TO INFINITY AND BEYOND: TOWARD A LOCAL INSTRUCTION THEORY FOR COMPLETED INFINITE ITERATION

by

IULIANA RADU

A Dissertation submitted to the

Graduate School-New Brunswick

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

Graduate Program in Mathematics Education

Written under the direction of

Keith Weber

and approved by

New Brunswick, New Jersey

October, 2009

ABSTRACT OF THE DISSERTATION

To Infinity And Beyond: Toward A Local Instruction Theory For Completed Infinite Iteration

By IULIANA RADU

Dissertation Director:

Keith Weber

There is evidence that students' and mathematicians' images of and reasoning about concepts such as infinite unions and intersections of sets, limits, and convergent series often invoke the consideration of infinite iterative processes. Therefore, not reasoning normatively about infinite iteration may hinder students' understanding of fundamental concepts in mathematics. Research suggests that the vast majority of college students are inclined to use non-normative reasoning when presented with tasks that challenge them to imagine an infinite iterative process as completed and define its outcome. However, there is almost no research on how students' conceptions of completed infinite iteration can be refined in a normative direction.

Adopting a design research approach, the purpose of this thesis is to develop a local instruction theory for completed infinite iteration. This design entails multiple cycles through phases of development, implementation, and analysis. The study consisted of a sequence of two constructivist teaching experiments conducted with pairs of

mathematics majors, during which the students worked collaboratively through a variety of infinite iteration tasks. Each cycle consisted of 6 or 7 sessions lasting approximately 2 hours, together with Pre-test and Post-Test sessions. The data analysis consisted of multiple phases of iterative analysis of the videotaped sessions and written work and was guided by situated learning transfer theories.

The participants in this study employed a variety of types of reasoning when solving completed infinite iteration tasks, some normative and some not. One way that the students refined their reasoning on infinite iteration was by making references to previously solved tasks (in the presence of a concern for consistent reasoning across tasks), although the refinement was not always in normative directions. The challenge that a researcher faces consists in helping students use such references to establish “anchors” in normatively solved tasks to which they can then contrast the solution paths proposed for other tasks. A variety of approaches to responding to this challenge are explored. The study culminates with the formulation of a local instruction theory for completed infinite iteration, containing among other things a sequence of instructional activities and a rationale for their appropriateness.

Acknowledgements

I could not have completed this dissertation without the support of a large number of people. I would like to thank my advisor, Keith Weber, who introduced me to research in undergraduate mathematics education early in my doctoral studies by inviting me to join his research projects even before he was officially my advisor. His guidance during my last four years in the program and especially during all stages of working on the dissertation study was invaluable. I would also like to thank Carolyn Maher, my first year advisor, through whom I was exposed to the IML project and thus was able to experience first hand, for the first time, aspects of what it means to do qualitative research. I will never forget my first meeting with Carolyn during which, as a first year doctoral student who had never done research in mathematics education, I asserted with confidence that I would be using quantitative methods in my research; Carolyn was quick to reply that I could not know that yet. How right she was!

Warren Crown is another Rutgers faculty member who mentored me during my first three years in the program. Although he was not directly involved in my work on the dissertation study, his guidance and support during my first tentative steps in the world of teaching mathematics education courses and supervising prospective teachers was much appreciated. I would also like to thank Gerald Goldin, who indulged me for countless hours in discussing the infinity-related mathematics behind my dissertation study, and gave me quite a few ideas for how to create variations from existing infinite iteration tasks.

I also received much support from fellow students in the Ph.D. and Ed.D. programs in Mathematics Education at Rutgers, many of whom offered themselves as "lab rats" for my task-piloting experiments. Special thanks go out to Cathleen Rossman, Kathy Lin, and Cecilia Arias, my fellow "Dissertation Support Group" members, through whose eyes I could look at my study from an outsider's perspective. During my dissertation work, I found it immensely helpful to set daily goals and report to the group with respect to the weekly progress – brilliant idea, gals! Another thank you goes to the four mathematics majors who participated in this study. I appreciate the time and effort you dedicated to this project, and the perseverance with which you strove to find your way through the maze of infinite iteration tasks I threw at you.

I am grateful for the support offered by Harriet Schweitzer and Judy Bornstein, two wonderful educators from the Center of Math, Science, and Computer Education (Rutgers). Through the center, they offered me a graduate assistantship when I was in dire straits regarding financial support for my studies; without them I might not have finished my doctorate! Working at CMSCE also allowed me to try my hand at offering professional development courses to in-service mathematics teachers, which greatly helped my understanding of the public school system and the needs of K-12 teachers.

Last but not least I would like to thank my family. My husband Viorel supported me unconditionally through all the ups and downs inherent to a lengthy and challenging process such as a doctoral program, and on top of that, took on a great share of the house keeping and cooking responsibilities. I truly couldn't have asked for more. Much gratitude to my parents, two dedicated mathematics teachers who instilled in me a love of mathematics from an early age and supported me in everything I set out to do. Finally, a big thank you to my two sisters, Simona and Speranta, who were always there for emotional support and the occasional math talk. I love you all.

Notes On The Text

This dissertation contains excerpts from student interviews and classroom dialogue; these excerpts were typed verbatim. All students are referred to in the text with pseudonyms.

Chapters 3 through 6 contain stand-alone papers, which is why the heading format in these chapters is different from that used in Chapters 1 and 2.

Chapters 3 and 4 are written using the pronoun “we” instead of “I”. This decision reflects the fact that a person other than the author made significant contributions toward the two papers contained in these chapters.

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Chapter 1- Introduction

"The fear of infinity is a form of myopia that destroys the possibility of seeing the actual infinite, even though it in its highest form has created and sustains us, and in its secondary transfinite forms occurs all around us and even inhabits our minds." (Georg Cantor)

The concept of mathematical infinity permeates K-16 mathematics. The *Curriculum and Evaluation Standards for School Mathematics* issued by the National Council of Teachers of Mathematics (NCTM) in 1989 make direct reference to the infinite twice: once in Standard 5 for Grades 5-8 (Number and Number Relationships), when discussing the need for students to grasp the "infinite" quality of decimals, and then again in Standard 13 for Grades 9-12 (Conceptual Underpinnings of the Calculus), when referring to the study of limiting processes by examining infinite sequences and series and areas under curves. The same Standard 13 stresses that while "most of the mathematics discussed in the other 9-12 standards involves finite processes, […] the concept of limit and its connection with the other mathematical topics in this standard is based on infinite processes." (NCTM, 1989) Further, the infinite is part of the fabric of many more mathematical concepts. At a K-6 level, students may ponder about it in a variety of contexts, such as: while investigating the issue of the existence of a "largest natural number"; when realizing that the division process used for converting a fraction into a decimal may not always end after finitely many steps; when they are introduced to irrational numbers and their decimal form; when imagining lines, arrays and planes in plane geometry. In grades 7-12 students are introduced to infinite sequences and their limits, infinite series, and the limit of a function at a point; all of these topics are related to the concept of mathematical infinity.

While the currently accepted mathematical definitions for concepts such as infinite unions and intersections of sets, limits, convergent series, and infinitesimals avoid making direct references to infinite iterative processes, these definitions seek to form a static logical articulation that captures the dynamic essence of infinite iteration (Lakoff & Nunez, 2000). Furthermore, there is evidence that students' and mathematicians' images of and reasoning about such concepts often invoke the consideration of infinite iterative processes (Dubinsky et al., 2005). For example, if we think of the division process by which 1/3 is converted to a decimal, we have a countable process that produces, in sequence, the numbers 0.3, 0.33, 0.333, and so on. $0.\overline{3}$ (the decimal representation of 1/3) can be seen as being obtained *after all of the steps* of this process have been performed. Similarly, convergent infinite series or (countable) infinite unions can be thought of as the objects produced by completed infinite iteration (one of repeated summation, the other of repeated unioning, respectively). Therefore, being able to conceptualize an infinite iterative process as a totality, with *all* of its steps having been performed, can play an instrumental role in understanding fundamental concepts in mathematics.

Background and Research Questions

In the mathematics education literature, an infinite iterative process is defined in general terms to be a countable infinite sequence of actions that can be applied to a mathematical object (e.g., a number, a set, a geometric figure). Questions about infinite iterative processes typically ask students to consider applying this entire sequence of actions to an object and then to describe the resulting object (also referred to as *final state* or *state at infinity* in some studies). A good number of studies have shown that the vast majority of students fail to provide normative solutions to infinite iteration tasks that require them to imagine a completed infinite process and describe its outcome (Brown et al., 2008; Dubinsky et al., 2005; Dubinsky et al., 2008; Ely, 2007; Fischbein, Tirosh and Hess, 1979; Stenger et al, 2005). However, most studies regarding student reasoning about infinite iterative processes are mainly concerned with documenting existing student reasoning and not with improving students' reasoning. Furthermore, to my knowledge there are no studies that follow students' reasoning *over time* in order to explore ways in which students can build on their current reasoning regarding completing infinite iterative processes and move towards more normative conceptions of completed infinite iteration and its resulting outcome. The developmental research project (Gravemeijer, 1998) that I am reporting on in this dissertation aims to address, at least in part, these gaps in the literature.

The main research questions guiding my study are:

- What characterizes the normative state at infinity of an *arbitrary* infinite iterative process?
- What is the nature of mathematics majors' reasoning about infinite iterative processes and their states at completion?
- How can students come to develop a mathematically normative understanding of infinite iterative processes?

I address these questions through four stand-alone papers contained in Chapters 3-6. Given that each of these papers ends with its own Discussion section, there will be no additional discussion after the presentation of the four papers.

Dissertation Structure

To prepare the ground for the papers addressing the three research questions outlined above, in Chapter 2 I present a brief summary of the historical development of the treatment of infinity in mathematics. This historical review serves two purposes. First, it summarizes Cantor’s approach to treating infinity, which is referenced often throughout the four papers making up the main part of this dissertation. Second, it describes the mathematical community’s thousand-year-long progression from a *potential* treatment of infinity to an *actual* one. To have an actual view of infinity, one must be able to conceive of an infinite set as a static, well-defined mathematical object, existing in its totality, as opposed to a collection of elements that one continues to “build” with no end, which represents the potential view and has a dynamic feel to it. Some researchers argue that being able to conceive of an infinite iterative process as completed and defining a state at infinity corresponds to achieving an actual view of infinity (Dubinsky et al., 2005; Mamolo and Zazkis, 2008), which is the view adopted by the majority of today’s mathematicians. Therefore, a review of the historical development of mathematical infinity provides additional support for the importance of studying students’ reasoning on infinite iteration.

In Paper 1 (Chapter 3) I review the two main theories (in mathematics education research) that attempt to describe, from a cognitive point of view, what is involved in conceptualizing infinite iteration as completed and defining a resulting object. The first one employs the Action Process Object Schema (APOS) constructivist learning theory (Asiala et al, 1996) and describes the mental constructions that one might make in attempting to construct an infinite iterative process and its resulting object. The second one, proposed by Lakoff and Nunez (2000), employs a linguistic perspective and the method of mathematical idea analysis and posits that one can conceive of infinite iteration as completed and having a resultant state through the means of a conceptual metaphor based on a mapping between finite and infinite processes. I argue that the existing discussions of student reasoning about infinite iteration through the lens of either of these theories are unclear with respect to the mathematical foundation based on which they differentiate between normative and non-normative states at infinity. To resolve this ambiguity, I propose a formalization of the concept of state at infinity for the class of infinite iterative processes whose sequence of intermediate states can be described as a sequence of sets. I then exemplify how the proposed definition can be used to tackle completed infinite iteration problems. Lastly, I argue that the formalization of the concept of state at infinity proposed by this paper may help explain why students often struggle with infinite iteration tasks, and discuss the pedagogical implications of this formalization for instruction designed to help students refine their reasoning about infinite iteration in a normative direction.

In Paper 2 (Chapter 4) I report on my work with two mathematics majors during the first cycle of a two-cycle teaching experiment designed according to the principles of developmental research (Gravemeijer, 1998). My goals for this first cycle were 1) to characterize student reasoning on completed infinite iteration across a variety of tasks, and 2) to explore the effect on students' reasoning of a learning environment designed to be conducive to analogical reasoning. The data resulting from this first cycle confirmed what was reported by previous literature, which is that students' reasoning on infinite iterative processes is influenced by real-world considerations and that attempts to define a state at infinity are usually based on generalizing various aspects of the intermediate states of the process to the final state (most often leading to non-normative states at infinity). The participants in my study also displayed this kind of reasoning in the case of processes with undefined final states; therefore, this report extends the class of tasks for which student reasoning on infinite iteration has been documented. Additionally, my data suggests that a different type of reasoning may be triggered in students when the normative state at infinity is an open set in a topological space familiar to the students; in such cases, the participants in my study claimed that the accumulation points of the normative state at infinity needed to be part of it, in other words that the state at infinity needed to be a closed set. For easy reference, I will refer to this type of reasoning as limiting reasoning (LIM) throughout the rest of this chapter.

In the second part of Paper 2 I present evidence that the students in my study spontaneously made references to previously solved tasks or other mathematical contexts familiar to them in order to reformulate the current problem using a different context, or to refute or support an argument for the current task. Some of these references across tasks led the students to change their reasoning on one or both of the tasks involved in the task comparison, although the references across tasks did not always affect both students in the same way. I suggest that the manner in which the students refined their reasoning on completed infinite iteration while working through a complex collection of tasks is similar to how Wagner's (2006) transfer in pieces framework describes learning.

My goals for the second cycle of the teaching experiment (during which I worked with a different pair of mathematics majors) were to obtain a new set of student initial responses to the main types of tasks used in Cycle 1, and to explore ways in which I can challenge the non-normative types of student reasoning described in Paper 1, should they arise in the reasoning of the students in Cycle 2. Paper 3 (Chapter 5) describes three types of interventions I designed in order to encourage the students to question the validity of the LIM type of reasoning (which was indeed exhibited by Cycle 2 students). These interventions acknowledge that students have competing intuitions about infinity that may be triggered differently by different tasks. With this in mind, the interventions discussed in Paper 3 are designed to help students establish an "anchor" in one of the normatively solved tasks and revisit tasks that triggered non-normative reasoning by comparing the solution paths proposed for these to that of the anchor task. Students' reactions to these interventions are discussed in detail.

In Paper 4 (Chapter 6) I draw on the findings presented in Papers 2 and 3, as well as on the existing literature on infinite iteration, to propose a *local instruction theory* (in the sense of Gravemeijer, 1998) for the domain of completed infinite iteration. This local instruction theory (LIT) is developed under the assumption that the learning of a new concept or principle takes place in a manner similar to what Wagner (2006) calls transfer in pieces, and is formulated with a specific learning environment in mind, namely inquiry-based instruction focused on problem-solving (Maher, 2002; Richards, 1991).

The proposed LIT consists of a set of learning goals for the students, an envisioned learning route towards those goals, and corresponding instructional activities, which are each introduced with a rationale justifying their appropriateness and are accompanied by examples of possible student reactions from the empirical data I collected. The presentation of the LIT is followed by a discussion of the flexibility present in some of the LIT's elements. I conclude the paper with an overview of several directions for future research which could help improve the LIT.

Chapter 2 – The Historical Development of Mathematical Infinity

Introduction

“Infinity is something that never stops and keeps going. I wonder how anyone thought of infinity. Who knows?” – 6th grader (Cordeiro, 1988)

“Maybe, infinity exists in mathematics only…but then, mathematics becomes completely abstract.” – Jack, age 17 (Sierpinska, 1987)

“Infinity – you have it everywhere, and mathematics is just an attempt by man to classify the world […] The Universe is infinite” – Tom, age 17 (Sierpinska, 1987)

“Infinity is a very, very high number” – Nic, age 7 (Tall, 2001)

These quotes are examples of student answers when they were asked to talk about their understanding of the concept of infinity. The variety and vagueness of the answers may make one wonder whether all these students are referring to the same thing. However, this lack of clarity or uniformity in student answers should not be surprising; the concept of infinity puzzled mathematicians for thousands of years, and not even today is there a solid consensus with respect to what its place in mathematics should be or how it should be handled. Nevertheless, it is undeniable that a significant number of fundamental mathematical concepts (such as irrational numbers, sequences, limits, functions, geometrical concepts, and probability concepts) are related to the infinite. Some argue that the development of the infinity concept in one’s mind may parallel the historical development of the treatment of infinity in mathematics (e.g., Moreno & Waldegg, 1991). Therefore, reviewing the main stages through which the mathematical treatment of infinity passed over thousands of years may provide insight into how an individual may develop an understanding of infinity that is in agreement with today’s mathematical practices with respect to the infinite. In the remainder of this chapter I will provide a brief summary (a necessarily oversimplified account) of the millenia-long transformation of the mathematical infinite. To this end, I will touch on the treatment of infinity in the Greek culture, then on the distinction between two contrasting views of infinity as *potential* and *actual*, and finally on the contributions of Bolzano and Cantor to the treatment of the infinite in mathematics.

The Greeks and the Infinite

According to Moreno & Waldegg (1991), *the infinite* played three roles in the Greek culture: as a noun, it appeared in mythological or theological contexts, referring to the nature of the Gods. As an adjective, it was used to describe an "absolute", such as the Universe, Space or Time. Finally, as an adverb, it was used in reference to actions that *could be* "continued forever" (for example, one could keep adding 1 to an initial quantity, or one could keep making one step forward).

The word that the Greeks used for infinity was *apeiron*, which meant "out of control", "untidy" – something one would rather not deal with (Clegg, 2003). The Greeks were indeed uncomfortable dealing with infinity in mathematics, although they acknowledged the necessity of investigating this topic. Aristotle (384 BC – 322 BC) argued that because spatial magnitudes, motion and time were either finite or infinite (not finite), anyone studying physics had to "discuss the infinite and to inquire whether there is such a thing or not, and if there is, what it is" (Aristotle, 1983). In mathematics, it became obvious that one needed to deal with infinity when trying to find a common measure for the side of a square with side equal to 1 and its diagonal (Gardiner, 2002), or when considering the sum of 1, -1, 1, -1,… (which appeared to be equal to both 0 and 1, depending on how one grouped the terms of the sum) (Clegg, 2003).

The Potential Infinity Era

In his “Physics”, Aristotle introduced the idea of *potential infinity*, acknowledging that the sequence of counting numbers did not have an end, but denying that that supported the existence of infinity. To him, that simply showed that one could always find a number as large as one wanted (Lakoff & Nunez, 2000; Clegg, 2003). This view was echoed thousands of years later by the German mathematician Carl Friedrich Gauss[1] (1777 – 1835):

> “I protest against the use of an infinite quantity as an actual entity; this is never allowed in mathematics. The infinite is only a manner of speaking, in which one properly speaks of limits to which certain ratios can come as near as desired, while others are permitted to increase without bound.” (Gauss, 1831)

Viewing infinity as potential seemed to give rise to inconsistencies in the body of knowledge of the Greeks, such as Zeno’s famous paradoxes, which claimed that motion was only an illusion (for more details on Zeno’s paradoxes, see Fischbein, 2001). In order to address such inconsistencies, mathematicians and philosophers needed to search for new ways of looking at infinity and, consequently, at the world.

Bolzano’s Actual Infinity and Comparison Criteria

Bolzano (1781 – 1848) was among the first mathematicians to suggest that saying something was infinite was, in fact, the same as saying a particular set had infinitely many members (Moore, 1990). Therefore, it was necessary for *sets* to be defined more carefully as mathematical objects in order for infinity to be considered, mathematically, in its roles as adjective and noun. Bolzano’s work *The paradoxes of Infinity* (Bolzano, 1851) introduced a set as a collection of objects that could be perceived as a *whole*, taking the stress away from the constructive, element-wise process that can be associated with the construction of a set and putting it on the set's *unity* as an object. At that point, infinity could be seen as an *attribute* of a set. Bolzano argued that the set of points in space, the set of points in time, and the set of natural numbers were examples of infinite sets; he also attempted to define an infinite set of true statements, in a manner similar to Dedekind's (1831-1916) later endless set of thoughts (Moore, 1990). Thus, the treatment of infinity progressed from *potential infinity* to *actual infinity*.

Once infinite sets had been (somewhat) accepted as mathematical objects, it became necessary to devise a way to manipulate and compare them. Bolzano investigated the use of one-to-one (bijective) correspondences between two sets, when trying to compare their sizes (Clegg, 2003). He confirmed Galileo's (1564-1642) claim that each natural number corresponded to a perfect square, and vice versa (Kaplan & Kaplan, 2003); he also succeeded in showing that the set of numbers between 0 and 1 was in one-to-one correspondence with the set of numbers between 0 and 2, and that, in fact, any two intervals of real numbers could be put in such correspondence (Clegg, 2003). However, Bolzano believed that there existed "true" criteria for comparing sizes of sets, and these were the "subset" criteria – that is, every proper subset of a set has fewer elements than the set itself (Bolzano, 1851). Under this view, the set of perfect squares clearly had fewer elements than the set of natural numbers, even if the two sets could be put in bijective correspondence.

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[1] Gauss is ranked as one of history's most influential mathematicians, his most important contributions to mathematics being in the fields of number theory, analysis, and differential geometry.

Cantor's Contribution to the Study of Infinity

The historical development of the concept of infinity reached a turning point with the work of Cantor, a 19th century mathematician who built on Bolzano's work and devised a new way of comparing sizes of infinite sets. In order to be able to properly talk about comparing sets, Cantor felt it was necessary to revise the definition for the concept of a set. The version he proposed was the following:

> "By a set we are to understand any collection into a whole M of definite and separate objects *m* of our intuition and our thought." (Cantor, 1955)

The last part of Cantor's definition ("of our intuition and our thought") stresses the fact that a set is made up of objects linked conceptually, which is in agreement with Bolzano's attempt to distance the formal definition of the concept of a set from the intuitive sequential process associated with the construction of a set (Clegg, 2003).

Cantor's Approach to Size Comparison

Cantor wanted to find a mathematical way of expressing the rough intuition that a set's size was "a matter of what you can still see when you back off from the set to such an extent that you can still see its members, but can no longer make out what they are" (Moore, 1990, p. 115). With this in mind, instead of part-whole relationships, Cantor used one-to-one correspondences[2] between sets in order to compare their sizes: two sets A and B between which existed a bijection were considered *equipotent* or *equinumerous* (having the same number of elements, and denoted by $A \approx B$), while a set A from which one could build an injection into another set B (but not a bijection) was said to have fewer elements than B (denoted $A \prec B$) (Moreno & Waldegg, 1991). It's not difficult to see, both intuitively and formally, that these comparison criteria yield the same result as Bolzano's "subset" criteria when dealing with finite sets. However, under this new approach to set size comparison, it was possible for two infinite sets to have the same number of elements even if one of them was a proper subset of the other. Therefore, with Cantor's approach, the set of naturals[3] had the same size as the set of perfect squares (use $f(n) = n^2,\ n \in N$), the set of evens ($g(n) = 2n,\ n \in N$), or the set of primes ($h(n) = p_n$, where p_n is the n^{th} prime[4], $n \in N$).

At that point, within Cantor's framework, the *cardinality* (i.e., number of elements) of the set of natural numbers was greater than that of any finite set. The natural question to ask next was how this cardinality compared to that of other known sets, such as the set of integers (Z), the set of rationals (Q), or the set of reals (R). The remainder of this chapter will be concerned with how Cantor approached this question and with his theory of the transfinites (infinite cardinals and ordinals), as documented by Clegg (2003), Kaplan & Kaplan (2003), Lieber (1969), Moore (1990), Oppy (2006), and Zellini (2004).

Cantor's Approach to the Cardinalities of Z, Q, and R

In order to compare the cardinality of N with that of another set with Cantor's approach, one has to investigate whether it is possible to build a bijection between N and that other set. One way of doing this consists in finding the algebraic expression of a bijection between the two sets (as exemplified on page 6 when we used f(n)=2n to show

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[2] Gottlob Frege (1848 – 1925), arguably the greatest logician of all time, also adopted one-to-one correspondences for comparing sizes of sets.
[3] In this paper, the set of naturals means {1, 2,3,…} and it will be denoted by N. There are also texts that use N to denote {0, 1, 2, 3,…} and N^* to refer to {1, 2, 3,…}.

that N $\approx$ 2N). A more visually compelling way of defining such a bijection involves aligning the elements of the two sets in two parallel rows (or columns), such that each element from the domain set is positioned exactly above (or next to) the element in the codomain set that the function assigns to it. In what follows I will use the latter approach.

i) N $\approx$ Z

N:	1	2	3	4	5	6	7…
Z:	0	1	-1	2	-2	3	-3…

This alignment, apart from defining a bijection between N and Z, also allows us to *start counting* the elements of Z because now we have a beginning point. For this reason, sets that are finite or in bijection with N are called *countable*.

ii) N $\approx$ Q

We can write Q^+ (the set of all positive rationals) in the form of an "infinite matrix" in the manner described in Figure 1.

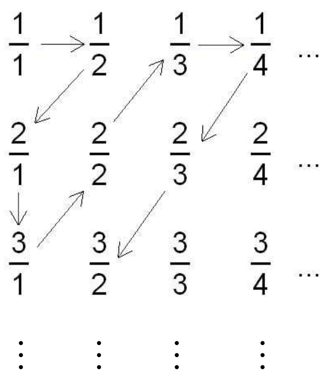

Figure 1. An enumeration of Q^+.

This diagram suggests an elegant way to count the elements of Q^+, the one suggested by the arrows. However, some numbers are counted twice this way (as certain fractions are equivalent, such as 2/2 and 4/4). This means that as one progresses with the

[4] In order for h to be well-defined, we need to show that p_n exists for any $n \in N$. The proof is at least 2,300

enumeration suggested by the diagram, one must skip those fractions whose values have already been mentioned. Thus we obtain a bijection between Q^+ and N. As Q^- (the set of all negative rationals) is equinumerous with Q^+ ($x \mapsto -x$ being a bijection from Q^+ to Q^-), it follows that Q^- must also be equinumerous with N (see footnote[5]). We have already seen that N is equinumerous with the evens and the odds, which means that we can pair Q^+ with the evens and Q^- with the odds. There is still a little problem: 0 (from Q) is not paired with any number from N. That can be resolved by "sliding" the entire enumeration of Q\{0} one position so that 1 (from N) remains unpaired, then pair it with 0 from Q. Now we have a bijection between N and Q, which makes Q countable. It is interesting to note that although Q is a densely ordered set[6], it still turned out to be countable, which is something that many find counter to what our intuition tells us.

iii) $N \prec R$

Cantor initially tried to prove that R had the same cardinality with N, but had trouble finding a bijection between the two. He eventually found an argument that these two sets did not have the same cardinality, which is now known as Cantor's diagonal argument. He first compared N with the real numbers in the interval (0, 1), using decimal representations for the latter. Assume that there exists a bijection f from N to (0, 1). Then we can arrange the elements of (0, 1) in the following way:

1 $0.a_{11}a_{12}a_{13}\ldots$
2 $0.a_{21}a_{22}a_{23}\ldots$
3 $0.a_{31}a_{32}a_{33}\ldots$
……………………

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years old and is attributed to Euclid (see Kaplan & Kaplan (2003) for a detailed exposition).

[5] Here we are using the fact that equinumerosity is transitive, i.e. that $A \approx B$ and $B \approx C$ implies $A \approx C$. This result follows from the fact that the composition of two bijections is a bijection.

[6] This means that for any two x < y in Q, there exists at least one element z in Q such that $x < z < y$ (here "<" refers to the natural order on Q).

where $a_{ij} \in \{0, 1, 2, ..., 8, 9\}$ for all $i, j \in N$. Now "construct" a number b in the interval (0, 1) in the following way: $b = 0.b_1 b_2 b_3 ...$, where $b_i = \begin{cases} 3, \text{ if } a_{ii} = 4 \\ 4, \text{ otherwise} \end{cases}$, for $i \in N$.

b is a real number somewhere between 0 and 0.5, therefore b should be one of the numbers in our enumeration of (0, 1). But this is impossible (as for any $n \in N$, b and f(n) differ with respect to the nth digit after the decimal point). This means a bijection between (0, 1) and N cannot exist.

This result, coupled with (0, 1) $\approx$ R (see footnote[7]) and with the fact that equinumerosity is transitive, proves that N is not equinumerous with R. There obviously exists an injection from N to R (the identity function, for example), so $N \prec R$ (see footnote[8]). Cantor thus proved that there were *at least* two types of infinity, that of N and that of R. This result brought about the question "are there any other types of infinity?" Cantor succeeded in answering this question affirmatively, and what is even more fascinating is that he proved it in two different ways.

The Cardinality of Power Sets

For any set A, there is a specific set associated with it made up of all the possible subsets of A. This set is called the *power set* of A, and is usually denoted by P(A). Formally, $P(A) = \{B \mid B \subseteq A\}$. If A is a finite set with n elements, it is not difficult to prove that the cardinality of P(A) is 2^n, which is obviously strictly greater than n. Is it

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[7] Use, for example, $f : (0,1) \to R$, $f(x) = \tan(\pi x - \frac{\pi}{2})$, which is bijective as per Calculus 1.

[8] See Kaplan & Kaplan (2003) for geometrical arguments of the fact that any two intervals of R have the same cardinality, which is that of R. Similar arguments are also provided for the fact that R^2 and R^3 (along with any 2-dimensional and 3-dimensional "intervals") have the same cardinality with R.

also true that $A \prec P(A)$ if A is an infinite set? Cantor was able to prove that that indeed was the case, using an argument similar to the diagonal argument presented in the proof for $N \prec R$, as follows. Let $f{:}A \rightarrow P(A)$ be an arbitrary function. We will show that this function cannot be surjective by finding an element of P(A) that is not in the range of f. Define M = $\{a \in A \mid a \notin f(a)\}$. In other words, M is a subset of A defined so that an element a of A is in M only if a is *not* a member of f(a). M is an element of P(A), but there is no $m \in A$ such that f(m) = M (as if that were the case, that m would have to be both in M and outside of M). Therefore f cannot be surjective, which means one cannot build a bijection between A and P(A). It is trivial to show one can build an injection from A to P(A) (e.g., for each a in A, define f(a) = {a}), so $A \prec P(A)$.

Using this result, one can create a sequence of sets of increasing infinite cardinalities by starting with an infinite set (for example N) and building power sets from it: $N \prec P(N) \prec P(P(N)) \prec P(P(P(N))) \prec \ldots$. This proves not only that there exist other types of infinities than those of N and R, but also that there exists an infinite number of infinities. Cantor succeeded in showing that P(N) $\approx$ R, but questioned whether other infinite cardinals (or *transfinite cardinals*, as he called them) existed between the size of N and that of P(N), or between the size of P(N) and that of its power set. His quest to "order his infinities" led him to consider ordinal numbers.

Ordinal Numbers and the Alephs

Cardinal numbers tell us "how many elements" a set has, while ordinal numbers tell us where we are in an ordered sequence. For example, as a cardinal number, 1 refers to a set with one element, while as an ordinal it refers to the first element in an ordered sequence. It's easy to confuse the two notions because with finite numbers, we use the same symbols for both cardinal and ordinal numbers. For a set M, Cantor denoted its ordinal number by $\overline{M}$ (to show that the nature of the objects in M is negated, or ignored), and its cardinal number by $\overline{\overline{M}}$ (to indicate that both the nature of the objects *and* their order were ignored) (Lieber, 1969).

The set of finite ordinals, together with its natural order, is a *well-ordered set*; that means it has a first element, and that each of its elements (except the last, if there is a last) has an immediate successor. Cantor's brilliant idea regarding ordinals consisted in creating a new ordinal to come after *all* finite ordinals: he called it ω (omega, the last letter of the Greek alphabet). This new ordinal can be interpreted as both the *first transfinite ordinal*, and also as the ordinal number of N (taken as a well-ordered set). With these in mind, it makes sense that Cantor defined $\omega + 1$ to be the next ordinal number after ω, as well as the ordinal number of sets of the type 1, 2, 3, …, a. Continuing in this manner, he added to the sequence of ordinals $\omega + 2$, $\omega + 3$ and so on, until $\omega + \omega$; this last ordinal refers, as you might have already guessed, to sets of the type $a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots$. If we denote $\omega + \omega$ by $\omega \cdot 2$ (although the "+" we use between transfinite ordinals is clearly different from regular addition), we can continue the ordinal sequence by $\omega \cdot 2 + 1$, $\omega \cdot 2 + 2$, and so on.

This is how Cantor constructed the well-ordered set of transfinite ordinals $\{\omega, \omega+1, \omega+2, \ldots, \omega \cdot 2, \omega \cdot 2+1, \ldots, \omega^2, \ldots, \omega^\omega, \ldots, \omega^{\omega^\omega}, \ldots\}$. Note that the sets whose ordinal numbers are in this set *all* have the same cardinal number as N (as exemplified in the note below[9] in the particular case of $\omega+1$). What Cantor showed about this set of transfinite ordinals is that it was *uncountable*, so its ordinal number could not be a combination of ω's. For this reason, Cantor had to invent a notation for its ordinal number, and he chose Ω. With this notation, the sequence of transfinite ordinals can be continued with Ω, $\Omega+1$, …, and the whole procedure outlined in the case of ω is applied again, and again, and again.

Now Cantor could finally return to the naming and ordering of his cardinals. The first type of infinity that he studied was that of N, and he denoted it by $\aleph_0$ ($\aleph$ being the first letter of the Hebrew alphabet). The cardinal of those sets whose ordinal number was Ω could not be $\aleph_0$, as they were not countable; Cantor denoted their common cardinal by $\aleph_1$. Continuing in this manner, Cantor obtained an infinite sequence of increasing cardinal numbers: $\aleph_0$, $\aleph_1$, $\aleph_2$, …, $\aleph_n$, … . But this unlimited sequence of transfinite cardinals "does not exhaust the conception of transfinite cardinal number. We will prove the existence of a cardinal number which we denote by $\aleph_\omega$ and which shows itself to be the next greater to all the numbers $\aleph_n$; out of it proceeds in the same way as $\aleph_1$ out of $\aleph_0$ a next greater $\aleph_{\omega+1}$, and so on, without end" (Cantor, as quoted by Lieber, 1969, p. 175). Thus, Cantor showed that the number of transfinite cardinals was infinite for the second time – the first proof involved the cardinality of power sets, discussed in 1.4.3[10].

The question at this point was whether the sequence of transfinite cardinals obtained from the power sets of N was the same as the sequence of the alephs. In particular, the hypothesis that the cardinal number of R (denoted by c)[11] was the same as

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[9] The well-ordered set {1, 2, 3,…, a} can be enumerated as a, 1, 2, 3, … . This enumeration suggests a bijection from N to {1, 2, 3, …, a}, defined by f(1) = a and f(n) = n-1 for n > 1.

[10] Cantor also developed a kind of arithmetic for transfinite cardinals, as detailed in Moore (1990)

[11] The cardinal number of P(N), shown to also be that of R, was denoted by c from "continuum".

$\aleph_1$ is called the *continuum hypothesis*, which Cantor tried hard to prove but never succeeded.

Reactions to Cantor's Theory

Cantor's theory of the transfinites was received with criticism by many of his contemporaries, including Poincaré (1854 - 1912) in France, the German Kronecker (1823 – 1891), and even the famous Gauss (1777 – 1855), who all thought it was illegitimate to think of or deal with *completed infinities*. Shortly after Cantor published the second volume of his work, Italian mathematician Burali-Forti (1861-1931) published what is now known as the Burali-Forti paradox: if we consider the set of *all* ordinals, which is well-ordered by construction and therefore has an ordinal number of its own, then this ordinal number must be greater than all the ordinals in the set, but at the same time be one of them, as the set contains all ordinals. Cantor's response to this paradox (which he claimed he thought of two years earlier) was to say that it must mean that certain collections of objects are not sets at all – he called them *inconsistent collections* (today mathematicians label such objects *classes*). Again, the study of the infinite forced mathematicians to turn their attention to how a set was defined. Paradoxes such as Burali-Forti's and Russell's [12] (the latter having nothing to do with the alephs) made it clear that it was time for another revision of the way sets were defined. This revision came in early 20th century in the form of the Zermelo-Fraænkel axiomatic approach to set theory, consisting of nine principles and usually denoted by ZF. Kurt Gödel and Paul Cohen showed, in 1940 and 1963, respectively, that the continuum hypothesis could be neither

[12] Russell's paradox has to do with the set of all sets that do not contain themselves. This set, call it M, both contains and does not contain itself.

proved nor disproved in ZF (with or without the axiom of choice[13]). Gödel also showed that ZF's (or ZFC's, if we add the axiom of choice) consistency cannot be proved from within ZF (ZFC). Among all these uncertainties, one thing is certain: this axiomatic set theory does not fall pray to Russell's paradox and Burali-Forti's paradox.

Today, Cantor's theory of the transfinites is accepted by most mathematicians, although not by all. At the moment there is no other well-developed candidate theory of the infinite. Furthermore, according to Kaplan & Kaplan (2003), there are current research programs that seek insight into finite (mathematical) situations by using recently developed transfinite numbers defined based on Cantor's work, but greater than all of Cantor's alephs. And so, the story of the infinite is still being written.

[13] Formulated by Zermelo in 1904, the axiom of choice says that given any set of pairwise disjoint non-empty sets, there exists at least one set that contains exactly one element in common with each of the non-empty sets. See Appendix F for a list of all axioms in ZFC.

Chapter 3 – Paper 1

Formalizing the State at Infinity of Completed Infinite Iteration

1. Introduction

While infinite iterative processes may be hypothetical and fanciful to many when considered in the context of the physical world as we know it (e.g., Gardner, 2001; Oppy, 2006), such processes play an important role in mathematics. Although the currently accepted mathematical definitions for concepts such as infinite unions and intersections of sets, limits, convergent series, and infinitesimals avoid making direct references to infinite iterative processes, these definitions seek to form a static logical articulation that captures the dynamic essence of infinite iteration (Lakoff & Nunez, 2000). Furthermore, there is evidence that students' and mathematicians' images of and reasoning about such concepts often invoke the consideration of infinite iterative processes (Dubinsky et al., 2005). As a specific example, one cognitive path to conceiving of the set obtained by performing union over countably many sets is to conceptualize it as the resulting object of an infinite iterative process (of repeated unioning) seen as completed. Similarly, $0.\overline{3}$ can be conceptualized as the resulting object of a completed repeated division process. The goal of this paper is to explore, at a more general level, the sense in which we can talk about the object resulting from applying infinitely[14] many actions to an initial object.

The mathematics education literature on the teaching and learning of mathematical topics related to infinity is extensive; however, research on infinite iteration has begun to emerge only in recent years. There have been a number of studies

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[14] Throughout this paper, by "infinitely many actions" I am referring to a countable, ordered infinite set of actions.

examining college students' reasoning on completed infinite iteration tasks (e.g., Brown et al., 2008; Dubinsky et al., 2008; Mamolo & Zazkis, 2008; Stenger et al, 2005). Collectively, the findings of these studies suggest that the vast majority of (college level) students produce non-normative solutions to tasks asking for the "resulting object" of various infinite iterative processes. However, none of the studies explicitly describes what characterizes a normative resulting object for an *arbitrary* infinite iterative process, and consequently what characterizes normative reasoning in this respect.

In this paper we propose to address this ambiguity by focusing on the class of infinite iterative processes whose steps can be represented by operations on sets. For this type of processes, we draw on the case of finite processes and on other standard mathematical concepts (such as infinite union) to formulate two internally-consistent, equivalent definitions for the notion of "resulting object of completed infinite iteration". We then exemplify how the proposed definitions can be used to tackle completed infinite iteration problems. Lastly, we argue that the formalization of the concept of "state at infinity" advanced in this paper may help explain why students often struggle with infinite iteration tasks, and discuss the pedagogical implications of this formalization for instruction meant to help students reason about infinite iteration in a normative manner.

2. Theories on Completed Infinite Iteration

In mathematics education research, there are two main theories that attempt to describe, from a cognitive perspective, what is involved in conceptualizing infinite iteration as completed and defining a resulting object. The first one employs the Action Process Object Schema (APOS) constructivist learning theory (Asiala et al, 1996) and describes the mental constructions that one might make in attempting to construct an infinite iterative process and its resulting object. The second one, proposed by Lakoff and Nunez (2000), employs a linguistic perspective and the method of mathematical idea analysis and posits that one can conceive of infinite iteration as completed and having a resultant state through the means of a conceptual metaphor based on a mapping between finite and infinite processes. The remainder of section 2 discusses each of these theories in detail.

2.1 The APOS-Based Theory

Researchers embracing the APOS perspective (e.g. Brown et al., 2008; Dubinsky et al, 2008; Stenger et al., 2005) propose that one is able to see infinite iteration as completed and construct a state at infinity (a resulting object) by moving from an action view of the iteration process to a process view and then to an object view. Brown et al. (2008) summarize this theoretical description nicely:

> "A process conception of infinite iteration develops as an individual coordinates multiple instantiations of a finite iterative process. When fully constructed, the individual is able to imagine the resulting infinite process as being complete, in the sense of being able to imagine that all steps have been carried out. Reflecting on the process, the individual may come to see it in its totality, that is, as a single operation in a moment of time. This may lead to an attempt to apply an action of evaluation to the process which, if successful, results in the encapsulation of the infinite iterative process. The encapsulation is a transcendent object that is understood to be outside of the process, and the object is identified with the state at infinity." (p. 23)

An empirically-based description (i.e., one based on observed student reasoning) of the construction of infinite iterative processes and their states at infinity is detailed in Brown et al. (2008) and is based on student reasoning on a set theory problem. Dubinsky et al. (2008) extend that description by analyzing student reasoning with respect to what they label a two-dimensional infinite iterative process, as it involves the coordination of three infinite iterations (the problem in Brown et al. involved the coordination of only two processes). In both of these studies it was found that the students who solved the tasks in a non-normative manner displayed an action or process view of the infinite iteration involved in the presented task.

2.2. The Basic Metaphor of Infinity

A second theory regarding completed infinite iteration comes from Lakoff and Núñez (2000). They propose that an infinite iterative process can be seen as completed if a metaphorical final state is added to it. This addition can be done by drawing a parallel between finite processes (that have a well-defined final state) and infinite processes. As both types of processes have an initial state and a clear procedure for obtaining the next state from an existing state, one can extend the parallel by imagining that, like finite processes, infinite processes also have a final, unique state that follows all intermediary states. This extension is what Lakoff and Núñez (2000) call the Basic Metaphor of Infinity (BMI). The metaphorical process thus obtained has infinitely many intermediary states and a metaphorical final state that is *unique*—that is, there is no distinct previous state within the process that both follows the completion stage of the process yet precedes the final state, and there is no other state of the process that both results from the completion of the process and follows the final state.

Although there have been no empirical studies designed specifically to test the applicability of the BMI to student reasoning about infinite iterative processes, several researchers employing an APOS framework (Brown et al., 2008. Dubinsky et al., 2008) did discuss their empirical data from the point of view of BMI as well, highlighting the differences between BMI and APOS. Next we will summarize these differences in the context of a specific infinite iteration task and argue that each of these theories, by itself, is deficient in characterizing what normative reasoning is in the context of infinite iteration tasks.

2.3 What Is Lacking?

Let us consider the Tennis Ball Problem, an infinite iteration task used in Dubinsky et al. (2008):

> Suppose that we have three bins of unlimited capacity, labeled holding bin, bin A, and bin T, with a dispenser button that when pushed, moves balls from the holding bin to bin A. The holding bin contains an infinite quantity of tennis balls, numbered 1, 2, 3,…. Half a minute before 12:00 noon, the dispenser is pressed and balls #1 and #2 drop into bin A, and ball #1 is moved instantaneously from A to T. A quarter of a minute before 12:00 noon, the dispenser is pressed again and balls #3 and #4 drop into bin A, with the smallest numbered ball in bin A immediately moved into T. In the next step, 1/8th of a minute before 12:00 noon, the dispenser is pressed and balls numbered #5 and #6 drop from the holding bin into bin A, with the smallest numbered ball in bin A immediately moved into T. If the pattern just mentioned continues, what will be the contents of bin A and bin T at 12 noon?

Dubinsky et al. (2008) explain that the "mathematically correct answer" is that bin A is empty and bin T contains all the balls, because for each ball there is a precise moment in time before noon when this ball is added to bin A and then a later moment in time (also before noon) when the same ball is moved from A to T and left there. According to Dubinsky et al., the initial stages of the reasoning involved in solving this problem (those leading to constructing an iteration through the set of natural numbers) are explained in a similar manner by the APOS-based theory and BMI. However, the authors note that the two theories diverge when it comes to the mechanism through which one defines a state at infinity, claiming that BMI's focus on the intermediate states of the Tennis Ball Problem (for each of which bin A is *not* empty) in order to metaphorically conceive of a resultant state may suggest a "non-empty" (and therefore mathematically incorrect) state at infinity for bin A.

However, we argue that the APOS-based approach seems to suffer from the same deficiency. Dubinsky et al. (2008) do note that under the APOS framework, how one conceives of the state at infinity for this problem depends on what action what decides to apply to the completed infinite iteration in order to encapsulate it into an object, and list two possible actions: one is a cumulative action, focused on answering the question of "how many balls in each bin at noon?", which is likely to lead to the conclusion that neither bin is empty; the other is an extensive action, focused on what happens with each ball throughout the process, and is likely to lead to the "bin A is empty" answer. Considering that, in the case of this problem, there are at least two conceivable ways to follow an APOS-based path to constructing a state at infinity, it is unclear why the APOS-based theory is more appropriate than BMI in explaining the cognitive aspects of mathematically correct reasoning with respect to infinite iteration.

The discussion above suggests that the applications of both of these theories to individual infinite iteration tasks can lead to multiple possible states at infinity: in the case of BMI, the resultant state at infinity depends on what aspects of the intermediate states of the process one focuses on in metaphorically creating a resultant state; in the case of APOS, the resulting state at infinity depends on what evaluating action one chooses to apply to the completed iteration. In order to claim that only one of these possible states at infinity is correct, the proponents of either of the two theories have added ad-hoc, task-specific assumptions. For example, in the case of the Tennis Ball Problem, Dubinsky et al. (2008) assume that from the fact that ball 4 is placed in bin T by one of the steps of the process and is unaffected by subsequent steps, it necessarily follows that ball 4 must also be in bin T at noon (*). Why would this be taken as true, except for the fact that it sounds reasonable based on our real-world experience? Obviously this reason is not enough to establish the aforementioned assumption as undeniable truth, as our real-world experience does not include completed infinite iteration. Experience in the physical world also teaches us to expect that if you start with an empty container and engage in an iterative process that at each step puts two balls in the container and removes one, at the end of finitely many steps the container has more balls than it did at any of the intermediate steps. The generalization of this fact from the physical world to infinite iteration would imply that a similar process with infinitely many steps, once conceptualized as completed, would result in a container with infinitely many balls in it. However, this assumption is not taken as true by Dubinsky et al., and it certainly does not lead to what the authors term "a mathematically correct solution" in the case of the Tennis Ball Problem. Not only do Dubinsky et al. not provide a rationale for the assumptions they employ in claiming a solution path to be "the correct one", they also omit to disclose what these assumptions (such as (*)) are – instead, such assumptions are used as if they are part of standard mathematical theory.

In sum, although both the APOS and BMI approaches can prove to be useful in explaining the cognitive aspects of the reasoning one may engage in to construct an infinite iterative process and an associated state at infinity, some of the existing discussions of infinite iteration through the lens of either of these theories are unclear with respect to the mathematical foundation based on which they differentiate between normative and non-normative states at infinity. To resolve this ambiguity, we believe it is first necessary to restrict the discussion only to infinite iterative processes that can be described by the same mathematical indicators. The processes involved in the tasks used in the aforementioned studies can all be described as processes whose intermediate states are sets of objects, which is why we propose to focus our discussion only on this type of processes. The following section proposes two equivalent definitions for the notion of state at infinity of an infinite iterative process of this kind.

3. Defining the State at Infinity

From here on, we will employ the term *infinite iterative process* to refer to an initial set S_0, together with an infinitely countable ordered set of actions A_i, where an action consists of finitely many operations on sets. S_n (the intermediate state obtained after step n) is defined recursively to be the result of applying A_n to S_{n-1}. For example, if we consider a process defined by

$$S_0 = \varnothing$$

$$\text{For any } n \geq 1 \ (n \in N),\ S_n = S_{n-1} \cup \{2 \cdot n\},$$

then A_5 (the action performed at step 5) consists of performing union between the sets {2, 4, 6, 8} and {10}, and S_5 (the object obtained after step 5) is the set {2, 4, 6, 8, 10}.

For an iterative process with finitely many steps, it is natural to define the object produced by the completed process as S_n, where n is the natural number representing the number of steps of the finite process. For infinite iterative processes, it is not readily apparent what the object produced by the completed process might be, as there is no last step. Phrases such as "the final state of a completed infinite process" and "the object obtained after infinitely many steps have been performed" are not defined in traditional college level mathematics courses. Although the BMI and APOS theories propose that this object must be unique and follow all the objects obtained at finite steps, they do not offer additional characterizations. In what follows we will give more precise meaning to this concept in the context of processes where the intermediate states are sets.

3.1 The R1-R3 Approach

Let us consider the finite process associated with the first 4 steps of the Tennis Ball Problem discussed in section 2.3. If we focus only on bin A, then the set representation of the finite process associated with the contents of bin A is:

$$S_0 = \varnothing$$

$$S_1 = \{2\}$$

$$S_2 = \{3, 4\}$$

$$S_3 = \{4, 5, 6\}$$

$$S_4 = \{5, 6, 7, 8\}$$

We could actually perform this process in the physical world and once all four steps are completed, we could simply look into the container, find that it contained balls 5, 6, 7, and 8, and define this set of balls to be the final state of the 4-step process. Backtracking a bit, let us examine how bin A came to contain these balls and *only* these balls. If we look at ball 5, it is in the container at the end of the process because it was put there by step 3 *and* none of the steps following step 3 removed it from the container. Similarly, although ball 2 was in the container after step 1 was performed, it is not part of the final state because there is a step (in this case step 2) that removed it from the container *and* none of the subsequent steps put ball 2 back in the container. This approach to explaining why an object is or is not an element of the final state of the process does not invoke the fact that this process has finitely many steps, and as such has the potential of being useful in defining a final state for an infinite iterative process. If we can partition the set of objects involved in the infinite iteration into two subsets, namely those to which the "ball 5" argument can be applied, on one hand, and those to which the "ball 2" argument can be applied, on the other, then we can define the final state to be the former subset (which can also be the empty set, in case the set of objects not belonging to the final set contains all of the objects under consideration).

The two types of arguments used above can be formulated abstractly in the following way:

(R1) An object x belongs to the final state of an infinite iterative process if $\exists n \in N$ such that $\forall m \in N, m \geq n$, we have $x \in S_m$. [In other words, if an object x is in some tail of the sequence of sets S_i, then it belongs to the final state.]

(R2) An object x *does not* belong to the final state of an infinite iterative process if $\exists n \in N$ such that $\forall m \in N, m \geq n$, we have $x \notin S_m$. [In other words, if an object x is *not* an element of any set in some tail of the sequence S_i, then *x* is *not* an element of the final state.]

It is important to note that it is necessary to formulate these "rules" as two separate statements (instead of having one "if and only if" rule for the belonging of an object to the final state) because the two conditions stated as the respective hypotheses of these two rules are not the only two possible scenarios. More specifically, it is possible for an object to belong to infinitely many of the intermediate sets produced by the process, and for the same object *not* to belong to infinitely many of the intermediate sets produced by the process. Therefore, R1 cannot be used to claim that the object in question is part of the final state, while R2 cannot be used to claim that the object is *not* part of the final state. As an example of this situation, consider the situation of a process whose intermediate states are {1}, {2}, {1}, {2}, … . This sequence of sets is similar to the numerical sequence 1, 2, 1, 2, … in the sense that they both oscillate between two states. The latter does not have a numerical limit, according to the standard definition of limits[15]. For this reason, a natural choice for treating infinite iterative processes for which such an oscillating object exists is to say that in this case, the final state is undefined. With this in mind, we formulate the definition of a final state at infinity for an infinite iterative process (to be denoted S_∞) to be the following:

> R1: An object x belongs to S_∞ if $\exists n \in N$ such that $\forall m \in N, m \geq n$, we have $x \in S_m$.
>
> R2: An object x *does not* belong to S_∞ if $\exists n \in N$ such that $\forall m \in N, m \geq n$, we have $x \notin S_m$.
>
> R3: If there exists an object x such that neither R1 nor R2 is applicable to x (“oscillating object”), then S_∞ is undefined. Otherwise, S_∞ is the collection of objects to which R1 applies.

This definition is consistent with the definition of a final state for finite processes and with the standard meaning given to infinite unions and intersections of sets (if conceptualized as the resulting objects of completed infinite iteration). Additionally, it formalizes the real-world intuition that an object has a continuous space-time path (i.e., if

[15] A real number sequence $\{x_n\}_{n \in N}$ is said to have a limit L if, for every $\varepsilon > 0$, there exists a natural number $m \geq 1$ such that for every $n \geq m$, $|x_n - L| \leq \varepsilon$.

an object maintains the same fixed location for all points in time $t_n = 1 - \frac{1}{2^n}$, then it should be at the same location at t=1, by continuity). This assumption is in agreement with what Allis & Koetsier (1995) propose in attempting to create a system in which the final state of an infinite iterative process is logically determined by the information provided by the intermediate states.

3.2. The Pointwise Convergence Approach

Defining a final state for an infinite iterative process whose intermediate states are sets can also be tackled with the help of characteristic functions and pointwise convergence[16] of a sequence of functions.

Given a set X and A a subset of X, the characteristic (or indicator) function of A with respect to X is a function $1_A : X \to \{0, 1\}$, defined by

$$1_A(x) = \begin{cases} 1, \text{ if } x \in A \\ 0, \text{ if } x \notin A \end{cases}$$

In the case of infinite iteration, each set S_n can be represented by its characteristic function (with respect to the whole set of objects that are involved in the process over all the steps). For example, in the case of the process mentioned earlier ($S_0 = \varnothing$ and $S_n = S_{n-1} \cup \{2 \cdot n\}$ for $n \geq 1$, $n \in N$), the characteristic function of S_5 (with respect to N) is $1_{S_5} : N \to \{0, 1\}$, $1_{S_5}(x) = \begin{cases} 1, \text{ if } x \in \{2,4,6,8,10\} \\ 0, \text{ otherwise} \end{cases}$.

[16] If $\{f_n\}_{n \in N}$ is a sequence of functions defined on the same domain D, and if there exists a function f defined on D such that for all $x \in D$, $\lim_{n \to \infty} f_n(x) = f(x)$, then we say that $\{f_n\}_{n \in N}$ converges pointwise to f.

Thus, the sequence of intermediate states of an infinite iterative process can be represented by an infinite sequence of characteristic functions, all with the same domain D (the set of all objects involved in the process). Then the final state of an infinite iterative process can be defined as follows:

> Let $\{S_n\}_{n\in N}$ be the sequence of intermediate states of an infinite iterative process and $D = \bigcup_{n\in N} S_n$. If $\{1_{S_n}\}_{n\in N}$ (with respect to D) is pointwise convergent to a function $f : D \to \{0, 1\}$, then the final state of the process is the set $S_\infty = \{x \in D \mid f(x) = 1\}$. If such a "limit function" does not exist, then by definition the final state of the process is undefined.

It is relatively simple to prove that the pointwise convergence definition for the notion of state at infinity and the R1-R3 definition are logically equivalent. A detailed proof of this equivalence can be found in Appendix C.

4. Applications

Armed with these definitions, we can start tackling tasks involving infinite iterative processes and investigate whether these definitions are applicable to them. Given the equivalence of the two definitions proposed in the previous sections, we will employ only the R1-R3 approach in solving these sample problems.

4.1. Where Did All the Marbles Go?

Consider the following problem:

> <u>The 10 Marble Problem.</u> Suppose there is a jar capable of containing infinitely many marbles and an infinite collection of marbles labeled 1, 2, 3, and so on. At step 1, marbles 1 through 10 are placed in the jar and marble 1 is taken out of the jar. At step 2, marbles 11 through 20 are placed in the jar and marble 2 is taken out; at step 3, marbles 21 through 30 are put in the jar and marble 3 is taken out; and in general, at step n marbles 10*n* -9 through 10*n* are placed in the jar and marble *n* is taken out. How many marbles are in the jar after all steps have been performed?

Mamolo and Zazkis (2008) encouraged college students to consider a timed version of this problem (i.e., the process was timed index such that infinitely many steps could be completed in 60 seconds). The researchers reported two types of common reactions to this problem. In one case, the students could not conceive of all the steps of the process having been completed, viewing the time interval over which the process was set to take place as “an endless 60 seconds”. In the other, the students noted that after step n the jar contained 9n marbles and reasoned that “at the end” the jar should contain “9 infinity” marbles, reasoning which was labeled by Mamolo and Zazkis as “rates of infinity”. These student modes of reasoning are in agreement with those reported by Dubinsky et al (2008) in the case of the Tennis Ball Problem.

In contrast with the student reasoning mentioned above, employing the R1-R3 approach to this problem suggests that we should investigate, for each of the marbles involved in the process, whether it “stabilizes” its position as inside or outside of the jar after finitely many steps. To do this, let us extract a set representation of the intermediate states of this process, as indicated by the contents of the jar after each finite step.

$S_0 = \varnothing$; $S_1 =$ {marble 2, marble 3, …, marble 10}; S_2 = { marble 3, marble 4, …, marble 20}; S_3 = { marble 4, marble 5, …, marble 30}. It can be proven by induction that S_n = { marble n+1, marble n+2, …, marble 10n}. Then for an arbitrary natural number k, marble k is *not* in any of the sets in a tail of the $\{S_n\}_{n\in N}$ above (for example, in the tail starting at rank k). By R2, marble k is not part of S_∞, if S_∞ indeed exists. As k was chosen arbitrarily, R2 can be applied to any of the marbles in the initial set of marbles we started with, which means that this process does not involve any oscillating objects (and so that the state at infinity is not undefined). As all marbles stabilize outside of the jar, “at the end” of the process the jar is empty.

4.2. Handling Oscillating Objects

Consider the following problem:

The Bin Swapping Tennis Ball Problem. You have two bins labeled A and B and infinitely many tennis balls labeled 1, 2, 3… .

- At steps 1 and 2 you place ball 1 in bin A and ball 2 in bin B, respectively.
- For $n \geq 3$, at step n you put ball n in the bin containing numbers of the same parity as n (i.e., if n is even, place ball n in the bin containing even-labeled balls)
- If $n = 3^k$ for some natural number k, then swap **the contents of the 2 bins after** you placed ball n in the appropriate bin.

After all the steps have been performed, what are the contents of the two bins?

This problem uses the previously encountered situation of an initially empty set of containers and objects being put in or taken out of them by infinitely many steps. We are asked what the contents of each of the two bins are “at the end”, so this time there are two locations of interest. Therefore, we need to investigate the existence of two final states, one corresponding to each of the two sequences of sets that describe the step-by-step contents of each bin.

It is relatively straight-forward to show that for an arbitrary natural number n, ball n is put in one of the bins at step n and from there on it is indefinitely moved from one bin to the other. Therefore, R1 and R2 do not apply to ball n with respect to the sequence of sets pertaining to bin A, and the same is true with respect to the sequence of sets pertaining to bin B. Hence each sequence of sets has at least an oscillating object and according to R3, the final state for each of them is undefined.

In contrast, the students we worked with proposed a very different solution path. As one of the goals of our work with mathematics majors was to investigate how students reason about completed infinite iteration across a variety of tasks, we did not provide the students with any abstract definitions for the state at infinity notion. In the absence of any such definitions, when presented with this task, the students noted that that at any intermediate step, one of the bins contained only odd-numbered balls and the other only even-numbered balls. Considering this parity-based partition of the set of already processed balls that was observed at the intermediate states, and the fact that after all steps had been performed all the balls from the original pile would have been distributed among the two bins, the students claimed that “at the end” one bin contained *all* the even-labeled balls while the other contained *all* the odd-labeled balls, but that it could not be determined which bin was the “even” one and which was the “odd”. It is possible that the students relied on their real-world experience with finite processes in an attempt to create a state at infinity for this particular task. It is true that if we consider a finite version of the infinite iteration involved in this problem, at the end of that process the two bins would indeed contain a partition of the processed balls into odd-labeled and even-labeled. The students may have generalized such “finite” intuitions to infinite iterative processes.

4.3. The Endless Spiral

Let us now consider an infinite iteration task set in a geometric context.

> The z^n Problem. Let z be a complex number of non-zero norm r less than 1, and consider an infinite iterative process that at step n ($n \in N$), computes the n^{th} power of z. Let P be the set of points in the plane that corresponds to all the complex numbers computed by the process after all steps (i.e., infinitely many steps) have been performed. Describe P. Does the origin of the plane belong to this set?

As the problem concerns the totality of complex numbers created by the completed process, a straightforward way of extracting a sequence of sets representation of this process involves describing, for each step n, the set of complex numbers created up to that step. This sequence of sets is:

$S_0 = \varnothing$; $S_1 = \{z\}$; $S_2 = \{z, z^2\}$; $S_3 = \{z, z^2, z^3\}$; …; $S_n = \{z, z^2 …, z^{n-1}, z^n\}$. Given that $S_n \subset S_{n+1}$ for any natural number n, we can easily deduce that there are no oscillating objects in the case of this process, so the process does have a state at infinity. If k is an arbitrary natural number, then z^k belongs to S_k and all subsequent sets. By R1, z^k belongs to S_∞. Therefore, $\{z^n \mid n \in N\}$. In order for the complex number 0 (corresponding to the origin of the two-dimensional plane R^2) to be part of S_∞, it would need to equal z^n for *some* natural number n, but there is no natural number n with this property (as the norm of z^n is r^n, which is a number strictly larger than 0, the norm of the complex number 0). By R2, $0 \notin S_\infty$, and the same is true of any other complex numbers not of the z^n form. With this in mind, we can describe P as an infinite, countable set of points that lie on a "downward spiral" that starts from P_z and spirals towards the origin of the plane in a counter-clockwise[17] manner (as $|z^n| < |z^{n+1}|$ for any natural number n and $\lim_{n\to\infty} |z^n| = \lim_{n\to\infty} |z|^n = 0$, so the higher the step number, the closer the created "point" is to the origin). The spiral, however, does not include the origin of the plane; it just comes infinitely close to it.

To our knowledge, this task has not been used by other researchers in studies regarding infinite iteration, which is why data regarding common student reactions to this task comes only from our work with mathematics majors. When we first piloted this task we were somewhat surprised to discover that some of the students had a strong belief that

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[17] Conventionally, angles are measured in a counter-clockwise manner starting from the positive x-axis, when represented in the context of the unit circle. According to De Moivre's formula, when two complex numbers are multiplied, the angle corresponding to the product is equal to the sum of the angles corresponding to each of the multipliers. These two facts combined explain the counter-clockwise spiral.

the S_∞ for this process was $\{z^n \mid n \in N\} \cup \{0\}$, arguing that $z^n \xrightarrow{n\to\infty} 0$ meant that after infinitely many steps had been performed, the process "reached 0". We hypothesize that this type of reasoning is due to an attempt to reason about a sequence of sets as if it were a sequence of numbers. This hypothesis is supported by Mamona-Downs (2001), whose findings suggest that some students view the limit (of a numerical sequence) as the last term of the sequence. Further discussion on this issue is contained in the Discussion section of this paper.

4.4. Acting on a Number

> The Powder Problem. Assume you have a scale capable of measuring weights with any degree of accuracy you desire. At step 1 you put ½ ounces of fine powder on it. At step 2 you *add* ¼ ounces of powder to the powder already on the scale. In general, at step n you add $(1/2)^n$ ounces of powder to the powder already on the scale. Imagine continuing adding powder on the scale in this manner ad infinitum. Assuming all steps have been performed, how much powder do you have on the scale?

The problem is concerned with determining the quantity of powder on the scale at the end of the process, which suggests that the answer, if defined, should be a numerical value and not a set of objects. Given that the R1-R3 approach to defining a state at infinity requires the intermediate states of the process to be represented by sets and applying this definition results in a state at infinity represented as a set of objects, it is not clear how one could proceed by employing this definition. Instead of trying to extract a set representation for the intermediate states of this process, a more natural choice would be to represent them by a sequence of numbers $\{s_n\}_{n\in N}$, where s_n represents the quantity of powder on the scale after step n has been performed. Thus we obtain the sequence

$s_1 = 1/2$, $s_2 = 1/2 + 1/4$, …, $s_n = \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k$, … . Then the amount of powder accumulated over infinitely many steps is $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$. This sum is defined in mathematics as the limit of the sequence of partial sum ($\lim_{n\to\infty} s_n$), which in our case is 1. Thus, there is 1 ounce of powder on the scale after all steps have been completed.

When presented with this task, our students were fairly quick in taking the solution path described above, and displayed a high degree of confidence in the correctness of their solution. This was in stark contrast with the uncertainty and cognitive conflict experienced by the same students when discussing the three tasks presented in sections 4.1-4.3 or the Tennis Ball Problem from section 2.3. A possible explanation for this discrepancy in student confidence levels across tasks may be that while infinite numerical sequences and their limits, as well as infinite series, are likely to have been discussed numerous times during the students' mathematical education, sequences of sets are not often addressed by mathematics courses until one's later college years. This may be why when attempting to create a meaning for the "limit set" of a sequence of sets, students may resort to the notion of limit of a numerical sequence, perhaps in an attempt to reduce the level of abstraction (as suggested by Hazzan, 1999).

4.5. Extracting the Relevant Information

As the two previous sections suggest, deciding whether the relevant information in an infinite iteration problem is best represented by a sequence of sets or a sequence of numbers is far from trivial. Furthermore, even when one decides that the intermediate states of a process are best represented by a sequence of sets, it may not be clear what that sequence should be. To explore this issue in more detail, let us consider the following problem:

> The n→n+1 Label Problem. Suppose you have a jar containing a non-labeled marble and outside of the jar you have an infinite collection of *labels* 1, 2, 3, and so on. At step 1, label 1 is attached to the marble in the jar. At step 2, label 1 is removed from the marble in the jar and label 2 is attached to the marble in the jar. In general, at step n, label n-1 is removed from the marble in the jar and label n is put in its place. Assume ALL steps have been performed. What are the contents of the jar at this point?

We are asked to describe the contents of the jar after all steps of this process have been performed. In an attempt to describe the contents of the jar at the end of each of the finite steps, one may choose to use the set representation {unlabeled marble}, {marble 1}, {marble 2}, …, {marble n},…. This seems to be a reasonable sequence, as at the end of the n^{th} step of the process the jar contains a marble labeled n. Applying the R1-R3 definition to this sequence of sets suggests an “empty jar” answer for this problem.

On the other hand, given that the steps of the process manipulate only a collection of labels and do not affect the original non-labeled marble in any way, a more natural choice for the set representation of the intermediate states would be ∅, {label 1}, {label 2}, and so on. Applying the R1-R3 definition to this sequence of sets indicates an “empty set” answer with respect to the label set, which means that after all steps have been completed the jar contains only the unlabeled marble.

In my own work I have encountered both types of reactions from students. In each case, the process of extracting the relevant information from the problem and creating a representation for it appeared to be influenced by the types of infinite iteration tasks previously solved by the students, which they directly referenced while working on the current task; furthermore, the students sorted the information in this problem into “crucial” and “irrelevant” categories gradually, concurrently with considering the applicability of a variety of solution paths used in the past. This manner of making sense of a novel problem resembles what Wagner (2006) proposes, which is that deciding what the mathematical structure of a problem is and whether it is structurally similar to a previously encountered problem is intimately connected with the problem-solving process itself.

5. Discussion

In the preceding sections we argued that the notion of *state at infinity of an infinite iterative process* has at times been used in the mathematics education literature in a vague manner, in the absence of a normative mathematical definition for what constitutes the correct state at infinity. We proposed to avoid such ambiguity by focusing on infinite iterative processes whose intermediate states can be represented as sets of objects, and formalized the state at infinity notion through two equivalent definitions. We further noted that these definitions are consistent with the standard definitions for infinite union and intersection of sets and explored their applicability to a variety of problems.

5.1. Other Possible Definitions

Although we justified our choices for the meaning given to the notion of state at infinity by drawing on reasoning about finite processes, infinite unions and intersections from set theory, and the meaning of limit of a numerical sequence, it is important to note that drawing on these areas of “standard mathematics” is nevertheless a choice, albeit one that may please from an aesthetical point of view. We acknowledge that it is possible to give other meanings to this notion, and indeed different people may well find another set of assumptions (different from R1-R3) to be the most reasonable. For example, one can choose to address the "oscillating object" situation by declaring that, by definition, a process that involves such an object has the empty set as its final state. If we call this R3', then the R1-R2-R3' collection of assumptions is internally consistent. Considering The 10 Marble Problem or The z^n Problem from the point of view of this set of assumptions would not affect the solutions already proposed, but would yield a different answer in the case of The Bin Swapping Tennis Ball Problem. Furthermore, the R1-R2-R3' approach would not be equivalent to the pointwise convergence definition.

It is also conceivable that some may propose definitions for the notion of state at infinity by attempting to capitalize on the intuition that the pattern observed in the cardinalities of the intermediate states should match the cardinality of the final state. To this end, one may suggest that an infinite iterative process (with intermediate states $\{S_n\}_{n\in N}$) has a final state if and only if there exists a set S and a natural number n such that for any m>n, we have $S_m = S$. With this definition, none of the "sequence of sets" problems we discussed allows a state at infinity. Ultimately, the assumptions on which one decides to build meaning for the state at infinity notion depend on the degree of alignment with standard mathematics notions one wishes to achieve. In the case of infinite iteration problems such as The 10 Marble Problem or The Tennis Ball Problem (discussed in Dubinsky et al., 2008; Ely, 2007; Mamolo and Zazkis, 2008), the R1-R3 approach produces final states consistent with what the aforementioned researchers have called normative states at infinity.

5.2. Cognitive Considerations

From a cognitive point of view, the R1-R3 approach suggests that reasoning about completed infinite iteration in a normative manner has two main components: 1) awareness that acting on a set through infinitely many actions is different from acting on a number through infinitely many actions; and 2) in the case of the "sequence of sets" processes, the focus should be on how the process affects *each individual object* involved in the process. We discuss each of these issues below.

Sets versus numbers. Consider The Powder Problem discussed in section 4.4. The process involved in it is of the "acting on a number" category. More specifically, it starts with the number 0 (the amount of powder on the scale before any steps were performed) and "edits" it infinitely many times by adding a non-zero number to it at each step. As discussed earlier, the resultant final state of this process is the number 1, according to the standard definition for the notion of convergent infinite series. If, on the other hand, the situation in the problem required us to keep track of all the values that this initial quantity "passed through" as it was edited by the process, and we were asked to describe the collection of all these finite-step values, the reasoning required by the problem would be of a quite different nature. In this case, the n^{th} intermediate state of the process would be best represented by the set of intermediate values "produced" by the process up to step n. Applying the R1-R3 approach to this set representation, the resultant final state of this process would be the infinite set containing all the elements of the partial sum sequence for the series in question, and this set *does not* contain the number 1.

Similarly, if we think of $\sum_{n=1}^{\infty} \frac{9}{10^n}$ as the final state of an infinite iterative process of repeated addition, then it is indeed the case that the limit of the sequence of numbers obtained as intermediate states by this infinite addition process (which is $.\overline{9}$) *is* the number obtained after all the steps have been completed. However, if we are not concerned about calculating such a sum and instead are interested only in describing the set of numbers obtained as intermediate states by the process (i.e., the numbers in the partial sum sequence associated with this series), then we would have to say that set is the set of all truncations of $.\overline{9}$, which *does not* contain $.\overline{9}$. Thus, being able to distinguish between a process that acts on a set and one that acts on a quantity and acknowledging the difference in treatment between these two situations is an important component of reasoning about infinite iteration in a normative manner (at least in the sense defined by R1-R3).

A lack of awareness with respect to the “sets versus numbers” distinction may be also responsible for the non-normative student answers reported by Brown et al. (2008) in the context of a task that asked the students to decide whether $\bigcup_{n=1}^{\infty} P(\{1, 2, 3, ..., n\})$ was equal to the power set of the natural numbers. Most of the participants in this study initially answered this question positively. The authors explain that all these students conceptualized $\bigcup_{n=1}^{\infty} P(\{1, 2, 3, ..., n\})$ by constructing an infinite iterative process with intermediate states P({1}), P({1, 2}), and in general P({1, …,n}), and hypothesize that the students might have seen P(N) as the last term of this sequence of power sets, term that could be “reached” by completed infinite iteration. This type of reasoning may be rooted in perceiving the limit of a numerical sequence as the state at infinity of an infinite iteration of the “acting on a number” type discussed above, and does not produce a normative solution in the case of the P(N) task because here the process in question produces, as an intermediate state, a set of sets. Exposing students to a task collection exploring the “sequence of sets versus sequence of numbers” distinction may help the students change their reasoning on the P(N) task in a normative direction.

Generalizations. It is important to note that when one attempts to create a meaning for the notion of state at infinity of an infinite iteration process, on a cognitive level one is likely to resort to a sequence of *generalizations*. What is being generalized and in what order each type of generalization is used has a significant impact on the emergent meaning of the state at infinity notion. To explore this issue in more detail, consider a variation of the Tennis Ball Problem discussed in section 2:

> Having an empty bin and infinitely many tennis balls (labeled 1, 2, 3, and so on) placed on the outside of the bin, you engage in a ball-moving process: at step n, you take balls 2n and 2n-1 from the pile outside of the bin and place them in the bin, after which you move ball 2n-1 out of the bin. After all steps have been completed, what is in the bin?

One of the mathematics majors to whom we presented this task proposed a solution path that consisted of the following main steps: 1) work out the contents of the bin for the first 3 steps; 2) based on the data from the first few steps, infer that after an arbitrary step n, the bin will contain all the even-numbered balls “processed” up to that step (n balls in all); 3) conclude that “at the end” the bin will contain infinitely many balls, namely all the even-numbered balls from the original pile. This solution path involves two generalizations. The first one uses the parity and cardinality patterns observed in the first 3 intermediate states of the process to claim that an arbitrary intermediate state must have corresponding parity and cardinality properties. The second generalization consists of extending the said properties from an arbitrary intermediate state to the state at infinity. Each of these generalizations focuses on observed patterns in global properties of the intermediate states (cardinality and parity): first time from specific intermediate states to an arbitrary one, and the second time from the arbitrary one to the state at infinity.

In contrast, a solution path consistent with the R1-R3 approach could consist of the following steps: 1) focus on a single ball, say 4, and note that from step 2 on, it is inside the bin; 2) generalize this single-ball property from ball 4 to an arbitrary even-labeled ball 2n; 3) use this to infer that ball 2n is inside the bin after all steps have been completed; and 4) coordinate multiple values of n (in N) to conclude that 2n spans 2N, that is that all even-numbered balls are inside the bin “at the end”. This solution path consists of two generalizations as well. The first one extends a local property of the intermediate states (the presence of ball 4 in a tail) to a similar local property of the intermediate states regarding an arbitrary even-labeled ball. The second uses a local property of the intermediate states (ball 2n belongs to them from rank n on) to deduce a local property of the state at infinity (it contains ball 2n).

In the case of the Odd/Even Ball Problem, the two sequences of generalizations lead to the same state at infinity (as shown in Figure 2). However, if the two sequences of generalizations are applied to The Tennis Ball Problem discussed in section 2.3, only the “local properties” sequence will lead to a normative state at infinity (see Figure 3). Empirical data from our own work, as well as data discussed by Dubinsky et al. (2008) and Mamolo and Zazkis (2008), suggest that the majority of college students are inclined to use the “global properties” sequence when presented with infinite iteration tasks, which may at least partially account for the many literature reports of non-normative student reasoning regarding infinite iteration.

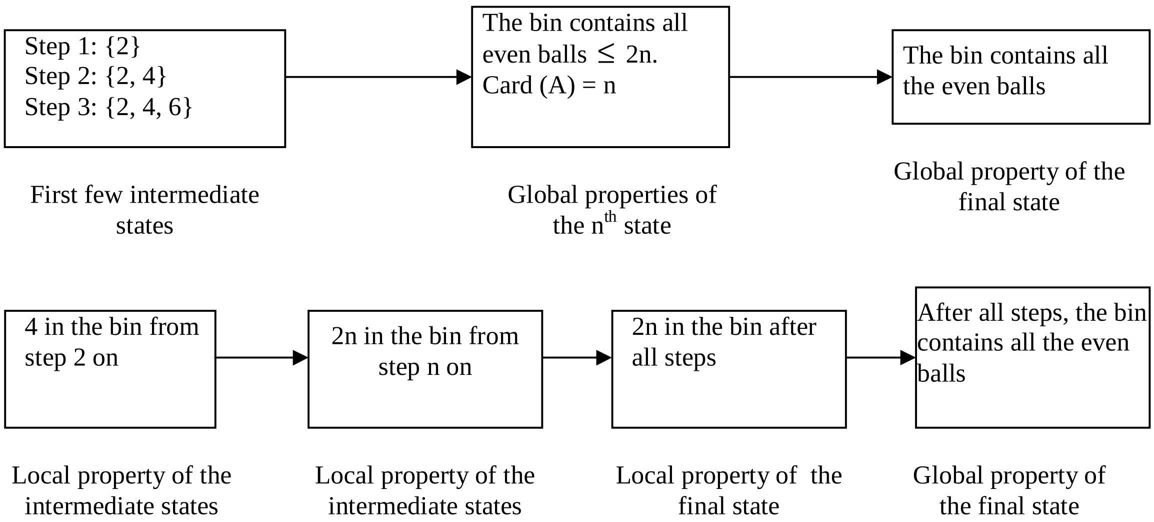

Figure 2. Two solution paths for the Odd/Even Ball Problem: the first is focused on global properties of the intermediate states, while the second focuses on local properties of the intermediate states.

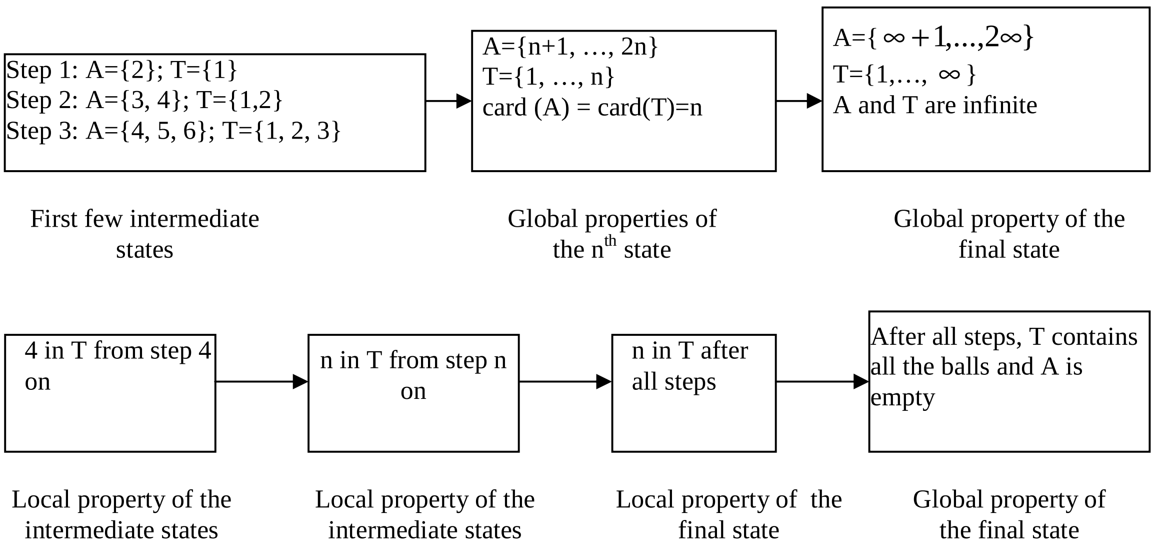

Figure 3. Two solution paths for The Tennis Ball Problem, leading to two different answer.

Encouraging students to explore the mathematical basis for each kind of generalization they make in the context of iteration tasks, as well as investigating different types of generalizations in the context of other mathematical topics and acknowledging that they may produce both valid and invalid solution paths in those contexts, can constitute a promising starting point towards helping students reason about infinite iteration in a normative manner.

Chapter 4 – Paper 2

Conceptual Changes in Mathematics Majors' Understanding of Completed Infinite Iterative Processes

1. Introduction

There has been extensive research on the teaching and learning of topics related to mathematical infinity. Numerous researchers have argued that the notion of infinity is problematic for students of all ages, and that the Cantorian treatment of infinity has proven especially difficult because students find it counterintuitive (e.g. Falk & Ben-Lavy, 1989; Fischbein, Tirosh & Hess, 1979; Tall, 1980; Wheeler & Martin, 1988).

More recently, researchers have begun investigating students' conceptions of infinite iterative processes. An infinite iterative process is a countable infinite sequence of actions that can be repeatedly applied to a mathematical object (e.g., a number, a set, a geometric figure). Questions about infinite iterative processes typically ask students to consider applying this entire sequence of actions to an object and then to describe the resulting object. Research has documented that the vast majority of students at all levels provide non-normative solutions to tasks that require them to define the outcome of an infinite iterative process (Brown, McDonald, & Weller, 2008; Dubinsky, Weller, Mcdonald, & Brown, 2005; Dubinsky, Weller, Stenger, & Vidakovic, 2008; Mamolo & Zazkis, 2008; Stenger, Vidakovic, & Weller, 2005). Several researchers argue that because infinite iteration has historically played a crucial role in the treatment of infinity in mathematics, students' failure to reason normatively about infinite iterative processes may hinder their learning of many of the mathematical concepts related to infinity (Dubinsky et al., 2005; Lakoff & Núñez, 2000).

Although there is a growing body of research on the difficulties that students have with infinite iterative processes, there has been almost no research exploring *how* students may come to reason about tasks involving infinite iteration in a normative manner. Our study aims to address this gap. More specifically, the two research questions addressed by this study are

- What types of arguments do mathematics majors produce in reasoning about infinite iterative processes and their states at infinity?
- Through what learning paths can mathematics majors come to reason about infinite iteration in a normatively correct manner?

2. Related Literature

2.1 Early Research on Infinite Iteration

Piaget and Inhelder (1956) reported on children's (ages 5-12) understanding of infinite divisibility by asking them to reason about the result of continually dividing geometric figures in half. The researchers concluded that only in the abstract (formal) operational thought stage (age 11-12) did children conceptualize the division process as infinite, and the "final elements" as points, finding that was also corroborated by Fischbein (1963).

Fischbein, Tirosh and Hess (1979) presented 470 students in grades 5-9 with situations in which geometric objects were divided repeatedly. The focus of this study was on examining to what extent students would view these processes as finite or infinite, rather than on what the resulting object of such processes would be. At all grade levels, the majority of students offered finitist views about the process (e.g., claiming "the process comes to an end after finitely many steps") rather than infinitist views (e.g., "the process never ends").

2. 2. The APOS Perspective on Infinite Iteration

Researchers embracing the Action Process Object Schema (APOS) learning theory (e.g. Brown et al., 2008; Dubinsky et al, 2008) propose that to construct an arbitrary infinite iterative process, one needs to first construct a process of iterating completely through the positive integers, which can be encapsulated into an object (conventionally labeled ∞) as one attempts to apply an action of evaluation to the process in trying to determine what comes next. Reaching a process view of an arbitrary infinite iterative process then involves coordinating this completed iteration through N with a transformation that assigns an object to each natural number; once seen as a totality, this process can be encapsulated into an object by applying an action of evaluation to it, the obtained object being "a state at ∞" and understood as beyond the objects that correspond to the natural numbers (a transcendent object).

Several studies examined college students' reasoning on versions of a problem that we called The Original Tennis Ball Problem in this study, and which states:

> Suppose you are given an infinite set of numbered tennis balls (1, 2, 3,…) and two bins of unlimited capacity, labeled A and B.
>
> At step 1 you place balls 1 and 2 in bin A and then move ball 1 to bin B.
>
> At step 2 you place balls 3 and 4 in bin A and then move ball 2 to bin B.
>
> At step 3 you place balls 5 and 6 in bin A and then move ball 3 to bin B.
>
> ……………………………………………………………………………………….
>
> This process is continued in this manner ad infinitum. Now assume that ALL steps have been completed. What are the contents of the two bins at this point?

Dubinsky et al. (2008) and Mamolo and Zazkis (2008) used timed versions of this task (i.e., the steps of the process were time-indexed such that infinitely many steps could be completed in finite time). Using an APOS perspective, both studies reported that the vast majority of the participants had a process view of the infinite iteration involved in the tasks. These students either could not conceptualize the infinite process as completed or produced non-normative final states for bin A, claiming that it contained infinitely many elements. More specifically, the students in the latter category attempted to describe the state at infinity for the process in question by replacing *n* with $"\infty"$ either in the algebraic expression indicating the cardinality of the n^{th} intermediate state or in the expression that represented the range of balls in bin A after n steps.

Brown and her colleagues (2008) explored college students' reasoning on infinite iteration by challenging them to prove or disprove $\bigcup_{n=1}^{\infty} P(\{1, 2, 3, \ldots, n\}) = P(N)$. All twelve students in the study attempted to make sense of the infinite union on the left of the equation by approaching it as an infinite iterative process and noticed that $\bigcup_{k=1}^{n} P(\{1, 2, 3, \ldots, k\}) = P(\{1, 2, 3, \ldots, n\})$, consequently initially deciding that the state at infinity of the infinite iterative process was indeed P(N). Subsequent interviews revealed that only one student had an object view of the infinite iterative process in this problem (infinite union) and was successful in disproving the given statement. The other students in the study gave explanations that were coded as showing an action or process view of the infinite union, and could not successfully complete the problem. The authors concluded that the APOS approach to constructing infinite iteration provides adequate terms for explaining the various stages of understanding reached by the students in making sense of $\bigcup_{n=1}^{\infty} P(\{1, 2, 3, ..., n\})$.

2. 3. The Basic Metaphor of Infinity

Lakoff and Núñez (2000) propose that an infinite iterative process can be seen as completed if a metaphorical final state is added to it. This addition can be done by drawing a parallel between finite processes (that have a well-defined final state) and infinite processes. As both types of processes have an initial state and a clear procedure for obtaining the next state from an existing state, one can extend the parallel by imagining that, like finite processes, infinite processes also have a final, unique state that follows all intermediary states. This extension is what Lakoff and Núñez (2000) call the Basic Metaphor of Infinity (BMI).

Lakoff and Núñez's (2000) framework does not offer guidance on what properties the metaphorical final state should have, only that it should exist, be unique, and follow each of the intermediate states. To our knowledge, no researchers have explicitly used BMI as a framework for their research on infinite iterative processes. Dubinsky and his colleagues are skeptical of the utility of BMI since students' use of metaphorical reasoning to answer questions about infinite iterative processes usually led them to conclusions that were not normatively correct (Brown et al, 2008; Dubinsky et al, 2008), although Ely (2007) argues that the BMI perspective sometimes offers a better perspective than APOS in making sense of students' non-normative behavior.

2. 4. What Is Normatively Correct Reasoning?

While students' solutions to tasks about infinite iterative processes have been addressed in several papers (e.g., Brown et al., 2008; Dubinsky et al., 2008; Mamolo &

Zazkis, 2008; Stenger et al, 2005), none of these papers has characterized what types of reasoning about the final state of an *arbitrary* infinite iterative processes are normatively correct. To avoid this ambiguity, in our study we used only infinite iterative processes for which the intermediary states are described as sets of objects. More precisely, in this study an infinite iterative process consists of an initial set S_0, together with an infinitely countable ordered set of actions A_i, where an action consists of finitely many operations on sets. S_n (the intermediate state obtained after step n) is defined recursively to be the result of applying A_n to S_{n-1}.

We argue that one can normatively define a state at infinity S_∞ (the set resulting after all A_i's have been performed) using the following rules:

(R1) If an object x is an element of every set in some tail of the sequence of sets S_i, then x is an element of S_∞. (More precisely, if $\exists m \in N \; s.t. \forall n \geq m, x \in S_n$, then $x \in S_\infty$).

(R2) If an object x is *not* an element of any set in some tail of the sequence A_i, then x is *not* an element of S_∞.

(R3) If there exists an object x such that neither R1 nor R2 is applicable to x, then S_∞ is undefined.

Applying this reasoning to the Original Tennis Ball problem (described in 2.2) shows that for any natural number n, ball n is in B_∞ because after step n, ball n is placed in bin B and never removed. Hence, as n was arbitrary, every ball eventually is in B_∞.

3. Research Methods

3. 1. Paradigm

This research was the first iteration of a design research study (Cobb, Confrey, deSessa, Lehrer, & Schauble, 2003; Gravemeijer, 1998). This type of research involves a cyclic process in which the researcher formulates “the significant disciplinary ideas and forms of reasoning that constitute the prospective goals or endpoints for student learning” (Cobb et al., 2003, p.11), after which a hypothetical learning trajectory and an associated instructional sequence are designed; the instruction is implemented with one or more students and this implementation is carefully observed and analyzed, then the hypothetical learning trajectory and instructional activities are subsequently revised based on this analysis. These steps are then repeated in a new cycle, with a new student or group of students

In this paper we report on the first cycle of this multi-cycle teaching experiment. Our instructional goals for this cycle were for the participants to be able to conceptualize an infinite iterative process as completed (that is, be able to imagine infinitely many steps as having been performed) and provide normative arguments regarding the state at infinity of an infinite iterative process across a variety of tasks. The sequence of tasks we used was semi-structured. Some tasks were designed before the study to address common non-normative answers documented in the literature, while others were added during the study in response to the specific types of arguments displayed by the participating students. In addition to the types of tasks already used by other researchers, we also used tasks in which the described process did not have a state at infinity (in the sense described in section 2). The rationale behind the creation of these variations was to obtain a collection of tasks that were similar enough to each other to inspire students to make connections between the types of reasoning used for each task, but also different enough to potentially trigger a variety of types of student arguments, which we hypothesized would create cognitive conflict for the students.

3. 2. Participants

A written pre-test containing The Original Tennis Ball Problem was administered to 14 mathematics majors[18] enrolled in a mathematics education course at a large university in northeastern United States. Out of the 13 students who provided non-normative answers to this problem, two[19] were randomly chosen. Both of these students, Max (a junior) and Tom[20] (a senior), had already completed Linear Algebra, four semesters of Calculus, and a course in proof techniques.

3. 3. Procedure

Following the written pre-test, Max and Tom were interviewed separately by the first author and asked to explain in detail their reasoning on the pre-test problem. (All problems used in this study and our proposed normative solutions can be found in Appendix A, along with examples of student solutions). The two students then worked collaboratively for a total of six problem-solving sessions lasting approximately two hours each, during which they progressed through the task sequence at their own pace. Students were encouraged to work collaboratively and provide justifications for their answers. At this stage, the interviewer neither informed them if their answers were

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[18] We decided to work with mathematics majors in order to ensure that the participants had a certain maturity regarding proof techniques, and also because a mathematics major would have already been exposed to a wide variety of mathematical concepts, which gave us more mathematical contexts to choose from for our task variations.

[19] We worked only with two students at a time in order to be able to closely follow each student's reasoning, both during each session and later during data analysis.

[20] All student names are pseudonyms.

correct nor tried to steer their reasoning in a particular direction. These sessions, as well as the initial interviews, were videotaped. The students then individually completed a written post-test containing tasks similar to those used in the main task sequence.

3. 4. Analysis

The pre-test interviews and the six problem-solving sessions, totaling 14 hours of video, were transcribed. These transcripts were initially analyzed in terms of the types of reasoning that students used to solve the problems they were given, how this reasoning changed over time, and what factors caused shifts in their reasoning. We noticed that shifts in reasoning occurred when the participants compared their reasoning for the problem they were currently solving with problems that they had encountered in the past. Consequently, we focused on the ways in which students used comparisons between problems in their reasoning and how this process contributed to their learning.

4. Results

4.1. Types of Student Reasoning

4.1.1 *Generalizing properties from the intermediary states to the final state*

When asked to define the final state of an infinite iterative process, there were instances in which both students first found what properties would be present after finitely many steps of the process had been performed, and then concluded that the final output would have these properties as well.

Generalizing cardinality. When asked what the final state of an infinite iterative process would be, the students initially tended to focus on answering the question "How many elements would be in the final state?" rather than "Which elements would be in the final state?" We present two instances of this type of reasoning.

Episode 1. For the Original Tennis Ball Problem, both Tom and Max, during their pre-test interviews, argued that bin A and bin B would each contain an infinite number of balls. Max noted that since each bin contained *n* balls after *n* steps, both bins should have infinitely many balls after all steps had been performed. Tom's reasoning was similar.

Arguments of this type have already been documented in the literature in the context of this problem or a similar one (e.g. Dubinsky et al, 2005; Dubinsky et al, 2008; Ely, 2007; Mamolo & Zazkis, 2008), so our students' initial reactions to these tasks did not come as a surprise. However, the variety of tasks used in our research allowed us to find instances of this type of argumentation in the context of other tasks as well, as described in the next episode.

Episode 2. Consider the case of The 1/2 Marble Problem, in which an infinite iterative process is defined such that the contents of a jar oscillate between {marble "1"} and {marble "2"}. The students were asked what was in the jar after all the steps had been performed. Max concluded that the jar contained exactly one marble whose label could not be determined because "you always have one marble in the jar, either 1 or 2".

Generalizing other properties. In some variations of The Original Tennis Ball Problem that we used, one way to describe the contents of the two bins (after an arbitrary finite step n) was through the means of a property shared by all the balls in a bin.

Episode 3. In Session 4 the students worked on The Bin Swapping Tennis Ball Problem, in which an infinite iterative process distributed countably many balls into two bins such that after *n* steps, one bin contained only the ball labeled 1, while the other contained the balls with labels 2, 3, …, *n*; however, the contents of the two bins were swapped at every odd step. Both Max and Tom reasoned that because of this observed pattern in the labels on the balls found in each bin after finitely many steps, "at the end" one bin must contain the ball labeled "1" while the other must contain all of the other balls (with labels in N\{1}). The fact that the process described by the problem caused the contents of the bins to be swapped at every odd step was not deemed to be of great importance by the students, who accounted for it by claiming that it was not possible to determine "which bin contains what" at the end, while still maintaining that a final state existed in the form of the partition (of the initial set of balls) mentioned above.

4.1.2. *The "reaching the limit" argument*

Another type of argument that the students employed repeatedly surfaced in the context of geometrical construction tasks, in which a set of points on the real line or in the two-dimensional plane was defined as the resulting object of an infinite iterative process. In such situations, there were several instances in which the students claimed that besides the union[21] of all the sets representing the intermediate states, the final state needed to also contain the limit points of this union.

Episode 4. In session 3 the students worked on The z^n Problem, in which z is a complex number of norm greater than 0 but less than 1 and an infinite iterative process is defined such that at step n the process defines a complex number $z_n = z^n$. The students were asked whether 0 belonged to the set of complex numbers produced by the completed process. Both students claimed that the sequence of complex numbers produced by the successive steps of the process converged to 0 and acknowledged that there was no natural number n such that z^n equaled 0. However, they could not agree on an answer to

[21] In all the geometrical construction tasks, the process was defined in such a way that any intermediate state was included in the subsequent one (so $S_0 \subseteq S_1 \subseteq ... \subseteq S_n \subseteq ...$).

the main question of the problem. It is in this context that the following dialogue took place:

> Max: If we finish the process we're at the limit. That's the only way you finish the process.
> Tom: When I think of the set produced by the process, I think of every z^n where n is a natural number…
> Max: The only way you can finish the process is if you reach the origin. 'Cause if you're not, then you're not done yet!

Tom claimed that the final state for this process was the collection of all complex numbers of the form z^n (where n is a natural number), while Max believed that the set defined by the completed process contained all the elements in $\{z^n \mid n \in N\}$ *and* the complex number 0, which is a limit point for $\{z^n \mid n \in N\}$. Later in that session Max commented that after all the steps of the process had been performed, "you're sitting on top of the origin", which he interpreted as 0 being "reached" by the process, and thus being part of the set of points produced by the process.

Episode 5. A surprising instance where this reasoning occurred was in the context of The 1+1/n Marble Problem (formulated by Tom in Session 1 while working on the 1/2 Marble Problem, which is described in episode 2). This problem posits that an infinite sequence of marbles exists where the n^{th} marble has the label $1+1/n$. At step 1, the first marble (labeled 2) is put in the jar. At step 2, the first marble is removed and the second (labeled 3/2) is placed in the jar. At step 3, the second is removed and the third (labeled 4/3) is placed in the jar. And so on. Both students strongly believed that at the conclusion of this infinite iteration, there would be a marble in the jar labeled 1, even though both acknowledged that no marble with that label was in the sequence of marbles manipulated by the process.

4.1.3. *Normatively correct solutions*

On some occasions, without prompting or direction from the interviewer, the students spontaneously generated normatively correct solutions to the questions they were asked.

Episode 6. The Vector Problem (the first task in Session 1) was designed with the purpose of having students shift their focus from global properties (such as cardinality) of the sets representing the intermediary states to the question of which specific objects were elements of each of the intermediary sets.

> The Vector Problem. Let $v = (1, 0, 0, ...) \in N^N$. You are going to "edit" this vector step by step.
>
> - Step 1: $v = (0, 1, 2, 0, 0, ...)$
> - Step 2: $v = (0, 0, 1, 2, 3, 0, 0, ...)$
> - Step 3: $v = (0, 0, 0, 1, 2, 3, 4, 0, 0 ...)$
>
> …………………………………..
> This process is continued ad infinitum. Now assume ALL steps have been completed.
>
> Describe v at this point.

This problem evoked normatively correct reasoning from both students. Tom commented that "for any entry, at some point it's going to go to 0 and stay there, so if you're done with your process it's just going to be the 0 vector." Max wondered momentarily whether the "final vector" wouldn't contain "all the natural numbers between the zeros," but then proceeded to explain that could not be the case as there would be no specific position in the vector at which the string of natural numbers could start. We hypothesize that the abstractness of The Vector Problem, and the fact that the problem focused students' attention on individual elements and positions rather than cardinalities, may have evoked this normatively correct reasoning.

This concludes our discussion of the main types of initial arguments displayed by the students in response to our tasks. Table 1 in section 4.4 provides a summary of the types of reasoning used initially by each student in response to each task (the tasks are listed in chronological order).

4.2. References to Other Tasks or Mathematical Contexts

The two students in this study often made references to problems or contexts other than the problem that they were working on at the time. These references involved comparing two or more problems in terms of what the students perceived as structural similarities or differences, and were used for three different purposes: a) to reformulate the current problem using a new context, in hope that the new context will bring additional insight; b) to revisit a previously addressed problem, when working on the current problem provided insight that put the correctness of the solution given to the previous task under question; and c) to refute or support an existing argument for the current problem.

4.2.1. *Reformulating the current problem*

Episode 7. Let us consider The Original 10 Marble Problem. This problem had a time index and was analogous to The Original Tennis Ball Problem: at the first step, marbles 1 through 10 were placed in a jar, after which marble 1 was removed from the jar; at step 2, marbles 11 through 20 were placed in the jar and marble 2 was removed from the jar, and so on. (See the Tasks document for the complete text of the problem).

Tom's initial response to this problem was that there were infinitely many marbles in the jar at the end of the process. After the interviewer asked him whether he could name a specific marble found in the jar at the end, he was troubled by the fact that he could not and proposed the following:

> Tom: I'm thinking right now of an analogy. You know the infinity hotel? This is sort of like if you have rooms 1 through 10, and you move 10 people in there, put 1 person in each room, then person in room 1 moves out, right? So you just redefine the question in terms of hotel rooms. It makes me think of a wave – people are moving down the corridor of the hotel. Once you get [to the end], the wave would be stopped. Because you're done with your operations, you're not performing anymore. It's sort of like you have this infinity, 'cause the wave is increasing. But that infinity of people you would never find if you walked down the corridor. For any room, that person would have been taken out.

While Tom did not resolve the cognitive conflict he experienced, his reformulation of The Original 10 Marble Problem using the context of Hilbert's Hotel provided a new way for him to explain his reasoning.

Note that in this case, the problem that Tom referenced was not part of the task sequence used in this study, but one encountered in a prior mathematics class. Other instances in which the students reformulated the problem at hand involved using real-life contexts (e.g., comparing the situation in The Original 10 Marble Problem with a savings account in which one repeatedly deposits $10 and takes out $1 and at some point is left with nothing), or entirely abstract terms (e.g., claiming that The 1/2 Marble Problem was equivalent to asking what the limit of the numerical sequence "1, 2, 1, 2, 1, 2,…" was).

4.2.2. *Revisiting a previously addressed problem from the task sequence*

Episode 8. Upon starting working on The Vector Problem (see episode 6), within seconds the students commented on its similarity to the Original Marble/Tennis Ball problems and the Hilbert's Hotel formulation of the marble problem, noting that the string of non-zero entries "moves across the vector" in a manner similar to the wave of people from the hotel formulation of The Original 10 Marble Problem. Having agreed on the "0 vector" answer to The Vector Problem, the two students revisited The Original 10 Marble Problem:

> Max: If that's true [referring to the "0" vector answer], then all the marbles are removed from the jar would be the right answer [to The Original 10 Marble Problem]. 'Cause you're removing them 1, then 2, then 3, eventually you would remove them all if you finish the process. You couldn't say any numbers that are in the jar, 'cause you're going to exhaust all the numbers. [The interviewer points out that during the pre-test, not being able to name any marbles in the jar did not mean to him that the jar was empty.] Yeah, but I kind of changed my mind. 'Cause even if you say that number is there at this step, at the next step it might not be there so if you do finish the process, there's not going to be nothing there. There can't be any numbers left, any natural numbers, so that can't happen.

The abstractness of The Vector problem, together with a formulation that had students focus on individual elements/positions as opposed to the cardinality of a set, appears to have helped the students move towards a more logical approach to the set of problems equivalent to The Original Tennis Ball Problem and resolve the cognitive conflict they exhibited during their respective Pre-test interviews.

4.2.3 *Referencing another problem to refute or support an existing argument*

<u>Episode 9</u>. As already discussed in episode 4, Max and Tom disagreed on The z^n Problem (with respect to whether the complex number 0 belonged to the set of numbers defined by the completed process): Max strongly believed that it did, while Tom argued the opposite. It is in this context that the following exchange took place:

> Max: Then how can you tell me this equals 1 [referring to The 1+1/n Marble Problem from episode 5], and you're trying to tell me that 0 [does not belong to the set]?
> Tom: Yeah, it is kind of inconsistent reasoning. Because when I gave that example with the balls, I was kind of thinking…
> Max: When we had this one, we ended up saying at the end we had 1 on the ball. Well, I say we have 0 right now. We're *at* the origin!
> Tom: Yeah, I mean, yeah, it depends…if we use that reasoning, which I recall being what we agreed on, then I guess you'd have to say that the origin, the zero

> vector is in your set. But on the other hand I am not really sure I agree with that [the solution given to The 1+1/n Marble problem] anymore.

Interestingly, while both students acknowledged some type of structural similarity between these two problems, they reacted in different ways to the acknowledgement: Max believed even more strongly in his “0 belongs to the final set” position on The z^n Problem, while Tom started questioning the validity of the previously agreed upon solution for The 1+1/n Marble Problem and continued to argue for “0 not part of the final set” in the case of The z^n Problem. Although in this episode the two students reacted differently to the reference made to another task, we note that both Max and Tom displayed concern for reasoning in a consistent manner across tasks.

4.3 Charting the Flow of Ideas

As the data discussed in 4.2 suggests, the students’ progress through the task sequence designed for this study was far from linear. There were numerous instances when previously addressed tasks were referenced or revisited, and problems or mathematical contexts from the students’ mathematical background were brought up as the students strived to develop a conception of completed infinite iteration that was, in their view, consistent across tasks. In this section we provide additional detail on how the students refined their reasoning *within tasks* and *across tasks*. Figure 4 contains a schematic representation of the students’ progress through The Midpoint Problem (which involved performing repeated division of a line segment that the students decided to call [0, 1], and asked the students to describe the entire collection of division points obtained after performing infinitely many steps). As Figure 4 indicates, at times the two students reacted differently to the same reference to another task. Figure 5 contains a representation of our view of the “web of connections” built by Max and Tom through the course of the study. As this diagram indicates, explicit connections were made among all but one of the problems in our task sequence. Back-references to five of the tasks led to solution changes (for the referenced tasks) on the part of either one or both students.

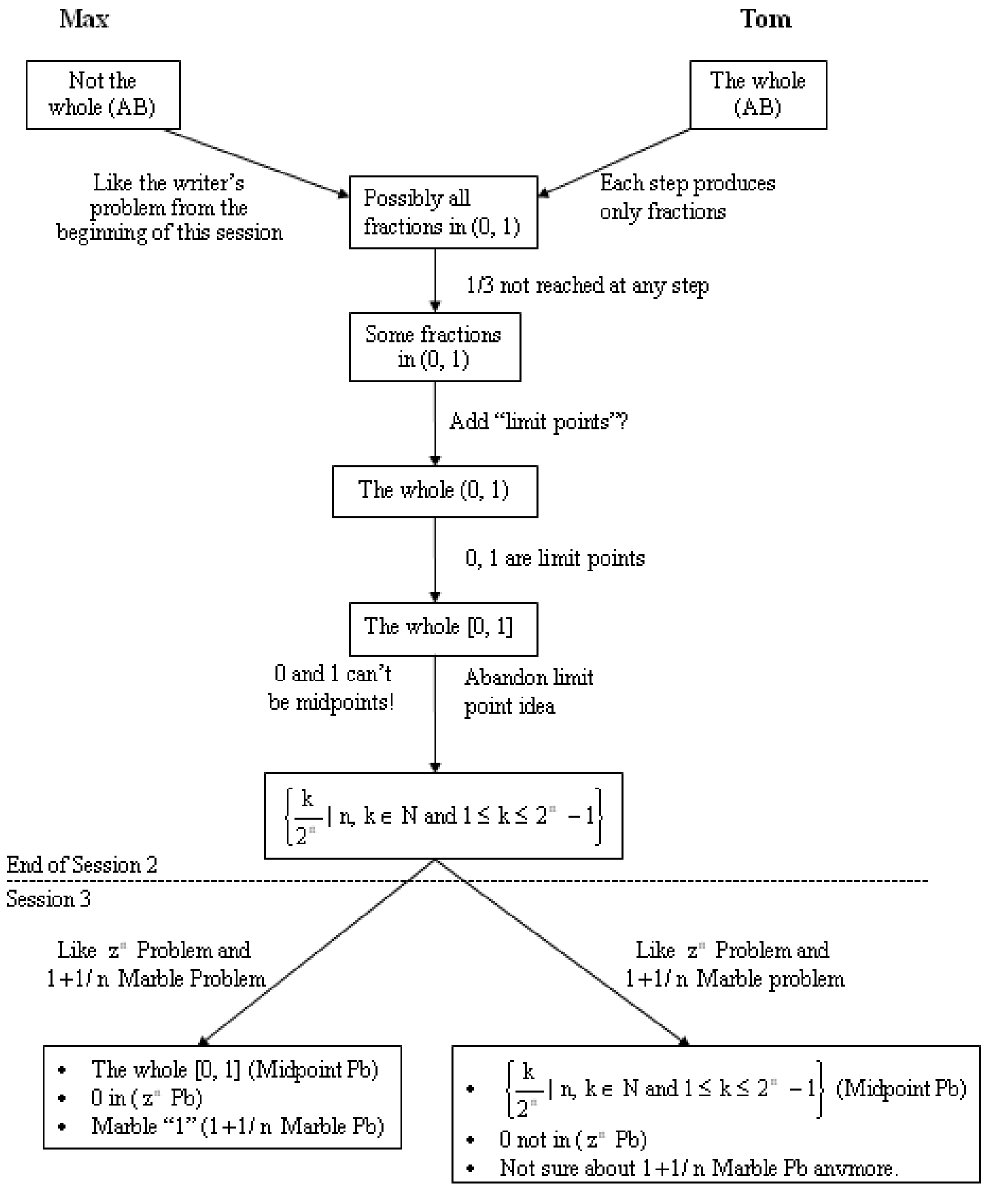
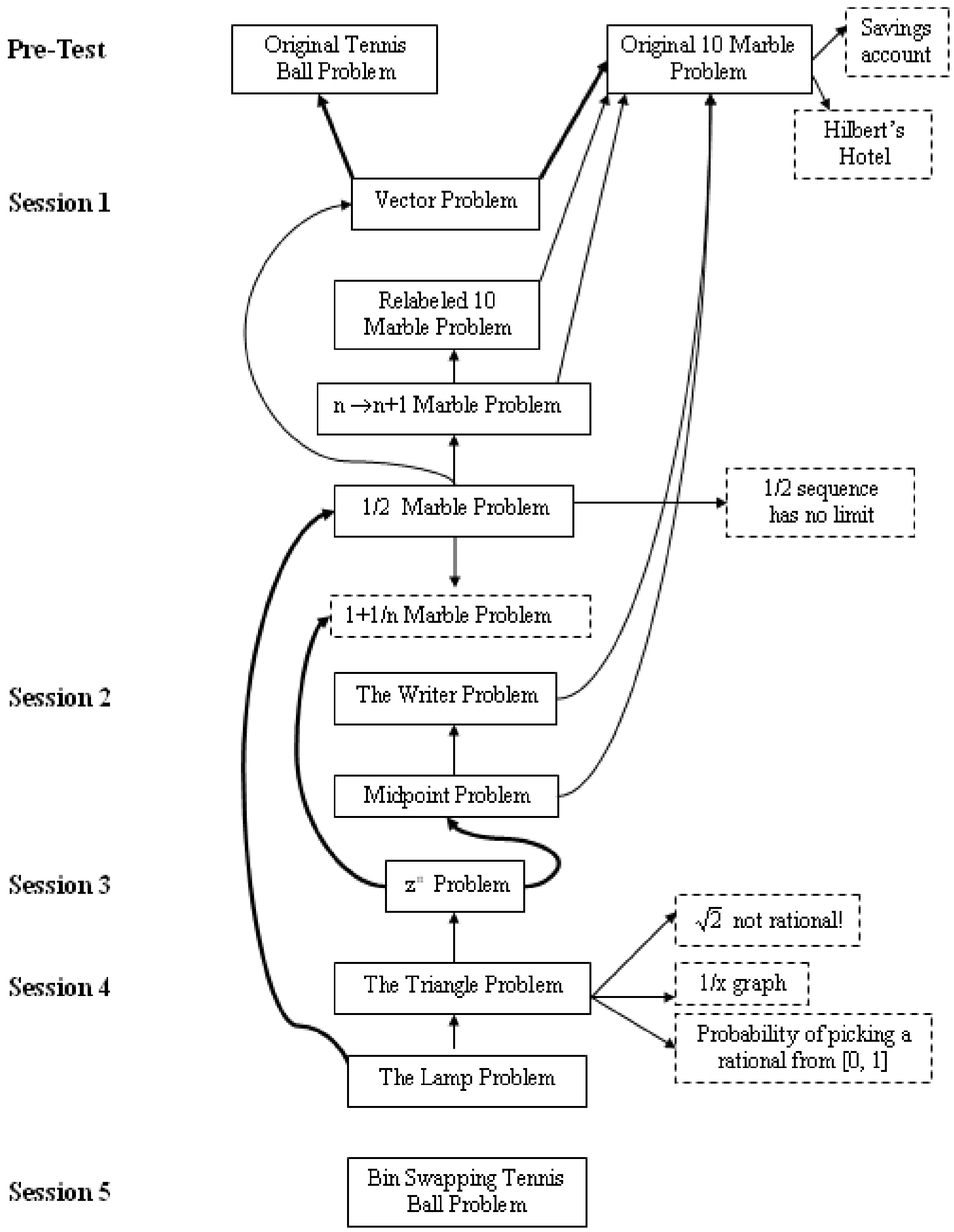


Figure 4. The students' progression through The Midpoint Problem.



Legend

$x \rightarrow y$: While discussing x, problem y was referenced

Bold arrow: the reference caused one or both of the students to change their solution to the referenced problem

Dashed line box: the problem in the box was proposed by one of the students

Figure 5. Student references across tasks

4.4. Our Intervention and Post-Test Results

At the end of five problem-solving sessions, both students displayed changes in their reasoning about infinite iteration, compared to their initial responses to our tasks. In the case of problems such as The Original 10 Marble Problem and The Original Tennis Ball Problem, both students had moved from non-normative to normative solutions. With respect to the geometrical construction problems, Tom had settled on normative solutions after originally being the one to propose the "reaching the limit" type of argument, while Max was increasingly drawn towards "reaching the limit" arguments as the study progressed. Finally, on "no final state" tasks both students exhibited "generalizing global properties" tendencies.

In the final session, the first author drew the students' attention back to The Original Tennis Ball Problem (which they had solved normatively by the end of the Session 1) and the assumptions that they had made when solving the problem—namely that a ball was/was not in a bin at the end of the process if it was/was not part of some tail of the sequence of intermediary states (in the sense discussed in section 2.4 of this paper). The students were then encouraged to revisit the other problems in the task sequence and investigate whether the final state they had defined in each case was consistent with these assumptions.

The ensuing reflection on the task sequence affected the students' conceptions of infinite iteration differently. For Tom, it led to normative solutions to all the tasks in the sequence. In contrast, Max accepted the aforementioned assumptions as sensible for the subclass of problems isomorphic to The Original Tennis Ball Problem, but claimed that they were "not applicable" or "incomplete" in the case of each task where considering them seemed to point to a different answer than the one he believed to be correct. Throughout this final discussion Max continued to display a concern for consistent reasoning across tasks, but was unable to formulate a set of assumptions that would produce a set of final states matching the ones he claimed were "the correct ones" for the whole task sequence.

Table 1 summarizes each student's initial and final positions on each task used in the problem solving sessions, as well as their answers on the Post-Test tasks.

Table 1. Students' initial and final responses to each task

Tasks	Tom	Max
The Original Tennis Ball/10 Marble Pb.	GEN-C $\rightarrow$ NORM	GEN-C $\rightarrow$ NORM
The Vector Problem	NORM	NORM
The Relabeled 10 Marble Problem	NORM	NORM
The $n \rightarrow n+1$ Marble Problem	NORM	NORM
The ½ Marble Problem	Undecided $\rightarrow$ NORM	GEN-C
The 1+1/n Marble Problem	GEN-C, LIM $\rightarrow$ NORM	GEN-C, LIM
The Writer Problem	NORM	NORM
The Midpoint Problem	LIM $\rightarrow$ NORM	Undecided
The z^n Problem	NORM	LIM
The Triangle Problem	NORM	LIM
The Lamp Problem	Undecided $\rightarrow$ NORM	GEN-O
The Bin Swapping Tennis Ball Pb.	GEN-O $\rightarrow$ NORM	GEN-O
Post-Test Problems		
Pb. $\approx$ The Original Tennis Ball Pb.	NORM	NORM
Pb. $\approx$ The Bin Swapping Tennis Ball Pb.	NORM	GEN-O
Pb. $\approx$ The Midpoint Problem	NORM	LIM

Legend
NORM – Normative solution
GEN-C – Generalizing the cardinality pattern observed among the intermediary states to the state at infinity (as discussed in section 6.1.1)
GEN-O – Generalizing patterns other than cardinality observed among the intermediary states to the state at infinity (as discussed in section 6.1.1)
LIM – "Reaching the limit" argument (as discussed in section 6.1.2)
$x \approx y$ – Problem x is isomorphic to problem y
$x \rightarrow y$ – Student changed his reasoning from x to y

5. Discussion

Students naturally employed a variety of reasoning strategies when reasoning about infinite iteration: the final state of an infinite process must share the same global properties as the intermediate states (episodes 1-3); the final state includes the limit points of the set of objects produced collectively by the intermediate steps (episodes 4 and 5); and normatively correct solutions where the final state contains the objects that were in some tail of the sequence of intermediate states (episode 6).

We hypothesize that the type of reasoning that students invoke when reasoning about infinite iterative processes may be dependent upon the contextual features of the problem being solved. We note that geometric tasks tended to elicit the “reaching the limit” reasoning, while the abstract vector task that directed the participants’ attention to specific elements and positions elicited normatively correct reasoning from both participants. This result was confirmed in a subsequent iteration of this design study (Radu, in preparation) and is consistent with the findings of Tirosh and Tsamir (1996); in the latter study, it was found that the way infinite comparison tasks were represented had a strong influence on the types of reasoning that students used and that some representations elicited normatively correct reasoning while others did not.

Table 1 illustrates that substantial learning occurred over the course of this learning cycle. In episode 1, our participants exhibited the same types of non-normative reasoning as described elsewhere in the literature (Dubinsky et al, 2008; Ely, 2007; Mamolo & Zazkis, 2008). By the end of the learning cycle, Tom arrived at normatively correct answers for every question that he had been given. He also answered every question on the post-test correctly. Although Max still used both normatively correct and incorrect reasoning by the conclusion of the cycle, he did change his solution to The Original Tennis Ball Problem in a normative direction and produced normatively correct solutions to all subsequent tasks that were isomorphic to this problem.

We do not believe an APOS-based explanation can account for Tom's learning. The APOS theory essentially takes a deficit approach to students' non-normative reasoning, suggesting that students fail to provide normatively correct solutions because they have not yet constructed process- or object-level understandings of important mathematical constructs, and that teaching students to reason normatively involves the explicit construction of these understandings. Further, Dubinksy and his colleagues discourage students from relying on their intuition when completing these tasks because their intuitions are often inaccurate (Brown et al, 2008; Dubinsky et al., 2005; Dubinksy et al, 2008). In episode 6, both students naturally used normatively correct reasoning without any explicit instruction to successfully solve the vector problem. This data support the conjecture that students *have the ability* to reason about infinitely iterative tasks in normatively correct ways. We suggest that normatively correct reasoning about infinite iterative processes frequently does not occur with students because the learning environments and task contexts elicit other modes of reasoning that are not normatively correct. Further, the improvement in Tom's learning resulted from his comparing the vector problem to the other problems he was solving (for instance, see episode 8). Tom appeared to have conflicting intuitions about infinity and some of these intuitions were indeed inaccurate (see episodes 1 and 3). However, much of Tom's learning consisted of his *capitalizing on the correct intuitions that he had*. Other researchers have noted that APOS instruction in other mathematical domains fails to utilize helpful intuitions that students bring to the table when approaching new mathematical topics (e.g., Confrey & Costa, 1996); our findings support such critiques. As a methodological note, we were able to observe the learning trajectories documented by our study because we examined student learning across tasks, while most studies observed students completing only one or two tasks.

We observed that students learned to reason normatively about infinite iterative processes by expanding the class of tasks that elicited this normatively correct reasoning and contracting the class of tasks that elicited other forms of reasoning. As Figure 5 documents, when the students were working on tasks, they spontaneously compared their reasoning to other tasks involving infinite iteration. This influenced how they attempted to solve the problem that they were working on, but at five different points in this study, this also led the students to reconsider the solutions they obtained to the previous problems. In episode 8, this process led both Tom and Max to change their solutions to The Original Tennis Ball Problems in a normative direction. We suggest what is happening here is that the two students are expanding the circumstances to which they believe the newly acquired mode of reasoning (i.e., the one they employed for The Vector Problem) is applicable. In episode 9, we see Tom questioning whether the non-normative reasoning he used on a previous task was ever appropriate. We suggest that here he is contracting the situations under which he would apply this type of reasoning.

We believe that Tom's reasoning can be modeled by Wagner's (2006) theory of "transfer in pieces". Wagner argues that many learning theories about transfer posit that students transfer knowledge from one problem to the next by *abstracting* general principles from one problem solving context and *applying* them to new contexts. Wagner offers an alternative perspective, arguing that students gradually learn to reason in normatively correct ways not by constructing general abstract principles, but by gradually refining an initially topical set of principles to *account for* (rather than ignore) new contexts, and thus *expanding the situations to which normatively correct reasoning is applicable*. If the construction of abstract principles occurs at all, it occurs after transfer has occurred many times. In this study, Tom slowly expanded his normatively correct reasoning and only abstracted reasoning principles at the end of the study with prompting from the interviewer.

Finally, we note that simply giving students interesting tasks would not be sufficient for all students to arrive at normatively correct solutions for all the tasks that they attempt. Despite desiring consistency in his reasoning, Max only arrived at normatively correct solutions for a subset of the tasks that he completed. In some cases, comparing the tasks he was completing to other tasks reinforced, rather than challenged, Max's non-normative reasoning (see episode 9). In future teaching experiments, we are seeking problems like the vector problem that will naturally elicit normatively correct modes of reasoning and then attempting to leverage solutions to these problems as an anchor to which students can compare their previous and subsequent reasoning.

Chapter 5 – Paper 3

Addressing Students' Non-normative Reasoning on Completed Infinite Iteration

1. Introduction

There has been extensive research on the teaching and learning of topics related to mathematical infinity. Numerous researchers have argued that the notion of infinity is problematic for students of all ages, and that the Cantorian treatment of infinity has proven especially difficult because students find it counterintuitive (e.g. Falk & Ben-Lavy, 1989; Fischbein, Tirosh & Hess, 1979; Monaghan, 2001; Tall, 1980; Wheeler & Martin, 1988).

More recently, researchers have begun investigating students' conceptions of infinite iterative processes. An infinite iterative process is a countable infinite sequence of actions that can be repeatedly applied to a mathematical object (e.g., a number, a set, a geometric figure). Researchers investigating infinite iterative processes typically ask students to consider applying this entire sequence of actions to an object and then to describe the resulting object. Research has documented that the vast majority of students at all levels provide non-normative solutions to tasks that require them to define the outcome of an infinite iterative process (Brown, McDonald, & Weller, 2008; Dubinsky, Weller, Mcdonald, & Brown, 2005; Dubinsky, Weller, Stenger, & Vidakovic, 2008; Ely, 2007; Mamolo & Zazkis, 2008; Stenger, Vidakovic, & Weller, 2005). Several researchers argue that because infinite iteration has historically played a crucial role in the treatment of infinity in mathematics, students' failure to reason normatively about infinite iterative processes may hinder their learning of many of the mathematical concepts related to infinity (Dubinsky et al., 2005; Lakoff & Núñez, 2000).

Although there is a growing body of research on the difficulties that students have with infinite iterative processes, there has been almost no research exploring *how* students may come to reason about tasks involving infinite iteration in a normative manner. One exception was Mamolo and Zazkis' (2008) attempt to guide students' reasoning on infinite iteration in normative directions by using instruction on the Cantorian approach to size comparison of infinite sets, although their instructional intervention had little success.

In my work, I have designed and implemented a multi-cycle teaching experiment (Cobb et al., 2003) in an effort to address this gap, during which I worked with pairs of mathematics majors (one pair per cycle). In Paper 2 I presented data obtained from the first cycle of the experiment, describing the main types of reasoning displayed by the participants in response to a variety of infinite iteration tasks, as well as documenting the students' natural tendency to use references across tasks in their arguments and examining how these references affected their reasoning. In the present paper, I will report data from both the first and second cycle in order to examine in more detail the aspects of the learning environment that triggered explicit changes in the students' reasoning. Special attention will be given to pedagogical decisions made by the researcher designed to encourage the students to question the validity of a non-normative type of reasoning.

2. Background

2.1. Goals of the Design Experiment

The main research questions that guided my work throughout the design experiment were:

1. What is the nature of mathematics majors' reasoning about infinite iterative processes and their states at completion?
2. How can students come to develop a mathematically normative understanding of infinite iterative processes?

Although tasks involving infinite iteration have been used in earlier research (Piaget and Inhelder, 1956; Fischbein, 1963; Fischbein, Tirosh and Hess, 1979), these studies employed a quantitative approach and reported mainly on the extent to which students would view the processes in question as finite or infinite and on what the nature of the "final state" described by the students was. As such, these studies did not provide much detail with respect to the intricacies of the various types of reasoning displayed by the students. The later qualitative studies on students' reasoning on completed infinite iteration (e.g., Brown et al., 2008; Dubinsky et al., 2008; Mamolo & Zazkis, 2008; Stenger et al, 2005) each used only one or two tasks, and collectively spanned a narrow range of tasks. To my knowledge, there is no study qualitatively examining the reasoning of *the same students* over a wide range of infinite iteration tasks or documenting *changes* in individual students' reasoning on infinite iteration over time. My work is characterized by both of these aspects.

2.2. Cycle 1 Overview

The data resulting from the first cycle of the teaching experiment (which was discussed in Paper 2) confirmed what was reported by previous literature, which is that students' reasoning on infinite iterative processes is influenced by real-world considerations and that attempts to define a state at infinity are usually based on generalizing various aspects of the intermediate states of the process to the final state (most often leading to non-normative states at infinity). The participants in my study also displayed this kind of reasoning in the case of processes with undefined final states (in the sense to be defined in section 3); therefore, this report extends the class of tasks for which student reasoning on infinite iteration has been documented.

Additionally, my data suggests that a different type of reasoning may be triggered in students when the normative state at infinity is an open set in a topological space familiar to the students; in such cases, the participants in my study claimed that the accumulation points of the normative state at infinity needed to be part of it-- in other words that the state at infinity needed to be a closed set. For example, one of the tasks presented students with a process that, at step n (with n a natural number), "output" z^n, where z was a complex number of non-zero norm less than 1. When the students were asked to describe the set of complex numbers produced by the completed process, one of them claimed that, apart from the members of $\{z^n \mid n \in N\}$, the complex number 0 was also part of the set produced by the completed process because $\lim_{n \to \infty} z^n = 0$, further arguing that completing the process meant "reaching the limit". This line of argumentation was employed by one or both students in the context of three other tasks. For easy reference, I will refer to this type of reasoning as limiting reasoning (LIM) throughout the rest of this paper.

In Paper 2 I also present evidence that the Cycle 1 students spontaneously made references to previously solved tasks or other mathematical contexts familiar to them in order to reformulate the current problem using a different context, or to refute or support an argument for the current task. Some of these references across tasks led the students to change their reasoning on one or both of the tasks involved in the task comparison, sometimes in normative directions and sometimes not. This suggests that the students were *transferring* aspects of the reasoning employed for one task to another based on perceived similarities between the two tasks. The manner in which these students refined their conceptions of infinite iteration while working through a collection of related tasks is supported by situated learning transfer theories, which are briefly reviewed next.

2.3. Related Transfer Literature

Traditional approaches to *knowledge transfer* propose that it takes place when abstract knowledge (obtained by identifying a pattern across multiple instances that is then turned into a decontextualized principle) is then applied to a novel situation that is determined to be an instance to which the abstract principle can be applied (Fuchs et al, 2003; Reed, 1993). The majority of empirical studies based on traditional approaches to transfer commonly involve presenting the subjects with a number of similar (from an expert's point of view) tasks and their solutions, called the *source tasks*, optionally followed by activities that are meant to help the subjects abstract the common structure of the presented problems and of their solutions. If subjects' solutions to a novel task that shares a similar structure to the source task contain certain elements that experts deem to be representative of the solution path of the source task, transfer is said to have occurred (e.g. Gick & Holyoak, 1980; Catrambone & Holyoak, 1989).

According to Lobato (2006), the classical transfer approach has been criticized on a number of grounds, including: a) relying on models of expert performance and thus ignoring the processes employed as individuals naturally draw on prior knowledge to make sense of and tackle novel situations; b) positing that decontextualized (abstract) knowledge is necessary for transfer to take place; and c) characterizing transfer as the static application of already acquired knowledge and thus not accounting for instances where individuals transform novel situations until they resemble an already familiar one.

In response to these critiques, researchers have attempted to formulate new transfer frameworks that take into account situated learning perspectives. In Lobato's (2003) *actor-oriented transfer perspective*, "transfer is treated broadly as the influence of a learner's prior activities on activity in novel situations, which entails any ways in which learning generalizes" (Lobato, 2006a, p. 111). Additionally, Pratt and Noss (2002) propose a transfer model based on the notion of *situated abstraction*. According to Pratt and Noss, learners "can abstract knowledge within settings and remain tied to the objects and relations within the situation, its tools, linguistic conventions, and structures" (p. 457). Finally, Wagner (2006) proposes a transfer model called *transfer in pieces*. According to this model, transfer of knowledge is a complex process during which an initially topical set of principles is constantly refined to *account for* (and not ignore) the new contexts of the new problems encountered. Thus, the acquisition of abstract knowledge can be seen as a *consequence* of transfer and not a required initial component for it to happen. Furthermore, according to Wagner, deciding what the mathematical structure of a problem is and whether it is structurally similar to a previously encountered problem is intimately connected with the problem-solving process itself; as a working set of principles is refined to account for new contexts, structural commonalities of the growing class of examples are gradually formulated, which in turn helps with the formulation of an abstract principle or set of principles applicable to the class.

As my goal with this design experiment was to investigate how students reason about completed infinite iteration as they progress through a complex collection of tasks and interact with them in a manner that respected their intuitions and proposed modes of reasoning, I did not see fit to adopt any of the classical approaches to transfer as a guiding framework for this study. Although the data obtained in this study contains elements that can be interpreted through each of the three situated learning perspectives described above, I chose to use Wagner's (2006) transfer in pieces approach in my analysis because it enables me to better examine the progressive refinement in the students' knowledge structures. However, Wagner does not directly address what teaching for transfer entails under his framework. In discussing certain researcher-controlled aspects of the learning environment in my study that influenced students' reasoning, I found Lobato, Ellis, and Muñoz' (2003) notion of *focusing phenomena* useful. This notion is defined as "features of the classroom environment that regularly direct students' attention toward certain (mathematical) properties or patterns when a variety of features compete for students' attention. […] Focusing phenomena emerge not only through the instructor's behavior but also through co-constructed mathematical language, features of the curricular materials, and the use of artifacts. The resulting mathematical object of focus and what students notice mathematically, are co-constituted through focusing phenomena and students' prior knowledge, experiences, and goals." (Lobato 2006a, p. 110-111). Although the notion of focusing phenomena did not play a role in the design of the study or the interventions that I made, I did find this notion useful in the analysis and description of my data.

2.4. Cycle 2 and Focus of This Paper

As described in Paper 2, during Cycle 1 students employed a variety of types of reasoning in reaction to the infinite iteration tasks presented to them, some of which were normative, and some of which were not. Further, in some cases, students referenced normative tasks to revise non-normative reasoning to normative reasoning, but in others, they referenced tasks solved in a non-normative manner to support the further application of non-normative reasoning in the context of a new task. Hence, a challenge for me as a researcher was to find ways to inspire the students to reference normatively solved tasks in productive ways.

My goals for the second cycle of the teaching experiment (during which I worked with a different pair of mathematics majors) were to obtain a new set of student initial responses to the main types of tasks used in Cycle 1, and to explore ways in which I could challenge the non-normative types of student reasoning described in Paper 2, should they arise in the reasoning of the students in Cycle 2. In this paper I report on data from Cycle 2 to address one question related to the second goal: "What types of focusing phenomena can help students examine the validity of the LIM reasoning?"

3. Characterizing Normatively Correct Reasoning on Completed Infinite Iteration

While students' solutions to tasks about infinite iterative processes have been addressed in several papers that will be discussed in more detail in the next section (e.g., Brown et al., 2008; Dubinsky et al., 2008; Ely, 2007; Mamolo & Zazkis, 2008; Stenger et al, 2005), none of these papers has characterized what types of reasoning about the final state of an *arbitrary* infinite iterative process are normatively correct. To avoid this ambiguity, in this study I used only infinite iterative processes for which the intermediary states are described as sets of objects. More precisely, in this study an infinite iterative process consists of an initial set S_0, together with an infinitely countable ordered set of actions A_i, where an action consists of finitely many operations on sets. S_n (the intermediate state obtained after step n) is defined recursively to be the result of applying A_n to S_{n-1}.

For such a process, I define a state at infinity (or final state) S_∞ (the set resulting after all A_i's have been performed) using the following rules:

(R1) If an object x is an element of every set in some tail of the sequence of sets S_i, then x is an element of S_∞. (More precisely, if $\exists m \in N \ s.t. \forall n \geq m, x \in S_n$, then $x \in S_\infty$).

(R2) If an object x is *not* an element of any set in some tail of the sequence A_i, then x is *not* an element of S_∞.

(R3) If there exists an object x such that neither R1 nor R2 is applicable to x, then S_∞ is undefined.

From here on, the phrase "normative solutions" will be used to refer to arguments that lead to states at infinity that are in agreement with this definition, usage which is consistent with what other researchers have called normative solutions to tasks involving infinite iteration.

To exemplify how this definition can be used, let us consider a problem that I called The Original Tennis Ball Problem in this study, versions of which were also used by Stenger et al. (2005), Ely (2007), and Dubinsky et al. (2008):

> Suppose you are given an infinite set of numbered tennis balls (1, 2, 3,…) and two bins of unlimited capacity, labeled A and B.
>
> At step 1 you place balls 1 and 2 in bin A and then move ball 1 to bin B.
>
> At step 2 you place balls 3 and 4 in bin A and then move ball 2 to bin B.
>
> At step 3 you place balls 5 and 6 in bin A and then move ball 3 to bin B.
>
> ……………………………………………………………………………………….
>
> This process is continued in this manner ad infinitum. Now assume that ALL steps have been completed. What are the contents of the two bins at this point?

<u>Solution</u>: Since there are two containers whose contents are affected by the process and I am asked what is in each of them "at the end", I need to define two states at infinity: A_∞ and B_∞. Let n be an arbitrary natural number. Assuming that the implied pattern in the problem is that at step n, the two lowest-numbered balls still available outside of the bins are placed in bin A, after which the lowest-numbered ball in bin A is moved to bin B, I can prove by induction that at step n, ball n is moved from bin A to bin B, and then is not affected by any of the subsequent steps. Applying R1 to ball n in relation to bin B, it follows that ball n is in bin B after all steps have been performed. As n was chosen arbitrarily, this argument holds for any of the balls, which means that all balls are in bin B and bin A is empty.

4. Infinity-Related Literature

4.1. Empirical Studies on Completed Infinite Iteration

In mathematics education research, there are two main theories that attempt to describe, from a cognitive point of view, what is involved in conceptualizing infinite iteration as completed and defining a resulting object. The first one employs the Action Process Object Schema (APOS) constructivist learning theory (Asiala et al, 1996) and describes the mental constructions (action-process-object) that one might make in attempting to construct a completed infinite iterative process and its resulting object. The second theory, proposed by Lakoff and Nunez (2000), employs a linguistic perspective and the method of mathematical idea analysis and posits that one can conceive of infinite iteration as completed and having a resultant state through the means of a conceptual metaphor based on a mapping between finite and infinite processes. Both of these theories are discussed in Papers 1 and 2 and their detailed description is beyond the scope of this paper.

There are several studies that adopt the APOS framework to investigate college students' reasoning on completed infinite iteration tasks (e.g., Brown et al., 2008; Dubinsky et al., 2008; Mamolo & Zazkis, 2008; Stenger et al, 2005). Collectively, the findings of these studies suggest that the vast majority of (college level) students produce non-normative solutions to tasks asking for the "resulting object" of various infinite iterative processes. More specifically, most of the participants in these studies either fail to conceive of infinitely many steps as having been completed (and thus reject the request for a final state as nonsensical), or attempt to define a final state by requiring that certain patterns observed in the sequence of intermediate states of the process hold for the final state (which in the majority of the cases results in a non-normative state at infinity).

4.2. Teaching Experiments on Infinity-Related Topics

On the topic of size comparison of infinite sets, three studies are worth mentioning. Tsamir (1999) examined the effect of a formal course in Cantorian Set Theory on prospective secondary mathematics teachers' views on size comparison of infinite sets. The results suggested that the teachers who had taken the set theory course were significantly more likely to choose 1-1 correspondence for set comparison over other strategies than the teachers who had not taken the course; however, in both groups of teachers, *most* teachers did not see any problem with using more than one criteria for comparing infinite sets, without displaying any awareness of contradictions that might arise this way.

A follow-up study (Tsamir, 2000) compares the effects of two different kinds of instruction on prospective teachers' view of comparison of infinite sets: a formal set theory course and an "Enrichment Cantorian Set Theory Course". The latter attempted to take into account the teachers' intuitions of infinity; to that end, the teachers were invited to investigate the legitimacy of the simultaneous use of four different methods when comparing finite sets (counting, 1-1 correspondence, inclusion, and intervals), as well as that of the use of different methods for comparing infinite sets (1-1 correspondence, single infinity, inclusion). The course also discussed the importance of consistency in mathematics. The teachers in the enrichment course and the teachers who had taken a formal Cantorian Set Theory course were then assessed through the same set of size comparison problems. The results suggested that the enrichment course was more successful than the formal course in helping teachers use only adequate methods for size comparison between infinite sets and reach awareness of the need to preserve consistency in a given mathematical system. Similarly, Tirosh (1991) reported that a 20-lesson program on basic notions of set theory that built on the student's intuitions of infinity and focused on raising the students' awareness of the inconsistencies in their own thinking raised the percentage of normative answers on size comparison tasks from 0% (before instruction) to 70%.

On the topic of infinite iterative processes, Mamolo and Zazkis (2008) examined college students' reasoning on a problem similar to The Original Tennis Ball Problem. [The two problems are similar in the sense that both involve processes that define a sequence of S_i's of ever-increasing cardinality, but for which S_∞ is the empty set; Mamolo & Zazkis' problem also had a time index that "forced" the whole process to be completed in finite time.] Before instruction, the students' answers were of mainly two types: the process never ends or there are infinitely many balls left in the container (supported by a claim that ∞ balls went out of the container and 9∞ stayed in, which the researchers called a "rates of infinity" argument). Instruction consisted in comparing the size of infinite sets through 1-1 correspondences, along with the presentation of a normative solution to the studied problem based on 1-1 correspondences. After instruction the students were asked to reconsider this problem in writing. The rates of infinity argument persisted, and the vast majority of the students who did manage to construct an explicit 1-1 correspondence between the in-going and out-going balls did not see it as meaning the container was empty "at the end". The rare students who did interpret the 1-1 correspondence as meaning an empty container mentioned that "they did not like it". These results indicate Mamolo and Zazkis' teaching intervention was seemingly unsuccessful. To my knowledge, this is the only teaching experiment in the literature that deals specifically with infinite iterative processes and their states at completion.

In summary, these studies suggest that when it comes to tasks involving infinity, instructional approaches relying solely on teaching the abstract principles behind normative reasoning have limited success. However, if such interventions are coupled with directing the students' attention to potential inconsistencies in their reasoning, discussing the source of their intuitions, and general discussions on the role of consistency in mathematics, the rate of success can improve.

5. Research methods

5.1. Paradigm

The data for this study comes from the first two cycles of a design research study (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Gravemeijer, 1998). This type of research involves a cyclic process in which the researcher formulates "the significant disciplinary ideas and forms of reasoning that constitute the prospective goals or endpoints for student learning" (Cobb et al., 2003, p.11), after which a hypothetical learning trajectory and an associated instructional sequence are designed; the instruction is implemented with one or more students and this implementation is carefully observed and analyzed, then the hypothetical learning trajectory and instructional activities are subsequently revised based on this analysis. These steps are then repeated in a new cycle, with a new student or group of students

5.2. Materials

The sequence of the tasks used in each cycle was semi-structured. Some tasks were designed before the study to address common non-normative answers documented in the literature, while others were added during each cycle (sometimes even during a session) in response to the specific types of arguments displayed by the participating students.

The tasks can be categorized in two ways. From the point of view of the word problem used, some tasks used pseudo-real-world contexts (e.g., The Original Tennis Ball Problem), while others were geometrical tasks (e.g., The Midpoint Problem, to be discussed in a later section). The tasks can also be classified with respect to the existence of the state at infinity (in the sense of the definition discussed in section 3).

The goal of the task creation process was to obtain a collection of tasks that were similar enough to each other to inspire students to make connections between the types of reasoning used for each task, but also different enough to potentially trigger a variety of types of student arguments, which might lead to cognitive conflict for the students and consequently to conceptual changes (as suggested by Piaget, 1963).

5. 3. Participants

For Cycle 2 of the design experiment, a written pre-test containing the Original Tennis Ball Problem, a no-final state task, and The Midpoint Problem (geometrical task) was administered to 25 mathematics majors[22] enrolled in two mathematics education

[22] I decided to work with mathematics majors in order to ensure that the participants had a certain maturity regarding proof techniques, and also because a mathematics major would have already been exposed to a wide variety of mathematical concepts, which gave us more mathematical contexts to choose from for our task variations.

courses at a large university in northeastern United States (see Appendix B for a complete list of all Pre-test problems and the tasks used in the problem-solving sessions, together with their normative solutions and descriptions of the student solutions encountered during Cycle 2). Out of the 23 students who provided non-normative answers to all three problems, six were randomly chosen for further screening. Each of these six met with me individually for approximately an hour during which they were asked to revisit the Pre-Test problems and explain in detail their reasoning on each task. Of these, two[23] were selected based on the clarity with which they could explain their reasoning. Both of these students, Chris and Todd[24], had already completed Linear Algebra, four semesters of Calculus, and a course in proof techniques.

Although the main data discussed in this paper comes only from Cycle 2, at times I will make reference to the Cycle 1 students (Max and Tom) in order to provide additional evidence of certain types of reasoning. These students' mathematical background was similar to that of the Cycle 2 students, and they participated in selection procedures and problem-solving sessions that were similar in nature and structure to those described for the Cycle 2 students. From here on, the phrase "the students" will refer to the Cycle 2 participants (Chris and Todd), unless specified otherwise.

5.4. Procedure

Following the Pre-Test interviews that completed the screening phase, Chris and Todd worked collaboratively for a total of seven problem-solving sessions lasting approximately two hours each, during which they progressed through the task sequence at their own pace. The students were encouraged to work together on tasks, provide

[23] I worked only with two students at a time in order to be able to closely follow each student's reasoning, both during each session and later during data analysis.

justifications for their answers, and assess the validity of their answers as well as those of their partner. At the same time, I did not validate or invalidate the answers given by the students. Instead, I asked clarifying questions regarding the students' answers, and if references to other tasks were made, I encouraged the students to elaborate on the aspects they perceived as similar or different across the two tasks. This behavior characterized most of my interaction with the students, with the exception of the types of interventions that are the focus of this paper.

At the beginning of each session, the students were interviewed separately and asked to summarize their solutions on the tasks discussed in the previous sessions. After the six sessions, the students each individually completed a think-aloud post-test containing tasks similar to those used in the main task sequence. All sessions (including the pre-test, individual interviews before each problem-solving sessions, and post-test) were videotaped. The students' involvement in the study spanned approximately a month.

5. 5. Analysis

All videotaped sessions of Cycle 2 (totaling approximately 19 hours of video) were transcribed. These transcripts were initially analyzed in terms of the types of reasoning that students used to solve the problems presented to them, how this reasoning changed over time, and what factors caused shifts in their reasoning. The second pass through the data focused on identifying episodes containing evidence of focusing phenomena (in the sense defined in section 2) and categorizing them based on what triggered the shift of focus on the part of the students in each case. The changes in student reasoning were interpreted using an adaptation of the *knowledge frames* found in Wagner's (2006) analytical framework. Wagner uses the term *frame* to refer to

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[24] All student names are pseudonyms.

knowledge structures used to orient an individual to a particular context; in other words, a "frame guides an individual's perception of what is important in a situation, thereby shaping ways of thinking or acting under the circumstances" (Wagner, 2006, p. 10). For each of the identified focusing phenomena, I used the associated episode to construct one or more knowledge frames identifying two types of knowledge resources that were available to the students during the solution activity for that particular task: one refers to aspects of the students' mathematical knowledge (acquired outside of the study), while the other describes the state of the students' emergent collection of topical rules regarding how to define a state at infinity, as suggested by the students' discussion during that particular episode.

It is important to note that these frames are intended to schematize *my interpretation* of the changes in the students' reasoning, and as such are necessarily interpretative in nature. Furthermore, they do not contain detailed representations of the problem-solving processes employed by the students; those aspects will be addressed in the accompanying discussion.

6. Results

In this section I discuss four different types of focusing phenomena present in Cycle 2 that challenged the students' LIM reasoning, discuss the rationale behind their design, and examine the effects they had on the students' reasoning. Three of them describe the main ways in which I participated in the students' discussion, while the forth is related to the task nature and task sequencing used in parts of this design experiment.

6.1. Focusing Phenomena: Task Sequencing

Although with each cycle I had a collection of base tasks at hand, the task sequence was adjusted on a continuous basis to address the types of reasoning exhibited by the students. The adjustments consisted both in changing the order of existing tasks and spontaneously inserting newly formulated tasks in places where I thought fruitful task comparison could refine the students' working principles in normative directions. Each intervention of this type was meant to produce a shift in focus of on the part of the students. In this section I describe one such intervention.

As discussed in Paper 2, in some of the geometric-context tasks that I created for this study, the normative S_∞ (expressed as a set of real numbers or complex numbers) did not contain its limit points (with respect to the default topology of the space in question). From here on I will refer to the processes described in such tasks as *limiting processes*. Both students in Cycle 1 reasoned at one point or another that the existence of a convergent sequence in the set of numbers constructed by a process required its limit to also be part of the final set (LIM reasoning) This type of student reasoning in the case of limiting processes may stem from the following: if we think of $\sum_{n=1}^{\infty}\frac{9}{10^n}$ as the final state of an infinite iterative process, then it is indeed the case that the limit of the sequence of numbers obtained as intermediate states by this infinite addition process *is* the number obtained after all the steps have been completed; however, if we are not concerned about calculating such a sum and instead are interested only in describing the set of numbers obtained as intermediate states by the process (i.e., the numbers in the partial sum sequence associated with this series), then we would have to say that set is the set of all truncations of $.\overline{9}$, which *does not* contain $.\overline{9}$. Therefore, the students' LIM reasoning might be due to an interpretation of the sequence of intermediate sets as a sequence of numbers, and not as a sequence of sets.

Episode 1. To test this hypothesis, in Cycle 2 I devised a sequence of four problems meant to direct the students' attention to the difference between these two situations; to this end, the sequence of problems contained problems in which an intermediate state was a set of labeled marbles (Problem 1), a set of labels (Problem 2), a number (Problem 3), and again a set of labeled marbles (Problem 4), except that this problem also had an artificial limiting aspect (i.e., the sequence of the numbers on the marble labels converged to 1). Figure 6 contains the full text of these problems, in the order in which they were presented to the students in Session 3.

Problem 1 (The n->n+1 Marble Problem). Suppose you have an empty jar and outside of the jar you have an infinite collection of marbles labeled 1, 2, 3, and so on. At step 1, you put marble "1" in the jar. At step 2, you put marble "2" in the jar and remove marble "1". In general, at step n, you put marble "n" in the jar and remove marble "n-1" from the jar. Assume ALL steps have been performed. What are the contents of the jar at this point?

Problem 2 ((The n->n+1 Label Problem). Suppose you have a jar containing a non-labeled marble and outside of the jar you have an infinite collection of *labels* 1, 2, 3, and so on. At step 1, the label "1" is attached to the marble in the jar. At step 2, the label "1" is removed from the marble in the jar and the label "2" is attached to the marble in the jar. In general, at step n, the label "n-1" is removed from the marble in the jar and the label "n" is put in its place. Assume ALL steps have been performed. What are the contents of the jar at this point?

Problem 3. (The 1+1/n Powder Problem) On a scale you have 2 ounces of powder. At step 1 you remove some so that you are left with 3/2 ounces. At step 2 you remove some more so that you are left with 4/3 ounces. In general, at step n you remove some powder so that you are left with $1+\frac{1}{n+1}$ ounces of powder. After all steps have been performed, how much powder do you have on the scale?

Problem 4. (The 1+1/n Marble Problem) Suppose you have an empty jar and outside of it you have an infinite number of marbles labeled with "$1+\frac{1}{n}$", where n is a natural number (so the first one is labeled "2", the second one "3/2", the third one "4/3", etc). At step 1, the marble labeled "2" is put in the jar. For any n>1, at step n you remove marble "$1+\frac{1}{n-1}$" from the jar and put marble "$1+\frac{1}{n}$" in the jar. Assume ALL steps have been performed. What are the contents of the jar at this point?

Figure 6. The "sets versus numbers" sequence (Session 3, Cycle 2).

This sequence of tasks was presented to the students in Cycle 2 after they had solved The Original Tennis Ball Problem normatively by turning away from generalizing the patterns observed in the cardinalities of the sets obtained at finite steps (which led to the initial conclusion that both bin A and bin B contained infinitely many elements "at the end"), and focusing instead on how the process affected each ball (object-based reasoning). The episode involving this shift of focus is addressed in Paper 2 in the context of Cycle 1, and occurred in a similar manner in Cycle 2.

Additionally, at the point of the introduction of the task sequence described in Figure 6, the Cycle 2 students had not worked on any limiting processes (in the sense defined at the beginning of section 6.1). Chris and Todd did not encounter much difficulty in producing solutions to problems 1 and 2 in this sequence by employing the object-based reasoning used in the context of The Original Tennis Ball Problem (with the "objects" being the set of balls in Problem 1 and the set of labels in Problem 2). Figure 7 contains my interpretation of the knowledge resources relevant to the students' discussion on these two tasks. The highlighted resources are those cued in by the solving activity, whereas the ones that are not highlighted represent resources that the students referenced while working on past tasks, but for which there is no evidence to have been abandoned by the students as invalid (i.e., resources that are "dormant" in the context of these tasks). Following Wagner's (2006) representation convention, each "window of ideas" is assumed to have high cuing priority with respect to the window in which it is enclosed. These conventions will also be used for the subsequent knowledge frames.

Phenomena Involving Infinity

Infinite Iterative Processes

Limits

Rule 1: The pattern in the cardinalities of the intermediate states needs to be consistent with the cardinality of the final state.

Rule 2: If an object is part of the intermediate sets from a finite step on (for the rest of the finite steps), it will be part of the final set.

Rule 3: If an object is part of an intermediate set but then is removed at a later step and not added back by any of the subsequent (finite) steps, then it will not be part of the final set.

Figure 7. The students' knowledge frame in the context of Problems 1 and 2 in the task sequence.

In The 1+1/n Powder Problem (Problem 3), an initial quantity of powder is affected by infinitely many powder-removal steps, such that after step n 1+1/n ounces of powder remain, and the problem asks how much powder is left after all the steps have been performed. The students proposed two solutions to this problem. One involved summing the amount of powder removed at each step over all the steps (infinitely many), and then subtracting this quantity from the original quantity of 2 ounces. The other involved computing $\lim_{n\to\infty}(1+1/n)$ and claiming that this is how much powder was left on the scale after all steps had been completed. There was almost no discussion regarding differences or similarities between this problem and the previous two problems. See Figure 8 for the knowledge frame corresponding to the students' discussion of this problem.

Phenomena Involving Infinity

Infinite Iterative Processes

Limits

Rule 1: The pattern in the cardinalities of the intermediate states needs to be consistent with the cardinality of the final state.

Rule 2: If an object is part of the intermediate sets from a finite step on (for the rest of the finite steps), it will be part of the final set.

Rule 3: If an object is part of an intermediate set but then is removed at a later step and not added back by any of the subsequent (finite) steps, then it will not be part of the final set.

Series

Figure 8. Knowledge frame for The 1+1/n Powder Problem.

When Problem 4 (The 1+1/n Marble Problem) was presented to the students, the following exchange took place:

> Todd: [turning pages back to look at the previous problems] Alright, we're sort of combining the two things that we said… maybe.
> I: What two things?
> Todd: The initial problem [n->n+1 Marble Problem] and the limit problem [1+1/n Powder Problem].
> Chris: You are combining certain elements, yes. […] Well, what do you think the possibilities are for what's in the jar at the end?
> Todd: Well, the possibilities are "marble" and "no marble",
> Chris: Is there a way that there can be no marble?
> Todd: I don't think so.
> Chris: Ok. Oh, I see, back to the first problem for a second. It does seem like an interesting point there, but…
> I: What's the interesting point?
> Chris: That there's no marbles in there because of the first problem, where we said there was nothing in there. In this case if you want to make the argument of nothing, you can say that you got an infinite number of marbles, then they'd all be removed in an infinite number of steps.
> Todd: I mean the argument is similar… you can say for any marble, you can get a step where that marble would be removed, and it would remain outside of the jar for all the time after…

> Chris: Yes, you could. […] The trick is, you have to find in here, what is the difference between problems 4 [1+1/n Marble Problem], 3 [1+1/n Powder Problem], and 1 [n->n+1 Marble Problem]? I believe it's a marble labeled 1 [referring to the final state of The 1+1/n marble Problem], because if there are infinitely many marbles labeled 1+1/n, there would be one at 1 plus 1 over infinity, which would be 1 [referring to 1+1/infinity]. So it's a marble with the label 1 on it, whether or not it's written 1 or 1 plus 1 over an absurdly large number.
> Todd: So we're combining two different types of inferences?
> Chris: Yes, but which one is the dominant inference? It can't be nothing and marble 1 at the same time. This is not to say that if this is "nothing is in the jar" then problem 3 [1+1/n Powder Problem] was wrong, or to say that if one marble is in the jar, that problem 1 was wrong. It's just a matter of which inference is the one that's applicable.

This excerpt is representative of the students' efforts throughout Cycle 2 to make progress on a novel problem (The 1+1/n Marble Problem) by comparing it to one or more of the previous problems, and attempting to discern which of those problems the novel one was "truly alike". This episode also brings support to Wagner's (2006) claim that the process of determining structural similarity between two problems is intimately connected with the process of refining topical principles into more general ones that can account for all the problems and contexts that the student encountered. As Chris and Todd wonder whether Problem 4 is more similar to Problem 1 or Problem 3, at the same time they investigate which of the two different "types of inference" (in Todd's words) used in those problems is applicable to the novel task. Such an investigation can result in an extension of the class of problems to which the students considered each type of inference to be applicable, or can lead to a refinement of the existing set of "types of inferences" to allow the inclusion of the novel problem.

In their quest to determine how The 1+1/n Marble Problem compared to The n->n+1 Marble Problem and The 1+1/n Powder Problem, the two students re-examined prior arguments in terms of the existence or non-existence of a "step at infinity" in the argument, as well as in terms of the nature of the objects manipulated by each process:

> Todd: In problem 1, these are all things you get during steps using the natural numbers. In this problem [1+1/n marble Problem], the thing we're claiming at the end [that marble labeled 1 is in the jar] is not something you can get to with the natural numbers. Does that make a difference or not?
> Chris: Well, in problem 3 [Powder Problem], you claimed, as well as I did, that there was 1 ounce left on the scale at infinity. And that's because after steps corresponding to each natural number, you had removed 1 ounce… after you went through the whole infinite number of natural numbers. Now, at the same time… are you really claiming the same thing [in The 1+1/n Marble Problem]? I don't know.
> Todd: The other thing is, Problem 3 and problem 4 are different in the sense that you can't talk about removing things in the same way.

As suggested by this excerpt, this is the point at which the two students started questioning whether it's legitimate to formulate an argument based on the existence of a step outside of the steps corresponding to the natural numbers, and thus not directly mentioned in the description of the process. In the same context, Todd raised the issue of the manner in which "things are removed" in the two processes: in the Powder problem, what is being removed at each step is part of the original state (the amount of powder on the scale at the beginning of the process), whereas in The 1+1/n Marble Problem what is being removed from the jar at each step is not part of the original state of the jar, which was "empty". This suggests that the juxtaposition of these tasks did lead the students to reflect on the distinction between intermediate states as sets and as numbers. Later, in session 6, Todd raised this issue again in the context of another limiting process task (where the intermediate states were sets) by altering that problem to produce its "number" equivalent and claiming that the two problems were significantly different. However, despite displaying some level of understanding regarding the difference between these two types of problems, the two students continued to use the LIM type of reasoning for all subsequent limiting process problems. Thus, this part of the data suggests that differentiating between an intermediate state expressed in the form of a set and an intermediate state that can be represented as a number whose magnitude is of interest in the problem is not a sufficient condition for students' normative reasoning with respect to limiting process tasks. The diagram in Figure 9 contains the hypothesized knowledge frame associated with the students' solution activity with respect to The 1+1/n Marble Problem.

Phenomena Involving Infinity

Infinite Iterative Processes

Limits

Rule 1: The pattern in the cardinalities of the intermediate states needs to be consistent with the cardinality of the final state.

Rule 2: If an object is part of the intermediate sets from a finite step on (for the rest of the finite steps), it will be part of the final set.

Rule 3: If an object is part of an intermediate set but then is removed at a later step and not added back by any of the subsequent (finite) steps, then it will not be part of the final set.

Series

Rule 4: If a *limiting process*:
- **There is a step at infinity, following all finite steps.**
- **The "limit" object is added to the final set by the step at infinity.**

Figure 9. The students' knowledge frame in relation to The 1+1/n Marble Problem.

6.2. Focusing Phenomena: Highlighting Inconsistency in the Students' Reasoning

As documented by Paper 2, there were numerous instances in Cycle 1 when one or both students made references across tasks based on perceived similarities among these tasks, and often these references affected student reasoning in multiple ways.

Abundant evidence of student use of references across tasks was found in Cycle 2 as well. However, in each cycle there were also a few instances in which one or both students made a claim in the context of a specific problem and supported it by a set of reasons, but then ignored the same line of reasoning in the context of a later problem where that type of reasoning was, in my view at least, applicable. In some of these cases I brought the students' attention to this matter and encouraged them to investigate whether the earlier type of reasoning was applicable to the new problem. In doing so I attempted to create a situation of cognitive conflict for the students and hoped that in response the students would reflect more deeply on what characterizes the highlighted type of reasoning and under what conditions it could be applied.

Episode 2. In Cycle 2, following the sessions in which the students reached normative reasoning on problems isomorphic to The Original Tennis Ball Problem, as well as the session in which the task sequence described in 6.1 was tackled, students were presented with The z^n Problem (session 4). In this problem, z is a complex number of norm greater than 0 but less than 1 and an infinite iterative process is defined such that at step n the process defines a complex number $z_n = z^n$. The students were asked whether 0 belonged to the set of all complex numbers produced by the completed process. Both Chris and Todd claimed that 0 was indeed part of the final set, while acknowledging that z^n was not 0 for any natural number n. The argument for their claim was that since $\lim_{n\to\infty} z^n = 0$, at the "step at infinity" 0 was produced by the process. This mysterious step at infinity was not clearly defined by the students, but was claimed to be "the last step in the process", coming after all the natural numbers, but not a natural number.

One session later, the students were presented with The Midpoint Problem, which describes a repeated segment division process (by halving), and asks how the set of all "midpoints" constructed after all steps have been completed compares with the original segment [AB]. The Cycle 2 students proposed to assume the original segment to be [0, 1], and after some discussion agreed with each other that numbers of the form $\frac{k}{2^n}$ (where n is a natural number and k a natural number less than 2^n) were part of the final set. Figure 10 contains my interpretation of the knowledge frame associated with the discussion on The Midpoint Problem up to this point.

Phenomena Involving Infinity

Infinite Iterative Processes

Limits

Rule 1: The pattern in the cardinalities of the intermediate states needs to be consistent with the cardinality of the final state.

Rule 2: If an object is part of the intermediate sets from a finite step on (for the rest of the finite steps), it will be part of the final set.

Rule 3: If an object is part of an intermediate set but then is removed at a later step and not added back by any of the subsequent (finite) steps, then it will not be part of the final set.

Series

Rule 4: If a *limiting process*:
- o There is a step at infinity, following all finite steps.
- o The "limit" object is added to the final set by the step at infinity**.**

Figure 10. The students' knowledge frame as they worked through The Midpoint Problem. Highlighted principles represent those cued by this problem before my intervention.

It is in this context that the following exchange took place:

> Todd: So it seems in terms of the first point [the one right to the right of A] can get arbitrarily close to A. But if we say the segment is from 0 to 1, do we include

> all the real numbers [between 0 and 1]? 'Cause I was thinking like $1/2^n$, there are some numbers you can't write as a multiple of that. […] Like any number that is not rational, between 0 and 1.
> Chris: That's certainly true, but there are plenty of rationals that can't be written that way either.
> Todd: True, yeah.
> I: Can you think of a rational?
> Todd: A rational that can't be written [as a multiple of a $1/2^n$], like 1/3?
> Chris: That's pretty good.
> Todd: So I mean you can get arbitrarily close to 1/3, but I don't know if that matters…
> Chris: You can get arbitrarily close to 1/3, but there's absolutely no way that I see that you can actually hit 1/3, because there is nothing that actually limits towards it in this case.
> I: Actually, there is. You can build a sequence that converges to 1/3 of numbers only of this kind, the kind you described before [$\frac{k}{2^n}$].

A discussion followed regarding how one can construct a sequence with limit 1/3 using only numbers of the form $\frac{k}{2^n}$. Once students agreed that that was possible, I pointed out the limit argument they had used to claim that 0 was part of the final set in The z^n Problem and encouraged the students to discuss its applicability here.

> Chris: I can see that it's possible you can make a case for rationals, that it would even converge upon those, I can see making a case that it converges upon irrationals even, for that matter. Although I am willing to grant that maybe possibly you can get rationals [to be in the final set] by this converging sequence… irrationals, by doing rational division, you're not going to get that, I don't care if that happens to be the limit or not, it's not happening.
> Todd: I think the number of numbers that are not given by this formula [$k/2^n$] is so much bigger than the number of numbers that are. […] To get arbitrarily close to this [1/3], we're going to have a bunch of points next to it getting close to it, which suggests that there's more numbers gotten by this [pointing to the $k/2^n$ description of the final set] than there are gotten by not that. But there's more irrational numbers than rational numbers between 0 and 1, so…

As this excerpt suggests, the students realized that accepting the limit rule as applicable to claim that 1/3 was in the final set (in the case of The Midpoint Problem) required them to also consider whether the irrationals between 0 and 1 were part of the final set by the same argument. Knowledge resources that had not been cued until then, such as the fact that the set of irrationals is larger than the set of rationals, or an understanding of irrationals as “not reachable by a sequence of rationals”, were all of a sudden brought to the front of the discussion. This development caused a refinement of the knowledge frame associated to the solution activity with respect to The Midpoint Problem (see Figure 11).

Phenomena Involving Infinity

Infinite Iterative Processes

Limits

Rule 1: The pattern in the cardinalities of the intermediate states needs to be consistent with the cardinality of the final state.

Rule 2: If an object is part of the intermediate sets from a finite step on (for the rest of the finite steps), it will be part of the final set.

Rule 3: If an object is part of an intermediate set but then is removed at a later step and not added back by any of the subsequent (finite) steps, then it will not be part of the final set.

Series

Rule 4: If a *limiting process*:
- **There is a step at infinity, following all finite steps.**
- ~~**The “limit” object is added to the final set by the step at infinity.**~~
- **Some of the “limit” objects, but not necessarily all, are added to the final set at the step at infinity.**

There are more irrationals than rationals.

The irrationals cannot be “reached” by rational division.

Figure 11. The students’ knowledge frame as they worked through The Midpoint Problem, after my intervention.

The students seemed to be conflicted at this point because they were not comfortable claiming that numbers other than those of the form $\frac{k}{2^n}$ (especially irrationals) were in the final set, which was the claim that the application of the limit rule seemed to point towards, but at the same time trusted the limit rule solution they produced for The z^n Problem. Todd explicitly referred to previous problems in describing his predicament, saying that “if we were not to add 1/3 [to the final set of The Midpoint Problem], then a lot of things would have to change, maybe this [pointing to The z^n Problem], because all of our answers were like, the sequence converges.”

The conflicted situation in which the students were at this point led them to consider more carefully what the “limit rule” was and what its relevance was to The Midpoint Problem:

> Todd: Ok, so we have 3 choices. We can reject the convergence inference thing, or we can say the inference thing is right and 1/3 is there, or we can make the inference rule more specific so that 1/3 does not have to be in there but everything else [solutions to previous problems] remains correct.

Although the students’ cognitive conflict was not entirely resolved by the end of that session, this episode is important for a number of reasons. First, it suggests that analyzing similarities between tasks and their respective solutions does not always refine students’ topical principles in what experts call normative directions. In the instance presented in this episode, the solution provided by the students to The z^n Problem was not a normative one, while the solution to The Midpoint Problem that Chris produced individually in the Pre-Test and replicated collaboratively with Todd in Session 4 before my intervention was in fact normative. I had intervened in the manner described in this episode in hope that by comparing the two problems and their respective proposed solutions, the students would reconsider the validity of the “limit rule” and modify their solution to The z^n Problem in a normative direction. Instead, the students attempted to reconcile the solutions to the two problems by starting to doubt their approach to The

Midpoint Problem. Second, this episode highlights how this type focusing phenomena, together with various aspects of the problem that experts might consider surface features, can cue in latent knowledge resources and potentially affect the students' topical principles. Third, it provides clear evidence of the students' concern for reasoning consistently across and within tasks (although from an expert's point of view they failed to do so).

6.3. Focusing Phenomena: Establishing an Anchor Task

After the completion of six sessions in which a total of 16 different problems were discussed, the data suggests that the two students in Cycle 2 could reason normatively about tasks similar to The Original Tennis Ball Problem (except for The 1+1/n Marble Problem), but employed LIM arguments in the case of the geometrical tasks involving limiting processes. For the duration of these six sessions my only interventions were of the types discussed in sections 6.1 and 6.2. During the last session I decided to take on a more active role. Each of the two focusing phenomena described next attempted to establish the normative type of reasoning displayed by the students in the context of one specific task as a point of reference against which the solution paths to other problems could be contrasted. The assumption behind these attempts was that if the students were confident in the correctness of the normative solution given to a task, then potential conflicts between that type of reasoning and different ones displayed in the context of other tasks would cause the students to question the validity of the these other types of reasoning.

6.3.1. *Abstracting reasoning from a normatively solved task*

Episode 3. In the last session I directed the students' attention back to The Original Tennis Ball Problem and asked the two students to repeat the argument they had settled on in one of the beginning sessions of Cycle 2. After the two students summarized a solution similar to that presented in section 2 of this paper (without mentioning any R1/R2), I asked what the argument was for "ball 5 in bin B at the end". As the students argued that ball 5 was in bin B because it was placed there at step 5 and it was not affected by any of the subsequent steps, I asked whether it was a reasonable assumption to assume that any object that was placed in a bin at a finite step and then left there by all subsequent steps would be in the bin after all steps had been completed. The students found it reasonable, and we called this assumption R1 (see section 3 for a detailed definition of R1). Starting from soliciting an argument for "ball 5 not in bin A", I formulated R2 in a similar manner, with the students agreeing that it was a reasonable assumption to make. Following this discussion, I encouraged the students to consider whether the solutions they had given to each of the other tasks respected these two assumptions (R1 and R2). In doing so, I attempted to facilitate the transition of the type of "object-specific" reasoning suggested by R1 and R2 from being a *model of* the reasoning employed for The Original Tennis Ball Problem to becoming a *model for* the type of reasoning that can be used for any task that involves a completed infinite iteration whose intermediate states are sets (in the sense of Gravemeijer, 1999).

The students' discussion of the sequence of tasks in light of R1 and R2 went on uneventfully for all the tasks similar to The Original Tennis Ball Problem, with the students encountering no problem in pointing out at what point in each argument each of the two assumptions (R1 and R2) was used. When the discussion reached The 1+1/n Marble Problem, the following discussion took place:

> Chris: Yeah, I mean there is something that makes it slightly different. It would require at least another inference rule. But it doesn't involve those [R1 and R2] at least, somewhat.
> Todd: Well, you could, but I think the answer would be the opposite [from their previously agreed upon answer]
> Chris: It wouldn't complete the answer.
> Todd: We ended up saying the jar contained the marble with label 1, right?
> Chris: Yes, I said that.
> Todd: So we're using R2 to say that no marble with label 1+1/n is in the jar, because we can give you a step where that marble is removed and it's gone. But then to say that marble 1 is in the jar, we did some weird things.
> I: Can you use R1 to claim that marble 1 is in the jar at the end?
> Todd: Well, there's no finite step where a marble with label 1 is put in the jar, so we can't use that rule, I think.
> Chris: Well, there certainly isn't a finite step. I believe the claim was that "at infinity" marble 1 was put in the jar and since there is no step beyond that, it wouldn't be removed. I'm not saying it's because of those rules [R1 and R2] necessarily. It does apply R2 in that all the marbles with labels 1+1/n are removed at specific finite steps. It does not give us the complete conclusion we had, but it does tell us that nothing else could be in there, except maybe that marble with label 1.

This excerpt suggests that the students thought The 1+1/n Marble Problem was "slightly different" from the ones discussed until then, and that another "inference rule" had to be added to R1 and R2 in order to accommodate the "step at infinity" and obtain the "complete answer" (which, in the students' view, was that a marble with label 1 was in the jar at the end). After some discussion, the students realized that R2 could be applied to claim that the object "marble labeled 1" was not in the jar at the end, and thus that the addition of a new "inference rule" had to be accompanied by a modification of R2 in order to keep the rule system internally consistent. However, the students were not successful in producing such a modification of the rule system because it was not clear how the effect of the “step at infinity” on the final state could be characterized in a manner not dependent on the contextual details of a specific problem.

As discussion of the remaining tasks progressed, Chris continued to display discomfort towards the solutions suggested by an R1/R2 approach to the limiting process problems. Chris had been the main proponent of the “step at infinity” concept throughout the whole Cycle 2; failing to amend the rule system to incorporate the step at infinity, Chris reluctantly agreed to the solutions suggested by R1/R2 to the limiting process tasks, but made it clear he “did not like it” and “didn’t believe it was right”. On the other hand, Todd started to display an accepting attitude towards R1 and R2, claiming “these are my friends”. During the course of Cycle 2, Todd had not displayed much confidence in the solutions proposed by Chris to tasks other than those similar to The Original Tennis Ball Problem, even stating at some point that working on these tasks was similar to being asked to solve a problem in an axiomatic system for which he was not told what the axioms were. R1 and R2 provided some sense of “knowing what the axioms are” for Todd, but he also claimed he “did not like” the solutions suggested by the application of R1 and R2 in the case of the limiting process tasks, although his discomfort was visibly less intense than Chris’. On the Post-Test however, Todd produced normative solutions to all the problem types, while Chris did so only for the problem similar to The Original Tennis Ball Problem and returned to the non-normative reasoning patterns he displayed during the sessions for the rest of the problems.

In summary, this focus-shift intervention through which I attempted to establish the pattern of reasoning used for The Original Tennis Ball Problem as an anchor to which the students could relate the rest of the problems was only partially successful. The students’ strong intuitions of a “step at infinity” (Max in Cycle 1 and Chris in Cycle 2) caused these two students to see limiting process tasks as significantly different from the other problems and thus did not find The Original Tennis Ball Problem as relevant to them. The other two students (Tom in Cycle 1 and Todd in Cycle 2) produced normative solutions to all the tasks contained in the Post-Test, which included one limiting process task.

6.3.2. *Parallel processes*

In light of Max’s (Cycle 1) view of the R1/R2 set of assumptions as inadequate for treating limiting process tasks, I began to think about a different way to suggest an “anchor task” to the students: not by abstracting a mode of reasoning from a task, but by building on the students’ intuitions.

Episode 4. In Cycle 2, during the final session, when the review of the tasks reached The 1+1/n Marble Problem and the students indicated that they did not think R1 and R2 constituted the appropriate rule system for tackling this problem, the following discussion took place:

> I: I have a little exercise in imagination that I’ve just thought of. Let’s assume that I’m going to have each of you perform a process, at the same time. You [Chris] are going to work with red marbles, and you [Todd] are going to work with yellow marbles. Each of you has as many marbles as the natural numbers, but they’re not labeled in any way, I am just telling you how many you have. Imagine that you have them arranged in front of you starting with a first, and going on indefinitely to the right. Now each of you is holding a jar, and each of you is in front of your first marble. The process is: you take the first marble, you put it in the jar; then you make a step to the right, you toss out the marble in the jar and you put in [the jar] the one in front of you. And so on. You do this at the same time, Chris with the red ones and Todd with the yellow ones. Now we assume you managed to complete all the steps. What do you think you have in the jar, each of you?
> Chris: There wouldn’t be anything, in mine at least.
> Todd: I guess so.

> Chris: I see where this is leading, I think. If I can say there's nothing in there, then… [Mumbling to himself] There is a contradiction there, yes.
> I: Now I want to make an addition to the game. You're going to perform the same process, at the same time, except that I'm going to ask you to close your eyes. Then I'm going to put labels on the balls, but you don't see what the labels are. On one set of balls I'm going to put the 1+1/n labels, on the other set of balls I'm going to put the "1, 2, 3, …" labels. But you have your eyes closed, and you start performing the process I described before, processing the balls in order from the first one. Let's assume you're done performing all the steps, and you can open your eyes. What's in your jar, each of you?
> Chris: I don't care what the labels are, I don't have anything in there and that does reject the argument I had before, and that's the end of that then.
> Todd: Sounds good. There's nothing in both our jars.
> I: And what if your eyes were open? Let's say you do it a 3rd time, you can look at the labels, but the performed process is the same. Does that make a difference?
> Todd: I don't think it makes a difference.
> Chris: I know. I've already come to terms with that just in the past minute, that that answer [marble labeled 1 is in the jar at the end] doesn't hold. Well, *I guess in any problem where I was using a step at infinity, I'll just toss that out, because that step does not exist then.* [my emphasis]

This discussion suggests that the consideration of this "parallel process game" directed the students' focus towards the conditions that needed to be in place in a problem in order to be able to apply an argument similar to that used to claim bin A was empty "at the end" in The Original Tennis Ball Problem. This game appears to have successfully established The n->n+1 Marble Problem (solved normatively by the students in Session 3 and discussed briefly in section 6.1 of this paper) as an anchor for The 1+1/n Marble Problem, and unlike my R1/R2 anchoring attempt described in Episode 3, this intervention made the similarity between the two problems make immediate intuitive sense to the students. As indicated by the excerpt above, Chris interpreted this newfound similarity between these two problems as an indication that the step at infinity did not exist, which he claimed invalidated all the solutions to other tasks in which he was making use of such a step. Similarly, when I engaged Max (Cycle 1) in this "parallel game" during the member checking procedure that took place months after the end of the problem solving sessions, he reacted to it by commenting on the irrelevance of the labels with respect to the outcome of each of the processes described in the game, and thus refined his reasoning on The 1+1/n Marble Problem (which at the end of Cycle 1 was of the LIM reasoning). Max (Cycle 1) and Todd (Cycle 2) had been the main proponents of the existence of a step at infinity, in their respective cycles; thus, their reaction to the parallel game suggests that it can be a powerful tool in challenging the students' LIM reasoning.

7. Discussion

This paper described several types of focusing phenomena (in the sense of Lobato, Ellis, and Muñoz, 2003) used in the context of a design experiment on the topic of completed infinite iteration. They are: a) juxtaposing processes with intermediates states represented by a sequence of sets and tasks with intermediate states represented by a sequence of numbers, in order to turn the students' attention toward the distinction between these two types of problems (Episode 1); b) highlighting a reasoning pattern used previously by the students and encouraging them to consider its applicability in the context of a new problem (Episode 2); c) abstracting a reasoning pattern from a normatively solved problem and encouraging students to consider its applicability to other problems (anchoring through abstraction -Episode 3); and d) juxtaposing two tasks by removing and then adding back some of their contextual details, which led the students to perceive the two processes as isomorphic and consequently establish an anchor in a normatively solved task (anchoring by building on intuitions – Episode 4).

As indicated by the data presented in the context of each episode, each of these focusing phenomena affected the students' reasoning and consequently their set of topical principles, sometimes in normative directions and sometimes not. This is to be expected under a transfer framework that posits that the manner in which an individual decides what the important features of a problem are or how he or she perceives similarities among tasks is highly dependent on the mathematical "baggage" he or she brings along. For example, during one of the sessions, Chris (Cycle 2) mentioned that his intuition of a "step at infinity" was quite likely inspired by the concept of "node at infinity" that he was learning about in a Graph Theory course taken at the time of the study. Similarly, in Episode 3, I presented data that suggested that the students' refinement of the set of topical principles was influenced by their conceptions of irrationals as "unreachable by a sequence of rational numbers", as well as by considerations about size comparison between the set of rationals and the set of irrationals.

The data collected across the two cycles of this experiment indicates that the following hold true for each of the four participating students, regardless of the differences in their mathematical background: 1) cognitive conflicts (either noted by the students or highlighted by me) led the students to modify their reasoning on one or more tasks; 2) anchoring interventions (either on an abstract or intuitive level) can each lead students to refine their reasoning on completed infinite iteration in a normative direction, although those capitalizing on some of the students' existing intuitions may be more successful that the ones taking an abstract approach to anchoring. Thus, refining one's reasoning in a normative direction with regard to infinite iterative processes does not necessarily have to involve sidestepping *all* of one's intuitions with respect to infinity, as suggested by previous research. For example, Mamolo and Zazkis (2008) build on Tsamir's (1999) work and claim that in matters related to infinity, "what is desirable is an instructional approach that will help students separate their ‘realistic’ and intuitive considerations from conventional mathematical ones” (p. 180). Based on my data, I argue that at times it is possible to capitalize on students’ intuitions in helping them to reach normative conceptions of concepts related to infinity; it is the instructor’s job to identify *which* of the students’ intuitions can be used in a fruitful manner in triggering conceptual changes in students.

Chapter 6 – Paper 4

A Local Instruction Theory for the Domain of Completed Infinite Iteration

1. Introduction

Although constructivist perspectives provide a useful framework for thinking about how mathematics is learned, they do not offer a specific vision of how mathematics should be taught (Simon, 1995). Design research (Cobb et al, 1993) aims to bridge the gap between general theory and practice by offering domain-specific guiding frameworks that can be used by teachers to develop instruction fit for the situation in their classrooms. Gravemeijer (1998) refers to such a domain-specific framework as a called a *local instruction theory* (LIT), and is the product of a cyclic process in which an instructional sequence regarding a specific mathematical topic is designed, tested, and repeatedly revised in a classroom setting through a series of cycles.

This paper proposes a LIT for the domain of completed infinite iterative processes. Informally, an infinite iterative process can be described as a countable infinite sequence of actions that can be repeatedly applied to a mathematical object (e.g., a number, a set, a geometric figure). Questions about infinite iterative processes typically ask students to consider applying this entire sequence of actions to an object and then to describe the resulting object. Research has documented that the vast majority of students at all levels provide non-normative solutions to tasks that require them to define the outcome of an infinite iterative process (Brown, McDonald, & Weller, 2008; Dubinsky, Weller, Mcdonald, & Brown, 2005; Dubinsky, Weller, Stenger, & Vidakovic, 2008).

However, there is almost no research investigating how students can reach a normative conception of completed infinite iteration. The LIT described in this paper aims to contribute to research addressing this gap.

The proposed LIT was developed through a two-cycle teaching experiment conducted with pairs of mathematics majors (one pair per cycle). Its description includes a set of learning goals for the students, an envisioned learning route towards these goals, a sample instructional sequence meant to help students progress towards the learning goals, and guidelines with respect to the nature of the learning environment for which it was envisioned. As with any LIT, what I propose in this paper is not to be taken as a ready-to-use instructional tool, but rather used as a guide for teachers who can then create their own adaptations of it based on the needs of their classroom.

2. Background

The concept of infinity permeates K-16 mathematics, being related to questions ranging from the elementary "Is there a largest natural number?" to the more difficult "What is a set?" One of its aspects, namely infinite iterative processes, has historically played a crucial role in the development of the treatment of infinity in mathematics (Clegg, 2003; Moore, 1990). Although the currently accepted mathematical definitions for concepts such as infinite unions and intersections of sets, limits, convergent series, and infinitesimals avoid making direct references to infinite iterative processes, these definitions seek to form a static logical articulation that captures the dynamic essence of infinite iteration (Lakoff & Nunez, 2000). Furthermore, there is evidence that students' and mathematicians' images of and reasoning about such concepts often invoke the consideration of infinite iterative processes (Dubinsky et al., 2005). Therefore, students' failure to reason normatively about infinite iterative processes may hinder their learning of many of the mathematical concepts related to infinity.

The mathematics education literature contains several studies investigating students' conceptions of infinite iterative processes. These studies (which will be discussed in more detail in section 4) suggest that the vast majority of students at all levels provide non-normative solutions to tasks that require them to define the outcome of an infinite iterative process (Brown, McDonald, & Weller, 2008; Dubinsky, Weller, Mcdonald, & Brown, 2005; Dubinsky, Weller, Stenger, & Vidakovic, 2008; Ely, 2007; Mamolo & Zazkis, 2008; Stenger, Vidakovic, & Weller, 2005). However, there is almost no research concerned with investigating how one can help these students refine their reasoning about infinite iteration in a normative direction.

In order to prepare the ground for describing my proposed LIT for the domain of completed infinite iteration, it is necessary to describe in more precise terms what I mean by an infinite iterative process and its state at infinity. In this study an infinite iterative process consists of an initial set S_0, together with an infinitely countable ordered set of actions A_i, where an action consists of finitely many operations on sets. S_n (the intermediate state obtained after step n) is defined recursively to be the result of applying A_n to S_{n-1}. I define the state at infinity for such an infinite iterative process (to be denoted S_∞) to be the unique object identified by the following set of rules:

(R1) If an object x is an element of every set in some tail of the sequence of sets S_i, then x is an element of S_∞. (More precisely, if $\exists m \in N \text{ s.t. } \forall n \geq m, x \in S_n$, then $x \in S_\infty$).

(R2) If an object x is *not* an element of any set in some tail of the sequence A_i, then x is *not* an element of S_∞.

(R3) If there exists an object *x* such that neither R1 nor R2 is applicable to *x* ("oscillating object"), then S_∞ is undefined. Otherwise, S_∞ is the collection of objects to which R1 applies.

This R1-R3 approach to formalizing the notion of state at infinity of an infinite iterative process produces states at infinity that are in agreement with what Brown et al. (2008), Dubinsky et al. (2008), Ely (2007), and Mamolo and Zazkis (2008) have called normative states at infinity for the processes involved in these studies. A detailed discussion describing the rationale behind the formalization of the state at infinity notion in this manner, as well as a discussion of its pedagogical implications, can be found in Paper 1.

3. Theoretical Perspective

3.1. Design Research

The study that I am reporting on was conducted in the form of design research. *Developmental research* (Gravemeijer, 1998) (also called *design research* by Cobb et al., 2003) refers to a cyclic process in which an instructional sequence regarding a specific mathematical topic is designed, tested, and revised in a classroom setting, with the goal of developing a local instruction theory for that topic. According to Gravemeijer (1998), "developmental research consists of a mixture of curriculum development and educational research in which the development of instructional activities is used as a means to elaborate and test an instructional theory" (p. 277).

More specifically, the researcher specifies "the significant disciplinary ideas and forms of reasoning that constitute the prospective goals or endpoints for student learning"

(Cobb et al., 2003, p.11), then formulates an envisioned learning route towards the learning goals, accompanied by a sequence of activities meant to support student learning in the manner indicated by the envisioned learning route. These two components can be inspired by existing literature on student learning of the given topic, as well as by more general theories regarding the development of mathematical ideas. The activity sequence and the associated conjectured learning route are then tested with one or more students, and then revised according to the actual student learning that occurred in the classroom. These steps are then repeated in a new cycle, with a new student or group of students.

The insight into student learning gained from the repeated testing of revised instructional sequences can then be used to formulate a *local instruction theory* associated with the topic in question. Gravemeijer (1998) notes that apart from a suggested instructional sequence, the local instruction theory also contains a set of learning goals for the students, a rationale for the choice of the instructional activities, as well as the kinds of discourse that are encouraged and the norms of participation that are established (in agreement with Cobb et al., 2003).

It is important to note the reflexive nature of the relationship between theory and instructional design: an initial general theory of the development of mathematical ideas, coupled with the existing literature on the topic of interest, informs the development of preliminary learning goals for students, an envisioned learning route and a preliminary instructional sequence. Upon observing actual student reasoning in response to the preliminary sequence of activities, the learning goals, learning route and associated activities are refined, and thus the working theory of how one may come to reach normative conceptions of that specific topic is refined. This feedback cycle between theory and the implementation of an instructional sequence with students is much more complex than the above description suggests. Based on student feedback, researchers may alter the instructional sequence during an individual session, or in the interval between two consecutive sessions of the same cycle, or during the preparation stages for the next cycle. Thus, the three theoretical components of the working local instruction theory (goals, learning route, and associated instructional sequence) are subject to change multiple times even within one single cycle.

3.2. A Transfer-Based Approach to Learning

My exploratory work with two mathematics majors on the topic of completed infinite iteration was inspired by Tirosh's (1991) observation that, while reasoning on tasks related to size comparison of infinite sets, the students in her study spontaneously made references to tasks previously discussed in class and used them to justify their responses on the current problem. Tirosh comments further that analogical reasoning can help students achieve conceptual changes, citing Strauss and Perlmutter (1986) and Clement (1987).This observation, together with Piaget's (1963) position that cognitive conflict can trigger conceptual changes, gave me a starting point for the design of the first few activities to use with students, which consisted of infinite iteration tasks already used in the literature and variations of them that I created. As the first two students I worked with progressed through these initial tasks and their variations, they did spontaneously notice similarities between the problem they were currently working on and previously addressed tasks, and in some cases changed their reasoning on one or more of the tasks involved in the comparison.

This aspect of the student reasoning I observed in my exploratory work led me to adopt Wagner's (2006) transfer framework. Wagner proposes a transfer of knowledge model called *transfer in pieces*. According to this model, transfer of knowledge is a complex process during which an initially "topical" set of principles is constantly refined to *account for* (and not ignore) the new contexts of the new problems encountered. Thus, the acquisition of abstract knowledge is a *consequence* of transfer. Furthermore, according to Wagner, deciding what the mathematical structure of a problem is and whether it is structurally similar to a previously encountered problem is intimately connected with the problem-solving process itself; as a working set of principles is refined to account for new contexts, structural commonalities of the growing class of examples are gradually formulated, which in turn helps with the formulation of a an abstract principle or set of principles applicable to the class.

Wagner's transfer in pieces framework served as a lens through which I analyzed the students' reasoning throughout each cycle, and continually informed the task design process. This framework served the role of the general theory regarding the development of mathematical ideas mentioned by Gravemeijer (1998) as one of the components of a LIT.

3.3. The Learning Environment

The environment I envisioned for my study was informed by that characterizing a problem-centered, inquiry-based classroom (Richards, 1991; Maher, 2002; Francisco & Maher, 2005). In such an environment, the participating students are encouraged to work together on tasks, to provide justifications for their answers, to question the correctness of their answers as well as those of their peers, and to consider generalizations and extensions of the given problems. At the same time, in this type of environment the researcher neither validates nor invalidates the answers given by the students, and does not force the students' investigations in predetermined directions. Instead, the researcher may ask clarifying questions regarding the students' answers, or point out a conflict in the ideas presented and encourage students to discuss how the conflict can be resolved.

Given my adoption of Wagner's (2006) transfer in pieces framework, apart from the researcher behavior described above, I acted in one additional way in my interaction with the students. If the students made references to previously solved tasks or other familiar mathematical contexts in order to justify a claim regarding the current problem, I encouraged the students to reflect further on what ways the referenced task was alike or different from the current one, and what relevance the perceived similarity or difference between tasks had on to the current problem. This type of researcher behavior, together with those described in the previous paragraph, will be referred to as *default researcher behavior* in the next sections of the paper. Occasionally I engaged in behavior that was meant to address aspects of students' reasoning that were specific to the domain of infinite iteration. These interventions will be described in detail in section 5, in the context of the instructional activities in which they occurred. They can be seen as instances of Lobato, Ellis, and Muñoz's (2003) *focusing phenomena*, which are defined as "features of the classroom environment that regularly direct students' attention toward certain (mathematical) properties or patterns when a variety of features compete for students' attention." (Lobato 2006a, p. 110).

4. Relevant Literature

4.1. Intuitions of Infinity

The mathematics education literature on the teaching and learning of topics related to infinity suggests that students' intuitions regarding infinity often are not in agreement with the Cantorian treatment of the infinite. Fischbein, Tirosh, and Hess (1979) investigated students' intuitions of infinity, where by "intuitions of infinity" the authors referred to what one *feels* as being true or self-evident, as opposed to what one accepts as being true as a consequence of logical analysis. In this study, students in grades 5-9 were presented with problems involving infinite iterative processes (in a geometrical context) and size comparison of infinite sets. By comparing students' answers across grade levels and across achievement levels, the authors concluded that classroom mathematics teaching does not affect students' intuitions of infinity, which were found to be mainly of the "finitist" type (i.e., claimed that the various geometrical construction processes described by the problems had to come to an end after finitely many steps, or that the evens were less numerous than the natural numbers). Furthermore, it was observed that on certain questions the more mathematically advanced students gave a higher percentage of finitist answers than the lower achievers, which was explained by the authors by the claim that mathematical education "strengthens logical thinking, i.e. the finitist schemes of our natural manner of thinking" (p. 36).

Fischbein, Tirosh and Hess' (1979) claim that students' intuitions about infinity are misleading (in the sense that they are not in agreement with Cantorian theory) are also echoed by Fischbein, Tirosh, & Melamed (1981) and Tall (1980). Lakoff & Núñez (2000) explain the misleading nature of our intuitions by claiming that our intuitions and reasoning schemas are built on experiences with finite entities, and that when confronted with a new situation, our first impulse is to listen to our intuition, and then to employ our reasoning. In terms of numbers and sets, most people's intuitions say that a set should have a larger number of elements than any of its proper subsets, because this is true in the case of finite sets. Fischbein (2001) advances an alternative explanation for the misleading nature of our intuitions with respect to infinity, one based on the nature of our *mental models* (i.e., mental representations of an entity which replace, in the reasoning process, the original entity, in order to stimulate and ease the solving endeavor). According to Fischbein, when one thinks of a point or a line, it is likely that one pictures the point as a small dot on paper and the segment as a drawn stripe connecting two small dots. Employing these models, it is difficult for one to accept that a segment contains infinitely many points or that two segments of differing lengths contain an equal number of points (Fischbein, 2001; Tall, 1980).

4.2. Research on Infinite Iteration

Researchers have recently started to investigate student reasoning on tasks that required them to conceive of an infinite iterative process as completed and define its state at completion. Although tasks involving infinite iteration have been used in earlier research (Piaget and Inhelder, 1956; Fischbein, 1963; Fischbein, Tirosh & Hess, 1979), these studies employed statistical methods and reported mainly on what extent students would view the processes in question as finite or infinite and on what the nature of the "final state" described by the students was, and as such did not provide much detail with respect to the intricacies of the various types of reasoning students displayed.

In mathematics education research there are two main theories that attempt to describe, from a cognitive point of view, what is involved in conceptualizing infinite iteration as completed and defining a resulting object. The first one employs the Action Process Object Schema (APOS) constructivist learning theory (Asiala et al, 1996) and describes the mental constructions that one might make in attempting to construct an infinite iterative process and its resulting object. Brown et al. (2008) summarize this theoretical description nicely:

> "A process conception of infinite iteration develops as an individual coordinates multiple instantiations of a finite iterative process. When fully constructed, the individual is able to imagine the resulting infinite process as being complete, in the sense of being able to imagine that all steps have been carried out. Reflecting on the process, the individual may come to see it in its totality, that is, as a single operation in a moment of time. This may lead to an attempt to apply an action of evaluation to the process which, if successful, results in the encapsulation of the infinite iterative process. The encapsulation is a transcendent object that is understood to be outside of the process, and the object is identified with the state at infinity." (p. 23)

The second theory, proposed by Lakoff and Nunez (2000), employs a linguistic perspective and the method of mathematical idea analysis and posits that one can conceive of infinite iteration as completed and having a resultant state through the means of a conceptual metaphor based on a mapping between finite and infinite processes. As both types of processes have an initial state and a clear procedure for obtaining the next state from an existing state, one can extend the parallel by imagining that, like finite processes, infinite processes also have a final, unique state that follows all intermediary states. This extension is what Lakoff and Núñez (2000) call the Basic Metaphor of Infinity (BMI).

On an empirical level, there are several studies that adopt the APOS framework to investigate college students' reasoning on completed infinite iteration tasks (e.g., Brown et al., 2008; Dubinsky et al., 2008; Mamolo & Zazkis, 2008; Stenger et al, 2005). Collectively, the findings of these studies suggest that the vast majority of (college level) students produce non-normative solutions to tasks asking for the "resulting object" of various infinite iterative processes. More specifically, most of the participants in these studies either claimed that it does not make sense to assume that infinitely many steps as can be completed (and thus perceive tasks asking for a "final state" as internally contradictory), or attempt to define a final state by requiring that certain patterns observed in the sequence of intermediate states of the process hold for the final state (which in the majority of the cases results in a non-normative state at infinity). Following an APOS-based analysis of qualitative data containing student reasoning on such tasks, the authors of the aforementioned studies hypothesize that these two types of non-normative reasoning are due to an action or process view of the infinite iteration involved in the tasks. These studies and the student reasoning reported by them will be discussed in more detail in the sections describing my proposed LIT for the domain of completed infinite iteration.

Although non-normative reasoning (from an expert' s point of view) with respect to infinite iteration seems to in fact be the norm among students, there is very little research on what types of interventions may help students refine their reasoning on infinite iteration in a normative direction. Mamolo and Zazkis (2008) attempt such an intervention in a study that exposed college students to a "paradoxical" task (an infinite iterative process that involved putting balls in and out of a container such that the cardinal number of the n^{th} intermediate state was 9n, but S_∞ was the empty set). Before instruction, the students' answers were of mainly two types: "the process never ends" or "the state at infinity contains infinitely many elements" (supported by a claim that ∞ balls went out of the container and 9∞ stayed in, which the researchers called a "rates of infinity" argument). Instruction consisted in comparing the size of infinite sets through 1-1 correspondences, along with the presentation of a normative solution to the studied problem based on 1-1 correspondences. After instruction the students were asked to reconsider the same problem in writing. The rates of infinity argument persisted, and the vast majority of the students who did manage to construct an explicit 1-1 correspondence between the in-going and out-going balls did not see it as meaning the container was empty "at the end". The rare students who did interpret the 1-1 correspondence as meaning an empty container mentioned that "they did not like it". This suggests that interventions consisting solely of teaching the abstract principles behind what is considered to be normative reasoning with respect to infinite iteration can be relatively unsuccessful.

5. A Refined Local Instruction Theory

The LIT that I propose for the domain of completed infinite iteration contains a set of learning goals for the students, an envisioned learning route, and a sample instructional sequence designed under the assumptions on teaching and learning outlined in section 3. For each step in the envisioned learning route I will discuss a rationale, propose one or more activities that may help with the implementation of that step, and build on examples from my empirical data to conjecture a set of expected types of student reasoning in the context of each activity.

In order to easily refer to several aspects of the instructional activities or facilitator behavior that are specific to my study, I coined a few terms that are not used anywhere else in the mathematics education literature on infinite iteration, which I will describe next. In what follows, the notation "S_n" refers to the set representing the n^{th} intermediate state of the process under discussion.

Types of Processes

- I use the term *paradoxical process* to refer to an infinite iterative process with a well-defined S_∞ for which the pattern in the cardinalities of the S_n's is inconsistent with the cardinality of S_∞ (more precisely, $\lim_{n\to\infty} \text{card}(S_n) \neq \text{card}(S_\infty)$). By *non-paradoxical process* I refer to one with $\lim_{n\to\infty} \text{card}(S_n) = \text{card}(S_\infty)$.
- I employ the phrase *limiting process* to refer to processes whose S_n's are embedded in a space whose default topology is likely to be familiar to mathematics majors (such as R or R^2), and whose S_∞ does not include all its accumulation points (in other words, it is not a closed set).
- By *no-final-state process* I refer to processes whose sequence of S_n's contains at least one oscillating object (in the sense defined in the R1-R3 approach).

Types of Student Reasoning

- I use the abbreviation GEN-C to refer to a type of reasoning exhibited by the students which consists of attempting to define a state at infinity by requiring that an observed pattern in the cardinalities associated with the sequence of S_n's be consistent with the state at infinity. For example, claiming that a process for which card(S_n) = n (for any n) needs to have a state at infinity with infinitely many elements falls under this type of reasoning.
- Similarly, I denoted by GEN-O any reasoning that involved defining a state at infinity by generalizing patterns in global properties of the intermediate states *other* than cardinality. For example, noting that each intermediate state of a process contained only odd numbers and requiring that the state at infinity contain only odd numbers as well (for this sole reason) is categorized as GEN-O.
- Finally, in the case of a limiting process, I denoted by LIM the type of reasoning that involves claiming that the state at infinity of a process must contain the limit of any convergent sequence contained in the state at infinity (in other words, claiming that S_∞ needs to be a closed set with respect to the default topology of the space in question).

The types of students reasoning with respect to completed infinite iteration identified above are discussed in detail in Papers 2 and 3.

5.1. Goals

In the incipient stages of my work on completed infinite iteration, my learning goals for the students were for them to be able to imagine infinitely many steps (of an arbitrary process) as having been completed, and to display consistency in the manner of approaching defining a state at infinity across a variety of tasks. Analyzing the reasoning of the students from the first cycle helped me refine my learning goals for the students to explicitly address three categories of tasks that, in the eyes of my students, required different types of reasoning in order to define a state at infinity. Completing two cycles of design research led me to formulate the list of four learning goals described below.

Goal 1. Completion

The students will be able to conceive of infinitely many steps as having been performed. In other words, the students will be able to imagine that for each natural number, the step with that number has been reached, and thus perceive the process as completed. Reaching this stage is also seen as a necessary part of reasoning normatively about infinite iteration by researchers embracing the APOS framework (Brown et al., 2008; Dubinsky et al., 2008). Having done so, the students will be able to start exploring ways to define a state at infinity.

Goal 2. Focus on object-based reasoning across a variety of contexts

Across a variety of tasks, the students will investigate the existence of a state at infinity by focusing on each individual object involved in the process and investigating whether it does or does not belong to a tail of the sequence of the intermediate states of the process (in the sense described by the R1-R3 approach outlined in section 2).

Goal 3. Develop treatment for oscillating-object processes

Upon exploring a variety of processes that involve at least one oscillating object (in the sense described in Paper 1 and also in section 2 of this paper), the students will display modes of reasoning that are consistent across these types of tasks *and* do not conflict with the modes of reasoning proposed in the context of processes that allow a state at infinity in the sense of R1-R3. Although in Paper 1 I proposed that the case of processes involving oscillating objects be treated by declaring the state at infinity to be undefined, I will also consider that the students successfully met this goal if they propose to treat such cases in a different way as long as the consistency conditions formulated above are met.

Goal 4. Formulate an overarching definition for the concept of state at infinity

Upon exploring ways to define a state at infinity through a collection of tasks spanning both pseudo-real-world and geometrical contexts and processes that do or do not involve oscillating objects, the students will come to see all the explored tasks as instances of the same type of problem in terms of the mathematical indicators that can be used to describe them (i.e., tasks that involve an infinite sequence of sets for which one is asked to define a final set). Building on this similarity criterion, the students will coordinate all the local definitions for state at infinity to create a unique definition, applicable to all the presented tasks.

5.2. A Framework for the Refined Local Instructional Theory

Table 2 describes the envisioned learning route towards the goals outlined in the previous section, together with a short description of the rationale justifying the appropriateness of each step. The table is followed by a detailed discussion of each step which expands on the rationale for each step, then proposes sample activities for the implementation of that step and conjectured student responses to the proposes activities. The conjectures with respect to student reactions to the activities discussed are mainly formulated based on my work with two pairs of mathematics majors (Max and Tom during Cycle 1, Todd and Chris during Cycle 2); occasionally I will also refer to pilot data I collected informally by distributing written questionnaires containing the tasks in question to Calculus students at a liberal college in upstate New York, and by piloting tasks on mathematics education graduate students in my department.

Table 2. A framework for a local instruction theory for completed infinite iteration.

	Step	Rationale/Purpose
0.	Documenting the students' initial reasoning over a variety of infinite iteration tasks	The students' prior experiences with infinity-related topics may inspire a variety of types of reasoning regarding states at infinity (both "within task" and "within student"). Each student's initial responses are to be taken into consideration in the planning of the next steps.
1.	Exploring non-paradoxical, non-limiting infinite iteration tasks that have a state at infinity	This step is designed for students who cannot conceive of infinitely many steps as having been performed. By working on infinite iteration tasks for which they are comfortable describing a state at infinity, the students may begin to develop a meaning for "after all steps have been performed" and can be encouraged to reflect on the manner in which a state at infinity was defined in such a case.
2.	Facilitating focus shift from "how many" to "which ones"	This step is suggested for students who exhibit GEN-C and/or GEN-O reasoning on the initial assessment. The focus shift may be achieved by manipulating the contextual details of a previous task in order to suggest to the students the use of a novel mental model. If normative reasoning is reached on this task and the focus shift to "which ones" is successful, previous tasks should be revisited in light of this.

3.	Exploring no-final-state tasks	This step is designed to be used after the students reach normative reasoning on non-limiting tasks that have final states, with the purpose of investigating if students still resort to GEN-C or GEN-O types of reasoning after abandoning them in the context of problems that had a well-defined state at infinity.
4.	a. Exploring infinite iteration in a geometrical context	The purpose of this step is to investigate whether certain aspects of the geometrical context evoke in students a different image of the natural numbers than the previous tasks, namely one where a complete iteration through N includes a final "step at infinity". If this is the case, students are likely to exhibit LIM reasoning.
	b. Tackling LIM - Sequence of numbers versus sequence of sets	This step is designed to encourage the students to reflect on the difference between a process where the sequence of intermediate states is one of numbers and a process where said sequence is a sequence of sets. A simplification of the latter to the former may be the cause for the LIM type of reasoning.
5.	Building structural similarity classes	This step is designed to capitalize on the students' natural tendencies to make references across tasks, suggesting a more structured activity in which comparisons among tasks are to be done across the whole task sequence. As it is important to investigate the manner in which the students perceive the tasks as alike or dissimilar, comparison criteria should not be provided.
6.	Establishing an anchor in a normatively solved task	This step is designed to turn the students' attention to a task they had solved normatively and are confident about, highlight the type of reasoning used there, and facilitate exploration of the applicability of that mode of reasoning to the rest of the task sequence.
7.	Defining the state at infinity of an infinite iterative process	This step is designed to encourage the students to identify elements common to all the presented tasks, and use this set of common elements to formulate a definition for a state at infinity independent of the contextual details of each task.

Step 0. Documenting the students' initial reasoning

Rationale

For students up to and including advanced mathematics majors, it is unlikely that the students' mathematical background includes tasks demanding to define a state at infinity for explicit infinite iterative processes with intermediate states represented by sets. However, research such as that reported in Brown et al. (2008) suggests that students sometimes attempt to make sense of infinite unions through the means of an infinite iterative process, seeing the infinite union as the state at infinity of such a process. Students may also reason about infinite iterative processes when working with infinite sequences, series, and limits. For this reason, students may attempt to make sense of phrases such as "after all (infinitely many) steps have been performed" or "after the infinite process is finished" using conceptions of infinity borrowed from the aforementioned contexts. The purpose of this step is to identify *how* the participating students make use of such conceptions in making sense of a completed infinite iteration and its state at infinity (if any), and outline the main types of reasoning employed by the students. As this is an assessment of the students' initial reactions to infinite iteration tasks, it is recommended that the students be assessed individually so that there is no influence from a peer's answers. Additionally, the researcher should limit himself or herself to asking clarifying questions and encouraging the student to provide detailed justifications for his or her answers, and not attempt to lead the student's reasoning in any predetermined direction. This "assessment" step is crucial in determining what the next step should be.

Sample activities

For this step I suggest a collection of three tasks, covering the main types of problems to be used in the study: a non-limiting, paradoxical task with a well-defined state at infinity (The Original Tennis Ball Problem); a no-final-state task (The 1/2 Marble Problem); and a limiting geometrical task with a well-defined final state (The Midpoint Problem). The Pre-Test I used in Cycle 2 is shown in Figure 12.

1. The Original Tennis Ball Problem (OTBP)

Suppose you are given an infinite set of numbered tennis balls (1, 2, 3, …) and two bins of unlimited capacity, labeled A and B.

At step 1 you place balls 1 and 2 in bin A and then immediately move ball 1 to bin B.
At step 2 you place balls 3 and 4 in bin A and immediately move ball 2 to bin B.
At step 3 you place balls 5 and 6 in bin A and immediately move ball 3 to bin B.

This process is continued in this manner ad infinitum. Now assume that ALL steps have been completed. What are the contents of the two bins at this point?

2. The 1/2 Problem

Suppose you have an empty jar and outside of it you have two marbles labeled 1 and 2. At step 1, marble labeled 1 is put in the jar. At step 2, marble 1 is removed from the jar and marble 2 is put in the jar. At step 3, marble 2 is removed from the jar and marble 1 is put back in the jar. In general, at step n, the marble currently in the jar is replaced by the marble that was outside of the jar. Assume ALL steps have been performed. What are the contents of the jar at this point?

3. The Midpoint Problem

Consider segment AB. We divide segment AB into two equal parts, denoting its midpoint by H. Then we divide each of the segments AH and HB into halves, denoting their respective midpoints by P and Q. Imagine continuing to divide segments in this manner, labeling the obtained midpoints at each step. Assuming that all of the steps have been performed, we denote by $\mathcal{M}$ the set of all midpoints obtained through this process. Describe $\mathcal{M}$.

Figure 12. Step 0 sample activities.

Conjectured student responses

1. The Original Tennis Ball Problem.

Previous research on this problem (Dubinsky et al, 2008; Ely, 2007)), my pilot work, and the implementation of two cycles of this study suggest that the vast majority of student answers to this problem fall into two categories. One of them contains answers that suggest that students are not comfortable reasoning about infinitely many steps having been completed and are not sure how to reason about a state corresponding to a completed process (as was the case of Todd from Cycle 2). The second type of answers belong to students who seem to be comfortable thinking about infinitely many steps having been performed, and claim that both bin A and bin B contain infinitely many balls (GEN-C reasoning). Additional details may include “bin B contains balls from 1 to ∞ and bin A balls from ∞+1 to 2∞”, or “bin B contains balls from 1 to ∞/2 and bin A balls from ∞/2+1 to ∞” (GEN-O reasoning), as suggested by the data discussed by Dubinsky et al. (2008). In the case of some students, an answer of this type may coexist with the inability to name any specific balls in bin A. Tom and Max from Cycle 1 and Chris from Cycle 2 gave answers belonging to this second category. Although in my pilot work I encountered a few students producing normative solutions to this problem (2 out of 43), previous research suggests they are the exception and not the norm (see Ely, 2007).

2. The 1/2 Marble Problem

To my knowledge there is no published research documenting student reasoning on this problem, or problems like this in which there is at least an oscillating object in students’ intermediate states. According to the two cycle iterations I performed for this study, a likely student reaction to this task is to claim that after all steps have been completed, the jar contains exactly one marble but it is not possible to determine whether it is marble 1 or marble 2 (Max from Cycle 1, Chris and Todd from Cycle 2). Possible justifications for this claim include noting that the jar contains exactly one of the two marbles after each finite step and consequently that “at the end” the same must be true

(GEN-C); alternatively, one may note that the parity of step n determined which marble was in the jar after that step and try to extend the notion of parity to "infinity" in order to determine which marble was in the jar "at infinity" (GEN-O). Another possible student reaction to this task might be to claim that the problem has three possible answers (0 marbles, exactly one marble, and both marbles in the jar at the end), without finding good enough reasons to commit to any of them.

3. The Midpoint Problem.

This problem was adapted from Fischbein, Tirosh and Hess (1979). Students may argue that $\mathcal{M}$ covers the whole segment [AB] (or (AB)), supporting this claim by the fact that the distance between two consecutive midpoints converges to 0 as the number of the steps performed goes to infinity. Upon closer inspection, students may realize that points corresponding to irrational numbers (if [AB} is taken to be [0, 1]) are not produced at finite steps. In this case, the students may either claim that $\mathcal{M}$ contains only points of the form $\frac{k}{2^n}$, with n a natural number and k a natural number less than $2^n - 1$ (Chris, Cycle 2), or claim that all points on segment [0, 1] are part of $\mathcal{M}$ because those not of the form $\frac{k}{2^n}$ are produced by the process "in the limit" or "at the last step" or "at the step at infinity" (LIM reasoning; Tom's initial suggestion, Cycle 1).

Step 1. Exploring non-paradoxical, non-limiting infinite iteration tasks with a well-defined state at infinity

Rationale

In the vernacular, the term "infinite" is often used to mean "endless". This fact, coupled with the reasonable assumption that steps performed in the "real world" take a non-trivial segment of time, may make it difficult for students to tackle problems that require them to imagine infinitely many steps having been performed. For such students, this demand is synonymous with being asked to reason about what happens “at the end of an endless process”, which appears to be internally contradictory.

It is possible that some of the students tested in Step 0 displayed discomfort with the phrase “after all steps have been completed”, objecting that it is not possible to complete all the steps for reasons such as those outlined above. In this case, I suggest presenting the students with an infinite iterative process whose completion may be easier to conceive of – one whose state at infinity is in agreement with easily noticeable global properties of the sequence of intermediate states. With such a task, it is likely that students will define a final state for the process by generalizing said global properties, but such a task offers a context in which students can at least start imagining a point “after all finite steps” without feeling conflicted about it. Thus incipient meaning for “after all steps have been completed” is being developed. Researcher questioning in the context of this step, as well as subsequent steps in the local instructional theory, can address the focus on global properties of intermediate states, if this is indeed part of the students’ reasoning.

Sample activity

In Cycle 2, one of the students (Todd) displayed discomfort with imagining infinitely many steps having been performed, especially in the context of The Original Tennis Balls Problem and The 1/2 Marble Problem. When Chris explained that for him, “after all steps have been completed” meant “having gone through every possible number that can exist", Todd replied that he could not imagine it because it would take infinite time. For this reason, I started session 1 of that cycle with a modified version of The Original Tennis Ball Problem, which in this report I will call The Odd/Even Tennis Ball Problem for easy reference (see Figure 13). This problem differs from The Original Tennis Ball Problem in two respects. First, the balls manipulated by a certain step are not affected by any of the other steps, which makes the process much easier to "picture" by the students (as noted by Chris and Todd, the Cycle 2 students). Second, its final state (all evens in bin A and all odds in bin B) follows the parity and cardinality patterns displayed by the intermediate states, so unlike OTBP, there is nothing paradoxical about it.

The Odd/Even Tennis Ball Problem

Suppose you are given an infinite set of numbered tennis balls (1, 2, 3,…) and two bins of unlimited capacity, labeled A and B.

- At step 1 you place balls 1 and 2 in bin A and then immediately move ball 1 to bin B.
- At step 2 you place balls 3 and 4 in bin A and immediately move ball 3 to bin B.
- At step 3 you place balls 5 and 6 in bin A and immediately move ball 5 to bin B.

……………………………………..

[In general, at step n you place balls 2n and 2n-1 in bin A and then immediately move ball 2n-1 to bin B.]

This process is continued ad infinitum. Now assume that ALL steps have been completed. What are the contents of the two bins at this point?

Figure 13. Step 1 sample activity.

Conjectured student responses

I have used this task during Cycle 2 (Session 1) and also informally with fellow graduate students in mathematics education on which I occasionally piloted tasks. In both cases I introduced the task following a student's objection to the possibility of completing infinitely many steps in the context of OTBP. The likely student response to this task is to describe the contents of the two bins following the first 3-4 steps and note that after each of these steps bin A contains only even numbers and bin B only odd numbers. The next step might be to generalize this observation to an arbitrary step n, and state that after step n bin A contains all even numbers up to n and bin B all odd numbers up to n. This step was often skipped by the students I observed working on this problem, who usually generalized the odd/even pattern observed in the first 3-4 steps directly to the state at infinity, claiming that "at the end" bin A would have *all* even numbers and bin B would contain *all* odd numbers (a combination of GEN-O and GEN-C). When I asked the Cycle 2 students to write down as detailed a solution as possible following their discussion on this problem, they also provided arguments that indicated some focus on what happens to an individual ball. For example, Chris noted that "you can say that because at each step, the even ball remains in A and the odd ball moves to B and those two specific numbers never get touched again, they would stay that way" (Cycle 2, session 1).

Apart from the default researcher behavior, once the students commit to a solution to this problem the interviewer can continue questioning in a few possibly fruitful directions. One possible direction would be to note that since the students attempted a solution to this problem, they must have found a way to interpret the phrase "after all steps have been completed", and encourage them to describe what this interpretation was. A follow-up question could encourage the students to describe the aspects of this problem that made it different enough from OTBP such that "after all steps have been completed" was problematic there but not here. By comparing the two problems through this lens the students may start reflecting on whether the given interpretation of the phrase in question acquired in the non-paradoxical task could be transferred to OTBP.

Secondly, the students can be asked more follow-up questions with respect to the claimed state at infinity. The researcher may ask how one knows that, say, ball 88 is in bin A. The students may say that the process is defined in such a way that A contains only evens and B contains only odds, so at the end, after all balls have been processed, bin A must contain all evens (GEN-O reasoning); since 88 is even, ball 88 must be in bin A. At this point a potentially fruitful question would be to ask students to provide an argument for ball 88 being in bin A at the end without referring to other balls. It is expected that the students will point out that at step 44, ball 88 was put in bin A and then it was not removed from there by subsequent steps. Similarly, the students could be asked to provide an argument for why ball 87 is not in bin A at the end. Such questions may facilitate the later transition of this mode of "object-specific" reasoning from a *model of* the reasoning inspired by this particular problem to a *model for* the type of reasoning that can be used for any task that involves a completed infinite iteration whose intermediate states are sets (in the sense of Gravemeijer, 1999).

Step 2. Facilitating focus shift from "how many" to "which ones"

Rationale

This step is designed to be used with students who have already started to create some personal meaning for the phrase "after all steps have been completed", as indicated by their pre-test interviews or reasoning during Step 1 activities. Its purpose is to direct the students' focus to how the infinite iterative process affects each of the individual objects it manipulates. As suggested by previous studies (Dubinsky et al., 2008; Mamolo & Zazkis, 2008), when presented with The Original Tennis Ball Problem, students who can conceive of infinitely many steps as having been completed tend to focus on answering *how many* balls are in each bin "at the end", instead of focusing on where each ball ends up. The aforementioned studies also reported that when researchers attempted to bring the students' attention to an individual ball and challenged the students to name one ball found in bin A “at the end”, a typical student answer was “I can’t name one, but I still believe there are infinitely many balls in bin A”. These findings are in agreement with Ely (2007), and I received similar responses from the students in both cycles during the Pre-Test interviews.

Before beginning the study I report on, I hypothesized about a possible reason why the type of shift-of-focus intervention reported in the literature did not cause much change in students’ reasoning. In the real world, if you have a container in which there is a strictly growing pile of balls, after finitely many steps the container will contain a quantity of balls larger than what was in the container at any of the previous steps. Assuming that this would hold true for a process with infinitely many steps is not unreasonable. It is possible that the students asked to reason about The Original Tennis Ball Problem employed a mental model (in the sense of Fischbein, 2001) as the one described above in reasoning about the hypothetical situation in this problem. In the presence of such a model, asking the student to think about only one ball at a time does not change the fact that the student may do so while still seeing the ball as part of a growing pile of balls, which brings with it significant “real-world baggage”. For this reason, I attempted to formulate a problem which not only made use of more abstract terms than The Original Tennis Ball Problem, but also had the potential of inspiring the students to use a mental model different from an ever-growing pile of objects.

Sample activity

As discussed in Paper 1, one way to formalize the definition of the state at infinity of an infinite iterative process involves the use of point-wise converges of a sequence of characteristic functions. Using this definition for solving OTBP, one would have to construct a sequence of characteristic functions for bin A and one for bin B, both sequences having as domain the set of natural numbers. A function defined on N can be uniquely represented through a vector of infinite length that contains the elements of its image, listed in order: (f(1), f(2), f(3), …). Not knowing whether the students were familiar with point-wise convergence, I decided to reformulate OTBP using vectors resembling those associated with the characteristic functions relevant to OTBP.

This is how The Vector Problem (Figure 14) was born.

<u>The Vector Problem</u>. Let $v = (1, 0, 0, ...) \in N^N$. You are going to "edit" this vector step by step.

- Step 1: $v = (0, 1, 2, 0, 0, ...)$
- Step 2: $v = (0, 0, 1, 2, 3, 0, 0, ...)$
- Step 3: $v = (0, 0, 0, 1, 2, 3, 4, 0, 0 ...)$

…………………………………..

This process is continued ad infinitum. Now assume ALL steps have been completed.

Describe v at this point.

Figure 14. Sample activity for Step 2.

Conjectured student responses

Using a combination of GEN-O and GEN-C reasoning, the students could argue that "at the end", the vector would contain infinitely many zeros followed by all the natural numbers, which in turn are followed by another sequence of infinitely many zeros. In fact, this was the first suggestion of Max from Cycle 1. Similarly, Todd (Cycle 2) wondered whether "after all steps have been completed, there is an infinite number of zeros preceding the first non-zero digit". Max's suggestion was challenged by his partner, Tom, who claimed that "for any entry, at some point it's going to go to 0 and stay there, so if you're done with your process it's just going to be the 0 vector"; the discussion that ensued between the two students quickly convinced Max that there was no position in the final vector at which the sequence of natural numbers could begin. Similarly, in Cycle 2, Chris argued that since the vector had infinitely many positions, once there are infinitely many zeroes "in it", there is no space left for the 1, 2, 3, … sequence. This line of reasoning appeared to convince Tom, who commented that it was similar to what he was beginning to think, namely that any position in the vector will eventually be zero.

In summary, although in both cycles there were initial student suggestions containing GEN-O and/or GEN-C reasoning, upon discussing the details of the proposed arguments each pair of students reached consensual normative solutions to The Vector Problem, without any intervention on my part. Furthermore, and perhaps more importantly, the students in both cycles mentioned perceived similarities between this problem and OTBP. Paper 2 describes in detail how the students in Cycle 1 spontaneously referenced OTBP after reading The Vector Problem, then upon reaching normative reasoning on The Vector Problem revisited OTBP and produced a normative solution for it and a related problem. Subsequent problems of the type "$|S_i| \leq |S_{i+1}|$ for all $i \in N$ but $S_\infty = \varnothing$" were also solved normatively. For this reason, this focus-shift intervention proved to be highly successful in highlighting aspects of the problem that were fundamental to its structure and thus inspiring students to choose a normative argument over an intuitive one rooted in real-world considerations.

As described above, the students I worked with spontaneously commented on similarities between The Vector Problem and OTBP and used these perceived similarities to refine their reasoning on OTBP. However, given the small number of students used in my study, I acknowledge it is also possible that students may solve The Vector Problem in a normative manner but not spontaneously make any connections between it and previous tasks. If this is the case, the researcher has the option of asking the students whether The Vector Problem reminds them of any other problem encountered in the past. An even more direct type of researcher intervention could consist of directing the students' attention towards OTBP and asking them whether it is in any way similar to The Vector Problem. Given that I have not encountered the need to use such an intervention, I do not have enough experimental data to suggest more specific guidelines with respect to possible courses of action for the researcher. Such a situation can be the focus of a future research project, as could working with students whose collaborative efforts do not lead to a normative solution to The Vector Problem.

Step 3. Exploring no-final-state tasks

Rationale

This step is designed to be used with students who have already demonstrated normative reasoning in the context of paradoxical or non-paradoxical processes that allowed a state at infinity. Just like getting acquainted with a new concept may be done through investigating both examples and non-examples of the concept, developing a meaning for the state at infinity of an infinite iterative process may be helped by investigating both processes that allow such a state and processes where the state at infinity is undefined.

Sample activities

The sample pre-test discussed in Step 0 contained one such task, The 1/2 Marble Problem (see Figure 15). This task can be revisited in Step 3, which gives the students the opportunity to explore the applicability of the newly acquired modes of reasoning to a new context and type of problem. Additionally, the researcher can investigate whether the students' work on the problems in steps 1 and 2 had any influence on their reasoning on this problem.

The 1/2 Marble Problem**.** Suppose you have an empty jar and outside of it you have two marbles labeled 1 and 2. At step 1, marble labeled 1 is put in the jar. At step 2, marble 1 is removed from the jar and marble 2 is put in the jar. At step 3, marble 2 is removed from the jar and marble 1 is put back in the jar. In general, at step n, the marble currently in the jar is replaced by the marble that was outside of the jar. Assume ALL steps have been performed. How many marbles are in the jar at this point?

The Lamp**.** You have a lamp with a switch; the lamp is turned off. At any step $n \geq 1$, you turn the lamp on if n is odd; otherwise, you turn off the lamp. Assuming all the steps have been performed, is the lamp on or off?

The Bin Swapping Tennis Ball Problem. Suppose you are given an infinite set of tennis balls (labeled with 1, 2, 3, …) and two bins of unlimited capacity, A and B. At step n ($n \geq 1$):

- If n is **odd**, swap the contents of the 2 bins
- If n = 4k-3 or 4k for some $k \geq 1$, then place ball n in bin A
  Otherwise, place ball n in bin B

For each $n \geq 1$, step n was finished at time $t_n = 1 - \frac{1}{2^n}$ (the process started at t=0). What are the contents of each bin at t=1?

Figure 15. Step 3 sample activities.

Conjectured student responses

As detailed in the discussion under Step 0, encountered student responses to The 1/2 Marble Problem include claiming that at the end there is exactly one marble in the jar because that is the case after each finite step (GEN-C reasoning), but that it cannot be determined whether this marble is marble 1 or marble 2 as during the intermediate states, the marbles keep alternating between being inside and outside of the jar. Alternatively, the students may suggest that the "correct answer" is one of three possibilities (no marbles in the jar, exactly one marble in the jar of undetermined label, or both marbles in the jar), without expressing a preference for any of these options. It is also possible that some students will simply say that they have no idea how to reason about this problem.

In the case of the GEN-C reasoning (leading to claiming exactly one marble is in the jar at the end), it is possible that one of the students may note that in OTBP, the pattern observed in the cardinalities of the intermediate states did not hold for the state at infinity. If not, this is an observation the researcher can make, thus potentially helping the students in their exploration of this “cardinality-based” type of reasoning and the situations to which they perceive it to be applicable. As was the case with Max in Cycle 1, students’ reaction to such an observation may be that OTBP is an entirely different type of problem (than The ½ Marble Problem) in the sense that “it runs through all the numbers [labeled balls] and exhausts them”, whereas in The ½ Marble Problem there are only two numbers/marbles involved which are used repeatedly; on these grounds, the students may claim that the agreed-upon solution to the OTBP does not weaken in any way the GEN-C type of reasoning in the context of The 1/2 Marble problem. This is the point where it is likely for students to run out of ideas on how to make further progress on this task.

The task can be reformulated using a context where the cardinality aspect is less obvious. An example of how this can be done is The Lamp Problem (figure 15). I used this task in both cycles as a follow-up to The 1/2 Marble Problem (although not presented immediately after that task). In both cases, the students spontaneously commented on its similarity to The ½ Marble Problem, claiming that “lamp on” could be interpreted as “marble 1 is in the jar” and “lamp off” as “marble 2 is in the jar”. In response to The Lamp Problem, initial responses were mostly of the type “you cannot determine whether it’s on or off at the end”, with some of the students adding that while the status of the lamp was undeterminable, it needed to be one of the only two possible statuses for a lamp, on or off. At times I encouraged the students to use comparisons between the two tasks to refine their reasoning on The 1/2 Marble Problem. In response to such a suggestion, Max (Cycle 1) noted that since it was not possible for the lamp to be both on and off at the same time (because of real-world considerations), it necessarily followed that it was not possible for both marbles to be in the jar at the end, in The 1/2 Marble Problem. He reasoned similarly to conclude that since a lamp could not be neither on nor off (again making use of a real-world assumption in the process), it followed that the possibility of an empty jar in The 1/2 Marble Problem could be eliminated. Further evidence that contextual details can affect students' reasoning can be found in Chris' (Cycle 2) comment that he was more sure about "the lamp is either on or off but can't determine which one" than about "there is exactly one marble in the jar but can't determine which one".

Another possible student reaction to The Lamp Problem is exemplified by Tom's (Cycle 1) reasoning. After some in-depth discussion with Max on this problem and its relationship to The ½ Marble Problem, Tom commented that he did not find any satisfactory answers for The Lamp Problem. His argument was that both "the lamp is neither on nor off" and "the lamp is both on and off" appeared to him to be contradictions, and arguments that either one of these two statuses was "the final one" were weak because the lamp did not "stabilize" on either. Tom concluded that "there might not be a lamp!", which was the first student comment hinting at the possibility of a non-existent state at infinity.

To further investigate what other problem aspects of no-final-state tasks triggered the use of GEN-C or GEN-O reasoning in students, I created The Bin Swapping Tennis

Ball Problem (see Figure 15). This is a variation of OTBP in the sense that a set of balls labeled with the natural numbers is distributed among two bins (A and B), in such a way that at any finite step, the odd-labeled balls processed up to that step are in one bin and the even-labeled balls in the other. However, the process forces the contents of the two bins to be swapped at every odd step, so the growing collection of already-processed odd-labeled balls oscillates between bin A and bin B (same for the even-labeled collection). The students in both cycles claimed that after all steps had been completed, one bin would contain all the odd-labeled balls and the other bin all the even-labeled balls, but it could not be determined which bin contained what types of balls (GEN-O reasoning). Furthermore, the discussion of this problem did not involve any references to previous no-final-state tasks, and the students from both cycles expressed much more confidence in this answer than they were in those given to The 1/2 Marble Problem and The Lamp Problem.

Although in both cycles the students commented that The 1/2 Marble Problem's question was similar to asking what the limit of the numerical sequence "1, 2, 1, 2,…" was and answered the latter question with "the limit does not exist", when asked to translate this answer into the language of The 1/2 Marble Problem, the students invariably answered that it meant "there is exactly one marble in the jar, can't determine which one". Thus, it appears that after agreeing on normative, and arguably counterintuitive reasoning on problems such as OTBP, the students resorted again to intuitive, real-world-based considerations when presented with no-final-state tasks, for which they could not apply the type of reasoning agreed upon in the context of OTBP.

In summary, the expected student responses to this collection of three no-final-state tasks may be categorized as falling in the GEN-O and GEN-C categories. In the absence of more direct researcher interventions in the student discussion of these tasks (which I refrained from in both cycles), students may maintain these types of reasoning for no-final-state tasks until activities that will be discussed under Steps 5, 6, and 7 challenge them.

Step 4a. Exploring infinite iteration in a geometrical context

Rationale

The data discussed so far, as well as a significant body of transfer literature (Catrambone & Holyoak, 1987; Fuchs et al., 2003; Novick and Holyoak, 1991), provides evidence that having students complete tasks with the same underlying structure but embedded in different contexts can elicit vastly different types of reasoning. All but one (The Vector Problem) of the tasks discussed in the previous steps were formulated using real-world contexts. The purpose of this step is to investigate whether embedding the objects manipulated by the process in a topological space quite familiar to the students (the real line or the real plane) inspires the students to propose ways of reasoning about completed infinite iteration not exhibited in the context of previous tasks.

Sample activities

Figure 16 contains the text of two geometric context problems I used in both cycles. The Midpoint Problem is adapted from Fischbein, Tirosh and Hess (1979) and it involves a process which "produces" points on the real line. I created The z^n Problem as a follow-up task for The Midpoint Problem in light of the Cycle 1 students' responses to this task (to be discussed shortly).

The Midpoint Problem. Consider segment [AB]. We divide segment [AB] into two equal parts, denoting its midpoint by H. Then we divide each of the segments [AH] and [HB] into halves, denoting their respective midpoints by P and Q. Imagine continuing to divide segments in this manner, ad infinitum. Assuming that all of the steps have been performed, we denote by $\mathcal{M}$ the set of all "midpoints" obtained through this process. Describe the set $\mathcal{M}$.

The z^n Problem. Let $z_0 = 1$ and z = r(1+i) (where r is a real number). Using these two numbers, new complex numbers are obtained through the following step by step process:

- Step 1: obtain z_1, where $z_1 = z_0 \cdot z$.
- Step 2: obtain z_2, where $z_2 = z_1 \cdot z$.

……………………………………….

- Step n: obtain z_n, where $z_n = z_{n-1} \cdot z$.

……………………………………….

This process is continued ad infinitum. Assume all the steps have been performed. Find a value for r such as the set of **all** the complex numbers obtained through the completed process is

i) finite
ii) infinite and unbounded
iii) infinite and bounded
Is 0 part of the set of complex numbers defined by the completed process?

Figure 16. Step 4a sample activities.

Conjectured student responses

As discussed under Step 0, a common initial student reaction to The Midpoint Problem is to claim that the set produced by the process covers the whole (AB) (or [AB]), without providing much justification for it other than perhaps the fact that the distance between two consecutive midpoints converges to 0 as the number of the steps performed goes to infinity (as was the case with Cycle 2' Chris and Todd in their written answers to the Pre-Test). These answers match what is reported in Fischbein, Tirosh and Hess (1979); in that study, the vast majority of the students (grades 5-9) claimed that for any random point C chosen on segment [AB], there had to be a finite step at which that point would be produced by the process.

Further reflection on this task (either individually or through discussion with partner) may lead the students to realize that if the segment [AB] is taken to be [0, 1] (which can be done without loss of generality), only the points corresponding to numbers of the form $\frac{k}{2^n}$ (with n a natural number and k a natural number less than $2^n - 1$) could be produced at a finite step. Upon this realization, one can either decide that $\mathbb{M}$ is equal to $\left\{ \frac{k}{2^n} \mid n, k \in N \text{ and } 1 \leq k \leq 2^n - 1 \right\}$ (the normative solution), or claim that $\mathbb{M}$ is the whole segment [0, 1] because any number in this segment not of the form $\frac{k}{2^n}$ is converged upon by a sequence of $\frac{k}{2^n}$-type numbers, and thus reached by the process (LIM reasoning).

The students used a variety of terms to refer to the point at which these non-$\frac{k}{2^n}$ numbers are added to $\mathbb{M}$: "in the limit" (Tom, Cycle 1), "at the last step" (Max, Cycle 1), and "at the step at infinity" (Chris, Cycle 2). Student beliefs about the nature of rationals and irrationals, as well as about the respective sizes of these two sets, may lead some students to move from claiming that $\mathbb{M}$ is the whole segment [0, 1] to claiming that it contains only the rationals in [0, 1]. Justifications for such a claim may include that "the irrationals cannot be reached through a sequence of rationals" (Chris, Cycle 2), or that the irrationals cannot be part of $\mathbb{M}$ because if they were, the set of numbers added to $\mathbb{M}$ at "the last step" of the process would be larger than that containing the points created by the totality of the finite steps, which may be assumed to be "unacceptable" by the students (Max, Cycle 1).

I created The z^n Problem (see Figure 16) in response to the students' LIM arguments to The Midpoint Problem (during Cycle 1). The rationale behind this problem was to provide the students with a context less likely to activate the students' prior knowledge regarding the nature of rationals, irrationals, and different sizes of infinity. Additionally, I hypothesized that the sections of this problem where no limiting process was involved (i. and ii.) would evoke normative "final states" from the students, defined as the collection of complex numbers produced by the totality of the finite steps. Such reasoning might then be transferred to the limiting process part (section iii.).

The students in both cycles encountered no difficulty in finding values for r that would satisfy, in turn, each of the three cases mentioned by the problem. More specifically, the students argued that a norm of 1 or 0 for z produced a finite "final set", whereas a norm greater than 1 led to an infinite and unbounded final set and a non-zero norm that was less than 1 led to an infinite and bounded final set. However, when considering the question whether 0 belonged to the final set in the "infinite and bounded" case, three of the four students (across both cycles) claimed that it was, the reason for this claim being that $\lim_{n\to\infty} z^n = 0$ and that completing all the steps of the process required "reaching this limit". Only Tom (Cycle 1) appeared to think of the final set as "the collection of each z^n, where n is a natural number".

In sum, in the absence of non-default researcher interventions the conjectured student responses to these two tasks cover both normative and LIM-based reasoning. The researcher may challenge the students' LIM arguments in a number of ways. For example, if any of the students used the LIM type of reasoning for one of the tasks but not the other, either the researcher or one of the students can use this opportunity to start discussion regarding the conditions necessary in order for the LIM reasoning to be applicable to a problem, and consequently question why LIM was not applied to both tasks; this, coupled with the students' concern for maintaining consistent reasoning across tasks (as discussed in Papers 2 and 3), may lead the students to revise their prior reasoning on one or both of these tasks. In both of the cycles I conducted, such discussion was initiated by the students. The relevant data from Cycle 1 is discussed in Paper 2 while the discussion from Cycle 2 is presented in Paper 3. Additional researcher interventions to challenge the LIM type of reasoning will be discussed under steps 4b and 6.

Step 4b. Tackling LIM -Sequence of numbers versus sequence of sets

Rationale

The data obtained in Cycle 1 led me to hypothesize that the LIM type of student reasoning in the context of limiting tasks may be a consequence of the students' attempts to reduce the level of abstraction. Hazzan (1999) posits that when presented with new concepts, learners may attempt to reduce their level of abstraction by reducing the complexity of the mathematical entities involved, for example by replacing a set with one of its elements.

In my study, the tasks discussed so far ask the students to consider a sequence of sets and investigate the existence of a "limit set" for it. Considering the K-16 curriculum in the United States, it can be argued that the participants in this study were much more familiar with numerical sequences than with infinite sequences of sets. Therefore, the students' reduction of the sequence of sets in a task to a numerical sequence is a real possibility.

Assuming that a student takes such an “abstraction reduction” approach, the LIM reasoning can be explained by the following distinction. If we think of $\sum_{n=1}^{\infty}\frac{9}{10^n}$ as the final state of an infinite iterative process, then it is indeed the case that the limit of the sequence of numbers representing the intermediate states of this infinite addition process *is* the number obtained after all the steps have been completed, and thus the “final state” of the process. However, if we are not concerned with calculating such a sum and instead are interested only in describing the set of numbers obtained as intermediate states by the process (i.e., the numbers in the partial sum sequence associated with this series), then we would have to say that set is the set of all truncations of $.\overline{9}$, which *does not* contain the limit of the numerical sequence, namely $.\overline{9}$. Therefore, the students’ LIM reasoning may be due to an interpretation of the sequence of intermediate states as a sequence of numbers, and not as a sequence of sets. Successfully drawing the students’ attention to this distinction may lead them to rethink the components of LIM reasoning and discuss in more depth its applicability to the tasks encountered up to that point.

Sample activities

To test the aforementioned hypothesis, for Cycle 2 I devised a sequence of three problems meant to direct the students’ attention to the distinction between “sequence of sets” and “sequence of numbers” tasks. This mini task contains one “sequence of numbers” task (Problem 2) placed between two “sequence of sets” tasks. Figure 17 contains the full text of these problems, in the order in which they were presented to the students in Session 3.

Problem 1 (The n->n+1 Marble Problem). Suppose you have an empty jar and outside of the jar you have an infinite collection of marbles labeled 1, 2, 3, and so on. At step 1, you put marble "1" in the jar. At step 2, you put marble "2" in the jar and remove marble "1". In general, at step n, you put marble "n" in the jar and remove marble "n-1" from the jar. Assume ALL steps have been performed. What are the contents of the jar at this point?

Problem 2. (The 1+1/n Powder Problem) On a scale you have 2 ounces of powder. At step 1 you remove some so that you are left with 3/2 ounces. At step 2 you remove some more so that you are left with 4/3 ounces. In general, at step n you remove some powder so that you are left with $1+\frac{1}{n+1}$ ounces of powder. After all steps have been performed, how much powder do you have on the scale?

Problem 3. (The 1+1/n Marble Problem) Suppose you have an empty jar and outside of it you have an infinite number of marbles labeled with "$1+\frac{1}{n}$", where n is a natural number (so the first one is labeled "2", the second one "3/2", the third one "4/3", etc). At step 1, the marble labeled "2" is put in the jar. For any n>1, at step n you remove marble "$1+\frac{1}{n-1}$" from the jar and put marble "$1+\frac{1}{n}$" in the jar. Assume ALL steps have been performed. What are the contents of the jar at this point?

Figure 17. The "sets versus numbers" sequence.

Conjectured student responses

The Cycle 2 students noted that Problem 1 (The n->n+1 Marble Problem) was similar to OTBP and solved it normatively, arguing that the jar will be empty after all steps have been completed because each marble placed in it is removed at a later point and not used again. When presented with the "sequence of numbers" task (Problem 2, The 1+1/n Powder Problem), the students proposed two different solution paths. In this problem, an initial quantity of powder of 2 ounces is affected by infinitely many powder-removal steps, such that after step n 1+1/n ounces of powder remain, and the problem asks how much powder is left after all the steps have been performed. One of the proposed solutions involved summing the amount of powder removed at each step over all the steps (infinitely many), and then subtracting this quantity from the original quantity of 2 ounces. The other involved computing $\lim_{n\to\infty}(1+1/n)$ and claiming that this is how much powder was left on the scale after all steps had been completed. The students did not initiate any discussion regarding comparisons between this problem and the previous problem.

Similarly to The n->n+1 Marble Problem, in The 1+1/n Marble Problem we encounter again a process that manipulates an ω-type set of marbles by starting with the first and placing each marble in and (a step later) out of the jar. However, in this problem the n^{th} marble in the sequence is labeled 1+1/n instead of n (which was the case with The n->n+1 Marble Problem). The Cycle 2 students commented that this problem seemed to be a combination of the previous two problems, and that one needed to decide which of the two types of reasoning used in the previous problems was “the dominant one”. In comparing this problem to The 1+1/n Powder Problem, the students noted that the solution path involving summing up “what was removed” used for the latter problem did not seem to be applicable to The 1+1/n Marble Problem, also adding that “things are not removed in the same way” (across the two problems). Considering the applicability of the two remaining solution paths (the normative solution given to The n->n+1 Marble Problem and the “taking the limit” approach from The 1+1/n Powder Problem) to The 1+1/n Marble Problem, the students decided that the former could be used to claim that none of the marbles from the 1+1/n-labeled marbles could be in the jar at the end, while the latter could be used to claim that a marble labeled “1” was in the jar. For a more thorough discussion with respect to the students’ reasoning in the context of the mini task sequence presented in this step see Paper 3.

In sum, this mini sequence of tasks can be expected to: reinforce the students’ confidence in the normative reasoning first produced for OTBP (through Problem 1);

normatively solve a "sequence of numbers" task (Problem 2); and draw comparisons across all three tasks in the process of solving Problem 3. The discussed data suggests that the students may become aware of certain structural differences between Problem 2 and Problem 3; it also indicates that such awareness may not always be enough to prevent LIM reasoning in the context of "sequence of sets" tasks. In fact, it provides evidence that the inclination to use LIM reasoning can be so strong in some students that it prevails over other considered solution paths even with a pseudo-limiting process such as the one contained in The 1+1/n Marble Problem.

Step 5. Building structural similarity classes

Rationale

The collection of tasks used in this study is designed such that each[25] is a contextualized version of the abstract task "Given an infinite iterative process with each intermediate state represented by a set of objects, create a meaning for the notion of 'final set'." Reaching consistent meaning (and consequently consistent reasoning) across tasks requires that as one progresses through the tasks, one gradually comes to see these tasks as similar in the sense described above. Wagner's (2006) transfer framework proposes that "the perception of structure in a problem situation must be understood as actively constructed in the solution activity itself" (p. 11). I subscribe to this point of view, which is why this step, designed to encourage the students to reflect more deeply on the ways in which they perceive the encountered tasks as alike or different, is to be used after the students had ample time to consider each task, both by itself or in comparison with other tasks if so inclined, and settle on a solution for it.

[25] The 1+1/n Powder Problem is an exception, being a problem where the most natural representation for an intermediate state was a number.

Sample activity

One way to implement this step consists in providing students with a complete list of the tasks they worked on during the previous sessions, and ask them (either orally or through written instructions) to work together on grouping the tasks based on what they perceive as structural similarities. Given that the purpose of this activity is to investigate what the students' views of the relationships among the tasks are after working through all the tasks, criteria for comparison should not be provided. Here is an example of the instructions I used in Cycle 1:

> Group these problems based on similarities in their mathematical structure. Create as many categories as you think are necessary, and assign each problem to a category. Describe the defining characteristics of each category (How do you decide if a given problem belongs to a certain category?).

Conjectured student responses

As I concur with Wagner's (2006) view that one's view of a problem's structure develops as one works to construct a solution for the problem, I expect the students to place tasks for which they settled on similar types of reasoning in the same group. Specifically, it is likely that the students form one category for the tasks where they used GEN-O or GEN-C reasoning to claim that certain aspects of the final state were determined by the process, while other aspects were undeterminable (The 1/2 Marble Problem, The Lamp Problem, and The Bin Swapping Tennis Ball Problem). Another category may contain the tasks where the students invoked some type of limiting reasoning (The z^n Problem, The Midpoint Problem, The 1+1/n Marble Problem, and The 1+1/n Powder Problem). Problems where the students employed object-specific reasoning (in concordance with what I labeled R1 and R2) will likely form the remaining category (The Original Tennis Ball Problem, The Odd/Even Tennis Ball Problem, The

Vector Problem, and The n->n+1 Marble Problem). Within each category students may propose subcategories based on other task aspects they deem of interest. For example, the students in Cycle 2 further split the LIM category according to the patterns observed in the cardinalities of the intermediate states of a process, while the Cycle 1 students did not focus on this aspect.

All four students encountered difficulties in formulating the defining characteristics for each category of tasks. At times they formulated characteristics that still contained contextual details (such as “marble”, “jar”, “time index[26]”) and thus were not representative of all tasks claimed to be in that category. Some of the proposed criteria were expressed in more abstract terms and lacked task-specific details, but employed phrases with vague meaning (e.g., “problem defines a process that approaches a particular state and this ‘limit state’ is consistent with the process”). This is not entirely surprising, considering that the relevant mathematical features of a mathematical situation (from an expert’s point of view) are only obvious to those who are acculturated mathematically to view the situation in a normative way. The empirical data suggests that at this stage, some students’ reasoning may still be a collection of several topical principles, and their perception of the structural relationships among the tasks may still be at least partially rooted in contextual details.

Step 6. Establishing an anchor in a normatively solved task

Rationale

As conjectured above, it is likely that at step 5 the students will place all normatively solved tasks in one category. The goal of step 6 is to highlight the mode of

[26] Some of the tasks I used over the two cycles had a “time component” – that is, the process was time indexed such that infinitely many steps could be performed in finite time.

reasoning used throughout this category and encourage the students to consider its applicability to the other two categories.

Sample activity

My implementation of this step involved a researcher-led discussion. In the last session of each cycle, I directed the students' attention back to The Original Tennis Ball Problem and asked them to repeat the argument they had settled upon in one of the beginning sessions (which was the normative one). After the two students summarized it, I asked what the argument was for "ball 5 in bin B at the end". As the students argued that ball 5 was in bin B because it was placed there at step 5 and it was not affected by any of the subsequent steps, I asked whether it was a reasonable assumption to assume that any object that was placed in a bin at a finite step and then was left there by all subsequent steps would be in the bin after all steps had been completed. The students found it reasonable, and we called this assumption R1 (see section 2 for a detailed definition of R1). Starting from soliciting an argument for "ball 5 not in bin A at the end", I formulated R2 in a similar manner, with the students agreeing that it was a reasonable assumption to make. Following this discussion, I encouraged the students to consider whether the solutions they had given to each of the other tasks respected these two assumptions.

Conjectured student responses

In both cycles, as expected, the discussion went on uneventfully for all the tasks in the OTBP category, with the students encountering no problem in pointing out at what point in each argument each of the two assumptions (R1 and R2) was used. When the discussion moved to the first problem in the LIM category (which was The 1+1/n Marble

Problem, in both cycles), the students were quick to note that R2 could be applied to each marble in the (1+1/n)-labeled sequence to claim it was not part of the final set. However, two of them (one from each cycle) claimed that in the case of this problem, the R1/R2 assumptions did not lead to the “complete answer”, as they did not provide any support for a marble labeled “1” to be in the jar at the end. One of these students (Max, Cycle 1) proposed that the set of assumptions needed an addition, a third assumption to support the “marble labeled 1 is in the jar at the end” claim, not realizing that such an additional assumption would create a conflict with R2. The other student who believed the R1/R2 set was not adequate for this problem (Chris, Cycle 2) realized that an additional assumption would require the re-writing of R2, which he did not know how to approach, but still claimed that there had to be a way to rewrite these assumptions so that both the LIM solution to The 1+1/n Marble Problem and the already discussed solutions to the OTBP category would be supported by them. It is interesting to note an interplay here. When students reach a situation in which their answer violates a set of proposed assumptions, it can either be the assumptions or their solutions that are at fault. They seem to be assuming it is the assumptions, which is not irrational

When the tasks in the “partially undetermined final state” category were discussed, the same two students claimed that the R1/R2 set of assumptions needed to be amended in order to support the aspects that they claimed *could* be determined about the final state of these tasks, such as the claim that in The 1/2 Marble Problem, at the end the jar had to contain exactly one marble.

In summary, the students may have mixed reactions to the activity in this step. The abstraction of R1/R2 from the reasoning used in OTBP, coupled with an investigation of these assumptions' applicability to the rest of the tasks, can lead some of the students to refine their prior solutions to the tasks in the LIM and UND categories in normative directions, and in doing so to reach a level of abstraction from which common structural elements across all tasks become apparent. Such was the case of Tom (Cycle 1), as well as Todd (Cycle 2). Alternatively, some students may attempt to modify the set of assumptions in order to obtain one compatible with the three types of reasoning they had settled on during the problem-solving sessions (one for each category); however, it is likely that they will fail to formulate such a set of assumptions using language independent of contextual details, and as such may begin to question the LIM and GEN-O/GEN-C types of reasoning.

Apart from the default researcher interventions, during this step I experimented with a few other types of interventions, meant to anchor a LIM task to one which the students had solved normatively. Unlike the main anchoring activity presented in this step, which can be seen as one requiring the students to engage in formal, abstract reasoning, these sub-interventions aimed to capitalize on the students' intuitions. The main idea behind these "anchoring" attempts is to create a parallelism between two processes, one for which the students defined a normative final state and one for which the LIM reasoning persisted. The parallelism I am referring to can be achieved by asking one of the students to imagine he is performing the first process, while his partner is performing the second, *at the same time*. One example of this activity, which I named a "parallel game", juxtaposes The n->n+1 Marble Problem and The 1+1/n Marble Problem in such a way that the contextual nature of what is written on the labels becomes more evident. The activity led the strongest proponent of the LIM type of reasoning to declare that "in any problem where I was using a step at infinity, I'll just toss that out, because that step does not exist then" (Chris, Cycle 2). A detailed description of this episode can be found in Paper 3.

The data suggests that while both "anchoring" approaches can be successful in helping students to refine their reasoning in normative directions, both can also be unsuccessful. Their degree of success with a particular student may be highly dependent on whether the student is mainly a semantic or syntactic thinker (in the sense of Weber and Alcock, 2004), which is why both approaches should be used during this step.

Step 7. Defining the state at infinity of an infinite iterative process

Rationale

This step is designed to build on the "model of/model for" transition that was initiated in Step 6. Its goal is to encourage students to formulate a definition for the state at infinity of an infinite iterative process that is applicable to all the problems in the task sequence.

Sample activity

The students could be given written instructions along the lines of:

> Consider an arbitrary infinite iterative process whose intermediate states are sets. Formulate a definition for its "state at infinity". Is your definition applicable to all the tasks we discussed? If yes, do the states at infinity suggested by this definition match the ones you believe to be correct in the case of each task? If the definition is not applicable to some of the tasks, is there any way to refine it to make it applicable?

Conjectured students responses

The students are expected to build on the R1/R2 discussion from the previous session and realize that these two statements can provide the basis for such a definition. However, there are tasks in the case of which these statements cannot be applied for one or more of the objects manipulated by the process (e.g., The 1/2 Marble Problem). A complete definition for the state at infinity needs to handle these cases as well, which is why the students are expected to add an additional statement to R1 and R2. This part of the defining activity may prove to be quite challenging for the students. As mentioned earlier (Step 6), three of the four participants in my experiment attempted to add a “rule” that accommodated their GEN-C and GEN-O reasoning, but failed to do so. As an optional component of this activity, the researcher may suggest to the students to consider the example of the definition of limit of a numerical sequence in relation to divergent sequences.

5.3. Activities Database

Apart from the activities presented in detail in the discussion of each step of the LIT, I employed several other tasks over the course of the two cycles of my design research experiment. For the sake of brevity these activities are not detailed in this paper. All of the activities and tasks that I created and used in this study are summarized by Table 3, in relation to the individual step of the LIT that they are relevant to. The full text of each of the tasks can be found in Appendices A (for Cycle 1) and B (for Cycle 2), accompanied by normative solutions (from an R1-R3 perspective) and examples of student solutions.

Table 3. Summary of the tasks and activities relevant to each step. Italicized text describes an activity that involves multiple problems and may require a more active role on the part of the researcher, as described in 5.2.

	Step	Activities
0.	Documenting the students' initial reasoning over a variety of infinite iteration tasks	The Original Tennis Ball Problem The Original 10 Marble Problem The ½ Marble Problem The Midpoint Problem
1.	Exploring non-paradoxical, non-limiting infinite iteration tasks that have a state at infinity	The Odd/Even Tennis Ball Problem The Decimating Marble Problem
2.	Facilitating focus shift from "how many" to "which ones"	The Vector Problem *Revisiting prior and new problems in light of the shift*: The Original Tennis Ball Problem The Original 10 Marble Problem The Writer Problem The Relabeled 10 Marble Problem The n→n+1 Marble Problem The 1+1/n Marble Problem
3.	Exploring no-final-state tasks	The ½ Marble Problem The Lamp Problem The Bin Swapping Tennis Ball Problem
4.	a. Exploring infinite iteration in a geometrical context	The Midpoint Problem The z^n Problem The Triangle Problem Halves Truncations Halves and Reals
	b. Tackling LIM - Sequence of numbers versus sequence of sets	Sequence of: The n→n+1 Marble Problem The n→n+1 Label Problem The 1+1/n Powder Problem The 1+1/n Marble Problem
5.	Building structural similarity classes	*Ask students to group tasks according to perceived similarities*
6.	Establishing an anchor in a normatively solved task	*Parallel games* *Investigate applicability of R1-R2 (abstracted from normatively solved task) to other problems*
7.	Defining the state at infinity of an infinite iterative process	*Discuss criteria based on which all tasks are similar. Based on this criteria, students are asked to formulate a definition for state at infinity applicable to all tasks*

6. Discussion

6.1. LIT Summary and Accompanying Observations

In sum, my work with mathematics majors on the topic of completed infinite iteration was guided by four learning goals for the students: 1) to be able to conceive of infinitely many steps as having been completed; 2) to define a state at infinity by focusing their reasoning on how the process affects each individual object involved in the process, and maintain this focus across a variety of tasks; 3) to develop a uniform treatment (across tasks) of oscillating-object processes which was consistent with the object-based focus outlined in Goal 2; and 4) to formulate an overarching definition for state at infinity applicable to all the tasks considered throughout the problem-solving sessions.

The proposed local instructional theory for the domain of completed infinite iteration outlined by this paper includes an articulation of these goals and is guided by situated-learning transfer theories (Wagner, 2006) and inquiry-based instruction principles (Richards, 1991; Maher, 2002). It also includes an envisioned learning route and corresponding instructional activities, which are each introduced with a rationale justifying their appropriateness and are accompanied by examples of possible student reactions inspired by the empirical data I collected. This LIT proved successful in helping students reason normatively about problems that are documented in the literature as challenging for the students (such as The Original Tennis Ball Problem), and partially successful in helping them reach normative reasoning on limiting processes and no-final-state tasks. For more details on its effectiveness see Papers 2 and 3.

As it is the case with any local instructional theory, my proposed LIT is not meant to be taken as ready-to-use instructional material, but rather as a guide to one's teaching that can and should be adapted to one's own philosophy of teaching and learning, one's position with respect to what it means to reason about completed infinite iteration in accordance with standard mathematical theory, and the individual needs of the students one works with. In attempting such an adaptation, one should keep in mind the following two limitations of the study. First, the proposed LIT was designed based on data collected from working with mathematics majors, and as such some of its suggestions may not be appropriate for younger students or college level students not majoring in mathematics. Additionally, given the low number of participants in this study, one should not assume that the LIT necessarily addresses *all* possible types of student reasoning on infinite iteration tasks.

It is important to note that the presented LIT allows a potential implementer a significant degree of flexibility even in the absence of major modifications. For example, although some steps in the envisioned learning route are conditioned by the successful completion of previously listed steps, it is still possible to modify the step order to some degree. There is no significant reason for which limiting processes should be explored *after* exploring no-final-state tasks. As explained in the discussion of Step 3, I chose to expose students to no-final-state tasks at that particular point in the instructional sequence in order to investigate whether the students' focus on how the process affects individual objects, exhibited during activities pertaining to what I listed as Step 2, would be preserved in the context of no-final-state tasks. However, one could formulate a similar rationale for exposing students to LIM processes right after the activities of Step 2.

Additionally, the “sequence of sets versus sequence of numbers” module (Step 4b) could be positioned a few steps earlier as well.

6.2. Directions for Future Research

6.2.1. *Variations on question phrasing*

One aspect of the discussed instructional activities that can be modified in a number of ways involves the manner in which the question referring to “the state at infinity” is phrased. In the existing literature discussing completed infinite iteration tasks I have found two different types of questions. To better describe these two question types, let us use the case of OTBP for exemplification purposes. The version of this problem used in Dubinsky et al. (2005) involves asking the reader to imagine the infinite process as “finished” and describe the contents of each bin at that point. The second type of question involves adding a time index to the infinite iterative process such that infinitely many steps can be completed in a time interval of finite length, and ask the reader to describe the contents of the two bins after that time interval has passed (Dubinsky et al., 2008; Mamolo and Zazkis, 2008). In my study, I opted for the phrase “assume all steps have been completed” to refer to the completion of the process, and used the time-index question only when “borrowing” a time-indexed task found in the literature. In my work throughout the two cycles of the design research experiment I have tried to remain fairly consistent throughout the tasks with respect to the way in which this question was phrased, in order to reduce the number of variables that could affect student reasoning. For this reason, the empirical data I collected does not provide enough information with respect to the effects (if any) of the manner in which the “state at infinity” question is phrased on students’ reactions to the task. Nevertheless, this is one aspect of the instructional activities that allows room for further experimentation, adding to the flexibility of the LIT guidelines.

Before experimenting with these three ways of phrasing the "state at infinity" question, one should take into account the following considerations. Firstly, the phrase "assume the (infinite) process is finished" may conflict with some students' perception of the term "infinite" as meaning "endless", and thus may lead them to perceive this type of question as internally contradictory. Secondly, the time index approach to questioning about a state at infinity has been documented as producing mixed reactions in students. Mamolo and Zazkis' (2008) used a time index that allowed infinitely many steps to be performed in 60 seconds, and reported that some of the students viewed that finite time interval as an "endless 60 seconds". This suggests that the addition of a time index may not necessarily help students conceive of infinitely many steps as having been completed. On the other hand, Dubinsky et al. (2008) reported that of the 14 students whose reasoning they examined closely with respect to a timed OTBP, the only one who produced a normative solution seemed to have been helped by the time component, indicating that in the absence of a time component this student might also struggle to produce a normative solution. My own empirical data contains more mixed reactions to the time component: the Cycle 1 students commented that the time component "did not matter" as it did not affect how the objects involved in the process were manipulated. On the other hand, Cycle 2 students actually used the time component of one of the problems to justify the existence of a "step at infinity" that supported the LIM type of reasoning. In sum, one can expect a wide variety of student reactions to the way a task is referring to the state at infinity, and using each of the three types discussed above for the same task or a small sub-collection of tasks otherwise isomorphic to each other may enrich student discussions.

6.2.2. *Addressing student reasoning on oscillating object processes*

As suggested by the student reasoning discussed under Step 3 of the LIT, three of the four students involved in this study consistently displayed GEN-O reasoning with respect to processes involving oscillating objects; these students claimed that these processes had a state at infinity, some aspects of which *could be* determined from the information in the problem while other aspects *could not*. The fourth student (Tom, (Cycle 1) displayed discomfort with claiming anything the state at infinity in the case of The 1/2 Marble Problem and The Lamp Problem, and commented that the question "Assuming all the steps have been completed, what is the status of the lamp?" may be interpreted as implying the existence of a "final state". My intention with this phrasing was to maintain consistency with the question phrasing used in the previous problems and not hint that there might be anything different about this set of problems, but it is indeed possible that such a phrasing be interpreted by students as suggesting the existence of a final state. Future research can investigate student reactions to an alternative phrasing that may address this issue, such as "What can be said about the status of the lamp/the content of the jar after all steps have been completed?"

There are additional possible explanations for the students' insistence with respect to the existence of a state at infinity for "oscillating object". One of them concerns the positioning of these tasks in the task sequence after an initial collection of tasks that did have a state at infinity (in the R1-R3 sense). Working on those initial tasks could condition the students to expect that any infinite iteration has a final state, and to perceive their “job’ as only identifying what its characteristics are in each case, and *not* determining its existence. Additionally, real-world intuitions can also suggest that a final state needs to exist: in the real world it is true that a lamp can be only on or off, or that a marble that is being put in and taken out of a jar ad infinitum needs to end up either on the outside or the outside of the jar, because there is nowhere else for it to go.

With these possible explanations in mind, one could add an additional module to the LIT that investigates well-defined concepts familiar to the students considered in a context where the produce an “undefined” answer. For example, division by 0 (say, in the context of real numbers) is undefined because there is no sensible way to define it that would preserve the meaning of division as the inverse of multiplication. This does not mean that division and multiplication are not well-defined operations, only that there are numerical values for which the result of division is undefined. Similarly, the concept of limit of a numerical sequence is well-defined, but there are many sequences for which applying that definition does not produce a limit, and in those cases we say the limit is undefined or alternatively, that the sequence is divergent. Lastly, one could also investigate the concept of the image of a set S under a function f where there exist elements in S for which f is not defined; in such a situation, we say that f(A) is not defined. Exploring such situations could cue, in the students’ minds, an entire knowledge frame of “undefined” instances from their previous mathematical experiences, which may have high *cuing priority* (as defined by diSessa, 1993) once students are exposed to oscillating object processes. Drawing on the “undefined” knowledge frame may increase the likelihood that students consider the possibility of an undefined final state for certain infinite iterative processes. An “undefined” set of activities may be employed either before or after students consider oscillating object processes for the first time.

6.2.3. *Exploring the role of consistency*

An additional module not specifically related to infinite iteration could consist of exploring the role of consistency in mathematics. Considering that one way in which the students in my study refined their reasoning on infinite iteration consisted of making comparisons across tasks they perceived as similar and addressing potentially inconsistent reasoning across tasks (as identified by the students themselves or by me), it is important to note that the effect of such task comparisons on students’ reasoning depends in part on the students’ perception of what it means to “reason consistently”, their concern for consistency, and their skills in detecting conflicting reasoning across tasks. The students in my study were concerned with reasoning in a consistent manner across tasks, as indicated by direct comments they made in this respect, but often disagreed with each other with respect to whether the types of reasoning employed in two different tasks were consistent with each other; at other times, agreeing on an instance of what the students perceived to be inconsistent reasoning across tasks affected the students’ reasoning in different ways, as discussed in Paper 2.

For the reasons outlined above, the LIT may benefit from the addition of a “consistency” module, meant to both investigate the students’ views of what it means to reason in a consistent manner and to give them a chance to reflect on this topic at length. Employing such a module in the context of the LIT is somewhat similar to what Tsamir (2000) did in a teaching experiment meant to help pre-service teachers to reason about size comparison of infinite sets in a normative manner (i.e., based on 1-1 correspondence). Tsamir reports that discussing the importance of consistency in mathematics with the teachers, as well as inviting them to investigate the legitimacy of the use of three different methods for comparing the size of infinite sets, proved helpful in helping the teachers move towards normative reasoning. In the case of infinite iteration, a “consistency” module adapted after Tsamir’s work could invite students to discuss the legitimacy of two different methods for defining a state at infinity (e.g., the “generalizing global properties” path versus “generalizing local properties” one, as discussed in Paper 1). This part of the module could be preceded by a similar one involving a mathematical topic not related to infinite iteration, preferably one that students are knowledgeable about, so that the students can explore the meaning of consistency without worrying about the counterintuitive nature of infinity.

6.2.4. *A different approach*

As discussed under Step 2 of the LIT, the abstractness of the Vector Problem may have been partially responsible for triggering normative reasoning on this task in all four of the students in this study. This aspect of my empirical data provides support for Mamolo and Zazkis’ (2008) suggestion that reaching normative reasoning on infinite iteration may be helped by “an instructional approach that will help students separate their ‘realistic’ and intuitive considerations from conventional mathematical ones” (p. 180). Similar views are expressed by Tsamir (1999) and Dubinsky and Yiparaki (2000). This begs the question, what would happen if, instead of letting students create a definition for the “state at infinity” concept through involvement with a variety of specific tasks, many of which are framed in terms of pseudo-real-life situations, the instructional sequence would begin with an entirely abstract task? That is, the students’ first challenge could instead be an invitation to create an abstract definition for the concept of “limit of a sequence of sets” by building on the standard notion of limit of a numerical sequence; afterwards the students could explore the class of infinite iteration tasks described in this paper in light of the newly created definition, and at the same time refine the initial definition if the situations present in the given tasks suggest a more sensible meaning for the “limit set” concept. Such an approach has the potential of focusing the students’ reasoning on an abstract approach to infinite iteration from the very beginning, and thus may decrease the chances of the students’ relying on reasoning based on real-life considerations and potentially reinforced in the context of many of the pseudo-real-life tasks.

My data suggests that initial non-normative reasoning developed in the context of one task can be indeed reinforced by the subsequent consideration of other tasks that the students may see as situations to which the type of reasoning in question could be applied. In the interviews I conducted with each of the participating students during the *member checking* (Creswell, 1998) procedure, each of the two students who displayed LIM reasoning until the end of the study commented that activities such as the ones described under Step 6 of the LIT did cause them to question the validity of the LIM reasoning, but that they continued to maintain the correctness of LIM because doing otherwise would have meant “giving up too much” (meaning, changing their solutions for too many of the previously-addressed tasks). The “abstract reasoning first” approach described in the previous paragraph could avoid the development of student attachment to non-normative reasoning over time. However, asking students to create a definition for the concept of “limit of a sequence of sets” may prove to be an extremely challenging task, even for mathematics majors. Although defining is an important part of doing mathematics, students are rarely asked to define a concept in standard mathematics classes; instead, they are accustomed to having definitions given to them and may not perceive “defining” as being part of “doing mathematics” (e.g., de Villiers, 1998; Freudenthal, 1991; Larsen & Zandieh, 2008) Therefore, it is possible that students may encounter difficulties with the approach proposed above. If this is the case, one could encourage the students to work through the activities described under Steps 1 and 2 of the LIT, and *then* challenge them to create a definition for the “state at infinity” concept. LIM and no-final-state tasks could then be explored in light of the outcome of the “defining” activity. The data from this study cannot be used to speculate on how students would react to the two instructional sequences sketched above. Further research on their appropriateness could help paint a more complete picture of the possible learning paths through which students can reach normative understandings of completed infinite iteration.

Appendix A: Cycle 1 Tasks

Note: Each problem is followed by a normative solution and examples of student solutions. The Cycle 1 students are named Max and Tom (pseudonyms).

Pre-test Version 1

The Original Tennis Balls Problem. Suppose you are given an infinite set of numbered tennis balls (1, 2, 3,…) and two bins of unlimited capacity, labeled A and B.

At step 1 you place balls 1 and 2 in bin A and then immediately move ball 1 to bin B.
At step 2 you place balls 3 and 4 in bin A and immediately move ball 2 to bin B.
At step 3 you place balls 5 and 6 in bin A and immediately move ball 3 to bin B.
……………………………………………………………………………….

This process is continued ad infinitum. Now assume that ALL steps have been completed. What are the contents of the two bins at this point?

Solution:

Step number	Balls in Bin A	Balls in Bin B
1	1, 2 **2**	 **1**
2	2, 3, 4 **3, 4**	1 **1, 2**
3	3, 4, 5, 6 **4, 5, 6**	1, 2 **1, 2, 3**
n	**n+1, … , 2n**	**1, 2, …, n**

Let n be an arbitrary natural number. At step n, the ball labeled n, which has already been placed in bin A at a previous step, is being moved to bin B. None of the subsequent steps affects ball n. By R1, after all steps have been performed, ball n is in bin B. As n was chosen arbitrarily, this reasoning can be applied to any ball. Therefore, all balls are in bin B after all steps have been performed, which means that bin A is empty.

Student Responses:

Since after each step of the process the two bins contain an equal number of balls, after all steps have been completed the number of balls in bin A will be equal to the number of balls in bin B. Note that from one step to the next the number of balls in each bin increases, so after all steps have been completed each bin will contain infinitely many balls. However, it is not possible to name any specific ball that is in bin A. [Max’s and Tom’s initial response; revisited and changed to a normative solution in Session 1.]

<u>Hilbert's Hotel</u>. You have a hotel with an infinite number of rooms, numbered 1, 2, 3, … . The hotel is full, meaning that there is a guest in each room. Is there anything you can do to accommodate a new guest, if it's not allowed to have two guests in one room?

<u>Solution</u>: Let us label the original guests according to the number of the room they're in, so the guest presently in room n will be called guest n. In order to free up one room, we can move each guest to the "next" room, namely ask guest n to move to room n+1. If we ask this of *all* the "old" guests, then room 1 remains empty, and the new guest can be placed there.

<u>Student responses</u>

Max initially suggested that if we denote the set of natural numbers by {1, 2, …, n}, where n was "infinity", then the new guest can be put in room n+1, and since infinity+1=infinity, that room would exist as part of the hotel. When asked to explain in more detail the nature of the "n" that represented "infinity", he responded he wasn't quite sure how to explain that, and in order to avoid the part he could not explain he looked for a different solution path, which was the one I presented in the Solution section. He also produced normative solutions to follow-up questions that challenged him to accommodate infinitely (countably) many new guests.

Pre-test Version 2

<u>The Original 10 Marble Problem</u>. Suppose there is a jar capable of containing infinitely many marbles and an infinite collection of marbles labeled 1, 2, 3, and so on. At time t = 0, marbles 1 through 10 are placed in the jar and marble 1 is taken out. At t = 0.5, marbles 11 through 20 are placed in the jar and marble 2 is taken out; at t = 0.75, marbles 21 through 30 are put in the jar and marble 3 is taken out; and in general at time $t = 1 - 0.5^n$, marbles $10n + 1$ through $10n + 10$ are placed in the jar and marble $n + 1$ is taken out. How many marbles are in the jar at time t = 1?

<u>Solution</u>: Time t=1 represents a point in time when all the steps of the process have already been performed (as $t = 1 - 0.5^n$ is less than 1 for any natural number n). Let k be a fixed arbitrary natural number. Let $n_1 = \left\lceil \frac{k-1}{10} \right\rceil$. At time $1-0.5^{n_1}$, marble k is put in the jar. If we denote $n_2 = k-1$, then at time $1-0.5^{n_2}$ marble k is taken out of the jar. It can be proven by induction that marble k is not affected by the steps of the process coming after time $1-0.5^{n_2}$. By R2, marble k is not in the jar at t=1.

As k was chosen arbitrarily, this reasoning can be applied to any of the marbles. in the collection of marbles manipulated by the process. Therefore, at t=1 the jar is empty.

<u>Student Response</u>:

Define a (numerical) sequence such that its nth term represents how many marbles are in the jar at time $t = 1 - 0.5^n$, so $a_n = 9(n+1)$. The number of marbles in the jar at t=1 is the limit of this sequence, which is infinity, so at t=1 you have infinitely many marbles in the jar. It is not possible to name a specific marble in the jar, but there's sort of an accumulation of marbles in the jar. At every moment of our operation the number of marbles in the jar is getting larger. And because it's getting larger at every step without bound, then the number of marbles at t=1 is infinite. [Tom's initial response; revisited and changed to a normative solution in Session 1.]

<u>The Puppy Problem.</u> You have infinitely many children (numbered 1, 2, 3, …), each playing with a puppy. A new child arrives. Is there anything you can do in order to give the new child a puppy to play with, if you can't get a new puppy?

This problem is isomorphic to Hilbert's Hotel in Pre-Test Version 1. Tom solved it normatively, as he did follow-up questions involving "5 new kids" and "infinitely (countably) many new kids".

Session 1

<u>The Vector Problem</u>. Let $v = (1, 0, 0, ...) \in N^N$. You are going to "edit" this vector step by step.

- Step 1: $v = (0, 1, 2, 0, 0, ...)$
- Step 2: $v = (0, 0, 1, 2, 3, 0, 0, ...)$
- Step 3: $v = (0, 0, 0, 1, 2, 3, 4, 0, 0, ...)$

  ……………………………………

This process is continued ad infinitum. Now assume ALL steps have been completed. Describe v at this point.

<u>Solution</u>: Using induction one can show that after n steps (where n is an arbitrary natural number), v consists of a sequence of n zeroes, followed by the sequence of the first n+1 natural numbers listed in increasing order, and has zeroes on the rest of the positions. Let $M_n = \{k \in N \mid v_k \neq 0 \text{ at step n}\}$ and $M = \{k \in N \mid v_k \neq 0$ after all steps have been completed$\}$. Let k be an arbitrary natural number. At step k v_k becomes 0 and the kth position is not affected by subsequent steps, so $k \notin M_k$ and $k \notin M_p, \forall p > k$. By R2, $k \in M$. As k is an arbitrary natural number, we can conclude that $\forall k \in N$, $k \notin M$. This means that M=$\varnothing$, hence after all steps have been completed v is the zero vector in N^N.

<u>Student Responses:</u>

a) Because after step n the vector contains n zeroes followed by the first n+1 natural numbers followed by an infinitely long tail of zeroes, after all steps have been completed the vector will be infinitely many zeroes followed by all the natural numbers in order followed by infinitely many zeroes. [Max's initial suggestion, which he abandoned soon after as a result of his conversation with Tom].

b) Let n be an arbitrary natural number. At step n the nth position of the vector becomes 0 and stays that way for all subsequent steps. Therefore, after all steps have been completed, this n^{th} position is 0. Since n was chosen arbitrarily, this can be said about any of the positions, so v is the zero vector after all steps have been completed. (Max and Tom's co-constructed solution)

The Relabeled 10 Marble Problem. Suppose there is a jar capable of containing infinitely many marbles and an infinite collection of marbles labeled 1, 2, 3, and so on (outside of the jar). At time $t = 0$, marbles 1 through 9 are placed in the jar and marble 1 is relabeled by adding a 0 after 1 (obtaining 10). At $t = 1/2$, marbles 11 through 19 are placed in the jar and marble 2 is relabeled by adding a 0 after 2 (obtaining 20); in general at time $t = 1 - (1/2)^n$, marbles $10n + 1$ through $10n + 9$ are placed in the jar and marble $n + 1$ is relabeled by adding a 0 at the end of its current label. What are the contents of the jar at time $t = 1$?

Solution: Let n be an arbitrarily chosen natural number not divisible by 10. The ball originally labeled n is placed in the jar at some point $t=1 - (1/2)^k$ where $k \le n-1$, and at $t = 1-(1/2)^{n-1}$ it is relabeled by having a "0" added to its end. After this first relabeling, the label on this marble becomes $\overline{n0}$. At $t = 1-(1/2)^{10n-1}$, this marble has another 0 added to its label, becoming $\overline{n00}$. Using induction one can prove that $\{n10^k - 1 \mid k \in N\}$ represents the collection of the times at which the ball originally labeled n is relabeled. As the cardinality of this set is equal to that of the natural numbers, it follows that after all steps have been completed the ball originally labeled n will have a label made up of n followed by infinitely many zeroes (formally, one can use R1 to argue the existence of each individual 0 in the "final label" of the ball originally labeled n).

It follows that after all steps have been completed, the jar contains infinitely many balls: one for each ball whose label was originally not divisible by 10, now labeled by the original number followed by infinitely many zeroes.

Note that if we ignore the sub-steps making up each individual step in this process, the contents of the jar after each finite step appear to be the same with the corresponding ones in the marble problem in Pre-test Version 2 (where we argued that the application of R1 and R2 led to the claim that the jar is empty after all the steps have been completed). This observation stresses the importance of specifying which abstract objects one chooses to work with, when the text of the problem lends itself to multiple interpretations.

Student Response:

Choosing to interpret the problem as "relabeling marble n by adding a 0 to the number on its label is the same action with removing marble n and putting in marble $\overline{n0}$", we obtain a problem identical to The Original 10 Marble Problem (see above). [Max, Tom]

The n→n+1 Label Problem. Suppose you have a jar containing a non-labeled marble and outside of the jar you have an infinite collection of *labels* 1, 2, 3, and so on. At step 1, label 1 is attached to the marble in the jar. At step 2, label 1 is removed from the marble in the jar and label 2 is attached to the marble in the jar. In general, at step n, label n-1 is removed from the marble in the jar and label n is put in its place. Assume ALL steps have been performed. What are the contents of the jar at this point?

Solution: Unlike the marble problem in the 2nd version of the Pre-Test, where the objects manipulated by the process were labeled marbles, here the objects are the labels themselves (the non-labeled marble can be considered as being part of the background on which these label switches are taking place).

Let n be an arbitrarily chosen natural number. Label n is put on the marble in the jar at step n, and removed from that marble at step n+1. The steps after step n+1 do not affect label n, so for all steps after step n+1 label n continues to be outside of the jar. By R2, label n is not on the marble in the jar after all steps have been completed. This reasoning can be applied to any label from the collection of labels originally outside of the jar, so none of these labels can be on the marble in the jar at the end. Therefore, after all steps have been completed the jar contains the non-labeled marble that was in the jar before the process began.

Note that if one decides to work with labeled marbles as the objects in this problem, then the reasoning described above can be used to claim that the jar is empty after all steps have been completed. However, we consider this choice to be less natural than the first one we described, as at no step in this process is the non-labeled marble removed from the jar.

Student Response:

Choosing to assume that relabeling the marble from n to n+1 is the same action as removing marble n and adding marble n+1, one can interpret this problem as being identical to one where at step n, marble n would be removed from the jar and marble n+1 would be put in. In this case, no marble can be in the jar after all steps have been completed as any individual marble is taken out of the jar at a specific step and not put back in by the subsequent steps. Therefore, the jar is empty "at the end". [Max, Tom]

The 1/2 Marble Problem. Suppose you have an empty jar and outside of it you have two marbles labeled 1 and 2. At step 1, marble labeled 1 is put in the jar. At step 2, marble 1 is removed from the jar and marble 2 is put in the jar. At step 3, marble 2 is removed from the jar and marble 1 is put back in the jar. In general, at step n, the marble currently in the jar is replaced by the marble that was outside of the jar. Assume ALL steps have been performed. What are the contents of the jar at this point?

Solution: For marble 1, there is no natural number such that at step n marble 1 is put in the jar and continues to remain in the jar for all subsequent steps (for if there were such a number n, at step n+1 marble 1 is removed from the jar, which contradicts the choice of n). Therefore, R1 is not applicable to marble 1, so we cannot conclude that marble 1 is in the jar at the end. Reasoning similarly, one can conclude that R1 is not applicable to marble 2, which is why we cannot claim that marble 2 is in the jar at the end.

Also, there is no natural number n such that after step n marble 1 is out of the jar, and continues to remain out of the jar for all remaining steps. For this reason, R2 cannot be applied to marble 1, so we cannot say that marble 1 is *not* in the jar at the end. For a similar reason, we cannot say that marble 2 is *not* in the jar at the end.

By R3, it follows that the final state is undefined.

Student Response:

a) At each step there is either marble 1 or marble 2 in the jar, so at the end the jar will contain either marble 1 or marble 2, but it cannot be determined which one as none of the marbles "stabilizes" inside or outside of the jar. [Max]

b) There are 3 possibilities for what's in the jar at the end: no marbles in the jar, one marble in the jar, or both marbles. Cannot determine which one is correct. [Tom]

The 1+1/n Marble Problem. [*suggested by Tom*] Suppose you have an empty jar and outside of it you have an infinite number of marbles labeled with $1+\frac{1}{n}$, where n is a natural number (so the first one is labeled 2, the second one 3/2, the third one 4/3, etc). At step 1, marble labeled 2 is put in the jar. For any n>1, at step n you remove marble $1+\frac{1}{n-1}$ from the jar and put marble $1+\frac{1}{n}$ in the jar. Assume ALL steps have been performed. What are the contents of the jar at this point?

Solution: Let n be an arbitrary natural number. At step n+1, the marble labeled $1+\frac{1}{n}$ is removed from the jar. For any k>n+1, step k affects only marbles $1+\frac{1}{k-1}$ and $1+\frac{1}{k}$, neither of which can be the marble labeled $1+\frac{1}{n}$ (as both $1+\frac{1}{k-1}$ and $1+\frac{1}{k}$ are less than $1+\frac{1}{n}$ when k>n+1). Therefore, marble $1+\frac{1}{n}$ remains outside of the jar for all steps after n+1. The condition for applying R2 to this marble is satisfied; hence, I conclude that marble $1+\frac{1}{n}$ is not in the jar after all steps have been performed.

As n was chosen arbitrarily, this reasoning applies to each marble in the set of marbles manipulated by the process. Therefore none of these marbles is in the jar at the end, so the jar is empty.

Student Response:

To find out what's in the jar at the end, one needs to take the limit of the $1+\frac{1}{n}$ sequence, which is 1, so at the end there is a marble labeled 1 in the jar. [Max, Tom]

Session 2

<u>The Writer</u>. Tristram Shandy, the hero of a novel by Laurence Sterne, starts writing his biography at age 40. He writes it so conscientiously that it takes him *one week* to lay down the events of *one day*. If he is to document each day of his life and the pace at which he writes remains constant, can you envision a situation in which his autobiography can be completed?

<u>Solution</u>: Let us explore the possibility of his autobiography being completed in a finite number of weeks, call it N. Then, according to the scenario presented in the problem, at age 40 years + N weeks, Tristram has finished documenting the first N days of his life. As N is much smaller than 40 * 365 + 7N (the number of days he has already lived), it is obvious that not each day of his life has been documented, which contradicts the assumption that the autobiography is complete. Therefore, the autobiography cannot be finished in a finite number of days.

I will now explore the hypothetical scenario of a life spanning infinitely many days (and consequently infinitely many weeks). There is a one-to-one correspondence between the ordered set of weeks in Tristram's life starting after he turned 40, and the ordered set of days in his life (from birth). Therefore, for any natural number n, day n in the writer's life is documented during the n^{th} week (counted starting from age 40). As n was chosen arbitrarily, this is true of any day in the author's life. Assuming we're at a point where such a life has already been lived (so infinitely many days/weeks have been lived), we can say that the autobiography is complete.

<u>Student Response</u>:

a) If the writer has an infinitely long life, then the set {1, 2, 3, 4,…} represents the days counted from the writer's birth and {7, 14, 21, 28,… } represents the days in the weeks passed after the writer turned 40). There is a one-to-one correspondence between these two sets, which means that after infinitely many days have been lived, each day of the writer's life has been documented. [Max]

b) Let n (a natural number) be the number of a specific day in the writer's life (counted from birth). If the writer lives an infinitely long life, then there is also an n^{th} week (counted from when he turned 40), during which day n is documented. Thus, at the end of this infinitely long life, each day (from birth) would have been documented. [Tom]

<u>The Midpoint Problem</u>. Consider segment [AB]. We divide segment AB into two equal parts, denoting its midpoint by H. Then we divide each of the segments AH and HB into halves, denoting their respective midpoints by P and Q. Imagine continuing to divide segments in this manner. Assuming that all of the steps have been performed, we denote by $\mathcal{M}$ the set of midpoints obtained through this process.
Now, starting also with segment AB, we perform the same process as before, except that the "halving" action is replaced by splitting a segment in 3 equal parts. Assuming that all

of the steps have been performed, we denote by $\mathcal{T}$ the set of points obtained through this process.
Which of these two sets ($\mathcal{M}$ and $\mathcal{T}$) is more numerous?

<u>Solution</u>: Without loss of generality, let us assume that segment [AB] is the segment corresponding to the interval [0, 1] on the real line. From here on, I will refer to various points on this segment by using the names of the real numbers associated with them. Under this convention, I can say that ½ is produced at step 1, ¼ and ¾ at step 2, and so on.

<u>Claim</u>: For any natural number n, $M_n = \left\{\frac{k}{2^n} \mid k \in N, 1 \le k \le 2^n - 1\right\}$ (I denote by M_n the set of points produced by steps 1 to n).

This is easily verified for n=1, as at step 1 we add only ½ to M, which is consistent with $M_1 = \left\{\frac{1}{2^1}\right\}$.

Now let us assume that for some natural number n, $M_n = \left\{\frac{k}{2^n} \mid k \in N, 1 \le k \le 2^n - 1\right\}$. At step n+1, we create a new midpoint between each two consecutive points in $M_n \cup \{0, 1\}$ (seen as a set ordered n terms of magnitude). Let $\frac{k}{2^n}$ and $\frac{k+1}{2^n}$ be such a pair of consecutive points, where k is an arbitrary natural number chosen between 0 and $2^n - 1$.
The interval $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$ can also be written as $\left[\frac{2k}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right]$, so its midpoint is $\frac{2k+1}{2^{n+1}}$.
Therefore, $M_{n+1} = \left\{\frac{2k}{2^{n+1}} \mid k \in N, 1 \le k \le 2^n - 1\right\} \cup \left\{\frac{2k+1}{2^{n+1}} \mid k \in N, 0 \le k \le 2^n - 1\right\}$, which is $\left\{\frac{k}{2^{n+1}} \mid k \in N, 1 \le k \le 2^{n+1} - 1\right\}$. This completes the proof of the claim.
To decide which numbers from the interval [0, 1] belong to M, let us denote by x an arbitrary number from this interval.

1) If x is of the form $\frac{k}{2^n}$, where n and k are natural numbers and $1 \le k \le 2^n - 1$, then x belongs to M_n. As $M_1 \subset M_2 \subset ... \subset M_n \subset ...$, x is also an element of M_k for all $k \ge n$. Therefore, we can apply R1 and conclude that x belongs to M.

2) If x is not of the form $\frac{k}{2^n}$ (where n and k are natural numbers and $1 \le k \le 2^n - 1$), then x is not in M_1 and also not in any of the subsequent sets M_k, $k > 1$. By R2, x is not in M.

In conclusion, $M = \left\{\frac{k}{2^n} \mid n, k \in N \text{ and } 1 \le k \le 2^n - 1\right\}$.

Reasoning similarly, one can show that $T = \left\{ \frac{k}{3^n} \mid n, k \in N \text{ and } 1 \le k \le 3^n - 1 \right\}$.
Both M and T are infinite sets of fractions, their cardinalities being both equal to that of N; therefore, M and T are equinumerous.

Student Response:

a) M includes $\left\{ \frac{k}{2^n} \mid n, k \in N \text{ and } 1 \le k \le 2^n - 1 \right\}$, which is the set of points "reached" by the process at various finite steps. Regarding an arbitrary number *x* in [0, 1] that is not of this form, there exists a sequence of numbers $\{x_n\}_{n \ge 1}$ of the $\frac{k}{2^n}$ form that converges to *x* (e.g., for each n, define x_n to be chosen from among the midpoints created at step n as the one that is closest to *x*). Because of this, after all steps have been completed, *x* is "reached' by the process, so *x* is part of M. As *x* was chosen arbitrarily, we can conclude that M = [0, 1]. [Max and Tom]

Session 3

The z^n Problem. Let $z_0 = 1$. At step 1, you multiply it by z = r(1+i) (where r is a real number) and denote the result by z_1. At step 2, you multiply z_1 by z and denote the result by z_2. In general, at step n you multiply z_{n-1} by z and denote the result by z_n. Assume all the steps have been performed. Find a value for r such as the set of all the complex numbers produced by this process is:

i) finite
ii) infinite and unbounded
iii) infinite and bounded. In this case, does 0 belong to the set of all complex numbers produced by this process?

Solution: Let P be the set of complex numbers produced by the process, and P_n be the set of numbers produced by steps 1 to n. Thus $P_n = \{z_1, z_2, ..., z_n\} = \{z^1, z^2, ..., z^n\}$ One can easily see that $P_1 \subset P_2 \subset ... \subset P_n \subset ...$. Applying R1 to any z^k (where k is a natural number), we conclude that z^k belongs to P. R2 can be applied to any complex number not of the form z^k where k is a natural number, so such numbers do not belong to P. Therefore $P = \{z^n \mid n \in N\}$

i) In order for P to be finite, we need to ensure that "$z^n \ne z^k$ for any distinct natural numbers n and k" does not hold. If |z|>1 then $|z^n| > |z^k|$ for n>k so any two powers of z are different from each other. Also, if 0<|z|<1 then $|z^n| < |z^k|$ for n>k, so two different powers of z cannot be equal in this case either. The only cases left are |z|=0 or |z|=1.

- In order for |z| to be zero, r needs to be 0. In this case P ={0}

- To have |z|=1, we need r to be $\pm\frac{\sqrt{2}}{2}$ (as the norm of 1+i is $\sqrt{2}$). In this case, the points in the plane associated with the numbers in P lie on the unit circle. For $r=\frac{\sqrt{2}}{2}$, the angle made by OP_z with the positive x-axis is 45°. As 45*8=360, we have $z^8=1$ and $z^9=z$, so P contains 8 distinct complex numbers whose associated points in the plane are evenly spread around the unit circle. A similar situation occurs for $r=-\frac{\sqrt{2}}{2}$.

ii) As already pointed out in the solution to i), if |z|>1 or 0<|z|<1 then any two powers of z are different from each other. This ensures that P is infinite. In order for it to be unbounded, we need to ensure that the set of norms of the complex numbers in P is unbounded. For |z|>1, $\lim_{n\to\infty}|z^n|=\lim_{n\to\infty}|z|^n=\infty$. So |z|>1 is the necessary and sufficient condition for P to be infinite and unbounded.

iii) As already discussed, if 0<|z|<1 then P is infinite and $\lim_{n\to\infty}|z^n|=\lim_{n\to\infty}|z|^n=0$, so the set of norms of the numbers in P is included in the interval [0, 1], therefore P is infinite and bounded. In order for 0 to be part of P, we would need to find a natural number n such that $z^n=0$. But for any natural number n, $|z^n|=|z|^n>0$ (as |z|>0), so the equation $z^n=0$ does not have any solutions among the natural numbers. Therefore, 0 does not belong to P. However, the sequence $\{z^n\}_{n\in N}$ converges to 0, so 0 is part of the closure of P (with respect to the standard topology on R^2).

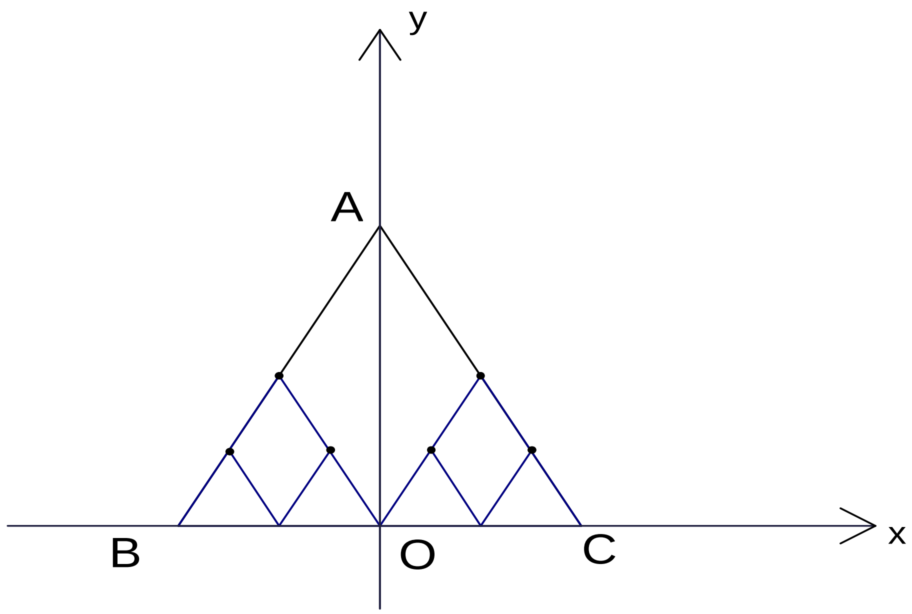
<u>Student Response</u>:

For i) and ii) both Max and Tom offered solutions similar to what I presented above. For iii), after agreeing that $|z|$r needed to be between 0 and 1 in order for P to be infinite and bounded, the students disagreed on whether 0 was part of P or not.
Tom: P is the collection of numbers created by the totality of the finite steps, so $P=\{z^n \mid n\in N\}$, which does not include 0.
Max: $\lim_{n\to\infty}z^n=0$ so after all steps have been completed, you are at the limit, so the limit needs to be included in P; $P=\{z^n \mid n\in N\}U\{0\}$.

Session 4

<u>The Triangle Problem</u>. You have an equilateral triangle, ABC. At step 1 you construct the midpoints of each of its “upper” sides, and then use each of these 2 points as a vertex for a new equilateral triangle having one side along the basis of the original equilateral triangle, and an area of ¼ of the area of the original triangle (see figure below). You then repeat this procedure with *each* of the new equilateral triangles built at the previous step.

This process is continued in this manner. Assuming that all the steps have been completed, denote by P the set of ALL midpoints constructed through this process.



a) If we use the system of coordinates displayed below (with OC representing 1 unit on the x-axis and AO representing 1 unit on the y-axis), can you determine if the point (17/1024, 1/16) is in set P?
b) Can you describe a procedure through which you can choose a random point from set P?
c) Does P contain any points from segment [BC?]

<u>Solution</u>: Let P_n denote the set of points constructed *at step n* of the process, where n is a natural number. By using induction, one can easily show that for an arbitrary n,

$$P_n = \left\{ \left(\frac{k}{2^n}, \frac{1}{2^n} \right) | \, k \in Z, \text{ k odd and } 1 - 2^n \le k \le 2^n - 1 \right\}.$$

a) There are no natural numbers n and k with $1 \le k \le 2^n - 1$ and k odd such that (17/1024, 1/16) = $\left(\frac{k}{2^n}, \frac{1}{2^n} \right)$. One simple way to argue the truth of this claim is to note that 17/1024 and 1/16 do not have the same denominator (and 17/1024 cannot be simplified further), whereas the fractions in pairs of the form $\left(\frac{k}{2^n}, \frac{1}{2^n} \right)$ do. Therefore, there is no natural number n such that (17/1024, 1/16) belongs to P_n. By R2, (17/1024, 1/16) does not belong to P.

b) By R1, $\bigcup_{n\in N} P_n \subseteq P$. By R2, P does not contain any elements that do not belong to $\bigcup_{n\in N} P_n$. Therefore, $\bigcup_{n\in N} P_n = P$, so $P = \left\{\left(\frac{k}{2^n}, \frac{1}{2^n}\right) \mid n, k \in N,\ k \text{ odd and } 1-2^n \le k \le 2^n - 1\right\}$.
Assuming one has a way of choosing a random natural number n, a procedure for choosing a random element from P is:

1. Choose a random natural number n.
2. Choose a random odd integer k such that $1-2^n \le k \le 2^n - 1$.
3. Output the pair $\left(\frac{k}{2^n}, \frac{1}{2^n}\right)$.

Alternatively, since P is countable, one could consider an enumeration of P, choose a random natural number n, and output the pair found at the n^{th} position in the enumeration.

c) Let (x, 0) be an arbitrary point from segment [BC]. $\frac{1}{2^n} \neq 0$ for any natural number, which means that (x, 0) does not belong to P_n, $\forall n \in N$. By R2, (x, 0) does not belong to P. It is interesting to note however that no matter how you choose a point (x, 0) on the segment [BC], there exists a sequence in P that converges to (x, 0). In other words, [BC] is included in the closure of P (with respect to the standard topology on R^2).

Student Responses:

Same as above for a) and b). For c), the students disagreed:
Tom: P is all the points created by the finite steps taken together, so

$$P = \left\{\left(\frac{k}{2^n}, \frac{1}{2^n}\right) \mid n, k \in N,\ k \text{ odd and } 1-2^n \le k \le 2^n - 1\right\}$$

Max: Given that at step n we're creating points at a distance of $\frac{1}{2^n}$ from the x-axis, after finishing all the steps we are on the x-axis because $\lim_{n\to\infty} \frac{1}{2^n} = 0$. If we were above the x-axis we wouldn't be done with the process. Being on the x-axis means having reached all the points on it, so [BC] is included in P, along with

$$\left\{\left(\frac{k}{2^n}, \frac{1}{2^n}\right) \mid n, k \in N,\ k\ odd\ and\ 1-2^n \le k \le 2^n - 1\right\}.$$

The Lamp Problem. You have a lamp with a switch; the lamp is turned off. At any step $n \ge 1$, you turn the lamp on if n is odd; otherwise, you turn off the lamp. Assuming all the steps have been performed, is the lamp on or off?

Solution: This problem is isomorphic to The 1/2 Marble Problem from Session 1. The final state is undefined.

<u>Student Responses</u>:

a) After each finite step the lamp is either on or off, so at the end the lamp is either on or off but it cannot be determined which one it is. [Max]

b) I don't find any answer to be satisfactory for this problem. Because you could say that it's both on and off, but that's a contradiction. I was entertaining briefly the idea that it's neither on nor off, but that's also a contradiction, for the same reason. And if you say that it's on, then for any point that it's on, I can show you a point that it's off, and put the conclusion that it's on at the end under serious doubt, and I can also put the conclusion that is off under serious doubt. So it's doubtful that it's on and off, it's doubtful that it's neither on nor off, it's doubtful that it's on, it's doubtful that it's off, so it's like, at some point I'm started to entertain the point that there might not be a lamp! [Tom]

Session 5

<u>The Bin Swapping Tennis Ball Problem</u>. Suppose you are given an infinite set of tennis balls (labeled with 1, 2, 3, …) and two bins of unlimited capacity, A and B. At steps 1 and 2 you place ball 1 in bin A and ball 2 in bin B, respectively. At step n ($n \geq 3$):

- If n = 4k-3 or 4k for some $k \geq 1$, then place ball n in bin A
- Otherwise, place ball n in bin B
- **Furthermore,** if n is odd, swap the contents of the 2 bins (after placing ball n in the appropriate bin)

For each $n \geq 1$, step n was finished at time $t_n = 1 - \frac{1}{2^n}$ (the process started at t=0). What are the contents of each bin at t=1?

<u>Solution</u>: Using induction one can show that for any $n \geq 1$ (n being a natural number), after n steps have been performed one of the bins contains only the ball labeled with 1, while the other contains the balls with labels from 2 to n. More precisely, if $n \equiv 1 (\mathrm{mod}\, 4)$ or $n \equiv 2 (\mathrm{mod}\, 4)$, A is the bin containing ball 1 while bin B contains the balls with labels from 2 to n; if $n \equiv 0 (\mathrm{mod}\, 4)$ or $n \equiv 3 (\mathrm{mod}\, 4)$, bin B contains ball 1 while bin A contains the balls 2 to n.

Let us examine bin A. Let k be an arbitrary natural number. For the ball labeled k, there is no step n at which it is put in bin A and it stays there for all subsequent steps (as once ball k is placed in a bin, it keeps being moved from one bin to the other for the rest of the process). Therefore, we cannot apply R1 to conclude that ball k is in bin A after all steps have been performed. For similar reasons, we cannot apply R2 to conclude that ball k is *not* in bin A after all steps have been completed. By R3, the final state of bin A is undefined. Similar reasoning can be used to come to the same conclusion about bin B.

Student Responses:

Using induction you can show that after an arbitrary step n, ball 1 is by itself in one of the bins and the balls {2, …, n} are in the other bin. Depending on n's relationship with multiples of 4, for a specific n it can also be determined which bin has ball 1 and which bin has the rest. After all steps have been completed all the balls would have been processed, so ball 1 is by itself in one bin and the rest of the balls [i.e. N\{1}]in the other bin. However, one cannot determine which bin contains what, because the contents of the bins are changed at every odd step. [Max and Tom]

Reflection on Tasks. The students were given a list of all problems considered in sessions 1-5. The list was followed by the text "Group these problems based on similarities in their mathematical structures. Create as many categories you think are necessary, and assign each problem to a category. Describe the characteristics of each category (how do you decide if a given problem belongs to a certain category?)." The students performed this task individually, in writing.

Post-Test (administered through written test)

Problem 1. You have a jar and infinitely many tennis balls, labeled by $(-1)^{n+1}n$ (that is, the first ball is labeled 1, the second -2, the third 3, and so on). At t=0 the jar is empty.

- At t=1/2 you place ball labeled 1 in the jar
- For each n>1, at $t=1-\frac{1}{2^n}$ you remove ball with label $(-1)^n(n-1)$ from the jar and put in ball labeled $(-1)^{n+1}n$.

a) What are the contents of the jar at t=1?

b) Is this problem different from the $1+\frac{1}{n}$ Marble Problem (Session 1, problem 5), in terms of its mathematical structure? Explain your answer.

Solution: a) Let k be an arbitrary natural number. Ball with label $(-1)^{k+1}k$ is put in the jar at $t=1-\frac{1}{2^k}$ (step k) and removed at time $t=1-\frac{1}{2^{k+1}}$ (step k+1). The steps coming after step k+1 do not affect this ball, so the ball continues to remain outside of the jar for all steps after step k+1. By R2, the ball labeled $(-1)^{k+1}k$ is not in the jar at t=1. This line of reasoning can be used for any natural number k, and consequently for any ball from that collection of balls that were outside of the jar at t=0. Therefore, none of these balls is in the jar at t=1. Since these balls are the only objects under consideration, the jar is empty at t=1.

b) Firstly, this problem has a time index, whereas the $"1+\frac{1}{n}"$ problem did not, instead asking what the contents of the jar were after all steps had been performed. As $1>1-\frac{1}{2^n}$ for any natural number n, time t=1 comes after all steps have been performed; therefore, I consider that the two ways in which the two questions are phrased are essentially equivalent.
The second difference between the two problems comes in the way the initial set of balls is labeled. It appears that these labels are not essential information in the problem; the only thing that matters is that we have a countable collection of balls of the order type ω, and that the process manipulates these balls in order, starting with the initial element, and without revisiting an element once it is processed.
For these reasons, I consider these two problems to be equivalent.

Student responses:

While both Cycle 1 students concluded that the jar is empty at t=1 because each ball involved in the problem is removed from the jar at some finite step and never put back in again (Tom also mentioned R2 in his justification). However, the students reacted differently to part b) of this problem. Tom claimed that the two problems were identical because R2 could be applied to each to claim an empty jar "at the end"; Max argued that although it was true that each of the objects specified in each problem were removed from the jar at some finite step and not put back afterwards, the difference between the two problems consisted in the fact that the convergent sequence of labels in the case of the 1+1/n problem resulted in a ball labeled "1" in the jar at t=1, whereas in this problem the sequence of labels was divergent, hence the empty jar.

Problem 2. You have two bins labeled A and B and an infinite number of balls labeled 1, 2, 3… .

- At steps 1 and 2 you place ball 1 in bin A and ball 2 in bin B, respectively.
- For $n \geq 3$, at step n you put ball n in the bin containing numbers of the same parity as n (i.e., if n is even, place ball n in the bin containing evens)
- If $n = 3^k$ for some natural number k, then swap the contents of the 2 bins **after** you placed ball n in the appropriate bin.

a) After all the steps have been performed, what are the contents of the two bins?
b) Note that as you go through the process, the contents of the two bins are swapped less and less often. Does this fact make this problem different from The Bin Swapping Tennis Ball Problem (session 5)? Explain your answer.

Solution: a) Using induction one can show that after any finite number of steps n, one bin contains all the evens smaller than or equal to n, while the other contains all the odds less than or equal to n. However, the contents of the bins are swapped every time a ball labeled with a power of 3 is processed, which means that after a ball is placed in a bin for the first time, subsequent steps move it to the other bin and then back again, and so on. For this reason, R1 cannot be applied to claim that any particular ball is in bin A at the end. Similarly, R2 cannot be applied to claim that any particular ball is *not* in bin A at the end. By R3, the final state of bin A is undefined. The same is true for bin B.

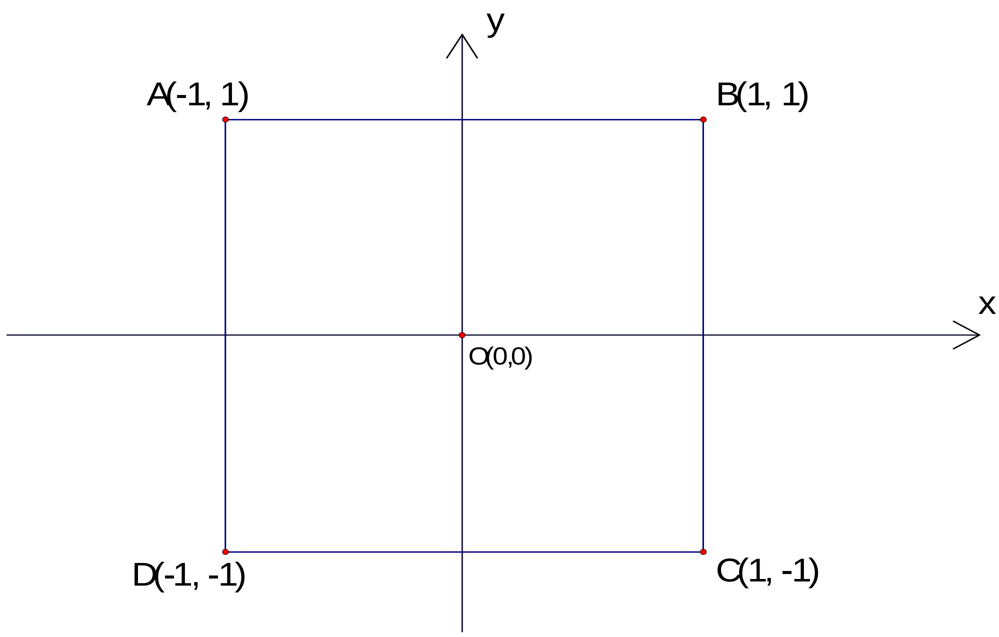
b) The fact that the contents of the two bins are swapped less and less often as the process progresses does not change the fact that for any step n, there is a k>n such that at step k, the contents of the bins are swapped. This is the essential aspect of the problem, which prevents us from applying either R1 or R2. Therefore, this difference from The Bin Swapping Tennis Ball Problem (session 5) makes the two problems only superficially different, which means that the same line of reasoning can be applied to both.

<u>Student responses:</u>

Tom answered a) in a manner similar to what I described as the normative solution. Max worked out the first 9 steps of the problem, noted that for each finite step one bin contained the odd balls while the other contained the even balls processed up to that point, and claimed that "at the end", all odd balls are in one bin and all even balls in the other bin, but that it could not be determined which bin contained what. His representations of the bins for the first 9 steps suggests that instead of swapping the bin contents, Max chose to change the labels of the bins, approach employed during one of the problem-solving sessions (Session 5) as well. During that session, the two students commented that the two interpretations of the problems ("switching the labels of the bins" versus "swapping the contents of the bins" were equivalent).

<u>Problem 3</u>. Consider the square ABCD shown in the figure below (by "square" I am referring to the totality of the 4 segments shown below and **not** to the area encompassed by these 4 segments).

- Start at the center (O(0, 0)). At step 1, go upward and draw a segment of length ½. Your pencil ("cursor") is now at point (0, ½). We'll say it is "pointing upward", as this is how it was left after the drawing it just performed.
- For $n \geq 2$, at step n, turn your cursor 90° clockwise and draw a segment of length $2-\frac{1}{2^{n-2}}-\frac{1}{2^n}$. After performing step n, your cursor is left at the end of the segment you've just drawn, pointing in the direction in which the drawing was performed.



a) After all steps have been completed, what is the position of your cursor?

b) Consider the set of points in the plane lying on the "drawing" constructed by this process (after all steps have been performed). What is the relationship between this set and the set of points that make up square ABCD (disjoint/intersecting/one includes the other)? Explain your answer.

<u>Solution</u>: a) A few simple calculations reveal that for any natural number n, the endpoint of the segment drawn at step n lies inside the square at a distance of $\frac{1}{2^n}$ units from the edge of the square towards which the segment is drawn. The position of the cursor after step n is defined as being the endpoint of the segment drawn at that step. Therefore, the cursor lies inside the square after each finite step. However, no position is visited twice by the cursor, so if we imagine a jar in which at each step we put the point in plane representing the position of the cursor after that step, we can apply R2 to claim that none of the intermediary positions of the cursor can be its position after all steps have been performed. At the same time, R1 cannot be applied to any point in the plane. By R3, the final position of the cursor is undefined.

b) R1 can be applied to any point lying on one of the segments built at a finite step to claim that that point is an element of the set of points lying on the outwards spiral drawn by this process. Using induction one can prove that for any natural number n, the segment drawn at step n is strictly contained by the interior of the square. For this reason, no point on an edge of the square ABCD can lie on a segment built at a finite step; by R2, such a point is not part of the set of points produced by the completed process. Therefore, the spiral and the square are two disjoint sets of points. However, as with each step the spiral gets within a distance of $\frac{1}{2^n}$ units from the square, the distance between the two sets is 0.

Student responses:

Tom reasoned that the process did not determine a “final position” for the cursor as the cursor does not stabilize in any fixed point from a finite step on, and thus R1 could not be applied. Further, Tom claimed that the set of points constructed by the completed process was a union of infinitely (countably) many intervals, each having as many points as the real numbers, thus the final set was a set equal in size with the real numbers but disjoint from the set of points making up the square ABCD (while acknowledging that the distance between the set of points created after n steps and the square approached 0 as n approached “positive infinity”).

Max, on the other hand, reasoned that “at the end”, the position of the cursor could not be determined exactly but that there were four possible positions for the cursor (namely, the four corners of square ABCD). Additionally, Max claimed that the set of points produced by the completed process included all four segments making up square ABCD.

Appendix B: Cycle 2 Tasks

Note: Each problem is followed by a normative solution and examples of student solutions. The Cycle 2 students are named Chris and Todd (pseudonyms).

Pre-test

The Original Tennis Ball Problem. Suppose you are given an infinite set of numbered tennis balls (1, 2, 3,…) and two bins of unlimited capacity, labeled A and B.

- At step 1 you place balls 1 and 2 in bin A and then immediately move ball 1 to bin B.
- At step 2 you place balls 3 and 4 in bin A and immediately move ball 2 to bin B.
- At step 3 you place balls 5 and 6 in bin A and immediately move ball 3 to bin B.

………………………………..

[In general, at step n you place balls 2n and 2n-1 in bin A and then immediately move ball n to bin B.]

This process is continued ad infinitum. Now assume that ALL steps have been completed. What are the contents of the two bins at this point?

See Appendix A for normative/student solutions. The Cycle 2 students did not provide additional types of reasoning from what is described in Appendix A.

The Midpoint Problem. Consider segment [AB]. We divide segment [AB] into two equal parts, denoting its midpoint by H. Then we divide each of the segments [AH] and [HB] into halves, denoting their respective midpoints by P and Q. Imagine continuing to divide segments in this manner. Assuming that all of the steps have been performed, we denote by $\mathcal{M}$ the set of all midpoints obtained through this process. What is the relationship between set $\mathcal{M}$ and segment [AB] ("equal" or "one strictly included in the other")?

Solution: See Appendix A for normative solution and some examples of student responses.

Student responses:

Chris solved this problem normatively during the Pre-test, and he proposed the same solution (leading to M = $\left\{\frac{k}{2^n} \mid n, k \in \text{N and } 1 \le k \le 2^n - 1\right\}$) when this problem was revisited in Session 5. Todd followed his lead, not having a clear idea for how to go about this problem. When I brought up the "reaching the limit" argument produced by the students in a previous session in the context of The z^n Problem, and pointed out that any number on the [0, 1] interval was a limit for a well-chosen sequence of "midpoints", Chris began to question his solution to The Midpoint Problem. After discussing the issue with Todd, he decided that the limiting reason could be used to claim that all rationals on

[0, 1] were part of M, but that it could be used to claim the same thing for the irrationals, as "the could not be reached by a sequence of rationals".

The 1/2 Marble Problem. Suppose you have an empty jar and outside of it you have two marbles labeled 1 and 2. At step 1, marble labeled 1 is put in the jar. At step 2, marble 1 is removed from the jar and marble 2 is put in the jar. At step 3, marble 2 is removed from the jar and marble 1 is put back in the jar. In general, at step n, the marble currently in the jar is replaced by the marble that was outside of the jar. Assume ALL steps have been performed. What are the contents of the jar at this point?

<u>Solution:</u> See Appendix A for normative solution and some examples of student responses.

<u>Student Responses:</u>

Initially (Pre-test), Todd had trouble imagining this process after all steps had been completed, while Chris claimed that "at the end" the jar had to contain one marble but that it could not be determined which one. When this problem was discussed in the context of the Lamp Problem, Chris claimed that one had to determine whether infinity was odd or even in order to determine the contents of the jar at the end. At that point both he and Todd were considering the possibility of no marbles, one marble, or both marbles in the jar at the end. Chris reasoned that infinity could not be both odd and even, so the possibility of two marbles was eliminated for this reason; similarly, Chris reasoned that infinity could not be neither odd nor even, and this eliminated the possibility of an empty jar. Thus, the only remaining possibility was to have exactly one marble in the jar at the end; the students did not know what else to do to determine "which marble" and claimed that there might not be a way to determine that.

Session 1

The Odd/Even Tennis Ball Problem. Suppose you are given an infinite set of numbered tennis balls (1, 2, 3,…) and two bins of unlimited capacity, labeled A and B.

- At step 1 you place balls 1 and 2 in bin A and then immediately move ball 1 to bin B.
- At step 2 you place balls 3 and 4 in bin A and immediately move ball 3 to bin B.
- At step 3 you place balls 5 and 6 in bin A and immediately move ball 5 to bin B.

……………………………………..

[In general, at step n you place balls 2n and 2n-1 in bin A and then immediately move ball n to bin B.]

This process is continued ad infinitum. Now assume that ALL steps have been completed. What are the contents of the two bins at this point?

<u>Solution</u>: There are no oscillating objects so the process has a final state. By R1, after all steps have been completed each even-numbered ball is in bin A and each odd-numbered ball is in bin B.

<u>Student responses:</u>

The Cycle 2 students solved this problem by noting that after each finite step, bin A contained the even balls processed up to that step and bin B the odd balls processed up to that step. Coupling that with the observation that "once a ball is processed by one step it is not affected by the subsequent steps" and the fact the finishing the process meant procession all balls, the students concluded that when the process is completed, bin A contains all evens and bin B all odds.

The Original 10 Marble Problem. Suppose there is a jar capable of containing infinitely many marbles and an infinite collection of marbles labeled 1, 2, 3, and so on. At time t = 0, marbles 1 through 10 are placed in the jar and marble 1 is taken out. At $t = 0.5$, marbles 11 through 20 are placed in the jar and marble 2 is taken out; at $t = 0.75$, marbles 21 through 30 are put in the jar and marble 3 is taken out; and in general at time $t = 1 - 0.5^n$, marbles $10n + 1$ through $10n + 10$ are placed in the jar and marble $n + 1$ is taken out. How many marbles are in the jar at time $t = 1$?

<u>Solution:</u> See Appendix A for normative solution and some examples of student responses.

<u>Student responses:</u>

Although both students in Cycle 2 originally argued that the jar would contain infinitely many marbles at t=1, they both refined their original solution toward normative ones. Some interesting details here with respect to the manner in which each student made explained his version of what I consider to be a normative solution.

Chris referred to a one-to-one correspondence between the set of points in *time* representing when the steps were performed, and the removal of a specific marble from the jar: as each marble had a corresponding point in time before t=1 at which it was removed from the jar and not put back afterwards, Chris concluded the jar was empty. It is interesting to note that initially Chris felt that the time component of the problem made this one-to-one correspondence possible, although later in the same session he commented that there is also a one-to-one correspondence between the set of marbles and the set of steps.

Todd revealed an interesting mental image he was using to reason about this problem: he imagined an infinitely long sequence of blue "slots" within the jar, and used the convention that when marble n was placed in the jar, it was placed in the n^{th} slot, and when marble n was removed from the jar, slot n turned from blue to red and could not be used again. Todd concluded that at t=1 the jar was empty because he reasoned that all the slots would turn red by that point.

Session 2

The Vector Problem. Let $v = (1, 0, 0, ...) \in N^N$. You are going to "edit" this vector step by step.

- Step 1: $v = (0, 1, 2, 0, 0, ...)$
- Step 2: $v = (0, 0, 1, 2, 3, 0, 0, ...)$
- Step 3: $v = (0, 0, 0, 1, 2, 3, 4, 0, 0...)$

……………………………………..

This process is continued ad infinitum. Now assume ALL steps have been completed. Describe v at this point.

Solution: See Appendix A for normative solution and some examples of student responses.

Student Responses:

Both Chris and Todd produced normative solutions to this task by the end of the session it was presented to them (Session 2). The beginning of their discussion on this problem revealed some interesting strategies. Chris argued that if you take a vector of finite length, say 5, and edit it in the way suggested in this problem, by the end of step 5 you end up with the zero vector; he used this observation to claim that with a vector that has infinite length, after as many steps as there are natural numbers you end up with the zero vector as well. This initial type of reasoning displayed by Chris could be considered to be of the GEN-O type.

The 1/2 Marble Problem (Revisited). Suppose you have an empty jar and outside of it you have two marbles labeled 1 and 2. At step 1, marble labeled 1 is put in the jar. At step 2, marble 1 is removed from the jar and marble 2 is put in the jar. At step 3, marble 2 is removed from the jar and marble 1 is put back in the jar. In general, at step n, the marble currently in the jar is replaced by the marble that was outside of the jar. Assume ALL steps have been performed. What are the contents of the jar at this point?

Solution: See Appendix A for normative solution and some examples of student responses.

Student Responses:

Initially (Pre-test), Todd had trouble imagining this process after all steps had been completed, while Chris claimed that "at the end" the jar had to contain one marble but that it could not be determined which one. When this problem was discussed in the context of the Lamp Problem, Chris claimed that one had to determine whether infinity was odd or even in order to determine the contents of the jar at the end. At that point both he and Todd were considering the possibility of no marbles, one marble, or both marbles in the jar at the end. Chris reasoned that infinity could not be both odd and even, so the possibility of two marbles was eliminated for this reason; similarly, Chris reasoned that infinity could not be neither odd nor even, and this eliminated the possibility of an empty jar. Thus, the only remaining possibility was to have exactly one marble in the jar at the end; the students did not know what else to do to determine “which marble” and claimed that there might not be a way to determine that.

Session 3

The n→n+1 Marble Problem. Suppose you have an empty jar and outside of the jar you have an infinite collection of marbles labeled 1, 2, 3, and so on. At step 1, you put marble “1” in the jar. At step 2, you put marble “2” in the jar and remove marble “1”. In general, at step n, you put marble “n” in the jar and remove marble “n-1” from the jar. Assume ALL steps have been performed. What are the contents of the jar at this point?

Solution: Let n be an arbitrary natural number. At step n, marble n is placed in the jar, and at step n+1 it is removed from the jar. None of the subsequent steps (after step n+1) affect marble n, so by R2 marble n is not in the jar at the end. As n was chosen arbitrarily, this is true of any of the marbles, therefore the jar is empty at the end.

Student responses:

Chris and Todd solved this problem normatively (using the phrase “one-to-one correspondence” explicitly in their solution).

The n→n+1 Label Problem. Suppose you have a jar containing a non-labeled marble and outside of the jar you have an infinite collection of *labels* 1, 2, 3, and so on. At step 1, the label “1” is attached to the marble in the jar. At step 2, the label “1” is removed from the marble in the jar and the label “2” is attached to the marble in the jar. In general, at step n, the label “n-1” is removed from the marble in the jar and the label “n” is put in its place. Assume ALL steps have been performed. What are the contents of the jar at this point?

Solution: See Appendix A for normative solution and some examples of student responses.

Student responses:

Unlike the Cycle 1 students, the Cycle 2 students did not Choose to assume that relabeling the marble from n to n+1 was the same action as removing marble n and adding marble n+1. Chris and Todd reasoned that as each label is removed from the unlabeled at a specific step never to be used again, at the end the jar contained the unlabeled marble and nothing else.

The “1+1/n” Powder Problem. On a scale you have 2 ounces of powder. At step 1 you remove some so that you are left with 3/2 ounces. At step 2 you remove some more so that you are left with 4/3 ounces. In general, at step n you remove some powder so that you are left with $1+\frac{1}{n+1}$ ounces of powder. After all steps have been performed, how much powder do you have on the scale?

Solution: Assuming that the amount of powder left “at the end” is defined and is determined uniquely by the removal actions described in the problem, let us denote it by

L. The amount of powder removed by the totality of the steps is $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$, which is equal to 1 (can be easily shown through conventional Calculus techniques). Therefore, L = 2-1 = 1 ounce.

<u>Student responses:</u>

Chris produced the solution mentioned above, while Todd claimed that the amount of powder left on the scale "at the end" was the limit of the $1+\frac{1}{n}$ sequence, which he determined to be 1. Todd was not sure why he thought this limit was relevant to the problem.

The "1+1/n" Marble Problem. Suppose you have an empty jar and outside of it you have an infinite number of marbles labeled with "$1+\frac{1}{n}$", where n is a natural number (so the first one is labeled "2", the second one "3/2", the third one "4/3", etc). At step 1, the marble labeled "2" is put in the jar. For any n>1, at step n you remove marble "$1+\frac{1}{n-1}$" from the jar and put marble "$1+\frac{1}{n}$" in the jar. Assume ALL steps have been performed. What are the contents of the jar at this point?

<u>Solution:</u> See Appendix A for normative solution and some examples of student responses.

<u>Student responses</u>:

The Cycle 2 students wondered whether this problem was more structurally alike to The n→n+1 Marble Problem or The 1+1/n Powder Problem. Todd suggested that perhaps the solution path they employed for The n→n+1 Marble Problem was applicable. Chris acknowledge that was a possibility but claimed that this problem was more similar to The 1+1/n Powder Problem, and therefore that the fact that the sequence of numbers on the marble labels converged to 1 indicated that at the end, the jar contained a marble with the label "1" on it.

Session 4

The z^n Problem. Let $z_0 = 1$ and z = r(1+i) (where r is a real number). Using these two numbers, new complex numbers are obtained through the following step by step process:

- Step 1: obtain z_1, where $z_1 = z_0 \cdot z$.
- Step 2: obtain z_2, where $z_2 = z_1 \cdot z$.

…………………………………………….

- Step n: obtain z_n, where $z_n = z_{n-1} \cdot z$.

……………………………………….

Assume all the steps have been performed. Find a value for r such as the set of **all** the points in the plane that correspond to the complex numbers obtained through this process is

i) finite
ii) infinite and unbounded
iii) infinite and bounded
Q: Does 0 belong to the set of all complex numbers produced by this process?

Solution: See Appendix A for normative solution and some examples of student responses.

Student responses:

Similarly to Cycle 1 students, Cycle 2 students solved this problem normatively with respect to finding the values of r that the problem inquires about. Regarding the additional question under iii), Chris approached it by pointing out that $\lim_{n\to\infty} z^n = 0$, which he interpreted as meaning that 0 is indeed added to the set of numbers produced by the completed process, and "the step at infinity" was the step responsible for this addition. Todd displayed similar tendencies, although less convincingly. Todd attempted to argue that 0, which was $\lim_{n\to\infty} z^n$, was indeed "reached" by the process by pointing out that in The Original 10 Marble Problem, t=1 was the limit of the sequence of points in time mentioned by the process ($t_n = 1 - 0.5^n$); his argument was that that problem's question about t=1 implied that the "limit could indeed be reached", which is why 0, the limit of z^n, was reached by the number-creating process as well.

The Bin Swapping Tennis Ball Problem, Version 1. Suppose you are given an infinite set of tennis balls (labeled with 1, 2, 3, …) and two bins of unlimited capacity, A and B. At step n ($n \geq 1$):

- If n is **odd**, swap the contents of the 2 bins
- If n = 4k-3 or 4k for some $k \geq 1$, then place ball n in bin A
  Otherwise, place ball n in bin B

For each $n \geq 1$, step n was finished at time $t_n = 1 - \frac{1}{2^n}$ (the process started at t=0). What are the contents of each bin at t=1?

Solution: Using induction one can show that for any $n \geq 1$ (n being a natural number), after n steps have been performed one of the bins contains the even balls processed up to that point, while the other contains the odd balls processed up to that point. More precisely, if $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$, A is the bin containing the odd balls while bin B contains the even balls; if $n \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$, bin B contains the odds while bin A contains the evens.

Let us examine bin A. Let k be an arbitrary natural number. For the ball labeled k, there is no step n at which it is put in bin A and it stays there for all subsequent steps (as once ball k is placed in a bin, it keeps being moved from one bin to the other for the rest of the process). Therefore, we cannot apply R1 to conclude that ball k is in bin A after all steps have been performed. For similar reasons, we cannot apply R2 to conclude that ball k is *not* in bin A after all steps have been completed. By R3, the final state of bin A is undefined. Similar reasoning can be used to come to the same conclusion about bin B.

Student responses:

The Cycle 2 students worked out the first few steps of the process and noted that after each of those steps there was an odd/even partition of the already processed balls between the two bins. Then they claimed that this implied that at the end, one bin will contain all the evens while the other will contain all the odds, but it could not be determined which bin contained what. Therefore, the only thing that *could* be determined about the final state was that each bin contained infinitely many balls: one the odds, and one the evens. The students' further discussion on this problem contained a debate regarding how the contents of each bin compared to the other, in terms of size. The students indicated that they weren't sure whether these two "infinities" were the same size, as claiming that they were would imply that the infinity of the set of natural numbers was "even".

Session 5

The Bin Swapping Tennis Ball Problem, Version 2

Suppose you are given an infinite set of tennis balls (labeled with 1, 2, 3, …) and two bins of unlimited capacity, A and B. At step n ($n \geq 1$):

- If n is **odd**, swap the **labels** of the 2 bins
- If n = 4k-3 or 4k for some $k \geq 1$, then place ball n in bin A
  Otherwise, place ball n in bin B

For each $n \geq 1$, step n was finished at time $t_n = 1 - \frac{1}{2^n}$ (the process started at t=0). What are the contents of each bin at t=1?

Solution: This problem is open to interpretation, as it seems to involve two processes – one manipulates balls, the other manipulates labels. I interpret the problem as asking what the contents of the two physical (fixed) bins are, irrespective of what the labels on the bins might be. Let us assume, for easy reference, that at the beginning of the process the bin labeled A is on the left and the bin labeled B is on the right, from the point of view of an imaginary person who would manipulate the balls. Under this assumption, it is easy to show that after any finite step, all even balls processed up to that point are in the left bin, while all odd balls processed up to that point are in the bin on the right. As the process does not involve any ball removal, the contents of the two bins (left and right) as the number of performed steps increases create two "inclusion chains" (meaning that for each bin, the contents after step n are included in the contents after step n+1). Therefore, applying R1 to an arbitrary ball n, at the end of the process we can claim that that ball is in the left bin if n is even, otherwise being in the right bin. Therefore, at the completion of the process the left bin contains all the even-numbered balls, while the right one contains all the odd-numbered ones.

Regarding the process manipulating the labels, it is simple to show that both objects in question (label A and label B) are oscillating objects, therefore by R3 the outcome of this process is not defined.

Student responses:

The Cycle 2 students claimed there was no difference between this problem and The Bin Swapping Tennis Ball Problem, Version 1, and argued about it in the same way as they did for that problem.

The Lamp. You have a lamp with a switch; the lamp is turned off. At any step $n \geq 1$, you turn the lamp on if n is odd; otherwise, you turn off the lamp. Assuming all the steps have been performed, is the lamp on or off?

Solution: See Appendix A for the normative solution and some examples of student responses.

The Midpoint Problem (Revisited). Consider segment [AB]. We divide segment [AB] into two equal parts, denoting its midpoint by H. Then we divide each of the segments [AH] and [HB] into halves, denoting their respective midpoints by P and Q. Imagine continuing to divide segments in this manner. Assuming that all of the steps have been performed, we denote by $\mathcal{M}$ the set of all midpoints obtained through this process. What is the relationship between set $\mathcal{M}$ and segment [AB] ("equal" or "one strictly included in the other")?

Solution: See Appendix A for normative solution and some examples of student responses.

Student responses:

Chris solved this problem normatively during the Pre-test, and he proposed the same solution (leading to M = $\left\{\frac{k}{2^n} \mid n, k \in \mathrm{N} \text{ and } 1 \leq k \leq 2^n - 1\right\}$) when this problem was revisited in Session 5. Todd followed his lead, not having a clear idea for how to go about this problem. When I brought up the "reaching the limit" argument produced by the students in a previous session in the context of The z^n Problem, and pointed out that any number on the [0, 1] interval was a limit for a well-chosen sequence of "midpoints", Chris began to question his solution to The Midpoint Problem. After discussing the issue with Todd, he decided that the limiting reason could be used to claim that all rationals on [0, 1] were part of M, but that it could be used to claim the same thing for the irrationals, as "the could not be reached by a sequence of rationals".

Session 6

Halves. Consider the segment [0, 1]. Let $a_0 = 1$ and a = ½. Using these two numbers, new real numbers are obtained through the following step-by-step process:

- Step 1: obtain a_1, where $a_1 = a_0 \cdot a$.
- Step 2: obtain a_2, where $a_2 = a_1 \cdot a$.

…………………………………………

- Step n: obtain a_n, where $a_n = a_{n-1} \cdot a$.

…………………………………………

Assume all the steps have been performed. Is 0 included in the set of real numbers produced by the completed process?

Solution: By induction one can easily prove that $a_n = \left(\frac{1}{2}\right)^n$. The intermediate states for this process are $S_0 = Ø$, $S_1 = \left\{\frac{1}{2}\right\}$, $S_1 = \left\{\frac{1}{2}, \frac{1}{4}\right\}$, …, $S_n = \left\{\frac{1}{2}, \frac{1}{4}, ..., \frac{1}{2^n}\right\}$,… .

For any natural number n, $\frac{1}{2^n}$ belongs to a tail of this sequence of intermediate states, so by R1 $\frac{1}{2^n}$ belongs to the final state. As the intermediate states do not contain any numbers not of the form $\frac{1}{2^n}$, by R2 such numbers do not belong to the final state.

Therefore, the set produced by the completed process is $\left\{\frac{1}{2^n} \mid n \in N\right\}$.

Student responses:

Chris and Todd reasoned that, apart from $\left\{\frac{1}{2^n} \mid n \in N\right\}$, the final set also contained 0 which, being equal to $\lim_{n\to\infty} \frac{1}{2^n}$, was reached by the "step at infinity".

Truncations. Consider an iterative process which at step n ($n \geq 1$) defines a real number a_n as the n-digit truncation of π. Is π in the set of points constructed by the completed process (i.e., after all steps have been performed)?

Solution: By R1, the set of numbers (points on the number line) created by the completed process contains the set $\{\pi\text{'s } n-\text{digit truncation} \mid n \in N\}$. π is not in any tail of the sequence of intermediate states of this process, so by R2 π does not belong to the final state.

Student responses:

Chris noted that the sequence of truncations of π converges to π, and similarly to the previous problem, claimed that π was added to the final set of this process by the step at infinity. Thus, for Chris the final state consisted of the set of all truncations of π *and* π.

Todd noted that he first interpreted the question as "Imagine you have π's decimal expansion written in its entirely on a piece of paper, and that decimal expansion is covered with another piece of paper. At each step of the process you slide the cover one position to the right, thus revealing a truncation of π. After all steps have been completed, have you revealed π? ". The students agree that this is indeed a different problem and that it is the case that in this process, π has been revealed at the end of the process. However, in spite of acknowledging the different nature of the two problems, both students argued that in the case of the original problem π was part of the final state.

Halves and Reals. Let $b \in [0,1]$ be a real number. Consider an iterative process that at each step n ($n \geq 1$), defines a real number $a_n = \frac{k}{2^n}$, where $1 \leq k \leq 2^n - 1$ and $\left|\frac{k}{2^n} - b\right| = \min_{1 \leq i \leq 2^n - 1}\left\{\left|\frac{i}{2^n} - b\right|\right\}$.

Assume all steps have been performed. Is b part of the set of points defined by the completed process?

Solution: Using the ε, N definition for the limit of a numerical sequence, one can prove that $\lim_{n\to\infty} a_n = b$. However, unless b is of the form $\frac{k}{2^n}$, ($1 \leq k \leq 2^n - 1$), b is not part of any of the intermediate states of this process, so by R2 it is not part of the final state. For an arbitrary natural number n, a_n is part of the tail of intermediate states that starts at position n, so by R1, a_n is part of the final state. Thus the final state is $\{a_n \mid n \in N\}$ (which includes b if b is for the form $\frac{k}{2^n}$, $1 \leq k \leq 2^n - 1$).

Student responses:

Upon becoming convinced that $\lim_{n\to\infty} a_n = b$ (without producing a formal proof, however), the Cycle 2 students treated a few different cases, built around what kind of number b was. The cases were: 1) $\frac{k}{2^n}$, ($1 \leq k \leq 2^n - 1$); 2) 0 or 1; 3) a rational number not of the form from case 1; and 4) an irrational number. Chris claimed that in case 1) b was certainly part of the final state as it was produced at a finite step. For case 2), the students referenced The Halves Problem and claimed that when b was 0 the current problem became that problem, so b was indeed a part of the final state. For b=1, a similar reason was invoked to claim that b was part of the final state, although surprisingly Chris said he was less comfortable about that part that he was in the b=0 case. For case 3), after a long discussion both students used limiting reasoning to claim that b was indeed part of the final state. Finally, in case 4) the students used LIM reasoning again to claim that b was part of the final state. It is important to note that the students indicated varying degree of confidence in their answer across the last 3 cases. Both students were confident in their answer for case 1. Chris was comfortable with his answer for case 2, less comfortable in case 3, and even less comfortable in case 4. Todd, on the other hand, commented that in each of the last 3 cases, the claim that b was part of the final state was based on the same principle (namely, the a_n sequence converges to b), so he was equally uncomfortable with all 3 of them.

Session 7

Reflection on Tasks. The students were given a list of all problems considered in sessions 1-6. The list was followed by the text "Group these problems based on similarities in their mathematical structures. Create as many categories you think are necessary, and assign each problem to a category. Describe the characteristics of each category (how do you decide if a given problem belongs to a certain category?)." This task was performed collaboratively through discussion with the partner.

Post-Test (administered through individual interviews)

Problem 1. You have a jar and infinitely many tennis balls, labeled by $(-1)^{n+1}n$ (that is, the first ball is labeled 1, the second -2, the third 3, and so on). At t=0 the jar is empty.

- At t=1/2 you place ball labeled 1 in the jar
- For each n>1, at $t=1-\frac{1}{2^n}$ you remove ball with label $(-1)^n(n-1)$ from the jar and put in ball labeled $(-1)^{n+1}n$.

a) What are the contents of the jar at t=1?

b) Is this problem different from the $1+\frac{1}{n}$ Marble Problem (Session 3), in terms of its mathematical structure? Explain your answer.

Solution: See Appendix A for the normative solution and some examples of student responses.

Student Responses:

Both Cycle 2 students solved part a) normatively. In part b) both students commented that the time component did not have any effect on the process as it did not change how the balls were being handled. Additionally, Todd commented that although the labels are different between the current problem and The $1+\frac{1}{n}$ Marble Problem, the application of R2 to each ball/marble led to the same answer, an empty jar. Chris provided similar comments when this problem was first treated at the beginning of the post-test, but at the end of the post-test when I asked him one more to summarize his position on The $1+\frac{1}{n}$ Marble Problem, he returned to the "marble labeled 1 in the jar" claim that he supported during the problem-solving session in which this problem was discussed.

Problem 2. You have two bins labeled A and B and an infinite number of balls labeled 1, 2, 3… .

- At steps 1 and 2 you place ball 1 in bin A and ball 2 in bin B, respectively.
- For $n \geq 3$, at step n you put ball n in the bin containing numbers of the same parity as n (i.e., if n is even, place ball n in the bin containing evens)
- If $n = 3^k$ for some natural number k, then swap the contents of the 2 bins **after** you placed ball n in the appropriate bin.

a) After all the steps have been performed, what are the contents of the two bins?
b) Note that as you go through the process, the contents of the two bins are swapped less and less often. Does this fact make this problem different from The Bin Swapping Tennis Ball Problem (session 5)? Explain your answer.

Solution: See Appendix A for the normative solution and some examples of student responses.

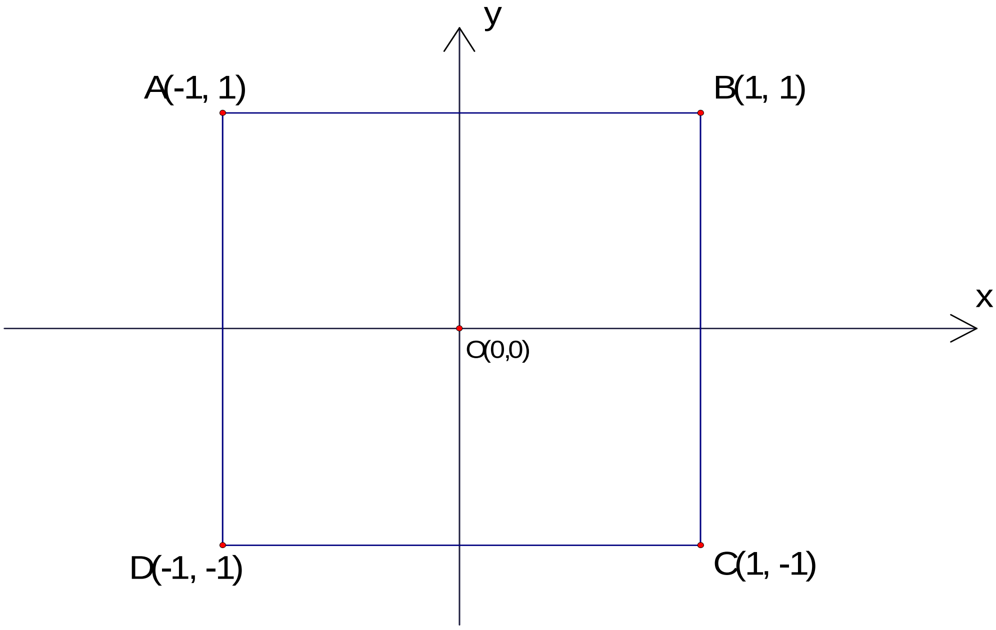
Student responses:

The responses given by Cycle 2 students mirrored those given by the Cycle 1 students, with Todd matching Tom's answer that under the R1-R2 assumptions nothing could be said about the final state, and Chris matching Max's answer that the set of balls was partitioned among the two bins in a manner matching the finite step partitions (odd versus even, in this case), but that it could not be determined which bin contained the odds and which the evens.

Problem 3. Consider the square ABCD shown in the figure below (by "square" I am referring to the totality of the 4 segments shown below and **not** to the area encompassed by these 4 segments).

- Start at the center (O(0, 0)). At step 1, go upward and draw a segment of length ½. Your pencil ("cursor") is now at point (0, ½). We'll say it is "pointing upward", as this is how it was left after the drawing it just performed.
- For $n \geq 2$, at step n, turn your cursor 90° clockwise and draw a segment of length $2-\frac{1}{2^{n-2}}-\frac{1}{2^n}$. After performing step n, your cursor is left at the end of the

segment you've just drawn, pointing in the direction in which the drawing was performed.



a) After all steps have been completed, what is the position of your cursor?

b) Consider the set of points in the plane lying on the "drawing" constructed by this process (after all steps have been performed). What is the relationship between this set and the set of points that make up square ABCD (disjoint/intersecting/one includes the other)? Explain your answer.

<ins>Solution:</ins> See Appendix A for the normative solution and some examples of student responses.

<ins>Student responses</ins>:

Todd answered part a) by specifying that he considered as an intermediate state "the set of the current cursor position", and argued that R2 can be used to claim that any point in the plane *does not* belong to the final set associated with these intermediate states. Thus, the final state was the empty set, which he interpreted as meaning that the cursor has no defined position after the process is completed. For part b), Todd argued that for any point on square ABCD, R2 can be applied to claim that it is not part of the set of points in the plane defined by the completed process, and thus that this set and the square were disjoint.

Chris, on the other hand, for part a) argued that at the step at infinity the cursor would be in one of the corners of the square, but that it could not be determined which one. As such, he claimed that that corner is part of the set of points produced by the process, and thus that this set and the set of points on ABCD was certainly not disjoint. Chris added that given that no step followed the step at infinity and that the “drawing” did not “reach” the square before the step at infinity, it followed that the two sets intersected in only one point (a corner).

Appendix C: Equivalence of Two Definitions for State at Infinity

Theorem. Let $\{S_n\}_{n\in N}$ represent the intermediate states of an arbitrary infinite iterative process. The process admits an R1-R3 final state S_∞ if and only if S_∞ is its pointwise convergence final state.

Proof: Let $D = \bigcup_{n\in N} S_n$ and for any natural number n, denote by f_n the characteristic function of S_n with respect to D.

($\Rightarrow$) Let S_∞ be the final state defined by the R1-R3 approach. Define $f{:}D \to \{0, 1\}$ by $f(x) = \begin{cases} 1, \text{ if } x \in S_\infty \\ 0, \text{ otherwise} \end{cases}$. In order to show that S_∞ is the pointwise convergence final state of the process, it is enough to show that f_n converges pointwise to f.

Let x be an arbitrary element of D. If $x \in S_\infty$ then R1 applies to x, which means that there exists a natural number k such that $\forall m \geq k$, $x \in S_m$. Then the sequence $f_n(x)$ stabilizes to the value "1" from rank k on, so we can conclude that $f_n(x) \xrightarrow{n\to\infty} 1$. But f(x)=1, so $f_n(x) \xrightarrow{n\to\infty} f(x)$. Similarly, if $x \notin S_\infty$ then f(x)=0 and R2 applies to x, which means that there exists a natural number k such that $\forall m \geq k$, $x \notin S_m$. Therefore, the sequence $f_n(x)$ stabilizes to 0 from rank k on. It follows that $f_n(x) \xrightarrow{n\to\infty} 0$, which combined with f(x)=0 completes this part of the proof. We have shown that for an arbitrary x in D, we have $f_n(x) \xrightarrow{n\to\infty} f(x)$, which means that f_n converges pointwise to f.

($\Leftarrow$) Let S_∞ be the final state defined by the pointwise convergence approach. Define $f(x) = \begin{cases} 1, \text{ if } x \in S_\infty \\ 0, \text{ otherwise} \end{cases}$. We know that f_n converges pointwise to f. Let x be an arbitrary element of D. Then $f_n(x)$ is a convergent numerical sequence, which combined with the fact that the only possible values in this sequence are 0 and 1, allows us to conclude that from a rank k on, $f_n(x)$ stabilizes to one of these values. Since for any n, f_n is the characteristic function of S_n with respect to D, it follows that object x either belongs to all sets in a tail of $\{S_n\}_{n\in N}$ *or* does not belong to any of the sets in a tail of $\{S_n\}_{n\in N}$. Thus, either R1 or R2 is applicable to x, which means that the R1-R3 final state exists. Let us denote it by S_R. We need to show that $S_R = S_\infty$.

$x \in S_\infty$ iff $f_n(x) \xrightarrow{n\to\infty} 1$ iff $\exists k \in N$ such that $\forall m \geq k, f_m(x) = 1$ iff $\exists k \in N$ such that $\forall m \geq k, x \in S_m$ iff $x \in S_R$ (by R1). Q.E.D.

Corollary. An infinite iterative process has an undefined final state by the R1-R3 definition if and only if it has an undefined final state by the pointwise convergence definition.

Proof: Assume the process has an undefined final state according to the R1-R3 approach. If it actually had a final state by the pointwise definition, then according to the theorem proven above, it should also have one according to R1-R3. But this is not the case, so the pointwise convergence final state is also undefined. The converse is proven similarly. Q.E.D.

Appendix D: Cycle 1 Transcripts

Note: The Pre-Tests were done with a student at a time. In each regular problem-solving session, the two participants are Max and Tom (pseudonyms).

Max Pre-test
4/2/08

00:00 [Max works on two questions I decided not to use in the study because they were not explicitly describing an infinite iterative process.]

14:16 [Max starts work on The Original Tennis Ball Problem].

Max: [after working out the first few steps in writing] The two bins have an equal number of elements… So the content of the two bins after you do an infinite [number of steps]…like all the natural numbers? Well, they both would have an infinite number of balls…equal to the natural numbers.

I: Why?

Max: Because you put two balls in [bin A] at each step and take one out…and you would get the same result if you put 3 in bin A at each step and move one out, even if you had 4 in A and 2 in B after step 2….

17:21 I: Can you name a few balls in bin A and a few balls in bin B "at the end"?

Max: This bin [pointing to bin B] will have the first half of all balls you put in; this bin [pointing to bin A] will have the second half.

I: I am not sure what that means. Can you give me one specific ball in bin A?

Max: I can't give you that because bin B is gonna have all naturals in it…so that would be any natural number…I mean all the natural numbers".

I: Why?

Max: Because you're doing this an infinite number of times…so you have to do it so that you take all of them out of here [pointing to bin A]."

18:55 I: So, at what step is ball 50 put in bin B?

Max: Step 50.

I: So then, about bin A… you said there are infinitely many balls in there … which one would be there?

Max: I can't tell you any…well, it's still going to be N, but it's going to be like…see, this [pointing to bin B] has all the natural numbers in there, so then this [pointing to bin A]…are you saying this is going to be greater than N? It can't be greater than N…but it can't really be zero, 'cause you're still moving two balls at a time so you always have just as many here [pointing to bin A].

20:40 I: Can an object be in two different locations at a certain moment in time?

Max: No, no. But you're always moving two balls into here [bin A] and putting one over here [bin B], so you'll still going to have the same number of balls in each bin.

21:40 I: I am not sure I understand how all balls can be in bin B but some of them in bin A.

Max: Well, you never run out of numbers… so the process isn't gonna really stop but I mean if it did … well, you know, it can't really stop, you know what I'm saying…'cause I mean the natural numbers don't end, so you'll always put balls in here but any natural number is going to come in here at some point.

22:25 [I introduce a similar problem (not on Pre-Test). He uses draft paper to take notes. The problem is the tennis ball variation that splits the natural numbers in odds and evens between the two bins (put 2n and 2n-1 in bin A, move 2n-1 to bin B. I just describe the process; Max interrupts me before I ask the question for this problem].

Max: So basically I'm going to have all the odd numbers in bin B and all even numbers in bin A".

I: When?

Max: When the process is complete.

I: You see, these 2 processes [this and the one in the Original Tennis Ball problem] are quite similar in the sense that there are as many steps as there are natural numbers, and in the original tennis ball problem you said "the process cannot be finished". How is it that you can "finish" this one?

Max: You can't finish this either, but I'm saying that if you compare the 4 bins, they're each going to have the same number of balls…

I: You mean, "after all steps have been completed?"

Max: Yes; the difference between the 2 problems is in which balls will be in each bin, but all 4 bins will have the same number of elements.

25:48 I: How come you're comfortable talking about "that point" [after all steps have been completed] in that situation [the variation], but here [pointing to the original one] you said the process "doesn't stop"?

Max: Neither process stops, but I'm saying you would have the set {1, 3, 5…} [in bin B, in the variation problem]

I: Ok, if we can talk about that point, after all the steps, how many elements would you have in here [pointing to where he wrote {1, 3, 5,…}?

Max: An infinite number of elements…which is really the same size as {1, 2, 3…}".

I: If you want to say that there is a point at which that set contains an infinite number of tennis balls, we must be at that point, after all the steps have been completed, 'cause if we stopped after the first trillion steps, that wouldn't be an infinite set…

Max: Right.

I: So you are considering that point, after all the steps.

Max: Yes.

I: And you are definite that the balls are split that way [odds and evens, in the variation problem].

Max: Yes.

I: Ok, let's move on to the next problem.

27:13 [Max starts working on the Infinity Hotel Problem; mumbles to himself while reading the problem]

Max: You can do that! How about we denote set of natural numbers as {1, 2, 3, … n}, where n is infinity … the last natural number, which you can't say specifically, but it's like infinite … the last one. Put the new guest in n+1. inf +1 is infinity so you can put in a new guest.

I: If n is the last natural number, n+1 is also a natural number so how can n be the largest natural number? Not sure what you mean by this n...

Max: No, not like a natural number, but at the end of the natural numbers… Don't know how to explain… [pause]

31:55 Max: [suddenly] How about we tell everyone to shift down a room and we put this guest in the first room. Is that allowed?

I: You tell me. If we do this, will this shifting process ever be completed?

Max: Yes, if they all did it at the same time, they would be moved.

33:30 I: I see. What if you get an infinite number of new guests?

Max: [after pause] It's possible! We can order the rooms in 2 rows (odds and evens). Then shift everyone from room n to room 2n, so that the side with odd numbers remains empty and the new guests can be put in there.

I: Ok… and for this one, same question as before, is it possible to complete this shifting process?

Max: It depends on how long it takes them… it's gonna take some people…. When you get to say like the 1000^{th} person is going to have to go to the 2000^{th} room. It depends on how far the rooms are apart. I mean might take a while, but yeah it's possible to complete it theoretically…if they took no time to actually walk down the corridor then yeah, you can do it.

I: Can you tell me more about how you imagine, in your mind, this shifting being done?

Max: Well, if we take into consideration that it takes longer and longer to move for people in higher numbered rooms, than this process is never finished. But if we don't consider time than it is finished instantly.

42:10 I: So regarding the tennis ball problem earlier [The Original Tennis Ball Problem], can you think about it [the process] as done instantly?

Max: That's how I *was* thinking about: I know it does it [ball-moving] a lot of times but I was just thinking about what the result was at the end.

I: Ok, we can wrap it up here.

Max: Was I right? I'm pretty sure I'm right… I'm positive.

[End of session]

Tom Pre-Test
4/4/08

0:00 [Tom works on two questions I decided not to use in the study because they were not explicitly describing an infinite iterative process.]

24:10 [Tom starts work on The Original 10 marble Problem (timed version)].

Tom: It seems like for finitely many operations, it doesn't matter how you index the time… the whole concept of time is a bit superfluous, as at the first step we put 10 marbles in and remove 1, and the next one we put another 9 marbles, at the next another 10 marbles…if all you're concerned with is how many marbles you have in the jar, you could wait 3 weeks, you know, and it wouldn't make a difference. But now we're talking about the limit of this so…because it's one thing to talk about finitely many operations and then to talk about the limit. […] I could see how it could potentially change the problem the fact that we're talking about time equals 1, as opposed to some number closed to 1 that was truncated.

29:30 Tom: Maybe if we have a sequence that for every term is an expression of the size…[…]. The way we defined the sequence in the end it goes to infinity…in the sense that it's monotonic increasing and it's not upbounded anywhere. So…I think there should be as many marbles in the jar as there are natural numbers.

I: At time t=1?

Tom: Clearly there aren't finitely many marbles in the jar at t=1 because if there were finitely many marbles that would mean our sequence a_n would converge to a particular number that would be the size of the number of marbles at t=1. But we already know that the sequence diverges. So now, you know, the size of the marbles is infinite. Now the question is "is it infinite in the sense of the natural numbers are infinite, or infinite in the sense that the real numbers are infinite…but the thing is, we're talking about marbles, you know, like objects, like 1, 2, 3, 4, 5…so I can't see there being as many marbles as there are real numbers. […] So I'm thinking this is as many marbles as natural numbers.

32:55 I: I see… well, does "as many as natural numbers" mean "all the marbles"?

Tom: You can build a bijection between the marbles in the jar [at t=1] and N…

I: Is marble labeled 1 in the jar at t=1?

Tom: Marble 1 is never in the jar…I mean, it's not in the jar at t=1. Because you take marble 1 out at some point, and the way we defined our operation it never goes back again, because you deal with the marbles 1 through 10 and then you don't look back.

I: What about marble 2?

Tom: [pauses, chuckles] I'm beginning to see…because you can ask the question for any marble…cuz you're taking marble 1 out, and marble 2 out, and marble 3 out…at every step, I mean at some point, for any marble, say the k^{th} marble, at some point that marble gets taken out. So if you go on forever, presumably then every marble gets taken out. But, yeah…the thing is you're adding…you're putting 10 marbles in and taking one out, so at every step you have 9 more marbles than before. Ok…Alright, now I'm not sure.

36:30 Tom: Let's consider again the problem after a finite number of steps… after a finite number of steps you would run out of marbles to put in and you could argue that you can be left with the jar empty as you would continue marbles out but not putting anything in.

I: I see… but that would mean you're reinventing the process by changing the nature of the steps, as you would have some steps where you're not putting anything in, you're just taking marbles out. Let's assume that's not allowed, to change what's done at each step.

Tom: You have infinitely many marbles, so you're never going to run out, so the need to stop won't arise, so you're never going to stop and you won't have to reinvent your operation, you're just gonna keep on going. So no matter how many marbles you remove, there's always going to be the next 9. For example you took marble 1 out, but then there's marbles 2 through 10 that are there. Then you take marble 2 out, but I can say that marbles 3 through 11 [sic] are there. Whenever you remove a marble, the next 9 are always there in your jar. So you're always going to have marbles in your jar. This process of adding ten [marbles] and taking out one [marble] at every step…there's sort of an accumulation of marbles in the jar… so at every moment of our operation the number of marbles in the jar is getting larger. And because it's getting larger at every step without bound, then you can say that the number of marbles at t=1 is infinite.

40:05 I: Ok, so one of your explanations points toward "an infinite number of marbles in the jar at t=1", while the other points out that a lot of marbles are taken out during the process, so what exactly is left?

Tom: For a second there, I was thinking maybe this is the null set. Because you could argue that for any marble a_k, that marble is taken out. So if every marble is taken out then at some point you're not going to have any marbles in the jar. If every one is taken out, then that's the null set. But I'm not really convinced of that. I guess I'm having trouble getting to t=1; I'm sort of thinking about the process as you're approaching 1, but I'm having trouble getting an idea of what the jar would be at 1. In a sense what the jar is at 1 is sort of different than what it's becoming as you're getting to 1. t=1 would be when're done, so to speak, but you're never done. You could argue you would never get to t=1 if you were to do this process. So if we were sitting at t=0 we would never get to t=1 so we would never have a jar at t=1 to physically look at and count them to see how many marbles there are. I mean, the marbles in the jar at t=1 would be a very abstract concept. […] At least in the real world I don't see a jar of marbles in this sense really existing. I mean if you were to start from t=0…maybe there's an abstract concept of what the jar of marbles is that you can have an idea in your head.

I: Can you look at this situation ignoring the real world?].

43:10 Tom: I'm thinking right now of an analogy. When I took Math 300… you know the infinity hotel? This is sort of like if you have rooms 1 through 10, and you move 10 people in there, put 1 person in each room, then person in room 1 moves out, right? It makes me think of a wave – people are moving down the corridor of the hotel; […] but it's getting bigger as it slides across. If you have this increasing wave going down, that would be as you're approaching 1 [t=1], not when you're at 1. Cuz once you get to 1, the wave would be stopped. Because you're done with your operations, you're not performing anymore. But I have no idea what that would be, because there's no end of the corridor and you're never going to stop. So… it's sort of like you have this infinity…cuz the wave is increasing… so if you get to 1, then at that point you have infinitely many people in the hotel. But that infinity of people you would never find if you walked down the corridor. For any room, that person would have been taken out. […] I was thinking you would have this situation at t=1 and that I would freeze time at that particular moment and do as much exploring at that particular time. There wouldn't be anybody in any room. But logically it makes sense to me that that would be this infinity of people because the no. of people in the hotel increases by 9 after every operation so if you're done with every operation you have this infinity of people. But they're infinitely far away from me. It's sort of like you have this point at infinity at the end, that I'm never gonna get to where everybody is…

I: Ok, I think I understood what you mean. Let's go on to the next problem.

49:50 [Tom starts work on The Puppy Problem, which is actually the Infinity Hotel problem simply put in a different context].

Tom: [after taking time to read the problem to himself]. Right. Let's pair the new child with puppy 1, then child 1 with puppy 2, child 2 with puppy 3, etc. 'Cause N and N union one more are equally large.

I: What if 5 new children arrive?

Tom: It would be the same thing… I remember a problem in math 300 that a finite set union with natural numbers still is as big as the natural numbers.

55:33 I: How are you thinking about this reassignment process? All done at once, or one by one?

Tom: Sequentially… this child comes along, then I arrive and I say ok, now I'm switching the connections, so this one gets moved over, then you take this child and this one gets moved over…you just keep going down the line[…] So you have this process that you're going down the line doing this. But then you can think of it after you're done, so to speak. Done being when all the connections have been switched [pointing to this drawing].

I: Can we talk about "that state" [of being done], being that you're switching the connections one by one… can you be done with this switching?

Tom: I mean, when you take the limit of a sequence, and you say it converges to a particular limit, then you can argue you never are really done converging to that limit because there's still another term in your sequence. But you can still define what it means for a sequence to converge to a particular number. […] You can say that this is sort of a limit for this process.

57:30 I: I see. And what if we have an infinite number of new children arriving?

Tom: As far as I recall, the set of integers and the set of naturals are equal in size. How about we label the new kids with negative integers… from 0 to minus infinity? Then we go through the sequence of puppies and distribute them one to the sequence of existing children, one to the sequence of new children, and so on. [In writing, he tries to find an exact formula for the "switching"].

I: I am clear on how you do the reassignments, that's ok. One last question, does the marble problem remind you of any other problem other than that hotel problem you mentioned?

Tom: No.

[End of session]

Session 1, File A
4/6/08

00:00 [Start work on The Vector problem; the students read the problem individually].

Tom: So the number of dimensions [of the vector] is equal to the size of the natural numbers… [pause] This seems similar to the marble problem [referring to The Original 10 Marble Problem he worked on during the Pre-Test]. [Addressing Max] Do you know that one?

[Tom proceeds to explain to Max what the marble problem is about, as Max worked on the tennis ball problem in the pre-test.]

Tom: The point is that at some point, you know, marble 1 is taken out, marble 2 is taken out… at every point there's a marble that's taken out, right, for any one…

Max: So eventually they're all taken out and there's nothing left in the jar?

Tom: Yeah, something like that, so I was thinking…the way this pattern seems to be set up (he means the pattern in the vector)…you're sort of moving across. […] The first entry starts off at 1 then goes to 0 and presumably it stays at 0. The second entry starts at 0, goes to 1 then goes back to 0, and presumably it stays at 0. So I think for any entry, at some point it's going to go to 0 and stay there; so if you're done with your process it's just going to be the 0 vector.

Max: You think so?

Tom: Maybe.

Max: Well you're still going to have all the natural numbers in between them, aren't you?

Tom: If a particular entry goes to 0, then that means there's the numbers 1, 2, 3 and so forth… after it, but that's in the process…

2:50 I: To make sure you understand the problem in the same way… how about you write explicitly what the vector looks like at Step n?

[Pause, both students are looking at their sheets and scribbling, occasionally mumbling.]

Max: You're going to have 0's then the natural numbers and then more 0's…

[Tom doesn't indicate to have heard that – he writes that at step n, the vector is (0, 0, …,1, 2,…, n+1, 0, 0,…), with 1 being on the n+1 position].

Max: So after the final step, it's going to be the 0 vector… [pause] I agree with that.

Tom: Wait a minute. There are going to be n entries that are non 0 at any step in the process, but…

Max: But if it goes to infinity, you're still going to have 0.

Tom: Yeah, you're going to have… every position is going to be 0. I mean, if you take any position, that position goes to 0 at some point.

Max: Yeah, so the final position is 0. The 0 vector.

Tom: Right.

6:00 I: So in what ways was this similar to the marble problem?

[Tom explains again the process in the marble problem to Max in more detail. The question of the problem is not directly specified – both students assume it's about what happens "at the end".]

10:35 Max: So you're saying you're going to end up taking out all the marbles, have nothing left in the jar.

Tom: When I did the problem, I think it was 2 days ago, I had conflicting ideas about the problem. One, that it makes sense that there's nothing in the jar. On the other hand, there is this mass of marbles that accumulates and accumulates and accumulates and then what, it just disappears?

Max: I think that either way, the answer is valid, isn't it?

I: You mean, the two answers can coexist, even though they're saying different things about the final state?

Max: What if it's like…'cause you're dealing with two infinities, and if you're subtracting, you can't really do that…

I: What do you mean by two infinities?
Max: I'm saying this is like undefined, 'cause you have infinity minus infinity. So you could answer each way, it would be right, right?

I: Well, infinity minus infinity is undefined because in that general form it's not specified what kind of infinity we're talking about. If you have two numerical sequences that are specified (the limit of each being infinity), you could find out what the limit of the difference is, couldn't you?

Max: If that's true, then all the marbles are removed from the jar would be the right answer. […] Cuz you're removing them 1, then 2, then 3, eventually you would remove them all if you finish the process.

I: That makes sense. What about the other argument, why doesn't that make sense? A few days ago each you thought the other one made sense too.

Max: 'Cause you couldn't say any numbers that are in the jar, 'cause you're going to exhaust all the numbers.

I: You also said that last time [that you can't name any left in the jar/bin] But then it didn't mean to you that the jar/bin was empty; you said, "infinite number, but can't tell which ones".

Max: Yeah, but I kind of changed my mind.

I: Why?

Max: 'Cause even if you say that number is there at this step, at the next step it might not be there so if you do finish the process, there's not going to be nothing there. There can't be any numbers left, any natural numbers, so that can't happen.

I: Tom, what do you think about this?

14:43 Tom: I'm having trouble thinking about where the marbles are actually going, because through the process they get larger and larger and larger, though the entire process, but then all of a sudden you just sort of jump to the end at 1 [t=1], and then suddenly they're gone. So it's like well, if they're consistently getting larger at every step in the process, throughout the entire process, and then suddenly they're not there it's like wait a minute, they went *somewhere*…yeah you're taking marbles away but for any one you take away there's 9 [sic] more going in, so…I think it's like having a savings account, putting $10 in and taking one out, and doing this over and over again, and then having 0 at some point. You know it's like wait a minute…somehow that doesn't make sense.

15:50 Max: I'm fine with just saying there's nothing left, you know…nothing left in the account.

Tom: I mean, because on the one hand I think we have a pretty solid argument that there are no marbles left in the jar. You can take an arbitrary marble, or an arbitrary door in the hotel [referring to the Infinity Hotel he referenced during the Pre-test] and you can actually prove that there's nobody there, in general for any position. So therefore, there's nobody at every position which is therefore the same as the null set**.** It's a pretty solid argument but it's really not intuitive, that you would have a size that gets really really big, arbitrarily large, and then suddenly disappear, you know, when you get to the end.

16:55 I: When you say you have a solid argument, do you feel this is a rigorous proof?

Tom: if somebody asked me to write an argument for why there were no marbles in the jar at t=1 I think I could do it.[…] Whereas I don't think I could come up with one for why it should be infinity, because I have no idea where they are, because they're not in the rooms of the hotel.[…] My first intuition was that there were infinitely many marbles, but where are they, I mean which ones are they**,** I think would be the question. Because if you say marble 2371 is part of the infinity, no, it's not because it's been removed. You can say that about any marble. But I still have this intuition that if it's getting larger there should be infinitely many…

18:10 I: Does this remind you of anything in Calculus where something was happening in a specific way, and all of a sudden something changes abruptly?

Max: You mean, like a critical point on a graph, where something changes from increasing to decreasing?

Tom: Before, when you were saying another example of an unexpected change, I was thinking about the step function…that you're moving along at a particular number, a constant, and all of a sudden you jump to something else.

I: Yeah, something like that. Or you can picture function with a vertical asymptote where the graph approaches it from the left, increasing, and then to the right of that line, including the point where the line intersects the x-axis, the function becomes 0.

23:20 [I ask them to write a convincing argument summarizing their current solution for The Original 10 Marble Problem. Max is in charge of writing. They attempt to use a proof by induction, and are a bit unclear of why they need induction. Their P(k) is "marble k is not in the jar at the end". They assume that the rule "at step n we put marbles 10n+1 to 10n+10 in and remove marble n" is given, when in fact that is the only part that should be proven by induction, if any. Later we discuss that this formula was indeed given in the text of the problem, so it's really not necessary to use induction at all.]

40:00 [Start work on the Relabeled 10 Marble Problem].

Tom: Does it matters whether you have a time index or not?

Max: No. We just want to consider what happens when it's finished.

I: Let's discuss this, does it make any difference to have the time index?

Max: It wouldn't make no difference to me. We're just ignoring time and look at the last step. I just want to know what happens when it's done.

Tom: It's weird because if we don't explicitly state the time thing…because when we say it's done, it's like what does "done" imply? That we have a process that happened "before", "before" also being a concept of time. So…because if we don't have time explicitly stated, then really who's to say that this step happened after this one…

I: Well, there is a default order on the natural numbers, and based on that we know in which order the steps are happening.

Max: it's not going to change the end result so we can just ignore it.

Tom: We can just proceed without the time index until we run into a problem and if we don't, then we don't need the time index.

Max: Well, if we just study the process after it ends, look at the last step and determine what we have at that point….

Tom: There is no last step, because for any natural number you can add one and get…

Max: Yeah, I know that, I don't mean the last natural number. I mean the final state, not a last step among the steps described in the process.

48:24 Max: You are left with 10, 20, 30,… which corresponds to N marbles [he uses N to mean aleph zero].

I: Let's make sure you understand the problem as I meant it. Can you write down what you have in the jar after step n?

Tom: [writes a general formula for what happens at step n: 10n+1 through 10n+9 go in and n+1 has a 0 added] So when you put 91-99 in – that's when 10 becomes 100…

56:00 Max: So when you get to the 10th step, the 10 is going to become 100. So you won't have it in there no more, that's what you're saying. Ok, so that's wrong [what he was saying before, that you're left with 10, 20, 30…]. Ok. So really we're just going to have big numbers eventually. I guess eventually you'll still have none, cuz I mean every number is going to get multiplied by 10…

Tom: We can do n=99…

Max: Wait a second…I guess, nothing in there? We're going to end up with numbers bigger and bigger… so I guess what we really have…we got to have nothing in the jar again? I mean all the numbers are going to get bigger and bigger so we'll never have like a 1 in there, a 2… If you see 100 in there now, the 100 is going to be gone …[…] So I guess we have to say there's none… an empty jar.

Tom: I'm still not clear where all the marbles are. If you have marbles 1-9, 1 becomes 10, then you have 2 through 10, then you put 11 though 19, so now you have 2 though 19 but 2 becomes 20 so now you have 3 through … isn't it the same as…?

Max: Yeah, it's the same thing, remember that from the marbles [pointing to the Original 10 Marble Problem on paper]

I: The same as what?

Tom: Well, I remember writing out a few steps in the other marble problem, so for the first step…[softly - I'm getting the different operations confused] you have the first step and you put in marbles 1 though 10 and take away the first one. That leaves you marbles 2 though 10. But in this problem you have 2 though 10 as well cuz you put in marbles 1 though 9 and 1 becomes 10. The difference is before we were adding explicitly 10 marbles 1 though 10 and we just threw away one. Now we have marbles 1 though 9 and we just take the label off off 1 and put a 10 on it (or zero at the end). So it's a little like you're getting rid of the 1. It depends on how you think of it, like…because marble 1 becomes marble 10…I guess you can think of it as it's a physical marble that exists that's the same no matter what label you put on it, you're just labeling it with a different number. Or you could say it physically turns in…transforms itself from an entity that's marble 1 to an entity that's marble 10. So in a sense you're throwing out marble 1 anyway, I guess you could say … in the sense that marble 1 no longer exists as a marble that's labeled 1. So you still have 2 though 10, and I think at every each step it's the same number of marbles labeled… I mean after you made the transformation. So like if you're here then you have 2 though 10 and if you're here then you have 3 through 19… let me see, that doesn't sound right.

Max: 3 through 29… Hmm, at the end of 2nd step? 3 through 20.

Tom: I mean, the operation is defined a little different but I mean it's the same [as the Original 10 Marble Problem]… in the end, at the end of every step…like if you had a jar, right, that was the jar of the marbles for the last problem, and another jar with the marbles for this problem, if you were to do the first step for each one and then look at the marbles at the end of every step, it would be the same marbles with the same labels in each one at every step. So the answer to this is the same as the answer to this because they're equivalent … I mean the effect of the operation is equivalent.

1:01:05 I: How come we get an empty jar at the end if we keep adding 9 marbles at a time and we're not removing any?

Tom: It's because the relabelling is the same as removing marbles.

1:02:50 [Start work on The n->n+1 Label Problem, which involves a jar with an unlabeled ball, on which at the first step one puts a label with 1 on it, then at the 2nd step that label is removed and label with 2 on it is put on the ball, etc; the question is what the contents of the jar are after all steps are completed]

Max: [immediately] At the end? You'd have no ball. Well, you can't say what the number is so…

I: So no ball?

Max: You want the number on the ball?

I: No, whatever is left in the jar at the end. What are the contents…

Tom: No, I don’t really think there’s a ball there. I mean at every step of the process there’s a ball there, but I don’t think at the end there’s a ball.

1:04:35 I: Can you think of a way to rephrase the problem to get an equivalent one, in terms of putting in and getting out balls?

[They work together to rephrase it in terms of put 1 in, then put 2 in get 1 out, put 3 in get 2 out, etc.]

1:09:15 I: Ok, how about if instead of using the sequence of labels 1, 2, 3, and so on, we use 1, 0, 1, 0 for relabeling? So At after step 1 the label on the ball says 1, after step 2 the new label says 0, after step 3 the new label says 1, and so on.

Max: And you want to know what the end number is going to be? And we gotta say it’s neither 0 nor 1.

I: Is this problem equivalent to the other problem? [the previous one, The n-> n+1 Label Problem]

Max: The ball at the end is going to be neither 0 nor 1… well it’s either 0 or 1 but we can’t say which one.

Tom: I don’t think it’s equivalent to these because we never have a ball 2 here but we do have ball 2 here and we have a ball 2 there (referring to the previous 2 problems). So this one is… it’s like those sequences… the ones that bounce up and down…

Max: Yeah, like the sequence that goes 1 0 1 0…and then you have to say that the sequence doesn’t converge and the series doesn’t converge either.
So that means it’s not finite so there’s no solution…[…] so it’s neither 0 nor 1.
[…] Well, if it’s not 0 nor 1 it can’t be a ball…

Tom: I honestly have no idea. Because it’s not like we can… like before with the marbles, we could say this particular marble isn’t in the jar at the end cuz of the way the operation is defined. But here it’s like at the end you can ask is ball 1 in the jar? Well, you know, I can’t argue that it isn’t because ball 1 is labeled 0, but then whenever it’s labeled 0 it’s labeled 1 later, so I don’t really know of any argument to show that ball 1 isn’t in the jar or that ball 0 isn’t in the jar. Then, you know, by the same token, you can’t say that it is for certain cuz it’s always being relabeled.

Max: Well we know it can’t be neither 0 nor 1 because it’s one of those divergent harmonic series…

1:12:45 Tom: Maybe we can redefine the problem…that this problem is equivalent to asking …suppose we have some sequence where… a_n is equal to 1 for n odd, 0 for n even. What is the limit of a_n as n goes to infinity? I think that's equivalent, right? This is a problem we've done in 311, the sequence doesn't converge. I guess it's a little different cuz it's sort of like asking what's the term when n is infinity?

I: Wait, what's different from what?

Max: This one the process ends [referring to the n->n_1 Label Problem]… this time the process does not end [referring to this 0/1 Label Problem]. I mean it will end after the N [he means aleph zero] steps but you can't determine which one it is…

I: Well, both processes are of the same length, being indexed by the natural numbers. If one ends, shouldn't the other end as well?

Max: I'm just saying when it ends, you can't have either of these numbers on the ball. […] I'm saying it does end; neither of those numbers can be on the ball. It's like Tom's problem about the sequence a_n, which is not convergent.

Tom: I think with this problem we're asking what ball we have in the jar at the end. I think the only thing you can say is what's the limit of the sequence. But there is no limit. So then what would that mean …because if we were to say for this [ball pb] say hypothetically that the answer is that there aren't any balls in the jar, neither 1 nor 0 nor any other ball… you know, you could say that…because that is an answer, that is a definite state of the jar, I mean it exists, you could have a jar with no balls in it. But the sequence doesn't converge…it's not that it converges to 0 or 1…it doesn't converge, there's no condition or state that the sequence exists at when you do that.

1:17:15 I: Ok, so how does "the limit of the theoretical sequence does not exist" translate into the language of the jar and balls?

Max: I'd say that assuming the process ended and 1 is the ball, then do one more step and 0 is the ball, so really the process never ends so you can't have a 1 or a 0 on the final ball. Something like that?

I: Then, if it's not 0 and it's not 1, does that mean the jar is empty?

Max: It's not exactly empty but it can neither be 0 nor 1. Does that mean that it's empty or what?

I: Well, you can argue that there's no point in the process where it's anything different from 0 or 1, and you're claiming that 0 is not there and 1 is not there…

Max: So there's nothing there at the end.

I: Then you mean it's empty?

Max: Right, it's not 1 or 0 so then it's nothing.

Tom: Yeah, it could be. Yeah…right, I think you'd have to say that there's nothing in the jar.

I: Before [in the previous problems], whenever you claimed the jar was empty we argued that any ball/number was not there because that something was removed and then never revisited.

Tom: I was thinking…suppose, what sort of problem would we have to set up if we wanted ball 1 in the jar at the end but not ball 0? We could have a process where you have ball 1, ball 1, ball 1, like a constant sequence an=1 for all n. Or you could have ball 0, ball 0, ball 0 for all n, then you could say that ball 0 is at the end. So that's how we would know if there was a ball 1 or a ball 0 at the end, or you could redefine it so you had a ball being relabeled in such a way that you have a sequence that converges to 1 or converges to 0. Like you know, 1+1/n or something. But this…

I: Actually let's consider that. So how do you want to rephrase it?

Tom: Yeah, so maybe you have a sequence, you know a_n=1+1/n, and then you say that at every step the ball's label is 1+1/n for whatever n you're taking about. So maybe the first thing is labeled, you know, 2…and then the next step is labeled, uh, let me see…

I: Tom, is this problem you're proposing of the relabeling type?

Tom: Yes. So at the end you have ball 1 in the jar and nothing else, because a_n goes to 1 as n goes to infinity.

Max: So you're saying you're starting at 2 and going to 1? Well in that case you'd end up with 1, would be the last, yes. But that's not related to this…

I: Wait, do you both agree that "in the end" we have a ball labeled 1?

Both: Yes.

I: Well, do you have a label with 1 on it in the set of labels you're using in the steps of the process described by the problem?

Tom: No, not in the process. But the limit…

Max: When you take the limit to infinity you get 1.

Tom: The end of the process represents the limit itself of your sequence, of your process of relabeling. Of course, you know, relabeling goes on forever, but you know, after you're done with that you get to 1. Of course, 1+1/n is never equal to 1 but it converges to 1.

1:27:33 I: Are you equally comfortable saying that a ball with label 1 is in the jar at the end as saying that 1 is the limit of the numerical sequence 1+1/n?

Max: Yes, very comfortable.

[Tom nods yes.]

I: Ok, let's go back to comparing Tom's pb with the 1 0 1 0 … problem.

Max: In this case (1/0 case) there's a marble in the jar, it's either 1 or 0 but it's not 1 or 0 so it's really like a paradox, I don't think it really has a solution, you can't really say that, because it's either 1 or 0 but it's not 1 or 0, so…that's about all we can say.

I: What happened to the "empty jar" idea from before?

Max: It's not empty. It's never empty cuz it's divergent, so it's never gonna be empty.

Tom: It's like…because I was thinking, maybe the reason I thought of this example was ok, so if we're considering the possibility that there no balls in the jar, what would be a condition where we could say that there's a ball in the jar, like what would we have to have in the problem in order to be able to conclude that there is a ball in the jar. Because if we have infinitely many steps, then that means in order for a particular ball to be in the jar at the end, you would have to have some sort of sequence that converges to that ball, to that number. So then if we say that you can only say that a ball is in the jar in the end after infinitely many steps if you can find a sequence in the steps that converges to that number in some fashion, then you can say that there is no 1 or 0 in the jar in this problem because there is no sequence…because your sequence a_n does not converge to 1 or 0.

I: Again, if no 0 and no 1, does that mean it's empty?

Max: It's not empty…[…] there's a ball in there but it's 1 or 0 but it's not 0 or 1.

I: Max, do you mean that there's a ball in there but it's not defined what label is on it?

[End of Session 1, File A]

Session 1, File B
4/6/08

00:00 Tom: You can think about the process in two different ways. One, that a physical ball is in the jar and you keep reaching in and changing labels, thus never taking out the ball. Two, that at each step you take the physical ball out of the jar, relabel it, and put it back in the jar. The 2nd view makes it easier to say that the jar might be empty at the end.

00:55 I: Ok, let's recap the possible answers so far: one is "a ball in the jar but we don't know what label is on it"; the second is "the jar is empty. Any preference for one over the other?

Tom: I am more inclined to say there isn't anything in the jar, although it's also possible that there is no solution.

I: See, before we had a situation with one ball in the jar at all times (n-> n+1 Label Problem) where you were comfortable with "no ball in the jar at the end", so the fact that there's one ball in the jar after each step does not necessarily mean there has to be one in the jar at the end.

Max: Right, 'cause since this is divergent, there is no defined end point…the ball's still there.

I: Ok, final preferences.

Tom: I would say there isn't anything in the jar.

Max: I want to say there is a ball in the jar [with no defined label on it].

7:00 Tom: [pointing to n->n+1 pb] this problem…you can define a sequence like we did for these 2 and say that you know, at step 1 the label is 1, at step 2 the label is 2, at step 3 the label is 3… so you just have an = n which is a divergent series…sequence rather…so you can say that if you can redefine the pb in terms of sequences where the sequence models every step perfectly and the sequence is divergent then there isn't anything in the jar, maybe…

Max: Well, there are two types of divergence… one went through the natural numbers, the other one oscillated between 2 numbers… so the situations are not similar.

Tom: I'm just noticing that in both cases we're saying that …I mean really what's happening is you know here you have a sequence 1, 2, 3,… that diverges and this sequence also diverges…

I: Ok, let's leave it at that for now.

[End of session]

Session 2, File A
4/20/08

00:00 [Start work on the first problem, which is recap from last time. We start with the n->n+1 Label Problem. They read the problem individually].

Tom: Right, so we said nothing is in the jar at the end, as for any n ball n is removed at step n+1. [Max agrees]

4:20 [Moving on to recap Problem no. 2 on the sheet, the 0/1 Label Problem which is now phrased using 1/2 instead of 0/1. The students take some time to remember the problem.]

Max: Well, there's a marble in the jar but you can't say which one it is.

Tom: I remember us going back and forth forever on this problem and not getting anywhere. […] At every step there's a marble, that's true…so if there's this marble at every step in the problem going on forever then I think it's reasonable to conclude there would be a marble in the jar except, you know, the sequence 1 2 1 2 1 2 … is divergent so I guess it depends on what conditions we have for what it means for a marble to be in the jar when the steps are finished. I think that's what led me to do the example of 1/n [he means 1+1/n] , you know, with the relabeling…Like, do you have to have some sort of convergent sequence for your labels in order for a marble to be in the jar at the end, or is it possible to have a marble in the jar at the end even if the sequence that you're using is divergent…which does not really make sense to me that that would be the case. I don't know…[I: if a sequence is divergent…] then I don't think there could be a marble in the jar at the end that has a particular label on it that we could say that's what it is. I could only really see a marble being in the jar with a label that we know what it is if there's some sort of sequence that converges to that. That's my intuition at least.

Max: Yeah, there's a marble in the jar but it's… we don't know what the label is.

7:15 I: Remember the previous marble problem in which we put 10 in, we remove the lowest number, for which you said "the jar is empty at the end". So there we had an ever increasing number of marbles in the jar and still we had none at the end, so are you sure that "one marble in the jar at all times" means "one marble at the end"?

Max: in this case there's still a marble in the jar…in the other case you're going through all the numbers and you exhaust them but in this case you just switch between two numbers.

I: Do you mean it matters which marbles are in the jar…not just the fact that I always have something in, but *which* ones I have?

Max: Yeah.

9:37 I: So, do you guys agree on what you want to write?

Max: It's either 1 or 2, but it's not 1 nor 2. That's the only way you can really answer it.

Tom: That's actually a contradiction…

Max: That's a contradiction but that's what it is.

Tom: Actually I'm beginning to think that there's nothing in the jar.

Max: Oh, there *is* something in the jar… You just don't know what number it is then…it's either 1 or 2 but you don't know which…it's undeterminable also, you know what I mean? […] You can't really finish all the steps…

Tom: It's a little bit different than the problem we had before about putting 10 marbles in and taking one out because…we may have already said this but it's progressive. You know, so like once a marble is removed it's not put back again so we can forget about it. And the marbles that are removed…so marble 1 is removed, marble 2 is removed, marble 3 is removed…they're sort of next to each other. Whereas in this problem, when a marble is removed it's put back in again. Like I couldn't say there's nothing in the jar because marble 1 is removed or marble 2 is removed because at some point later in the process they're both put back in at some point. It's sort of difficult to say there's nothing in the jar because they're being put back in again at other steps. So…

11:50 Max: It's not empty. There's a marble in the jar…

Tom: I'm not 100% certain it's not empty. It could be empty I'm just not… I don't know that I could prove that. It wouldn't surprise me if it turned out to be empty but I don't know…

Max: It can't be empty

Tom: I mean, my intuition about the first problem with the 10 marbles and taking one out was that that couldn't be empty either but it turned out to be so…

Max: That's different

Tom: I mean my intuition was wrong before so we'll see…

Max: In that case you would run out of numbers…in this case you always have 1 or 2, they don't go nowhere so…
Tom**:** There's a possibility we've been ignoring. They could both be in the jar. Because the thought just occurred to me, you know you could take… so if you have a sequence going 1 2 1 2 1 2 etc, suppose we took a subsequence that was just the 1 position, so 1 1 1 1…, and you took another subsequence that was 2 2 2 2 2…[…] then you could say well you have a subsequence that converges to 1 and a subsequence that converges to 2, so maybe you could argue that both of them are in the jar at the end.

Max: I don't think that they are both in the jar. I think there's just one marble in the jar.

14:03 I: Ok, so how do you want to leave it for now?

Max: One marble whose label we don't know.

Tom: I'm least comfortable with both in jar; equally comfortable with 'no marbles in the jar' or 'one with undetermined label'.

I: Are you 100% sure that "the answer" is one of the 3 mentioned answers?

Both: Yes.

17:35 [Recapping the "1+1/n" Marble problem.]

Max: Marble 1 is left in the jar.

Tom: I remember discussing this in connection to the second problem. Yeah, I remember being particularly comfortable with this one. I remember I gave this as an example to explain my thinking … I sort of gave this as an example because I was sure in my own mind that it would be 1 so like this would be an example of a problem where you could say that marble 1 was in the jar at the end or something. I was trying to formulate my idea that to have a marble in the jar you have to have a convergent sequence to that number.

I: Ok, I think we recapped enough. We'll work on some new problems today.

21:00 [Start work on the writer question.]

Max: [right after reading the question] I guess you could do that.

Tom: Let's look at how long it takes the writer to write his biography up to the age 40 (the age he is when he starts writing).

Max: we're not worried about how old he gets, right?

Tom: Well, at some point he's going to die I mean…
Max: Are you sure?

I: That's not stated in the problem, so it's up for debate.

Max: I would say…eventually you're going to have … since the sets are equal…

I: Which sets?

Max: For instance you go like 1 week, 2 weeks, 3 weeks, 1 day, 2 days, 3 days…they have the same number of elements in each set so…

I: I see, same numbers but the unit of measurement is different.

Max: right, but that does not really matter. It's just like the set of even natural numbers is the same size as the set of natural numbers, so I would say the same thing… a week is the same as a day if you just keep going.

I: If you keep going in what way?

Max: … in the set… [gestures indicating a one to one correspondence between the set of days and set of weeks].

23:17 Tom: If the guy lives long enough to finish documenting his first 40 years, he would then be writing ABOUT writing his biography, which would be kind of weird.

Max: So you could complete it…

[Tom starts writing down the information in the problem using a time line. He then starts calculating how long it takes to record the events in a year, and then in the first 40 years of the writer's life.]

27:42 Max: But we're not worried about his age

Tom: Yeah, I guess we're assuming this guy is immortal or something, or at least like a turtle or something…

28:18 Tom: No, I don't think that he would be able to finish his autobiography, I mean if you require him to finish his biography including past the age 40 up until and including how old he is currently because what we have is… I mean it takes a week to finish the events of a day. So let's say at some point you were… his autobiography was a complete record of his life up to 1 minute prior to finishing. So let's say he was a particular age and he had his whole biography done except he had to write down the last minute of his life and then he would be finished with his biography. Well, the way it's set up it's going to take longer than a minute for him to write down the record of that minute. So he would then at some point be farther away than he was at the beginning. And you can say that no matter how small the increment of time is. You can say well what if he's a second away from finishing his biography, it's still going to take him longer than a second to write down the events of that second. So I don't think you'll ever going to be able to …

29:55 Max: I'm just saying if you look at the days… 1, 2, 3, and if you look at the weeks or whatever you do… you can say the weeks is days so you're going 7, 14, 21…

I: Why 7, 14?

Max: I'm saying for weeks that go by, like 7 days in a week, I'm just using like the same unit for each set… the first set is the number of days, the second is the weeks, so I'm saying like 7 days, 14 days… So I'm saying since each of these sets has the same number of elements, that it could be completed.
[…] Cuz like you can always say like, for each number that I write, you can always have a number here (pointing to the 7,14… set), so you can keep going…I'm saying it can be completed.

Tom: So for any day in his life, there's a particular amount of time…so it takes him…so this is cumulative, this second line. So this would be, like if you had a complete thing, this would be useful cuz at any day, you can just say at the 365th day, it's going to take him a total amount of this time to do it.

32:45 Tom: I'm sorry, what was your argument again?

Max: So what I'm saying is that like…so you go 1, 2, 3 days, and you have like how long it takes to write about those days. What I'm saying is that since…if you have 4, 5, 6 million in a set…for every number you're going to have a number down here (on the line indicating how long it takes to write about those days), so you always have a number go with it, so that you always have that these sets have the same number of elements, is what I'm saying, therefore it can be completed.

Tom: Ok, so this would require him to live … He would have to live to an infinite age. He would have to be immortal to do this, but if he were immortal he could do it because it wouldn't matter that it takes a week to write a day. So, right…I guess I was thinking that this person we're talking about would have sort of a normal life span… I was thinking very practically minded, like if you really had to write you own biography clearly this wouldn't work out. Yeah, I mean so this is a little bit like the first problem with the 10 marbles and taking one out in that… you know you could argue that for any day in his life, there is a point at which he would finish that day ("finish" in the sense of "write about"). So therefore, he would finish every day of his life at some point or another, so he would finish his biography.

35:41 Tom: This is sort of like the first problem with the marbles because you could argue it's sort of counterintuitive that there wouldn't be anything left in the jar… if it's an ever increasing amount of marbles in the jar then it seems like if you're trying to remove every marble it's like you're losing the war because every time you do that, 9 more marbles appear in the jar. Nevertheless, you do actually remove all of them at the end. This seems like the same sort of thing that you know, you're losing the war in the sense that you know, every time you get a day done, there's, you know, you've just lived another week so that's 6 more days; so, you can think of it that way, that every time you finish a day 6 more days pop up. But, you are finishing every day successfully. So, you can say that this is the same sort of problem as the problem with the 10 marbles and taking one out.

36:47 I: I see. I have a question about this. How do you feel talking about the point "when all the days in his immortal life are over"? Before [in The Original 10 Marble Problem] we assumed each step took half the time of the previous step; does the current context gives them more trouble?

Max: It's not a problem for me.

Tom: It's really counterintuitive. Obviously he's never going to finish it in a finite amount of time. But if you allow this process to go on forever, then you could say, you know, in the limit, he finishes the book. So the only situation would be if you continue this process forever, then after you're done with this process, you know taking your limit, the book would be finished.

38:57 I: How do you visualize "the point" at which the book is finished – any specific mental images?

Max: I just think "go to the end". I only see the start and the finish. I don't worry about what happens in between. I see the finish as he completes it. I'm just talking like…the 2 sets and say it's gonna finish. I don't really have a mental image.

Tom: Like after you've finished, so you know, this person being immortal wouldn't have died. So in principle you would think you could find this person and talk to him, but I don't think that you could because we're not talking about a particular point in the process, we're talking about, you know, the limit of the process, after you're done with your infinitely long period of time. So what you would have then is a situation where his whole life would have been lived and documented so, you know, I'm sort of like… How do I explain this? I'm thinking that, you know, if you visualized this guy's life as sort of like a timeline, and you said this is his current age, and this is how long he's lived, the period of time that he's both lived and documented, and this is his whole time, this is the period that he's still trying to document at a finite point in the process…then this would just continue this way, and if you take the limit then you just have you know, a line, an Euclidean line that's infinitely long (drawing a line until the edge of the page)…and then you say that the entire thing is you know just like that (extending the marking that indicates those days have been documented until the end of the line). And then you can ask "what's the current like…what's the present?" So you could say this is the present time here (pointing to a specific point on the line). But here there's no present time. So if there's no present time, if you were like searching the universe at any time you wouldn't find him anywhere in there because there's no present time at which he exists.

42:07 I: Can you point on that drawing where we are, assuming we're at a point where the biography is completed.

Max: [Max points to the right end of the drawn line.] I say we're right at the end.

Tom: Having the process completed means having a full array drawn but it has no final point and we, at the point where the book is completed, wouldn't be on the line.

48:00 [Start work on the Midpoint problem. The students take some time to read the problem individually]

Max: I'm just thinking this is kind of like … decimals…kind of like when you go to a decimal… an infinite number between any 2 decimals that's bigger than the natural numbers… I don't think you could say…

Tom: I don't know…so, first …because, if we say first of all that in the first process of…you have a line and your line segment and then you take your midpoint, and then the midpoints of those 2 segments… it seems intuitive to me that at some point you grab every point in your entire segment.

Max: I don't agree with that at all.

Tom: Even after infinitely many [steps]?

Max: Right…between any two points you still have like an infinite number of …[…] But with decimal numbers, you never have that finiteness… that's the difference, it's a larger size, right? It's greater than the natural numbers so you can't count them all. This infinity between 2 decimal places…

Tom: The interval on the number line for 0 and 1…

Max: Like, you can make A and B 0 and 1….

Tom: A and B are never going to be in this set… because A and B are the endpoints for your line segment… so there's no point in this process at which A would be a midpoint. So, it would just be … after you're done with this process you would have your points A and B and then you would have like nothing in between.
I: Do you mean every point reached by the process is removed from the segment?

Tom: Yeah, so you go through the process described in the first paragraph and you know, you remove every midpoint and you put it in set M. Then you're going to have this point is removed, this point is removed…like all points are removed. So after you're done with this process….my idea is that you would remove every point in the segment at some point except for A and B.

53:40 Tom: the open interval (A, B) is actually equal to M, I'm pretty sure.

Max: I guess you could say that… if you really finish the process, which is hard to do.

I: Why is this one hard to finish? You "finished processes" before, right?

Max: Well, that was natural numbers… I'm not sure with decimals numbers you can finish them… can you?

I: I am beginning to think you use the term "finish" in a different way than I do, so let me clarify. Do you use "finish" to mean "reaching all points on the line segment". When I say "can you finish", I mean "can you see this process as completed?" or "can you think at the point after which steps have been performed?" Is imagining "the end" more difficult in this problem than in the previous problem?

Max: Yes.

56:05 Tom: I'm noticing… if you say that A is 0 and B is 1, like the interval from 0 to 1 on the real number line, and you perform this process, all of the numbers you are removing are rational numbers. Right? Because first you remove one half, then you remove ¼ and ¾…

Max: Well, what about sqrt(2)/2? That would be in there…

Tom: Yeah, that is my concern…

I: In there where?

Max: Between 0 and 1…

Tom: So if you're really removing every point, that means at some point we would have to remove sqrt(2)/2, which isn't a rational number but it looks like every number that we remove is a rational number… if, you know, you're from 0 to 1.

Max: Right, the fractions…well, that's because the fractions are the same size as the natural numbers, I think… so we gotta look at all fractions…
Tom: Maybe it could do that ["remove" sqrt(2)/2] in the limit? I don't know…

I: Are you removing ALL rational numbers?

Max: Well, wait a minute, we're just dividing in half, so we are JUST doing fractions…

57:45 Max: Yes, we can complete the steps, and M and T are equal in size. […] I'm saying that M and T have the same number of points. It's kind of like the last problem where we have like the sets and the sets are both you know the size of the natural numbers since we added the fractions…

I: How do you know that both of the sets [of midpoints and tripoints, respectively] are the size of the naturals?

Max: It's kind of like before, because you have like 2, 4, 6 is the same as 1, 2, 3, you know what I'm saying, like the sets have the same number of elements. They're both the size of N. It's kind of like these two sets of fractions are the size of N. We're going to have that M and T are equal in size.

Tom: I was just sort of trying to list out the steps for the case of the 0, 1. And I realized that 1/3 is sort of missed… as well as 1/5. Because at the first step you get rid of ½, and at the second step you get rid of ¼ and ¾, at step 3 you get rid of 1/8, 3/8, 5/8, 7/8… Like I ignore repetitions… so at step 2 you would also be removing 2/4 but that's ½ from the first step so I ignored it.

Max: So it would be all fractions.

Tom: Well, not all fractions cuz 1/3 isn't in this process anywhere. […] Because 1/3 isn't going to like appear on one of these steps, because the denominator of a fraction is a power of 2. […] So I don't think there's any numerator you could , like… so I don't think 1/3 is going to be a number in this [in set M].

1:02:45 I: Can you formulate a convincing argument that 1/3 is not produced by the process?

[Tom is checking, in writing, that it's not possible to obtain 1/3 at any step of the process by showing that the equation (2k-1)/(2^n)=1/3 has no solutions in the natural numbers.]

Tom: So you wouldn't ever have 1/3 removed at any finite step. […] Not at any finite time. I'm thinking… is there like maybe in the limit? Maybe if I were to pick specific numbers at each step maybe I could construct some sort of sequence that converges to 1/3, potentially.
Max: That's true, sure. You can absolutely do that. In that case M and T are still equal.

I: Max, why do you think that Tom is interested in knowing whether he can construct a sequence out of these "midpoints" that converges to 1/3?

Max: I'm not sure how he's trying to go about it…

Tom: I'm trying to… so, my initial sense was that every number was removed after A and B. Because it made intuitive sense to me… so you would get something like this, right? You know after your first step. [he draws little circles around the midpoints already constructed, to indicate "holes" left in the segment if those points are removed]. And you would get these would be removed, and then these would be removed… If you keep continue doing this forever, just if you were to look at these visually…

I: It looks like there's more and more holes, right?

Tom: Yeah, if you do this you're gonna end up with infinitely many holes, so I was guessing maybe at some point you get this, you know? But then what this means is that every point here is removed, you know. So this is a line segment. So then to quantify that we can just do the interval between 0 and 1, cuz I don't think it particularly matters what our endpoints are. We could do -3 and 5 but I don't think it would really change out answer. Just for simplicity we do 0 and 1, and now I'm trying to make sure that every number is removed between 0 and 1. So we confirmed that 1/3 is never removed in any finite step, which means that if every point is going to be removed that means that there are going to be a lot of points that aren't removed in finite steps that have to be removed in the limit. Yeah, so maybe 1/3 could still be removed like at that infinity point, like in the limit.

1:10:20 I: Ok, so Tom claims it is possible to construct a sequence of "midpoints" converging to 1/3. Is the same true for sqrt(2)/2?

Max: Ok, so that's an irrational number. The answer is… I'm going to say no. […] That's right. Because that's an infinite decimal.

I: What's an infinite decimal? You mean a non-repeating decimal?

Max: Right. That means that even if you picked a point that came before it, there's also an infinite number of points in between that and sqrt(2)/2, that has more elements than N, the set of natural numbers.

I: Well, 1/3, in decimal form, is also an infinite decimal, albeit repeating, so what makes the 1/3 situation different from that of sqrt(2)/2?
[Max is having trouble remembering the rule for conversion of repeating decimals into fractions. I fill in Tom on our discussion. Max reinforces that indeed sqrt(2)/2 can't be a limit to a convergent sequence of midpoints.]

1:14:20 Tom: I guess it is less intuitive that sqrt(2)/2 would be removed because it's an irrational number but 1/3 is a rational number. But actually I think I have a process that might remove every point. I'm not… So what I was thinking was, like just for 1/3, so 1/3 is between ¼ and 3/8. So I took just that part of the line, from ¼ to 3/8. Ok, so 1/3 is between 5/16 and 3/8, right here. So you could take just that part of the line, you know, from 5/16 to 3/8. So then I'm not sure…so then you take a midpoint right? And 1/3 is either in this interval here or this interval here. I'm not really sure which one it is, but it's going to be in one or the other. So let's say it's in this one [chooses one of the segments]. So then you would take this interval, 3/8… so this would be I don't know, whatever you want to call it. And then… now there's an interval here, 1/3 is either here or here. Let's say it's here. […] If you choose the left endpoints of these chosen segments that 1/3 is in, you get a sequence of midpoints converging to 1/3.

I: Can we use the same idea to converge to sqrt(2)/2?

Tom: Yeah. That was the idea, that in principle this should work for any number on your number line. Or at least… I don't know, I guess 1/3 is kind of special in that it's not any particular number in a finite step. So let me see if it would work for ½. If you have like your set of all numbers on this interval between 0 and 1 that aren't removed in a finite number of steps, right? So like 1/5 would also be one of these numbers, I'm pretty sure. Sqrt(2)/2 would also be a number. Then you could just use this process to construct a sequence that converges to that number.

Max: Yes, you can converge to that number.

Tom: So then you could define a sequence that converges to your number, so you could argue that in the limit that number is removed from your interval.

Max: Yeah, you can converge towards a number, but you can't get to the next one before that number… But you're still going to end up with M and T having the same number elements. […]. You can converge to any number, true, but you can never get to the number that is before the number... you can never get…

1:19:20 I: Ok, let's recap. You both agree that we can find a sequence of midpoints that converges to ANY number between 0 and 1. Tom claims that because of this, we need to add that randomly chosen number to set M, say that it is part of it when the process is completed.

Max: If that's true, then directly M and T are equal.

1:25:33 Max: Right, cuz you get your number in your sequence, all that you do before that irrational number, but then you could always get an infinite number of points between them still, and the size of the numbers in between is greater than the natural numbers, like the number of the natural numbers. So it's like you have the natural numbers and you have that thing that's over here that's like bigger.

I: Do you mean we shouldn't add irrational numbers to set M?

Max: I am not sure if you can do that. I mean I never counted that… Cuz you see, I never knew how to look at that. Cuz I don't know how to imagine it, like when you go beyond the natural numbers, cuz that's a whole other…that's different, you know? I don't know how to exactly consider that. But I want to say I don't think you can have that point in there, that irrational number. You never get to it, you know what I'm saying? I mean you can go in a direction towards it, but whereas you go towards it you still have that infinity between them.

1:29:20 Max: Alright, if you want to know what's in M, if an irrational number is actually in M… it depends on whether you can finish the process when we're dealing with that larger size of infinity. I guess it's how you consider that, I guess you could say you could finish it. Yeah, it's a different size but I guess you can still finish it. I would say… I guess you could have the irrational numbers in M.

1:30:22 Tom: I was thinking… well, two things occurred to me. One, that if we say that if we have a sequence that converges to some point like 1/3, then that being the case that means that it's in M at the end of our process, so if we say that then I think that A and B, the endpoints, are also in M. Because I could say that, you know, 1/8, 1/16, 1/32, 1/64… is a sequence that converges to 0, and 7/8, 15/16, and 31/32, that's a sequence that converges to 1. So I think logically then A and B…like if what I am saying about the sequences converging is true, then 0 and 1 have to be in M as well.

[End of Session 2, File A]

Session 2, File B
4/20/08

00:19 Max: I guess you still have for both times the open interval A to B, therefore they're equal [referring to sets M and T]. Including the irrational numbers.

I I: Ok, so you want to include the irrational numbers in M?

Max (forcefully): YES!

I: Ok, how about T, do you include irrationals there too?

Max: Yes.

Tom: I guess I got very into my own thinking of the problem in terms of removing points and putting them in M. I was looking at what the problem actually says about being midpoints. I was just thinking that for 1/3 to be in M, what that means is that at some point is going to have to be a midpoint of some interval. But I think we've already shown that actually in particular, 1/3 is never a midpoint in a finite number of steps.

03:00 Max: I'm just going to say that M and T are equal and they are both the open interval AB.

I: Max, I am not sure you heard what Tom was sating; he claimed that A and B are "limit points" as well, meaning we can find a sequence of midpoints to converge to each of them.

Max: Oh ok – the closed interval AB [describing sets M and T].

I: Which is the whole thing?

Max: Yes! Wait… 0 is a midpoint? (twice)

I: Tom already said that 0 is not a midpoint. But I thought you were including… you seem to be including some numbers because they were limit points.

Max: I'm including them because you're gonna have every number between A and B. But the thing is you're never gonna have exactly 0.

I: Well, you never really reach sqrt(2)/2 either, but you included it [in M and T] as a limit point. And 0 and 1 are limit points too.

5:13 Max: Those are the only options? 0 is not going to be a midpoint. […] when it's done…see, that's the thing, that's what I'm saying: it's harder to think about these decimal numbers. I am not sure how you're supposed to do that.

I: Well, sqrt(2)/2 is not a midpoint either. Do you want to include it in M?

5:55 Max: No. I think we have… regardless, the sets are equal sizes. I'm going to say… Maybe we can just say, you know…we're going to have a set of 1/2^n [for M]… I don't know how to do this, I don't know how to write this. Something like that, you know… then you're going to have 1/3^n and so on [for T], you know… I'm not sure that's the proper way to do it. And then we get that they're the same sizes.

I: You wrote only fractions in those two sets, does that mean you are not putting irrationals in any of the sets now?

Max: No, I am not. I am not. No 0, no 1, no irrational numbers. It's going to have these numbers in the set, and they're equal sets he means sets M and T are equal in size]. Kind of like a correspondence…

9:46 Tom: I don't know… . I'm conflicted. I'm beginning to think that maybe it's… because either A or B, they're never really midpoints, because to be midpoints there would have to be some point out here [points to the outside of the segment]. So A and B could still be in your interval… like *not* in M at the end of the process conceivably. And you would maybe have 1/3 here [not removed from the interval]. So maybe 1/3 might not be in M. I don't know… I'm having some second thoughts about this.

11: 30 I: If you wanted to use set notation to represent what is happening in this problem, could you do that?

[Tom writes $M_0 = \varnothing$, $M_1 = \{1/2\}$, $M_2 = \{1/4, 2/4, 3/4\}$…, with the convention that M_k represents the set of midpoints constructed by the process up to and including step k.]

I: So for M_k, what would be its elements?

Max: Let's write M_3 too, to make it easier to recognize the "pattern" for the general term.

[Tom then writes $M_k = \left\{\frac{1}{2^k}, \frac{2}{2^k}, ..., \frac{2^k - 1}{2^k}\right\}$]

15:20 I: Ok, so how do you define M, the set produced by the completed process, based on the individual M_k's?

Max: Why don't you do the union of all those sets, M_1 to M_k?

Tom: Well, we're defining it so that M_1 is a subset of M_2, M_2 is a subset of M_3, M_3 is a subset of M_4… so we can just talk about M_k. Like M_k contains all of the M_i's up to k. So we can ignore all of the sets up to and including M_{k-1} cuz M_k contains all of those already.

Max: We need the union, right? M_1 union M_2 up to M_k?

I: The thing is all steps have been completed, so what do you mean by “up to k”?

Max: what’s wrong with the union?

Tom: Just that it’s… The union of M_1 and M_2 is equal to M_2. The union of M_2 and M_3 is equal to M_3. So, we don’t need to include… we don’t need M_1 or M_2 or M_3… like if I wanted…

Max: how about we say k is the limit of …? No, we don’t want to say that… How about k is defined as max(size(N)), like in Matlab. [by N he means the set of natural numbers].

19:00 Max: Can we define k like k=1 to infinity? […] How about we just write the union of k equal 1 to infinity? We know we’re repeating stuff, but at least we get everything in there. And you know, even if we repeat stuff we still got the last one in there so we got everything covered. And it gets the job done.

I: If we do it this way, do you think this way of representing the problem, do you think that’s accurately describing the situation in the problem?

Tom: [faintly] Yeah.

[Max nods yes in a convincing manner.]

I: If it is, do we have irrational numbers in M?

Max: No, it’s all fractions.

Tom: Well, it’s a set of fractions but not all fractions. It’s fractions of the form 2… let’s see how we can define this. I mean we have a definition of what sort of fractions…

I: Is it possible to refine the definition of M further so that it doesn’t include an infinite union?

Tom: Starting from the format we derived here [pointing to the part of the paper where he was earlier solving an equation to decide whether 1/3 is a midpoint…

[More discussion about the proper way to represent M in this way – some details here but not that significant, not related to infinity. Max writes
$M = \left\{ \frac{n}{2^k} \mid n, k \in \text{N and } n \le 2^k - 1 \right\}$.]

29:15 I: Ok, so if that’s M… are you guys entirely sure that this is M? So no more irrationals, no more 1/3…

Tom: Yeah!

Max: Absolutely!

I: What about T? Would it be hard to write a similar expression for T?

Max: We just replace 2 with 3?

I: That's all we do?

Max: Yup. They're still the same size! [referring to M and T]

[Tom starts writing out T_0, T_1, T_2, then writes out T_k as $T_k = \left\{\frac{1}{3^k}, \frac{2}{3^k}, ..., \frac{3^k - 1}{3^k}\right\}$ and $T = \left\{\frac{m}{3^k} \mid m, k \in N \text{ and } m \leq 3^k - 1\right\}$]

31:20 I: So then, if we get to the main question, which is which one is larger…?

Max: They're the same size.

Tom: This strikes me that they should be the same size…
Max: It's kind of like… pair them up…

I: How can you pair them up? Can you find a specific pair-up?

Max: Sure. This one with this one, this one with this one… [points to paper, not clear which pair-up he's referring to].

I: Can you explicitly define a function from M to T that explains the pair up?.

[They start work on that and don't find a function right away.]

Max: They're still the same size.

34:23 I: So Max, I am interested to know why you're so convinced they're the same size, given that right now you don't have a specific function to show one to one correspondence.

Max: Well, you could come up with a function I guess, because you just start from the beginning and you just start matching them up.

I: What do you mean by "start at the beginning"?

Max: I’m just saying you can match the first one of M with the first one in T and then keep going.

I: That implies, or that assumes that we have a specific order in which we’re listing the elements of each set. Which order would that be?

Max: We kind of wrote out both of their orders [pointing to the formal expressions of the 2 sets, which do not involve infinite unions].

Tom: Right, except that we might run out of one before we run out of the other…
I: What do you mean run out of one?

Tom: Like, with the k… I think we can agree that it’s the same k in M and T, right? I mean… I don’t know. In the sense that k is an arbitrary natural number that does not have any restrictions on it. 3^k-1 is greater than 2^k-1. So… the denominators… like if it was just 3^k and 2^k, then yeah you could do a bijection. You could do 2^1 goes to 3^1, 2^2 goes to 3^2. So that’s not really the problem. The problem is n here and m there. Could you do a bijection from n to m? It’s like, couldn’t you just do 1 to 1, 2 to 2, 3 to 3, whatever…except that, that’s why we said 3^k-1 greater than 2^k-1.

I: I’m not sure that was the kind of pairing that Max was suggesting. I thought that he was suggesting a different one. He was saying, we have a specific order in which we list them. Right? By what order are you going to list them, Max?

Max: Well, I was doing like what I did with the other problem. You know, just start with the first elements, and keep pairing them up…
I: Do you care to list the elements? You said the first elements, the second elements… can you list them?

[Max starts listing some elements for M and his elements have a fixed k in them.]

I: What is k, as we are talking about set M at the end of the process, so how is k defined?

[He writes N={1, 2, …, k}]

I: Tom, do you agree with that?

Tom: No, as that defines only a finite step.

40:23 I: Maybe Max does not use k to mean a regular natural number but “something at the end of” the natural numbers.

Max: Yeah, that’s what I want to do.

Tom: Oh… I'm pretty sure that's a contradiction. I'm not exactly sure I remember the proof for why that doesn't work but … I remember 311…

Max: I know, cuz you say n+1, bam, you got another one… But what I'm saying is we don't gotta go that far. Oh, you're talking about this notation…well, forget that notation. Let's just say… M = T. Well, not necessarily… I don't mean that, I mean like M and T have the same number of elements.

41:22 I: Let's recap. Max claims there is a way to describe a pairing between M and T by pairing first with first, second with second… I'm asking in what order are elements in M and T listed so that we can clearly see what the first is, what the 2nd is, etc.

Max: Well, starting from A and towards B, list all the midpoints.

I: So, we're listing them in increasing order, for left to right?

Max: That's right.
I: What would be the first one?

Max: [mumbles, scribble something, pauses] You can't do that…

Tom: I think I'm beginning to realize what you're saying…

Max: Ok, yeah, they are the same cardinality then, but that cardinality is different than like the natural…oh, they're not, I'm doing fractions here. The fractions… that is equal to the natural numbers. Ooooh…see, they're both sets of fractions.

I: But they're not *all* the fractions, right? So the fact that some of the fractions are not in these sets has no effect on the cardinality?

Max: No, it doesn't.

Tom: So M and T are both sets of rational numbers, but neither of them is finite. So M is an infinite set and T is an infinite set, but they're both strict subsets of the rational numbers. So the cardinality of M has to be bigger than the cardinality of any finite set, but its cardinality can't be bigger that the cardinality of the rational numbers. And the same can be said for T. I think that… I mean there's no cardinality that is between a finite set and the rational numbers. Right? That a set is either a finite set or is equivalent in size to the rational numbers or is greater than the cardinality of rational numbers. […] so the cardinality of M is less than or equal to the cardinality of the rational numbers and it has to be greater than the cardinality of any finite set. And you can say the same thing for T. So my point is that M and T are both … are having the same cardinality of the rational numbers by process of elimination. Cuz if it's not the cardinality of the rational numbers then it has to be finite. But we've already said that it's not. So since the cardinality of both of these things is that of the rational numbers, then, you know…

Max: I agree! You explain very well [addressing Tom]!

[End of session]

Session 3, File A
4/26/08

00:00 [Start work on The z^n problem. The students take some time to understand what the problem is saying, what the z_i's are (initially each spends time by himself trying to understand the text of the problem). Max asks what bounded/unbounded means. Tom explains what it means for a real sequence to be bounded above, below, and bounded in general.]

06:50 I: Make sure you understand what z^n is.

Max: Yeah, we got it, z^n is "$z*z_0*z_1*z_2*\ldots*z_{n-1}$".

I: Tom, do you agree?

[They debate on that for quite a while. At times they're not really understanding each other. They finally agree that z_n is z^n.]

18:20 [Getting ready to start considering what r should be to get a finite set.]

I: Can you give me an example of an arbitrary complex number in the set obtained "after all steps have been performed"?

Max: It depends on what r is.

I: Let's say r=3.

Max: Then it's infinite and unbounded [referring to the set produced by the completed process.]

Tom: You would have like…you would have as many complex numbers as there are natural numbers. So if r>1, then r^n goes to infinity.

20:15 [Tom proceeds to compute the powers of 1+i. Computes up to the 8^{th} power.]

25:10 Tom: Is this even productive? Let me see, what were we trying to do here?

I: Let's see, why are you trying to find the powers of 1+i?

Tom: I was trying to figure… because we want to… so we're assuming that all the steps have been performed, and this is trying to get an idea of what that meant, like what does $r^n(1+i)^n$ look like, you know, in the limit, as I'm going to infinity. You know, if r is bigger than 1, then r^n goes to infinity, so I just wanted to know what $(1+i)^n$ did when n goes to infinity.

26:00 I: I see. Are you both clear on the bounded vs. unbounded part, what that means?

Tom: Does it mean the sequence z^n would be bounded? The analogue to bounded in the real space would be like the absolute value of z^n would be less than or equal to some constant number I guess…

[Max challenges some of Tom's calculations regarding the first 8 values of $(1+i)^n$]

28:17 I ask them to discuss what bounded vs. unbounded means. They discuss what the absolute value of a complex number is and how it represents the length of the vector. Initially they say a set is bounded if you can draw two horizontal lines between which all points of the set are. They also write that we must have a real number M such that $|z^n| \leq M$. I point out that if the latter is true, then the 2 horizontal lines are not enough to impose that condition. Max suggests that maybe we need a box around the set. We agree that we can use that as our definition for bounded. Tom asks if we need a box or a circle. I ask them if the two definitions are equivalent, and after some discussion they decide they are.

38:12 [Start work on the finite case.]

Max: what about r=0?
Tom: That would certainly be "a case".

I: In this case, how many elements does the set produced by the completed process have?

Max: That would be 2 [elements]: z_0=1 and the rest are 0.

I: Can you find another value for r such that the final set is finite?

Max: Sure, r = 1.

Tom: Hmm…

Max: Cuz we're going to have a couple terms and then we're going to get the terms back…

Tom: we will?

Max: I'm pretty confident. Let me try.

Tom: Then z^n would be equal to $(1+i)^n$, and …

Max: [scribbling] I see a pattern of 4 in the values of $(1+i)^n$, which is why this [the case of r=1] is going to be a "four-thing".

[They do more calculations for $(1+i)^7$, $(1+i)^8$, the latter of which they get to be 16.]

42:00 Max: And that [the final set for r = 1] is finite? It can't be finite […] that's infinite…
[Tom continues to calculate powers of $(1+i)^n$, and tries to find patterns among certain powers.]

45:25 I: So, what's your current guess with respect to whether for r=1, the produced set is finite or infinite?

Tom: I'm pretty sure this is going to be infinite. I mean, you have infinitely many terms.

I: Is this set bounded or not?

Max: Unbounded. r has to be less than 1 to be bounded.

I: It might be useful to look at how the produced points "change quadrants".

Max: You're still going to go all over the place…it's going to get bigger and bigger.

47:48 Max: It's like a spiral…

[Tom is drawing some kind of a "square spiral", connecting "consecutive" points by straight lines.]

Tom: Look at the first 8 points [for r = 1], the points can be connected by straight lines (horizontal/vertical).

Max: Maybe this is because we're considering only integers, and if we considered all the numbers like with the t parameter, we would get a circular curve.

51:05 Tom: I'm just thinking the length of the vector 1+i is… like the norm is sqrt(2). So if r is 1, then the norm of z^n is sqrt(2)^n. So, as n goes to infinity, sqrt(2)^n is going to go to infinity. So the length of your vector z^n is increasing, you know…so it wouldn't be bounded. I don't think.

Max: if r is greater than or equal to 1, it's going to be unbounded. That's not right?

I: I don't know, you tell me.

55:05: Max: Oh, we need r^n less than z^n. Oh man, know what I'm saying? The norm of z^n needs to be less than r^n? […] We need r^n to be less than this in order to be bounded. r^n has to be less than this term, z^n.

I: You mean the absolute value of $(1+i)^n$?

57:40 Tom: I was just trying to think, like, so… this is for r=1, and it sort of spiraling out, infinite and unbounded. But if we were to pick something like… so, the length of 1+i is root 2, so if we picked r = 1/sqrt(2), so we might have like a circle, maybe… Because I'm thinking if r was a sufficiently small positive number, like 1/1000, then the spiral should go to the origin. I mean, if you make r small enough so that r^n goes to 0 faster than $(1+i)^n$ spirals out…

[Tom explains this twice more, for Max.]

1:02:26 Max: I can give you an r that works [he means in order for the final set to be infinite and bounded] If you get a small number it will work. Actually, any number less than … a number less than sqrt(2) works.

I: But for r=1, didn't you get infinite and unbounded?

Max: Ok, let me check.

[Meanwhile, Tom worked individually, in writing, and derived the expression of the absolute value of z^n to be $|r|^n (\sqrt{2})^n$.]

1:05:10 I: Let's see, if r is equal to that (1/sqrt(2)), what do you get [as the final set]?

Tom: You have a circle… that is, all of the terms for your sequence will be of norm equal to 1. I'm not sure exactly how it looks, but it would go around in your spiral but the spiral would always be length 1 around the origin. […] All of the points would be on the circle…I don't know if you have all the points on the circle, you might but I don't know.

I: That's a good question. Max, did you hear that?

Max: Oh, this is the finite case!

Tom: Well, the circle has infinitely many points… more specifically… I don't know, that's interesting. If it's a circle, maybe you can have finitely many points if spiraling around the circle the points start overlapping. I don't know.
Max: Oh, the terms are increasing, but the other thing is decreasing, so it's keeping it in that norm.

Tom: Let me think… the only thing not rational in $\sqrt{2}^{n}(1+i)^{n}$ is the square root… and if we had only integers in the coordinates of the points produced by the process we could claim that points on the circle with non-integer coordinates are not reached by the process.

1:09:24 [I encourage the students to think about the points produced in the plane, in the order in which they are produced by the process.]

I: Are the angles between two consecutive vectors of the process equal?

Max: Yes, we have the same angle between two consecutive vectors.

I: What is that common angle?

Max: Umm… what is arcsin of 1/sqrt(2)?

Tom: 45 degrees.

Max: Right. [He starts going around the circle with the pen, 45 degrees each time. Tom is a bit confused by the fact that our z^n vectors are not in the x+iy form, but rather $r^{n}(1+i)^{n}$.]

I: What is the relationship between the angle of 2+3i and the angle of 6(2+3i)?

Both: The same angle.

1:13:48 I: So what does this mean about the points in the process, are they overlapping or not, when we're on the unit circle?

Max: Absolutely!

I: How many elements are there in the final set?

Max: We're talking about the whole circle… 8!

[Tom slowly comes to the same conclusion (counts around the circle).]

I: I see you're counting. Is there a way to find out that there's 8 points without the specific counting?

Max: 360/45.

1:16:08 I: I see. So, is this (1/sqrt(2)) is the only value for r in order to get a finite set?

Tom: I don't know. We might try r=-1 just to see what happens.

Max: -1? Or -1/sqrt(2)?

Tom: No, just -1. Or I guess we could try the -1/sqrt(2). Oh that would be a spiral that goes out.
Max: But the problem is when we square, when we cube we would be getting different values.

Tom: Right, the sign is flipping back and forth. So, it would be what we already have… what we just figured out with the 8 elements for the circle, except that we would have $(-1)^n$ in our expression for z^n.

Max: But it would still hit on the same points. I think it's still these 8 points.

Tom: Oh duh, I'm sorry. Cuz if you have a point (x, y), a coordinate grid, then the point (-x, -y) is… you know, you have your odd symmetry, so if you were to take the point (1, 1), and then (-1, -1), and you were to flip it upside down, the points would sort of go on top of each other. So like, because you have 8 points in the particular places that they are… well let me see, at least for the… cuz you have 4 points that are in each quadrant… there are 4 points on the lines, so like for the 4 points that are actually in quadrants, multiplying by -1 just puts them into another point that's… and then if you have a point on the x-axis or the y-axis multiplying by -1 puts it to the other point on the other side. So every point is mapped into a point that's already there, so you still have 8.

Max: Really? It's just maximum eight. Possibly 4…although I would doubt 4.

1:19:00 [The students are not sure 8 and 2 are the only possibilities for the cardinality of a finite set. Max wonders if for r=-1/sqrt(2), we will get 4 or 6 points. Tom suggests it will still be 8 eights, but visited "in the opposite direction". Max starts calculating each of the first 8 points, algebraically, to decide in which quadrant it lands. I also ask them to find by what angle we're "jumping" each time. With the 2 methods combined, they become convinced in this case we still get 8 points, and it's the same 8 points as before.]

1:30:44 I: Can you choose a different z such that the produced set has 4 points?

Max: So… we need an associated angle of 90 degrees. I ask what complex number gives us an angle of 90.

[End of Session 3, File A]

Session 3, File B
4/26/08

00:00 Max: Oh, it would be i, without anything in front of it. Cuz (1, 1) is 45, so if we have like (0, 1) we're on the 90. That's right, right? If we need a new complex number we need to say z should be i.

I: And if you got –i as your complex number…

Max: That's still 4 [points], that is 270 [degrees]

I: And if you got -1 as your complex number to start with?

Max: That's still an angle of 90 so… Oh, the angle is 180. That gives you 2 points?!??!

I: Tom, what do you think?

Tom: The sequence is 1, -1, 1, -1,… So this produces only 2 points. And for z=1, it's only 1 point.

3:30 I: Ok, back to the original problem. What about the part that asks about [a final set that is] infinite and bounded?

Max: In that, r is less than 1/sqrt(2). Of course, in absolute value [referring to r].

[Tom nods in agreement.]

5:10 I: So, for such an r, is 0 part of the set of complex numbers produced by the process?

Tom: No, not in the sense that…

Max: Well, that's where the sequences converges to…

Tom: Yeah, I mean that's the limit, but you couldn't find a particular index for z_n where z_i is equal to the 0 vector. I mean, assuming that the value for r is greater than 0.

Max: Oh, I'm pretty sure you do include that [the complex number 0 in the final set].

6:25 Tom: I mean, I can see that we're converging to 0, but for like for an index, say z_{100} or z_{1238}...

Max: Right, that won't be 0.

Tom: Right. But no… but even if you complete all the steps, and you have z_i for every i in the natural numbers, none of those z_i's will be the 0 vector. They could be very close to the 0 vector, but… that's assuming that r is greater than 0.

Max: if you complete the steps, you'll be at 0.

Tom: Yeah, that's the limit, but that's not part of the sequence.

Max: It's not?

9:23 Max: Are we just trying to determine whether the origin is in the set? I guess it's not?

Tom: Right, that was my answer.

Max: I think it's possible to finish it, then you'd be left at the origin. So if you're at the origin, you're at the origin.

I: I guess that's the question, what do we mean by finishing?

Max: If we finish, we'd…

Tom: The limit…

Max: Yeah, we finish at the limit. If we take the limit, we're right there. The limit is 0, so yeah, 0 is included. It's different than last time! [referring to The Midpoint Problem]

I: Why is it different?

Max: Because it was way too many points last time. Last time we were trying to get in between decimals, which we can't do. But now we're just going to a fixed number, we're getting closer and closer, and we converge right to that point. That's possible.

I: But last time we had sequences converging to 1/3, but at the end of the day you said that 1/3 was not part of the set.

Max: 1/3 is irrational.

I: No, 1/3 is rational.

Max: Oh, that's right.

Tom: But I was thinking in terms of what it means to finish. Infinity is not actually an element of the natural numbers.

Max: Well, it's finite.

I: What's finite?
Max: it's not finite, but we can look at the end, without bothering with the middle stuff... At the end we're left at the origin. You know, you just follow the limit.

11:22 Tom: If I were to write down the limit of this thing, I suppose I could write something like $z_\infty = 0$. Maybe it's not strictly correct, but conceptually that's what we're trying to say, that $z_\infty = 0$.

I: You could also write the limit of the length of z_n is 0.

Max: Yeah, if we were to write down that vector, you'd end up with 0.

I: I think we all agree here that if we're talking indeed about what the limit number and the limit length are, that would be the 0 complex number and the 0 length. But do you think that's the same question as "is 0 as a complex number in the set produced by this process?"

Max: In that case it's probably a different question.

Tom: So we're talking of the set of all vectors… If it's the set of all elements, like z_i, then it's just every z_i for i a natural number. So you wouldn't like…I mean you would have uncountably many z_i's…

I: Uncountably many?

Tom: I'm sorry, countably many, equivalent in size to the natural number. No, I don't think the 0 vector would be in there. Because you never really get to the limit. If it's just the set of z_i's, then all of those z_i's will go to 0, but they're not… no particular one is the 0 vector.

Max: Then how can you tell me this equals 1 [points to paper, on which he wrote $\lim_{n \to 1}\left(1+\frac{1}{n}\right)=1$ in relation to The 1+1/n Problem discussed in Session 1], and you're trying to tell me that ….0?

Tom: Yeah, it is kind of inconsistent reasoning. Because when I gave that example with the balls [The 1+1/n Marble Problem], I was kind of thinking…

Max: When we had that one, we ended up saying at the end we had 1 on the ball. Well, I say we have 0 [in the final set] right now. We're AT the origin!

14:16: I: How can we reconcile these two approaches? I remember that with the 1+1/n marble problem, you both had no doubt that after the process is completed we have a ball with label 1 in the jar.

Max: Same thing. That's why I say we're at the origin!

Tom: Yeah, I mean, yeah, it depends…if we use this reasoning, which I recall being what we agreed on, then I guess you'd have to say that the origin, the zero vector is in your set. But on the other hand I am not really sure I agree with that anymore.

I: Along the same lines, what about the midpoint problem? There, although 1/3 was a limit for a sequence in the final set, you decided that 1/3 was not in the final set.

15:45 Max: We want to say 1/3 is in the set. The set is the open interval from 0 to 1. For both M and T. I think it was M and T…

I: Let me clarify, are you changing your mind about that problem?

Max [pointing at Tom]: Well, if he also wants to…

[Tom needs help in remembering the Midpoint problem; I give details of the text of the problem.]

Tom: right, so in that case [Midpoint Problem], the reasoning was that if a number from [0, 1] was the limit of a sequence in the [final] set, that number also needed to be in the set, although it was not a midpoint at any step. I think by the end of that session we decided that wasn't the case, no?

Max: Well, what we decided was…. Well, we ended up concluding that it wasn't the case, that the set contained only fractions.

Tom: So then, we could take what we decided there, and what we said with the balls of 1+1/n, and ask the same question, because there…

Max: I'm absolutely positive that the last one [ball in the 1+1/n problem] has a 1 on it.

Tom: Oh, ok. Because there are two sides to it…

Max: You can finish the process. If you do, you're left with that. I want to finish this the same. I want to include the origin.

I: So what dou propose Max, do you mean "ball with 1 on it: in the 1+1/n problem, the origin is in the set for the current problem, and the (0, 1) or [0, 1] interval for the Midpoint problem?

18:08 Max: Yeah, I think we gotta go all the way, man. We gotta take these limits and go. So I say yes.

Tom: I don't know…

Max: I know one time we said it was the interval (referring to [0, 1]). I want to keep it that way.

18:20 Tom: I'm just thinking we need to decide this problem first [pointing to the 1+1/n problem].

I: You think that's the easiest to decide?

Max: This is the easiest.

Tom: because if we decide this problem, then we just apply that reasoning to the other problems.

Max: Let's do that.

Tom: If there is a ball in the jar that has label 1 [in the 1+1/n problem], then that means that the 0 vector is in the set in our process [in The z^n Problem], because you would have to apply the same reasoning here as we did there, and it would also mean that the entire interval would be removed [in the Midpoint Problem] because if you have a sequence of midpoints that converges, you know, we number this limit a midpoint and that number is also removed by the reasoning here and the reasoning there. But if you say there's no ball with 1 on it at the end, then that means that the 0 vector is not an element of the process here and it also means that 1/3 is not removed from the interval, as well as any other point of the interval that's not a midpoint. So, I guess it depends on whether you think that… I don't know… I guess it has to do with whether infinity is going to be treated as an element in our set. You know, like, so in order to say that there's an element in our sequence z_n here, where some z_i is equal to the 0 vector, you have to say that z sub infinity is the 0 vector, so infinity would have to be an element in the set of indexes.

20:56 I: Well, what do you think the phrasing used in these problems means, in terms of whether infinity is an element of the set numbering the steps?

Tom: No… I mean, in general I don't talk about infinity as being an element of a set. I mean, because you know the natural numbers doesn't have infinity as an element of the set, nor does any other set that I've learned about.

Max: I always thought of that as a closed interval… I mean open interval.

Tom: Yeah, if you're taking an interval say from 5 to infinity, you have an open interval on the side of it cuz infinity is not in the set of real numbers. That's what makes me think now that I thought about it, that I was initially wrong about the ball having the label 1 at the end. Because that would require that… I guess it depends on how you phrase the question. Like if you're talking about the set of all balls at each step, as opposed to when you finish the process.
I: Personally, when I use the phrase "all the steps have been performed", I am referring a point where we can say about any natural number that the step with that number has been performed.

23:45 Max: So we're back to the starting point, and we know we're at the origin.

Tom: Actually, that makes me think that origin isn't in the thing, actually…

Max: If we finish all the s… process? We've finished the process!

Tom: Yeah, I know, but the way … you know, we're defining the process, that is the process being finished, then you're just talking about, you know, for any natural number. If you pick a natural number at random, that natural number is finite, so that finite number, say 4287 or whatever, z sub that number is going to be some vector, and that vector isn't going to be the 0 vector. So then you could say that, you know, let n be an arbitrary natural number, given that natural number z_n isn't the 0 vector, so no z_n is the 0 vector, so the 0 vector isn't in the process.

I: You mean, isn't among the numbers produced by the process?

Tom: Yes.

Max: If you finish the process, you're left at the origin. If you're at the origin, you gotta include it.

Tom: Yeah, but…

I: I'm playing devil's advocate. When we write 0 and 1 like that [writing "(0, 1)"], right? What does that mean?

Max: That does not include 0 and 1, it's an open interval.

I: Right, but you know, we are pretty much at 0, just not including 0. Meaning, we are arbitrarily close to 0, meaning that from this set I can get points as close to 0 as I want, but I'm still not including 0 in this segment. So in this situation at least, being arbitrarily close to 0 and having 0 in the set are not exactly the same thing, based on this notation. So then where does that leave us, with respect to our problems?

26:05 Max: M and T are the closed interval [0, 1].

[Tom asks to be reminded what M and T were.]

Max: And we're left with the origin included in the set. Well, this is the obvious one (referring to 1+1/n pb). I mean it has to be 1. That's the limit of the geometric series, sequence or whatever.

Tom: So to me, to finish the process means that for any natural number n, the step n has been finished. So then whatever n that we happen to be talking about, then we can show the complex number associated with that n is not the 0 vector. So you pick an n, and I show you the complex number that that's associated with and I show you furthermore it's not the 0 vector. And that will work no matter what n you choose. […] So then the question is, is the origin produced by this process? In order for the origin to be produced by the process, the process as opposed to the limit, it would have to be equal to some…

Max: The process finishing is the same as the limit. Taking the limit is the same as finishing the process. So if we finish the process we're at the limit. That's the only way you finish the process.

Tom: When I think of the process, I think of every z_n where n is a natural number.

Max: The only way you can finish the process is if you reach the origin. Cuz if you're not, then you're not done yet!

28:40 I: Let's focus our discussion a bit. If I denote $A = \{z^n \mid n \in N\}$ and $\overline{A} = A \cup \{z_\infty\}$, where $z_\infty = \lim_{n\to\infty} z^n = 0$. Then the question comes down to whether you want A or $\overline{A}$, based on the phrasing "all the steps have been performed".

Max: It's $\overline{A}$.

Tom: And I think that it's A.

32:30 Max [reading from paper]: "Assume all the steps have been performed". That makes it clear that you're going to end up with that (pointing to $\overline{A}$).

Tom: I think it makes it clear it's the other one.

Max: I don't think you know how to finish all the steps! The only way to finish is to get here!

Tom: I think we talked about this before, if you have like an open interval from like 0 to 1, and your steps were, you know, 1/n, then you would go to the origin and…

Max: But in that case you're not finishing all the steps.
Tom: Yeah, I would. So I could say your process is you know… your steps are 1/n, for n in the natural numbers… or say for n>=2, cuz we're between 0 and 1, where 0 and 1 are not in the interval. Then I could say ok, finish the process. Ok, right?

Max: Well, finish it! You can't finish till you're at 0.

Tom: Yeah, exactly. 0 wasn't in the set. It doesn't mean that it's not the limit of 1/n.

Max: Well, it's on the edge. The limit is 0.

I: I think you both agree that the limit is 0. He's just saying finishing the process means considering all of those points of the form 1/n, put together in a set. And because no 1/n is equal to 0, he's not putting 0 in that set.

Max: This is different than normal math. Cuz the 0 vector has to be in the set.

I: I don't think your opinions of "conventional math" differ necessarily, it's maybe just the interpretation of what it means to "complete all the steps" that's different, with Max wanting to put the limit of a process in the set of points constructed by the process, and Tom being against it.

Max: That's right. I'm 100% sure.

[End of session]

Session 4, File A
5/4/08

00:00 [Start work on the Triangle Problem. It takes a while for them to comprehend the text of the problem.]

Max: The points created by the process are "all over the place" so there's probably a complicated formula that describes the set constructed by the completed process.

Tom: Maybe it depends on whether you give A, B, and C appropriate coordinates.

[Discussion on what the system of coordinates mentioned by the problem is. Tom understood it pretty quickly but Max took longer to catch up.]

14:00 Tom: The y-coordinate of the points created this way seems to follow the pattern $1/2^n$ where n represents the step at which the point was created, so in order for a point with y-coordinate 1/16 to be in the "final set", it would have to be created at step 4.

[Together they write out the points produced by the first 3 steps.]

16:30 Max: So the point is that you can't have the x one that small and the y one that big (pointing to the given point, for part a)). Cuz the smallest one, they are equal…and x just gets bigger. […] That's not in the set. […] x cannot be smaller than y. So that point is not in set P.

Tom: Because the line from the origin to the midpoint of AC has slope 1, right, so then every midpoint… any point that's on that line in fact, x and y are going to be equal to each other… so none of those points are going to be a possibility… and there are no points in this area here where x is less than y, so…

22: 50 [Start work on part b), which asks them to describe a procedure that would spit out a random point in set P. Pause, some thinking done individually.]

Max: So… the points obtained on the positive side of the triangle at step n are from $(1/2^n, 1/2^n)$ to $(2^n - 1/2^n, 1/2^n)$ (where the numerator of the x-coordinate goes through all the odd numbers in the range {1, …, $2^n - 1$}). Then, to obtain a random point in P, you only need to pick a random natural number and plug it "in here".

I: I notice that your formula refers only to the "positive" side of the triangle. How many random natural numbers would you have to choose in order to obtain a random point in the set P?

Tom: One natural number… [thinking some more]. Actually, by choosing a first n, you choose a step number, but then you have to also choose which of the points created at that step to take.

27:15 I: Imagine that you're trying to program a computer and you have to write the procedure for the computer to follow, assuming the computer can randomly choose a natural number.

Max: How about telling the computer to randomly choose a natural number n and then to create an array of size 2^{n-1}, as that's how many points we create at step n, on the positive side of the triangle.

I: Don't we need a random point from the entire set P?

Max: Then you can create an array of size 2^n. Then, tell the computer to choose a random index for the array and output the number at that index.

44:35 [Tom suggests to choose an integer k such that the absolute value of 2k-1 is less than 2^n. Then the output point is $(2k-1/2^n, 1/2^n)$.]

48:45 Max: [jumping to part c), which hasn't been discussed yet] So I guess the question [part c)] is asking are we going to go all the way, including the origin… or any of the points on BC. The answer is we have to be consistent so we're saying "Yes". The entire…the closed interval between B and C [is part of the set produced by the process]

.

51:10 Start work on point c), which asks whether there are any points on segment BC that are in set P. This is the first time that Tom reads this part and his first reaction is to start laughing.]

52:24 Max: P contains the entire closed interval [referring to BC], plus all these other points above it.
Tom: I'm thinking… right, so if you have a particular point, you know, a particular midpoint, then that means you can define… that there's a line that this point… or a line segment rather that this point is on, where there are endpoints for that particular line segment… So for this particular one, (¾, ¼), then you have your line segment from C to the midpoint of AC, right? That is the midpoint of that line segment, which is why it is in P. So if you say that there is some point on the line BC that is in P, that means I should be able to find some sort of line segment that that point is a midpoint of.

I: Do you agree with that? (addressing Max)

Max: Yeah. I'm just saying cuz like the y-coordinate is ½^n, so all you gotta do is take the limit of that and you'd end up at 0. So that means that…

Tom: So I'm saying like ok, let's say that there's a point on the line BC that's in P. So let's take that point. What are the two endpoints of the line segment that it's on?

Max: Well if y is zero, then any point on this line satisfies it.

[I suggest Max misunderstood Tom's question and rephrase it.]

54:10 Max: Any segment… Well, it would be after the process is finished. You wouldn't be able to draw it but… It's going to be the entire closed interval cuz you're gonna have your triangles all the way down the line … when it's finished, it's going to be all points. We have no height so we're left with just points on BC.

Tom: that's the thing, like if you have a point on the line BC, and you can't find a particular line segment that it's a midpoint of, by definition it's not in P.

Max: Well, if you take the limit, and the limit is 0, that means you're ON the x-axis, which is segment BC.

Tom: Except that that kind of gets back to my earlier question, about what the line segment is.
Max: Well, we know from last time that we're going to include the entire closed interval.

I: Was that decided in the last session?

Max: I was certain on my end, and I'm staying with it here. I'm even more confident this time. That's an easy limit. (referring to 1^2n).

I: What makes you more confident?

Max: Maybe cuz I know I'm gonna end up with limit, and the limit is zero again.

I: Wasn't the z^n limit easy too?

Max: Right. I guess I'm just saying confident. I'm 100% each time, so I guess it's to say… I mean how can you argue against the limit?

55:55 Tom: But, except that…. Because I can say that we have a definition for the set of P, and so then we can talk about what sort of properties an element of P has…

Max: Look at what happens when it goes to infinity…

Tom: I see that ½^n goes to 0, I'm saying that that's not really that important, at least in terms of the question. Because you can… alright, so… so you can say… alright so suppose that P has some element that's on the line BC, well then that particular point the y-coordinate would be 0, and so then you would have to have some sort of…there would be a line segment that it's a midpoint of, right? So that in order… now so that line segment you have to say well, does that line segment have positive length? So is it a line segment if its length is 0? Cuz that's what it would have to be, it would have to be the midpoint of a line segment where the two endpoints are the point itself. It would be like saying that the origin is the midpoint to the line segment from the origin to the origin.

Max: You're not at the origin.

Tom: Well, I'm not saying the origin…

Max: You are when you finish the last… when you finish the process… after the finished process, the last step you're going to end up at the origin.

Tom: I'm saying that in order for there to be a point on the interval from B to C, and have that be an element of P, it would have to be a midpoint from a line segment where the two endpoints are the same point.

Tom: But I'm saying, does that make sense, to say that a line segment is one point?
Tom: I mean, is that even allowed?

Max: Yes!

Tom: No, I don't know…

Max: Yes, absolutely!

I: So one segment can be just 1 point?

Max: Yes… when we are done, we are left at a point.

57:48 I: It seems the root of your disagreement is that you have different definitions for set P. For Tom, an element in P must be a midpoint for a segment that has an end on BC and an end above BC. So if a point on BC is claimed to be in P, one must find a segment of that type for which that point is a midpoint.

Max: Well, if you wanted to go back and write out the steps for like step n-1, in relation to step n, then sure that would be the segments for before you do the last step.

I: I don't follow… what do you mean by last step?

Max: The last step I mean when the process is finished…

Tom: … the limit.

Max: …right, you finish the limit, you've done every midpoint on the line… every point on the open interval BC… closed interval.

Tom: Except for one thing. If you have like the limit of a sequence, then the limit itself doesn't have to be in the domain of the sequence. Right? So you can define a sequence that converges to a particular number where that number is not in the domain of the sequence.

I: Tom, do you mean in the range of the sequence?

Tom: Right, I'm saying you could have a sequence of rational numbers that converges to an irrational number. You know, just because you have a process that converges to, you know a particular set of points on the line BC, it doesn't mean that those points are in P just because it's the limit of a particular process.

1:00:18 Max: Well, in this case, we're going to end up on … you divide all the points, you're going to end up on the x-axis, and you gotta include the points!

Tom: Well, if that's true, then I could say that sqrt(2) is a rational number because it's the limit of a rational sequence. It's the same reasoning!

Max: I can prove that sqrt(2) is irrational!

Tom: Yeah, exactly! No, but I'm saying that if I were to take your reasoning and apply it to the sequence that converges to sqrt(2), I could use your reasoning in combination with that sequence to prove that sqrt(2) is rational. Right? Because you have a sequence of all rational numbers that converges to sqrt(2) so then…

Max: Well, show me that sequence!

Tom: Honestly, I don't really completely recall the sequence, I know there is one, but I don't actually recall…

Max: I think it would be different than this.

Tom: Let me make sure I understand correctly. So if you have a process, like ½^n…

Max: If you're doing truncation that means you're getting error, don't you?
I: Right, but it's a sequence that converges to that; you don't stop at 1000 digits or whatever. It's the whole sequence.

Max: Ah, right. So ignore that. I really think that don't really apply. You're gonna end up sitting on the origin… sitting on the x-axis. So where you're gonna go?

Tom: No, that's the thing. I can approach the x-axis and never get there.

Max: So you think you're not gonna get there? Even though you're on top of it?

Tom: Actually, I'm certain of it. It's like, you have, you know, your function like 1/x, you can say that that approaches the x-axis as x goes to infinity, but I think it's pretty well known that there is no x value that solves the equation 1/x=0.

Max: […] I know it approaches 0… the limit is 0.

Tom: Yeah. That doesn't mean that there is a y value [sic] such that 0=1/x. You can't solve that, obviously. So just because the graph, or the function, approaches the x-axis, it doesn't mean that there are any points, you know, on the x-axis…

1:03:00 Max: That's because in normal math, you're not really finishing the process. I think that if we like finished the process, that means we've reached the graph.

Tom: So then this isn't normal math….?
Max: That's right.

Tom: Oh, ok…

Max: I remember I said that last time too.

I: So you're saying we're redefining acceptable mathematical procedures?

Max: It's sort of different. We're looking at finishing a process that you usually don't finish… but in this case we're finishing it.

Tom: I guess that's the thing… I see this as being something you could do normal math with. I guess we're using the term "normal math" kind of loosely but…

Max: I think we're looking at what happens when the process is finished.

I: [to Max] So you see this is as divergent from normal procedures of mathematics that you learned so far?

Max: Possibly, yeah.

I: So, in this math you're saying, we get to the limit and in the other math we don't get to the limit?

Max: Well, this is a little different. 1/x it goes that way (pointing to the right), this case we're just working within an interval here, so it's a little different. I mean we can't keep going…

I: You can still "go" in a certain direction, just that it's a different direction than that of 1/x.

Max: [after some pause and gesturing] Yeah, it contains the entire segment BC.

Tom: I don't know…

I: And Tom, you're saying that it doesn't contain it…?

Tom: Right.

1:05:15 Tom: Uh… I have one point. If we require that a line segment have a positive length, like strictly greater than 0, then I'll say I'm 100% positive there are no points in the set P that are on line BC. But if you say that you can have a line segment that is one point and that that point is a midpoint of itself or something like that, then ok, maybe if you allow that, then you could say that maybe there is a point on the line BC. But then it's an open question, I'm not saying that it is, just that I would be willing to consider it in that case.
I: Ok, so you're not sure about the conventions having to do with that part of geometry, so based on those answers you might take a different position?

Tom: Right, so like if we say you know, oh by the way, a line segment that's one point counts as an actual segment, then you know, alright fine… I guess I was thinking of a line segment as being something having to have a length greater than 0.

I: So if for now we agree that a segment has to have a length different from 0, then you'd say that BC is not contained in set P?

Tom: Yes, yes.

I: What about you Max?

Max: Well, if the length is greater than 0, if you restrict it that way, then you couldn't. No you could not, not if you restrict it that way. But eventually the norm is going to be 0.

I: If we finish the process, you're saying…?
Max: Right.

I: Ok, let's leave this problem at that, so summarize your positions, each on your own sheet.

1:07:50 I: Actually, while he's writing that, Max, I have another question for you. So, you gave a procedure for the computer to randomly spit out a point in P. So, how about you look at the procedure again and see… does it ever spit out a point on segment BC?

Max: Oh yes.

I: It does?

Max: Yes, it would spit…at the last step it would spit out the whole closed interval.

I: Let's look at the procedure. The procedure here – this is what you guys agreed on.

Max: Well, the random natural number n… well yeah, if it's the last number in the set of natural numbers, yeah…

I: Well, you told it to pick a *natural number*, right?

Max: Alright, then … no, we have to take the limit in order to get on BC.
I: But then, does that mean that your procedure is incorrect?

Max: No, we just take the limit of the procedure.

I: I'm confused because I thought the procedure was designed to spit out any element of P, and now you're saying you have to take the limit of the procedure?

Max: I'm not worried about that… I know what you… if you do it with the computer, if you do a random n, then you're not going to have a 0 y-value, no.

I: I see what you're saying. Let me try to bring a different angle. So here's the thing. So you're saying a point on segment BC is part of P, right? Now, I think we talked before… if we're looking at the number of points on segment BC and the number of points in set P above segment BC, which one is more numerous? I mean, which one is larger?

Max: Oh, it would definitely be the segment BC.

Tom: The size of the set of points on BC is equal to the size of R (the real numbers), and the set of points in P above segment BC is equal in size to N, which is a smaller size than that of R.

1:11:48 I: So, if you're doing a procedure that randomly picks a point [from P], which one is more likely to come up: one on segment BC or one above segment BC? If our procedure is correct, that it randomly gets something from P?

Max: It would be like 99% on segment BC.

Tom: In fact… actually… I would almost say it's 100% certain. It's like if you have the interval of the real numbers and you pick a number from the interval of real numbers at random, what you're basically saying is that that number is going to be irrational… I mean, it would have to be a rational number… Well, I mean, so if you… this is sort of an analogy I guess… asking to pick a point at random, and to say what's the probability that it's a point above the line BC, that would be at least a little bit like asking … if you had the interval of the real numbers and you pick a point at random, what's the probability of picking a rational number, as opposed to picking an irrational number?

I: Uh-um – and what's the probability?

Tom: Well, I'm actually not sure, but I'm pretty certain…

Max: Probably it's very close to 0…

Tom: It's either very close to 0 or it is 0, I'm not sure which one it is… I haven't studied that…

Max: It's very small.

I: Ok, so you're saying very small or 0, right?

Both: Yeah.

I: So then, most of the times when… I mean almost all the time when we apply your procedure we should get a point on segment BC, right? Because your procedure randomly spits out…

Max: Yeah, absolutely.

I: Ok, So let's look at the procedure again. When do you see it spitting out something from segment BC?

Max: If you just do a random n it's not going to happen … it's not going to happen at all with that formula. It only happens if we consider… Yeah, cuz that formula don't consider the finished process.

I: It doesn't?

Max: Yeah, cuz you can't get to the limit of…

I: Then, do you want to redo the procedure? Cuz you know, obviously the procedure was about the whole set P, right? I said "a random point in set P"…

Max: Well, that's what you can do with a computer! It doesn't do the limit for us, you know … you can't fix the formula! We gotta go … we gotta look at… not use a formula with a random n. We gotta use a formula after we finish all the n's… which we can't really do with a formula. Sure, you take the limit I guess when you get to the n.

1:16:30 Tom: Well, actually, now that I think about it, you can use the procedure in step b) as an argument for why there wouldn't be any points on the interval BC [in P]… if you say that the process in b) is correct. Because if this procedure is correct, then I could say every point has this form. So that means that if there's a point from interval BC that's in P, that means there is some number in the natural numbers somewhere that would give you ½^n is equal to 0. But obviously there is no such natural number, so … the point is the y-coordinate is always going to be greater than 0, you know, using this process, so…

Max: I'm just saying the formula is not right then.

I: So we need to change the procedure?

Max: Kind of.

I: Ok, so how do you want to change it?

Max: well, I'm not sure how to write it in math terms…

I: Just tell us the general idea… think out loud …

Max: Can we just call it the union from like n=1 to infinity?

I: Well, but it needs to be programmable, right, so how are you going get the computer…?

Max: Well, the computer can't do it.

Tom: So it's impossible in principle for a computer to do it.

Max: Right. But using simple mathematics like taking the limit you can do it.

Tom: Max, you could claim that the computer cannot implement a procedure for spitting out a number in our set P because that would involve performing an infinite number of steps. That's NOT the position I'm taking, but Max could adopt that position.

1:22:45 I: Ok, how about a set of instructions to give to a human, to spit out a point in P?

Max: You can't do it!

I: Why not?

Max: A human can't do that.

Tom: what, pick a random natural number?

Max: Well, any number that they pick it's not going to put in the origin.

I: Yeah, that's true.

Max: You've gotta finish the process. And that's no finite number! There's no finite number that can do that.

I: So you're saying, computer or not, human or not, this procedure does not really refer to the finished process.

Max: No, it refers to every step in between.
I: In between what?

Max: It doesn't… . You can't finish it doing that. You just keep going and going. If you take the limit, you can get to the end. And that's all we gotta consider. We don't gotta consider all this garbage in the middle. We know we got at the end…

I: Well, we need to consider those too [produced points above BC]. Those are part of set P as well…

Max: Are part of set P but… I mean it's kind of interesting but what we're trying to figure out is this and we know that's part of the set… the whole closed interval BC.

1:26:12 [Start work on lamp problem.]

Max: [after 5 seconds from seeing the problem] Oh, this is the same as the "change the ball from 1 to 2".

I: Let's see if it's the same.

Max: The lamp is neither on nor off. You can't solve this.

I: Do you remember what you said about the ½ pb?

Max: Yeah, I said the ball is either labeled 1 or 0, but not 1 nor 0.

I: So it was that there is a ball, but we don't know what label it has.

Max: Yeah, that's right. You can't determine.

I: And I think another choice you guys had for that one, cuz you… maybe you were sure about that one but I think Tom also said "or there's nothing in the jar", right? That was an option as well?

Max: There is a ball in the jar but you don't know the number.

I: Did you have a 3rd option for the ½ pb, "having both… two balls in the jar"?

Max: That might have been an option but it's not true. […] You can't determine this. The lamp is either on or off but it's neither on nor off so you can't determine which it is.

I: So which answer from the ball pb. would correspond to what you're saying about this problem?
Max: Well, it's either on or off but it's not on nor off…. That was my answer to the ball problem: it's either 1 or 2 but it's not 1 nor 2… it's a ball but you don't know which label is on it. So it's either on or off but you don't know which it is.

I: And Tom? By the way, what would be the equivalent of "there's no ball in the jar"?

Max: Well, the bulb burns out or something.

I: Well, I guess that's a reasonable answer considering that it's running for an infinite amount of time.

Max: No, that's not very good. I guess you don't have an alternative which means we can throw that option out. So we're left with two options…

I: Which was "two balls"…or what? As in both… actually I don't remember if it was… yeah, as in "both 1 and 2".

Max: How can it be on and off at the same time?

Tom: It couldn't, it's a contradiction.

Max: Alright, so we'll throw that away. We're just left with one option!

Tom: No, we're left with two.

Max: Oh, you think… you know it can't be either on or off! That's why! You can't determine it!

Tom: Alright, so it's a contradiction to say it's both on and off. But you can say it's on, or you can say it's off, or you can say that the problem that they're asking us to solve doesn't make sense.

Max: Well, it doesn't. That's why you can form…you can pretty much form whatever answer you want.

Tom: Right, but I'm saying there's 3 options: you can say it's on, it's off, or this question does not, like…

Max: We can only say it's on or off but it's neither on nor off.

Tom: If we were to say that the lamp is on at the end of this process, right? If we were to just say that… You know, you can then ask the question, through the process that we're performing, any time that it's on it means that at the very next step it's going to be off.

Max: Right, you can never prove it's on or off. Impossible.
That's right you can't determine it, because you can't prove either way.

I: How do you know you're not just missing something?

Max: Oh, I know.

I: Ok, so Max never misses…

Max: There's no way it's on and there's no way it's off. There are no other choices so you can't determine it. That's your only two choices and it can't be either one of them!

Tom: I keep thinking of that sequence that alternates between plus and minus 1. You know, 1, -1, 1, -1, …. I think you can prove that that sequence doesn't converge. So I don't know… that suggested to me, at least …

[End of Session 4, File A]

Session 4, File B
5/4/08

00:00 [From File A: Tom: So I don't know… that suggested to me, at least …]
… in terms of this problem. I don't want to… Cuz I remember, I think it was 2 or 3 sessions ago I was actually in agreement with Max on the whole infinity thing, like I was saying… You know, I originally came up with 1+1/n and said, you know, if you have a sequence that finishes, then there is a ball with label 1 at the end of the process. You know, the analogy is kind of tricky I think… Because I remember I then went back on that and said that … that there isn't a ball with label 1 at the end.

I: This is about the1+1/n problem?

Tom: Yeah. I kind of want to use the analogy with the sequence plus and minus 1…. I'm not really sure that it completely holds or not.

I: So then… about that one, about 1+1/n, if it's not a ball with label 1, then what is it?

Tom: Well, that's the thing, like…

Max: It must be label 1.

Tom: Well, let me see. I don't know, like… with the problem we just saw, with the equilateral triangle, there my position essentially was that… cuz if you faced question c) as, you know, "describe the points that are on the line BC", or something like that, I would have eventually probably come to the same answer I did, but it would be different in that the question would be assuming that there were points, you know, of that type [he means, points in P that were also on BC]. So, I sort of see question 2 like that… "assuming all steps have been performed, is the lamp on or off"… that's sort of like "assuming, you know, that it makes sense to talk about what happens after all the steps have been performed", you know. So…

I: Well, in the previous tasks it made sense to talk about a point at the end of the process, so I guess you can interpret the question as more like "what are you going to say about that point?"

I: Max? Your thoughts on what he said?

Max: Well, I still think the same thing that I thought all along.

I: Ok, so for the 1+1/n we have "1" at the end"…

Max: Yeah, definitely.

I: and for the 1, 2, 1, 2, …,

Max: It's undeterminable… what label is on the ball.

I: And for the lamp?

Max: It's either on or off but you don't know which one. You don't know whether it's on or off…you don't know… you can't determine it.

I: Ok, so you're saying the process does not determine what's happening at that point, at t=1, if we're going to formulate it in terms of time.

Max: Right.

I: So if you go back to the 1, 2, 1, 2 problem, if we're gonna apply the same reasoning…

Max: Right. You don't know what the label… you don't know which ball it is, but there's a ball in there.

I: So you're saying there's a ball, but we don't know which label. So you're saying the label is not determined by the process, but the fact that there is one ball in there is determined. How are you sure that the fact that there is one ball is determined, if here we couldn't determine anything about the lamp, but there you want to…

Max: Well, cuz we took the norm of the vector was 1…

I: In the past we had cases where the sequence of the norm of the vector was not indicative of the norm of the "final vector".

Max: Ok, throw that out. I don't care about the norm. I'm just saying there's gotta be a ball in there.

I: But why does there have to be a ball, if you don't care about the norm, then why cuz the norm is what tells you about the quantity.
Max: The norm is 1.

I: The norm is 1 at each step, but I just gave you an example of what happens with the norm during the process…

Max: In this case it doesn't end… in this case it's never gonna be 0.

I: Maybe, but how do you know that it's going to maintain something…

Max: It's an oscillation. You have to put an external force to stop that, which you can't do here. You know what I mean? It's like a pendulum swinging back and forth, it never stops.

I: So you still want to have a ball in there?

Max: Oh yes!

Tom: I was just thinking, with this problem… There aren't really any answers with this problem that I found satisfactory. Because you could say that it's both on and off, but that's a contradiction. I was entertaining briefly the idea that it's neither on nor off, but that's also a contradiction, for the same reason. And if you say that it's on, then for any point that it's on, I can show you a point that it's off, and put the conclusion that it's on at the end under serious doubt, and I can also put the conclusion that is off under serious doubt. So it's doubtful that it's on and off, it's doubtful that it's neither on nor off, it's doubtful that it's on, it's doubtful that it's off, so it's like, at some point I'm started to entertain the point that there might not be a lamp! […] If there was a lamp, it would have to be either on or off, like, applying the normal logic that we have about lamps [I: and the normal physics], that there's an exclusive OR, that it's either on or off, so if there was a lamp it would be either on or off, but… if you could show that it's neither on nor off, then there's no lamp.

Max: I don't think the lamp can go anywhere. The lamp is still going to be there. You're kind of right, I mean it's not determinable, so we know that it's either on or off but it's neither on nor off, that's where the contradiction is, that's why it's undeterminable.

I: I am confused about Max's answer about the ball problem in relation to his answer to the lamp problem; how come with the ball pb, the process still determined something (that there is a ball in the jar), and the only undetermined part is the label on it.

Max: Not the tendency…. But the end result does. You know that the end result is going to be norm 0 for the1/2 problem, and you know the end result here is going to be norm 1. So you don't care about the middle steps, we're just looking at the finished process, right? So all we care about is what we end up with. And we know what we end up with.

I: Well so maybe sometimes you can look at the finished process, maybe sometimes you can't. Is that an option?

Max: Yeah, I usually just try to look at the end, I don't care about what happens in between. I'm taking the process, I'm telling what happens, and I know what happens when it's done. I'm telling you what the state is at the end, and in this case we can't determine that. That's what I'm saying.

Max: The finite… All this determines [referring to the intermediate stages] is what happens when you're done. And what happens when you're done does determine it. So like what I'm saying is like… with the marble, you end up with the 0 vector so you know there's no marble in the jar. In this case you end up with a 1-vector so you know there's a marble in there, but you just don't know what is, if it's like (0 1) or (1 0) or how you label the vector… you know that it's one of them. So you know that the norm is 1 so that means you have a marble. You just don't know what the label is. You know that there's a marble in the jar! (forcefully)

12:55 I suggest they write summaries of their answers, at the same time asking if they feel they have the same answer.

Max: Do you agree with my answer?

Tom: Your answer being that it's undeterminable?

Max: That's right. What's your answer, that there's no lamp? [all 3 laugh]

Tom: Well, I am nowhere near sure of any answer that I would have for this problem as number 1, but I had to give an answer I would say that there is no lamp that has the property that it's either on or off.

Session 5 (one file)
5/11/08

00:00 [Start work on the modified tennis ball problem (which is a time-version problem where the contents of the bins are swapped every other time; after step n one bin contains {1} and the other {2, 3, 4, …, n}.]

[The students collaboratively work out the first 7 steps. After 7 steps, bin A {2, …, 7}, and bin B has {1}.]

12:00 Max: Oh, so the contents of each bin are going to be "ball 1 in bin B, everything else in bin A".

[Tom wants to do a few more steps, to get a feel of the problem.]

17:30 Tom: I mean… we're sort of doing the swapping by changing the names [of bins]… This collection of balls is going from A to B, back to A, and going back and forth between A and B, but all of these balls are sort of staying together.

Max: So what you have, you have that the one bin has bin 1, the other bin has bins 2 through infinity [sic]; you just don't know which bin is A or B, but you know one bin is going to have this number and the other bin is going to have the rest of the numbers.

I: So, what are the contents of the bins after step 9?

Max: Bin A will have ball 1, bin B will have balls 2 through 9.

I: Ok, and after 10?

[Max offers to do step 11.]

I: For an arbitrary step n, how many cases would we have in stating what the contents of the bins are?

[They work on that and are able to state the 4 cases, based on the relation between n and multiples of 4.]

25:50 I: So after all steps are completed, what do you have?

Max: At the end of the process? Well you don't know which bin it will be, but you know that… you know that ball 1… you just know that one bin is going to have ball 1, the other bin is going to have 2 through n [N?], know what I'm saying?

I: Through n what?

Max: Well, natural numbers…every other number. So you have ball 1 … and the other balls in the other bin. You just don't know which one is A and which one is B.

I: Ok, but you're sure that one of them has only one ball numbered 1, and the other one has 2 to infinity?

Max: That's right.

I: Is that… would you say that too, Tom?

Tom: I'm sorry… you have 1 in one of the bins and all of the rest in the other one? Yeah…

I: And you don't know either which bin would be the one with 1?

Tom: Uh… it's flipping back and forth…

Max: You can't determine that…

Tom: Yeah… so… I mean, not really.

Max: I mean, you can't determine which bin. You just know one bin's like this and one bin's like this.

Tom: Yeah, I think.

I: So about the final state, you know one thing for sure, that the balls are split in that way, 1 is by itself somewhere in one bin, and the rest are together in the other bin. Just that you don't know which is which?

Max: Yeah, that's right.

Tom: Yeah… [less confident than Max]

I: Ok. I think we can leave that at that. Can you want to write a summary of that?

[The rest of this session involves students individually working on summarizing their solutions to each task addressed so far in the study and splitting the whole

set of tasks tackles in this cycle into groups based on perceived structural similarities; this is done in writing.]
[End of session]

Session 6, File A
5/18/08

0:00 [I ask the students to read the other's answers to the categorization task, and ask each other questions about them if anything is unclear. They take some time to read each other's answers, in silence.]

7:04 Tom: I like how you [Max] invoked the "vector characteristic" to answer e). Tom asks Max why i) (the Original Tennis ball Problem) is in the same category as a) (the Vector problem) and e) (n-> n+1 Marble Problem, timed version).

Tom: In my mind, there (referring to original tennis problem) it's a little different, because you have one bin that's empty, you have another bin that's equal in size to the natural numbers. But in a) and e) you don't have that, I mean there's no set of elements that is equal in size to the natural numbers in a) or e)…

Max: You're saying a) and e) are the same, right?

Tom: Yeah, because in both you have the vector…

Max: Why did I put i) in there?

Tom: Yeah, like why is i) in there? I was just curious… is that because of the empty bin?

Max: For i), you can use a vector representation as well, for the contents of bin A, which "ends" with the zero vector.

11:35 Tom: And about your next category, in which you put d) (the ½ marble problem), h) (the lamp problem), and j) (the modified tennis ball problem). Your answer to d), "either 1 or 2, but it's neither 1 nor 2", is a contradiction.

Max: I meant it's one of them but you don't know which one.

13:10 Tom: I also like your answer to c), the Writer problem. [Max answered that problem but highlighting a one-to-one correspondence between the set of days and the set of weeks.]

I: This Writer problem, couldn't this be solved also by using a vector representation that ends up with the zero vector? Does that mean it should be added to the previously discussed category?

Max: I did not approach c) that way.

I: I know. What I am trying to say is that if you create task categories based on which solution path you decide to use, then it might not be clear to a second person what your categories are; if they read your instruction, they wouldn't necessarily put the Writer problem in the category you did. For example, The n->n+1 marble problem, can you solve it through a "one-to-one correspondence between 2 sets" approach? And if yes, does that mean it should go in the same category as the Writer problem? I'll let you think about that.

24:00 I: Tom, I have a question for you regarding your categories. Can you explain what you mean by "a limit state that is consistent with the process"? How do you define that?

Tom: For example, in the case of problem b) (the Triangle problem), we are converging to segment BC but points on BC are not included in the "limit state", so this is an example of "the limit state is *not* consistent with the process".
For example the function f(x) = 1/x, as x goes to infinity, the values of the function converge to 0, but there is no finite x at which the function becomes 0. It's similar.

I: Can your "limit state not consistent with the process" be rephrased as "limit state not reached at a finite step of the process", would that be the same?

Tom: [hesitantly] Yeah…

I: What about a) (the Vector pb) as an example of a problem where the limit state is consistent with the process, can you explain why that one is in there?

Tom: I am not sure now, that I think about it… [pause, looks at the problem].

29:00 I: Let's see, your third category was… the alternating ones?

Tom: Right, the ones that aren't… so like, even in my first two categories, in both cases you are converging to a particular situation, whereas in 3), I don't think any of these problems actually are. See, e) (n->n+1 marble pb) is not in there, as using the vector representation you can show that you have the empty jar "at the end".

37:40 [Start the part where I play a bigger teacher role.]

I: Let's talk together through the collection of tasks. How about we start with the Original Tennis Ball Problem, since I remember you having agreed entirely on this one. Do you remember what you said about the content of bin A, after the process is completed?

Max: Empty.

Tom: Right.

I: And what was the reason for that?

Max: Well, when the process is complete, every ball has been moved from bin A and put into bin B.

I: I see. Can you go into more detail on that?

Tom: Well, we can say if you pick any ball that's in bin A, then at some point that ball is going to be put in bin B.

I: And you can specify exactly at what point it's going to be put in bin B?

Tom: Right, I think the text of the problem tells us that.

39:40 I: Ok, let's analyze this argument. What are its main components? First we take a random natural number n, we show that after step n the ball labeled n has been moved from bin A to bin B, and then claim that since n was chosen randomly, this is true for any ball; therefore there are no balls in A "at the end". When we claim that "ball n not in bin A after step n" implies that "ball n not in bin A 'at the end'", we use something that holds true at each of the finite stages (that ball n is not in bin A) and claim that this something needs to hold for the final state as well. But we saw, even with this problem, that some aspects of the finite states do not hold for the final one. In this case, for example, the fact that bin A and bin B have an equal number of elements. We saw that this is true at each of the finite states, but was not true about the final state. Then, what assures us that this other type of generalization from the finite states to the final one, regarding ball n not being in bin A, is valid? The solution you summarized seems to be making use of an assumption that if an object is (or is not) part of the intermediary states from a finite step on, then that object is (or is not) part of the final set. So, for the solution you agreed upon for this problem to be valid, we need to have the following assumptions, or principles, taken as true:

P1: An object is part of the "final state" if at some finite step it becomes part of an intermediary state, and continues to be part of all of the subsequent intermediary states.

P2: An object is *not* part of the "final state" if at some finite step it is not part of an intermediary state, and continues *not to be* part of each of the subsequent intermediary states.

You've made use of these principles in solving this problem (OTBP), but without making direct reference to them. Does this make sense so far?

Max: Yeah.

I: Tom?

Tom: Right.

48:20 I: Ok, let's see then, which of these principles are used to claim bin A is empty at the end, for the original tennis ball problem?

Max: The second one.

I: What about for bin B? For bin B we said that contains what?

Max: That would be the first one [principle].

I: Wait, tell me first… what does bin B contain?

Max: At the end? It contains all the balls, you're using the first principle.

49:30 I: What about problem e) (n->n+1 problem)?

Max: The jar is empty.

I: For this one, which of the principles are we using?

Max: Using the second one.

[Tom nods in agreement.]

50:55 I: What about c)? [the Writer Problem]

Tom: That's the first principle. Because once you write about a day, right, then you can say it's in your set of days that you've written about, that are recorded… and once it's in there, it stays in there.

Max: If we start with the set of days that need to be written about, and remove days, then we would be applying P2. I guess it makes more sense to "add the days" (start with the empty set), as with the other approach the set of days to write about keep changing, as the writer lives on and more days are added.

52:30 I: What about f) (the 1+1/n pb)?

Max: Ok. Well, at t=1, ball "1" is in the jar. That's what I said at least.

I: Ok, and which principles are applied in this solution?

Max: Well, it would be… you're removing them all, eventually… well if you let the 1/n as this thing and move them all, you're just left with "1". Oh, so I guess you're increasing the n, as you increase the n you just end up with "1" at t=1. I'm positive.
Tom: So, in order to say that a ball is in the jar for this problem, at the end, we have to say that at some finite point it was put in the jar and stays in there for every step after that?

I: If you want to use the first principle, then yes.

Tom: Ok. So we wouldn't be able to use that for this problem. Or we wouldn't be able to use that principle to say that "1" is in the jar, because it's not put in at any finite step.

I: Did you follow that, Max?

Max: Yeah.

I: So you agree that we can't use the first principle to claim that "1" is in the jar?

Max: Oh. Neither of those would work.

I: So I think we agree that the first principle doesn't apply. What about the second one?

Tom: If we wanted to say that marble "1" isn't in the jar?

Max: We can say at step equals 1, we're going to have marble "2" but then we won't have it again, then we're going to have marble "3/2" so that won't be in there after step 2… […] I mean after step 1 you're getting rid of marble 2, then you're getting rid of 3/2, then you're getting rid of… […]. So you're starting at 2 and you're going down, and eventually you're going to be at 1. So everything between 2 and 1 is going to be eliminated, so you can look at it that way. At t=1 we're going to have marble 1 in the jar.

I: Let's backtrack a bit. So far, we don't have any principle that we can use to claim that 1 is in the jar, right? We decided that the first principle cannot be applied.

Max: That's right. What about the second principle.

I: Yes, what about the second principle? The second principle lets you say that something is not in the jar, it doesn't help you say that something is in the jar at the end.

Max: Well, it says that… well I mean, you already looked at that for the “1/n” part. The “1” part is always there so you keep that separate. So if you look at the “1/n” part, yeah eventually you’re going to be left at 0, so you just have the “1” left. You can’t get rid of that one. The “1” here.

I: Do you have a marble with the label “1” on it in this problem?

Max: At t=1 there is one, in the jar.

I: Let’s see, how are you applying the principles?

Max: I am applying them half way.

I: You’re applying them half way? What do you mean?

Max: I am applying them on 1/n. I take the limit of this, I take “the limit of 1” plus “the limit of 1/n”, so I’m just left with the limit of 1 plus 0, so I end up with 1. I’m splitting it up.

I: I see, you’re splitting it up. I got that approach. Tom, what do you say?

56:42 Tom: Um, I don’t think that either of the principles that were stated so far are… like I don’t think that you can use either principle to say that marble 1 is in the jar.

I: Ok, can you use either principle to say anything about the final state?

Tom: Well, you can use the second principle to say that every marble of the form 1+1/n is not in the jar at the final stage. Because 1+1/n is removed at a particular finite step and once removed it stays out. You can say that for any marble of that form. You can say that there aren’t marbles of this type in the jar at t=1.

I: Ok, so you can say that. Can you say that the jar is empty?

Tom: Well no, I suppose… Based on our principles there’s nothing more we can say, I mean conceivably there could be an elephant in the jar.

Max: He basically stated that we’re going to be left with marble “1”. He basically said, we’re going to eliminate all of them, so that means we’re going to have just “1” in there.

Tom: Actually, I wasn’t introducing a marble “1”.

Max: Well, that’s all you got left after you get rid of this part.

Tom: Right, but I'm saying why does there have to be a marble "1"? Why does there have to have a marble to label "1", why does that marble have to exist?

Max: Well, 'cause when you finish the process, you eliminated all of them numbers. You ended up with the zero vector or whatever, so you just have "1" over here.

59:07 Max: So "1" is always in the jar. We can look at this as two objects, and the first object is always the "1".

Tom: Oh, I see what you're saying. I'm not sure I agree with it, but he's sort of saying like you know, if you have a ball with a label 1+1/n, then that's equal to a ball with label 1 plus a ball with label 1/n or something like that? I don't know…

I: How about we look at the sequence of labels as saying "2", "3/2", "4/3", and so on.

Max: It's the same thing.

1:01:30 I: Here's a suggestion. I'll start outlining a solution for this problem [1+1/n Marble Problem], and you stop me if you think I am making an incorrect statement or if you don't follow any of the inferences. The set of marbles that I'm working with in the 1+1/n problem is in one-to-one correspondence with the natural numbers, where 1 goes to 1+1/1, 2 goes to 1+1/2… right?
So if I use this correspondence and I'm using the same kind of reasoning that I did for 1 goes to 2, 2 goes to 3, 3 goes to 4, etc, which is the n->n+1 marble problem…

Max: I don't think that's the right… I don't think you want to use that for this problem.
I: Well, where am I going wrong? I used exactly the same kind of argument as before, and showed which principle I am using at each step.

Max: No, no, no.

I: Which step is wrong?

Max: There's something wrong there, man.

I: So I'm tricking you, right?

Max: Yeah!

Tom: No, I see your point…

Max: All you got to do is take the limit and we end up with "1".

Tom: Because in order to say that marble "1" is in the jar, we'd have to be able to apply our first principle but we can't, so therefore it's not in the jar, right? I mean…

1:06:33 Max: There's something wrong, man.

Tom: Right… so first you have to show…

Max: You're saying what is NOT in the jar, but you're not saying what IS in the jar.

I: Well, because in order to be able to say that something is in the jar, I need to be able to apply the first principle, that's the onle that lets me say something is in the jar, right?

Max: It's not put into the jar at any finite step, that's the problem. It's only there at the end. It's not there… But we can say like anything between 2, from 2… including 2 but not 1… (writing (1, 2]). These are not in the jar.

I: What do you mean by that notation?

Max: From 1 to 2, including 2 and not including 1.

I: Is that an interval of real numbers?

Max: It's an interval, yeah.

I: In our problem the numbers on the labels are fractions though, so I'm not sure…

Max: You know what I mean. Start at 2, go to 3/2… so everything above 1, these are all eliminated. So we say that those are not in the jar… I guess using the second principle.

I: Yeah, that's what I'm doing here.

Max: But you're not saying what IS in the jar.

I: Correct, because I cannot say about anything that it IS in the jar. What we discussed so far is that the only way I can say something is in the jar at the end is if we apply…

1:08:30 Max: It's not at any finite step. […] We need some new principles.

I: That's the thing. If you want to say that something is in the jar for a different reason, then you need to modify the principles… So then you might want to modify P1 to say that this is ONE case in which an object can be [in], and then there must be an additional principle based on which something is in the jar. But if that's the case, you need to tell me what principle that is, and why does that principle make sense?

Max: It's a simple limit, does that make sense?
I: The question is, why do you want to take the limit in this problem? Who says that we're supposed to be taking the limit? Based on what?

Max: It's sitting right there, what else are you going to do with it? Well, if you finish the process, you're going to have to look at it as n approaches infinity, and as it does that, the label on the ball tends towards 1. So if you finish the process, you're just going to be left with "1".

I: So Max, again clarifying our positions. I agree with you that the number on the ball, in magnitude, approaches 1. It's a number closer and closer to 1. It's not clear to me why you want to take the limit of those numbers, though.

Max: At t=1, that's what you have, if you finish the process.

I: If you want to do it that way, then let's take the problem before, [n-> n+1 problem] … So the label on the n^{th} ball is "n", right? As n approaches infinity, n approaches infinity… so if you want to talk about the limit of the sequence of labels, the limit is infinity. How about in that problem, I say "if you're really finishing the process, then you must have a ball with this symbol (∞) in the jar"? Is that argument ok?

Max: The jar is empty in that case.

I: But you see, I thought I'm using exactly the same principle as you just told me.

Max: Well, no. Ok, man. Look. It's different. You have one term there [n->n+1 problem], I have 1+1/n.

Tom: It doesn't matter though, like I could write 1+1/n as (n+1)/n.

Max: Well, this limit is 1.

I: Ok then, then how about we don't even have that "1" [referring to the 1+1/n problem]? We have only "1/n". Not two terms, one term.

Max: Then it would be "0" at the end.

I: But you said the difference is that that's two numbers, this is one number. Now it's not even…

Max: Well that's 0, this is 1, so 1+0=1.

I: Yeah, we agree about the limits.

Max: In this case [n->n+1 problem] we… sort of the vector thing, and eventually we push them all out, and we have zero at the end.
I: Tom, what do you think about this? Where are you at this point?

1:11:50 Tom: I was thinking about the proposed principle that you know, if you… in order to say that marble "1" is in the jar for some reason other than this, you know, have some sort of principle about a limit converging to a particular number or state, if we were to do that we'd have to go back to every problem we've already done and re-examine every one of those to make sure that, you know, everything is consistent and it makes sense. Because once our principles are different, then it's potentially a completely different set of answers. So you have a set of answers for all these problems for these 2 principles, but you have a different set of answers for these principles modified with Max's principles. It's almost like they're different… I mean they're the same problems stated but you're using different principles so it's like a different game.

I: I see, you'd be changing the rules of the game. So now the question is, which rules should we choose? I agree with you that if we change the rules of the game (that's why I wanted to specify those, to be really clear what the rules are) then we would have to revisit every problem. So at this point, what do you think… do you think these two principles are enough, or do we need to add anything?

Max: Well, we obviously need to do something.

Tom: I mean… I don't know.

Max: It doesn't hold for this one (referring to 1+1/n problem).

Tom: But that's the thing; it seems like… I mean why do we want so much for marble "1" to be in the jar? I mean it sort of makes sense, it's plausible to me that it could be…

Max: There's no way the jar could be empty.

Tom: I mean, um, it's sort of like… it seems like sort of an artificial principle we're tacking on…

I: The limit one?

Tom: Right, Max's principle, the third one about marble "1" being in the jar, because… I suppose you could introduce a 3rd principle and modify this one so that it's consistent with it and then redo every problem… But I'm not sure what the motivation would be, because these principles (referring to the original two principles) are consistent in and of themselves, like you could answer every problem with these principles, right… so maybe in some certain context it might make sense to have Max's principle. You know, that would be a different game, you know. I mean you could do that… if you could set up 3 principles that are consistent with one another that would allow marble "1" to be in the jar, that's fine, in the sense that it is a consistent game. I'm not sure what the motivation of that would be.

1:14:40 I: Well, let's try to entertain this limit principle. Let's try to decide whether to add it. So Max, if we add that principle, do you think that would change your answer to the n->n+1 problem?

Max: No, I'd still say that in that case the jar is empty.

I: What do you think, Tom?

Tom: I think it would change the answer. Because we're saying our principle is that…

I: Yeah, let's state the principle clearly. What is the principle, Max?

Max: What is the principle?

I: Yeah, the one you want to add; the new one.

Max: I know what it means, I'm not sure… Oh man… Well ok… Well, look at… you just look at what happens to the label as n approaches infinity, which is like taking the limit…

Tom: Well, you have to have a sequence first.

I: Right, so in order for that to happen it has to be problems of this type, where it's only one ball in the jar at one time, have only one sequence happening, and you're saying if the sequence of the labels has a… finite limit or any kind of limit… what do you want to say?

Max: I guess it would be a finite limit.

I: Only a finite limit?

Max: That's right! Because if it would be infinite then we'd have a different case. In this case, since it's finite the end state is the finite limit.

Tom: But… I have a thing. If you have a particular problem you can look at it in terms of sequences, potentially, if it's the sort of problem where you'd be able to do that… but you don't have to. I mean there are other problems we did where, you know, I think you had something with sets for one of these problems but I didn't use sets to answer it. So there are different ways of answering a problem…

Max: You gotta get the same answer….
Tom: But Max's principle seems to be dependent on a particular way of looking at the problem, with sequences. So for any problem that has a sequence, I can try to solve it some other way… even if I got the same answer, because I'm not using sequences I wouldn't be able to use Max's principles, so I would be restricted to these two and I would then get a different answer.

I: So you're saying that would lead to inconsistencies just like with the criteria…

Tom: Yeah, potentially.

1:17:10 Max: That's why those two are not the only two we need. So we got the finite limit case now. Since this limit is finite, we're going to say that the end state of the marble is the finite limit… equal to that, which is "1".

I: Ok, let's entertain that idea.

Max: In the case where it's not finite, then we end up with something else.

1:17:43 I: I got an idea. Let's take the marbles from the previous problem (the ones labeled 1+1/n) and add an alternate label to each one: the marble labeled "1+1/n" gets the alternate label "n" (separated from the original label by a "/"). Imagine the marbles as arranged from left to right, starting with "1+1/1 / 1" and continuing indefinitely to the right. If we look at the alternate labels and start "processing" the marbles from left to right by putting each in a jar, then tossing it out and putting the next one in line in the jar, we get the n-> n+1 problem for which you agreed the answer is "empty jar" (for the state of the jar when the process is completed).

Both: Right.

I: But if we handle the marbles in the same way as before, but we look at the original labels (1+1/n), we get the 1+1/n problem, for which Max claims we have marble "1" in the jar after all steps have been performed.

Max: Yes, that is my position.

1:25:10 I: Now Max, imagine you're at a distance from the infinite sequence of marbles. which are arranged in front of you, starting with "1+1/1 / 1" on the left and going on to the right. Tom and I are next to the sequence of marbles and I tell Tom which "game" to play (that is, which label to pay attention to when processing the marbles), but you [Max] cannot hear what I told Tom. From that distance, where you can see the marbles arranged in front of you but can't see the labels), can you tell which game Tom is playing, one he starts playing?

1:27:15 Max: No. But you see, at t=1 though, if he's playing the first game, which is 1+1/n, he's going to have "1" at t=1. If he's playing the second game, the ball is going to disappear at t=1.

I: Ok, so you're saying even though from a distance you can't tell the difference between the two games, at t=1 something magical is going to happen and finally you're going to be able to tell which is which, 'cause something is going to be different.

Max: Yeah, cuz in one case it's going to be a ball, in one case it's not going to be a ball, so that's how I'll know.

1:28:12 I: Ok, now Max, imagine that now we're playing a game where one set of balls is yellow, the other is red. You don't have any information on the labels on the balls, you don't even know if there are any. Do you have any reason to believe that at t=1 the outcome of the two games (one with the set of red balls, one with the set of yellow balls) would be different, not having any information on the labels?

[End of Session 6, File A]

Session 6, File B
5/18/08

00:50 Max: Oh, if he just has two infinite sets of balls? Oh sure, he's going to exhaust all the balls, yes.

I: In both cases?

Max: Yeah, if he has two sets of balls, yes.

I: I now decide to put labels on the two sets: the "1+1/n" set of labels on the yellow balls, the "n" labels on the red balls, and ask Tom to process each set of balls in the same way as before, when there were no labels. Will the addition of the labels change the outcome of the 2 games?

1:52 Max: Yes.

I: Yes?

Max: Yes.

Tom: No. It doesn't matter whether there are labels or not. So right, so there wouldn't… I mean, the jar would be empty if we apply these principles.

11:15 I: Ok, time to recap each individual position.] So at this point, what do you think Tom?

Tom: I think we should just stick with the two principles as we have them.

I: Ok, and if we stick with these principles, what does that mean for this problem (referring to the original 1+1/n problem)?

Tom: Um, the jar is empty.

I: [addressing Max] And you want to say that if the limit of the label…

Max: If the limit is finite, then we have something in the jar, that is ball “1” in the jar. If it’s infinite, then in that case there’s nothing in the jar. So that’s two different answers, and I like them both.

I: Ok. So you want to have number “1” in the jar, based on that principle.

Max: That’s right, yes.

[We end this discussion here, as it seems each student has made up his mind with respect to this position.]

15:10 I: Ok, moving on to problem d) (The 1/2 pb).

Max: Well, neither of the principles applies. Neither of those two hold (referring to the 2 principles).

Tom: Right.

Max: We’ve got 2, then we’ve got 1 again…

I: Let’s see, let’s focus our attention on ball 1. Can we apply the first principle for ball 1?

Max: It’s there for every odd numbered step, so no.

I: Ok, so we cannot say that it is in the jar. Can we apply the second principle?

Both: No.

I: And the same I guess goes for ball 2.

Tom: Right.

I: What about Max’s principle, can we apply that?

Max: Well, we have a divergent sequence. [short pause] We know there IS a marble in the jar. That's a different case… we got another case.

I: Let's recap. We cannot say that 1 is in the jar, we cannot say that 1 is NOT in the jar, we cannot say that 2 is in the jar, we cannot say that 2 is NOT in the jar. So far we're clear, right? And we can't apply your principle of taking the finite limit because the finite limit doesn't exist. So now, then based on what principle are you saying that there is a marble in the jar? So far we don't have a principle here that tells me that there is a marble in the jar.

Max: Well, for any $t<1$, we can determine whether it's 1 or 2. At $t=1$ it becomes indeterminate, but there's still one of them in the jar. Well, we can't apply the first principle.

I: Right, so then what are you applying? I'm not clear…

Max: Well, I'm not applying any of these, I'm applying my own principle.

I: Oh, so you want to add to the list of principles again?

Max: Add another one, yes.

I: What does this one say?

Max: You know what my answer is…

I: Perhaps, but it would be useful to state a more precise principle, just in case we don't understand exactly…

Max: The principle? Well, I don't know how to word it, but if you have an oscillating… I call it an oscillating thing…

Tom: You mean a sequence that's not convergent?

Max: Right, ok. It's going between 1 and 2. When it does that, you can't determine it… you cannot determine the state at $t=1$. You just know that there is one of them in the jar.

I: So that would be for a very specific type of problems, you're saying for problems where I have…

Max: There would be d) and the lamp problem, d) and h). It's the lamp thing, the same thing: on, off, on, off.

18:10 I: Uh-uh. So then for h), how would you apply that principle for h), for the lamp?

Max: Same one – you just replace 1 and 2 with on and off.
I: Right but if you want to say…let's say I don't know this problem, apply this principle for the lamp.

Max: Ok, well, you just look at it… for t<1, you know whether it's on or off at any time, but once t=1 you can't determine whether it's on or off.

I: I'm not sure how you're applying… your principle is this…

Max: What's my principle?

I: I though your principle is only for when you have a sequence of numbers involved, cuz you said "non-convergent sequence".

Max: Well, it says "oscillation between two states".

I: Ok, "oscillation between two states". Let's put it that way. Then… what was the conclusion of the principle?

Max: At t=1, the state is indeterminate… it's one of those states, but indeterminate… at t=1.

I: Ok, so it's one of those states, but indeterminate. So then applying that principle, we would say for the lamp that it's one of those, but indeterminate, and for the balls, we would say it's one of 1 or 2, but it's indeterminate.

Max: That's right, that's correct.

I: [addressing Tom] What do you say about that?

19:52 Tom: I think we're adding a lot of principles…

Max: We don't got a choice!

Tom: Well… I mean…

I: Well, if we were to use… obviously the finite limit principle does not apply anyway, cuz our problem doesn't match its conditions… so if we were to use only these 2 principles, what conclusion would we get to?

Max: We already…

Tom: We wouldn't have a conclusion…

Max: Cannot even apply either of…

I: Yeah, so we stated why we can't claim the conclusion of P1, we can't claim the conclusion of P2, for either of the balls, so…

Max: That's why we made our own addition [to the principles].

I: Yeah, so I cannot say anything about "1", whether it's in or out, I can't say anything about "2"… and basically these principles don't let me say anything. So then what if I formulate my conclusion this way: for this problem, I'm going to say that nothing is determined about the final stage, for the lamp. Is that the same as your answer? Cuz based on my principles it doesn't let me say anything. So I'm going to say the final stage is not defined by the process.

Max: Right – it has to be one of those 2 but you don't know which one.

Tom: Well no… I don't think you're saying that it has to be one of the two.

I: Right, there's a difference between what I'm saying and what you're [Max] saying. I'm simply saying the final stage is not defined by the process, without adding any further information.

Max: Well, it *has* to be one of those two.

Tom: But why?

Max: Well, what else? Those are the only two states you have. I mean you got a lamp that's either on or off. That's the only two states. You can't have any other condition. It can't be like half on or something. It's just indeterminate.

Tom: See, the thing is, if it's one or the other… I mean it can't be that ball 1 is in the jar at the end, because then we would have to apply the first principle and we can't. We couldn't say that ball 2 is in the jar cuz we'd have to apply principle 1 and we can't. The same thing goes for the second principle. So, I mean…

I: Right, and we didn't have any principle that said we have an unspecified object in the jar, so what if we just look at the sequence 1, 2, 1, 2… nevermind jars, nevermind processes. We just have this sequence, 1, 2, 1, 2… if I ask you "what is the limit?", if I put it in those terms. If I say "what can you tell me about the limit of this process", what do you say?

Tom: There is no limit, it is divergent.

I: Right, the limit doesn't exist. You're not telling me the limit is either 1 or 2 but I don't know which one; you're just telling me the limit doesn't exist. Then if we put it in terms of the jar and marble 1, marble 2, marble 1, marble 2, … why do you want to say it's either 1 or 2 but we don't know which one?

Max: Um… it's either 1 or 2 but you don't know which one.

I: Yes, the question is why do you want to say that? When I was talking only about the limit of the numerical sequence, you just said that limit doesn't exist. You're not telling me it's either 1 or 2 but I don't know which one, you're telling me it doesn't exist.

Max: Well, you know that it has to be either 1 or 2.

I: It comes down to the question why does it have to be either 1 or 2?

Max: That's the only two conditions.

I: I see, but infinity can be tricky, so you need to think carefully what principles you're applying if you want to claim anything at all about the final stage.

Max: I'm applying the "t= 1 state" principle.

24:00 Tom: Maybe, like, you're thinking more in terms of the physical real world, you know like if you actually had a lamp sitting here on the table…

Max: So what's your answer?

Tom: That you can't determine anything at all based on our two principles.
But I mean, these two principles are very theoretical, I mean they're stated that way. I was thinking like if you think about it in terms of like a real physical lamp and there are two states that the lamp can be in and it's alternating between one or the other, then it seems sort of reasonable in sort of a practical sense to assume that it's always going to be either one or the other…

Max: That's right, but it is!

Tom: That's a very practical way of looking at it, which is fine, except that these aren't very practical principles, they're theoretical principles.

I: Well, if we are to take "physical world" aspects into consideration, then perhaps we shouldn't talk about completing an infinite process as most physicists would argue that such a task is impossible. So the very fact that we're attempting to describe a state "at the end" of an infinite process suggests we're already removed from the real world and therefore should be careful with using "real life" assumptions such as "a real lamp is either on or off, at any point in time".

29:07 I: So, if we want to use just these 2 principles, and Max's "finite limit" principle doesn't apply, then the only thing we can say is that it's not defined, or it's not determined by the process.

Max: That's right.

I: If you want to say that there is one ball… one physical ball in there, for the 1, 2, …
Max: There's a ball in there! There's a ball in there!

I: Then it's not clear which principle you're using, that's all I'm saying.

Max: Right. I'm using my own principles.

I: That's… that's the oscillating principle?

Max: Yes.

I: Which said what, that if…

Max: That at t=1, it's either 1 or 2, but it's indeterminate…

I: You need to say what the assumptions of the principle are. If something holds, then… what is that "something holds" part? What is the hypothesis of the principle?

Max: Well,…

I: So we have the situation with replacing balls, and if we have an oscillating sequence associated with this replacement process, and we have one ball at a time at each step… is that what you're thinking of? Then, at the end of the process, there is going to be one ball of undetermined label.

Max: That's right.

I: And why do you choose this principle?

Max: It makes sense because that is what… because it's right!
[…] It makes sense because it cannot be proved false!

32:22 I: So then based on your [Max's] principle, the answer should be " there is one ball but we don't know which one".

Max: That's right.

I: And you Tom, do you want to embrace Max's principle?

Tom: Let me think… I'm leaning towards not…

34:00 [Moving on to problem j), the modified tennis ball problem where at each step you have "1" in one bin and the rest of the already processed balls in the other bin].

I: For that one, we saw that at any finite stage, one of the bins had ball 1 and the other bin had all the others that were processed up to that step. Except that the contents of the bins were switched occasionally.

Max: That's right. Yes. So we just don't know which in bin A and which in bin B, only that one of them in bin A, one of them in bin B… it's indeterminate. Same as last time.

I: I thought your principle was about an oscillating sequence… and only one element at a time…

Max: A A B B… same thing.

I: So now your sequence is of a different kind, it's a sequence of sets?

Max: Ok…

I: I'm asking you, I don't know. I'm trying to see… so far your principle was only about a numerical sequence.

Max: Oh, it works for all sequences… […] We know we have ball "1" and balls up to n, minus ball 1… so we're just looking at the end process and we know that we have those two quantities in the bins; we just don't know what label we have on the bins.

I: So just by applying these two, let's see… what would be the conclusion just by applying only the main principles [P1 and P2].

Max: You can use that to determine which balls are in the bins. But I'm talking about trying to figure out which one is labeled bin A, which one is labeled bin B, right?

I: Actually… the way I phrased the problem, I'm not changing the labels of the bins, I'm changing the contents. So the bins are stationary bins, with their labels remaining the same…

Max: It's the same thing, man. I mean if you're switching the bins… it's easier to change the labels.

I: Just imagine that you're grabbing everything (referring to the balls in one bin) and moving it to the other bin (and vice-versa). So the labels stay the same, I'm not changing the labels, I'm changing the contents. In this case, can we apply any of these principles for anything? The two main principles.

Max: Well, no…

Tom: So we're switching the contents of the bins…

I: If we're looking only at bin A, can we apply any of the principles to claim anything? Only about bin A.

Max: No.

Tom: No.

I: So we can't say about any ball that it's in bin A, we cannot say about any ball that it's not in bin A.

Both: That's right.

I: Same thing about bin B?

Max: That's right.

I: So if we're applying only these two principles, what would be the answer to this problem?

Max: Well, you couldn't determine…

Tom: Same as the lamp problem… and d) (the ½ pb).

Max: But I don't agree with that. If you apply only P1 and P2 you can't determine it; they don't apply so you can't determine it.

I: Right, so would you say that the final stage is what?

Max: In this case, the final stage… you don't know nothing about it.

I: It's undetermined, undefined… by the process?

37:44 Max: Using my principle, you know the quantities of both bins, you just don't know which quantity is in bin A and which quantity is in bin B.

I: So tell me again how you're applying your principle, cuz I'm not clear…

Max: Well, I'm looking at the finished process, and I know my two quantities, but you don't know which bin they're in.
I: Max, it's not clear how you apply the principle. I understood *that* part (referring to what Max has just said). See, what you formulated before had a very specific hypothesis and thus it seemed to be applicable only to a small class of problems.Is the current problem part of that class?

Max:That's really similar. In this case, we can look at ball "1" as being "1", and the natural numbers minus ball "1" as being like "2", so we're oscillating between "1" and "2". But actually since we have 4k+1, 2, 3, …

I: But we have two containers now. Do you want to apply the principle twice, once for each container?

Max: Sure, yeah.

I: Ok, so let's talk about bin A only.

Max: Bin A? Alright. But we also have the thing where it was constant for two steps, but that's a small detail. So in bin A we're going to have something like 1, 1, 2, 2, … then bin B would go like 2, 2, 1, 1, … know what I mean? That's how it works. So they're both indeterminate.

I: […] OK, so then applying your principle it's going to be that for each bin, it's going to be one of these two cases, but we don't which one.

Max: Right. If "1" is in bin A then "2" is in bin B, or the other way around.

I: Ok, so with that principle it's that, and with only the other principles it's not defined at all. Tom, what do you think about this?

Tom: I don't think you can determine… I don't think you can apply either of the two principles. Um… I don't know, it's…

I: So we already discussed what the answer would be with only these 2 principles, and with his extended one.

Tom: Yeah. It seemed sort of reasonable, based on the way the problem was defined… I don't remember exactly how it was defined, but I remember it seeming reasonable to me that one ball would be in one bin, the rest are in the other bin, and that the labels of the bin are not determined, but…. I guess we wouldn't really be able to say that, using only our two principles… or would we? I don't know…

Max: We wouldn't. Do you want to set my principle?

41:35 I: Let's recap what your answer was for the 1/2 marble problem.

Tom: Right, it was "not determined". Yeah, then it's the same thing.
Yeah, I'll go with just those [P1 and P2].

42:10 [Moving on to discussing problems where what's being obtained at each step of the process is a set of points (on the real line or in the plane), and each set is included in the set obtained at the next step). Looking first at problem b) (the Triangle problem).]

Max: That's right. Ok man, so we had different answers here. I'm going to say yes, the entire segment BC is included. That's the only way you can finish the process.
I: Ok, let's think carefully about which principles are used here.

Max: Well, I'm sticking with my answer. We're going to need another principle to cover these type of problems.

43:45 I: Let's structure our discussion here a bit so that we're not all over the place. We're going to start with the two core principles and see what the answer is with that, and then add principles as necessary and see if the answer changes.

Max: Ok.

I; So in this problem after a given step, the set of points contains the set of points obtained after the previous step. So we get a chain of sets with respect to inclusion, with the first being included in the second, etc…

Max: Sure. We can apply the first principle.

Tom: I remember we defined a set explicitly with, you know, the two…

I: With a formula?

Tom: Yeah, with a formula for what the set was… so any element of that set that we defined, we would be able to apply the first principle for. And then we can also say that no element of the interval BC would be in our set from the first principle, 'cause it's never… like no point from the interval BC is… so here we apply P2.

Max: Oh, I see. I'm going to need my amendment then.

I: We'll get there in a bit. So using only these two principles, we would get all points of the form, something over 2^n, where the top part was of a specific kind that we discussed… so all fractions of that kind… Oh, I guess here we need two coordinates, right… it was ½^n and a lot of other things for the x-coordinate. […]

Ok, so we would only get the points of that form, which don't ever get to be on the segment BC. Agree so far? Using only the first two principles.
Tom: Yeah.

Max: Right.

I: Ok. Now Max, do you want to… you want to add a new principle?

46:13 Max: Well, I want to say that the entire segment BC is included.

I: Ok, so obviously we need to change our principles for that…

Tom: It would have to have something to do with the fact that the size of the set that we defined last time, you know, all of the midpoints not including interval BC… I think it's equivalent in size to the natural numbers… I think we said that?

I: I don't think you did. But that's correct.

Tom: Right. So that being the case, the interval BC is equivalent in size to the real numbers, which is a different cardinality, so you're shifting, like… we have different sizes, you know, for the midpoints above interval BC, whereas the interval BC… so you'd have to…

I: Right, you would be adding a lot, not just one thing.

Tom: Right.

Max: I thought we talked about that last time.

Tom: Right. So your principle would have to have something to do with that, would have to bring up that… I mean would have to reconcile the difference in size.

Max: That's right.

I: So what is this new principle? What principle do you want to use to claim that segment BC is in there?

Max: I don't know if I can make an argument for that… We talked about this. It was determined that it didn't matter for some reason… the size of… I knew it was a lot more points on BC than we had fractions…

I: Ok, so maybe putting that aside for now… well we still don't have the points on segment BC using just these two principles because they're not. So the question is, what is the extra principle you want to use to claim that points on BC are in here?

Max: Well, I'm just saying that if you gotta finish the process, you're going to end up at the true points, which means you can't divide them anymore, which means you have points of 0 length.

Tom: Well, line segments of 0 length.

Max: Well, I'm talking about…midpoints are points.
Tom: Yeah. But like you have a line segment, and the midpoint is a point on that line segment that has a particular definition, right? So I guess you would have to say… I'm only thinking of line segments that have positive length. So I guess you could argue if you have one point, then that point is a midpoint of itself, I don't know….

Max: True. That's right.

49:12 I: It would be nice if you can formulate this new principle in more general terms, perhaps this way it would be applicable to other situations as well and you wouldn't have to keep adding new ones.

Max: Well for everything before this I'm positive that it works.

I: You mean that it's a consistent theory… in that sense.

Max: Yeah, I know that these solutions are right. On this one I just question it a little bit.

I: All I'm saying for this one we need a specific principle… it would be nice to have it formulated in more general terms. So that we can apply it to more situations than one. Look at P1 and P2, do you see how we can apply them to all of the problems we discussed so far?

Max: Well, but the problem is they end up being useless for half the problems. For half the problems, we had to make our own amendment.

Tom: Well, no.

I: Well, they turned out to be useless in the sense that using just them…

Max: You can't determine.

I: Is that an unacceptable situation in mathematics, to say that something is undefined or not determined by the situation in a problem?

Max: Not necessarily. BUT… we can make our own stuff that also can work in like this other place.

I: You mean, let's assume an additional principle to be true and just build in a different direction?

Max: Right.

50:57 I: Ok, so again the question is what new principle do you want to use here? It would be nice to formulate it as general as possible so that it's applicable to other problems… as we have two more problems to discuss so I hope whatever you're going to formulate it's going to be applicable for those as well.

Max: I'd rather look at the other problems.

[Moving on to problem k), the Midpoint Problem.]

I: In terms of formulating your principle, Max, something that Tom said when we worked on this M and T problem, that might help you.

Max: Oh man, so you're saying I'm wrong?

I: I'm not saying you're working, I'm just trying to help you formulate your principle. If you remember, when you worked on the M and T problem and you were trying to decide whether it's the full [0, 1] segment or just some fractions of a specific form, if you remember Tom was saying, you can find a sequence that converges to any number on the segment [0, 1]. And I was saying well, let's consider that question, the answer was "yes", for any point on the segment [0, 1] you could find a sequence of midpoints… in the already existing set, that converges to that. And then the question was, well, does that mean we should add those other points as well just because there is a sequence of points converging to that particular point?

Tom: Right. Well, the answer to that question would have a lot to do with how you answered f), the thing with 1+1/n, right? So depending on how you answered f), that would affect how you answered k), with the midpoints and tripoints.

52:45 I: Ok. Max, does that help you? […]

Max: We can find something that converges to it, but I just don't know if you have enough numbers to do it.

I: If I remember correctly, Tom devised a specific procedure that explains how for a given number on segment [0, 1], you can build a sequence converging to it using only midpoints, so fractions of the form k/2^n blah blah.

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Max: Right, he did. I just don't know if you have enough numbers to reach it, cuz of the difference in cardinality, you know.

Tom: I don’t know, I mean…

Max: That would be the only reason why you couldn’t have them. If there was no issue there, then yeah you’d have the whole segment.

I: Issue where?

Tom: I think he’s saying if there’s no issue with my…

Max: You don’t have enough fractions to meet the number of points. But if you include the entire segment that would be every point, including like irrational things… But if you include every point, you don’t have enough fractions to do that…

I: To do what, to approximate it by a sequence of fractions or to cover…?

Max: No, to actually get the entire segment, to cover the entire segment.

I: As in, with the fractions to cover the segment?

Max: Yeah.

I: For that, I think you guys already discussed that that’s not possible as it’s obviously different cardinalities.

Tom: Yeah, right.

Max: Then how are you going to have the segment BC?

I: I don’t know, that’s what I was asking.

Max: That was my issue with that, when we first discussed it.

Tom: I think that when I was thinking about this, remember… I think you just said the thing with having the intervals to construct a sequence that converges to a real number on the interval [0, 1]… I think the reason you’re allowed to do that for any real number is the fact that we’re allowed to reuse fractions if you’re converging to a different number.

I: I think Max is talking about being able to cover the whole segment [0, 1] only with fractions of that type, that are midpoints.

Tom: Oh no, you can’t.

I: Is that what you’re saying, Max?

Max: [who is thinking about the Triangle problem again] I'm just saying that, if you just have points on fractions…all you have is the fractions there, and what I'm saying is that on BC you end up with decimal points. I don't know how I can go from fraction to decimal.

Tom: Well, that's long division, I mean…

I: I think he means the sequence of fractions converging to the…

Max: The problem I have that at the end, I don't think how we can go to BC, cuz that would be like all the decimals. Between any two fractions we know we have a size greater than the number of fractions. So I'm just saying, that was the only problem I had.

I: So does that make you doubt your answer of [0, 1], the full interval [0, 1] (referring to the midpoint problem)?

Max: Oh yeah, that would definitely make me doubt it.

I: So then, are we still stating another principle to claim that BC is included in the final set, or not?

Max: Well, if we were to ignore the size of them and assume that they're equal… which we know they're not but …

I: No, let's not assume something that we already know it's not true.

Max: Then I'd have to say you cannot include the entire segment (referring to BC in the triangle problem).

I: But can you include any points (from segment BC) at all, Max?

Max: No, 'cause if you include one point you include all of them.

I: Why?

Max: You cannot include just any… because you're moving down. So if you were to move to that final place… Oooooh… Oh man, that's a tough one. If I think that way, I have to include all the points.

I: Which way? I'm totally lost.

Max: When you finish all the steps, you have to be on that segment. It contains a lot of points on BC.

I: So wait, it contains a lot of points on BC but not all? […] Or it contains only fractions on BC?

Max: Yeah!

Tom: I'm just asking this out of curiosity. If you were to take a level on this, you have two midpoints, then you have 4, then you have 8, then you have 16… so on a given level, you have powers of 2. But if you were to count them you would have 2, then 4, then 8, right. So that would be like 2^1, 2^2, 2^3, 2^4…

I: You mean how many points you have on each level, right?

Tom: Right, right. I guess my question would be, what's… Because if we are to operate under Max's idea, that at some point you reach the interval BC, you have this idea that the number of points at that level would be 2 to the big N, somehow. So then, what's the size of that set in relation to the real numbers?
Like 2 to the big N, 2 to infinity…

I: Well, that's the cardinality of the power set of N, which is actually equal to R, in size.

Tom: So that would be really convenient for your thing (addressing Max), but I don't really agree with that anyway.

I: If indeed we have that step where you have a 2 to the N… but the debatable thing is whether you actually have that step.

Tom: Yeah, I don't think that you do; I'm still operating under the two principles only. But I'm just saying that if you were to do Max's thing, then…

Max: I like what he's saying then.

I: So the same thing would apply for the midpoint/halving problem.

Max: Yeah, they're all the same. How many points do we have after step n? We have 2^n. And that means we include the origin (referring to the complex number problem).

I: Easy, let's not jump that much. With the halving problem, at each step you create… at step 1 you create 1 point, at step 2 you create 2 points… at step n you create 2^(n-1), and they keep being added to the other points, I mean you consider all the sets together, so it's not only those but also the ones constructed at the other steps as well.

Max: So if you have that many points that includes the whole segment.

I: If you do have 2 to the big N, that's the cardinality of the full segment.

Max: Of the real numbers. Oh yeah!

I: By the way – that does not necessarily mean that it includes the full segment because the set of irrationals in that segment has the same cardinality as R and as segment [0, 1], but it's not the full segment [0, 1]. So just the fact that you're saying that two sets are the same cardinality and one is included in the other, it doesn't necessarily mean they're equal. Just like with the odds and all the natural numbers – they're the same cardinality, one is included in the other, they're not equal sets.

Max: They're the same size!

I: They're the same size, they're not equal sets!

Max: Well, I want to answer this question [Triangle problem]. The set does contain points from BC.

Tom: No… I don't think so.

I: So you're saying it contains some points but not all points?

Max: Do I have to specify that?

I: Yeah…

Max: I'm 100% sure there are some points on the segment.

I: What did the question say?

Max: The question is, does it contain points from the segment BC? And that answer is "yes".

I: Ok. So it does contain some points but not all points, is that your position?

Max: I guess so.

I: And your position (addressing Tom)?

Tom: That there aren't any points on interval BC [in the final set].

I: And that would be applying just these two principles?

Tom: Yeah, just those two, yeah.

1:03:10 I: So Max, one last time, what principle allows you to claim that the set constructed by the completed process contains points on BC? If you use Tom's suggestion from previous sessions (that for any convergent sequence in the "final set", its limit should be in the final set), then if we add this as an additional principle this would require that the "final set" is a closed set… would be another way to say it.

Max: That would be the entire segment then 'cause you can converge to any point.

I: Right, if you want to have that principle, that would force you to take the entire segment. You cannot have just some points.

Max: That's what I originally said. Well, I'm definitely going to include points from that segment, so if I gotta take the whole segment, I will.
[…] It's a lot of points… Well, I know you're on segment BC. It's gotta be a lot of points… the entire segment? It can't be the entire segment. No, it cannot.

I: So then this is not the principle, the one I phrased.

Max: I guess not.

I: So then what lets you add just some points?

Max: Only the fact that we don't have enough to add all the points. Cuz we're working with that fraction stuff and we can only have 2^n points, right? But there's greater than that number of points…

I: Again, if you consider that it is indeed the case that at some point you need to have 2 to big N points…

Max: You do!

I: But that's the thing, why?

Max: Why? That's the only way to finish the process!

I: That's a statement with no back-up, no principle. What does it mean "that's the only way to finish the process"?

Max: When I look at a problem, I look at what's done at the end, I don't look at all this junk in the middle. I know when I'm done, all I have is my one segment. Know what I'm saying?

I: Ok, so you're always looking at the limiting process.

Max: Yes!

I: So see, I guess we do kind of come back to the previous issue; you're taking a limit here, and again the question is, is this what this problem really entails? Looking at the set produced by the finished process, does it mean taking a limit of some kind?

Max: Yes.

I: And you're saying yes; ok, I understand your choice.

Max: And it takes me to segment BC.

I: Ok. And you're saying (addressing Tom)… ?

Tom: … "nothing on segment BC", using only with these two principles [P1 and P2]

I: Ok.

1:08:28 Tom: I'm thinking that if you say that there are some points in interval BC…

Max: I have to say it's all the points?

Tom: Yeah, pretty sure.

Max: Well, that's what I said.

I: Yeah, but it seems you're not sure now…

Max: I just don't know if it's the entire segment. I just know it's some points from BC. […] But he's saying that there's no points. So at least I am partially right.

1:09:32 [I ask them to go back to the M/T problem, keeping in mind their respective positions on the triangle problem.]

Max: I was consistent here. So if it's the entire segment then yeah, it's M and T, but if it's not the entire segment then they're not equal to [0, 1]… to that closed interval. Obviously he's going to say that they're not.

Tom: Right.

Max: I'm going to say that it is then… I'm going to include the entire segment again.

I: So if you're including the entire segment there, how come you're not… [for the Triangle problem]?

Max: I am – I am including BC.

I: Oh, all of it?

Max: I have to.

I: Before you said just some, you mentioned you're partially right because you said some… So now all of it, right?

Max: M and T are equal. I know M equals T, therefore they're equal to [0, 1]. Yeah, I'm going to have to go with that.

I: Ok, so it's the full segment [0, 1], and finally for that z^n problem?

Max: It includes the origin.

I: Based on your taking the limit…?

Max: Yeah baby.

I: And Tom?

Tom: I'm going to say no. It's just the points that are in the sequence that is converging to the origin, but not the origin, based on the two principles.

1:11:35 Tom: I think what Max wants to do is look at f), you know the marble with 1+1/n, 'cause you can write like a good, solid principle like one of these two for f), that I think he can use for a lot of the other problems where we disagree. Because problem b) is using the principle that we haven't stated for f), whatever it is, which can also be used in b), because you can define a sequence of midpoints that converges to any element of interval BC, I'm thinking.

Max: Right, so we're going to include the entire segment.

I: Are you sure it's exactly the same situation, given that in b), the triangle problem, the process builds an ever-growing set from which we don't remove any points at any step, while in the 1+1/n problem, there is some removal going on?

1:14:40 Max: Well, I'm just saying I'm not sure it can be the entire segment.

I: So you're still undecided.

Max: It cannot be the entire segment. […] I can tell you some points that are in the set.
I: Ok, tell me some points.

Max: On this axis, we said OC was length 1, so we would have 0, ½, ¼, 1/8, …

I: Fractions ½^n?

Max: Yeah, all them points. But the problem is there is other points in between those points not included in the 2^n format.

I: Ok. So do you want to say that we're getting the points of the form ½^n on segment BC? With the plus and the minus signs.

Max: Right. 'Cause there's an infinite number of points in between all those points, That's why you can't include the whole segment.

I: Ok, so you want to take only those, ½^n.

Max: Yeah, that's better.

I: Ok. And for the M and T problem?

Max: Same thing, they're not equal to the entire closed interval then.

I: So they're not equal, it's just ½^n… actually not just ½^n, it's ½^n and the top part can be from 1 to…

Max: … to 2^n-1, increment by 2. The odd numbers. […] That's right, it's not the closed interval, but M and T are the same size. That's right, M and T are not equal to the closed interval [0, 1].

I: If they're not equal, are they the same in size?

Max: Oh man. […] Yeah, they're the same size. Yes!

I: For T, can you tell me the specific form of a point?

Max: [starts writing to answer that question.]

I: [addressing Tom] And for you, M and T are … ?

Tom: … not equal to [0, 1], yes.

I: And you have a specific form for the points in T as well?

Tom: Let me see. So first you would define the process of taking midpoints and the process of taking tripoints, you can formulate it so that you can say at any finite step these are the points in M and these are the points in T. And you can say, by your first principle, that these points actually are in M and T, you know when you're finished. But then you'd also be able to say that particular points like the sqrt(2) aren't in M or T, because they are not added at any finite step.

I: So are the two sets equal in size?

Tom: I actually don't remember... Wait a minute. M and T are infinite sets. I think they have to be equal in size. Because they're both also equal in size to the natural numbers, so yeah, they'd have to be equal in size.

[End of session]

Appendix E: Cycle 2 Transcripts

Note: The Pre-Tests were done with a student at a time. In each regular problem-solving session, the two participants are Chris and Todd (pseudonyms).

Todd Pre-Test
10/29/08

0:00 [Todd starts work on the Original Tennis Ball Problem.]

Todd: So there's A, B… I'm just going to draw them. So the tennis balls are numbered, so you can identify them I guess?

I: Uh-um.

Todd: I kind of remember this problem [from written pre-test]. I'm just going to do it over then… Oh, that was the question, it says "now assume all steps have been completed". I guess a lot of people, including me, were confused about that. The way I interpret it was, like, you're at a given step, and I think maybe it means all the steps prior to that had been done in ordered… or does it not mean that?

I: It could mean that, except it says the process is continued ad infinitum. What does that mean? It means you have infinitely many steps. How many? As many as the natural numbers. So for each natural number, you have a step. When I say "all the steps have been performed", it means that whatever natural number you're going to choose, the step with that number has been performed. So you're at a point where you can say that about any natural number, that the step with that number has been performed.

Todd: Ok, then it says what are the contents of the bins at this point?

So I just kind of said… First I did small cases, so what happens at step 1? Put 1 and 2 in bin A, so I drew 1 and 2, and then it says move ball 1 to bin B, so I drew 2 in bin B. That's like the beginning and end of a step, so I just considered at the end of a step… So for step 2, you get 3 and 4 for bin A, and then 2 in bin B. So that's the end of step 2. So I just did that a couple times, and you see 1, 2, 3, and 4, 5, 6. I guess the total number of balls is 2 times the number of steps. And then if you look at bin B at the end of each step, you have all the balls up to and including that step. And in bin A you have, at a certain step, all the balls from step plus 1, to 2 times step.

I: Go ahead, write that down.

Todd proceeds to write what he said, using x for the step number.

5:24 Todd: And I think I stopped there… yeah.

I: Yeah, that's what you did last time [on the written pre-test]. So this is for finitely many steps. Because when you say x, x is a natural number. That means you're considering what's happening after finitely many steps. The question is asking what are the contents of the bins after infinitely many steps have been performed (as many as the natural numbers). Is there any way you can look at that?

Todd: Um… I guess I could say, oh, there's tennis balls. In terms of the number of balls that are in bin A and the number of balls in bin B, at any step they're going to be equal.

I: To each other?

Todd: There's not much more I can say specifically. I guess I didn't answer the question… I don't know if you can complete all steps.

I: Ok.

Todd: No, you can complete all steps, but you can't look at it after it's been done.

I: Wait, you can complete them as in "you can finish all of them"?

Todd: Well, you can say it would continue, and then if you stop it at a point you can be like, here's what in the bin.

I: So this would be at a point where you stop the process?

Todd: Well, this whole stop/continue process thing… I mean just assume the process has been done for any number you can think of, and then if you look at any specific step number, you can look at it and say something about it. But this whole thinking as happening in time, and then going to the end of it…. There's no end to it so…

7:43 I: Ok, but what if each step took 0 seconds, it didn't take any time. Then infinitely many steps would be performed in 0 plus 0 plus 0… how many times? An infinite number of times. It would be like an infinite series, each term of the sequence you're adding is 0. That equals 0 (the sum). That means you're done in 0 seconds adding the whole thing. This sum represents the time needed to finish all the steps. Then it wouldn't take an infinite amount of time to finish it. It would take 0 seconds.

Todd: If each step has 0 time… but I mean then there's no difference between not doing it and doing it infinitely many times…

I: Well, there would be a difference in the sense that the contents of the bins would change.

Todd: Well, I'm still facing the same problem which is like… I guess you have every natural numbered ball in B, and then… The problem is that the balls in A and the balls in B are different. So every ball that's in B has at what point bin in A. But then there are balls in A that are not in B yet.

I: At a specific step?

Todd: Yeah. Then the problem is, if you say this runs for every natural numbers, eventually every natural number would be in B. But the problem is that at any step there's an equal number of balls in A as there are in B; they're going to go to B eventually but they're not there yet. I guess I just have a problem saying at the end of this thing what's going to happen, because what happens at the end of the natural numbers… I don't know what happens at the end, that's not how I think about it…

11:15 I: Ok, so because of that you're not sure what to say about "the end"… whether it exists or not?

Todd: Well you can say something like… I don't know, I just have a problem with this "all steps have been completed thing", because you can't…

I: Ok, let's draw a line. I'm going to modify that problem a bit. The beginning part is the same: I take balls 1 and 2 and put them in A, and then move ball 1 to B. But then at step 2 I'm going to take balls 3 and 4, put them in A and then I move ball 3 to B. Before I was moving ball number 2, right?

Todd draws a diagram depicting the movement of the balls for steps 1 and 2.

Todd: [softly] Even and odd numbers?

I: Can you tell me what step 3 would be?

Todd: I guess you'd have 2, 4, 6 [in A] and 1, 3, 5 [in B]?

I: U-um. Now, with this one, if I say all the steps have been performed, which means for any natural number, the step with that number has been performed, would you be able to answer the same question, what are the contents of the bins after all steps have been performed?

Todd: Yeah. Well, it would be sort of similar to the first answer, but it would be a lot nicer to say. I would just say that B would be all the odd numbers and A all the even numbers.

I: Why is that nicer?

Todd: It's more satisfying… Oh here's another reason. The contents of A and B do not change once the steps have been completed, there's no more… So at each step we put in 2 more balls in A and then move 1 to B. But we don't take balls that have been through a step before and move them again. So for instance you take 1, 2, 3 [referring to the Original Tennis Balls Problem], those were once in A and then were moved to B. [In the variation], A and B have fixed contents – once 2 is in A, it's not going to move out again.

I: Ok. That makes it easier?

Todd: Yeah. If I was going to say something after all the steps have been completed… I still don't know how I'm thinking about that, but at least you know that all the balls in A will be 2 times a number, and all the balls in B will be 2k+1, so that's like odd. Whereas in the first problem, I can't say anything… So let's say you choose a step in the first problem and you say here are balls that are in A, the problem is when you go beyond that they are no longer in A, they'll be in B. Whereas if you do the same type of thing in the 2nd problem it's going to be, these balls are in A, and then as you continue with steps 3 times as big, they're still going to be in A. So it's easier to say after all steps have been completed what are the contents because they don't change.

15:50 I: So would you say you're somewhat comfortable saying, for the 2nd problem, that you have odds in one bin and evens in the other bin, or entirely comfortable?

Todd: I'd say I'm entirely comfortable… yeah.

I: And for the first one you're not comfortable saying anything about what happens after all steps have been completed?

Todd: Not after all…after any step I think I did that right. The thing is eventually, everything is going to be in B so you can't say anything… the information about what's in A and what's in B depends on the step [pointing to OTBP], and the information about what's in A and what's in B depends on the numbers in A and B [pointing to the variation].

I: The numbers in… what do you mean?

Todd: It's going to be in A if it is divisible by 2.

I: Oh, so you have a way to characterize what goes into A and a way for B… for the 2nd problem.

Todd: The way that I thought of characterizing the numbers in A and B in the first problem depends on the step number, and I guess when you say like after all steps have been completed…

I: You don't have any step number?

Todd: Right. That's why I just said if you look at any given step then this is what's in them, but other than that… I guess you could say all numbers in A… No, you can't.

I: What were you going to say?

Todd: I was going to say… but I don't know if I think that way now… is that all the numbers in A are bigger than B. But there's nothing about each individual number itself that will tell you it's going to be in A definitely after all steps have been completed.

I: Ok, so you don't have an easy way to characterize… what property should a number fulfill to be in A after all steps have been completed?

Todd: Right.

17:40 Todd starts work on the ½ Marble Problem.

I: It's the same problem you had on the written pre-test.

Todd draws a container representing a jar.

Todd: We might have to talk about the same stuff again, I don't know. Yeah, I did it the same way. At the end of step 1, 3, 5, marble 1 is in the jar, and at the end of 2, 4, 6, marble 2 is in the jar. So if you are given a step number, you can tell which marble is in the jar. But again, it's like a property of the number of the steps, so when you're given the whole "all steps have been performed" thing, there's no way to say what's in the jar. Because after all steps have been completed… I still don't know about that.

I: Let's take it to mean the same thing it means in the second version of the problem on the other side, right? Where you said you were entirely comfortable answering that question? I mean, it's referring to a process that has as many steps as that, and there you were comfortable answering the question after all steps had been completed. So we take it to mean exactly the same thing. Obviously the actions of the process are different, what you do at each step is different. But in terms of how many steps you have, and how you look at a point that's supposed to be after all of them, just what you do [at each step] is different.

Todd: Well, I'm saying my approach to it, how I did it back a week ago, was the same as my approach to this one [OTBP], which was to say like at each step, here's what's in it… based on the number of the step.

I: Can you write a summary of that?

Todd: Sure. 'Cause here [variation of OTBP] you're saying oh, the number of the tennis balls in A and B is a characteristic of… or if you take a number of a ball, which bin it's in is a characteristic of that balls itself. Whereas here [1/2 Marble Pb.] it's not the same thing… or I can't think about it the same way to make it the same idea.

Todd proceeds to write down a summary of what he has just explained.

I: Ok, so there's nothing that you can say about the contents of the jar after all the steps have been performed.

Todd: Well there's nothing I can say about the state of the jar. I'm sure other people might…

I: Ok.

24:25 [Starting work on the Midpoint Problem.]

Todd draws segment AB, commenting that he didn't take geometry in high school.

Todd: I'm just doing the first couple of steps so that I can get a handle on this…

I: Sure.

Todd: At every step, if you think of each one like a segment, you divide each segment in half and you get more segments. So M is the set of midpoints. I don't remember exactly what I wrote, but I think I said… Here's the number of the step, here's the number of points you have. I think I said the total number of points. So it is something like… At step 1 you divide it in half, you get 1 midpoint. At step 2…

I: Feel free to keep track of how many points you have after each step.

Todd: I think it's going to be something like 2^n. At step 2 you have 3. At step 4 you have 7. So I guess it's $2^n - 1$. Ok, so… and then I'm not sure about what M is.

28:20 I: Right… so what you have there, you're telling me how many elements are in M after n steps.

Todd: Right. And then, I don't know if I said something about the size, like where they're located. I think I said something else, but I don't think it was very informative.

I: You did say something else. I'm not going to remind you what it is.

Todd: I think it was about running out of the letters of the alphabet.

I: Right. Yes, I don't care that much about the letters. I just mentioned the letters so that you can start drawing the picture. You kind of pick out points from the segment AB. Which points did you pick out, at the point where all steps have been completed?

Todd: So I guess… [pause].

I: Actually you did write something else, which was informative.

Todd: Was it about the size?

I: Yes.

Todd: That's not really about the midpoints themselves…

I: Well, it is. When you talk about a set, if you say something about its size, that is a characteristic of it. So you do describe it in some way.

Todd: Was it the size of the segments that I was talking about?

I: No… But how about the size of M after all steps have been completed?

30:12 Todd: Ok, so after all steps have been completed… I guess if you look at any two midpoints, I guess they get arbitrarily close to each other…

I: Well, any two midpoints are fixed, so I am not sure what you mean by "they get arbitrarily close to each other". You mean you keep constructing more and more midpoints and they get closer and closer… ?

Todd: The closest two midpoints as you go… I should continue that thought just for posterity. If you look at any two midpoints that are next to each other, and if you look at this step by step, as steps get bigger, they get arbitrarily close to each other.

I: So you mean… the distance between two consecutive midpoints, as you perform more and more steps, that distance gets smaller and smaller.

Todd: Yeah, the midpoints are not moving, but if you get more and more of them… right. Which I guess would mean that… since you need to close the distance by adding midpoints, you would get a lot more points. And then the question is how many, or which points…

I: I guess both questions are interesting.

Todd: Right. It would be nice if I could say something like "oh, all points".

I: What do you mean by all points?

Todd: Right… I would be getting into things I am not sure about. [pause]

I: Well, let's take it one at a time. What would be the size of M after all steps had been completed?

Todd: Well, it's not finite… it's infinite. I'm not really good with cardinality or anything like that. But ok, I'll just keep talking. So M is infinite, I think.

I: Could you outline a proof of why M is infinite? Informally… how would you go about that?

Todd: [pause].

I: Keep in mind that you already have a formula for the number of points after a specific number of steps.

Todd: Right, that's good. Could I construct a proof? Maybe not… but I'll try. Well, here's a number of midpoints that we have, and we can make it greater than that by increasing n [the number of the steps already performing] until the number of midpoints exceeds whatever target we set. I don't know if that actually proves that M is infinite…

34:28 I: Uh-um. What would be the structure of this proof? In Math 300 you learned various types of proofs, right?

Todd: Well, it's very similar to epsilon-delta type of things…

I: Well, epsilon-delta type of things means you're doing something with limits. But in terms of the structure of the proof, is it a direct proof, is it proof by induction, proof by contradiction, that type of thing?

Todd: I think it has contradictory elements to it. If you assume it's fixed [referring to the assumption that the number of elements in M after all steps had been completed is a natural number]… I guess it would be contradiction.

[Todd spends the next 5 minutes specifying the details of such a proof (by contradiction) to argue that M (after all steps had been completed) is infinite. He operates under the assumption that the size of M "at the end" is the limit of $2^n - 1$ as n goes to infinity, without explicitly stating this assumption.]

39:22 I: Ok, so we have that about the size of M. Now what about exactly which points are part in M? We know how many they are, but you can't completely characterize a set only by its cardinality. […] We know that it's infinitely many points…

Todd: Like, is it all points in AB… would be in M?

I: Yeah, that would be the question.

Todd: I know… I'm trying not to talk about it [laughs; really long pause]. So I know very vaguely that two sets have the same cardinality if you can write a bijective function between them. So, I guess I would need a function… if I was going to say something like "every point in AB is in M", I would have to have a function… like getting a point in AB, what step number would it take to get in M. Maybe you can write it like that, 'cause all steps are done. If you can say that

that's true for any point [in segment AB]… I have to remember which way the function goes. Well, if it's bijective I don't know if it matters.

44:00 I: Keep in mind that you said that when you build a bijective function between two sets, that just shows that they have the same cardinality. It doesn't show that the sets are equal. Right? Between the set of natural numbers and the set of even numbers you have a one-to-one correspondence, but one is included in the other. They have the same cardinality, but with infinite sets you can have one as a proper subset of the other. So if you manage to build a one-to-one correspondence between points in AB and points in M, that just shows they would be of equal cardinality, not that they're equal as sets. But I think you're saying something different… that you're going to build a function from the set of natural numbers, which is the set of steps, to segment AB? And f(n) is going to be the point, or the set of points, added to set M at that step… something like that?

Todd: Well, that's a good idea. I think what I said was kind of backwards – if you get a point [from segment AB], the function would give you the number of the step… but that's probably hard to do.

I: Well, let's try it that way. Because the way I said it, f(n) would be a set of points, not a single point, because at every step you're adding more than one points (except for the first step).

46:00 Todd: I guess a point along AB is like a distance… Can I just say A is 0 and B is 1?

I: Sure.

Todd: So I guess it would be all the real numbers in (0, 1)?

I: What, the points on segment AB?

Todd: Yeah… not closed, so like open.

I: So the open interval (AB). I was not very specific when I said segment AB, I did not say whether it was closed or not.

Todd: I said open because if M contains midpoints, and A and B are not midpoints…

I: Oh wait, so when you say the open interval (0, 1), you're just trying to say that with this notation, this is what (AB) becomes, or trying to say the set M is (0, 1)?

Todd: Oh…

I: I'm not trying to steer you in any direction, just trying to make sure I understand what you're saying.

I: The way I was thinking about it was that the function would give you… it would be like you take a midpoint, and at what step does it become part of M? This midpoint is not going to be A or B… There's never going to be a midpoint in A, 'cause there's no interval [around A].

I: Ok, so you mean there's no specific step at which A becomes a midpoint.

Todd: Right. The open interval (0, 1) will be the domain of the function, and I'm going to be looking at what [step] number I have to get to to get that midpoint in M.

I: Ok, so you want to define a function on (0, 1), with values in what?

Todd: I think I said the real numbers.

I: Didn't you say that when you apply the function, it should give you the number of the step?

Todd: Oh… I guess it's from the reals to the natural numbers.

49:30 I: And you're not trying to build any kind of function, but a bijective one?

Todd: Oh… that was a bad idea. You can say M has the same cardinality as AB, I don't know if it's true, but we're not looking at that.

I: Ok, so you just want to build a function, not a special type of function.

Todd: I don't think it's going to be bijective anyway. Unless you say that it's going to be the first [step] number where it's true that that midpoint is in M. Because if you take the point H (midpoint of (AB)), any step after 1 will satisfy the point H.

I: Actually, it's not going to be bijective anyway, because at step 2 you add two points, right? That means f of this point [pointing to ¼] is going to be 2, and f of this point [pointing to ¾] is going to be 2, so the function is not injective.

[Off-topic discussion – Todd wonders if a certain math professor will see this video and be disappointed by Todd 's performance. I reassure him that will not be the case.]

52:30 Todd: If you think of a given point [from segment [AB]], put it into the function and get a number n, it has to be true that at a prior step… no. Anyway, there has to be two points equidistant from it. There has to be some sort of recursive type… I don't know if I can write it…

I: Ok, let me ask this question. Now that you denoted this segment by [0, 1]. That means you can refer to a point by a value [number]. So then, the point with value 1/3, is that in M?

Todd: I guess I want to say no, but I don't actually… [longer pause]. Maybe. Because you're going to have points like ½ and ¼ after step 2, so it's in the range… it is possible. And as you keep dividing…

I: It gets more and more crowded around 1/3?

Todd: [laughs] Yeah, but that doesn't prove it. At first I was going to say no, but now I'm not so sure. I mean, let's say this is ¼ and this is ½. 1/3 is somewhere between the two points… and it's going to be closer and closer to the center.

I: What center?

Todd: Maybe it's not… It will be closer to the center [midpoint] between these 2 [1/2 and ¼], but then when you put the next midpoint in, it might not be close to the center anymore. It's not very mathematical though…

I: That's fine. Ok, you can leave that there because later on we're going to investigate that more.

56:43 [Start of intuition/confidence ratings – in writing]

[End of session]

Chris Pre-Test
10/31/08

00:00 [Start work on original tennis ball problem. Chris: works through the first 3 steps, writing out which balls are in each bin after each step.]

2:27 Chris: So I am noticing that exactly half of the balls at any given step, up to the halfway point would be in B, and after the half-way point would be in A. So I would just generalize this as… after step n, everything up to n/2 would be in box B.

[I question whether it's "up to n/2", he reconsiders that and changes his answers to "everything from 1 to n are in B and all remaining balls are in A… and this would consist of those that are greater than n and less than or equal to 2n".]

Chris: So at step n, that's what they would be. So I believe that's as far as I thought to go with this. I don't know if there's anything further, as in 'all steps have been completed'. I believe that says what's in each bin at this point.

I: Ok. How do you interpret that question, or… that phrase, "after all steps have been completed"? How many steps do you think that's referring to?

Chris: Well, in terms of how many steps, for "all steps have been completed", it says… in this manner to infinity, but to interpret something to infinity and yet assign a number to it would be interesting, because then if n was an infinite number of steps, I'd have to say 1 to infinity are in B and everything greater than infinity and less than 2 infinity are in A. So I just have to label it "n". "All steps"… the way that I consider that is, is there a way to summarize it so that you give a finite number of steps however large it may be, you can get the value for what is in each bin, and beyond that you can generalize from looking at it what it would do towards infinity I suppose… not in actual infinity.

I: So when you say "towards infinity"…

Chris: Yes, it's a potential…

I: … as the number of the performed steps increases to infinity…

Chris: Yes, the limit as n increases to infinity, I suppose you can figure this way.

6:30 I: Ok, so you wrote what happens after n steps. And how are you generalizing that to "after all steps"?

Chris: Well after all steps, the first half of… um dealing with infinity as a finite amount… Um, after all steps, not just some number n but for however infinitely many steps there may be… the first half of … Sorry, it just sounds ridiculous for me to say that, but the first half of the numbered balls would be in B and the second half would be in A. It's generalizing it but… don't know if that works very well with describing something that's infinite.

[I comment that "the first half" from his explanation is not that well-defined for an infinite set, unlike the case of a finite set.]

Chris: Generalized to infinity, I don't know specifically how I would generalize that in any sort of term except sort of that process where I can take a finite large number and just expand it as I go, but in more general terms for actual infinity, I don't know how I would describe that.

I: When I used that phrase, "after all steps have been completed", when I meant by it is that we're assuming we're at a point where no matter how you choose a random natural number n, we're past the point where that step was performed. And I can say that for any natural number. We're past all of them.

Chris: We're past all the possible… Well, that's an interesting thought, because then I suppose it would be similar to saying that the first infinite number of balls was in B and the second infinite… but that would assume a certain countability to the infinite number in both sets. At least that's the way I'm visualizing the situation.

9:43 I: Let me ask you this way. When you said that the first half of the balls are in B, and the second half in bin A, how large or how big are the sets that you put in each?

Chris: The set size… well at infinity the set… I would love to say… I suppose I would have to classify infinity into sets of order of infinity, and I don't know enough to do that probably in a normative fashion. The way that I would think of it … The natural number from… the positive integers I suppose I could say also… are infinite in number, as are the negative integers. In my mind at least, even though they're both infinite, they're equal sets of infinite numbers.

I: So you mean in terms of sizes?

Chris: Yes, in terms of size of set, they're both equal. Obviously the values are not equivalent. They're the same size, so… the total set of all integers, not including 0 for this case 'cause I specifically excluded it, would be if I were to say that it would be size of one set plus the size of the other, which would be I guess in my terms the next order of infinity, although I don't know if they call it that. There would be one infinity in B and a second infinity in A, that would combine to a larger order of infinity, which sounds ridiculous but I believe there's something like it but I don't know.

I: So let's start with the infinity of the natural numbers. That's the initial set that we start with, right?

Chris: Yes.

I: So when you're saying that half of them are in bin B and half in bin A… how large did you say those sets are? You said they are equal in size, right? And what is that common size?

Chris: Well, since it's an infinite number of steps, if I were to take infinity as some sort of, which I can't, finite amount, then it would be half of infinity, but how do you do such a thing? So, I mean in terms of the total set of infinite balls, each obviously is half of that. So it's infinity in B, and an infinite number in A, and… what do you call that, ordinality? Yeah… the ordinality of A would be contained in… the value of each ball would be twice the number of each that was in B. There would be a corresponding match.

I: Do you mean a one to one correspondence between the two sets?

Chris: Yes, there is a one to one correspondence. And the value of… there's no such thing. That's not true! I'm trying that, but that's not true. There is a one-to-one correspondence, but the value of the number, there's no such thing.

13:35 I: So, just to clarify, the size of the sets in each bin, after all steps have been completed… is that finite or infinite?

Chris: In terms of the total number of balls, if you added them together, it's semi-finite. I call it semi-finite because you can determine it if you had… if you know the set of the infinite number of balls that are placed total into both…

I: Which is the set of the natural numbers, cuz that's what we start with. So that's an infinite set.

Chris: It would be half of that set, but since that set is not… In terms of that you can find the total number in each bin as half of set, however since infinity itself is not, at least at my level of understanding, it's also infinite at the same time, in each.

I: Ok, so in each we have an infinite amount of balls, and when you put them together we get the natural numbers.

Chris: Yes, which is in itself a larger infinity than the other, in my view of it anyway.

15:00 [I comment that so far he has been talking about how big the sets (in each bin) are, but that it's not clear which balls are in which bin after all steps have been completed.]

Chris: If each set is infinite though, then the balls in B obviously would have to be the natural numbers less than or equal to infinity… for B.

[I comment that each natural number is less than or equal to infinity, so wouldn't that phrase refer to all natural numbers?]

Chris: The way that I think of this in my head, I suppose, is almost… I have thought about it this way before… is infinity as having an order to it somewhat. So…

I: You mean to quantify how big it is?

Chris: Yes, I can quantify in terms of relationship in my mind to it… because obviously, to me at least, the set of the positive integers and the negative integers are infinite in and of themselves, and the set of integers combined, plus 0 even, is an infinity which by that definition I have supplied at least, is larger than both. So, in this case I was describing that in B, which is the first infinity, and A, which is the second half of the complete infinity of steps taken… I'm saying that if we actually knew the set size of each, then obviously the number of a ball in B would be less than or equal to the number of the set size. I was generalizing that somewhat to say that if B has an infinite number and A has an infinite number, then they would combine to a total of… I hesitate to say twice an infinite number, but in terms of the set I defined as infinite, if you take that description then yes, it's twice the infinite number.

19:00 [I comment that we can take that description of infinity, but it's still unclear how to decide whether a given natural number ends up in A or B.]

I: Let's say you choose 1005. Where is that, after all steps have been completed?

Chris: It's an interesting point. At about 3 seconds as you were saying that, I realized that you could… at least in my mind, all of them could be in B at any given point.

I: At any given point…?

Chris: At any given point given sufficiently large infinity, which is sufficiently large for its own value. Obviously at a finite step, you could say that if you had… I don't know which number you said but 1005 sounds reasonable… then after step 1005, before that point it would be in A, and after that point it would be in B for as many steps as you like. And since infinity is obviously larger than 1005, I'd have to say at infinity it is in fact in B and not A. So if I were to generalize this process and you take an infinite number [of steps] and you took a finite ball, then it would have to be in B because at that point… ok, you would have to say that it would be in B at that point because as you generalize this to infinity any finite number is less than infinity so it would have to be in B, no matter how far you go at it. Because there's always a step that you can take that would put it there. There's always enough numbers that you can assign to balls that would make it, so that that would be in the first half of whatever finite step you wanted, and if it's in B for some finite value n then for the infinite step it would have to be in B.

21:40 I: So when you say "the infinite step", are you thinking about a specific step that has that "label"?

Chris: I don't say that very often, but yes, I actually do think of it that way sometimes because it makes it simpler for me than to think of a potential infinity. I know that if my infinity at some point isn't large enough to satisfy, I can always bump it up further up at any point and keep going. I know that infinity is countless, in a very general sense. I use it somewhat as a physical value, because any finite number is less than an infinite number, just because any natural number is finite and there are an infinite number of those, so you can always generate another one if you wanted to say that one finite number was the largest number… you can keep generating, that's clear enough.

I: Let's recap. You made an argument for why 1005 would be in B after all steps would be completed. By that would mean that for any natural number n, that step has already passed, and we can say that about any natural number.

Chris: yes, because infinity is going to be larger than any finite n.

I: You also said that you can generalize this reasoning that you just did for 1005, you can do that for any natural number n. You can say… you can find a specific step at which that ball gets in B.

Chris: Which would probably… actually definitely be step n. That a numbered ball n would go into B. At step 1, 1 goes in… at step 2, 2… it would go, whatever step you want, for any finite step, it would go into B at that point.

23:55 I: Does that mean that after all steps have been completed, we can say that all the balls that we started with are in bin B?

Chris: As far as I'm thinking about it, at least for this very second, yes, you can say that they're all in B at that point. At a finite step you can put it into one or the other, but no matter what you place in A, at some step, some infinite step, it will go into B.

I: Some infinite step?

Chris: I know, I keep labeling it that way. I don't mean to label it in such awkward fashion…

I: I'm not protesting the language, but before you said that at step n that ball has been put in bin B. So that n is a finite number. And now you're saying at an infinite step it goes to bin B, that's why I was trying to reconcile the two.

Chris: Oh, I was not trying to say anything different than "at step n it will go into B". […] No matter what number you choose, at that step number it [ball numbered n] will go into B.

[I ask if he can get equally specific about the answer regarding bin A.]

Chris: If I'm dealing with finite steps… which I know is not the point of this necessarily… but if I deal in finites, obviously A has at some step n, exactly equal to B. If I can make the case that any number in B… any number in the collection of natural numbers, is in B, after an infinite number of steps, then in… Here I am, holding infinity as a finite value again, then there would be nothing in A at that point. At the same time that you're adding one ball to B, at the finite level, you're also adding on to A, technically. […] The number in A, after any finite step, is still equal to the number in B, after any finite step. So at infinity, part of me is saying that there's nothing there, because an infinite number is already in B. But part of me that has been thinking about infinity for years, fortunately or unfortunately, depending on the point of view you may have… is saying that no, you can't just say that because there's an infinite number in B, you can't just say that there's nothing in A, because infinity itself as I'm thinking of it, is not by itself everything. Especially when I think about it the way I defined it with an infinite number of … but that's *my* definition, I'm not claiming it's generalized that way.

28:17 [I note that with regard to bin B, his argument did not only claim that there's an infinite number of balls in there after all steps have been completed, but also mentioned *which* balls would be in there.]

Chris: For some reason I'm reminded of something I looked at… don't know why I was looking at it, I think it was connected to something I was reading at the moment… Um, it's a strange example I don't know if I even want to deal with that.

I: No, go ahead. I mean, as long as it has relevance.

Chris: I know, I'm trying to think about that before I open my mouth. I mean, it does have relevance as far as I'm concerned right now. What was it exactly though? I have to be able to describe it. It was an odd situation though. I guess I'm going to have to name it by name 'cause describing it I'm going to mess up the story along the way. I remember reading something about the Hilbert's hotel problem.

I: Uh-um – I know what that is.

Chris: That saves me a lot of energy right now. Because I don't have to worry too much about if I miss a detail, because it's something vaguely understood. But there's no room in the hotel at that point, you can say it's completely full, but if you shift everything over by one, then all of a sudden you have an infinite number open again. Sort of the situation I'm running into at the moment, because I can say everything at one point is there, but then I can say ok, go one step further, now something is in there again, in both. So I can say that every number at some point will go into B, and at the same time every number will be in B, because it is infinite, if you take the definition that infinity encompasses every possible number. But A on the other hand, if I was to say was completely empty, would also… It's just raising sort of a wall in my mind, because it seems to violate the terms of this where every time you add something, you add… and since there's an infinite number of balls to be drawn from… You can't just say that because every number can be in B at infinity, that infinity is small enough to be counted and just stuck in B.

31:15 I: Let's recap your position on this problem. I think you said there's infinitely many balls in each bin, and we can make some kind of argument for …

Chris: … for any ball being in B, but at the same time you can make the argument that there must be something remaining in A due to the nature of infinity.

32:17 [Starting work on the 1/2 problem.]

[Chris recaps what he remembers to have done for this problem on the written pre-test, which is state what was in the jar after step n, where n was a natural number. He states that there was not enough time for him to explain how to generalize that "to infinity".]

Chris: Well, let's see… [reads the problem to himself]… I diagram two marbles, 1 and 2. Ok, so there's only two, there's not an infinite number of marbles, there's just an infinite step. Aah… ok. Well, obviously, at step 1, it's in 1, at step 2, 2 is in there, at step 3, 1 is there, so… At every odd finite step n, the jar contains marble 1, and at every even finite step, the jar contains [marble] 2. Now that's finite. Now beyond that… if you don't know whether infinity… so infinity could go on for whatever, you can make it even or odd. So you can make the argument at least, that at the infinite point, after all steps are completed, you could almost say that 1 and 2 are in there at the same time, or they're not in there?

I: you mean neither is in there?

Chris: I mean both and neither.

35:28 I: Ok, so the possible answers you're considering are… one is both marbles are in there, the other one is none of the marbles. Is there any other possible answer?

Chris: You could say one or the other, but to classify infinity as a specific number in this case can always be extended because it's not… I mean the size in my mind of odd positive… I should call them natural numbers for the sake of it. But off natural numbers is equal, in my mind, to the size of even. Which, unfortunately, even though they're both infinity, technically it suggests my infinity to be an even value… if they're going to be equal.

I: Alright, 'cuz you can split it in half…?

Chris: Because if infinity is some sort of odd, then infinity plus 1… but then infinity could be raised, so what is the definition of infinity plus 1? Ok… let's see. If I go through finite steps, obviously I can figure out which is which. If you gave me a number, it's easy to tell which is which. But at an infinite number of steps you have no idea whether it's even or odd, which is why unfortunately I am reminded of "is the cat in the box alive or dead when you don't look at it?" Ok, 1 and 2. So… since I can't generalize… I can generalize for finite. I can't generalize for infinite, that 1 by itself is there. If you say an infinite number of steps is completed, then if we were to talk about it in terms of some arbitrary value, you can say depending on what you're taking it. Well technically you can also say that at an infinite number of steps, we just keep going so there is no stop point to it. So at the same time, until you stop to observe it, which makes us feel more like that experiment, you don't know which one is in there. At the same time, if you don't know which one is in there, let's just say that it's continuing continuously… but if I were to assume it continues just continuously, it's still going right now because an infinite number of steps each taking a finite length of time I would say, will continue assumingly infinitely.

39:02 Chris: If you were to stop at some point, obviously it would not contain one and it would contain the other. If you were to say ok, this is enough for right now, hang on there. Grab the thing, you can see which one it is. At the same time, I can say ok, go back, take this back and go, have them start again for an equal length of time… Not an equal length of time but an infinite length of time… some random length of time, number of steps unknown, but it could go on forever, is the idea.

Until you observe it you don't know which one is in and which one is out. You know one's in, one's out, but you have no idea which it is. Now, at no finite point can both be in or both be out, because the part where both are in or both are out is a half step.

I: Right…

Chris: Well, for continuous things I would care but not for this. So, the argument that I have for it being… if I say at once both and neither [whistles]. I can't even keep my words straight. If it was at once both and neither, then I'm bordering on the realm of philosophy. I don't know if I have an argument specifically that would say that… it's just sort of a suspicion that I have because you can't observe the experiment at that level. I mean…

41:15 I: Let me ask you this. Is there anything at all that you can claim about what's in the jar, if you consider infinitely many steps having being performed?

Chris: So it's a done infinity, a stopped infinity, an actual infinity, which is the way I like thinking about the darn thing. One or the other is in there… I believe.

I: So you can say something about the cardinality of the set then? See before I was asking you about how big the sets were, and you told me that both of them were infinite, right? Now here can we say something about the cardinality of the set?

Chris: Well, inside the jar is 1 or 2, if all the steps are… Well then I get into the same kind of trap I was in the prior problem. If I stop it somewhere, is it infinite? But if you stop at an infinite number, an actual infinity instead of potential… and I know I'm using words I've just heard before.

I: I'm familiar with those words, and I think you're using them the way they are used in the math education literature so that's ok.

Chris: Well, I never saw any of that, but I've seen them before at least, it's not just picking a word…

I: So again, let's define "after all steps have been completed" as "whatever n natural number you choose, that step has passed". So we're at a point where we passed all the natural numbers. Is there anything at all that we can claim about what's in the jar?

43:23 Chris: Can I say that it has 1 alone? That assumes to me a finite number of steps. But if all steps are completed and the experiment is completely stopped, and it's done, infinite steps have been done, then technically your infinity is something that you basically counted, sort of.

I: Well, in mathematics you have the term countable; the infinity of the natural numbers is called countable… meaning that you can order them in a way that you can imagine yourself counting through them.

Chris: Oh yes, I can certainly imagine myself, but… is there anything that can be said about how much is in there? I would like to say… the intuitive side says that one or the other is in there, even if I can't tell which. The intuitive side tells me that?

I: Is there any other side pointing in a different direction?

Chris: Of course there is [laughs].

I: You don't seem very happy with the answer…

Chris: No, I don't like the fact that it exists there because it breaks everything into pieces.

I: Ok, so intuitively you would say that but it feels as if it refers to a finite number of steps.

Chris: Yes, because there would be an actual finite value to your… it's countable, but is it finite, it's countable, is it finite…? The distinction between those two I think is almost so ingrained from years of people saying infinite is infinite and finite is finite and you can't deal with the two in a… I mean, people have treated infinite sometimes as a finite number and I've seen it, I just don't… It still brings back the same questions every time that I deal with it as an actual value. I don't know, I couldn't even figure out why at this point… I'm sure somewhere along the way it was drilled very well into me that you can't deal with infinity the same way you deal with finity.

45:45 I: Ok, let's wrap up this problem. So intuitively you gave me an answer, do you want to make that your final answer about "after all steps have been completed," or just say "no, I can' say anything"?

Chris: If it's a countable number of steps, I would stick with the fact that one or the other is in there, but…

I: Again, just to be clear and clarify what we mean by countable, in mathematics countable can be infinite as well.

Chris: Yes, I do understand that needs to be clear. Infinite number of steps is a countable infinity because it has been completed as the problem says… all steps have been performed. So I'm assuming a countable infinity. So at that point, since it is a countable number of steps then it contains one or the other, but I surrender to anything that wants to help with which one it is, and I don't think that there really is specifically a way of telling which one, because if you experiment than you've done a finite number of steps.

47:25 [Starting work on the Midpoint Problem. Draws diagram containing the AB segment and the "midpoints" produced by the first 3 steps.]

Chris: I drew a slightly less detailed diagram the first time [referring to the written pre-test], but that's ok. The idea is that I know that each division can then almost in itself be treated as… A to H and then H to B… and then it's this lovely recursive set here. So at the first case you obviously have 1 midpoint, at this step you have 2, at the next step you would have… well, 2 plus the one, I should say…

I: Yeah, newly added…

Chris: Then I would go 4 +2+1, and then so fourth it would be 8, 4, 2, 1… and of course obviously you get 3, 7, 15… and each of these… It reminds me lovely of the tower of Hanoi. $2^n - 1$… for finite steps, finite. That's the number of things in M at a finite set of steps. Now after all steps is another matter entirely.

I: So first let me ask you this. M defined as the set of all midpoints after all steps have been completed – is that finite or infinite?

49:27 Chris: After all steps have been completed… well, obviously at any finite step at least, there are more divisions you can make. If you take an infinite number of steps… well, it's sort of a strange thing, but if I wanted to say that between A and B was something else instead of labeled by letters… that may be. I could reasonably claim at least as an analogue that it was basically a number line between 0 and 1 and take your midpoints. It's the same sort of thing, doesn't have quite the abstraction level but that's ok for now.

I: I'm fine with that.

Chris: In that case then anything between A and B, or 0 and 1… any finite number of steps… how would I even want to quantify that? Well, obviously I could quantify it with this finite thing [referring to $2^n - 1$], but I don't want to deal with that currently.

I: Well, you have there the number of the points in the set after a specific step. And you see that it's a sum, right? Could you write something for the size of the set after infinitely many steps?

Chris: Well, let's see – 1, 2, 4… I mean I could write an infinite sum, but does that really necessarily clarify… ? Because at each step you're adding to what you had before 2 to the 1, plus… No, 2 to the 0 plus 2 to the 1, and so in that case it would be 2 to the n for n=0 to infini… [does not finish the word].

52:00 I: That's an acceptable notation, you used that in Calculus, right?

Chris: Yes, that's acceptable. Obviously that's not going to converge upon infinity as a geometric series. It can't converge, it's getting bigger every time. You can summarize the cardinality of the set that way, but is that really… Let me think about that for a second. Let's see, n is 0, that's one step? Eh… I don't know if I like that, because if I want n to be the number of steps… I should shift my index. It was making me look at it kind of weirdly because then I would have to say the top would be infinity. But now I've got infinity minus 1 at the top... No, I'm not going to write that as the top portion, but I mean if I wanted to add this, and assume for a moment that one could actually hit infinity at that point and you could actually extend it far enough to do it, then the first way it would be some… God help me for this one, 2 to the infinity, but now it's 2 to the infinity minus 1, which is a beautiful number in and of itself. I'm just going to pretend I didn't write that for the time being…

53:57 I: Ok, let's back to the original question. Is this set infinite or finite, according to what you wrote there?

Chris: The set should be infinite at an infinite number of steps…

I: *After* an infinite number of steps… after infinitely many steps?

Chris: Yes, after infinitely many steps, ok.

I: Now, in terms of what goes into it, as an infinite set, I see that what you wrote in the original pre-test is that it was the entire segment between A and B, but not including A and B?

Chris: Not including A and B because there is… If you take the way that I am visualizing it as similar to a number line, that's where I actually grab that from, sort of. I took that because I said ok, there are an infinite set of rational points between 0 and 1; irrational points… hmm, could you make the case that they're in there?

I: In where?

Chris: In M… obviously it's in the segment. Every midpoint, if you take the length of whatever thing you want, is some number whichever it happens to be, some countable number for the length. So you have 1 for the length of this let's say [length of AB], over some small L… So for each division, you will have something, k/L… Every midpoint between 0 and 1 is going to be a rational fraction where… Do I want k less than L? Yes I do. Now obviously the choices of L are infinite by the choices of… no… yes, infinite, if I say that L is not equal to 1. If I divide by 1 then I'm in big trouble, it's not there. All the natural numbers not including 1 [for the choice of L].

57:02 I: Can L be 3?

Chris: [pause] No, it can't be 3 because it's not… That's true, it can't be that either. But even counting the numbers that L can be equal to as $2^1, 2^2$ and so forth, 2 to the … [writes 2^∞]. I won't deal with that for now.

I: Ok, so L can be of that form, a power of 2.

Chris: Yes, it can in fact be of that form. 3 is obviously not one of those. But even that set is an infinite set of choices in its own right. Because this value up here, your exponent, is infinite. So as a result of that, the way that I'm seeing is that, since there are an infinite number of possibilities to divide by… Is that infinite k? It's actually strange, because of the choice of 2 [in the place of L] there's only one choice [for k], for the case of 4 you can say 1, and you can say 3. 2 is not really needed in that case. Then for the case of 8, you can say 1, 3, 5, 7… so then k is odd, I suppose I should say that. So good. Well that set of odd positive is in fact something that is usually termed as infinite. And in this case, for each value of L, there are as many, if not more k's. So you can get, very vaguely speaking, an infinite number of said ratios… that would still be less than 1.

I: Ok, so that's the argument for "set M is infinite", right?

Chris: Yes, that is my argument for it.

I: And not only that, but you have a specific form for the numbers that go into it.

Chris: Yes, well if I wanted to write this on something to be extremely clear, I would put it… Let's see, set M equals… Well your question says A to B, but if I translate this, then it would be k/L such that k is contained in the odd greater than 0, and L equals 2^n, where n is contained in the set of naturals.

I: And k less than L, right?

1:01:13 Chris: Yes, I did forget that little step… and k less than L.

I: Ok. So here [pointing to written pre-test] you said that this set is equal to… [the whole open segment (AB)]

Chris: I was slightly over generalizing when I said that, because no, it does not equal what I said there. I realize at this point that [then] I was in too much of a rush to stop and say no, it does not equal 1/3, no, it does not equal 1/5… Now of course the set of M for any finite n will now be defined as follows from 0 to 1. But now will it be for an infinite number as your n divisions?

I: Well, what you wrote there, is that for "after all steps have been completed", or for a finite n?

Chris: Well, everything k/L is going to be in there, where k is less than L, k is an odd number, and L is of that form, where n is some… Yeah that was "for all", I did not specify.

I: So I take it you don't agree anymore with [the answer on the written pre-test].

Chris: No, I do not agree with that… wait a minute, I did not visualize this as a number line [referring to his reasoning on the problem on the written pre-test], so between A and B here, eventually if you keep dividing, would it actually hit everything? No it still wouldn't hit the 1/3 point, but if you keep dividing into halves and into pieces, would you still hit the point?

I: Which one, 1/3?

Chris: Well I don't think you hit the point by going strictly by halves, but would you hit the point if you kept dividing in half… I don't think you'd actually ever hit that, but that's actually a classic definition of a limit to infinity… It is an infinite number of points that would be in there, but it doesn't contain everything between the two [endpoints of segment AB]. So no, I do not agree with that [previous answer]. Let's make sure I have my facts straight. No, I don't agree that it's every point in there, but it is in fact an infinite set consisting of this sort of thing [pointing to k/L set description of M] instead of this [pointing to written answer from original written pre-test]. This argument is very flawed, as a good amount of my first drafts are.

1:05:00 [I ask Chris to rate in writing (on a scale from 1 to 10) his confidence level in each of his answers to the 3 problems, as well as the extent to which his answers are in agreement with his intuitions.

[End of session]

Session 1
11/12/08

0:00 [Start work on the odd/even tennis ball problem (no bin swapping)]

Todd: Is this the one we did the first time [on pre-test]?

Chris: It's similar [referring to the OTBP], but there is one strategic difference.

I: Actually, for him [referring to Todd], I might have given him this version, as a variation.

Chris: Ok. That's nice to know. Alright then… It's certainly different than the one I saw the first time [referring to OTBP]. Well, at this point, I can see one difference between the one I saw the last time. Unlike last time, where you put 1, 2 in and then moved the first one, then put 3, 4 in and moved the second… now ball 2 doesn't move because there is no way for it to move [out of bin A]. It doesn't go anywhere. That's terrific. Well, I can at least say that everything in A is even and everything in B is odd. Now, the size of the set is not entirely as clear anymore.

I: Which set and at which point? After a specific step, or after all steps have been completed?

Chris: After all steps. After a specific step it's not much of a problem.

2:35 I: Before we go any further, can you each say how you think about the phrase "after all steps have been completed"? You told me individually, but maybe you should make sure that you mean it the same way?

Todd: I guess I'll go first, then you can give the right answer. I have no idea what that actually means, so…

Chris: Ok.

I: But when you [Todd] first thought about this problem, you actually gave me some answer for this problem, so how did you interpret it for this specific problem, last time?

Todd: Once the balls are in place they don't move, so… It's a property of each ball, which bin it's in. So you can say this ball is divisible by 2 so it's going to be in A.

Chris: Ok. By all steps being completed I mean an infinite number of steps… Somehow I still feel weird saying there is a countable and finite infinity, but… for an infinite number of steps, it's not a certain [step]. It's as if you have gone through every possible number that can exist.

I: Natural number?

Chris. Yeah, I'm assuming we're talking about natural numbers.

Todd: Well that's the thing, I was like I can't imagine 'cause like the time… it would take infinite time. And she's like, what if each step takes 0 time? And I was like, then I don't know.

4:20 Chris: Well then the whole thing would be completed very quickly. Um, ok. In that case I can think of infinity as being something that can be completed. Speaking of it is not entirely as easy as conceiving of that idea. Ok, what are the contents of the two bins? Well, would you [addressing Todd] agree that the contents of A would be all even numbers, positive?

Todd: Yeah.

Chris: And B would obviously be the odd numbers.

Todd: Yeah. So when you said the size of the set, what did you mean?

Chris: Well, I was dealing with the problem where it was 3, 4, move 2 [the OTBP]. At the end I made the argument that if you complete an infinite number of steps, they [the bins] both should have the same number of balls in each, but you can also say that because it's an infinite number of steps, you can also make the case that every single ball would eventually leave bin A at that time, so A would be empty.

Todd: If you give me a number of a ball, we can show the step where the ball would be in bin B. So in a certain sense every number is in bin B, but at any given point…

Chris: Yeah. So in that case, if there's nothing in one, then everything is in the other. But that's not this [odd/even pb.] problem. But that's what I meant by the "the size of it". As for this [odd/even pb.], the set sizes are going to be equal at least, but ball 2 can never leave [from bin A], balls 4, 6, can never leave. There's no defined step in this case for moving them. As for where to go from here, I'm not entirely certain.

6:55 I: Todd , can you recap your position on this [odd/even pb.], what you told me last time? I think we got his [Chris's position], are you in agreement with that?

Todd: Yeah. I guess I'm pretty comfortable saying even balls are in bin A, odd balls in bin B. I don't remember what else I said. I think I said it was easier for me to think about this problem than the other problem [OTBP].

I: Yeah. I think you gave me a reason for that. Do you remember what that reason was?

Todd: Well, it's the same thing with like, the contents of A, B in this problem don't change when you change steps. Or they do change, but any ball that's in A or B stays in A or B.

I: Uh-um. So you mean after a specific step, after a ball it's put in a specific bin, it doesn't leave at a later point?

Todd: Right. 'Cause that's the problem with the previous problem [OTBP]. Nothing that's in bin A originally stays in bin A. So when you say all steps have been completed, it's really hard to think about what happens.

I: Ok. Let's finish with this [odd/even pb] problem, and then if you want to discuss the other problem some more, we can do that then.

Todd: So are we [addressing Chris] just going to say that the contents in the bins are all the even and all the odd natural numbers?

Chris: Well I would say that, as long as that's an agreement, I'm comfortable enough with that. That the contents are going to be even in A, odd in B.

I: If you're to write a convincing argument for this answer, what would that be?

8:45 Chris: I have an idea how I would go about it, but…

Todd: I mean in general, at step n you place balls 2n and 2n-1 in bin A and bin B, respectively.

Chris: Yes. I mean, you know that every even number is of that form [2n], assuming n is an integer. You can say that because at each step, the even ball remains in A and the odd ball moves to B and those two specific numbers never get touched again, it would stay that way. So only evens are placed in A and only odds are placed in B, then it would make sense to say that everything in A is even and everything in B is odd.

11:12 I invite them to write down a summary of the solution they agree upon. Todd offers to be the one to write down the summary.

Todd: [writing and speaking out loud]] Based on the rules, for any step n, at the end of the step you get ball 2n in bin A and ball 2n-1 in bin B. Once the balls are in place, they don't move.

Chris: Yeah, well if at step n, [ball] 2n goes and [ball] 2n-1 goes, whatever additional step you're going to be at, it will never be the same [ball] number again [that will be processed]. So when the step is done, those balls are stuck, they can't move. So everything in A will be even and everything in B will be odd.

14:18 Well, there is a difference between saying "every element in A is even" and "it contains all the even numbers". Which one are you claiming?

Chris: I'm claiming that all the even numbers are there [in bin A]. It's just a way of specifying that since all the natural numbers are placed into some bin, and since there is no number that is even and odd in the natural numbers… .If all balls are in A or B and every element in A is even and every element in B is odd, then all the evens are in A and all the odds are in B. That's more formal than anything I've said the whole day [laughs].

I invite them to rate their confidence and intuition in the given solution, for this problem.

16:15 [Start work on (timed) original 10 marble problem)]

They read the problem individually.

Chris: [more to himself] At t=1… Well, let's see, $t = 1-(1/2)^n$… that's unfortunate. Yup.

[Both students think to themselves, scribble.]

Todd: I'm not sure what's in the jar at t=1. I don't know if you can get to t=1 exactly.

I: You don't know what, if you can get to t=1?

Todd: Well, is $(.5)^n$ ever equal to 1? I mean equal to 0.

Chris: If you mean whether it gets to 1 at some point, the limit [of $1-(.5)^n$] as n goes to infinity is 1.

Todd: Ok, so not exactly.

Chris: At infinity it would be 1, by the definition of limit anyway.

Todd: Is it ever actually 1?

I: What do you mean by "ever"?

Todd: Like, is there some n such that t_n is 1?

I: Well, what do you think?

Todd: I can't do algebra, but I'm going to say no.

I: Why is that?

Todd: Because it's the exponential function… no, it's a power function, so you can never hit 0.

I: Oh, so what you're subtracting from 1 does not equal 0 for n a natural number?

Todd: Yeah, but that's a cheap technique.

21:00 Chris: It's valid as far as you thinking about it… it's not cheap in that form. There is no one natural number you can plug in there and say ok, that number [0.5^n equals 0]. Now, as it goes to infinity though, you would I hope agree that the limit is 1.

Todd: Yeah.

Chris: So what would you say about the marbles?

Todd: Um… all the marbles, at time n, from 1 to n+1 are missing. But the jar contains marbles n+2 to 10n+10.

I: What is that, after step n?

Todd: Yeah, at the end of it.

I: Wait, so say that again, after step n you have… ?

Todd: From n+2 to 10n+10. So marble n+1 is taken out at the end of step n. So at step 0 you put in marbles 1 though 10, 1 is taken out. So you get n+2 to 10n+10.

Chris: I'm not entirely clear. Let's label the first step, at time 0. n is 1. So you're saying 1 though 10 are put in, then 1 is removed.

Todd: Right. So at n=0, you get 9 marbles in the jar, then at the next step you have 18 [in the jar].

Chris: Basically what you're saying is that what's left at any step in the jar would be…

Todd: 9(n+1).

Chris: Yeah, ok. And of course, taken out would be balls n through…

Todd: So at the first step n=0 ball 1 leaves, so that's [ball] n+1 [that is removed at step n]. […] Does this check out?

24:50 Chris: The numbers are right… I mean after the first step, 1 would be removed after adding 1 through 10, so 2 through 10 would be there. After the 2nd step, 1 and 2 would be gone and 3 through 20 would be there.

Todd: So I guess if you let n get really big, which I guess you have to to get t about 1, you'd get

Chris: Right… Well, what do you think would be in the jar?

Todd: Well, it's not what's in the jar, it's how many marbles are in the jar?

I: Is that the same question or a different question?

Todd: Well, which marbles I can't tell you, because if you let n big enough, any given marble is not in the jar, right? [addressing Chris]

Chris: Don't ask me like I'm a professor here!

I: What do you mean by "n big enough"?

Todd: At step n, marbles 1 through n+1 are not in the jar. So if you get a marble, you can get a step when the marble is not in the jar. So "how many marbles" is a lot easier to answer than "which marbles", I think.

I: Do you agree with that Chris?

Chris: Yeah, fair enough. At any given step you can tell what's in and what's not in.

I: Well, he [Todd] means at t=1.

Chris: Yeah, after all steps have been completed.

I: Do you feel that's the same thing, asking about what happens at time t=1 and asking "after all steps have been completed"?

Chris: Well, since t is of the form 1-$(1/2)^n$, since no specific n that I can give will get t=1, after all steps have been completed, an infinite number of steps, then t would equal 1 because n would be infinite at that point. It's not a very good way of saying that but… I want to say that it would hit t=1 at infinity. To say that you can have a time 0 and a time 1 and have it never hit time 1 is really counterintuitive to me. It has to hit it if you define it in some way, as time keeps moving. […] After an infinite number of steps would be completed, time would be 1. That's the way I can visualize this anyway. That when n hits infinity, then the time is 1.

28:05 I: The question is, what does it mean when n hits infinity? In your regular math classes I don't think that phrase is defined…

Chris: No, that is certainly not used.

I: Usually, when we write small n we mean a specific natural number. A natural number by definition is not infinity. I guess when you say "when n hits infinity" it's not clear what you mean by that.

Chris: Fair enough. I know that the limit of the time when n approaches infinity is 1-0, which is 1. I would say that it would take an infinite number of steps to hit t=1.

I: Ok, so infinitely many steps would have to be performed to get to 1.

Chris: Yes. Infinitely many time intervals would also be passed over in that time. Infinitely many steps is one step further than that, because the way I thought about it was that each ball would move at t=1-$(1/2)^n$. For example, when n=0, the time is 0 and ball 1 would move out of the jar. Since there is an infinite number of natural numbers starting at 0, of the form n+1, and there is an infinite number of times t_n= 1-$(1/2)^n$… Since there's an infinite number of both of those… The number of balls in that infinity is equal to the number of time intervals in this infinity, so every ball would be removed when time is 1.

Todd: Oh I had the completely opposite answer…

Chris: Yeah, I know. That's why I was asking what you thought on that first.

I didn't think that way at first…

Todd: So the argument is basically, because you can give an n such that a ball is removed, after all steps all balls would be removed?

Chris: Yes, that's my argument for the time being.

Todd: Oh, alright.

31:37 I: Todd , what did you say?

Todd: I said the opposite thing. If you look at all the different steps, the number of balls in the jar at each step is 9 times the number of steps plus 1, so…

Chris: Well, what are you thinking now? I'm just curious.

Todd: Bummer.

I invite the students to individually rate their confidence and intuition for their respective solutions to this problem.

Todd: I guess I was saying, if you look at any particular step, the number of marbles is growing, so…

I: From what step to the next?

Todd: Yeah. I mean…

Chris: Ok. I don't know why I'm asking this, but… If the ball removed was not n+1 but… Let's say at step 1 you remove ball 2 and at step 2 ball 12. Or any other set of numbers, not 1, 2, 3, but let's say 2, 4, 6, or whatever. What would you say would be left at t=1?

I: So you still add 10 each time, 1 through 10 first… ?

Chris: Yeah. Same balls added, just the number of the ball removed is different. Because when 1 is removed, then 2 is removed, at least for every step you're counting through the natural numbers at some rate. If you skipped some numbers, what would be left?

I: So let's consider your question, Chris. Did you understand it, Todd ?

34:25 Todd: Well, which numbers do you add and which do you remove?

Chris: Let's pretend that this is like the first problem and you just remove 1, 11, 21, and so on. What would you say would be left after all steps were completed at that point?

Todd: It would be all the numbers that are not 1, 11, 21, …

Chris: All numbers not ending in 1, I guess that's what you're after. Why would the other ones not be removed though?

Todd: Same logic as for the first question. Once we decide to leave them there, they stay. They're stuck.

Chris: Ok. So what makes it different if it comes up through the natural numbers, 1 through infinity, you're removing 1, 2, 3. You're removing the same amount of marbles at each step.

Todd: Here's where we disagree, because I'm saying in terms of how many…

Chris: I know. I'm just trying to…

Todd: I see what you're saying. When you're removing them 1, 2, 3, then you can't say these ones are definitely going to be there.

Chris: Right. I mean the same number of marbles at each step is being removed, 1 removed and 9 being placed. So ok. You were going to ask something?

Todd: Well, a variation on your problem.

36:23 I: Wait, so you guys agree on Chris's variation (with 1, 11, 21, being removed)?

Chris: On the variation of 1, 11, 21, I agree that the numbers not ending in 1 would still be in there at t=1. I agree with that because there's no way to remove them. They're placed in there and you can't take out any number not ending in 1.

I invite them to individually rate [in writing] their confidence and intuition on Chris's variation [1, 11, 21,… removed].

I: Ok, so now we're back to the original problem?

Todd: The thing is we both agree on the variation, we just disagree on how that applies to what we're doing [the original marble problem].

I: Well, which part of Chris's version are you trying to apply to the original problem?

Chris: I know how I'm applying it. If you take [out] numbers that aren't consecutive so that you can't remove a certain set, sure there would be numbered balls left [in the jar] at the end of the time, but if you count through the natural numbers after putting them in, you're basically taking out everything that you're putting in. That's how I would apply that variation. The specific way that this is phrased, is 1, 2, 3 [being removed], not 1, 11, 21, it changes things. He [Todd] is thinking it doesn't matter which ball out of the 10 it is [that is being removed], because 9 [balls] would still be left there. It's just that the argument is not as obvious, because you *can* remove each one. So it's not obvious that 9(n+1) would still be there.

38:54 I: Would still be there after all steps had been completed?

Chris: Yeah.

I: But then what would be n?

Chris: Well, at each step I'm saying.

I: Would you agree that after a step n, you have 9(n+1) marbles in the jar?

Chris: That I do agree with.

Todd: Let's say we do the first step and we get balls 1 through 10 [and put them in the jar], and then we take out the last ball we put it. It's basically the same thing you did, so you're going to get basically…

Chris: You'll get the same version. I agree that if you took 1, 11, 21,… [Chris's version] or 10, 20, 30 [Todd 's version] …

I: Wait, so what's his [Todd 's] version?

Chris: He changed it from all ending in 1 to all ending in 0 to be [in the jar] at the end. I mean each variation like that I would agree with. For each step, there would be 9 balls left over. After all steps are completed, because none of the numbers not ending in 0 can't be removed, they would still be there. I would agree there would be balls left over in every situation you can imagine of this, except the case where you count up every natural number, and that's what you're removing. That's the only case on which I can't agree with you.

41:33 Todd: Ok, well here's the second variation. Let's say we take balls 1 through 10, we put them in. We have gremlins taking the marbles out, and we're trying to trick them to leave marbles in. So we take the number 10 and erase the 0 and write a 0 on marble with label 1. So they take marble 1 out, but that is in fact marble 10. Then we change the labels back. Same thing for 20 and 2, at the next step.

Chris: So you're basically tricking them to… I see the argument you're making, that it doesn't matter which one you remove, it will be exactly the same situation.

Todd: That's a more mature way of saying it.

Chris: That's ok, I'm not exactly serious most of the time. I agree with most of that, I just… If you're counting up, I mean, would you agree that there is an infinite number of times between 0 and 1, in that time interval?

Todd: yeah.

Chris: Then there will be one that corresponds to each natural number, plus 0. And there is an infinite number of those. And each marble is a natural number, and there are an infinite number of those. I'm saying that there is a one-to-one correspondence of each time interval to the ball, and the only way that prevents the situation of having balls left in the jar, is that each ball is counted. You can find a finite step at which each one [ball] is removed. It's not like where you start at 2 and go through everything, and 1 is just going to sit there. If I said you put 1 through 10 in and took out 2, then 11 through 20 in and took out 3, 1 has to be there [in the jar] still. I would say in that case 1 is still left. It's because there is an equal number of natural numbers to time intervals, because of the way the time intervals are defined. That's why I'm saying that nothing is left.

45:11 I: So if the problem was not phrased with the time intervals, would that change your answer.

Chris: Well, it depends on what you change the time interval to.

I: I don't change it to anything, I don't even mention it. I just say "after all steps have been completed".

Chris: After all steps have been completed? That makes it a little more interesting. I'd have to give a similar answer to that, that everything would be out of the jar, because everything would eventually be counted. But the time interval gives me a very specific grounding for that, because I can say that each time interval on this geometric interval here is getting smaller and closer to 1 each time. Each time interval, there are the same number of those as there are of marbles. It's a lot more abstract to say it when you don't have a time interval or something to connect it to.

I: Well, you have step numbers.

Chris: Yes, I know, but an infinite number of steps as it were… It's a lot easier to express, not so much to do perhaps, but to express that an infinite number is equal to an infinite number when they're based on the same set. The size of the set of natural numbers is equal to the size of the set of natural numbers. It's the same set. So it's easy to compare that. It's a lot more difficult to compare when you say, ok here's an infinite number of a set, and we're moving everything… You have the same number of steps, I guess, which does compare somehow. I don't why that's different then, because I just saw that connection. Infinite n, n is a step… the size of the natural numbers is the size of the number of steps. And the number of marbles is the same, which I guess would be the same exactly, with nothing left [in the jar]. Which would mean that everything is outside of the jar at t=1, or in the OTBP, everything would be in bin B at the end.

48:17 I: Todd ?

Todd: I don't know, I'm still hung up on it.

Chris: That's alright. I did enough of this at the [pre-test] interview. This is one I've at least convinced myself of.

I: This is somewhat different from what you said at the interview.

Chris: Oh, it's a little bit different. I mean I was getting there at the end of the interview, I didn't actually get there though.

Todd: I'm just hung up because at every step you have a net gain of 9 marbles, and there doesn't seem there is any way to take more than one marble out [at a step], so…

I: I think we all agree that the number of marbles from one step to the next increases by 9.

Chris: Yeah.

Todd: To use math terminology, you have a monotonically increasing function that expresses the number of marbles in the jar, and so I want to say that there are infinitely many marbles in the jar [at the end].

Chris: That reminds me of something else actually. It's just a different story. I won't go into details if you heard it before. Have you heard of Zeno's paradox?

Todd: Is that the one… ?

Chris: Achilles and the tortoise are running in a race. He gets a head start of a certain amount. And at each time, the tortoise is still ahead of Achilles, even though Achilles is about 10 times faster. I guess you can phrase that in a different sort of way. It sounds like you're saying that they start at the same point though and the number of balls are just growing too quickly to be taken away.

Todd: It's sort of the opposite of that though – what if the hare is trying to chase Achilles and he [Achilles] is way faster?

I: Right, so the slow one is behind the fast one?

Chris: Right, it is the opposite of that, I don't know why I was reminded of that, but it reminds me of it because it seems to be a situation where no matter how much you do, in Achilles' case, you're not catching that tortoise according to Zeno. Of course, with the concept of limits and infinite, I believe he came to the conclusion that eventually he would pass the tortoise, which makes sense to your intuition at least, I mean eventually he will pass, if given enough time. But it doesn't make sense because at each step he's still behind. I know there is a different between the two cases, but this is what I was reminded of. In this case [O10Mp], even though the number of balls is increasing, the time interval is decreasing a lot faster. The interval between two consecutive steps.

52:20 I: Well, you said before that if the question was phrased without the time interval, you would still make the same argument.

Chris: I would make the same argument but I'm not exactly worried about making the argument to myself at the moment. I'm trying to see if there is something that by talking this out I can see if I'm completely wrong [laughs]. I'm trying to give this in as many ways I can think of to express why I believe it.

I: So Todd , do you feel Chris's story is similar to our problem? You said it's the opposite – so is it relevant at all to our problem?

Todd: There are elements of it. I mean I didn't think about time until right now. So I got to start thinking about that. I see it [Chris's story] as Achilles is faster than the tortoise and Achilles is behind, but in this case it's like the tortoise is the marbles being removed and it starts out actually behind Achilles, who is in this case the balls added. It's a strange analogy. I didn't think about this whole time situation.

53:55 I: Well, why do you think that makes a difference?

Todd: I don't know if it does. I haven't actually thought about it so I can't really say. It seems that if you take time into consideration, at time 0 to time .5 you gain 9, and at time .5 to .75 you gain 9 again, so you're actually speeding up the difference between…

Chris: I understand that was probably not the best example [the story]. But… how many natural numbers are there? This is not as stupid a question as it sounds. Because you can say here is your starting line, I put the tortoise here and Achilles here [same starting point, at the left end of a segment]. So here's the start, and here [drawing a vertical line at the right end of the horizontal line segment and writing ∞ after it] is the infinite finishing line. So Achilles may get here [to this end] faster, but is he actually measuring the number of steps? You said there's infinitely many time intervals between 0 and 1. So even if he [the tortoise] goes 1, 2, 3, 4, 5,… , after infinitely many steps he will get here [to the vertical line] anyway, and Achilles might have stopped at that point. It sounds counterintuitive at that point, but the number of natural numbers is equal to the number of natural numbers. Achilles can't go beyond it, tortoise can't go beyond it. Tortoise counts every step as a natural number, after an infinite number of steps he would reach that final set cardinality.

56:03 Todd: That argument doesn't hold that much weight for me though. Because let's say we got n^2 over n. Both of those go to infinity.

Chris: Yes, correct, but they're not going to the same infinity.

Todd: So you're saying it's more like 9n over n? The thing is, how many balls can Achilles put in the jar total? He can only put a certain number in there.

I: You mean after a specific step or after all steps?

Chris: After all steps. He can't put anything that's not in the natural numbers. Yes, there are infinitely many natural numbers, but he can't go beyond it. And if you're saying that infinite number of steps, which is equal to the number of natural numbers, have been completed, then the tortoise must be done removing balls. He had to. So if you're saying that Achilles has put in more balls than the tortoise has removed, by 9 times or so, then you're saying that the number of natural numbers is actually 9 infinity. Fine, but then the tortoise must go 8 more infinity. 'Cause he's got to count every one of them.

I: And I have to ask what you mean by 9 infinity.

Chris: Well, I'm pretending for a moment that… I mean if you're assuming that infinity is a number that the tortoise reached. Let's say he completed all the steps but Achilles still has put 9 times that many balls in the jar; then the tortoise has not reached infinity yet because he's not done. There's still more numbers he can count through. That's not an infinite number of steps.

58:05 I: When you say the tortoise reached infinity, you mean it went through all the natural numbers?

Chris: It went through every natural number, yes.

I: What other steps are there for the tortoise to go through if we're saying it goes only through the natural numbers?

Chris: I'm not sure I understand the question, if you could repeat the question.

I: Just a second, I'm waiting for Todd to be with us. [Todd is scribbling something]

Todd: I was just trying to write more functions that describe… [pause]

I: Functions that describe what?

Todd: I got the number of marbles in the jar as a function of n, right? It's 9(n+1), after step n. Then you can do the ratio of marbles that are in [the jar] to the marbles that are out. Maybe if we make two jars, like in the first problem.

I: Why do you want to take the ratio?

Todd: If there's some way of thinking that I can have some intuition about, then I need to change what I'm thinking about. It's the same system, you're just thinking about it in a different way, maybe that will give you more insight. Like I've run out of … I know that argument [Chris's, for the O10MP] and I know the argument of well, here's the number of marbles and it's increasing, but there's no way to…

I: To decide between the two of them?

Todd: Yeah, I mean that made sense [Chris's], but this made sense too [the cardinality one].

I: Are you more attracted to one over the other, or are they equally…?

Chris: I won't be offended if you don't agree with me.

1:00:30 I invite the students to individually rate the confidence and intuition for these two solutions to the O10MP.

Chris: [addressing Todd] I'm sure you heard at some point that that equals 1 [pointing to what he had written down, which is 0.99999999999999…].

Todd: Yeah.

Chris: Why does that equal 1?

Todd: Uh……………………..

Chris: I'm sorry, I know this sounds like a lot of questions with obvious answers. But why would you say that that's 1? I'm hoping that this breaks… Because I feel at least in this sort of thing, that there is that final step I suppose which might push you to believe my argument. I mean, there is no way of knowing what it is at infinity, thought. I mean you can say that the limit as that approaches an infinite number of digits is 1… it gets closer and closer each time. You can say that it's $1-(1/10)^n$.

I: So that would be only a truncation of 0.999…

Chris: Yeah. It reminds me of this up here, $t=1-(1/2)^n$. It still eventually approaches 1 in both cases, you don't seem to disagree with that. But at one point does it actually reach one though?

Todd: Well, not at any particular point, but we just sort of say it's 1.

Chris: It's a convention, that's certainly true, but I mean if I were to stop after150 9's, you wouldn't say it's 1. You'd say it's really close, but it's not 1. So at what point does it actually reach 1 and why is that a convention that we actually use?

1:05:55 Todd: well, it's only 1 if it doesn't end.

Chris: Right, how many digits is that?

Todd: Infinitely many.

Chris: Yeah, I'd say so. Since you're counting digits though by 1, 2, 3, which is the natural numbers, it works at infinity though. So there is a way to visualize infinity at least as, even thought it keeps going as far as we can count, there is a way to view that there is some point where you can say ok, we can treat it all as one big thing, as one specific entity. Like you can treat 5 as one specific thing, obviously if you're taking the infinite number and saying it equals 1, you're treating that [the infinite tail] as one group. You can say that an infinite number can be treated as a number, sort of.

I: When you say an infinite number, I assume you mean a number with an infinite number of digits after the decimal point.

Chris: Yeah, or an infinite number of places. I'm trying to think of an argument that will work here, but I've pretty much said as much as I can pull out of nowhere on this.

1:06:20 I: Todd , is this relevant to you, in terms of the other problem [O10MP]?

Todd: Well, not that much in terms of #2. Well I don't see the interpretation of it, how you set it up in terms of what Chris is saying. Any marble won't be in the jar at some point, therefore there's nothing in the jar. That's true, but at the same time the number of the marbles in the jar never decreases.

Chris: I perfectly understand that.

Todd: What I was trying to get to is the ratio of the marbles in the jar to the marbles in the jar is 9(n+1) to n+1. So what is the limit of this as n goes to infinity?

Chris: 9, I know what you're saying. Is that really so much of a limit, because basically you're saying the limit as n goes to infinity of 9?

I: Well, it's the limit of a constant sequence, that's fine.

Chris: I understand, it's a constant limit, but… [laughs].

Todd: So I'm thinking of it as a 2-jar problem. I liked what you said about the first problem, the order of the two bins. Eventually every tennis ball in the first question goes to bin B, but the order of the number of balls in each bin is the same. Well, here I'm saying the order… it's 9 to 1, between the inside and the outside.

Chris: Well, when does… ? You know that ball 1 is removed at step 1. What is the largest marble remaining once 1 is removed?

Todd: 10.

Chris: Ok, it's going to be 10 times the step number. Ok. So when is 10 removed?

At step 10, right?

Todd: Yes.

Chris: What's the largest number [in the jar] then?

Todd: 100?

Chris: Yeah.

Todd: I agree with you on that…

Todd: I'm saying that what number you're going to pull out of the set of natural numbers, the removal has to get there otherwise infinitely many steps haven't been completed.

1:09:52 Todd: Right. But I'm saying, just like in the tennis ball problem [the original one], were you comfortable saying after infinitely many steps had been completed, bin A was empty?

Chris: I wasn't comfortable saying that A was empty, but I got there eventually because there is an argument to make for that.

Todd: Alright. Can we agree to disagree?

I invite the students to rate their confidence and intuition for the OTBP, for the answer that each believed to be correct at that point.

[End of session]

Session 2
11/19/08

0:00 [Start work on the Vector Problem. The students read the problem individually and spend some time thinking about it, without saying much. At my suggestion, they check with each other whether they are interpreting the text of the problem (especially the "…" that ends each of the vectors described in the first 3 steps) in the same way.]

2:15 Todd: So after all steps have been completed, there is an infinite number of zeroes preceding the first non-zero digit, basically?

Chris: You can say that. I thought you were getting at that, anyway.

I: Can you say what the 4th step is, just to make sure you understand the pattern in the same way?

Todd: 4 zeroes and then 1, 2, 3, 4, 5, and then the rest is zeroes.

I: Do you agree with that Chris?

Chris: Yeah. [Addressing Todd .] I see where you're getting that, because each step increases, it changes all the first n positions to 0, and then the next n+1 go from 1 to n+1, and then zeroes after. However, because this can only go up to… I mean it can go on for an infinite length, this vector has infinitely many positions, but if you have infinitely many zeroes, can anything really go after it? The 1, 2, 3, 4, to infinity.

Todd: I see, so once you let there being an infinite number of zeroes, it's all zeroes?

Chris: You can say that.

Todd: Or you never get to 1, if you want to say it that way.

Chris: Let's say that we have this here [writes

$Step\ \infty : (0, 0, 0,...)\ 1$

on the worksheet]. Do you ever get to this part where you have 1, or does the vector end there?

Todd: I want to say no, I want to say it doesn't get to 1 ever.

Chris: At the moment I'm inclined to agree with that, so ok. I'm trying to see if there is a way that 1 is actually in there. I mean, thinking about it in general, you have infinite zeroes, then you have 1 through infinity+1 [rolls eyes and whistles], then you have, then infinity minus two infinity zeroes [writes

$\infty\ 0's, 1-\infty+1, \infty-2\infty\ 0's$

on the worksheet.] This is obviously not purely mathematical. So the way I'm seeing now is it's all zeroes; it does not account for any digits, which seem to be being added… *seem to be.*

I: What do you think, Todd ?

Todd: After doing the previous problems, I agree with this… they're all zeroes.

6:48 I: And how would you write a proof for that?

Chris: If the vector is in N^5…

[He writes

1 (0, 1, 2, 0, 0)

2 (0, 0, 1, 2, 3)

3 (0, 0, 0, 1, 2)

4 (0, 0, 0, 0, 1)

5 (0, 0, 0, 0, 0)

on the worksheet.]

I'm taking a truncation. After 5 steps… there is no space [for non-zero digits to be part of the vector]. [Back to the original problem] So after the size of natural numbers steps… that's the length of the vector, it can only be as long as the set of natural numbers. And [the vector from] step n has n zeroes in front. So after all steps, all the 1, 2, 3, should be gone by that point.

Todd: That's similar to what I was thinking – give a position, and eventually it will be zero.

Chris: After n steps, the spot of the first non-zero digit is n+1. So if you actually completed all the steps, which is the size of the natural numbers, then the first non-zero number would be one beyond that, which is not in the size of the vector. So they should all be zero. I feel I'm saying the same thing repeatedly, though perhaps in different ways. I'm not sure I'm closer to a proof.

9:20 [I mention that in Math 300, one proof technique they used when trying to prove something was true for each element in a set, was to choose a random element of that set and prove the statement for it, and then claim the statement holds for any element of the set since the element used for the proof was chosen arbitrarily.]

Todd: I like what you did, choose an arbitrary vector length. But I guess you could also choose an arbitrary place value…

I: What do you mean by place value? Position?

Todd: Yeah… index.

Chris: A place in the sequence, ok, and you can tell when it will be zero.

Todd: The point is to show they're all zero, so maybe we should say it's non-zero, and then show that can't be.

Chris: I guess you can say take the value at position n in N… because it still has to be in that size of the vector. After step n, the value is 0 at position n. Since the vector is only as long as the natural numbers, it should make sense that after a step has been performed corresponding to each number in the natural numbers, everything should be 0.

Todd: Well, it [the value at position n] is 0 [at step n] and it doesn't change from 0.

Chris: Right, I should say that. [Writing] And is 0 from step n onward. So it certainly doesn't change. Since the vector v is of length equal to the size of the set N, and there will be steps performed corresponding to each value in N, for all positions in v, there is a to make each value 0. So after all steps have been completed, all positions are zero. I don't necessarily know if that's a very convincing argument for that. I got an intuitive sense that it's convincing, but that doesn't mean that I'm not missing some mathematical step somewhere.

17:26 [I invite the students to rate intuition and confidence level for the Vector Problem.]

Chris: I guess the only other way to do it would be to take each case: N^1, then N^2, and so on. Obviously with N^1, there is only one step, (0). In N^2, (0, 1) then (0, 0). [mumbling]. Obviously as you add a length to it, and you take the set of numbers you can use being in that smaller set, they all turn to 0. So with an infinite one I don't see it being any different.

I prompt Chris to explain this new approach to solving the Vector Problem from the very beginning, as I am not sure what he is doing.

Chris: The variation I had is truncating the size of the vector… the number of dimensions. Let's say the vector is length 1, has 1 dimension. Then the first position will be 0 and won't change, so that's it for that one. If you let it be 2 dimensions, the first step will have (0, 1), the 2nd step will be (0, 0). I guess the variation is varying the number of dimensions from 1 to the cardinality of the natural numbers. At each different set of dimensions, at the step equal to the number of dimensions, it goes to a zero vector of that number of dimensions. So for N^1 you have 1 step, for N^2 2 steps, for N^3, 3 steps. At the final step in each case, you get the 0 vector. If we're going to say all steps are completed with the size equal to the cardinality of the natural numbers, then you'll get the 0 vector of that many dimensions.

22:20 [I misunderstand Chris's new approach to solving the given Vector Problem, thinking he's trying to compose a variation of the problem, not of the solution, so he explains what he means some more.]

24:00 I: Do you guys understand what I thought Chris was trying to do? I thought he was saying, let's start with a zero vector of one dimensions, so just one position that is a 0. At the next step that becomes (0, 1). Then (0, 0, 1, 2). At the next step, (0, 0, 0, 1, 2, 3).

Chris: Increasing length, I see. […] I did a vertical section, she's doing a diagonal section.

I: Right. So if we do it that way, after all steps had been completed, what would be my vector? Can you think about this variation [of the problem]?

Chris: [addressing Todd] What do you think of this diagonal variation?

Todd: Um… I guess it's different because you don't have the trailing zeroes at the end. But I think you can probably still apply this technique to it [pointing to Chris's notes where the "treat cases of finite length and see in how many steps the vector becomes all 0's" method is documented].

I: How would you apply that technique?

Todd: Well, after all steps have been completed, you will have a vector of infinite dimension…

Chris: Actually, in that case, I'm not sure we can apply my technique… I mean in this case [pointing to the original problem], the vector is the same length at each step. But here [referring to my new version of the problem], it's going from 2, to 4, to 6, and so forth. So at step n it's a vector if 2n dimensions. So I don't know if that's necessarily the same thing.

27:30 Todd: But it's infinite though [referring to the vector after all steps have been completed, in this new version of the problem], right?

Chris: Yeah, I mean the vector is still only… The way I am visualizing it, at infinity though, you would have a vector of infinite zeroes and the set of natural numbers following it. I don't know why that would be different [from the original vector problem]. I suppose it's because there is no bound on the dimensions of the vector. Because here [the original Vector Problem] there's a limit on it, it just isn't a finite limit; it can't be larger than the size of the natural numbers. But in this case, with the variation, it's increasing each time to 2 times the step. So I imagine at infinity, the vector would have twice the number of dimensions that these vectors here [pointing to the 3 vectors described in the first 3 steps of the original Vector Problem] have. It's not clear.

Todd: Maybe, I don't know. It's like looking at infinite zeroes, infinite N, then infinite zeroes again [in the original problem]. That would be like 3 infinity. You can still apply this [Chris's "finite vector" technique].

Chris: The thing is though, now it becomes slightly different because if you have a vector… I don't know, let's say this is Step n here, you have n zeroes followed by [the numbers] 1 through n. And the dimension of this equals 2n.

Todd: So you're saying here [the original vector Problem] we're looking at any position in the vector because it's of infinite length no matter what, but with the variation it's not infinite so you can't take the same…?

Chris: I mean, see at step n here, the vector is 2n dimensions. For the original problem, at step n the vector is always the same length. The number of dimensions is big N.

Todd: So you can't use this technique [Chris's "finite vector cases" technique] then 'cause it relies on taking an arbitrary…

Chris: It relies on taking a specific dimension for a vector. So it's similar to that [original problem], except that at each step it [the length of the vector] increases. [Writing].

31:52 I: What did you write?

Chris: I always feel strange writing that, but at the infinite step I have infinite zeroes preceding 1 through infinity, because for each step you have twice the number of the step dimensions, so I don't see why at infinity it would be different. There's no bound placed on the length of the vector.

I: By the way, when you say "step at infinity", or "the infinite step", can you clarify what you mean by that? You said you feel a bit weird saying that.

Chris: I always feel strange because I said that for about 2 or 3 sessions… To say there is such a step… I mean there obviously is a step where everything is completed. It's just the definition that's always given is that you can always go one past it no matter what it is, so it's dealing with an actual infinity. I guess by infinite step, I mean when everything is completed. There has been a certain countable number that has been completed, even though it's not finite.

I: Ok. Todd , when you heard him say "infinite step", is that what you thought he meant?

Todd: Um… Sort of.

Chris: I have a strange set of notations, don't worry about that.

Todd: Is it like 0's of arbitrary length, basically?

Chris: I wouldn't call it arbitrary. I mean can you envision a point where… I mean you can envision 1 step having taken place, 2 steps, 3 steps. Can you envision an infinite number of steps having been completed?

Todd: I can't visualize it, but I can presume it happened.

Chris: Well, I guess that's the point. When I put the infinity symbol there, I mean to say that everything has been completed, we're actually physically at the point at infinity. That's it. There is no more of this number n, n+1. This infinity is not an n, it's done.

34:30 Chris: The reason I can do that and is internally consistent with me though… Have you [addressing Todd] taken Math Reasoning before, so you know these symbols for the sets, integers and all that?

I suggest he use draft paper for this part.

Chris: I'll just say for now that we'll take the positive integers, $Z_{>0}$. What's the size of that set?

Todd: Aleph null?

Chris: [laughs]. So you already know what I'm saying, that's fine. I don't remember exactly how to write aleph null. [Writes "∞"

next to $Z_{>0}$, then writes $Z_{<0}: \{-1,-2,-3,...\}$]. Cardinality of… [pointing to $Z_{<0}$]?

Todd: Same deal. They have the same cardinality as the natural numbers.

Chris: Obviously. Then, if we take the complete set $(Z_{<0} \bigcup Z_{>0} = Z_{\neq 0})$… So, the cardinality of this… If we got half of this being infinite, the other half being infinite, then the union is 2 times infinity. I know it's not exactly accurate but in my mind at least I have this view that this infinity (of $Z_{\neq 0}$) is of a higher order than these two (of $Z_{<0}$ and $Z_{>0}$). The set of positive integers and the set of negative integers are obviously smaller than the set of all integers without zero.

I: As in, they are proper subsets of…?

Chris: Yes, so that's the way I'm viewing this symbol of infinity (∞), just because I don't use the alephs, I don't remember them entirely. But this infinity there [that of the vectors described in the original Vector Problem] is equal to this one here [pointing to $Z_{>0}$], and I have no problem saying there is an infinite number of zeroes followed by the numbers 1 through infinity [for the "final vector" of my variation] as it is an infinity like this one here [pointing to $Z_{\neq 0}$].

38:45 I: Todd , what is your opinion on this variation? How do you want to go about it? Do you agree with Chris's [approach]?

Todd: He's saying that it's infinite zeroes followed by the natural numbers, basically? And that's what you get after all steps have been completed?

Chris: Yes, that's what I'm saying. Because the vector isn't bounded by this length [pointing to a N^N vector]. It's bounded by twice that.

Todd: Oh.

Chris: It's bounded, but it's not bounded by the natural numbers, it's bounded by twice that size.

I: Do you agree with that Todd ?

Todd: That's interesting. I don't know…

I: Do you see where he's getting that?

Todd: I see.

I: Chris, before you said my variation was a diagonal section, right? Well, if I'm taking the diagonal section of the infinite list of vectors of infinite length [obtained by listing the vectors in the original problem one below the other, in the order indicated by the steps], the length that I reach at any finite step is always less than the length of each vector in the original list, which is equal to the size of the natural numbers. How can I get to 2 times that size if I am always inside this "infinite rectangle"?

40:55 Chris: Yes, I think I do, but I might be missing that. I suppose I don't exactly see where that's going at this point but I mean, the bound on each individual step is the number of dimensions times the step number. I mean here [in the original Vector problem], at any given step you can just add another zero and add a number to the end [of the non-zero sequence of entries], and it doesn't violate any of the prior patterns to do that. I mean it would seem to violate things where the 1, 2, 3, would all disappear to get all zeroes in this case, but that's because there's a set length for the vector, a number of dimensions. It's like the 1, 2, 3 are being pushed over the length of the table. In this case [my variation], it's like you have a set of this length, zeroes up to here and numbers 1 through the step numbers up to there. Next step you say ok, let's add two more spaces to this table, shift the numbers 1 through n over, put a zero here and put the next number there [at the end of the vector]. It doesn't have a stopping point so to speak, 'cause you can keep expanding that for any step, including the infinite step. But that's just my visual nature.

I: Why would it still be expanding after the infinity point, didn't you say before that the infinite step is after all steps?

Chris: Well, *to* that point. Let's say it's one step before infinity, which is sort of absurd, but say that it is. For the final step, it would still expand two more, and you can put two more things in there, and that would be the end of that. It doesn't expand after that.

I: Well, if that step exists. If it doesn't exist, it might not be helpful to think about that. […] Todd , what were you going to say?

44:00 Todd: I was trying to cross off… Like we had two possible explanations for the original problem. The approach we had here [for the original problem] was to take position n in vector v. The problem is this vector [in my diagonal variation] does not have infinite length, a set length. So we can't use this technique to find out what's going on. What I was going to say is that this one [Chris's "finite length cases" approach] does not seem to be out.

Chris: I'm starting to see what you are getting at with the diagonal, or I think. It's not a fully formed idea but I think I'm starting to get to that.

Todd: It's another one of those things where there's two competing interpretations and I just can't…

I: Chris, did you follow what he said? He was saying that you guys had two ways to argue for a zero vector [in the original problem]. One was the one you wrote about [choose an arbitrary position in the vector and show it stabilizes to 0], another was to choose a specific fixed length, go down the list of [infinitely long] vectors and look only at the vectors truncated to that length, and after finitely many steps you will get a truncated zero vector.

Todd: If you say it like that it might not even apply. Because if you choose a big enough size [for the finite length of your truncations], it's not going to exist in your variation, initially. Like you're truncating to size 10, you're only looking at dimension 2, then 4 in your steps, so you'd have to start at the 10th step.

I: Right, so the first ones wouldn't be up to that length. But you would have plenty afterwards to fit that length.

47:10 Todd: Right, so as written I don't think it would work. You can adjust it to work, but the question is, is that valid reasoning?

Chris: Well, the I way I see it is that you can say that for the case when the dimension is 1, clearly they're all zero after step 1, for 2 you can say it's true… and by saying that it's true for some arbitrary finite n in the set of naturals, can you verify that it would work for n+1 then? If you can do that, then you can induce the whole set of natural numbers.

Todd: Well, for a fixed n, I can tell you at what step you can take a truncation of the vector that would be the 0 vector. Is that good enough though?

[Some back and forth discussion is taking place between Chris and Todd about the applicability of Chris's technique, but it's not clear whether they're talking about it in the context of the original problem or my variation.]

Chris: Does that apply to that [my variation]? The reason I started to question myself on the dimension of the vector in her situation, I can't really explain that. But does the infinite step actually happen? For some reason all this talk of sizes, of infinities and diagonals immediately invoked Cantor in my mind. I'm not entirely clearly on what you mean by a diagonal contained within it.

Todd: If you want to talk about how to get the vectors in her variation from the list of [infinitely-long] vectors in the original problem, you just draw a diagonal [through the list of vectors from the original problem].

Chris: Ok, that's the diagonal. Right. And you're saying because the size of this vector is bounded with the cardinality of the natural numbers, how can it ever go beyond that? Well, they're all contained within that and I understand where you're getting the point that after all steps it's still within that, but I don't believe that it is, because I believe at some point it actually breaks out of that.

52:30 I: Ok, And how would you describe that point?

Chris: How would I describe it? When all steps have been completed, it's countable at least, so you can take it from step 1 to the step where all have been completed, half way between the two is where… I mean, because at some point you're saying that an infinite number of steps have been completed in this case, without the diagonal. And it's got that same number of steps is equal to the number of dimensions at a certain step. About half way in between… I can't really reason where that would be… but at that point it would have half of that infinite number of dimensions at that step would be zero and then it would be 1 through whatever step number it happens to be. Beyond that point though, you're still allowed to add two more dimensions to it for the next step, because you haven't reached the infinite step, so it increases the size no matter what. I mean it's correct to say that some of these steps are in fact contained in the original problem. I don't believe that they're contained all the way through an infinite number of steps. There is a point about half way between the two… the only reason I can't really quantify it, is because if you take an infinite number of steps, then half of that infinity are zeroes and half are 1 through half of infinity. At that point it breaks, but what step do you classify half of infinity as?

I: That's a big question.

Chris: I believe there is a break point there, because the restriction on the dimension of the vector isn't there, as it is in this problem [the original vector problem].

55:32 I: Ok. Todd , are you following what he's saying? Do you understand what he means by a break point?

[Chris explains his theory about the break point one more time to Todd .]

Todd: So the ramification for "it doesn't break out" is that…

Chris: Then my solution that it's [infinitely many] zeroes and then 1 through infinity, that would be completely out because there is no space for all that… for her variation.

[…]

I: So let me make sure that I understand what you're saying, Chris. In my variation, after all steps have been completed, you'd have a vector where for the first big N positions you'd have zeroes, and for the next big N positions we'd have all the natural numbers.

Chris: Yes.

I: And Todd , what's your position on this?

Todd: Can my position be "no comment"?

I: I guess it would be helpful if you gave a reason for why you're confused, if you're indeed confused.

Todd: Um… [longer pause]

I: Is it because you have two conflicting arguments or you have none?

59:15 Todd: Well, I'm saying this argument [the "choose a random position and show it stabilizes to 0" one] isn't valid here, in your variation.

Chris: I agree with that.

Todd: And then this one [Chris's "finite length" method], it's not clear that it is valid in your variation.

[More back and forth discussion between Chris and Todd that does not bring anything new.]

1:04:02 Chris: I guess that really is the heart of my argumenTodd: because you can divide it [the "infinite" matrix of vectors from the original problem] by such a diagonal [separating the zeroes from whatever comes after them], the left side matches how I thought about the original problem, and the right side is still there so you can put that at the end, because it [the length of the vector] is not bounded.

I: […] And Todd ? I'm still trying to get a position out of you.

Todd: I don't know how to explain it.

Chris: How are you thinking about it? Try something. I'm perfectly open to being wrong.

1:06:25 I: Are you thinking about it in any other way than Chris did? If you're not, that's fine.

Todd: It's just the diagonal thing, that's what I wanted to understand. What it really shows me is that the second proof does not work [for the variation] just because you can construct a zero vector of arbitrary length. That alone is not enough.

[Rating intuition and confidence for both the original vector problem and my variation.]

1:07:53 [Start work on the ½ Marble Problem. Students reading the problem individually, Chris mumbling to himself]

I: Todd , do you remember this problem?

Todd: I remember I said I had no idea.

I: I think you were saying something more.

Todd: I guess you can say that there is a marble in the jar, but because the identity of the marble depends on the step, you can't say which one.

I: It's interesting that that's different from what you said before [during the pre-test interview]. You said there's nothing you can say about after all steps had been completed, and we did this right after we did the odds/evens variation. You said that for this one we can't say anything was because there [with the odds/evens] you can easily see a pattern that did not depends on the number of the step, but here you couldn't see such a pattern, which is why you could not say anything about the end.

Todd: If you had pressed me, I would have said there is a marble but can't tell which one. It's not that I thought there wasn't a marble in it. I'm pretty sure at that point I thought there was a marble.

I: I see. So your answer was more about not knowing which one it was. And Chris?

1:10:03 Chris: I don't remember my position [during the pre-test], but as I think about it now, I concur entirely in the fact that I do not know specifically which one is in there.

I: But do you want one [marble] to be in there?

Chris: Do I want one, both or none? I was toying with the idea that there were possibly both or none or in there… along with only one marble in there. Unlike the other problems, in this one the set of natural numbers is not implied anymore, except for the number of steps. I can't draw connections of size of sets, which obviously I like to do. At every odd step [marble] 1 is in, and at every even step [marble] 2 is in, but for infinity what can one say about that? Because infinity is not even or odd.

[…]

Chris: Now that I've actually said it to myself that way, I can say that both cannot be in the jar at the same time at least, so it's either none or one [marble]. Because if they're both in, that implies that infinity is both even and odd, and that's ridiculous.

I: Ok, and if it's "none", what would that imply?

Chris: If it's none, that would mean infinity is neither of the two [odd or even].

I'm not nearly as convinced of that as I am of the fact that you can't have infinity being both odd and even. To say that it's neither… is weird also. So… I guess I could conclude one [marble] is in there, so I can at least rule those out. It's one of the two [marbles], but I can't tell you which.

I: Ok, it looks like you agree on an answer of the type, there is one marble in there and we don't know which one. Now, since you're not sure how to proceed further, let me ask you this. Can you try to rephrase this problem in a way that would allow you to say precisely that you have exactly one marble in the jar and you would also know which one. So by changing the rules of the game…

1:13:33 Chris: I don't think neither of us has trouble saying that there's one [marble] in there… is that right?

Todd: I became less sure once you started with there's none, there's two… and I was like "get out of here".

I: And then he argued against those options.

Todd: I know, but now I'm…

Chris: I contradict myself all the time. […] Do you have a reason why there should be both or none?

Todd: No, I just thought it was enticing to say that. I'm not sure there's going to be one, now…

Chris: Well, can you see a case where there would be both?

Todd: All steps have been performed, right? Is there a last step or not?

Chris: All steps have been performed means that there must be a last step. According to my view.

I: What do you mean by a last step? A final state of the jar, or a final step of the quality of the ones described in the process?

Chris: You can also say that it's a final state, but I would also argue there has to be a last step also, even if you don't know what that number could possibly be. If you say infinitely many steps have been completed, you're saying that every single thing in the set of natural numbers has had a step labeled by it, so there does have to be a final step, even if you have a hard time finding what that number is and finding it and writing it down. I'd say there is a last step though.

1:15:53 I: If you cannot write it down, how else would you define it so that we know what you mean by it? For any of the steps described by the process, we always have a step coming after it, so we wouldn't call it last.

Chris: If all steps have been completed, that very last step… There has to be a last physical step, even though I can't quantify it with a specific number. […] If you say that a finite number of steps have been performed, there's obviously a last step. If you're treating an infinite number of steps as something that can be completed, even though you can't put a number to it, then I believe it as well has a last step to it. I don't know how to describe it further.

I: Ok, let's leave it at that for the time being. I think you were asking Todd whether he can envision two marbles being in the jar after all steps.

Todd: I didn't say I had an argument. I said it sounded pretty sweet.

Chris: It helps in the problem if we can rule it out or if we can't.

1:18:53 Todd: That final state… you'd have to reach it by taking the steps outlined here, right? So… there's no step that leaves you with two marbles in the jar.

Chris: Right, no step has two marbles in the jar.

Todd: So there's no way to reach a final state with 2 marbles. Same for no marbles.

Chris: Right, you can't use the intermediate steps.

1:19:47 I: I'd like to point something out. For the original Marble Problem, where you put marbles 1 through 10 in [the jar] and remove marble 1, then put in 11 through 20 and remove 2… Your answer was that after all steps have been completed, you'd have an empty jar. So now if we look at how many we have at each step, and how many we have at the end, it doesn't seem to be consistent with what we have at each step. We always have a non-zero finite number of marbles in the jar after each intermediate step, and then at the end your conclusion was that we had zero. So to say that at each step here we have exactly one marble, therefore we can't have zero at the end… if you're just talking about the cardinality of the set at each step, I gave you a counterexample where that was not the case. Just this argument by itself. So is that enough to rule out the "two" and "zero" [as options for the cardinality of the final state]?

[Chris recaps his argument for the original 10 Marble problem, then comes back to the ½ Marble problem to revisit the issue of what parity infinity is.]

Chris: Because every finite number is even or odd, is it enough to say that infinity is even or odd? Which is how I came to the semi-conclusion that we have [marble] 1 or [marble 2] in the jar.

Todd: Let's try and make an equivalent game.

Chris: That's what we're supposed to do, find one to make this more determinate.

[…] That means the steps still have to be infinite.

I: Yeah, I still want to have that part. You can change any other part. [pause]. This might be a good time to stop. You can think about this for next time.

[End of session]

Session 3
11/26/08

0:00 [Start work on n-> n+1 Marble Problem]

Todd: This is going to be one of those things where there is nothing in the jar.

I: First make sure you understood the problem, that you understand it the same way… […] Let me ask you a question, after step 50 what do we have in the jar?

Todd: Marble 50, that's it.

I: And which marbles have we finished processing, meaning we're not going to use them again.

Todd: Marble 1 through marble 49.

Chris: I agree, alright.

I: Ok, it seems you understand it the same way. What do you what want to say about it?

Chris: Well, it's clear to me that there are two options: there is either nothing, or there is one [marble in the jar after all steps].

2:30 Todd: I'd like to make a strong argument for "nothing".

Chris: Ok.

Todd: Let's say you wanted to make a formula for when marble whatever is never going to be there ever again, and it's just step "number of marble plus one", so you want to do that for all natural numbers and then there are no marbles that are in the jar.

Chris: That's a pretty strong argument, I like that, that's why I thought nothing. The only reason that one [marble] is even a possibility to me, it's because for any finite step there is one left [in the jar], so there is always going to be one, so at the very end it's conceivable that there is one "marble" in there. But I don't particularly agree, I'm just saying that it could be the case. I would agree with that though, that they are all gone because there is a step at which every single natural number would be removed.

Todd: Ok. Or you could do like, let there be a marble in the jar number whatever [after all steps], and then that's a contradiction because it was removed at step…

Chris: That's a good argument for it, as there will be a point at it will be removed so it won't be there at the end.

3:50 I: Do you want to try to write a convincing argument for whatever position you want to take? Let's let Todd do the honors this time since he didn't write much last time. I don't care much about the format of the proof, as long as its structure is clear.

Todd: [writing] Let n be the number of a marble. At step n+1, that marble leaves the jar and never returns.

Chris: What can n be?

Todd: n can be any natural number.

Chris: So n is a natural number. So now you have some information that you can use to judge the fact that it won't be there after a certain time. So for every n this holds, so they will all be gone.

Todd: Right, so do you want to say it like that [in writing]?

6:53 I: Well, before you make a statement about all of the marbles, why do you feel you can say that marble labeled n is not in the jar after all steps? Here you're telling me the step at which it is removing from the jar, and then that it is not added [to the jar] by any of the subsequent steps. Do you feel that is enough to make the claim that after all steps have been completed, marble n is in the jar?

Chris: Certainly for any arbitrary n, the step n+1 which is in the same set of natural numbers will remove it.

I: Yeah, I agree with that statement.

Chris: But do I believe that is enough?

I: Yeah, the question is how do you use those statements to claim that at the end, meaning after all steps have been completed, we're not going to have marble n, for that particular n, in the jar?

Todd: So you're saying it's not good enough?

I: Well, I'm trying to get a sense of whether that's convincing enough for you to make this claim.

Chris: [Addressing Todd] Do you believe that if you say what we've just said, that that's enough? Do you believe that's enough to prove it, or there is some element to make it more solid, or did we not prove anything?

I: So, you certainly proved something for that fixed number n, that the marble with that label is not in the jar after step n+1, and then it's not in the jar for step n+2, n+3, and so on, all the following natural numbers. So you know what to say about marble n after each finite step. So the question is how can we make a statement about after all steps? 'Cause all we know so far is what's happening after each finite step.

9:30 Chris: I have an idea of how to attack that particular question. Considering it's natural number, you can use some form of induction and claim that… I mean obviously after step 2, marble 1 will be gone. You can verify that at step n+1, n will be gone. Although you would have to make a hypothesis and prove that using the other one. But anyway, that's a possibility of how to prove it for all the natural numbers, and since these are only natural numbers, that would be enough. I personally believe it's enough, but…

I: [addressing Chris] Do you feel that addresses my question? Because with induction, you can prove the statement "for any n in the natural numbers, marble n is not in the step after step n+1". So you would be proving more rigorously a statement about where each marble is in respect to the jar, after each finite step. You already made this statement. What I am asking about is in regard of the next inference, where you use what you know about the whereabouts of each marble after each finite step, and you want to make a statement about where each marble is after all steps have been completed. This seems to be a qualitatively different stage. I'm asking about this jump from information about the finite steps to claiming something about after all steps. Based on what can we make this jump?

Chris: I've heard of this before, unfortunately… [pause]

I: Let's take another example in mathematics, a real-valued function which is not continuous at 1. You can have all the information you want for the values of the function as x approaches 1, but you're not going to be able to say what f(1) is based on the values of the function for x different from 1. So here, especially if we think about the timed version: we know what's happening with the contents of the jar at times approaching 1. So if our situation here is in any way like the discontinuous function, then we wouldn't be able to make that claim [about after all steps].

Chris: Not by itself, anyway. I know what you're saying, I hear it.

I: Todd , did you follow what I said?

13:40 Todd: I think so. I'm not as intuitive as he is though. […] What if we said that there is a second jar of things that were discarded from the first jar. And you can show that that contains every marble. Is that anything additional, or not really?

I: Well, that would be a pretty similar situation. Now when you look at the second jar, what do you know? Only what's happening after each finite step. Again you're making a jump to claim something about the end state, so again. What allows you to make that jump for either of them?

Chris: Yeah… alright. It does remind me of something though. Say that you have two true statements and they lead you to the conclusion of the third statement. And you believe that's true because of the first two. Well, you can have somebody who believes the first true statements are true, but doesn't believe your conclusion anyway. Sort of an abstract way of discussing this. With infinity we're always presented with something similar to that.

I: Right. In logic you have a set of premises, and then based on some mechanism you infer a conclusion. And when you use a valid type of inference, then from true premises you get a true conclusion. But that was based on some inference rules accepted in that theory. But someone else comes and says I don't like your rules, I want to define new inference rules. Then starting from your set of true premises, if you use different inference rules, you're going to get to a different true conclusion. So I guess that's my question here, what's our inference rule to jump from the finite steps to the end state?

16:37 Chris: Well, I know on my part it's an intuitive thing. Do I have a mathematical reason for it? Nothing I can formulate currently… maybe in a few minutes I'll be able to.

I: Ok. So intuitively you feel this inference should be valid.

Chris: Because I can compare the size of the number of marbles to the size of the natural numbers.

I: Todd , do you have any comments on this inference rule?

Todd: [shakes head left and right meaning "no"].

I: Ok, so so far you want to go with that kind of inference because you feel it intuitively makes sense.

Chris: I am comfortable leaving it that way.

I: Ok, finish writing up this argument.

Todd ; Are we going to make a claim about the size of the steps and the size of the set of marbles?

Chris: That's the way that I'm visualizing it but I mean…

Todd: That's really similar to what we did with the …

Chris: Yes, certainly very similar to the other arguments we made before.

[The students finish writing up what they proposed before as an argument for this problem.]

22:40 Todd: [Reading what he wrote down] Let n in the natural numbers be the number of the marble. At step n+1 marble n leaves the jar, never to return. After all steps have been completed, marble n remains out of the jar. By choosing an arbitrary n, we can extend the argument to all marbles, because they are labeled with the natural numbers. Thus, after all steps have been completed, the jar is empty.

[Intuition and confidence rating]

23:45 [Start work on the n-> n+1 Label Problem]

Chris: [laughing, after reading the problem] I like that variation. Sounds very familiar.

Todd: So we can't say there's nothing in the jar?

Chris: I would imagine there is a marble in there.

Todd: Right.

Chris: Now the question is, is there a label in there?

I: [addressing Chris] Can you explain why this problem sounds familiar?

Chris: One of the variations from the previous sessions, when you put marbles 1 through 10 in the jar, and remove [marble] 1; he [Todd] mentioned writing a 0 at the end of a label. But it's not really the same problem. Obviously here the marble remains in the jar, instead of removing it. At least we agree the marble is in the jar so I can at least write the marble is still in the jar, for now.

I: Do you agree with that, Todd ?

Todd: I think the marble is in the jar.

I: When you say "the marble" what do you mean, you mean the unlabeled marble?

Todd: There is a marble in the jar and I'm talking about the marble that's in the jar.

Chris: Yes, I agree, because that marble does not move. Now the label changes but the marble is still there.

26:48 Todd: So are these actually different, these two situations?

I: You mean the problem before?

Todd: Yeah, this one and the one before.

Chris: There the marble is removed out of the jar at some point, but for sure now we know that a marble is in the jar.

I: If we think about a first jar being related to the first problem [n->n+1 Marble problem] and a second jar for this problem [n->n+1 Label Problem], would you say after each finite step that the contents of the two jars are the same?

Chris: Yeah, I guess it's a similar thing. The content after each step is the same, yeah. But the marble doesn't leave the jar [in the second problem], it's just the label leaving. So here we have marble in the jar, labels are leaving; in the other case the marble and the label are leaving.

Todd: So we think there is a marble in the jar after all steps have been completed. But do we think there is a label?

Chris: Um… [pause]

I: Do you have a guess for that, Todd , at this point?

Todd: Because I was thinking, if there is a marble in the jar here [n->n+1 Label Problem], then that is totally wrong [pointing to the n->n+1 Marble Problem]. But then I realized that the marbles and the labels are not coming apart here at any step [in the n->n+1 Marble Problem].

29:30 Chris: Right. So we know the marble is there [in the n->n+1 Label Problem].

Todd: For the second one, let's just take out the word "marble", and just talk about marbles. Then it's identical to this one [n->n+1 Marble Problem], where the result was there is nothing in the jar. So using the same logic we used here, there would be no labels in this jar. But maybe we're totally wrong.

I: So Todd , when you say "using the same kind of logic", can you spell out what that would be in the case of the current problem?

Todd: Let's take away the word "marble" so that all is being put in and removed is the numbers 1, 2, 3 and so on. That's sort of the same as marbles with numbers on them. So then we would say, let n in the natural numbers be the number on a piece of paper. The piece of paper leaves the jar at step n+1 never to return, blah, blah. Then there would be no pieces of paper with numbers on them in the jar once all the steps had been completed.

I: Ok, so then what would be in the jar [after all steps]?

Todd: Since there is always a marble in the jar, I would say [at the end] there is a marble with no label.

I: OK. What do you think, Chris?

Chris: Like he said, the marble shouldn't matter in this instance. Because it's basically, just the labels entering and leaving. It's the same problem. The only difference technically is the way it's phrased. Because I agree with [the solution given to problem] 1, I certainly agree with 2. I have a perfect explanation for why the marble is there. For the label I have no explanation.

32:47 Todd: Ok. But the problem I have is that, like she was saying, for any finite step, these two are the same…same situations. So…

Chris: I agree with that.

I: I guess that kind of brings us back to what we said before, that if two people agree on a set of statements that are true, it doesn't necessarily mean they'll come to the same conclusion. So here we can say that two people looking at the two jars [associated to problem 1 and problem 2, respectively] would say that they contain the same thing, after each finite step. Even though there is a difference in what we do in between, during the steps. Is this problematic for you guys, that you're about to make different claims about what's in each jar at the end.

Chris: The only reason I make that claim myself [in regard to Problem 2] is because the marble technically doesn't leave the jar at any point, even though labels are being added to and removed from it. Whereas in the other problem, the marble with the first label, the marble with the second label, leave and do not come back. Here [Problem 2] the marble is independent I suppose. It doesn't really make a difference; the marble will be in there [at the end].

I: Ok. Do you want to write an argument for this?

[Chris writes a summary of the argument, with Todd 's assistance. It's a parallel argument to what they did for the first problem, except that now it's referring to the set of labels.]

37:40 Chris: […finishing writing the argument] So after all steps, the marble remains without an attached label.

I: Todd , do you have anything to add to it, or is that ok?

Todd: Yeah…. it's consistent with the first time.

I: Do you feel that taking a different approach here would be invalidating your solution to the first problem?

Todd: The inferences we're making in both cases are questionable… maybe. It depends on whether there is a significant difference between them or not.

Chris: Well, it shows that the inference we're making is based on the numbers, and now on whether they're attached to a marble, which is nice to know. But it would be inconsistent I think if we said something different for this second problem. Because the only difference is whether the number is not attached to the marble or not attached. Whether a number is in the jar at the end is still the same problem. The marble is going to be there anyway.

[Intuition and confidence rating]

39:40 Todd: There's no like retroactive confidence? Now that I've seen this, what I said for the first one is like… [making noise and gesture to indicate the confidence related to the 1st problem went down].

I: You mean your confidence went down for the other one?

Todd: Yeah.

I: If that's the case, rate it again for Problem 1. Don't change the previous rating, but give it a new rating.

40:35 [Start work on the 1+1/n Powder Problem]

Chris: [upon reading the problem] I don't like repeating fractions… well, maybe it's not repeating fractions, but you are using fractions repeatedly.

Chris: [addressing Todd] How much do you think is left I guess [after all steps], before we start arguing why?

Todd: I don't really have an idea.

Chris: Ok, fair enough.

Todd: Greater than or equal to 1.

Chris: That's clear enough to me. Well, can you say it's less than or equal to 2?

Todd: Yeah, let's say that.

Chris: Well, we actually got it between a lower and upper bound. Excellent.

Todd: I'm going to use the abbreviation AAS for "after all steps", 'cause I've been writing it a lot.

44:20 Chris: [writing] I'm just calculating how much is being removed at each step, specifically. Yeah, that agrees with what I thought. Let me go to step 3 just to be sure. Let me make sure I'm not generalizing too soon. I'm subtracting how much powder we have at the end of a step from the amount that was there after the previous step. I'm assuming that's a valid technique for the moment. So at each step, at step n you're removing $\frac{1}{n(n+1)}$. Ok, now I've got all the information I wanted. Good.

I: Do you see what he's doing, Todd ?

Todd: Uh-um.

I: Do you see a reason for doing that? How do you think that helps?

Todd: [addressing Chris] Are you trying to sum them?

Chris: I could do that. I wasn't thinking of it. I was trying to see what was coming out at each step. I know how much is left at each step, but I like seeing both sides of it, just in case I want to check something along the way. You could sum them, if you wanted to. If you wanted to make the claim that I believe, which is 1 ounce at the end, you could add these from 1 to infinity and it should add to 1.

I: So what is the series, can you write it out?

Chris: The series would be the sum from n equals 1 to infinity of 1 over $n^2 + n$. Ok, so if the claim is "1 ounce", then the sum should equal 1. [addressing Todd] Obviously you weren't thinking of that, so what were you thinking?

You already claimed it's between 1 and 2, do you have any guess where it lands?

49:30 Todd: I mean I was going to take the limit of that as n goes to infinity, but that's not really…

Chris: Well, that's fine.

I: The limit of what?

Todd: The limit of the amount of powder you're left with at each step, as n goes to infinity.

Chris: Well, what do you think the limit is?

Todd: I was going to say 1. n is getting really bad, that [1/n] is getting really small… it's going to go "poof".

I: I have a question, apart from what the limit is. Why do you think you should be taking the limit?

Todd: well, I'm not saying that it's right. This is what I'm thinking…

I; Right, but the question is, what made you think about the limit?

Todd: Well, I just imagined n getting really, really big, infinitely big. I guess it's the whole "n in the natural numbers" thing, because after all steps you would have gone through all the natural numbers, so the numbers would be getting really big, so as n gets big, what happens. But I don't like that anymore now, because of the whole, we're not allowed to make an inference…

Chris: Well, we're allowed.

Todd: It's not clear it works.

Chris: Well, that's another matter entirely.

51:48 I: Do you feel that making this claim and taking this limit is the same kind of inference as before, say in Problem 1 [n->n+1 Marble Problem]?

Todd: That's a good question [pause]. Actually, no. No, not really.

I: [addressing Chris] What I was asking him [Todd] was if claiming that the limit of this expression is what we're looking for requires the same kind of inference that we talked about in Problem 1.

Chris: It does take that inference, but this one feels more grounded in Calculus, because the limit concept has been drilled in us for years at this point. It does involve some inference; taking the limit does not necessarily mean that at infinity it is that. That's the same type of inference, as I can see it. I have no idea what you said [addressing Todd], because I was doing the steps of that summation, only to find that the way I broke it down makes it unhelpful to me.

54:37 I: Why is that?

Chris: If I break down $1/(n^2 + n)$ into partial fractions, I get

$\sum_{n=1}^{\infty}\frac{1}{n} - \sum_{n=1}^{\infty}\frac{1}{n+1}$, which is two stinking harmonic series. That is not going to help me get an answer.

Todd: That's good!

I: Actually, that might help. Todd , do you think that helps?

Todd: Yeah.

[Discussion follows concerning properties of series/partial fraction technique, a bit of recap from Calculus. Chris concludes that the sum of the original series, $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ is 1.]

1:02:15 Chris: So if this claim is true… [addressing Todd] so your claim is that the limit of what's left [on the scale after each step] is 1. I'm basically taking the limit of how much is being removed. They should come out to the same number, if we're on the right track. At least for now, I don't see any reason why there's not 1 [ounce] left [on the scale, after all steps].

I: Todd , what do you think about his general approach, to compute the sum of what's being removed. That's certainly a different approach from yours. Do you think that's a valid approach to the problem?

Todd: Ok, so let's talk about the difference between our two methods.

Chris: At least in theory they should be equivalent. I mean you're taking the limit of what's left. So basically at every given point [step], it [what is left on the scale and what was removed up to that point] is always going to add to 2 [ounces].

So they should be equally valid approaches. Although I did think of the limit approach first, and then I thought of something different.

1:05:30 I: why did you give up on the limit idea?

Chris: I didn't give up on it, I was just calculating these [differences] and then I saw a pattern and then thought it should equal 1, and then I wanted to sum them and went through it.

I: So if that sum is indeed 1, does that give you the answer for what's left after all steps?

Chris: Since it started with 2 [ounces] and I'm removing 1 through all steps, then the answer is 2-1. This 1 [the sum] by itself is not the answer, I have to subtract it from 2. This [pointing to the limit approach] is more straight-forward I believe, because it's acting from the point of view of what's there [on the scale]. It just seems that this [the sum approach] is more clear why that works out, because you can see the telescope. It's a little bit less clear with the limit approach, even in my mind is as correct.

I: Well, you said before there might be a problem with the limit approach because when you have the limit of a sequence, it doesn't mean that the terms of the sequence become that limit. Todd , is that a problem for you?

Todd: Remember when we were talking about time, and it was getting closer to 1, and I was like it's never actually 1, and he was like "Todd , stop talking!" and I said fine. I feel the same way here.

Chris: I don't remember saying exactly that, but anyway. If it exists at infinity, then the limit will be what it is there, but you can't guarantee with the limit that it is at infinity… available.

1:08:32 Todd: So the inference that we're making is that because the limit is this as n goes to infinity, after all steps have been completed it [the amount of powder on the scale] will be what the limit is.

Chris: Yes, because it exists at infinity, even though you can't physically perform that amount of steps. Since it does exist at infinity, there is a place where all steps have been completed… It's not like a limit as you approach a disconnecting point of a function, it actually does exist there.

I: How about the limit of 1/x as x approaches infinity? The limit exists, right?

Chris: Yeah, as long as the denominator does not hit 0, the function exists. At infinity it will actually hit zero, one could say, by the limit. However, if you were to take the limit in the opposite direction, as x goes to 0, well the limit is infinity.

Todd: So we're going to say it's a valid inference, basically?

I: Which one?

Todd: This one: because the limit of 1+1/n as n goes to infinity is 1, then after all steps we have 1 ounce on the scale.

Chris: Alright, so what you're asking me to be clear on is whether this limit can determine this because it actually can exist at infinity, that all steps have been completed meaning infinitely many steps have been completed, it can exist and is not some weird disconnecting point. If that's what you're saying, then yes, I believe it's a valid [inference], because it can exist at an actual infinity. If it was undefined at infinity then we'd have a problem. You can conceivably take infinitely many steps and get a finite value, it's not going to be something undefined.

[Intuition and confidence rating in process.]

1:14:15 [Start work on 1+1/n Marble Problem. Students read the problem individually and think about it silently for a bit]

Chris: I'm interested to see what you [addressing Todd] would say about this problem, I mean once you have a guess about what to say.

Todd: [turning pages back to look at the previous problems] Alright, we're sort of combining the two things that we said… maybe.

I: What two things?

Todd: The initial problem [n->n+1 Marble Problem] and the limit problem [1+1/n Powder Problem].

Chris: You are combining certain elements, yes.

I: Ok, so problems 1 and 3?

Todd: Yes.

Chris: Well, what do you think the possibilities are for what's in the jar at the end?

Todd: Well, the possibilities are "marble" and "no marble", because all marbles have labels, so if there is a marble, there is a label. I guess you could also have label, but no marbles.

Chris: I don't believe that's possible here.

Todd: Can you have an unlabeled marble?

Chris: Do you want it?

Todd: No, I don't really want it.

Chris: Then no.

Todd: Ok, so possibilities would be "labeled marble" (LM) and "no marble" (NM). And now the question is, the label, is it going to be 1, or not 1?

Chris: You gave me an idea that you think it's going to be 1 because that was your initial guess.

1:18:00 I: Are these the only two possibilities?

Todd: We think so.

Chris: We have one marble, no marble… Is it possible to have more than one marble in there? I don't believe there's two, but do you believe that's a possibility worth looking into?

Todd: Um… no.

I: How about 3?

Chris: I mean multiple marbles… if you're going to imagine 3, you need to imagine 2 in there first.

I: So is that one of the options or no?

Todd: No. It's either going to be one labeled marble or no marbles.

I: Just to make sure you understand the problem, can you write out what's in the jar for the first 2-3 steps.

[Todd writes the contents of the jar for the first 5 steps.]

Chris: Is there a way that there can be no marble?

Todd: I don't think so.

Chris: Ok. Oh, I see, back to the first problem for a second. It does seem like an interesting point there, but…

I: What's the interesting point?

Chris: That there's no marbles in there because of the first problem, where we said there was nothing in there. In this case if you want to make the argument of nothing, you can say that you got an infinite number of marbles, then they'd all be removed in an infinite number of steps.

Todd: I'm not actually saying that.

Chris: I know, I just say I could. I haven't committed my opinion yet.

1:21:42 Todd: I mean the argument is similar… you can say for any marble, you can get a step where that marble would be removed, and it would remain outside of the jar for all the time after…

Chris: Yes, you could. I can see that argument, but I can also see another argument. If you wanted to compare this problem to the problem we just did 1+1/n Powder problem]… it seems this packet was designed in such a way. You can say that since last time we had 2, then 3/2, then 4/3 remaining and so forth, that after an infinite number of steps it would land on 1. The trick is, you have to find in here, what is the difference between problems 4 [1+1/n Marble Problem], 3 [1+1/n Powder Problem], and 1 [n->n+1 Marble Problem]. Obviously you would probably agree with me that it's not going to be no marbles in the jar and a marble labeled 1, at the same time. It's a contradiction. Ok. Would you like part of my argument?

Todd: Well, what's your position first?

Chris: I believe it's a marble labeled 1, because if there are infinitely many marbles labeled 1+1/n, that there would be one at 1 plus 1 over… eh, I don't want to use…

Todd: Infinity?

Chris: Infinity, yeah, which would be 1 [referring to 1+1/infinity].

So it's a marble with the label 1 on it, whether or not it's written 1 or 1 plus 1 over an absurdly large number.

1:24:05 Todd: Ok, so here's the argument. Here [referring to n->n+1 Marble Problem], the numbers in the jar are going to keep increasing. There's no number that can be last, because there's always a larger one. Here [referring to 1+1/n Marble Problem] we're bounded by 1 - the numbers can't be smaller than 1.

Chris: There is a trick though, because you have an infinite number of balls. Yes, if you are taking it by grains of sand, ounces and all that, then yes, you can say it is bounded by 1. But you have to agree first that there is a ball that would actually equal 1 at some point.

I: You mean that would be labeled 1?

Chris: Yes, be labeled 1.

Todd: That's like this question [referring to the 1+1/n Powder Problem], right? So we're combining two different types of inferences?

Chris: Yes, but which one is the dominant inference? It can't be nothing and marble 1 at the same time. This is not to say that if this is "nothing is in the jar" then problem 3 [1+1/n Powder Problem] was wrong, or to say that if one marble is in the jar, that problem 1 was wrong. It's just a matter of which inference is the one that's applicable. I believe it's the one that there is a number of the form 1+1/n at infinity, which is 1.

Todd: So is it possible after all steps have been completed, the number would be 1?

Chris: [laughs] That's my position.

Todd: Is it possible that the number is 1, and there is nothing in the jar? In the sense that that was the last ball that was removed?

Chris: Well, at step n you put the nth ball in, so at the infinite step, if you want to substitute infinity for n, you put the marble 1+1/infinity in the jar, which is 1. Is there a step beyond infinity?

Todd: So we're basically saying there's no more marbles that we can put in.

Chris: There's no way to remove that marble 1 once it's in, because it was put in at that time [infinite step]. Of course you could argue that that one gets removed too, but…

1:26:36 I: Well, remember from last time that when we say after all the steps, we mean for each natural number there is a step with that number, and we're at a point where we passed all the steps.

Chris: Right, I'm just saying that at that step where the ball equals 1, where n is at the end of natural numbers and it's at infinity itself… [addressing Todd] assuming you agree with me that 1+1/n can get 1.

I: well, can it get 1 for n a natural number?

Chris: Not for a specific natural number, no. I'm trying to say if you don't believe that then I might as well not even go there yet because my argument is going to sound like nothing.

Todd: So after all steps we get to the last natural number, and we replace it [the marble in the jar] with [marble] 1.

I: Again, let's be clear what you mean by the last natural number.

Todd: I don't mean it's the last natural number… I mean like there's a step where we replace that ball with…

Chris: 1 plus 1 over [laughs] question mark, and it will equal 1.

So if 1 is the last ball in there, can there actually be a step to remove it? Because if the marble is labeled 1, you've hit all steps as far as I've seen.

Todd: So no?

Chris: I think you can't have a step that makes the marble less than 1. Which should be mildly clear, because there is no natural number that makes 1+1/n less than 1. At best you can claim that the limit as n approaches infinity is 1, but you can't claim that it's going to be under 1.

1:29:23 I: So now, entertaining the idea that there is going to be a marble labeled 1 at the end, if the label doesn't become 1 for any n natural numbers, that means that this thing doesn't happen after a specific finite step. Then claiming that marble with label 1 is in the jar seems to be using a kind of inference different from what we've done so far. For example, for the original tennis ball problem, we ended up with an empty bin A and bin B having all the balls. What was the argument for ball 3 being in bin B? Well, it got there at a specific finite step and for all subsequent steps it stayed there. So that's how we claimed that something was in bin B. Here [current problem] you're introducing a new way of claiming that something is in the jar that is not based on a specific finite step.

Chris: I have some way of explaining that. At step n you remove not the ball 1+1/n. In the tennis ball problem at step n you removed ball n. Here you're removing the one from a step prior, you're not removing the number n itself, so that one could still be there. It would have to be the only marble left. They wouldn't all be removed because that last step, whatever it happens to be equal to, won't have a step beyond it to remove it.

I: Ok. Todd , what do you think about that?

Todd: That's not what we said in Problem 1.

Chris: I think it what we said… at least what I said. I certainly was thinking it, maybe I wasn't very clear. I think it worked because it operated over the natural numbers.

Todd: Wait, what are we talking about here?

Chris: Alright. With the tennis balls or the marbles or whatever, we have numbers on them from 1 to infinity. So, when do you remove marble 1, in Problem 1 [1+1/n Marble Problem]?

Todd: At step 2.

Chris: The tennis ball problem was different [from Problem 1] in that… Sorry, go back to the original tennis ball problem, 'cause this is not going to apply to Problem 1. In the original tennis ball problem, at step 1, 1 and 2 were put in bin A and then 1 moved to B. Then 3 and 4 put in A and 2 moved to B. SO each time, the ball with the step number was moved over [from A to B]. So if you have the size of natural numbers as the number of steps you have, and you are moving a ball corresponding one to one with each step, nothing is going to be there [after all steps]. Here though [1+1/n Marble Problem], you're not removing marble 1+1/n at step n, you're removing it at step n+1. So it doesn't completely match, so there is one marble left. Makes me rethink Problem 1 a little bit, but that's because I don't remember the problem exactly.

I: Uh-um.

1:33:40 Chris: The reason I mentioned the tennis balls is that the correspondence was one to one, but if you take infinity as some ending point for this, [marble] 1+1/n is still there because the step beyond it is not there.

I: Isn't that the case with Problem 1 though? Let's look at problem 1.

Chris: Let's see. At step n you put marble n in, remove [marble] n-1. Oh, I suppose you could say that there is a marble of this thing in it [after all steps]… it could have the little infinite on it.

Todd: Ok, so in the tennis ball problem there were balls 1 through n in the out bin [bin B]. With the jar problem [1+1/n Marble Problem], marble 1+1/n is still in at the end of step n.

Chris: Same as problem 1 [n->n+1 Marble Problem]. Which is fine… I over-generalized in this problem I believe, I wasn't using my one to one correspondence properly, which is the basis of every argument I've used in here so far, I believe. The one to one correspondence or something wasn't being removed.

Todd: So for this problem 4 [1+1/n Marble Problem], because of the one to one correspondence, there is a marble with a number in the jar.

Chris: Yeah, that's my argument. If you want to think of a counter-argument or an additional concurrence, go ahead.

Todd: Ok, here's the second question. In problem 1, these are all things you get during steps using the natural numbers. In this problem [1+1/n marble Problem], the thing we're claiming at the end [that marble labeled 1 is in the jar] is not something you can get to with the natural numbers. Does that make a difference or not?

Chris: Well, in problem 3, you claimed, as well as I did, that there was 1 ounce left on the scale at infinity. And that's because after steps corresponding to each natural number, you had removed 1 ounce… after you went through the whole infinite number of natural numbers. Now, at the same time… are you really claiming the same thing? I don't know.

1:36:50 Todd: The second thing is, problem 3 and problem 4 are different in the sense that you can't talk about removing things in the same way. This is much more like the limit thing that I was using…

I: What is more like the limit thing, problem 4?

Todd: Right, because Chris's argument for Problem 3 was using subtraction, and here it doesn't apply because the balls are just labeled, you can's subtract things.

Chris: You're not subtracting actual values, although technically you can say that the label on the ball changes by the exact amounts I was doing before.

Todd: Is that an ok inference?

Chris: Well, I believe it is, but don't take my word for it. There are better mathematicians than me out there. As proven by the fact that I believe I whiffed on Problem 1 so horribly as to miss that the one to one correspondence missed a marble.

Todd: So this [1+1/n Powder Problem] does not use the one to one correspondence, you're saying?

Chris: It does somewhat, but it doesn't apply so to speak. Well, you know it's greater than 1 [the amount of powder left at the end on the scale], what else would you say though? I mean you're not covering to remove at any given point 0 ounces – that would have been a one to one correspondence. I don't think it applies to that problem probably. It sort of is the one to one correspondence but not exactly. The tennis ball was purely the one to one correspondence purely, also the original 10 marble problem, where the marbles were being removed at half a second, at ¾ of a second through the time… that was a one to one. This is not a one to one deal.

I: Ok, because it's offset by 1?

Chris: It's offset by 1, which is just enough in some famous cases to create a whole new realm of… [laughs and pauses].

I: We need to call it a day for now. Todd , I understood Chris's position on this; how do you want to leave this problem? We're going to revisit it, but for now what do you want to say about it?

Todd: I sort of agree, there is a marble in the jar with label 1 on it [facial expression indicates doubt].

1:39:53 [Intuition and confidence rating for Problem 4. Chris wants to rerate Problem 1]

[End of session]

Session 4
12/3/08

0:00 [Start work on The z^n Problem]

[In preparation for this problem, I discuss with the students the geometric interpretation of the multiplication of two complex numbers, as well as what it

means for a set of points to be bounded in the plane.]

7:00 [Students individually read The z^n Problem, think about it silently for a bit]

Todd: Ok, so if r is $\frac{1}{\sqrt{2}}$?

Chris: I know what you're getting at. I take it you're asking about the finite [case]. Do you mean equal to $\frac{1}{\sqrt{2}}$, or less than that?

Todd: So less than $\frac{1}{\sqrt{2}}$ would be ok?

Chris: You can choose whatever you want, I'm just trying to understand what you're saying.

Todd: I just chose $\frac{1}{\sqrt{2}}$.

I: Why did you choose $\frac{1}{\sqrt{2}}$?

Todd: If I'm right, that gives you a norm of 1 for z. So we're starting just with 1 [drawing it on a unit circle]. So then z is here…

Chris: That should be a 45 degree angle.

11:55 Chris: Let me make sure that works. Let's see, z is just going to rotate. If you want to make this nice and clean you just take a z and rotate it 45 degrees each time without changing the length. That's certainly finite. It's countable, in fact that's easily countable.

I: How many elements does it have?

Chris: If I counted correctly, there are 8. Even though the number of vectors goes over… it's the same basic vectors, you repeat them over again.

Todd: That was my question, if that's actually finite.

Chris: Because it asks for all points in the plane that correspond to it, the point in the plane is the same regardless of how you multiply to get it. So that's finite.

I: Do you write a summary of it?

[Each student writes a summary of this discussion.]

13:53 Chris: It sounds like you chose r that way because the norm [of z] becomes 1.

I: I guess the next question in, why did you want the norm to be 1?

Chris: Right.

Todd: If the norm is not 1, then it's either getting smaller or bigger. You're going to get either a spiral sing looking thing, or a spiral going outwards. Is there a way to get a norm of 1 without having 35 degrees, or no?

Chris: Well considering that the vector is already predetermined at 45 degrees, you can't really… I mean you can have something with a norm of 1 with a different r, but not with that vector [the z given in the problem].

15:13 I: So if you wanted to change the angle you could choose a different vector [for z], change the problem a bit.

Chris: [addressing Todd] What are you trying to… ?

Todd: Well, if you wanted to have infinite and bounded, with a norm of 1, right? Like a spiral looking sink would be infinite and bounded.

Chris: Well you can keep multiplying and you wouldn't run out of possibilities for points, that is true.

I: Which case are we talking about now?

Chris: Infinite and bounded. We jumped the infinite and unbounded case, that was a bit obvious. It appears that we both came to the conclusion that any number greater than $\frac{1}{\sqrt{2}}$ for r would give an infinite and unbounded variety. 'Cause the norms are going to keep increasing. [Addressing Todd] I am assuming that's what you were getting at. Is that a fair assumption?

Todd: Right, well, I was thinking that if you change the angle to 1…

I: The angle is 1? 1 degree?

Todd: 1 degree… it doesn't matter what the measure is…

Chris: Well, it [the set of points defined by the completed process] is still finite if it [the angle associated with z] is 1 degree.

Todd: Exactly, but I was wondering if there is an angle [for z, which is chosen to have norm 1] such that the set is infinite but bounded.

Chris: You mean that it would keep going around and around? That would be 360 divided by an irrational number.

I: That's a good question! Actually, can we find a vector [for z] like that? But first let's finish this version [original problem, part i)]

[Students finish writing up summaries of their answers to part i), claiming that r has to be $\frac{1}{\sqrt{2}}$ in order for the set to be finite]

19:42 I: Is that r the only choice so that we have a finite set of points?

Todd: Um… I want to say yeah, let me think.

Chris: [Finishing writing] Ok, anyway. For that particular vector, that may be the only one. Actually no, I had one earlier what am I saying?

Todd: What about 0?

Chris: I had that earlier. […] If you want to, you can also say $-\frac{1}{\sqrt{2}}$ … it would do the same thing really.

I: So do you think for that one [r= $-\frac{1}{\sqrt{2}}$] you would still get 8 points?

Chris: Well, it might take longer to say there.

I: What do you mean longer?

Chris: [draws unit circle]. z_1 would be here, then rotate and get here, again here… yeah, it would get the same thing. I thought because it was taking a different angle it might take longer, I don't know why I thought that. I didn't exactly see the point being at exactly 180 plus 45 degrees. But yeah, it comes out to the same number of points and in fact the same points. So let me write also r equals 0 or r equals $-\frac{1}{\sqrt{2}}$.[…] I don't think there's another way to do it without having the norm 0 or the norm 1.

23:50 I: Ok.

Chris: So for positive r this is going to rotate by 45 degrees each time, and for negative r it is going to rotate in that nightmarish fashion, but not too bad.

I: Ok, let's do the second one.

Chris: Infinite and unbounded? Well, r greater than $\frac{1}{\sqrt{2}}$, anything would work there I believe. Or r less than $-\frac{1}{\sqrt{2}}$. The norms of the subsequent vectors would increase without bound. If the norms increase without bound, the angle really doesn't matter 'cause it's just going to keep extending anyway. It would be enough to say that it's unbounded in the plane if it's unbounded on a line even.

I: You mean you would look, for example, at the x-axis or the y-axis and see how many different points were created on that one?

Chris: You can say that. For example if this norm was slightly bigger [than 1], let's say z was over here [pointing to drawing], I could say take this and take 8 [steps] later it would be over here [further away from the origin, on the first bisector]. If it's not even bounded on that line [first bisector], it can't be bounded in the plane.

I: Uh-um.

Chris: [addressing Todd] Infinite and bounded – do you have an answer?

Todd: Yeah, [r] not 0 and absolute value is less than $\frac{1}{\sqrt{2}}$.

I: Ok, so a short explanation of why that yields a set that is infinite and bounded?

27:07 Chris: [addressing Todd] You had a good explanation before, I think.

Todd: It spirals in but only on these 8 lines, I guess [pointing to lines making angles with the x-axis that are multiples of 45 degrees].

Chris: No matter which version we have, it's going to be on the 8 lines because of the vector chosen, but yeah. I mean, I think you're saying that the norm is going down, it's decreasing. I'll do the writing on this one. So the norm of each vector will be bounded by 1… greater than or equal to 0 and less than or equal to 1. So the norms will be bounded, it will be an infinite number of points but it will still be bounded.

I: Ok, so that argues for the bounded-ness part.

Chris: Yes, it doesn't argue for the infinite part. But the norms will decrease at each step never to repeat the same norm. So I guess that will be an argument for the infinite portion, because they're never repeated at any point.

I: Ok. Now when you wrote $0 \leq z_i \leq 1$… um, can the norm be 0?

Chris: Can the norm be 0? I don't believe it can be 0, at a finite step at least. Do you want us to consider that?

30:00 I: Yes, I want you to consider that. I guess a different form of the question would be, is the 0 vector part of the set created by the completed process?

Chris: Well, assuming that by "all steps have been performed" you mean that all natural number steps have been performed… Basically the norm in each case is given by $r\sqrt{2}$… the absolute value of it.

I: Wait, that's the norm of the original vector.

Chris: Yes, and at each step that would be to the n^{th} power. So the limit of that at infinity would have to be… I guess I could argue that the 0 vector would be in there at some point> it would be at the very last step, but it would be. Because the limit of the norm would approach 0 and it would hit there, I believe. But I don't have a solid foundation for the argument other than limit Calculus.

I: What do you think, Todd ? Is (0, 0) part of the set of points in the plane created by the completed process?

Todd: I like the… I think the limit of that [the sequence of norms, in part iii)] is 0.

I: I think we all agree that the limit is 0. It's just a matter of how to interpret that. So we agree that the limit is 0, now the question is if this means that we are adding (0, 0) to the set of points created by the completed process.

Todd: Um, I don't know…

32:35 I: In considering this, are you thinking about any of the previous problems at all?

Chris: Somewhat. There is… I mean I could consider them, let me see if I can come up with a better argument.

Todd: It's similar to the ounces of powder.

I: In what way?

Todd: In one of the arguments [for The 1+1/n Powder Problem] we said that the limit would be 1, and therefore 1 ounce was left on the scale.

Chris: If we chose an r such that the norm of the first vector would be ½, then you could make it like in the problem where the marbles in the jar were removed at time $1-(1/2)^n$.

I: That was the time step, right?

Chris: I know it was the time step, but a comparison could be made… I'm trying to remember the problem exactly.

I: I do have the worksheets [from the previous sessions], here you go.

Is it the one where 10 marbles were put in at a time, and removing one?

Chris: Yeah, that [$1-(1/2)^n$] was the time step, but the idea was though that because time went from 0 to 1 and it had to hit 1 at some point, that meant that $(1/2)^n$ had to eventually hit 0 at some point, because that's how the time step was defined. So that's how I can rationalize that if $r\sqrt{2}$ is at ½, due to the same set being used for choices of powers of n, it would have to hit 0 at some point. That of course takes the assumption that my assumption in this was correct, but I can compare the problems at least.

35:50 I: Right. Now what if r is not given a value such that we get norms of ½, ¼, 1/8… Would you make the same argument?

Chris: Well, I know that if the norm [of z] is less than 1, the sequence of norms converges to 0, and I will say that yes, it will hit 0. If it's ½ or less, it will certainly hit 0. I believe it will hit 0 anyway, but I don't have a problem that I can recall easily that says that, or as a comparison.

I: Todd , what do you say about this argument?

Todd: I agree.

I: With each part?

Todd: I want to say everything, but I like the whole idea to look at the sheet that you wrote and say "uh-ah!"

I: The sheet I wrote?

Todd: It's implicit in the problem here [The Original 10 Marble Problem], where it asks about time t=1, which means that t=1 is possible. So we're going to make the appeal to people who are smarter than us.

I: [laughs] Well, if the parallelism is valid!

Todd: I like the whole "less than or equal to ½" [as a value for the norm of z], because it's, what do you call that, the sandwich... the squeeze theorem?

38:00 I: So what if [the norm of z] is greater than ½ but less than 1? What happens then? I guess you cannot use the squeeze approach.

Todd: So something that is bigger than ½ is ¾, right? So it's 3/2 times ½? We could take the 3 out…

Chris: I don't know if that's good or bad. I liked it when it was 1 over something. Now with something like ¾, $\left(\frac{3}{4}\right)^n$ has to approach 0. Well let's see, 3^n over 4^n, the limit of that as n approaches infinity. That kind of leaves you at…

Todd: I don't have a problem saying the limit of that is 0. The question is the whole making an inference about what's happening to the points produced by the process. Like, is there a reason why we want to add [to the set of points produced by the process] the limit of things that go to 0?

Chris: Well, let me think… Is anything less than 1, multiplied with itself, going to give 0 at some point?

I: When you say at some point, do you mean at a finite point?

Chris: Neah, at a finite point it's not going to happen. [pause; mumbling to himself]. It certainly converges to 0, I just can't figure how I'm going to … I'm finding the differences between 1 and the powers of ¾. I'm saying ¾ is remaining, the other way might be easier to justify for these.

I: To justify what?

Chris: If the limit of 1-$\left(\frac{3}{4}\right)^n$ goes to 1, then [the powers of] ¾ would go to 0 and hit 0. [more quiet calculations] That is not going to get me any closer to this 'cause you have to assume it both ways anyway.

I: It's clear that this limit [of 1-$\left(\frac{3}{4}\right)^n$] is 1… you're just not sure how to interpret this, right?

Chris: Yeah, I mean the limit of 1-$\left(\frac{3}{4}\right)^n$ is 1, the limit of the number itself to the power is 0… I don't say a deeper argument to say that it hits 0, but I believe it hits 0.

43:33 I: Ok. So if we were to write the set of points, to write M equals, in set notation, and then to write the set produced by this process, what would it look right? You know what I mean? Something like that. There [in part 1], we had

$M = \left\{1, \frac{1}{\sqrt{2}}(1+i), ..., \left(\frac{1}{\sqrt{2}}(1+i)\right)^7\right\}$. That's the entire set. Now for this [part iii)]. Since it's an infinite set you can't write all of them. We would include all the powers of z [writing $\{1, z, z^2, ...z^n, ...\}$ … but would that be enough?

Todd: Should we add 0 at the end, that's what you're saying?

I: The question is, using this notation, does this include the 0 vector? Let me make it more formal [writing $\{z^n \mid n \in N\}$]. If I use this notation, is 0 (as a complex number) included here?

[Both students pause.]

I: So in order to have 0 included here, you would need to have one of these, z^n with n a natural number, equal 0.

Chris: Right. well, for any finite n, it won't hit 0.

I: So then this, $\{z^n \mid n \in N\}$, does not accurately describe the set of points produced by the completed process?

Chris: I believe it leaves out the final point.

Todd: So there's 3 possibilities: this [$\{z^n \mid n \in N\}$] includes 0, this does not include 0 but it's good 'cause we don't want it to include 0, or it [the set of points produced by the completed process] does include 0 and to make the answer complete we have to union 0 to $\{z^n \mid n \in N\}$.

I: Let's take it step by step. Are you done deciding whether 0 is part of the set [of points produced by the completed process]?

Chris: it should be there for all norms less than 1.

I: Now, the second step, does what I wrote [$\{z^n \mid n \in N\}$] include the 0 vector?

Todd: I don't think so.

Chris: No, for any finite n it certainly doesn't, so I don't think I can say that that has 0 in it.

I: So you want to do something like that? [adding "U{0}" to the right of $\{z^n \mid n \in N\}$]

Chris: I'd say that's fine.

48:25 I: Ok, so this is fine?

Chris: I think so, but I don't know so. There is the possibility that 0 was included in the first writing without the union. But I don't see how, if it's defined as each individual z^n.

Chris: Is the number infinity in natural numbers?

Todd: Saying "all steps have been performed" means including another step that is not included that is not finite, or no?

Chris: No… I'm assuming that it's the same thing as before, it includes all the natural numbers of which there are an infinite number, but…

I: When you finish going through all these dot dot dot at the end [pointing to $\{1, z, z^2, ... z^n, ...\}$], what I mean is, again, for any natural number…

Chris: … it's already been done.

Todd: So this [$\{1, z, z^2, ... z^n, ...\}$] and this [$\{z^n \mid n \in N\}$] are the same, so this would be all that there is, so if 0 is outside of that, how can we say 0 is in M?

Chris: Now that I'm going over this again, 0 might have actually been contained in $\{z^n \mid n \in N\}$, without the union.

I: What do you think, Todd ?

Todd: It's so arbitrary. I have no idea. I shouldn't say arbitrary, but I don't know.

Chris: All steps have been performed means you went through an infinite number of finite steps. Infinity is not actually a number contained in the natural numbers.

I: So if n is a finite number, then z^n is not the 0 vector, 'cause the norm is not 0. So this notation [$\{z^n \mid n \in N\}$] includes only vector of this kind.

Chris: Ok. Do I believe that 0 is a vector that can be reached after all steps have been completed? Well, do you [addressing Todd] believe that 0 can be reached after all steps? Or does it stop at some point?

Todd: It doesn't stop at a point, but it doesn't mean that it has to contain 0.

Chris: That is true, it could have a way not to contain 0.

I: Todd , can you think of an example of a set of numbers that gets really close to a specific value but that set does not contain that value?

Todd: Well, going to high school math, any asymptote would be like that, that value is not in the function but the function gets really close to it.

53:28 I: Right. So for example this set, $\{1/n \mid n \in N\}$.

Chris: It certainly goes to 0, it doesn't mean 0 is in there.

Todd: So when we say the limit of something, does it mean there is a finite number such that [a term of the sequence equal 0]? No, it doesn't.

Chris: That's for sure. You can certainly find cases where a number is the limit, but it's not there in the function you're looking at.

Todd: Also like a vertical asymptote, when the function has a denominator, if it's undefined at 0…

Chris: When you go towards 0 it goes to infinity. But neither 0 nor infinity is actually a point that you can give a finite number to. At infinity it is 0, though.

Todd: At n=0 it is undefined, there's definitely not a number.

Chris: Certainly not at 0. At infinity though, that value is 0.

I: When you say at infinity, do you mean for small n equals infinity?

Chris: I'm trying not to say that n is at that point. I'm getting a little interference from the point at infinity notation from graph theory, where you can have a point placed on it. So I believe there is a point at which that is 0 and I believe that 0 is in there [set M from part iii)], but I can't give a finite number for it. Sounds like a contradiction. I'm starting to believe 0 was in the original notation [$\{z^n \mid n \in N\}$], even though there is no n I can give that will do it. At infinity it will be. Is infinity in the set of natural numbers? If the natural numbers had 1 member, it would be {1}, if it had two members, it would be {1, 2}, if it's an infinite number is infinity in the set, no of course not.

I: Ok. So if infinity is not part of N, then we know exactly what we mean by this notation [$\{z^n \mid n \in N\}$].

Chris: Then no, 0 is not in $\{z^n \mid n \in N\}$.

I: Ok, so then you would need to do the union with {0} to get the whole set of points produced by the process. Is that where you stand?

Chris: Yeah, sounds fine.

57:35 [At my suggestion, students start considering Todd 's earlier question: what angle for z should one choose, for z of norm 1, to produce an infinite and bounded set of points by the completed process?]

Chris: Well, if you got every point on the circle, you would get an infinite set of points. However, that means you have to go through every single angle. If you divide 360 [degrees] by a rational number for an angle, it's not going to work as you will be repeating the same things. […] So it has to be an irrational number at least, if not transcendental, if it exists.

I: Here you're talking about the angle, right, what the value of the angle should be?

Chris: Yeah. It's certainly not an integer value. That would only give you 360 [points] at the most. And it can't be a rational value because that leaves the irrational things out.

I: Well, you said it can’t be a rational value because it leaves the irrational things out. But the rational values are still infinite.

Chris: Yeah, I know. Still…

I: What’s a rational value, let’s say 1/5 of a degree?

Todd: No, that’s no good.

I: How many points would you get if that was the angle?

Todd: 360 times 5.

I: Ok.

Chris: You could argue that the angle should at least be less than 1 degree to get everything [obtain all points on the unit circle].

Todd: I don’t think so. It just has to not repeat. So integer multiples of it have to be non-repeating.

I: Ok, so what can we infer from that?

Todd: So it has to be an angle between 0 and 360.

Chris: It would still have to be irrational.

I: Let’s see, let’s assume we have an irrational number of degrees, how would you prove that the process doesn’t repeat points?

Chris: ‘Cause no integer number times an irrational number of degrees will equal 360, because that would mean that the irrational number is a ratio. […] Clearly if you’re taking a value of Pi degrees, it certainly will not hit 360 degrees, no matter what rational number you choose. Nothing over Pi is 360… no integer. So it will not hit that point again.

1:02:56 Todd: Wait, so Pi and 360 Pi is not the same angle?

Chris: I don’t think it is. For a number to have the same angle as Pi, it would have to have a natural number n such that n Pi equals some k times 360 plus Pi. And k would be an integer also… a natural. [writes an equation, shows has no solutions].

Now your problem is to find something of norm 1 which has angle Pi, in degrees. I mean obviously it exists. I guess I’d say cos (Pi degrees) +I sin (Pi degrees). That would work.

I: So you proved this way that none of the points in the process get to land on a previous point. So let me ask you a question. Are we covering the whole circle with this process?

Chris: I don't know. It's certainly infinite and bounded. I don't know if it's covering the whole circle. Every [point on the circle whose corresponding angle is a] multiple of Pi is clearly hit. Is there a way to hit a rational degree though, with an initial angle of Pi? Is there any way that you can get 1 degree?

Todd: No, that would mean Pi is rational right? So you're saying that then n times Pi would be n/k, but that's not possible.

Chris: Yeah. Obviously Pi is transcendental enough so that this is not going to happen. So in this case it [the process] certainly doesn't cover the whole circle, because it's only covering irrational points, multiples of irrational points, which will never be rational numbers. So it covers only irrational numbers and that's enough. Is there a way to cover all points? Not by choosing a rational number.

1:09:15 [Intuition and confidence for each part of The z^n Problem and for this variation, Todd 's variation.]

1:10:55 [Start work on The Bin Swapping Tennis Ball Problem, timed version]

[The students read the problem individually, discuss the rules for ball placement to make sure they interpret them the same way.]

Todd: So does the ball placement happen before or after the swap?

I: It's in the order that is listed.

Chris: [reading] What are the contents of each bin at t=1? So basically after all steps have been completed, what is in each bin? This is going to take some notation.

Todd: Let's try a couple of them [steps].

I: By the way, when I say swap the contents, it means that at that point, whatever is in each bin you take it out at the same time, and you put each content in the other bin.

Chris: I know, so A is now in B and B is now in A. You're basically transferring the labels from one to the other.

I: I prefer if you're thinking about it as changing the contents. I said you swap the contents, not the labels.

Chris: I understand. I don't see much difference here, but maybe I will later.

1:14:42 [More work on figuring out the first 8 steps.]

Chris: After we hit step 8 we can probably generalize from there because everything is in a modular form. […] And now you have 2, 4, 6, 8, and 1, 3, 5, 7. It's an interesting algorithm of sorting your even and odds.

Now at time equals 1, what are the contents? [Addressing Todd] What can you say about the contents? Forget which bin at the moment, what can you say about what's in either bin, I guess?

Todd: So n has to go to infinity? So are we doing this whole infinity divided by 2 maybe? It seems like at each step the bins have… not quite equal contents [referring to the cardinality of each bin], but they're really close.

Chris: That's true, but they don't need to have equal contents. You know that there is odds in one and evens in another, at every step; there is no intermingling of the two. So you can probably say securely that all the odds are in one and all the evens are in the other.

Todd: Yeah. Alright.

I: By the way, if you had to prove that there is no step at which the odds and evens are mixed, what proving technique would you have to use?

Chris: It's awfully convoluted. I don't know how I would prove this. I almost did a proof by algorithm, but I don't think that quite works.

I: By algorithm?

Chris: Basically showing an algorithm that says something is… sort of an inductive process.

I: When you need to prove something for any n, that naturally invites…

Todd: Induction?

1:21:50 I: Ok, I don't want you to necessarily go through all the details. So assuming that we used a combination of induction and reasoning by cases to prove that indeed after each finite step n, there will be evens in one bin and odds in the other. I guess based on the parity of n, or how it is with respect to 4k, we can also say what's in each bin. So let's assume we did all that. What else can we do about the general question, t=1?

Chris: Well, we can at least believe that there will be evens in one bin and odds in the other. As for what else at the complete time, at t=1, everything that is in the natural numbers is in a bin. Everything even is in a bin and everything odd is in the other. Now can I say which is A and which is B? No, not very nicely, because that requires knowing whether infinity is of 1, 2, 3, 4 mod 4.

I: Todd , does the nodding mean you agree completely?

Todd: I'm ok with modular whatever. I understand what you're saying. Alright, so what problems is this similar to?

Chris: The time is from the marble jar. It has… 4k-1 and 4k-3 is not even remotely similar to… and the swapping of the bins is something that I haven't seen up to this point.

Todd: Well, the first tennis ball problem [session 1, problem 1], which was all the evens are in one and all the odds are in the other, and they don't change.

Chris: Yeah. The difference is that this time we're changing the contents of the bins. Even though all we're doing really is changing the label of the bin that the evens are in.

I: I guess there's a slight difference between thinking about it that way and thinking that you swapped the contents. Because in one situation you leave the physical balls where they are and change something else, in the other situation you move the physical balls. Just in case that makes a difference, at the time being let's just stick to how the problem is phrased.

Chris: I wasn't changing the labels, I was changing the contents over at this point, but I'm just saying it's the same as the other type of thinking.

I: So Todd , I guess you can put down that one as well, I think it was Problem 1 in session 1.

1:26:15 Todd: I mean is that going to give us the number of balls, after all steps? I mean we made a statement about the number of balls in each bin for the other problem. […] The size of the set of balls in each bin.

I: With that other problem, you could also say which ones were in each bin. You had a characteristic by which to identify the balls in each bin.

Todd: Right, but that's not applicable here.

Chris: I can't get any further…

Todd: Is it possible that you just can't say?

Chris: You mean at the final step, what's in there? I believe that you can't say, because that would require infinity to have the characteristic of a single number.

Todd: Was there a problem where there were two different bins where we couldn't say which balls were in which one?

Chris: There was a problem similar to that where we couldn't say exactly what was in. I'm thinking of the marble jar with the two balls, at step 1 putting 1, at step 2 putting 2 in and removing 1. It's almost the same as saying the outside of the jar is something else [a second jar]. Ok, if you keep swapping them back and forth, which one is it? Can you really tell at infinity whether it's even and thus 2, odd and thus 1, or neither of the two.

I: So Todd , when you said "is it possible we cannot tell?", what did you mean? We cannot tell in which bin the evens are, or we cannot tell anything about the final state?

Todd: I meant that we can't tell where the evens are.

I: Ok, so you can tell that you're going to have odds and evens separated, but you can't tell in which bin each would end up.

1:29:18 Todd: Right. I think using the argument from the other tennis ball problem with no swapping, we can say that the set of balls in each bin is infinite in size, even though we can't tell what's in each one.

Chris: It would be nice if they all stayed in one thing, but because they're swapping, it's nearly impossible to tell what bin the evens and odds are in, after all steps.

Todd: Ok, so we can definitely say evens and odds are separated.

Chris: Yes.

Todd: But can we say the size of the set of balls in each bin is infinite?

Chris: Yes, you could say that?

Todd: But do you want to say that?

Chris: It certainly is infinite in both bins. I would agree with that. Do both bins have the same size?

Todd: At the finite steps it doesn't matter…

I: If you say all evens are in one and all odds are in one, is there a one to cone correspondence between evens and odds?

Chris: There would be if infinity had some sort of boundary, at some point. I mean you can say that it's one more…

I: Right, you can say for each odd, the associated even is going to be the one above it.

Chris: But that's only going to be true if infinity has an even number of numbers, which is not exactly provable.

Todd: It's a lot easier to say that at the end they're separated.

Chris: I mean by the same token you can say that the last one is an odd ball, and then you can say there is an even one that follows it, and then an odd one that follows it. If infinity is even, yes there is a one to one correspondence, but if it's odd, there isn't anymore.

Todd: So we can't say which bin has even or odd.

Chris: And the sizes of the two bins are infinite, but maybe not equal. Which is strange, but whatever. I could be not equal for all that matters. I mean the set of natural numbers is also infinite, I'm certainly not going to say that that's greater than what is contained in the set of natural numbers.

I: Ok, you want to leave it that way?

Chris: I guess.

[Start rating for intuition and confidence.]

Todd: I'm pretty confident in the things that I said we can say, I'm not confident in the things I said we can't say.

[End of session]

Session 5
12/6/08

0:00 [Start work on a timed version of the Bin Swapping Tennis Ball Problem, in which the swapping is done with the labels, not the contents of the two bins.]

I: This came up last time, one of you said the two versions are the same thing. Now I'm using the way that Chris wanted to interpret it before, and the question is if you would approach the same way. Something that would be interesting to me is, assuming that you have a picture of something put somewhere then moved and things like that, does that picture/movie change at all [from one version of the problem to the other]? Todd ?

Todd: I don't actually remember how I visualized it last time.

I: Try to visualize it now [in both cases].

Todd: Ok. You [addressing Chris] go first then.

Chris: This is not too different for me, considering last time I was the one who suggested it was the same thing. Upon thinking about it, I rethought about the first 4 steps and the same thing did happen The motion picture in my head changed a little bit but not to the point where it would change any of the results, as far as I'm concerned. Yeah, it's the same thing as far as I'm concerned. The way I visualize switching the contents of the bins was, the balls stay where they are and the boxes flip. 'Cause I can't imagine putting them in and out, swapping the balls. I'm not moving the contents, I'm moving the boxes.

I: Todd , how are you thinking about it?

Todd: It doesn't really matter to me. I wasn't thinking of bins moving though. I think there was another problem where there is first marbles and then we put labels on the marbles. I was thinking about that and realized that in the cases where you're only moving the labels, the marbles don't even matter so much. So you can think of it just with labels. So using that metaphor, the balls would be like jars, and we just stick labels on them, the labels are the ones that are really important.

5:13 I: So what is the important part in this one [current problem], if you're saying something becomes less important?

Todd: No that the balls become less important, because the questions is what happens to the balls, but sort of… the numbers become where the balls were.

I: Ok. So how do you go about it? Is there any difference at all from last time, in the way you'd solve it?

Chris: Everything else appears to be the same, except… if you were to diagram it, instead of an A and a B column, you would have… the content of the bin almost.

I: [pointing to worksheet from last time] This is the way you did it last time. So if you were to draw again the first few steps [for the current problem], would you use the same diagram, or a different one?

Todd: I think it would be the same.

Chris: Well, it could be done that way. I was saying that if you switch the labels, I guess I would draw something similar, let me see. Step 1 would have 1 A, nothing in B. Step 2 would be 1 A, 2 B. Then 1, 3, in B, 2 A. Although you could do it the same way [as before] and keep the columns filled, it doesn't seem to make as much sense to do it that way, 'cause now the bin is staying the same and the label is moving. He's right that, as far as I am concerned, this does make it work discrete in the case of the labels, because now there is no thought in my mind that there's a chance that the balls are either all in one bin or outside of the bins, because they never move. It's the only difference I would see though. In the diagram I would do it this way for simplicity, switching the label on the bin but keeping the column of things the same. Otherwise I think everything comes out the same as we said last time.

8:30 I: Todd , do you agree with that?

Todd: That made sense, It's pretty obvious that the odds and evens are going to stay…

Chris: Well, according to this diagram at least, all the odds are on the left side, all the evens are on the right side. There's certainly not going to be a time when they move, it's just the labels…

I: Ok. In terms of your confidence and intuition for this one, are they any different than last time?

[Rating is done in writing, individually.]

I: Chris: for the problem last time, if we said that there is no way to move the bins because they are glued to the ground and you cannot move the labels either. So the only thing you can move is the balls. Would it change the movie in your mind?

Chris: I can understand swapping the contents. It doesn't change the way I view the problem, just the image of how they're moved, but that doesn't really affect the problem.

I: And Todd , if I specify that you can't move the bins or the labels, does that change the way you visualize the process in your mind, for the problem from last time?

Todd: Last time I was thinking about swapping the balls.

I: Ok. So I guess we are done with this one.

11:42 [Start work on The Lamp Problem, The students read the problem individually]

Todd: [Addressing Chris] It seems to be like the two marbles that are put in the

jar.

Chris: Yeah. I thought about that.

I: [Addressing Todd] Can you describe it a bit more? 'Cause there were a lot of marble problems…

Todd: There is a jar and two marbles, 1 and 2. Step 1 you put 1 in, step 2 you put 2 in take 1 out. So at the end of an odd step there's the 1 marble, at the end of an even step there's the 2 marble. What happens after all steps have been completed?

I: So in what ways is it similar to the current one? Can you be more specific about which aspects of this one correspond to which aspects from the other one?

Todd: If you think like… "the lamp is on" is "marble 1 is in the jar", "the lamp is off" is "marble 2 is in the jar".

Chris: You can say that. I wouldn’t disagree with that.

Todd: Which is bad, ‘cause I don’t think we solved that one.

I: Do you remember what your last word on that one was?

Todd: I have no idea. Because the state of the thing depends on the number of the step.

Chris: I’m sure we said something along the lines of, we couldn’t tell what was what because we don’t know if infinity is an even or odd step.

I: Did you say then that you could say anything at all [about the contents of the jar after all steps]?

Chris: I seem to recall we said that one or the either was in. It couldn’t be both and it couldn’t be neither. In this case it is especially true because [the lamp] can’t be on and off at the same time. With the marbles you can have both in or both out, you can have that. It’s really difficult to have a lamp on and off at the same time, at least physically. Unless the bulb is blown, but that’s not what we’re shooting for here. There’s nothing else that can be said about it.

15:17 I: Then let’s see. The answer you had for the other one was, there is exactly one marble in the jar, but we don’t know if it’s marble 1 or marble 2. How would that translate for this [lamp] problem?

Chris: We know it’s on or off, but we don’t know which of the two. Which I believe holds, although I am not sure about the marble problem; I am not doubting myself on this [lamp problem] so far.

I: If you remember, when you did the 1, 2, 1, 2 marble problem, one suggestion I had was if you can think of any way to modify this problem so that you would know for sure you had one problem in the jar after all steps had been completed, and you would know for sure which marble that was. Since then, do you think we’ve done any problems like that? Were there any where we had only one marble in the jar after each step, and then in the end you claimed that there was one marble in the jar with a specific label on it?

Todd: There is the one where the marbles were labeled 1+1/n or something like that.

Chris: Right. We knew there was a marble in the jar [at the end] at least.

Todd: But we claimed the number on the marble was the limit of the labels.

Chris: Yeah. Which would be 1 in that case. We knew there was one marble, because no matter how small that little piece got, there was always 1 still being added on the label. There would at least be something, even if somehow it went through every natural number in existence, that 1 would still be there, so we knew that there was something there.

I: Ok. Well, does that problem help you in any way with this one? Or does it help you to verify the answers you gave at the end of last time?

Chris: I don't know if it clarifies anything for me any further, really.

I: Well, what aspects did that problem have that you could claim for sure that there was a marble in the jar at the end and which one it was, that are not present here?

Chris: Well, in the prior problem there would always be a marble with a label at any step or even after all steps, there was something left over that didn't depend on the fractional additional to the label… what was it, 1+1/n. The 1 did not depends on the 1/n, so there would always be a label for such a marble.

19:08 I: But here [The ½ Marble Problem] we could change the labels into a 1+0 and 1+1, so we would also have a beginning 1 that would not depend on the number added afterwards.

Chris: That is true. But then again, you have a 1 which… In that case that's a very strange thing because 1 [from the ½ Marble Problem] would have to represent either on or off [for the lamp], depending on which way you chose it.

Todd: That sequence does not converge to 1.

Chris: It certainly does not converge upon anything.

I: Which sequence?

Chris: 1, 2, 1, 2,…

Todd: Or 0, 1, 0, 1,…

Chris: Does not converge to anything so it doesn't say what would be there.

I: So I guess that aspect that I was asking about for would be that you need to have a convergent sequence on the labels, to be able to claim there was [exactly] one marble at the end and which one?

Todd: That problem, The 1+1/n Marble Problem, was a combination of the "how many ounces of powder on the scale" plus the marble in the jar problem.

I: What's the marble in the problem in the jar problem? Maybe the one where we're replacing 1 with 2, 2 with 3, 3 with 4… ?

Chris: If that was at the beginning of that section, then yeah.

Todd: But I think we said something about what happens to the marbles even though there was no convergent sequence of labels.

I: For which one, the first one, with 1, 2, 3, …?

Todd: Yeah.

I: Chris, do you remember the answer to that one?

Chris: I believe we said that nothing was left at the end of it. There was a label that would correspond to every step in the process. But in this case with the lamp… Yeah, I guess it would be the convergence of the sequence I was doing, not that much the independence of the "1". Yeah, still, I'd say this one you can't tell because it varies between two specific points, either on or off. There's no real way to say well it's getting closer each time to being more off than on; it's just not defined that way.

22:24 I: Ok. So in the context of the 1+1/n Marble problem, what if it was not 1+1/n, it was just 1/n on the labels, at each step?

Chris: That converges as a sequence. So I would say the same thing, because it converges to something, and it's a finite something, 0. […]

Todd: So that goes to 0?

Chris: I would hope so, or I have a bad mathematical foundation.

Todd: So you would have a marble with label 0?

Chris: Well, that's the way I thought about it anyway, I can't guarantee it's the truth.

Todd: So then we can say, if there is a reachable end state…

Chris: If there is a finite end state that you can actually see converging to, it becomes very clear what the answer should be.

I: And Todd , what would you say about this 1/n Marble variation I introduced?

Todd: If we said 1+1/n goes to 1, we should say 0+1/n goes to 0 in the end.

I: When you say to 0, do you mean you would have a marble with 0 on it?

Todd: Yeah, because we said we would have a marble with 1 on it [for the 1+1/n Marble Problem], so we should have a marble with 0 on it.

Chris: I agree with that.

I: Ok. And do we have a final word on this one, the lamp?

Todd: So the one where we put 1 in, removed it, put 2, removed it, put 3, that [sequence of labels] doesn't go to anything, so in the end there's nothing in there.

I: Well, 1, 2, 1, 2 doesn't go to anything.

Chris: That wasn't exactly what I said. I was saying there was a step to remove each marble from the jar, never to be placed in again. And there is a one to one correspondence of these. This one [lamp] I have no idea whether it's on or off. Except that I know one or the other is true.

28:10 Todd: So it's not the case that there is nothing in the jar?

I: What jar? He's talking about the lamp. Yeah, what would be the equivalent of "there's nothing in the jar" in this case?

Todd: There is a lamp.

Chris: [jokingly] A finger pushed half way between on and off.

Todd: How about it's neither on nor off?

Chris: I don't know if I would have said that. Do you mean to say that it's not on and it's not off? Because that's what you just said. Just making sure that's what you said. That's not what I'm thinking of, but at least I'm clear on what is being said.

I: [Addressing Todd] So if neither of those states is possible for the end, what state is it in?

Todd: I don't know. [Addressing Chris] Do you remember when we had the one to one correspondence thing to explain why there's nothing in the jar on some problems? And then the argument for the 1+1/n problem was, yeah you take away the marbles but that is a reachable end state.

Chris: It doesn't diverge, it converges upon some finite value.

Todd: But if you have something that doesn't converge, does that mean that it's…

Chris: It depends on how it's described. If you say you're adding to 1 ounce of powder on the scale, you add something defined by the harmonic series. So you add ½, 1/3, ¼, and so on. I mean that doesn't converge to anything. The sequence does, but the series doesn't. So in that case you can't say, so it depends. So you can't just say that just because it doesn't converge in one case, it won't converge in another. You can't say there is no possible one to one correspondence if it diverges.

31:42 I: Ok, let me ask you this question. One problem is 1/n labels on the marbles, and if instead of those labels we have the $1/2^n$ labels, do you feel these two problems are different in any way? Because the series associated with one sequence is convergent, the other one is not.

Chris: Just because one does not converge as a series, it does not mean it doesn't converge as a sequence. Now if we're talking about time, in this case convergence of the sequence is all that matters. But if we're talking making a series for a problem where you're adding at each step, as opposed to just placing a label at that step… I guess the difference is if you're taking a marble and you have a label on it, then you can deal with just convergence of the sequence; if you're adding or changing something like ounces of powder, then it has to depend more on the series, because you're adding and subtracting based on that convergence.

I: Ok, 'cause in one case we forget entirely about the previous marble, but in the powder example, we build on the previous amount.

Chris: Yeah, it's not forgotten, it's still stored there.

I: Ok. How about we write something about the lamp?

35:03 Todd: I wrote what was said about the finite end states from the marble problems, I was thinking it may be relevant to the lamp problem but I don't think it is.

Chris: I think we should write two different things. If you say neither on nor off, that differs from what I believe about the situation, which is that it's either on or off and I can't tell which it is.

Todd: I was just entertaining the possibility, I don't actually…

Chris: How about you make it clear what you do believe at the moment, so I can judge if I agree with it.

Todd: So the first one, the ½ Marble Problem, I said there's a marble in there but I don't know which one. This one [lamp] you can't say something like that. I guess you could say there is a lamp, but we don't know if it's on or off. That's like saying there's a jar but I don't know which marble is in it. It seems that's a different thing to say that.

I: Ok, so you're saying the first answer [to the ½ Marble Pb.] does not translate well into this context [lamp].

Todd: I don't think. I can say less about this one [lamp] than I said about the last one.

I: Ok, how about you each write a summary of your own answer?

[Students write their answers individually, then write intuition and confidence for the lamp problem.]

42:05 [Start work on the Midpoint Problem]

I: Both of you saw this problem in the pre-test interview. And I believe you have some discussing to do because you had different approaches to it.

[longer pause, students are reading the problem.]

Chris: Oh this is slightly different [from the version in the pre-test], ok.

Todd: Is it?

Chris: What is the relationship between the set and the segment? Not what is the set described as, but what is the relationship between the set of all points and the segment.

I: Ok. How about we let Todd start the discussion on this one?

Todd: See, the first time I did it I did something different than I did the second time, and now I don't remember what I did the first two times.

I: Whatever it was, I guess you can start fresh if you don't remember?

[Todd starts scribbling something.]

I: Chris, what's the meaning of those vertical lines [drawn over segment AB]?

Chris: Oh, I was just making sure I have some visual form of where everything is, how the pattern is developing anyway.

I: Do you remember what you did for the interview?

Chris: I know how I described the set. I described the set as all…

I: Wait, before you say that, we said we'll let Todd go first. But keep that thought.

Chris: No, I've got it.

46:47 Todd: Ok, so all the segments are equidistant from each other, after each step. The distance between two consecutive points is the same. So I was looking at the equation for the distance between two consecutive points after step n, it would be $1/2^n$.

I: Do you agree with that, Chris?

Chris: Yeah, that's fine. I'll take that. That's similar to how I defined M in the first place. That if I took it from 0 to 1 instead of from A to B, it would be all points corresponding to $1/2^n$… No, that was wrong anyway. But that is how I did it the first time. I defined it as all points $1/2^n$ for some n in the natural numbers, But of course that gives you ½, ¼, and leaves off everything else. So I suppose at that point I said any number of the form 1 through n-1, over 2^{n-1}. That's how I defined it the second time. But this time, I guess I would define it the same way this time, but the question is different. It's not really asking me how I would define it so much as the relationship between the set and the segment.

I: Maybe your answer to the first question would suggest the answer to this question.

Chris: If we define M as the set of all midpoints… Should we give that answer yet?

I: Let's see. Todd , do you have a feel for how you want to answer the question in this problem?

Todd: No, I'm really bad about this sort of problem.

I: Do you understand the question?

Todd: I mean, you want to ask like, is there a one to one correspondence between the elements of the set M and the segment?

I: Well, that would be a question about their sizes, but I asked a question about sets. So when you have two sets A and B you can ask how they relate to each other, you can ask whether they are disjoint, or if they intersect, or is one included in the other, or are they equal, meaning every element in one is an element in the other, and vice-versa. So how do the two sets relate to each other in terms of inclusion?

[More discussion on notation for inclusion, strict inclusion, etc.]

53:21 Todd: So it seems in terms of the first point [the one right to the right of A] can get arbitrarily close to A. But if we say the segment is from 0 to 1, do we include all the real numbers?

Chris: If you want to define 0 to 1, then you should include all the reals.

Todd: Ok, 'cause I was thinking like $1/2^n$, there are some numbers you can't write as a multiple of that.

I: Like what?

Todd: Like any number that is not rational, between 0 and 1.

Chris: That's certainly true, but there are plenty of rationals that can't be written that way either.

Todd: True, yeah.

I: Can you think of a rational?

Todd: A rational that can't be written, like 1/3?

Chris: That's pretty good.

Todd: So I mean you can get arbitrarily close to them, but I don't know if that matters…

I: To what?

Todd: To 1/3.

Chris: What's the closest… You can get arbitrarily close to 1/3, but there's absolutely no way that I see that you can actually hit 1/3, because there is nothing that actually limits towards it in this case.

55:39 I: Actually, there is. You can build a sequence that converges to 1/3 of numbers only of this kind, the kind you described before.

Chris: Well, I suppose you could. Yeah, you could do ¼ and then smaller pieces.

I: I am going to describe a process through which we can build a sequence of midpoints converging to 1/3. Let's say the first term of the sequence is ½. Then I would choose the midpoint that I just created [at step 2] that is closest to 1/3, which happens to be this. So this is a_1, this is a_2. Then at the next step again I choose the closest midpoint to 1/3. It might not be monotonously converging, might jump around [1/3] a bit, but it is converging. As you said, the difference between two consecutive points [after step n] is $1/2^n$. You can prove it is converging to 1/3.

Chris: Yeah, it wraps around… true.

I: What about an irrational number, do you feel we could do the same thing, find a convergent sequence to an irrational way the same way? Let's say $\frac{\sqrt{2}}{2}$, which is in this interval.

Todd nods yes.

Chris: [pause] Yeah. Yes, it is in there.

58:17 I: Before we consider this convergence question, can you write a summary of what you discussed before, what you think describes the set M?

Todd: Um, I think it's true that any two consecutive points become arbitrarily close to each other.

I: Ok, and you also said what the distance between two such points [after n steps] is. Is there anything else you think is true? Chris, you were saying that when you were asked to describe M, you had a specific form for the points?

Todd: I liked the notation $k/2^n$, where k is from 1 to n.

I: What do you want to do with that notation Todd , how do you want to use it?

Todd: I guess you want to say is this equal or is this included in segment AB?

Chris: I would say strictly included. I don't think you can ever hit an irrational number with this process. You can't generate one, that's for sure. You can make a case that you could generate rational numbers with it.

I: You mean any rational number, like 1/3?

Chris: You can make a case, even though it's not a very good one. Like the one that was offered that at least it converges upon it, whether it actually hits it or not.

Without going into the convergence question, which I thought was to be discussed after this.

I: Let's finish this $k/2^n$ thing that Todd suggested.

Todd: The distance between any two consecutive points becomes arbitrarily close to 0.

1:02:09 I: Ok. What about the $k/2^n$, you said something about that. If you're trying to describe the points obtained through the collection of infinitely many steps that are described in this process, how would you describe that using set notation?

Todd: M equals curly thing [writes "{"]… [Addressing Chris] Do you like $k/2^n$?

Chris: Yeah, that will work. I have no problem with that notation.

Todd: $k/2^n$, where k equals 1 to n-1, and n is a natural number.

I: Is 7/8 part of that set?

Chris: Not exactly. k can go from 1 to 2^n-1.

Todd: Ok, right.

Chris: Now 7/8 is certainly in there.

I: Do you think that describes all the points in M, M being the set of all midpoints created by the completed process?

Chris: Given that that is how they are generated, they have to be of that form no matter what the point may be. It has to be that because that's the process being done.

Todd: Well, it's true for any finite step. So after all steps there is a jump to…

Chris: Well, it's for n all natural numbers. You can't have something that it's not formed that way.

1:05:30 I: Ok, now that we're done with that part, if you remember the inward spiral [from part iii) of The z^n Problem]. What we're forming here at each step, we have a set that includes the set produced by the previous step, because we keep all the already created points and add one [new point]. I think you guys agreed that the sequence z^n converges to 0, 0 as a complex number. So we have a sequence of [complex] numbers that converges to 0 and you wanted 0 to be added to the set of points produced by the completed process for this reason, that z^n converged to 0.

Todd: Right.

I: Well here [Midpoint Problem], in our growing set of points produced by this process, we can extract a sequence in the manner that we've just discussed that converges to 1/3. So if here [The z^n Problem] you want to add 0 [to the final state] based on the reasoning that this is the limit of this sequence of complex numbers [z^n], why don't you want to add 1/3 here [Midpoint Problem], which is the limit of this sequence of real numbers? And then, if we do that for 1/3, that's going to be true for other numbers as well, because we can extract another sequence of midpoints to converges to 1/5, 1/7, and actually to irrational numbers as well… in the same manner.

Todd: [Addressing Chris] That is what you were saying, you can see it happening for rationals… ?

Chris: I can see that it's possible you can make a case for rationals, that it would even converge upon those, I can see making a case that it converges upon irrationals even, for that matter. Although I am willing to grant that maybe possibly you can get rationals by this converging sequence… irrationals, by doing rational division, you're not going to get that, I don't care if that happens to be the limit or not, it's not happening.

1:09:37 I: Let's take it one at a time. Do we agree that you can extract a sequence of midpoints that converges to 1/3?

Chris: Yeah, I agree with that.

I: Do we agree that we can extract a sequence of midpoints that converges to $\frac{\sqrt{2}}{2}$

Chris: Yeah, it can probably converge to that.

I: Ok, so if we have a sequence of midpoints converging to 1/3, and a sequence of midpoints converging to $\frac{\sqrt{2}}{2}$, then using the same reasoning as here [The z^n Problem] where you're adding 0 to the set of points produced by the process, don't you want to add 1/3 and $\frac{\sqrt{2}}{2}$?

Todd: Ok, so to converge to one of these numbers that is not of this form [$k/2^n$], we want to look at a sequence of numbers that is getting closer and closer to…

I: Well, I mentioned that because that's what you guys did for The z^n Problem. So if the reason for which you added 0 to your set in that problem was because 0 was the limit of that sequence of complex numbers and when the process is completed that limit is reached, then I'm trying to apply that reasoning here [Midpoint Problem]. If you don't want to apply it here, I guess the question would be, what's different between the two problems?

Todd: Here's why. Here [pointing to the inward spiral pertaining to The z^n Problem] we've got a bunch of different points converging to one number. Here, [Midpoint Problem] if we choose a number like 1/3, we've got a sequence of numbers getting closer and closer to it. But the problem is, here [The z^n Problem] we're looking at a sequence converging to a number, here [Midpoint Problem] there's a given number that we're looking at. I think the number of numbers that are not given by this formula [$k/2^n$] is so much bigger than the number of numbers that are.

Chris: That makes sense.

1:12:32 Todd: To get arbitrarily close to this [1/3], we're going to have a bunch of points next to it getting close to it, which suggests that there's more numbers gotten by this [pointing to the $k/2^n$ description of M] that there are gotten by not that. There's more irrational numbers than rational numbers between 0 and 1, so…

I: Is that a statement, are you saying the number of irrational numbers is greater than the number of rational numbers on the segment from 0 to 1?

Todd: I think so.

I: What do you think, Chris?

Chris: never thought of counting those. But considering that the irrationals complete the line between 0 and 1 when the rationals are already there… Of course a lot of points are already there but there's so much space in between. I'm not sure I can say for certain which one is bigger than the other. I certainly don't want to get into that sort of thing.

I: Ok, let's not compare them then. I mean before we get to the question of all the rational numbers, let's just focus on only one number which is this 1/3. We can worry about the rest later. 1/3 is the limit of this sequence that is certainly included in M. If here [The z^n Problem] you chose to add this number to the final state because it was the limit, if that was the only reason why you chose to add it, then the same reasoning should apply there [Midpoint Problem], unless there's some extra aspect to this problem [pointing to The z^n Problem] that doesn't hold in this problem [Midpoint Problem]. Just like with a theorem, when you decide whether to apply it in a certain context, you need to have some hypothesis that holds in each context where you want to apply it, before you can claim the conclusion. So here [The z^n Problem], if all you need to add this additional point 0 [to the final state] is to have a sequence of points in the end set converging to that point, if that's the hypothesis of your "theorem" in a way, then here we already have that holding: we have a sequence of points that we already know is included in set M, converging to 1/3. So if that's the theorem we're using here [z^n Problem], then its hypothesis holds in this case [Midpoint Problem] as well. That's what I'm getting at.

Chris: I get it, I know what you're saying. Except it still applies in one case and not in the other.

1:15:30 I: Ok, so let's identify some aspects that are present [z^n Problem] here but not here, so that I cannot apply this principle [Midpoint Problem].

Todd: So we're saying 1/3 is in M or not? Because it seems like it should be.

I: Based on what?

Todd: Based on this [pointing to the diagram that refers to the procedure for constructing a sequence of midpoints converging to 1/3], that seems good.

[Addressing Chris] But you're going to say no?

Chris: I'm going to say no because I don't think there's a way to do that. Even with the convergence, I don't think it applies to this case. Regardless of the fact that there can be one convergence, and there are even probably multiple convergences that to 1/3 in there. That same sequence that converges to 1/3 can converge to any number of other things as well, no matter how you define it.

I: Actually, one sequence cannot have two limits.

Chris: You can say one sequence converges to 1/3, but at the same time there is enough space between it that you can say it converges to something else. I'm not going to say it converges to two different things. By choosing the numbers you're trying to get, you can have the same sequence converging to two different things. Depending on how well you refine the sequence.

I: There's actually a theorem in Real Analysis that says that one sequence cannot have two different limits.

Chris: Exactly, and that's why it can't converge to either of them.

I: But if by the way I define it I can show it converges to 1/3, then by exclusion it cannot converge to any other number.

Chris: right, I'm saying it can't converge to either of them but it can converge to both, so therefore it can't exist that way.

I: So you're saying actually that sequence [described earlier as converging to 1/3] does not converge to 1/3?

Todd: Or alternatively, it's impossible to have a sequence [of points from M] that converges to 1/3?

Chris: You can find something that gets closer to 1/3 but to say it converges to 1/3 is more than I am willing to go through.

1:18:23 [Discussion on what the definition of the limit of a sequence is and how it can be shown that the procedure described earlier does produce a sequence of midpoints that converges to 1/3.]

1:26:20 Chris: I still don't think /3 is in there.

I: Ok.

Chris: And there isn't a way to get me to think that it is.

I: Ok, but then to reconcile these two problems, I want to know why this principle can't be applied here. If this is what the principle is, that you just need to find a sequence of points in the final set that converges to that point [in order to add that point to the final set]. Because if we agreed that this set of a_i's converges to 1/3, then we need to add 1/3 [to the final set]. So if there's a strong preference for not including 1/3, we need to see why the principle doesn't apply. Todd , what do you think?

Todd: [laughing] Can I just go back over the worksheet and go like no, no?

Nevermind.

I: Go ahead, tell me. Which one?

Todd: All of them. No, no [making gestures that indicate crossing off.]

I: Which ones do you feel would be invalidated by this? We haven't even said anything is wrong, I just brought up a question.

Todd: Well, I don't know.

I: Why would you cross out everything from before? If we were to add 1/3 here, or if we were not to add it?

Todd: If we were not to add it, then a lot of things would have to change. Maybe this [pointing to The z^n Problem], unless I can think of a reason, which I don't think I can. All of our answers were like, the sequence converges…

I: Tell me specifically, which ones?

Todd: All the things where we looked at a sequence that converges and we said after all steps, it's actually what the sequence converges to.

I: Which ones were those?

Todd: Powder Problem, Marble Problem…

I: The 1+1/n Marble Problem?

Todd: Yeah.

I: Ok, that's two so far. What about the one 1 goes to 2, 2 goes to 3 [The n->n+1 Marble Problem], where we were replacing an existing marble with a marble with the next natural number on it, would that be invalidated?

1:28:30 Todd: That sequence goes to the infinite, and we didn't say that there is an infinite ball. So I don't think that one we'd have to [change solution for].

I: Ok, So it wouldn't affect the reasoning for that one.

Todd: No, but it's possible we used faulty reasoning for that as well.

I: Ok. So right now you're 50-50 for either of the two directions?

Todd: Well, since I can't say how this is wrong, I think this argument [the convergence one] suggests that 1/3 is in there.

I: Well, this argument suggests only that from the way I constructed this sequence of midpoints, it converges to 1/3. Now it's up to you if you want to apply this principle [that the final set needs to include its limit points], and if you don't want to apply the principle you need to show me why it's not applicable in this case.

Todd: So how many choices do we have here? We can reject that whole inference thing [pointing to the diagram for The z^n Problem]…

Chris: Which I'm not doing, but you're free to do.

[Todd starts writing a list of choices.]

I: Todd , for each choice that you list, I also want you to say if you think that affects the answer to other problems. Chris can think about that as well.

Chris: Not adding 1/3. It's not there.

I: Ok. If you're not adding 1/3, do you feel that influences your answers for any of the other problems?

Chris: No. 1/3 is not there, and every other limiting case was strictly accurate.

Todd: Ok, so we have 3 choices. We can reject the inference thing, or we can say the inference thing is right and 1/3 is there, or we can make the inference rule more specific so that 1/3 does not have to be in there but everything else [previous] remains correct.

Chris: Could be.

I: But you do want to be consistent with the principles you apply across problems, otherwise it's like you're inventing the inference rules each time [for each problem]. In that case all answers would become arbitrary.

Todd: Well I am inventing all the inferences that I can make [laughs]. It's totally arbitrary. I don't know.

I: You can invent whatever you want, but then you need to apply it consistently across problems.

Todd: Right. There were some problems that like when you look at another problem, it makes you realize there are things you didn't say about them that maybe you should have said. I don't know.

I: Ok. Any other ideas of how to figure out this thing or we can call it a day? It is close to ending time.

Todd: [Addressing Chris] Do you want to end it?

Chris: That's fine.

Todd: I just can't think of anything, so we might as well.

I: Todd , do you have any personal preference towards any of the 3 choices?

Todd: No. […] So the choices are to include the limit in the set [for the Midpoint Problem], to not include it, which would suggest that this thing is weird or wrong or arbitrary, or there is a difference between the two problems [that allows us to make that inference in The z^n Problem but not in the Midpoint Problem], which we haven't figured out yet.

I: So no inclination towards any of them?

Todd: I don't really have an intuition. I have confidence that this is alright to say, but not intuition… just because other people seem to be doing it. There was a problem that was like what happens at t=1 after all steps have been completed.

I: How about you rate your inclination for each of the 3 choices. Chris, you can rate the choice you're going for, 'cause I think you have more of an idea which one you want. Do you want to give us a summary of your position?

Chris: I guess. Position on this is that M is still going to be strictly included [in segment AB] because any rational can be included in that set if you really want to make that argument, but irrationals are not going to be included. And that's just that. I don't have the mathematical foundation to prove that, but I know that can't be there. If it was, then I am in the wrong place.

Todd: So you're saying 1/3 is in there?

Chris: I am allowing it to be there because I can't say that it isn't by what I've said before. I will allow it to be there very vaguely. But I am still not going to let go that irrationals cannot come from rationals, limited or otherwise.

I: So maybe you should rate two different answers. One would be that not even 1/3 is in there, M is only numbers of this kind [$k/2^n$]. Would you be more comfortable with that answer?

Chris: I'd be more comfortable with a lot other answers than what I'm saying because I still don't like that 1/3 is in there. But I can see an argument for it and I'm not going to be agree with it or disagree with it at this point. My argument is purely that M cannot be the whole segment Ab because the irrationals cannot be in there.

[End of session]

Session 6
12/10/08

0:00 [Start work on the Halves Problem.]

[Students read the problem individually.]

I: Did you guys understand the problem?

Both: Yeah.

I: Alright, Todd goes first.

Todd: I'm pretty sure I understand the problem. Is it going to be $1/2^n$?

I: What's going to be $1/2^n$?

Chris: Yeah, I suppose it is.

Todd: Finding a formula for what number is obtained at step n.

I: At this point can you write M using the set notation?

Chris: Ok, a_n is $1/2^n$, so that's fine. Is 0 included in the set of real numbers [produced by the completed process]? Well, I took that to mean is 0 ever produced by this, if you want to make it a set, that would probably fit more with the question. 0 is produced at the last step anyway, so to me it didn't matter, I simplified the problem to "is it produced", but fair enough.

I: So did you simplify it to "is 0 produced at a finite step"?

Chris: No, not that. Is it produced at any step in the process by the time everything is complete?

Todd: Bummer.

Chris: Why?

Todd: I don't know. That's unfortunate.

Chris: Does that mean you think it [0] is in there, it's not in there, or you have no idea whether it's in there or not?

Todd: I don't know.

Chris: You have to commit to one of the 3 options. Is there another option?

Todd: Let's talk about it. I mean I can do it by analogy, except that…

4:07 I: Analogy to what?

Todd: To other problems.

I: Can you pick one?

Todd: You can do the Midpoint Problem or you can do the Vector Problem [he's referring to The z^n Problem from 2 sessions earlier].

I: Let's take the vector problem.

Chris: It looks more similar to that, at least.

I: Similar in what way?

Chris: Because there you're trying to see if the 0 vector is ever produced by the process. Not as similar to the Midpoints, although it is there also because it's going ½, ¼, 1/8 and so forth. It's just taking one midpoint each time, not multiple.

I: Ok, so Todd , let's go with that analogy.

Todd: The vector problem was done by analogy to something you [addressing me] wrote, not that we wrote. It was something like $1-1/2^n$, and then after all steps have been performed we were at time t=1 [referring to the timed version of the Original 10 Marble Problem], so I said we'll assume it [z^n where z is a complex number of norm less than 1] will become 0 [after all steps].

Chris: I don't recall that being part of it.

I: Actually I think he [Todd] did bring it up then.

Chris: Did he?

I: If you remember, when you made the argument for 0 [being part of the set produced by the completed process, for The z^n Problem], you broke it into two cases: if the norm of the original vector z was less than ½ then you could use that time thing, if it was greater…

5:52 Chris: Yeah, that's true. Fair enough, I did bring that up.

Todd: That was kind of cheating. […] All these things are defined by other people but if I don't know how they defined it, how can I… ?

I: What things?

Todd: Everything. All of math.

I: Right, but once you start with some axioms and with specific types of inferences that are considered to be valid, then if any of those people make a claim, you can check using those initial axioms and types of reasoning if it's correct or not. You do need to build on something; if you don't have anything you agree on at the beginning then you can't have any kind of meaningful discussion.

Todd: So imagine like I am solving a problem and I don't know the axioms, and you're asking me something, how can I know…?

Chris; Well, you have some axioms.

I: So what would be some axioms?

Chris: I don't know what axioms he has to be honest, but you have some mathematical axioms in there. Whether they are applicable is another matter, but you have something. And if you don't, then by solving a problem you're creating your own axioms.

Todd: That's a problem, because they might not be the same axioms.

I: Between the two of you?

Todd: Between me and math.

I: Have you heard of Non-Standard Analysis? There are a lot of ways of defining certain things.

Chris: The axioms have to come from somebody saying something. Your mathematical system could work – doesn't matter if it fits with people use in mathematics in general, but it could work.

Todd: But here we're trying to use the axioms that are agreed upon by everyone, right? Alright, this is a wild discussion.

8:16 I: It's a good point. Unless you have some axioms in common, then it might not be clear what you're basing your inferences on and what he is, if you have the same foundation. So this might be a good time to think about this. Like you said, once you solve a problem and you make some claims about the situation in that problem and you say that this is the reasoning we should employ, then you are kind of laying down some axioms. Maybe they are not even that clear to you but in order for that answer to be argued as correct, you already assume some things, and then it's up to you to make them explicit. Once you make them explicit and say based on these assumptions and these types of reasoning I define as valid I am making this claim, then no one is going to be able to argue against that. They can say I don't like your assumptions and will make different assumptions, they can say that, but within your system, your answer will make sense.

Todd: My problem is that my system is sort of based on semi-circular reasoning.

I: If your system is not consistent, then yes, you have a problem. You need to build a system that is not going to contradict itself.

Todd: Well, part of the system is like, I assume that some of the questions you're asking are not lies.

I: How about you build a system that does not rely on what I say? […] Maybe you're trying to say why you made some choices based on what you perceived to be important in these problems. So do you have any mathematical assumptions that you feel you're using in this problem?

11:26 Todd: Well, I mean we can have an assumption but I don't know if it's right.

Chris: Well, it's an assumption at the moment so don't worry about it.

I: That's the point, what does it mean to be right? With the axioms of Euclidean geometry, if someone doesn't like them they can change them and build a different type of geometry. It doesn't mean one is correct and one is incorrect. Just starting from different assumptions we build a different kind of theory.

Todd: Well with numbers like natural numbers and rational numbers and irrational numbers, you need to know what the assumptions are when you make those numbers, right?

I: Yeah, there are multiple ways of building those numbers.

Todd: 'Cause some of the questions we're asking basically boil down to that, right?

I: To what, to what we mean by natural, rational, and irrational numbers?

Todd: Well, or how you can construct them or something like that.

Chris: The way I'm inferring a lot of these things, in a lot of the problems I've used the one-to-one correspondence countability argument.

I: Can you say what it is?

Chris: What the argument is? In some cases, like the… I don't know what problems it would have to do with. There was something with grains of sand, and it was removed…

I: Powder?

Chris: That's right. It was defined by… What was it?

[I give him the worksheets from previous sessions.]

Chris: Timed marble problem, that's it. $1\text{-}1/2^n$ is the time in that, and I made a correspondence to that.[…] I map it so that when the time is $1\text{-}1/2^n$, marble n+1 is removed. Each numbered marble would be removed at a time that is countable in the time set. I used that argument of bijectivity in order to prove a lot of the things I said. This is the inference I'm using though. I don't know if it's applicable, but it's an inference I'm using.

I: If you want, we can formulate all the problems from today like that, we can say that at time $1\text{-}1/2^n$ we're going to perform step n as it is defined in that problem. Does that help you?

15:12 Chris: I mean, I'm just saying that it's an example of an inference I used to solve some of the problems. It's just something that intuitively or cognitively made sense. So I mean your axioms don't have to be accurate with most conventional mathematics although it would be nice. I don't know if mine is, I'm just using what I have. So don't worry if you think your axioms are mathematically invalid, you can only go with what you have to work with.

I: I guess it comes down to the question, what does it mean "mathematically invalid"? Again, if your system is inconsistent, yes, that is a problem. But if it's different from another axiomatic system, that's fine, you have a reason that compelled you to build it in a different direction. […] Once each of you formulates specific axioms, you should try to be consistent across problems. I guess it wouldn't be that helpful if you're building a different system for each problem.

Todd: That's the problem, the way I solved the timed marble problem [referring to the Original 10 Marble Problem] is different from the way I solved other problems.

I: Well, what's similar and what's different, sometimes it is hard to define. […] I guess it is up to you to define what it means that two problems are the same, or that two types of reasoning are the same.

19:04 Chris: For this problem at hand even [referring to Halves], what would you [addressing Todd] say, is 0 in there? You don't have to give a reason at the moment.

Todd: I'm not sure though, if it's in. 'Cause you're just like, what happened in Calculus…

Chris: So you were given the argument of limits and you took that to be true and valid? I mean obviously if you are going to take the sequence of a_n, it's going to limit to 0. But are you taking that to mean 0 is in the final set? I've gotten into trouble in the past going along with this limit argument which is actually not the way I've seen a lot of these problems.

I: When you say in the past you mean… ?

Chris: Yeah, I mean like the vector I said it was 0, I don't think of it as a limit. I actually think that $1/2^n$ at infinity actually is 0. I don't use a limiting process to say that. I actually go by the assumption that that is 0.

I: Well, but then you go by the assumption that we defined what it means at infinity.

Chris: Yeah, I know, I'm using a lot of assumptions I wasn't speaking of. To make it more common between us I would use the limit argument. I guess I should say now that's not exactly the way I was going at it. I was using that as sort of a convenience tool. But the last problem last time kind of said to me that you can't be doing that if that's not what're you're doing. But anyway the argument I would make is that 0 is included [in the final set for the Halves Problem], because after all steps have been completed, 1 over infinity I've taken it to be 0. Valid or not, that's what I've taken it to be so I believe it's included.

Chris: I guess another question about this problem would be, what is the reason for including the value at infinity? Because all the problems say "after all steps have been performed"; it's up to you to decide what that means.

Chris: It's up to me to assume that all steps include the set that corresponds to every natural number. That's the assumption I was working on also. You have to realize that's an assumption also.

23:00 I: Todd , what do you think of that assumption that Chris just mentioned? That we need to add this value at infinity [to the final set].

Todd: So you're saying that because we're saying all steps have been completed, that means it's at infinity?

Chris: Yeah, that's the line of reasoning I'm using.

Todd: Ok. I could make an argument for both answers.

I: When you say both answers, which two reasons are you referring to?

Todd: You could make the argument that there is no n such that $1/2^n$ is 0, therefore 0 is not included in the set. Or you could make the argument that by doing infinitely many steps it comes arbitrarily close, and when you think about all steps, it actually is 0, because this means take that plus whatever the limit is.

I: Uh-um. And do you have a preference for any of that?

Todd: If I am to be consistent with other things we said, then I would say that 0 is included.

I: Other things would be The z^n Problem?

Todd: Sure.

I: Well, the way you wrote it [he wrote $\{\frac{1}{2^n} \mid n \in N\}$], does that include 0?

Based on that standard notation from your math classes, is 0 included in that set?

Todd: I don't think 0 is included [in the set described as $\{\frac{1}{2^n} \mid n \in N\}$]. I think we said that union 0 in The z^n Problem.

I: Do you want to use that here?

Todd: We can do that.

27:00 [Start work on the Truncations problem.]

[Students read the problem individually. Short discussion on what is meant by truncations, what the first few are, how many digits of Pi each has memorized.]

Chris: Is π in the set of points constructed by the completed process? My assumption is, will π be the last step, when all steps are completed? I mean 3 is a point, 3.1 is a point, 3.14 is a point, will π itself be in there?

Todd: I like this question. I think it's cool.

I: Do you feel it's different from the question before?

Todd: Yes.

I: In what ways?

Todd: Because when you step outside looking at an individual step, you can just think of them as just going.

I: And being done going?

Todd: Well, you can't be done going.

I: Don't I say "after all steps"?

Todd: Right. But I don't have to look at it. Like π is never done going, just like I don't have to think of the natural numbers as being done going.

I: But isn't that true of the previous problem as well? The number that you were constructing were ½, ¼, 1/8, and if you were to draw points, it would start from here and get these points that get closer and closer together. And here [Truncations problem] if you start drawing points, it would look pretty similar, just that they would be going towards the righTodd: 3, 3.1, etc, so in terms of the points that you're drawing, doesn't it "go on" the same way?

31:25 Todd: The only problem is I don't think of π as points. […] I wasn't thinking like that at all, on the number line. I was thinking just like adding the next digit along each time.

Chris: That's actually the way I thought of it, I didn't think of the number line argument, but that's fine, it doesn't matter to me as long as you have a concept of it.

Todd: Because you don't have to worry about convergence. Well you do, and that's the whole question.

Chris: I mean it's obviously going to converge. If it doesn't converge to π, I'd like to hear what it converges to.

Todd: It doesn't have to converge, because it actually is π.

I: When you say "it", what do you mean?

Todd: Each part of the process you're not saying like it's not π and it becomes π. Well you are. The question is not saying you have something that's not close to π; this is something like this is a part of π and now you're making a bigger part of π.

Chris: You're making it closer to the full value, you're not truncating it you mean?

Todd: Right.

I: Just keep in mind that the question is about the set of all points/numbers. So after 3 steps, you have a set of 3 points, you don't have only the last one. After 4 steps you have 4 points, or 4 numbers.

33:51 Chris: I've got an argument in my mind, I just don't want to influence the discussion before…

Todd: It seems pretty intuitive to me that a_n would have an infinite truncation.

I: What does that mean, infinite truncation?

Todd: Or "not truncated".

I: Did you say "the end would have an infinite truncation"?

Todd: Yes.

I: So you said at the end? At the end of what?

Todd: After all steps have been completed. I said an infinite truncation, but that's the same as not truncated.

I: So when you say you'd have that, you mean you'd have it in the set we're constructing?

Todd: Um…

I: At each step we have a set we have constructed by all the previous steps. And then when you're done you have a set. What exactly do we have in that set?

Chris: Is it in there somewhere? Obviously it's never going to be the first step, [number] 3 – that's not π. 3.1, second step, that's not π. So is there a step at which it will be there?

I: Let me ask a question about "is there a step"…

Chris: I don't mean a finite step. As I've said before, I do actually see a step at infinity as being part of this. This might just be having taken a semester of graph theory with a point at infinity actually being a point that you can draw something to and reach. So when I say a step, I mean step at infinity included. So for me, because there is a step at infinity, I can make an argument that π is in there. But I don't want to elaborate too deeply on that until his argument has been given fully so that I can at least consider it.

37:03 Todd: So using your one-to-one thing, we can say the first truncation is obtained at step 1 and has 1 value, step 2 would have 2 values…

Chris: Step 2 would have 3.1, it would create one with that, at least. So a_2 has 2 digits, a_3 has 3 digits. Is that what you're saying, that the index number corresponds to that [number of digits in the truncation created at that step]?

Todd: It seems to be the argument that you were making before, or not exactly?

Chris: Not entirely. You're saying that the number 1 can correspond to this [the number 3] because of the number of digits, 2 can correspond to this [3.1]. It's a similar argument, but not exactly what I was going for.[…] I would correspond it directly, each number to each digit value. For example, 1 would correspond to the digit 3, 2 to 1, 3 to 4, and so forth [in general n corresponds to the n^{th} digit in the decimal expansion of π.

I: Keep in mind that unlike that initial vector problem where we were editing the same vector every time [The vector Problem from session 2], here we're adding a new number to the set [at each step].

Chris: Right. But since I am including this step at infinity, I am saying that because there is a one-to-one correspondence between the natural numbers I am indexing on and the number of digits in π, and there are infinitely many natural numbers, then there will a step, the step at infinity, at which π is produced. But I'm not saying that the one thing is the only thing that's in the set at that point. I'm saying that at the step at infinity, that is what is produced.

I: Alright, so π is the last addition to the existing ones.

Chris: Right. If it's not the last step, then what could you add at the step after that to create something beyond π? So it has to be the last step, if it exists.

That's my argument. Yours [addressing Todd] is 1 is [corresponding] to this, 2 is to this… How does this say that is in the set, if you agree with that?

Todd: When I thought about the problem, I wasn't thinking about points. The set of points is different from asking "if you reveal all the digits of π, is it π?"

I: But keep in mind that I'm asking what is the set of points produced by this process?

Chris: Well, what would you say? It sounds like you're saying it would be, because you're saying that if you reveal everything [all the digits], then there's π.

43:50 [Rating confidence and intuition for the Halves Problem, which I forgot to ask them to do at the end of that problem.]

Chris: Ok, where were we? [Back to the Truncations problem] I took it you're corresponding the n here [in a_n] to the number of digits [in the truncation of π produced at step n]. So the way I took it is you said there is a step a… tilda [$a_{\sim}$], let's say, at which point you will have π here in its infinite decimal expansion [i.e. π is produced by the process at step $a_{\sim}$]. So there is a step at which that will happen… finite or infinite, that's your choice.

I: And then the set of numbers [produced by the completed process] would be the numbers across the rows.

Todd: But this is different from saying, if you just starting revealing the digits of π, you will be revealing π [at the end]. These are two different questions.

Chris: I don't see the difference in the questions, but…

46:00 I: Let's clarify, what are the two different questions, Todd ?

Todd: One is like, if you're revealing all the digits of π, then at the end you would have revealed π.

I: What do you mean by revealing? You mean that you start with a 3, and then you add one more digit, then one more digit, to the same number?

Todd: The natural numbers are just a slide that is being pulled back.

Chris: Ah, I see.

I: Ah, so let's assume that π is already written there [in its entirety] and it was covered, and you slide the cover [to the right, one digit at a time]?

Todd: Right.

Chris: I see what you're saying. Obviously if you remove it all the way you have π, but you're saying the difference is, are there enough steps in the natural numbers even at infinity to do it?

Todd: Here's the problem, it's a separate problem from what you're doing.

Chris: I see, that's the boundary that's occurring. To me there's no boundary there, because of the correspondence between digits of π and natural numbers, it doesn't matter because there will be a step, there are a countable number of steps for which π will be revealed.

Todd: This is the problem I have, because you can say you keep this [the sliding of the cover] to infinity, and then you can say, at what final step did it get?

Chris: I was saying that's why I couldn't understand why they are two different problems.

Todd: So you assume that the infinite step in the set of all finite step, the set of natural numbers?

Chris: I'm assuming because it's countable, the set of natural numbers, which is the set of steps by the way, a_1, a_2, a_3… Considering they're indexed on the same set, that it [π] will be produced because there will be a step at which that will happen. The last step.

48:13 Todd: The other problem is, are you saying there is like a last digit of π?

Chris: I'm not saying there's a last finite digit of π. I'm still saying there's an infinite number of digits. Because the set of natural numbers is infinite, and you can countable infinity.

I: So Todd , let's recap what your two questions were. One of them is, if we have π written out with all its digits but it's covered, and at each step I'm sliding the cover one more position to the right, the question is if after infinitely [countably] many steps we have π revealed in front of us. Do you have an answer for that?

Todd: Yes. π is indeed revealed.

I: Ok. And what was the other version of the question?

Todd: The other question is [pointing to the worksheet to the Truncations Problem], is π in the set of points [produced by the process that spits out the n-digit truncation of π]? Because that's sort of like, has questions about finiteness.

I: What do you mean about finiteness?

Todd: All the points are finite points [finitely many digits].

Chris: It is of course your assumption that after all steps, it is still finite.

Todd: I think the answer to both of these questions is yes.

I: [addressing Todd] So you think they are different questions but you'd answer yes to both of them?

Todd: Yes.

I: Would the reasoning be the same?

Todd: No. The first question is like kind of obvious. If you reveal the digits of π, then you're revealing π, yes that's what you're doing. The second question is an assumption about what this means, "after all steps", does that include some sort of step at infinity?

51:17 I: From what I understand from Chris's argument, by the phrase "all steps" he also means a step at infinity, and that's the step at which the whole π becomes part of the set. Todd , is that one of your assumptions as well?

Todd: Yeah, that's fine. I mean, it seems intuitively obvious but I don't have a reason, and I think that's a good reason.

Chris: That's fine, you don't need a step at infinity if you don't want it to be there.

Todd: The problem is, if you don't have a step at infinity then you can't say π is in the set.

Chris: No, I would argue the same thing, if you don't have a step at infinity then π is not there. But because I require that step to do that, that's the way my argument is structured at least.

[Students individually write a summary of their arguments.]

Todd: This proof is going to be "because I say so".

Chris: Remember what I did at the end of last session, I wasn't using a whole ton of mathematical reasoning at that point. The stubborn reflex kicked in and took over.

53:57 I: While Todd writes that down… Chris, do you remember in the first session, when we were discussing this phrase, "after all steps have been performed", I was explaining it as "for any natural number n, there is a step with that number n that has passed". Do you feel your assumption of a step at infinity is consistent with that description?

Chris: Well, with the step at infinity, at the step at infinity certainly all natural number steps for any finite n have been completed. But since the natural numbers are an infinite set, I do believe that the step at infinity is part of it because you can't… I mean you can count every natural number but you can't explicitly list every natural number without… You can't get to the largest natural number. Of course you can list it, that makes it countable. I believe it's not inconsistent with that.

[Rate confidence and intuition for Truncations Problems.]

56:37 I: I have a question that is not on the sheet, I've just thought of it now. Let's imagine this: that I'm defining an iterative process such that at step n, I am to say out loud that number. I need to perform this process. Let's say I'm starting now, so I'm saying "one", then "two", so I'm basically counting through the natural numbers. This is the process. Let's assume we can talk about all steps having been performed. At that point what would I have accomplished? As in, what did this process produce? See, we always talk about what's left after the process is done.

Chris: So at the end what numbers have been said, basically? Well, I know where my argument leads me.

Todd: Ok.

I: I want to assume I had enough time to perform all the steps. If we assume I said number n at time $t=1-1/2^n$, then we are at t=1 and I am doing going through this process.

Chris: Alright, so what has been said?

Todd: I don't know. I feel like you saying numbers is the same as going through… here's natural numbers, just 1 plus the previous natural number, and that's all of them, then you've said all of them.

1:03:45 Chris: [addressing Todd] So what do you believe has been said after all steps?

Todd: I want to say you said all the natural numbers. Similar to here [the "reavealing" version of the Truncations problem]; even though at each step you're only counting a finite number, when you're counting all the steps then you got the entire set.

I: Chris, what do you say?

Chris: I don't know. It would be really convenient to me to use that one-to-one correspondence of times to numbers said aloud, because each time involves the same n you're saying aloud [e.g., step 1 happens at $1-1/2^1$). Since there is a correspondence between each time, you would have said all natural numbers. At time t=1, you would have gone through an infinite number of steps. So you would have said the complete set of natural numbers, 'cause that's what you're counting.

I: Am I saying anything at t=1?

Chris: Well, in order to hit t=1 you have to be at infinity, so I'm assuming you're saying something along the lines of "actual infinity", but I don't know what number that corresponds to.

I: But that's not defined by the process; the only rule you know is that at step n, I'm saying n.

Chris: I know. I have an intuitive step that says it, but I don't have anything defined if we stay within that strict construction. That was an assumption on my part that actual infinity is there, because it's required to reach t=1. You've at least said every natural number because you can't hit t=1 without saying them all at the very least.

1:06:40 I: So Once I reach I said them all, and the additional question that I had was, did I say anything else other than all natural numbers?

Chris: I don't know if you said anything else because that's not defined in the process explicitly; in my mind it is.

Todd: That's a good answer.

I: Now let's do a parallel between this "saying n out loud" process and the π [Truncations] problem. As Chris is counting through the numbers, you Todd go through this process [Truncations]. So whenever he says a natural number, you perform the step with that number in this process [Truncations]. So then, when are you going to write π is the question? In order to have π in the set produced by the process, you Todd need to write it at some point. And for every step here [Truncations] we have a step there ["saying n out loud"] and the other way around. I guess that's the question that I'm getting aTodd: how is π going to appear there if all Chris says is only natural numbers, he does not say anything at t=1 or he's saying it's not defined by the process. When Chris says a natural number, you're going to write the n-digit truncation of π. When are you going to write π if all Chris does is going through the natural numbers?

Todd: The assumption we had was we are going to write π at the infinite step.

Chris: That's part of my assumption although that's slightly awkwardly phrased given the current context.

Todd: I was going to say I write all the digits of π, and then you go "phew" and I say "π", at time t=1 [both laugh]. That would be an interpretation.

I: Here's the thing. At t=1 Chris can do whatever he wants out of his own will but the process does not define anything for him to do at t=1, how come it is defining something to do for you?

Todd: Because I want it to be that way?

Chris: I guess I mean the step after the infinite number of steps have been completed. Technically the step at infinity, but it's the step right before that. That could just be the last natural number in the set of natural numbers. Unfortunately that is not a very defined number. So for my convenience I call it step at infinity.

I: Well, N being an inductive set, we know that for any element in an inductive set, we need to have a next one. So if you put in an element that is the element at infinity, then you must also have infinity plus 1, and that one must have a successor and so on.

Chris: I know… I don't like the thought of going into further set theory and alephs. But in the set of natural numbers, the number of digits in π and the index of π… I mean my terminology is awkward because infinity is an awkward thing to discuss. If there is an infinite number of digits in π and an infinite number of indexing numbers, then after every natural number index has gone, every digit would have been said also. Now of course, by the process definition [saying n out loud] I don't have something to say. Well, what's the step number that this happens at? This is where I invoked saying there is a step at infinity because that's the only way I could get around the fact I don't have a way to say that within the definition of it. I do understand that's outside the definition to do that… slightly. It sounds like the definition is off by 1 [laughs].

I: Ok. And Todd , do you still want to say π at t=1 just because you want to?

Todd: That's actually what my reasoning is. I seem to be saying π here, it's really, really similar to π, and it seems I have been saying it for a long time. I don't know.

I: Well, you will have been saying truncations for a long time.

Todd: Yeah. I still have this problem… [longer pause]

I: Ok. We are ready to go on to the next problem. I guess the question would be, is there any change in your ratings for problem 2 [Truncations] due to my counting out loud through N thing, or no?

[Chris changes his rating]

1:13:26 [Start work on the Halves and Reals Problem]

[Students read the problem individually]

Todd: So this is basically a restatement of the Midpoints question.

Chris: It's similar enough. The minimum part is going to take a bit to wrap my mind around.

[More discussion clarifying the text of the problem.]

1:18:33 Chris: Is b a part of that set? Obviously if b is of the form $k/2^n$, it's in there. That's trivial. If b is of that form, then $k/2^n$ –b would be 0, so that's going to be the minimum distance, so b is going to be there [in the final set].

[Both writing a summary of this.]

I: Todd , do you agree with that?

Todd: I do.

I: Ok. Chris, you figured out this much, how about Todd figures out the next part?

Chris: [addressing Todd] What do you want to be the next part? Do you want it to be the endpoints, 0 and 1? Or the rationals, or the irrationals?

Todd: I was just going to take b not of the form $k/2^n$, but I guess we need to be more specific. Ok, so there's 0, there's 1… I think for 0 we sort of answered a similar question [looking through the worksheet].

Chris: Oh, so if you're taking b to be 0, is it the same problem as Problem 1 [Halves]? Yup, pretty much, as far as I can see.

Todd: It's the same set minus 0, for sure. It seems like it is a different process though [from Problem 1, Halves]. Here [Halves] we're multiplying a thing by another thing…

I: Maybe the operations through which you're getting these specific numbers [the sequence a_n] are different [when comparing problem Halves to problem halves and Reals].

Todd: So I'm asking the question, does that matter?

I: That's for you to decide.

Todd: [faintly] I don't think it matters.

Chris: I would say they're the same problem, for b=0.

1:24:31 Todd: So it doesn't matter then that the process is slightly different?

Chris: They're both iterative processes, both defined with steps $n \geq 1$.

Todd: Ok, so they should be the same situation then.

Chris: Should be. So you're going to say that 0 is in there [the final set], then.

Todd: Do you want to say, by the same argument as in Problem 1?

I: If you think it's the same argument, sure.

[Both writing summaries of case b=0].

Chris: Now I'm going to consider b=1. Obviously you have the same sort of thing going upwards. You're going to take ½, ¾, 7/m, and so on. I'm comfortable saying that if b=1, then b is in M [final set].

Todd: What's weird about this is that we're looking at midpoints, right? Seems the process makes it clear that 0 wouldn't be included as a midpoint.

Chris: You're defining as a midpoint in the process… It doesn't necessarily mean that everything in that set is going to be an obvious midpoint. I mean you're not going to argue with me that a finite step would produce 0.

Todd: No, I am not.

Chris: Just because 0 is not a midpoint doesn't mean it will not be generated by the process of creating midpoints. As far as I see, it doesn't require that, for 0 and 1 anyway. If you want to say that $1/\pi$ is not a midpoint, I don't know [about $1/\pi$ being in the set M or not] . I am less certain of that then I am about 0 and 1.

1:28:35 I: Well, for the complex numbers [The z^n Problem], the complex numbers defined at finite steps always had norms greater than 0, and you decided that 0, with a norm of 0, was part of the final set. So in that sense, specific characteristics that are true of the elements produced at finite steps seem not to hold for all of the elements that you claim are in the final set. In that situation at least.

Chris: Yeah, they're not required necessarily to hold. Although there are specific reasons for each case.

I: So what about Chris's argument that 1 is in the set, if b is 1?

Todd: Good.

I: Did you write it up?

Chris: Now let's see, rationals.

I: Todd 's turn.

Todd: Alright, let's let b equal p/q, some sort of rational number, p and q different from 0, and b not of the form $k/2^n$.

Chris: If you don't believe it's in there, you can say it's not in there [final set].

Todd: [addressing me] You provided this argument that you've got this cool sequence that converges…

I: Let's put it this way. For b a rational number not of the $k/2^n$ form, do you think the sequence a_n converges to b? And we're talking about the standard definition of limit.

Todd: Yes.

Chris: I agree it converges.

I: Ok. Now does that have any relevance, do you want to use that in any way or not?

Todd: Alright, so we agree that it converges to b, defined like this.

I: I have another question. Does it converge in a monotone manner?

Todd: I think it jumps around the limit.

I: What do you think, Chris?

Chris: Thinking about 1/3 [as the value of b]. a_1 is ½, a_2 is ¼, a_3 is 3/8. So clearly it jumps for sure.

I: Ok, so it does converge to 1/3 but it jumps around.

1:36:24 Todd: Alternatively, you can look just at the odd numbered steps, if you wanted to, and then it can converge from one side.

Chris: It doesn't have to do that though. Actually, that's not problematic, but it doesn't correspond to all steps n, that's for sure.

I: You could split it into two subsequences, say c_n and b_n, one converging from below and one from above.

Chris: Yes, both converge, but it's just a matter of the correlation between each step and the terms of the sequence that's relevant to this problem. I mean the countability is still the same, but it's less clear that once you've completed all the steps you've converged on 1/3, whether or not you hit it. Whereas if you take it being a_1, a_3 and so forth converging this way, it's less to count through, sort of… intuitively. Either way, it works.

I: Just a quick question. So we can potentially split the sequence into two subsequences, where c_n contains the elements less than 1/3 and b_n with the elements greater than 1/3. Since the whole sequence, before the split, converges to 1/3, we know that any subsequence has to converge to the same number. Now if we think only of c_n, the one on the left [of 1/3], would that make the question easier? If my problem was that for 1/3, I'm building this sequence, c_n, converging to it from the left. Would that make it easier to decide whether 1/3 is in the final set than thinking about a_n, which jumps around 1/3?

Todd: Here's what I'm thinking. We made an argument for b=1. What was the actual wording of the argument?

Chris: I said that when all steps have been completed, $1/2^n$ is 0. So $1-1/2^n$ is 1 at that same point.

Todd: So we could say $b-c_n$ is 0.

Chris: Well, at infinity you could say that. That would get around the rationals, sort of. I buy it that rationals are in there [the final set]. All the rationals, not just pieces.

1:40:18 I: So does $b-c_n$, as a sequence, converge to 0? c_n is the sequence converging to b from the left.

Chris: It converges to 0, yes.

I: So then, can we use the argument you used for 1 [being in the final set] in order to say that b is reached in this [current] case?

Chris: well, it's not as obvious for me to perform because I have the assumption that $1/2^n$ is 0 at that step, but you can say that because the converging sequence is something of the form $k/2^n$… It's almost dependent on the $1/2^n$ case. $k/2^n$ is slightly more difficult to characterize. I can characterize $1/2^n$ as being 0, but what's 2 times that? I mean you could say it's 0, but theoretically you can also say that for some k that's really large, almost infinite… I don't know. It's just a matter that $1/2^n$ I can say it's 0, an infinitesimal point at 0. What I would do for $2/2^n$, $3/2^n$, I don't know. That's the problem I have with that argument I'm using. It's applicable in some cases. When I try to extend it, I don't have such an obvious path. I mean I can see it would converge for everything, but I don't know if that means it's actually there.

Chris: Well, intuitively, I would say the rationals are in there. I didn't think that last time, but I certainly think that now. I could even give some case to the irrationals although I don't really like that. I have no real reasoning to go any further with it than that, that's the problem. It's purely intuitive at this point, that the rationals [are in the final set] because of the convergence, but at the same time… I can't show it much further than that.

I: And Todd , what do you think about this, comparing his argument for 1 being in there with what you were suggesting, that $b-c_n$ converging to 0 as a reason for b being in the final set is the same argument as for 1 [being in the final set]?

Todd: It's the same argument.

I: So based on that, would you be as equally comfortable claiming that this b that c_n converges to has to be part of the final set as you were for b=1?

Todd: I mean, after all steps you're going to get a sequence that converges to b. That sequence is a self-representation of b, so we're going to say b is in there.

I: I'm not clear what you mean by "the sequence is a representation of b".

Todd: That's something I thought of just now. It's like the number .9999… is the number 1, but it's never actually 1.

Chris: Well, it's not 1 at any finite number of digits.

I: If you have infinitely many digits, then it is 1.

Chris: There's that infinite step that pushes it. But that's a series, this [in our problem] is a sequence.

I: Right, if you have a convergent series, then it does have a finite sum, and in that we can say that that series is a representation of that number. If you have a sequence whose term you're not summing, then I'm not sure what you mean by the "sequence is a representation of this number", other than "this is the limit of the sequence".

1:45:45 Todd: Right. Well, each element of the sequence is a sum of very small points. So like a_3 is 3/8, that's a sum of lengths.

Chris: You could do that. Fine.

I: Oh, so you want to look at this sequence as the partial sum sequence of another sequence?

Todd: But is the same as saying that this sequence has a limit that is itself a sum?

I: If you look at this sequence as a partial sum sequence, and if this sequence is convergent, then the associated series is convergent, equal to a finite number, so if before that's what you meant by "b is represented by this sequence", I can make sense of it that way. […] So does that clarify anything for you, Todd ?

Todd: So we get a series whose sum is 1/3. Each of these [pointing to the points in the sequence c_n converging to 1/3 from the left.] is an element in the final set.

Chris: Each term of the infinite sequence c_n is a partial sum.

I: Right. So the limit of the sequence c_n is the sum of the infinite series.

Todd: But we still haven't said anything about…

Chris: Well, in any case do you believe 1/3 is in the final set?

Todd: We could say that, but nothing we said so far made it [clear].

Chris: Yeah, I know, but you have to start with a tentative opinion. You can't go forward in finding a reason for or against if you don't pick one to try.

Todd: Fine, for the moment let's say 1/3 is in the [final] set of points.

Chris: Ok, why do you believe that?

Todd: We can make an argument for that because… [pause]. The infinite series is just representing what the sequence is at a particular step of the sequence, is that right?

Chris: Yeah, the partial sums are telling the sequence.

Todd: The fact that this sequence, you can think of it as an infinite sum of the series representing the points in the sequence does not mean that the limit of the sequence has to be in the final set of points. […] Well, using that thing we said before, 1/3 should be in there [final set].

I: A sequence is an ordered set, yes? If we're just going to write the set of points in the sequence, for example for b=0, then our set is numbers of the form$1/2^n$ where n is a natural number. 0 is not in here because I cannot write it as $1/2^n$ for n a natural number. This is the set of elements of the sequence So I guess it comes down to the phrasing of the problem, is it asking you only about the set of elements in the sequence, or does it refer also to points outside of the set of elements of the sequence?

1:56:16 Todd: The problem is, if you take it as a matter of interpretation, it becomes arbitrary.

I: Well again, if you formulate an additional assumption that maybe is not clear in that phrasing, and say based on this assumption I'm going to have to go with this answer and then you use that assumption consistently across the problems, I have no problem with that.

Todd: [Addressing Chris] So your assumption is you can't construct irrational numbers from fractions?

Chris: It was. You certainly can't construct it by finite steps out of rational numbers. But again, as you were talking, I was going back and forth. Well you could, and then you can't. In this particular case, I believe because it's one particular point, I believe it can be contained [in the final set]. In the other case [Midpoints Problem] I guess I would have to extend that a little bit, but still hearing that intuition saying no, you can't do that. But yeah, I guess I would have to say they're all in there by the convergence argument.

I: So if we agree that an irrational is not reached by a sequence of rational point at a finite step, how is 1/3 reached by the sequence of $k/2^n$ numbers?

Chris: Well, it [the sequence] converges to that value [1/3].

I: Right, but also does the process-constructed sequence for b= $\frac{\sqrt{2}}{2}$.

Chris: I know, that's why said it [b irrational] is in there [final set] this time. I flipped that [from last time]. It's still fighting me in my brain. I don't have a good argument against it to say no, it's not. […] No matter how you choose b, since we did b being 0, 1, $k/2^n$, rationals, irrationals, then b is in there, in set M.

I: Ok. And would you have a common argument for all of them?

Chris: No, I can't really make a common argument. I mean you can find a convergent sequence for each. In the case of b of the form $k/2^n$ that seems unnecessary. For everything else [0, 1, rationals, and irrationals], I can make that statement and it wouldn't be too problematic. Although for 0 and 1 I have something I like more, I can make the general argument that the convergence would work for all four.

1:59:59 I: So you mean the argument for 0 and 1 has more intuitive feel than the rest?

Chris: Yes, 0 and 1 has much more intuitive feel the way I argued it because of the assumptions I made in that case. As for the rationals and irrationals, I don't have a specific value for something, I assume infinity would give me that.

I: Ok. And Todd ? Are you sure about anything on this problem?

Todd: I'm sure that the points of the form $k/2^n$ are in there [final set]. 0 and 1, you can say they're in there.

I: Are you less sure than for $k/2^n$?

Todd: Yeah, it's like you're crossing a threshold here. For $k/2^n$ it's explicitly stated in the problem, this [for 0 and 1] it's based on an assumption. Based on the sequence convergence argument. Actually, all these [cases other than $k/2^n$] are using the convergence argument. That's why I'm saying all these things are the same, because if I'm using the same argument for all, the answer should be the same.

[End of session]

Session 7
12/12/08

0:00 [Start of recap of problems from past sessions. The students were given a list of all problems solved up to that point.]

[I bring up the Original Tennis Ball Problem and ask the students to recap the reasoning they provided for claiming Bin A was empty and Bin B contained all the balls after all steps had been completed. After they do, I point out that in order for this reasoning to hold water, the assumptions detailed in R1 and R2 need to be made. The two students agree. I encourage the students to go through the rest of the problems on the list and see if the reasoning they provided for each was in agreement with R1 and R2.]

11:00 I: I would be interested in hearing which of the rest of the problems you think are similar to this Original Tennis Ball Problem. I'm not saying similar in what way, so it's up to you what the relevant criteria are.

Todd: I think the second one is in there [The Original 10 Marble Problem].

Chris: I agree with that one, I've already seen that one.

I: So what do they have in common, what criteria are you looking at?

Chris: What I'm looking at is the fact that there is a step defined by the time $1-1/2^n$ at which you can say a marble with that label is no longer in the jar anymore [referring to The Original 10 Marble Problem]. That's the similarity I noted at least, and because you can find a time and a step for each marble to be removed, then it's a similar problem to the first one [The Original Tennis Ball Problem]. I also said that the jar was empty at time t=1 [referring to The Original 10 marble Problem], so the problem even has the same result as the result of the other problem.

I: In terms of the two rules we have here (R1 and R2), to claim the jar is empty in Problem 2, did you make use of any of these rules, and if yes which one?

Todd: R2?

Chris: I was going to say it, but yes.

I: Did you make use of R1 at any point?

Chris: No… I mean yes, the marbles are being removed [from the jar] and put into something technically, but that doesn't really have a bearing on the question of the problem.

I: Ok. And did you use any other rules that we should add to this list?

Chris: I think R2 is the only one required to say that, because there's a step that matches for everything.

16:05 I: Ok. So we got that one down. What else, can you find anything else [similar to these two]?

Chris: Well, I know the next one down the list that I have, but…

Todd: Number 5? [The n->n+1 Marble Problem]

Chris: Yeah, that's the one I saw. Ok, so number 5 it is.

I: This one is similar because…

Chris: This is the same as the problem for the timed marbles {Original 10 Marble Problem], except that instead of giving an explicit time for each step, it just says at step 2 place 2 in the jar and remove 1, at step 3 place 3 in the jar and remove 2. We're just doing these steps. And we're also not putting balls 1 through 10 in.

I: And what rules are we using for Problem 5 [The n->n+1 marble Problem]

Todd: The R2 assumption, because we can give a step at which a given marble will be removed and doesn't come back. So it's gone after all steps.

But I think we also need to clarify… You [addressing Chris] were talking about how this happens after… So the first marble is removed at step 2 [in The n->n+1 Marble Problem]. So this happens later than the natural numbers?

Chris: Oh, so it does not associate exactly with the step that it's numbered? [meaning the removal of marble labeled n is not done at step n].

Todd: Yeah, I think you mentioned that.

Chris: Yeah, I did.

I: Does that matter? In terms of how these principles are formulated (R1 and R2), it says "at a specific step", not "at a step that corresponds to the number on the marble". Do you think we should modify this principle, or leave it this way and not care about at which step, as long as it's a specific finite step?

Todd: That's a good question.

Chris: I'm of the opinion to leave it, because I don't think there's anything wrong with that. Still it's a specific step and it's finite. The fact that the number is offset by 1 that does not change anything.

Todd: Yeah, I agree.

20:20 I: Ok. What else, do we have anything else [in the category of problems similar to The Original tennis Ball Problem]?

Chris: Number 6 [The n->n+1 Label Problem]. A similar problem. Instead of marbles with numbers on them, it’s a marble in the jar with a label in it, and it’s similar except that the label is removed… same steps, except that instead of removing the marble and putting the next one in, it’s taking the same marble and putting a new label on it.

I: Uh-um. And what rules are we using for this one?

Chris: I believe we’re using both.

Todd: To say what?

I: Right, what was your answer for this problem?

Chris: That there was no label on the marble…

Todd: … but the marble is in the jar.

Chris: yeah, the marble is never removed from the jar.

I: Can you say specifically where you’re using R1 and R2?

Chris: R1 keeps the marble in the jar: it’s placed in, never leaves, so it will be there [after all steps]. And R2 is for the labels, because a label is removed at a certain step never to be put back.

I: Ok. So far, so good. What else?

Chris: I found a few more that I would consider…

Todd: You said we can use more than one criteria to compare.

Chris: Well, as long as it’s a similar problem.

I: Do you want to move on to a different category? I would like to stick to a set of criteria that will give me a partition of the set of problems.

Chris: I wasn’t really considering that.

I: [addressing Todd , who’s drawing a horizontal line below the list of problems similar to The Original Tennis Ball Problem] What’s that line, are you done with this category?

Todd: I’m already done with that category, yes.

Chris: I wasn't, but feel free to feel we're done. Well, do you find anything else similar?

Todd: If you want to use it [the set of two principles, R1 and R2] for the Modified Tennis Ball Problem?

Chris: I was going to use it for both Modified Tennis Ball Problems (version 1 and 2).

Todd: You can also use it for The 1+1/n Powder Problem to say, you never remove 1 ounce of powder so it will be there at the end.

Chris: I don't know if I'd use it.

I: Are we still in the same [first] category?

Chris: Yes.

Todd: The 1+1/n Marble Problem.

Chris: Yes, that's another one.

Todd: So for me, [problems] 8 [The 1+1/n Marble Problem], 10, and 11 [the two versions of the Modified Tennis Ball Problem].

24:38 I: What's 8?

Chris: 8 is the 1+1/n Marble Problem. Yeah, I mean there is something that makes it slightly different. It would require at least another inference rule. But it doesn't involve those at least, somewhat.

Chris: Yeah.

Todd: Well, you could, but I think the answer would be the opposite [from their previously agreed upon answer]

Chris: It wouldn't complete the answer.

I: So go ahead, how can you apply these two rules?

Todd: We ended up saying the jar contained the marble with label 1, right?

Chris: Yes, I said that.

I: And Todd agreed with you.

Todd: So we're using R2 to say that no marble with label 1+1/n is in the jar, because we can give you a step where that marble is removed and it's gone. But then to say that marble 1 is in the jar, we did some weird things.

I: Can you use R1 to claim that marble 1 is in the jar at the end?

Todd: Well, there's no finite step where a marble with label 1 is put in the jar, so we can't use that rule, I think.

Chris: Well, there certainly isn't a finite step. I believe the claim was that at infinity marble 1 was put in the jar and since there is no step beyond that, it wouldn't be removed. I'm not saying it's because of those rules [R1 and R2] necessarily. It does apply R2 in that all the marbles with labels 1+1/n are removed at specific finite steps. It does not give us the complete conclusion we had, but it does tell us that nothing else could be in there, except maybe that marble with label 1.

27:10 I: But for marble with label 1, can't we apply rule 2 again? Starting with step 1, let's look at step 1. At step 1, marble with label 1 is not in the set. For all subsequent steps, marble with label 1 is not in the set. Applying R2, that means marble with label 1 is not in the set after all steps. Is that a valid application?

Chris: It's a valid application… Well considering I was using a step at infinity to put that in, it's not really a finite step anyway. I don't think that actually conflicts too much with it.

I: So far, with those two rules that we have, you're saying we can't get the answer you guys had.

Chris: No.

I: What do you suggest we do?

Todd: I think the way we were considering it at the time was probably to separate the label 1 plus the label 1/n, so then you couldn't actually apply rule 2. That's cheap though. Well, the way we were visualizing was there was a marble with label 1, and then at each step you were sticking 1/n to that…

Chris: Well, that's one way to do it.

Todd: But that changes the problem. So let's say there is only one marble and it's in the jar. You can take it out each time if you want, but the number 1 is written on it, and at each step n you write on it "+1/n".

I: Do you feel that changes the problem?

Todd: I don't even know if that changes it. It doesn't change the fact that rule 2 still applies to it. So we haven't gotten around it.

I: You have to be careful about what kind of objects you're talking about. These rules are phrased in very general terms. When you apply them to specific problems, you need to define for yourself, what type of set is produced by this process? What kind of elements does it have? What kind of mathematical object are the elements? So you need to define what you're extracting from the problem, what the important information in the problem is.

31:05 I: So what I'm getting from this discussion is that if we want to continue to be able to apply rule 2 through all of these problems, if we apply it in this case it's telling us that marble with label 1 is not in the jar at the end, and marbles with labels 1+1/n are also not in the jar at the end.

Chris: Marbles with labels 1+1/n [are not in the jar at the end] certainly.

I: Right, so what are your objects here? If an object is a marble with a label with a number written on it, then any of the marbles we discussed fall in that category, including marble with label 1 and those with labels 1+1/n. Then we can apply rule 2 and claim that marble with label 1 is not in there. If you want to claim that marble 1 is in there, then you have to change the rules. We cannot have 3 rules where 2 of them contradict each other, then it's not a consistent theory. So we cannot just add a third rule. We would need to modify rule 2. If we modify it, then we would need to go back to problem 1 and see if we can still claim the same thing with the new set of rules. Not only problem 1 but all the ones you put in the same category.

Chris: I don't know, because it's not at a finite step that marble with label 1 is put in anyway. So I don't think rule 2 is going to contradict it. It's not removed at a finite step because it's not even placed in at a finite step.

[Discussion clarifying the phrasing of rule 2, and what conditions need to be fulfilled in order for it to be applied to an object.]

36:48 Chris: I got the idea of what that [R2] is saying now, so… as long as that can be said. So it's saying if you find a step at which it's not in there and it's never added to that. This is where it makes a difference to me because if I'm saying that it [marble with label 1] is added at the final step which is not a finite step, then I don't see how that [marble labeled 1 being in the jar at the end] is a contradiction [with the conclusion of R2].

I: I should have been more specific. When I say the object is not in the set from a specific step on, I'm referring only to finite steps.

Chris: Ok. That's certainly a different interpretation of R2 than I would like it to be.

Todd: You're saying that interpretation precludes some sort of infinite step?

Chris: It doesn't preclude an infinite step. Maybe it does actually. If it's all steps defined by the process, then the infinite step is contained therein. So that [clarification of R2] would kind of stop that in its tracks.

I: So all the steps that appear in these rules are finite steps.

Chris: Yeah… I mean the trick here I guess is that I always thought of it that the infinite step is the last step but that requires that the infinite step be contained in these problems. Which goes back to my interpretation that it is, but that in itself means it might not be consistent with the rules everyone else is going by right now.

I: We can consider changing these rules.

Chris: As far as the problems that have been discussed so far go, adding the step at infinity into this does not change it because I was already considering it in there, but depending on the way it's phrased, it might change it.

40:30 I: Let's see, if you want to change R2, how would that go?

Chris: I'm not certain about how I would go about saying this. Because the step at infinity, there is no step in this process beyond it. Even though it [marble labeled 1] is not added at a finite step, which would mean that R2 would apply it to it… it's added at that final step, but I don't know how exactly to go about that.

Todd ; Here's how I would go about it. R2 blah blah blah… is not in there, it's not added at any finite step, unless the infinite step puts the object in the resulting set [obtained after all steps].

I: Whenever the hypothesis of a theorem holds, then you can apply it and the conclusion is true. So you cannot say "unless", as that would contradict the conclusion of the previous part of R2.

Chris: So maybe "if an object is not in the set at step n and would not be added at the step at infinity"… but that's still kind of touchy though.

Todd: You need to define what that means though.

Chris: I know, it has to be clearly defined, which makes a lot of difficulty because the intuitive nature of it… as with all things that are intuitive to a person, are probably the hardest things to define.

I: See, here [in R1 and R2], whatever steps or objects I am referring to are specified by the problem and I know exactly what's happening at each step. If you want to say something is added "at infinity", then you need to explain in general terms how the process defines what is added at infinity, because I have no way of knowing what that means in a general context.

Todd: Here's my suspicion. You have a sequence of marble labels that at infinity would go to 1. Is that the basis for saying that the infinite step puts marble labeled 1 in the jar?

Chris: Because the limit is there?

Todd: Well, I don't you don't think of it that way.

Chris: For right now, until something else is placed there, it might be the only way I have to get around that.

Todd: Do you really put infinity as "n" and then say that's 1?

Chris: That's the way I've always thought of it. That's something I've been doing for so many years, I probably don't even know much algebra at the point I started doing that, so I don't really know where I got that anymore.

44:26 Todd: Ok. When we went into problem 8 [The 1+1/n Marble Problem], we had resolved problems 5 [The n->n+1 Marble Problem] and then 7 [The 1+1/n Powder Problem]. The thing is we can apply these rules [R1 and R2] to the Powder problem and come up with an answer, but we can't apply these rules to this problem and come up with an answer. It seems the combination of two things ends up being not what you'd expect.

Chris: I'm not against defining the step at infinity as a limit in process, as it's not entirely different from what I've been doing. But that would change R2 slightly. But as long as that would be the definition of what happens at the step at infinity, we can live with it. At least it would be defined. Now whether or not we'd have to change that definition at some point, that's another matter entirely.

45:35 I: Before you think about how to further change R2, I have a little exercise in imagination that I've just thought of. Let's assume that I'm going to have each of you handle the same process, at the same time. You [Chris] are going to work with red marbles, and you [Todd] are going to work with yellow marbles. Each of you has as many marbles as the natural numbers, but they're not labeled in any way, I am just telling you how many you have. Imagine that you have them arranged in front of you starting with a first, and going on indefinitely to the right. Now each of you is holding a jar, and each of you is in front of your first marble. The process is like this: you take the first marble, you put in the jar; then you make a step to the right, you toss out the marble in the jar and you put in the one in front of you. And so on. You do this at the same time, you with the red ones and you with the yellow ones. Now we assume you managed to complete all the steps. What do you think you have in the jar, each of you?

Chris: There wouldn't be anything in mine at least.

Todd: I guess so.

Chris: I see where this is leading, I think. If I can say there's nothing in there, then there is no step at infinity where it would be removed.

Todd: Well, there is no step where you have no balls in there.

I: If you wanted to claim you have an empty jar after all steps, do these rules [R1 and R2] help you?

Todd: Well, they're not labeled, but they're sort of labeled because they belong to the natural numbers associated with them.

Chris: [Mumbling to himself] There is a contradiction there, yes.

50:40 I: Now I want to make an addition to the game. You're going to be playing the same game, at the same time, except that I'm going to ask you to close your eyes. Then I'm going to put labels on the balls, but you don't see what the labels are. On one set of balls I'm going to put the 1+1/n labels, on the other set of balls I'm going to put the "1, 2, 3, …" labels. But you have your eyes closed, and you start playing the same game. Let's assume you're done performing all the steps, and you can open your eyes. What's in your jar, each of you?

Chris: I don't care what the labels are, I don't have anything in there and that does reject the argument I had before, and that's the end of that then.

Todd: Sounds good. There's nothing in both our jars.

I: And what if your eyes were open? Let's say you do it a 3rd time, you can look at the labels, but the rules of the game are the same. Does that make a difference?

Todd: I don't think it makes a difference.

Chris: I know. I've already come to terms with that just in the past minute, that that answer doesn't hold. Well, I guess in any problem where I was using a step at infinity, I'll just toss that out, because that step does not exist then.

52:56 Todd: Alright, let's say that there's two balls, a red ball and a yellow ball. And I arrange all the red balls in the even number positions, and the yellow balls in the odd numbered position. And then you go through them and put them in jar and take them out. Is that the same as the ½ Marble Problem?

I: Wait, so I have in front of me a sequence arranged as the natural numbers, one is red, one is yellow, one is red, one is yellow, like that? And you're playing the game I described before?

Todd: Right, and you have nothing in the jar at the end.

Chris: That unfortunately changes a lot of…

I: Yes, you have nothing in the end there. So what are you trying to say with that one?

Todd: It sounds like it's the same as the… No, it's not the same. The game is different, but really similar. Possibly the same, with the 1 and 2 labeled marbles going in and out of the jar [The ½ Marble Problem]. I don't know if it's the same though.

I: Chris, what do you think, are those two games the same?

Chris: Yeah, from what I'm hearing basically. Now if that applies and the step at infinity does not continue, then we can basically say that problems 1, 2, 3, 4, 5, 6, 8, 10, 11, 12…

Todd: But this game is flawed, I think, because…

Chris: Yeah, but those assumptions… all those answers [to the problems mentioned above], we can say that there is nothing there.

Todd: But when we think about the ½ Marble Problem, we can't apply that rule.

Chris: I mean by the rule you just defined, that if you take red, yellow, red, yellow, then it's always going to be removed at some step and you can figure out which one that specific one is removed…

Todd: But here's the difference, in the thing I proposed, we're using different balls, but in the ½ Marble Problem it's the same two balls. So there's no step at which either is removed.

Chris: It's not those, but by that they're all empty though, still. By what you've just said you've actually extended those rules without realizing it. By that rule, which seems to be there by what you said…

55:45 I: I think you guys are not talking about the same thing, so that's why I want to interrupt you, just to clarify. Todd gave that example with infinitely many balls where the yellows were even and the odds were red. If I'm going to perform the process as described, tossing each individual ball out of the jar at a specific time, then I can say the jar is empty at the end by using rule 2 [R2]. Because each object [marble] is uniquely identified by it position in the sequence and is an object different from the next one, which is at a different position in the sequence. And then you said, isn't this the same as The ½ Marble Problem? But then I think you followed up by saying, it's not the same because in the ½ Marble Problem we have only two objects and we keep switching between them, while in the other one we have infinitely many objects, even if we have only two colors. So in the latter we have a color that is repeating, but not an object that is repeating. So that's why we need to be careful about what objects we're looking at, is the object an individual marble or a color? So keeping that in mind, you need to decide if this problem [proposed by Todd , with the two colors] allows you to say that in the ½ Marble Problem there's nothing in the jar [at the end].

Chris: I don't think it lets you say that. This is where you seemed you were going with it but I don't believe it does that.

Todd: Yeah, I don't think it means anything [regarding the ½ Marble Problem].

58:15 I: Let's structure the discussion a bit. Let's finish with this 1+1/n Marble Problem. Where are you now?

Chris: I would say that there isn't something with the label 1 in there [jar] at that point. By R2, because there is no finite step at which it is added, so R1 can't be applied to it.

I: Ok, so R1 we can't apply, R2 we can apply. Todd ?

Todd: I agree.

I: Ok, let's go to this ½ Marble Problem since you brought it up.

I: Can you apply any of these rules [R1 and R2]?

Chris: Well, nothing gets in there without being removed… at a finite step.

Todd: No.

Chris: You can't, because either objects stays in or stays out.

I: Now what was your answer for this one, do you remember?

Todd: I think we said something silly like there is a marble but we don't know which label is on it.

Chris: We don't know which marble is in there, but there was one in there.

I: Is that consistent with these rules?

Chris: It doesn't really apply those rules.

Todd: If you apply them it doesn't tell you anything.

I: Ok, so how do you want to leave your answer?

Chris: I don't have a better answer for it, for what marble is in there.

Todd: Isn't it possible that there is no marble?

Chris: I think that because of the definition of the problem, there is always a marble being placed in before you remove the other one, so technically that situation is never a possibility. If you removed it first and then placed one in, it's possible. But in order for it to be empty at the end, you need to have somewhere in the step, even at a half-step, where there would be nothing.

1:02:15 I: You have two objects that are in contention here [for being in the jar at the end], right? In order to be able to claim you have nothing in the jar at the end , you need to be able to claim for each individual ball that it's not in the jar at the end. And the only tool we have for that is R2, and you're saying you cannot apply it.

Chris: Well, I was giving a less R1/R2 approach to it. At that point you would least need something, even if half-way between a step… if it's not empty at any particular point within a step, it can't be empty at the end.

I: Ok, so you're saying the order in which the addition/removal of balls is done…

Todd: How about the problem with infinitely many balls [red and yellow balls, alternating] if you put one in before removing the previous one... would you still say the jar is empty at the end?

Chris: No, because technically there is a way to run out of balls. But you cannot run out of 1, 2, 1, 2… There's always a ball to replace it with.

I: I have a question. So what was your answer right now?

Chris: That there is one marble, don't know what label it is.

I: Ok, so here's my question about that. So far, whatever answer you had was supported by these two rules. Right now we cannot apply any of these two these problem. But you're not saying that the final state is completely undetermined – you're saying there are some parts of it that are undetermined, but I know for sure there is a ball in the jar. You know "how many".

Todd: Well, it is.

I: You just said we cannot apply any of them [R1 and R2]…

Todd: Yes, to the question of "which one". But if you define as an object how many balls are in the jar, then for a given step there is a marble in the jar and for all subsequent steps there is a marble in the jar, so by R1 there is a marble in the jar at the end.

I: What is your object defined as?

Todd: The objects is like, the state of being one marble in the jar.

I: Ok, but those are different entities. You're saying the cardinality remains the same, even if the specific objects that give that cardinality change.

[…]

1:08:00 Chris: In any case, that inference that was suggested is inconsistent anyway. At the end of any step there is a marble in there. That's also true for Problem 1, 2, 5, 6… It's similar for all those. There is always a marble in the jar, and we still said they were empty for 5 and 6, I think.

Todd: So I guess we need to specify more clearly what an object is…

Chris: The thing is, there must be a way to define [problems] 5 and 6 to be empty so that being something in the jar at the end for the ½ Marble Problem is not going to interfere with an empty jar for problems 5 and 6. Either that, or our answer [for the ½ Marble Problem] is just inconsistent and can be worked on. […] It's either something different in the problem that makes it different, or we can't say both at the same time. […] Technically, the way I've been thinking about this I said that there is something in there. I didn't have a solid reason for it, but if that is inconsistent as we've seen, then I really have no reason to know whether it's nothing, something, or both object. I have no reason to say any of them. That's actually what I said the first time I was in here, that I had no idea. I settled on one just for the sake of… I said that but I didn't have any confidence.

I: Ok, so how do you want to phrase your answer now, after this discussion?

Chris: [addressing Todd] How do you want to phrase your answer?

Todd: Now we can't say anything.

Chris: That's where I'm sitting currently. I don't have a rule that I can use… I agree with the rules we uses for 1, 2, 5, 6, so I can't throw those out yet, but I have nothing to say then for problem 4 [1/2 Marble Problem].

I: Ok, so then we can just say that the final state is not determined by the process?

Chris: Well, not determined by "our process". By our foundation.

I: Ok. Are we in a new category [of problems] here?

Chris: Not unless we hit the modified tennis ball problems.

Todd: So we went through 1, 2, 5, and 6, and then looked at 8 and changed our minds. And now we say nothing anymore for Problem 4.

Chris: Yeah, which is a change.

1:13:00 I: Ok, so what did you want to discuss now, the modified tennis ball problems?

Chris: Yeah, that would be good. I'm not looking forward to certain problems in this set now, because I still have the same feel… I don't have a system that supports it. [Regarding the modified tennis ball problems]. As we went through them, I believe we said Problems 10 and 11 were the same.

Todd: No.

Chris: I believe they were though. Because it was swapping the contents of the bins versus swapping the labels of the bins, and I don't think either one of those is going to make this problem any different.

I: Just remember, be careful how you're defining the objects [mentioned in rules R1/R2]. What are your sets, what are the elements of the sets?

[…]

I: Do you remember what your answer was for each of these problems?

Chris: For both of these we said that the evens were in one bin, the odds in the other, and we didn't know which was which.

I: Ok, let's see what rules we can apply.

Chris: The problem with [applying] R1 and R2 right now is that the set changes. Now that I see this schema that we're building, I see that Problem 11 [Modified Tennis Ball Problem in which the labels of the bins are swapped at every odd step, but the contents of the bins are not swapped] is easily satisfied because the labels are changed, but the bin is still the same.

I: Ok, let's talk about that one [Problem 11] then.

Chris: The left-hand bin starts as A and gets balls 1, 3, 5, …. Although we know the odds are in the left-hand bin and the evens in the right one, but we don't know which labels are on the bin [at the end].

I: So which rules are applied for that answer? Todd ?

Todd: We can apply R1, for each left and right [bins].

I: So here there're are two different types of sets. One is sets of balls, actually two sets of balls, one for the left bin and one for the right bin, and then another set would be the label on the left, and the label on the right. So you're applying R1 for what?

Todd: The first set of sets, the balls itself.

I: And what are you doing for the labels?

Chris: We can't apply R1 or R2 so it's not defined. And there's no real extra axiom that would help unless we make up a ridiculous axiom.

1:18:23 I: Ok, so that was the second version of the Modified Tennis Ball Problem [Problem 11 in their task list]. What about the first one?

Chris: The first one, now we know that evens and odds are in a bin. [Addressing Todd] What can you say about this problem, without swapping labels [Problem 10]? Can you say where each ball is? I mean I think you can say where the balls are, you can't say whether they're in A or B and there's no left hand and right hand because they're being moved.

Todd: Well, we can't apply R1 or R2 to the contents of A and B.

Chris: To the contents of each bin, if we don't worry about the label. We can apply R1 to say all the odds are by themselves and all the evens are by themselves in a bin.

Todd: Ok.

I: Well, let's see to which set we're applying R1. Before we defined a set of balls as "the set of balls in the left bin". Are you keeping that definition or no?

Chris: We're taking an arbitrary definition and saying, all of the balls labeled even numbers and all the balls labeled odd numbers are in different groups, but we don't know what bin they're in. We just know that they're going to be in the same bin.

I: I'm still now sure how you're defining the set.

Chris: I know, I'd have to make another physical distinction, but that's not in the problem itself.

I: Right. Well, if you're not to try to change the problem, if you are to use the definition of sets from before… we have these two containers, they are stationary, the labels on them are stationary, so if we define the sets as one being what's in bin A and one what's in bin B, can we apply rules R1 and R2?

Chris: No, only to the balls if you take them out of that context.

I: So you can't apply them to the contents of an individual bin.

Chris: No. [Reading the text of the problem] Oh, "what are the contents of each bin at t=1"? But then you can't tell what is in A, what is in B, if we're going to take it at that general level.

1:23:21 I: So if we define the sets as the contents of the stationary bins, what are you saying about that? Todd ?

Todd: R1 and R2 don't apply, so the answer would be "undetermined".

I: Ok, do you want to move on to another problem?

Chris: I don't think there are any more of this particular type.

I: Did we discuss the [1+1/n] Powder problem?

Chris: I don't think that applies, because it's not a discrete object being removed. A piece of something is being taken away. Even though it could be made into a problem like that, the way it's formulated is not exactly that.

I: I guess you could look at 1 ounce of powder as an infinite collection of increasingly smaller pieces that are being removed

Chris: Yeah, and you could say how much is removed. But I mean, the jar in the 1+1/n Marble Problem being empty, if they were that similar, that would specify that there would be no powder left, which is obviously not the case.

I: Not really, because we do start with 2 ounces of powder. You can think about 1 ounce at the botTodd , and on top of it another 1 ounce, and this top one is the one from which we remove the quantities removed at each step.

Chris: And R1 would hold for that [botTodd 1 ounce] and R2 would hold for the bits of powder on top.

Chris: [addressing Todd] Do you want to rephrase the problem that way, or you want to classify it as something different?

Todd: I'd rather not classify it in the same category.

I: Ok, so we're starting a new category?

1:27:22 Todd: Can we talk about the lamp problem?

I: Oh, in that category is that?

Todd: Undetermined.

Chris: Based on what we said for problem 4 [The ½ Marble Problem].

I: Ok. So what about the Powder one, do you want to make that a new category?

Chris: Yeah, Problem 7. […] [Addressing Todd] What do you say is left after all steps?

Todd: 1 ounce of powder.

Chris: And why would you say that?

Todd: Didn't you say that if you sum up what you're removing, you get 1?

Chris: That's true.

Todd: So let's just use that.

Chris: That works, I like it. I mean that formulation with the set works too, as 1 [ounce of powder] never leaves.

[Both students agree with these two approaches to the Powder problem.]

1:31:03 I: Can we do the Infinite Vector Problem, to get rid of that one?

Chris: You mean to do it now?

I: Right. Do you want to put it into any of the existing categories?

Chris: No, certainly not. It's just going to be the zero vector [referring to the answer on which they agreed previously].

I: Why is that a different category, what's different about it?

Todd: You can think R1 I think.

Chris: Well, that requires sets, sort of, so that doesn't work. But you're altering the vector at each step so that in position n, everything up to that position is 0, and every position will be run through with a corresponding step, so they will be 0 by the end. I don't know how that's related… . I mean it's similar but it doesn't use the rules…

I: Well, if you want to be able to apply these principles, then you need to talk about some kind of set.

Chris: I guess you can say it's the set of positions in the final vector, but that is kind of strange.

Todd: Or the numbers of the positions of the zero elements in the vector. So that defines a set at each step n for which we can apply R1.

Chris: I guess you can define it that way. Trying to force it into a set-based idea is going to be very painful for me. I guess you can say it's the set of all positions in the vector that are value 0. And there is a step at which each position is added to that set.

I: Yeah, that seems to work. And what would you claim that set becomes after all steps?

Chris: All positions in the vector… all natural numbers.

Todd: We can do that.

1:37:47 Chris: Ok, Let’s see. Problems 9 and 13 through 16 remain.

13 [Midpoints] and 16 [Halves and Reals] are similar, 14 [Halves] and 9 [The z^n Problem] are the same, and 15 [Truncations] is kind of its own thing, so let’s knock 9 out of the way.

I: Ok, what do you want to start with?

Chris: [addressing Todd] Would you care to start with the vector types, the midpoint types, or the truncations?

Todd: So you think the Truncations are different from everything?

Chris: Well, do you see anything that it fits with? I guess you could say it almost fits with the same kind of argument we used for Problem 3 [Infinite Vector Problem]. Each position in the vector would be contained in the set… the set would contain all digits of π. It’s harder to define. It’s not quite as simple as that one.

I: Well, the problem does define a set [the set of points constructed by the completed process]. So after step 1 we have {3}, after step 2 we have {3, 3.1}, after step 3 {3, 3.1, 3.14}…

Chris: Yeah, I guess it doesn’t fit that much [with the situation in Problem 3].

I: If we’re talking about these sets, how would you apply the rules?

Chris: I don’t know how you would apply the rules here necessarily. It [π] certainly wouldn’t be removed but it would be added in the last step, so neither of these rules is really necessary for that.

I: Todd ?

Todd: Is the object a set, or is the object the elements of the set?

I: At each step you have a set of numbers, so after all steps have been completed you will have obtained a set of numbers.

Todd: I don’t know if you can apply R1 and R2 then.

Chris: I don’t think those apply to it necessarily, I mean they apply to it but they don’t apply for what we’re looking for.

Todd: So if π was in here for a finite step we could apply R1, but it’s not. Saying we can’t apply R1 for π doesn’t mean that it [π] is not in there.

I: But can we apply R1 for anything?

Chris: I mean for 3, for 3.1, for 3.14, for each truncation. We just can't do it for the question of, is π in the set at the end.

Todd: Well, the only way we can use R1 to say that π is in the set would be to say that π is there for some finite step, but we can't say that.

Chris: Well, even without R1 and R2, there was an argument that we used [in the past] that had π in there. We don't have to use only R1 and R2, that would be a really small set of axioms.

1:41:45 I: Can you apply R2 for anything?

Chris: Well, you can say that π is not in the set for a long time, but if we're arguing that it is in there [at the end], then there has to be something that says R2 does not apply to this.

I: Well, the way it is stated, R2 does apply to π to claim that it is not in there.

Chris: Because there is no finite step…

I: Right, exactly like we had the marble with label 1 not in the jar [in the 1+1/n Marble Problem].

Chris: If there was such a thing as the very last natural number, π would be defined on that step. If there was such a definition.

I: So again, if you want to claim that π is in the final set, you would have to modify R2.

Chris: Well, I still believe it's there, at actual infinity, but I'm stuck.

Todd: But it's not a question of at infinity, it's a question of what it means "after all steps".

Chris: After infinitely many steps have been completed, then an infinite number of digits [from π] would be revealed. Or infinite number of digits of π would be added to the set.

I: But it's not digits that are added to our set, it's full numbers.

Chris: Yeah, it's a number with infinitely many digits that's added.

I: When you say an infinite number of digits have been revealed and that is π, that's not the same as saying for any digit in the decimal expansion of π, there is a truncation that will have that digit and then all the truncations coming after it will have that digit. We agree on that much, right?

Chris: Yeah.

I: That just means that digit is reached and then contained by all subsequent truncations. But you're saying an infinite number of digits of π in the same number, which is different from having truncations that pass the position.

Chris: I'm just trying to say a truncation that involves an infinite number of digits, but that's not a truncation. […] I know the size of the natural numbers is equal to the size of the number of digits in π. So there should be a one-to-one correspondence that should be able to be completed. It should be added at the last step but I don't have such a definition. And that would unfortunately break R2. So the correct answer is going to be put on hold by a faulty R2.

1:44:56 I: Ok. Todd ?

Todd: For any specific digit in the expansion of π we can find a set that contains numbers that have that digit, but that's not the same thing as saying π is in there.

[…] I think you can change this problem so you end up getting π, but the way it is written you don't.

Chris: I believe it [π] is there but I can't justify it with R2.

I: How would you change it?

Chris: I don't know.

I: I think Todd actually proposed something last time. You were imagining that π was already written and you had a cover and at each step you were revealing a digit [moving to the right]. After all steps you could claim π was revealed. But see, here these digits are already part of the same number. But here [Truncations] I'm building different numbers, I'm building a collection of numbers, not only one number. […]

Chris: Ok.

I: So do you agree that if the problem was what Todd said, you would end up with π?

Both: Yeah, ok.

I: So see, here [Todd 's version] I'm building one single number, here [Truncations] I'm building a set. That's the main difference. Chris, do you feel that should give us π [in the context of Truncations]?

Chris: That's what you would get, I just don't have a way of giving that.

Todd: There's two separate questions. [Addressing Chris] One would be, using those rules [R1 and R2], do we definitely agree that using those, π is not in there?

Chris: If it has to happen at a finite step, something that is contained in the natural numbers, unless you argue that infinity is contained in the natural numbers, then no, it's going to work. Now do you argue that, is another matter.

1:48:55 Todd: I like these axioms – they are my friends.

I: What do you like?

Todd: R1 and R2.

Chris: I don't know what to say then. It [π] probably wouldn't be then because of those rules, but… I don't have a way to add a rule. I don't know how to alter it either without breaking mathematics.

I: How about another parallel game? While Chris performs the red ball game from before [described while discussing the 1+1/n Marble Problem], you [Todd] go through the process in this problem [Truncations]. As he performs step 1, you [Todd] perform step 2 in this problem, so you create set A_1={3}. So you're creating your sets, he's doing the marble replacement. After he is done with that thing and ends up with an empty jar, what do you end up with? What are all the numbers you created by that point?

Chris: By correspondence that should be that π is contained in that set, but that also requires that there was a last ball at that point [in my game].

I: [Addressing Chris] That you're handling?

Chris: Yes. There is no such ball, so that makes it unfortunate.

Todd: Right, 'cause we would need to have a last element in the set, but…

Chris: Correct, but there is no such theorem that says there is a greatest element in the set [of natural numbers].

I: Ok. So for the time being, do you want to leave this problem in any way?

Chris: Well, to stay consistent with the rules we have, I have to say no, it's not [π is not in the final set], but I don't like it. That's not right.

I: Ok, and Todd ?

Todd: Yeah, I mean… I like the argument you're making about the correspondence, but I don't know… I don't like this answer [π is no tin the final set], but I'll leave it that way.

I: If you were to rate your confidence and intuition in saying that π is not in the final set?

Todd: My intuition is nothing right now. Same for confidence.

Chris: I don't like it. I'm not confident in that solution, I don't even believe that solution, so very low levels for confidence and intuition. I don't have a better rule in the next few minutes.

I: Ok, I think we should leave it that way. For the remaining problems, I think the discussion will go in a similar manner.

Chris: Unfortunately. […] I don't know if we would get much further. We discussed those problems recently and I don't think we changed significantly from there, and if it broke one of those rules, we would probably be hard pressed to find a rule that would say, hey it's there. Even in another session I don't think I could figure out how to fix all these problems and those rules.

1:55:28 I: Let's talk only about one problem then, let's take the complex number problem [The z^n Problem].

Chris: I don't think it contains 0 anymore.

Todd: Dude, me too! Great.

Chris: See, I've already reversed my position on so many problems because of those rules, and because of them I would have to change the complex number problem to not include 0, so I'm going to say it for any other problem I have to say it for that one.

I: Ok. So Todd , why did you change my mind?

Todd: I don't like to change my mind. See, the consequence of saying this one does not contain 0 [The z^n Problem] would be similar to saying this one is not 0 [0 not part of the set produced in the Halves problem]. But the problem with that some of these questions suggest that fact that problem 14 would have to be 0.

I: Why is that?

Todd: Problem 2. The general form of the time is $t=1-1/2^n$, and then "after all steps" which is the question in Problem 14 is a similar question to asking, how many marbles are in the jar at time t=1 [referring to The Original 10 Marble Problem]?

Chris: Well, we didn't really invoke that, we used these rules and disregarded time when I solved it this time.

Todd: That's true. But time t=1 is equivalent to $t=1-1/2^n$ after all steps, right?

[More discussion about the difference between a process that edits one number, such as the time process, and a process that builds a set.]

Chris: In any case, for those two problems in my mind you have to say you can't include 0 by those rules [R1 and R2]. I'm saying that mostly because I don't have a rule that would fix that problem, currently. In fact I am fighting with it.

I: Ok, I understand that you both don't like that you had to change that. It might not be permanent but for the time being, if you're saying you're going to apply these rules to all problems, you need to say that 0 is not in the final set for these two problems [9 and 14].

1:59:45 Chris: The only other problems were the Midpoints and…

Todd: I changed my mind about Midpoints, I don't think you can construct rational numbers by this process.

Chris: By finite steps I guess you can't. You mean rationals that are not of the $k/2^n$ form. Which is where I started my position on that problem [laughs].

Todd: Right, and 0 and 1 either. It would just be $k/2^n$ where k is 1 through 2^n-1 and n is all the natural numbers.

I: Ok. […] And for the Reals and Halves problem, if b is not of the $k/2^n$ form, what is produced by that process?

Chris: It's the closest $k/2^n$ to b, at each successive step [the collection of those numbers].

I: Ok, so here are we using any of the rules, is this answer consistent with the rules?

Chris: Yeah, it's consistent.

I: Are we using R1 for anything here?

Chris: For the Midpoints, yes. For Halves and Reals don't really need to… I guess you do need to use it, fine. We're not using R2 'cause nothing is being removed from the set.

Todd: Well, we're using R2 to answer the question about b being part of the set produced by the process, when b is not of the form $k/2^n$.

[End]

Appendix F: **Zermelo-Fraenkel Axiomatic Set Theory (with the Axiom of Choice)**

1. Axiom of extensionality: Two sets are the same if and only if they have the same elements.
2. Axiom of empty set: There is a set with no elements. We will use {} to denote this empty set.
3. Axiom of pairing: If *x*, *y* are sets, then so is {*x*,*y*}, a set containing *x* and *y* as its only elements.
4. Axiom of union: Every set has a union. That is, for any set *x* there is a set *y* whose elements are precisely the elements of the elements of *x*.
5. Axiom of infinity: There exists a set *x* such that {} is in *x* and whenever *y* is in *x*, so is the union *y* U {*y*}.
6. Axiom of separation (or subset axiom): Given any set and any proposition P(*x*), there is a subset of the original set containing precisely those elements *x* for which P(*x*) holds.
7. Axiom of replacement: Given any set and any mapping, formally defined as a proposition P(*x*,*y*) where P(*x*,*y*) and P(*x*,*z*) implies *y* = *z*, there is a set containing precisely the images of the original set's elements.
8. Axiom of power set: Every set has a power set. That is, for any set *x* there exists a set *y*, such that the elements of *y* are precisely the subsets of *x*.
9. Axiom of regularity (or axiom of foundation): Every non-empty set *x* contains some element *y* such that *x* and *y* are disjoint sets.
10. Axiom of choice (Zermelo's version): Given a set *x* of mutually disjoint nonempty sets, there is a set *y* (a *choice set* for *x*) containing exactly one element from each member of *x*.

(Enderton, 1977)

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Acknowledgment of Previous Publications

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Radu, I. & Weber, K. (under review). Conceptual Changes in Mathematics Majors' Understanding of Completed Infinite Iterative Processes. Submitted to Educational Studies in Mathematics.

Curriculum Vita

Iuliana Radu

Education

October, 2009	Rutgers, The State University of New Jersey, New Brunswick, NJ Mathematics Education, Ph.D.
June, 2003	Bucharest University, Bucharest, Romania Applied Computer Science, M.S.
June, 2000	Bucharest University, Bucharest, Romania Mathematics, B.S.

Professional Experience

2006-2008	**Professional Development Instructor**. As a graduate assistant at the Center for Math, Science and Computer Education (Rutgers University), I led professional development workshops on topics concerning the teaching of mathematics using technology (audience K-12 in-service mathematics teachers)
2005-2006	**Teacher Supervisor**. Acting as a university supervisor for Rutgers University, I supervised pre-service mathematics teachers (high-school level) during their teaching internships
2003-2006	**Mathematics Education Teaching**. As a teaching assistant at the Graduate School of Education (Rutgers University), I assumed full responsibility for the course “Teaching Mathematics in the Elementary School” (for pre-service elementary school teachers)
2001-2003	**Mathematics Teaching**. As a Visiting Mathematics Instructor at Marlboro College (Vermont), I assumed full responsibility for the following courses and tutorials: Elementary Math Learning System (a basic skills course), Linear Algebra, Calculus I, Calculus II, Biostatistics, Real Analysis, GRE Strategies

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