PSEOUDOHOLOMORPHIC QUILTS AND KHOVANOV HOMOLOGY

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ABSTRACT OF THE DISSERTATION

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We generalize the symplectically-defined link homology theory developed by Paul Seidel and Ivan Smith to an invariant of tangles. We obtain a group-valued invariant, a functor-valued (or symplectic-valued functor) invariant and an A_∞functor-valued one for tangles. We provide evidence for the equivalence of this invariant with Khovanov's combinatorially defined invariant by showing the equivalence for flat (crossingless) tangles and their cobordisms. We also obtain an exact triangle for the Seidel-Smith invariant similar to that of Khovanov.

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Dedication

To the memory of my mother

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Chapter 1

Introduction

1.1 Background

The topic of this thesis lies at the intersection of low dimensional topology, representation theory and symplectic geometry.

Symplectic geometry studies phase spaces originally arising from physical systems. However in modern mathematics one studies more abstract phase spaces which arise as the moduli spaces of geometric structures on low dimensional manifolds. Low dimensional topology studies spaces of dimensions two, three and four up to topological equivalence and the embeddings of curves and surfaces into such spaces. In low dimensional topology one is interested in finding invariants which can tell topologically different spaces apart. As an example, knots are embeddings of the circle into the three dimensional space and links are embeddings of a finite number of circles into this space. Knot theory was not considered to be a genuine part of mathematics at first but since its relations to other branches of mathematics started to be discovered in nineteen-eighties, it has been the subject of extensive research. The reason behind these relations is that knotting, linking and braiding occur in various areas of mathematics such as representation theory, symplectic geometry, algebraic geometry and that is why there are link invariants coming from all these branches.

Every link can be obtained as the closure of a braid and braids form a group. That is how knot theory is related to representation theory. The first such connection was discovered by V. Jones [12] where he studied the representations of braid group on Hecke algebras and obtained a polynomial knot invariant, the celebrated Jones polynomial. Jonese's work inspired a flurry of activity whose notable outcomes were Witten's Chern-Simmons TQFT [41] and Reshetikhin-Turaev knot invariants [28]. The latter one obtains a knot invariant from the representation theory of any quantum group. In this framework the Jones polynomial corresponds to quantum $\mathfrak{sl}_2(\mathbb{C})$.

The introduction of categorification into low dimensional topology was another turning point in the subject. Categorification is the process of replacing sets-theoretic structures with category-theoretic ones. A category of objects potentially has more structure than a set of elements because there are morphisms between its objects. In the year 2000 M. Khovanov [14] introduced a link invariant which assigns a doubly graded vector space to a given knot. This invariant categorifies the Jones polynomial in the sense that its graded Euler characteristic equals the Jones polynomial. It turns out that Khovanov homology has more structure than Jones polynomial and is able to tell some links apart which the Jones polynomial is not able to. Another work in this direction was knot Floer homology [26] which categorifies the Alexander polynomial of a knot and extends to an invariant of three and four manifolds.

The main difference between these two instances of categorification is that Khovanov homology is given purely combinatorially but Heegaard-Floer homology was originally defined using Lagrangian Floer homology. A combinatorial description of knot-Floer homology was obtained later [20]. This raises the question of whether there is a geometric model for Khovanov homology or in other words categorification of the Jones polynomial using differential geometry. In addition to the phenomenological aspect of this question, finding such a geometric picture would make computation of Khovanov homology for families of knots possible and would bring extension of this invariant to three and four manifolds within closer reach.

Symplectic geometry possesses powerful tools which can be used in categorification. Among these are Lagrangian Floer cohomology, Fukaya categories and Mau-Wehrheim-Woodward functor. Floer cohomology was one of the first indications of the algebraic structure behind symplectic geometry. To each pair of Lagrangian submanifolds L, L' of a compact symplectic manifold, satisfying extra technical conditions, Floer cohomology assigns a graded abelian group $HF(L, L')$ whose Euler characteristic equals the intersection number of the two submanifolds. Therefore Floer cohomology categorifies intersection number. Fukaya category Fuk(M) of a symplectic manifold M is an algebraic object which contains information about the Floer chain complex of each pair of Lagrangian submanifolds of M. To any Lagrangian submanifold L of the product $M^- \times N$ of two symplectic manifolds, Mau, Wehrheim and Woodward [21] assign an A_∞functor $\Phi_L^\#$ from Fuk(M) to Fuk(N). This provides us with a tool for obtaining categorical invariants of manifolds as follows. Roughly speaking to each d-dimensional manifold X one assigns a symplectic manifold $M(X)$ which is the moduli space of geometric structures of a specific type on X. For example to each Topological surface one can assign the moduli space of Flat connections on it. Then for any $(d+1)$-dimensional cobordism Y between X_1 and X_2 one obtains a Lagrangian submanifold $L_Y \subset M(X_1)^- \times M(X_2)$ which consists of pairs of structures on X_1 and X_2 which extend over to Y. Since every d-dimensional manifold can be decomposed into elementary pieces using Morse theory, this method assigns a sequence of Lagrangian submanifolds, a Lagrangian correspondence, to each such manifold. One can either compute the Floer cohomology of this correspondence or apply the Mau-Wehrheim-Woodward construction and obtain an A_∞functor.

An attempt at constructing a symplectic-geometric categorification of the Jones polynomial, or in other words a geometric model for Khovanov homology, was made by P. Seidel and I. Smith [35]. Their model gives a knot invariant using symplectic geometry which is conjecturally equivalent to Khovanov homology. The essential ingredient in their construction is a representation

$$h : Br_m \to \pi_0(Symp(\mathcal{Y}_m)) \tag{1.1}$$

of the braid group into the symplectic mapping class group of certain algebraic varieties $\mathcal{Y}_m$. For a link K given as the closure of a braid $\beta \in Br_m$, the Seidel-Smith invariant is the Floer cohomology

$$HF(L, h_\beta(L))$$

of L and $h_\beta(L)$ where L is a specific Lagrangian submanifold of $\mathcal{Y}_m$.They prove that this group is independent of the choice of the braid representation β. Later C. Manolescu [17] gave a more explicit description of the invariant and equipped the chain complex with a second grading, showing that the Euler characteristic of this chain complex equals the Jones polynomial. However it is not known if this grading descends to a grading on cohomology. The representation (1.1) is in fact the monodromy representation of a fiber bundle

$$\chi : \mathcal{S}_m \to \mathrm{Conf}_{2m}$$

over the configuration space with fiber $\mathcal{Y}_m$. Here $\mathcal{S}_m$ is the transverse slice to the orbit under the adjoint action of a nilpotent matrix of the Jordan form (m, m) in the Lie algebra $\mathfrak{sl}_{2m}(\mathbb{C})$. Here we can see a connection to the Lie groups SL_n. The number of blocks in the Jordan matrix whose adjoint orbit is used, is related to the fact that the invariant one obtains is expected to be related to the quantum $\mathfrak{sl}_2$ link invariant. Ciprian Manolescu used this analogy to obtain link invariants which are expected to be related to $\mathfrak{sl}_n$ link invariants [18]. They specialize to the Seidel-Smith invariant when $n = 2$. There is a program for generalizing this construction to all Reshetikhin-Turaev invariants. See below.

There are two main problems with the above constructions. Firstly while Khovanov homology and algebraic categorifications of $\mathfrak{sl}_n$ link polynomials are doubly graded, the symplectic categorifications have only one grading. The conjectural relation expects the symplectic invariants to equal their corresponding algebraic invariant after the collapse of the bigrading on the latter one. The second problem is that this equivalence has not been established. Roughly speaking the symplectic invariants might have higher differentials coming from higher pseudoholomorphic polygons which are not present in Khovanov homology.

1.2 Foreground

In this thesis we construct a generalization of the Seidel-Smith invariant to even tangles. Tangles are, roughly speaking, knots with endpoints and the invariant assigned to them has more algebraic structure. An m-braid is a function from the set of m points to itself and Seidel and Smith assign a mapping to such a function. An (m, n) tangle is a correspondence between a set of m points and a set of n points and what we assign to it is a Lagrangian correspondence. To any elementary (m,n)-tangle T we assign a Lagrangian correspondence L_T between $\mathcal{Y}_m$ and $\mathcal{Y}_n$. If T is a braid, we assign to it the graph of the symplectomorphism h_T defined by Seidel and Smith. The elementary tangles other than braids are caps and cups. To a $(m,m+2)$ cap we assign a vanishing cycle over the diagonal $\Delta \subset \mathcal{Y}_m \times \mathcal{Y}_m$. The Lagrangian assigned to a cup is the transpose of this vanishing cycle. See section 5.1. Now any given (m,n)-tangle $\mathcal{T}$ can be decomposed into a composition of elementary ones

$$\mathcal{T} = T_k T_{k-1} \cdots T_1.$$

To $\mathcal{T}$ we assign the *generalized* Lagrangian correspondence

$$\Phi(\mathcal{T}) = (L_{T_k}, L_{T_{k-1}}, \cdots, L_{T_1})$$

between $\mathcal{Y}_m$ and $\mathcal{Y}_n$.We then prove that up to isomorphism of generalized correspondences, $\Phi(\mathcal{T})$ is independent of the decomposition of $\mathcal{T}$ into elementary tangles.

This way we obtain two invariants for each (m,n)-tangle T. The first one is a functor $\Phi_T^{\#}$ from the generalized Fukaya category of $\mathcal{Y}_m$ to that of $\mathcal{Y}_n$. The category used here is an enlargement of the Fukaya category of a Stein manifold to include a special class of noncompact Lagrangians. The second one is a graded abelian group, denoted $\mathcal{HSS}(T)$, which is, roughly, the Floer cohomology of $\Phi(T)$. For this second invariant to be well-defined we first have to deal with the compactness of the involved moduli spaces. The reason is that the Lagrangians assigned to caps and cups are not compact. We prove compactness using standard (but not very well-known) arguments on Lagrangians in manifolds with contact type boundary. In sections 2.9 and 2.10 we put together necessary tools for construction of Floer homology of noncompact Lagrangians in Stein manifolds. From these and the Functoriality Theorem of [39] we get the following.

Theorem 5.2.3. *$\mathcal{HSS}(T)$ is well-defined and is independent of the decomposition of T into elementary tangles.*

In Chapter 7 we provide further evidence for the equivalence of the two invariance. We prove that the two invariants are equal for flat tangles, we establish an exact triangle for the Seidel-Smith invariant and finally we construct a homomorphism from Seidel-Smith to Khovanov homology with flattened grading.

This work is of interest within symplectic geometry as well. The aforementioned conjecture would follow if one can endow the Fukaya category of the symplectic manifolds $\mathcal{Y}_m$ (which are Milnor fibers) with an extra piece of structure namely a second grading. A crucial property of the Fukaya category of these manifolds is that it is generated by finitely many elements. One can hope to obtain such a second grading by mimicking the construction of homological grading on Khovanov homology.

Joel Kamnitzer [13] has proposed a method for categorifying all link polynomials from quantum groups. In this picture, for a complex reductive group G, the symplectic fibration used by Seidel and Smith (which is in fact the adjoint quotient map) is replaced by a fibration whose total space is the Beilinson-Drinfeld Grassmannian. This Grassmannian is, roughly speaking, the moduli space of $G^\vee$-bundles on $\mathbb{P}^1$ which are trivial on the complement of a finite set of points. Here $G^\vee$ is the Langlands dual of G. When two such points approach each other, one has a similar situation to that of Seidel-Smith where two eigenvalues come together. Kamnitzer proves a local neighborhood theorem analogous to that of Seidel and Smith.

1.3 Organization of chapters

In Chapter 2 we review the fundamentals of Lagrangian Floer theory and its quilted version.

In Chapter 3 we review Khovanov's invariant for tangles from [15] and [14].

In Chapter 4 we review the construction of Seidel's and Smith's Floer theoretic link invariant from [35] and then in Chapter 5 we generalize this invariant to tangles.

In Chapter 6 we enhance the group-valued invariant of Chapter 4 into an A_∞functor valued invariant. In other words to each (m, n)-tangle we assign an A_∞functor between the Fukaya categories of the manifolds $\mathcal{Y}_m$ and $\mathcal{Y}_n$.

Finally in Chapter 7 we study the symplectic invariant further and provide new evidence for its equivalence with Khovanov invariant.

Chapter 2

Quilted Lagrangian Floer cohomology

In this chapter we review the definition of Lagrangian Floer cohomology from [3] and [23] and its quilted version from [38]. The book [22] is the standard reference for the case of closed curves. Let L_0, L_1 be two Lagrangian submanifolds of a symplectic manifold M. If some extra conditions (to be discussed below) are satisfied by M and L_0, L_1 then Floer cohomology assigns an abelian group $HF(L_0, L_1)$ to the pair L_0, L_1.

2.1 Lagrangian correspondences

Definition 2.1.1. *A symplectic manifold (M, ω) consists of a differentiable manifold M together with a closed nondegenerate differential two-form ω on M. We denote $(M, -\omega)$ by M^-.*

Lagrangian submanifolds are the most important class of submanfolds of a symplectic manifold. They provide the right setting for studying boundary value problems in symplectic manifolds.

Definition 2.1.2. *A Lagrangian submanifold of a symplectic manifold (M^{2n}, ω) is n dimensional submanifold of M such that the restriction of ω to L is zero.*

For example if $\phi : (M, \omega) \to (N, \omega')$ is a symplectomorphism, i.e. if $\phi^*\omega' = \omega$ then the graph of ϕ is a Lagrangian submanifold of $M^- \times N$. The notion of a symplectomorphism between two symplectic manifolds is a rather restrictive one. For example the two manifolds have to have the same dimension. Lagrangian correspondences provide a more flexible notion of correspondence between symplectic manifolds.

Definition 2.1.3. *A Lagrangian correspondence L between two symplectic manifolds M_0 and M_1 is a Lagrangian submanifold of $M_0^- \times M_1$. The transpose L^t of L is defined to be the same set regarded as a Lagrangian submanifold of $M_1^- \times M_0$.*

Just as symplectomorphisms, there is a notion of composition for Lagrangian correspondences. In fact there are two such notions.

Definition 2.1.4. *If $L_{0,1}$ is a Lagrangian correspondence between M_0, M_1 and $L_{1,2}$ is a correspondence between M_1, M_2 then the geometric composition $L_{0,1} \circ L_{1,2}$ is defined as*

$$L_{0,1} \circ L_{1,2} := \{(m, m'')|\exists m' \in M_1 s.t.(m, m') \in L_{0,1}, (m', m'') \in L_{1,2}\}$$

which is a subset of $M_0 \times M_2$.

Definition 2.1.5. *This composition is embedded if $L_{0,1} \times L_{1,2} \subset M_0 \times M_1 \times M_1 \times M_2$ intersects the diagonal $M_0 \times \Delta_{M_1} \times M_2$ transversely and the projection $\pi_{0,2} = \pi_0 \times \pi_2$ embeds the intersection into $M_0 \times M_2$. In this case the composition is a Lagrangian submanifold of $M_0^- \times M_2$. Here π_i is the projection onto M_i.*

Since both $L_{0,1} \times L_{1,2}$ and $M_0 \times \Delta_{M_1} \times M_2$ are Lagrangian submanifolds of $M_0^- \times M_1 \times M_1^- \times M_2$, the composition $L_{0,1} \circ L_{1,2}$ is a Lagrangian submanifold of $M_0^- \times M_2$ if it is embedded. In order to remedy the problem of Lagrangian correspondences whose composition is not embedded, Wehrheim and Woodward [39] introduced the notion of generalized Lagrangian correspondence.

Definition 2.1.6. *A generalized Lagrangian correspondence between symplectic manifolds M, M' consists of a sequence $M = M_0, M_1, \cdots, M_n = M'$ of symplectic manifolds and a sequence $\underline{L} = (L_{0,1}, \cdots L_{i,i+1}, \ldots L_{n-1,n})$ such that $L_{i,i+1}$ is a Lagrangian correspondence between M_i and M_{i+1}. In case $M' = pt$, we call $\underline{L}$ a generalized Lagrangian submanifold of M. We call $\underline{L}$ compact if each $L_{i,i+1}$ is compact.*

There is an alternative notion of composition for Lagrangian correspondences.

Definition 2.1.7. *The algebraic composition* $\underline{L}\#\underline{L'}$ *of two generalized Lagrangian correspondences* $\underline{L} = (L_k, L_{k-1}, \ldots, L_1)$ *from* M_0 *to* M_1 *and* $\underline{L}' = (L'_l, L'_{l-1}, \ldots, L'1)$ *from* N_1 *to* N_2 *is the correspondence* $(L_k, L_{k-1}, \ldots, L_1, L'_l, L'_{l-1}, \ldots, L'1)$ *from* M_0 *to* M_2.

According to [39] Section 2.2, the *symplectic category* is the category whose objects are monotone symplectic manifolds (including exact ones) and whose morphisms are equivalence classes of generalized Lagrangian correspondences. The equivalence relation on morphisms is generated by the following two relations. Firstly,

$$(L_0, L_{0,1}, \ldots, L_{i,i+1}, \ldots, L_{n-1,n}, L_n)$$

is equivalent to

$$(L'_0, L'_{0,1}, \ldots, L'_{i,i+1}, \ldots, L'_{n-1,n}, L'_n)$$

if each $L_{i,i+1}$ is Hamiltonian isotopic to $L'_{i,i+1}$ in $M_i^- \times M_{i+1}$. Secondly

$$(L_0, L_{0,1}, \ldots, L_{i-1,i}, L_{i,i+1}, \ldots, L_n)$$

is equivalent to

$$(L_0, L_{0,1}, \ldots, L_{i-1,i} \circ L_{i,i+1}, \ldots, L_n)$$

whenever the composition $L_{i-1,i} \circ L_{i,i+1}$ is embedded. The idea of symplectic category goes back to Weinstein [40] where he considered only Lagrangian correspondences as morphisms. Since the composition of two Lagrangian correspondences might not be embedded, he did not obtain a genuine category.

Definition 2.1.8. *A* $d+1$ *dimensional symplectic valued topological field theory is a functor from the category (d-manifolds, cobordisms) to the symplectic category. Here "cobordism" means cobordism modulo isotopy. A genus zero symplectic valued topological field theory is a functor from the category of* $d-2$ *dimensional submanifolds of* $\mathbb{R}^d$ *with boundary and cobordisms between them to the symplectic category.*

2.2 Maslov Index

Let $\mathcal{L}(n)$ be the set of all Lagrangian subspaces of $\mathbb{C}^n$. We have $\mathcal{L}(n) \cong U(n)/O(n)$.

Definition 2.2.1. *If $\gamma : S^1 \to \mathcal{L}(n)$ is a loop define its Maslov index to be*

$$\mu(\gamma) = \deg(\det^2 \circ \gamma).$$

One can assign a Maslov index to a pair of paths of Lagrangian subspaces as follows. This definition is due to Robbin and Salamon [30]. Let Λ be a path of Lagrangians in $\mathbb{C}^n$ such that

$$\Lambda(t) = \text{image}\, Z(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} : \mathbb{R}^n \to \mathbb{C}^n.$$

To Λ we associate the quadratic form given by

$$Q_{t_0}(\Lambda)(v) =< X(t_0)u, \dot{Y}((t_0)u > - < Y(t_0)u, \dot{X}(t_0)u > \tag{2.1}$$

for $v = Z(t)u$. Let Λ' be another path of Lagrangians in $\mathbb{C}^n$ such that

$$\Lambda'(t) = \text{image}\, Z'(t) = \begin{pmatrix} X'(t) \\ Y'(t) \end{pmatrix} : \mathbb{R}^n \to \mathbb{C}^n.$$

Definition 2.2.2. *A crossing is a t_0 such that $\dim \Lambda(t_0) \cap \Lambda'(t_0) > 0$. The relative crossing form of such a crossing t_0 is defined to be*

$$\Gamma_{t_0}(\Lambda, \Lambda') := Q_{t_0}(\Lambda)|_{\Lambda'(t_0)\cap\Lambda(t_0)} - Q_{t_0}(\Lambda')|_{\Lambda(t_0)\cap\Lambda'(t_0)}. \tag{2.2}$$

Definition 2.2.3. *The Maslov index of the pair Λ, Λ' is defined to be*

$$\mu(\Lambda, \Lambda') = \frac{1}{2} \operatorname{sign} \Gamma_{-\infty}(\Lambda, \Lambda') + \frac{1}{2} \operatorname{sign} \Gamma_{+\infty}(\Lambda, \Lambda') + \sum_{t\, crossing} \operatorname{sign} \Gamma_t(\Lambda', \Lambda). \tag{2.3}$$

Let E be a trivial $\mathbb{C}^n$ bundle over the strip $\mathbb{D} = [0,1] \times \mathbb{R}$. Let F be a subbundle of $E|_{\partial\mathbb{D}}$ with Lagrangian fibres. For any choice of trivialization Φ of E we get two maps $\gamma_0, \gamma_1 : \mathbb{R} \to \mathcal{L}(n)$ defined by $\gamma_i(t) = \Phi(F_{(i,t)})$.

Definition 2.2.4. *Define the Maslov index of (E, F) to be*

$$\mu(E, F) = \mu(\gamma_1, \gamma_2). \tag{2.4}$$

2.3 Pseudoholomorphic polygons

Recall that an almost complex structure on a manifold M is a field of endomorphisms of the tangent bundle of M, $J \in \Gamma(\mathrm{End}(TM))$, such that $J \circ J = -\mathrm{id}$. Let (M, ω) be a symplectic manifold.

Definition 2.3.1. *An almost complex structure J on M is compatible with ω if and $\omega(\cdot, J\cdot)$ is a Riemannian metric on M.*

Definition 2.3.2. *A surface with strip like ends $(S, \mathcal{E}^+, \mathcal{E}^-)$ consists of*

- *a Riemann surface S together with two sets $\mathcal{E}^+, \mathcal{E}^- \subset \partial\bar{S}\backslash S$ called outgoing and incoming ends*

- *for each outgoing end $\zeta \in \mathcal{E}^\pm$ a biholomorphic map*

$$\iota_\zeta : [0,1] \times \mathbb{R}^+ \to S$$

and for each incoming end $\zeta \in \mathcal{E}^\pm$ a biholomorphic map

$$\iota_\zeta : [0,1] \times \mathbb{R}^- \to S$$

such that ι_i^+ (resp. ι_i^-) maps the boundary of $[0,1] \times (0,\infty)$ to ∂S and

$$\lim_{t \to +\infty} \iota_\zeta(s,t) = \zeta$$

(resp. $\lim_{t\to-\infty} \iota_\zeta(s,t) = \zeta$). We assume that endpoints on each boundary component of S are ordered cyclically so for each endpoint ζ we have boundary components $c_+(\zeta)$ and $c_-(\zeta)$. We also assume that $\mathcal{E}^+(S) \cup \mathcal{E}^-(S) = \bar{S}\backslash S$.

Given such a surface, a *family of almost complex structures* parameterized by S is a map J that assigns a C^∞ almost complex structure $J(x)$ on M compatible with ω to each point $x \in S$ such that

$$(\iota_\zeta^* J)(s,t) = J_\zeta(t) \tag{2.5}$$

does not depend on s. Let $c_1, \ldots, c_m$ be the boundary components of S and let $L_1, \ldots, L_m$ be Lagrangian submanifolds of M.

Definition 2.3.3. *A J-holomorphic m-gon with boundary condition $L_1, \ldots, L_m$ is a map $u \in W^{1,p}(S, M)$, for some fixed $p > 2$, satisfying*

U1) $J(u(x)) \circ du(x) = du(x) \circ j$ *for all* $x \in S$

U2) $u(c_i) \subset L_i$, *for* $i = 1, \ldots, m$.

The case $m = 2$ is of special importance. Let $\mathbb{D}$ denote the strip $[0,1] \times \mathbb{R}$. The equation (2.5) translates into the fact that $J = J(t)$ is a path of C^∞ almost complex structures on M compatible with ω. Denote by $\mathcal{J}$ the set of all such paths on M. The equation **U1** translates into

$$\partial_s u(s,t) + J(t)(u(s,t))\, \partial_t u(s,t) = 0.$$

Let L_1, L_2 be two lagrangian submanifolds of M. We assume L_1 and L_2 intersect transversely. If both Lagrangians are compact then there is a Hamiltonian symplectomorphism of M which makes L_1 and L_2 transverse to each other. Let $x, y \in L_1 \cap L_2$. A J-holomorphic strip joining x and y is a map $u : \mathbb{D} \to M$ satisfying **U1** and **U2** as well as

U3) $\lim_{t\to\infty} u(s,t) = y$, $\lim_{t\to-\infty} u(s,t) = x$.

Each such u induces a map $v : \mathbb{D}^2 \to M$ which is J-holomorphic on the interior of $\mathbb{D}^2$ and $v(\sqrt{-1}) = y, v(-\sqrt{-1}) = x$. Let $\mathcal{M}(x, y, J)$ denote the set of all such J-holomorphic strips. It inherits a topology as a subset of $W^{1,p}(S, M)$. To each such u one can assign a Maslov index (See Section 2.2)

$$\mu(u) = \mu(u^*TM, F) \tag{2.6}$$

where F is given by the pullback of TL_1 and TL_2 to $\partial\mathbb{D}$. Let $\mathcal{M}_k(x, y, J)$ denote the subset of $\mathcal{M}(x, y, J)$ containing elements of Maslov index k.

Definition 2.3.4. *The area of u is*

$$A(u) = \int_{\mathbb{D}} u^*\omega.$$

The energy of u is

$$E(u) = \int_{\mathbb{D}} ||du||^2$$

where the norm is given by the metric $\omega(\cdot, J\cdot)$.

2.4 Gromov Compactness

Let J be of class C^∞ and let $\{u_n\}$ be a sequence in $\mathcal{M}(x, y, J)$ with bounded area. As before we denote the (pointwide) norm of the one form du in the metric $\omega(\cdot, J\cdot)$ by $||du||$. If $||du_n||$, as a function on $\mathbb{D}$, is uniformly bounded it follows from elliptic bootstrapping that a subsequence of u_n converges (in C^∞ topology) on compact subset of $\mathbb{D}$ to a J-holomorphic map (See e.g. Theorem B.4.2 in [22]). In general one just has a bound on the energy of the curves and so $||du_n||$ might not be uniformly bounded at all points. This results in the bubbling phenomenon.

Definition 2.4.1. *Call a point $z_0 \in \mathbb{D}$ singular if there is a sequence of points $z_n \to z_0$ such that $||du_n(z_n)|| \to \infty$.*

Let S_1 (resp. S_2) be the set of singular points on the interior (resp. boundary) of $\mathbb{D}$. If $z_0 \in S_1$ (resp. S_2) define

$$v_n(z) = u_n(z_n - z/c_n) \tag{2.7}$$

on a neighborhood $B_r(0)$ in $\mathbb{C}$ (resp. $\mathbb{H}$). We have $||dv_n(z)|| = ||du_n(z_n - z/c_n)/c_n||$ so with an appropriate choice of c_n, v_n has uniformly bounded derivative on $B_r(0)$ so converges to a curve $v : B_r(0)(resp. B_r(0) \cap \mathbb{H}) \mapsto M$. It follows from the invariance of area that $v(1/z)$ has finite area so by removal of singularities theorem its singularity at 0 is removable so v is defined on whole of S^2 (resp. $\mathbb{D}$). It can be seen that

$$E(v) = \lim_{\epsilon \to 0} \lim_{n \to 0} \int_{B_\epsilon(z_0)} v_n^* \omega. \tag{2.8}$$

We denote this quantity by $E(z_0)$.

If $S = S_1 \cup S_2$ denotes the set of all singular points for u_n then $||du_n||$ is uniformly bounded on $\mathbb{D} \backslash S$ so a subsequence of u_n converges in C^∞ topology to a holomorphic curve $u : \mathbb{D}\backslash S \to M$ on compact subsets of $\mathbb{D}\backslash S$. We have $E(u_n) = \int_{\mathbb{D}\backslash B_r(S)} u_n^*\omega + \int_{B_r(S)} u_n^*\omega$ so

$$\lim E(u_n) = E(u) + \sum_{z \in S} E(z).$$

Since there is a $\hbar > 0$ such that each holomorphic sphere in M has energy greater than or equal to $\hbar$ ([22], Proposition 4.14), S is a finite set. The limit map can be thought of as a set of maps $(u, v_1, \ldots, v_k)$ where $k = \#S$ and $u : \mathbb{D}\backslash S \to M$, $v_i : S^2 \to M$ are J-holomorphic. The maps v_i are the rescaled maps (2.7).

There is another phenomenon arising, from noncompactness of $\mathbb{D}$, called broken flow lines.

Theorem 2.4.2 ([4]). *If $\{u_n\}$ is a sequence of elements of $\mathcal{M}(x, y, J)$ of bounded energy then there are*

- *a finite number of points $x_1, \ldots, x_{l-1} \in L_0 \cap L_1$,*
- *a finite subset $S = \{z_1, \ldots, z_k\}$ of singular points in $\mathbb{D}$.*
- *sequences $\{t_n^\alpha\}$ for $\alpha = 1 \ldots l$*

such that $u^\alpha(s, t) := u(s, t + t_n^\alpha)$ converges (in C^∞ topology), on any subset of $\mathbb{D}$ of the form $K\backslash\{z_1, \ldots, z_k\}$ with K compact, to an element of $\mathcal{M}(x_{\alpha-1}, x_\alpha, J)$ where $x_0 = x, x_l = y$.

The maps u^α are called broken flow lines. Using a gluing argument (as in Chapter 10 of [22]) one can show that for any broken flow line or strip with bubbles there is a seqeunce of the elements of $\mathcal{M}(x, y, J)$ converging to that curve in the sense of the Lemma 2.4.2 above. Therefore one can compactify the topological space $\mathcal{M}(x, y)$ by adding such limit curves.

2.5 Transversality

This section is devoted to the proof of the following theorem.

Theorem 2.5.1. *There is a subset $\mathcal{J}_{reg}$ of second category in $\mathcal{J}$ such that $\mathcal{M}_k(x, y, J)$ is a (noncompact) smooth manifold of dimension k for $J \in \mathcal{J}_{reg}$*

Let $\mathcal{B}$ denote the set of all $u \in C^\infty(\mathbb{D}, M)$ such that u satisfies **U2** and **U3**. Let $\mathcal{B}^{k,p}$ be the set of all $u \in W^{k,p}(\mathbb{D}, \mathbb{M})$ satisfying **U2** and **U3**. The tangent space to $\mathcal{B}$ at u is $T_u\mathcal{B} = \Omega^0(\mathbb{D}, u^*TM)$. Let $\mathcal{J}^l$ be the set of all paths of almost complex structures of class C^l on M. The tangent space $T_J\mathcal{J}^l$ to $\mathcal{J}^l$ at a point J can be identified with the set of all paths $Y : [0, 1] \mapsto C^l(\mathrm{End}\, TM)$ satisfying the following two conditions:

$$\omega(Yu, v) + \omega(u, Yv) = 0$$

and

$$Y(t)J(0) + J(t)Y(t) = 0$$

for all $t \in [0, 1]$. The set $\mathcal{J}^l$ has the structure of a Banach manifold with a chart at a point J is given by $Y \to J \exp(-JY)$.

Let $g_J(u, v) = \frac{1}{2}(\omega(u, Jv) + \omega(v, Ju))$ and ∇ be its Levi-Civita derivative. The modified derivative $\tilde{\nabla}_v X = \nabla_v X - \frac{1}{2}J(\nabla_v J)X$ is J-linear. If $u \in \mathcal{B}$ and ξ is a section of u^*TM define

$$\Phi(\xi) : u^*TM \mapsto \exp_u(\xi)^*TM$$

to be the parallel transport along geodesics $\gamma(s) = \exp_{u(z)}(\xi(z))$. Set

$$\mathcal{F}_u(\xi) = \Phi_u^{-1}(\xi)\, \bar{\partial} \exp_u(\xi)$$

and define

$$D_u : \Omega^0(\mathbb{D}, u^*TM) \mapsto \Omega^{0,1}(\mathbb{D}, u^*TM)$$

by $D_u\xi = d\mathcal{F}(0)\xi$. Define $\mathcal{J}^l_{reg}$ to be the set of all J of class C^l such that for every J-holomorphic strip u, the operator D_u is surjective and denote by $\mathcal{J}_{reg}$ the intersection of all these sets. An application of the inverse function theorem shows that for $J \in \mathcal{J}_{reg}$, $\mathcal{M}(x, y, J)$ is a finite dimensional manifold. Define

$$\mathcal{M}(x, y; \mathcal{J}^l) = \{(u, J) \in \mathcal{B}^{k,p} \times \mathcal{J}^l | u \in \mathcal{M}(x, y; J)\}.$$

We use the following facts in the proof of 2.5.1.

Theorem 2.5.2 (Oh [24], Floer-Hofer-Salamon [5])). *Let $u \in \mathcal{M}(x, y)$ then the set Θ_u consisting of all $(s, t) \in \mathbb{D}$ for which*

- *$u(\{s\} \times [0,1])$ intersects $u(\mathbb{R} \backslash \{s\} \times [0,1])$,*
- *$u(\{s\} \times [0,1]) \cap (L_0 \cap L_1) = \emptyset$ and*
- *$\frac{\partial u}{\partial s}(s,t) \neq 0$ for all $t \in [0,1]$*

is open and nonempty in $\mathbb{D}$.

Theorem 2.5.3. *(Unique Continuation) If $u_1, u_2 : B_\epsilon(0) \mapsto \mathbb{C}^n$ satisfy $\partial_t u_i + J\partial_s u_i = 0$ where J is an almost complex structure and $u_1 - u_2$ vanishes to infinite order at 0 then $u_1 = u_2$ on B_ϵ.*

Proof. Consider the bundle E on $\mathcal{B}^{k,p} \times \mathcal{J}^l$ whose fiber at (u, J) equals

$$L^p(\Omega^{0,1})(\mathbb{D}, u^*TM)$$

and the section $\mathcal{F}(u, J) = \bar{\partial}_J u$ of E. We have to show that the differential of this section given by

$$D\mathcal{F}(u, J)(\xi, Y) = D_u\xi + \frac{1}{2}Y(u) \circ du \circ j$$

is injective at points of the intersection of the $\mathcal{F}$ with the zero section i.e. at points (u, J) where $\mathcal{F}(u, J) = 0$. Because D_u is Fredholm, it is sufficient to show that the image of $D\mathcal{F}$ is dense whenever $\mathcal{F}(u, J) = 0$. If this is not true then by Hahn-Banach theorem there is a nonzero L^q section η of $\Omega^{0,1}(\mathbb{D}, u^*TM)$ such that

$$\int_{\mathbb{D}} < \eta, D_u\xi >= 0$$

and

$$\int_{\mathbb{D}} < \eta, Y \circ du \circ j >= 0$$

for any ξ and Y so $D_u^*\xi = 0$. We show that $\eta(z_0) = 0$ for any $z_0 = (s_0, t_0) \in \Theta_u$ so by unique continuation theorem 2.5.3, η is identically zero. If this is not true then because $du \neq 0$ there is $Y_0(t)$ such that $< \eta(z_0), Y_0(t_0) \circ du(z_0) \circ j >> 0$. There is no other $t \in [0,1]$ such that $u(s_0, t_0) = u(s_0, t)$ so the inverse image $u^{-1}(u(z_0))$ consists of finitely many points $\{z_0 = (s_0, t_0), z_1, ..., z_k = (s_0, t_k)\}$ where the s_i are distinct. Let ρ be a bump function on $[0,1]$ concentrated in a small neighborhood V of t_0. Then $\int_{\mathbb{D}} < \eta, \rho Y_0 \circ du(z_0) \circ j >> 0$ which is contradiction. $\square$

Lemma 2.5.4. *If $\pi : \mathcal{M}(x, y; \mathcal{J}^l) \to \mathcal{J}^l$ is the projection on the second factor and $(u, J) \in \mathcal{M}(x, y; \mathcal{J}^l)$ then $\ker D_u = \ker d\pi(u, J)$ and so $\mathcal{J}^l_{reg}$ consists of regular values of π.*

Proof. The tangent space to $\mathcal{M}(x, y; \mathcal{J}^l)$ at (u, J) consists of (ξ, Y) such that $D_u\xi + \frac{1}{2}Y(u) \circ du \circ j_{\mathbb{D}} = 0$ and $d\pi_{(u,J)}(\xi, Y) = Y$.

$\square$

Corollary 2.5.5. *For l large enough, $\mathcal{J}^l_{reg}$ is of second category in $\mathcal{J}^l$.*

Proposition 2.5.6. *If $p > 2$ then $\mathcal{M}(x, y; \mathcal{J}^l)$ is a separable Banach submanifold of $\mathcal{B}^{k,p} \times \mathcal{J}^l$.*

Taubes' argument We now generalize the corollary 2.5.5 to the C^∞ case. Define $\mathcal{J}_{reg,k}$ to be set of all paths of almost complex structures J such that if u is any J-holomorphic strip with $\|du\|_{L^\infty} \leq k$ then D_u is onto. We have $\mathcal{J}_{reg} = \bigcap_{k>0} \mathcal{J}_{reg,k}$. We will show that $\mathcal{J}_{reg,k}$ is open and dense in $\mathcal{J}$ for any k so $\mathcal{J}_{reg}$ is of second category in $\mathcal{J}$. First note that $\mathcal{J}_{reg,k}$ is open in $\mathcal{J}$ for if J_n is a sequence in the complement of $\mathcal{J}_{reg,k}$ then there is a sequence of J_n-holomorphic curves u_n such that $\|du_n\| \leq k$ and D_{u_n} is not onto. Now a subsequence of u_n converges to a J-holomorphic curve u where J is the limit of (a subsequence of) J_n. The map u satisfies $\|du\| \leq k$ and D_u which is the limit of D_{u_n} is not surjective.

Now we prove that $\mathcal{J}_{reg,k}$ is dense in $\mathcal{J}$. Let $J \in \mathcal{J}$ by corollary 3.3 there is a sequence $J^l \in \mathcal{J}^l{}_{reg} = \bigcap_{k'>0} \mathcal{J}^l{}_{reg,k'}$ such that $\|J - J^l\|_{C^l} < 2^{-l}$ for l large enough. Now because $\mathcal{J}^l_{reg,k}$ is open in C^l topology, there is a r_l such that if $\|J'_l - J_l\|_{C^l} < r_l$ then $J'_l \in \mathcal{J}^l_{reg,k}$. Choose J'_l to be C^∞. So $\|J'{}_l - J_l\|_{C_l} < \min\{r_l, 2^{-l}\}$ and $J'_l \in \mathcal{J} \cap \mathcal{J}^l{}_{reg,k} = \mathcal{J}_{reg,k}$.

2.6 Orientation of the moduli spaces

If the Lagrangians L_0, L_1 have some extra structure one can equip the manifolds $\mathcal{M}(x, y)$ with an orientation.

Definition 2.6.1. *A spin structure on an orientable manifold L^n is a principal $Spin(n)$ bundle $\tilde{P}$ and a two-fold covering map $\pi : \tilde{P} \to P$ where P is the principal $SO(n)$ bundle associated to TM. The map π is required to fit into the following commutative diagram.*

$$\begin{tikzcd} Spin(n) \arrow[r] \arrow[d] & \tilde{P} \arrow[d] \\ SO(n) \arrow[r] & P \end{tikzcd}$$

A manifold admitting a spin structure is called a spin manifold.

For an orientable manifold L to be spin it is necessary and sufficient that the second Stiefel-Whitney class of L vanish. In particular if L admits an almost complex structure whose first Chern class vanishes

Theorem 2.6.2 ([7],[34]). *If Lagrangians L_0, L_1 are spin then the manifolds $\mathcal{M}(x, y)$ are orientable. Moreover each pair of spin structures on L_0, L_1 induces an orientation on the moduli spaces in a canonical way.*

This theorem is also valid in the case of k Lagrangians and the moduli space of pseudoholomorphic k-gons with boundary on the Lagrangians (Section 11 in [34]).

2.7 Definition of Lagrangian Floer cohomology

There is an action of $\mathbb{R}$ on $\mathcal{M}_k(x,y,J)$ given by $u(s,t) \to u(s,t+\tau)$. The action is free for $k>0$. This follows from unique continuation 2.5.3. Define

$$\widetilde{\mathcal{M}}_1(x,y,J) = \mathcal{M}_1(x,y,J)/\mathbb{R}.$$

We impose the following two assumptions on L_0, L_1.

L1) Monotonicity: there exists a $c>0$ such that for any smooth disc u with boundary on L_i,

$$\mu(u) = c \cdot A(u)$$

where $A(u) = \int_{\mathbb{D}} u^*\omega$ is the area of u and μ is its Maslov index.

L2) The image of $\pi_1(L_0)$ in $\pi_1(M)$ is torsion.

Lemma 2.7.1. *Under the assumptions **L1** and **L2** all elements of $\mathcal{M}_k(x,y,J)$ (for k fixed) have the same energy.*

Proof. If $u_1, u_2 \in \mathcal{M}_k(x,y,J)$ then $\gamma = u_1(0,.) * u_2(0,.) \in \pi_1(L_0)$. It follows from **L2** there is a disc D in M and a number l such that $\gamma^l = \partial D$. Now D together with u_1 and u_2 gives a disc v with boundary on L_1. By **L1** we have $\mu(v) = cE(v) = c(E(u_1) - E(u_2) + lE(D))$ but $\mu(v) = \mu(u_1) - \mu(u_2) + k\mu(D) = k - k + lcE(D)$. So $E(u_1) = E(u_2)$. □

Corollary 2.7.2. $\widetilde{\mathcal{M}}_1(x,y,J)$ *consists of finitely many points.*

Proof. If $\{u_n\}_{n=1}^{\infty}$ is a sequence in $\widetilde{\mathcal{M}}_1(x,y,J)$ then by Gromov compactness, u_n converges to another element of $\widetilde{\mathcal{M}}_1(x,y,J)$ because, by monotonicity, a broken flow line has Maslov index ≥ 2 and so does a sphere bubbling off. No discs could bubble off because by 2.7.1, elements of $\widetilde{\mathcal{M}}_1(x,y,J)$ have minimal energy and the energy is additive. So $\widetilde{\mathcal{M}}_1(x,y,J)$ is compact zero dimensional and hence finite. □

Set $CF(L_0, L_1) = \mathbb{Z}/2\mathbb{Z} < L_0 \cap L_1 >$. If L_0 and L_1 satisfy the assumptions of Theorem 2.6.2 we set

$$CF(L_0, L_1) = \mathbb{Z} < L_0 \cap L_1 > .$$

Define $\partial^J : CF(L_0, L_1) \to CF(L_0, L_1)$ by

$$\partial^J x = \sum_{y \in L_0 \cap L_1} \#\widetilde{\mathcal{M}}_1(x, y, J) y.$$

In the $\mathbb{Z}/2\mathbb{Z}$ case the count is mod 2. In the oriented case $\widetilde{\mathcal{M}}_1(x, y, J)$ is oriented and zero-dimensional and the count is the sum of the signs of its elements.

Theorem 2.7.3. *If L_1, L_2 satisfy* ***L1*** *and* ***L2*** *and in addition the minimal Maslov number of L_1 and L_2 is greater than or equal to 3 then $\partial^J \circ \partial^J = 0$.*

In such a case we define $HF(L_1, L_2, J) = \ker \partial^J / \operatorname{im} \partial^J$.

Proof. We have

$$< \partial\partial x, z >= \sum_{y \in L_0 \cap L_1} \#\widetilde{\mathcal{M}}_1(x, y, J) \cdot \#\widetilde{\mathcal{M}}_1(y, z, J).$$

By Gromov compactness $\widetilde{\mathcal{M}}_2(x, z, J)$ could be compactified to a manifold

$$\overline{\mathcal{M}}_2(x, z, J)$$

such that $\partial\overline{\mathcal{M}}_2(x, z, J)$ consists of broken flow lines or discs bubbling off. If minimal Maslov number is ≥ 3, it could be shown no discs could bubble off: The case $x \neq z$ follows from monotonicity and additivity of the Maslov index. If $x = z$ then in the limit we might obtain a constant strip at x and a disc with boundary on L_0 or L_1 through x. The dimension of the set of all such discs is the minimal Maslov number of L_i (≥ 3) minus the dimension of the group of automorphisms of disc which fix a given point (which is two). So this dimension is greater than or equal to 1 hence this set could not be in the boundary of $\widetilde{\mathcal{M}}_2(x, z, J)$. So $\partial\overline{\mathcal{M}}_2(x, z, J) \subset \bigcup_{y \in L_0 \cap L_1} \widetilde{\mathcal{M}}_1(x, y, J) \times \widetilde{\mathcal{M}}_1(y, z, J)$. A gluing argument establishes the reverse inclusion. $\square$

Independence of J could be shown by constructing chain maps $\phi_{J,J'}$ and $\phi_{J',J}$ from $CF(L_0, L_1)$ to itself. See Theorem 3.6 in [31] for the closely related case of Floer homology for Hamiltonian fixed points.

2.8 Grading

The topic of this subsection is the absolute grading of Floer cohomology groups. We discuss the special case which is of concern in this thesis, i.e. the case of $\mathbb{Z}$ grading. The reader is referred to [32] for more detail. Let (M^{2n}, ω) be a symplectic manifold. We know that for a Lie group G and its maximal compact subgroup K, the isomorphism classes of G bundles on a topological space X are in one-to-one correspondence with those of K bundles over X. The symplectic group $Sp(2n)$ has $U(n)$ as maximal compact subgroup therefore upon choosing a compatible almost complex structure the structure group of TM can be reduced to $U(n)$. Denote by P the principal $U(n)$ bundle we get from TM in this way.

Definition 2.8.1. *An infinite Maslov cover of M is a $SU(n)$ bundle $\tilde{P}$ over M such that $\tilde{P} \times_{SU(n)} \mathbb{C}^n = TM$.*

Since the universal cover $\widetilde{U(n)}$ of $U(n)$ has $SU(n)$ as maximal compact subgroup, we get an equivalent definition by replacing $SU(n)$ with $\widetilde{U(n)}$ in the above definition.

Lemma 2.8.2. *If the structure group of TM can be further reduced to $SU(n)$ then M has an infinite Maslov covering.*

Proof. Let $g_{\alpha,\beta}$ and $g'_{\alpha,\beta}$ be transition functions of TM as a $U(n)$ resp. $SU(n)$ bundle over the same covering of M. Then the transition functions $\{(g_{\alpha,\beta}, t) | \quad g_{\alpha,\beta} = e^{it} g'_{\alpha,\beta}\}$ define a $\widetilde{U(n)}$ bundle $\tilde{P}$ over M for which we have $\tilde{P} \times_{\widetilde{U(n)}} \mathbb{C}^n = TM$. □

The isomorphism classes of such covers are in one to one correspondence with $H_1(M)$. Let $\mathcal{L} \to M$ be the bundle whose fiber at $x \in M$ is the Lagrangian

Grassmanian of T_xM. A Maslov cover $\tilde{P}$, induces a covering $\tilde{\mathcal{L}} \to \mathcal{L}$ given by $\tilde{P} \times_{SU(n)} \widetilde{Lag}(n)$. Each Lagrangian submanifold L of M determines a section s_L of $\mathcal{L}$. A *grading* of L is a cover $\tilde{s}_L$ of s_L. If L_0, L_1 are two Lagrangian submanifolds of M that intersect transversely we can assign an absolute grading to each intersection point x as follows. Let $\tilde{\Lambda}_i = \tilde{s}_{L_i}$ for $i = 0, 1$. Let $\tilde{\Lambda}(t)$ be a path joining $\tilde{\Lambda}_0$ to $\tilde{\Lambda}_1$ and let $\Lambda(t)$ be its projection. Define

$$\deg(x) = \mu^{path}(\Lambda, T_xL_0) + \frac{n}{2} \tag{2.9}$$

where μ^{path} is the Maslov index for paths. It does not depend on the choice of the liftings $\tilde{s}_{L_i}$ because of the naturality of μ^{path}. In general if the canonical bundle is not trivial, one can obtain only a $\mathbb{Z}/N$ grading for some $N \in \mathbb{N}$.

There is an equivalent way of describing this grading. If the canonical bundle of M is trivial, it has a global section (or trivialization) η. The global section η gives us a map $\alpha_M : M \to S^1$ given by

$$\alpha_M(x) = \frac{\eta(u_1, \ldots, u_n)^2}{|\eta(u_1, \ldots, u_n)|^2} \tag{2.10}$$

for any orthonormal basis $u_1, \ldots, u_n$ of T_xM. For each Lagrangian submanifold L we can define a *phase function* $\alpha_L : L \to S^1$ by

$$\alpha_L(x) = \frac{\eta(v_1, \ldots, v_n)^2}{|\eta(v_1, \ldots, v_n)|^2} \tag{2.11}$$

for any orthonormal basis $v_1, \ldots, v_n$ of T_xL.

Alternative Definition 2.8.3. *A grading of L is a choice of a real valued function $\tilde{L}$ such that $\alpha_L = \exp(2\pi i \tilde{L})$.*

For a pair of transversely intersecting graded Lagrangians L_0, L_1 and $x \in L_0 \cap L_1$ one can set

$$\deg(x) = \frac{n}{2} + \tilde{L_0}(x) - \tilde{L_1}(x). \tag{2.12}$$

We denote by $L\{m\}$, L with its grading shifted by m, i.e. $\tilde{L}\{m\} = \tilde{L} - m$. A grading for a diffeomorphism ϕ from M to itself is a choice of a function

$\tilde{\phi} : M \to \mathbb{R}$ such that $\exp(2\pi i \tilde{\phi}(x)) = \alpha_M(x)/\alpha_M(\phi^{-1}(x))$. $\phi(L)$ has a preferred grading given by

$$\widetilde{\phi(L)}(x) = \tilde{L}(\phi^{-1})(x) + \tilde{\phi}(\phi^{-1}(x)). \tag{2.13}$$

A choice of grading for ϕ induces a grading on the graph Γ of ϕ:

$$\tilde{\Gamma}(x, \phi(x)) = \tilde{\phi}(x). \tag{2.14}$$

Let $\underline{L} = (L_{0,1}, \dots, L_{n-1,n})$ be a Lagrangian correspondence and assume that $L_{i,i+1} \subset M_i^- \times M_{i+1}$. Assume we have chosen a trivialization η_{M_i} for the canonical bundle of M_i.

Definition 2.8.4. *A grading on $\underline{L}$ is a lift $\tilde{L}_{i,i+1}$ of $\alpha_{L_{i,i+1}}$ for each i where the phase functions $\alpha_{L_{i,i+1}}$ are with respect to $\eta_{M_i^-} \wedge \eta_{M_{i+1}}$.*

If M_i, $i = 0, 1, 2$ have trivial canonical bundle and we have chosen trivializations η_{M_i} for each i then gradings $\tilde{L}_{0,1}$ and $\tilde{L}_{1,2}$ determine a grading on

$$L_{0,2} = L_{0,1} \circ L_{1,2}$$

with regard to the trivialization $\eta_{M_0} \wedge \eta_{M_2}$, given by

$$\tilde{L}_{0,2}(m, m'') = \tilde{L}_{1,2}(m', m'') + \tilde{L}_{0,1}(m, m') \tag{2.15}$$

where m' is the unique point such that $(m, m', m', m'') \in (L_{0,1} \times L_{1,2}) \cap M_0 \times \Delta_{M_1} \times M_2$ provided that the composition is embedded.

Assume we have chosen a trivialization η_M for the canonical bundle of M. The canonical bundle of M^- is the dual of $\bigwedge^{\dim M} TM$. Thus we can take $\sqrt{-1}\eta^{-1}$ as a trivialization of the determinant bundle of M^-. So the phase function of a Lagrangian $L \subset M$ is the negative of the inverse of the phase function of the same Lagrangian as a subset of M^-. (We denote the later one by L_-.) Therefore a grading $\tilde{L}$ of L as a Lagrangian submanifold of M induces, in a natural way, a grading of L as a submanifold of M^-. This new grading equals $k/2 - \tilde{L}$ for some integer k. We choose $k = n = \dim L$ therefore we have

$$\tilde{L}_- = \frac{n}{2} - \tilde{L}. \tag{2.16}$$

2.9 Stein manifolds

The symplectic manifolds used in the construction of the symplectic tangle invariant are not compact. However they possess plurisubharmonic functions which enable us to prove compactness for pseudoholomorphic surfaces. First of all a Stein manifold is exact as defined below.

Definition 2.9.1. *A symplectic manifold* (M, ω) *is called exact if there is a one form* θ *on* M *such that* $\omega = d\theta$*. A Lagrangian submanifold* L *of such a manifold is called exact if there is a function* K *on* L *such that* $\theta|_L = dK$*.*

In particular if $H^1(L) = 0$ then L is exact. An application of Stokes theorem shows that a pseudoholomorphic disk with boundary on an exact Lagrangian submanifold has zero energy and so is constant. Therefore disc bubbling does not occur when one is concerned with the Floer cohomology of such Lagrangians.

Definition 2.9.2. (M, ψ) *is called a Stein manifold if* M *is a complex manifold and* ψ *is a proper function which is bounded below and such that* $-dd^c\psi$ *is a symplectic form on* M*.*

For a subset $N \subset M$, we denote by $N_{\leq c}$ (N_c, $N_{\geq c}$) the intersection of N with sublevel (level, superlevel) sets of ψ. Also set $\theta = -d^c\psi$.

Definition 2.9.3. *A Lagrangian submanifold* $L \subset M$ *is called c-allowable if it is exact and* $\psi|_L$ *does not have any critical points in* $M_{\geq c}$*. It is called allowable if it is c-allowable for some c. A generalized correspondence* $(L_{k,k-1}, \ldots, L_{1,0})$*, with* $L_{i,i-1} \subset M_i^- \times M_{i-1}$*, is called allowable if the* M_i *are Stein and all the* $L_{i,i+1}$ *are allowable.*

Note that this implies that L intersects the level sets of ψ transversely at infinity. Compact Lagrangian submanifolds of Stein manifolds are evidently allowable. Let $Z = \nabla\psi$ be the Liouville vector field on M. Denote by $\kappa : \mathbb{R} \times M \to M$ the flow of Z.

Definition 2.9.4. *A Lagrangian L in a Stein manifold M is said to have a conical end if there is a constant c such that L intersects M_c transversely and $\kappa([0,\infty) \times (L \cap M_c))$ equals $L_{\geq c}$.*

An exact Lagrangian with conical end is clearly allowable. Our next task is to assign a Lagrangian with conical end to an allowable Lagrangian which can replace the latter when Floer cohomology is concerned. We first need some definitions.

Definition 2.9.5. *A Lagrangian submanifold Λ of the symplectic manifold $M_{\leq c}$ is called κ-compatible if it intersects M_c transversely (possibly empty) and there is an $\epsilon > 0$ such that $\kappa\left([-\epsilon, 0] \times \Lambda_c\right) = \left({}_{c-\epsilon\leq}\Lambda_{\leq c}\right)$.*

Definition 2.9.6. *A Hamiltonian isotopy induced by a time-dependent function H_t is called conical if there is a constant c such that $H_t \circ \kappa(r,x) = rH_t(\kappa(0,x))$ for all $r \geq 0$, $t \in [0,1]$ and x in $M_{\geq c}$.*

Let ϕ_t be a one parameter family of symplectomorphisms of M and $L \subset M$ a Lagrangian submanifold. The isotopy $\phi_t(L)$ is called *exact* if $\phi_t^*\theta|_L = \theta|_L + dK_t$ for any $t \in [0,1]$ where K_t is a function on L depending smoothly on t. We have the following facts from [16] Section 5.

Lemma 2.9.7. *i) Any exact Lagrangian in $M_{\leq c}$ which intersects the boundary transversely can be exact-isotoped, relative to boundary, in $M_{\leq c}$ to a κ-compatible one.*

ii)If Λ_t is a Lagrangian isotopy in $M_{\leq c}$ such that all Λ_t intersect the boundary transversely and Λ_0, Λ_1 are κ-compatible then there is another isotopy Λ'_t with the same endpoints such that $\Lambda_t \cap M_c = \Lambda'_t \cap M_c$ for any $t \in [0,1]$ and all Λ_t are κ-compatible. If Λ_t is exact, Λ'_t can be chosen to be so.

We include the proof for completeness.

Proof. [16], Lemma 5.2 i)Let L be such a Lagrangian. Choose $r < c$ such that $K = L_{\geq -r}$ deformation retracts onto ∂L. $\theta|_K$ is closed and vanishes on the boundary so $\theta|_K = df$ for some f on K which vanishes on the boundary. Extend

f to a smooth function h on $M_{\leq c}$ vanishing on the boundary and set $\theta_t = \theta - tdh$ for $t \in [0,1]$. Let Z_t be a vector field such that $\iota_{Z_t}\omega = \theta_t$. Since $d\theta_t = \omega$, Z_t is symplectic and defines an embedding $\kappa_t : M_c \times [0,\infty] \to M_{\leq c}$. Let U be a neighborhood of M_c which lies in the image of k_t for all t and deformation retracts onto M_c. Set $X_t = \partial\kappa_t/\partial t \circ (\kappa_t)^{-1}|_U$. It is a symplectic vector field which vanishes on ∂M. $\iota_{X_t}\omega$ is closed and vanishes on boundary so equals dH_t for some H_t on U vanishing on π_c. Each X_t can be extended to a Hamiltonian vector field $\tilde{X}_t$ on $M_{\leq c}$. Let ϕ_t be the flow of the time dependent vector field $(\tilde{X}_t)$. We have $\phi_t \circ \kappa_0 = \kappa_t$ on a neighborhood of M_c so $\phi_1^*\theta = ({\kappa_0}^{-1})^*{\kappa_t}^*\theta = \theta_1$ near M_c. Therefore

$$\phi_1^*(\theta|_L) = \theta_1|_L = 0 \tag{2.17}$$

near the boundary of L and so $\phi_1(L)$ is κ-compatible. By (2.17), $(\phi_t(L))$ is exact.

ii) It is just a parameterized version of i).

□

Lemma 2.9.8. *Any exact Lagrangian isotopy, relative to boundary, in $M_{\leq c}$ which consists only of κ-compatible Lagrangians can be embedded into a Hamiltonian isotopy.*

The proof is similar to the proof of the fact that a symplectomorphism of zero flux is Hamiltonian.

Definition 2.9.9. *Let L be a c-allowable Lagrangian in M, we take $\Lambda = L_{\leq c}$, isotope Λ to a κ-compatible Λ' and denote by $\mathfrak{C}_c(L) \subset M$ the Lagrangian with conical end associated to Λ'. This means that $\mathfrak{C}_c(L)_{>c}$ is the image of L_c under the Liouville flow.*

It follows from the above lemmas that $\mathfrak{C}_c(L)$ is well-defined up to conical Hamiltonian isotopy. Moreover if $L_{\leq c}$ and $L'_{\leq c}$ are exact isotopic, relative to boundary, then $\mathfrak{C}_c(L)$ and $\mathfrak{C}_c(L')$ are conical Hamiltonian isotopic.

If the M_i are Stein and the Lagrangian correspondences have conical ends then it is easy to see that their composition has a conical end as well. Therefore we have the following.

$$\mathfrak{C}_c(L_{0,1}) \circ \mathfrak{C}_c(L_{1,2}) \simeq \mathfrak{C}_c(L_{0,1} \circ L_{1,2}) \tag{2.18}$$

Definition 2.9.10. *The Stein category has Stein manifolds as objects. Morphisms are given by equivalence classes of allowable generalized Lagrangian correspondences $(L_{0,1}, \ldots, L_{n-1,n})$. The equivalence relation on correspondences is the one in the symplectic category with the difference that we restrict the first kind of equivalence to include only exact isotopies.*

2.10 Floer cohomology for Lagrangian correspondences

Let $\underline{L}$ be a generalized Lagrangian correspondence as in the last section. By adding a trivial Lagrangian correspondence (i.e. the diagonal) if necessary we can assume that $n = 2k+1$ is odd. Define

$$\mathcal{L}_0 = L_0 \times L_{1,2} \times \cdots L_{2k-1,2k} \tag{2.19}$$

and

$$\mathcal{L}_1 = L_{0,1} \times L_{2,3} \times \cdots \times L_{2k+1} \tag{2.20}$$

which are Lagrangian submanifolds of $M = M_0^- \times M_1 \times \cdots M_n^-$. If $\mathcal{L}_0$ and $\mathcal{L}_1$ satisfy **L1** and **L2** from section 2.7 one can associate to $\underline{L}$ the Floer cohomology group

$$HF(\underline{L}) := HF(\mathcal{L}_0, \mathcal{L}_1). \tag{2.21}$$

In order for Floer cohomology to define a well-defined map on the symplectic category, we must understand the effect of composition of Lagrangian correspondences on Floer cohomology. The following important *Functoriality Theorem* is proved in [39].

Theorem 2.10.1 (Wehrheim, Woodward [39]). *Let* $\underline{L} = (L_{0,1}, L_{1,2}, \ldots, L_{n-1,n})$ *be a generalized Lagrangian correspondence between manifolds* $M_0, \ldots, M_n$ *such that for some* $0 < j < n$ *the composition* $L_{j,j+1} \circ L_{j-1,j}$ *is embedded. Denote*

$$\underline{L}' = (L_0, \ldots, L_{j,j+1} \circ L_{j-1,j}, \ldots, L_n).$$

Assume the M_i *are compact and monotone with the same monotonicity constant and all* $L_{i,i+1}$ *as well as* (2.19) *and* (2.20) *are monotone. Assume in addition that each* $L_{i,i+1}$ *is oriented, relatively spin, graded and its minimal Maslov number is at least three. Then with the induced grading and relative spin structure on* $L_{i,i+1} \circ L_{i-1,i}$ *there is a canonical isomorphism*

$$HF(\underline{L}) \cong HF(\underline{L}') \tag{2.22}$$

of graded abelian groups.

2.11 Floer cohomology for exact Lagrangian correspondences

Here we discuss Floer cohomology for exact Lagrangians whose detail is a bit different from that of compact Lagrangians discussed in sections 2.7 and 2.10. The Lagrangians used in the definition of our invariant are exact. Let $\underline{L}$ be a generalized Lagrangian correspondence between Stein manifolds and let $\mathcal{L}_0$ and $\mathcal{L}_1$ be as in (2.19) and (2.20). We want to see when

$$HF(\underline{L}) := HF(\mathfrak{C}_c(\mathcal{L}_0), \mathfrak{C}_c(\mathcal{L}_1)) \tag{2.23}$$

for some c large enough is well-defined.

First we assume that there is a conical Hamiltonian isotopy of M which makes $\mathcal{L}_0$ and $\mathcal{L}_1$ intersect transversally at finitely many points. We call this assumption *finite intersection* of the Lagrangians. It holds if one of the Lagrangians is compact. More generally it holds if one of the correspondences $L_{i,i+1}$ is compact and all the others are proper in the following sense.

Definition 2.11.1. *A correspondence* $L \subset M^- \times M'$ *is called proper if for each point* $y \in M'$ *the set* $\{x \in M | (x, y) \in M'\}$ *is compact.*

We denote the isotoped Lagrangians by the same notation. Let J_i be a compatible almost complex structure on M_i. If M_i is Stein then we require J_i to be invariant under the flow of Liouville vector field outside a compact set. We call such an almost complex structure *asymptotically invariant*. From the J_i we get an almost complex structure

$$J = (-J_0, J_1, \ldots, (-1)^{n-1} J_n)$$

on M. Let $x, y \in \mathcal{L}_0 \cap \mathcal{L}_1$ and let J_t be a one parameter family of almost complex structures on M for which there is an $r_0 > 0$ such that $J_t = J$ outside $M_{\leq r_0}$ for all t. Denote by $\mathcal{M}(x, y)$ the moduli space of the strips

$$u : \mathbb{R} \times [0, 1] \to \mathcal{Y}$$

such that $\frac{\partial}{\partial t} u(s,t) = J_t \frac{\partial}{\partial s} u(s,t)$ and $u(s,i) \in \mathcal{L}_i$ for $i = 0, 1$ and $u(-\infty, t) = x, u(\infty, t) = y$. $\mathbb{R}$ acts on $\mathcal{M}(x, y)$ by shifting the parametrization.

Consider the abelian group $CF(\mathcal{L}_0, \mathcal{L}_1)$ generated freely by the intersection points of the two Lagrangians. The following differential makes $CF(\mathcal{L}_0, \mathcal{L}_1)$ into a chain complex and the Floer cohomology $HF(\mathcal{L}_0, \mathcal{L}_1)$ is defined to be the cohomology of this complex.

$$\partial x = \sum_{y \in \mathcal{L}_0 \cap \mathcal{L}_1} \# \mathcal{M}_1(x, y)/\mathbb{R} \tag{2.24}$$

Here $\mathcal{M}_1(x, y)$ is the one dimensional part of the moduli space. The count is a priori in $\mathbb{Z}/2$. To be able to define Floer cohomology groups as $\mathbb{Z}$ one needs coherent orientations on the moduli spaces cf. 2.6.2. For this invariant to be well-defined one has to take care of the following issues: transversality of moduli spaces, nonexistence of bubbling, compactness and invariance under Hamiltonian isotopy. In the following discussion we assume that the first two criteria hold and focus on the last two. First compactness.

Lemma 2.11.2. *Assume (M, ψ) is a Stein manifold, $\mathcal{L}_0, \mathcal{L}_1$ are C-allowable Lagrangians for some $C \geq r_0$ and that $M_{\leq C}$ contains the intersection points of the Lagrangians. Then for any Riemann surface S with boundary and any J-holomorphic map $u : S \to M$ with $u(\partial S) \subset \mathcal{L}_0 \cup \mathcal{L}_1$, the image of u lies in $M_{\leq C}$ (which is independent of S and u).*

Proof. [2], [25] For $u \in \mathcal{M}(x, y)$, $\psi \circ u$ cannot have a maximum on the interior of the curve outside $M_{\leq C}$ by the maximum principle. This is because u is J holomorphic outside $M_{\leq r_0}$. Assume it has a maximum on a boundary point p i.e. $\max \psi \circ u = \psi \circ u(p) = R > C$. We can pick holomorphic coordinates on S in a neighborhood of p and therefore regard u, in a neighborhood of $p = (s_0, 1)$, as a function on some rectangle $[l_1, l_2] \times [1 - \delta, 1]$. We have $d\psi(\frac{\partial}{\partial s} u)(s_0, 1) = 0$. So $\frac{\partial}{\partial s} u(s_0, 1) \in T\mathcal{L}_1 \cap TM_R$. By assumption the intersection is transverse therefore it is Legendrian so $d\psi(J \frac{\partial}{\partial s} u(s_0, 1)) = 0$. But we have $\frac{\partial}{\partial t} u = J \frac{\partial}{\partial s} u$ so $d\psi(\frac{\partial}{\partial t} u)(s_0, 1) = 0$. This contradicts the strong maximum principle which implies $d\psi(\frac{\partial}{\partial t} u) > 0$. □

This enables us to apply the rescaling argument to show that the limit of a bounded energy sequence of such curves is either a broken trajectory or a curve with sphere or disc bubbles. Therefore we can compactify $\mathcal{M}(x, y)$ by adding these limiting curves to it. If in addition both M and the Lagrangians are exact, no bubbling occurs and so the sum in (2.24) is finite and we get $\partial^2 = 0$.

Proposition 2.11.3. *i) Let L, L' be two Lagrangians in a Stein manifold (M, ψ) which are C-allowable and satisfy the finite intersection condition. Then the Floer cohomology $HF(\mathfrak{C}_c(L), \mathfrak{C}_c(L'))$ is well-defined. It is independent of c when $c > C$ and $L \cap L' \subset M_{\leq c}$.*

ii) If $\{\phi_t\}_{t \in [0,1]}$ is a Hamiltonian flow such that $\cup_t \phi_t(L) \cap L'$ is contained in $M_{<C'}$ then for $c > C'$, $HF(\mathfrak{C}_c(L), \mathfrak{C}_c(L'))$ is canonically isomorphic to

$$HF(\mathfrak{C}_c(\phi_1(L)), \mathfrak{C}_c(L')).$$

Proof. Part i) follows from Lemma 2.11.2 along with the results in section 2.9. Independence of c follows from Lemma 2.11.2. For part ii) let H be the Hamiltonian function inducing ϕ and H' be a function which equals H on $M_{\leq C'}$ and is zero outside $M_{\geq c}$. If ϕ' is the flow of H' then $\mathfrak{C}_c(\phi_1(L)) = \mathfrak{C}_c(\phi'_1(L))$. Since ϕ' is compactly supported, with the help of Lemma 2.11.2 we can apply the usual invariance argument using the solutions of perturbed Cauchy-Riemann equation. □

Proposition 2.11.4. *If the M_i are Stein, $\underline{L}$ is allowable and satisfies the finite intersection condition and each $L_{i,i+1}$ is relatively spin then* (2.22) *holds.*

Proof. It is easy to see that the generators for the two Floer groups are in one-to-one correspondence. Take $x, y \in \mathcal{L}_0 \cap \mathcal{L}_1$ and let $\mathcal{M}_\delta(x, y)$ be the moduli space of pseudoholomorphic strips $u = (u_0, \dots u_n)$ where

$$u_i : [0, 1] \times \mathbb{R} \to M_i$$

if $i \neq j$ and $u_i : [0, \delta] \to M_j$ and with boundary condition for u given by $\underline{L}$ and with x and y as asymptotic points. Let $\mathcal{M}'(x, y)$ be the moduli space of strips

$$(u_0, \dots, u_{i-1}, u_{i+1}, \dots, u_n) : [0, 1] \times \mathbb{R} \to M_0 \times \cdots M_n$$

with boundary condition $\underline{L}'$ and asymptotic points x, y. Since the M_i are Stein and the Lagrangians are allowable and satisfy the finite intersection property then Lemma 2.11.2 implies that the holomorphic curves in $\mathcal{M}_\delta(x, y)$ and $\mathcal{M}'(x, y)$ stay in a compact submanifold of $M_0 \times \cdots \times M_n$ (which doesn't depend on δ) and so the proof proceeds as in [39] to show that for δ small enough the two moduli spaces are bijective. Note that exactness of the $L_{i,i+1}$ implies monotonicity and rules out bubbling so we do not need the assumption on the minimal Maslov index in 2.10.1. □

Remark 2.11.5. *Assume we have two Hamiltonian isotopic Lagrangian correspondences L_0, L_1 and another correspondence L' such that both compositions $L_0 \circ L'$ and $L_1 \circ L'$ are embedded. In general we do not know if $L_0 \circ L'$ and*

$L_1 \circ L'$ are Hamiltonian isotopic or not. However the proof of the Functoriality Theorem implies that Floer cohomology is invariant under such an isotopy.

Chapter 3

Khovanov homology for tangles

3.1 Tangles

A tangle T is defined to be a compact one-dimensional submanifold of (a diffeomorphic image of) $\mathbb{C} \times [0,1]$ such that $i(T) := T \cap (\mathbb{C} \times \{0\}) \subset \mathbb{R} \times \{0\}$ and $t(T) := T \cap (\mathbb{C} \times \{1\}) \subset \mathbb{R} \times \{1\}$ and both sets are finite. The second assumption makes $i(T)$ and $t(T)$ ordered sets. In this thesis we deal only with tangles with an even number of initial points and end points. Such tangles are called *even tangles.* If $\#i(T) = 2m, \#t(T) = 2n$ we say T is an (m,n)-tangle and write mTn. We also allow m and/or n to be zero.

Definition 3.1.1. *Two tangles T, T' are called equivalent if there is a continuous family T_t of tangles for $t \in [0,1]$ such that $T_0 = T$ and $T_1 = T'$ and the order of $i(T_t)$ and of $t(T_t)$ is fixed.*

Two tangles T_1, T_2 can be composed (concatenated) if $t(T_1) = i(T_2)$. Two equivalence classes $[T_1]$ and $[T_2]$ of tangles can be composed if $\#t(T_1) = \#i(T_2)$ and composition is done using the ordering on $t(T_1)$ and $i(T_2)$. Composition of tangles is denoted by juxtaposition. We will use the notation id_m, $\cap_{i;m}$, $\cup_{i;m}$

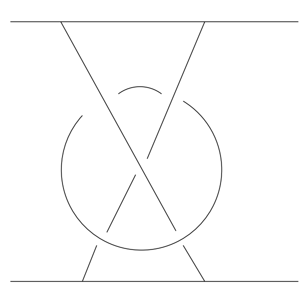

Figure 3.1: A $(1,1)$-tangle

and $\sigma_{i;m}$ for the elementary tangles in Figures 3.2 and 3.3 where m denotes the number of the strands. We might ignore m when there is no confusion. When we say a tangle T is equivalent to, say, $\cap_{i;m}$, we implicitly have a one to one correspondence between $i(T)$ and $\{1, 2, \ldots, 2m-2\}$ and also between $t(T)$ and $\{1, 2, \ldots, 2m\}$ in mind.

A *decomposition* of T is a sequence of tangles

$$n_0 T_1 n_1 T_2 \ldots n_{l-1} T_l n_l \qquad n_0 = m, n_l = n \tag{3.1}$$

such that T is equivalent to $T_1 T_2 \cdots T_l$. A Morse-theoretic argument shows that any T can be expressed (not uniquely) as a composition of elementary tangles. *Crossingless matchings* (section 4.5) are a special class of $(0, n)$ or $(n, 0)$-tangles. Given a set of $2n$ points on the plane, a crossingless matching is a set of n nonintersecting curves each joining a pair of the given points. In the context of tangles a crossingless matching is regarded as a subset of $\mathbb{C} \times [0, 1]$.

Definition 3.1.2. *Let $\mathcal{C}_n$ be the set of isotopy (in $\mathbb{C}$) classes of crossingless matchings between $2n$ points on the real line all of whose arcs lie in the upper half plane.*

The cardinality of $\mathcal{C}_n$ equals the nth Catalan number.

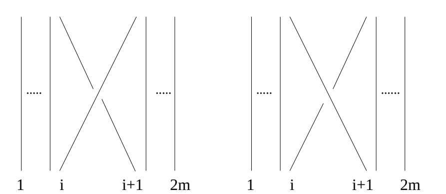

Figure 3.2: The braids σ_i and ${\sigma_i}^t$

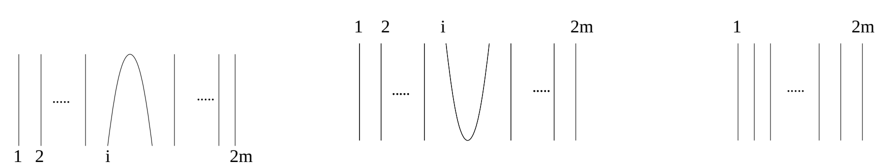

Figure 3.3: $\cap_{i;m}$, $\cup_{i;m}$ and id_m

One can define a category **Tang** whose objects are natural numbers and $\hom(m, n)$ consists of equivalence classes of (m, n)-tangles. **Tang** has a monoidal structure given by putting two tangles kTl and mTn "side-by-side" to obtain a $(k+m, l+n)$-tangle. We denote this by $T \oplus T'$. To each (m,n)-tangle T there is assigned a "mirror image" T^t which is a (n,m)-tangle. There is a generators and relations description of **Tang** due to Yetter [42] whose proof relies on Reidemeister's description of plane diagram moves.

Lemma 3.1.3 (Yetter [42])**.** *The following are all the commutation relations between elementary tangles where "=" means equivalence. If $|i-j| > 1$ we have:*

$$\sigma_i \sigma_j = \sigma_j \sigma_i \tag{3.2}$$

$$\cap_i \cup_j = \cup_j \cap_i \tag{3.3}$$

$$\cap_i \sigma_j = \sigma_j \cap_i \qquad \cup_i \sigma_j = \sigma_j \cup_i, \tag{3.4}$$

and for any i we have:

$$\sigma_i \cup_i = \cup_i \tag{3.5}$$

$$\sigma_i \sigma_i^t = id \tag{3.6}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tag{3.7}$$

$$\cap_{i;m} \cup_{i+1;m} = id_{m-1} \tag{3.8}$$

$$\sigma_i \cup_{i+1} = \sigma_{i+1}^t \cup_i \qquad \sigma_i^t \cup_{i+1} = \sigma_{i+1} \cup_i . \tag{3.9}$$

Given two decompositions of two equivalent tangles, the following lemma provides a natural way of converting one to the other.

Lemma 3.1.4. *If T, T' are two equivalent tangles and $D : T = T_1 T_2 \cdots T_m$, $D' : T' = T_1' T_2' \cdots T_m'$ are decompositions into elementary tangles then one decomposition can be converted to the other by a sequence of the moves described in Lemma 3.1.3.*

Proof. We can regard each decomposition as being inside $\mathbb{C} \times [0,1]$. We can also find Morse functions f, f' on T and T' respectively such that the decomposition of T (resp. T') by critical sets of f (resp. f') yields D (resp. D'); in other words f has critical points separating each T_i from T_{i+1} and no more. There is a diffeomorphism ϕ of $\mathbb{C} \times [0,1]$ such that $\phi(T) = T'$. The reason is that because T and T' are equivalent, there is a family $T(t)$ of tangles with the same boundary points such that $T(0) = T, T(1) = T'$. Because strands of $T(t)$ never intersect, we can obtain a time dependent vector field on $T(t)$ by differentiation. This vector field can be smoothly extended to $\mathbb{C} \times [0,1]$. Let ϕ be the time one map of this vector field. Then $\phi(T) = T'$.

Let $f'' = \phi^* f'$ and $T_i'' = \phi^{-1}(T_i')$. So we get a decomposition $D'' : T_1'', \cdots, T_m''$ of T induced by f''. According to Cerf theory [1], there is a family f_t of smooth functions such that $f_0 = f, f_1 = f''$ and f_t is Morse except for finitely many times $t_1, \cdots, t_k$ and at each t_i a pair of canceling critical points is introduced or deleted. Since T is one dimensional, each t_i has the effect of either (1) merging two adjacent handles (i.e. two adjacent elementary tangles) or decomposing one into two or (2) the effect of the move (3.8) above. In the first case the effect of merging two adjacent handles and then separating into two new ones is equivalent to one of the moves in Lemma 3.1.3.

Now between each t_i, t_{i+1}, each handle can be isotoped to an equivalent one. The only way this can change the decomposition is by changing the value of f_t at critical points and thereby changing the order of the critical points in the decomposition. This also has the effect of commuting the handlebodies in the decomposition.

$\square$

The invariant that we define in this thesis is an invariant of *oriented tangles*. An oriented tangle comes with an orientation of each one of its components. Two example are shown in Figure 3.4. When considering commutation relations between tangles we ignore the orientation.

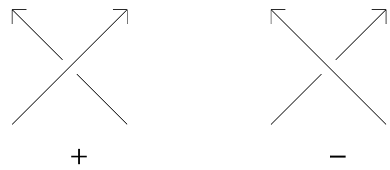

Figure 3.4: Oriented braids σ^+ and σ^-

3.2 The TQFT

Khovanov homology is based on a 1+1 dimensional TQFT $\mathcal{F}$ whose definition we review here. 1+1 dimensional TQFTs are in one-to-one correspondence with Frobenius algebras. Khovanov [14] defines the Frobenius algebra $\mathcal{V}$ to be equal to $H^*(S^2)\{-1\}$ (i.e. the cohomology of S^2 with its grading shifted down by one) as a ring. Let $\mathit{1}, \mathcal{X}$ be degree -1 and degree 1 generators of $\mathcal{V}$ respectively. We define comultiplication by

$$\Delta(\mathcal{X}) = \mathcal{X} \otimes \mathcal{X} \qquad \Delta(\mathit{1}) = \mathit{1} \otimes \mathcal{X} + \mathcal{X} \otimes \mathit{1}. \tag{3.10}$$

The unit map $\iota : \mathbb{Z} \to \mathcal{V}$ by $\iota(1) = \mathit{1}$. The trace map is defined by

$$\epsilon(\mathcal{X}) = 1 \qquad \epsilon(\mathit{1}) = 0. \tag{3.11}$$

It is evident that multiplication is given by

$$m(\mathit{1}, \mathcal{X}) = m(\mathcal{X}, \mathit{1}) = \mathcal{X} \qquad m(\mathcal{X}, \mathcal{X}) = 0 \qquad m(\mathit{1}, \mathit{1}) = \mathit{1}. \tag{3.12}$$

Definitions above are made by choosing a basis for $H^*(S^2)$. In section 7.3 we give a definition which does not need the choice of a basis. The TQFT $\mathcal{F}$ assigns to each closed one dimensional manifold (i.e. a circle) the vector space $\mathcal{V}$, to each cap the unit map $\iota : \mathbb{Z} \to \mathcal{V}$, to each cup the trace map $\epsilon : \mathcal{V} \to \mathbb{Z}$, to each pair of pants the multiplication $m : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$, and to each reverse pair of pants the comultiplication $\Delta : \mathcal{V} \to \mathcal{V} \otimes \mathcal{V}$.

3.3 Tangle cobordisms and the rings H^m

We denote the Cartesian coordinates on $\mathbb{C} \times [0,1] \times [0,1]$ by (z, t, s). For a subset $A \subset (\mathbb{C} \times [0,1] \times [0,1])$ we set

$$\partial_i^v A = A \cap (\mathbb{C} \times [0,1] \times \{i\})$$

and

$$\partial_i^h A = A \cap (\mathbb{C} \times \{i\} \times [0,1]).$$

Definition 3.3.1. *Let T_0, T_1 be two (m,n)-tangles. A cobordism between T_0 and T_1 is a smoothly embedded surface S in $\mathbb{C} \times [0,1] \times [0,1]$ s.t.*

$$\partial_i^v S = T_i$$

for $i = 0, 1$. We also require S to be the product of ∂_i^h or ∂_i^v with a small subinterval in a neighborhood of each face of $\mathbb{C} \times \partial([0,1] \times [0,1])$.

The identity cobordism between T and itself is denoted by $\mathbf{1}_T$. Tangle cobordisms can be composed in two ways. First the *vertical composition*: if S, S' are cobordisms between T_0, T_1 and T_1, T_2 then we get a cobordism

$$S' \circ S = \frac{S' \cup S}{\partial_0^v S' \sim \partial_1^v S} \tag{3.13}$$

between T_0 and T_1. Secondly the *horizontal composition*: if S is a cobordism between kT_0l and mT_1n, and S' is a cobordism between $lT_0'L$ and $nT_1'N$ then we get a cobordism

$$S'S = \frac{S' \cup S}{\partial_0^h S' \sim \partial_1^h S} \tag{3.14}$$

between $kT_0' \circ T_0L$ and $mT_1' \circ T_1N$. The last assumption in the definition of a cobordism ensures that compositions are smooth embedded surfaces.

For the purpose of this paper we just need to consider a special class of tangle cobordisms.

Definition 3.3.2. *For a crossingless matching $a \in \mathcal{C}_m$, the minimal cobordism between $a^t a$ and id_m is the one which is given by merging the corresponding strands* *of* a^t *and* a *from the outermost one to the innermost one as depicted in Figure 3.5. We denote this minimal cobordism by* S_a*.*

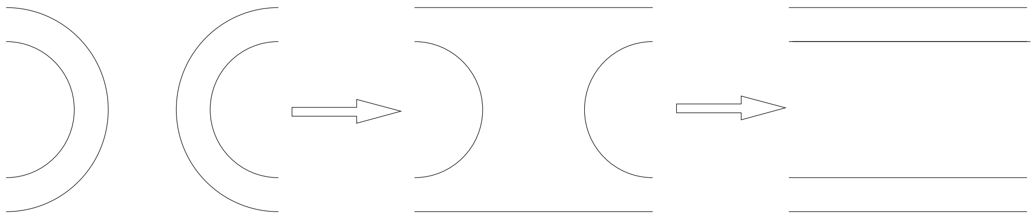

Figure 3.5: A minimal cobordism

For $a, b \in \mathcal{C}_m$ set

$$ {}_aH^m_b = \mathcal{F}(a^t b)\{m\} \tag{3.15}$$

and

$$ H^m = \bigoplus_{a,b\in\mathcal{C}_m} {}_aH^m_b. \tag{3.16}$$

Note that $a^t b$ is a disjoint union of circles so ${}_aH^m_b = \mathcal{V}^{\otimes k}$ where k is the number of the circles. Multiplication

$$ {}_aH^m_b \otimes {}_cH^m_d \to {}_aH^m_d $$

is defined to be zero if $b \neq c$. If $b = c$, let S_b be the minimal cobordism between $b^t b$ and id_m. The cobordism $\mathbf{1}^t_a S_b \mathbf{1}_c$ is a surface without corners so we get a map $\mathcal{F}(\mathbf{1}^t_a S_b \mathbf{1}_c) : {}_aH^m_b \otimes {}_bH^m_c \to {}_aH^m_c$ which gives us multiplication.

We now recall a recursive decomposition of H^m from [29]. Denote by $\mathcal{C}'_m$ the subset of $\mathcal{C}_m$ consisting of elements which contain $\cap_1$, i.e. elements which contain an arc between points 1 and 2, and denote by $\mathcal{C}''_m$ its complement. (1 can be replaced with any $1 \leq i \leq 2m-1$.) $\mathcal{C}'_m$ is in one-to-one correspondence with $\mathcal{C}_{m-1}$. We have a map $\mathcal{C}''_m \to \mathcal{C}_{m-1}$, $a \mapsto a'$, given by joining the two strands of a that stem from 1 and 2. Let $\mathcal{CC}^1_m \subset \mathcal{C}''_m \times \mathcal{C}''_m$ be the subset of all (a, b) such that the arcs passing through points 1 and 2 in $a^t b$ form a single circle. If (a, b) is in the complement of $\mathcal{CC}^1_m$ then the arcs passing through points 1 and 2 in $a^t b$ form two circles.

Let $a, b \in \mathcal{C}'_m$. If $\bar{a}$ denotes a with the $\cap_1$ removed then we have

$$ {}_aH^m_b = \quad {}_{\bar{a}}H^{m-1}_{\bar{b}} \otimes \mathcal{V}\{1\}. $$

This contributes a summand of $H^{m-1} \otimes \mathcal{V}\{1\}$ to H^m. Set

$$ \tilde{H}^m = \bigoplus_{a \in \mathcal{C}'_m, b \in \mathcal{C}''_m} \mathcal{F}(a^t, b)\{m\}. $$

The embedded circle C in $a^t b$ which passes through points 1 and 2 contributes a factor of $\mathcal{V}\{1+i\}$ to $\tilde{H}^m$ where i is the number of other pairs of points $2k-1, 2k$ which C passes through. We can set

$$ \tilde{H}^m = \bar{H}^m \otimes \mathcal{V}\{1\} $$

where $\mathcal{V}\{1\}$ is the "local" contribution of the circle containing $\cap_1$ or $\cup_1$. This means that if $a \in \mathcal{C}'_m, b \in \mathcal{C}''_m$ and we modify the strands of $a^t b$ passing through 1 and 2 only in a small neighborhood of the points 1 and 2 then we alter only the second factor in $\bar{H}^m \otimes \mathcal{V}\{1\}$. Also denote by $\tilde{H}^m_1$ and $\tilde{H}^m_2$ the contribution of $\mathcal{CC}^1_m$ and its complement to H^m. Again we can write $\tilde{H}^m_1 = H^m_1 \otimes \mathcal{V}\{1\}$ and $\tilde{H}^m_2 = H^m_2 \otimes \mathcal{V}\{1\} \otimes \mathcal{V}\{1\}$ where $\mathcal{V}\{1\}$ resp. $\mathcal{V}\{1\} \otimes \mathcal{V}\{1\}$ are "local"contributions from the single circle resp. the two circles formed by arcs passing through 1 and 2. Therefore we get

$$ H^m = \Big(\big(H^{m-1} \oplus \bar{H}^m \oplus \bar{H}^m \oplus H^m_1 \big) \otimes \mathcal{V}\{1\} \Big) \bigoplus H^m_2 \otimes \mathcal{V}\{1\} \otimes \mathcal{V}\{1\}. \quad (3.17) $$

as abelian groups.

3.4 The Khovanov invariant for flat tangles

Definition 3.4.1. *A flat tangle is a tangle which can be embedded into the plane i.e. a tangle without crossings.*

For a flat (m, n) tangle T we define

$$ \mathcal{Kh}(T) = \bigoplus_{a \in \mathcal{C}_m, b \in \mathcal{C}_n} \mathcal{F}(a^t T b)\{n\}. \quad (3.18) $$

Obviously $H^m = \mathcal{Kh}(id_m)$ as abelian groups. The abelian group $\mathcal{Kh}(T)$ for an (m,n)-tangle T has the structure of a (H^m, H^n)-bimodule which is given by

$$\mathcal{F}(\mathbf{1}_a S_b \mathbf{1}_T \mathbf{1}_c) : {}_a H^m_b \otimes \mathcal{F}(b^t T c) \to \mathcal{F}(a^t T c) \tag{3.19}$$

for each a, b, c and zero map ${}_a H^m_b \otimes \mathcal{F}(c^t T d) \to \mathcal{F}(a^t T d)$ if $b \neq c$.

For the unlinked union $T \sqcup S^1$ we have

$$\mathcal{Kh}(T \sqcup S^1) = \mathcal{Kh}(T) \otimes \mathcal{V} = \mathcal{Kh}(T)\{1\} \oplus \mathcal{Kh}(T)\{-1\} \tag{3.20}$$

If S is a cobordism between two flat (m,n)-tangles T_0, T_1, it induces a bimodule map

$$\mathcal{Kh}(S) : \mathcal{Kh}(T_0) \to \mathcal{Kh}(T_1) \tag{3.21}$$

which is given on each component by

$$\mathcal{F}(\mathbf{1}_a^t S \mathbf{1}_b) : \mathcal{F}(a^t T_0 b) \to \mathcal{F}(a^t T_1 b). \tag{3.22}$$

The fact that $\mathcal{Kh}(S)$ is independent of the isotopy class of S follows from the same property for the TQFT $\mathcal{F}$. It is obvious that

$$\mathcal{Kh}(S \circ S') \quad = \quad \mathcal{Kh}(S) \circ \mathcal{Kh}(S') \tag{3.23}$$

Lemma 3.4.2 ([15]). *If lTm and $mT'n$ are flat tangles then*

$$\mathcal{Kh}(T_0 \circ T_1) = \mathcal{Kh}(T_0) \otimes_{H^m} \mathcal{Kh}(T_1)$$

as (H^l, H^n)-bimodules. I f $T_0 S_0 T_0'$ and $T_1 S_1 T_1'$ are minimal cobordisms then

$$\mathcal{Kh}(S_0 S_1) = \mathcal{Kh}(S_0) \otimes_{H^m} \mathcal{Kh}(S_1). \tag{3.24}$$

3.5 The Khovanov invariant in general

In this section we present Khovanov's invariant for general tangles in a roundabout way which is shorter and suitable for our purpose. For a general tangle T, $\mathcal{Kh}(T)$ is a chain complex of graded bimodules over the rings H^m so it is doubly graded. For a flat tangle T the chain complex

$$\cdots \to 0 \to \mathcal{Kh}(T) \to 0 \to \cdots \tag{3.25}$$

with $\mathcal{K}h(T)$ in (second or homological) degree zero. We denote upward shift in first by $\{i\}$ and downward shift in second grading by $[i]$. The only elementary braids which are not flat are the braids $\sigma^+_{i;m}$ and $\sigma^-_{i;m}$. Consider the chain complexes

$$C^+_{i;m} \qquad \cdots \to 0 \to \mathcal{K}h(id_m) \xrightarrow{\alpha} \mathcal{K}h(\cup_{i;m}\cap_{i;m})\{-1\} \to 0 \to \cdots \tag{3.26}$$

$$C^-_{i;m} \qquad \cdots \to 0 \to \mathcal{K}h(\cup_{i;m}\cap_{i;m}) \xrightarrow{\beta} \mathcal{K}h(id_m)\{-1\} \to 0 \to \cdots \tag{3.27}$$

where the domain of maps α and β are in (second or homological) degree zero. The map α is $\mathcal{K}h(S_i) = \oplus_{a,b\in\mathcal{C}_m}\mathcal{F}(\mathbf{1}^t_a S_i \mathbf{1}_b)$ where S_i is the minimal cobordism between $\cup_{i;m}\cap_{i;m}$ and id_m. The map β is obtained in the same way from S^t_i which is a cobordism between id_m and $\cup_{i;m}\cap_{i;m}$. The -1 degree shift is to make the map α of (the first) degree zero.

Let $\sigma^+ = \sigma^+_{i;m}$ and $\sigma^- = \sigma^-_{i;m}$ be as in the Figure 3.4. Khovanov defines

$$\mathcal{K}h(\sigma^+) = C^+_{i;m}\{-1\} \tag{3.28}$$

$$\mathcal{K}h(\sigma^-) = C^-_{i;m}[1]\{2\}. \tag{3.29}$$

Now let $n_0T_0n_1T_1n_2\cdots n_kT_kn_{k+1}$ be a decomposition of a tangle T into elementary tangles.

Definition 3.5.1.

$$\mathcal{K}h(T) := \mathcal{K}h(T_0)\otimes_{H^{n_1}}\mathcal{K}h(T_1)\otimes_{H^{n_2}}\cdots\otimes_{H^{n_k}}\mathcal{K}h(T_k) \tag{3.30}$$

In [15], Khovanov defines his invariant using the cube of resolutions and obtains the above equation as a consequence. He also shows that $\mathcal{K}h(T)$ is independent of the decomposition and is invariant under isotopies of T. If L is a link, the homology of $\mathcal{K}h(L)$ is the original Khovanov homology [14] of L with its first grading reversed. We set

$$\overline{\mathcal{K}h}^i(T) = \bigoplus_{k+j=i} H(\mathcal{K}h(T))^{j,k}. \tag{3.31}$$

Seidel and Smith [35] conjecture that their invariant $\mathcal{HSS}$ is equal to $\overline{\mathcal{Kh}}$.

Lemma 3.5.2. *i) We have*

$$\mathcal{Kh}(\cup_{i;m}\cap_{i;m}) = \Big(\Big(H^{m-1}\bigoplus \bar{H}^m \bigoplus \bar{H}^m \bigoplus H_1^m\Big)\otimes \mathcal{V}\{1\}\otimes\mathcal{V}\Big)\bigoplus H_2^m\otimes\mathcal{V}\{2\}.$$

ii)On the first four direct summands, the map $\alpha : \mathcal{F}(a^t id_m b) \to \mathcal{F}(a^t \cup_{i;m} \cap_{i;m} b)$ is given by the comultiplication $\Delta : \mathcal{V}\{1\} \to \mathcal{V}\{1\}\otimes\mathcal{V}$ tensored with the identity map. On the last one it is given by the multiplication $m : \mathcal{V}\{1\}\otimes\mathcal{V}\{1\} \to \mathcal{V}$ tensored with the identity.
iii)On the first four direct summands, the map $\beta : \mathcal{F}(a^t \cup_{i;m} \cap_{i;m} b) \to \mathcal{F}(a^t id_m b)$ is given by the multiplication $m : \mathcal{V}\otimes\mathcal{V} \to \mathcal{V}$ tensored with the identity map. On the last one it is given by the comultiplication Δ tensored with the identity.

iv)

$$\begin{aligned}\overline{\mathcal{Kh}}(\sigma_k^+) &= \left(H^{m-1}\oplus\bar{H}^m\oplus\bar{H}^m\oplus H_1^m\right)\otimes\mathcal{V}\{-1\}\bigoplus H_2^m\otimes\mathcal{V}\{2\}\\ \overline{\mathcal{Kh}}(\sigma_k^-) &= \left(H^{m-1}\oplus\bar{H}^m\oplus\bar{H}^m\oplus H_1^m\right)\otimes\mathcal{V}\{3\}\bigoplus H_2^m\otimes\mathcal{V}\{2\}\end{aligned}$$

Proof. i) Follows easily by comparison to (3.17).
ii) This is because in the first four summands the cobordism merges two circles into one and in the last one it decomposes a circle into two.
iii) Similarly because in the first four summands the cobordism decomposes one circle into two and in the last one it merges two circles into one.
iv) Since $m : \mathcal{V}\otimes\mathcal{V} \to \mathcal{V}$ is surjective we have

$$H^1(C_{i;m}^-) = \frac{H_2^m\otimes\mathcal{V}\{1\}\otimes\mathcal{V}\{1\}\{-1\}}{H_2^m\otimes\operatorname{Im}\Delta} = H_2^m\otimes\frac{\mathcal{V}\otimes\mathcal{V}\{1\}}{\operatorname{Im}\Delta}.$$

$\mathcal{V}\otimes\mathcal{V}\{1\}/\operatorname{Im}\Delta$ is isomorphic to $\mathbb{Z} < \mathit{1}\otimes\mathit{1}, \mathit{1}\otimes\mathcal{X} - \mathcal{X}\otimes\mathit{1} > \{1\}$. Therefore

$$H^1(C_{i;m}^-) = H_2^m\{-1\}\oplus H_2^m\{1\} \cong H_2^m\otimes\mathcal{V}.$$

The map Δ is injective and the kernel of $m : \mathcal{V}\{1\}\otimes\mathcal{V} \to \mathcal{V}\{1\}\{-1\}$ equals $\mathbb{Z} < \mathcal{X}\otimes\mathcal{X}, \mathit{1}\otimes\mathcal{X} - \mathcal{X}\otimes\mathit{1} >$. Tensoring with $\mathcal{X}\otimes\mathcal{X}$ has the effect of shifting degree by two, and tensoring with $\mathit{1} \otimes \mathcal{X} - \mathcal{X} \otimes \mathit{1}$ does not shift the degree. Therefore

$$H^0(C^-_{i;m}) = \Big(H^{m-1}\{1\} \bigoplus \bar{H}^m\{1\} \bigoplus \bar{H}^m\{1\} \bigoplus H^m_1\{1\}\Big) \otimes (\mathbb{Z}\{2\} \oplus \mathbb{Z})$$

which is isomorphic to $\big(H^{m-1} \bigoplus \bar{H}^m \bigoplus \bar{H}^m \bigoplus H^m_1\big) \otimes \mathcal{V}\{2\}$.

Therefore

$$\begin{aligned} \overline{\mathcal{Kh}}(\sigma^-_k) &= (H^0(C^-_{k;m}) \oplus H^1(C_{k;m})\{1\})\{+1\} \\ &= \big(H^{m-1} \oplus \bar{H}^m \oplus \bar{H}^m \oplus H^m_1\big) \otimes \mathcal{V}\{2+1\} \bigoplus H^m_2 \otimes \mathcal{V}\{2\} \end{aligned}$$

Now for σ^+_i we have

$$\begin{aligned} H^0(C^+_{i;m}) &= H^m_2 \otimes (\ker m : \mathcal{V}\{1\} \otimes \mathcal{V}\{1\} \to \mathcal{V}\{2\}\{-1\}) \\ &= H^m_2 \otimes \mathbb{Z} < \mathcal{X} \otimes \mathcal{X}, \mathit{1} \otimes \mathcal{X} - \mathcal{X} \otimes \mathit{1} > \{2\} \\ &\cong H^m_2 \otimes \mathcal{V}\{3\}. \end{aligned}$$

$$\begin{aligned} H^1(C^+_{i;m}) &= \big(H^{m-1} \bigoplus \bar{H}^m \bigoplus \bar{H}^m \bigoplus H^m_1\big) \otimes \operatorname{coker}(\Delta : \mathcal{V}\{1\} \to \mathcal{V}\{1\} \otimes \mathcal{V})\{-1\} \\ &= \big(H^{m-1} \bigoplus \bar{H}^m \bigoplus \bar{H}^m \bigoplus H^m_1\big) \otimes \{\mathit{1} \otimes \mathit{1}, \mathit{1} \otimes \mathcal{X} - \mathcal{X} \otimes \mathit{1}\} \\ &\cong \big(H^{m-1} \bigoplus \bar{H}^m \bigoplus \bar{H}^m \bigoplus H^m_1\big) \otimes \mathcal{V}\{-1\} \end{aligned}$$

$$\begin{aligned} \overline{\mathcal{Kh}}(\sigma^+_k) &= (H^0(C^+_{k;m}) \oplus H^1(C_{k;m})\{1\})\{-1\} \\ &= \big(H^{m-1} \oplus \bar{H}^m \oplus \bar{H}^m \oplus H^m_1\big) \otimes \mathcal{V}\{-1\} \bigoplus H^m_2 \otimes \mathcal{V}\{3-1\} \end{aligned}$$

$\square$

Chapter 4

The symplectic invariant of tangles

In this chapter we review the construction of Seidel and Smith [35].Denote by Conf_m the space of all unordered m-tuples of distinct complex numbers $(z_1, \cdots, z_m)$. Denote by Conf_m^0 the subset of Conf_m consisting of m-tuples which add up to zero, i.e. $z_1 + \cdots + z_m = 0$.

4.1 Transverse slices

In this section we review Section 2 of [35]. The basic reference for this material is [36]. Let G be a complex semisimple Lie group and consider the adjoint action of G on its Lie algebra $\mathfrak{g}$. The adjoint quotient map $\chi : \mathfrak{g} \to \mathfrak{g}/G$ sends each element of $\mathfrak{g}$ to its orbit in $\mathfrak{g}/G$. A theorem of Chevalley (See [9], Chapter 23) asserts that the algebraic quotient $\mathfrak{g}//G$ can be identified with $\mathfrak{h}/W$ where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and W the associated Weyl group. Therefore χ can be regarded as assigning to each $y \in \mathfrak{g}$ the eigenvalues (or equivalently coefficients of the characteristic polynomial) of the semisimple part of y.

Definition 4.1.1. *A transverse slice S for the adjoint action at $x \in \mathfrak{g}$ is a locally closed subvariety $\mathcal{S}$ of $\mathfrak{g}$ containing x which is invariant under the action of the isotropy subgroup G_x, such that $\dim S$ equals the codimenion of the orbit of x and such that the morphism*

$$G \times S \to \mathfrak{g}, \quad (g, s) \mapsto \mathrm{Ad}_g\, s$$

is smooth.

It is evident that such an $\mathcal{S}$ intersects the orbit of any y sufficiently close to x

transversely. If K is an open submanifold of G containing the identity such that T_eK is complementary to $\{y \in \mathfrak{g} : [x, y] = 0\}$ then it can be easily seen that any other transverse slice at x lies (locally) in the image of the map

$$\mathrm{Ad} : K \times S \to \mathfrak{g}. \tag{4.1}$$

The Jacobson-Morozov lemma [11] tells us that if $x \in \mathfrak{g}$ is nilpotent then there are elements $y, h \in \mathfrak{g}$ such that

$$[x, y] = h \qquad [x, h] = -2x \qquad [y, h] = 2y.$$

Now we specialize to $\mathfrak{g} = \mathfrak{sl}_{2m} = \mathfrak{sl}_{2m}(\mathbb{C})$. In this case $W = S_n$. Consider the vector field K on $\mathfrak{g}$ given by $K(z) = 2z - [h, z]$. It defines a $\mathbb{C}^*$ action on $\mathfrak{g}$ given by $\lambda_r(z) = r^2 e^{-\log(r)h} z e^{\log(r)h}$ for $r \in \mathbb{C}^*$. The vector field K vanishes at x so x is a fixed point of λ_r. A slice at x is called *homogeneous* if it is invariant under λ_r. Take x to be a nilpotent Jordan matrix of the form (m, m). It can be written as

$$\begin{pmatrix} \mathbf{0}_{2\times 2} & \mathbf{1}_{2\times 2} & & & \\ \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{1}_{2\times 2} & & \\ \vdots & & & \ddots & \\ \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \cdots & \mathbf{0}_{2\times 2} & \mathbf{1}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \cdots & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} \end{pmatrix} \tag{4.2}$$

where $\mathbf{1}_{2\times 2}$ and $\mathbf{0}_{2\times 2}$ are the 2×2 identity matrices respectively. In this case y and h can be chosen to be the following matrices.

$$h = \begin{pmatrix} (m-1)\mathbf{1}_{2\times 2} & & & \\ & (m-3)\mathbf{1}_{2\times 2} & & \\ \vdots & & (m-5)\mathbf{1}_{2\times 2} & \\ & & & \\ & & & (-m+1)\mathbf{1}_{2\times 2} \end{pmatrix} \tag{4.3}$$

$$y = \begin{pmatrix} \mathbf{0}_{2\times 2} & & & & & \\ (n-1)\mathbf{1}_{2\times 2} & \mathbf{0}_{2\times 2} & & & & \\ \mathbf{0}_{2\times 2} & 2(n-2)\mathbf{1}_{2\times 2} & \mathbf{0}_{2\times 2} & & & \\ \vdots & \ddots & & & & \\ \mathbf{0}_{2\times 2} & \cdots & & 2(n-2)\mathbf{1}_{2\times 2} & \mathbf{0}_{2\times 2} & \\ \mathbf{0}_{2\times 2} & & & \cdots & (n-1)\mathbf{1}_{2\times 2} & \mathbf{0}_{2\times 2} \end{pmatrix} \tag{4.4}$$

Let $\mathcal{S}_m$ be the set of matrices in $\mathfrak{sl}_{2m}$ of the form

$$\begin{pmatrix} y_1 & \mathbf{1}_{2\times 2} & & & \\ y_2 & & \mathbf{1}_{2\times 2} & & \\ \vdots & & & \ddots & \\ y_{m-1} & & & & \mathbf{1}_{2\times 2} \\ y_m & & & & \mathbf{0}_{2\times 2} \end{pmatrix} \tag{4.5}$$

Here $y_1 \in \mathfrak{sl}_2$ and $y_i \in \mathfrak{gl}_2$ for $i > 1$. It is easy to see that $\mathcal{S}_m$ is a homogeneous slice to the orbit of x ([35], Lemma 23). χ restricted to Conf^0_{2m} is a differentiable fiber bundle ([35], Lemma 20). We denote the fiber of χ over t by $\mathcal{Y}_{m,t}$, i.e. $\mathcal{Y}_{m,t} = \chi^{-1}(t)$. If $t = (\mu_1, \ldots, \mu_{2m}) \notin \mathrm{Conf}^0$, by $\mathcal{Y}_{m,t}$ we mean $\mathcal{Y}_{m,t'}$ where $t' = (\mu_1 - \sum \mu_i/2m, \ldots, \mu_{2m} - \sum \mu_i/2m)$. Let E^μ_y denote the μ-eigenspace of y.

Lemma 4.1.2 ([35], Lemmata 25 and 25)**.** *For any $y \in \mathcal{S}_m$ and $\mu \in \mathbb{C}$ the projection $\mathbb{C}^{2m} \to \mathbb{C}^2$ onto the first two coordinates gives an injective map $E^\mu_y \to \mathbb{C}^2$. Any eigenspace of any element $y \in \mathcal{S}_m$ has dimension at most two. Moreover the set of elements of $\mathcal{S}_m$ with 2 dimensional kernel can be canonically identified with $\mathcal{S}_{m-1}$ and this identification is compatible with χ.*

Proof. If the first claim is not true then the intersection of E^μ_y with $\{0\}^2 \times \mathbb{C}^{2m-2}$ is nonzero. Applying the $\mathbb{C}^*$ action we see that the same holds for $E^{r^2\mu}_{\lambda_r(y)}$. As r

goes to zero, $\lambda_r(y) \to x$ so we get $\dim\ker x > 2$ which is contradiction. From this injectivity we see that each element of $\ker y$ is determined by its first two coordinates so if $\dim\ker y = 2$ then $y_m = 0$ and vice versa. The subset of such matrices is identified with $\mathcal{S}_{m-1}$. $\square$

For a subset $A \subset \mathfrak{sl}_{2m}$, let $A^{sub,\lambda}$ (resp. $A^{sub3,\lambda}$) be the subset of matrices in A having eigenvalue λ of multiplicity two (resp. three) and two Jordan blocks of size one (resp. two Jordan blocks of sizes 1,2) corresponding to the eigenvalue λ and no other coincidences between the eigenvalues. Here are two results describing neighborhoods of $\mathcal{S}_m^{sub,\lambda}$ and $\mathcal{S}_m^{sub3,\lambda}$ in $\mathcal{S}_m$.

Lemma 4.1.3 ([35], Lemma 27). *Let $D \subset \mathrm{Conf}^0_{2m}$ be a disc consisting of the $2m$-tuples $(\mu-\varepsilon, \mu-\varepsilon, \mu_3, \ldots, \mu_{2m})$ with $\varepsilon \in \mathbb{C}$ of small magnitude. Then there is a neighborhood U_μ of $\mathcal{S}_m^{sub,\mu}$ in $\mathcal{S}_m \cap \chi^{-1}(D)$ and an isomorphism ϕ of U_μ with a neighborhood of $\mathcal{S}_m^{sub,\mu}$ in $\mathcal{S}_m^{sub,\mu} \times \mathbb{C}^3$ such that $f \circ \phi = \chi$ on $\mathcal{S}_m \cap \chi^{-1}(D)$ where $f(x,a,b,c) = a^2+b^2+c^2$. Also if $N_y\mathcal{S}_m^{sub,\mu}$ denotes the normal bundle to $\mathcal{S}_m^{sub,\mu}$ at y the we have*

$$\phi(N_y\mathcal{S}_m^{sub,\mu}) = \mathfrak{sl}(E_y^\mu) \oplus \zeta_y \tag{4.6}$$

where ζ_y is the trace free part of $\{\mathbb{C}\cdot 1 \subset \mathfrak{gl}(E_y^\mu)\} \oplus \mathfrak{gl}(E_y^{\mu_3}) \oplus \ldots \oplus \mathfrak{gl}(E_y^{\mu_{2m}})$.

Proof. For $y \in \mathcal{S}_m^{sub,\mu}$, let $\mathcal{S}_y$ be a subspace of $T_y\mathcal{S}_m$ complementary to $T_y\mathcal{S}_m^{sub,\lambda}$ which depends holomorphically on y. These subspaces together form a tubular neighborhood of $\mathcal{S}_m^{sub,\mu}$ in $\mathcal{S}_m$. Since $\mathcal{S}_m$ and $\mathfrak{sl}_{2m}^{sub,\mu}$ intersect transversely, $\mathcal{S}_y$ is also a transverse slice at y for the adjoint action on $\mathfrak{sl}_{2m}$. We can produce another family of transverse slices $\mathcal{S}'_y$ by setting $\mathcal{S}'_y = \mathfrak{sl}(E_y^\mu) \oplus \zeta_y$ which equals the trace free part of $\mathfrak{gl}(E_y^\mu) \oplus \mathfrak{gl}(E_y^{\mu_3}) \oplus \cdots \oplus \mathfrak{gl}(E_y^{\mu_{2m}})$. The reason is that $[y, \mathfrak{sl}_{2m}] = 0 \oplus \mathfrak{sl}^0_{2m-2}$ where the first component consists of zero in $\mathfrak{sl}_2$ and the second one consists of matrices with zeros on the diagonal and the right hand side is transverse to $\mathcal{S}'_y$.

Now $\mathcal{S}_y$ is isomorphic (as an open complex manifolds) to $\mathcal{S}'_y$ for each y with an isomorphism that moves points only inside their adjoint orbits (and hence is compatible with χ). We can choose these isomorphisms to depend holomorphically on y. Each $\mathfrak{gl}(E_y^{\mu_i}) \subset \mathfrak{sl}_{2m}$ for $i > 2$ can be canonically (without choice of a basis) identified with $\mathbb{C}$. Lemma 4.1.2 tells us that E_y^{μ} can be canonically identified with $\mathbb{C}^2$ so $\mathfrak{sl}(E_y^{\mu})$ is identified with $\mathfrak{sl}_2$. It follows that $\mathcal{S}'_y \cong \mathfrak{sl}_2 \oplus \mathbb{C}^{2m-2}$. The desired ϕ is the composition of the two isomorphisms in this paragraph. □

Remark 4.1.4. *If y has two linearly independent μ_1 eigenvectors as well as two linearly independent μ_2 eigenvectors and with no other coincidences between the eigenvalues, we can repeat the above argument to obtain*

$$\phi(N_y(\mathcal{S}_m^{sub,\mu_1} \cap \mathcal{S}_m^{sub,\mu_2})) = \mathfrak{sl}(E_{\mu_1}) \oplus \mathfrak{sl}(E_{\mu_2}) \oplus \zeta. \tag{4.7}$$

So ϕ gives an isomorphism between a neighborhood of $\mathcal{S}_m^{sub,\mu_1} \cap \mathcal{S}_m^{sub,\mu_2}$ in $\mathcal{S}_m$ and $(\mathcal{S}_m^{sub,\mu_1} \cap \mathcal{S}_m^{sub,\mu_2}) \times \mathbb{C}^3 \times \mathbb{C}^3$.

Consider the line bundle $\mathcal{F}$ on $\mathcal{S}_m^{sub3,\mu}$ whose fiber at $y \in \mathcal{S}_m^{sub3,\mu}$ is $(y-m)E_{y_s}^{\mu}$ where y_s is the semisimple part of y. To $\mathcal{F}$ one associates a $\mathbb{C}^4$ bundle $\mathcal{L} = (\mathcal{F}\backslash 0) \times_{\mathbb{C}^*} \mathbb{C}^4$ where $z \in \mathbb{C}^*$ acts on $\mathbb{C}^4$ by

$$(a, b, c, d) \to (a, z^{-2}b, z^2c, d). \tag{4.8}$$

$\mathcal{L}$ decomposes as

$$\mathcal{L} \cong \mathbb{C} \oplus \mathcal{F}^{-2} \oplus \mathcal{F}^2 \oplus \mathbb{C}. \tag{4.9}$$

Fibers of $\mathcal{L}$ should be regarded as transverse slices in $\mathfrak{sl}_3$. Upon choosing suitable coordinates on such a transverse slice (at the zero matrix) the function χ equals the function $p : \mathfrak{sl}_3 \to \mathbb{C}^2$ given by

$$p(a, b, c, d) = (d, a^3 - ad + bc). \tag{4.10}$$

p is also well-defined as a function $\mathcal{L} \to \mathbb{C}^2$ because b and c are coordinates on line bundles which are inverses of each other. Denote by $\tau(d, z)$ the set of solutions of $\lambda^3 - d\lambda + z = 0$.

Lemma 4.1.5 ([35], Lemma 28)**.** *Let $P \subset \mathrm{Conf}_{2m}^0$ be the set of $2m+2$-tuples*

$$(\mu_1, \ldots, \mu_{i-1}, \tau(d, z), \mu_{i+3}, \ldots, \mu_{2m+2}). \tag{4.11}$$

where d and z vary in a small disc in $\mathbb{C}$ containing the origin. There is a neighborhood V of $\mathcal{S}_m^{sub3}$ in $\mathcal{S}_m \cap \chi^{-1}(P)$ and an isomorphism ϕ' from V to a neighborhood of zero section in $\mathcal{L}$ such that $p(\phi'(x)) = (d, z)$ if

$$\chi(x) = (\mu_1, \dots, \mu_{i-1}, \tau(d, z), \mu_{i+3}, \dots, \mu_{2m+2}).$$

4.2 Relative vanishing cycles

Let X be a complex manifold and K a compact submanifold. Let g be a Kähler metric on $Y = X \times \mathbb{C}^3$ (not necessarily the product metric) and denote its imaginary part by Ω. Consider the map $f : X \times \mathbb{C}^3 \to \mathbb{C}$ given by $f(x, a, b, c) = a^2 + b^2 + c^2$ and denote by ϕ_t the gradient flow of $-\operatorname{Re} f$. Let W be the set of points $y \in Y$ for which the trajectory $\phi_t(y)$ exists for all positive t.

Lemma 4.2.1. *W is a manifold and the mapping $l : W \to X$ given by $l(y) = \lim_{t\to\infty} \phi_t(y)$ is well-defined and smooth. We have $\Omega|_W = l^*\Omega|_X$. The function f restricted to W is real and nonnegative.*

Proof. The first two assertions follow from stable manifold theorem (Theorem 1 in [8]) and the rest from the fact that gradient vector field of $-\operatorname{Re} f$ is the Hamiltonian vector field of $\operatorname{Im} f$. □

Set $V_t(K) = \pi^{-1}(t) \cap l^{-1}(K) = l|_{\pi^{-1}(t)}^{-1}(K)$ which is a manifold for t small. It follows from Morse-Bott lemma that $V_t(K)$ is a 2-sphere bundle on K for t small. To generalize the invariant to tangles we will need a slightly more general version of the above construction in which K is noncompact and the metric equals the product metric outside a compact subset. (See section 5.1.) The resulting vanishing cycle equals (symplectically) the product bundle outside a compact subset.

4.3 Fibered A_2 singularities

Assume we have the same situation as in the Lemma 4.1.5, i.e. let $\mathcal{F}$ be a holomorphic line bundle over a complex manifold X and define Y to be $(\mathcal{F}\backslash 0)\times_{\mathbb{C}^*}\mathbb{C}^4$ where the $\mathbb{C}^*$ action is as in the formula (4.8). Let Ω be an arbitrary Kähler form on Y and by regarding X as the zero section of Y, Ω restricts to a Kähler form on X. Let (a,b,c,d) be the coordinates on fibers of $Y\to X$ and (d,z) coordinates on $\mathbb{C}^2$. Let the map $p: Y\to\mathbb{C}^2$ be as in Lemma 4.1.5. Let $Y_d = p^{-1}(\mathbb{C}\times\{d\})$ and $p_d : Y_d\to\mathbb{C}$ be the restriction of p. Set $Y_{d,z} = p^{-1}(d,z)$. For $d\neq 0$ the critical values of p_d are $\zeta_d^{\pm} = \pm 2\sqrt{d^3/27}$.

Let K be Lagrangian submanifold of X. Using the relative vanishing cycle construction for the function p_d we can obtain a Lagrangian submanifold L_d of Y which is a sphere bundle over K. (This construction works when Y is a nontrivial bundle over X as well.) There is another way of describing this Lagrangian as follows. Let $\mathrm{Y}\cong\mathbb{C}^4$ be the fiber of $Y\to X$ over some point of X and let $p:\mathrm{Y}\to\mathbb{C}^2$ be as before. The restriction of the $\mathbb{C}^*$ action to S^1 is a Hamiltonian action with the moment map $\mu(a,b,c,d) = |c|^2-|b|^2$. Define

$$C_{d,z,a} = \{(b,c) : \mu(b,c) = 0, a^3-da-z=-bc\}\subset \mathrm{Y}_{d,z} \tag{4.12}$$

which is a point if $a^3-da-z=0$ and a circle otherwise. The three solutions of this equation correspond to the critical values of the projection $q_{d,z}:\mathrm{Y}_{d,z}\to\mathbb{C}$ to the a plane. In the situation of Lemma 4.1.5 they correspond to the three eigenvalues of Y. Let $\alpha(r)$ be any embedded curve in $\mathbb{C}$ which intersects these critical values (only) if $r=0,1$. Define

$$\Lambda_\alpha = \bigcup_{r=0}^{1} C_{d,z,\alpha(r)} \tag{4.13}$$

which is a Lagrangian submanifold of $\mathrm{Y}_{d,z}$ (with Kähler form induced from $\mathbb{C}^4$). Let $\mathbf{c},\mathbf{c}',\mathbf{c}''$ be as in the Figure 4.1 where dots represent the critical values of $q_{d,z}$. We can associate to K a Lagrangian submanifold $\Lambda_{d,\alpha}$ of Y by defining

$\Lambda_{d,\alpha} = (Y|K) \times_{S^1} \Lambda_{\mathbf{c}}$. Seidel and Smith prove that these two procedures give the same result ([35], Lemma 40):

Lemma 4.3.1. *If the Kähler form on Y is obtained from a Kähler form on X, a Hermitian metric on $\mathcal{F}$ and the standard form on $\mathbb{C}^4$ then $L_d = \Lambda_{d,\mathbf{c}}$.*

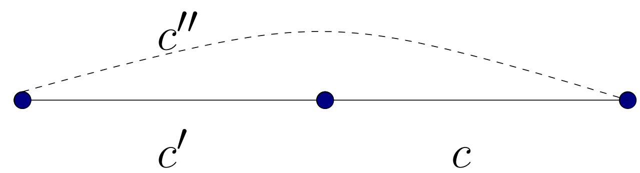

Figure 4.1:

4.4 Symplectic structure

The symplectic structure that Seidel and Smith use on $\mathcal{S}_m$ is not the standard structure on $\mathcal{S}_m \cong \mathbb{C}^{4m-1}$. This is to obtain well-defined parallel transport maps for the fibration χ whose fibers are noncompact. We need a plurisubharmonic function whose fiberwise critical point sets project properly under χ. Fix $\alpha > m$. Let $\xi_i(z) = |z|^{2\alpha/i}$ for $z \in \mathbb{C}$ and $i = 1, \dots m$. These functions are only C^1 but by adding compactly supported functions η_i we can obtain C^∞ functions $\psi_i = \xi_i + \eta_i$ on $\mathbb{C}$. We choose η_i such that $-dd^c\psi_i > 0$. Let ψ be the function on $\mathcal{S}_m$ whose value at $y \in \mathcal{S}_m$ is

$$\sum_{i=1}^{m} \sum_{\mu,\nu \in \{0,1\}} \psi_i((y_i)_{\mu,\nu}).$$

Here the y_i are the components of y as in (4.5). We can choose η_i so that ψ is an exhausting plurisubharmonic function on $\mathcal{S}_m$ which gives us the symplectic form $\Omega = -dd^c\psi$. Outside a set which is the product of the complement of a compact set in each coordinate plane, we have

$$\Omega = 4 \sum_{i=1}^{m} (\alpha/i)(\alpha/i - 1) \sum_{\mu,\nu \in \{0,1\}} |(y_{1i})_{\mu,\nu}|^{2\alpha/i} d(y_{1i})_{\mu,\nu} \wedge d\overline{(y_{1i})_{\mu,\nu}}.$$

By restriction we obtain Stein structures on each $\mathcal{Y}_{m,t}$. The addition of the functions η_i prevents ψ from being homogeneous with respect to the λ_r action but as $r \to \infty$ the functions $\eta_i(r^i z)$ are supported on smaller and smaller neighborhood of origin so $\eta_i(r^i z)/r^{2\alpha}$ go to zero and so we get the asymptotic homogeneity of ψ, i.e.:

$$\lim_{r\to\infty} \frac{\psi \circ \lambda_r}{r^{2\alpha}} = \xi. \tag{4.14}$$

Since the fibers $\mathcal{Y}_{m,t}$ are noncompact, existence of parallel transport maps for the fiber bundle $\chi|_{\mathrm{Conf}^0_{2m}}$ is not guaranteed. Let β be a curve in Conf_{2m}. Let H_β be the horizontal lift of $\dot{\beta}$ and $Z_{\beta(s)}$ be the projection of $\nabla\psi(\beta(s))$ to $\mathcal{Y}_{m,\gamma(s)}$. Seidel and Smith obtain a *rescaled parallel transport* map $h^{res}_\beta : \mathcal{Y}_{m,\beta(0)} \to \mathcal{Y}_{m,\beta(1)}$ which is given by integrating the vector field

$$H_\beta - \sigma Z_\beta \tag{4.15}$$

and then composing by the time σ map of $Z_{\beta(1)}$, where σ is a positive constant (depending on β). The map h^{res}_β is a symplectomorphism defined on arbitrarily large compact subsets of $\mathcal{Y}_{m,\beta(0)}$. For this procedure to work, one needs the fiberwise critical point set of ψ to project properly under χ. The homogeneity property (4.14) ensures this. See [35] Section 5A.

If $\mu \in \mathbb{C}^{2m}/\mathbf{S}_{2m}$ has only one element of multiplicity two or higher, which we denote by μ_1, denote by $\mathcal{D}_{m,\mu}$ the set of singular elements of $(\chi^{-1}(\mu) \cap \mathcal{S}_m)$ i.e.

$$\mathcal{D}_{m,\mu} = (\chi^{-1}(\mu) \cap \mathcal{S}_m)^{sub,\mu_1}. \tag{4.16}$$

Let $\mathcal{D}_m$ be the union of all these $\mathcal{D}_{m,t}$ regarded as a subset of $\mathcal{S}_m$. It inherits a Kähler metric from $\mathcal{S}_m$. We have the map $\chi : \mathcal{D}_m \to \mathbb{C} \times \mathbb{C}^{2m-2}/\mathbf{S}_{2m-2}$. By forgetting the first eigenvalue, the image of χ can be identified with Conf_{2m-2}. Lemma 4.1.2 tells us that if $\mu_1 = 0$ then $\mathcal{D}_{m,\mu}$ can be identified with $\mathcal{Y}_{m-1,\mu\setminus\{\mu_1\}}$. It can be shown ([35] Section 5A) that χ is a differentiable fiber bundle and we have rescaled parallel transport maps

$$h^{res}_\beta : \mathcal{D}_{m,\beta(0)} \to \mathcal{D}_{m,\beta(1)} \tag{4.17}$$

for any curve β in Conf_{2m-2}. These parallel transport maps are compatible with those for $\mathcal{Y}_{m-1,\mu\setminus\{0\}}$ under the identification above provided that β lies in Conf^0_{2m-2}. This is because of the special (product) form of the symplectic structure.

4.5 Lagrangian submanifolds from crossingless matchings

Let $\mu \in \mathrm{Conf}_{2m}$. A *crossingless matching* D with endpoints in μ is a set of m disjoint embedded curves $\delta_1, \dots, \delta_m$ in $\mathbb{C}$ which have (only) elements of μ as endpoints. See Figure 4.2. To D we associate a Lagrangian submanifold L_D of $\mathcal{Y}_{m,\mu}$ as follows. Let $\{\mu_{2k-1}, \mu_{2k}\} \subset \mu$ be the endpoints of δ_k for each k. Let γ be a curve in Conf^0_m such that $\gamma(t) = \{\gamma_1(t), \gamma_2(t), \mu_3, \mu_4, \dots, \mu_{2m}\}$, $\gamma_i(0) = \mu_i, i = 1, 2$ and as $s \to 1$, $\gamma_1(t), \gamma_2(t)$ approach each other on δ_1 and collide. For example if $\delta_1(t)$ is a parametrization of δ_1 s.t $\delta_1(0) = \mu_1, \delta_1(1) = \mu_2$ the we can take $\gamma(t) = \{\delta(t/2), \delta(1-t/2), \mu_3, \dots, \mu_{2m}\}$. Set $\bar{\mu} = \mu\setminus\{\mu_1, \mu_2\}$, $\mu' = \gamma(1)$.

If $m = 1$ then relative vanishing cycle construction for $\chi : \mathcal{S}_1 \to \mathbb{C}$ with the critical point over $\gamma(1) = 0$ gives us a Lagrangian sphere L in $\mathcal{Y}_{1,\gamma(1-s)}$ for small s. Using reverse parallel transport along γ we can move L to $\mathcal{Y}_{1,\mu}$ to get our desired Lagrangian submanifold. Now for arbitrary m assume by induction that we have obtained a Lagrangian $L_{\bar{D}} \subset \mathcal{Y}_{m-1,\bar{\mu}}$ for $\bar{D}$ which is obtained from D by deleting δ_1. Now $\mathcal{Y}_{m-1,\bar{\mu}}$ can be identified with $D_{m,\tau}$ where $\tau = (0, 0, \mu_3 - (\mu_1+\mu_2)/(2m-2), \dots, \mu_{2m} - (\mu_1+\mu_2)/(2m-2))$. Use parallel transport to move $L_{\bar{D}}$ to $D_{m,\gamma(1)}$. The later one is the set of singular points of $\mathcal{Y}_{m,\gamma(1)}$ so Lemma 4.1.3 tells us that we can use relative vanishing cycle construction for $L_{\bar{D}}$ to obtain a Lagrangian in $\mathcal{Y}_{m,\gamma(1-s)}$ for small s. Parallel transporting it along γ back to $\mathcal{Y}_{m,\mu}$ we obtain our desired Lagrangian which is topologically a trivial sphere bundle on $L_{\bar{D}}$. We see that L_D is diffeomorphic to a product of spheres. Different choices of the curve γ result in Hamiltonian isotopic Lagrangians. The same holds if we isotope the curves in D inside $\mathbb{C}\setminus\mu$.

Figure 4.2: Two crossingless matchings

4.6 The Seidel-Smith invariant

Now we can define the Seidel-Smith invariant. Since each manifold $\mathcal{Y}_{m,\nu}$ is a submanifold of the affine space $\mathcal{S}_m$ and has trivial normal bundle, its Chern classes are zero. This together with the fact that $H_1(\mathcal{Y}_m) = 0$ implies that the canonical bundle of $\mathcal{Y}_{m,\nu}$ is trivial and so has a unique infinite Maslov cover. We start by choosing global sections $\eta_{\mathcal{S}_m}$ and $\eta_{\mathfrak{h}/W}$. Then we choose trivializations for regular fibers of $\chi_{\mathcal{S}_m}$ characterized by $\eta_{\mathcal{Y}_{m,t}} \wedge \chi^* \eta_{\mathfrak{h}/W} = \eta_{\mathcal{S}_m}$. If we choose a grading for $L \subset \mathcal{Y}_{m,t_0}$ and β is a curve in Conf_{2m} starting at t_0, one can continue the grading on L uniquely to $h_{\beta|_{[0,s]}}(L)$ for any s. Therefore the grading of L uniquely determines that of $h_\beta(L)$.

Let $\mathcal{D}_+$ be the crossingless matching at the left hand side of picture 4.2. If a link K is obtained as closure of a braid $\beta \in Br_m$, let $\beta' \in \mathrm{Conf}_{2m}$ be the braid obtained from β by adjoining the identity braid id_m.

Definition 4.6.1.

$$\mathcal{HSS}^*(K) = HF^{*+m+w}(L_{\mathcal{D}_+}, h^{res}_{\beta'}(L_{\mathcal{D}_+}))$$

Here w is the writhe of the braid presentation, i.e. the number of positive crossings minus the number of the negative crossings in the presentation. Since the manifold is convex at infinity and the Lagrangians are exact, the above Floer cohomology is well-defined. Independence from choice of β is proved in [35], section 5C. Well-definedness of the invariant developed in this thesis gives an alternative proof. See Theorem 5.1.8.

Chapter 5

Generalization to tangles

In this Chapter we generalize the invariant of Seidel and Smith to tangles. This material can be found in [29] as well.

5.1 The functor associated to a tangle

Let

$$n_0 T_1 n_1 T_2 \ldots n_{l-1} T_l n_l \tag{5.1}$$

be a decomposition of an oriented tangle T into elementary tangles. Set $\nu_j = i(T_j)$ and $\nu_{l+1} = t(T_l)$ We have $\nu_i \in \mathrm{Conf}_{n_i}$ for $i = 0, \ldots, l$. To each T_i we want to associate a Lagrangian correspondence $L_{i,i+1} = L_{T_i}$ between $\mathcal{Y}_{n_i,\nu_i}$ and $\mathcal{Y}_{n_{i+1},\nu_{i+1}}$. In this way we can associate to T a generalized Lagrangian correspondence

$$\Phi(T) = (L_{0,1}, L_{1,2}, \ldots, L_{n-1,n})\{-m - w\} \tag{5.2}$$

from $\mathcal{Y}_n$ to $\mathcal{Y}_m$. Here m and w are the number of cups and the writhe (number of positive crossings minus the number of negative ones) of the decomposition respectively.

If T_k is an elementary braid in Br_{2m}, we set L_{T_k} to be $\mathrm{graph}(h^{res}_\beta)$ regardless of the orientation of the braid. Of course we can extend this definition to any braid. We note that the symplectomorphisms h^{res}_β are defined only on compact submanifols of the $\mathcal{Y}_m$, however this does not cause any problem since we are going to take the $\mathfrak{C}_c$ of these Lagrangians. We will make use of an alternative description of these symplectomorphisms as Dehn twists around vanishing cycles.

This description is not indispensable for our purpose but makes the presentation simpler. Let V_i be the relative vanishing cycle for the map f in Lemma 4.1.3 where ith and $(i+1)$th eigenvalues (μ_i, μ_{i+1}) of ν_k come together at some point μ.

There is a theorem of T. Perutz (generalizing an earlier result of P. Seidel) which describes monodromy maps around singularities of symplectic Morse-Bott fibrations as Dehn twists. Recall that a symplectic Morse-Bott fibration (also called Lefschetz-Bott fibrations) over a disc D consists of an almost complex manifold (E, J), a closed two-form Ω on E and a J-holomorphic map $\pi : E \to D$ such that the critical point set of π is a submanifold of E and the complex Hessian matrix of π is nondegenerate. The form Ω is required to be a symplectic form when restricted to each fiber.

Theorem 5.1.1 (Perutz [27], Theorem 2.19)**.** *Let $\pi : E \to D$ be a symplectic Morse-Bott fibration over the closed disc D which has only the origin as singular value. If in addition the fibration is normally Kähler, i.e. a neighborhood of the critical point set of π is foliated by J-complex normal slices such that J restricted to each slice is integrable and the restriction of Ω to each fiber is Kähler, then the monodromy map around the origin is Hamiltonian isotopic to the fibred Dehn twist τ_V along the vanishing cycle V for the map π.*

Therefore using the local picture of the Lemma 4.1.3 we see that if we have a subset $B \subset \mathcal{Y}_m$ for which the naive (non-rescaled) parallel transport map $h_{\sigma_i}|_B$ is well-defined then

$$h_{\sigma_i} \cong \tau_{V_i}. \tag{5.3}$$

The reason is that since the naive parallel transport is well-defined for all points of B, we can shrink the rescaling parameter in (4.15) to zero and thereby isotope $h^{res}_{\sigma_i}$ to h_{σ_i}.

Lemma 5.1.2. *Let $\nu = \{\mu_1, \ldots, \mu_{2m}\} \in \mathrm{Conf}^0_{2m}$ and $\gamma : [0,1] \to \mathfrak{h}/W$ such that $\gamma(0) = \nu$ and as $s \to 1$, μ_{2i-1} and μ_{2i} approach each other linearly and collide at* *some μ. Then $h^{res}_{\sigma_i}$ restricted to $\mathcal{S}^{sub,\mu}_m$ is the identity. Similarly if $\mu_{2k-1}, \mu_{2k}, \mu_{2k+1}$ come together at some μ, then σ_i and σ_{i+1} act trivially on $\mathcal{S}^{sub3,\mu}_m$.*

Proof. We use the picture of Lemma 4.1.5. Both statements follow from (7.32) and the fact that fibred Dehn twists are identity outside a small neighborhood of the spherical fibres. □

Let V_{ix} denote the (S^2) fiber of V_i over x. We grade τ_{V_i} in such a way that

$$\tau_{V_i} V_{ix} = V_{ix}\{1\} \tag{5.4}$$

and the grading function vanishes outside a neighborhood of V_i. This grading is unique. (Lemma 5.6 in [32])

Remark 5.1.3. *Monodromy actions of braid group on symplectic manifolds were first constructed in [16]. Parallel transport maps h^{res}_{β} form a homomorphism from $\pi_1(\mathrm{Conf}_{2m}) = Br_{2m}$ into the π_0 of $Symp(\mathcal{Y}_m)$. In particular symplectomorphisms associated to elementary braids satisfy Artin's commutation relations. (Symplectic manifolds used in [16] are compact with boundary. The manifolds $\mathcal{Y}_m$ in [35] that we use here can be obtained from them by attaching an infinite cylinder.)*

If $T_i = \cup_{j;m}$, we define a Lagrangian $L_{\cup_{j;m}}$, regardless of the orientation of $\cup_{k;m}$, as follows. The result depends on a real parameter $R > 0$. To simplify the notation we set $k = j, l = j+1$. With ν_i as given above let $\nu = \nu_i = \{\mu_1, \ldots, \mu_{2m}\}$. Let γ be a curve in Conf^0_{2m} such that $\gamma(0) = \nu_i$ and as $s \to 1$, μ_k and μ_l approach each other linearly and collide at a point μ'. For example we can take

$$\gamma(t) = \{\mu_1, \ldots, \mu_k + t(\mu_l - \mu_k)/2, \ldots, \mu_l - t(\mu_l - \mu_k)/2, \ldots, \mu_{2m}\}$$

provided that $\mu_k + t(\mu_l - \mu_k)/2$ does not intersect the other μ_i. Set $\nu^{k,l} = \nu \backslash \{\mu_k, \mu_l\}$, $\nu' = \gamma(1)$. We use Lemma 4.1.3 to identify a neighborhood of $\mathcal{S}^{sub,\mu'}_m$ in $\mathcal{S}_m$ locally with $\mathcal{S}^{sub,\mu'}_m \times \mathbb{C}^3$. This induces a Kähler form and hence a metric on $\mathcal{S}^{sub,\mu'}_m \times \mathbb{C}^3$. We perturb the complex structure outside a compact ball of radius ρ (to be chosen below) so that outside that set the resulting metric equals the product metric. Now we use the relative vanishing cycle construction for the whole $\mathcal{S}_m^{sub,\mu'}$. It yields (after restriction) a sphere bundle $V = V_{\gamma(1-s)}(\mathcal{S}_m^{sub,\mu'}) \subset \mathcal{Y}_{m,\gamma(1-s)}$ for small s with projection $\pi : V \to \mathcal{Y}_{m,\nu'} \cap \mathcal{S}_m^{sub,\mu'_k}$. The relative vanishing cycle construction can be used because the metric equals the product metric outside a compact set.

We denote the image of V under parallel transport map along $-\gamma$, i.e.

$$h^{-1}_{\gamma|_{[0,1-s]}}(V) \subset \mathcal{Y}_{m,\nu}$$

by the same notation V. Composing π with the parallel transport map $h^{-1}_{\gamma|_{[0,1-s]}}$ we obtain a projection $\pi : V \to \mathcal{Y}_{m,\nu'} \cap \mathcal{S}_m^{sub,\mu'}$ which is a S^2 bundle. By Lemma 4.1.3, $\mathcal{Y}_{m,\nu'} \cap \mathcal{S}_m^{sub,\mu'}$ can be identified with $\mathcal{D}_{m-1,\nu'}$ from (4.16). Let δ be a geodesic in $\mathrm{Conf}_m^{\mathcal{S}}$ joining ν' to $\nu^{k,l}$. We can use parallel transport map (4.17) along the curve δ to map $\mathcal{D}_{m-1,\nu'}$ to $\mathcal{D}_{m-1,\nu^{k,l}\cup\{0,0\}}$. The latter can be identified with $\mathcal{Y}_{m-1,\nu^{k,l}}$. So we obtain a fibration $\pi : V \to \mathcal{Y}_{m-1,\nu^{k,l}}$. We can use this map π to pull V back to $\mathcal{Y}_{m-1,\nu^{k,l}} \times \mathcal{Y}_{m-1,\nu^{k,l}}$. Let $\cup_{j;m}$ be its restriction to the diagonal. It is a Lagrangian submanifold of $\mathcal{Y}^-_{m,\nu_i} \times \mathcal{Y}_{m-1,\nu_i^{k,l}} = \mathcal{Y}^-_{m,\nu_i} \times \mathcal{Y}_{m-1,\nu_{i+1}}$. Let $\psi = \psi_1 + \psi_2$ be the plurisubharmonic function on $\mathcal{Y}^-_{m,\nu_i} \times \mathcal{Y}_{m-1,\nu_{i+1}}$. We can choose ρ in such a way that the inverse image of $\psi = R$ lies inside the ball of radius ρ. We have a projection $\pi : L_{\cup_{j;m}} \to \Delta \subset \mathcal{Y}^-_{m-1,\nu_{i+1}} \times \mathcal{Y}_{m-1,\nu_{i+1}}$.

As in the case of Lagrangians from crossingless matchings, replacing the curve γ with another curve in the same homotopy class (with fixed endpoints) results in a new $L_{\cup_{j;m}}$ which is Lagrangian isotopic to the former one. Sine the first homology group of this Lagrangian is zero, this isotopy is exact.

We grade the $L_{\cup_j}$ as follows. Lemma 5.1.7 below tells us that fibers of $L_{\cup_j}$ and $L_{\cup_{j+1}}$ over each point of the diagonal intersect transversely at only one point. We choose the grading in such a way that the absolute Maslov index of this intersection point (with regard to the two S^2 fibres) equals one. We can use Lemma 4.1.3 and isotope $L_{\cup_j}$ to $\Delta \times \sqrt{z}S^2$ for some small $z \in \mathbb{C}$. For any two Lagrangians L, L' we have $\alpha_{L\times L'} = \alpha_L \times \alpha_{L'}$. Also the two form $\eta_{T^*S^2}$ when restricted to S^2 is a volume form on S^2. Therefore we get $\alpha_{L_C} = \alpha_\Delta \cdot \frac{z}{|z|}$. The function α_Δ is constant. This means that the grading on $L_{\cap_j}$ is determined by the choice of a branch of $\arg(z)$. In particular such a grading is a constant function:

$$\tilde{L}_{\cup_j} = c_j. \tag{5.5}$$

We choose this c_j to be the same for every j. This together with the formula (2.12) (for $n = 2$) imply that the absolute Maslov index of each fiberwise intersection point in $L_{\cup_j} \cap L_{\cup_{j+1}}$ equals 1. Construction for $\cap_j$ is similar.

The following lemma insures that the Lagrangian we assign to crossingless matching agrees with that of Seidel and Smith.

Lemma 5.1.4. *If $C \in \mathcal{C}_m$ is a crossingless matching and arcs of C are isotopic to $T_1, T_2, \ldots, T_m$ where each T_i is either a cap or a cup then L_C is isotopic to $L_{T_1} \circ \cdots \circ L_{T_m}$.*

Proof. We use induction on m. The base case is vacuous. For the induction step we note that our construction is the same as the induction step in the construction of Seidel and Smith (Section 4.5 above) except for the base of the fibration. □

In order for Φ to define a functor, we must verify that the above correspondences satisfy the same commutation relations as the tangles they are associated to. First we have the following cf. Remark 5.1.3.

Lemma 5.1.5. *We have $L_{\sigma_i} L_{\sigma_{i+1}} L_{\sigma_i} = L_{\sigma_{i+1}} L_{\sigma_i} L_{\sigma_{i+1}}$ and if $|i-j| > 1$, $L_{\sigma_i} L_{\sigma_j} = L_{\sigma_j} L_{\sigma_i}$.*

Lemma 5.1.6. *We have*

$$L_{\cap_i} L_{\sigma_j} \simeq L_{\sigma_j} L_{\cap_i} \qquad L_{\cup_i} L_{\sigma_j} \simeq L_{\sigma_j} L_{\cup_i} \tag{5.6}$$

if $|i-j|>1$ *and for any* i *we have:*

$$L_{\cap_i} L_{\sigma_i} \simeq L_{\cap_i}\{1\} \qquad L_{\sigma_i} L_{\cup_i} \simeq L_{\cup_i}\{1\} \tag{5.7}$$

$$L_{\sigma_i} L_{\cup_{i+1}} \simeq L^t_{\sigma_{i+1}} L_{\cup_i} \qquad L_{\sigma_{i+1}} L_{\cup_i} \simeq L^t_{\sigma_i} L_{\cup_{i+1}} \tag{5.8}$$

where "$\simeq$" means exact isotopy.

Proof. Let $\beta : [0,1] \to \mathrm{Conf}^0_{2m}$ be a braid such that $\beta(0) = \beta(1) = \nu$. Note that in the construction of $L_\cap$ if we replace the curve γ with $\beta * \gamma$ and also replace δ with $\delta * \alpha$ where α joins $\nu^{(k)}$ to $\nu \backslash \{\beta(2k-1), \beta(2k)\}$, the construction will yield $h_\beta(\cap_{k;m})$. In general the basepoint $\nu^{(k)}$ or $\nu \backslash \{\beta(2k-1), \beta(2k)\}$ is of no importance so we can assume α to be constant. If β equals σ_k then $\beta * \gamma$ joins μ_{2k-1}, μ_{2k} and fixes the other eigenvalues so the construction will yield the same $L_{\cap_k}$. If $\beta = \sigma_{k+1}$ then $\beta * \gamma$ the result will be the same as starting with $\cap_{i+1}$ and using $\beta = \sigma_k$. These two facts imply the above isotopies ignoring grading. As another proof which shows equality of graded Lagrangians, we appeal to (5.4) which implies (5.7). To prove (5.8), we use Lemma 5.8 in [32] which asserts if L_0, L_1 are two graded Lagrangian spheres whose intersection consists of only one point of Maslov index one and with the grading of the Dehn twists and chosen as above, we have

$$\tau_{L_0} L_1 = \tau^{-1}_{L_1} L_0 \tag{5.9}$$

as graded Lagrangians. Now we use Lemma 4.1.5 to translate the picture into that of section 4.3. So we can identify $L_{\cup_i}$ with $\Lambda_{\alpha_1} \times_{S^1} \Delta$ and $L_{\cup_{i+1}}$ with $\Lambda_{\alpha_2} \times_{S^1} \Delta$. Because h_{σ_i} and $h_{\sigma_{i+1}}$ act trivially on Δ by 5.1.2, we can identify them with Dehn twists around Λ_{α_1} and Λ_{α_2} respectively. Our choice of grading (5.5) for $L_{\cup_i}$ implies that the hypothesis of (5.9) are met. This immediately implies (5.8). Note that because Λ_{α_i} are two dimensional, changing the order of them does not change the Maslov index of the intersection point. $\square$

Lemma 5.1.7. *We have the following commutation relations where "$\simeq$" means exact isotopy.*

$$L_{\cap_i}\ L_{\cup_j} \simeq L_{\cup_j}\ L_{\cap_i} \quad \textit{if} \quad |i-j|>1. \tag{5.10}$$

$$L_{\cap_{i;m}}\ L_{\cup_{i+1;m}} \simeq L_{id_{m-1}}\{1\} \qquad \textit{for any } i. \tag{5.11}$$

Proof. To prove (5.10) let $\nu = \{\mu_1, ..., \mu_{2m}\}$ and ν', ν'', ν''' be ν minus $\{\mu_{2i-1}, \mu_{2i}\}$, $\{\mu_{2j-1}, \mu_{2j}\}$ and $\{\mu_{2i-1}, \mu_{2i}, \mu_{2j-1}, \mu_{2j}\}$ respectively. Therefore

$$L_{\cap_{i;m}} \subset \mathcal{Y}_{m-1,\nu'} \times \mathcal{Y}_{m,\nu} \qquad \text{and} \quad L_{\cup_{j;m}} \quad \subset \mathcal{Y}_{m,\nu} \times \mathcal{Y}_{m-1,\nu''}.$$

We have projections $\pi_i : L_{\cap_{i;m}} \to \mathcal{Y}_{m-1,\nu'}$, $\pi_j : L_{\cup_{j;m}} \to \mathcal{Y}_{m-1,\nu''}$ given in the construction of the these Lagrangians. We have

$$\begin{aligned}
L_{\cap_{i;m}} L_{\cup_{j;m}} &= \{(m_1, m_2) | \exists\, m', (m_1, m') \in L_{\cap_{i;m}}, (m', m_2) \in L_{\cup_{j;m}}\} \\
&= \{(m_1, m_2) | \exists\, m', m_1 = \pi_i(m'), m_2 = \pi_j(m')\}. \\
L_{\cup_{j;m}} L_{\cap_{i;m}} &= \{(m_1, m_2) | \exists\, m', (m_1, m') \in L_{\cup_{j;m'}}, (m', m_2) \in L_{\cap_{i;m}}\} \\
&= \{(m_1, m_2) | \exists\, m', \pi_j(m_1) = m', \pi_i(m_2) = m'\} \\
&= \{(m_1, m_2) | \pi_j(m_1) = \pi_i(m_2)\}
\end{aligned}$$

By the remark after the Lemma 4.1.3, these projections are given by projection to the first and second factor in $\mathfrak{sl}(E_{\mu_1}) \oplus \mathfrak{sl}(E_{\mu_2}) \oplus \zeta$ and so $\pi_i \pi_j = \pi_j \pi_i$ from which the equality of the compositions follows.

As for (5.11) we must show that $L_{\cap_i} \circ L_{\cup_{i+1}}$ equals the diagonal $\Delta \subset \mathcal{Y}_m \times \mathcal{Y}_m$. Using Lemma 5.1.6 we see that

$$h_{\sigma_i \sigma_{i+1}} L_{\cap_i} = L_{\cap_{i+1}}.$$

So we are reduced to showing that $L_{\cap_i} \cap h_{\sigma_i \sigma i+1}(L_{\cap_i}) \simeq \Delta$. Let $\nu = (\mu_1, ..., \mu_{2m+2})$ be such that $\mu_i + \mu_{i+1} + \mu_{i+2} = 0$. Also denote $\bar{\nu} = (\mu_1, ...\mu_{i-1}, 0, \mu_{i+3}, ..., \mu_{2m+2})$ and

$$\hat{\nu} = (\mu_1, ..., \mu_{i-1}, 0, 0, 0, \mu_{i+3}, ..., \mu_{2m+2}).$$

We take $\mathcal{Y}_{m+1} = \mathcal{Y}_{m+1,\nu}$ and $\mathcal{Y}_m = \mathcal{Y}_{m,\bar{\nu}}$. The latter can be identified with $\mathcal{D}_{m+1,\hat{\nu}}$ which is $(\chi^{-1}(\hat{\nu}) \cap \mathcal{S}_{m+1})^{sub3,0}$. Lemma 4.1.5 gives us a $\mathbb{C}^4$-bundle $\mathcal{L}$ over $\mathcal{S}^{sub3,0}_{m+1}$ with a local symplectomorphism ϕ from $\mathcal{L}$ to a neighborhood U of $\mathcal{S}^{sub3,0}$ in $\mathcal{S}_{m+1}$. We identify $\mathcal{Y}_{m+1,\nu}$ with its image under ϕ and so $\Delta \subset \mathcal{L} \times \mathcal{S}^{sub3,0}_{m+1}$. $\mathcal{Y}_{m,\bar{\nu}}$ lies in the zero section and $L_{\cap_i}$ is the pullback of a sphere bundle over the zero section to Δ. Lemma 5.1.2 tells us that $h_{\sigma_i\sigma_{i+1}}$ is identity when restricted to Δ therefore $h_{\sigma_{i,i+1}}(L_{\cap_i})$ is another sphere bundle over Δ. We show that fibers of this two bundles over each point of Δ intersect at only one point so $L_{\cap_i} \cap h_{\sigma_i\sigma_{i+1}}(L_{\cap_i})$ is Lagrangian isotopic to Δ. Since these Lagrangians have vanishing first cohomology groups this isotopy is exact.

Let $L_1$and L_2 be fibers of $L_{\cap_i}$ and $h_{\sigma_i\sigma_{i+1}}(L_{\cap_i})$ over a point of Δ. So

$$L_2 = h_{\sigma_i\sigma_{i+1}}(L_1). \tag{5.12}$$

By Lemma 4.3.1, $L_{\cap_i}$ is isotopic to $\Lambda_{d,\alpha_1} = \Delta \times_{S^1} \Lambda_\alpha$ so L_1 can be identified with Λ_{α_1}. When we perform $h_{\sigma_i\sigma_{i+2}}$, the leftmost and the rightmost zeros in Figure 4.1 (which are the "fiberwise" eigenvalues) get swapped; therefore $\mathbf{c}$ is sent to $\mathbf{c}'$ and so $L_2 = \Lambda_{\mathbf{c}}$. Since $\Lambda_{\mathbf{c}} = \bigcup_{r=0}^{1} C_{d,z,\mathbf{c}(r)}$ and $C_{d,z,a}$ is given by $C_{d,z,a} = \{(b,c) : |b| = |c|, a^3 - da - z = -bc\}$, we see that since the curves $\mathbf{c}$ and $\mathbf{c}''$ intersect at only one point (i.e. the first root of $a^3 - ad - z = 0$) then $L_1 = \Lambda_{\alpha_1}$ and $L_2 = \Lambda_{\alpha_2}$ intersect at only one point so the proof is complete.

We chose the gradings (5.5) for $L_{\cap_i}$ to be a constant function c and be the same for all i. Let S_{ix} be the fiber of $L_{\cap_i}$ over x. Then the fiber of $L_{\cap_{i;m}} \circ L_{\cup_{i+1;m}}$ over x equals $S_{ix} \circ S^t_{i+1x}$. The grading of S^t_{i+1x} equals $1 - \tilde{S}_{i+1x}$ by (2.16). Therefore the grading of the point $S_{ix} \circ S^t_{i+1x}$ is $c + 1 - c = 1$ which implies (5.11).

$\square$

Theorem 5.1.8. *The assignment* Φ *in* (5.2) *is a functor from the category of even tangles to the symplectic category.*

Proof. Follows from 3.1.4 and 5.1.5 to 5.1.7. We see that difference in grading for each commutation relation gets canceled by the change in the writhe plus number of cups. The only commutation relations which involve grading shift are (5.7) and (5.11) which happen to be the only ones involving change in $-m-w$. For (5.11), $-m$ plus the degree shift is equal on both sides of the equation. Note that if T contains $\cap_i(\sigma_i^\epsilon)^\pm$ where $\epsilon = 1, -1$ and $\pm$ is the writhe of σ_i then $\pm\epsilon$ has to be equal to -1. This implies that $-w$ plus the grading shift is equal on both sides of (5.7). □

Remark 5.1.9. *The functor Φ can be viewed as a (graded) genus zero symplectic valued topological field theory which is defined only for even tangles.*

5.2 The group valued invariant

We can obtain tangle invariants at two levels from the symplectic valued topological field theory Φ: a functor valued invariant and a group valued one. The functor valued invariant is discussed in 6. The group valued tangle invariant is defined as follows.

Definition 5.2.1.

$$\mathcal{HSS}(T) = \bigoplus_{\substack{C \in \mathcal{C}_m \\ C' \in \mathcal{C}_n}} HF(L_C^t, \Phi(T), L_{C'}) \tag{5.13}$$

$$\mathcal{CSS}(T) = \bigoplus_{\substack{C \in \mathcal{C}_m \\ C' \in \mathcal{C}_n}} CF(L_C^t, \Phi(T), L_{C'}) \tag{5.14}$$

We will, in section 7.5, put extra conditions on the chain complex (5.14) for T a flat tangle. Each summand in the above direct sum is equal to the Floer cohomology of the Lagrangians

$$\begin{aligned} \mathcal{L}_0 &= L_C \times L_0 \times L_{1,2} \times ... \times L_{2k-1,2k} \\ \mathcal{L}_1 &= L_{0,1} \times L_{2,3} \times ... \times L_{2k+1} \times L_{C'} \end{aligned}$$

in $\mathcal{Y} = \mathcal{Y}_{n_0}^- \times \mathcal{Y}_{n_1} \times ... \times \mathcal{Y}_n$. If ψ_i is the plurisubharmonic function on $\mathcal{Y}_i$ then $\mathcal{Y}$ is a Stein manifold with the plurisubharmonic function $\psi = \Sigma\psi_i$.

Lemma 5.2.2. *The Lagrangians $\mathcal{L}_i$, $i = 0, 1$ are allowable.*

Proof. The Lagrangians are exact since they are simply connected submanifolds of exact manifolds. A point $m = (m_0, ..., m_{2k+1})$ of tangency of $\mathcal{L}_0$ to a level set of ψ is a critical point of $\psi|_{\mathcal{L}_0}$. Since ψ is the sum of the plurisubharmonic functions on each $\mathcal{Y}_i$, m_i is a critical point of $\pi_i^*\psi|_{L_{i,i+1}}$. If $L_{i,i+1}$ is noncompact, it is a vanishing cycle on the diagonal either in $\mathcal{Y}_m \times \mathcal{Y}_{m+1}$ or in $\mathcal{Y}_{m+1} \times \mathcal{Y}_m$. In the first case m_i is a critical point of $\pi_i^*\psi = \psi_i$ on $\mathcal{Y}_m$. Since $\mathcal{Y}_i$ and ψ_i are algebraic, the critical point set is compact. The second case is similar. □

Theorem 5.2.3. *For any tangle T, $\mathcal{HSS}(T)$ is well-defined and is independent of the decomposition of T into elementary tangles.*

Proof. The finite intersection condition holds since the Lagrangian correspondences $L_{i,i+1}$ are proper (cf. 2.11.1). We take the parameter R in the construction of the $L_{\cap_i}$ and $L_{\cup_i}$ so that all the intersection points are included in $M_{\leq R}$. Therefore by Proposition 2.11.3 the above Floer cohomology is well-defined. Note that since our Lagrangians are products of 2-spheres, they have a unique spin structure so the Floer cohomology groups above are modules over $\mathbb{Z}$. (cf. Theorem "Fs" in [7].) Independence of the decomposition follows from the Functoriality Theorem and theorem 5.1.8. □

It is clear that if K is a $(0, 0)$-tangle, i.e. a link, then the above invariant equals the original invariant of Seidel and Smith (4.6.1).

Chapter 6

The functor valued invariant

6.1 Fukaya categories and generalized Fukaya categories

According to [21] the *generalized Fukaya category* $\mathrm{Fuk}^{\#}(M)$ of a symplectic manifold M is an A_∞ category whose objects are compact generalized Lagrangian correspondences between a point and M and morphisms between two such objects $\underline{L}_0$ and $\underline{L}_1$ are the elements of Floer chain complex for $\underline{L}_0^t \# \underline{L}_1$. The A_∞ structure on $Fuk^{\#}(M)$ is given by counting holomorphic "quilted polygons". More precisely the maps

$$\mu^d : CF(\underline{L}_0, \underline{L}_1) \otimes \cdots \otimes CF(\underline{L}_{d-1}, \underline{L}_d) \to CF(\underline{L}_0, \underline{L}_d) \tag{6.1}$$

are given by counting quilted $(d+1)$-gons whose boundary conditions are given by the $\underline{L}_i$.

Here we need an enlargement of (generalized) Fukaya category of a Stein manifold to include noncompact admissible Lagrangians. For a Stein manifold M we denote by $\mathrm{Fuk}^{\#}(M)$ the A_∞ category whose objects are allowable proper generalized Lagrangian correspondences between M and a point. Symplectic manifolds involved in these correspondences can be either Stein or compact. Two such correspondences $\underline{L}_0$ and $\underline{L}_1$ satisfy the finite intersection property and we define the set of morphisms between them to be $CF(\mathfrak{C}_c(\underline{L}_0^t \# \underline{L}_1))$. Here we choose c so as to include the intersection points of the Lagrangians. The A_∞ structure is given in the same way as for the generalized Fukaya category of compact manifolds. Lemma 2.11.2 insures that the moduli spaces involved are compact.

6.2 A_∞functor associated to a Lagrangian correspondence

Let L be a Lagrangian correspondence between two symplectic manifolds M and N. Mau, Wehrheim and Woodward [21] associate an A_∞functor

$$\Phi_L^\# : \mathrm{Fuk}^\#(M) \to \mathrm{Fuk}^\#(N). \tag{6.2}$$

At the level of objects it is given by

$$\Phi_L^\#(\underline{L}) = \underline{L}\#L.$$

The higher maps

$$\Phi_L^{\#d} : CF(\underline{L}_0, \underline{L}_1) \otimes \cdots \otimes CF(\underline{L}_{d-1}, \underline{L}_d) \to CF(\Phi_L^\# \underline{L}_0, \Phi_L^\# \underline{L}_d)$$

are given by counting quilted discs.

6.3 The functor valued invariant

For an (m, n) tangle T, let $\Phi(T) = (L_{0,1}, L_{1,2}, ..., L_{n-1,n})$ be as in (5.2). We obtain an A_∞ functor

$$\Phi_T^\# : \mathrm{Fuk}^\#(\mathcal{Y}_m) \to \mathrm{Fuk}^\#(\mathcal{Y}_n) \tag{6.3}$$

which is given at the level of objects by $\Phi_T^\#(\underline{L}) = \underline{L}\#\Phi(T)$ for each $\underline{L} \in \mathrm{Fuk}^\#(\mathcal{Y}_m)$. If K is a link then we obtain a functor

$$\Phi_K^\# : \mathrm{Fuk}^\#(pt) \to \mathrm{Fuk}^\#(pt). \tag{6.4}$$

Chapter 7

Results on the symplectic invariant

7.1 Some computations

In this section we compute $\mathcal{HSS}$ for elementary tangles. Set

$$\mathcal{V} = H^*(S^2)\{-1\}.$$

Remember that $\oplus$ denotes unlinked disjoint union. Let $\sigma^+_{i;m}$ and $\sigma^-_{i;m}$ be as in the Figure 3.4. If Lagrangians $L, L' \subset X \times \mathbb{C}^3$ are obtained from Lagrangians $K, K' \subset X$ by the relative vanishing cycle construction then we can isotope L and L' to $K \times S^2$ and $K' \times S^2$ inside a compact subset. Therefore we get

$$HF(L', L') = HF(K \times S^2, K' \times S^2) = HF(K, K') \otimes H^*(S^2) \tag{7.1}$$

For a tangle T it follows from the above formula that

$$\mathcal{HSS}(T \oplus \bigcirc) = \mathcal{HSS}(T) \otimes H^*(S^2)\{-1\} = \mathcal{HSS}(T) \otimes \mathcal{V}. \tag{7.2}$$

The -1 degree shift here comes from the cup in $\bigcirc$. More generally it follows from the commutation relations in section 5.1 that for any two tangles $kTl, mT'n$ we have

$$\Phi^\#(T \oplus T') = \Phi^\#(T \oplus id_m) \circ \Phi^\#(id_l \oplus T') = \Phi^\#(id_k \oplus T') \circ \Phi^\#(T \oplus id_n). \tag{7.3}$$

This gives an injection

$$\mathcal{HSS}(T \oplus T') \hookrightarrow \mathcal{HSS}(T) \otimes \mathcal{HSS}(T') \tag{7.4}$$

For links L, L' this becomes an isomorphism

$$\mathcal{HSS}(L \oplus L') = \mathcal{HSS}(L) \otimes \mathcal{HSS}(L'). \tag{7.5}$$

Following Khovanov [15] we define

$$H^n = \mathcal{HSS}(id_n) = \bigoplus_{C,C' \in \mathcal{C}_n} HF(L^t_C, L_{C'}). \tag{7.6}$$

It follows from (7.1) and 5.1.4 that each summand in the above direct sum equals $H^*(S^2)^{\otimes k}$ where k is the number of circles made by attaching C^t to C'. Therefore (7.6) equals Khovanov's H^n as an abelian group.

Lemma 7.1.1. *We have*

$$\begin{aligned}
\mathcal{HSS} \quad & (\sigma^{\pm}_{i;m}) = \\
& \Big(\big(H^{m-1} \oplus \bar{H}^m \oplus \bar{H}^m \oplus H^m_1\big) \otimes \mathcal{V}\{1 \mp 2\}\Big) \bigoplus H^m_2 \otimes \mathcal{V}\{2\}. \\
\mathcal{HSS} \quad & (\cup_{i;m}) = (H^{m-1} \otimes \mathcal{V}) \bigoplus \oplus_{a \in \mathcal{C}''_m, b \in \mathcal{C}_{m-1}} HF(L_{a'}, L_b) \\
\mathcal{HSS} \quad & (\cap_{i;m}) = (H^{m-1} \otimes \mathcal{V}\{1\}) \bigoplus \oplus_{a \in \mathcal{C}''_m, b \in \mathcal{C}_{m-1}} HF(L_{a'}, L_b).
\end{aligned}$$

Proof. The last two equalities are immediate consequences of (7.2). The computation for the first equality is similar to that of the rings H^m. The only difference comes from the existence of braids $\sigma^{\pm}_{i;m}$. As stated before if we modify the strands of a^t passing through 1 and 2 in a small neighborhood of these two points then only the factors $\mathcal{V}\{1\}$ and $\mathcal{V}\{1\} \otimes \mathcal{V}\{1\}$ in (3.17) change. In the first four direct summands of the decomposition (3.17) the component of $a^t \sigma^{\pm}_{i;m} b$ containing the braid contributes the following.

$$HF(L_{\cap_i} L_{\sigma^{\pm}_{i;m}}, L_{\cup_i})\{-w(\sigma^{\pm}_{i;m})\} = HF(L_{\cap_i}\{\mp 1\}, L_{\cup_i})\{\mp 1\} = \mathcal{V}\{1 \mp 2\}$$

The second to last equality is because $\sigma^+_{i;m} = \sigma_{i;m}$ and $\sigma^-_{i;m} = \sigma^{-1}_{i;m}$ when we ignore the orientation.

In the last direct summand in (3.17) the component of $a^t \sigma^{\pm}_{i;m} b$ containing the braid looks locally like the tangle T_0 depicted in Figure 7.1. (The figure depicts the case of $\sigma^+_{i;m}$ i.e. the crossing in the figure is $\sigma^+_{i;m}$). It is equivalent to $\cap_i \cap_{i+2} (\sigma^{\pm}_{i;m}) \cup_i \cup_{i+2}$. We can use the commutation relation (5.8) to get

$\cap_i \cap_{i+1} {\sigma^{\pm}_{i;m}}^{-1} \cup_i \cup_{i+2}$. Therefore the contribution of T_0 is

$$\begin{aligned} HF \quad & (L_{\cap_i} L_{\cap_{i+1}} L_{\sigma^{\pm}_{i;m}}{}^{(-1)}, L_{\cup_i} L_{\cup_{i+2}})\{w(\sigma^{\pm}_{i;m})\} = \\ HF \quad & (L_{\cap_i} L_{\cap_{i+1}}\{\mp 1\}, L_{\cup_i} L_{\cup_{i+2}})\{w(\sigma^{\pm}_{i;m})\} = \\ HF \quad & (L_{\cap}, L_{\cup})\{1\} = \mathcal{V}\{2\}. \end{aligned}$$

Where we have used commutation relations (5.7) and(5.11).

$\square$

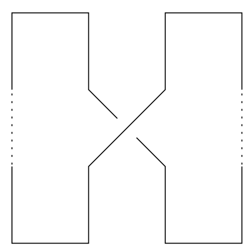

Figure 7.1: Proof of Lemma 7.1.1

7.2 Maps induced by cobordisms

In this section we study the maps induced by minimal cobordisms on $\mathcal{HSS}$. Using elementary Morse theory one can decompose any cobordism between two tangles into three elementary types. We will assign a cobordism map $\mathcal{HSS}$(S) to each such elementary cobordism S. Given a general cobordism S between two tangles T and T', we decompose it into elementary cobordisms

$$S = S_l \circ S_{l-1} \circ \cdots \circ S_1$$

and define $\mathcal{HSS}(S)$ to be the composition $\mathcal{HSS}(S_l) \circ \mathcal{HSS}(S_{l-1}) \circ \cdots \circ \mathcal{HSS}(S_1)$. Since such a decomposition is not unique, one can potentially get different maps from different decompositions. We will not address this problem here. Let T, T' be (l, m) and (m, n) tangles respectively.

Type I. Cobordisms, equivalent to trivial cobordism, between equivalent tangles. The (iso-)morphism assigned to such a cobordism is given by the Functoriality theorem. C.f. Theorem 5.2.3.

Type II. Birth or death of an unlinked circle:

$$S_{\bigcirc}: \quad T \bigcirc T' \longrightarrow TT' \tag{7.7}$$

$$S^t_{\bigcirc}: \quad TT' \longrightarrow T \bigcirc T'. \tag{7.8}$$

Equation (7.2) gives us a canonical isomorphism

$$\mathcal{HSS}(T \bigcirc T') \cong \mathcal{HSS}(TT') \otimes_{\mathbb{Z}} \mathcal{V}.$$

We define the map

$$\mathcal{HSS}(S_{\bigcirc}) : \mathcal{HSS}(T \bigcirc T') \longrightarrow \mathcal{HSS}(TT') \tag{7.9}$$

induced by the cobordism $S_{\bigcirc}$ to be $id \otimes \varepsilon$ and the map

$$\mathcal{HSS}(S^t_{\bigcirc}) : \mathcal{HSS}(TT') \longrightarrow \mathcal{HSS}(T \bigcirc T') \tag{7.10}$$

to be $id \otimes \iota$. Here ε and ι are the trace and unit maps from 3.2.

Type III. Saddle point cobordisms:

$$S_{\cap_i}: \quad T \cap_i \cup_i T' \longrightarrow TT' \tag{7.11}$$

$$S_{\cap_i^t}: \quad TT' \longrightarrow T \cap_i \cup_i T'. \tag{7.12}$$

Let $\Phi(T), \Phi(T')$ be the generalized Lagrangian correspondences associated to them. We define the following cobordism maps as follows.

$$\mathcal{CSS}(S_{\cap_i}) : \mathcal{CSS}(T \cup_i \cap_i T') \to \mathcal{CSS}(TT') \tag{7.13}$$

$$\mathcal{CSS}(S^t_{\cap_i}) : \mathcal{CSS}(TT') \to \mathcal{CSS}(T \cup_i \cap_i T') \tag{7.14}$$

The homomorphism (7.13) is defined to be the relative invariant associated to the quilt in Figure 7.2. The homomorphism (7.14) is the relative invariant of

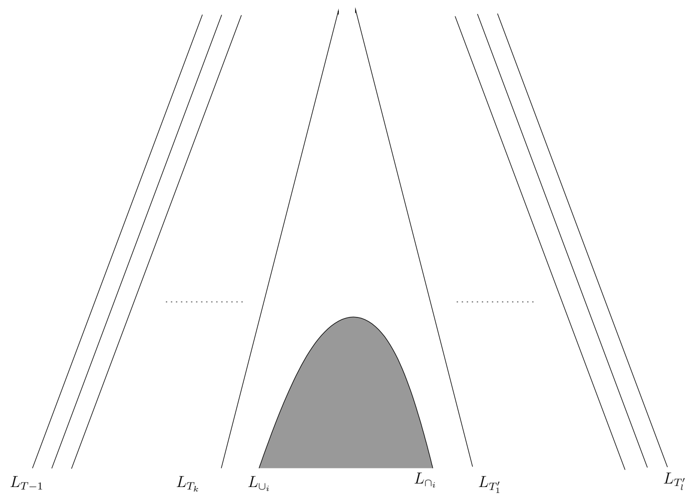

Figure 7.2: Quilt associated to a saddle cobordism

the transpose of this quilt. We can use the local picture of the Lemma 4.1.5 to to obtain a local isomorphism $\mathcal{Y}_{m_j} \cong \mathcal{Y}_{m_j-1} \times \mathcal{Y}_1$ and the Lemma 4.3.1 to get $L_{\cup_i} \cong \mathcal{Y}_{m-1} \times S^2$. Thus $\mathcal{CSS}(S_{\cap_i})$ becomes the tensor product of the relative maps of the quilts in Figure 7.2. Therefore the degrees of the relative maps of the quilt Q_0 and Q_0^t can be taken to be zero. See also Lemma 7.6.2 below. (In fact the we do not need this local picture to show thatthe degree of $CF(\mathbb{Q}_0^t)$ is zero.) However because of the -1 degree shift coming from the extra cap in $T \cup_i \cap_i T'$ we get

$$\deg \mathcal{CSS}(S_{\cap_i}) = 1 = \deg \mathcal{CSS}(S_{\cap_i}^t). \tag{7.15}$$

Minimal cobordisms (C.f. Definition 3.3.2) are a special combination of saddle cobordisms which deserve special attention. Let $a \in \mathcal{C}_l, b \in \mathcal{C}_m$ and $c \in \mathcal{C}_n$. Recall from section 3.3 that we denote the minimal cobordism $b^t b \to id$ by S_b. From S_b we get the cobordism $\mathbf{1_a 1_T S_b 1_{T'} 1_c}$ between $aTbb^tT'c$ and $aTT'c$. We associated to this cobordism the quilt depicted in Figure 7.2 and we denote it by Q_b. The relative invariant associated to Q_b gives a homomorphism of chain complexes

Figure 7.3: Decomposing the quilt associated to a saddle cobordism

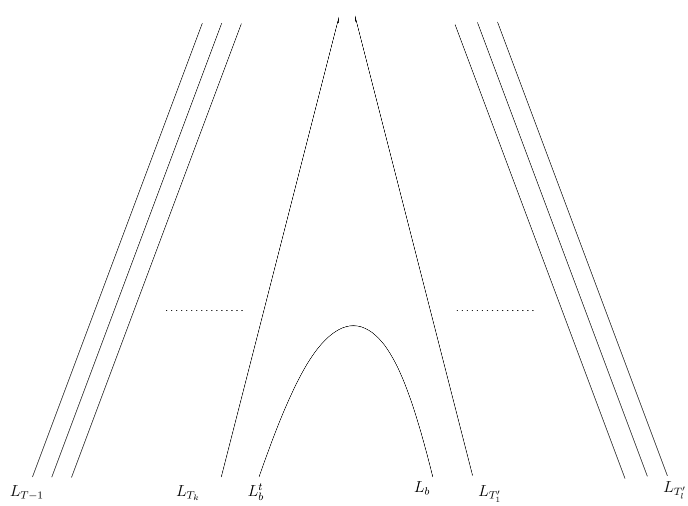

Figure 7.4: Quilt associated to a minimal cobordism

$$CF(Q_b): \quad CF(L_a^t, \Phi(T), L_b) \otimes_{\mathbb{Z}} CF(L_b^t, \Phi(T'), L_c) \longrightarrow CF(L_a^t, \Phi(T \circ T'), L_c). \tag{7.16}$$

as well as a homomorphism of graded groups

$$\mathcal{HSS}(S_b) := HF(Q_b) : HF(L_a^t, \Phi(T), L_b) \otimes HF(L_b^t, \Phi(T'), L_c) \to HF(L_a^t, \Phi(T \circ T'), L_c). \tag{7.17}$$

Summing over all such a, b and c we get maps

$$\mathcal{CSS}(S_b): \quad \mathcal{CSS}(T) \otimes_{\mathbb{Z}} \mathcal{CSS}(T') \longrightarrow \mathcal{CSS}(TT') \tag{7.18}$$

and

$$\mathcal{HSS}(S_b): \quad \mathcal{HSS}(T) \otimes_{\mathbb{Z}} \mathcal{HSS}(T') \longrightarrow \mathcal{HSS}(TT'). \tag{7.19}$$

If Q_b^t is obtained from Q_b by reversing the incoming and outgoing ends we get

$$\mathcal{HSS}(S_b^t) : \mathcal{HSS}(T \circ T') \to \mathcal{HSS}(\Phi(T)) \otimes_{\mathbb{Z}} \mathcal{HSS}(T'). \tag{7.20}$$

It follows from the formula for the degree of the relative map of a quilt (Formula (33) in [39]) that

$$\deg \mathcal{CSS}(S_b) = 0 = \deg \mathcal{CSS}(S_b^t). \tag{7.21}$$

In the case that $T = T' = id_1$ we get maps

$$m_{symp} : H^{*+1}(S^2) \otimes H^*(S^2) \to H^{*+1}(S^2) \tag{7.22}$$

and

$$\Delta_{symp} : H^{*+1}(S^2) \to H^{*+1}(S^2) \otimes H^*(S^2). \tag{7.23}$$

Lemma 7.2.1. *We have $m_{symp} = m$ and $\Delta_{symp} = \Delta$.*

Proof. Let f_0, f_1, f_2 be three Morse functions on a Riemannian manifold M. Fukaya and Oh [6] (See also [10]) prove that if we equip the cotangent bundle of M with the almost complex structure induced by the Levi-Civita connection on M then, for generic choice of the f_i, there is an orientation preserving diffeomorphism between the moduli space of pseudoholomorphic triangles connecting intersection points of the df_i and the moduli space of pair of pants trajectories between the corresponding critical points of $f_0 - f_1, f_1 - f_2, f_2 - f_0$. The moduli space of triangles in T^*S^2 with the almost complex structure from $\mathcal{Y}_1$ is zero dimensional and cobordant to the moduli corresponding to the structure induced by Levi-Civita connection. So the signed sum of the elements of the two are equal. Therefore (7.22) equals the wedge product on $H^*(S^2)$ so $m_{symp} = m$. The same arguments show that Δ_{symp} corresponds to the operation given by counting inverted Y's in Morse homology so upon choosing generators $1, X$ for $H^*(S^1)$ we get $\Delta_{symp}(1) = 1 \otimes X + X \otimes 1$ and $\Delta_{symp}(X) = X \otimes X$. Therefor $\Delta_{symp} = \Delta$. In fact the above maps m_{symp} and Δ_{symp} give a basis-free definition of the homomorphisms m, Δ.

$\square$

In order to understand the behavior of the maps (7.17) and (7.20) in general we need the following lemma which is a relative versions of "Kunneth formula for Floer homology" in [35].

Lemma 7.2.2. *Assume $Y = X \times \mathbb{C}^3$ is a Kähler manifold and X is given the induced metric. Let K, K', K''' be Lagrangian submanifolds of X and L, L'', L''' be obtained from them by relative vanishing cycle construction. Then we have the following commutative diagram*

$$\begin{tikzcd}
HF(L,L') \bigotimes \arrow[d, equal] & HF(L',L'') \arrow[rr] \arrow[d, equal] & & HF(L,L'') \arrow[d] \\
HF(K,K') \otimes H^*(S^2) \bigotimes & HF(K',K'') \otimes H^*(S^2) \arrow[rr] & & HF(K,K'') \otimes H^*(S^2)
\end{tikzcd}$$

where the lower horizontal maps factor through tensor products.

Proof. We first obtain a localization result for the holomorphic triangles u with boundary on L, L', L''. Then we isotope these Lagrangians to $K \times S^2, K' \times S^2, K'' \times S^2$ within that neighborhood . Now we have $u = (u_0, u_1)$ where u_0 is a triangle in X with boundary condition K, K', K'' and u_1 is a triangle in $\mathbb{C}^2$ with boundary on S^2, S^2, S^2. The lemma follows. $\square$

Note that the horizontal composition SS' of two cobordisms S and S' equals $(S\ id) \circ (id\ S')$.

Lemma 7.2.3. *For two minimal cobordisms S_a and S_b we have*

$$\mathcal{HSS}(S_a\ id) \circ \mathcal{HSS}(id\ S_b) = \mathcal{HSS}(id\ S_b) \circ \mathcal{HSS}(S_a\ id). \tag{7.24}$$

Proof. From [39] we know if Q and Q' are two quilts which can be composed vertically (i.e. along the strip-like ends) then $HF(Q \circ Q') = HF(Q) \circ HF(Q')$. We also know that $HF(Q)$ is (at the cohomology level) invariant under the isotopy of the quilt Q. The lemma follows from these two facts together with the isotopy in Figure 7.2. $\square$

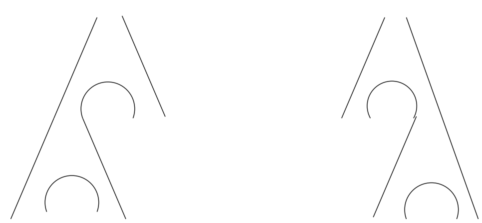

Figure 7.5: Isotopy between the composition of two quilts

Lemma 7.2.4. *Let T_0, T_1, T'_0, T'_1 be tangles such that $T_i \simeq T'_i$ for $i = 0, 1$ then we have the following commutative diagram*

$$\begin{array}{ccccc} HF(a^t, \Phi(T_0), b) & \bigotimes & HF(b^t, \Phi(T_1), c) & \longrightarrow & HF(a^t, \Phi(T_0 \circ T_1), c) \\ \downarrow & & \downarrow & & \downarrow \\ HF(a^t, \Phi(T'_0), b) & \bigotimes & HF(b^t, \Phi(T'_1), c) & \longrightarrow & HF(a^t, \Phi(T'_0 \circ T'_1), c) \end{array}$$

where the vertical maps are isomorphisms.

Proof. The vertical isomorphisms were constructed by showing that Lagrangian correspondences assigned to elementary tangles satisfy the same commutation relations as the corresponding tangles. So they are given by Hamiltonian isotoping the corresponding Lagrangians and the Functoriality theorem. For the first kind, the Hamiltonian isotopy induces a diffeomorphism between the corresponding moduli spaces of quilts. The second kind is an instance of "shrinking strips in quilted surfaces " and commutativity is given, in general settings, by Theorem 5.4.1 in [39].

□

7.3 H^m module structure

We define the symplectic analogue of the rings H^n as

$$H^n_{symp} = \mathcal{HSS}(id_n) = \bigoplus_{a,b \in \mathcal{C}_n} HF(L^t_a, L_b). \tag{7.25}$$

The product map from $HF(L^t_a, L_b) \otimes HF(L^t_c, L_d)$ to $HF(L^t_a, L_d)$ is given by zero if $b \neq d$ and is given by the map $HF(Q_b)$ otherwise.

Lemma 7.3.1. *For any $a, b \in \mathcal{C}_m$ we have $HF(L^t_a, L_b) \cong H^*(S^2)^{\otimes k} \cong \mathcal{V}^{\otimes k}\{k\}$ where k is the number of circles in $a^t b$.*

Proof. The Lagrangian L_a equals the composition of the Lagrangians associated to its arcs and similarly for L_b so $\Phi(a^t) \# \Phi(b) = \Phi(a^t b)$. Therefore

$$\begin{aligned} HF(L^t_a, L_b) &= HF(\Phi(a^t), \Phi(b))\{m\} = HF(\Phi(a^t b))\{m\} \\ &= \mathcal{HSS}(\Phi(k\bigcirc))\{m\} = H^*(S^2)^{\otimes k}. \end{aligned}$$

□

Proposition 7.3.2. *For any n we have*

$$H^n_{symp} \cong H^n$$

as graded rings.

Proof. The above lemma gives the isomorphism as graded abelian groups. Since for any $a, b \in \mathcal{C}_m$, $a^t b$ is equivalent to a number of unlinked circles, lemmas 7.2.4 and 7.2.2 give isomorphism of ring structures.

□

Therefore we drop the subscript *symp* in H^n_{symp} from now on. For an (l, m)-tangle T, $\mathcal{HSS}(T)$ has a structure of a (H^l, H^m) bimodule as follows. We have

$$\mathcal{HSS}(T) = \bigoplus_{b \in \mathcal{C}_l, c \in \mathcal{C}_m} HF(L_b, \Phi(T), L_c).$$

The ring ${}_aH^l_b$ acts on $HF(b, \Phi(T), c)$ from left by the map $\mathcal{HSS}(S_b)$ (in (7.19)). So does ${}_cH^m_d$ from right by the map $\mathcal{HSS}(S_c)$. We set the left action of ${}_aH^l_{b'}$ on $HF(b, \Phi(T), c)$ to be zero if $b \neq b'$ and similarly for the right action. This way we obtain an (H^l, H^m)-bimodule structure on $\mathcal{HSS}(T)$.

Remark 7.3.3. *Note that since the cobordism maps $\mathcal{CSS}(S_b)$ are of degree zero, the chain complex $\mathcal{CSS}(T)$ can be regarded as a chain complex of (H^l, H^m)-bimodules.*

Lemma 7.3.4. *With the same notation as in* (3.17) *we have*

$$\begin{aligned}
\mathcal{HSS}(id_m) &= H^m = \mathcal{Kh}(id_m) \\
\mathcal{HSS}(\cap_{i;m}) &= \mathcal{Kh}(\cap_{i;m}) \\
\mathcal{HSS}(\cup_{i;m}) &= \mathcal{Kh}(\cup_{i;m}) \\
\mathcal{HSS}(\sigma^{\pm}_{i;m}) &= \overline{\mathcal{Kh}}(\sigma^{\pm}_{i;m}) = \\
& \left(H^{m-1} \oplus H^{m\prime} \oplus H^{m\prime} \oplus H^m_1\right) \otimes \mathcal{V}\{1 \mp 2\} \bigoplus H^m_2 \otimes \mathcal{V}\{2\}
\end{aligned}$$

as H^m modules.

Proof. The first three equations follow from the fact that $\mathcal{Kh}$ and $\mathcal{HSS}$ for disjoint union of k circles are equal to $\mathcal{V}^{\otimes k}$. For the las one, the first equality was proved in 7.1.1 and the second one in the lemma 3.5.2.

□

7.4 Functoriality of the invariant for flat tangles

Let T and T' be (l,m) and (m,n) tangles respectively. Consider the map ψ_s

$$\begin{tikzcd} \mathcal{HSS}(T) \otimes_{\mathbb{Z}} \mathcal{HSS}(T') \arrow[d, equal] \\ \bigoplus_{a,b,b',c} HF(L_a^t, \Phi(T), L_b) \otimes_{\mathbb{Z}} HF(L_b'^t, \Phi(T'), L_c) \arrow[d, "\psi_s"] \\ \bigoplus_{a,c} HF(L_a^t, \Phi(T' \circ T), L_c) \cong \mathcal{HSS}(T' \circ T) \end{tikzcd}$$

which is zero if $b \neq b'$ and equals $\mathcal{HSS}(\mathbf{1}_a \mathbf{1}_T S_b \mathbf{1}_{T'} \mathbf{1}_c)$ otherwise. Here, as before, S_b is the minimal cobordism between bb^t and id_m. The abelian group $\mathcal{HSS}(T) \otimes_{\mathbb{Z}} \mathcal{HSS}(T')$ has the structure of a (H^l, H^n)-bimodule and ψ is a (H^l, H^n)-bimodule map. If $x \in HF(L_a^t, \Phi(T), L_b)$, $y \in HF(L_b'^t, \Phi(T'), L_c)$ and $\xi \in {}_bH^m_{b'}$ then

$$\begin{aligned} \psi_s(x\xi \otimes y) &= \mathcal{HSS}(\mathbf{1}_a \mathbf{1}_T S_{b'^t} \mathbf{1}_{T'} \mathbf{1}_c)\mathcal{HSS}(\mathbf{1_a 1_T S_b 1_{b'} 1_{T'} 1_c})(\mathbf{x}, \mathbf{y}) \\ \psi_s(x \otimes \xi y) &= \mathcal{HSS}(\mathbf{1}_a \mathbf{1}_T S_b \mathbf{1}_{T'} \mathbf{1}_c)\mathcal{HSS}(\mathbf{1_a 1_T 1_b S_{b'^t} 1_{T'} 1_c})(\mathbf{x}, \mathbf{y}) \end{aligned}$$

It follows from (7.24) that these two are equal and so ψ_s factors through a map of bimodules $\mathcal{HSS}(T) \otimes_{H^m} \mathcal{HSS}(T') \to \mathcal{HSS}(T \circ T')$ which we still denote by ψ_s.

Lemma 7.4.1. *If T and T' are flat then ψ_s gives an isomorphism*

$$\psi_s : \mathcal{HSS}(T) \otimes_{H^m} \mathcal{HSS}(T') \cong \mathcal{HSS}(T \circ T').$$

Proof. Proof is exactly the same as that of Theorem 1 in [15]. The map ψ is the direct sum of the maps

$${}_a\psi_c : \oplus_b HF(L_a^t, \Phi(T), L_b) \bigotimes_{H^m} \oplus_{b'} HF(L_{b'}^t, \Phi(T'), L_c) \to HF(L_a^t, \Phi(T \circ T'), L_c)$$

We have $\oplus_b HF(L_a^t \Phi(T), L_b) \cong \mathcal{HSS}(a^t T)$ and

$$\bigoplus_{b'} HF(L_{b'}^t, \Phi(T'), L_c) \cong \mathcal{HSS}(T'c)\{n\}$$

as left and right H^m-modules respectively. We also have $HF(L_a^t, \Phi(T \circ T'), L_c) \cong \mathcal{HSS}(aTT'c)\{n\}$. Therefore the argument is reduced to showing that

$$\mathcal{HSS}(a^tT) \otimes_{H^m} \mathcal{HSS}(T'c) \cong \mathcal{HSS}(a^tTT'c).$$

Now a^tT and $T'c$ are $(0,m)$ and $(m,0)$-tangles respectively so $a^tT = a' \oplus i\bigcirc$ and $T'c = c' \oplus j\bigcirc$ where $a', b' \in \mathcal{C}_m$ and i and j are the number of circles in a^tT and $T'c$ respectively. Thus we have $\mathcal{HSS}(a^tT) = \mathcal{HSS}(a') \otimes \mathcal{V}^i$, $\mathcal{HSS}(T'c) = \mathcal{HSS}(c') \otimes \mathcal{V}^j$ and $\mathcal{HSS}(a^tTT'c) = \mathcal{HSS}(a'c') \otimes \mathcal{V}^{i+j}$. So we need to show that

$$\mathcal{HSS}(a') \otimes_{H^m} \mathcal{HSS}(c') = \mathcal{HSS}(a'c').$$

We have $H^m \otimes_{H^m} H^m = H^m$ and if we multiply this identity with the idempotent $1_{a'}$ from left and by $1_{c'}$ from right we get the desired result. $\square$

We will need a slight generalization of this result.

Definition 7.4.2. *A tangle T is called semiflat if for all crossingless matchings a, b, the tangle a^tTb is flat.*

Lemma 7.4.3. *If T, T' and $T \circ T'$ are semiflat then ψ_s is an isomorphism.*

Proof. Note that for any crossingless matching a, b, $\mathcal{HSS}(aTb)$ has the structure of a right ${}_bH_b^m$ module and this module structure agrees with the one induced from H^m module structure on $\mathcal{HSS}(T)$. The same is true for $\mathcal{HSS}(bT'c)$ as a left ${}_bH_c^m$ module. We will show that

$$\psi_s : \mathcal{HSS}(aTb) \otimes_{{}_bH_b^m} \mathcal{HSS}(bT'c) \to \mathcal{HSS}(aT \circ T'c)$$

is an isomorphism for any a, b and c. Since the multiplication map

$$HF(L_a^t, \Phi(T), L_b) \otimes_{\mathbb{Z}} HF(L_b'^t, \Phi(T'), L_c) \xrightarrow{\psi_s} HF(L_a^t, \Phi(T' \circ T), L_c)$$

is zero if $b \neq b'$ this proves the lemma. It follows from the assumption that $\Phi(aTb)$, $\Phi(b'T'c)$ and $\Phi(aT \circ T'c)$ are isomorphic to $\Phi(\bigcirc^i)$, $\Phi(\bigcirc^j)$ and $\Phi(\bigcirc^k)$ respectively in the symplectic category for some i and j. By "shrinking strips in a quilt" (Theorem 5.4.1 in [39]), we have the following commutative diagram where the lower map is induced by the same cobordism as the top map.

$$\begin{tikzcd}
\mathcal{HSS}(aTb) \otimes_{{}_bH^m_b} \mathcal{HSS}(bT'c) \arrow[r, "\psi_s"] \arrow[d] & \mathcal{HSS}(aTT'c) \arrow[d] \\
\mathcal{HSS}(\bigcirc^i) \otimes_{{}_bH^m_b} \mathcal{HSS}(\bigcirc^j) \arrow[r] & \mathcal{HSS}(\bigcirc^k)
\end{tikzcd}$$

The lower homomorphisms is a composition of maps induced by cobordisms in which either two circles join at some n points or one circle decomposes to n circles. In each case each point of join or decomposition corresponds to an element of b. The first kind of map induces a homomorphism

$$\mathcal{V} \otimes_{\mathcal{V}^{\otimes n}} \mathcal{V} \to \mathcal{V}$$

given by the product m and the second type induces a map

$$\mathcal{V} \to \mathcal{V} \otimes_{\mathcal{V}} \otimes \mathcal{V} \otimes_{\mathcal{V}} \cdots \otimes_{\mathcal{V}} \mathcal{V}$$

given by the coproduct Δ. Each map is clearly an isomorphism.

□

Corollary 7.4.4. *For any flat tangle T we have*

$$\mathcal{HSS}(T) = \overline{\mathcal{Kh}}(T) = \mathcal{Kh}(T).$$

Proof. This follows from (3.30), 7.4.1 and 7.3.4. □

Lemma 7.4.5. *Let T, T' be flat (m, n)-tangles and S a cobordism between T and T' which equals a composition of minimal cobordisms. Then we have*

$$\mathcal{HSS}(S) = \mathcal{Kh}(S). \tag{7.26}$$

Proof. By (7.24) we can assume that S consists of a single minimal cobordism. Therefore we have $T = T_1 cc^t T_2$ and $T' = T_1 id T_2$ for a crossingless matching c and S equals $\mathbf{1}_{T_1} S_c \mathbf{1}_{T_2}$. For any $a \in \mathcal{C}_m$ and $b \in \mathcal{C}_n$, $a^t T_1$ equals a crossingless matching $a_1^t \in \mathcal{C}_m$ disjoint union with some k circles. The same is true for $T_2 b$

i.e. $T_2 b$ equals $b_2 \in \mathcal{C}_n$ disjoint union with l circles. So, the problem is reduced to showing that the map

$$\mathcal{HSS}(S) = HF(Q_c) : HF(L^t_{a_1}, L_c, L^t_c, L_{b_2}) \otimes \mathcal{V}^{k+l} \longrightarrow HF(L^t_{a_1}, L_{b_2}) \otimes \mathcal{V}^{k+l}$$

equals $\mathcal{Kh}(S)$. But

$$HF(L^t_{a_1}, L_c, L^t_c, L_{b_2}) = HF(L^t_{a_1}, L_c) \otimes HF(L^t_c, L_{b_2}) = {}_{a_1}H_c \otimes {}_cH_{b_2}$$

and $HF(L^t_{a_1}, L_{b_2}) = {}_{a_1}H_{b_2}$. Therefore the lemma follows from the equality of the ring structures on H^m and H^m_{symp} (Lemma 7.3.2).

□

7.5 Vanishing of the differential for flat tangles

Lemma 7.5.1. *Let $C_1, C_2 \in \mathcal{C}_m$ be two crossingless matchings. Then we can choose Floer data in such a way that the Floer chain complex $CF(L_{C_1}, L_{C_2})$ has differential equal to zero.*

Proof. We prove by induction on m. If $m = 1$ then there is only one crossingless matching and the Floer chain complex equals $CF(S^2, S^2)$ where S^2 is the zero section in $\mathcal{Y}_m = T^*S^2$. We can Hamiltonian isotope the zero section to a Lagrangian L s.t. L intersects the zero section at only two points. For example we can take L to be the graph of the one-form df where f is the height function on the zero section. In this case the Floer differential has to be zero because otherwise $HF(S^2, L)$ will not be equal to $H^*(S^2)$. This can also be seen by considering the Maslov indices of intersection points.

Now assume the statement holds for all crossingless matchings in $\mathcal{C}_k$ for $k < m$. Let α_1 be an arbitrary arc in C_1 and μ_1, μ_2 its endpoints. There are two cases. Either there is an arc α_2 in C_2 joining p and q or there is no such arc. Proof for these two cases are similar to the proofs of the Kunneth formula and the Thom isomorphism for Floer homology [35]. In the first case let $\bar{C}_i$ be obtained from C_i by deleting α_i, $i = 1, 2$. Then we can use lemma 4.1.3 and then isotope the induced metric into the product metric. So L_{C_i} gets isotoped to $L_{\bar{C}_i} \times S^2$. We choose a time dependent almost complex structure $\bar{J}_t$ on the base which is a compactly supported perturbation of its standard structure $\bar{J}_0$. We choose the almost complex structure on the total space to be equal to the product $\bar{J}_t \oplus J_{\mathbb{C}^3}$ in a small neighborhood U_0 of the zero section and equal to $\bar{J}_0 \oplus J_{\mathbb{C}^3}$ outside an open set U_1 containing $\bar{U}_0$. This way we can obtain an almost complex structure which is both regular and has similar properties to the product structure inside U_0. Since our pseudoholomorphic strips are confined to U_0, we have

$$CF(L_{C_1}, L_{C_2}) = CF(L_{\bar{C}_1}, L_{\bar{C}_2}) \otimes CF(S^2, S^2).$$

So the claim follows from the induction hypothesis and the argument for the base case. In the second case let α_2 be the unique arc in C_2 which has μ_2 as an endpoint and let μ_3 be its other end point. Now we can use lemma 4.1.5 to identify L_{C_i} with $L_{\bar{C}_i} \times_{S^1} \Lambda_{\alpha_i}$ where Λ_α is the lagrangian sphere associated to the curve α as defined in section 4.3. We choose the almost complex structure in a way similar to that of the first case above. Note that there are two possible configurations of the curves α_i.

In either case Λ_{α_1} and Λ_{α_2} intersect at only one point p corresponding to μ_2. So we have

$$CF(L_{C_1}, L_{C_2}) = CF(L_{\bar{C}_1}, L_{\bar{C}_2}) \otimes \mathbb{Z} < p > .$$

Let u be a holomorphic strip joining to intersection points of L_{C_1} and L_{C_2}. So we have $u = (u', u'')$ where u' is the projection to the first factor. By the induction hypothesis, u' is constant. Projection to the second factor is a holomorphic strip in $\mathbb{C}$ which has its boundary on α_1 and α_2. Such a finite energy curve has to be constant by the exponential convergence property of pseudoholomorphic strips. Therefore u'' is also constant so we get the desired result.

□

Lemma 7.5.2. *Let T be a flat (m, n)-tangle. We can choose the Floer data in such a way that the Floer chain complex whose cohomology is $\mathscr{HSS}(T)$ has* *differential equal to zero.*

Proof. Let $T = T_1 \cdots T_{k-1}$ be a decomposition of T and let $T_0 \in \mathcal{C}_m$ and $T_k \in \mathcal{C}_n$. Let L_{T_i} is a correspondence between $\mathcal{Y}_{m_i}$ and $\mathcal{Y}_{m_{i+1}}$. We use induction on $m = \sum m_i$. The case $m = 1$ was treated in Lemma 7.5.1. If T_1 is the identity tangle then $CF(L_{T_0}, L_{T_1}, \cdots, L_{T_k}) = CF(L_{T_0}, L_{T_2}, \cdots L_{T_k})$. So we can assume that T_1 is a cup. Therefore the both L_{T_0} and L_{T_1} are obtained by relative vanishing cycle construction from Lagrangians in $\mathcal{Y}_{m_0-1}$ and $\mathcal{Y}_{m_0-1} \times \mathcal{Y}_{m_1}$. Therefore we can use the same argument as in the proof of 7.5.1 for the induction step. $\square$

Definition 7.5.3. *For a flat (m,n)-tangle T we require the chain complex*

$$\mathcal{CSS}(T) = \bigoplus_{a \in \mathcal{C}_m, b \in \mathcal{C}_n} CF(L_a^t, \Phi(T), L_b)$$

to be given by Floer data in lemma 7.5.2.

From 7.5.2 and 7.4.1 we get the following.

Corollary 7.5.4. *Let T and T' be (l,m) and (m,n) flat tangles respectively. We have*

$$\mathcal{CSS}(T \circ T') = \mathcal{CSS}(T) \otimes_{H^m} \mathcal{CSS}(T').$$

7.6 Exact triangle for the symplectic invariant

In this section we prove an exact triangle for the Seidel-Smith invariant which is analogous to skein relations for knot polynomials. The tool we use is the exact triangle for Lagrangian Floer homology. This exact triangle was discovered by Seidel [33] for Dehn twists. We use a generalization of this triangle to fibred Dehn twists due to Wehrheim and Woodward [37]. Let M be a symplectic manifold and $C \subset M$ a spherically fibred coisotropic fibring over a base B. We denote the fibred Dehn twist around C by τ_C. The embedding $(\iota \times \pi)C$ is a Lagrangian submanifold of $M^- \times B$. By the abuse of notation we sometimes denote this submanifold by C.

Let Q_0 be the quilt in the Figure 7.6. The exact triangle in [37] establishes a quasi-isomorphism between the Floer chain complex $CF(L, \tau_C L')$ and the cone of the morphism $f := CF(Q_0)$, i.e. the relative map associated with Q_0.

$$f = CF(Q_0): \quad CF(L,\ (\pi \times \iota)C^t, (\iota \times \pi)C,\ L')\{-\frac{1}{2}\dim B\} \longrightarrow CF(L, L'). \tag{7.27}$$

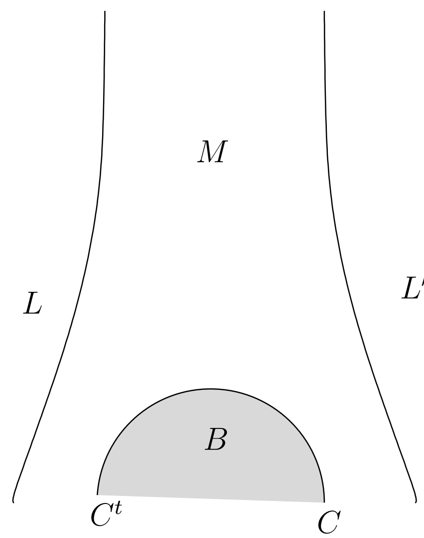

Figure 7.6: The quilt used in the exact triangle

More precisely we have the following.

Theorem 7.6.1 (Wehrheim, Woodward [37]). *If C has codimension at least two and the triple (L_0, L_1, C) is monotone and has Maslov index greater than or equal 3 then there is a quasi-isomorphism $(h\{1\}, k)$ from*

$$Cone(f) = CF(L, C^t, C, L')\{-\frac{1}{2}\dim B + 1\} \bigoplus CF(L, L')$$

to $CF(L, \tau_C L')$.

At the A_∞level one has the following exact triangle in $\mathrm{DFuk}^{\#}(M)$.

$$\begin{tikzcd}
 & & \operatorname{graph}\tau_C \arrow[dll] \\
C^t\#C\{-1/2\dim B\} \arrow[rrr] & & & \Delta_M \arrow[ul]
\end{tikzcd}$$

If $\underline{L} = (L_k, L_{k-1}, \ldots, L_1)$ is any generalized Lagrangian submanifold of M then by applying the A_∞ functor $\Phi^\#_{\underline{L}} = \Phi^\#_{L_k} \circ \cdots \circ \Phi^\#_{L_1}$ to (7.6) we get the following exact triangle in $\mathrm{DFuk}^\#(M)$.

$$\begin{tikzcd}
 & & \operatorname{graph}\tau_C\#\underline{L} \arrow[dll] \\
C^t\#C\#\underline{L}\{-1/2\dim B\} \arrow[rrr] & & & \underline{L} \arrow[ul]
\end{tikzcd}$$

Therefore theorem 7.6.1 holds, without any change, if L, L' are generalized Lagrangian submanifolds of M. One can prove this fact without using Fukaya categories. We use the following.

With the same assumptions as in 7.6.1 let $M = M_1 \times M_2$, $B = B_1 \times M_2$, and C be of the form $C_1 \times M_2$ where C_1 is a sphere bundle over B_1. Further assume that there are Lagrangian submanifolds $L_i \subset M_i$ and $L'_i \subset M'_i$ for $i = 1, 2$ such that $L = L_1 \times L_2$ and $L' = L'_1 \times L'_2$. Let $d = -\frac{1}{2}\dim B$ and $d_1 = -\frac{1}{2}\dim B_1$. Consider the map

$$CF(\bar{Q}_0): \quad CF(L_1,\ C_1^t, C_1,\ L'_1)\{-1/2\dim B_1\} \longrightarrow CF(L_1, L'_1). \tag{7.28}$$

Corollary 7.6.2. *Then $CF(L, \tau_C L')$ is quasi-isomorphic to*

$$Cone(\bar{Q}) \otimes CF(L_2, L'_2) \tag{7.29}$$

and we have a commutative diagram

$$\begin{tikzcd}
CF(L,\ C^t, C,\ L')\{d\} \arrow[r, "CF(Q_0)"] \arrow[d, "\{\frac{1}{2}\dim M_2\}"] & CF(L, L') \arrow[d] \\
CF(L_1, C_1^t, C_1, L'_1)\{d_1\} \otimes CF(L_2, L'_2) \arrow[r, "CF(\bar{Q}_0)"] & CF(L_1, L'_1) \otimes CF(L_2, L'_2)
\end{tikzcd}$$

Proof. We observe that $\tau_C = \tau_{C_1} \times id_{M_2}$. The homomorphism $CF(Q)$ is isomorphic to the tensor product of the maps induced by the two quilts in the Figure 7.7. The quilt on the left is $\bar{Q}$ and the quilt on the right induces the identity map. There is a grading shift $CF(L_2, \Delta^t_{M_2}, \Delta_{M_2}, L'_2) = CF(L_2, L'_2)\{\frac{1}{2}\dim M_2\}$ coming from (2.16). □

Proposition 7.6.3. *With the same assumptions as in Theorem 7.6.1, let* $\underline{L}, \underline{L}'$ *be two generalized Lagrangian submanifolds of* M*. Then* $CF(\underline{L}^t, \mathrm{graph}(\tau_{\mathrm{C}}), \underline{L}')$ *is quasi-isomorphic to the cone of the map*

$$CF(\underline{L}^t, C^t, C, \underline{L}')\{-1/2\dim B\} \longrightarrow CF(\underline{L}^t, \underline{L}') \tag{7.30}$$

Proof. Let $\underline{L} = (L_n, \cdots, L_k)$ and $\underline{L}' = (L_{k-1}, \cdots, L_1)$ where $L_i \subset M^-_{i+1} \times M_i$ and $M_k = M$. We can assume, by adding identity Lagrangian correspondences if necessary, that

$$\mathcal{L}_0 = L_n \times L_{n-2} \times \cdots \times L_k \times L_{k-1} \times \cdots \times L_2$$

and

$$\mathcal{L}_1 = L_{n-1} \times L_{n-3} \times \cdots \times \mathrm{graph}(\tau_C) \times \cdots \times L_1.$$

We have $\mathcal{L}_1 = \tau_{C'}\,(L_{n-1} \times L_{n-3} \times \cdots \times L_1)$ where $C' = M_{n+1} \times \cdots \times M_{k+1} \times C \times M_{k-1} \times \cdots \times M_1$ which fibers over $M_{n-1} \times \cdots \times B \times \cdots \times M_1$. The result follows from 7.6.2 by taking $M_1 = M = M_k$ and M_2 to be the product of the rest of the manifolds M_i.

□

Corollary 7.6.4. $CF(L, \tau_C^{-1}L')$ *is quasi-isomorphic to the cone* $\mathrm{Cone}\,(CF(Q^t_0))\,\{-1\}$.

Proof. This is a standard argument. If $l = \dim L$ then we have $CF^*(L, \tau_C^{-1}L') = CF^*(\tau_C L, L') = CF^{l-*}(L', \tau_C L)^\vee = \mathrm{Hom}(CF^{l-*}(L', \tau_C L), \mathbb{Z})$. It follows from 7.6.1 that this is quasi-isomorphic to

$$CF^{l-*}(L',\ (\iota \times \pi)C, (\pi \times \iota)C^t,\ L)\{1\}^\vee \oplus CF^{l-*}(L', L)^\vee$$

Figure 7.7: Decomposition of the cone in Corollary 7.6.2

(with appropriate differential). This in turn equals

$$CF(L, L') \bigoplus CF(L,\ (\iota \times \pi)C^t, (\pi \times \iota)C^t,\ L')\{-1\}.$$

$\square$

Now we use Theorem 7.6.1 (or more precisely 7.6.3) to obtain an exact triangle for the Seidel-Smith invariant. In the case under study $M = \mathcal{Y}_l$ and the spherically fibred isotropic is $C = L_{\cap_i}$. The coisotropic submanifold C is a sphere bundle over $B = \mathcal{Y}_{l-1}$. By Theorem 5.1.1, the fibred Dehn twist τ_C along C equals the monodromy map h_{σ_i} and so for any Lagrangian $L \subset M$ we have

$$L_{\sigma_i} \circ L \simeq \tau_C L. \tag{7.31}$$

Therefore using the local picture of the Lemma 4.1.3 we see that if we have a subset $B \subset \mathcal{Y}_m$ for which the naive (non-rescaled) parallel transport map $h_{\sigma_i}|_B$ is well-defined then

$$h_{\sigma_i} \cong \tau_{V_i}. \tag{7.32}$$

The reason is that since the naive parallel transport is well-defined for all points of B, we can shrink the rescaling parameter in (4.15) to zero and thereby isotope $h^{res}_{\sigma_i}$ to h_{σ_i}. Let kTl and $lT'm$ be tangles, $\sigma_i^+, \sigma_i^- \in Br_{2l}$ elementary braids and $T^{\pm} = T\sigma_i^{\pm}T'$. We observe that if a and b are crossingless matchings and we take

$\underline{L} = (a^t, \Phi(T))$ and $\underline{L}' = (\Phi(T'), b^t)$ then the map f in (7.27) is the same as the cobordism map (7.13).

Theorem 7.6.5. *Let e be the difference between the number of negative crossings in TT' and $T \cup_i \cap_i T'$ (with the latter oriented arbitrarily). Then $\mathcal{CSS}(T^-)$ is quasi-isomorphic to the cone of*

$$\mathcal{CSS}(T \cup_i \cap_i T')\{1-2e\} \overset{CSS(\mathbf{1}_T S_{\cap_i} \mathbf{1}_{T'})}{\longrightarrow} \mathcal{CSS}(TT') \tag{7.33}$$

and $\mathcal{CSS}(T^+)$ is quasi-isomorphic to the cone of

$$\mathcal{CSS}(TT')\{-1\} \overset{CSS(\mathbf{1}_T S^t_{\cap_i} \mathbf{1}_{T'})}{\longrightarrow} \mathcal{CSS}(T \cup_i \cap_i T')\{-2-2e\}. \tag{7.34}$$

Proof. We note that the sign conventions for positive braids and positive Dehn twists are opposites of each other. Since the degree of the map (7.13) equals 1, we apply the degree shift $\{-1\}$ to its target to obtain a map of degree zero. We have $w(T^-) = w(TT') - 1 = w(T \cup_i \cap_i T') + 2e - 1$ so we obtain (7.33).

In this case of T^+ we have $w(T^+) = w(TT') + 1 = w(T \cup_i \cap_i T') + 2e + 1$ and the cobordism map is of degree 1. $\square$

We have the following diagrams.

$$\begin{tikzcd}
 & \mathcal{CSS}(T^-) \arrow[dl] & \\
\mathcal{CSS}(T \cup_i \cap_i T')\{1-2e\} \arrow[rr] & & \mathcal{CSS}(TT') \arrow[ul, "\{1\}"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathcal{CSS}(T^+) \arrow[dl] & \\
\mathcal{CSS}(TT')\{-1\} \arrow[rr] & & \mathcal{CSS}(T \cup_i \cap_i T')\{-2-2e\} \arrow[ul, "\{1\}"']
\end{tikzcd}$$

These exact triangles are the same as those for Khovanov homology after the collapse of the bigrading as described in [19].

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