also discovered strong evidence showing that having 3 firefighters *every other turn* (which would imply $f(t) = 1.5$ in our future definition) will *not* be sufficient for a finite solution of the firefighter problem. Therefore, it strongly suggests that there is a clear dividing line of 1.5 in the two-dimensional grid case, where any iteration of the problem with strictly greater than 1.5 firefighters assures the admittance of a finite solution, but *no* iteration involving 1.5 or fewer firefighters allows the admittance of a finite solution.

While all three of the problems in this thesis rely heavily on computers, they do so not in a haphazard way, but mainly as a tool to do a large amount of bookkeeping. It is always necessary to have rigor in mathematics, and a lot of the debate that is present with the use of computers can be centered around the fact that some see no distinction between rigor and formality. Nowhere in this document is rigor absent; the use of computers does not indicate otherwise. It is worthwhile to notice the parallels between the work done in this thesis and the ground-breaking work done by the great Allen Newell and Herb Simon on the *Logic Theory Machine* [34] where they in effect take the following five rules,

$$p \vee p \to p$$
$$p \to q \vee p$$
$$p \vee q \to q \vee p$$
$$p \vee q \vee r \to q \vee p \vee r$$
$$(p \vee q) \to (p \to r \vee q),$$

which they assume as true, and then let the computer "get to work". What resulted was the ability of the computer to prove 38 of the first 52 theorems in *Principia Mathematica.* In this thesis, the computer is also given things that are true (and proved rigorously in this thesis), such as Theorem 2.6.1, stating that the sequences $\{f_\Delta(n)\}$ are eventually *pseudoperiodic* (see Section 2.6). With that information, this author has written a computer program in the `Java` programming language that not only allows the user to find the eventual behavior of the sequence $\{f_\Delta(n)\}$, but also *prove* that this is indeed the correct behavior.

The principal message the author would like to impart to the reader is this: whatever humans can do computers can do better, provided they have human assistance. A large amount of mathematics has followed the following three-part structure:

1. A proves a mathematical theorem.

2. B finds a better way to prove the theorem, and in the process discovers further concepts and theories that are worth pursuing.

3. B publishes a better proof, and later on Person B (perhaps with A as a collaborator) publishes further papers on the advancements made.

A large part of this thesis follows this structure, with the author taking the role of B. However, the author already has a main collaborator: the computer. It's hard to imagine much of the work in this thesis getting done without the raw power of the computer. It's not a power unrestrained, however – for as much as computers have advanced and progressed recently, it is useful only when it has a guiding hand to lead it, and this will certainly remain true for a long time.

Chapter 2 contains material which is written in [44]. Chapter 3 contains material which is included in [45] and [43]. Chapter 4 contains material which is included in [35] and [42].

Chapter 2

Avoiding Differences

2.1 Motivation and History

The motivation for this chapter is a result by Peter Shor from his graduate-school days, where he gives a counterexample to the Triangle Conjecture (see [48]). Given the alphabet $\Sigma = \{x, y\}$, the Triangle Conjecture of Perrin and Schützenberger [36] concerns *codes* which are subsets of the set

$$\mathcal{A}_m = \{x^i y x^j \mid i + j < m\}$$

where, of course, $m > 0$. The individual components $x^i y x^j$ are called *atoms*, as they will never be considered broken down any further. For the atom $x^i y x^j$, we call i the *prefix* and j the *suffix*. The conjecture is so named because the elements of $\mathcal{A}_m$ can be arranged graphically as a triangle, as can be seen for $m = 10$ in Figure 2.1.

Definition (Kleene Star [28]) Given a set A of atoms, $A^\star$ is defined as the smallest set satisfying the following conditions.

1. $\varepsilon \in A^\star$, where ε is the empty string.

2. $A \subseteq A^\star$.

3. If $v, w \in A^\star$, then $v \cdot w \in A^\star$, where $\cdot$ is the concatenation operation.

Definition Given a set A of atoms, define A^n to be the set of words that are obtained as concatenations of exactly n elements of A, or

$$A^n = \{w \in A^\star \mid |w| = n\}.$$

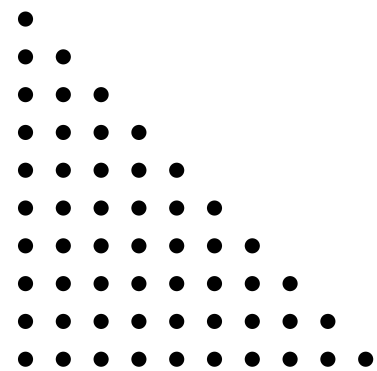

Figure 2.1: Representation of atoms $x^i y x^j$ as a triangle. Each point represents an atom $x^i y x^j$, where the x-coordinate represents i and the y-coordinate represents j.

Definition A subset $A \subseteq \mathcal{A}_m$ is a *code* if any word that can be formed by concatenation of atoms of A can be decomposed uniquely as a concatenation of atoms. Algebraically speaking, A is a code if the free monoid on A exhibits unique factorization. Combinatorially speaking, A is a code if $|A^n| = |A|^n$ for all $n \geq 0$.

Example The sets $A_m = \{yx^i \mid i < m\}$ and $B_m = \{x^i y x^{m-i-1} \mid i < m\}$ are codes for all $m \geq 0$. Verification of these facts is left to the reader.

Example The set $A = \{xyx, xyx^2, yx\}$ is not a code, for

$$xyx^2yx = xyx \cdot xyx, and$$
$$xyx^2yx = xyx^2 \cdot yx.$$

The Triangle Conjecture states that if $A \subseteq \mathcal{A}_m$ is a code, then $|A| \leq m$. In the initial paper where they introduced the Triangle Conjecture, Perrin and Schützenberger proved the following theorem.

Theorem 2.1.1 (Perrin-Schützenberger [36]). *If $X \subseteq \mathcal{A}_m$ is a code and the projections of X on each of the two coordinates are both equal to $\{0, 1, \ldots, r\}$ for some r, then $|X| \leq m$.*

Additionally, Pin and Simon [37] proved the following special cases of the Triangle Conjecture.

Theorem 2.1.2 (Pin-Simon [37])**.** *Let $X \subseteq \mathcal{A}_m$ be a code such that either the set of prefixes or the set of suffixes of X has size at most two. Then $|X| \leq m$.*

Theorem 2.1.3 (Pin-Simon [37])**.** *Let $X \subseteq \mathcal{A}_m$ be a code such that one of the following statements are satisfied.*

1. *There is exactly one prefix of X that has two or more suffixes.*

2. *There is exactly one suffix of X that has two or more prefixes.*

Then $|X| \leq m$.

Shortly thereafter De Felice [11] proved the following theorem.

Theorem 2.1.4 (De Felice [11])**.** *If $X \subseteq \mathcal{A}_m$ is a finite code that occupies at most three rows i, j, k satisfying $i < j < k$ and $j - i = k - j$, then $|X| \leq m$.*

Hansel [24] found an upper bound to the size of a code $X \subseteq \mathcal{A}_m$ by simply counting the different possible words that can be created from *all* atoms in $\mathcal{A}_m$. From this, he was able to prove the following theorem.

Theorem 2.1.5 (Hansel [24])**.** *The number of distinct words that can be constructed from n atoms from $\mathcal{A}_m$ is at most*

$$\left(\left(1 + \frac{1}{\sqrt{2}}\right) m\right)^n.$$

Hence, if $X \subseteq \mathcal{A}_m$ is a code, then $|X| \leq \left(1 + \frac{1}{\sqrt{2}}\right) m$.

An initial attempt can be made toward the Triangle Conjecture by considering what happens if we restrict the range of prefixes that can occur in our code. It would be symbolic suicide to consider just an arbitrary set of prefixes $\{i_1, i_2, i_3, \ldots, i_k\}$ for a code in $\mathcal{A}_m$, so to start we will consider the case where the set of prefixes is $\{0, 1, \ldots, \alpha m\}$, where αm is an integer. We will prove a slight strengthening of Theorem 2.1.5.

Theorem 2.1.6. *If the set of prefixes of $X \subseteq \mathcal{A}_m$ is contained in $\{0, 1, \ldots, \alpha m\}$ for some $0 \leq \alpha \leq 1$, then the number of distinct words that can be constructed from n atoms from X is at most*

$$\left(\left(\frac{\alpha + 1}{2} + \frac{\sqrt{1 + 2\alpha - \alpha^2}}{2}\right) m\right)^n.$$

Proof. We will follow the procedure in [24]. `Mathematica` [1] was used for all of the symbolic manipulation in this section. Fix m and let $T = \{x^i y x^j \mid i + j + 1 \leq \alpha m\}$.

We are interested in counting the elements in the set

$$T_{ij}^n = \{a_1 a_2 \dots a_n \in T_n \mid a_1 a_2 \dots a_n = x^i y \cdots y x^j\}.$$

Toward this end, we define the following numbers.

$$\begin{aligned}
t_{ij}^n &= |T_{ij}^n| \\
t_n = |T_n| &= \sum_{i=0}^{\alpha m} \sum_{j=0}^{\alpha m} t_{ij}^n \\
u_n &= \sum_{i=0}^{\alpha m} t_{i0}^n
\end{aligned}$$

Lemma 2.1.7. *There is a bijection between the sets*

$$T_{ij}^n \Longleftrightarrow \bigcup_{k=0}^{\alpha m - j - 1} T_{i0}^{n-1} \times T_{kj}^1 \cup \bigcup_{k=1}^{\alpha m - 1} T_{ik}^{n-1} \times T_{\alpha m - j - 1, j}^1.$$

Proof. Given a word $x^i y x^{i_2} y \cdots y x^{i_n} y x^j$, we perform the bijection by peeling off the right atom, where we make sure to remove as many x's as possible from x^{i_n}, while always making sure that the atom we remove is an element of $\mathcal{A}$. If we are able to remove $x^{i_n} y x^j$, which is the most possible, then this is an element of T_{kj}^1 for some $0 \leq k \leq \alpha m - j - 1$, since we made sure that $k + j + 1 \leq \alpha m$. What remains is an element of T_{i0}^{n-1}. Otherwise, we peel off as much as possible, guaranteeing that the atom removed is the largest size allowable, which means it is an element of $T_{m-j-1,j}^1$. Similarly, what remains is an element of T_{ik}^{n-1} for some $1 \leq k \leq \alpha m - 1$. Note that k cannot be zero in this case, since we took care of that condition first. □

Corollary 2.1.8.

$$t_{ij}^n = (\alpha m - j) t_{i0}^{n-1} + \sum_{k=1}^{\alpha m - 1} t_{ik}^{n-1}. \tag{2.1}$$

Proof. Follows from Lemma 2.1.7 considering that $\left|T_{ij}^1\right| = 1$ for any specific value of k and of j. □

By summing (2.1) over all possible indices i and j, we obtain

$$t_n = mt_{n-1} + \left(\left(\alpha - \frac{\alpha^2}{2}\right) m^2 - \frac{\alpha m}{2}\right) u_{n-1}.$$

Additionally, by letting $j = 0$ and by summing equation (2.1) over all possible indices i, we obtain

$$u_n = t_n + \alpha m u_{n-1}.$$

At this point, we have all the necessary ingredients to compute t_n, as this is simply an enumeration scheme with two quantities (t and u). We can express this enumeration scheme in matrix form.

$$M = \begin{bmatrix} m & \left(\alpha - \frac{\alpha^2}{2}\right) m^2 - \frac{\alpha m}{2} \\ 1 & \alpha m \end{bmatrix}.$$

This results in a characteristic polynomial of

$$\chi_M(x) = x^2 - m(\alpha + 1)x + \frac{\alpha m}{2}(\alpha m + 1)$$

Yielding a recurrence of

$$t_n = m(\alpha + 1)t_{n-1} - \frac{\alpha m}{2}(\alpha m + 1)t_{n-2}$$

With initial conditions $t_0 = a_0$ and $t_1 = a_1$ – for the reader to determine as it does not affect our asymptotic analysis – we get a generating function of

$$g_t(x) = \frac{a_0 - (a_0(\alpha + 1)m - a_1)x}{1 - (\alpha + 1)m + \frac{\alpha m}{2}(\alpha m + 1)x^2}$$

If we let

$$\Delta = [(\alpha + 1)m]^2 - 2\alpha m(\alpha m + 1)$$

by factoring the denominator and using partial fractions, we obtain

$$\begin{aligned} g_t(x) &= \frac{a_0 - (a_0(\alpha + 1)m - a_1)x}{1 - (\alpha + 1)m + \frac{\alpha m}{2}(\alpha m + 1)x^2} \\ &= \frac{a_0 - (a_0(\alpha + 1)m - a_1)x}{\left(1 - \frac{1}{2}\left[(\alpha + 1)m + \sqrt{\Delta}\right] x\right)\left(1 - \frac{1}{2}\left[(\alpha + 1)m - \sqrt{\Delta}\right] x\right)} \\ &= \frac{1}{\sqrt{\Delta}} \frac{\frac{a_0}{2}\left[(\alpha + 1)m + \sqrt{\Delta}\right] - (a_0(\alpha + 1)m - a_1)}{1 - \frac{1}{2}\left[(\alpha + 1)m + \sqrt{\Delta}\right] x} \\ &\quad - \frac{1}{\sqrt{\Delta}} \frac{\frac{a_0}{2}\left[(\alpha + 1)m - \sqrt{\Delta}\right] - (a_0(\alpha + 1)m - a_1)}{1 - \frac{1}{2}\left[(\alpha + 1)m - \sqrt{\Delta}\right] x} \end{aligned}$$

This yields the final result.

$$t_n = \frac{1}{\sqrt{\Delta}}\left(\frac{a_0}{2}\left[(\alpha+1)m+\sqrt{\Delta}\right]-(a_0(\alpha+1)m-a_1)\right)\left(\frac{1}{2}\left[(\alpha+1)m+\sqrt{\Delta}\right]\right)^n$$
$$-\frac{1}{\sqrt{\Delta}}\left(\frac{a_0}{2}\left[(\alpha+1)m-\sqrt{\Delta}\right]-(a_0(\alpha+1)m-a_1)\right)\left(\frac{1}{2}\left[(\alpha+1)m-\sqrt{\Delta}\right]\right)^n$$

After quite a bit of analysis, we obtain the limit

$$t_n = O\left(\frac{\alpha+1}{2}+\frac{\sqrt{1+2\alpha-\alpha^2}}{2}\right).$$

□

Note that the result is consistent with that of Hansel, since

$$\lim_{\alpha\to 1}\frac{\alpha+1}{2}+\frac{\sqrt{1+2\alpha-\alpha^2}}{2}=1+\frac{1}{\sqrt{2}},$$

which corresponds to the case where $\alpha = 1$.

2.2 The counterexample to the Triangle Conjecture

One small definition is first needed before we present the counterexample and its proof, for the literature varies on its meaning.

Definition Given sets X and Y of integers, the *difference set* $X - Y$ is defined as

$$X - Y = \{|x-y| \mid x \in X, y \in Y, x \neq y\}.$$

The counterexample to the Triangle Conjecture that Shor found is the following, arranged intentionally.

yx^{14}			
yx^{13}	x^3yx^6	x^8yx^6	
yx^7	x^3yx^4	x^8yx^4	$x^{11}yx^4$
yx^1	x^3yx^2	x^8yx^2	$x^{11}yx^2$
y	x^3y	x^8y	$x^{11}y$

The proof of why the above is a code is the true motivation for this chapter: say for example we have a word of the form $x^iyx^jyx^k$ which can be decomposed in two different ways, as

$$x^iyx^jyx^k = x^iyx^{j_1} \cdot x^{j_2}yx^k, \text{ and}$$
$$x^iyx^jyx^k = x^iyx^{j_3} \cdot x^{j_4}yx^k.$$

It is then required that $j_1 + j_2 = j_3 + j_4$, or $j_1 - j_3 = j_4 - j_2$. Therefore, it is sufficient to show that the difference set of $\{0, 3, 8, 11\}$, which are the prefixes of all the atoms, is disjoint from each of the difference sets of $\{0, 1, 7, 13, 14\}$, $\{0, 2, 4, 6\}$, and $\{0, 1, 2\}$, which are the suffixes of all the atoms of a given prefix. It is routine to check that these difference sets are indeed disjoint, and as this argument can be extended to words of any length, it is established that this is a counterexample to the Triangle Conjecture.

Remark The bounds in equation (2.2) are independent of m, for Shor also demonstrated a construction to create more counterexamples from previously-existing codes. Specifically, if $X \subseteq \mathcal{A}_m$ is a code, then by letting

$$X_0^{\uparrow} = \{x^{2i}yx^{2j} \mid x^iyx^j \in X\}$$
$$X_1^{\uparrow} = \{x^{2i}yx^{2j+1} \mid x^iyx^j \in X\},$$

then $(X_0^{\uparrow} \cup X_1^{\uparrow}) \subseteq \mathcal{A}_{2m}$ is also a code. Notice, however, that through this process

$$\frac{X}{m} = \frac{X_0^{\uparrow} \cup X_1^{\uparrow}}{2m}$$

so the lower bound ratio in equation (2.2) remains constant.

We can expand on the previous remark.

Definition Given a code X, we define the *k-expansion of X*, $X^{\uparrow k}$, as

$$X^{\uparrow k} = \bigcup_{a=0}^{k-1} X_a^{\uparrow k},$$

where

$$X_a^{\uparrow k} = \{x^{ki}yx^{kj+a} \mid x^iyx^j \in X\}.$$

Lemma 2.2.1 (Code Expansion Lemma)**.** *If $X \subseteq \mathcal{A}_m$ is a code, then $X^{\uparrow k} \subseteq \mathcal{A}_{km}$ is also a code.*

Proof. Let P_1 be the set of prefixes of X (which is equal to $X^{\uparrow 1}$) and let $\{S_i \mid i \in P_1\}$ be the set of suffixes for each prefix. Since X is a code, then we know that $P_1 - P_1$ is disjoint from $S_i - S_i$ for all i. Letting P_k be the set of prefixes of $X^{\uparrow k}$, we see that $P_k = kP_1$. We can now define the family of suffixes as $\{S_i' \mid i \in P_1\}$ since there is a one-to-one correspondence (multiplication by k) between the prefixes in P_1 and the prefixes in P_k. Note that the only multiples of k that are in $S_i' - S_i'$ are the ones that are in $S_i - S_i$, which completes the proof. □

Only a small amount of work has been done since Shor's counterexample. Since the Triangle Conjecture is refuted, a possible recourse is to consider the quantity

$$\gamma = \lim_{n \to \infty} \left(\frac{\text{size of largest code in } \mathcal{A}_n}{n} \right).$$

Shor's counterexample demonstrated that $\gamma \geq \frac{16}{15}$, and Hansel's counting argument that was expounded on in Section 2.1 shows that

$$\frac{16}{15} \leq \gamma \leq 1 + \frac{1}{\sqrt{2}}. \tag{2.2}$$

An interesting question would be to find the exact value of γ, but (2.2) is the state of the art.

Remark What Shor showed was the following: if a set P of prefixes is given and a family $\{S_p \mid p \in P\}$ of suffixes is given such that the difference set of P is disjoint from the difference set of S_p for all $p \in P$, then the family of atoms defined by

$$X = \bigcup_{p \in P} \{x^p y x_j \mid j \in S_p\}$$

is a code. It is worth noting that the converse is not true; for example, the following is a code but does not satisfy "Shor's Property".

$$\{x^1yx^1, x^1yx^2, x^2yx^8, x^2yx^9\}.$$

Following the ideas from Shor's counterexample, one way to find potential codes would be as follows: we start with m and prefixes prescribed by $P = \{p_1, \ldots, p_k\}$, each less than m. Then we can find the largest code in $\mathcal{A}_m$ with prefixes in P by finding the largest subset of $[m - p_i - 1]$ avoiding $P - P$ for each i, which would form the suffixes for atoms starting with $x^{p_i}y$. Indeed, going back to Shor's example, $\{0, 1, 7, 13, 14\}$ is a subset of maximum size of $[14]$ avoiding $\{0, 3, 8, 11\} - \{0, 3, 8, 11\} = \{3, 5, 8, 11\}$, $\{0, 2, 4, 6\}$ is a subset of maximum size of $[11]$ and $[6]$ avoiding $\{3, 5, 8, 11\}$, and $\{0, 1, 2\}$ is a subset of maximum size of $[3]$ avoiding $\{3, 5, 8, 11\}$.

This describes the importance of finding large subsets of $[n]$ that avoid prescribed differences in a prescribed set Δ, which for the purposes of the Triangle Conjecture is itself a difference set of integers. For now, we will only be considering the *size* of the largest subset of $[n]$ avoiding Δ, which we will denote by $f_\Delta(n)$. We will study another very similar quantity.

Definition $f_\Delta(n)$ is the size of the largest subset T of $[n]$ such that T avoids differences in Δ. Generally speaking, $f_\Delta(I; S)$ is the size of the largest subset T of I such that T avoids differences in Δ and $S \cap T = \emptyset$. We also define $f_\Delta(I) = f_\Delta(I; \emptyset)$ and $f_\Delta(n; S) = f_\Delta([n]; S)$. We say that a set A is a (Δ, S)*-set* if A avoids elements in S and differences in Δ. If $S = \emptyset$ then we will call A a Δ*-set*. We also say that a set $A \subseteq I$ is a *candidate* for $f_\Delta(I; S)$ if A avoids elements in S and differences in Δ, and $|A| = f_\Delta(I; S)$.

The first question relating to these quantities seemed to have been posed by Motzkin (see [7]) , when he asked about the quantity

$$\mu(\Delta) = \lim_{n\to\infty} \frac{f_\Delta(n)}{n},$$

which is also equal to $\lim\limits_{n\to\infty} \dfrac{f_\Delta(n; S)}{n}$ for any finite S. Cantor and Gordon [7] determined $\mu(\Delta)$ for $|\Delta| \leq 2$ and proved that $\mu(\Delta)$ is always rational. Haralambis [25] extended these results by determining $\mu(\Delta)$ for the following cases.

- $\Delta = \{1, j, k\}$ where j is even and $k = n(j + 1) + \overline{k}$ for $0 \leq \overline{k} \leq j$.
- $\Delta = \{1, j, k\}$ where j is odd and either k is odd or $k \geq \binom{j}{2}$.
- $\Delta = \{1, 2, j, k\}$ *except* where $j \equiv 0 \pmod 3$ and $k \equiv 1 \pmod 3$.
- $\Delta = \{1, 2, 3n, 3n + 5\}$ where $n \geq 2$.

- $\Delta = \{1, 3, 4, k\}$ where $k \equiv 2 \pmod 7$.

Gupta [21] gave more results, including the first results involving the determination of $\mu(\Delta)$ for an infinite family of Δ with $|\Delta| \to \infty$. A major shortcoming of the results obtained so far is that they are mainly *ad hoc* and do not give any insight into the underlying structure. This chapter will give major insight into the quantity $\mu(\Delta)$ by considering the sequences $\{f_\Delta(n)\}_1^\infty$. Through this consideration we will be able to give an algorithm that will compute $\mu(\Delta)$ for any given Δ and will hence allow for further investigations into the exact values of, for example, $\mu(\Delta) = \{2, 4, j, k\}$ due to formal parameter analysis and combinatorial methods. Additionally, a major lemma (Lemma 2.8.1) will be given that provides a useful upper bound on $\mu(\Delta)$, which, combined with the lower bounds provided by Cantor, Gordon, Haralambis, and Gupta, provide sharp or small double-sided bounds for $\mu(\Delta)$. Furthermore, the framework given in this chapter for dealing with $\mu(\Delta)$ readily extends to generalized versions, which will be discussed lightly in this chapter and have the real potential for future research.

2.3 Basic Properties of $f_\Delta(I; S)$

We now turn our focus on the discrete quantities $f_\Delta(I; S)$ and $f_\Delta(n; S)$. We start with a few simple but important lemmas. If proofs are not provided, then they are left to the reader and promise to be easy exercises.

Lemma 2.3.1. $f_\Delta(1; S) = \mathbf{1}_{1 \notin S}$.

Lemma 2.3.2. *If $A \subseteq I$ is a (Δ, S)-set and B is a candidate for $f_\Delta(I; S)$, then $|A| \leq |B|$.*

Lemma 2.3.3. *The set*

$$\mathcal{I}_\Delta(n, S) = \{A \subseteq [n] \mid A \textit{ is a } (\Delta, S)\textit{-set}\}$$

is an independence system over $[n]$ (see [49]), satisfying

1. *$\emptyset \in \mathcal{I}_\Delta(n, S)$, and*

2. *$B \in \mathcal{I}_\Delta(n, S), A \subseteq B \Rightarrow A \in \mathcal{I}_\Delta(n, S)$.*

Lemma 2.3.4. $f_\Delta(I;S) \geq f_\Delta(I) - |S|$.

Proof. If A is a candidate for $f_\Delta(I)$, then $A \setminus S$ is a (Δ, S)-set, so if A' is a candidate for $f_\Delta(I;S)$, then $|A'| \geq |A \setminus S| = f_\Delta(I) - S$. □

Lemma 2.3.5. $|f_\Delta(I;S) - f_\Delta(I;S')| \leq \max\{|S|, |S'|\}$.

Proof. We have the clear inequalities

$$f_\Delta(I;S) \leq f_\Delta(n)$$
$$f_\Delta(I;S') \leq f_\Delta(n).$$

Likewise, from Lemma 2.3.4 we have that

$$-f_\Delta(I;S) \leq |S| - f_\Delta(n)$$
$$-f_\Delta(I;S') \leq |S'| - f_\Delta(n).$$

Combining the previous two statements, we have

$$f_\Delta(I;S) - f_\Delta(I;S') \leq |S'|$$
$$f_\Delta(I;S') - f_\Delta(I;S) \leq |S|$$

which yields the result we seek. □

Lemma 2.3.6. *For any integer k, $f_\Delta(I;S) = f_\Delta(I \pm k; S \pm k)$.*

Proof. If $A \subseteq I$ is a (Δ, S)-set, then $A - k$ is a subset of $I - k$ and is a $(\Delta, S - k)$-set, as differences are unaffected if all members of a set are shifted by the same amount. Specifically, the map $A \mapsto A - k$ is a cardinality-preserving bijection from the subsets of I that are (Δ, S)-sets and the subsets of $I - k$ that are $(\Delta, S - k)$-sets. '$-$' can be replaced by '$+$' in the previous two sentences for the same effect. □

2.4 The Fundamental Recurrences

We are mainly concerned with finding $f_\Delta(n)$, but we require the extra parameter S as it allows us to state and prove the following recurrence equation, of which there is no similar equation without the extra parameter.

Theorem 2.4.1. *If* $1 \in S$*, then*

$$f_\Delta(n; S) = f_\Delta(n-1; S-1).$$

If $1 \notin S$*, then*

$$f_\Delta(n; S) = \max\{f_\Delta(n-1; S-1), 1 + f_\Delta(n-1; \Delta \cup (S-1))\}.$$

Proof. If $1 \in S$, then $f_\Delta(n; S) = f_\Delta(\{2, \ldots, n\}; S \setminus 1) = f_\Delta(n-1; S-1)$ from Lemma 2.3.6. If $1 \notin S$, let C be a candidate for $f_\Delta(n; S)$. We will condition on the event that $1 \in C$ or not. If $1 \in C$, then $C \setminus \{1\}$ is a maximal subset of $\{2, \ldots, n\}$ avoiding differences in Δ. However, $C \setminus \{1\}$ must also avoid elements in $\Delta + 1$, for if $\delta \in C$ and $\delta \in \Delta + 1$, then $\delta - 1 \in \Delta$ and the fact now that $1 \in C$ and $\delta \in C$ contradicts the fact that C avoided differences in Δ. Therefore, $|C \setminus \{1\}| = f_\Delta(\{2, \ldots, n\}; (\Delta + 1) \cup S) = f_\Delta(n-1; \Delta \cup (S-1))$ so $|C| = 1 + f_\Delta(n-1; \Delta \cup (S-1))$. If $1 \notin C$, then C is a set of size $f_\Delta(\{2, \ldots, n\}; S) = f_\Delta(n-1; S-1)$. Since C is to be the larger of the two possibilities, we take the maximum. $\square$

This concept can be extended further by generalizing what it means to *avoid a difference* and *avoid a specific element*. If we say that a set A avoids a difference d, then it implies that there is no $x, y \in A$ such that $y - x = d$. Very slightly rephrased, it means that there is no subset $\{x, y\} \subseteq A$ with $x + d = y$. Viewed in this manner, the following generalized definition is clear.

Definition Given a set $D = \{d_1, d_2, \ldots, d_k\}$ with $d_1 < d_2 < \cdots < d_k$, we say that a set A *avoids the generalized difference* D if

$$\forall x \in A \qquad \{x, x + d_1, x + d_2, \ldots, x + d_k\} \not\subseteq A.$$

If $\mathfrak{D}$ is a family of sets, then we say that A *avoids differences in* $\mathfrak{D}$ if A avoids the generalized difference D for all $D \in \mathfrak{D}$.

Similarly, if we say that A avoids an element $x \in S$, then it can be slightly rephrased to say that $\{x\} \not\subseteq A$. This yields the generalized definition.

Definition A set A *avoids* $X = \{x_1, \ldots, x_k\}$ if $X \not\subseteq A$. If $\mathfrak{S}$ is a family of sets, then we abuse notation and say that A *avoids (elements in)* $\mathfrak{S}$ if A avoids X for all $X \in \mathfrak{S}$.

With the generalized definitions, we can now define the generalized analogue to $f_\Delta(I;S)$.

Definition If $\mathfrak{S}$ and $\mathfrak{D}$ are families of sets, then $f_{\mathfrak{D}}(I;\mathfrak{S})$ is the size of the largest subset of I that avoids differences in $\mathfrak{D}$ and elements in $\mathfrak{S}$. We also define $f_{\mathfrak{D}}(n;\mathfrak{S}) = f_{\mathfrak{D}}([n];\mathfrak{S})$ and $f_{\mathfrak{D}}(I) = f_{\mathfrak{D}}(I;\emptyset)$.

Definition If $\mathfrak{S}$ is a family of sets, then

$$\mathfrak{S} - 1 = \{S - 1 \mid S \in \mathfrak{S}\}$$
$$(\mathfrak{S} - 1)^\star = \{S - 1 \mid S \in \mathfrak{S}, 1 \notin S\}.$$

With this new definition, we now have the generalized version of the recursion above.

Theorem 2.4.2. *If* $\{1\} \in \mathfrak{S}$ *then*

$$f_{\mathfrak{D}}(n;\mathfrak{S}) = f_{\mathfrak{D}}(n-1;(\mathfrak{S}-1)^\star).$$

If $\{1\} \notin \mathfrak{S}$*, then*

$$f_{\mathfrak{D}}(n;\mathfrak{S}) = \max\{f_{\mathfrak{D}}(n-1;(\mathfrak{S}-1)^\star), 1 + f_{\mathfrak{D}}(n-1;\mathfrak{S}-1)\}.$$

Proof. Similar to the proof of Theorem 2.4.1, but it is worth explaining why $(\mathfrak{S}-1)^\star$ is in the recurrence instead of simply $\mathfrak{S}-1$. For the first part of the recurrence, we assume that $\{1\} \in \mathfrak{S}$, so a candidate for $f_{\mathfrak{D}}(n;\mathfrak{S})$ vacuously avoids any other set $X \in \mathfrak{S}$ that contains 1, so we may disregard those sets. With the previous sentence in mind, the proof follows exactly in the same manner as Theorem 2.4.1. $\square$

We will mention now the cyclic variants, which will be expounded on later in the chapter.

Definition $f^c_\Delta(n;S)$ is defined as the largest subset of $\mathbb{Z}_n$ that avoids differences in Δ and elements in S, where subtraction is done modulo n. Similarly, $f^c_{\mathfrak{D}}(n;\mathfrak{S})$ is the largest subset of $\mathbb{Z}_n$ that avoids generalized differences in $\mathfrak{D}$ and generalized elements in $\mathfrak{S}$, again where all operations are done modulo n.

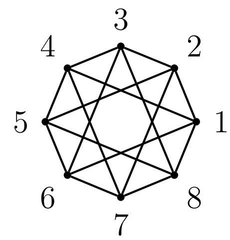

Figure 2.2: Graph $C_{8,\{1,3\}}$

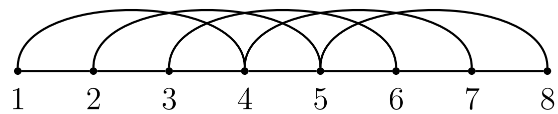

Figure 2.3: Graph $U_{8,\{1,3\}}$

2.5 A Graph Theory Connection

Astute readers may have noticed that this problem bears a strong resemblance to the problem of finding the independence number of circulant and unhooked-circulant graphs, defined as follows (following the terminology of [5]).

Definition Given a set S of integers, the *circulant graph* $C_{n,S}$ is the graph on vertex set $V = \mathbb{Z}_n$ such that $u \sim v$ if and only if $u - v \in S$, where the arithmetic is done modulo n.

Definition Given a set S of integers, the *unhooked-circulant graph* $U_{n,S}$ is the graph on vertex set $V = [n]$ such that $u \sim v$ if and only if $|u-v| \in S$, where normal arithmetic is used.

As examples, we present $C_{8,S}$ and $U_{8,S}$ with $S = \{1, 3\}$ in Figures 2.2 and 2.3.

There is a direct relationship between finding $f_\Delta(n)$ and finding the independence number of $U_{n,S}$. As a consequence, we can give graph-theoretical arguments to answer questions about $f_\Delta(n)$ by looking at $U_{n,S}$, and similarly questions about $f^c_\Delta(n)$ by considering $C_{n,S}$. In this example, we prove a theorem that extends the results of Brown and Hoshino (see [5]). The main focus of their paper (which also gave a very nice application to music, involving the number of different chords one could play) was the following result involving *independence polynomials*.

Definition Given a graph G on n vertices, the *independence polynomial* $I(G, x)$ is defined as

$$I(G, x) = \sum_{k=0}^{n} i_k x^k,$$

where i_k is the number of independent sets of G with precisely k vertices.

Theorem 2.5.1 (Brown-Hoshino [5])**.**

$$I(C_{n,[d]}, x) = I(C_{n-1,[d]}, x) + xI(C_{n-d-1,[d]}, x) \textit{ for all } n \geq 2d + 2.$$

Proof. See [5]. □

From Theorem 2.5.1, one can get a formula for the independence number of $C_{n,[d]}$. We go further, and prove the following theorem.

Theorem 2.5.2. *Define* $[k, l] = \{k, k + 1, \ldots, l\}$ *and given* n*, let* $n = q(k + l) + r$*. Then, the independence number of* $U_{n,[k,l]}$ *is* $qk + \min(r, k)$*. Equivalently,* $f_{[k,l]}(n) = qk + \min(r, k)$*.*

Proof. We will first consider the initial case where $n \leq k+l$. Clearly, if $1 \leq n \leq k$ then $U_{n,[k,l]}$ has no edges so the independence number would be n. Now we will show that the independence number is k when $n = k + l$, which would imply that the independence number is k when $k < n < k + l$.

Lemma 2.5.3. *The largest subset of* $[k + l]$ *avoiding differences in* $[k, l]$ *is of size* k*.*

Proof. Clearly, the set $\{1, 2, \ldots, k\}$ is a candidate. What remains now is to show that there is no bigger set. To this end, we will utilize Hall's Theorem [4]. Let $L \subseteq [k + l]$ be such that $A = \{a_1, a_2, \ldots, a_r\}$ with $a_1 < a_2 < \cdots < a_r$ avoids differences in $[k, l]$. Construct the bipartite graph $G = (A, B)$, where $B = \{b_1, \ldots, b_i\}$. For $a_i \in A$ and $b_j \in B$ we connect via the following rule.

$$a_i \sim b_j \iff a_i = j \text{ or } |a_i - j| \in [k, l].$$

If we could match A, then we would be done, as it would imply that $|A| \leq |B| = k$. We must show that Hall's Condition is satisfied. For a contradiction, let $S \subseteq A$ be such that $|N(S)| < |S|$ and assume S is minimal.

Claim If $a_i \in S$ then $a_i > k$.

Proof of claim. If $a_i \leq k$, then of course $a_i \sim b_{a_i}$. If there is no other vertex in S adjacent to b_{a_i}, then $S \setminus \{a_i\}$ also fails Hall's Condition, contradicting the minimality of S. However, if there is another vertex $a_{i'} \sim b_{a_i}$ in G, then that would imply that $a_i - a_{i'} \in [k, l]$, which contradicts the condition that A avoided differences in $[k, l]$. $\square$

From the claim, we may assume that for all $a_i \in S$, $k+1 \leq a_i \leq k+l$. However, S can only contain one element out of the following sets, of which there are k.

$$\begin{gathered} \{k+1, 2k+1\} \\ \{k+2, 2k+2\} \\ \vdots \\ \{l, k+l\} \\ \{l+1\} \\ \{l+2\} \\ \vdots \\ \{2k\} \end{gathered}$$

$\square$

Now we present an important extension to our lemma that says that the extremal pattern repeats itself.

Lemma 2.5.4. *For any positive integer n, the largest subset of $[n(k+l)]$ avoiding differences in $[k, l]$ is of size nk.*

Proof. The proof of this lemma is very similar to the proof of the previous one, but we present it in its entirely for completeness, while also trying to mimic the previous proof. Fix n. A candidate for our set (arranged intentionally) is

$$C_n = \left\{ \begin{array}{cccc} 1, & 2, & \cdots & k, \\ k+l+1, & k+l+2, & \cdots & 2k+l, \\ \vdots & \vdots & \vdots & \vdots \\ (n-1)(k+l)+1, & (n-1)(k+l)+2, & \cdots & (n-1)(k+l)+k \end{array} \right\}.$$

Let $\overline{C_n} = [n(k+l)] \setminus C_n$. Let $A \subseteq [n(k+l)]$, with $A = \{a_1, a_2, \ldots, a_r\}$ and $a_1 < a_2 < \cdots < a_r$ be such that A avoids differences in $[k,l]$. Construct the bipartite graph $G = (A, B)$, where $B = \{b_j \mid j \in C_n\}$. For $a_k \in A$ and $b_j \in B$ we connect via the following rule.

$$a_i \sim b_j \iff a_i = j \text{ or } |a_i - j| \in [k,l].$$

If we could match A, then we would be done, as it would imply that $|A| \leq |B| = nk$. We must show that Hall's Condition is satisfied. For a contradiction, let $S \subseteq A$ be such that $|N(S)| < |S|$ and assume S is minimal.

Claim If $a_i \in S$ then $a_i \in \overline{C_n}$.

Proof of claim. If $a_i \in C_n$, then of course $a_i \sim b_{a_i}$. If there is no other vertex in S adjacent to b_{a_i}, then $S \setminus \{a_i\}$ also fails Hall's Condition, contradicting the minimality of S. However, if there is another vertex $a_{i'} \sim b_{a_i}$ in G, then that would imply that $a_i - a_{i'} \in [k,l]$, which contradicts the condition that A avoided differences in $[k,l]$. $\square$

From the claim, we may assume that for all $a_i \in S$, $q(k+1) \leq a_i \leq q(k+l)$ for some $1 \leq q \leq n$. However, S can only contain one element out of the following sets for each

$q, 1 \le q \le n$, of which there are nk.

$$\{q(k+1), q(2k+1)\}$$
$$\{q(k+2), q(2k+2)\}$$
$$\vdots$$
$$\{ql, q(k+l)\}$$
$$\{q(l+1)\}$$
$$\{q(l+2)\}$$
$$\vdots$$
$$\{q(2k)\}$$

□

We have now shown what we wanted for each multiple of $k+l$.

Lemma 2.5.5 (Persistence Lemma)**.** *If $f_\Delta(kn, S) = km$ for infinitely many k, then $\mu(\Delta) = \frac{m}{n}$.*

Proof. Exercise. □

With Lemma 2.5.5, we have completed the proof of Theorem 2.5.2. □

2.6 Behavior of $f_\Delta(n)$ as $n \to \infty$

As mentioned before, in order to compute $f_\Delta(n)$ it is necessary to compute $f_\Delta(n; S)$ for other sets S based on the recursion. For example, to compute $f_{\{3,5\}}(n) = f_{\{3,5\}}(n, \emptyset)$ it is necessary to consider the system of recurrences listed in Figure 2.4. Using these recurrences, it is straightforward to compute the first 25 elements of the sequence $\{f_{\{3,5\}}(n)\}$.

$$1, 2, 3, 3, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, 10, 10, 11, 11, 12, 12, 13$$

There is obvious structure in this sequence past a certain point. The structure becomes clear when we consider the difference sequence, defining $f_{\{3,5\}}(0) = 0$.

$$1, 1, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1$$

$$\begin{aligned}
f_{\{3,5\}}(n;\emptyset) &= \max\{f_{\{3,5\}}(n-1;\emptyset), f_{\{3,5\}}(n-1;\{3,5\})+1\} \\
f_{\{3,5\}}(n;\{1\}) &= f_{\{3,5\}}(n-1,\emptyset) \\
f_{\{3,5\}}(n;\{2\}) &= \max\{f_{\{3,5\}}(n-1;\{1\}), f_{\{3,5\}}(n-1;\{1,3,5\})+1\} \\
f_{\{3,5\}}(n;\{1,2\}) &= f_{\{3,5\}}(n-1;\{1\}) \\
f_{\{3,5\}}(n;\{1,3\}) &= f_{\{3,5\}}(n-1;\{2\}) \\
f_{\{3,5\}}(n;\{2,4\}) &= \max\{f_{\{3,5\}}(n-1;\{1,3\}), f_{\{3,5\}}(n-1;\{1,3,5\})+1\} \\
f_{\{3,5\}}(n;\{3,5\}) &= \max\{f_{\{3,5\}}(n-1;\{2,4\}), f_{\{3,5\}}(n-1;\{2,3,4,5\})+1\} \\
f_{\{3,5\}}(n;\{1,2,3\}) &= f_{\{3,5\}}(n-1;\{1,2\}) \\
f_{\{3,5\}}(n;\{1,3,5\}) &= f_{\{3,5\}}(n-1;\{2,4\}) \\
f_{\{3,5\}}(n;\{1,2,3,4\}) &= f_{\{3,5\}}(n-1;\{1,2,3\}) \\
f_{\{3,5\}}(n;\{2,3,4,5\}) &= \max\{f_{\{3,5\}}(n-1;\{1,2,3,4\}), f_{\{3,5\}}(n-1;\{1,2,3,4,5\})+1\} \\
f_{\{3,5\}}(n;\{1,2,3,4,5\}) &= f_{\{3,5\}}(n-1;\{1,2,3,4\})
\end{aligned}$$

Figure 2.4: Enumeration scheme for computing the sequence $\{f_{\{3,5\}}(n)\}$.

The difference sequence above is a periodic sequence with period 2 and offset 6. As it turns out, this is true of all such sequences.

Definition A sequence is called *pseudoperiodic* if its difference sequence is periodic.

Definition Given sets Δ and S, the *closure* $\mathfrak{C}_\Delta(S)$ is the smallest family of sets satisfying the following conditions.

1. $S \in \mathfrak{C}_\Delta(S)$, and
2. For all $X \in \mathfrak{C}_\Delta(S)$, $X-1 \in \mathfrak{C}_\Delta(S)$ and $\Delta \cup (X-1) \in \mathfrak{C}_\Delta(S)$.

Equivalently, $\mathfrak{C}_\Delta(S)$ can be built up recursively by defining $\mathfrak{C}^1_\Delta(S) = \{S\}$ and

$$\mathfrak{C}^n_\Delta(S) = \{X-1 \mid X \in \mathfrak{C}^{n-1}_\Delta(S)\} \cup \{\Delta \cup (X-1) \mid X \in \mathfrak{C}^{n-1}_\Delta(S)\}.$$

and letting $\mathfrak{C}_\Delta(S) = \mathfrak{C}^{n^\star}_\Delta(S)$ where $n^\star$ is the least n satisfying $\mathfrak{C}^n_\Delta(S) = \mathfrak{C}^{n+1}_\Delta(S)$.

Theorem 2.6.1. *For any sets Δ and S, the sequence $\{f_\Delta(n;S)\}_{n=1}^\infty$ is an eventually pseudoperiodic sequence.*

Lemma 2.6.2. *Let $k = \max(\Delta \cup (S-1))$. For all $S' \in \mathfrak{C}_\Delta(S)$, $|f_\Delta(n;S) - f_\Delta(n;S')| \leq k$.*

Proof. Follows from Lemma 2.3.4. $\square$

Proof of Theorem 2.6.1. Fix Δ and S, and let $\mathfrak{C}_\Delta(S) = \{S_1, S_2, \ldots, S_k\}$. We may assume that $S_1 = \emptyset$ since it is always true that $\emptyset \in \mathfrak{C}_\Delta(S)$. We define the *state at n* to be

$$s_n = (a_{n,1}, a_{n,2}, \ldots, a_{n,k}),$$

where $a_{n,i} = f_\Delta(n; X_i) - f_\Delta(n; X_1) = f_\Delta(n; X_i) - f_\Delta(n; \emptyset)$. Note that this implies that $a_{n,1} = 0$ for all n. Tt follows from Lemma 2.6.2 that $|a_i| \leq k$ for all k. The main recursion defines a function F on the space of all possible states to itself. Since this space is finite (it has size at most $(2k+1)^{k-1}$) it implies that our sequences of states

$$s_1, F(s_1), F^2(s_1), F^3(s_1), \ldots$$

is eventually periodic, implying that the sequence $f_\Delta(n;S)$ is eventually pseudoperiodic. $\square$

Remark As with most proofs of this sort, the bounds achieved on the offset or the period of these eventually pseudoperiodic sequences are horrible. It is proved in [7] that the period is at most $2^{\max \Delta}$. However, experimental evidence (see Section 2.10) indicates that a period on the order of the sum of the elements of Δ is possible.

What we will be more interested in, however, is the following corollary.

Corollary 2.6.3. *For any set Δ, $\mu(\Delta)$ exists and is equal to $\frac{p}{q}$, where q is the length of the period of $\{f_\Delta(n)\}_{n=1}^\infty$ and p is the increase in $\{f_\Delta(n)\}_{n=0}^\infty$ over its period.*

Proof. Direct from Theorem 2.6.1, as to find $\mu(\Delta) = \lim_{n\to\infty} \dfrac{f_\Delta(n)}{n}$, it suffices to only start considering the sequence $\{f_\Delta(n)\}$ when it becomes purely pseudoperiodic. $\square$

2.7 The Cyclic Extension

In this section we will briefly investigate the cyclic version of $f_\Delta(n)$, which involves avoiding differences modulo n. Therefore, we view the set as a subset of $\mathbb{Z}_n$, and not simply a subset of $[n]$. To count $f^c_\Delta(n)$, we must introduce another parameter based on the following observation: if we want a similar recurrence as the non-cyclic case, then we would consider the scenario when 1 is removed from the situation. Consider trying to find $f^c_{\{2,3\}}(9)$ and consider the situation when 1 is removed, as described in Figure 2.5.

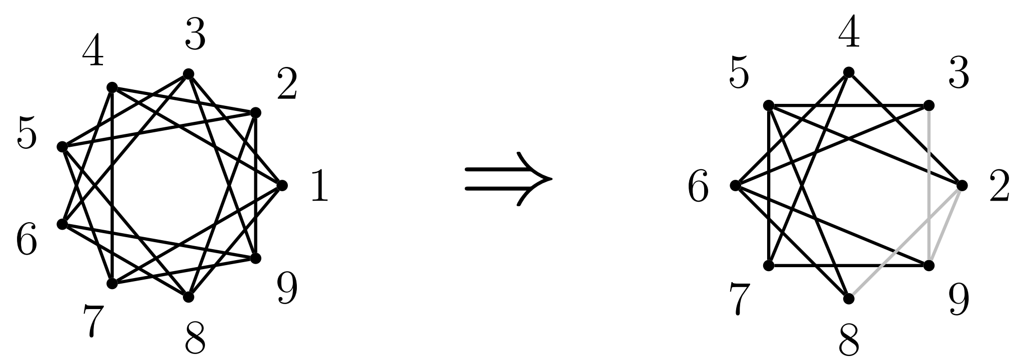

Figure 2.5: Taking $U_{9,\{2,3\}}$ and removing vertex 1.

The lighter edges in the lower graph above represent the edges that have been affected by the removal of vertex 1. The removal of the lighter edges produces a graph that is isomorphic to $UC_{\{2,3\}}(8)$. With the lighter edges, the graph yielded is defined as follows.

Definition Given sets Δ and Δ', the *two-sided circulant graph* $C_{\Delta,\Delta'}(n)$ on n vertices is the graph with vertex set $[n]$ and edges $E(G) = E_1(G) \cup E_2(G)$ defined as follows.

$$E_1(G) = \bigcup_{d \in \Delta} \{\{i, i+d\} \mid 1 \leq i \leq n-d\}, and$$

$$E_2(G) = \bigcup_{d' \in \Delta'} \{\{i, i+d \pmod n\} \mid n-d < i \leq n\}.$$

Just as before, we can generalize the definition and define $C_{\Delta,\Delta'}(I)$, which is the graph with vertex set I and similarly-defined edges. The reader can convince him or herself that $C_{\Delta,\Delta'}(I)$ is subgraph of $C_{\Delta,\Delta'}(n)$ induced by I, where $n = \max I$.

Note that $C_\Delta(n) = C_{\Delta,\Delta}(n)$. In the example above, we see that the removal of vertex 1 from $C_{\{2,3\}}(9)$ produced a graph isomorphic to $C_{\{2,3\},\{1,2\}}(8)$ (with the isomorphism $i \mapsto i-1$). It is not a coincidence that $\{1,2\} = \{2,3\} - 1$.

Lemma 2.7.1. *The removal of any vertex from $C_\Delta(n)$ produces a graph isomorphic to $C_{\Delta,\Delta-1}(n-1)$. The removal of vertex 1 from $C_{\Delta,\Delta'}(n)$ produces a graph isomorphic to $C_{\Delta,\Delta'-1}(n-1)$.*

Proof. Exercise. $\square$

Since there is no global uniformity anymore with the two-sided circulant graph, we must add an additional parameter Δ' when dealing with its relationship to the original quantity of interest, $f^c_\Delta(n)$.

Definition $f^c_{\Delta,\Delta'}(I;S)$ is the size of the largest independent set in $C_{\Delta,\Delta'}(I)$. $f^c_{\Delta,\Delta'}(I) = f^c_{\Delta,\Delta'}(I;\emptyset)$and $f^c_{\Delta,\Delta'}(n;S) = f^c_{\Delta,\Delta'}([n];S)$.

Remark The previous definition can be rephrased in terms of avoiding differences, but it is cleaner to associate the quantity $f_{\Delta,\Delta'}(n)$ with the two-sided circulant graphs.

With this operation in mind, along with the relationship between the circulant graphs $C_\Delta(n)$ and the quantity $f_\Delta(n)$, we have the following recurrence, similar in nature to the fundamental one of Theorem 2.4.1.

Theorem 2.7.2. *If* $1 \in S$ *then*

$$f^c_{\Delta,\Delta'}(n;S) = f^c_{\Delta,\Delta'-1}(n-1;S-1)$$

otherwise,

$$f^c_{\Delta,\Delta'}(n;S) = \max\{f^c_{\Delta,\Delta'-1}(n-1;S-1), 1 + f^c_{\Delta,\Delta'-1}(n-1;\Delta \cup (n-\Delta') \cup (S-1))\}.$$

Proof. Follows the proof of Theorem 2.4.1. □

The problem with the theorem above, at first glance, is that there is no bound on the number of parameters; S grows without bound as it depends on n. Previously S depended only on Δ so we could give a bound on how many different parameters we would need to keep track of. This can be solved by having a different recurrence, with a second S' parameter whose function is similar to the Δ' parameter.

Definition $f^c_{\Delta,\Delta'}(n;S,S') = f^c_{\Delta,\Delta'}(n;S \cup ([n] \setminus S'))$.

Theorem 2.7.3. *If* $1 \in S$ *and* $n \notin S'$*, then*

$$f^c_{\Delta,\Delta'}(n;S,S') = f^c_{\Delta,\Delta'-1}(n-1;S-1,S')$$

otherwise,

$$f^c_{\Delta,\Delta'}(n;S,S') = \max\{f^c_{\Delta,\Delta'-1}(n-1;S-1,S'), 1 + f^c_{\Delta,\Delta'-1}(n-1;\Delta \cup (S-1), \Delta' \cup S')\}$$

Note that a small extra condition was included, which can be avoided by certifying that $\max S' < n$. Since S' is solely affected by Δ' in the recurrence, this is a sufficient condition since we will usually start out with $S' = \emptyset$. Therefore, the strategy for computing the cyclic version of $f_{\Delta}(n)$ is

1. Compute $f_{\Delta,\Delta'}(n; S)$ for $1 \leq n \leq \max \Delta'$ using Theorem 2.7.2.

2. Compute $f_{\Delta,\Delta'}(n; S, S')$ for $n > \max \Delta'$ using Theorem 2.7.3.

We can link the two recurrences by noting that

$$f_{\Delta,\Delta'}(n; S, S') = f_{\Delta,\Delta'}(n; S \cup (n - S' + 1))$$

which gives us a complete overall strategy for computing any number of values of $f_{\Delta,\Delta'}(n; S)$ in total linear time. The drawback, as usual, is that there is a large (but fixed) computation that needs to be done (in $O(2^{\max \Delta})$ time) and we will need $O(2^{\max \Delta})$ space to keep track of the current state.

2.8 An application: the Triangle Conjecture, revisited

The first counterexample was found by Shor to the Triangle Conjecture, and computer programs can easily find further counterexamples. Additionally, as Lemma 2.2.1 demonstrated, any counterexample to the Triangle Conjecture can be extended to produce larger counterexamples via transformations such as

$$x^i y x^j \to x^{2i} y x^{2j} \text{ and } x^{2i} y x^{2j+1}.$$

This larger counterexample has the same ratio relative to m as the original example, but the underlying set of prefixes has changed. Are there infinite families of counterexamples with the same set of prefixes? This would be an asymptotic version of the Triangle Conjecture and can be stated succinctly as follows with the terminology used in this chapter.

Theorem 2.8.1 (Asymptotic Version of the Triangle Conjecture). *For any set X,*

$$\mu(X - X) \leq \frac{1}{|X|}.$$

Proof. Let $X = \{x_1, x_2, \ldots, x_k\}$ and consider the infinite family of sets

$$\begin{array}{ccccccccc}
\{ & x_1 & , & x_2 & , & \cdots & , & x_n & \} \\
\{ & x_1+1 & , & x_2+1 & , & \cdots & , & x_n+1 & \} \\
\{ & x_1+2 & , & x_2+2 & , & \cdots & , & x_n+2 & \} \\
\{ & x_1+3 & , & x_2+3 & , & \cdots & , & x_n+3 & \} \\
\{ & \vdots & , & \vdots & , & \vdots & , & \vdots & \}
\end{array}$$

Let $A_i = X + i$ and assume for a contradiction that for a fixed ε there is an arbitrarily large N also satisfying $N > \frac{a_k}{\varepsilon}$ with a set $B \subseteq [N]$ that avoids differences in $X - X$ and that

$$|B| > \left(\frac{1}{|X|} + \varepsilon\right) N.$$

From the choice of N, we have

$$|B| > \frac{N}{|X|} + a_k$$

and so

$$|B| - a_k > \frac{N}{|X|} > \frac{N - a_k + 1}{|X|}.$$

Let $B' = \{b \in B \mid b > a_k\}$ and consider the sets $\mathcal{A} = A_0, A_1, A_2, \ldots, A_{N-a_k}$. Note that $|B'| > N - a_k$. Each of the elements in B' appear in $|X|$ different sets in $\mathcal{A}$. However, no two elements of B' can appear in the same set in $\mathcal{A}$. Therefore, we have a contradiction by the Pigeonhole Principle. $\square$

With Theorem 2.8.1 in mind, it follows that you cannot *a priori* fix a set of prefixes and then obtain arbitrarily large counterexamples to the Triangle Conjecture with those prefixes. This strongly suggests that the true value of γ is $\frac{16}{15}$, the lower end of the range. In fact, experimental evidence indicates that that counterexamples decrease in ratio as m gets larger; this will be discussed further in the conclusion.

2.9 Towards an alternate proof of Szèmeredi's Theorem

The famed theorem of Roth [46] states that given δ, for large enough N any subset of $[N] = \{1, \ldots, N\}$ with size at least δN must contain a three-term arithmetic progression, where $\delta \to 0$ as $N \to \infty$. Roth's original theorem required that

$$\delta \gg \frac{1}{\log \log N}$$

and a considerable amount of effort has been put into improving the bound on δ. Szemerédi's Theorem [51] extended Roth's Theorem for any length arithmetic progression, and hence settled the weaker form of the Erdős-Turán conjecture. With Szemerédi's Theorem in the books, much research has been devoted to finding the quantities $r_k(n)$, the largest subset of $[n]$ that is k-*free*, meaning it does not contain any k-term arithmetic progression. There is a decent amount of motivation for finding these numbers, as demonstrated with these examples dealing with 3-free sets.

Example Let $q(n)$ denote the minimum number of queens needed on the main diagonal of a $n \times n$ chessboard so that all squares are either occupied or under attack.

Theorem 2.9.1 (Cockayne-Hedetniemi 1986 [8])**.** $q(n) = n - r_3(\frac{n}{2})$.

Example Naïve multiplication of two $n \times n$ matrices requires $O(n^3)$ multiplications, but the best algorithm so far for matrix multiplications, developed by Coppersmith and Winograd [10], requires $O(n^{2.376})$ multiplications, and it requires the use of "large" 3-free sets, as shown by the following theorem.

Theorem 2.9.2 (Salem-Spencer 1942 [47])**.** *Given $\varepsilon > 0$, there exists M_ε such that for any $M > M_\varepsilon$, there is a 3-free set $B \subseteq [\frac{M}{2}]$ such that $|B| > M^{1-\epsilon}$.*

The machinery in this chapter allows us to consider the quantity $r_{k,D}(n)$, which is defined as the size of the largest subset of $[n]$ that avoids k-term arithmetic progressions with difference at most D. With the terminology introduced in this chapter we could write

$$r_{k,D}(n) = f_{\{\{1,2,\ldots,k-1\},\{2,4,\ldots,2(k-1)\},\ldots,\{D,2D,\ldots,D(k-1)\}\}}(n),$$

but we will use the more convenient notation $r_{k,D}(n)$ instead. Note that $r_k(n) = r_{\{k,\frac{n-1}{k-1}\}}(n)$. We also define the quantity

$$\mu_{k,D} = \lim_{n\to\infty} \frac{f_{k,D}(n)}{n}.$$

With Corollary 2.6.3 in mind, the relationship with Roth's Theorem is then clear.

Theorem 2.9.3. *The following are equivalent:*

1. *Szemerédi's Theorem, and*

2. *For all k, $\mu_{k,D} \to 0$ as $D \to \infty$.*

Proof. ($\rightarrow$) If $\mu_{k,D} \to \overline{\mu}$ where $\overline{\mu} > 0$, then it implies the existence of a subset $X \subseteq \mathbb{N}$ with positive density $\overline{\mu}$ that avoids k-arithmetic progressions of difference D for all D. Hence it avoids all k-arithmetic progressions.

($\leftarrow$) Assume that $\mu_{k,D} \to 0$ as $D \to \infty$ and let $X \subseteq \mathbb{N}$ have positive upper density δ. Let $D^\star$ be such that $\mu_{k,D^\star} < \frac{\delta}{4}$. Using Corollary 2.6.3 (being a limit statement), let N be given $\varepsilon = \frac{\delta}{4}$, and let $N' > N$ be such that $|X \cap [N']| > \frac{\delta}{2}$. From Corollary 2.6.3 it follows that $|X \cap [N']|$ contains a k-arithmetic progression of common difference at most D, and hence X must contain this same arithmetic progression. □

We can use our machinery – the recursion from Theorem 2.7.2 – to find $r_{k,D}(n)$ and the resulting sequences can be analyzed. Full accompanying `Mathematica` and `Java` code can be found in [38]. As an introductory example, the first 25 terms of $r_{3,1}(n)$ – which is equal to $f_{\{\{1,2\}\}}(n)$ in the notation of Theorem 2.7.2 – is

$$1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, 9, 10, 10, 11, 12, 12, 13, 14, 14, 15, 16, 16, 17, \ldots$$

The pattern here is not hard to spot, and can be seen more clearly by looking at the sequence of successive differences.

$$1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$$

This suggests that some of the candidates of $r_{3,1}(n)$ are the ones that contain all elements not congruent to 0 modulo 3, which turns out to be true. Therefore, we can conclude so far that $\mu_{3,1} \geq \frac{2}{3}$, and it can easily be verified that $\mu_{3,1} = \frac{2}{3}$.

From Theorem 2.7.2, we know that $\mu_{k,D}$ exists for all positive k and D.

Corollary 2.9.4. *If* $\{r_{k,D}(n)\} = \{f_{\mathfrak{D}}(n;\mathfrak{S})\}$ *is pseudoperiodic with period* p *and* $\mu_{k,D\mu}$ *is given from Theorem 2.6.3, then the following statements hold.*

1. *There is a set* $A \subseteq \mathbb{Z}_p$ *that avoids differences (modulo* p*) in* $\mathfrak{D}$ *and elements (modulo* p*) in* $\mathfrak{S}$*.*

2. *There is no set* $B \subseteq \mathbb{Z}_{p'}$ *that avoids differences (modulo* p'*) in* $\mathfrak{D}$ *and elements (modulo* p'*) in* $\mathfrak{S}$ *such that* $\frac{|B|}{p'} > \mu$*.*

Below is a table of the results obtained in the specific case of avoiding three-term arithmetic progressions using experimental means. Cyclic set witnesses – the sets described in Corollary 2.9.4 – were also searched for and given, except in the case $k = 12$. An automated theorem-prover has been implemented in this case (see [38]), and proofs have been given confirming the exact values of α_k for $k \leq 9$.

Additionally, conjectured values of $\mu_{k,D}$ have been found for various other values of k and D.

2.10 Results and Further Study

This chapter linked the Triangle Conjecture and Shor's Counterexample to the more general problem of determining $f_\Delta(n)$, defined as the size of the largest subset of $[n]$ that avoids differences in Δ. The quantity $f_\Delta(n)$ was investigated fully, as was its counterpart $f_{\mathfrak{D}}(n)$. Additionally, cyclic variants of these two quantities were also investigated, although not as fully.

This chapter exhibited the fact that the sequence $\{f_\Delta(n)\}|_1^\infty$ is a pseudoperiodic sequence but no bound on the period was given, apart from the large bound given for free from the proof of Theorem 2.6.1. We can compute (and prove – see [41]) the values $\mu(\Delta)$ for the sequences $\{f_\Delta(n)\}|_1^\infty$ for small values of $\mu(\Delta)$. Theorem 2.5.2 solves the problem for all singleton Δ.

Corollary 2.10.1 (Corollary to Theorem 2.5.2)**.**

$$\mu(\{d\}) = \frac{d}{2d} \textit{ for any } d.$$

k	μ_k	Cyclic set witness
1	$\frac{2}{3}$	$\{1,2\}$ in $\mathbb{Z}_3$
2	$\frac{2}{3}$	$\{1,2\}$ in $\mathbb{Z}_3$
3	$\frac{4}{8}$	$\{1,2,6,7\}$ in $\mathbb{Z}_8$
4	$\frac{4}{9}$	$\{1,2,4,5\}$ in $\mathbb{Z}_9$
5	$\frac{4}{9}$	$\{1,2,4,5\}$ in $\mathbb{Z}_9$
6	$\frac{4}{9}$	$\{1,2,4,5\}$ in $\mathbb{Z}_9$
7	$\frac{4}{9}$	$\{1,2,4,5\}$ in $\mathbb{Z}_9$
8	$\frac{4}{9}$	$\{1,2,4,5\}$ in $\mathbb{Z}_9$
9	$\frac{4}{10}$	$\{1,2,4,5\}$ in $\mathbb{Z}_{10}$
10	$\frac{4}{11}$	$\{1,2,4,9\}$ in $\mathbb{Z}_{11}$
11	$\frac{8}{24}$	$\{1,2,4,5,13,16,19,20\}$ in $\mathbb{Z}_{24}$
12	$\frac{56}{177}$	$\star$
13	$\frac{6}{19}$	$\{1,2,4,13,15,16\}$ in $\mathbb{Z}_{19}$
14	$\frac{6}{19}$	$\{1,2,4,13,15,16\}$ in $\mathbb{Z}_{19}$
15	$\frac{6}{19}$	$\{1,2,4,13,15,16\}$ in $\mathbb{Z}_{19}$
16	$\frac{6}{19}$	$\{1,2,4,13,15,16\}$ in $\mathbb{Z}_{19}$
17	$\frac{6}{19}$	$\{1,2,4,13,15,16\}$ in $\mathbb{Z}_{19}$

Figure 2.6: Table giving values of $\mu_{3,D}$ for $1 \leq D \leq 17$. Cyclic set witnesses (see Corollary 2.9.4) are also given for all values except $D = 12$.

Additionally, the following lemma, although stated here, has certainly been known for a while.

Lemma 2.10.2. *As exhibited by the set of odd numbers, if Δ contains only odd numbers, then $\mu(\Delta) = 1/2$.*

Besides the above lemma and the results stated in Section 2.2, not much else is known about the value $\mu(\Delta)$ for other various families Δ. Additionally, while finding the value of $f^c_\Delta(n)$ is NP-Complete (see [9]), it is unknown whether the same is true in the non-circular case, although this author claims it to be true. Nevertheless, it does not automatically imply that the problem of finding $\mu(\Delta)$ is also NP-Complete

		k											
		3	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
	1	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$	$\frac{8}{9}$	$\frac{9}{10}$	$\frac{10}{11}$	$\frac{11}{12}$	$\frac{12}{13}$	$\frac{13}{14}$
	2	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{8}{9}$	$\frac{8}{9}$	$\frac{10}{11}$	$\frac{10}{11}$	$\frac{12}{13}$	$\frac{12}{13}$
	3	$\frac{4}{8}$	$\frac{8}{12}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{20}{23}$	$\frac{10}{11}$	$\frac{10}{11}$	$\frac{12}{13}$	$\frac{12}{13}$
	4	$\frac{4}{9}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{26}{30}$	$\frac{10}{11}$	$\frac{10}{11}$	$\frac{12}{13}$	$\frac{12}{13}$
	5	$\frac{4}{9}$	$\frac{4}{7}$	$\frac{16}{24}$	$\frac{22}{30}$	$\frac{6}{7}$							
	6	$\frac{4}{9}$	$\frac{4}{7}$										
	7	$\frac{4}{9}$	$\frac{6}{11}$										
	8	$\frac{4}{9}$	$\frac{6}{11}$										
D	**9**	$\frac{4}{10}$											
	10	$\frac{4}{11}$											
	11	$\frac{8}{24}$											
	12	$\frac{56}{177}$											
	13	$\frac{8}{19}$											
	14	$\frac{8}{19}$											
	15	$\frac{8}{19}$											
	16	$\frac{8}{19}$											
	17	$\frac{8}{19}$											

Figure 2.7: Values of $\mu_{k,D}$.

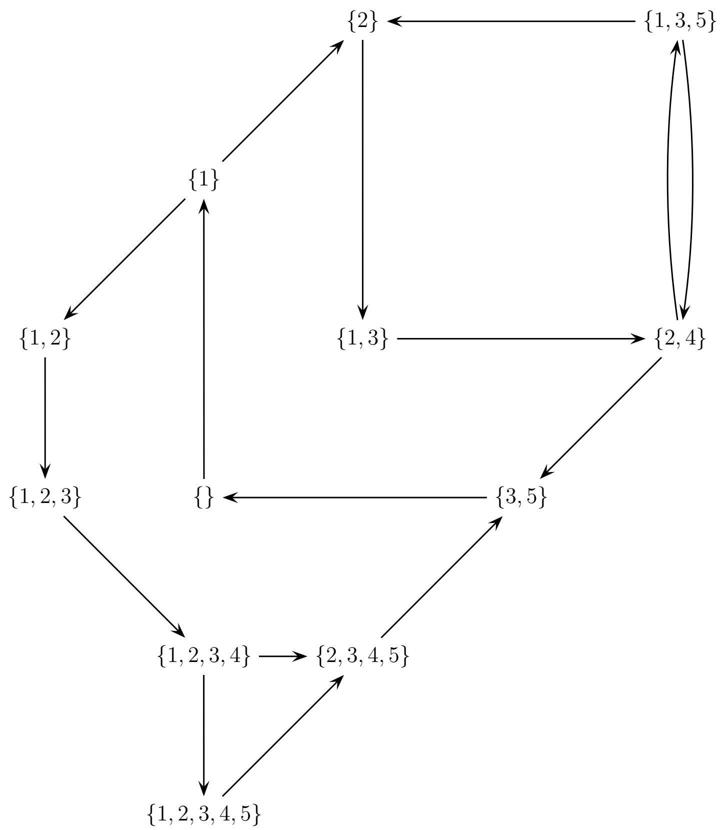

Figure 2.8: A directed graph representing the enumeration scheme for calculating $f_{\{}3, 5\}(n)$.

or NP-Hard. To this end, it certainly seems reasonable that symbolic methods could be used initially to find $\mu(\Delta)$ for three-member sets $\{i, j, k\}$, and from there finding a formula for $\mu(\Delta)$ depending solely on the elements of Δ.

Additionally, toward the goal of reducing the upper bound on the pseudoperiod of the sequences $(f_\Delta(n))|_1^\infty$, it may be worthwhile to consider the digraphs that are obtained by considering each parameter S as a vertex in the graph and connecting $S \to S'$ if S is used in the recurrence equation involving S'. For example, the graph obtained by considering the enumeration scheme for calculating $f_{\{}3, 5\}(n)$ is shown in Figure 2.8. Insights into the structure of this graph and how it could be utilized while the recurrence is "in motion" (a pebbling problem of sorts – see [30]) would be very helpful in lowering the bound on the pseudoperiod. From experimental results obtained so far on a wide variety of values Δ, the author wishes to conjecture the following.

Conjecture 1. *The pseudoperiod of* $\{f_\Delta(n)\}$ *is bounded from above by* $\sum \Delta$.

Finally, there is the question of the Triangle Conjecture itself, which now has to be modified to ask what γ is. Equation 2.2 gives the current known bounds, and so the author wishes to formally conjecture the true bound.

Conjecture 2. $\gamma = \frac{16}{15}$.

Specifically, the author thinks more is true that lends credence to the fact that counterexamples of the Triangle Conjecture are simply hiccups of sorts in creation that correct themselves as $m \to \infty$ (as justified by Theorem 2.8.1). To this end, we see that through Shor's construction of multiplying the size of a code, we make the following definition.

Definition Let

$$\mathfrak{C} = \{X \subseteq \mathbb{N} \mid X \text{ is a counterexample to the Triangle Conjecture}\}.$$

The *Triangle Conjecture Counterexample partially-ordered set* (TCC poset) is the partially-ordered set on $\mathfrak{C}$ where $X \prec Y$ if and only if Y is obtained from X by Shor's multiplication method.

Remark Officially a counterexample X to the Triangle Conjecture requires the parameter m, which specifies the set $\mathcal{A}_m$ that X is a subset of. However, given a set X the m can be found easily; $m = \max\{i + j + 1 \mid x^i y x^j \in X\}$.

Conjecture 3. *All minimal elements* X *of the TCC poset satisfy* $|X| = m + 1$.

This would imply that $\gamma = \frac{16}{15}$ assuming that Shor's counterexample is indeed the minimal counterexample with respect to m, which at this point should be possible to accomplish on today's computers.

Additionally, there is a large graph-theoretic aspect of the Triangle Conjecture that has not been discussed much in this chapter, or elsewhere for that matter. Defining a *right isosceles triangle in* $\mathbb{Z}^2$ as a triple of points $(i_1, j_1), (i_1, j_2)$, and (i_2, j_1) as three points that determine a right isosceles triangle, we have the following lemma.

Lemma 2.10.3. *If $X \subseteq \mathcal{A}_m$ is a code, then interpreting X as points in $\mathbb{Z}^2$, X does not contain any right isosceles triangle.*

Therefore, it would be interesting to find the size of the largest subset of $\mathcal{A}_m$ that avoids all isosceles right triangles. Toward this end, it would help to compute the number of isosceles triangles in $\mathcal{A}_m$, which produces this conjecture.

Conjecture 4. *There are $f(m)$ isosceles triangles in $\mathcal{A}_m$, where*

$$f(m) = \begin{cases} \frac{15}{36}m^3 + \frac{7}{8}m^2 + \frac{1}{12}m - \frac{3}{8} & \text{if } m \text{ is odd} \\ \frac{15}{36}m^3 + \frac{7}{8}m^2 + \frac{1}{12}m & \text{if } m \text{ is even} \end{cases}.$$

It can be verified that $f(m)$ will always be an integer for positive m. Using the deletion method (see [2]), one can they attempt to get upper bounds on the number of isosceles triangle-free subsets of $\mathcal{A}_m$, and hence new upper bounds.

Chapter 3

Spanning Trees in Grid Graphs

3.1 Introduction

The Matrix Tree Theorem of Kirchhoff, a generalization of Cayley's Theorem from complete graphs to arbitrary graphs [49], gives the number of spanning trees on a labeled graph as a determinant of a specific matrix. If $A = (a_{ij})$ is the adjacency matrix of a graph G, then the number of spanning trees can be found by computing any cofactor of the Laplacian matrix of G, or specific to the (n,n)-cofactor.

$$\text{Number of spanning trees of } G = \begin{vmatrix} a_{12} + \ldots + a_{1n} & -a_{12} & \cdots & -a_{1,n-1} \\ -a_{21} & a_{21} + \cdots + a_{2n} & & -a_{2,n-1} \\ \vdots & & \ddots & \vdots \\ -a_{n-1,1} & -a_{n-1,2} & \cdots & a_{n-1,1} + \cdots + a_{n-1,n} \end{vmatrix}$$

Since determinants are easy to compute, then the Matrix Tree Theorem allows for the computation for the first few numbers in the sequence of spanning trees for families of graphs dependent on one or more parameters. However, the downside of the Matrix Tree Theorem is that it can only produce a sequence of numbers, and cannot *a priori* assist in finding out the recurrence involved with said sequence, or even determine if such a recurrence exists. In this chapter, the initial motivation is the following families of graphs, which will be defined in the next section.

1. $k \times n$ grid graphs, with $n \to \infty$.

2. $k \times n$ cylinder graphs, with $n \to \infty$.

3. $k \times n$ torus graphs, with $n \to \infty$.

All of the families of graphs mentioned above belong to the more general class of graphs of the form $G \times P_n$ or $G \times C_n$, where P_n and C_n denote the path and cycle graph on n vertices, respectively. For each of these classes, a general method is obtained for finding recurrences for all of the above families of graphs, and explicit recurrences are found for many cases. The only drawback, as it stands, is the amount of computational power needed to obtain these recurrences, as the recurrences are obtained through characteristic polynomials of large matrices. The result is at least 50 new sequences of numbers with *complete information*, meaning recurrences and generating functions, plus improvements on the best-known recurrences known for other sequences.

3.2 History and Outline

The main source of the historical results is a paper [16] and website [15] by Faase, where the main motivation is to count the number of hamiltonian cycles in certain classes of graphs. Later, in 2000, Desjarlais and Molina [12] discuss the number of spanning trees in $2 \times n$ and $3 \times n$ grid graphs. In 2004, Golin and Leung [19] discuss a technique called *unhooking* which will be used in this chapter to reduce the problem of counting spanning trees in cylinder graphs to the problem of counting spanning trees in grid graphs.

In the first two papers and this chapter, the general idea is the same: our goal is to find the number of spanning trees, but the method we use requires us to also count certain related objects. The paper by Faase appeals to the Transfer-Matrix Method, used widely in statistical mechanics (for more about the Transfer-Matrix Method, see [49]). The main difference between this chapter and [12] is the direct application of the Cayley-Hamilton Theorem [29] to obtain recurrences for the sequences. Overall, the results from this chapter yield sequences for the number of spanning trees in the graphs $G \times P_n$ and $G \times C_n$ for any graph G. Along with these sequences, our methods find the minimal recurrence, generating function, and closed-form formulae for all of these sequences. As a consequence, we also find the sequences and recurrences for many other

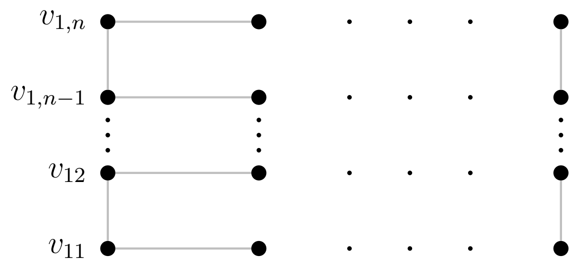

Figure 3.1: Vertex naming conventions for the grid graph.

types of subgraphs.

The bulk of the chapter focuses on the steps involved in finding the transition matrix for a given graph. In doing so, we will have to count other, related spanning forests with special properties.

3.3 Notation

All of the graphs we will be dealing with depend on two parameters, which we will call k and n. In all cases, we will think of k as fixed and $n \to \infty$.

Definition The $k \times n$ *grid graph* $G_k(n)$ is the simple graph with vertex and edge sets defined as

$$V\left(G_k(n)\right) = \{v_{i,j} \mid 1 \leq i \leq k, 1 \leq j \leq n\}$$
$$E\left(G_k(n)\right) = \{v_{i,j}v_{i',j'} \mid |i - i'| + |j - j'| = 1\}$$

In order to keep the diagrams clean, Figure 3.1 shows the vertex naming conventions we will use.

When showing examples, usually of spanning trees or spanning forests, we will always show the underlying graph in one form or another. A concrete example is given in Figure 3.2: we will use black edges for edges in the subgraph exemplified; all unused edges will show up in light grey.

When dealing with grids of arbitrary size, we will mainly be interested in the very right-most end of the grid, so we will represent the rest of the graph we do not care about by a gray box, as shown in Figure 3.3.

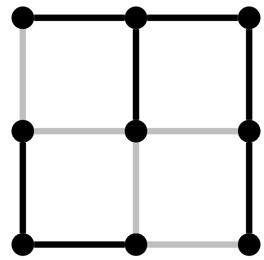

Figure 3.2: Showing a spanning forest of $P_2 \times P_2$ and its underlying graph.

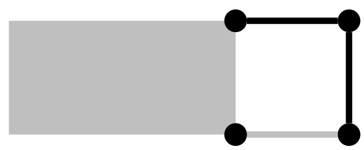

Figure 3.3: An example of a spanning tree/forest of $P_2 \times P_k$ where we only care about the right-hand side.

Definition The $k \times n$ *cylinder graph* $C_k(n)$ can be obtained by "wrapping" the grid graph around, specifically by adding the edges

$$E\left(C_k(n)\right) = E\left(G_k(n)\right) \bigcup \{\{v_{1,i}, v_{n,i}\} \mid 1 \leq i \leq k\}.$$

Remark Note that $C_k(n) = P_k \times C_n$.

Definition The $k \times n$ *torus graph* $T_k(n)$ can be obtained by "wrapping" the cylinder graph around the other way, specifically by adding the edges

$$E\left(T_k(n)\right) = E\left(C_k(n)\right) \bigcup \{\{v_{i,1}, v_{i,k}\} \mid 1 \leq i \leq n\}$$

Remark Note that $T_k(n) = C_k \times C_n$.

Throughout this paper, we will be dealing with partitions of the set $[k] = \{1, 2, \ldots, k\}$. We denote by $\mathcal{B}_k$ the set of all such partitions, and $B_k = |\mathcal{B}_k|$ are the Bell numbers. We will impose an ordering on $\mathcal{B}_k$, which we will call the lexicographic ordering on $\mathcal{B}_k$.

Definition Given two partitions P_1 and P_2 of $[k]$, for $i \in [k]$, let X_i be the block of P_1 containing i, and likewise Y_i the block of P_2 containing i. Let j be the minimum value of i such that $X_i \neq Y_i$. Then $P_1 < P_2$ iff

1. $|P_1| < |P_2|$ or

2. $|P_1| = |P_2|$ and $X_j \prec Y_j$, where $\prec$ denotes normal lexicographic ordering on sets of integers.

For example, $\mathcal{B}_3$ in order is

$$\mathcal{B}_3 = \{\{\{1,2,3\}\},\{\{1\},\{2,3\}\},\{\{1,2\},\{3\}\},\{\{1,3\},\{2\}\},\{\{1\},\{2\},\{3\}\}\}$$

However, we will use shorthand notation for set partitions, for example

$$\mathcal{B}_3 = \{123, 1/23, 12/3, 13/2, 1/2/3\}.$$

Since our examples will only deal with $k < 10$, we will not have to worry about double-digit numbers on our shorthand notation.

We will find many recurrences in this paper, all pertaining to the number of spanning trees of the graphs mentioned above. Since we will be dealing with each type of graph separately, we will always denote by T_n the number of spanning trees of whatever graph we are dealing with at the moment, which will be unambiguous.

3.4 Grid Graphs: The Example For $k = 2$.

What follows is mainly from [12] and is the inspiration for the other results on grid graphs. We would like to find a recurrence for T_n, which for now will represent the number of spanning trees in $G_2(n)$. If we started out with a spanning tree on $G_2(n-1)$, then Figure 3.4 shows the three different ways to add the additional two vertices to still make a spanning tree on $G_2(n)$.

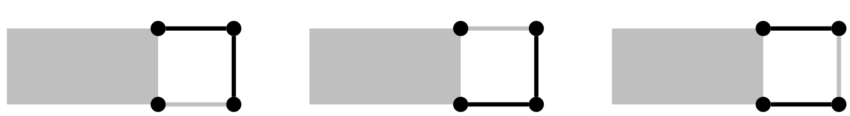

Figure 3.4: The three possible ways to extend a spanning tree of $P_2 \times P_{n-1}$ to a spanning tree of $P_2 \times P_n$.

However, there is also a way to create a spanning tree on the $2 \times n$ grid from something that isn't a spanning tree on $G_2(n-1)$. Let $x = v_{n-1,1}$ and $y = v_{n-1,2}$ be

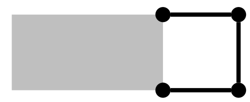

Figure 3.5: The only way to extend a special spanning forest of $P_2 \times P_{n-1}$ to a spanning tree of $P_2 \times P_n$

the end vertices on $G_k(n-1)$. If we have a spanning forest on $G_2(n-1)$ with the property that there are two trees in the forest and x and y are in distinct trees, then Figure 3.5 shows the only way to append edges to create a spanning tree in $G_2(n)$.

Therefore, in counting T_n it is useful to also count F_n, which we define as the number of spanning forests in $G_2(n)$ consisting of two trees with the additional property that the end vertices $v_{n,1}$ and $v_{n,2}$ are in distinct components. From the preceding two paragraphs we can now obtain the recurrence

$$T_n = 3T_{n-1} + F_{n-1}$$

and through similar reasoning we can also find the recurrence

$$F_n = 2T_{n-1} + F_{n-1}$$

At this point, let us note that we have enough information to find T_n (or F_n) in time linear in n. However, our goal is to provide explicit recurrences for T_n alone. If we let v_n denote the column vector

$$v_n = \begin{bmatrix} T_n \\ F_n \end{bmatrix},$$

and if we define the matrix A by

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix},$$

then we satisfy

$$Av_{n-1} = v_n.$$

With the starting conditions

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The characteristic polynomial of A is

$$\chi_\lambda(A) = \lambda^2 - 4\lambda + 1$$

so by the Cayley-Hamilton Theorem, we satisfy

$$A^2 - 4A + 1 = 0.$$

This can be re-written as

$$A^2 = 4A - 1$$

and if we multiply by the vector v_n on the right we obtain

$$\begin{bmatrix} T_{n+2} \\ F_{n+2} \end{bmatrix} = 4 \begin{bmatrix} T_{n+1} \\ F_{n+1} \end{bmatrix} - \begin{bmatrix} T_n \\ F_n \end{bmatrix}.$$

Hence, we now see that T_n and F_n satisfy the same recurrence.

$$T_{n+2} = 4T_{n+1} - T_n$$

$$F_{n+2} = 4F_{n+1} - F_n$$

with starting conditions

$$\begin{array}{cc} T_1 = 1 & T_2 = 4 \\ F_1 = 1 & F_2 = 3 \end{array}.$$

We now have all the information we need to obtain more information, such as the generating function and, finally, a closed-form formula for T_n. All of these items can be found in [12].

3.5 The General Case For Grid Graphs.

We want to use the same ideas for general k, but it requires a bit more bookkeeping. To extend the idea of F_n in the previous section, we need to consider partitions of $[k] = \{1, 2, \ldots, k\}$ and the forests that come from these partitions.

Definition Given a spanning forest $\mathcal{F}$ of $G_k(n)$, the *partition induced by* $\mathcal{F}$ is obtained from the equivalence relation

$$i \sim j \iff v_{n,i}, v_{n,j} \text{ are in the same component of } \mathcal{F}.$$

For example, the partition induced by a spanning tree of $G_k(n)$ is $123\cdots n$ and the partition induced by the forest with no edges is $1/2/3/\cdots/n-1/n$.

Definition Given a spanning forest $\mathcal{F}$ of $G_k(n)$ and a partition P of $[k]$, we say that $\mathcal{F}$ *is consistent with* P if

1. The number of components in $\mathcal{F}$ is precisely $|P|$.

2. P is the partition induced by $\mathcal{F}$.

Definition Given a graph G on k vertices and a partition P of $[k]$, let $\tau_G(n;P)$ be the number of spanning trees of the graph $G \times P_n$ consistent with P. We will often omit G when it is clear from the context, or irrelevant. Recall that we have an ordering of partitions, so we will define $\tau_G(n;i) = \tau_G(n;P_i)$. $\tau_G(n) = \tau_G(n;\{[n]\})$, the number of spanning trees of $G \times P_n$.

In the previous section, since $B_2 = 2$, we were counting two things: T_2 , which corresponds to $\tau_{P_2}(n)$, and F_n, which corresponds to $\tau_{P_2}(n;1/2)$. Therefore, for arbitrary k we are now tasked with counting B_k different objects at once, so we are to find the $B_k \times B_k$ matrix that represents the B_k simultaneous recurrences between these objects.

Definition Define by E_n the set of edges

$$E_n = E(G_k(n)) \setminus E(G_k(n-1))$$

Note that $|E_n| = 2k-1$ edges.

Given some forest $\mathcal{F}$ of $G_k(n-1)$ and some subset $X \subseteq E_n$, we can combine the two to make a forest of $G_k(n)$. If we are only interested in the number of components in the new forest and its induced partition, then we only need to know the same information from $\mathcal{F}$, and this is all independent of n.

Definition Given two partitions P_1 and P_2 in $\mathcal{B}_k$, a subset $X \subseteq E_n$ *transfers from* P_1 *to* P_2 if a forest consistent with P_1 becomes a forest consistent with P_2 after the addition of X.

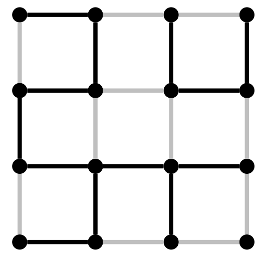

Figure 3.6: A spanning forest of $G_4(4)$ where, from left to right, the edges transfer as follows: $1/23/4 \to 1234 \to 12/34 \to 12/3/4$.

Example Figure 3.6 shows a spanning forest of $G_4(4)$ where, from left to right, the edges transfer from $1/23/4$ to 1234, from 1234 to $12/34$, and from $12/34$ to $1/2/34$.

Therefore, for a graph G with $|G| = k$, we can define the $B_k \times B_k$ matrix A_G by

$$A_G(i,j) = |\{X \subseteq E_{n+1} \mid X \text{ transfers from } P_j \text{ to } P_i\}|.$$

The 2×2 matrix in the previous section is A_{P_2}. Brute-force search with straightforward `Mathematica` code [40] can produce more matrices, such as the transition matrix for $P_3 \times P_n$ shown in Figure 3.7 and $P_4 \times P_n$ in Figure 3.8.

$$\begin{bmatrix} 8 & 3 & 3 & 4 & 1 \\ 4 & 3 & 2 & 2 & 1 \\ 4 & 2 & 3 & 2 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 2 & 2 & 1 \end{bmatrix}$$

Figure 3.7: The transition matrix A_{P_3}.

A_5, A_6, and A_7 have also been found; they are shown in [40]. Once these matrices are known, then everything about the sequence of spanning trees can be found. The following table shows some results obtained for grid graphs; results obtained for arbitrary graphs of the form $G \times P_n$ for all graphs G with at most five vertices are in [40], and results are continuously being computed for larger graphs with results posted as they arrive. The website will be continually updated as the author computes these matrices for larger graphs.

$$\begin{bmatrix}
21 & 8 & 9 & 11 & 8 & 14 & 11 & 15 & 3 & 3 & 4 & 3 & 4 & 5 & 1 \\
9 & 8 & 6 & 4 & 4 & 6 & 5 & 8 & 3 & 3 & 4 & 2 & 2 & 2 & 1 \\
6 & 4 & 9 & 4 & 4 & 4 & 4 & 4 & 3 & 2 & 2 & 3 & 2 & 2 & 1 \\
3 & 0 & 0 & 3 & 1 & 2 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
9 & 4 & 6 & 5 & 8 & 6 & 4 & 8 & 2 & 3 & 2 & 3 & 4 & 2 & 1 \\
1 & 0 & 0 & 1 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
3 & 1 & 0 & 1 & 0 & 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 4 & 6 & 4 & 3 & 4 & 3 & 4 & 3 & 2 & 2 & 2 & 2 & 2 & 1 \\
5 & 4 & 4 & 3 & 4 & 6 & 3 & 4 & 2 & 3 & 2 & 2 & 2 & 2 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
5 & 3 & 6 & 3 & 4 & 4 & 4 & 4 & 2 & 2 & 2 & 3 & 2 & 2 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
4 & 3 & 4 & 3 & 3 & 4 & 3 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 1
\end{bmatrix}$$

Figure 3.8: The transition matrix A_{P_4}.

3.6 A sample of results

$G_2(n)$: ([12]) $T_n = 4T_{n-1} - T_{n-2}$ Sequence: $\{1, 4, 15, 56, 209, \ldots\}$ (OEIS A001353)   Generating Function: $\frac{x}{1-4x+x^2}$
$G_3(n)$: ([15]) $T_n = 15T_{n-1} - 32T_{n-2} + 15T_{n-3} - T_{n-4}$ Sequence: $\{1, 15, 192, 2415, 30305, \ldots\}$ (OEIS A006238)   Generating Function: $\frac{3x(1+49x+1152x^2)}{1+24x-24x^2+x^3}$
$G_4(n)$: ([15]) $T_n = 56T_{n-1} - 672T_{n-2} + 2632T_{n-3} - 4094T_{n-4} + 2632T_{n-5} - 672T_{n-6} + 56T_{n-7} - T_{n-8}$ Sequence: $\{1, 56, 2415, 100352, 4140081, \ldots\}$ (OEIS A003696)   Generating Function: $\frac{x(x^6-49x^4+112x^3-49x^2+1)}{x^8-56x^7+672x^6-2632x^5+4094x^4-2632x^3+672x^2-56x+1}$
$G_5(n)$: ([15], with improvements from this paper) $T_n = 209T_{n-1} - 11936T_{n-2} + 274208T_{n-3} - 3112032T_{n-4} + 19456019T_{n-5}$ $-70651107T_{n-6} + 152325888T_{n-7} - 196664896T_{n-8} + 152325888T_{n-9}$ $-70651107T_{n-10} + 19456019T_{n-11} - 3112032T_{n-12} + 274208T_{n-13}$ $-11936T_{n-14} + 209T_{n-15} - T_{n-16}$ Sequence: $\{1, 209, 30305, 4140081, 557568000, \ldots\}$ (OEIS A003779)   Generating Function: See [40]

Figure 3.9: Full sequence information for $G_2(n), G_3(n), G_4(n)$, and $G_5(n)$.

$G_6(n)$: (new)

$T_n = 780T_{n-1} - 194881T_{n-2} + 22377420T_{n-3} - 1419219792T_{n-4}$
$+55284715980T_{n-5} - 1410775106597T_{n-6} + 24574215822780T_{n-7}$
$-300429297446885T_{n-8} + 2629946465331120T_{n-9} - 16741727755133760T_{n-10}$
$+78475174345180080T_{n-11} - 273689714665707178T_{n-12} + 716370537293731320T_{n-13}$
$-1417056251105102122T_{n-14} + 2129255507292156360T_{n-15} - 2437932520099475424T_{n-16}$
$+2129255507292156360T_{n-17} - 1417056251105102122T_{n-18} + 716370537293731320T_{n-19}$
$-273689714665707178T_{n-20} + 78475174345180080T_{n-21} - 16741727755133760T_{n-22}$
$+2629946465331120T_{n-23} - 300429297446885T_{n-24} + 24574215822780T_{n-25}$
$-1410775106597T_{n-26} + 55284715980T_{n-27} - 1419219792T_{n-28} + 22377420T_{n-29}$
$-194881T_{n-30} + 780T_{n-31} - T_{n-32}$

Sequence: $\{1, 780, 380160, 170537640, 74795194705, \ldots\}$ (OEIS A139400)

Generating Function: See [40]

Figure 3.10: Full sequence information for $G_6(n)$.

3.7 Extending to Generalized Graphs of the Form $G \times P_n$

For the results above, it was not necessary that the graph we were dealing with was a grid. We could have repeated the same process as above for any sequences of graphs G_n defined by

$$G_n = G \times P_n$$

for some predefined graph G. In fact, the `Mathematica` code in [40] handles any such general case. Therefore, it leads to the following theorem.

Theorem 3.7.1. *Let a graph G be given with k vertices, and define the sequence of graphs $\{G_n\}$ by $G_n = G \times P_n$. Then there is a $B_k \times B_k$ matrix M and a vector v, both taking on integer values, such that*

$$T_n = M^n v[1]$$

where T_n is the number of spanning trees in G_n. Furthermore, $M^n v[i]$ lists the number of spanning forests consistent with P_i in G_n.

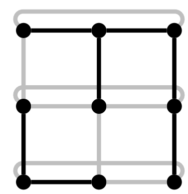

Figure 3.11: An example of a spanning forest of $C_3(3)$. The inclusion of either $v_{1,1}v_{3,1}$ or $v_{1,2}v_{3,2}$ admits a spanning tree.

Corollary 3.7.2. *Let a graph G be given with k vertices, and consider the sequence $\{T_n\}$. Then T_n satisfies a linear recurrence of order B_k.*

3.8 Extending to Cylinder Graphs

In this section we will discuss the changes necessary to extend the above arguments to find recurrences for cylinder graphs and generalized cylinder graphs. We shall take advantage of the "unhooking" technique covered in [19]. The technique is a reduction from a cylinder graph to a grid graph. Recall that the vertex sets of $C_k(n)$ and $G_k(n)$ are the same.

Definition For a given k, we define $\mathcal{E}_k$ by

$$\mathcal{E}_k = E(C_k(n)) \setminus E(G_k(n))$$

If we unhook (i.e. remove) the edges in $\mathcal{E}_k$ then what we have left is precisely $G_k(n)$. Now we have to consider what structures in $G_k(n)$ yield a spanning tree in $C_k(n)$ by the addition of some subset of edges from $\mathcal{E}_k$. Since we are going to add edges that go from one end of the grid to another, we must look at both ends of the grid now, as opposed to only looking at one end. For example, Figure 3.11 shows a spanning forest of $G_3(3)$ that will never yield a spanning tree of $G_3(n)$ for any $n > 3$ through the method described in the previous sections, but this spanning forest would create two different spanning trees of $C_3(3)$ through the addition of either edge $v_{1,1}v_{3,1}$ or $v_{1,2}v_{3,2}$.

Therefore, we can keep the same basic idea used with grid graphs, with some modifications. We must now keep track of how our spanning forest affects the vertices *at each end.*

Definition Given a spanning forest $\mathcal{F}$ of $G_k(n)$, the partition P of $[2k]$ *induced by* $\mathcal{F}$ is obtained from the equivalence relation

$$i \sim j \iff v_i, v_j \text{ are in the same component of } \mathcal{F}$$

where we identify the vertices $v_1, v_2, \ldots, v_k$ with $v_{1,1}, v_{1,2}, \ldots, v_{1,k}$, respectively, and the vertices $v_{k+1}, v_{k+2}, \ldots, v_{2k}$ with $v_{n,1}, v_{n,2}, \ldots, v_{n,k}$, respectively.

Definition Given a spanning forest $\mathcal{F}$ of $G_k(n)$ and a partition P of $[2k]$, we say that $\mathcal{F}$ *is cylindrically consistent with* P if

1. The number of trees in $\mathcal{F}$ is precisely $|P|$.

2. P is the partition induced by $\mathcal{F}$.

For example, the forest shown in Figure 3.11 is cylindrically consistent with the partition 12/3456. It's important to know what partition a certain forest of $G_k(n)$ is cylindrically consistent with, as that determines how many different ways edges can be added to achieve a spanning tree of $C_k(n)$. Since each spanning tree of $C_k(n)$ is uniquely determined by the underlying spanning forest of $G_k(n)$ and the extra edges from $\mathcal{E}_k$, we have all the information we need to count the number of spanning trees of $C_k(n)$.

Definition For a given k, the *tree-counting vector* d_k is the vector, indexed by the partitions of $[2k]$, such that $d_k(i)$ is the number of ways that edges from $E(C_k(n)) \setminus E(G_k(n))$ can be added to get from a forest cylindrically consistent with partition i to a spanning tree of $C_k(n)$. Notice that this is independent of n.

For example, it can be verified that d_2 is given in Figure 3.12.

To count the number of spanning trees for $C_k(n)$ we can produce the $B_{2k} \times B_{2k}$ matrix in the same way as we did for the grid graphs, and using this matrix we can find the number of spanning forests of $G_k(n)$ consistent with each of the partitions of $\mathcal{B}_{2k}$, which can be expressed as a vector of length B_{2k}. Then, when we take the dot product of this vector with d_k, we obtain the number of spanning trees of $C_k(n)$. For example,

1234	1
1/234	1
12/34	2
134/2	1
123/4	1
14/23	2
124/3	1
13/24	0
1/2/34	1
1/23/4	1
1/24/3	0
12/3/4	1
13/2/4	0
14/2/3	1
1/2/3/4	0

$$d_2 = (1, 1, 2, 1, 1, 2, 1, 0, 1, 1, 0, 1, 0, 1, 0)$$

Figure 3.12: The tree-counting vector d_2 in detail.

it can be verified that Figure 3.13 is the transition matrix for $C_2(n)$. The initial vector is

$$v = (1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)$$

We then obtain

$$(Av) \cdot d_2 = 12$$

$$(A^2v) \cdot d_2 = 75$$

$$(A^3v) \cdot d_2 = 384$$

$$\vdots$$

which yields the sequence of the number of spanning trees on $C_2(n)$.

Similar to the process with grids, there is nothing specific here to the simple cylinder graph – these methods can be used to obtain sequences for graph families of the form

$$\begin{bmatrix}
3 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 3 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 3 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 1 & 0 & 1 & 1 & 1
\end{bmatrix}$$

Figure 3.13: The transition matrix for $C_2(n)$.

$G \times C_n$ for arbitrary G. However, due to the rapid growth of B_{2k}, the ability to find the appropriate matrices becomes somewhat difficult starting at graphs with five vertices. Nevertheless, we still have the following theorem.

Theorem 3.8.1. *For a given graph G on k vertices, there is a $B_{2k} \times B_{2k}$ matrix M and a vector v of length B_{2k} such that*

$$(M^n v) \cdot d_k$$

is the number of spanning trees of the graph $G \times C_n$.

Corollary 3.8.2. *For a given graph G on k vertices, the number of spanning trees $\{T_n\}$ of $G \times C_n$ satisfies a linear recurrence of order at most B_{2k}.*

Although the sequence for $C_2(n)$ is already known, these methods used were able to obtain *new* sequences for $C_3(n)$ and $K_3 \times C_n$, which is stated in Figure 3.14.

$C_2(n)$: ([15], with improvements) $T_n = 10T_{n-1} - 35T_{n-2} + 52T_{n-3} - 35T_{n-4} + 10T_{n-5} - T_{n-6}$ Sequence: $\{1, 12, 75, 384, 1805, \ldots\}$ (OEIS A006235) Generating Function: $\frac{x(x^4+2x^3-10x^2+2x+1)}{(x^3-5x^2+5x-1)^2}$
$C_3(n)$: (new) $T_n = 48T_{n-1} - 960T_{n-2} + 10622T_{n-3} - 73248T_{n-4} + 335952T_{n-5} - 1065855T_{n-6}$ $+2396928T_{n-7} - 3877536T_{n-8} + 4548100T_{n-9} - 3877536T_{n-10} + 2396928T_{n-11}$ $-1065855T_{n-12} + 335952T_{n-13} - 73248T_{n-14} + 10622T_{n-15}$ $-960T_{n-16} + 48T_{n-17} - T_{n-18}$ Sequence: $\{1, 70, 1728, 31500, 508805, \ldots\}$ (OEIS to be submitted) Generating Function: See [40]
$K_3 \times C_n$: (new) $T_n = 58T_{n-1} - 1131T_{n-2} + 8700T_{n-3} - 29493T_{n-4} + 43734T_{n-5}$ $-29493T_{n-6} + 8700T_{n-7} - 1131T_{n-8} + 58T_{n-9} - T_{n-10}$ Sequence: $\{3, 318, 12960, 410700, 11870715, \ldots\}$ (OEIS to be submitted) Generating Function: $\frac{3x(1+48x-697x^2-2474x^3+9918x^4+62x^5-2045x^6+96x^7+5x^8)}{(-1+29x-145x^2+145x^3-29x^4+x^5)^2}$

Figure 3.14: Full sequence results for spanning trees of $G \times C_n$ for certain G.

3.9 An Application: Divisibility Sequences

This section exhibits an application of the methods described so far in this chapter: *all* sequences produced by counting spanning trees of grid graphs are *divisibility sequences.* It was not a thought in this author's mind to think of divisibility sequences, but once it was proposed by Richard Guy [22], a complete combinatorial proof revealed itself fairly easily. This section is devoted to this combinatorial proof.

Intuitively, the grid graph $G \times P_n$ is created by placing n copies of G side-by-side and then connecting corresponding vertices in each copy by a path. A *spanning forest* is an acyclic subgraph of G. A *spanning tree* of a graph G is an acyclic connected subgraph of G. If G is disconnected, then G contains no spanning trees and the same can be said for $G \times P_n$ for any n. Recall that we let $\tau_G(n)$ denote the number of spanning trees of

$G \times P_n$, and often we will omit the subscript G when there is no ambiguity. We will be interested in special types of spanning forests.

Definition A *right-justified spanning forest* of $G \times P_n$ is a spanning forest with the property that every component of the spanning forest contains at least one vertex of $\{v_{n,i} \mid 1 \leq i \leq \nu\}$. Similarly, we can define a *left-justified spanning forest* of $G \times P_n$ as a spanning forest with the property that every component of the spanning forest contains at least one vertex of $\{v_{1,i} \mid 1 \leq i \leq \nu\}$

If F is a right-justified (resp. left-justified) spanning forest of $G \times P_n$, then the *partition induced by* F is a partition of $[\nu]$ defined by the equivalence relation

$$i \sim j \iff v_{n,i} \text{ and } v_{n,j} \text{ (resp. } v_{1,i} \text{ and } v_{1,j}\text{) belong to the same component of } F. \tag{3.1}$$

We will abuse notation and say that v_i and v_j are in the same block of a partition, when officially we mean that i and j are in the same block.

We will also be interested in counting $\tau_G(n; P)$, which is the number of right-justified spanning forests of $G \times P_n$ which induce the partition P. Note that the number of spanning trees is $\tau_G(n; \{[n]\})$. Again, the subscript G will usually be omitted. In Section 3.5, we established a general method for counting $\tau_G(n)$ by counting $\tau_G(n; P)$ for all possible values of P. The ideas behind this enumeration scheme will be extremely helpful for the main result of this section.

We will often be dealing with spanning trees of $G \times P_{2n}$, where G has ν vertices. Note that we can split up a spanning tree T of $G \times P_{2n}$ into three separate parts, specifically

1. **The left half**, $\textsc{lh}(T)$, which is the subgraph induced by the vertices

   $$\{v_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq \nu\}.$$

   Note that this is a right-justified spanning forest of $G \times P_n$.

2. **The right half**, $\textsc{rh}(T)$, which is the subgraph induced by the vertices

   $$\{v_{i,j} \mid n < i \leq 2n, 1 \leq j \leq \nu\}.$$

Note that this can be viewed as a left-justified spanning forest of $G \times P_n$ through the vertex map $v_{i,j} \mapsto v_{i-n,j}$.

3. **The middle edges**, $\textsc{mid}(T)$, which is defined by

$$E(G \times P_{2n}) \cap \{v_{n,j}v_{n+1,j} \mid 1 \leq j \leq \nu\}.$$

   Note that we can view $\textsc{mid}(T)$ as a subset of $[\nu]$, and will do so frequently.

As an example, Figure 3.15 demonstrates the breakdown of a spanning tree of $P_4 \times P_4$ into the left half, the right half, and the middle edges.

When dealing with such a spanning tree T, we will use prime ($'$) notation to refer to vertices that are on the "other side" of T. Specifically, if T is a spanning tree of $G \times P_{2n}$, then $v'_{i,j} = v_{2n-i,j}$. Note that $v'' = v$. We will also extend the prime notation to edges, so if $e = xy$, then $e' = x'y'$.

Given a spanning tree T we will define $P_L = P_L(T)$ as the partition induced by the left half of T, viewing it as a right-justified spanning tree. Similarly, we will define $P_R = P_R(T)$ as the partition induced by the right half of T, viewing it as a left-justified spanning tree. We hope the reader will not be confused by the fact that $P_L(T)$ is obtained by looking at the right-hand side of $\textsc{lh}(T)$ and vice-versa. In Figure 3.15, $P_L = \{\{1,2,3\},\{4\}\}$, $\textsc{mid}(T) = \{2,4\}$, and $P_R = \{\{1,2,3,4\}\}$.

Additionally, $\mathcal{P}_n$ denotes the family of set partitions of $[n]$, and $B(n) = |\mathcal{P}_n|$ is the n^{th} Bell Number. $\mathcal{P}_n(k)$ denotes the family of set partitions of $[n]$ with exactly k blocks.

Definition A sequence $\{b_n\}|_1^\infty$ is a *divisibility sequence* if

$$n|m \rightarrow b_n|b_m \text{ for all } n, m. \tag{3.2}$$

Additionally, the sequence is a *strong divisibility sequence* if

$$gcd(b_n, b_m) = b_{gcd(n,m)} \text{ for all } n, m. \tag{3.3}$$

For now we will consider the case where $m = 2n$; the methods that are used to show that $a_n|a_{2n}$ can be expanded to cover any other multiple. If we take an arbitrary spanning tree of $G \times P_n$ then it can be decomposed uniquely into $\textsc{lh}(T)$, $\textsc{rh}(T)$, and $\textsc{mid}(T)$.

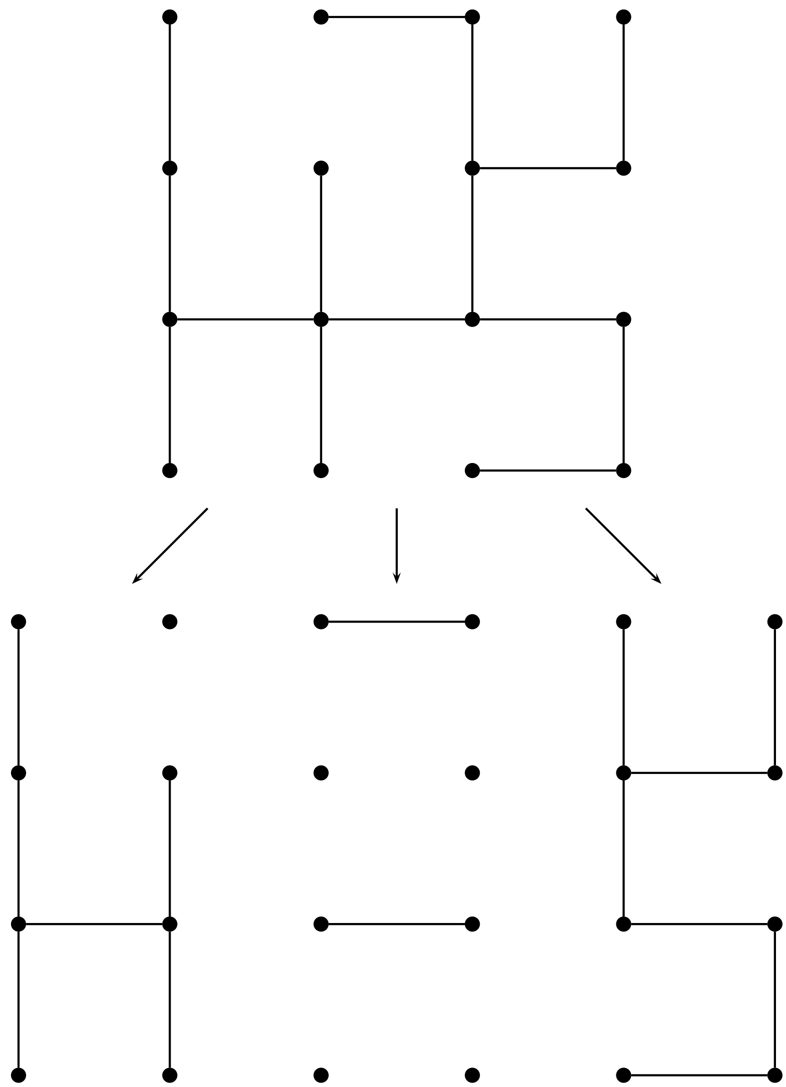

Figure 3.15: How a tree T decomposes into $\textsc{lh}(T)$, $\textsc{mid}(T)$, and $\textsc{rh}(T)$.

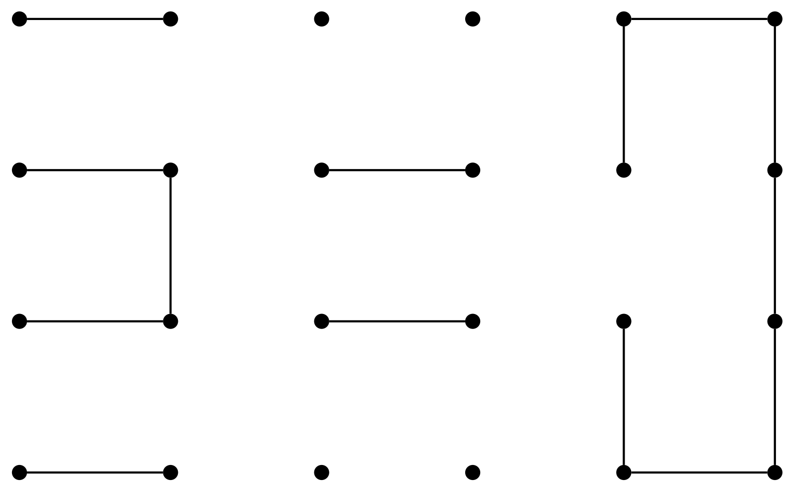

Figure 3.16: An example of the three parts of appropriate sizes that cannot combine to a spanning tree.

Additionally, if P_L has p_l blocks and $|\textsc{mid}(T)| = m$, then it follows that P_R must have $p_r = m - p_l + 1$ blocks. Note also that the partition P_R cannot be *any* partition with $m - p_l + 1$ blocks; for example, Figure 3.16 shows a left tree, right tree, and set of middle edges that cannot be combined to make a spanning tree, even though the relationship $p_r = m - p_l + 1$ still holds.

To this end we make the following definition.

Definition Given a partition P and a set of edges $\textsc{mid}$ (which is viewed as a subset of $[\nu]$), we call a partition P' *compatible with* P *and* $\textsc{mid}$ if

1. $|P'| = |\textsc{mid}| - |P| + 1$, and

2. For any two $a, b \in \textsc{mid}$, if a and b are in the same block of P, then a and b are in separate blocks of P'.

We denote by $\textsc{comp}(P, \textsc{mid})$ the set of partitions compatible with P and $\textsc{mid}$.

The second condition is necessary because if there were two edges in $\textsc{mid}$ that were both in the same block in each of the two partitions, then the combined graph would have a loop, and hence not be a tree. This is exemplified in Figure 3.16.

By conditioning on the size of the partition on the left-hand side and the number of middle edges, we can obtain a formula that relates $\tau_G(n)$ and $\tau_G(2n)$.

Lemma 3.9.1 (Counting the Number of Spanning Trees)**.**

$$\tau_G(2n) = \sum_{p=1}^{\nu} \sum_{P \in \mathcal{P}_\nu(p)} \sum_{k=1}^{\nu} \sum_{\text{MID} \in \binom{[\nu]}{k}} \tau_G(n; P) \sum_{P' \in \text{COMP}(P, \text{MID})} \tau_G(n; P'). \tag{3.4}$$

The fact that the above quantity on the right-hand side is divisible by a_n is immediate from the following lemma, which is the cornerstone of this section.

Lemma 3.9.2 (Split-Merge Lemma)**.** *Fix k, the number of edges, and p, the number of parts in the partition on the left-hand side. Then*

$$\sum_{\text{MID} \in \binom{[\nu]}{k}} \sum_{P \in \mathcal{P}_\nu(p)} \tau_G(n; P) \sum_{P' \in \text{COMP}(P, \text{MID})} \tau_G(n; P') = \binom{k-1}{p-1} \tau_G(n) \sum_{P \in \mathcal{P}(e)} (\prod P) \tau_G(n; P), \tag{3.5}$$

where $\prod P'$ is the product of all of the sizes of the parts in P'.

We shall give a bijective proof of Lemma 3.9.2. Toward the bijection, we define the following special sequence of edges.

Definition With $T = T_0$, suppose (i, j) is lexicographically least so that

1. v_i and v_j are both incident to middle edges,
2. i and j are in the same block of P_L, and
3. On the path from v_i to v_j there is an edge e such that $T' = T - e + e'$ is still a tree.

Let x_1x_2 be the first such edge satisfying the above condition (3) on the path from v_i to v_j. Defining $T_1 = T_0 - x_1x_2 + x_1'x_2'$, we can repeat the process to obtain a sequence of trees $T_1, T_2, \ldots$ and edges $e_1, e_2, \ldots$.

Remark Assuming that P_L is not the finest partition, such edges can always be found. This can be done by an inductive argument on $|P_L|$, for example.

Remark Every time an edge x_1x_2 is selected through the definition above, the block of P_L containing i and j will be split into two different blocks, and hence as a consequence two blocks of P_R will be merged into one block. We can then define the following finite sequences.

Definition (Continued) From the remarks, we see that the sequence of trees $T_1, T_2, \ldots$ and edges $e_1, e_2, \ldots$ is finite, and specifically ends at T_{p_r-1}, e_{p_r-1}. Given a tree T, we denote by $\text{L}\!\to\!\text{R}(T)$ the sequence of edges $e_1, e_2, \ldots, e_{p_r-1}$. Being in a symmetric situation with respect to P_L and P_R, we can similarly define the set $\text{R}\!\to\!\text{L}(T)$.

The following lemma is crucial to our bijection.

Lemma 3.9.3. *Letting $T' = T_{p_r-1}$, every edge of $\text{L}\!\to\!\text{R}(T)$ is in $\text{R}\!\to\!\text{L}(T')$.*

Proof. Fix T. Without loss of generality, it suffices to show that $e_1' \in \text{R}\!\to\!\text{L}(T')$. Suppose that e_1 was on the path between vertices v_i and v_j $(i < j)$ and consider the block of the partition in $P_R(T')$ that contains v_i'. If i is the smallest element in this block, then it is certain that $e_1' \in \text{R}\!\to\!\text{L}(T')$. In the case that there is a smaller element a in that block, consider the point during the process where we are about to remove e_1 and add e_1'. Part of the graph is shown in Figure 3.17, where solid lines indicate edges, dashed lines indicate edges that potentially may be there, and dotted lines indicate edges that aren't there.

With the figure in mind, we can now finish up our proof by considering, being that this is still a tree, how x_3' is connected to x_1. Specifically, the unique path from x_3' to x_1 either utilizes the edge x_3x_4 or does not. If x_3x_4 is not used, then the path can be viewed in this manner, as shown in Figure 3.18.

In this case, we have a contradiction as we cannot move e_1 to the right side, as this would cause a cycle. Similarly, if x_3x_4 was used in the path, then again there would be a contradiction for e_1 would not have been the lexicographically-least edge available to move to the right, as shown in Figure 3.19.

$\square$

From the Lemma 3.9.2, we can now prove the divisibility property.

Proof. We associate with the left-hand side the set of spanning trees T of $G \times P_{2n}$ with k middle edges and p parts in $P_L(T)$. We associate with the right-hand side a spanning tree of $G \times P_n$ (taking care of the $\tau_G(n)$ term) and a right-justified spanning forest F inducing a partition $P \in \mathcal{P}(e)$ with the following two added conditions.

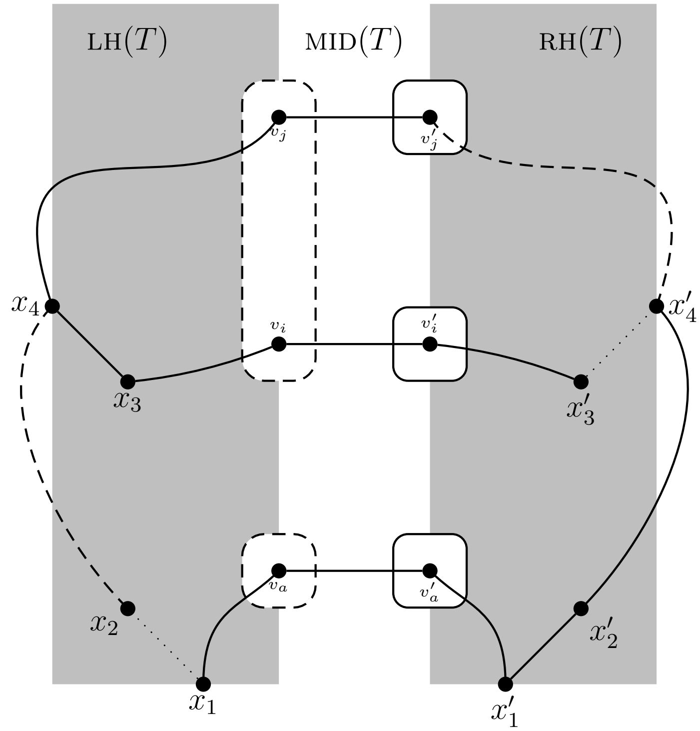

Figure 3.17: The situation in the proof of Lemma 3.9.3.

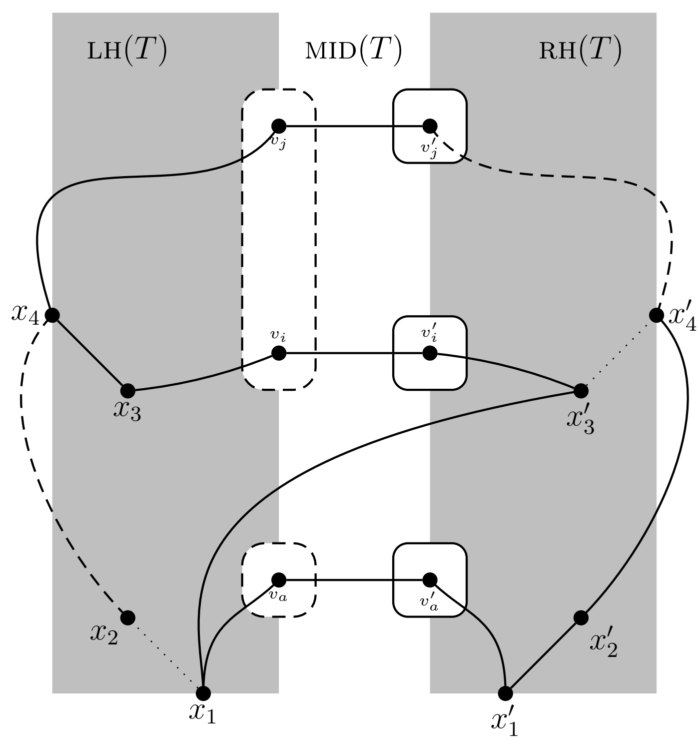

Figure 3.18: Case 1: The path from x_3' to x_1 does not involve the edge v_iv_i'.

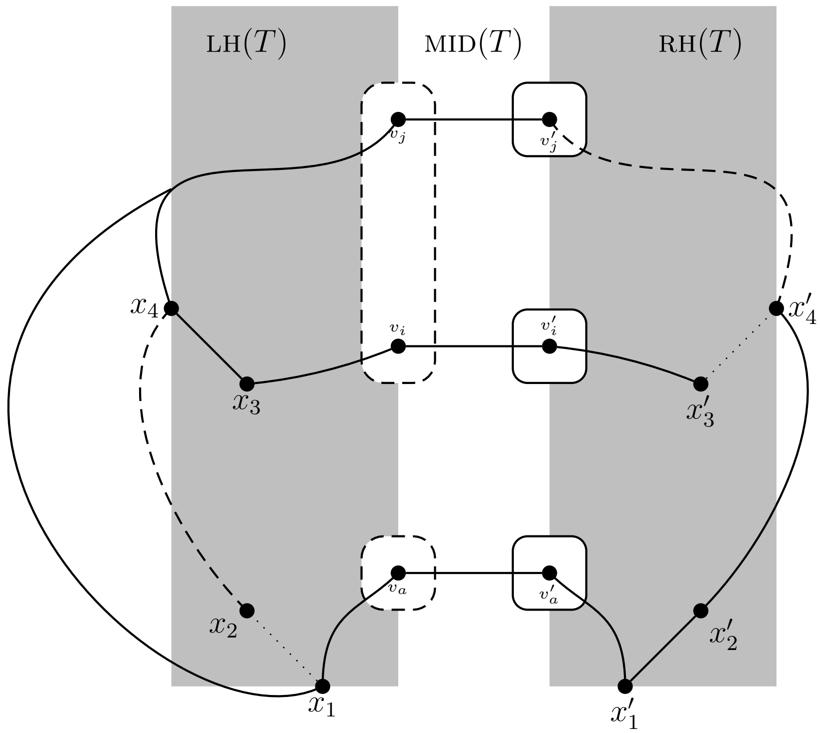

Figure 3.19: Case 2: The path from v'_i to x_1 passes through $v_iv'_i$.

1. In each block of P, one specific vertex is marked. This takes care of the $\prod P$ term.

2. A global identifier N between 1 and $\binom{k-1}{p-1}$ is assigned. This takes care of the $\binom{k-1}{p-1}$ term.

The bijection is then as follows: starting with a spanning tree T from the collection representing the left-hand side of the equation, move all of the edges in L→R(T), which consists of $p-1$ edges, over to the right-hand side of T to create T'. We can view the left half of T' as our spanning forest F and our right half as a spanning tree. Additionally, due to the way the edges were moved, each block of the partition induced by the spanning forest is incident to exactly one middle edge, so mark each vertex that is incident with a middle edge. $\square$

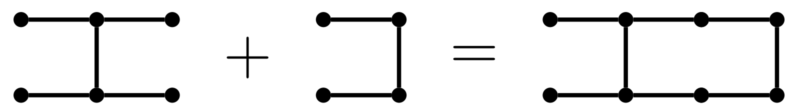

Figure 3.20: How trees are added.

3.10 Conclusions and Conjectures

From investigations, we have a few conjectures.

Conjecture 5. *For the matrix M given in Theorem 3.7.1, the characteristic polynomial $\chi_\lambda(M)$ factors over the integers into monomials whose degree is always a power of 2.*

Conjecture 6. *For any graph G, all recurrences for $\{\tau_G(n)\}_{n=1}^{\infty}$ satisfies a linear recurrence whose coefficients alternate in sign.*

Conjecture 7. *The recurrence of minimum order for the grid graph $G_k(n)$ has order 2^{k-1}.*

Conjecture 8. *The recurrence of minimum order for the graph $K_k \times P_n$ has order k.*

For the time being, we will only prove the special case of Conjecture 7 for the grid graphs $G_2(n)$. We will give a combinatorial proof that we hope can be adjusted accordingly to the higher cases. To aid in the proof, we will introduce the concept of *grid addition*, which is simply a shorthand way of creating the union of two grids.

Definition If G_1 is a subgraph of $P_k \times P_{n_1}$ and G_2 is a subgraph of $P_k \times P_{n_2}$, then $G_1 + G_2$ is a subgraph of $P_k \times P_{n_1+n_2}$ defined as the graph obtained by identifying the right-most vertices of G_1 with the left-most vertices of G_2. Any overlapping edges remain as one.

Example Figure 3.20 shows the addition of a tree on $G_2(3)$ with a tree on $G_2(2)$ to obtain a subgraph of $G_2(4)$.

Theorem 3.10.1. *The number of spanning trees of the graphs $G_2(n)$ satisfies the linear recurrence $T_n = 4T_{n-1} - T_{n-2}$ with the initial conditions $T_1 = 1$, $T_2 = 4$.*

Figure 3.21: How to interpret T_{n-2}.

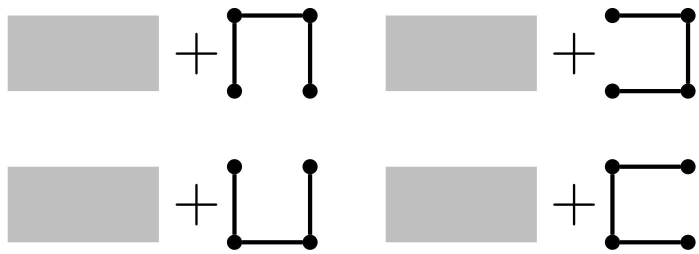

Figure 3.22: How to interpret $4T_{n-1}$.

Proof. Showing the initial conditions is a minor exercise. We will prove this recurrence in the equivalent form $T_n + T_{n-2} = 4T_{n-1}$. Let $\mathcal{T}_k$ denote the set of spanning trees of the graph $G_2(k)$. We will associate T_{n-2} with the set $\mathcal{T}_{n-2}$ with an addition at the end, as shown by Figure 3.21.

In this way, we can think of $\mathcal{T}_{n-2}$ as being trees of $G_2(n)$. Similarly, as Figure 3.22 shows, we will associate $4T_{n-1}$ with the set of trees from $\mathcal{T}_{n-1}$ with each of the four trees of $G_2(2)$ added at the end.

If we have a tree from $\mathcal{T}_n$, then we can decompose it depending on what the ending of the tree looks like. Figure 3.23 shows all of the possibilities, along with their decompositions. Note that the decompositions are of the same form as we dictated for $4T_{n-1}$.

Similarly, if we have a tree from $\mathcal{T}_{n-2}$ modified as explained above, then Figure 3.24 shows the decomposition. Again, note that the decompositions are of the same form as we dictated for $4T_{n-1}$.

The reader can verify that the map described is invertible, yielding the desired bijection. $\square$

Overall, this chapter demonstrated a concrete method for finding the recurrence and full information for the sequence $\{\tau_G(n)\}_{n=1}^{\infty}$ that counts the number of spanning trees

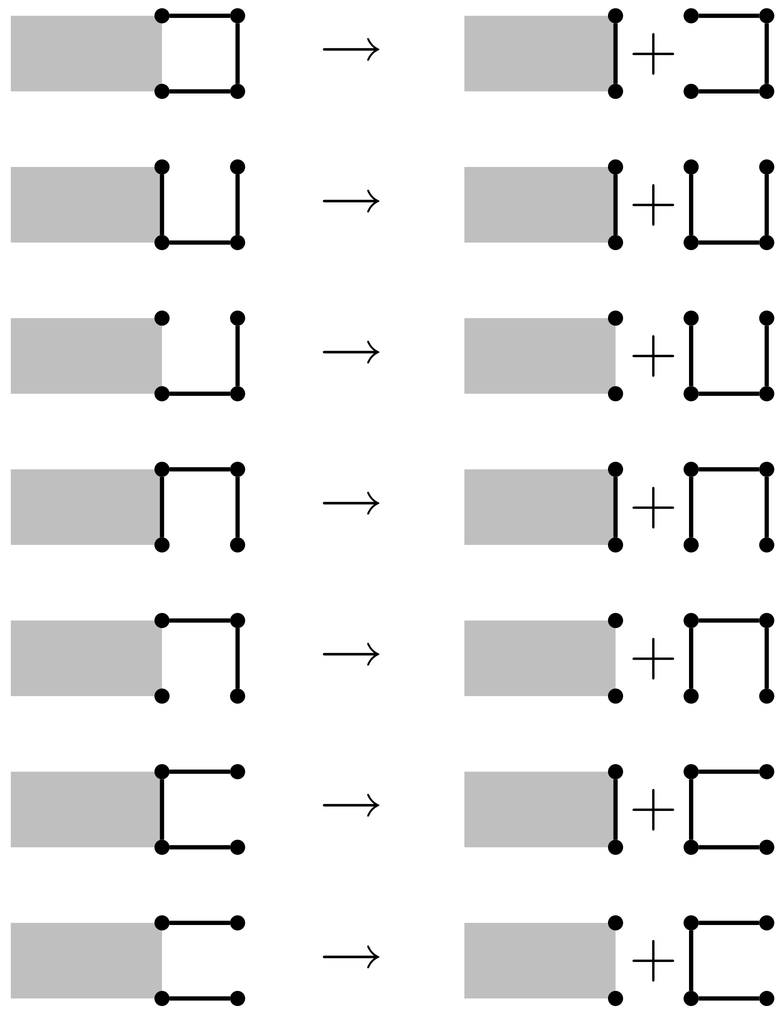

Figure 3.23: How to decompose certain elements of $\mathcal{T}_n$ into elements in $4\mathcal{T}_{n-1}$.

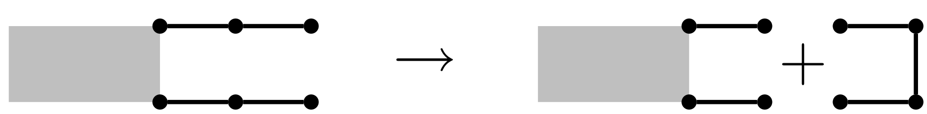

Figure 3.24: How to decompose certain elements of $\mathcal{T}_n$ into elements of $\mathcal{T}_{n-2}$.

of the grid graph $G \times P_n$. Similar methods have been demonstrated for graphs of the form $G \times C_n$ and

Chapter 4

The Firefighter Problem

4.1 Introduction and terminology

The *firefighter problem* is a dynamic problem introduced by Hartnell [27], that can be described as follows: given a connected graph G, a vertex r is initially set on fire. At the beginning of each discrete time period $t \geq 1$, a number of firefighters are available to be positioned at different vertices in G that are currently not on fire nor already have a firefighter positioned. For this paper, we shall represent the number of firefighters available at each time $t \geq 1$ by a function $f(t)$. These firefighters remain on their assigned vertices and thus prevent the fire from spreading to that vertex. At the end of each time period, all vertices that are not defended and are adjacent to at least one vertex on fire will catch the fire and become burned. Once the vertex is burned or defended, it remains that way permanently.

If G is a finite graph, the process ends when one of the following occurs.

(i) The fire is *contained*, meaning the fire is unable to spread, and there are still vertices in G that are neither burned nor defended.

(ii) The fire spreads until every vertex in G is either burned or defended.

If G is infinite, then (i) could still happen but (ii) is replaced by

(ii′) The fire cannot be contained, meaning the fire spreads indefinitely.

The firefighter problem was considered on a variety of graphs, including finite grids (MacGillivray and Wang [31], Wang and Moeller [53]), infinite grids (Develin and Hartke [13], Wang and Moeller [53], Fogarty [18]) and trees (MacGillivray et.al. [17], Hartnell [27]). Other related publications [1,3,4,6-8] are listed in the reference section.

The firefighter problem can be viewed in a more general context as a *monotonic irreversible k-threshold with vaccinations* (see [14]) process for $k = 1$ and many questions still remain for these generalizations. In these types of processes, the vertices on the underlying graph can take on either of the two values 0 or 1, corresponding to unburnt and burnt, and the value of these vertices can change over time. This process is *monotonic* because the set of vertices on fire (having value 1) at time t is a subset of the vertices on fire at time $t+1$. It is *irreversible* because once a vertex catches fire, it is on fire permanently. It is a 1-*threshold process* because an undefended vertex only needs to have one of its neighbors to be on fire at time t for it to catch on fire at time $t+1$. This is understandable in a firefighting setting, where adjacency to fire is all that is usually need to catch on fire shortly. Increasing the threshold factor k is more useful in an epidemiological setting, where association to a sick person is often not enough for an individual to contract a disease, yet being around enough people who are sick would be enough to contract the disease. Additionally, this process is *with vaccination* since there are *firefighters* (or *vaccinations* in the public-health setting) that can be placed on unburnt vertices and allow that vertex to remain unburnt permanently.

In this chapter, we will consider the two-dimensional infinite grid graph $G = \mathbb{L}_2$ defined by

$$V(G) = \mathbb{Z} \times \mathbb{Z},$$

$$E(G) = \{\{(m,n),(m',n')\} \mid |m-m'| + |n-n'| = 1\}.$$

Suppose we are given a function $f(t)$ representing the number of firefighters available for deployment at each time period t, our goal is to determine if it is possible to position the firefighters on the vertices of $\mathbb{L}_2$ such that at some finite time t', the fire is unable to spread any further. For our purposes, we shall only consider functions $f(t)$ that are periodic in t. Thus, we can state our problem formally as

CONTAINMENT

INSTANCE: A rooted graph $(\mathbb{L}_2, r)$ and a periodic function $f(t)$.

QUESTION: Is there a finite t' such that by positioning $f(t)$ firefighters at each time period t, the fire can be contained after t' time periods.

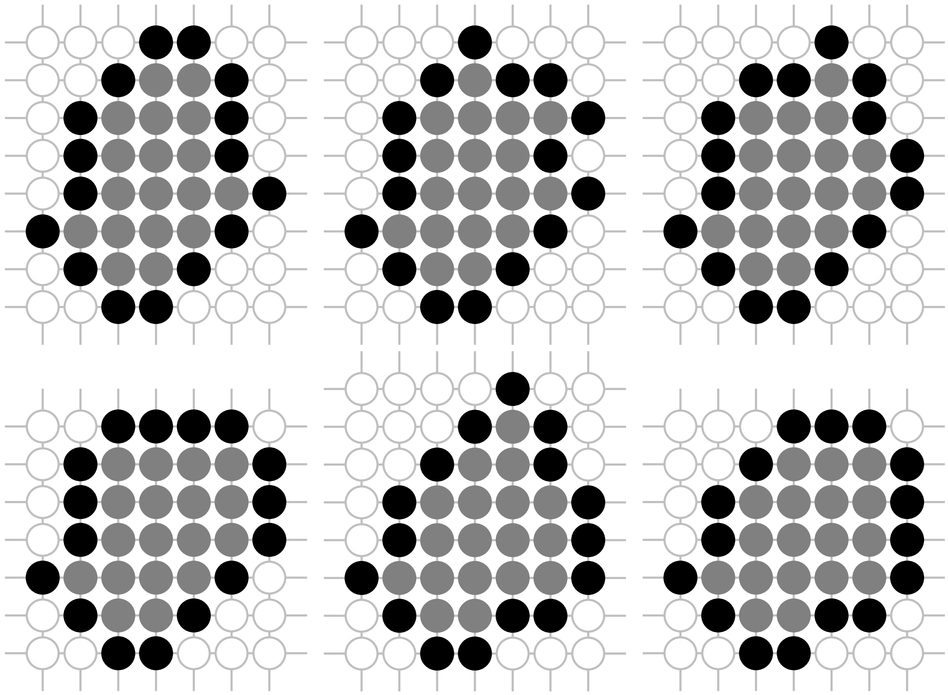

Figure 4.1: The six non-isomorphic minimal solutions to the firefighter problem with $f(t) = 2$. It is left to the reader to determine where the fire started in each scenario!

Most of the existing literature considers $f(t)$ to be a constant function (usually $f(t) = 1$) independent of t. Specifically, Wang and Moeller [53] showed that one firefighter per time period ($f(t) = 1 \ \forall t$) is insufficient to prevent the fire from spreading indefinitely while $f(t) = 2$ for all t suffices, in which case a minimum of 8 time periods are required to succesfully contain the fire. An alternative proof (using a computer program) to the minimum number of time periods required when $f(t) = 2$ for all t was provided by Develin and Hartke [13], who also established that a minimum of 18 vertices in $\mathbb{L}_2$ would be burnt before containment can be achieved. The six non-isomorphic minimal solutions taking 8 turns and burning 18 vertices are shown as follows, found using `Mathematica` software that can be found at [39].

One way to generalize the firefighter problem introduced by Hartnell is to allow the fire to start initially at a finite number of vertices in $\mathbb{L}_2$ rather than a single root r. This was considered by Fogarty [18] when it was shown that $f(t) = 2$ for all t is sufficient to contain a fire that starts at any *finite* number of vertices in $\mathbb{L}_2$. For the remainder of this paper, we shall consider the firefighter problem where the fire could start initially at either a single vertex or a finite collection of vertices in $\mathbb{L}_2$.

The results by Wang and Moeller [53], Develin and Hartke [13] and Fogarty [18] described above provide the motivation. We would like to know if $f(t)$ is not a constant function, and the *average* (whose notion will be made precise below) number of firefighters available per time period is a number between 1 and 2, is there a finite t' such that by positioning $f(t)$ firefighters at each time period t, the fire can be contained after t' time periods?

To make the notion of the average number of firefighters per time period precise, let $f : \mathbb{N} \to \mathbb{N} \cup \{0\}$ be a periodic function with period p_f. Define

$$N_f = \sum_{t=1}^{p_f} f(t) \quad \text{and} \quad R_f = \frac{N_f}{p_f}.$$

Thus, if the number of firefighters available for deployment at each time period is given by f, then R_f tells us the *average number* of firefighters available for deployment at each time period. We will frequently identify f with a sequence of its period. For example, we write $f = [2, 1, 2, 2]$ to correspond to the function defined as

$$f(t) = \begin{cases} 2 & \text{if } t \equiv 1 \bmod 4, \\ 1 & \text{if } t \equiv 2 \bmod 4, \\ 2 & \text{if } t \equiv 3 \bmod 4, \\ 2 & \text{if } t \equiv 0 \bmod 4. \end{cases}$$

Observe that $R_f = 1.75$ in this example.

Definition For any function $f : \mathbb{N} \to \mathbb{N} \cup \{0\}$, define $f^{-1} : \mathbb{N} \to \mathbb{N}$ by

$$f^{-1}(n) = \min \left\{ j \in \mathbb{N} \mid \sum_{t=1}^{j} f(t) \geq n \right\}.$$

In other words, $f^{-1}(n)$ can be thought of as the time t when the n^{th} firefighter becomes available for deployment.

Note that $f^{-1}(n)$ is a nondecreasing function of n.

Definition For a finite set $S \subset \mathbb{Z} \times \mathbb{Z}$ and some $(x, y) \in \mathbb{Z} \times \mathbb{Z}$, define

$$d(S, (x, y)) = \min\{|x' - x| + |y' - y| \mid (x', y') \in S\}.$$

Definition For any periodic function f and $S \subset \mathbb{Z} \times \mathbb{Z}$, we say that there is a *containment certificate of f for S* if and only if there exists a set $C_S(f) \subset \mathbb{Z} \times \mathbb{Z} \times \mathbb{N}$ that satisfies the following conditions.

1. For all $t \in \mathbb{N}$, $f(t) \geq |\{(x, y, j) \in C_S(f) \mid j = t\}|$;
2. For all $(x, y, t) \in C_S(f)$, $d(S, (x, y)) \geq t$;
3. The number of vertices that have at least one path in $\mathbb{L}_2$ to a vertex in S without passing through any vertex (x, y) where $(x, y, t) \in C_S(f)$ for some $t \in \mathbb{N}$ is finite.

We extend the definition slightly and call a containment certificate *convex* if it satisfies a fourth condition.

4. For every $(x, y, t) \in C_S(f)$, there is a path from a vertex in S to (x, y) of length $D(S, (x, y))$.

Hopefully the use of the word *convex* will not frustrate the reader. A containment certificate can be extended readily to a closed curve in $\mathbb{R}^2$ (this will be expounded on later in the chapter) and the notion of convexity only partially derives from this curve. While there are convex containment certificates that yield concave curves, it is true that any non-convex containment certificate produces a concave curve in $\mathbb{R}^2$.

Suppose that the set of vertices in S are initially set on fire and $f(t)$ represents the number of firefighters available for deployment at time t. A containment certificate of f for S, if it exists, contains all the information on where and when each available firefighter is deployed such that the spread of the fire can eventually be contained at some finite time t'. For example if $(8, 9, 4) \in C_S(f)$, then we would place a firefighter on $(8, 9)$ at time $t = 4$. Condition 1 of the containment certificate ensures that there are at most $f(t)$ firefighters deployed at time t. Condition 2 ensures that (x, y) is not already on fire when a firefighter is deployed there at time t. Condition 3 gurantees

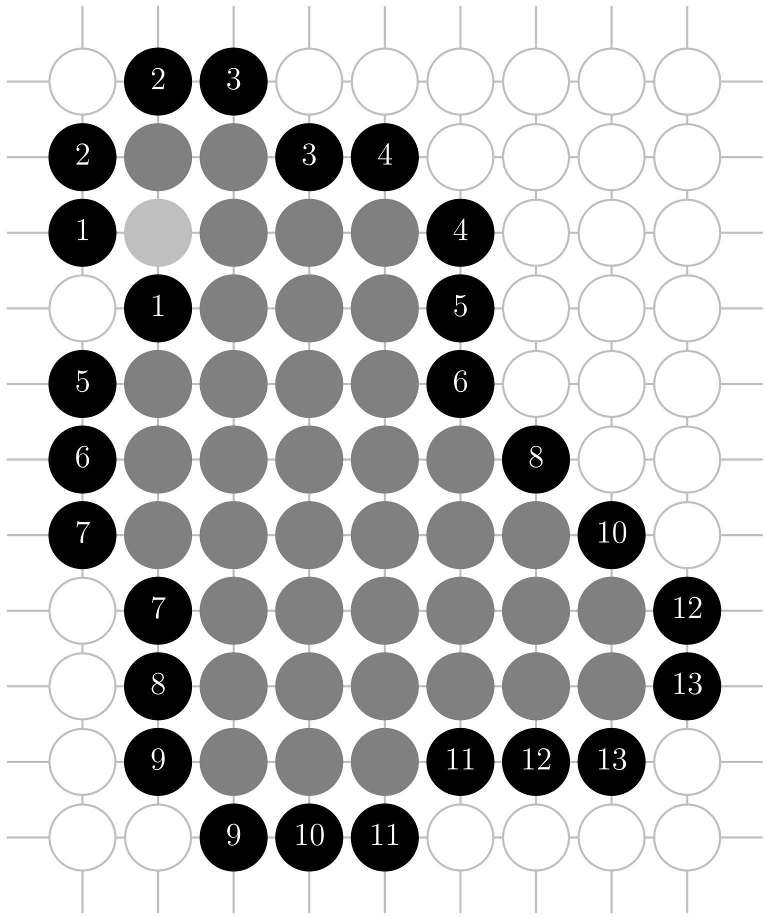

Figure 4.2: An example of a concave scenario. The initial fire is the vertex that is lighter than the others.

that there exists some $t' \geq \max\{t \mid (x, y, t) \in C_S(f)\}$ such that the number of vertices on fire at times $t \geq t'$ is a constant, meaning that the fire is indeed contained.

Suppose $C_S(f)$ is a containment certificate of f for S. For each $n \in \mathbb{N}$, define

$$\begin{aligned} C_S^{>n}(f) &= \{(x, y, t) \in C_S(f) \mid t > n\}; \\ C_S^{=n}(f) &= \{(x, y, t) \in C_S(f) \mid t = n\}; \\ C_S^{<n}(f) &= \{(x, y, t) \in C_S(f) \mid t < n\}. \end{aligned}$$

We will consider two partial orders associated with periodic functions.

Definition

$$f \preceq g \iff \sum_{t=1}^{k} f(t) \leq \sum_{t=1}^{k} g(t) \quad \forall k \in \mathbb{N}.$$

Additionally, we say that g *dominates* f if $f \preceq g$.

Definition

$$f \preceq^* g \iff \exists n \in \mathbb{N} \text{ such that } \sum_{t=1}^{k} f(t) \leq \sum_{t=1}^{k} g(t) \quad \forall k \geq n.$$

Additionally, we say that g *eventually dominates* f if $f \preceq^* g$.

Observe the fact that g dominates f implies g eventually dominates f. It is useful to note that to establish $f \preceq g$ for periodic f and g, it suffices to show that

$$\sum_{t=1}^{k} f(t) \leq \sum_{t=1}^{k} g(t) \quad \text{for all } 1 \leq k \leq lcm(p_f, p_g).$$

Several specific periodic functions will be used frequently in this paper. Their definitions and notations are introduced below.

Definition For any $n, k \in \mathbb{Z}^+$, $g_{n,k}$ is the periodic function with period n defined by

$$g_{n,k}(t) = \begin{cases} 0 & \text{if } t \not\equiv 0 \bmod n, \\ k & \text{if } t \equiv 0 \bmod n. \end{cases}$$

In other words, $g_{n,k} = [\overbrace{0, 0, ..., 0}^{n-1}, k]$.

Definition For any integer $n \geq 2$, $Z_n = g_{n,z_n}$ where

$$z_n = \begin{cases} \frac{3n}{2} + 1 & \text{if } n \text{ is even,} \\ \\ \frac{1}{2}(3n+1) & \text{if } n \text{ is odd.} \end{cases}$$

Note that for each n, z_n is defined to be the smallest positive integer such that $R_{Z_n} > 1.5$.

Definition For any integer $n \geq 1$,

$$F_n(t) = \begin{cases} 1 & \text{if } t \equiv k \bmod 2n+1, \text{ where } k \in \{1, 2, ..., n\} \\ \\ 2 & \text{if } t \equiv k \bmod 2n+1, \text{ where } k \in \{0, n+1, n+2, ..., 2n\}. \end{cases}$$

In other words, $F_n = [\overbrace{1, 1, ..., 1}^{n}, \overbrace{2, 2, ..., 2}^{n+1}]$. Note that $p_{F_n} = 2n + 1$ and $R_{F_n} > 1.5$ for all $n \geq 1$.

Definition If f is a periodic function and i is any non-negative integer, f_{+i} is the *i-translate* of f, defined by

$$f_{+i}(t) = f(t + i) \quad \text{for all } t \geq 1.$$

Note that $f_{+0} = f$. We are now ready to state the main result of this chapter.

Theorem 4.1.1. *Suppose a finite set $S \subset \mathbb{Z} \times \mathbb{Z}$ of vertices are initially set on fire. If the number of firefighters available for deployment per time period is given by a periodic function f such that $R_f > 1.5$, then there exists a containment certificate of f for S.*

Remark The above theorem gives no conclusion about containment of the fire if the function f is such that $R_f \leq 1.5$. We will discuss this briefly at the end of the chapter.

In Section 4.2, we will prove several lemmas regarding some of the periodic functions defined above. The main result is proven in Section 4.3 and the chapter concludes in Section 4.5 with a brief discussion on possible future work. Most of the results through Section 4.3 come from [35]. Results after Section 4.3 come from [42].

4.2 Several lemmas

We first show that the relation $\preceq^*$ is transitive.

Lemma 4.2.1. *If f, g and h are periodic functions such that $f \preceq^* g$ and $g \preceq^* h$, then $f \preceq^* h$.*

Proof. Let $n_1, n_2 \in \mathbb{N}$ be such that

$$\sum_{t=1}^{k} f(t) \leq \sum_{t=1}^{k} g(t) \quad \forall k \geq n_1 \quad \text{and} \quad \sum_{t=1}^{k} g(t) \leq \sum_{t=1}^{k} h(t) \quad \forall k \geq n_2.$$

Let $n = \max\{n_1, n_2\}$. We have

$$\sum_{t=1}^{k} f(t) \leq \sum_{t=1}^{k} h(t) \quad \forall k \geq n$$

and thus $f \preceq^* h$. $\square$

Lemma 4.2.2. *For any periodic function f, we have $g_{p_f,N_f} \preceq f$.*

Proof. Note that g_{p_f,N_f} and f have the same period. If $k < p_f$ then we have

$$0 = \sum_{t=1}^{k} g_{p_f,N_f}(t) \leq \sum_{t=1}^{k} f(t)$$

since f must take on non-negative values. If $k = p_f$ then

$$\sum_{t=1}^{p_f} g_{p_f,N_f}(t) = \sum_{t=1}^{p_f} f(t)$$

and so by definition we have $g_{p_f,N_f} \preceq f$. $\square$

Lemma 4.2.3. *If f is a periodic function that is non-decreasing on its period, then $f \preceq f_{+i}$ for all $i \in \mathbb{Z}^+$.*

Proof. Let $i \in \mathbb{Z}^+$. Since f and f_{+i} have the same period, it suffices to show

$$\sum_{t=1}^{n} f(t) \leq \sum_{t=1}^{n} f_{+i}(t) \quad \text{for all } n \leq p_f.$$

Case 1: Suppose $n+i \leq p_f$. In this case, as f is non-decreasing, we have $f(t) \leq f(t+i)$ for all $t = 1, 2, ..., n$, implying

$$\sum_{t=1}^{n} f(t) \leq \sum_{t=1}^{n} f(t+i)$$

and thus $f \preceq f_{+i}$.

Case 2: Suppose $n+i > p_f$. Note that

$$\begin{aligned}\sum_{t=1}^{n} f(t+i) = \sum_{t=i+1}^{n+i} f(t) &= \sum_{t=i+1}^{p_f} f(t) + \sum_{t=p_f+1}^{n+i} f(t) \\ &= \sum_{t=i+1}^{p_f} f(t) + \sum_{t=1}^{n+i-p_f} f(t).\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{t=1}^{n} f(t) &= \sum_{t=1}^{n+i-p_f} f(t) + \sum_{t=n+i-p_f+1}^{n} f(t) \\ &\leq \sum_{t=1}^{n+i-p_f} f(t) + \sum_{t=i+1}^{p_f} f(t) \quad \text{(since } f \text{ is non-decreasing)} \\ &= \sum_{t=i+1}^{n+i} f(t) = \sum_{t=1}^{n} f_{+i}(t)\end{aligned}$$

and we are done. □

Lemma 4.2.4. *If f is a periodic function such that $p_f \geq 2$ and $R_f > 1.5$, then $Z_n \preceq f$ for some $n \geq 2$.*

Proof. Take $n = p_f$. □

If we want to compare two periodic functions f and g, then as stated before we would have to compare f and g up to $lcm(p_f, p_g)$, which could be as large as $p_f p_g$. The following lemma adds a hypothesis but the end result allows us to simply compare the two functions up to one specific value.

Lemma 4.2.5. *Let g be a periodic function that is non-decreasing on its period and f be a periodic function such that $p_f \geq p_g$ and*

$$\sum_{t=1}^{p_f} f(t) < \sum_{t=1}^{p_f} g(t).$$

Then $f \preceq^* g$, meaning there exists $n \in \mathbb{N}$ such that

$$\sum_{t=1}^{k} f(t) \leq \sum_{t=1}^{k} g(t) \quad \textit{for all } k \geq n.$$

Proof. We first prove the following claim.

Claim: For each $k = 1, 2, 3, \ldots,$

$$\sum_{t=kp_f+1}^{(k+1)p_f} f(t) < \sum_{t=kp_f+1}^{(k+1)p_f} g(t).$$

Proof of Claim: Let $kp_f + 1 = k'p_g + r$, with $0 < r \leq p_g$. Then we have

$$\begin{aligned}
\sum_{t=kp_f+1}^{(k+1)p_f} g(t) &= \sum_{t=r}^{r+p_f-1} g(t) \\
&= \sum_{t=1}^{p_f} g_{+(r-1)}(t) \\
&\geq \sum_{t=1}^{p_f} g(t) \quad \text{by Lemma 4.2.3} \\
&> \sum_{t=1}^{p_f} f(t) = \sum_{t=kp_f+1}^{(k+1)p_f} f(t).
\end{aligned}$$

So from the above claim, the following function

$$h(k) = \sum_{t=1}^{kp_f} g(t) - \sum_{t=1}^{kp_f} f(t)$$

is a strictly increasing function in k. Define k^* by

$$k^* = \min\{k \in \mathbb{N} \mid h(k) > N_f\}.$$

Now let $n = k^* p_f$. This is the n that we require in order to prove the lemma. To see this, suppose $k \geq n$ and $k = a_k p_f + b_k$, where $0 \leq b_k < p_f$. Then

$$\begin{aligned}
\sum_{t=1}^{k} f(t) &= \sum_{t=1}^{a_k p_f + b_k} f(t) \\
&\leq \sum_{t=1}^{a_k p_f + p_f} f(t) \\
&= \sum_{t=1}^{(a_k+1)p_f} f(t) \\
&= \sum_{t=1}^{a_k p_f} f(t) + \sum_{t=a_k p_f+1}^{(a_k+1)p_f} f(t)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t=1}^{a_k p_f} f(t) + \sum_{t=1}^{p_f} f(t) \\
&= \left(\sum_{t=1}^{a_k p_f} g(t) - h(a_k) \right) + \sum_{t=1}^{p_f} f(t) \\
&\leq \left(\sum_{t=1}^{a_k p_f} g(t) - h(k^*) \right) + \sum_{t=1}^{p_f} f(t) \quad \text{(since } a_k \geq k^*\text{)} \\
&< \left(\sum_{t=1}^{a_k p_f} g(t) - \sum_{t=1}^{p_f} f(t) \right) + \sum_{t=1}^{p_f} f(t) \\
&= \sum_{t=1}^{a_k p_f} g(t) \leq \sum_{t=1}^{a_k p_f + b_k} g(t) = \sum_{t=1}^{k} g(t).
\end{aligned}
$$

The proof of the lemma is thus complete. ☐

Using Lemma 4.2.5 we can prove the next lemma easily.

Lemma 4.2.6. *For each $n \geq 2$, $F_{n^2} \preceq^* Z_n$.*

Proof. Note that F_{n^2} is periodic, $p_{F_{n^2}} = 2n^2 + 1 \geq n = p_{Z_n}$ and

$$\sum_{t=1}^{2n^2+1} F_{n^2}(t) = n^2 + 2(n^2+1) = 3n^2 + 2.$$

If n is even, then

$$\sum_{t=1}^{2n^2+1} Z_n(t) = 2n\left(\frac{3n}{2} + 1\right) = 3n^2 + 2n.$$

On the other hand, if n is odd, then

$$\sum_{t=1}^{2n^2+1} Z_n(t) = 2n\left(\frac{3n+1}{2}\right) = 3n^2 + n.$$

In either case, we have

$$\sum_{t=1}^{2n^2+1} F_{n^2}(t) < \sum_{t=1}^{2n^2+1} Z_n(t)$$

and thus by Lemma 4.2.5, $F_{n^2} \preceq^* Z_n$. ☐

Lemma 4.2.7. *Given any periodic function f such that $p_f \geq 2$ and $R_f > 1.5$, there exists some $n \geq 2$ such that $F_{n^2} \preceq^* f$.*

Proof. Suppose f is periodic, $p_f \geq 2$ and $R_f > 1.5$. By Lemma 4.2.2, $g_{p_f,N_f} \preceq^* f$. Note that $R_{g_{p_f,N_f}} = R_f > 1.5$, so by Lemmas 4.2.4 and 4.2.6, for some $n \geq 2$,

$$F_{n^2} \preceq^* Z_n \preceq^* g_{p_f,N_f}.$$

Applying Lemma 4.2.1 to

$$F_{n^2} \preceq^* Z_n \preceq^* g_{p_f,N_f} \preceq^* f$$

completes the proof. $\square$

4.3 Proof of main result

We first state two lemmas without proof.

Lemma 4.3.1. *Suppose X_1 and X_2 are both finite subsets of $\mathbb{Z} \times \mathbb{Z}$ such that $X_1 \subseteq X_2$. For any function f, if $C_{X_2}(f)$ is a containment certificate of f for X_2, then $C_{X_2}(f)$ is also a containment certificate of f for X_1.*

Definition For any $d \in \mathbb{N} \cup \{0\}$, define

$$S_d = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid |x| + |y| \leq d\}.$$

Lemma 4.3.2. *For any $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $(x, y) \notin S_d$,*

$$d(S_d, (x, y)) = |x| + |y| - d.$$

Now for any $n \in \mathbb{N}$, recall that $F_n = [\overbrace{1, 1, ..., 1}^{n}, \overbrace{2, 2, ..., 2}^{n+1}]$ is a periodic function with period $2n + 1$. Let

$$F_n^2 = [\overbrace{1, 1, ..., 1}^{n}, \overbrace{2, 2, ..., 2}^{n+1}, \overbrace{1, 1, ..., 1}^{n}, \overbrace{2, 2, ..., 2}^{n+1}].$$

Note that F_n^2 is periodic with period $2(2n + 1)$ and $F_n^2 \preceq F_n$. Let $p = 2(2n + 1)$ and define the function G_p of period p by

$$G_p = [\overbrace{1, 0, 1, 0, ..., 1}^{p-1}, p + 1].$$

It is easy to see that $G_p \preceq F_n^2$.

Lemma 4.3.3. *For any $n, d \in \mathbb{N}$, let $p = 2(2n+1)$. There exists a containment certificate of G_p for S_d.*

Proof. Consider the following eight sets.

$$
\begin{aligned}
A_0 &= \bigcup_{i=1,\ i \text{ odd}}^{2(p+1)^3(d+p)-1} \left\{ \left(\frac{i-1}{2}, -\left(d+p+\frac{i-1}{2}\right), i \right) \right\}, \\
A_1 &= \bigcup_{i=1}^{d+p} \bigcup_{k=1}^{p} \left\{ (-(i-1)p-k, -(d+p), ip) \right\}, \\
A_2 &= \bigcup_{i=1}^{d+p} \left\{ (-p(d+p)-i, -(d+p)+i, ip) \right\}, \\
A_3 &= \bigcup_{i=1}^{(p+1)(d+p)} \bigcup_{k=1}^{p} \left\{ (-(p+1)(d+p), (i-1)p+k, (d+p+i)p) \right\}, \\
A_4 &= \bigcup_{i=1}^{(p+1)(d+p)} \left\{ (-(p+1)(d+p)+i, p(p+1)(d+p)+i, (d+p+i)p) \right\}, \\
A_5 &= \bigcup_{i=1}^{(p+1)^2(d+p)} \bigcup_{k=1}^{p} \left\{ ((i-1)p+k, (p+1)^2(d+p), ((p+2)(d+p)+i)p) \right\}, \\
A_6 &= \bigcup_{i=1}^{(p+1)^2(d+p)} \left\{ (p(p+1)^2(d+p)+i, (p+1)^2(d+p)-i, ((p+2)(d+p)+i)p) \right\}, \\
A_7 &= \bigcup_{i=1}^{N} \left\{ \begin{array}{l} ((p+1)^3(d+p), -(i-1)p-k, ((d+p)(p+2+(p+1)^2)+i)p) \mid \\ \\ 1 \leq k \leq p \text{ and } (i-1)p+k \leq ((p+1)^3+1)(d+p)+2 \end{array} \right\},
\end{aligned}
$$

where

$$
N = \left\lceil \frac{((p+1)^3+1)(d+p)+2}{p} \right\rceil .
$$

We claim that $A = \bigcup_{i=0}^{7} A_i$ is a containment certificate of G_p for S_d.

Figure 1 illustrates the positions corresponding to the set $A = \bigcup_{i=0}^{7} A_i$.

Recall that an element (x, y, t) in a containment certificate can be thought of as the time t where a firefighter is positioned at (x, y). To show that the first condition in the definition of a containment certificate is satisfied, it is easier to describe the elements of the eight sets in terms on their positions on $\mathbb{Z} \times \mathbb{Z}$ and when these positions are taken

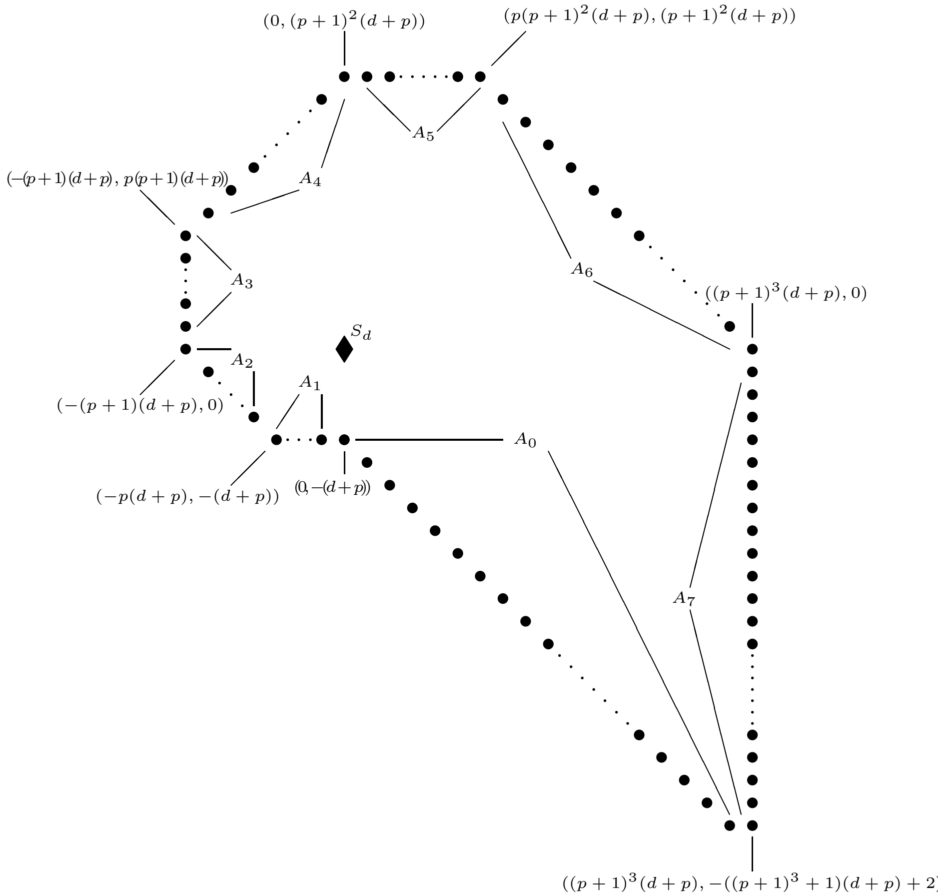

Figure 4.3: A global view of the containment certificate described in the proof of Lemma 4.3.3.

up by the firefighters. Note that $G_p(t) = 1$ for all odd t, $G_p(t) = p+1$ if $t = kp$ for some $k \in \mathbb{N}$ and $G_p(t) = 0$ otherwise.

1. At each odd $t = 1, 3, ..., 2(p+1)^3(d+p)-1$, a firefighter is positioned at $(\frac{t-1}{2}, -(d+p+\frac{t-1}{2}))$. This corresponds to the set A_0.

2. At each $t = ip$, $i = 1, 2, ..., d+p$, we have $p+1$ firefighters available, p of which have positions given by A_1 (forming a horizontal line) and the remaining one has position given by A_2 (forming a diagonal line).

3. At each $t = (d+p+i)p$, $i = 1, ..., (p+1)(d+p)p$, we have $p+1$ firefighters available, p of which have positions given by A_3 (forming a vertical line) and the remaining one has position given by A_4 (forming a diagonal line).

4. At each $t = ((p+2)(d+p)+i)p$, $i = 1, ..., (p+1)^2(d+p)$, we have $p+1$ firefighters available, p of which have positions given by A_5 (forming a horizontal line) and the remaining one has position given by A_6 (forming a diagonal line).

5. At each $t = ((d+p)(p+2+(p+1)^2)+i)p$, $i = 1, ..., N$, we place p firefighters at positions given by A_7. This forms a vertical line and the positioning ends when this vertical line meets with the diagonal line formed by firefighters whose positions corresponds to the set A_0.

We next check the second condition in the definition of a containment certificate.

Case 1: Suppose $(\frac{i-1}{2}, -(d+p+\frac{i-1}{2}), i) \in A_0$ for some $i \in \{1, 3, ..., 2(p+1)^3(d+p)-1\}$. By Lemma 4.3.2,

$$\begin{aligned} d\left(S_d, \left(\frac{i-1}{2}, -\left(d+p+\frac{i-1}{2}\right)\right)\right) &= \left|\frac{i-1}{2}\right| + \left|-\left(d+p+\frac{i-1}{2}\right)\right| - d \\ &= \frac{i-1}{2} + \left(d+p+\frac{i-1}{2}\right) - d \\ &= p+i-1 \geq i \quad \text{(since } p \geq 6\text{).} \end{aligned}$$

Case 2: Suppose $(-(i-1)p-k, -(d+p), ip) \in A_1$ for some $i \in \{1, 2, ..., d+p\}$ and some $k \in \{1, ..., p\}$. By Lemma 4.3.2,

$$\begin{aligned} d(S_d, (-(i-1)p-k, -(d+p))) &= |-(i-1)p-k| + |-(d+p)| - d \\ &= (i-1)p+k+(d+p)-d \\ &= ip+k \geq ip. \end{aligned}$$

Case 3: Suppose $(-p(d+p)-i, -(d+p)+i, ip) \in A_2$ for some $i \in \{1, 2, ..., d+p\}$. By Lemma 4.3.2,

$$\begin{aligned} d(S_d, (-p(d+p)-i, -(d+p)+i)) &= |-p(d+p)-i| + |-(d+p)+i| - d \\ &= p(d+p)+i+(d+p)-i-d \\ &= p(d+p+1) \geq ip. \end{aligned}$$

Case 4: Suppose $(-(p+1)(d+p), (i-1)p+k, (d+p+i)p) \in A_3$ for some $i \in \{1, 2, ..., (p+1)(d+p)\}$ and $k \in \{1, ..., p\}$. By Lemma 4.3.2,

$$\begin{aligned} d(S_d, (-(p+1)(d+p), (i-1)p+k)) &= |-(p+1)(d+p)| + |(i-1)p+k| - d \\ &= (p+1)(d+p)+(i-1)p+k-d \\ &= pd+p^2+d+p+ip-p+k-d \\ &= pd+p^2+ip+k \geq (d+p+i)p. \end{aligned}$$

Case 5: Suppose $(-(p+1)(d+p)+i, p(p+1)(d+p)+i, (d+p+i)p) \in A_4$ for some $i \in \{1, 2, ..., (p+1)(d+p)\}$. By Lemma 4.3.2,

$$\begin{aligned} & d(S_d, (-(p+1)(d+p)+i, p(p+1)(d+p)+i)) \\ = \;& |-(p+1)(d+p)+i| + |p(p+1)(d+p)+i| - d \\ = \;& (p+1)(d+p)-i+p(p+1)(d+p)+i-d \\ = \;& (p+1)^2(d+p)-d \\ = \;& p^2d+p^3+2pd+2p^2+p \\ \geq \;& p^2d+p^3+2pd+2p^2 \\ = \;& (p+2)(d+p)p \\ = \;& (d+p+(p+1)(d+p))p \\ \geq \;& (d+p+i)p. \end{aligned}$$

Case 6: Suppose $((i-1)p+k,(p+1)^2(d+p),((p+2)(d+p)+i)p) \in A_5$ for some $i \in \{1,...,(p+1)^2(d+p)\}$ and $k \in \{1,...,p\}$. By Lemma 4.3.2,

$$\begin{aligned}
d(S_d,((i-1)p+k,(p+1)^2(d+p)) &= |(i-1)p+k|+|(p+1)^2(d+p)|-d \\
&= ip-p+k+(p^2+2p+1)(d+p)-d \\
&= ip+k+p^2d+p^3+2pd+2p^2 \\
&\geq ip+p(pd+p^2+2d+2p) \\
&= ((p+2)(d+p)+i)p.
\end{aligned}$$

Case 7: Suppose $(p(p+1)^2(d+p)+i,(p+1)^2(d+p)-i,((p+2)(d+p)+i)p) \in A_6$ for some $i \in \{1,...,(p+1)^2(d+p)\}$. By Lemma 4.3.2,

$$\begin{aligned}
& d(S_d,(p(p+1)^2(d+p)+i,(p+1)^2(d+p)-i)) \\
= \;& |p(p+1)^2(d+p)+i|+|(p+1)^2(d+p)-i|-d \\
= \;& (p+1)^3(d+p)-d \\
= \;& p^3d+p^4+3p^2d+3p^3+3pd+3p^2+p \\
\geq \;& p^3d+p^4+3p^2d+3p^3+3pd+3p^2 \\
= \;& (p^2d+p^3+3pd+3p^2+3d+3p)p \\
= \;& ((p+1)^2+p+2)(d+p)p \\
= \;& ((p+2)(d+p)+(p+1)^2(d+p))p \\
\geq \;& ((p+2)(d+p)+i)p.
\end{aligned}$$

Case 8: Suppose $((p+1)^3(d+p),-((i-1)p+k),((d+p)(p+2+(p+1)^2)+i)p) \in A_7$ for some $i \in \{1,....,N\}$ and $k \in \{1,...,p\}$. By Lemma 4.3.2,

$$\begin{aligned}
& d(S_d,((p+1)^3(d+p),-((i-1)p+k))) \\
= \;& |(p+1)^3(d+p)|+|-((i-1)p+k)|-d \\
= \;& (p+1)^3(d+p)+(i-1)p+k-d \\
\geq \;& (p^3+3p^2+3p+1)(d+p)-p-d+ip \\
= \;& (p+d)(p^3+3p^2+3p)+ip \\
= \;& (p+d)p(p+2+(p+1)^2)+ip
\end{aligned}$$

$$= \quad ((d+p)(p+2+(p+1)^2)+i)p.$$

Thus, the second condition in the definition of a containment certificate is satisfied. To see that A satisfies the third condition, let us consider the closed curve (in $\mathbb{R}^2$) determined by A by "connecting the dots", meaning we draw a line segment between two adjacent points (x, y, t) and $(x', y', t') \in A$ that satisfy

$$\max\{|x - x'|, |y - y'|\} = 1.$$

Note that this produces a polygon P with nine sides. P separates $\mathbb{R}^2$ into an interior and an exterior. Since the interior has finite area as a subset of $\mathbb{R}^2$, there are only a finite number of lattice points in the interior. Also, note that S_d is a subset of the interior, thus any point on the exterior must cross P in order to reach any point is S_d. This implies that the only vertices that have at least one path to a vertex in S_d without passing through any vertex in A are precisely the lattice points in the interior of P, which is finite. $\square$

Lemma 4.3.4. *Suppose f and g are two periodic functions such that $f \preceq^* g$. If there is a containment certificate of f for S_d for all $d \geq 0$, then there is a containment certificate of g for S_d for all $d \geq 0$.*

Proof. Since $f \preceq^* g$, there exists $n \in \mathbb{N}$ such that

$$\sum_{t=1}^{k} f(t) \leq \sum_{t=1}^{k} g(t)$$

for all $k \geq n$. Since there is a containment certificate of f for S_d for all $d \geq 0$, let $C_{S_{n+d+1}}(f)$ be a containment certificate of f for S_{n+d+1}. We will use $C_{S_{n+d+1}}(f)$ to construct a containment certificate of g for S_d. We order the elements in $C_{S_{n+d+1}}(f)$ on the third coordinate such that

$$C_{S_{n+d+1}}(f) = \{(x_1, y_1, t_1), (x_2, y_2, t_2), ..., (x_r, y_r, t_r)\},$$

where $t_1 \leq t_2 \leq ... \leq t_r$. It is now easy to see that for all $j \geq 1$,

$$\sum_{t=1}^{t_j} f(t) \geq j.$$

Now define $C_{S_d}(g)$ to be

$$C_{S_d}(g) = \{(x_j, y_j, g^{-1}(j)) \mid 1 \le j \le r\}.$$

Note that elements in $C_{S_{n+d+1}}(f)$ and $C_{S_d}(g)$ differ only the third coordinate. To prove that $C_{S_d}(g)$ is indeed a containment certificate of g for S_d, we check the three conditions in the definition of a containment certificate.

Condition 1: Note that

$$\begin{aligned} |\{j \in \mathbb{N} \mid g^{-1}(j) = i\}| &= \text{number of } j \text{ such that } \min\{k \mid \textstyle\sum_{t=1}^{k} g(t) \ge j\} = i \\ &= g(i). \end{aligned}$$

Thus $C_{S_d}(g)$ satisfies the first condition since there are exactly $g(i)$ elements in $C_{S_d}(g)$ where that the third coordinate is i.

Condition 2: For the second condition, first consider the case where $(x_j, y_j, t_j) \in C^{\le n}_{S_{n+d+1}}(f)$. This implies $t_j \le n$. We want to show that $d(S_d, (x_j, y_j)) \ge g^{-1}(j)$. We claim that $g^{-1}(j) \le n$. Suppose, for a contradiction that $g^{-1}(j) > n$. By the definition of g^{-1}, this implies that

$$\sum_{t=1}^{n} g(t) < j.$$

However,

$$\sum_{t=1}^{n} f(t) \ge \sum_{t=1}^{t_j} f(t) \ge j \Rightarrow \sum_{t=1}^{n} g(t) < j \le \sum_{t=1}^{n} f(t),$$

which contradicts $f \preceq^* g$. So $g^{-1}(j) \le n$. Since $(x_j, y_j, t_j) \in C_{S_{n+d+1}}(f)$,

$$d(S_{n+d+1}, (x_j, y_j)) \ge 1 \Rightarrow d(S_d, (x_j, y_j)) > n \ge g^{-1}(j)$$

and we are done. Next consider the case where $(x_j, y_j, t_j) \in C^{>n}_{S_{n+d+1}}(f)$. We claim that $g^{-1}(j) \le t_j$. Suppose, for a contradiction that $g^{-1}(j) > t_j$. By the definition of g^{-1}, this implies that

$$\sum_{t=1}^{t_j} g(t) < j.$$

However,

$$\sum_{t=1}^{t_j} f(t) \ge j \Rightarrow \sum_{t=1}^{t_j} g(t) < j \le \sum_{t=1}^{t_j} f(t),$$

which contradicts $f \preceq^* g$ since $t_j > n$. So $g^{-1}(j) \leq t_j$. Since $(x_j, y_j, t_j) \in C_{S_{n+d+1}}(f)$,

$$d(S_d, (x_j, y_j)) > d(S_{n+d+1}, (x_j, y_j)) \geq t_j \Rightarrow d(S_d, (x_j, y_j)) > g^{-1}(j)$$

and we are done. Thus $C_{S_d}(g)$ satisfies the second condition in the definition of a containment certificate.

Condition 3: The third condition follows naturally because $C_{S_{n+d-1}}(f)$ is a containment certificate and the positions (x_j, y_j) determined by $C_{S_d}(g)$ and those determined by $C_{S_{n+d+1}}(f)$ are exactly identical. □

We are now ready to prove our main result.

Theorem 4.3.5. *Suppose a finite set $S \subset \mathbb{Z} \times \mathbb{Z}$ of vertices are initially set on fire. If the number of firefighters available for deployment per time period is given by a periodic function f such that $R_f > 1.5$, then there exists a containment certificate of f for S.*

Proof. Suppose f is a periodic function such that $R_f > 1.5$. If $p_f = 1$, this means that $f(t) \geq 2$ for all t. Fogarty [18] has shown that this is sufficient to contain the fire that starts at any finite set S. Suppose $p_f \geq 2$. By Lemma 4.2.7, there exists some $n \geq 2$ such that $F_{n^2} \preceq^* f$. Since $F_{n^2}^2 \preceq^* F_{n^2}$ and $G_p \preceq^* F_{n^2}^2$ where $p = 2(2n^2 + 1)$, we have $G_p \preceq^* f$.

Now let

$$d = \max\{|x| + |y| \mid (x, y) \in S\}.$$

By Lemma 4.3.3, there exists a containment certificate of G_p for S_d. By Lemma 4.3.4, since $G_p \preceq^* f$, there also exists a containment certificate of f for S_d, $C_{S_d}(f)$. Since $S \subseteq S_d$, by Lemma 3.1, $C_{S_d}(f)$ is also a containment certificate of f for S. □

From this, we have a simple corollary that extends the space of functions for which there are containment certificates.

Corollary 4.3.6. *If*

$$\liminf_{n \to \infty} R_f(n) > 1.5,$$

then there is a containment certificate of f for any finite S.

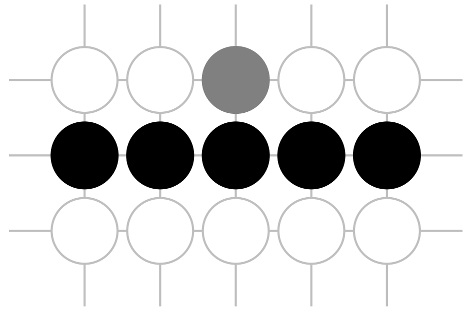

Figure 4.4: A snapshot of a part of a containment scenario.

Proof. Let $l = \liminf_{n\to\infty} R_f(n)$ and let r be a rational number satisfying $1.5 < r < l$. Additionally, let N be so that $R_f(n) > r$ for all $n > N$. Let F be any periodic function with ratio equal to r and consider the following function g, defined as

$$g(t) = \begin{cases} 0 & \text{for } 0 < t < N \\ f(t-N) & \text{otherwise} \end{cases}$$

Note that there exists a containment certificate for g for it is equivalent to having a containment certificate for F. Additionally, note that $g \preceq f$. Therefore, by Lemma 4.3.4, there is a containment certificate for f. $\square$

4.4 Lower bounds for convex containment certificates

The general containment certificate shown in Figure 4.3 admittedly looks a little bizarre, but it comes as an artifact from the intention to place the firefighters as close to the fire as possible. However, it isn't necessary to always place the firefighters adjacent to the fire to have the same effect. For example, consider Figure 4.4, a close-up of the following simplistic situation that is the start of a firefighting scenario.

In Figure 4.4, five fighters are placed, all below the single fire. Due to the structure of the infinite grid $\mathbb{Z}^2$, we can shift the middle firefighter down one step without any change in the possibility of full containment, as exemplified in Figure 4.5.

Being unconcerned about minimizing the total number of vertices that are burned, the fighters can allow the fire to spread one step below, but that is as far as it can go

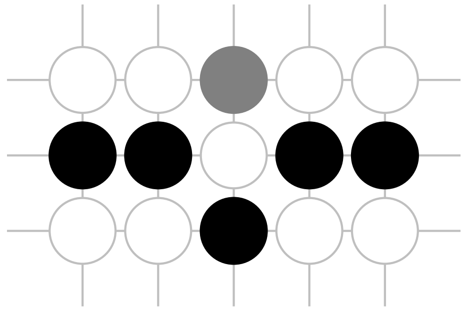

Figure 4.5: A legitimate modification made from Figure 4.4.

in that direction. The way the fire spreads otherwise is exactly the same as it was in the initial placement.

Indeed, we can go further and start with a containment certificate – i.e. a situation where the fire has already been contained – and then try to "expand" the containment wall as much as possible. This can be done easily with the help of a computer, but we first would like to enumerate all of the ways pieces of wall could form. We can do so easily with the help of a mathematical computer assistant such as `Mathematica` (code is given at [39]), which was used in this case to find, using brute force, the following list of 128 different ways a four-firefighter length of wall can be positioned relative to the fire.

We can take advantage of this fact and extend this idea over all possible situations that are seen in a containment certificate. Assume that we are dealing with a convex containment certificate. The important part of the containment scenario is then a wall around the set of fires; there may be firefighters completely "consumed" by the fire on the inside of the outer wall, but those are irrelevant here. If we start at one fighter on the outer wall and travel around the fire clockwise, we can view the next four fighters in this wall and these four fighters will be in one of the 128 scenarios that were enumerated using `Mathematica`. These 128 scenarios are shown in Figure 4.6 and Figure 4.7 and follow the "right hand rule" in that we think of following the wall starting at the middle of each example, and we view the fire as being to the right as we are going along the wall.

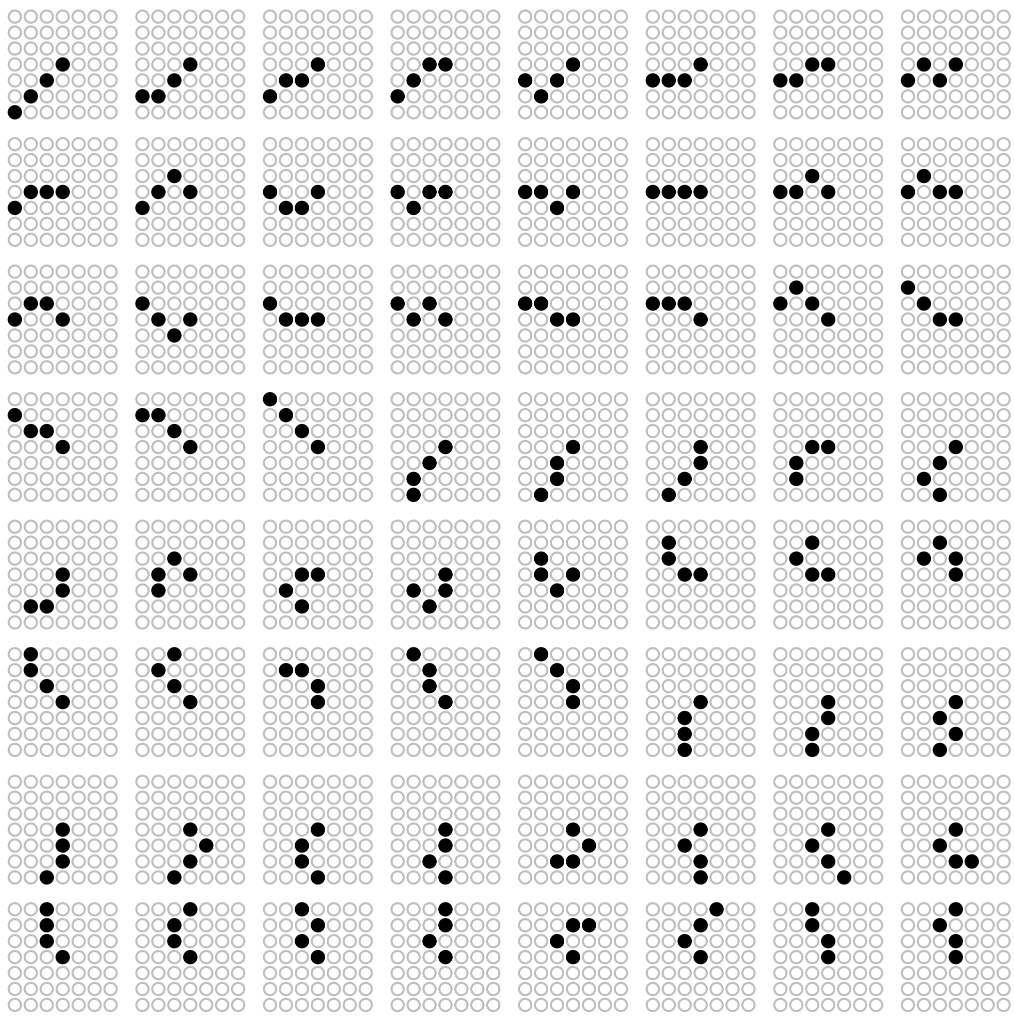

Figure 4.6: The first 64 of the 128 possible positions a group of four fighters could be in.

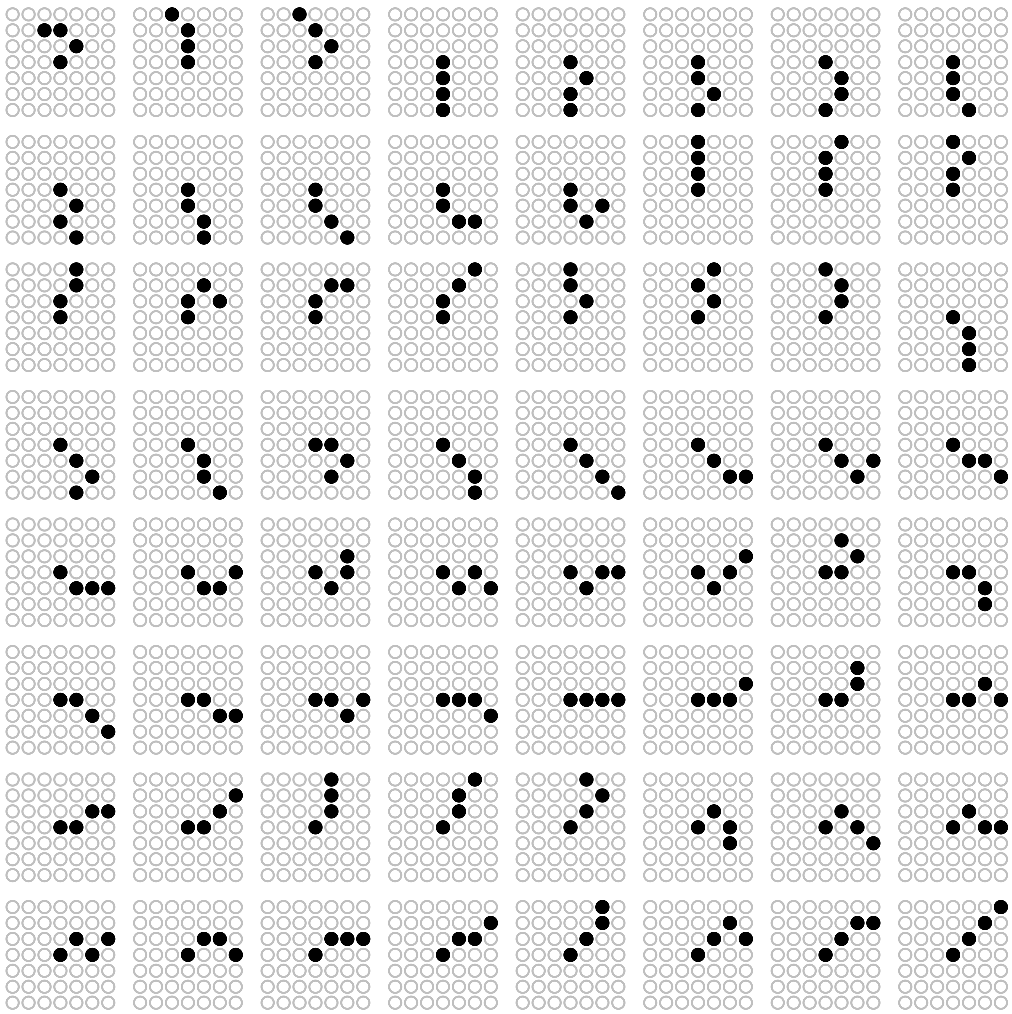

Figure 4.7: The second 64 of the 128 possible positions a group of four fighters could be in.

Using the simple idea from the beginning of this section, we are able to modify some of these 128 scenarios to take care of this fact and "push" the wall out as far as possible while still containing the fire. This mapping is shown in Figures 4.8, 4.9, 4.10, 4.11, 4.12, 4.13, 4.14, and 4.15.

Notice that in each of the transformations, only the middle two pieces out of the four are relocated, ensuring that the wall will still be contiguous after each stage. Hence, we can repeat this process throughout, and given that the mapping of local transformations is idempotent, this process will eventually converge. It turns out we always converge into a well-behaved structure.

Definition A *diamond containment certificate* with *corners* (x_S, y_S) and (x_N, y_N) is a containment certificate whose firefighters are the points $(x, y) \in Z^2$ satisfying one of the following two conditions.

- $\left|\frac{x-x_S}{y-y_S}\right| = 1$ and $\left|\frac{x-x_N}{y-y_N}\right| \geq 1$, or
- $\left|\frac{x-x_S}{y-y_S}\right| \geq 1$ and $\left|\frac{x-x_N}{y-y_N}\right| = 1$.

Remark Although it was not specified in the definition, the set of firefighters in a diamond containment certificate will only be nonempty if $2x_S, 2y_S, 2x_N$, and $2y_N$ are all integers. However, for the rest of this chapter, we will assume that x_S, y_S, x_N, and y_N are all integers, as all arguments will hold in the other cases.

Theorem 4.4.1. *Any convex containment certificate $\mathcal{C}$ can be mapped to a diamond containment certificate $\mathcal{C}'$ that works for the same initial configuration and the same number of turns.*

Proof. To obtain the diamond, all one needs to do is start at some section of the containment certificate and apply the mapping to the consecutive parts of the "sliding window" around the containment certificate, until there are no more non-identity transformations left (see [39] for a demonstration). The only configurations that map to themselves are the ones that precisely are part of some diamond containment certificate. What is obtained at the end of the process, considering that we started with a

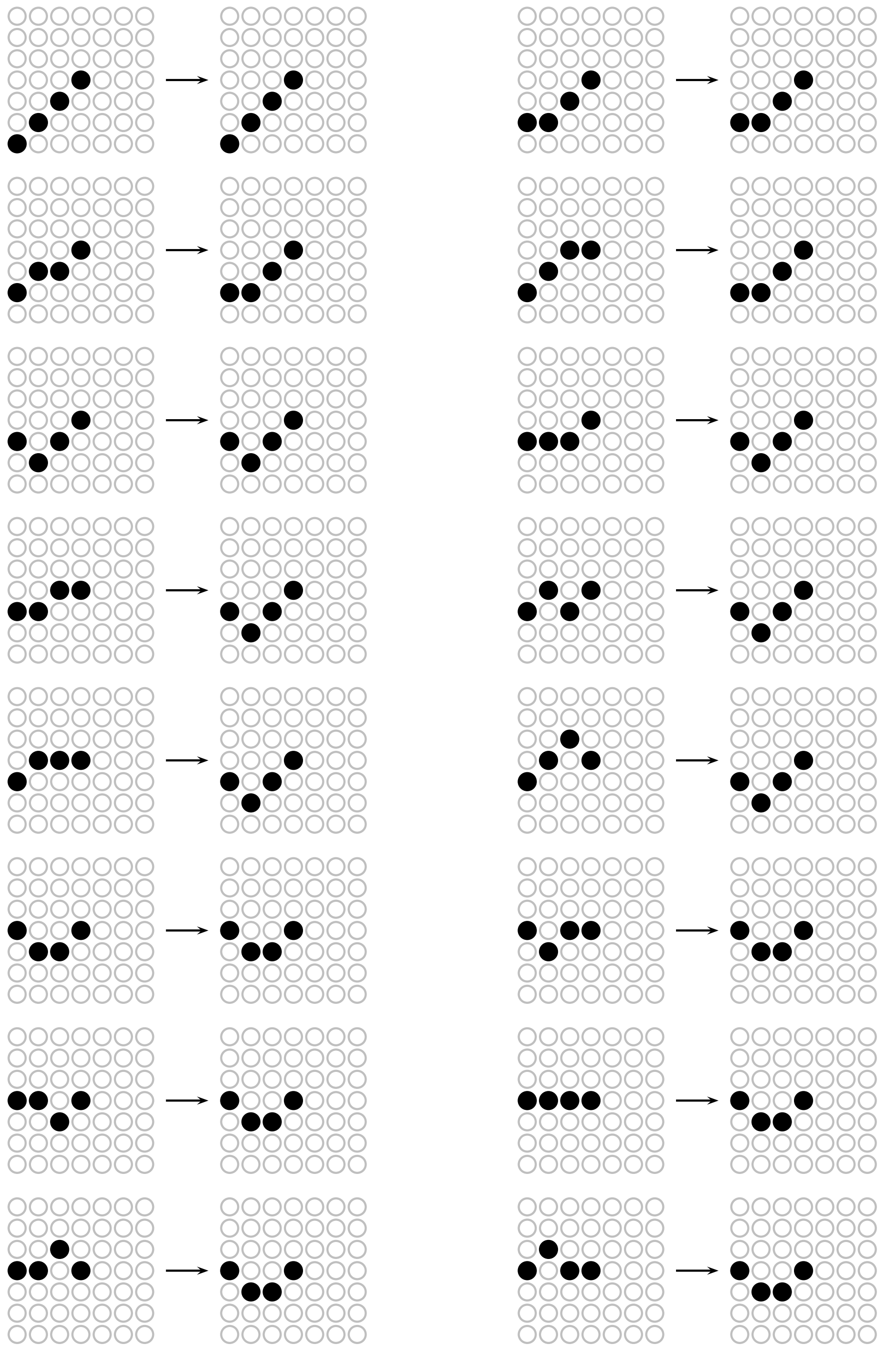

Figure 4.8: The wall transformation mapping for cases 1 through 16.

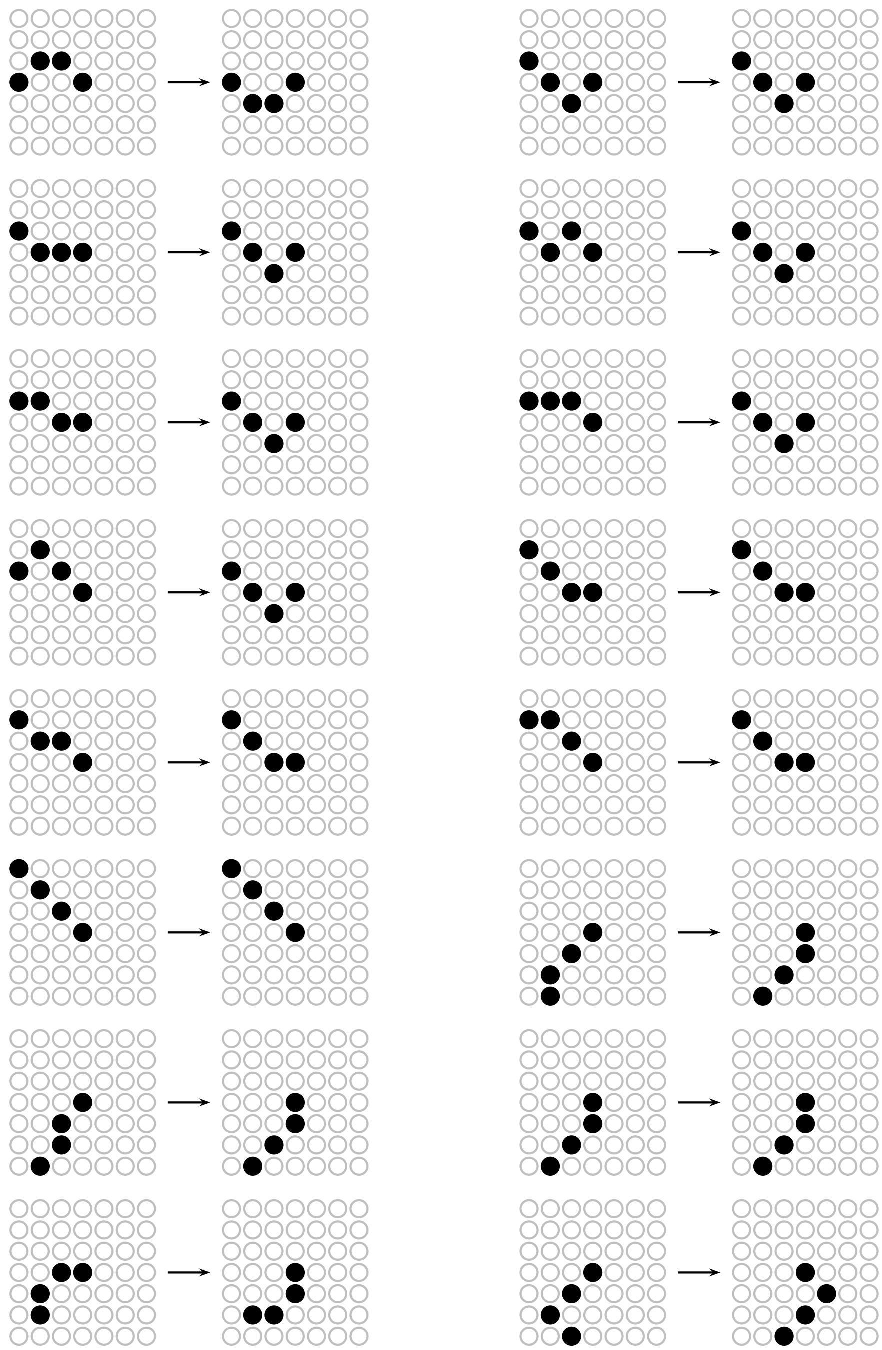

Figure 4.9: The wall transformation mapping for cases 17 through 32.

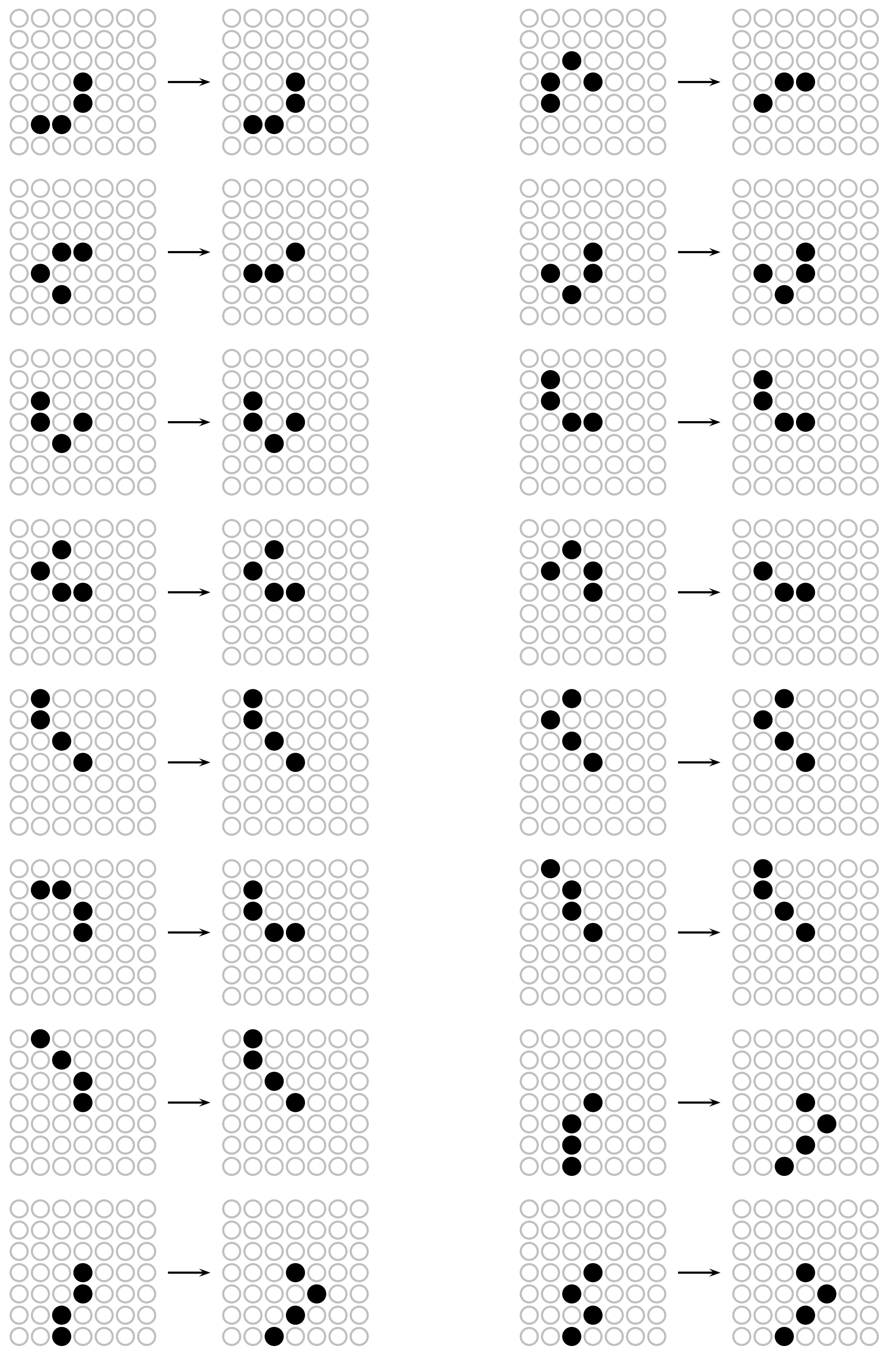

Figure 4.10: The wall transformation mapping for cases 33 through 48.

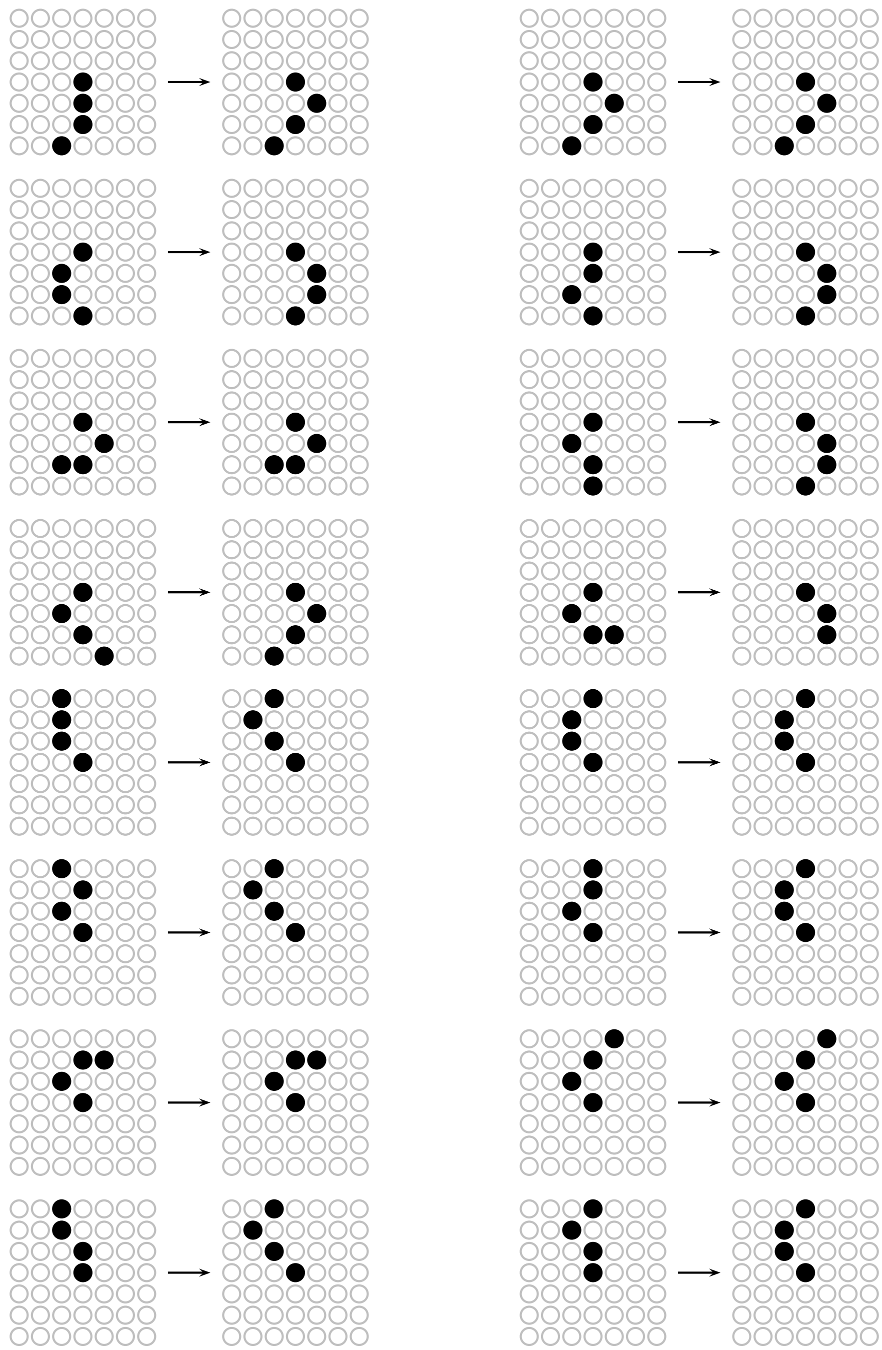

Figure 4.11: The wall transformation mapping for cases 49 through 64.

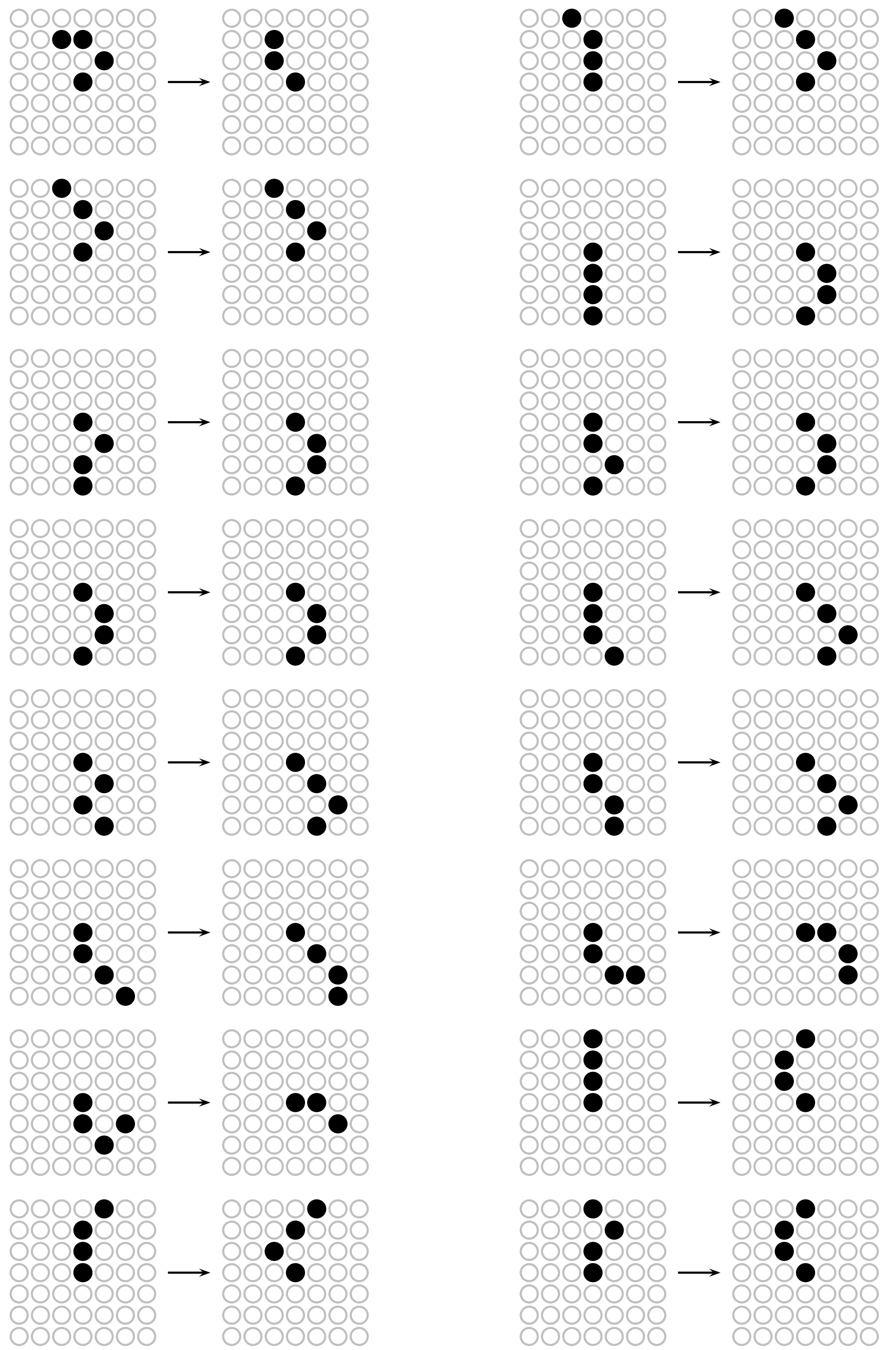

Figure 4.12: The wall transformation mapping for cases 65 through 80.

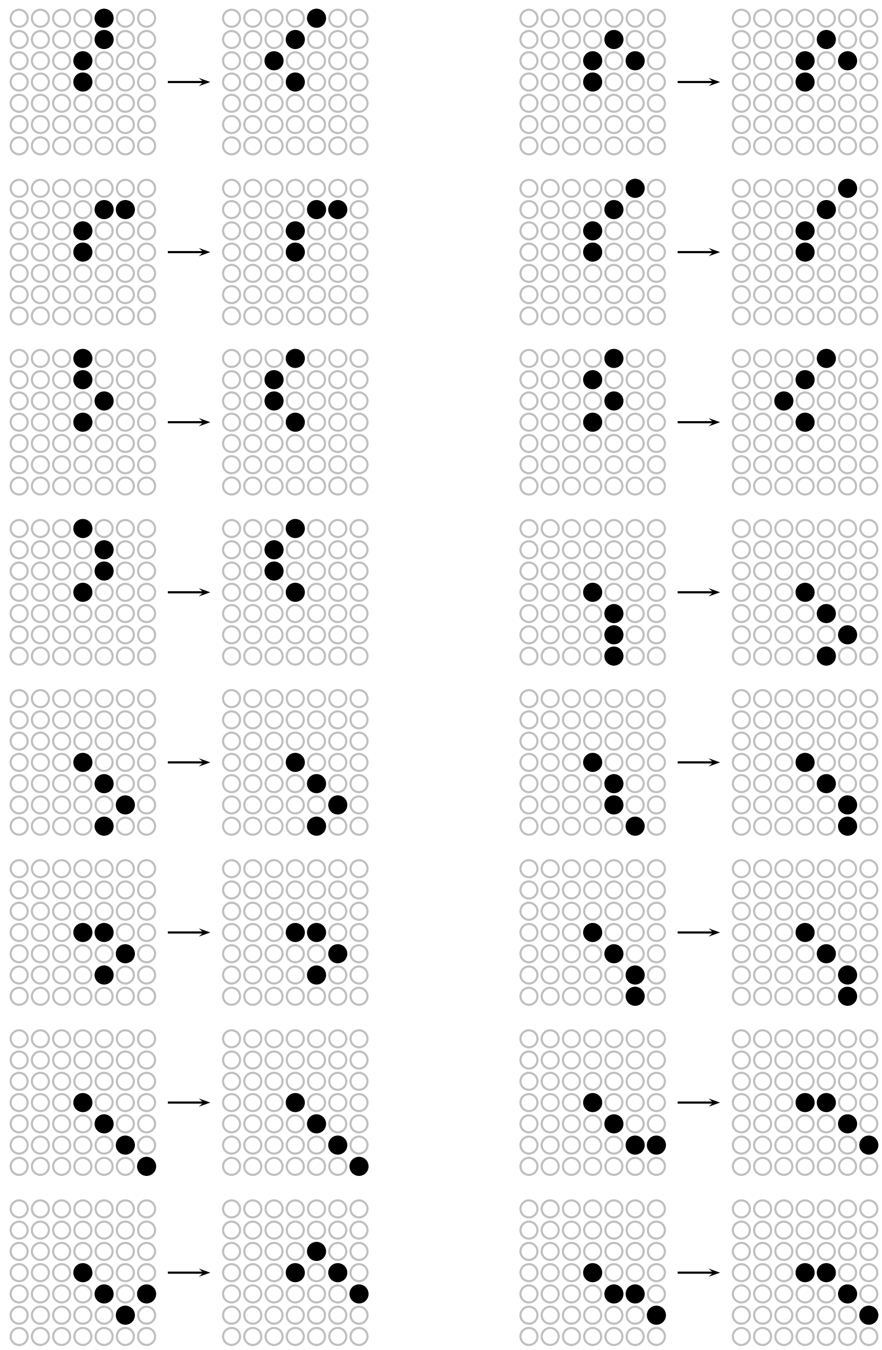

Figure 4.13: The wall transformation mapping for cases 81 through 96.

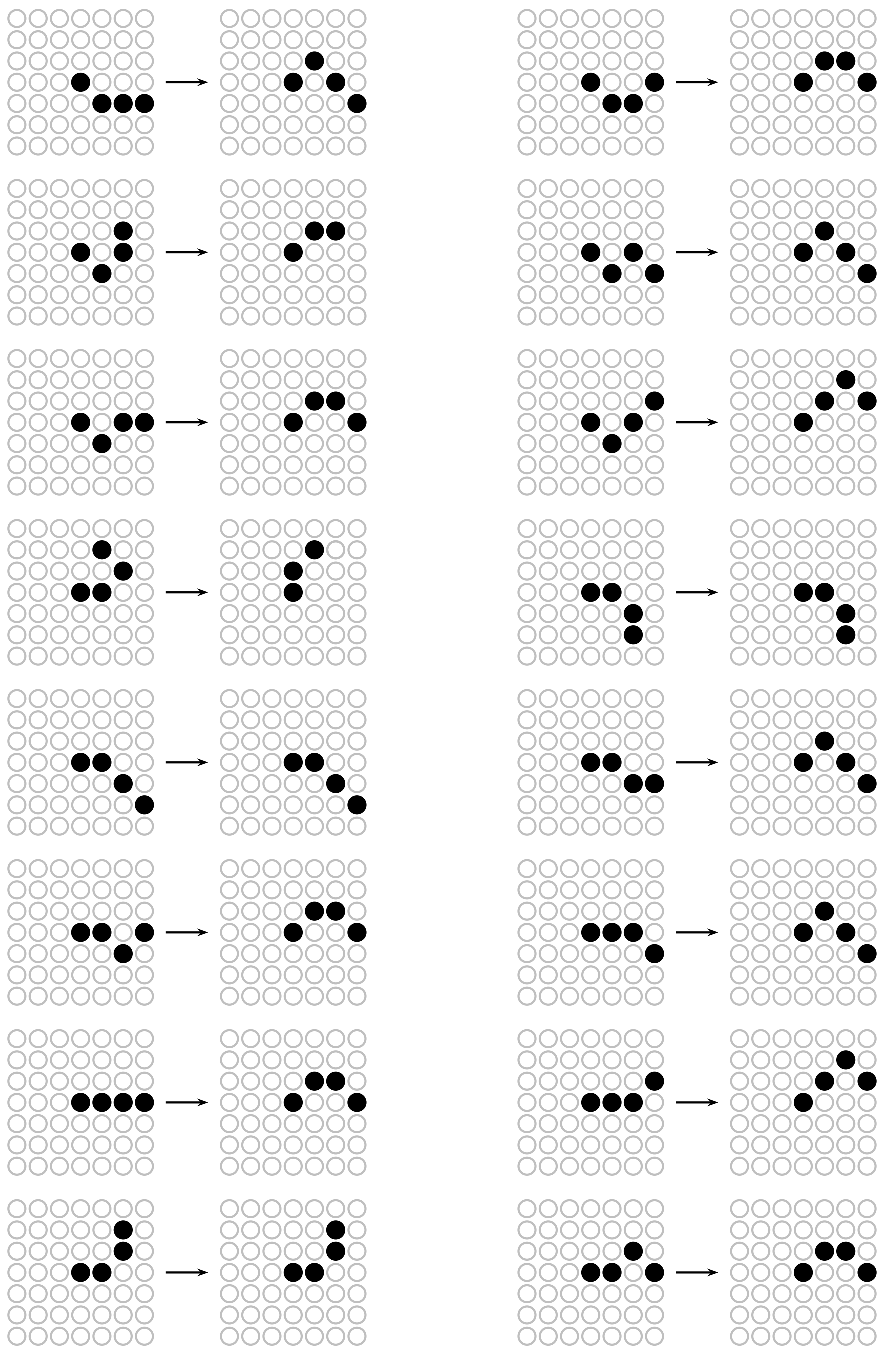

Figure 4.14: The wall transformation mapping for cases 97 through 112.

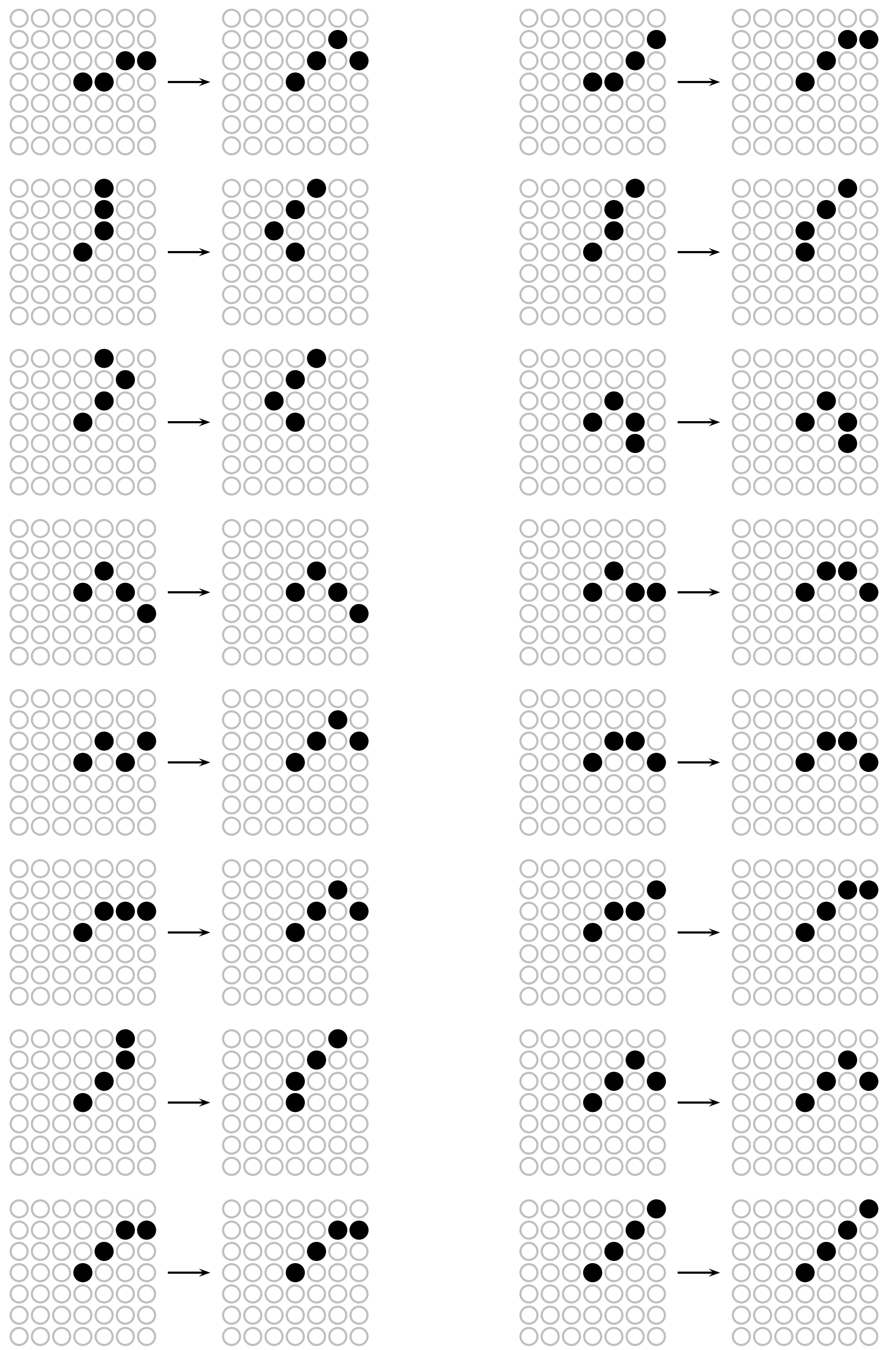

Figure 4.15: The wall transformation mapping for cases 113 through 128.

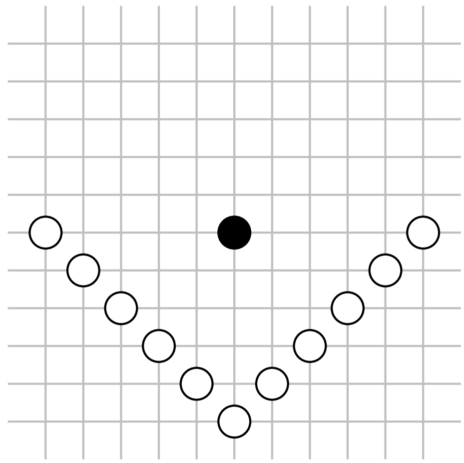

Figure 4.16: How the generic situation looks for having a diamond containment certificate (already predetermined) contain a fire starting at a single point.

fully-contained wall and the wall's integrity never changes during a transformation, is a diamond containment certificate.

If the initial containment certificate is convex, then any application of the mapping that transforms part of the containment certificate still upholds the rules of the containment certificate, so is still a solution to the initial configuration. □

From Theorem 4.4.1, we can now show that there are functions f with $R_f = 1.5$ that can not produce a convex containment certificate by showing that it can not produce a diamond containment certificate. We will show this specifically with the function $f = [3, 0]$.

Corollary 4.4.2. *There is no diamond containment certificate for $f = [3, 0]$. As a consequence, there is no convex containment certificate for f, also.*

Proof. We will assume that the fire starts at one point, (x, y), and a diamond with corners (x_S, y_S) and (x_N, y_N) is a diamond containment certificate. Additionally, we may also assume without loss of generality that the corner (x_S, y_S) is the first corner that the fire will approach, and we may also assume that $x \geq x_S$. Hence, part of our situation looks like Figure 4.16, where the containment certificate is shown.

With this scenario, we will have a need for a total of $2(y - y_S) + 1 + (x - x_S)$

firefighters by turn $(x-x_S)+(y-y_S)$, where the $2(y-y_S)+1$ firefighters are needed at positions $(x_1, y), (x_1+1, y-1), \ldots, (x_S, y_S), (x_S+1, y_S+1), \ldots, (x_2, y)$ and the $(x-x_S)$ firefighters are needed at positions $(x_2+1, y+1), \ldots, (x_2+x-x_S, y+x-x_S)$. However, notice at this point that we will also need $(x-x_S)+(y-y_S)$ firefighters to protect the upper-left wall, as for each two turns past $(x-x_S)+(y-y_S)$, two more firefighters are needed, one on the lower-left and one on the lower-right wall, and the upper-left firewall, which needs to be protected, also increases in length by one. Therefore, it is necessary that

$$3\left\lceil \frac{(x-x_S)+(y-y_S)}{2} \right\rceil - (2(y-y_S)+1+(x-x_S)) \leq (x-x_S)+(y-y_S),$$

or equivalently,

$$3\left\lceil \frac{(x-x_S)+(y-y_S)}{2} \right\rceil \leq 3(y-y_S)+2(x-x_S)+1. \tag{4.1}$$

We can check (4.1) by considering the following three cases.

- $x = 2l, y = 2k$: We have

$$\begin{aligned} 2\left\lceil \frac{2k+2l}{2} \right\rceil &\leq 6k+4l+1 \\ 3(k+l) &\leq 6k+4l+1 \\ 0 &\leq 3k+l+1 \end{aligned}$$

  which is true, for it must be that $|y| > |x|$ and so $|k| > |l|$.

- $x = 2l, y = 2k+1$: We have

$$\begin{aligned} 2\left\lceil \frac{2k+2l+2}{2} \right\rceil &\leq 6k+3+4l+1 \\ 3(k+l+1) &\leq 6k+4l+4 \\ 0 &\leq 3k+l+1 \end{aligned}$$

  which is still true, for even though $|y| > |x|$, it still implies here that $|k| \geq |l|$.

- $x = 2l + 1, y = 2k$: We have

$$2\left\lceil \frac{2k+2l+2}{2} \right\rceil \leq 6k + 4l + 2 + 1$$

$$3(k + l + 1) \leq 6k + 4l + 3$$

$$0 \leq 3k + l$$

  which is true for the same reasons as above.

From the calculations above we see that even though we might be able to have enough firefighters at turn $(x - x_S) + (y - y_S)$, we will not be able to protect the upper-left part of the diamond in the future, when the whole upper-left section of the firewall hits it at once. The calculations show that no matter what, we will not have enough firefighters in total to protect all of the areas that need protected at that time, and hence there are no diamond containment certificates, and so no convex containment certificates, for $[3, 0]$. □

4.5 Discussion and conclusion

For a given periodic function f and set $S \subset \mathbb{Z} \times \mathbb{Z}$, if a containment certificate of f for S exists, it is not necessarily unique. In fact, our initial efforts to prove Theorem 4.3.5 resulted in the construction of a containment certificate of the function $F_n = [\overbrace{1, 1, ..., 1}^{n}, \overbrace{2, 2, ..., 2}^{n+1}]$ for the set S_d, for every $n \geq 1$ and $d \geq 0$. Of course, with Lemmas 4.2.7 and 4.3.4, we are still able to arrive at Theorem 4.3.5. The containment certificate of F_n differs significantly from the containment certificate of G_p for S_d presented in Lemma 4.3.3. Our decision to present the containment certificate of G_p for S_d in this chapter is based on its relative simpler form and ease of checking the three conditions of a containment certificate.

In this chapter, we have established that if f is a periodic function with $R_f > 1.5$, then for any $d \geq 0$, there always exists a containment certificate of f for S_d. But what about periodic functions f with $R_f \leq 1.5$? Attempts have been made, for example, with the function $f = [2, 1]$ but with no success. Even in the simplest case when the fire breaks out at just a single vertex of $\mathbb{L}_2$, we were unable to determine if there is a containment certificate of $f = [2, 1]$ for S_0. Through our many attempts, however, we believe that such a containment certificate does not exist.

Conjecture 9. *There is no containment certificate of $f = [2, 1]$ for S_0.*

In this light, if we define the number R as

$$R := \inf\{k \in \mathbb{R} \mid \forall f \text{ with } R_f = k \text{ there exists a } C_S(f) \text{ for any finite } S\}$$

then the research mentioned in Section 1 showed that $1 \leq R \leq 2$, and this chapter has shown that $1 \leq R \leq 1.5$. So, it leads to the following question.

Question 1: What is R, exactly?

Note that if Conjecture 1 holds, then it would answer Question 1, and the answer would be 1.5. It is clear, however, that new machinery beyond what is covered in this chapter will be necessary to answer this question.

We wish to note, however, that containment certificates exist for "periodic" functions with ratios less than 1.5. The reason for the quotation marks will become clear soon. Consider first the function

$$g = [4, 0, 0, 0, 0, 0, 0, 0].$$

Clearly there is a containment certificate of g for S_0. However, by the way we defined g we would have $R_g = 0.5$, which is much less than 1.5. We can extend this example further to obtain ratios as close to 0 as possible where containment certificates still exist.

For a more subtle second example, consider the function

$$f = [2, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2].$$

This function has a containment certificate for S_0, as shown in Figure 4.17.

With the above example, we have reached a point at turn 8 where we were able to just hold off the fire indefinitely. Hence we could place one fighter per turn at this stage indefinitely without increasing the number of "exposed" vertices that could

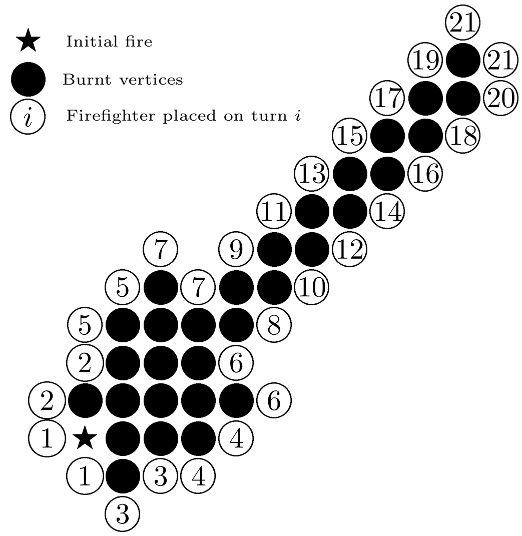

Figure 4.17: A not-so-nice containment certificate.

catch on fire the next turn. Although the two examples above are valid examples in the context of the chapter, they don't contain the spirit of our chapter. Rather than finding functions with a certain ratio where containment certificates exist, we are interested in the question of whether *all* functions with a given ratio admit containment certificates.

One final thing to notice is that the restriction on the periodicity of the function can probably be relaxed. For any arbitrary function $f : \mathbb{N} \to \mathbb{N}$, it will still be true that there exists a containment certificate of f for any finite S if f eventually dominates a F_n for some n. Given $f : \mathbb{N} \to \mathbb{N}$ and $n \in \mathbb{N}$, we define the *running ratio* of f at n to be

$$R_f(n) := \frac{\sum_{t=1}^{n} f(t)}{n}.$$

The author believes that the following conjecture is true.

Conjecture 10. *If* $R_f(n) > 1.5$ *for all* n *and*

$$\liminf_{n\to\infty} R_f(n) > 1.5,$$

then there is a containment certificate of f *for any finite* S.

Finally, the author wishes to note that this chapter stemmed from questions arising from epidemiology and that many extensions to this problem can be thought of by thinking of the problem in this manner. In this simplified model of disease spread, the vertices of the graph represent individuals in the population, and the edges represent relations that may allow for disease spread. Therefore, the results in this chapter could be translated into disease control for a population whose social structure is a grid and for a disease that strikes neighbors the next time period after a person is infected. While this is a very simplistic and unlikely setting for population structure and disease spread, we invite readers to extend these results to more general types of graphs and more interesting fire/disease behaviors that are more realistic. For example, the first modification that could be made to this problem is to add a probability parameter p to the scenario, which would be the probability that an unprotected vertex would catch fire given that a neighbor is on fire. Another possible modification would be to modify the graph as t increases, presumably to represent the changes in inter-person behavior as a day goes by: one is rarely likely to catch a disease from a co-worker at four in the morning!

Chapter 5

Conclusions and Further Work

In sum, this thesis detailed examples of situations where the use of the computer can be of crucial use and importance in solving certain combinatorial problems. Behind all of the work that the computer has done, it is important to remember the need for a well-crafted plan to be formulated, typically in the form of an enumeration scheme. Traditionally most of this work falls under the banner of Experimental Mathematics, and it should not be forgotten that just like in experiments in the other physical and social sciences, the design and planning takes up 90% of the time and effort. The conjectures stated in the concluding subsections of the previous chapters will be presented here again, along with potential research plans that are suitable even for interested undergraduates.

5.1 Avoiding Differences

Research Plan 1. This research plan would focus solely on the algorithm devised for finding and proving the behavior of $\{f_\Delta(n)\}$ and the value of $\mu(\Delta)$ for a given value of Δ. The algorithm described in this thesis is decidedly inefficient at dealing with large sets Δ. Related to this route is the question of what the pseudoperiod of $\{f_\Delta(n)\}$ really is, as bounds for the pseudoperiod (and even the offset) would go long ways to determining how far out in the sequence we need to look at to find the pseudoperiod.

Conjecture 1. *The pseudoperiod of $\{f_\Delta(n)\}$ is bounded from above by $\sum \Delta$.*

Boris Bukh [6] recently claimed that a paper on *tilings*, which are finite sets X such that there admits a set T such that $\{X + t \mid t \in T\}$ is a partition of $\mathbb{N}$ (see [52] and [50]), disproves this conjecture. However, this author believes this not to be the case, as these cyclic set witnesses aren't necessary tilings nor derived from them. Nevertheless, the author still wishes to state the conjecture, even though it may already be refuted.

Research Plan 2. One can use the computer programs in [41] to find more counterexamples, beyond Shor's, to the Triangle Conjecture. Is there anything in common with all of these counterexamples? To the author, with Theorem 2.8.1 in mind, it seems that such counterexamples are simply "growing pains" of sorts. Since the initial growing pains are the worst, it suggests the following conjecture.

Conjecture 2. $\gamma = \frac{16}{15}$.

To this end, analyzing the TCC Poset would be fruitful.

Conjecture 3. *All minimal elements X of the TCC poset satisfy $|X| = m + 1$.*

Research Plan 3. Most of the analysis was focused on $\{f_\Delta(n)\}$, but similar analysis on the other quantities $\{f_\Delta^c(n)\}$, $\{f_\mathfrak{D}(n)\}$, and $\{f_\mathfrak{D}^c(n)\}$ can be helpful if there is different behavior. More broadly, an analysis should be done on enumeration schemes $\{f_1, f_2, \ldots\}$ where *all* recurrences are of the form

$$f_i(n) = \max\{f_j(n-1), 1 + f_k(n-1)\}.$$

It is clear that the pseudoperiodicity of these sequences $\{f_i\}$ follow from this structure, but is there a characterization looming?

Research Plan 4. Recall that the simplest example of a set of atoms that was not a code was represented as an isosceles triangle in the plane. Similarly, a characterization of the simplest example (or examples) involving four atoms can be made, and for larger sets. It is then an interesting extremal graph theory problem to find the largest size of a set of atoms that avoids these structures. The following is obviously true and should not be difficult to prove.

Conjecture 4. *There are $f(m)$ isosceles triangles in $\mathcal{A}_m$, where*

$$f(m) = \begin{cases} \frac{15}{36}m^3 + \frac{7}{8}m^2 + \frac{1}{12}m - \frac{3}{8} & \text{if } m \text{ is odd} \\ \frac{15}{36}m^3 + \frac{7}{8}m^2 + \frac{1}{12}m & \text{if } m \text{ is even} \end{cases}.$$

5.2 Spanning Trees in Grid Graphs

Research Plan 5. There is certainly plenty of consistent behavior with the characteristic polynomials of these matrices that are created, and it may be worthwhile to analyze the structure of the matrices themselves.

Conjecture 5. *For the matrix M given in Theorem 3.7.1, the characteristic polynomial $\chi_\lambda(M)$ factors over the integers into monomials whose degree is always a power of 2.*

Conjecture 6. *For any graph G, the recurrence $\{\tau_G(n)\}$ satisfies a linear recurrence whose coefficients alternate in sign.*

Towards analyzing specific matrices, it would be useful to restrict attention to specific classes of grid graphs, such as mentioned in the following conjectures.

Conjecture 7. *The recurrence for the grid graph $G_k(n)$ has order 2^{k-1}.*

Conjecture 8. *The recurrence for the graph $K_k \times P_n$ has order k.*

5.3 The Firefighter Problem

Research Plan 6. This thesis came close to completely solving the basic firefighter problem in the two-dimensional grid, and it would be great to solve it completely.

Conjecture 9. *There is no containment certificate of $f = [2, 1]$ for S_0.*

Conjecture 10. *If $R_f(n) > 1.5$ for all n and*

$$\liminf_{n\to\infty} R_f(n) > 1.5,$$

then there is a containment certificate of f for any finite S.

The reason the first conjecture of these two is still here is because we still cannot be certain that a non-convex containment certificate does not exist.

Research Plan 7. This thesis introduced the novel idea of modifying the containment certificates to obtain lower bounds. Are there similar modifications that can be done in other scenarios? The list of 128 transformations was specifically designed to take advantage of the idea that having the firefighters on the diagonal is the most efficient way to deal with the two-dimensional grid. This can be thought of a "dual" notion, but how to formulate it precisely? Additionally, what other examples are there of this duality?

5.4 A Parting Statement

To whoever reads this: no matter what stage of life you are in, make sure that whatever you do, you can say the following two things about it.

1. "I'm good at it."
2. "I love it."

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