CHIRAL ALGEBRAS OF $(0,2)$ MODELS

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ABSTRACT OF THE DISSERTATION

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Sigma models with $(0,2)$ supersymmetry in two dimensions possess quasi-topological sectors characterized by chiral algebras. In this thesis, we study these chiral algebras and explore their nonperturbative aspects.

The chiral algebras of $(0,2)$ models emerge when one considers the cohomology of local operators with respect to one of the supercharges, and provide infinite-dimensional generalizations of the chiral rings of $(2,2)$ models. Perturbatively, they enjoy rich mathematical structures described by sheaves of chiral differential operators. Nonperturbatively, however, they vanish completely for certain $(0,2)$ models with no left-moving fermions. Examples include the models in which the target spaces are the complete flag manifolds of compact semisimple Lie groups.

The vanishing of the chiral algebra of a $(0,2)$ model implies that supersymmetry is spontaneously broken in the model, which in turn suggests that no harmonic spinors exist on the loop space of the target space. We analyze this supersymmetry breaking using holomorphic Morse theory on the loop space in the case where the target space is $\mathbb{CP}^1$. As expected, we find that instantons interpolate between pairs of perturbative supersymmetric states, thereby lifting them out of the supersymmetric spectrum.

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Chapter 1

Introduction

Supersymmetric sigma models in two dimensions have played central roles in a number of important physical and mathematical developments during the past three decades. One of the key concepts in these developments is that of the chiral rings of sigma models with $(2,2)$ supersymmetry [1]. These finite-dimensional cohomology rings are basic ingredients of topological sigma models [2, 3], and intimately connected to Gromov–Witten invariants [4, 2], Floer homology [5], and mirror symmetry [6], among others.

While the chiral rings of $(2,2)$ models are clearly very interesting, we also know that some of the beautiful structures of two-dimensional supersymmetric sigma models arise in essentially infinite-dimensional contexts; most notably, the elliptic genera [7, 8]. It is then natural to ask whether there are infinite-dimensional generalizations of the chiral rings. The answer is "yes." Such generalizations are provided by the *chiral algebras* of sigma models with $(0,2)$ supersymmetry [9].

The chiral algebra of a $(0,2)$ model is the cohomology of local operators with respect to one of the supercharges, graded by the right-moving R-charge, and equipped with a product structure inherited from the operator product expansion (OPE). As a consequence of $(0,2)$ supersymmetry, its elements vary holomorphically on the worldsheet, so it has a structure similar the chiral algebra of a conformal field theory—the algebra generated by holomorphic fields. Just like topological $(2,2)$ models are characterized by their chiral rings, there are quasi-topological twists of $(0,2)$ models that turn them into holomorphic field theories characterized by their chiral algebras.

Classically, the chiral algebra is isomorphic, as graded vector spaces, to the direct

sum of the Dolbeault cohomology groups of a certain infinite series of holomorphic vector bundles over the target space. Quantum mechanically, this structure gets deformed by quantum corrections. Since the chiral algebra, like the chiral rings, is invariant under deformations of the target space metric, one can compute it in the large volume limit where the theory is weakly coupled. The novelty is that, unlike the chiral rings, it receives perturbative corrections because the contributions from bosonic and fermionic fluctuations do not cancel due to the lack of left-moving supersymmetry. This leads to the very interesting subject of perturbative chiral algebras.

At the level of perturbation theory, the physics of a sigma model is determined by the local geometry of the target space. On the other hand, we can always deform the metric and make it flat locally without affecting the chiral algebra. Combining these observations, we reach a surprising conclusion: the perturbative chiral algebra can be described locally by sigma models with flat target spaces and, therefore, reconstructed by gluing these *free* theory descriptions patch by patch over the target space.

This fact was exploited by Witten [10] to show that the perturbative chiral algebra of a twisted model with no left-moving fermions can be formulated as the cohomology of the sheaf of chiral differential operators, introduced by Malikov et al. [11] and studied extensively [12, 13, 14, 15, 16] earlier in mathematics. In this picture, the moduli of the perturbative chiral algebra are encoded in the different possible ways of gluing the relevant free theories, whereas the anomalies of the theory manifest themselves in the obstructions to doing so consistently. In the case of models with left-moving fermions coupled to the tangent bundle of the target space, it was shown by Kapustin [17] that the perturbative chiral algebra is given by the cohomology of the sheaf of chiral de Rham complex [11]. The theory of perturbative chiral algebras has been further developed by Tan [18, 19, 20, 21] along these lines.

Nonperturbatively, instantons can change the picture radically [22]. A particularly striking example is the $(0, 2)$ model with no left-moving fermions whose target space

is the complete flag manifold of a compact semisimple Lie group G. The perturbative chiral algebra of this model is infinite-dimensional and possesses the structure of an affine $\mathfrak{g}$-module at the critical level [11, 23, 24]. In the presence of instantons, however, the chiral algebra *vanishes*.

A hint that such a vanishing phenomenon might exist comes from the conjecture made by Stolz [25] in 1996 and also independently by Höhn, which asserts that the elliptic genus of a supersymmetric sigma model with no left-moving fermions vanishes if the target space X admits a metric with positive Ricci curvature. In the paper [25], Stolz gave a heuristic argument for this conjecture, which goes as follows. Let us assume that the natural scalar curvature of the free loop space $\mathcal{L}X$ of X is computed, at a given point, by the integral of the Ricci curvature of X along the corresponding loop. Then $\mathcal{L}X$ has positive scalar curvature if X has positive Ricci curvature. By analogy with the Lichnerowicz theorem, this would imply that $\mathcal{L}X$ has no harmonic spinors. Meanwhile, supersymmetric states of the theory may be identified with harmonic spinors on $\mathcal{L}X$ [8]. Since the elliptic genus counts the number of bosonic minus fermionic supersymmetric states at each energy level, it would vanish then.

If Stolz's idea turns out to be correct, then the positivity of the Ricci curvature implies not only the vanishing of the elliptic genus, but also that the theory has no supersymmetric states; in other words, supersymmetry is spontaneously broken. Flag manifolds have positive Ricci curvature, hence supersymmetry must be broken in the $(0, 2)$ model in question. Supersymmetry breaking is indeed triggered whenever the chiral algebra becomes trivial with instanton corrections. The model with target space $X = \mathbb{CP}^1$ is simple enough so that one can see, by considering the holomorphic version of Morse theory [26] on $\mathcal{L}X$ [5], that instantons actually connect pairs of perturbatively supersymmetric states and lift them.

In fact, this vanishing of the chiral algebra of the flag manifold model is a special case of a more general vanishing theorem for nonperturbative chiral algebras—or

rather, "theorem" with quotation marks—that holds for a larger class of target spaces: the chiral algebra of a $(0, 2)$ model with no left-moving fermions vanishes nonperturbatively if the target space admits a rational curve with trivial normal bundle. This nonperturbative vanishing "theorem" is the main result of this thesis, which we will "prove" by a physical argument.

The nonperturbative aspects of the chiral algebras of $(0, 2)$ models and their relations to loop space geometry remain mysterious. I hope that the results presented in this thesis will shed some light on this subject.

This thesis is organized as follows. In Chapter 2, we formulate $(0, 2)$ models and discuss the general features of their chiral algebras. In Chapter 3, we demonstrate how the perturbative chiral algebra of a twisted model can be reconstructed by gluing free theories over the target space. In Chapter 4, we establish the nonperturbative vanishing "theorem." In Chapter 5, we study the supersymmetric spectrum of the $\mathbb{CP}^1$ model and argue that the model exhibits supersymmetry breaking from the viewpoint of holomorphic Morse theory on loop space. Finally, in Chapter 6, we present some possible directions for future research.

Chapter 2

Chiral Algebras of $(0,2)$ Models

In this chapter we discuss the general features of the chiral algebras of $(0,2)$ models. We begin by formulating the models that we will study in this thesis.

2.1 $(0,2)$ Models

A two-dimensional sigma model admits a $(0,2)$ supersymmetric extension if the target space is *strong Kähler with torsion*, or *strong KT* for short [10]. A complex manifold is called strong KT if it admits a hermitian metric whose associated $(1,1)$-form ω satisfies $\partial\bar{\partial}\omega = 0$. Kähler manifolds are in particular strong KT because Kähler forms obey $\partial\omega = \bar{\partial}\omega = 0$. Below we assume that the target space is Kähler, deferring the treatment of the strong KT case to Appendix.

Let Σ be a Riemann surface and let X be a Kähler manifold of dimension d. The bosonic sigma model with worldsheet Σ and target space X is then a field theory of smooth maps $\phi\colon \Sigma \to X$. Imposing $(0,2)$ supersymmetry on it requires the introduction of two right-moving fermions, ψ_+ and $\bar{\psi}_+$. They are sections of a square root of the antiholomorphic canonical bundle $\overline{K}_\Sigma$ of Σ, taking values respectively in the pullback by ϕ of the holomorphic and antiholomorphic tangent bundles T_X and $\overline{T}_X$ of X:

$$\psi_+ \in \Gamma(\overline{K}_\Sigma^{1/2} \otimes \phi^* T_X), \qquad \bar{\psi}_+ \in \Gamma(\overline{K}_\Sigma^{1/2} \otimes \phi^* \overline{T}_X). \tag{2.1}$$

Now, let me explain how a $(0,2)$ supersymmetric theory is constructed using this minimally $(0,2)$ supersymmetric field content.

To begin, we implement $(0, 2)$ supersymmetry. We do this by picking sections ϵ_-, $\bar{\epsilon}_-$ of $\overline{K}_\Sigma^{-1/2}$ and postulating the transformation generated by $-i\epsilon_- Q_+ + i\bar{\epsilon}_- \overline{Q}_+$ as follows:

$$\delta\phi^i = -\epsilon_- \psi_+^i, \qquad \delta\phi^{\bar{i}} = \bar{\epsilon}_- \bar{\psi}_+^{\bar{i}},$$

$$\delta\psi_+^i = i\bar{\epsilon}_- \partial_{\bar{z}}\phi^i, \qquad \delta\bar{\psi}_+^{\bar{i}} = -i\epsilon_- \partial_{\bar{z}}\phi^{\bar{i}}. \tag{2.2}$$

The supercharges Q_+, $\overline{Q}_+$ then satisfy the anticommutation relations

$$\{Q_+, Q_+\} = \{\overline{Q}_+, \overline{Q}_+\} = 0,$$

$$\{Q_+, \overline{Q}_+\} = -i\partial_{\bar{z}} \tag{2.3}$$

and generate the $(0, 2)$ supersymmetry algebra together with the generators H, P of translations, M of rotations, and F_R of R-symmetry. Under the last symmetry, ψ_+ and $\bar{\psi}_+$ are assigned charges -1 and $+1$; thus Q_+ has charge -1 and $\overline{Q}_+$ has charge $+1$.

In order to construct a $(0, 2)$ supersymmetric action, we choose a Kähler metric g on X and consider the local operator $g_{i\bar{j}}\psi_+^i\partial_z\phi^{\bar{j}}d^2z$, with $d^2z = 2idz \wedge d\bar{z}$, which is a $(1, 1)$-form on Σ and has R-charge -1. Then the action

$$S = \frac{1}{2\pi}\int_\Sigma d^2z\{\overline{Q}_+, g_{i\bar{j}}\psi_+^i\partial_z\phi^{\bar{j}}\} \tag{2.4}$$

is invariant under R-symmetry and, by virtue of the relation $\overline{Q}_+^2 = 0$, under the symmetry generated by $i\bar{\epsilon}_- \overline{Q}_+$ provided that $\bar{\epsilon}_-$ is antiholomorphic (so commutes with the ∂_z inside). That this action is also invariant under $-i\epsilon_- \overline{Q}_+$ for antiholomorphic ϵ_- becomes manifest if we rewrite it as

$$S = \frac{1}{2\pi}\int_\Sigma d^2z\{Q_+, g_{i\bar{j}}\partial_z\phi^i\bar{\psi}_+^{\bar{j}}\}. \tag{2.5}$$

Expanding the anticommutators and using the Kähler condition, one can check that the two actions (2.4) and (2.5) both coincide with

$$S = \frac{1}{2\pi}\int_\Sigma d^2z(g_{i\bar{j}}\partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}} + ig_{i\bar{j}}\psi_+^i D_z\bar{\psi}_+^{\bar{j}}). \tag{2.6}$$

The covariant derivative D_z is here the ∂-operator coupled to the pullback of the Levi-Civita connection Γ on X. Explicitly, $D_z\bar{\psi}_+^{\bar{i}} = \partial_z\bar{\psi}_+^{\bar{i}} + \partial_z\phi^{\bar{j}}\Gamma_{\bar{j}\bar{k}}^{\bar{i}}\bar{\psi}_+^{\bar{k}}$.

We may also add a topological invariant to the action. Let $B = B_{IJ} d\phi^I \wedge d\phi^J$ be a closed two-form on X. Then the functional

$$S_B = \int_\Sigma \phi^* B \qquad (2.7)$$

depends only on the cohomology class $[B]$ and the homology class $\phi_*[\Sigma]$. As such, it is invariant under any continuous transformations, especially the supersymmetry transformations and the R-symmetry. The topological action S_B vanishes in perturbation theory where one considers only homotopically trivial maps.

We have obtained a $(0, 2)$ supersymmetric action. To complete the construction, we need to make sure that a sensible quantum theory based on it exists. It turns out that X must satisfy two topological conditions for this. First, X must be spin, or equivalently, its first Chern class must be zero modulo 2:

$$c_1(X) \equiv 0 \pmod 2. \qquad (2.8)$$

As we will see, this condition ensures that the fermion parity $(-1)^F$ is well defined, which is necessary to distinguish bosonic and fermionic states. Second, the second Chern character $\mathrm{ch}_2(X) = c_1(X)^2/2 - c_2(X)$, which is also a half of the first Pontryagin class $p_1(X)$, must be zero:

$$\frac{1}{2} p_1(X) = 0. \qquad (2.9)$$

This is the condition for the cancellation of sigma model anomalies [27, 28], the obstruction to finding a well-defined path integral measure.

From the viewpoint of the the free loop space $\mathcal{L}X$ of X, the space of smooth maps from the circle S^1 to X, the first condition means that $\mathcal{L}X$ is orientable [29, 30]. The second condition, on the other hand, may be interpreted as the condition for $\mathcal{L}X$ to admit spinors [31]. The Dirac operator on $\mathcal{L}X$ [7, 8] will play an important role when we study the cohomology of $(0, 2)$ models in Chapter 5.

The action (2.6) plus the topological action (2.7) defines the simplest version of

$(0, 2)$ model. The supercharges are realized as

$$Q_+ = \oint d\bar{z}\, g_{i\bar{\jmath}} \psi_+^i \partial_{\bar{z}} \phi^{\bar{\jmath}}, \qquad \overline{Q}_+ = \oint d\bar{z}\, g_{i\bar{\jmath}} \partial_{\bar{z}} \phi^i \bar{\psi}_+^{\bar{\jmath}}, \tag{2.10}$$

and satisfy the reality condition $Q_+^\dagger = \overline{Q}_+$.

With a holomorphic vector bundle E over X, we can extend this model by adding left-moving fermionic fields valued in $\phi^* E$ and $\phi^* \overline{E}$. This extended $(0, 2)$ model is called the heterotic model. Since the left-movers contribute to the sigma model anomaly in the opposite way as the right-movers do, their presence changes the anomaly cancellation condition to $p_1(T_X)/2 = p_1(E)/2$. This condition is trivially satisfied if $E = T_X$; in this case, the heterotic model actually possesses $(2, 2)$ supersymmetry. It is also possible to add superpotentials [32]. For brevity, we will not consider these extensions in this thesis. For the perturbative aspects of the heterotic model, we refer to Tan [18, 19, 20, 21].

2.2 Chiral Algebras

Having formulated $(0, 2)$ models, let us now defined their chiral algebras.

Let Q be one of the supercharges, say $Q = \overline{Q}_+$. We consider the action of Q in the space of local operators, given by commutator on bosonic operators and anticommutator on fermionic operators. This Q-action increases R-charge by one, and squares to zero by the relation $Q^2 = 0$. Given a $(0, 2)$ model, therefore, we can define the Q-cohomology of local operators graded by R-charge.

Since $\partial_{\bar{z}}$ is Q-exact by the $(0, 2)$ supersymmetry algebra, $\partial_{\bar{z}}$ acts trivially in the Q-cohomology: if $\mathcal{O}$ is Q-closed, then $\partial_{\bar{z}} \mathcal{O} = [\{Q, iQ^\dagger\}, \mathcal{O}]$ is Q-exact. The Q-cohomology classes thus vary holomorphically on Σ. Moreover, there is a natural product structure on the Q-cohomology induced by the OPE:

$$[\mathcal{O}_i(z)] \cdot [\mathcal{O}_j(z')] = [\mathcal{O}_i(z) \cdot \mathcal{O}_j(z')] \sim \sum_k c_{ij}{}^k(z - z')[\mathcal{O}_k(z')]. \tag{2.11}$$

The Q-cohomology of local operators equipped with this OPE structure defines the

chiral algebra of the $(0,2)$ model. We denote it by $\mathcal{A}$ and its charge q subspace by $\mathcal{A}^q$:

$$\mathcal{A} = \bigoplus_q \mathcal{A}^q. \tag{2.12}$$

The chiral algebra possesses the defining properties of a chiral algebra in the sense of conformal field theory, except that the grading by conformal weight is missing.

The chiral algebra forms an interesting quasi-topological sector of the $(0,2)$ model. Consider the n-point function of Q-closed local operators, $\mathcal{O}_1, \cdots, \mathcal{O}_n$:

$$\langle \mathcal{O}_1(z_1, \bar{z}_1) \cdots \mathcal{O}_n(z_n, \bar{z}_n) \rangle \tag{2.13}$$

If one of the local operators, $\mathcal{O}_i$, is Q-exact and written as $\mathcal{O}_i = \{Q, \mathcal{O}_i'\}$ for some $\mathcal{O}_i'$, then the n-point function becomes $\langle \{Q, \mathcal{O}_1 \cdots \mathcal{O}_i' \cdots \mathcal{O}_n\} \rangle$. Computed with Q-invariant action and path integral measure, this correlation function becomes the integral of a "Q-exact form" over the field space and vanishes. The n-point function (2.13) therefore depends only on the cohomology classes of $\mathcal{O}_i$. In particular, it is a holomorphic (or more precisely, meromorphic) function of the insertion points

So far, we have considered the chiral algebra of a fixed $(0,2)$ model described by a fixed action. A different choice of the action of course leads to a different chiral algebra in general. Imagine deforming the theory by perturbing the action:

$$S \to S + \epsilon S'. \tag{2.14}$$

For this deformation to preserve the fermionic symmetry generated by Q, the perturbation S' must be Q-closed. If S' is Q-exact, on the other hand, then the chiral algebra of the deformed theory is isomorphic to that of the original theory. This conclusion is obtained by expressing the matrix elements of Q as path integrals on a cylinder of infinitesimal length. One can then show that the action of Q in the deformed theory is represented in the original theory by $Q + \epsilon[Q, \Lambda]$ for some operator Λ. Such a deformation is equivalent to the conjugation

$$Q \to e^{-\epsilon\Lambda} Q e^{\epsilon\Lambda}, \tag{2.15}$$

under which the chiral algebra is mapped to an isomorphic algebra, with isomorphism given by $\mathcal{O} \mapsto e^{-\epsilon \Lambda} \mathcal{O} e^{\epsilon \Lambda}$.

In our action the target space metric appears inside a Q-commutator, so any small change of the metric results in a Q-exact perturbation. Therefore, the chiral algebra is invariant under deformations of the target space metric that are compatible with the complex structure. The chiral algebra certainly depends on the complex structure, for this enters the definition of the supersymmetry transformations.

2.3   Twisting

We have seen that the $(0, 2)$ model action (2.4) is invariant under the supersymmetry transformation (2.2) if the parameters ϵ_-, $\bar{\epsilon}_-$ are antiholomorphic sections of $\overline{K}_\Sigma^{-1/2}$. When Σ is topologically nontrivial, however, it is very possible that this bundle does not admit any global antiholomorphic sections. In such a case the chiral algebras of a $(0, 2)$ model on Σ, as defined in the previous section, does not exist. But since what we actually need to construct the chiral algebra is one of the two supercharges, it would be nice if we could somehow save one in return for giving up the other. This is achieved by "twisting" the model.

Let us modify the spins of the right-moving fermions, so that ψ_+ and $\bar{\psi}_+$ become a $(0, 1)$-form and a scalar on Σ. To make this point clear, we rename these fields as

$$-\psi_+^i \to \rho_{\bar{z}}^i, \qquad -i\bar{\psi}_+^{\bar{\imath}} \to \alpha^{\bar{\imath}}. \tag{2.16}$$

From the supersymmetry transformation (2.2), we see that this modification turns Q_+ into a $(0, 1)$-form and $\overline{Q}_+$ into a scalar. The parameter $\bar{\epsilon}_-$ is now a function on Σ, so we can choose it to be constant. In this way, we obtain the *twisted model* which possesses a fermionic charge Q on any Riemann surface Σ. The twisted model has another fermionic symmetry, generated by Q_+, when $\overline{K}_\Sigma^{-1}$ admits antiholomorphic sections.

Since we only demand the twisted model to have one scalar fermionic charge, Q, we

can actually formulate them for any hermitian manifold X which may or may not be Kähler. To do this, we simply introduce fermionic fields

$$\rho \in \Gamma(\overline{K}_\Sigma \otimes \phi^* T_X), \qquad \alpha \in \Gamma(\phi^* \overline{T}_X), \tag{2.17}$$

and define the action of Q by

$$[Q, \phi^i] = 0, \qquad \qquad [Q, \phi^{\bar{\imath}}] = \alpha^{\bar{\imath}},$$
$$\{Q, \rho_{\bar{z}}^i\} = -\partial_{\bar{z}} \phi^i, \qquad \{Q, \alpha^{\bar{\imath}}\} = 0. \tag{2.18}$$

Then the twisted model with worldsheet Σ and target space X equipped with metric g is by definition the theory described by the action

$$S = \int_\Sigma d^2 z \{Q, -g_{i\bar{\jmath}} \rho_{\bar{z}}^i \partial_z \phi^{\bar{\jmath}}\} + \int_\Sigma \phi^* B. \tag{2.19}$$

When X is Kähler and g is a Kähler metric, this theory is obtained by twisting the corresponding $(0, 2)$ model and setting $\rho_{\bar{z}}^i = -\psi_+^i$ and $\alpha^{\bar{\imath}} = -i\bar{\psi}_+^{\bar{\imath}}$. If $\overline{K}_\Sigma^{1/2}$ is moreover trivial, the two models are equivalent because the twisting does nothing in that case.

To understand the effect of the twisting on the chiral algebra, let us first introduce some terminology. We say that a local operator $\mathcal{O}$ has dimension (n, m) if $\mathcal{O}(0)$ inserted at the origin transforms under a rescaling $z \to \lambda z$, $\bar{z} \to \bar{\lambda} \bar{z}$ as

$$\mathcal{O}(0) \to \lambda^{-n} \bar{\lambda}^{-m} \mathcal{O}(0). \tag{2.20}$$

The integers n and m are the holomorphic and antiholomorphic dimensions of $\mathcal{O}$, respectively. Thus, a local operator of dimension (n, m) transforms like $\partial_z^n \partial_{\bar{z}}^m$. Classically, our $(0, 2)$ models are conformally invariant and the space of local operators is graded by dimension, as well as R-charge.

After the twisting, Q generates a global fermionic symmetry that commutes with

conformal transformations. Classically the latter are also symmetries, so their generator, the energy-momentum tensor, should commutes with Q. Indeed, we have

$$T_{zz} = g_{i\bar{j}}\partial_z\phi^i\partial_z\phi^{\bar{j}},$$

$$T_{\bar{z}\bar{z}} = g_{i\bar{j}}\partial_{\bar{z}}\phi^i\partial_{\bar{z}}\phi^{\bar{j}} + g_{i\bar{j}}\rho^i_{\bar{z}}D_{\bar{z}}\alpha^{\bar{j}}, \tag{2.21}$$

$$T_{z\bar{z}} = T_{\bar{z}z} = 0,$$

and these are all Q-closed with the equation of motion $D_z\alpha^{\bar{j}} = 0$. Thus, classically conformal transformations act naturally on the chiral algebra. It follows in particular that the classical chiral algebra is graded by dimension.

In fact, $T_{\bar{z}\bar{z}}$ is not just Q-closed; it is actually Q-exact since we can rewrite it as $T_{\bar{z}\bar{z}} = \{Q, -g_{i\bar{j}}\rho^i_{\bar{z}}\partial_{\bar{z}}\phi^{\bar{j}}\}$. This is an important feature of the twisted model that is a consequence of the fact that the infinitesimal diffeomorphism $z \mapsto z + \epsilon v^z(z, \bar{z})$ almost commutes with the fermionic symmetry, but fails by a quantity involving $\partial_{\bar{z}}v^z$ and not $\partial_{\bar{z}}v^{\bar{z}}$, $\partial_z v^z$, or $\partial_z v^{\bar{z}}$. So, as far as the computations of $T_{z\bar{z}}$, $T_{\bar{z}z}$ and $T_{\bar{z}\bar{z}}$ are concerned, we may treat the diffeomorphism as though it commutes with Q and take the variation of the action under it inside the Q-commutator.

The Q-exactness of $T_{\bar{z}\bar{z}}$ means that antiholomorphic reparametrizations act trivially on the chiral algebra. Especially, the antiholomorphic rescaling $\bar{z} \to \bar{\lambda}\bar{z}$ leaves cohomology classes invariant. Since this acts on the cohomology class represented by a Q-closed local operator $\mathcal{O}$ of dimension (n, m) by $[\mathcal{O}(0)] \mapsto \bar{\lambda}^{-m}[\mathcal{O}(0)]$, we must have $[\mathcal{O}] = 0$ unless $m = 0$. Therefore, the chiral algebra of the twisted model is classically supported by Q-closed local operators of dimensions $(n, 0)$ with nonnegative integers n.

Quantum mechanically, the action gets renormalized and the energy-momentum tensor receive corrections. As is well known [33, 34, 35], the conformal invariance is broken and $T_{z\bar{z}}$ is no longer zero if the Ricci curvature of the target space is nonzero. Still, $T_{z\bar{z}}$ and $T_{\bar{z}\bar{z}}$ remain Q-exact at the level of perturbation theory. The reason is that the perturbative renormalization of the twisted model can be done while preserving the fermionic symmetry, using counterterms that are Q-closed operators. By rotational

symmetry, these counterterms have dimension (n, n) with $n > 0$. Such Q-closed local operators are actually all Q-exact, as shown in the next section.[1]

With $T_{z\bar{z}}$ and $T_{\bar{z}\bar{z}}$ trivial in the chiral algebra, it follows that the chiral algebra is still supported by Q-closed local operators of dimension $(n, 0)$. Moreover, the grading by holomorphic dimension is preserved even though the theory may no longer be conformally invariant. This is because if a local operator has classically dimension (n, m), then perturbatively n and m may be shifted, but the spin $n - m$ can only take integers and hence is protected from small quantum corrections.

Regarding Q as a BRST operator, we have found that the twisted model is perturbatively a holomorphic field theory, in which the antiholomorphic degrees of freedom completely decouple, and characterized by the chiral algebra graded by holomorphic dimension as well as R-charge. The OPE now takes the form

$$[\mathcal{O}_i(z)] \cdot [\mathcal{O}_j(z')] \sim \sum_k \frac{c_{ij}{}^k [\mathcal{O}_k(z')]}{(z - z')^{n_i + n_j - n_k}}, \tag{2.22}$$

where $[\mathcal{O}_k]$ have holomorphic dimensions n_k. Beyond perturbation theory, the grading by dimension may be broken.

The twisting thus allows us to define the chiral algebras with interesting holomorphic structure for a larger class of worldsheets and target spaces including those that are not compatible with $(0, 2)$ supersymmetry. However, there is a price to pay. Twisting the right-moving fermions amounts to tensoring with $\overline{K}_\Sigma^{-1/2}$ the bundle to which D_z couples. This changes the anomaly cancellation condition to $\mathrm{ch}_2(X) + c_1(\Sigma)c_1(X)/2 = 0$, or

$$\frac{1}{2}p_1(X) = \frac{1}{2}c_1(X)c_1(\Sigma) = 0. \tag{2.23}$$

(Here the pullback to $\Sigma \times X$ is implicit in the expression $c_1(X)c_1(\Sigma)$.) So the twisting

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[1]Here we have assumed that the counterterms are local operators. When the worldsheet Σ is curved, however, we need to introduce a metric on it in order to define a meaningful cutoff length. Coupling the theory to the worldsheet metric gives rise to a counterterm that is proportional to the Ricci curvature of Σ [35]. In that case, the theory suffers from gravitational and Weyl anomalies due to the asymmetry in the left- and right-moving central charges [36, 37]. Such c-number anomalies do not affect the action of Q on operators and hence the chiral algebra.

introduces an additional anomaly which requires $c_1(X)c_1(\Sigma)/2$ to vanish. We must therefore consider worldsheets with $c_1(\Sigma) = 0$ if the target space has $c_1(X) \neq 0$.

2.4 Classical Chiral Algebras and Quantum Corrections

Now let us take a closer look at the chiral algebra of a twisted model. We first consider the classical chiral algebra, then find possible forms of quantum corrections.

Since ρ and $\bar{z}$-derivatives of any fields have positive antiholomorphic dimensions, these do not enter the classical chiral algebra. Furthermore, z-derivatives of α can be replaced with other fields using the equation of motion $D_z\alpha = 0$. Thus, a general local operator of charge q and dimension n can be written, as a sum of operators of the form

$$\mathcal{O}(\phi, \bar{\phi})_{j \cdots k \cdots \bar{l} \cdots \bar{m} \cdots \bar{\imath}_1 \cdots \bar{\imath}_q} \partial_z \phi^j \cdots \partial_z^2 \phi^k \cdots \partial_z \phi^{\bar{l}} \cdots \partial_z^2 \phi^{\bar{m}} \cdots \alpha^{\bar{\imath}_1} \cdots \alpha^{\bar{\imath}_q}, \tag{2.24}$$

with the total number of ∂_zs equal to n.

Identifying $\alpha^{\bar{\imath}}$ with $d\phi^{\bar{\imath}}$, the local operator (2.24) of charge q and dimension n can be regarded as a $(0, q)$-form with values in a certain holomorphic vector bundle V_X^n over X, constructed from the holomorphic tangent bundle T_X and its dual $T_X^\vee$. For example, local operators of charge q and dimension zero are of the form $\mathcal{O}_{\bar{\imath}_1 \cdots \bar{\imath}_q} \alpha^{\bar{\imath}_1} \cdots \alpha^{\bar{\imath}_q}$; thus they are $(0, q)$-from on X and $V_{X,0} = 1$, the trivial bundle. For local operators of charge q and dimension one, we have two possibilities, $\mathcal{O}_{j\bar{\imath}_1 \cdots \bar{\imath}_q} \partial_z \phi^j \alpha^{\bar{\imath}_1} \cdots \alpha^{\bar{\imath}_q}$ and $\mathcal{O}^j_{\bar{\imath}_1 \cdots \bar{\imath}_q} g_{j\bar{k}} \partial_z \phi^{\bar{k}} \alpha^{\bar{\imath}_1} \cdots \alpha^{\bar{\imath}_q}$; thus they are $(0, q)$-forms of $V_{X,1} = T_X \oplus T_X^\vee$. At dimension two, we have five independent local operators of charge q. We list them in the case of $q = 0$: $\mathcal{O}_{j\bar{\imath}_1 \cdots \bar{\imath}_q} \partial_z^2 \phi^j$, $\mathcal{O}_{jk\bar{\imath}_1 \cdots \bar{\imath}_q} \partial_z \phi^j \partial_z \phi^k$, $\mathcal{O}_j{}^k{}_{\bar{\imath}_1 \cdots \bar{\imath}_q} \partial_z \phi^j g_{k\bar{l}} \partial_z \phi^{\bar{k}}$, $\mathcal{O}^{jk}{}_{\bar{\imath}_1 \cdots \bar{\imath}_q} g_{j\bar{l}} \partial_z \phi^{\bar{l}} g_{k\bar{m}} \partial_z \phi^{\bar{m}}$, and $\mathcal{O}^j{}_{\bar{\imath}_1 \cdots \bar{\imath}_q} \partial_z (g_{j\bar{k}} \partial_z \phi^{\bar{k}})$. Thus $V_{X,2} = T_X \oplus T_X^\vee \oplus S^2 T_X \oplus (T_X \otimes T_X^\vee) \oplus S^2 T_X^\vee$, where S^k is the symmetric kth power. In general, V_X^n is given by the series

$$\sum_{n=0}^{\infty} q^n V_{X,n} = \bigotimes_{k=0}^{\infty} S_{q^k}(T_X) \bigotimes_{l=0}^{\infty} S_{q^l}(T_X^\vee). \tag{2.25}$$

Here $S_t(V)$ of a vector bundle V is defined by $S_t(V) = 1 + tV + t^2 S^2 V + \cdots$. We set $V_X = \bigoplus_{n=0}^{\infty} V_{X,n}$ and denote by V_X^q the space of $(0, q)$ forms with values in V_X.

From the commutation relations (2.18) and the equation of motion $D_z \alpha^{\bar{\imath}} = 0$, we see that Q acts on local operators as the $\bar{\partial}$-operator. Classically, the qth Q-cohomology group of local operators of dimension n is therefore isomorphic to the qth Dolbeault cohomology group $H^q_{\bar{\partial}}(X, V_{X,n})$. Therefore, we find

$$\mathcal{A} \cong \bigoplus_{q=0}^{d} \bigoplus_{n=0}^{\infty} H^q_{\bar{\partial}}(X, V_{X,n}) \tag{2.26}$$

as graded vector spaces.

Quantum mechanically, this classical result is deformed by quantum corrections. The following two general principles are useful to keep in mind. First, quantum corrections can only destroy—or "lift"—cohomology classes, not create new ones, if the target space is compact and quantum effects are small enough. Second, quantum corrections lift cohomology classes in pairs.

Perturbatively, the structure of the chiral algebra is not very much different from the classical case. The basic property of sigma model perturbation theory is that it is local on the target space. The short distance singularities in perturbation theory come from the fluctuations around constant maps. The perturbative renormalization can thus be performed with the knowledge about the behavior of the theory near each constant map, in other words, locally at each point on the target space. From the locality, it follows that Q still acts on $V_{X,n}$ as a differential operator. Therefore, perturbatively

$$\mathcal{A} \cong \bigoplus_{q=0}^{d} \bigoplus_{n=0}^{\infty} H^q_Q(X, V_{X,n}), \tag{2.27}$$

with Q being a deformation of the $\bar{\partial}$-operator by perturbative corrections.

There is one important consequence of the perturbative corrections. Let us consider the perturbative action of Q on T_{zz}. The generator ∂_z of holomorphic translations commutes with Q. Since $T_{z\bar{z}}$ is perturbatively Q-exact, we have

$$\left[Q, \oint dz\, T_{zz} + \oint d\bar{z}\, T_{z\bar{z}}\right] = \oint dz\, [Q, T_{zz}] = 0, \tag{2.28}$$

which implies that $[Q, T_{zz}] = \partial_z \theta$ for some θ. The conservation of R-charge, dimension, the power counting on the target space, and the requirement that $\partial_z \theta$ be Q-closed leave only one possibility at one loop:

$$\theta \propto R_{i\bar{j}} \partial_z \phi^i \alpha^{\bar{j}}. \tag{2.29}$$

Thus we expect that the perturbative corrections destroy T_{zz} by making it no longer Q-closed, while lift $\partial_z \theta$ by making it Q-exact. If $c_1(X) = 0$, then θ is Q-exact and we can modify T_{zz} to retain a Q-closed energy-momentum tensor. But if $c_1(X) \neq 0$, there is no way to remove this conformal anomaly. So the perturbative chiral algebra is a little exotic in this case: it is like the chiral algebra of a conformal field theory, but one without an energy-momentum tensor! We will show in Chapter 3 that perturbatively $[Q, T_{zz}]$ is indeed given by a nonzero multiple of $\partial_z(R_{i\bar{j}} \partial_z \phi^i \alpha^{\bar{j}})$. This reflects the one-loop beta function that is proportional to the Ricci curvature of the target space.

Nonperturbatively, the physics is no longer local on the target space. This leaves the possibility of more radical deformations to the chiral algebra. To assert what kind of deformations are possible, we first need to discuss where the nonperturbative corrections come from. We will do so in the next section, but before that, let me explain why the two principles that we have stated above are true.

First of all, notice that similar principles hold for the space of supersymmetric states. If the target space is compact, there is a finite gap between the smallest eigenvalue of the laplacian $\{Q, Q^\dagger\} = H - P$ and zero. Thus, small quantum effects cannot push nonsupersymmetric states with $H - P > 0$ down to $H - P = 0$, but can only lift supersymmetric states by giving them a very small $H - P$. Also, nonsupersymmetric states come always in boson-fermion pairs related by the supercharges. So whenever an approximate supersymmetric state is lifted by quantum corrections, there should be another of opposite statistics that is lifted together.

For the chiral algebra, the argument for the first principle goes as follows. Classically, the theory is conformally invariant and there is actually an isomorphism between

the Q-cohomology of local operators and the space of supersymmetric states via the state-operator correspondence. For a local operator to represent a nonzero cohomology class after small quantum corrections are included, to begin with it must be very close to successfully representing a nonzero cohomology class classically. Then, the corresponding classical state must have very small $H - P$. However, this is impossible if the target space is compact.

The argument for the second principle is a bit longer. Let us represent the quantum-corrected action of Q by the operator $Q + \epsilon Q'$ in the classical theory. Here ϵ is a small parameter that controls the strength of quantum effects. For $(Q + \epsilon Q')^2$ to be zero, $\{Q, Q'\} = 0$. Suppose that $\mathcal{O}$ is a local operator that is Q-closed but not Q-exact, and let us see what happens when the Q-cohomology class $[\mathcal{O}]$ is lifted by the quantum corrections. (In what follows $\{Q, \mathcal{O}\}$ denotes commutator if $\mathcal{O}$ is bosonic and anticommutator if $\mathcal{O}$ is fermionic.)

First, suppose that $\mathcal{O}$ is not $(Q + \epsilon Q')$-closed, and one cannot find a correction that makes it $(Q + \epsilon Q')$-closed. Then $\{Q', \mathcal{O}\}$ is Q-closed since $\mathcal{O}$ is Q-closed. Further, it is not Q-exact. For if $\{Q', \mathcal{O}\} = -\{Q, \mathcal{O}'\}$ for some $\mathcal{O}'$, then $\{Q + \epsilon Q', \mathcal{O} + \epsilon \mathcal{O}'\} = 0$, but by assumption such $\mathcal{O}'$ does not exist. Thus $\{Q', \mathcal{O}\}$ represents a nontrivial Q-cohomology class. However, $\{Q', \mathcal{O}\} = \{Q + \epsilon Q', \epsilon^{-1} \mathcal{O}\}$, so $\{Q', \mathcal{O}\}$ is $(Q + \epsilon Q')$-exact.

Next, suppose that one can find a correction $\mathcal{O}'$ such that $\mathcal{O} + \epsilon \mathcal{O}'$ is $(Q + \epsilon Q')$-closed, but the corrected operator is $(Q + \epsilon Q')$-exact. Then $\mathcal{O} + \epsilon \mathcal{O}' = \{Q + \epsilon Q', \epsilon^{-1} \mathcal{O}''\}$ for some Q-closed $\mathcal{O}''$ such that $\mathcal{O} = \{Q', \mathcal{O}''\}$. As $\mathcal{O}$ is not Q-exact by assumption, $\mathcal{O}''$ is not Q-exact either. Thus $\mathcal{O}''$ represents a nontrivial Q-cohomology class. However, $\{Q + \epsilon Q', \mathcal{O}''\} = \epsilon \mathcal{O}$, so $\mathcal{O}''$ is not $(Q + \epsilon Q')$-closed. Moreover, one cannot find a correction that makes it $(Q + \epsilon Q')$-closed because, again, $\mathcal{O}$ is not Q-exact.

The above argument shows that whenever a Q-cohomology class $[\mathcal{O}]$ gets lifted by quantum corrections, either becoming no longer Q-closed or Q-exact, there is the associated Q-cohomology class, $[\{Q', \mathcal{O}\}]$ or $[\mathcal{O}'']$, which also gets lifted but by the other

way. This demonstrates the second principle.

2.5 Localization

The chiral algebra has an important property that follows from the invariance under deformations of the target space metric: it receives contributions only from instantons and fluctuations around them.

In a theory with a fermionic charge Q, *instantons* are bosonic field configurations at which the fermionic fields become Q-invariant. In our case, $Q = \overline{Q}_+$ and the instanton equation reads

$$\{Q, \psi_+^i\} = \partial_{\bar{z}} \phi^i = 0. \tag{2.30}$$

Thus, instantons are holomorphic maps from Σ to X. The space $\mathcal{M}$ of such maps is called the *instanton moduli space*. The above property of the chiral algebra can then be stated that the path integral computation of the chiral algebra localizes to $\mathcal{M}$. The reason that this localization takes place is that we can compute the chiral algebra in the limit where the volume of the target space is very large. But in the large volume limit, any path integral localizes to the zeros of the bosonic action

$$\int_{\Sigma} d^2z \, g_{i\bar{j}} \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{j}}, \tag{2.31}$$

hence to holomorphic maps. This localization principle is what makes the chiral algebra effectively computable.

Incidentally, there is another situation in which this localization applies: when the path integral computes the correlation function of Q-closed operators [3]. In this case the localization occurs because, in essence, away from $\mathcal{M}$ one can use the fermionic symmetry to set one of the fermionic fields to zero. Then the operators in the correlation function become independent of this fermionic field and the integration over it vanishes. This situation is not really relevant for us right now, though—we would like to ask, in the first place, whether a given local operator is Q-closed or not!

To keep track of the order of instanton contributions, we set the real part of the B-field to the Kähler form ω and assume for simplicity that it is normalized such that

$$\int_\Sigma \phi^* B = kt \tag{2.32}$$

for integers $k \geq 0$. We label the components of instanton moduli space $\mathcal{M}$ by their degree. The k-instanton moduli space $\mathcal{M}_k$ is the space of instantons of degree k. Correlation functions now decompose into different instanton sectors:

$$\langle \ldots \rangle = \sum_{k=0}^\infty e^{-kt} \langle \ldots \rangle_k. \tag{2.33}$$

Correspondingly, any operator A can be expanded in the instanton weight:

$$A = \sum_{k=0}^\infty e^{-kt} A_k. \tag{2.34}$$

Sigma model perturbation theory is the zero-instanton approximation. Instantons of degree zero are constant maps, and their moduli space $\mathcal{M}_0 \cong X$.

Instantons are closely related to the anomaly of R-symmetry. We can see this by constructing the path integral measure for the right-moving fermions.

The fermionic fields ψ_+ and $\bar{\psi}_+$ are sections of $\overline{K}_\Sigma^{1/2} \otimes \phi^* T_X$ and $\overline{K}_\Sigma^{1/2} \otimes \phi^* \overline{T}_X$, respectively, hence in order to define the fermionic path integral measure, we must introduce local frames on these bundles. This can be done by expanding the fermions in the eigenmodes of the laplacians as

$$\psi_+(z, \bar{z}; \phi) = \sum_s b_0^s v_{0,s}(z, \bar{z}; \phi) + \sum_n b_n^\alpha v_{n,\alpha}(z, \bar{z}; \phi),$$
$$\tag{2.35}$$
$$\bar{\psi}_+(z, \bar{z}; \phi) = \sum_r c_0^r \bar{u}_{0,r}(z, \bar{z}; \phi) + \sum_n c_n^\alpha \bar{u}_{n,\alpha}(z, \bar{z}; \phi),$$

where $\bar{u}_{0,r}$, $v_{0,s}$ are zero modes, $\bar{u}_{n,\alpha}$, $v_{n,\alpha}$ are nonzero modes, and b_0^s, c_0^r, b^n, c^n are grassmannian variables. The measure is then defined by the formal product

$$\prod_{r,s,n,\alpha} db_0^s dc_0^r db^n dc^n. \tag{2.36}$$

Because of the paring of nonzero modes, the nonzero mode part of the fermionic path integral measure (2.36) is neutral under the R-symmetry. On the other hand, the

zero mode part has charge equal to the number of ψ_+ zero modes $v_{0,s}$ minus the number of $\bar{\psi}_+$ zero modes $\bar{u}_{0,r}$, that is, minus of the index of the $\bar{\partial}$-operator twisted by $\phi^* T_X$. On a compact Riemann surface Σ, the index is given by

$$\int_\Sigma \phi^* c_1(X). \tag{2.37}$$

Therefore, R-symmetry is anomalous if $c_1(X) \neq 0$.

Due to this anomaly, the correlation function vanishes in a given instanton sector unless the total charge of the inserted operators is equal to the quantity (2.37) computed for maps in that sector. Furthermore, the R-symmetry is broken to the discrete $\mathbb{Z}_{2k}$ symmetry, where $2k$ is the greatest common divisor of $c_1(X)$. (Recall the condition $c_1(X) \equiv 0 \pmod{2}$.) As a result, it is possible for two local operators $\mathcal{O}_1$ and $\mathcal{O}_2$, whose charges differ by a multiple of $2k$, to define the same cohomology class in the *full* correlation function. Therefore, the grading of the chiral algebra by R-charge is broken to a $\mathbb{Z}_{2k}$-grading in the presence of instantons.

Chapter 3

Perturbative Chiral Algebras

In the previous chapter we have discussed the chiral algebras in general terms. We now focus on the perturbative approximation to the chiral algebras of the twisted models. The goal of this chapter is to understand how the perturbative chiral algebras can be reconstructed, to all orders in perturbation theory, by gluing certain free conformal field theories over the target spaces.

3.1 Čech–Q Isomorphism

The classical chiral algebra of a twisted model with target space X is given by the Dolbeault cohomology of the holomorphic vector bundle V_X over X, which is the direct sum of the bundles $V_{X,n}$ appearing in the series (2.25). The Čech–Dolbeault isomorphism states that we have

$$H^q_{\bar{\partial}}(X, V_X) \cong \check{H}^q(X, \mathcal{D}^{\bar{\partial}}_X), \tag{3.1}$$

where $\mathcal{D}^{\bar{\partial}}_X$ is the sheaf of holomorphic sections of V_X. Classically, the Čech–Dolbeault isomorphism therefore provides an alternative formulation of the chiral algebra.

Perturbatively, Q acts on sections of V_X as a deformation of the $\bar{\partial}$-operator. It is then plausible that we can furnish a similar sheaf-theoretic interpretation of the perturbative chiral algebras, via an isomorphism

$$H^q_Q(X, V_X) \cong \check{H}^q(X, \mathcal{D}^Q_X) \tag{3.2}$$

between the perturbative Q-cohomology of local operators and the cohomology of the

sheaf $\mathcal{D}_X^Q$ of perturbatively Q-closed sections of V_X. Does such a perturbative "Čech–Q isomorphism" exist?

To answer this question, we recall two key ingredients in the proof of the Čech–Dolbeault isomorphism: the $\bar{\partial}$-Poincaré lemma and the fact that $H^p(X, \mathcal{V}_X^q) = 0$ for all $p > 0$, where $\mathcal{V}_X^q$ are the sheaves of $(0, q)$-forms of V_X. We now argue that these properties of the sheaf of local operators carry over to the perturbative case.

First, we note that the $\bar{\partial}$-Poincaré lemma can be formulated as the vanishing of the higher cohomology groups on a topologically trivial open set; in other words, $H_{\bar{\partial}}^q(\mathbb{C}^d, V_X) = 0$ for $q > 0$. We need here the Q-Poincaré lemma, $H_Q^q(\mathbb{C}^d, V_X) = 0$ for $q > 0$. This follows immediately from the general principle that quantum corrections can only destroy, and never create, cohomology classes.

As for the assertion that $H^p(X, \mathcal{V}_X^q)$ vanish for all $p > 0$, this relies on the existence of a partition of unity. Suppose that we have a "classical" partition of unity $\{\rho_\alpha\}$ subordinate to an open cover $\{U_\alpha\}$, so that $\sum_\alpha \rho_\alpha(\phi, \bar{\phi}) = 1$. Quantum mechanically, we must renormalize the composite operators ρ_α, but if we do this consistently, then the resulting operators should add up to unity again. This gives a "quantum" partition of unity.

The perturbative Čech–Q isomorphism (3.2) can now be established by imitating the proof of the Čech–Dolbeault isomorphism. First, the Q-Poincaré lemma yields the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{D}_X^{Q,q-1} \longrightarrow \mathcal{V}_X^{q-1} \xrightarrow{\ Q\ } \mathcal{D}_X^{Q,q} \longrightarrow 0, \tag{3.3}$$

where $\mathcal{D}_X^{Q,q}$ are the sheaves of perturbatively Q-clsoed $(0, q)$-forms of V_X. This in turn

induces the long exact sequence of cohomology:

$$0 \longrightarrow H^0(X, \mathcal{D}_X^{Q,q-1}) \longrightarrow H^0(X, \mathcal{V}_X^{q-1}) \longrightarrow H^0(X, \mathcal{D}_X^{Q,q})$$

$$\longrightarrow H^1(X, \mathcal{D}_X^{Q,q-1}) \longrightarrow H^1(X, \mathcal{V}_X^{q-1}) \longrightarrow H^1(X, \mathcal{D}_X^{Q,q}) \longrightarrow \cdots$$

$$\vdots$$

$$\longrightarrow H^p(X, \mathcal{D}_X^{Q,q-1}) \longrightarrow H^p(X, \mathcal{V}_X^{q-1}) \longrightarrow H^p(X, \mathcal{D}_X^{Q,q}) \longrightarrow \cdots .$$

(3.4)

Now, since $H^p(X, \mathcal{V}_X^q) = 0$ for $p > 0$, the long exact sequence produces a chain of isomorphisms:

$$H^q(X, \mathcal{D}_X^Q) \cong H^{q-1}(X, \mathcal{D}_X^{Q,1})$$

$$\vdots$$

$$\cong H^1(X, \mathcal{D}_X^{Q,q-1})$$

$$\cong H^0(X, \mathcal{D}_X^{Q,q})/QH^0(X, \mathcal{V}_X^{q-1}).$$

(3.5)

Therefore, $H^q(X, \mathcal{D}_X^Q) \cong H_Q^q(X, V_X)$.

3.2 Chiral Algebras from Free Theories

Although we have established the Čech–Q isomorphism, this is not quite the end of the story. For the computation of the Čech cohomology groups $H^q(X, \mathcal{D}_X^Q)$, we need to know what the sections of $\mathcal{D}_X^Q$ look like, which are by definition Q-closed local operators of charge zero. Classically, they are simply holomorphic sections of the holomorphic vector bundles V_X. However, the perturbative corrections to the classical action of Q can be quite intricate.

To circumvent this problem, we adopt a different strategy: instead of defining the sheaves $\mathcal{D}_X^Q$ using the perturbative action of Q, we define them through isomorphic sheaves obtained by gluing *free* theories patch by patch over the target space. We now explain how one can reconstruct the Q-cohomology in this way, to all orders of perturbation theory.

Choose an open cover $\{U_\alpha\}$ of X such that $U_\alpha \cong \mathbb{C}^d$. The crucial observation is the following: over each topologically trivial open set U_α, one can deform the theory to make it locally free without affecting the Q-cohomology. Specifically, one flattens the target space metric and deforms the two-form gauge field to zero over U_α. Under such a deformation, the action of Q changes by conjugation, $Q \to e^{\Lambda_\alpha} Q e^{-\Lambda_\alpha}$. Let $\widetilde{Q}_\alpha$ be the restriction of this deformed supercharge to U_α.

With respect to some local trivializations, the holomorphic vector bundle V_X is described by holomorphic transition functions $f_{\alpha\beta}$. We consider new transition functions $\tilde{f}_{\alpha\beta} = e^{\Lambda_\alpha} f_{\alpha\beta} e^{-\Lambda_b}$. The bundle $\widetilde{V}_X$ constructed using $\tilde{f}_{\alpha\beta}$ is isomorphic to V_X. Furthermore, the operators $\widetilde{Q}_\alpha$ glue consistently to define a differential operator $\widetilde{Q}$ acting on the sections of $\widetilde{V}_h$. It is clear from the construction that the sheaf $\mathcal{D}_X^{ch}$ of $\widetilde{Q}$-closed sections of $\widetilde{V}_X$ is isomorphic to $\mathcal{D}_X^Q$ under the isomorphism $\widetilde{V}_X \cong V_X$. In particular, we have an isomorphism between their cohomology:

$$H^q(X, \mathcal{D}_X^Q) \cong H^q(X, \mathcal{D}_X^{ch}). \tag{3.6}$$

Thus, the Q-cohomology is isomorphic to the Čech cohomology of the sheaves $\mathcal{D}_X^{ch}$.

It may seem that all we have done is just relabeling various objects. The point is that the action of $\widetilde{Q}$ is determined locally by a free theory. We know how it acts on local operators exactly—it is just the $\bar{\partial}$-operator!

The sections of $\mathcal{D}_X^{ch}$ are easy to describe locally. They are holomorphic local sections of V_X, hence do not depend on $\bar{\phi}$. If we introduce the bosonic fields β of dimension one and γ of dimension zero by

$$\beta_i = \delta_{i\bar{j}} \partial_z \bar{\phi}^{\bar{j}}, \qquad \gamma^i = \phi^i, \tag{3.7}$$

then such local operators are functions of β, γ, and their derivatives. Their dynamics are governed by the free, chiral, conformal field theory with action

$$S = \frac{1}{2\pi} \int_\Sigma d^2z \, \beta_i \partial_{\bar{z}} \gamma^i, \tag{3.8}$$

called the *free $\beta\gamma$ system*. The action yields the OPE

$$\beta_i(z)\gamma^j(w) \sim -\frac{\delta_i^j}{z-w}, \tag{3.9}$$

with the $\beta\beta$ and $\gamma\gamma$ OPEs being regular. The sheaf $\mathcal{D}_X^{ch}$ of local operators of free $\beta\gamma$ systems is known as the sheaf of *chiral differential operators* [38, 13]. Therefore, we have found that the Q-cohomology of a $(0,2)$ sigma with target space X is given by the cohomology of a sheaf of chiral differential operators on X.

We emphasize that this result is exact in perturbation theory because free $\beta\gamma$ systems, being free, do not receive perturbative corrections. The perturbative corrections are now encoded in the transition functions $\tilde{f}_{\alpha\beta}$, which describe how these $\beta\gamma$ systems are to be glued together.

So the question is, what transition functions should we use? Of course, they are fixed up to isomorphisms once the theory is formulated globally, as it fixes the original sheaves $\mathcal{D}_X^Q$. But as we mentioned already, the problem is that it can be hard to determine $\mathcal{D}_X^Q$ in the first place, because doing so requires detailed knowledge of the perturbative corrections. Still, we can start with the local descriptions by $\beta\gamma$ systems and glue them with various choices of the transition functions. We are then effectively parametrizing the moduli of the chiral algebra by the way this gluing is done.

The guiding principle in finding a consistent set of transition functions is as follows. Perturbatively, the path integral measure is constructed by gluing local measures over the zero-instanton moduli space $\mathcal{M}_0 \cong X$, in such a way that the gluing preserves the OPE structure, R-charge, and scaling dimension across patches. The same must be true of the free theory description. In other words, the gluing must be done using the symmetries of the $\beta\gamma$ system that commute with charge and dimension, hence generated by currents of charge zero and dimension one.

For example, the current $J_V = -V^i \beta_i$ for a holomorphic vector field V has appropriate charge and dimension. The OPE between J_V and γ is

$$J_V(z)\gamma^i(w) \sim \frac{V^i(w)}{z - w}. \tag{3.10}$$

Thus J_V generates the infinitesimal diffeomorphism $\delta\gamma = V$. On β, it acts by

$$J_V(z)\beta_i(w) \sim -\frac{(\partial_i V^j \beta_j)(w)}{z - w}, \tag{3.11}$$

which means that classically β transforms as a $(1,0)$-form under this diffeomorphism. Quantum mechanically, the expression $\partial_i V^i \beta_j$ must be regularized and there is a correction to the transformation law.

If we have a holomorphic one-form B, then we can also make $J_B = B_i \partial_z \gamma^i$. This acts on β only, with the OPE

$$J_B(z)\beta_i(w) \sim -\frac{B_i(w)}{(z - w)^2} + \frac{(C_{ij}\partial_w \gamma^j)(w)}{z - w}, \tag{3.12}$$

where $C = \partial B$. The corresponding charge vanishes if B is ∂-exact. The target space of the free $\beta\gamma$ system being topologically trivial, B is ∂-exact if and only if $C = 0$, and for any closed holomorphic two-form C, there is a holomorphic one-form B such that $C = \partial B$. The symmetries of this type are thus in one-to-one correspondence with the closed holomorphic two-forms.

Now, the transition functions $\tilde{f}_{\alpha\beta}$ can be constructed by choosing a holomorphic vector field $V_{\alpha\beta}$ and a closed holomorphic two-form $C_{\alpha\beta}$ on each overlap $U_a \cap U_b$, finding holomorphic one-forms $B_{\alpha\beta}$ such that $C_{\alpha\beta} = \partial B_{\alpha\beta}$, and setting

$$\tilde{f}_{\alpha\beta} = \exp\left(-\oint dz \, V^i_{\alpha\beta} \beta_i + \oint dz \, B_{\alpha\beta,i} \partial_z \gamma^i\right). \tag{3.13}$$

For the gluing to be consistent, the transition functions must satisfy $\tilde{f}_{\alpha\beta}\tilde{f}_{bc}\tilde{f}_{ca} = 1$ on triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$. In terms of $V_{\alpha\beta}$ and $C_{\alpha\beta}$, this condition reads

$$V_{\alpha\beta} + V_{\beta\gamma} + V_{\gamma\alpha} = C_{\alpha\beta} + C_{\beta\gamma} + C_{\gamma\alpha} = 0; \tag{3.14}$$

that is to say, $V_{\alpha\beta}$ and $C_{\alpha\beta}$ must satisfy the cocycle condition of Čech cohomology.

The vector field parts of the transition functions determine how the U_αs are glued together into the target space X, whereas the one-form parts determine the other moduli of the chiral algebra. Since the latter leave γ invariant, the dimension zero subspace of the chiral algebra receive no perturbative corrections, except for possible deformations of the classical complex structure of the target space.

The choice of the transition functions is not unique. Suppose that we choose a different set of holomorphic vector fields, say $V'_{\alpha\beta}$. For this new choice to give a consistent gluing, the difference $V'_{\alpha\beta} - V_{\alpha\beta}$ must satisfy the cocycle condition, too. Moreover, if $V'_{\alpha\beta} - V_{\alpha\beta} = W_\alpha - W_\beta$ for some holomorphic vector fields W_α on U_α and W_β on U_β, then the change just amounts to acting $\mathcal{D}_X^{ch}(U_\alpha)$ with transformations generated by W_α. Thus, different ways of choosing $V_{\alpha\beta}$ that lead to distinguished sheaves are parametrized by elements of the first cohomology group $H^1(X, \mathcal{T}_X)$ of the sheaf $\mathcal{T}_X$ of holomorphic vector fields on X. This is the space of deformations of the complex structure on X.

Similarly, the different ways of choosing $C_{\alpha\beta}$ are parametrized by elements of the cohomology group $H^1(X, \Omega_X^{2;cl})$, where $\Omega_X^{2;cl}$ denotes the sheaf of closed holomorphic two-forms on X. To understand what this space represent, consider a locally defined $(2,0)$-form T. If we define $\mathcal{H} = dT$, then $\mathcal{H}$ is a locally defined closed form of type $(3,0) \oplus (2,1)$. Conversely, for any local closed form $\mathcal{H}$ of type $(3,0) \oplus (2,1)$, we can find a local $(2,0)$-form T such that $\mathcal{H} = dT$.[1] Thus we have the short exact sequence

$$0 \longrightarrow \Omega_X^{2;cl} \longrightarrow \mathcal{A}_X^{2,0} \overset{d}{\longrightarrow} \mathcal{Z}_X^{3,0} \oplus \mathcal{Z}_X^{2,1} \longrightarrow 0, \tag{3.15}$$

where $\mathcal{A}_X^{p,q}$ and $\mathcal{Z}_X^{p,q}$ are respectively the sheaves of (p,q)-forms and closed (p,q)-forms on X. Since $H^p(X, \mathcal{A}_X^{2,0}) = 0$ for $p > 0$, the long exact sequence of cohomology implies

$$H^1(X, \Omega_X^{2;cl}) \cong H^0(X, \mathcal{Z}_X^{3,0} \oplus \mathcal{Z}_X^{2,1})/dH^0(X, \mathcal{A}_X^{2,0}). \tag{3.16}$$

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[1] By the Poincaré lemma, locally $\mathcal{H} = d(U + V)$ for some $(2,0)$-form U and $\bar{\partial}$-closed $(1,1)$-form V. By the $\bar{\partial}$-Poincaré lemma, locally $V = \bar{\partial}W$ for some $(1,0)$-form W. Then $T = U + V - dW$.

This is the space of closed forms $\mathcal{H}$ of type $(3,0) \oplus (2,1)$ modulo those that can be written as dT with T a globally defined $(2,0)$-form.

This moduli arises because we could add to our action the classically conformal invariant Q-closed term

$$S_T = \int_\Sigma d^2z \{Q, T_{ij}\rho_{\bar{z}}^i \partial_z \phi^j\}. \tag{3.17}$$

This is not necessarily Q-exact because T may not be a globally defined $(2,0)$-form, so can affect the chiral algebra. However, we can subtract a globally defined $(2,0)$ form from T to make it locally zero. Under such a deformation, the chiral algebra is left invariant and the theory becomes locally free. Therefore, this term can be treated—and necessarily included—in the framework of sheaf of chiral differential operators.

Let us recapitulate what we have discussed so far in this chapter. Classically, the Q-cohomology is given by Dolbeault cohomology. One can exploit the Čech–Dolbeault isomorphism to recast it in the language of Čech cohomology. Quantum mechanically, the action of Q receives quantum corrections and in general, it is hard to capture the explicit dependence of the Q-cohomology on the moduli of the chiral algebra. Even though we have the Čech–Q isomorphism perturbatively, this itself does not help to make the problem tractable since the perturbative corrections enter the very definitions of the sheaves that are used in the Čech cohomology computations. To proceed, we use the fact that sheaf theory is local in nature and deform the theory to make it free locally on the target. The Q-cohomology can then be reconstructed by gluing free $\beta\gamma$ systems over the target space, while the moduli of the chiral algebra is now encoded in the way this gluing is done.

3.3 $\mathbb{CP}^1$ Model

To illustrate the use of the sheaf of chiral differential operators, and also to prepare for the discussion in Chapter 4, let us compute the perturbative chiral algebra of the

twisted model with target space $X = \mathbb{CP}^1$ for the first few dimensions.

Recall that quantum corrections can only destroy cohomology classes. Hence, if we wish to understand the perturbative chiral algebra, we should first know the classical chiral algebra. The classical chiral algebra of the $\mathbb{CP}^1$ model can be summarized concisely by the following formula for the Hodge numbers:

$$h^0(\mathcal{O}(n)) = \begin{cases} n+1 & (n \geq 0); \\ 0 & (n \leq -1), \end{cases} \qquad h^1(\mathcal{O}(n)) = \begin{cases} 0 & (n \geq -1); \\ -n-1 & (n \leq -2). \end{cases} \tag{3.18}$$

Here $\mathcal{O}(k)$ is the line bundle over $\mathbb{CP}^1$ whose first Chern class is equal to k; thus $T_X = \mathcal{O}(2)$ and $T_X^\vee = \mathcal{O}(-2)$. Let us see how perturbative corrections modify these classical conditions, using the tool developed in the previous section.

Let N and S be the "north" and "south" poles of X, respectively. Then X is covered by $\{U_N, U_S\}$, where $U_N = X \setminus \{S\}$ and $U_S = X \setminus \{N\}$. We put the $\beta\gamma$ system on U_S and the $\beta'\gamma'$ system on U_N, related to each other by

$$\gamma' = \frac{1}{\gamma}. \tag{3.19}$$

Since $H_{\bar{\partial}}^1(X, T_X) = H^1(X, \Omega_X^{2,cl}) = 0$, this is essentially the only way to glue the two systems.

Classically, β transforms as $\beta' = -\gamma^2\beta$, but we need to regularize this expression. A nice thing about dealing with a free theory is that the regularization of composite operators is readily performed. We will use conformal normal ordering:

$$\gamma^2\beta(z) = \lim_{w \to z} \left(\gamma^2(w)\beta(z) - \frac{2\gamma(w)}{w-z} \right). \tag{3.20}$$

The quantum transformation law for β that is compatible with this regularization is

$$\beta' = -\gamma^2\beta + 2\partial_z\gamma. \tag{3.21}$$

The anomalous term $2\partial_z\gamma$ ensures the transformations (3.19) and (3.21) to preserve the OPEs; without it, the $\beta'\beta'$ OPE is not zero.

We first look at the zeroth cohomology $\mathcal{A}^0$. Since these are given by zeroth Čech cohomology groups, cohomology classes are represented by global sections the sheaf of chiral differential operators on X.

In dimension zero, the relevant local operators are holomorphic functions. Since a holomorphic function on a compact manifold must be constant, the dimension zero subspace of $\mathcal{A}^0$ is one-dimensional and generated by the constant operator 1.

In dimension one, the possible local operators are those of the form $B(\gamma)\partial_z\gamma$ and $V(\gamma)\beta$, where B is a holomorphic one-form and V is a holomorphic vector field. Since global holomorphic one-forms do not exist on $\mathbb{CP}^1$, there are no cohomology classes of the former type. On the other hand, we have three independent holomorphic vector fields, ∂, $\gamma\partial$, and $\gamma^2\partial$. Corresponding to these, classically there are three cohomology classes represented by the local operators. These survive to the perturbative chiral algebra, with the quantum counterparts being

$$J_- = \beta = -\gamma'^2\beta' + 2\partial_z\gamma',$$
$$J_3 = -\gamma\beta = -\gamma'\beta', \qquad (3.22)$$
$$J_+ = -\gamma^2\beta + 2\partial_z\gamma = \beta',$$

generating the current algebra of $SL(2)$ at level -2:

$$J_3(z)J_3(w) \sim -\frac{1}{(z-w)^2},$$
$$J_3(z)J_\pm(w) \sim \pm\frac{J_\pm(w)}{(z-w)^2}, \qquad (3.23)$$
$$J_+(z)J_-(w) \sim -\frac{2}{(z-w)^2} + \frac{2J_3}{z-w}.$$

That the perturbative chiral algebra has these currents is a reflection of the fact that the target space admits an $SL(2)$-action; in fact, $\mathbb{CP}^1$ is the flag manifold of $SL(2)$.

Next, we consider the first cohomology $\mathcal{A}^1$. The first Čech cohomology groups are generated by the sections of $\mathcal{D}_X^{ch}(U_S \cap U_N)$ that cannot be written as a difference of sections of $\mathcal{D}_X^{ch}(U_S)$ and $\mathcal{D}_X^{ch}(U_N)$. Such sections can be regarded as global sections of $\mathcal{D}_X^{ch}$ with poles at S and N. We need poles at both S and N; otherwise the sections

can be considered as well-defined sections of $\mathcal{D}_X^{ch}(U_S)$ or $\mathcal{D}_X^{ch}(U_N)$ and hence vanishes in the cohomology.

In dimension zero, we already have $h^1(\mathcal{O}(0)) = 0$ classically, thus the dimension zero subspace of $\mathcal{A}^1$ is zero. In the Čech language, this means that we can always split a holomorphic function f with poles at S and N into holomorphic functions f_S with a pole at N and f_N with a pole at S.

In dimension one, we can try operators of the form β/γ^n having poles at S. However, they are all regular at N. The other possibilities are operators of the form $\partial_z\gamma/\gamma^n$. Requiring that they have poles at N, we find that only $\partial_z\gamma/\gamma$ can represent a nontrivial cohomology class. Indeed, it does. Dimension one sections of $\mathcal{D}_X^{ch}(U_S \cap U_N)$ are currents used to glue the $\beta\gamma$ and $\beta'\gamma'$ systems, so the sheaf of chiral differential operators would be trivial if none of them represents a cohomology class. Thus, the dimension one subspace of $\mathcal{A}^1$ is one-dimensional and generated by $[\{\partial_z\gamma/\gamma\}]$.

In dimension two, the sections with poles at both N and S are $\partial_z^2\gamma/\gamma$, $\partial_z^2\gamma/\gamma^2$, $(\partial_z\gamma)^2/\gamma$, $(\partial_z\gamma)^2/\gamma^2$, and $(\partial_z\gamma)^2/\gamma^3$. The last one does not lead to an independent cohomology class since we have the relation

$$\partial_z^2\gamma' = -\frac{\partial_z^2\gamma}{\gamma^2} + \frac{2(\partial_z\gamma)^2}{\gamma^3}. \tag{3.24}$$

This is consistent with the Hodge numbers $h^1(\mathcal{O}(-2)) = 1$ and $h^1(\mathcal{O}(-4)) = 3$.

The perturbative corrections induce another relation. Consider the energy-momentum tensor of the $\beta\gamma$ system

$$T_S(z) = (\beta\partial_z\gamma)(z) = \lim_{w \to z}\left(\beta(w)\partial_z\gamma(z) - \frac{1}{(w-z)^2}\right) \tag{3.25}$$

and that of the $\beta'\gamma'$ system $T_N = \beta'\partial_z\gamma'$. A straightforward computation shows

$$T_N - T_S = -2\partial_z\left(\frac{\partial_z\gamma}{\gamma}\right) \tag{3.26}$$

on $U_N \cap U_S$. Hence, the cocycle $-2\partial_z(\partial_z\gamma/\gamma)$ vanishes in the cohomology. This is the Čech cohomology counterpart of the relation $[Q, T_{zz}] = \partial_z\theta$, the existence of which was argued in Section 2.2.

There are no other perturbative relations that lift classical cohomology classes, so the dimension two subspace of $\mathcal{A}^1$ is three-dimensional. Using the cohomology classes obtained already, we can construct three classes: $[J_-\theta]$, $[J_3\theta]$, and $[J_+\theta]$.

We have shown that the dimension zero subspace of $\mathcal{A}^0$ is generated by $[1]$ and the dimension one subspace of $\mathcal{A}^1$ is generated by $[\theta]$, while the dimension one subspace of $\mathcal{A}^0$ is generated by $[J_-]$, $[J_3]$, $[J_+]$ and the dimension two subspace of $\mathcal{A}^1$ is generated by $[J_-\theta]$, $[J_3\theta]$, $[J_+\theta]$. Therefore, up to the first two nontrivial dimensions, we have found the isomorphism $\mathcal{A}^0 \cong \mathcal{A}^1$ induced by the map $1 \mapsto \theta$. It has been shown [38] that this isomorphism persists in higher dimensions as well.

3.4 Target Spaces with Nonzero First Chern Class

Next, let us consider the target space X with nonzero first Chern class, $c_1(X) \neq 0$. We would like to see if there is anything in the perturbative chiral algebra that is associated with $c_1(X)$. This example will also be relevant for the discussion in Chapter 4,

Let $\{U_\alpha\}$ be a good cover of X; thus, all nonempty finite intersections of the open sets U_α are diffeomorphic to $\mathbb{C}^d$. Choose holomorphic transition functions $f_{\alpha\beta}$ of the canonical bundle K_X on the overlaps $U_\alpha \cap U_\beta$. On triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$, the transition functions satisfy

$$\delta f_{\alpha\beta\gamma} = f_{\beta\gamma} f_{\gamma\alpha} f_{\alpha\beta} = 1. \tag{3.27}$$

Thus $\{f_{\alpha\beta}\}$ defines a cohomology class in $H^1(X, \mathcal{O}_X^\times)$, where $\mathcal{O}_X^\times$ is the sheaf of nowhere vanishing holomorphic functions on X. This is mapped to the first Chern class $c_1(X)$ under the homomorphism $H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z})$ induced by the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^\times \longrightarrow 1. \tag{3.28}$$

The map $\mathbb{Z} \to \mathcal{O}_X$ is the inclusion and $\mathcal{O}_X \to \mathcal{O}_X^\times$ is given by $f \mapsto \exp(2\pi i f)$.

To obtain a cohomology class in the chiral algebra, we map $f_{\alpha\beta} \mapsto \log f_{\alpha\beta}$. Naively, it seems that the right-hand side of the cocycle condition (3.27) is mapped to 0 and

$\{\log f_{\alpha\beta}\}$ defines a cocycle for the chiral algebra. However, due to the nontrivial topology of the manifold we only have

$$\delta \log f_{\alpha\beta\gamma} = \log f_{bc} + \log f_{ca} + \log f_{\alpha\beta} \in 2\pi i \mathbb{Z}. \tag{3.29}$$

In fact, $\{\delta \log f_{\alpha\beta\gamma}\}$ is a cocycle representing $c_1(X)$. So we make use of an additional degree of freedom: the worldsheet. We differentiate this equation by z and set

$$\theta_{\alpha\beta} = \frac{1}{2}\partial_z \log f_{\alpha\beta} = \frac{1}{2}f_{\alpha\beta}^{-1}\partial_z f_{\alpha\beta}. \tag{3.30}$$

Now $\{\theta_{\alpha\beta}\}$ represents a cohomology class of charge one and dimension one.

Let us find a representative of $[\{\theta_{\alpha\beta}\}]$ in the sigma model. First, we note that $\theta_{\alpha\beta}$ can be written as $\theta_{\alpha\beta} = W_\beta - W_\alpha$, where W_α is the quantity

$$W = \frac{1}{2}\partial_z \phi^i \partial_i \log \det |g_{i\bar{j}}| \tag{3.31}$$

computed in U_α. This does not mean that $\{\theta_{\alpha\beta}\}$ is trivial in Čech cohomology since W_α are not holomorphic. Rather, W_α transform by holomorphic transition functions. Thus, $\bar{\partial}W$ is globally well defined and represents an element of $H_{\bar{\partial}}^1(X, F_1)$, and this is the desired cohomology class. Using the formula $R_{i\bar{j}} = -\partial_i \bar{\partial}_{\bar{j}} \log \det |g_{i\bar{j}}|$, we find

$$\bar{\partial}W = \frac{1}{2}R_{i\bar{j}}\partial_z \phi^i \alpha^{\bar{j}}. \tag{3.32}$$

It is indeed a local operator of charge one and dimension one.

Classically, we obtain cohomology classes of higher dimensions by applying ∂_z repeatedly on this operator. Quantum mechanically, these classes are actually all lifted because, as we argued in Section 2.3, perturbatively corrections induce the relation $[Q, T_{zz}] = \partial_z \theta$ for some $\theta \propto R_{i\bar{j}}\partial_z \phi^i \alpha^{\bar{j}}$. Let us look at this phenomenon from the Čech cohomology perspective.

On each U_α, we put the $\beta_\alpha \gamma_\alpha$ system with energy-momentum tensor $T_\alpha = \beta_{a,i}\partial_z \gamma_\alpha^i$. The OPE between J_V and T_α gives

$$J_V(z)T_\alpha(w) \sim \frac{\partial_i V^i(w)}{(z-w)^3} + \frac{(\partial_w \partial_i V^i + V^i \beta_i)(w)}{(z-w)^2} + \frac{1}{2}\frac{\partial_w^2(\partial_i V^i)(w)}{z-w}. \tag{3.33}$$

We see that T_α is not invariant under the infinitesimal diffeomorphism $\delta\gamma = \epsilon V$:

$$\delta T_\alpha = \frac{\epsilon}{2}\partial_z^2(\partial_i V^i). \tag{3.34}$$

The finite form of this transformation is [13]

$$T_\beta - T_\alpha = \frac{1}{2}\partial_z^2 \log \det\left|\frac{\partial\gamma_\beta^i}{\partial\gamma_\alpha^j}\right| = \partial_z\theta_{\alpha\beta}. \tag{3.35}$$

Therefore, we have $[Q, T_{zz}] = \partial_z\theta$ with $\theta = \bar{\partial}W$.

Chapter 4

Nonperturbative Vanishing of Chiral Algebras

We have seen in the previous chapter that, although the perturbative corrections deforms the classical chiral algebra, the consequences are relatively minor. In particular, the gradings by R-charge and dimension are still preserved in perturbation theory, and the perturbative chiral algebra of a twisted model can be completely reconstructed from a sheaf of chiral differential operators.

Nonperturbatively, however, instantons violate R-charge if the target space X has $c_1(X) \neq 0$, and also the grading by dimension. These properties of instantons make it possible for them to induce more radical deformations. In this chapter, we will see a particularly striking example of such a nonperturbative deformation: instantons lift all of the cohomology classes and, therefore, the chiral algebra vanishes nonperturbatively.

4.1 Vanishing Chiral Algebras

To begin, let us discuss the general mechanism that renders a chiral algebra trivial. The starting point is the following observation: the chiral algebra is trivial if and only if there exists a local operator Θ such that

$$\{Q, \Theta\} = 1, \tag{4.1}$$

in other words, if and only if $[1] = 0$. For if $[1] = 0$, then $[\mathcal{O}] = [1] \cdot [\mathcal{O}] = 0 \cdot [\mathcal{O}] = 0$. Conversely, if $[\mathcal{O}] = 0$ for all Q-closed local operator $\mathcal{O}$, then $[1] = 0$ in particular.

The relation (4.1) cannot be induced by perturbative corrections. To see this, note that for this relation to hold Θ must have perturbatively charge -1 because R-symmetry

is not broken perturbatively. Then, Θ must contain at least one ρ and hence have positive antiholomorphic dimension, but since scaling dimension is perturbatively not violated either, $\{Q, \Theta\}$ cannot be equal to 1. Therefore, the vanishing of the chiral algebra, if occurs, is a purely nonperturbative phenomenon induced by worldsheet instantons.

Even nonperturbatively, the relation (4.1) is impossible if $c_1(X) = 0$. In this case, R-symmetry is conserved and thus Θ has charge -1. Furthermore, Θ must be perturbatively Q-closed since the vanishing can occur only nonperturbatively. Then, Θ must be perturbatively Q-exact because it has positive antiholomorphic dimension. However, we know that quantum corrections lift cohomology class in pairs, so this is a contradiction: [1] cannot be paired with $[\Theta]$, which is already zero.

The same line of reasoning leads to the conclusion that there must be an isomorphism between the bosonic and fermionic subspaces of the perturbative chiral algebras. For each perturbative cohomology classes must be paired with some other class with different statistics.

4.2 $\mathbb{CP}^1$ Model

The simplest example of a vanishing chiral algebra is provided by the twisted model with target space $X = \mathbb{CP}^1$. Since $c_1(\mathbb{CP}^1) = 2$, R-symmetry is anomalously broken to $\mathbb{Z}_2$ nonperturbatively in this model, with the charge violation given by $-2k$ at the k-instanton level. The twisted model exists when the worldsheet Σ has $c_1(\Sigma) = 0$, or equivalently, the canonical bundle K_Σ is trivial. We choose $\overline{K}_\Sigma^{1/2}$ to be trivial so that the twisted and untwsited models are isomorphic, with isomorphism given by tensoring the fermionic fields by a nowhere vanishing section of $\overline{K}_\Sigma^{1/2}$.

The perturbative chiral algebra of the $\mathbb{CP}^1$ model was studied in Section 3.3. As we saw there, its bosonic and fermionic subspaces are isomorphic in an interesting way: the cohomology classes [1] and $[\theta]$ play the role of "ground states" of the chiral algebra, on

which the other cohomology classes are constructed by acting with the "creation opera-tors," which are the bosonic cohomology classes. In view of this suggestive isomorphism and the R-charge violation, one may expect that degree one instantons "tunnel" from $[\theta]$ to $[1]$, thereby lifting these perturbative cohomology classes out of the chiral algebra. This is indeed the case.

We now show, to leading order in perturbation theory around degree one instantons, that the action of Q on θ induces the nonperturbative relation

$$\{Q, \theta\} = e^{-t}(s \cdot 1 + Q\text{-exact local operator}). \tag{4.2}$$

Here s is a nowhere vanishing section of K_Σ. The operators on the two sides have dif-ferent dimensions; such a relation is possible nonperturbatively since instantons violate scaling dimension. It follows that $\{Q, \Theta\} = 1$ with $\Theta = e^t s^{-1}\theta + \cdots$. The chiral algebra of the $\mathbb{CP}^1$ model therefore vanishes nonperturbatively.

If the relation (4.2) does exist, then the same relation should hold with θ replaced by any local operator representing the perturbative cohomology class $[\theta]$. This is simply because, in doing so, the right-hand side will differ by a Q-exact operator that vanishes perturbatively (thus at least of order e^{-t}). By the same token, if this relation holds for some representative, then it should also hold for θ. Our strategy is then to pick a representative on which the action of Q takes a particularly simple form. We will call this representative θ^∞.

For definiteness, we consider the case where Σ is the cylinder $S^1 \times \mathbb{R}$ described by a holomorphic coordinate w. The action of Q on θ^∞ can be read off from the matrix elements of $\{Q, \theta^\infty\}$ inserted at $w = 0$:

$$\langle \Psi_j | \{Q, \theta^\infty(0)\} | \Psi_i \rangle. \tag{4.3}$$

To compute such matrix elements, we propagate the states to the far future and the far past, at the same time rescaling them to cancel the exponential factors from the evolution operator. Then the matrix elements are represented by path integrals with

an insertion of the contour integral

$$\oint d\bar{w}\, G(\bar{w})\theta^\infty(0) \tag{4.4}$$

of the conserved current G for Q and asymptotic boundary conditions specified by the initial and final states corresponding respectively to $|\Psi_i\rangle$ and $|\Psi_j\rangle$.

For the purpose of computing the action of Q, it is actually more convenient to compactify the cylinder to the Riemann sphere $\widehat{\Sigma} = \mathbb{CP}^1$ by adding points at infinity and bring them to a finite distance by the conformal map $w \mapsto z = e^{-iw}$. Before we do this, however, we must *untwist* the theory because the twisted model with $X = \mathbb{CP}^1$ is anomalous on $\widehat{\Sigma}$. The initial and final states are then mapped to local operators inserted at $z = 0$ and ∞:

$$\left\langle \mathcal{O}_j(\infty) \oint d\bar{z}\, G(\bar{z})\theta^\infty(1)\mathcal{O}_i(0) \right\rangle. \tag{4.5}$$

Our theory is not really conformally invariant due to quantum corrections, so there are actually nonlocal contributions coming from the transformation of the renormalized action. These can be discarded as far as the leading term of $\{Q, \theta^\infty\}$ is concerned.

Now that the matrix elements are expressed as correlation functions on the sphere, we can exploit the approximate conformal symmetry to simplify the problem. Let us bring the vertex operators close to each other by a conformal transformation, combine them by OPE, and then go back to the original frame. This replaces the vertex operators by a sum of local operators of various dimensions located at $z = 0$. Therefore, the action of Q on θ^∞ is characterized to leading order by the correlation function

$$\left\langle \oint d\bar{z}\, G(\bar{z})\theta^\infty(1)\mathcal{O}(0) \right\rangle \tag{4.6}$$

evaluated for an arbitrary local operator $\mathcal{O}$ in the presence of degree one instantons.

In the present case, the one-instanton moduli space $\mathcal{M}_1$ is the space of biholomorphic maps from $\widehat{\Sigma} = \mathbb{CP}^1$ to $X = \mathbb{CP}^1$, which is isomorphic to the space of Möbius

transformations. At $\phi_0 \in \mathcal{M}_1$, the complex conjugates of $\bar{\psi}_+$ zero modes are holomorphic sections of $K_{\widehat{\Sigma}}^{1/2} \otimes \phi_0^* T_X \simeq \mathcal{O}(1)$, while $\bar{\psi}_+$ zero modes can be regarded as holomorphic $(0,1)$-forms of the same bundle. Since $h^0(\mathcal{O}(1)) = 2$ and $h^1(\mathcal{O}(1)) = 0$, there are two $\bar{\psi}_+$ zero modes and no ψ_+ zero mode in the one-instanton sector. We see that G and θ^∞ contain just the right number of $\bar{\psi}_+$s to soak up the $\bar{\psi}_+$ zero modes. The correlation function (4.6) can be nonvanishing when $\mathcal{O}$ is purely bosonic, as it should be if the constant part of $\{Q, \theta\}$ is nonzero. It is important here that we have no excess fermion zero modes; otherwise, we would need to bring down the interaction terms and the conformal invariance present at leading order would be broken.

Instantons are holomorphic maps, so typical one-instanton contributions are captured by taking $\mathcal{O}$ to be a local operator of the form

$$\mathcal{O}_{\phi\cdots\phi\bar{\phi}\cdots\bar{\phi}}(\phi, \bar{\phi}) \frac{1}{k_1!} \partial_z^{k_1} \phi \cdots \frac{1}{k_N!} \partial_z^{k_N} \phi \frac{1}{l_1!} \partial_z^{l_1} \bar{\phi} \cdots \frac{1}{l_{\bar{N}}!} \partial_z^{l_{\bar{N}}} \bar{\phi}. \tag{4.7}$$

In this case, only the "classical" part of $\mathcal{O}$ contributes; the correlation function (4.6) is given, to leading order, by setting $\phi = \phi_0$ and integrating over the fermion zero modes and the instanton moduli space:

$$e^{-t} \int d\mathcal{M}_1 \, d\eta_1 \, d\eta_2 \oint d\bar{z} \, G(\bar{z}) \theta^\infty(1) \mathcal{O}(0) \Big|_{\phi=\phi_0}. \tag{4.8}$$

Here $d\mathcal{M}_1$ is the measure on $\mathcal{M}_1$, and $d\eta_1 \, d\eta_2$ is the measure for the $\bar{\psi}_+$ zero modes. (Of course, the contour integral must be evaluated before dropping quantum fluctuations. The appearance of short distance singularities is essential for that!)

Suppose that the integrations over the nonzero modes and fermion zero modes are done, and we would now like to integrate over the one-instanton moduli space. A Möbius transformation $\phi_0 \in \mathcal{M}_1$ is commonly written as

$$\phi_0(z) = \frac{az + b}{cz + d}, \tag{4.9}$$

using complex parameters a, b, c, d such that $ad - bc \neq 0$. This parametrization makes $\mathcal{M}_1$ into the noncompact group $PGL(2, \mathbb{C})$.

A Möbius transformation can also be specified by the points X_0, X_1, $X_\infty \in X$ to which it maps 0, 1, $\infty \in \widehat{\Sigma}$. This latter parametrization provides a natural compactification of $\mathcal{M}_1$ to $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ and is more convenient. Using the formula

$$\frac{1}{k!}\partial_z^k \phi_0(0) = (X_0 - X_\infty)\Big(\frac{X_1 - X_0}{X_1 - X_\infty}\Big)^k, \tag{4.10}$$

we find that the classical part of $\mathcal{O}$ is given by

$$\mathcal{O}_{\phi\cdots\phi\bar\phi\cdots\bar\phi}(X_0, \overline{X}_0)(X_0 - X_\infty)^N\Big(\frac{X_1 - X_0}{X_1 - X_\infty}\Big)^\Delta(\overline{X}_0 - \overline{X}_\infty)^{\overline{N}}\Big(\frac{\overline{X}_1 - \overline{X}_0}{\overline{X}_1 - \overline{X}_\infty}\Big)^{\overline{\Delta}}, \tag{4.11}$$

where $(\Delta, \overline{\Delta})$ is the dimension of $\mathcal{O}$. Apart from the coefficient functions, the correlation function (4.6) thus depends on the integers k_i, l_i through $(N, \overline{N})$ and $(\Delta, \overline{\Delta})$ only.

We can simplify the correlation function by choosing θ^∞ neatly. Let us make the target space metric flat except in the neighborhood of ∞. With this deformation of the metric, the Ricci curvature becomes zero almost everywhere, but develops a sharp peak at ∞ with total area 4π. (Recall that the Ricci form represents $2\pi c_1$.) We denote this Ricci curvature by $R^\infty_{\phi\bar\phi}$ and set $\theta^\infty = R^\infty_{\phi\bar\phi}\partial_z\phi\alpha$. Then θ and θ^∞ represent the same cohomology class.

The correlation function now contains a delta function at $X_1 = \infty$, so the dependence of the classical part (4.11) on $(\Delta, \overline{\Delta})$ disappears after integrating over X_1. Integrating over X_∞ as well, we obtain

$$e^{-t}\int_{\mathbb{CP}^1} g_{\phi\bar\phi}d^2X_0\, A_{N,\overline{N}}^{\phi\cdots\phi\bar\phi\cdots\bar\phi}\mathcal{O}_{\phi\cdots\phi\bar\phi\cdots\bar\phi} \tag{4.12}$$

for some section $A_{N,\overline{N}}$ of $(T_X)^{\otimes N} \otimes (\overline{T}_X)^{\otimes \overline{N}}$. What is this section? It must be constructed from $g_{\phi\bar\phi}$ and $R^\infty_{\phi\bar\phi}$ since these are the only inputs from the target space geometry. Moreover, to lowest order in the large volume limit, their derivatives are small and should not enter. It follows that $A_{N,\overline{N}} = 0$ for $N \neq \overline{N}$.

Now, the crucial observation is the following: the zero-instanton moduli space being the target space itself, the integral (4.12) is nothing but the two-point function

$$\Big\langle e^{-t}\sum_{N\geq 0}\mathcal{A}_N(1)\mathcal{O}(0)\Big\rangle \tag{4.13}$$

with $\mathcal{A}_N \propto A_{N,N}^{\phi\phi\cdots\bar{\phi}\bar{\phi}\cdots}(g_{\phi\bar{\phi}}\partial_z\bar{\phi})^N(g_{\bar{\phi}\phi}\partial_{\bar{z}}\phi)^N$, computed to the leading order of perturbation theory! Assuming that there are no other one-instanton contributions, this shows that the action of Q on θ^∞ is given by

$$\{Q, \theta^\infty(z, \bar{z})\} = e^{-t}\sum_{N \geq 0}|z|^{2N-2}\mathcal{A}_N(z, \bar{z}). \tag{4.14}$$

The dependence on the insertion point has been recovered by a scale transformation.

Since $\mathcal{A}_N$ are perturbatively Q-closed local operators of (twisted) dimension (N, N), $\mathcal{A}_0$ is a holomorphic function on X, hence constant, while $\mathcal{A}_N$ for $N > 0$ are perturbatively Q-exact. Going back to the cylinder, we have thus established the relation (4.2), with the section s given by $e^{-2\tau}\mathcal{A}_0$ (times the inverse square of the nowhere vanishing section of $\bar{K}_\Sigma^{1/2}$ used in the twisting).

For other choices of the worldsheet, we compute the matrix elements of $\{Q, \theta^\infty\}$ between states living on the ends of a very short cylinder lying in Σ. By taking the cylinder out of the worldsheet and extending it along the time direction, such matrix elements can be expressed as matrix elements on an infinitely long cylinder. The problem thus reduces to the case of $\Sigma = S^1 \times \mathbb{R}$, and the conclusion is unchanged. However, the resulting section s depends on the details of how this mapping is done.

Having understood roughly how the relation (4.2) should arise in the presence of instantons, let us now make our argument more precise by carrying out the computation sketched in the above argument.

The first step is to evaluate the contour integral. This seemingly straightforward task is actually very tricky. We are looking for an antiholomorphic single pole $1/(\bar{z}-1)$ in the OPE

$$G(\bar{z})\theta^\infty(1) \propto (g_{\phi\bar{\phi}}\partial_{\bar{z}}\phi\bar{\psi}_+)(\bar{z})(R_{\phi\bar{\phi}}^\infty\partial_z\phi\bar{\psi}_+)(1). \tag{4.15}$$

It appears that the only way to obtain such a pole is to contract $\partial_{\bar{z}}\phi$ with $R_{\phi\bar{\phi}}$. This contraction, however, just leads to the classical action of Q and annihilates θ. We must find "hidden" quantum fluctuations producing additional antiholomorphic poles.

At this point, we recall that the fermionic fields couple to the pullback of the tangent bundle of X. Thus, the eigenmodes with which the fermionic fields are expanded carry within themselves the bosonic field. But away from the instanton moduli space, the bosonic field is itself subject to quantum fluctuations. As a result, the fermionic modes—even the zero modes—can produce short distance singularities against the bosonic nonzero modes!

To extract this bosonic dependence of the fermionic fields, consider a normal neighborhood $\mathcal{N}_1$ of $\mathcal{M}_1$ which is diffeomorphic to the normal bundle of $\mathcal{M}_1$. Let $\{x^\alpha\}$ be local coordinates on $\mathcal{M}_1$ (e.g., ones given by the parametrization (4.9)) and parametrize the normal directions by $\{y^\alpha\}$ in such a way that $y^\alpha = 0$ on $\mathcal{M}_1$. Given $\phi(z, \bar{z}; x, y) \in \mathcal{N}_1$, we denote its projection to $\mathcal{M}_1$ by $\phi_0(z; x)$. An instanton of degree one maps the points of $\widehat{\Sigma} = \mathbb{CP}^1$ to the points of $X = \mathbb{CP}^1$ in a one-to-one manner, so we can invert the function $\phi_0(z; x)$ to obtain $z(\phi_0; x)$. Using this, we express the bosonic field as

$$\phi(z, \bar{z}; x, y) = \varphi\big(\phi_0(z; x), \bar{\phi}_0(\bar{z}; x); x, y\big), \tag{4.16}$$

where $\varphi(\phi_0, \bar{\phi}_0; x, y) = \phi(z(\phi_0; x), \bar{z}(\bar{\phi}_0; x); x, y)$. The supercharge Q is a differential operator on the field space. Computing $[Q, \bar{\phi}]$ using the expression (4.16), we find

$$-i\bar{\psi}_+ = \frac{\partial\bar{\varphi}}{\partial\bar{\phi}_0}[Q, \bar{\phi}_0] + \frac{\partial\bar{\varphi}}{\partial x^r}[Q, x^r] + \frac{\partial\bar{\varphi}}{\partial y^\alpha}[Q, y^\alpha]. \tag{4.17}$$

The antiholomorphic section $[Q, \bar{\phi}_0]$ of $\phi_0^*\overline{T}_X$ is $-i$ times $\bar{\psi}_{+0}(\phi_0)$, the zero mode part of $\bar{\psi}_+$ at ϕ_0. Since $\partial_{\bar{z}}\bar{\phi} = (\partial\bar{\varphi}/\partial\bar{\phi}_0)\partial_{\bar{z}}\bar{\phi}_0$ and $\partial_{\bar{z}}\bar{\phi}_0 \neq 0$, we can write

$$\bar{\psi}_+ = \frac{\partial_{\bar{z}}\bar{\phi}}{\partial_{\bar{z}}\bar{\phi}_0}\bar{\psi}_{+0}(\phi_0) + \cdots. \tag{4.18}$$

So we have extracted partially the dependence of $\bar{\psi}_+$ on the bosonic fluctuations.

As desired, $\partial_{\bar{z}}\phi$ from G can now be contracted with $\bar{\psi}_+$ from θ^∞ to produce an antiholomorphic double pole. This gives

$$\oint d\bar{z}\, G(\bar{z})\theta^\infty(1) \propto \Big(R^\infty_{\phi\bar{\phi}}\frac{\partial_z\phi}{\partial_{\bar{z}}\bar{\phi}_0}\partial_{\bar{z}}\bar{\psi}_+\bar{\psi}_{+0}(\phi_0)\Big)(1) + \cdots. \tag{4.19}$$

The leading term is all we need. This is because the fermionic nonzero modes can be ignored to the lowest order computation, and modulo the fermionic nonzero modes and the equation of motion for the bosonic field, the leading term of (4.18) represents the zero mode part of $\bar{\psi}_+$. This term correctly reduces to $\bar{\psi}_{+0}(\phi_0)$ on $\mathcal{M}$ and, with the equation of motion, satisfies

$$D_z\left(\frac{\partial_{\bar{z}}\bar{\phi}}{\partial_{\bar{z}}\bar{\phi}_0}\bar{\psi}_+(\phi_0)\right) = R^{\bar{\phi}}{}_{\bar{\phi}\phi\bar{\phi}}\frac{\partial_z\bar{\phi}}{\partial_{\bar{z}}\bar{\phi}_0}\psi_+\bar{\psi}_+\bar{\psi}_{+0}(\phi_0), \tag{4.20}$$

which vanishes if the fermionic nonzero modes are dropped.

To complete the computation, we need to specify the path integral measure. For this, we can choose any Q-invariant measure. The derivation of the relation (4.6) relies only on the fact that $\mathcal{A}_0$ is nonzero. Since different measures differ by multiplication of invertible Q-closed operators, $\mathcal{A}_0$ is nonzero for any measures if it is so for some measure. Moreover, apart from the subtlety discussed above, the role of the nonzero mode integration is simply to give the bosonic and fermionic determinants, which are Q-invariant by themselves (though do not cancel since the left-moving supersymmetry is lacking). So we just need to consider the part $d\mathcal{M}_1\,d\eta_1\,d\eta_2$ for the instantons and the fermion zero modes.

Instantons are parametrized by the Möbius coordinates (4.9). Let us focus on the region of $\mathcal{M}_1$ where $d \neq 0$ and set $d = 1$ by an overall rescaling. We choose $d\mathcal{M}_1$ to be conformally invariant. A conformal transformation $z \mapsto z'$ acts on $\mathcal{M}_1$ by $\phi_0'(z') = \phi_0(z)$. The invariance under $\phi_0 \mapsto \phi_0'$ determines $d\mathcal{M}_1$ up to a factor:

$$d\mathcal{M}_1 = |a - bc|^{-4}d^2a\,d^2b\,d^2c. \tag{4.21}$$

The two $\bar{\psi}_+$ zero modes are generated from instantons by global superconformal transformations, so the zero mode part of $\bar{\psi}_+$ can be expanded as

$$\bar{\psi}_{+0}(\phi_0) = \eta_1\partial_{\bar{z}}\bar{\phi}_0 + \eta_2\bar{z}\partial_{\bar{z}}\bar{\phi}_0. \tag{4.22}$$

Then $d\mathcal{M}_1\,d\eta_1\,d\eta_2$ is Q-invariant. This follows from the fact that $d\mathcal{M}_1$ is invariant

under superconformal transformations, which is in turn a consequence of the conformal invariance.

In terms of the variables X_0, X_1, X_∞ describing the compactified one-instanton moduli space, $d\mathcal{M}_1$ is written as

$$dM_1 = \frac{d^2X_0\, d^2X_1\, d^2X_\infty}{|X_0 - X_1|^2|X_1 - X_\infty|^2|X_\infty - X_0|^2}. \tag{4.23}$$

Thus, the integral (4.8) vanishes when X_1 is integrated and set to ∞ unless the integrand behaves as $|X_1|^4$ for large $|X_1|$. From the expression

$$\mathcal{O}(0)|_{\phi=\phi_0} = \mathcal{O}_{\phi\cdots\phi\bar\phi\cdots\bar\phi}(X_0, \overline{X}_0)(X_0 - X_\infty)^N(\overline{X}_0 - \overline{X}_\infty)^{\overline{N}} \tag{4.24}$$

which is valid at $X_1 = \infty$ and

$$\frac{1}{k!}\partial_z^k\phi_0(1) = (X_1 - X_\infty)\left(\frac{X_0 - X_1}{X_\infty - X_0}\right)^k, \tag{4.25}$$

we see that it is necessary for the integral to be nonvanishing that the operator at $z = 1$ contains $\partial_z\phi_0\partial_{\bar z}\bar\phi_0$. The fermion zero mode integration gives exactly what is needed:

$$\int d\eta_1\, d\eta_2 \oint d\bar z\, G(\bar z)\theta^\infty\Big|_{\phi=\phi_0} \propto R_{\phi\phi}^\infty(\phi_0, \bar\phi_0)\partial_z\phi_0\partial_{\bar z}\bar\phi_0. \tag{4.26}$$

Hence, the mysterious expression (4.19) of the contour integral is turned into the pullback of the Ricci form by the instanton. Other choices of $\mathcal{O}$ do not give a nonvanishing result; those necessarily contain nonzero modes, which must be contracted with the fields within G or θ^∞ and leads to a decrease in the dimension of the operator at $z = 1$.

Putting all pieces together, we find that the integral (4.8) is proportional to

$$e^{-t}\int_{\mathbb{CP}^1\times\mathbb{CP}^1} d^2X_0\, d^2X_\infty\, \frac{\mathcal{O}_{\phi\cdots\phi\bar\phi\cdots\bar\phi}(X_0, \overline{X}_0)}{(X_\infty - X_0)^{2-N}(X_\infty - X_0)^{2-\overline{N}}}. \tag{4.27}$$

The X_1-integral gave the evaluation of $c_1(X)$ on the fundamental class $[X]$. The remaining integral is precisely of the form (4.12), with

$$A_{N,\overline{N}}^{\phi\cdots\phi\bar\phi\cdots\bar\phi}(X_0, \overline{X}_0) \propto g^{\phi\bar\phi}(X_0, \overline{X}_0)\int d^2Y\, Y^{2-N}\overline{Y}^{2-\overline{N}} \tag{4.28}$$

and $Y = X_\infty - X_0$. As expected, $A_{N,\overline{N}}^{\phi\cdots\phi\overline{\phi}\cdots\overline{\phi}}$ vanish for $N \neq \overline{N}$ by rotational symmetry. For $N = \overline{N}$, however, they diverge due to the contributions from the points where X_0 and X_∞ coincide or are infinitely far apart (or both).

One way to regularize the integral (4.28) is to choose $l > 0$ and impose the lower bound l and the upper bound $1/l$ on the distance between X_0 and X_∞ measured by the target space metric. This regularization distinguishes the "poles" of $\widehat{\Sigma}$ from other points, therefore breaks conformal symmetry. The residual symmetries are scale transformations and rotations. At $z = 0$, the dimension of a local operator is determined by the behavior under scale transformations and hence preserved. Sure enough, the only nonzero and finite term in $\{Q, \theta^\infty(0)\}$ is $\mathcal{A}_1(0) = (A_{1,1}^{\phi\overline{\phi}} g_{\phi\overline{\phi}} \partial_z \overline{\phi} g_{\overline{\phi}\phi} \partial_{\overline{z}} \phi)(0)$, which correctly has dimension $(1, 1)$. Away from $z = 0$, however, the symmetry that determines the dimension is a combination of scale transformation and translation. Since the latter are broken by the regularization, the grading by dimension is violated nonperturbatively. On the original cylinder translations are conserved, so the $(0, 2)$ supersymmetry algebra is still good.

We would like to see whether $\mathcal{A}_0 = A_{0,0}$ is a nonzero constant. For $l \ll 1$, the regularization amounts to cutting out from the integration domain the region in which $g_{\phi\overline{\phi}}(X_0, \overline{X}_0)|Y|^2 \geq l^2$. Then the regularized integral gives

$$\mathcal{A}_0(X_0, \overline{X}_0) \propto g^{\phi\overline{\phi}}(X_0, \overline{X}_0) \int_{l^2 g^{\phi\overline{\phi}}(X_0, \overline{X}_0)}^{\infty} \frac{d|Y|^2}{|Y|^4} \propto l^{-2}, \tag{4.29}$$

which is indeed nonzero and constant.

Finally, let us check that we have $Q^2 = 0$ to leading order at the one-instanton level. This can be seen by computing the action of Q on $G' = R_{i\overline{j}}^\infty \partial_{\overline{z}} \phi^i \alpha^{\overline{j}}$. By the same argument as for $\{Q, \theta^\infty\}$, one can show that $\{Q, G'\} = 0$, which in turn implies that $\{Q, G\} = -2e^{-t}\{Q, G_1\}$ for some G_1. We see that Q may no longer square to zero at the one-instanton level, but rather, satisfies $\{Q, Q\} = -2e^{-t}\{Q, Q_1\}$ with

$$Q_1 = \oint d\overline{z}\, G_1. \tag{4.30}$$

The operator $Q + e^{-t}Q_1$ does square to zero, though. Thus we have found that the supercharge receives instanton corrections.

The redefinition of Q do not spoil the relation $\{Q, \theta\} \sim e^{-t} \cdot 1$. The one-instanton correction just adds a perturbatively Q-closed operator $e^{-t}\{Q_1, \theta\}$ of charge zero to the right-hand side. For a generic target space metric, there is no perturbative Q-cohomology class of charge zero associated to the Ricci curvature. Thus $\{Q_1, \theta\}$ is Q-exact. This completes the demonstration of the relation (4.2).

4.3   Nonperturbative Vanishing "Theorem"

The property of the $\mathbb{CP}^1$ model that was crucial for establishing the vanishing of its chiral algebra is that R-symmetry is broken to $\mathbb{Z}_2$ without any excess fermion zero modes, namely, by two $\bar{\psi}_+$ zero modes and no ψ_+ zero modes in the one-instanton sector. The same property also implies the vanishing of the nonperturbative chiral algebra for other target spaces.

Consider a $(0, 2)$ sigma model with compact target space X of complex dimension d. In the case of $X = \mathbb{CP}^1$, there is only one $\mathbb{CP}^1$ in the target space that instantons can wrap, which is the target space itself. In general, X has many holomorphic curves of genus zero, called *rational curves*. Suppose that X has a rational curve C such that there are two $\bar{\psi}_+$ zero modes and no ψ_+ zero modes associated to the instantons wrapping it once. We would like to show that the one-instanton action of Q on θ renders the chiral algebra of this model trivial.

Let us compute the contribution to the action of Q on θ from the instantons wrapping C. Since there are always two $\bar{\psi}_+$ zero modes in the directions tangent to C, these should be all the fermion modes; especially, none should come from the normal directions. Then the computation is essentially the same as in the case of the $\mathbb{CP}^1$ model if we regard C as the target space. For instance, the fermion zero mode integration

turns the contour integral (4.26) again into the pullback of the Ricci form, but this time restricted to C, and the integration over the parameters of the instantons becomes an integration over $C \times C \times C$. Consequently, the contribution to $\{Q, \theta\}$ from C is a nonzero constant supported on C and zero outside (times a nowhere vanishing section of K_Σ), plus perturbatively Q-exact operators.

We have found that $\{Q, \theta\}$ contains a local operator of dimension $(0, 0)$ at the one-instanton level. This operator must be perturbatively Q-closed and hence a holomorphic function on X. But since X is compact, it must be constant—not just on C, but on the whole target space! Therefore, the chiral algebra must be trivial.

One may wonder how the existence of one rational curve C, whose contribution to $\{Q, C\}$ is confined on C itself, can possibly imply something about other regions of the target space. This point can be understood by looking at possible deformations of C to nearby rational curves.

Infinitesimal deformations of C are described by holomorphic sections of the normal bundle $N_{C/X} \cong T_X|_C / T_C$. By Grothendieck's theorem [39], $N_{C/X}$ splits into a direct sum of line bundles over $C \cong \mathbb{CP}^1$:

$$N_{C/X} \simeq \mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_{d-1}). \tag{4.31}$$

The integers n_1, ..., n_{d-1} are fixed by the requirement that there be no fermion zero modes in the normal directions. After complex conjugation, these zero modes become holomorphic zero- and one-forms of the bundle $\mathcal{O}(-1) \otimes (\mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_{d-1}))$ over $\widehat{\Sigma} = \mathbb{CP}^1$. According to the formula (3.18), then it must be that $n_1 = \cdots = n_{d-1} = 0$ and the normal bundle is trivial, $N_{C/X} \cong \mathcal{O}^{\otimes d-1}$. Therefore, generically, we expect that C can be translated in every direction in the target space. If the rational curves obtained by translations sweep the whole target space, their contributions to $\{Q, \theta\}$ can add up to a nonzero constant.

In conclusion, we have obtain the following nonperturbative vanishing "theorem":

the chiral algebra of a twisted model with no left-moving fermions vanishes nonperturbatively if the target space admits a rational curve with trivial normal bundle.

4.4   Flag Manifold Model

An example in which the above deformation argument is beautifully demonstrated is the $(0, 2)$ sigma model with target space the complete flag manifold G/T of a compact semisimple Lie group G, equipped with the canonical G-invariant complex structure. The flag manifold has vanishing first Pontryagin class, so this model is anomaly free, and $c_1(G/T) = 2(x_1 + \cdots + x_r)$ with $r = \operatorname{rank} G$, so R-symmetry is broken to $\mathbb{Z}_2$.

The perturbative chiral algebra of the flag manifold model has a rich structure as an affine $\mathfrak{g}$-module at the critical level [11, 23, 24]. The reason is that the affine $\mathfrak{g}$-currents lie in the perturbative chiral algebra due to the symmetry of the target space geometry. They necessarily generate the current algebra at the critical level because $c_1(G/T) \neq 0$; otherwise, there would be an energy-momentum tensor by the Sugawara construction.

Pick a rational curve C in the target space. The normal bundle of C is then trivial, since a global section can be constructed by choosing a point in the total space and acting with the action of G. Therefore, the chiral algebra is trivial nonperturbatively. Indeed, the G-action on C generates rational curves that cover the whole target space.

4.5   Supersymmetry Breaking

The vanishing of the chiral algebra of a twisted model a nontrivial consequence on the Hilbert space of the theory. To see this, let us define the Q-cohomology of states graded by R-charge. This is naturally a module over the chiral algebra: for a Q-closed local operator $\mathcal{O}$ and a Q-closed state $|\Psi\rangle$, we define the action of $[\mathcal{O}]$ on $[|\Psi\rangle]$ by

$$[\mathcal{O}] \cdot [|\Psi\rangle] = [\mathcal{O}|\Psi\rangle]. \tag{4.32}$$

When the theory is conformally invariant, the cohomology of states is isomorphic to the chiral algebra by the state-operator correspondence.

The cohomology of states is most interesting when the target space is Kähler. In this case, the supercharges of the underlying physical $(0, 2)$ model obey $Q_+^\dagger = \overline{Q}_+$, so we have the inequality

$$\{Q_+, \overline{Q}_+\} = H - P \geq 0. \tag{4.33}$$

A state has $H - P = 0$ if and only if it is annihilated by both Q_+ and $\overline{Q}_+$. States with this property are said to be *supersymmetric*. The restriction of Q to the supersymmetric states is identically zero by definition, whereas any Q-closed state $|\Psi\rangle$ that is not supersymmetric is Q-exact:

$$|\Psi\rangle = \frac{\{Q, Q^\dagger\}}{H - P}|\Psi\rangle = Q\Big(\frac{Q^\dagger}{H - P}|\Psi\rangle\Big). \tag{4.34}$$

Thus, the cohomology of states is isomorphic to the space of supersymmetric states.

Now suppose that the chiral algebra is trivial: $[1] = 0$. Then

$$[|\Psi\rangle] = [1] \cdot [|\Psi\rangle] = 0. \tag{4.35}$$

This equation says that Q-closed states are Q-exact, so the cohomology of states is trivial. Furthermore, in the Kähler case, this means that there are no supersymmetric states in the physical model. Therefore, perturbative supersymmetric states are all lifted by instantons and supersymmetry is spontaneously broken. In particular, the elliptic genus vanishes.

Chapter 5

Supersymmetry Breaking and Loop Space Geometry

So far, we have mainly considered the chiral algebra. We now turn to the cohomology of states and discuss its relation to the loop space of the target space. By studying the cohomology of the $\mathbb{CP}^1$ model using holomorphic Morse theory on loop space, we will be able to see how instantons actually pair up perturbative supersymmetric states and lift them out of the cohomology. This gives us further insights into the nonperturbative physics of $(0,2)$ models.

5.1 Cohomology of States as Cohomology of Loop Space

Consider the $(0,2)$ model with compact Kähler target space X, defined on the cylinder $\Sigma = S^1 \times \mathbb{R}$ with coordinates (σ, τ), where $\sigma \sim \sigma + 2\pi$. We equip Σ with a complex structure by setting $\partial_z = \partial_\sigma - i\partial_\tau$ and pick $\overline{K}_\Sigma^{1/2}$ to be trivial; then H and P are the generators of translations in τ and σ, respectively. With the B-field set to the Kähler form $\omega = ig_{i\bar{j}}d\phi^i \wedge d\phi^{\bar{j}}$, the action is given by

$$
\begin{aligned}
S &= \frac{1}{2\pi} \int_\Sigma d\sigma d\tau \{Q, g_{i\bar{j}}\psi_+^i (\partial_\sigma - i\partial_\tau)\phi^{\bar{j}}\} + \frac{1}{2\pi} \int_\Sigma \phi^*\omega \\
&= \frac{1}{2\pi} \int_\Sigma d\sigma d\tau \Big(g_{i\bar{j}}(\partial_\tau \phi^i \partial_\tau \phi^{\bar{j}} + \partial_\sigma \phi^i \partial_\sigma \phi^{\bar{j}}) + g_{i\bar{j}}\psi_+^i (D_\tau + iD_\sigma)\bar{\psi}_+^{\bar{j}} \Big).
\end{aligned}
\tag{5.1}
$$

We would like to identify the cohomology of states associated to this theory.

We will deal mainly with states in the Hilbert space, so it is most natural to proceed in the Hamiltonian formalism. We will do this by regarding τ as time. Then, at each time τ, the bosonic field $\phi \colon S^1 \times \mathbb{R} \to X$ specifies a point ϕ_τ in the loop space $\mathcal{L}X$ by $\phi_\tau(\sigma) = \phi(\sigma, \tau)$. Similarly, the fermionic fields specify $\psi_{+,\tau}$ and $\bar{\psi}_{+,\tau}$, which we

may identify respectively with vectors in $T_{\mathcal{L}X}|_{\phi_\tau} \cong \Gamma(\phi_\tau^* T_X)$ and $\overline{T}_{\mathcal{L}X}|_{\phi_\tau} \cong \Gamma(\phi_\tau^* \overline{T}_X)$, by $\psi_{+,\tau}(\sigma) = \psi_+(\sigma, \tau)$ and $\bar{\psi}_{+,\tau}(\sigma) = \bar{\psi}_+(\sigma, \tau)$. In what follows, we will fix a time τ and write these simply as ϕ, ψ_+, and $\bar{\psi}_+$. As is clear from this description, the theory may now be viewed as supersymmetric quantum mechanics on $\mathcal{L}X$. Let us canonically quantize it and see what we get.

Choose a local orthonormal frame $\{e_a, e_{\bar{a}}\}$ of $T_X \oplus \overline{T}_X$. The quantization identifies $-\delta S/\delta(\partial_\tau \phi)$ with the functional derivative $\delta/\delta\phi$ on $\mathcal{L}X$, so we have

$$
\begin{aligned}
\frac{\delta}{\delta\phi^i} &= -\frac{1}{2\pi} g_{i\bar{j}} \partial_\tau \phi^{\bar{j}} - \frac{1}{4\pi} \omega_{ia\bar{b}} [\psi_+^a, \bar{\psi}_+^{\bar{b}}], \\
\frac{\delta}{\delta\phi^{\bar{i}}} &= -\frac{1}{2\pi} g_{\bar{i}j} \partial_\tau \phi^{j} - \frac{1}{4\pi} \omega_{\bar{i}a\bar{b}} [\psi_+^a, \bar{\psi}_+^{\bar{b}}].
\end{aligned}
\tag{5.2}
$$

On the other hand, the fermionic fields are quantized to obey

$$
\{\psi_+^a(\sigma, \tau), \bar{\psi}_+^{\bar{b}}(\sigma', \tau)\} = 2\pi \delta^{ab} \delta(\sigma - \sigma').
\tag{5.3}
$$

This is the loop space version of the Clifford algebra $\{\Gamma^a, \Gamma^{\bar{b}}\} = \delta^{a\bar{b}}$, in which ψ^a, $\psi^{\bar{a}}$ play the roles of the gamma matrices Γ^a, $\Gamma^{\bar{a}}$ with extra continuous indices σ parametrizing the directions along the loop. States furnish a representation of this algebra, so they are spinors on $\mathcal{L}X$. Finally, the supercharges are quantized to

$$
\begin{aligned}
Q &= \frac{1}{2\pi} \int d\sigma\, g_{\bar{i}j} \bar{\psi}_+^{\bar{i}} (i\partial_\tau + \partial_\sigma) \phi^j = \int d\sigma\, \bar{\psi}_+^{\bar{i}} \left(-i\frac{D}{\delta\phi^{\bar{i}}} + \frac{1}{2\pi} g_{\bar{i}j} \partial_\sigma \phi^j \right), \\
Q^\dagger &= \frac{1}{2\pi} \int d\sigma\, g_{i\bar{j}} \bar{\psi}_+^{i} (i\partial_\tau + \partial_\sigma) \phi^{\bar{j}} = \int d\sigma\, \psi_+^{i} \left(-i\frac{D}{\delta\phi^{i}} + \frac{1}{2\pi} g_{i\bar{j}} \partial_\sigma \phi^{\bar{j}} \right),
\end{aligned}
\tag{5.4}
$$

where $D/\delta\phi$ is the covariant functional derivative. These may be thought of almost as the holomorphic and antiholomorphic halves of the Dirac operator on $\mathcal{L}X$; but not quite, because they got extra pieces coupling to the vector field ∂_σ.

So we have found that the Q-cohomology is the cohomology of the Dirac operator on $\mathcal{L}X$ deformed by the Killing vector generating the natural isometry of $\mathcal{L}X$, namely, the rotations of loops.

To understand the effect of the deformation, let Q_0, $Q_0^\dagger$ be the halves of the "bare" Dirac operator obtained by dropping the extra pieces from the supercharges. Actually,

this operator is ill defined—all vibration modes contribute equally and there are no continuum limits in which the cutoff on frequency is taken to infinity—but let us treat it as though it existed. Then, for a function $h\colon \mathcal{L}X \to \mathbb{R}$, we define the deformed Dirac operators Q_h, $Q_h^\dagger$ by

$$Q_h = e^{-h/2\pi} Q_0 e^{h/2\pi} = \int d\sigma\, \bar{\psi}_+^{\bar{i}} \Big(-i\frac{D}{\delta\phi^{\bar{i}}} - \frac{i}{2\pi}\frac{\delta h}{\delta\phi^{\bar{i}}} \Big),$$
$$Q_h^\dagger = e^{h/2\pi} Q_0^\dagger e^{-h/2\pi} = \int d\sigma\, \psi_+^i \Big(-i\frac{D}{\delta\phi^i} + \frac{i}{2\pi}\frac{\delta h}{\delta\phi^i} \Big),$$
(5.5)

which generate symmetries of the theory with action

$$S_h = \frac{1}{2\pi}\int_\Sigma d\sigma d\tau \Big\{ Q_h, -i\psi_+^i\Big(g_{i\bar{j}}\partial_\tau\phi^{\bar{j}} - \frac{\delta h}{\delta\phi^i}\Big)\Big\} + \frac{1}{2\pi}\int_{-\infty}^\infty d\tau \partial_\tau h.$$
(5.6)

The point of introducing such operators is that they reduce to the supercharges if we choose h appropriately. Therefore, the Q-cohomology is isomorphic to the cohomology of the spinor bundle of $\mathcal{L}X$.

The function h that does the trick is constructed as follows. First, we pick a base loop in each connected component of $\mathcal{L}X$. Then, given a loop ϕ, we choose a homotopy $\hat{\phi}\colon [0,1] \times S^1 \to \mathcal{L}X$ from the base loop (in the component that ϕ lies) to ϕ and set[1]

$$h(\phi) = \int_{[0,1]\times S^1} \hat{\phi}^*\omega.$$
(5.7)

Under a variation of the loop, h changes by

$$\delta h = \int d\sigma(-ig_{i\bar{j}}\delta\phi^i\partial_\sigma\phi^{\bar{j}} + ig_{i\bar{j}}\partial_\sigma\phi^i\delta\phi^{\bar{j}}).$$
(5.8)

Thus the deformed Dirac operators (5.5) indeed reduce to the supercharges (5.4).

5.2 Perturbative Cohomology of States

We have seen that the Q-cohomology can be interpreted as the cohomology of the Dirac operator on the loop space of the target space. So far, this connection between

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[1]Actually, h is not single valued if the cohomology class of ω do not vanish on some two-cycle. However, we can always go to the covering space of $\mathcal{L}X$ in which h becomes single valued and define the theory in this space. See [40].

the Q-cohomology and the loop-space cohomology remains hypothetical, as a rigorous construction of the "Dirac operator on loop space" is not known yet. Still, following this logic backwards, we can obtain a perturbative description that approximates the loop space cohomology by an infinite-series of well-defined cohomologies associated to the target space itself.

Perturbatively, we may compute the Q-cohomology by restricting to the subspace in which $H - P = 0$ to leading order; Q-closed states with $H - P > 0$ can be discarded since they are all Q-exact. From the quantum expressions of the supercharges, we find that $\{Q, Q^\dagger\} = H - P$ contains the potential energy

$$\frac{1}{2\pi} \int d\sigma \, g_{i\bar{j}} \partial_\sigma \phi^i \partial_\sigma \phi^{\bar{j}}. \tag{5.9}$$

This means that the energy required to "stretch" a loop in the target space is proportional to its length squared. In the large volume limit, this potential rises rapidly away from the zeros, thereby localizing low-lying states to the constant loops (for a fixed P). As the constant loops form a copy of X inside $\mathcal{L}X$, we expect these states to define sections of some vector bundles over X. This is an indication that the Q-cohomology can be treated perturbatively within the target space geometry.

The curvature of the target space becomes significant only when a loop has a measurable length. States supported on a tiny loop can then be treated in the Fock space of a free closed string. So let us construct this space.

Consider a small loop ϕ fluctuating around a point ϕ_0. We identify ϕ with a vector field $\varphi \in \phi_0^*(T_X \oplus \overline{T}_X)$ and expand it in the eigenmodes of D_σ, which reduces to ∂_σ at constant loops:

$$\varphi^a(\sigma) = \sum_{n \in \mathbb{Z} \backslash \{0\}} \phi_n^a e^{in\sigma}, \qquad \varphi^{\bar{a}}(\sigma) = \sum_{n \in \mathbb{Z} \backslash \{0\}} \phi_n^{\bar{a}} e^{in\sigma}. \tag{5.10}$$

Then we define the bosonic raising and lowering operators

$$\alpha_n^a = -\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial \phi_{-n}^{\bar{a}}} + n\phi_n^a\right), \qquad \alpha_n^{\bar{a}} = -\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial \phi_{-n}^a} + n\phi_n^{\bar{a}}\right),$$

$$\tilde{\alpha}_n^a = -\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial \phi_n^{\bar{a}}} + n\phi_{-n}^a\right), \qquad \tilde{\alpha}_n^{\bar{a}} = -\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial \phi_n^a} + n\phi_{-n}^{\bar{a}}\right),$$

(5.11)

They obey the familiar commutation relations $[\alpha_m^a, \alpha_n^{\bar{b}}] = [\tilde{\alpha}_m^a, \tilde{\alpha}_n^{\bar{b}}] = m\delta^{ab}\delta_{m,-n}$. Similarly, we parallel transport (in $\mathcal{L}X$) the fermionic fields to ϕ_0 and expand:

$$\psi_+^a(\sigma) = \sum_{n\in\mathbb{Z}} \tilde{\psi}_n^a e^{-in\sigma}, \qquad \bar{\psi}_+^{\bar{a}}(\sigma) = \sum_{n\in\mathbb{Z}} \tilde{\psi}_n^{\bar{a}} e^{-in\sigma}. \tag{5.12}$$

The fermionic modes obey the anticommutation relations $\{\tilde{\psi}_m^a, \tilde{\psi}_n^{\bar{b}}\} = \delta^{ab}\delta_{m,-n}$.

The states of the Fock space are obtained by applying on the ground states the raising operators, α_n, $\tilde{\alpha}_n$, $\tilde{\psi}_n$ with $n < 0$. The ground states are degenerate. They are given by

$$|0;\phi_0\rangle = \prod_{a=1}^{d} \tilde{\psi}_0^a \prod_{n=1}^{\infty} e^{-n\phi_{-n}^a \phi_n^{\bar{a}} - n\phi_n^a \phi_{-n}^{\bar{a}}} \tilde{\psi}_n^{\bar{a}} \tilde{\psi}_n^a \tag{5.13}$$

and those obtained from it by acting with $\tilde{\psi}_0^{\bar{a}}$. These states are annihilated by the annihilation operators, α_n, $\tilde{\alpha}_n$, $\tilde{\psi}_n$ with $n > 0$. The number of $\tilde{\psi}_0^{\bar{a}}$ give the R-charge of the state.

Although states on any one tiny loop cannot feel the curvature, we can also make the superpositions of states carried by many tiny loops spread over the target space. These are able to detect the curvature.

Suppose that we would like to make the superposition of ground states, all of the same charge q, using a wave function ψ:

$$|0;\psi\rangle = \sum_{\phi_0\in X} \psi(\phi_0)\tilde{\psi}_0^{\bar{a}_1}\cdots\tilde{\psi}_0^{\bar{a}_q}|0;\phi_0\rangle. \tag{5.14}$$

In view of the fact that the zero modes $\tilde{\psi}_0$ generate the Clifford algebra, the zero mode parts of $|0;\psi\rangle$ from different points must combine smoothly to define a spinor on X. The "Dirac sea" part, involving the infinite product of $\tilde{\psi}_n^a \tilde{\psi}_n^{\bar{a}}$ with $n > 0$, must also combine consistently, but this should be possible when the sigma model anomaly is absent. The

perturbative ground states are therefore in one-to-one correspondence with the spinors. Every time a raising operator is applied on a ground state, it creates an index of either T_X or $\overline{T}_X$. Thus, excited states are spinors twisted by powers of the tangent bundle.

Now, we expand the supercharges (5.4) to next-to-leading order. We find that they are written in the raising and lowering operators as

$$Q = \sum_{a=1}^{d}\left(-i\tilde{\psi}_0^{\bar{a}}\frac{D}{D\phi_0^{\bar{a}}} + \sqrt{2}\sum_{n\in\mathbb{Z}\setminus\{0\}}\sum_{a=1}^{d}\tilde{\psi}_n^{\bar{a}}\tilde{\alpha}_{-n}^{a}\right),$$
$$Q^{\dagger} = \sum_{a=1}^{d}\left(-i\tilde{\psi}_0^{a}\frac{D}{D\phi_0^{a}} + \sqrt{2}\sum_{n\in\mathbb{Z}\setminus\{0\}}\sum_{a=1}^{d}\tilde{\psi}_n^{a}\tilde{\alpha}_{-n}^{\bar{a}}\right).$$
(5.15)

Computing $\{Q, Q^{\dagger}\}$, we obtain

$$H - P = -\slashed{D}^2 + 2\sum_{a=1}^{d}\sum_{n\in\mathbb{Z}\setminus\{0\}}(\tilde{\alpha}_{-n}^{a}\tilde{\alpha}_n^{\bar{a}} + n\tilde{\psi}_{-n}^{a}\tilde{\psi}_n^{\bar{a}}).$$
(5.16)

Here $\slashed{D}$ is the Dirac operator on X. Hence the right-moving raising operators $\tilde{\alpha}_{-n}$, $\tilde{\psi}_{-n}$ increase $H - P$ by $2n$. Since the low-lying eigenvalues of $-\slashed{D}^2$ are of order g^{-1}, it follows that low-lying states, those that have $H - P = 0$ to leading order, do not have these excitations. Restricted to these states, the supercharges reduce to the halves of the Dirac operator. The Q-cohomology is thus given to next-to-leading order by the cohomologies of twisted spinor bundles over X.

The space of low-lying states are graded by the eigenvalues of P. In terms of the raising and lowering operators, P is written to leading order as

$$P = -\frac{d}{12} + \sum_{a=1}^{d}\sum_{n\in\mathbb{Z}\setminus\{0\}}(\alpha_{-n}^{a}\alpha_n^{\bar{a}} - \tilde{\alpha}_{-n}^{a}\tilde{\alpha}_n^{\bar{a}} - n\tilde{\psi}_{-n}^{a}\tilde{\psi}_n^{\bar{a}}).$$
(5.17)

The constant $-d/12$ comes from normal ordering and cancels the zero point energy. Let us look at the relevant bundles for small values of P. For $P = -d/12$, we have the ground states $|0;\psi\rangle$, the spinors. For $P = -d/12 + 1$, we have states of the form $\alpha_{-1}|0;\psi\rangle$; these are spinors twisted by $T_X \oplus \overline{T}_X$. For $P = -d/12 + 2$, we have $\alpha_{-2}|0;\psi\rangle$ or $\alpha_{-1}\alpha_{-1}|0;\psi\rangle$, hence spinors twisted by $(T_X \oplus \overline{T}_X) \oplus S^2(T_X \oplus \overline{T}_X)$. In general, low-lying states with $P = -d/12 + n$ are spinors twisted by the holomorphic vector bundle

$V_{X,n}$ given by the series (2.25). Therefore, we have found that the spinor cohomology of $\mathcal{L}X$ can be approximated by the direct sum of the cohomologies of the twisted spinor bundles $S \otimes V_{X,n}$.

As long as the volume of the target space is not strictly infinite, there are higher order corrections to the cohomology. For example, at the next order in perturbation theory Q contains a term proportional to $\tilde{\psi}_l^{\bar{i}} R_{\bar{i}j\bar{k}l} \alpha_m^j \alpha_n^{\bar{k}} \alpha_{-l-m-n}^l$. The action of Q on $|T_{zz}\rangle = g_{i\bar{j}} \alpha_{-1}^i \alpha_{-1}^{\bar{j}} |0; \psi\rangle$, with ψ harmonic, thus gives

$$\tilde{\psi}_0^{\bar{i}} R_{\bar{i}j\bar{k}l} \alpha_{-2}^j \alpha_1^{\bar{k}} \alpha_1^l |T_{zz}\rangle \sim R_{i\bar{j}} \alpha_{-2}^i \tilde{\psi}_0^{\bar{j}} |0; \psi\rangle \tag{5.18}$$

This relation suggests that, although the states $|T_{z\bar{z}}\rangle$ and $|\partial_z\theta\rangle = R_{i\bar{j}} \alpha_{-2}^i \tilde{\psi}_0^{\bar{j}} |0; \psi\rangle$ are Q-closed to next-to-leading order, perturbative corrections lift them at the next order. We recognize this as the Fock space counterpart of the relation $[Q, T_{zz}] = \partial_z\theta$.

Since P is exactly conserved quantum mechanically, the quantum corrections can only modify the Q-cohomology within each eigenspace of P. In particular, the perturbative corrections just deform the Dirac operator to another differential operator that restricts to $S \otimes R_n$ for all n. Also, the elliptic genus is a topological invariant and receives no quantum corrections.

5.3 Holomorphic Morse Theory on Loop Space

Now that we have a fairly clear picture of the perturbative cohomology of states, let us ask: what is the exact cohomology of states?

In the context of supersymmetric quantum mechanics, Morse theory provides a powerful tool for studying the supersymmetric spectrum [41]. To adapt this approach to our case, we need to make two generalizations. First, our models are quantum field theories in two dimensions, so we must deal with Morse theory on loop space along the lines of Floer [5]. Second, we must consider holomorphic Morse theory [26] since the supercharges exist in the right-moving sector only.

As a preliminary step to holomorphic Morse theory on the infinite-dimensional manifold $\mathcal{L}X$, let us consider the finite-dimensional case, supersymmetric quantum mechanics on X with action

$$S = \int d\tau \left(g_{i\bar{j}} \partial_\tau \phi_0^i \partial_\tau \phi_0^{\bar{j}} + g_{i\bar{j}} \tilde{\psi}_0^i D_\tau \tilde{\psi}_0^{\bar{j}} \right). \tag{5.19}$$

This theory is obtained from the $(0,2)$ model by simply killing the σ-dependence of the fields, and describes the dynamics of the zero modes. The Hilbert space of the theory is therefore the space of spinors. The supercharges are the antiholomorphic half of the Dirac operator $-i\overline{\mathcal{D}} = -i\tilde{\psi}_0^{\bar{a}} D_{\bar{a}}$ and its adjoint $-i\overline{\mathcal{D}}^\dagger = -i\tilde{\psi}_0^a D_a$, obeying the supersymmetry algebra $\{-i\overline{\mathcal{D}}, -i\overline{\mathcal{D}}^\dagger\} = H$. We would like to identify the $\overline{\mathcal{D}}$-cohomology.

Given a function $f\colon X \to \mathbb{R}$, let V_f be a vector field given by $\bar{\partial}f = i_{V_f}\omega$. A typical situation in which holomorphic Morse theory is applicable is when V_f is a holomorphic vector field. In this case, we define the deformed supercharges $-i\overline{\mathcal{D}}_f$ and $-i\overline{\mathcal{D}}^\dagger_f$ by $\overline{\mathcal{D}}_f = e^{-f}\overline{\mathcal{D}}e^f$ and $\overline{\mathcal{D}}^\dagger_f = e^f\overline{\mathcal{D}}^\dagger e^{-f}$. These generate symmetries of the deformed theory described by the action

$$S_f = \int d\tau \{-i\overline{\mathcal{D}}_f, -i\psi_+^i (g_{i\bar{j}}\partial_\tau \phi^{\bar{j}} - \partial_i f)\} + \int d\tau \partial_\tau f \tag{5.20}$$

and obey the deformed supersymmetry algebra $\{-i\overline{\mathcal{D}}_f, -i\overline{\mathcal{D}}^\dagger_f\} = H_f - i\mathcal{L}_{X_f}$, where

$$H_f = H - \frac{i}{2} D_a V_{f,\bar{b}} [\tilde{\psi}_0^a, \tilde{\psi}_0^{\bar{b}}] + g_{i\bar{j}} V_f^i V_f^{\bar{j}} \tag{5.21}$$

is the Hamiltonian of the deformed theory and

$$\mathcal{L}_{X_f} = V_f^i D_i + \overline{V}_f^{\bar{i}} D_{\bar{i}} + \frac{1}{2} D_a V_{f,\bar{b}} [\tilde{\psi}_0^a, \tilde{\psi}_0^{\bar{b}}] \tag{5.22}$$

is the Lie derivative on spinors with respect to the Hamiltonian vector field $X_f = V_f + \overline{V}_f$ of f. Since f is time independent, $\mathcal{L}_{X_f}$ commutes with H_f. Energy eigenstates are therefore eigenstates of $i\mathcal{L}_{X_f}$ as well.

We see that the deformation introduced the potential energy $\|V_f\|^2$. If we now consider the large volume limit, and at the same time rescale f in such a way that V_f

remains unchanged, then supersymmetric states (for a fixed value of $i\mathcal{L}_{X_f}$) localize to the critical points of f. Let us assume that f is a Morse function with nondegenerate, isolated critical points $\{x_a\}$, and look at the structure of the perturbative supersymmetric states.

Around each x_a, we can find Kähler normal coordinates $\{\phi_0^i\}$; thus $g_{i\bar{j}}(0) = \delta_{ij}$ and $\partial_k g_{i\bar{j}}(0) = \partial_{\bar{k}} g_{i\bar{j}}(0) = 0$, where $\phi^i = 0$ at x_a. By assumption, $D_{\bar{i}} D^j f(0) = \partial_{\bar{i}} \partial_{\bar{j}} f(0)$ and $D_i D^{\bar{j}} f(0) = \partial_i \partial_j f(0)$ vanish. On the other hand, we can use a unitary transformation to diagonalize $D^i D_j f(0) = \partial_{\bar{i}} \partial_j f(0)$. Then $\partial_i \partial_{\bar{j}} f(0) = \lambda_{a,i} \delta_{ij}$ with real eigenvalues $\lambda_{a,i}$, all nonzero by the nondegeneracy of the critical point. With these coordinates, the deformed supercharges become

$$
\begin{aligned}
-i\overline{\mathcal{D}}_f &= \sum_{i=1}^d -i\tilde{\psi}_0^{\bar{i}}(\partial_{\bar{i}} + \lambda_{a,i}\phi_0^i), \\
-i\overline{\mathcal{D}}_f^\dagger &= \sum_{i=1}^d -i\tilde{\psi}_0^i(\partial_i - \lambda_{a,i}\phi_0^{\bar{i}}),
\end{aligned}
\tag{5.23}
$$

and the Lie derivative becomes

$$
i\mathcal{L}_{X_f} = \sum_{i=1}^d \lambda_{a,i}\Big(\phi_0^i\partial_i - \phi_0^{\bar{i}}\partial_{\bar{i}} + \frac{1}{2}[\tilde{\psi}_0^i, \tilde{\psi}_0^{\bar{j}}]\Big),
\tag{5.24}
$$

to the leading order in the expansion in ϕ_0.

For a state to be annihilated by both $-i\overline{\mathcal{D}}_f$ and $-i\overline{\mathcal{D}}_f^\dagger$, it must be annihilated either by $\tilde{\psi}_0^{\bar{i}}$ and $\partial_i - \lambda_{a,i}\phi_0^{\bar{i}}$ or by $\tilde{\psi}_0^i$ and $\partial_{\bar{i}} + \lambda_{a,i}\phi_0^i$ to leading order, for each i. Thus, to this order, the perturbative supersymmetric states supported on x_a are

$$
|n_i; x_a\rangle = \prod_{i;\lambda_{a,i}<0} (\phi_0^{\bar{i}})^{n_i}\tilde{\psi}_0^{\bar{i}} \prod_{i;\lambda_{a,i}>0} (\phi_0^i)^{n_i} \prod_{i=1}^d e^{-|\lambda_{a,i}|\phi_0^i\phi_0^{\bar{i}}}\tilde{\psi}_0^i
\tag{5.25}
$$

with nonnegative integers n_i. These states have $H_f = i\mathcal{L}_{X_f} = \sum_{i=1}^d (n_i + 1/2)|\lambda_{a,i}|$ and R-charges equal to the Morse index of x_a (up to a shift by a constant common to all the critical points; Morse index is here defined by the number of the negative eigenvalues of $D^i D_j f$ at the critical point). Starting from these leading order expressions, we can construct supersymmetric states to all orders in perturbation theory.

To find the exact $\overline{D}_f$-cohomology, we compute the action of $\overline{D}_f$ on perturbative supersymmetric states. We can do this by representing the matrix elements of $\overline{D}_f$ by path integrals. Since these states are sharply peaked at the corresponding critical points, we can express the matrix element $\langle n_j; x_b | \overline{\mathcal{D}}_f | m_i; x_a \rangle$ as

$$\lim_{T \to \infty} \langle n_j; x_b | e^{-(H - i\mathcal{L}_{X_f})T} \frac{[\overline{\mathcal{D}}_f, f]}{f(x_a) - f(x_b)} e^{-(H - i\mathcal{L}_{X_f})T} | m_i; x_a \rangle. \tag{5.26}$$

This can in turn be represented by the path integral

$$\lim_{T \to \infty} e^{2i\mathcal{L}_{X_f}(|n_i; x_a\rangle)T} \int \mathcal{D}\phi_0 \, \mathcal{D}\tilde{\psi}_0 \, e^{-S_f} \Psi^{\dagger}_{n_j; x_b}(T) \frac{\tilde{\psi}_0^{\bar{i}} \partial_{\bar{i}} f}{f(x_a) - f(x_b)} \Psi_{m_i; x_a}(-T), \tag{5.27}$$

with $\Psi_{n_i; x_a}$ being the wavefunctions of $|n_i; x_a\rangle$.

In the large volume limit, the path integral localizes to the local minima of the bosonic part of the action. The bosonic action can be written as

$$\int d\tau \left\| \partial_\tau \phi^i \mp g^{i\bar{j}} \partial_{\bar{j}} f \right\|^2 \pm \int d\tau \partial_\tau f, \tag{5.28}$$

so the relevant configurations are trajectories satisfying

$$\partial_\tau \phi^i \mp g^{i\bar{j}} \partial_{\bar{j}} f = 0, \tag{5.29}$$

that is, ascending and descending gradient flows of f. For the Morse index to increase as going from x_a to x_b, we must have ascending gradient flows with the $-$ sign. These are instantons for which $\{\overline{\mathcal{D}}_f, \tilde{\psi}^{\bar{i}}\} = 0$.

The path integral (5.27) vanishes unless there is precisely one $\tilde{\psi}_0^{\bar{i}}$ zero mode and no $\tilde{\psi}_0^i$ zero mode. If this is the case, then the instanton moduli space is one-dimensional; the only $\tilde{\psi}_0^{\bar{i}}$ zero mode comes from time translations, under which the instanton equation is invariant. In the case of ordinary Morse theory, the integrations over the fermion zero mode and the instanton moduli space cancel $f(x_a) - f(x_b)$ in the denominator, while the integration over the nonzero modes gives ± 1 because the bosonic and fermionic determinants have the same absolute values due to supersymmetry. The path integral for holomorphic Morse theory are more complicated, for wavefunctions depend on the

powers of ϕ_0 and also there is no cancellation between the bosonic and fermionic determinants; for example, it vanishes unless $\sum_{i=1}^{d}(m_i + 1/2)|\lambda_{a,i}| = \sum_{i=1}^{d}(n_i + 1/2)|\lambda_{b,i}|$. If the path integral (5.27) does give a nonzero result, then we have a relation

$$\langle n_j; x_b|\overline{\mathcal{D}}_f|m_i; x_a\rangle \sim e^{-(f(x_a)-f(x_b))}. \tag{5.30}$$

This indicates that $|m_i; x_a\rangle$ is no longer $\overline{\mathcal{D}}_f$-closed, whereas $|n_j; x_b\rangle$ is no longer $\overline{\mathcal{D}}_f^{\dagger}$-closed, after the tunneling effects are taken into account. Therefore, these states are nonperturbatively lifted.

Let us return to our case. The $(0, 2)$ model with target space X is supersymmetric quantum mechanics on $\mathcal{L}X$, deformed by the function h. Furthermore,

$$V_h = -i \int d\sigma \, g^{i\bar{j}} \frac{\delta h}{\delta \phi^{\bar{j}}} \frac{\delta}{\delta \phi^i} = \int d\sigma \, \partial_\sigma \phi^i \frac{\delta}{\delta \phi^i} \tag{5.31}$$

is a holomorphic vector field on $\mathcal{L}X$, with X_h generating rotations. This suggests that everything we have said about the finite-dimensional case naturally generalizes to our case. However, h is not a Morse function—the critical point set of h is not a discrete set of points, but the whole manifold X in $\mathcal{L}X$. As a result, the space of perturbative supersymmetric states is the direct sum of certain twisted spinor bundles on X, a bit too large to compute all the matrix elements of Q between its members.

We know how to fix this problem already. Since the zero mode part of the theory is described by supersymmetric quantum mechanics on X, we can deform the supercharges to localize supersymmetric states further to the critical points of a Morse function!

As before, let f be a Morse function on X with nondegenerate, isolated critical points. We define

$$h_f = h + \int d\sigma \, \phi^* f \tag{5.32}$$

This function is known as the *symplectic action functional*. Its critical points are the 2π-periodic solutions of the Hamiltonian equation

$$\partial_\sigma \phi(\sigma) + X_f(\phi(\sigma)) = 0. \tag{5.33}$$

It is possible that there are nonconstant solutions, but we expect that such configurations can be killed by perturbing f. The only solutions are then the constant loops sitting at the critical points of f. Supersymmetric states of the supercharges Q_{h_f}, $Q_{h_f}^{\dagger}$ deformed by h_f are thus localized there.

The Hamiltonian vector field X_f lifts naturally to a vector field on $\mathcal{L}X$, which we denote by X_f as well. The deformed supercharges Q_{h_f}, $Q_{h_f}^{\dagger}$ obey

$$\{Q_{h_f}, Q_{h_f}^{\dagger}\} = H_{h_f} - P - i\mathcal{L}_{X_f}, \tag{5.34}$$

where H_{h_f} is the Hamiltonian of the deformed theory and the Lie derivative

$$\mathcal{L}_{X_f} = \int d\sigma \Big(V_f^i \frac{D}{\delta \phi^i} + \overline{V}_f^{\bar{\imath}} \frac{D}{\delta \phi^{\bar{\imath}}} + \frac{1}{2} D_a V_{f,\bar{b}} [\tilde{\psi}_+^a, \tilde{\psi}_+^{\bar{b}}] \Big). \tag{5.35}$$

The operators H_{h_f}, P, and $\mathcal{L}_{X_f}$ all commute. Hence, at each critical point x_a, we will obtain a tower of perturbative supersymmetric states labeled by P and $\mathcal{L}_{X_f}$.

We can readily work out the structure of perturbative supersymmetric states. In Kähler normal coordinates $\{\phi_0\}$ around x_a, the deformed supercharges are written as

$$
\begin{aligned}
Q_{h_f} &= \sum_{i=1}^{d} -i\tilde{\psi}_n^{\bar{\imath}} \Big(\frac{\partial}{\partial \phi_n^{\bar{\imath}}} - (n - \lambda_{a,i})\phi_n^i \Big), \\
Q_{h_f}^{\dagger} &= \sum_{i=1}^{d} -i\tilde{\psi}_n^i \Big(\frac{\partial}{\partial \phi_n^i} - (n + \lambda_{a,i})\phi_n^{\bar{\imath}} \Big)
\end{aligned}
\tag{5.36}
$$

to leading order. We rescale f so that $0 < |\lambda_{a,i}| < 1$. Then, the perturbative supersymmetric states are given by the ground states

$$|0; x_a\rangle = \prod_{i; \lambda_{a,i}<0} \tilde{\psi}_0^{\bar{\imath}} \prod_{i=1}^{d} e^{-|\lambda_{a,i}|\phi_0^i \phi_0^{\bar{\imath}}} \tilde{\psi}_0^i \prod_{n=1}^{\infty} e^{-(n-\lambda_{a,i})\phi_{-n}^i \phi_n^{\bar{\imath}} - (n+\lambda_{a,i})\phi_n^i \phi_{-n}^{\bar{\imath}}} \tilde{\psi}_n^{\bar{\imath}} \tilde{\psi}_n^i \tag{5.37}$$

and the excited states obtained from the ground states by applying the left-moving creation operators α_{-n}, and powers of ϕ_0^i or $\phi_0^{\bar{\imath}}$ depending on whether $\lambda_{a,i}$ is positive or negative. We have

$$i\mathcal{L}_{X_f} = \sum_{i=1}^{d} \lambda_{a,i} \Big(\phi_n^i \frac{\partial}{\partial \phi_n^i} - \phi_n^{\bar{\imath}} \frac{\partial}{\partial \phi_n^{\bar{\imath}}} + \frac{1}{2}[\tilde{\psi}_{-n}^i, \tilde{\psi}_n^{\bar{\jmath}}] \Big), \tag{5.38}$$

to leading order, so the ground state $|0; x_a\rangle$ has $H_{h_f} = i\mathcal{L}_{X_f} = \sum_{i=1}^{d} |\lambda_{a,i}|/2$ and $P = 0$. For excited states, each holomorphic index i adds $+\lambda_{a,i}$ and antiholomorphic index $\bar{\imath}$ adds $-\lambda_{a,i}$ to $i\mathcal{L}_{X_f}$.

As in the finite-dimensional case, the exact Q_{h_f}-cohomology is found by computing the matrix elements $\langle \Psi_j; x_b | Q_{h_f} | \Psi_i; x_a \rangle$ by the path integrals

$$\lim_{T \to \infty} e^{2(P + i\mathcal{L}_{X_f})(|\Psi_i; x_a\rangle)T} \int \mathcal{D}\phi\, \mathcal{D}\tilde{\psi}_0\, e^{-S_{h_f}} \Psi^{\dagger}_{j; x_b}(T) \frac{\bar{\psi}^{\bar{\imath}}_+ \partial_{\bar{\imath}} h_f}{h_f(x_a) - h_f(x_b)} \Psi_{i; x_a}(-T), \quad (5.39)$$

The path integrals localize to instantons,

$$\partial_\tau \phi^i - g^{i\bar{\jmath}} \frac{\delta h_f}{\delta \phi^{\bar{\jmath}}} = \partial_\tau \phi^i - i\partial_\sigma \phi^i - g^{i\bar{\jmath}} \partial_{\bar{\jmath}} f = 0. \quad (5.40)$$

These are deformations of holomorphic maps from Σ to X.

5.4 $\mathbb{CP}^1$ Model

We now apply the holomorphic Morse theory formalism to the $\mathbb{CP}^1$ model in order to study the supersymmetry breaking which the vanishing of the chiral algebra implies. We equip X with the Fubini-Study metric

$$g = \frac{s d\phi\, d\bar{\phi}}{(1 + |\phi|^2)^2}, \quad (5.41)$$

and choose the Morse function

$$f = \frac{s\lambda}{4} \frac{|\phi|^2 + 1}{|\phi|^2 - 1}, \quad (5.42)$$

where $0 < \lambda < 1$. Then the Hamiltonian vector field $X_f = \lambda(-i\phi\partial_\phi + i\bar{\phi}\partial_{\bar{\phi}})$ generates rotations on the target space, $\phi \to e^{-i\theta}\phi$.

The critical points of f are S at $\phi = 0$, with Morse index zero and the eigenvalue of the Hessian equal to λ, and N at $\phi = \infty$, with Morse index one and the eigenvalue equal to $-\lambda$. So, in the limit $s \to \infty$, we have the space of perturbatively supersymmetric states of charge zero at S, and the space of perturbatively supersymmetric states of

charge one at N. Apart from the charges, these spaces are isomorphic, making it possible for instantons to pair up all of their members and lift them.

In the present case, instantons going from S to N are $\phi \propto e^{-in\sigma+(n+\lambda)\tau}$ with $n \geq 0$ and those going from N to S are $\phi \propto e^{in\sigma-(n-\lambda)\tau}$ with $n \geq 1$. The fermion zero modes satisfy the equations

$$(\partial_\tau + i\partial_\sigma - \lambda)\bar{\psi}_+ = 0,$$
$$(\partial_\tau + i\partial_\sigma + \lambda)(g_{\bar{\phi}\phi}\psi_+) = 0.$$
(5.43)

These are solved by $\bar{\psi}_{+0} \propto e^{il\sigma+(l+\lambda)\tau}$ and $\psi_{+0} \propto g^{\bar{\phi}\phi}e^{im\sigma+(m-\lambda)\tau}$. The normalizability at $\tau = \pm\infty$ gives the conditions $-\lambda < l < 2n+\lambda$ and $\lambda < m < -2n-\lambda$ for instantons going from S to N, and $-2n+\lambda < l < -\lambda$ and $2n-\lambda < m < \lambda$ for instantons going from N to S. Imposing that there be one $\bar{\psi}_+$ zero mode and no ψ_+ zero mode leaves only $\phi \propto e^{\lambda\tau}$ and $\phi \propto e^{i\sigma-(1-\lambda)\tau}$ as instantons possibly giving nonvanishing contributions.

As a first example of an instanton effect, we consider the amplitude going from the ground state $|0; S\rangle$ at S to the other ground state $|0; N\rangle$ at N, both of which have $i\mathcal{L}_{X_f} = \lambda/2$ and $P = 0$. They are connected by the family of instantons

$$\phi_{\sigma_0,\tau_0}(\tau) = e^{\lambda(\tau-\tau_0-i\sigma_0)},$$
(5.44)

parametrized by σ_0, τ_0. These instantons propagate along lines attached to S and N. We will refer to them as *worldline instantons*.

Since the worldline instantons do not depend on σ, the computation essentially reduces to supersymmetric quantum mechanics on X. The path integral (5.39) gives

$$\int d\tau_0 \, d\sigma_0 \, e^{-S_h} \int d\sigma \, \partial_{\tau_0}\bar{\phi}_{\sigma_0,\tau_0} \frac{\delta h}{\delta\bar{\phi}}(\phi_{\sigma_0,\tau_0}) \propto \big(f(N) - f(S)\big)e^{-(f(N)-f(S))/2\pi}.$$
(5.45)

Here we have used the fact that neither the ground state wavefunctions nor the Morse function h depend on the phase of ϕ_{σ_0,τ_0}. So we obtain

$$\langle 0; N|Q_{h_f}|0; S\rangle \propto e^{-(f(N)-f(S))/2\pi}.$$
(5.46)

Therefore, we conclude that the worldline instantons induce a relation $Q_{h_f}|0;S\rangle \sim |0;N\rangle$ and lift the ground states.

Notice that R-charge is not violated in this relation. This is because the worldline instantons are "classical" instantons, in the sense that they capture the classical geometry of the target space. Since $|0;S\rangle$ and $|0;N\rangle$ are would-be supersymmetric states of $P = 0$ which should correspond to harmonic spinors in the original theory (before the deformation by the Morse function f), the lifting of these states is a reflection of the Lichnerowicz theorem: there are no harmonic spinors on a manifold with positive scalar curvature.

Once the ground states are lifted, one might be tempted to say that all the excited states constructed on them should also be lifted by the fluctuations around the same instantons. However, there is a symmetry that forbids the lifting of some of those states. Consider the $U(1)$ rotation that acts on the bosonic nonzero modes only. To the leading order of the large volume limit, the action is quadratic in fluctuations and this is a symmetry. Let us call this approximate symmetry the bosonic R-symmetry.

In the path integral computation of the matrix elements, only the zero mode part of Q_{h_f} contributed. Therefore, the bosonic R-charge commutes with Q_{h_f} to leading order. For example, the states $\phi_0^k \alpha_{-n}|0;S\rangle$ and $\bar{\phi}_0^k \bar{\alpha}_{-n}|0;N\rangle$ cannot be paired by the worldline instantons to leading order, even though they both have $P = n$ and $i\mathcal{L}_{X_f} = (k+3/2)\lambda$. On the other hand, $\phi_0^k \alpha_{-n}|0;S\rangle$ and $\bar{\phi}_0^{k+2}\alpha_{-n}|0;N\rangle$ will be lifted because they have the same P, $i\mathcal{L}_{X_f}$, and bosonic R-charge. Similarly, $\phi_0^{k+2}\bar{\alpha}_{-n}|0;S\rangle$ and $\bar{\phi}_0^k \bar{\alpha}_{-n}|0;N\rangle$ will be lifted. However, $\phi_0^k \bar{\alpha}_{-n}|0;S\rangle$ and $\bar{\phi}_0^k \alpha_{-n}|0;N\rangle$ are not lifted for $k \leq 1$. Since the former corresponds to a section of $K_X^{1/2} \otimes T_X \cong \mathcal{O}(1)$ and the latter corresponds to a $(0,1)$-form of $K_X^{1/2} \otimes T_X^\vee \cong \mathcal{O}(-3)$ in the original theory, this is consistent with the Hodge numbers $h^0(\mathcal{O}(1)) = h^1(\mathcal{O}(-3)) = 2$. In this way, we can show that worldline instantons lift all states in the Fock spaces that do not make appearance in the classical Q-cohomology.

Of course, higher order corrections break the bosonic R-symmetry. We have already encountered an example of this, where the Fock space counterpart of the operator relation $[Q, T_{zz}] = \partial_z \theta$ gives $[Q, g_{i\bar{\jmath}} \alpha^i_{-1} \alpha^{\bar{\jmath}}_{-1}] \sim R_{i\bar{\jmath}} \alpha^i_{-2} \tilde{\psi}^{\bar{\jmath}}_0$. Thus, the matrix element of Q_{h_f} between $\alpha_{-1} \bar{\alpha}_{-1} |0; S\rangle$ and $\alpha_{-2} |0; N\rangle$ should become nonzero after the higher order corrections are included. Although the two states have different $i\mathcal{L}_{X_f}$ to leading order, there is no contradiction here, as $i\mathcal{L}_{X_f}$ also gets corrected. For the same reason, $\alpha_{-k} |0; N\rangle$ for $k \geq 3$ should be all lifted because they correspond to $\partial_z^{k-1} \theta = [Q, \partial_z^k T_{zz}]$. Complex conjugation on the target space implies that $\bar{\alpha}_{-k} |0; S\rangle$ are lifted for $k \geq 2$.

So it seems that perturbative supersymmetric states that are not lifted by the worldline instantons or the fluctuations around them are $\alpha_{-1} |0; N\rangle$ and $\bar{\alpha}_{-1} |0; S\rangle$, and those constructed on them. The first two states are connected by the instantons

$$\phi_{\sigma_0, \tau_0}(\sigma, \tau) = e^{i(\sigma - \sigma_0) - (1-\lambda)(\tau - \tau_0)}, \tag{5.47}$$

going from N to S, which have the right behavior for $\alpha_{-1} |0; N\rangle \propto \phi'_{-1} |0; N\rangle$ and $\bar{\alpha}_{-1} |0; S\rangle \propto \bar{\phi}_{-1} |0; S\rangle$, where $\phi' = 1/\phi$; thus, they may lift these states. Although perturbatively the initial state has charge two and the final state has charge zero, this lifting is possible because degree one instantons violate R-charge by -2. Moreover, it is consistent with the bosonic R-charge which is assigned to nonzero modes and not to instantons.

This time, the path integral (5.39) contains the factor $\phi_1(T) \phi'_{-1}(-T) \propto e^{-2(1-\lambda)T}$ from the wavefunctions of positive energy states. This factor is canceled by $e^{2(P + i\mathcal{L}_{X_f})T}$ in front of the path integral, except for the contribution from the zero point energy $\lambda/2$ which should be included in the path integral measure (or canceled by considering normalized matrix elements). The rest of the computation proceeds as in the first example, and we obtain

$$\langle \Psi_f | Q_h | \Psi_i \rangle \propto e^{-(h(\phi_\infty) - h(\phi_{-\infty}))/2\pi}. \tag{5.48}$$

Therefore, $\alpha_{-1} |0; N\rangle$ and $\bar{\alpha}_{-1} |0; S\rangle$ are lifted. In contrast to the first example, this

nonvanishing amplitude is induced by *worldsheet instantons*, depending on σ as well as τ. These instantons wrap the target space once as they propagate from N to S, sweeping out its fundamental class.

Let us summarize what we have found. We have seen, by deforming the supercharges using a Morse function, that the Q-cohomology of states for $X = \mathbb{CP}^1$ is given approximately by the Fock spaces of closed strings with charge zero and one located respectively at S and N. At this point, the bosonic and fermionic supersymmetric spectra are manifestly isomorphic due to the degeneracy of the ground states. The classical geometry of the target space then induces "classical" instanton effects, which lift those states that do not enter the classical Q-cohomology. Some of the surviving states are lifted by the perturbative corrections to classical instantons. Finally, worldsheet instantons lift every other states that were not lifted by classical instantons or perturbative corrections, thereby breaking supersymmetry. These worldsheet instantons capture the geometry of the loop space $\mathcal{L}X$, just like worldline instantons—instantons in supersymmetric quantum mechanics on X—capture the geometry of X.

Chapter 6

Outlook

In this thesis, we have uncovered a surprising phenomenon in which the chiral algebras of certain $(0,2)$ models completely vanish nonperturbatively. The vanishing of the chiral algebra also implies supersymmetry breaking, and we have seen that holomorphic Morse theory on loop space is a useful tool in studying the instanton lifting behind it. Our results suggest several possible directions for future research. I would like to conclude this thesis by discuss two of them here.

One possible direction is the nonperturbative generalization of the sheaf-theoretic approach to the chiral algebras.

In Chapter 3, we saw that the perturbative chiral algebra of a twisted model is computed by the cohomology of the sheaf of $\beta\gamma$ systems on the target space. To establish this result, we used two properties of the twisted model and its chiral algebra. First is that perturbatively the action of Q is local on the target space and defines a differential operator acting on sections of a holomorphic vector bundle. Second is that the chiral algebra is invariant under deformations of the target space metric, so locally it can be described by a free theory.

Nonperturbatively, the counterpart of the first property is that the action of Q is local on the instanton moduli space $\mathcal{M}$. This is true because, instantons being nonpropagating, the short distance singularities of the quantum theory come only from the directions normal to $\mathcal{M}$ in the field space and the renormalization can be done locally on $\mathcal{M}$. Whether the second property generalizes to $\mathcal{M}$ is not clear, but it is plausible that there exists a sheaf-theoretic formulation of the chiral algebra that replaces the

sheaf of $\beta\gamma$ systems by another sheaf defined on the whole instanton moduli space, not just the zero-instanton component $\mathcal{M}_0 \cong X$. Such a formulation would allow us to compute the nonperturbative chiral algebra systematically, so this idea seems worth pursuing.

The other possible direction is an application of holomorphic Morse theory to the Höhn–Stolz conjecture.

A closer look at the argument in Chapter 5 reveals that the vector field V_f generated by the Morse function f does not need to be holomorphic, but rather, being holomorphic only in the neighborhood of the critical points of f is sufficient for the analysis to work. Naively, one can achieve this by deforming f appropriately around the critical points. If this is the case, the Höhn–Stolz conjecture is likely to follow in the Kähler case.

In the large volume limit, the space of supersymmetric states is approximated by the direct sum of Fock spaces localized at each critical point. These Fock spaces are all isomorphic as vector spaces, but carry different charges: looking at the expression of the ground states (5.37), we see that the Fock space at a critical point of Morse index q consists of states of charge q. This large volume approximation is usually the starting point of computing the exact spectrum of supersymmetric states. However, if we are only interested in the elliptic genus, we may use it to compute the exact answer.

Now suppose that the target space X has positive Ricci curvature. Then X has no harmonic spinors because the scalar curvature is also positive. This means that there are no supersymmetric states with $P = -d/12$ before we deformed the supercharge by the Morse function. On the other hand, the ground states of the Fock spaces localized around the critical points after the deformation are would-be supersymmetric states with $P = -d/12$. Since there are none such states, they must be lifted by instantons. This lifting is possible only if there are the same numbers of bosonic and fermionic ground states; otherwise they cannot be all paired up. But then, since the local Fock spaces are isomorphic to one another, there must be the same numbers of bosonic and

fermionic states at each energy level. Therefore, the elliptic genus vanishes.

Where did we use the condition that the Ricci curvature of the target space is positive? This is where the divergence of the quantum field theory comes in. The renormalization group of a sigma model generates a flow of the target space metric. At one-loop, the metric $g(\mu)$ renormalized at energy scale μ obeys the equation [33, 34, 35]

$$\mu\frac{dg_{i\bar{j}}(\mu)}{d\mu} = R_{i\bar{j}}(g(\mu)),\tag{6.1}$$

where $R_{i\bar{j}}(g)$ is the Ricci curvature of g. From this equation, we see that the metric becomes larger as the energy scale μ gets larger when the Ricci curvature is positive definite. Since the theory is weakly coupled when the target space has large volume, the theory is then well behaved—it is asymptotically free. So if the Ricci curvature is nonzero, physically it had better be positive. The Ricci curvature indeed arises when we normal order the higher order terms of the supercharges in the Fock space representation.

Appendix

$(0,2)$ Models with Strong KT Target Spaces

In this appendix, we construct $(0,2)$ sigma model actions with non-Kähler target spaces. Our strategy is to first define $(0,2)$ supersymmetric actions locally on a general complex target space X, then find the conditions on X for these local actions to consitently combine into a single global action.

Locally on the target space, choose a $(1,0)$-form $K = K_i d\phi^i$ and consider the action

$$S = -i \int_\Sigma d^2z \{\overline{Q}_+, [Q_+, K_i \partial_z \phi^i - \overline{K}_{\bar\imath} \partial_z \phi^{\bar\imath}]\}. \tag{A.1}$$

By $Q_+^2 = \overline{Q}_+^2 = 0$, this local action is invariant under $-i\epsilon_- Q_+ + i\bar\epsilon_- \overline{Q}_+$ for antiholomorphic sections ϵ_-, $\bar\epsilon_-$ of $\overline{K}_\Sigma^{1/2}$. Expanding the Q_+-commutator, we find

$$S = \int_\Sigma d^2z \{\overline{Q}_+, g_{i\bar\jmath} \psi_+^i \partial_z \phi^{\bar\jmath} - T_{ij} \psi_+^i \partial_z \phi^j\}, \tag{A.2}$$

with $g_{i\bar\jmath} = \partial_i \overline{K}_{\bar\jmath} + \partial_{\bar\jmath} K_i$ and $T = \partial K$. Expanding the $\overline{Q}_+$-commutator as well, we obtain the expression

$$S = \int_\Sigma d^2z (g_{i\bar\jmath} \partial_{\bar z} \phi^i \partial_z \phi^{\bar\jmath} + i g_{i\bar\jmath} \psi_+^i D_z \bar\psi_+^{\bar\jmath}) + i \int_\Sigma \phi^* T. \tag{A.3}$$

Here the covariant derivative is $D_z \bar\psi_+^{\bar\imath} = \partial_z \bar\psi_+^{\bar\imath} + (\partial_z \phi^{\bar\jmath} g^{\bar\imath l} \partial_{\bar k} g_{l\bar\jmath} - \partial_z \phi^j g^{\bar\imath l} \partial_{\bar k} T_{lj}) \bar\psi_+^{\bar k}$.

Globally, the $(0,2)$ sigma model action is constructed by gluing the local descriptions found above, consistently over the target space. It is clear that $g_{i\bar\jmath}$ should be interpreted as a hermitian metric on the target space, hence globally defined. The associated $(1,1)$-form $\omega = i\partial \overline{K} - i\bar\partial K$ obeys

$$\partial \bar\partial \omega = 0. \tag{A.4}$$

Therefore, X must be strong KT. With respect to this strong KT structure specified by K, the covariant derivative D_z is the pullback of the Bismut connection $\Gamma + H$, where Γ is the Levi-Civita connection on X and

$$H = \frac{1}{2}d(T + \overline{T}) = \frac{i}{2}(\bar{\partial}\omega - \partial\omega). \tag{A.5}$$

is the *torsion three-form*. Conversely, if X is strong KT, then one can always find locally a $(0,1)$-form K such that the target space of the action (A.1) is equipped with a given strong KT structure.[1] In the Kähler case, we can set $K = \partial\mathcal{K}$ with $\mathcal{K}$ a Kähler potential, whereby T vanishes.

Now, suppose that we have two local actions of the form (A.1) whose target spaces are open sets U and U' in X, equipped with the metrics that descend from the same strong KT structure on X. If K and K' are the corresponding local $(1,0)$-forms, then $\partial(\overline{K}' - \overline{K}) = \bar{\partial}(K' - K)$ on the overlap $U \cap U'$. This shows that $\bar{\partial}(K' - K)$ is a closed real $(1,1)$-form. Such a form is generated by $i\bar{\partial}\partial f$ from a "Kähler potential" f, hence

$$K' = K + \kappa + i\partial f \tag{A.6}$$

for some holomorphic one-form κ. On the other hand, the transformation $K \to K'$ is a symmetry of the action (A.1). Therefore, the two local actions can be glued together to describe a theory with target space $U \cup U'$. Applying this gluing procedure for an open cover of X, we obtain a would-be global action of the form (A.3).

We still need to check that this would-be global action is well defined for all maps, not just for those that can be captured by the local actions. The kinetic terms of the action (A.3) are manifestly well defined since they are written with the metric and the

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[1] Let ω be a real $(1,1)$-form obeying $\partial\bar{\partial}\omega = 0$. Since $\partial\omega$ is closed, by the Poincaré lemma we have $\partial\omega = d\tau$ locally for some two-form τ. Writing $\tau = \tau^{(2,0)} + \tau^{(1,1)} + \tau^{(0,2)}$, we find the equations $\partial\tau^{(2,0)} = \bar{\partial}\tau^{(1,1)} + \partial\tau^{(0,2)} = \bar{\partial}\tau^{(0,2)} = 0$ and $\partial\omega = \bar{\partial}\tau^{(2,0)} + \partial\tau^{(1,1)}$. By the $\bar{\partial}$-Poincaré lemma, $\tau^{(0,2)} = \bar{\partial}\alpha$ for some α. Then $\bar{\partial}(\tau^{(1,1)} - \partial\alpha) = 0$, so again by the $\bar{\partial}$-Poincaré lemma, $\tau^{(1,1)} = \partial\alpha + \bar{\partial}\beta$ for some β. On the other hand, by the ∂-Poincaré lemma $\tau^{(2,0)} = \partial\gamma$ for some γ. Now we have $\partial\omega = d(\bar{\partial}\alpha + \partial\alpha + \bar{\partial}\beta + \partial\gamma) = \partial\partial\bar{\partial}(\beta - \gamma)$, and the ∂-Poincaré lemma implies that $\omega = i\partial\overline{K} + \bar{\partial}(\beta - \gamma)$ for some $\overline{K}$. Since ω is real, $\beta - \gamma = -iK$.

torsion three-form, both of which are globally defined. However, the pullback term

$$S_T(\phi) = i \int_\Sigma \phi^* T \tag{A.7}$$

is apparently not well defined. For T changes under the gauge transformation (A.6) as $T \to T + \partial \kappa$, hence is not globally defined.

To make sense of this term, pick a map ϕ_α from each connected component $\mathcal{C}_\alpha$ of the space $\mathcal{C}$ of smooth maps from Σ to X. We write the two-cycles $\phi_{\alpha*}[\Sigma]$ in X simply as Σ_α; the two-cycle for a general map $\phi: \Sigma \to X$ is denoted by Σ. Now, given $\phi \in \mathcal{C}_\alpha$, choose a homotopy C from ϕ_α to ϕ. This may be regarded as a three-chain such that $\partial C = \Sigma - \Sigma_\alpha$. (If the maps are one-to-one, we can choose C to be a three-dimensional submanifold of X interpolating Σ and Σ_α.) Then the Stokes theorem gives

$$S_T(\phi) - S_T(\phi_\alpha) = i \int_{\partial C} T = i \int_C H^{(2,1)}. \tag{A.8}$$

This formula determines $S_T(\phi)$ in terms of $S_T(\phi_\alpha)$, the gauge invariant quantity $H^{(2,1)}$, and the homotopy C. If we choose another homotopy C', then $S_T(\phi)$ shifts by

$$i \int_{C'} H^{(2,1)} - i \int_C H^{(2,1)} = i \int_D H^{(2,1)}, \tag{A.9}$$

where $D = C' - C$ is a three-cycle. Thus, S_T becomes independent of the choice of C if $H^{(2,1)}$ vanishes in the cohomology or, since $H = H^{(2,1)} - id\omega/2$, if H does.

What we assign for the values of $S_T(\phi_\alpha)$ are part of the definition of the action. There are constraints, though, because if a subset of base maps $\{\phi_\alpha, \cdots, \phi_\beta\}$ induces a linear relation $c_\alpha[\Sigma_\alpha] + \cdots + c_\beta[\Sigma_\beta] = 0$, then there exits a three-chain U such that $\partial U = c_\alpha \Sigma_\alpha + \cdots + c_\beta \Sigma_\beta$ and

$$c_\alpha S_T(\phi_\alpha) + \cdots + c_\beta S_T(\phi_\beta) = i \int_U H^{(2,1)}. \tag{A.10}$$

Practically, these constraints do not cause much difficulty; we are usually interested in the components of $C^\infty(\Sigma, X)$ that contain holomorphic maps, and it is natural to set $S_T(\phi_\alpha) = 0$ for holomorphic ϕ_α.

Let us summarize. The $(0,2)$ sigma model action (A.3) is well defined if the target space X is equipped with a strong KT structure whose torsion three-form H vanishes in the cohomology. The definition requires a choice of the values of S_T at the base maps ϕ_α that satisfies the consistency condition (A.10) whenever $c_\alpha[\Sigma_\alpha] + \cdots + c_\beta[\Sigma_\beta] = 0$, but practically this degree of freedom is irrelevant. In fact, it suffices for $H/2\pi$ to represent an *integral* cohomology class:

$$[H/2\pi] \in H^3(X; \mathbb{Z}). \tag{A.11}$$

The integrality ensures that the action takes values in $\mathbb{C}/2\pi i\mathbb{Z}$ and, therefore, the path integral weight e^{-S} is well defined.

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