GRADIENT ESTIMATES FOR THE CONDUCTIVITY PROBLEMS AND THE SYSTEMS OF ELASTICITY

BY BIAO YIN

A dissertation submitted to the

Graduate School—New Brunswick

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

YanYan Li

and approved by

New Brunswick, New Jersey

October, 2009

ABSTRACT OF THE DISSERTATION

Gradient Estimates for the Conductivity Problems and the Systems of Elasticity

by Biao Yin

Dissertation Director: YanYan Li

We investigate the high stress concentration in stiff fiber-reinforced composites. By the anti-plane shear model, this problem can be transferred into the conductivity problems with multiple inclusions. Here we consider the extreme cases, i.e. the perfect and insulated conductivity problems. We obtain the optimal blow-up rates of the gradient in the perfect conductivity problems and an upper bound of the gradient in the insulated conductivity problems. We also study the related problems in elliptic systems including systems of elasticity and obtain some partial results.

Acknowledgements

I would like to extend my deepest appreciation to all people who have helped and inspired me during my graduate study at Rutgers University.

First of all, I especially want to thank my advisor, Prof. YanYan Li, for his enormous guidance, encouragements and suggestions throughout my research and study. His insight and enthusiasm in research has deeply inspired me. In addition, he is always accessible to his students and tries his best to help them. I feel very fortunate to work with him.

I would like to thank many professors who have taught me mathematics during my undergraduate at University of Science and Technology of China and my graduate at Rutgers, for their great efforts and inspirations. I am very grateful to my thesis committee members Professor Zheng-Chao Han, Professor Hyeonbae Kang, Professor Michael Vogelius, for their help on my research and my writing of this thesis. I am also indebted to Professor Haim Brezis for his interest in our work and his strong support in my research.

I am very thankful to my collaborators Ellen ShiTing Bao and Haigang Li, for many valuable discussions during our collaboration. I appreciate Professor YanYan Li, Ellen ShiTing Bao and Haigang Li for allowing me to include our joint work in my thesis.

I am indebted to my fellows and friends for providing a stimulating and fun environment in which to learn and grow. I am especially grateful to Aobing Li, Qinian Jin, Haoyuan Xu, Ren Guo, Michael Neiman, Ming Shi, Jin Wang, Yuan Zhang, Luc Nguyen, Tianlin Jin, Yunpeng Wang, Changzheng Li, Rongjie Lai, and many other friends in and outside of Rutgers for their constant support and friendship.

I owe a great deal to Rutgers University and the mathematics department for their financial support and technical assistance during my graduate study.

My deepest gratitude goes to my family for their everlasting love and support throughout my life; this dissertation is simply impossible without them. My parents Xiguo Yin and Meiju Zhan are my heroes in my heart. They worked hard to support the family and spare no effort to provide the best environment for my sister and me to grow up. Although my father is no longer with us, he is forever remembered. I am sure he shares our joy and happiness in the heaven. My sister Min is always lovely and nice. I feel very proud of her for her talent and success. My wife Jia, it is your love, support and understanding that make everyday a beautiful day for me. I thank you from the bottom of my heart.

Dedication

To my parents Xiguo Yin, Meiju Zhan, my sister Min Yin

To my wife Jia Wu

Table of Contents

Chapter 1

Introduction

In this thesis, we study elliptic partial differential equations arising from the study of composite materials, particularly, the stiff fiber-reinforced composites. We are interested in the stress intensity inside the composites since it provides important information for the damage analysis of the fiber composites. Different mathematical models are developed to deal with these problems. Here we introduce two different models and derive the corresponding partial differential equations which we will study in the following sections.

The first and the simplest model that we consider in the study of composite materials is the anti-plane shear model. Generally It asserts that the strain is achieved when the displacements in the material are zero in the plane of interest, but nonzero in the direction perpendicular to the plane. In the equilibrium case, the anti-plane displacement satisfies the partial differential equation for the conductivity problems. In the following we give a brief introduction of the conductivity problems and the extreme cases that we studied.

Let Ω be a domain in $\mathbb{R}^n$ with $C^{2,\alpha}$ boundary, $n \geq 2$, $0 < \alpha < 1$. Let $\{D_i\}$ $(1 \leq i \leq m)$ be m strictly convex open subsets in Ω with $C^{2,\alpha}$ boundaries, $m \geq 2$, satisfying

$$\begin{aligned} &\text{the principal curvature of } \partial D_i \geq \kappa_0, \\ &\varepsilon_{ij} := \operatorname{dist}(D_i, D_j) > 0, \quad (i \neq j) \\ &\operatorname{dist}(D_i, \partial\Omega) > r_0, \quad \operatorname{diam}(\Omega) < \frac{1}{r_0}, \end{aligned} \tag{1.1}$$

where $\kappa_0, r_0 > 0$ are universal constants independent of ε_{ij}. We also assume that the $C^{2,\alpha}$ norms of ∂D_i are bounded by some universal constant independent of ε_{ij}. This implies $\operatorname{diam}(D_i) \geq r_0^*$ for some universal constant $r_0^* > 0$ independent of ε_{ij}.

We state more precisely what it means by saying that the boundary of a domain, say Ω, is $C^{2,\alpha}$ for $0 < \alpha < 1$: In a neighborhood of every point of $\partial\Omega$, $\partial\Omega$ is the graph of some $C^{2,\alpha}$ function of $n-1$ variables. We define the $C^{2,\alpha}$ norm of $\partial\Omega$, denoted by $\|\partial\Omega\|_{C^{2,\alpha}}$, as the smallest positive number $\frac{1}{a}$ such that in the $2a-$neighborhood of every point of $\partial\Omega$, identified as 0 after a possible translation and rotation of the coordinates so that $x_n = 0$ is the tangent to $\partial\Omega$ at 0, $\partial\Omega$ is given by the graph of a $C^{2,\alpha}$ function, denoted as f, which is defined as $|x'| < a$, the $a-$neighborhood of 0 in the tangent plane. Moreover, $\|f\|_{C^{2,\alpha}(|x'|<a)} \leq \frac{1}{a}$.

Denote

$$\widetilde{\Omega} := \Omega \backslash \overline{\cup_{i=1}^m D_i}.$$

Given $\varphi \in C^{1,\alpha}(\partial\Omega)$, the conductivity problem can be modelled by the following equation:

$$\begin{cases} div(a_k(x)\nabla u_k) = 0 & \text{in } \Omega, \\ u_k = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $k = (k_1, \ldots, k_m)$ and

$$a_k(x) = \begin{cases} k_i \in (0, \infty) & \text{in } D_i, \\ 1 & \text{in } \widetilde{\Omega}. \end{cases} \tag{1.3}$$

It is well known that there exists a unique solution $u_k \in H^1(\Omega)$ of the above equation, which is also the minimizer of I_k on $H^1_\varphi(\Omega)$, where

$$H^1_\varphi(\Omega) := \{u \in H^1(\Omega) \mid u = \varphi \ on \ \partial\Omega\}, \qquad I_k[v] := \frac{1}{2}\int_\Omega a_k |\nabla v|^2.$$

In the context of composite materials, the domain Ω here would represent the cross-section of a fiber-reinforced composite, D_i $(0 \leq i \leq m)$ would represent the cross-sections of the fibers, $\widetilde{\Omega}$ would represent the matrix surrounding the fibers, and the shear modulus of the fibers D_i would be k_i and that of the matrix $\widetilde{\Omega}$ would be 1. Equation (3.1) is then obtained by using a standard model of anti-plane shear, and the solution u_k represents the out of plane elastic displacement. The most important quantities from an engineering point of view are the stresses, in this case represented by ∇u_k.

It is well known that the solution u_k satisfies $\|u_k\|_{C^{2,\alpha}(D_i)} < \infty$. In fact, if ∂D_i $(0 \leq i \leq m)$ are $C^{m,\alpha}$, we have $\|u_k\|_{C^{m,\alpha}(D_i)} < \infty$. Such results do not require D_i to be convex and hold for general elliptic systems with piecewise smooth coefficients; see e.g. theorem 9.1 in [18] and proposition 1.6 in [17]. For a fixed $0 < k < \infty$, the $C^{m,\alpha}(D_i)$-norm of the solution might tend to infinity as $\varepsilon_{ij} \to 0$. Babuska, Anderson, Smith and Levin [4] were interested in linear elliptic systems of elasticity arising from the study of composite material. They observed numerically that, for solution u to certain homogeneous isotropic linear systems of elasticity, $\|\nabla u\|_{L^\infty}$ is bounded independently of the distance ε_{ij} between D_i and D_j. Bonnetier and Vogelius [8] proved this in dimension $n = 2$ for the solution u_k of (3.1) in the limit case when two unit balls are touching at a point. This result was extended by Li and Vogelius in [18] to general second order elliptic equations with piecewise smooth coefficients, where stronger $C^{1,\beta}$ estimates were established. The $C^{1,\beta}$ estimates were further extended by Li and Nirenberg in [17] to general second order elliptic systems including systems of elasticity. For higher derivative estimates, e.g. an ε-independent L^∞-estimate of second derivatives of u_k in D_1, we draw attention of readers to the open problem on page 894 of [17]. In [18] and [17], the ellipticity constants are assumed to be away from 0 and ∞. If we allow ellipticity constants to deteriorate, i.e. $k = \infty$ or $k = 0$, the situation is different. In these two extreme cases of the conductivity problems, the electric field, which is represented by the gradient of the solutions, may blow up as the inclusions approach to each other, the blow-up rates of the electric field have been studied in [2, 3, 6, 19, 24, 25, 26].

In particular, when there are only two strictly convex inclusions, and let ϵ be the distance between the two inclusions, it was proved by Ammari, Kang and Lim in [3] and Ammari, Kang, H. Lee, J. Lee and Lim in [2] that, when D_1 and D_2 are balls of comparable radii embedded in $\Omega = \mathbb{R}^2$, the blow-up rate of the gradient of the solution to the perfect and the insulated conductivity problem is $\varepsilon^{-1/2}$ as ε goes to zero; with the lower bound given in [3] and the upper bound given in [2]. Yun in [24] generalized the above mentioned result in [3] by establishing the same lower bound, $\varepsilon^{-1/2}$, for two strictly convex subdomains in $\mathbb{R}^2$. Note that [3] and [2] contain also results for $k < \infty$.

In [6], we give both lower and upper bounds to blow-up rate of the gradient for the solution to the perfect conductivity problem in a bounded matrix, where two strictly convex subdomains are embedded. Our methods apply to dimension $n \geq 3$ as well. One might reasonably suspect that the blow-up rate in dimension $n \geq 3$ should be smaller than that in dimension $n = 2$. However we prove the opposite: As ε goes to zero, the blow-up rate is $\varepsilon^{-1/2}$, $(\varepsilon|\ln\varepsilon|)^{-1}$ and ε^{-1} for $n = 2,\ 3$ and $n \geq 4$, respectively. We also give a criteria, in terms of a linear functional of the boundary data φ, for the situation where the rate of blow-up is realized. Later in [7], we generate the results in [6] for the perfect conductivity problems in the presence of multiple closely spaced inclusions in a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$. We also establish an upper bound on the gradients for the insulated conductivity problems. More reecntly, Lim and Yun in [19] obtained further estimates with explicit dependence of the blow-up rates on the size of some inclusions for the perfect conductivity problem (see also [2] for results of this type).

Next, we consider the linearized elastic model in the study of composite materials. Linear elasticity is widely used in structural analysis and engineering design of composite materials, for details see [16, 22] and the references therein. In this model, the displacement at each point inside the material is a three dimensional vector which satisfies a system of partial differential equations.

Let Ω be a domain in $\mathbb{R}^n$, $\varphi \in H^1(\Omega)$, then the system of linear elasticity is as follows.

$$\begin{cases} \dfrac{\partial}{\partial x_h}(A_{ij}^{hk}\dfrac{\partial u_j}{\partial x_k}) = 0 & \text{in } \Omega, \\ u = \varphi. & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

Where the coefficients $A_{ij}^{hk} \in L^\infty(\Omega)$ satisfy the following condition

$$\begin{aligned} &A_{ij}^{hk} = A_{ji}^{kh} = A_{hj}^{ik}, \\ &\kappa_1\eta_{ih}\eta_{ih} \leq A_{ij}^{hk}\eta_{ih}\eta_{jk} \leq \kappa_2\eta_{ih}\eta_{ih}. \end{aligned} \tag{1.5}$$

It is well known that this system has a unique weak solution $u = (u_1, u_2, \ldots, u_n) \in H^1(\Omega)$.

The stress tensor σ_i^h and the strain tensor ϵ_j^k are defined by the following equations

$$\epsilon_j^k = \frac{1}{2}(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}); \qquad \sigma_i^h = A_{ij}^{hk}\epsilon_j^k \tag{1.6}$$

The conormal derivative is defined as follows

$$\frac{\partial u}{\partial \nu} = (A_{ij}^{hk}\frac{\partial u_j}{\partial x_k})N_h \tag{1.7}$$

where $N = (N_1, N_2, \cdots, N_n)$ is the outer normal unit vector on $\partial\Omega$

In the classical theory of linear elasticity for a homogeneous isotropic body, the coefficients are given by the following formula

$$A_{ij}^{hk} = \lambda\delta_{ih}\delta_{jk} + \mu(\delta_{ij}\delta_{hk} + \delta_{ik}\delta_{hj})$$

where λ is the first Lamé's parameter and μ is the shear modulus, with $2\mu \geq \kappa_1$ and $2\mu + n\lambda \leq \kappa_2$ to satisfy the ellipticity in (1.5).

Let D_1 and D_2 be two subdomains of Ω, denote

$$\widetilde{\Omega} := \Omega \setminus \overline{D_1 \cup D_2}$$

Suppose the Lamé pairs in $D_1 \cup D_2$ and $\widetilde{\Omega}$ are $(\widetilde{\lambda}, \widetilde{\mu})$ and (λ, μ) respectively, namely, the system coefficients are

$$A_{ij}^{hk} = (\lambda\chi_{\widetilde{\Omega}} + \widetilde{\lambda}\chi_{D_1\cup D_2})\delta_{ih}\delta_{jk} + (\mu\chi_{\widetilde{\Omega}} + \widetilde{\mu}\chi_{D_1\cup D_2})(\delta_{ij}\delta_{hk} + \delta_{ik}\delta_{hj})$$

Denote

$$\mathcal{L}_{\lambda,\mu}u := \mu\Delta u + (\lambda + \mu)\nabla(\nabla \cdot u).$$

Then system (1.4) can be written as the following

$$\begin{cases} \mathcal{L}_{\lambda,\mu}u = 0 & \text{in } \ \widetilde{\Omega} \\ \mathcal{L}_{\widetilde{\lambda},\widetilde{\mu}}u = 0 & \text{in } \ D_1 \cup D_2 \\ u|_+ = u|_- & \text{on } \ \partial D_1 \cup \partial D_2, \\ \frac{\partial u}{\partial \nu}|_+ = \frac{\partial u}{\partial \nu}|_- & \text{on } \ \partial D_1 \cup \partial D_2, \\ u = \varphi & \text{on } \ \partial\Omega. \end{cases} \tag{1.8}$$

where the subscript $\pm$ indicates the limit from outside and inside the domain, respectively.

By the above equation (1.7), the conormal derivative is

$$\frac{\partial u}{\partial \nu}|_+ = \lambda(\nabla \cdot u)N + \mu(\nabla u + \nabla u^T)N, \ \ \frac{\partial u}{\partial \nu}|_- = \widetilde{\lambda}(\nabla \cdot u)N + \widetilde{\mu}(\nabla u + \nabla u^T)N$$

It has been proved in [17] that when $0 < \lambda, \widetilde{\lambda} < \infty$ and $0 < \mu, \widetilde{\mu} < \infty$, the stress and strain are bounded independent of the distance ε between the two inclusions D_1 and D_2. Actually among others the $C^{1,\alpha}$ estimate is established in [17] independent of ε for general elliptic systems. But when the shear modulus $\widetilde{\mu} = \infty$ or $\widetilde{\mu} = 0$ in D_1 and D_2, the stress and strain may blow up as these two inclusions approach to each other. Based on the ideas we use for the conductivity problems, We are expecting to find the blow-up rates of the stress and strain for systems of linear elasticity as well.

We mainly focus on the systems of linear elasticity with extreme shear moduli in the fibers of the composite materials. As the first step, stimulating from [1], we derive the gradient estimates for the systems of linear elasticity with special boundary values on the closely spaced inclusions. Our methods are mainly L^2 estimates for elliptic systems. But we haven't achieved much for the systems of linear elasticity, as we will see in Chapter 4, the main problem still remains open.

This thesis is organized as follows. In Chapter 2, we study the perfect conductivity problems with two inclusions. In Chapter 3, we extend our results into multiple inclusions and we also study the insulated conductivity problems. In Chapter 4, we consider the elliptic systems and obtain some partial results.

Chapter 2

The perfect conductivity problems with two inclusions

In this chapter, we consider the perfect conductivity problems with only two inclusions. The results are from our paper [6].

2.1 Mathematical set-up and the main results

Let Ω be a bounded open set in $\mathbb{R}^n$ with $C^{2,\alpha}$ boundary, $n \geq 2$, $0 < \alpha < 1$, D_1 and D_2 be two bounded strictly convex open subsets in Ω with $C^{2,\alpha}$ boundaries satisfying the conditions in (1.1). Given $\varphi \in C^2(\partial\Omega)$, the perfect conductivity problem can be described as follows:

$$\begin{cases} \Delta u = 0 & in \;\; \widetilde{\Omega}, \\ u|_+ = u|_- & on \;\; \partial D_1 \cup \partial D_2, \\ \nabla u \equiv 0 & in \;\; D_1 \cup D_2, \\ \displaystyle\int_{\partial D_i} \frac{\partial u}{\partial \nu}\Big|_+ = 0 & (i = 1, 2), \\ u = \varphi & on \;\; \partial\Omega. \end{cases} \tag{2.1}$$

where

$$\frac{\partial u}{\partial \nu}\Big|_+ := \lim_{t \to 0^+} \frac{u(x + t\nu) - u(x)}{t}.$$

Here and throughout this paper ν is the outward unit normal to the domain and the subscript $\pm$ indicates the limit from outside and inside the domain, respectively.

The existence and uniqueness of solutions to equation (2.1) are well known, see the Appendix. Moreover, the solution $u \in H^1(\Omega)$ is the weak limit of the solutions u_k to equations (3.1) as $k \to +\infty$. It can be also described as the unique function which has the " least energy" in appropriate functional space, defined as $I_\infty[u] = \min_{v \in \mathcal{A}} I_\infty[v]$, where

$$I_\infty[v] := \frac{1}{2}\int_{\widetilde{\Omega}} |\nabla v|^2, \qquad v \in \mathcal{A},$$

$$\mathcal{A} := \left\{ v \in H^1_\varphi(\Omega) \middle| \nabla v \equiv 0 \ in \ D_1 \cup D_2 \right\}.$$

The readers can refer to the Appendix for the proofs of the above statements.

Denote

$$\rho_n(\varepsilon) = \begin{cases} \dfrac{1}{\sqrt{\varepsilon}} & for \ \ n = 2, \\ \dfrac{1}{\varepsilon|\ln \varepsilon|} & for \ \ n = 3, \\ \dfrac{1}{\varepsilon} & for \ \ n \geq 4. \end{cases} \tag{2.2}$$

Then we have the following gradient estimates for the perfect conductivity problem

Theorem 2.1.1. *Let $\Omega, D_1, D_2 \subset \mathbb{R}^n$, ε be defined as in (1.1), $\varphi \in C^2(\partial\Omega)$. Let $u \in H^1(\Omega) \cap C^1(\overline{\widetilde{\Omega}})$ be the solution to equation (2.1). For ε sufficiently small, there is a positive constant C which depends only on n, κ_0, r_0, $\|\partial\Omega\|_{C^{2,\alpha}}$, $\|\partial D_1\|_{C^{2,\alpha}}$ and $\|\partial D_2\|_{C^{2,\alpha}}$, but independent of ε such that*

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega})} \leq C\rho_n(\varepsilon) \tag{2.3}$$

Remark 2.1.1. *We draw attention of readers to the independent work of Yun [25] where he has also established the upper bound, $\varepsilon^{-1/2}$, in $\mathbb{R}^2$. The methods are very different. Results in this paper and those in [24] and [25] do not really need D_1 and D_2 to be strictly convex, the strict convexity is only needed for the portions in a fixed neighborhood (the size of the neighborhood is indepedent of ε) of a pair of points on ∂D_1 and ∂D_2 which realize minimal distance ε.*

To prove Theorem 2.1.1, we first decompose the solution u of equation (2.1) as follows:

$$u = C_1 v_1 + C_2 v_2 + v_3 \tag{2.4}$$

where $C_i := C_i(\varepsilon)$ $(i = 1, 2)$ be the boundary value of u on ∂D_i $(i = 1, 2)$ respectively, and $v_i \in C^2(\overline{\widetilde{\Omega}})$ $(i = 1, 2, 3)$ satisfies

$$\begin{cases} \Delta v_1 = 0 & in \ \widetilde{\Omega}, \\ v_1 = 1 \ \ on \ \partial D_1, \quad v_1 = 0 \ \ on \ \partial D_2 \cup \partial\Omega, \end{cases} \tag{2.5}$$

$$\begin{cases} \Delta v_2 = 0 & in\ \widetilde{\Omega}, \\ v_2 = 1 \ \ on\ \partial D_2, \quad v_2 = 0 \ \ on\ \partial D_1 \cup \partial\Omega, \end{cases} \tag{2.6}$$

$$\begin{cases} \Delta v_3 = 0 & in\ \widetilde{\Omega}, \\ v_3 = 0 \ \ on\ \partial D_1 \cup \partial D_2, \quad v_3 = \varphi \ \ on\ \partial\Omega. \end{cases} \tag{2.7}$$

Define

$$Q_\varepsilon[\varphi] := \int_{\partial D_1} \frac{\partial v_3}{\partial \nu} \int_{\partial\Omega} \frac{\partial v_2}{\partial \nu} - \int_{\partial D_2} \frac{\partial v_3}{\partial \nu} \int_{\partial\Omega} \frac{\partial v_1}{\partial \nu}, \tag{2.8}$$

then $Q_\varepsilon : C^2(\partial\Omega) \to \mathbb{R}$ is a linear functional.

Theorem 2.1.2. *With the same conditions in Theorem 2.1.1, let $u \in H^1(\Omega) \cap C^1(\overline{\widetilde{\Omega}})$ be the solution to equation (2.1). For ε sufficiently small, there exists a positive constant C which depends on n, κ_0, r_0, $\|\partial\Omega\|_{C^{2,\alpha}}$, $\|\partial D_1\|_{C^{2,\alpha}}$, $\|\partial D_2\|_{C^{2,\alpha}}$ and $\|\varphi\|_{C^2(\partial\Omega)}$, but is independent of ε such that*

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega})} \geq \frac{|Q_\varepsilon[\varphi]|}{C} \rho_n(\varepsilon) \tag{2.9}$$

Remark 2.1.2. *If $\varphi \equiv 0$, then the solution to equation (2.1) is $u \equiv 0$. Theorem 2.1.1 and Theorem 2.1.2 are obvious in this case. So we only need to prove them for $\|\varphi\|_{C^2(\partial\Omega)} = 1$, by considering $u/\|\varphi\|_{C^2(\partial\Omega)}$.*

Remark 2.1.3. *It is interesting to know when $|Q_\varepsilon[\varphi]| \geq \frac{1}{C}$ for some positive constant C independent of ε. Roughly speaking $Q_\varepsilon[\varphi] \to Q^*[\varphi]$ as $\varepsilon \to 0$, and this amounts to $Q^*[\varphi] \neq 0$. For details, see Section 2.*

Remark 2.1.4. *As we mentioned in Remark 3.1.1, the strictly convexity assumption of the two inclusions is not necessary. Indeed, our methods can also apply to more general case with arbitrary shape of the inclusions.*

For instance, in dimension $n = 2$, by a translation and rotation of the axis, without loss of generality we may denote the curve $\partial D_1 \cap B(0,r)$ as $x = f(y) - \frac{\varepsilon}{2}$ and the curve $\partial D_2 \cap B(0,r)$ as $x = g(y) + \frac{\varepsilon}{2}$ where $r \in \mathbb{R}$ is a fixed positive number which is independent of ε and $f(0) = g(0) = 0$, $g'(0) - f'(0) = 0$. Assume further that $g(y) - f(y) > 0$ for $(x,y) \in B(0,r) \setminus (0,0)$, which is equivalent to say

$$g(y) - f(y) = a_0 y^{2k} + \circ(|y|^{2k}), \tag{2.10}$$

for some $a_0 > 0, k \geq 1 \in \mathbb{Z}$.

Under this assumption, in $\mathbb{R}^2$*, let* $u \in H^1(\Omega) \cap C^1(\overline{\widetilde{\Omega}})$ *be the solution to equation (2.1). For* ε *sufficiently small, there exist positive constants* C *and* C' *where* C *depends on* n*,* a_0*,* r_0*,* $\|\partial\Omega\|_{C^{2,\alpha}}$*,* $\|\partial D_1\|_{C^{2,\alpha}}$ *and* $\|\partial D_2\|_{C^{2,\alpha}}$*,* C' *depends on the same as* C *and also* $\|\varphi\|_{C^2(\partial\Omega)}$*, but both are independent of* ε *such that*

$$\frac{|Q_\varepsilon[\varphi]|}{C'}\varepsilon^{-1/2k} \leq \|\nabla u\|_{L^\infty(\widetilde{\Omega})} \leq C\|\varphi\|_{C^2(\partial\Omega)}\varepsilon^{-1/2k}. \tag{2.11}$$

where k *is the smallest integer such that* $a_0 := (g-f)^{(2k)}\big|_{y=0} > 0$ *and* $Q_\varepsilon[\varphi]$ *is defined by (2.8).*

The proof is essentially the same except for the computation of $\int_{\widetilde{\Omega}} |\nabla v_1|^2$ *which should be* $\varepsilon^{-1+1/2k}$ *instead of* $\varepsilon^{-1/2}$ *(see Section 1.2).*

Theorem 2.1.1−2.1.2 can be extended to equations with more general coefficients as follows: Let n, Ω, D_1, D_2, ε and φ be same as in Theorem 2.1.1, and let

$$A_2(x) := \left(a_2^{ij}(x)\right) \in C^2(\overline{\widetilde{\Omega}})$$

be $n \times n$ symmetric matrix functions in $\widetilde{\Omega}$ satisfying for some constants $0 < \lambda \leq \Lambda < \infty$,

$$\lambda|\xi|^2 \leq a_2^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \qquad \forall x \in \widetilde{\Omega},\ \forall \xi \in \mathbb{R}^n,$$

and $a_2^{ij}(x) \in C^2(\overline{\Omega\backslash\omega})$.

We consider

$$\begin{cases} \partial_{x_j}\left(a_2^{ij}(x)\ \partial_{x_i}u\right) = 0 & in\ \widetilde{\Omega}, \\ u|_+ = u|_- & on\ \partial D_1 \cup \partial D_2, \\ \nabla u = 0 & in\ D_1 \cup D_2, \\ \displaystyle\int_{\partial D_i} a_2^{ij}(x)\partial_{x_i}u\nu_j\big|_+ = 0 & (i = 1, 2), \\ u = \varphi & on\ \partial\Omega. \end{cases} \tag{2.12}$$

where repeated indices denote as usual summations.

Here is an extension of Theorem 2.1.1:

Theorem 2.1.3. *With the above assumptions, let $u \in H^1(\Omega)\cap C^1(\overline{\widetilde{\Omega}})$ be the solution to equation (2.12). For ε sufficient small, there is a positive constant C which depends only on n, κ_0, r_0, $\|\partial\Omega\|_{C^{2,\alpha}}$, $\|\partial D_1\|_{C^{2,\alpha}}$, $\|\partial D_2\|_{C^{2,\alpha}}$, λ, Λ and $\|A_2\|_{C^2(\overline{\widetilde{\Omega}})}$, but independent of ε such that estimate (2.3) holds.*

Similar to the decomposition formula (2.4), we decompose the solution u of equation (2.12) as follows:

$$u = C_1V_1 + C_2V_2 + V_3 \tag{2.13}$$

where $C_i := C_i(\varepsilon)$ $(i = 1, 2)$ be the boundary value of u on ∂D_i $(i = 1, 2)$ respectively, and $V_i \in C^2(\overline{\widetilde{\Omega}})$ $(i = 1, 2, 3)$ satisfies

$$\begin{cases} \partial_{x_j}\left(a_2^{ij}(x)\ \partial_{x_i}V_1\right) = 0 \qquad in\ \widetilde{\Omega}, \\ V_1 = 1\ \ on\ \partial D_1, \quad V_1 = 0\ \ on\ \partial D_2 \cup \partial\Omega, \end{cases} \tag{2.14}$$

$$\begin{cases} \partial_{x_j}\left(a_2^{ij}(x)\ \partial_{x_i}V_2\right) = 0 \qquad in\ \widetilde{\Omega}, \\ V_2 = 1\ \ on\ \partial D_2, \quad V_2 = 0\ \ on\ \partial D_1 \cup \partial\Omega, \end{cases} \tag{2.15}$$

$$\begin{cases} \partial_{x_j}\left(a_2^{ij}(x)\ \partial_{x_i}V_3\right) = 0 \qquad in\ \widetilde{\Omega}, \\ V_3 = 0\ \ on\ \partial D_1 \cup \partial D_2, \quad V_3 = \varphi\ \ on\ \partial\Omega. \end{cases} \tag{2.16}$$

Define

$$\begin{aligned} Q_\varepsilon[\varphi] := &\int_{\partial D_1} a_2^{ij}(x)\ \partial_{x_i}V_3\ \nu_j \int_{\partial\Omega} a_2^{ij}(x)\ \partial_{x_i}V_2\ \nu_j \\ &- \int_{\partial D_2} a_2^{ij}(x)\ \partial_{x_i}V_3\ \nu_j \int_{\partial\Omega} a_2^{ij}(x)\ \partial_{x_i}V_1\ \nu_j, \end{aligned} \tag{2.17}$$

then $Q_\varepsilon : C^2(\partial\Omega) \to \mathbb{R}$ is a linear functional.

Theorem 2.1.4. *With the same conditions in Theorem 2.1.3, let $u \in H^1(\Omega) \cap C^1(\overline{\widetilde{\Omega}})$ be the solution to equation (2.12). For ε sufficiently small and $Q_\varepsilon[\varphi]$ defined by (2.17), there is a positive constant C which depends only on n, κ_0, r_0, $\|\partial D_1\|_{C^{2,\alpha}}$, $\|\partial D_2\|_{C^{2,\alpha}}$, λ, Λ and $\|A_2\|_{C^2(\overline{\widetilde{\Omega}})}$, but independent of ε such that estimate (2.9) holds.*

2.2 Proof of Theorem 2.1.1 and 2.1.2

As in the above section, we write $u = C_1v_1 + C_2v_2 + v_3$ as in (2.4). To prove our main theorems, we first estimate $\|\nabla u\|_{L^\infty(\widetilde{\Omega})}$ in terms of $|C_1-C_2|$, and then estimate $|C_1-C_2|$.

In this section we use, unless otherwise stated, C to denote various positive constants whose values may change from line to line and which depend only on n, κ_0, r_0, $\|\partial\Omega\|_{C^{2,\alpha}}$, $\|\partial D_1\|_{C^{2,\alpha}}$ and $\|\partial D_2\|_{C^{2,\alpha}}$.

Proposition 2.2.1. *Under the hypotheses of Theorem 2.1.1, let u be the solution of equation (2.1). There exists a positive constants C, such that, for sufficiently small $\varepsilon > 0$,*

$$\frac{1}{\varepsilon} \mid C_1 - C_2 \mid \leq \|\nabla u\|_{L^\infty(\widetilde{\Omega})} \leq \frac{C}{\varepsilon} \mid C_1 - C_2 \mid + C\|\varphi\|_{C^2(\partial\Omega)}. \tag{2.18}$$

To prove this proposition, we first estimate the gradients of v_1, v_2 and v_3. Without loss of generality, we may assume throughout the proof of the proposition that $\|\varphi\|_{C^2(\partial\Omega)} = 1$; see Remark 2.1.2.

Lemma 2.2.1. *Let v_1, v_2 be defined by equations (2.5) and (2.6), then for $n \geq 2$, we have*

$$\|\nabla v_1\|_{L^\infty(\widetilde{\Omega})} + \|\nabla v_2\|_{L^\infty(\widetilde{\Omega})} \leq \frac{C}{\varepsilon}, \quad \|\frac{\partial v_1}{\partial \nu}\|_{L^\infty(\partial\Omega)} + \|\frac{\partial v_2}{\partial \nu}\|_{L^\infty(\partial\Omega)} \leq C.$$

Proof: By the maximum principle, $\|v_1\|_{L^\infty(\widetilde{\Omega})} \leq 1$, and since v_1 achieves constants on each connected component of $\partial\widetilde{\Omega}$, and each connected component of $\partial\widetilde{\Omega}$ is $C^{2,\alpha}$ then the gradient estimates for harmonic functions implies that

$$\|\nabla v_1\|_{L^\infty(\widetilde{\Omega})} \leq \frac{C\|v_1\|_{L^\infty}}{\operatorname{dist}(\partial D_1, \partial D_2)} = \frac{C}{\varepsilon}.$$

Similarly, we can prove $\|\nabla v_2\|_{L^\infty(\widetilde{\Omega})} \leq C/\varepsilon$. The second inequality follows from the boundary estimates for harmonic functions. □

Before estimating $|\nabla v_3|$, we first prove:

Lemma 2.2.2. *Let $\rho \in C^2(\widetilde{\Omega})$ be the solution to:*

$$\begin{cases} \Delta\rho = 0 & \text{in } \widetilde{\Omega}, \\ \rho = 0 \quad \text{on } \partial D_1 \cup \partial D_2, & \rho = 1 \quad \text{on } \partial\Omega. \end{cases} \tag{2.19}$$

Then $\|\nabla\rho\|_{L^\infty(\widetilde{\Omega})} \leq C$.

Proof: Let $\rho_i (i = 1, 2) \in C^2(\Omega \backslash \overline{D}_i) \cap C^1(\overline{\Omega \backslash D_i})$ be the solution to:

$$\begin{cases} \Delta \rho_i = 0 & in\ \Omega \backslash \overline{D}_i, \\ \rho_i = 0 \ \ on\ \partial D_i, \quad \rho_i = 1 \ \ on\ \partial\Omega. \end{cases}$$

Again by the maximum principle and the strong maximum principle, we obtain $0 < \rho_1 < 1$ in $\Omega \backslash \overline{D}_1$. Since $\overline{D}_2 \subset \Omega \backslash \overline{D}_1$, we have $\rho_1 > 0 = \rho$ on ∂D_2. And since $\rho_1 = \rho$ on ∂D_1 and $\partial\Omega$, therefore $\rho_1 > \rho$ on $\widetilde{\Omega}$. Now because $\rho_1 = \rho = 0$ on ∂D_1 and $\rho_1 > \rho > 0$ on $\widetilde{\Omega}$, so

$$\|\nabla \rho\|_{L^\infty(\partial D_1)} \le \|\nabla \rho_1\|_{L^\infty(\partial D_1)} \le C.$$

Similarly,

$$\|\nabla \rho\|_{L^\infty(\partial D_2)} \le \|\nabla \rho_2\|_{L^\infty(\partial D_2)} \le C.$$

By the boundary estimate of harmonic functions, we know that $\|\nabla \rho\|_{L^\infty(\partial\Omega)} \le C$. Since $\Delta \rho = 0$ in $\widetilde{\Omega}$, $\partial_{x_i} \rho$ is also harmonic, by the maximum principle,

$$\|\nabla \rho\|_{L^\infty(\widetilde{\Omega})} \le \max\Big(\|\nabla \rho\|_{L^\infty(\partial D_1)}, \|\nabla \rho\|_{L^\infty(\partial D_2)}, \|\nabla \rho\|_{L^\infty(\partial\Omega)} \Big) \le C.$$

$\square$

Now, we estimate $|\nabla v_3|$:

Lemma 2.2.3. *Let v_3 be defined by equation (2.7), for $n \ge 2$, we have*

$$\|\nabla v_3\|_{L^\infty(\widetilde{\Omega})} \le C.$$

Proof: Since $v_3 = -\rho = \rho = 0$ on $\partial D_i (i = 1, 2)$, and $-\rho \le v_3 = \varphi \le \rho$ on $\partial\Omega$, we have, by the maximum principle,

$$-\rho \le v_3 \le \rho \quad \text{in } \widetilde{\Omega}.$$

It follows, for $i = 1, 2$, that

$$\|\nabla v_3\|_{L^\infty(\partial D_i)} \le \|\nabla \rho\|_{L^\infty(\partial D_i)} \le C.$$

By the boundary estimate,

$$\|\nabla v_3\|_{L^\infty(\partial\Omega)} \le C.$$

By the harmonicity of $\partial_{x_i} v_3$ and the maximum principle,

$$\|\nabla v_3\|_{L^\infty(\widetilde{\Omega})} \leq C.$$

$\square$

Remark 2.2.1. *Without assuming* $\|\varphi\|_{C^2(\partial\Omega)} = 1$*, we have*

$$\|\nabla v_3\|_{L^\infty(\partial D_1 \cup \partial D_2)} \leq C\|\varphi\|_{L^\infty(\partial\Omega)},$$

where C *has the dependence specified at the beginning of this section, except that it does not depend on* $\|\partial\Omega\|_{C^{2,\alpha}}$*. This is easy to see from the proof of Lemma 2.2.3.*

The above lemma yields the main result of [1].

Corollary 2.2.1. ([1]) *Let* B_1 *and* B_2 *be two spheres with radius* R *and centered at* $(\pm R \pm \frac{\varepsilon}{2}, 0, \cdots, 0)$*, respectively. Let* H *be a harmonic function in* $\mathbb{R}^3$*. Define* u *to be the solution to*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^3 \backslash \overline{B_1 \cup B_2}, \\ u = 0 & \text{on } \partial B_1 \cup \partial B_2, \\ u(x) - H(x) = O(|x|^{-1}) & \text{as } |x| \to +\infty. \end{cases}$$

Then there is a constant C *independent of* ε *such that*

$$\|\nabla(u - H)\|_{L^\infty(\mathbb{R}^3 \backslash \overline{B_1 \cup B_2})} \leq C.$$

Proof: By the maximum principle and interior estimates of harmonic functions, the C^3 norm of $u|_{B_{2R}(0)}$ is bounded by a constant independent of ε. Apply Lemma 2.2.3 with $\Omega = B_{2R}(0)$ and $\varphi = u|_{B_{2R}(0)}$, we immediately obtain the above corollary. $\square$

With the above lemmas, we give the

Proof of Proposition 2.2.1: $\text{dist}(\partial D_1, \partial D_2) = \varepsilon$, by the mean value theorem, $\exists\ \xi \in \widetilde{\Omega}$ such that

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega})} \geq |\nabla u(\xi)| \geq \frac{|C_1 - C_2|}{\varepsilon}.$$

By the decomposition formula (2.4),

$$\nabla u = C_1 \nabla v_1 + C_2 \nabla v_2 + \nabla v_3 = (C_1 - C_2)\nabla v_1 + C_2 \nabla (v_1 + v_2) + \nabla v_3.$$

Hence,

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega})} \le |C_1 - C_2| \|\nabla v_1\|_{L^\infty(\widetilde{\Omega})} + |C_2| \|\nabla (v_1 + v_2)\|_{L^\infty(\widetilde{\Omega})} + \|\nabla v_3\|_{L^\infty(\widetilde{\Omega})}.$$

By Lemma 2.2.2, since $v_1 + v_2 = 1 - \rho$ in $\widetilde{\Omega}$, we have

$$\|\nabla (v_1 + v_2)\|_{L^\infty(\widetilde{\Omega})} = \|\nabla (1 - \rho)\|_{L^\infty(\widetilde{\Omega})} = \|\nabla \rho\|_{L^\infty(\widetilde{\Omega})} \le C.$$

Using the fact we showed in the Appendix, $\|u\|_{H^1(\Omega)} \le C$, so $|C_1| + |C_2| \le C$. Therefore using also Lemma 2.2.1 we obtain,

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega})} \le \frac{C}{\varepsilon} \mid C_1 - C_2 \mid + C.$$

This proof is now completed. □

Later we will give an estimate of $|C_1 - C_2|$, which, together with Proposition 2.2.1, yields the lower and upper bounds of $\|\nabla u\|_{L^\infty(\widetilde{\Omega})}$ for strictly convex subdomains D_1 and D_2.

2.2.1 Estimate of $|C_1 - C_2|$

Back to the decomposition formula (2.4), denote

$$a_{ij} = \int_{\partial D_i} \frac{\partial v_j}{\partial \nu} \quad (i, j = 1, 2), \qquad b_i = \int_{\partial D_i} \frac{\partial v_3}{\partial \nu} \quad (i = 1, 2). \tag{2.20}$$

We first give some basic lemmas:

Lemma 2.2.4. *Let a_{ij} and b_i be defined as in (2.20), then they satisfy the following:*

1. $a_{12} = a_{21} > 0,\ a_{11} < 0,\ a_{22} < 0,$

2. $-C \le a_{11} + a_{21} \le -\frac{1}{C},\ -C \le a_{22} + a_{12} \le -\frac{1}{C},$

3. $|b_1| \leq C, \ |b_2| \leq C.$

By the fourth line of equation (2.1), C_1 and C_2 satisfy

$$\begin{cases} a_{11}C_1 + a_{12}C_2 + b_1 = 0, \\ a_{21}C_1 + a_{22}C_2 + b_2 = 0. \end{cases} \tag{2.21}$$

By solving the above linear system, using $a_{12} = a_{21}$ and $a_{11}a_{22} - a_{12}a_{21} > 0$ which follows from Lemma 2.2.4, we obtain

$$C_1 = \frac{-b_1 a_{22} + b_2 a_{12}}{a_{11}a_{22} - a_{12}^2}, \qquad C_2 = \frac{-b_2 a_{11} + b_1 a_{12}}{a_{11}a_{22} - a_{12}^2}, \tag{2.22}$$

and therefore,

$$|C_1 - C_2| = \frac{|b_1 - \alpha b_2|}{|a_{11} - \alpha a_{12}|}, \qquad \text{where} \ \ \alpha = \frac{a_{11} + a_{12}}{a_{22} + a_{12}} > 0. \tag{2.23}$$

Based on this formula, we will give the estimates for $|a_{11} - \alpha a_{12}|$ and $|b_1 - \alpha b_2|$, then the estimate for $|C_1 - C_2|$ follows immediately.

Proof of Lemma 2.2.4: (1) By the maximum principle and the strong maximum principle,

$$0 < v_1 < 1 \quad \text{in } \widetilde{\Omega}.$$

By the Hopf Lemma, we know that

$$\frac{\partial v_1}{\partial \nu}\Big|_{\partial D_1} < 0, \quad \frac{\partial v_1}{\partial \nu}\Big|_{\partial D_2} > 0, \quad \frac{\partial v_1}{\partial \nu}\Big|_{\partial \Omega} < 0.$$

Similarly,

$$\frac{\partial v_2}{\partial \nu}\Big|_{\partial D_1} > 0, \quad \frac{\partial v_2}{\partial \nu}\Big|_{\partial D_2} < 0, \quad \frac{\partial v_2}{\partial \nu}\Big|_{\partial \Omega} < 0.$$

Thus $a_{11} < 0$, $a_{12} > 0$, $a_{21} > 0$ and $a_{22} < 0$.

Also, since v_1 and v_2 are the solutions of equations (2.5) and equations (2.6), respectively, we have

$$\begin{aligned} 0 &= \int_{\widetilde{\Omega}} \Delta v_1 \cdot v_2 - \int_{\widetilde{\Omega}} \Delta v_2 \cdot v_1 = -\int_{\partial D_2} \frac{\partial v_1}{\partial \nu} \cdot 1 + \int_{\partial D_1} \frac{\partial v_2}{\partial \nu} \cdot 1 \\ &= -a_{21} + a_{12}, \end{aligned} \tag{2.24}$$

i.e. $a_{21} = a_{12}$.

(2) We will prove the first inequality, the second one stands with the same reason. By the harmonicity of v_1 in $\widetilde{\Omega}$,

$$a_{11} + a_{21} = -\int_{\widetilde{\Omega}} \Delta v_1 + \int_{\partial\Omega} \frac{\partial v_1}{\partial \nu} = \int_{\partial\Omega} \frac{\partial v_1}{\partial \nu} < 0.$$

By Lemma 2.2.1,

$$a_{11} + a_{21} = \int_{\partial\Omega} \frac{\partial v_1}{\partial \nu} \geq -C.$$

On the other hand, since $0 < v_1 < 1$ *in* $\widetilde{\Omega}$ and $v_1 = 1$ *on* ∂D_1, by the boundary gradient estimates of a harmonic function, $\exists\, B(\bar{x}, 2\bar{r}) \subset \widetilde{\Omega}$, such that $v_1 > 1/2$ in $B(\bar{x}, \bar{r})$, where $\bar{r}$ is independent of ε. Let $\rho \in C^2(\Omega \backslash \overline{D_2 \cup B(\bar{x}, \bar{r})}) \cup C^1(\partial\Omega \cup \partial D_2 \cup \partial B(\bar{x}, \bar{r}))$ be the solution of the following equation:

$$\begin{cases} \Delta \rho = 0 & in\ \Omega \backslash \overline{D_2 \cup B(\bar{x}, \bar{r})}, \\ \rho = 1/2 \quad on\ \partial B(\bar{x}, \bar{r}) & \rho = 0 \quad on\ \partial D_2 \cup \partial\Omega. \end{cases}$$

By the maximum principle and the strong maximum principle, $0 < \rho < 1/2$ in $\Omega \backslash \overline{D_2 \cup B(\bar{x}, \bar{r})}$. A contradiction argument based on the Hopf Lemma yields,

$$-\frac{\partial \rho}{\partial \nu} \geq \frac{1}{C} \qquad \text{on } \partial\Omega.$$

On the other hand, since $\rho \leq v_1$ on the boundary of $\Omega \backslash \overline{D_1 \cup D_2 \cup B(\bar{x}, \bar{r})}$, we obtain, via the maximum principle, $0 < \rho \leq v_1$ in $\Omega \backslash \overline{D_1 \cup D_2 \cup B(\bar{x}, \bar{r})}$. It follows, using $\rho = v_1 = 0$ on $\partial\Omega$, that

$$\frac{\partial v_1}{\partial \nu} \leq \frac{\partial \rho}{\partial \nu} \qquad \text{on } \partial\Omega.$$

Thus,

$$a_{11} + a_{21} = \int_{\partial\Omega} \frac{\partial v_1}{\partial \nu} \leq \int_{\partial\Omega} \frac{\partial \rho}{\partial \nu} \leq -\frac{1}{C}.$$

(3) Clearly,

$$0 = \int_{\widetilde{\Omega}} \Delta v_1 \cdot v_3 - \int_{\widetilde{\Omega}} \Delta v_3 \cdot v_1 = \int_{\partial\Omega} \frac{\partial v_1}{\partial \nu} \cdot \varphi + \int_{\partial D_1} \frac{\partial v_3}{\partial \nu} \cdot 1 = \int_{\partial\Omega} \frac{\partial v_1}{\partial \nu} \cdot \varphi + b_1.$$

Thus,

$$|b_1| = \Big| \int_{\partial\Omega} \frac{\partial v_1}{\partial \nu} \cdot \varphi \Big| \leq \int_{\partial\Omega} \Big| \frac{\partial v_1}{\partial \nu} \Big| \leq C.$$

Thus, we finished the proof. $\square$

2.2.2 Estimate of $|a_{11} - \alpha a_{12}|$

By a translation and rotation of the axis, we may assume without loss of generality that D_1, D_2 are two strictly convex subdomains in $\Omega \subset \mathbb{R}^n$ which satisfy the following:

$$(-\varepsilon/2, 0') \in \partial D_1,\ (\varepsilon/2, 0') \in \partial D_2,\ \varepsilon = \mathrm{dist}(\partial D_1, \partial D_2) = \mathrm{dist}(D_1, D_2). \tag{2.25}$$

Near the origin, we can find a ball $B(0,r)$ such that the portion of ∂D_i $(i = 1, 2)$ in $B(0,r)$ is strictly convex, where $r > 0$ is independent of ε. Then $\partial D_1 \cap B(0,r)$ and $\partial D_2 \cap B(0,r)$ can be represented by the graph of $x_1 = f(x') - \varepsilon/2$ and $x_1 = g(x') + \varepsilon/2$ respectively, where $x' = (x_2, \cdots, x_n)$. Thus $f(0') = g(0') = 0,\ \nabla f(0') = \nabla g(0') = 0,$ and $-CI \le \big(D^2 f(0')\big) \le -\frac{1}{C} I,\ \frac{1}{C} I \le \big(D^2 g(0')\big) \le CI.$

With these notations, we first estimate a_{ii} for $i = 1, 2$.

Lemma 2.2.5. *Let a_{ii} be defined by (2.20), then*

$$\frac{1}{C\sqrt{\varepsilon}} \le -a_{ii} \le \frac{C}{\sqrt{\varepsilon}}, \quad for\ n = 2,\ i = 1, 2.$$

Proof: It suffices to prove it for a_{11}. By the harmonicity of v_1, we have

$$0 = \int_{\widetilde{\Omega}} \Delta v_1 \cdot v_1 = -\int_{\widetilde{\Omega}} |\nabla v_1|^2 - \int_{\partial D_1} \frac{\partial v_1}{\partial \nu} = -\int_{\widetilde{\Omega}} |\nabla v_1|^2 - a_{11},$$

i.e.

$$a_{11} = -\int_{\widetilde{\Omega}} |\nabla v_1|^2.$$

Now we construct a function (here in $\mathbb{R}^2$, we let $x = x_1,\ y = x_2$)

$$\overline{w}(x, y) = -\frac{x - g(y) - \frac{\varepsilon}{2}}{g(y) - f(y) + \varepsilon} \tag{2.26}$$

on $O_r := \widetilde{\Omega} \cap \big\{(x, y) \big|\ |y| < r\big\}$. It is clear that $\overline{w}(x, y)$ is linear in x for fixed y and

$$\overline{w}\ |_{B(0,r)\cap\partial D_1} = 1; \quad \overline{w}\ |_{B(0,r)\cap\partial D_2} = 0,$$

so we have

$$\int_{f(y)-\frac{\varepsilon}{2}}^{g(y)+\frac{\varepsilon}{2}} |\partial_x \overline{w}(x,y)|^2 dx \le \int_{f(y)-\frac{\varepsilon}{2}}^{g(y)+\frac{\varepsilon}{2}} |\partial_x v_1(x,y)|^2 dx,$$

i.e.

$$\frac{1}{g(y)-f(y)+\varepsilon} \le \int_{f(y)-\frac{\varepsilon}{2}}^{g(y)+\frac{\varepsilon}{2}} |\partial_x v_1(x,y)|^2.$$

Integrating on y we get

$$\begin{aligned}\int_0^{r/2}\int_{f(y)-\frac{\varepsilon}{2}}^{g(y)+\frac{\varepsilon}{2}} |\partial_x v_1(x,y)|^2 dxdy &\ge \int_0^{r/2} \frac{1}{g(y)-f(y)+\varepsilon} dy \\ = \frac{1}{C}\int_0^{r/2} \frac{1}{y^2+\varepsilon} dy = \frac{1}{C\sqrt{\varepsilon}}.\end{aligned} \tag{2.27}$$

Thus

$$-a_{11} \ge \int_0^{r/2}\int_{f(y)-\frac{\varepsilon}{2}}^{g(y)+\frac{\varepsilon}{2}} |\partial_x v_1(x,y)|^2 dxdy \ge \frac{1}{C\sqrt{\varepsilon}}.$$

On the other hand, we can find $\psi \in C^2(\overline{\Omega})$ such that

$$\psi = 0 \ on \ \overline{O}_{r/8}, \quad \psi = 1 \ on \ \partial D_1 \backslash (\overline{O_{r/4}}), \quad \psi = 0 \ on \ \partial D_2 \backslash (\overline{O_{r/4}}),$$

$$\psi = 0 \ on \ \partial\Omega, \quad \text{and} \quad \|\nabla \psi\|_{L^\infty(\Omega)} \le C.$$

We can also find $\rho \in C^2(\overline{\Omega})$ such that

$$0 \le \rho \le 1, \ \rho = 1 \ on \ \overline{O}_{r/2}, \ \rho = 0 \ on \ \overline{\Omega}\backslash O_r \text{ and } |\nabla \rho| \le C.$$

Let $w = \rho\overline{w} + (1-\rho)\psi$, then $w = 1 = v_1$ on ∂D_1;$w = 0 = v_1$ on ∂D_2; $w = 0 = v_1$ on $\partial\Omega$ and $w = \overline{w}$ on $\overline{O}_{r/2}$. Then by the properties of ψ, ρ and the harmonicity of v_1, we have

$$\int_{\widetilde{\Omega}} |\nabla v_1|^2 \le \int_{\widetilde{\Omega}} |\nabla w|^2 \le \int_{\widetilde{\Omega}\cap O_{r/2}} |\nabla \overline{w}|^2 + C. \tag{2.28}$$

A calculation gives

$$\partial_y \overline{w} = \frac{g'(y)(g(y)-f(y)+\varepsilon) - (g(y)-x+\frac{\varepsilon}{2})(g'(y)-f'(y))}{(g(y)-f(y)+\varepsilon)^2}.$$

We will show $\int_{\widetilde{\Omega}\cap O_{r/2}} |\partial_y \overline{w}|^2 \le C$.

Indeed,

$$
\begin{aligned}
&\int_0^{r/2}\int_{f(y)-\frac{\varepsilon}{2}}^{g(y)+\frac{\varepsilon}{2}} |\partial_y \overline{w}(x,y)|^2 dxdy \\
&\leq 2\int_0^{r/2}\int_{f(y)-\frac{\varepsilon}{2}}^{g(y)+\frac{\varepsilon}{2}} \left(\frac{g'(y)^2}{(g(y)-f(y)+\varepsilon)^2}+\frac{(g(y)-x+\frac{\varepsilon}{2})^2(g'(y)-f'(y))^2}{(g(y)-f(y)+\varepsilon)^4}\right) dxdy \\
&= 2\int_0^{r/2}\frac{g'(y)^2}{g(y)-f(y)+\varepsilon}dy + 2\int_0^{r/2}\frac{(g'(y)-f'(y))^2}{g(y)-f(y)+\varepsilon}dy \\
&= C\int_0^{r/2}\frac{y^2}{y^2+\varepsilon}dy + C\int_0^{r/2}\frac{y^2}{y^2+\varepsilon}dy \\
&\leq C.
\end{aligned}
\tag{2.29}
$$

Then by (2.28) and (2.29)

$$
\begin{aligned}
|a_{11}| = \int_{\widetilde{\Omega}} |\nabla v_1|^2 &\leq \int_{\widetilde{\Omega}\cap O_{r/2}} |\nabla \overline{w}|^2 + C \\
&\leq C\int_0^{r/2}\int_{f(y)-\frac{\varepsilon}{2}}^{g(y)+\frac{\varepsilon}{2}} |D_x \overline{w}(x,y)|^2 dxdy + C \\
&= C\int_0^{r/2}\frac{1}{g(y)-f(y)+\varepsilon}dy + C = C\int_0^{r/2}\frac{1}{y^2+\varepsilon}dy + C \\
&\leq \frac{C}{\sqrt{\varepsilon}}.
\end{aligned}
\tag{2.30}
$$

The proof is completed. □

Similarly, we have

Lemma 2.2.6. *Let a_{ii} be defined by (2.20),*

$$\frac{1}{C}|\ln\varepsilon| \leq -a_{ii} \leq C|\ln\varepsilon|, \quad for\ n=3,\ i=1,2.$$

Proof: We consider

$$\overline{w}(x_1,x') = -\frac{x-g(x')-\frac{\varepsilon}{2}}{g(x')-f(x')+\varepsilon} \tag{2.31}$$

on $O_{r/2} := \widetilde{\Omega}\cap\{(x_1,x')|\ |x'|<\frac{r}{2}\}$. Use the same proof in Lemma 2.2.5, we have

$$\int_0^{r/2}\int_{f(x')-\frac{\varepsilon}{2}}^{g(x')+\frac{\varepsilon}{2}} |\partial_{x'}\overline{w}(x_1,x')|^2 dx_1 dx' \leq C.$$

Therefore, it suffices to verify that

$$\int_{\widetilde{\Omega}\cap O_{r/2}} |\partial_{x_1}\overline{w}(x_1,x')|^2 \sim |\ln\varepsilon|.$$

Indeed,

$$\int_{\widetilde{\Omega}\cap O_{r/2}} |\partial_{x_1}\overline{w}(x_1,x')|^2 = \int_{|x'|<r/2} \frac{1}{g(x')-f(x')+\varepsilon}dx' = \int_0^{r/2} \frac{t}{Ct^2+\varepsilon}dt \sim |\ln\varepsilon|.$$

This completes the proof. □

Lemma 2.2.7. *Let a_{ii} be defined by (2.20),*

$$\frac{1}{C} \le -a_{ii} \le C \quad for\ n \ge 4,\ i=1,2.$$

Proof: We only need

$$\int_{O_{r/2}} |\partial_{x_1}\overline{w}(x_1,x')|^2 = \int_{|x'|<r/2} \frac{1}{g(x')-f(x')+\varepsilon}dx' = \int_0^{r/2} \frac{t^{n-2}}{Ct^2+\varepsilon}dt \sim C.$$

The proof is completed. □

Lemma 2.2.8. *Let α be defined by (2.23), we have*

$$\frac{1}{C} \le \alpha \le C.$$

Proof: By the definition of α and using the second statement in Lemma 2.2.4, we are done. □

To summarize, we have

Proposition 2.2.2. *Let a_{ij} and α be defined by (2.20) and (2.23), we have*

1. $\frac{1}{C\sqrt{\varepsilon}} \le |a_{11}-\alpha a_{12}| \le \frac{C}{\sqrt{\varepsilon}} \qquad for\ n=2,$

2. $\frac{1}{C}|\ln\varepsilon| \le |a_{11}-\alpha a_{12}| \le C|\ln\varepsilon| \qquad for\ n=3,$

3. $\frac{1}{C} \le |a_{11}-\alpha a_{12}| \le C \qquad for\ n\ge 4.$

Proof: Since $a_{11}<0$, $a_{12}>0$, $a_{11}+a_{12}<0$ and $\alpha>0$, we have

$$|a_{11}| < |a_{11}-\alpha a_{12}| < (1+\alpha)|a_{11}|.$$

Combining the results of Lemma 2.2.5, Lemma 2.2.6, Lemma 2.2.7 and Lemma 2.2.8, the proof is completed. □

2.2.3 Estimate of $|b_1 - \alpha b_2|$

Proposition 2.2.3. *Let $b_1,\ b_2$, α and $Q_\varepsilon[\varphi]$ be defined by (2.20), (2.23) and (2.8), we have*

$$\frac{|Q_\varepsilon[\varphi]|}{C} \le |b_1 - \alpha b_2| \le C\|\varphi\|_{C^2(\partial\Omega)}.$$

Proof: Combining the third result in Lemma 2.2.4 and Lemma 2.2.8, we have

$$|b_1 - \alpha b_2| \le |b_1| + |\alpha||b_2| \le C\|\varphi\|_{C^2(\partial\Omega)}.$$

On the other hand, by the definition and the harmonicity of v_1 and v_2 and using Lemma 2.2.4, we obtain

$$\begin{aligned} |b_1 - \alpha b_2| &= \frac{|b_1(a_{22}+a_{12}) - b_2(a_{11}+a_{12})|}{|a_{22}+a_{12}|} \\ &\ge \frac{1}{C}\cdot \Big| \int_{\partial D_1} \frac{\partial v_3}{\partial \nu} \int_{\partial\Omega} \frac{\partial v_2}{\partial \nu} - \int_{\partial D_2} \frac{\partial v_3}{\partial \nu} \int_{\partial\Omega} \frac{\partial v_1}{\partial \nu} \Big| = \frac{|Q_\varepsilon[\varphi]|}{C}. \end{aligned}$$

This completes the proof. □

Now we are ready to prove our two main theorems:

Proof of Theorem 2.1.1-2.1.2: By Proposition 2.2.1 and (2.23), then using Proposition 2.2.2, 2.2.3, we are done. □

2.3 Estimate of $|Q_\varepsilon[\varphi]|$

In order to identify situations when $\|\nabla u\|_{L^\infty}$ behaves exactly as the upper bound established in Theorem 2.1.1, we estimate in this section $|Q_\varepsilon[\varphi]|$. To emphasize the dependence on ε, we denote D_1, D_2 by $D_{1\varepsilon}$, $D_{2\varepsilon}$, denote φ by φ_ε, and denote v_1, v_2, v_3 defined by equation (2.5), (2.6), (2.7) as $v_{1\varepsilon}$, $v_{2\varepsilon}$, $v_{3\varepsilon}$. In this section we assume, in addition to the hypotheses in Theorem 2.1.1, that along a sequence $\varepsilon \to 0$ (we still denote it as ε), $D_{1\varepsilon} \to D_1^*$, $D_{2\varepsilon} \to D_2^*$ in $C^{2,\alpha}$ norm, $\varphi_\varepsilon \to \varphi^*$ in $C^{1,\alpha}(\partial\Omega)$. We use notation $\widetilde{\Omega}^* = \Omega \backslash \overline{D_1^* \cup D_2^*}$, and assume, without loss of generality, that $D_1^* \cap D_2^* = \{0\}$. We will show that as $\varepsilon \to 0$, $v_{i\varepsilon}$ converges, in appropriate sense, to v_i^* which satisfies

$$\begin{cases} \Delta v_1^* = 0 & in\ \widetilde{\Omega}^*, \\ v_1^* = 1 \ \ on\ \partial D_1^* \backslash \{0\}, & v_1^* = 0 \ \ on\ \partial\Omega \cup \partial D_2^* \backslash \{0\}, \end{cases} \tag{2.32}$$

$$\begin{cases} \Delta v_2^* = 0 & in\ \widetilde{\Omega}^*, \\ v_2^* = 1 \ \ on\ \partial D_2^* \backslash \{0\}, & v_2^* = 0 \ \ on\ \partial\Omega \cup \partial D_1^* \backslash \{0\}, \end{cases} \tag{2.33}$$

$$\begin{cases} \Delta v_3^* = 0 & in\ \widetilde{\Omega}^*, \\ v_3^* = 0 \ \ on\ \partial D_1^* \cup \partial D_2^*, & v_3^* = \varphi^* \ \ on\ \partial\Omega. \end{cases} \tag{2.34}$$

First we prove

Lemma 2.3.1. *There exist unique* $v_i^* \in L^\infty(\widetilde{\Omega}^*) \cap C^0(\overline{\widetilde{\Omega}^*} \setminus \{0\}) \cap\ C^2(\widetilde{\Omega}^*)$, $i = 1, 2, 3$, *which solve equations* (2.32), (2.33) *and* (2.34) *respectively. Moreover,* $v_i^* \in C^1(\overline{\widetilde{\Omega}^*} \setminus \{0\})$.

Proof: The existence of solutions to the above equations can easily be obtained by Perron's method, see theorem 2.12 and lemma 2.13 in [12]. For reader′s convenience, we give below a simple proof of the uniqueness. We only need to prove that 0 is the only solution in $L^\infty(\widetilde{\Omega}^*) \cap C^0(\overline{\widetilde{\Omega}^*} \setminus \{0\}) \cap\ C^2(\widetilde{\Omega}^*)$ to the following equation:

$$\begin{cases} \Delta w = 0 & in\ \widetilde{\Omega}^*, \\ \ \ w = 0 & on\ \partial\widetilde{\Omega}^* \backslash \{0\}. \end{cases} \tag{2.35}$$

Indeed, $\forall\ \varepsilon > 0$, we have

$$|w(x)| \leq \frac{\varepsilon^{n-2} \|w\|_{L^\infty(\widetilde{\Omega}^*)}}{|x|^{n-2}}, \quad on\ \partial(\widetilde{\Omega}^* \backslash B_\varepsilon)(0).$$

By the maximum principle,

$$|w(x)| \leq \frac{\varepsilon^{n-2} \|w\|_{L^\infty(\widetilde{\Omega}^*)}}{|x|^{n-2}}, \quad \forall\ x \in \widetilde{\Omega}^* \backslash B_\varepsilon(0).$$

Thus $w \equiv 0$ in $\widetilde{\Omega}^*$. The additional regularity $v_i^* \in C^1(\overline{\widetilde{\Omega}^*} \setminus \{0\})$ follows from standard elliptic estimates and the regularity of the ∂D_i and $\partial\Omega$. $\square$

Lemma 2.3.2. *For $i = 1, 2, 3$,*

$$v_{i\varepsilon} \longrightarrow v_i^* \quad in \quad C^2_{loc}(\widetilde{\Omega}^*), \qquad as \quad \varepsilon \to 0, \tag{2.36}$$

$$\int_{\partial\Omega} \frac{\partial v_{i\varepsilon}}{\partial \nu} \longrightarrow \int_{\partial\Omega} \frac{\partial v_i^*}{\partial \nu}, \qquad as \quad \varepsilon \to 0, \qquad i = 1, 2, \tag{2.37}$$

$$\int_{\partial D_{i\varepsilon}} \frac{\partial v_{3\varepsilon}}{\partial \nu} \longrightarrow \int_{\partial D_i^*} \frac{\partial v_3^*}{\partial \nu}, \qquad as \quad \varepsilon \to 0. \tag{2.38}$$

Proof: By the maximum principle, $\{\|v_{i\varepsilon}\|_{L^\infty}\}$ is bounded by a constant independent of ε. By the uniqueness part of Lemma 2.3.1, we obtain (2.36) using standard elliptic estimates. By Lemma 2.2.3, $\{\|\nabla v_{3\varepsilon}\|_{L^\infty}\}$ is bounded by some constant independent of ε, so $\|\nabla v_3^*\|_{L^\infty} < \infty$. Estimate (2.37) and (2.38) follow from standard elliptic estimates. The proof is completed. $\square$

Similar to $Q_\varepsilon[\varphi_\varepsilon]$, we define

$$Q^*[\varphi^*] := \int_{\partial D_1^*} \frac{\partial v_3^*}{\partial \nu} \int_{\partial\Omega} \frac{\partial v_2^*}{\partial \nu} - \int_{\partial D_2^*} \frac{\partial v_3^*}{\partial \nu} \int_{\partial\Omega} \frac{\partial v_1^*}{\partial \nu}, \tag{2.39}$$

then $Q^* : C^2(\partial\Omega) \mapsto \mathbb{R}$ is a linear functional. Let $Q_\varepsilon[\varphi_\varepsilon]$ and $Q^*[\varphi^*]$ be defined by equation (2.8), (2.39), then, by the above lemmas,

$$Q_\varepsilon[\varphi_\varepsilon] \longrightarrow Q^*[\varphi^*], \qquad \text{as} \quad \varepsilon \to 0.$$

Corollary 2.3.1. *If $\varphi^* \in C^2(\partial\Omega)$ satisfies $Q^*[\varphi^*] \neq 0$, then $|Q_\varepsilon[\varphi_\varepsilon]| \geq \frac{1}{C}$, for some positive constant C which is independent of ε.*

In the following we give some examples to show that, in general, the rates of the lower bounds established in Theorem 2.1.2 are optimal.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with $C^{2,\alpha}$ boundary, $0 < \alpha < 1$, which is symmetric with respect to x_1-variable, i.e., $(x_1, x') \in \Omega$ if and only if $(-x_1, x') \in \Omega$, where $x' = (x_2, \cdots, x_n)$.

Let D_1^* be a strictly convex bounded open set in $\{(x_1, x') \in \mathbb{R}^n | x_1 < 0\}$ with $C^{2,\alpha}$ boundary, $0 < \alpha < 1$, satisfying $0 \in \partial D_1^*$ and $\overline{D_1^*} \subset \Omega$. Set $D_2^* = \{(x_1, x') \in \mathbb{R}^n | (-x_1, x') \in D_1^*\}$.

Let $\varphi \in C^2(\partial\Omega)\backslash\{0\}$ satisfy

$$\varphi_{odd}(x_1, x') := \frac{1}{2}\big[\varphi(x_1, x') - \varphi(-x_1, x')\big] \le 0 \text{ (or } \ge 0), \tag{2.40}$$

on $(\partial\Omega)^+ := \{(x_1, x') \in \partial\Omega | x_1 > 0\}$.

For $\varepsilon > 0$ sufficiently small, let

$$\begin{aligned} D_{1\varepsilon} &:= \big\{(x_1, x') \in \Omega \big| (x_1 + \frac{\varepsilon}{2}, x') \in D_1^*\big\}, \\ D_{2\varepsilon} &:= \big\{(x_1, x') \in \Omega \big| (x_1 - \frac{\varepsilon}{2}, x') \in D_2^*\big\}, \\ \varphi_\varepsilon &:= \varphi. \end{aligned}$$

Proposition 2.3.1. *Under the above assumptions, we have* $|Q_\varepsilon[\varphi]| \ge \frac{1}{C}$*, for some positive constant* C *independent of* ε*. Consequently,*

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^\infty(\widetilde{\Omega})} &\ge \frac{1}{C\sqrt{\varepsilon}} && for\ \ n = 2, \\ \|\nabla u_\varepsilon\|_{L^\infty(\widetilde{\Omega})} &\ge \frac{1}{C\varepsilon|\ln\varepsilon|} && for\ \ n = 3, \\ \|\nabla u_\varepsilon\|_{L^\infty(\widetilde{\Omega})} &\ge \frac{1}{C\varepsilon} && for\ \ n \ge 4, \end{aligned} \tag{2.41}$$

where u_ε *is the solution to equation (2.1).*

The above proposition can be easily obtained by the following lemma which gives a necessary and sufficient condition instead of condition (2.40) on φ for the lower bounds (2.41) to hold.

Let

$$(v_3^*)_{odd}(x_1, x') := \frac{1}{2}\big[v_3^*(x_1, x') - v_3^*(-x_1, x')\big], \tag{2.42}$$

we have

Lemma 2.3.3. *Under the same hypotheses in Proposition 2.3.1 except for the condition (2.40), let* $Q_\varepsilon[\varphi]$ *and* $(v_3^*)_{odd}(x)$ *be defined by equation (2.8) and (2.42), then the following statements are equivalent:*

1. *For some positive constant* C *independent of* ε*, we have* $|Q_\varepsilon[\varphi]| \ge \frac{1}{C}$*,*

2. $\int_{\partial D_2^*} \frac{\partial (v_3^*)_{odd}}{\partial \nu} \ne 0.$

Proof: By symmetry, the strong maximum principle and the Hopf Lemma, we can easily obtain

$$\int_{\partial\Omega}\frac{\partial v_1^*}{\partial\nu}=\int_{\partial\Omega}\frac{\partial v_2^*}{\partial\nu}<0.$$

Then

$$\begin{aligned}Q^*[\varphi]&=\int_{\partial\Omega}\frac{\partial v_1^*}{\partial\nu}\Big(\int_{\partial D_1^*}\frac{\partial v_3^*}{\partial\nu}-\int_{\partial D_2^*}\frac{\partial v_3^*}{\partial\nu}\Big)\\&=\int_{\partial\Omega}\frac{\partial v_1^*}{\partial\nu}\Big(\int_{\partial D_1^*}\frac{\partial (v_3^*)_{odd}}{\partial\nu}-\int_{\partial D_2^*}\frac{\partial (v_3^*)_{odd}}{\partial\nu}\Big)\\&=-2\int_{\partial\Omega}\frac{\partial v_1^*}{\partial\nu}\int_{\partial D_2^*}\frac{\partial (v_3^*)_{odd}}{\partial\nu}.\end{aligned}$$

Hence, $Q^*[\varphi]\neq 0$ if and only if $\int_{\partial D_2^*}\frac{\partial (v_3^*)_{odd}}{\partial\nu}\neq 0$. Then by Corollary 2.3.1, we complete the proof. □

Proof of Proposition 2.3.1: Note that $(v_3^*)_{odd}(0,x')=0$ by symmetry, and $(v_3^*)_{odd}$ is harmonic with $(v_3^*)_{odd}=\varphi_{odd}\leq 0$ (or ≥ 0) but not identically 0 on $(\partial\Omega)^+$. Now by using the strong maximum principle and the Hopf Lemma, it is clear that $\int_{\partial D_2^*}\frac{\partial (v_3^*)_{odd}}{\partial\nu}\neq 0$, Hence, by Lemma 2.3.3 and Theorem 2.1.2, we are done. □

Remark 2.3.1. *If $\varphi=\sum_{i=1}^n b_ix_i$ with $b_i\in\mathbb{R}$ and $b_1\neq 0$, then by Proposition 2.3.1 we have $|Q_\varepsilon[\varphi]|\geq\frac{1}{C}$. Therefore, by Theorem 2.1.1 and 2.1.2, the blow-up rates of $\|\nabla u\|_{L^\infty(\widetilde{\Omega})}$ are $\varepsilon^{-1/2}$ in in dimension $n=2$, $(\varepsilon|\ln\varepsilon|)^{-1}$ in dimension $n=3$ and ε^{-1} in dimension $n\geq 4$.*

Now instead of in a bounded set Ω, we consider in $\mathbb{R}^n$:

$$\begin{cases}\Delta u_\varepsilon=0 & in\ \mathbb{R}^n\backslash\overline{D_{1\varepsilon}\cup D_{2\varepsilon}},\\ u_\varepsilon|_+=u_\varepsilon|_- & on\ \partial D_{1\varepsilon}\cup\partial D_{2\varepsilon},\\ \nabla u_\varepsilon\equiv 0 & in\ D_{1\varepsilon}\cup D_{2\varepsilon},\\ \displaystyle\int_{\partial D_{i\varepsilon}}\frac{\partial u_\varepsilon}{\partial\nu}\Big|_+=0 & (i=1,2),\\ \displaystyle\limsup_{|x|\to\infty}|x|^{n-1}|u_\varepsilon(x)-H(x)|<\infty, & \end{cases}\tag{2.43}$$

where $H(x)$ is a given entire harmonic function in $\mathbb{R}^n$.

we have the following result regarding the lower bound for $|\nabla u_\varepsilon|$:

Proposition 2.3.2. *With the same assumptions on $D_{1\varepsilon}$ and $D_{2\varepsilon}$ as in Proposition 2.3.1, and let $H(x)$ be an entire harmonic function in $\mathbb{R}^n$ satisfying $H_{odd}(x_1, x') := \frac{1}{2}\big[H(x_1, x') - H(-x_1, x')\big] < 0$ (or > 0) on $\mathbb{R}^n_+ := \{(x_1, x') \in \mathbb{R}^n | x_1 > 0\}$, then for some positive constant C independent of ε, we have*

$$
\begin{aligned}
\|\nabla u_\varepsilon\|_{L^\infty(\mathbb{R}^n \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}})} &\geq \frac{1}{C\sqrt{\varepsilon}} && \text{for } n = 2,\\
\|\nabla u_\varepsilon\|_{L^\infty(\mathbb{R}^n \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}})} &\geq \frac{1}{C\varepsilon |\ln \varepsilon|} && \text{for } n = 3,\\
\|\nabla u_\varepsilon\|_{L^\infty(\mathbb{R}^n \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}})} &\geq \frac{1}{C\varepsilon} && \text{for } n \geq 4,
\end{aligned}
\tag{2.44}
$$

where u_ε is the solution to equation (2.43).

Proof: Step 1. First, we show that there exists a positive constant C independent of ε, such that for any small $\varepsilon > 0$,

$$
|x|^{n-1} |u_\varepsilon(x) - H(x)| \leq C, \qquad \forall\ x \in \mathbb{R}^n \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}}. \tag{2.45}
$$

(i) For any bounded open set $U \subset \mathbb{R}^n$ with C^1 boundary ∂U satisfying $\partial U \cap \overline{D_{1\varepsilon} \cup D_{2\varepsilon}} = \emptyset$, we have, in view of the first and the fourth lines in (2.43),

$$
\int_{\partial U} \frac{\partial u_\varepsilon}{\partial \nu} = \int_{U \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}}} \Delta u_\varepsilon = 0. \tag{2.46}
$$

(ii) We show that there exists a positive constant M independent of ε, such that

$$
\|u_\varepsilon - H\|_{L^\infty(\mathbb{R}^n \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}})} \leq M, \qquad \forall \text{ small } \varepsilon > 0.
$$

We only need to prove

$$
\|u_\varepsilon - H\|_{L^\infty(\mathbb{R}^n \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}})} \leq \sum_{i=1}^{2} (\max_{\overline{D}_{i\varepsilon}} H - \min_{\overline{D}_{i\varepsilon}} H). \tag{2.47}
$$

Since $\nabla u_\varepsilon = 0$ in $D_{1\varepsilon} \cup D_{2\varepsilon}$, u_ε is constant on each $D_{i\varepsilon}$, denoted as $C_i(\varepsilon)$. We know that

$$
\lim_{|x| \to \infty} (u_\varepsilon(x) - H(x)) = 0, \tag{2.48}
$$

and

$$
C_i(\varepsilon) - \max_{\overline{D}_{i\varepsilon}} H \leq u_\varepsilon - H \leq C_i(\varepsilon) - \min_{\overline{D}_{i\varepsilon}} H, \quad \text{on } D_{i\varepsilon}, \quad i = 1, 2. \tag{2.49}
$$

If (2.47) did not hold, say,

$$\sup_{\mathbb{R}^n}(u_\varepsilon - H) > \sum_{i=1}^{2}(\max_{\overline{D}_{i\varepsilon}} H - \min_{\overline{D}_{i\varepsilon}} H),$$

then, because of (2.48) and (2.49), there would exist $0 < a < \sup_{\mathbb{R}^n}(u_\varepsilon - H)$ such that $U := \{x \in \mathbb{R}^n \mid (u_\varepsilon - H)(x) > a\} \neq \emptyset$ satisfies $\partial U \cap \overline{D_{1\varepsilon} \cup D_{2\varepsilon}} = \emptyset$. We may assume, by the Sard theorem, that a is a regular value of $u_\varepsilon - H$, and therefore ∂U is C^1. By the Hopf lemma, $\dfrac{\partial(u_\varepsilon - H)}{\partial\nu} < 0$ on ∂U, and therefore

$$\int_{\partial U} \frac{\partial(u_\varepsilon - H)}{\partial\nu} < 0.$$

On the other hand, using (2.46) and the harmonicity of H in U, we have

$$\int_{\partial U} \frac{\partial(u_\varepsilon - H)}{\partial\nu} = -\int_{\partial U} \frac{\partial H}{\partial\nu} = -\int_U \Delta H = 0.$$

A contradiction.

(iii) Consider $w_\varepsilon(x) := u_\varepsilon(x) - H(x)$. Fix a constant $R_0 > 0$, independent of ε, such that $D_1^* \cup D_2^* \subset B_{R_0/2}(0)$, and let

$$\widetilde{w_\varepsilon}(y) := \frac{1}{|y|^{n-2}} w_\varepsilon\Big(\frac{y}{|y|^2}\Big), \quad 0 < |y| < \frac{1}{R_0}.$$

Then $\widetilde{w_\varepsilon}$ is harmonic in $B_{1/R_0}\backslash\{0\}$. By the last line of (2.43), there exists a positive constant $C(\varepsilon)$ such that

$$|\widetilde{w_\varepsilon}(y)| \leq C(\varepsilon)|y|, \quad 0 < |y| < \frac{1}{R_0}.$$

Therefore, $\Delta\widetilde{w_\varepsilon} = 0$ in B_{1/R_0} and $\widetilde{w_\varepsilon}(0) = 0$. By (ii), we have $|\widetilde{w_\varepsilon}| \leq C$, on $\partial B_{1/R_0}$, for some positive constant C independent of ε. Hence, $|\widetilde{w_\varepsilon}| \leq C$, $|\nabla\widetilde{w_\varepsilon}| \leq C$ in $B_{1/(2R_0)}$, then

$$|\widetilde{w_\varepsilon}(y)| \leq C|y|, \quad |y| < \frac{1}{2R_0}.$$

Therefore, also using (ii), (2.45) holds.

Step 2. For $R > R_0$, let $\Omega = B_R(0)$. Let $\varphi_\varepsilon := u_\varepsilon|_{\partial\Omega}$, then by Corollary 2.3.1 and

Theorem 2.1.2 it is enough to show, for some R, that $Q^*[\varphi^*] \neq 0$, where φ^* is defined at the beginning of this section. By symmetry, we have

$$Q^*[\varphi^*] = \int_{\partial\Omega} \frac{\partial v_1^*}{\partial \nu}\Big(\int_{\partial D_1^*} \frac{\partial v_3^*}{\partial \nu} - \int_{\partial D_2^*} \frac{\partial v_3^*}{\partial \nu} \Big).$$

Without loss of generality, we may assume $H_{odd}(x) > 0$ on $\mathbb{R}^n_+$. Recall that v_3^* is the solution of (2.34) with boundary data φ^*. In the following we use notation $(v_3^*)_h$ to denote the the solution of (2.34) with boundary data h. Since $Q^*[\varphi^*]$ is linear on φ^* and by symmetry $Q^*[H_{even}] = H[\varphi^*_{even}] = 0$, where $H_{even}(x) := H(x) - H_{odd}(x) = \frac{1}{2}\big[H(x_1, x') + H(-x_1, x')\big]$ and similar for φ^*_{even}, we may assume $H(x) = H_{odd}(x)$.

Now consider $w(x) = H(x) - (v_3^*)_H(x)$. Then $w(x)$ is harmonic in $\widetilde{\Omega}^*$ which is defined at the beginning of this section. By symmetry, $w(-x_1, x') = -w(x_1, x')$, $w(x) = H(x)$ on $\partial D_1^* \cup \partial D_2^*$ and $w(x) = 0$ on $\partial\Omega$. Therefore,

$$-2\int_{\partial D_2^*} H\frac{\partial w}{\partial \nu} = \int_{\widetilde{\Omega}^*} w(x)\Delta w(x) + \int_{\widetilde{\Omega}^*} |\nabla w|^2 = \int_{\widetilde{\Omega}^*} |\nabla w|^2 \geq 0.$$

On the other hand, $(v_3^*)_H = 0$ on ∂D_2^*, $(v_3^*)_H > 0$ on $(\partial\Omega)^+$ and, by the oddness of $(v_3^*)_H$, $(v_3^*)_H = 0$ on $\{(x_1, x') \mid x_1 = 0\}$. Thus, by the maximum principle and the strong maximum principle, $(v_3^*)_H > 0$ in $\widetilde{\Omega}^*$ and in turn, using the Hopf lemma, $\dfrac{\partial (v_3^*)_H}{\partial \nu} > 0$ on ∂D_2^*. Hence, using the harmonicity of H,

$$\begin{aligned}\max_{\partial D_2^*} H \int_{\partial D_2^*} \frac{\partial (v_3^*)_H}{\partial \nu} &\geq \int_{\partial D_2^*} H\frac{\partial (v_3^*)_H}{\partial \nu} \geq \int_{\partial D_2^*} H\frac{\partial H}{\partial \nu} - \int_{\partial D_2^*} H\frac{\partial w}{\partial \nu} \\ &\geq \int_{D_2^*} |\nabla H|^2 \geq \frac{1}{C},\end{aligned}$$

Therefore,

$$\int_{\partial D_2^*} \frac{\partial (v_3^*)_H}{\partial \nu} \geq \frac{1}{C},$$

for positive constant C independent of R.

For $s_\varepsilon := \varphi_\varepsilon - H$ on $\partial\Omega$, by step 1, there exists a constant $C > 0$ which is independent of ε and R, such that $\|s_\varepsilon\|_{L^\infty(\partial\Omega)} \leq CR^{1-n}$. By Remark 2.2.1, we have $\|\nabla (v_3^*)_{s^*}\|_{L^\infty(\partial D_1^* \cup \partial D_2^*)} \leq C\|s^*\|_{L^\infty(\partial\Omega)}$, thus,

$$\Big|\int_{\partial D_i^*} \frac{\partial (v_3^*)_{s^*}}{\partial \nu}\Big| \leq C\int_{\partial D_i^*} \|s^*\|_{L^\infty(\partial\Omega)} \leq CR^{1-n},$$

for some positive constant C independent of ε and R.

Therefore, for large enough R,

$$\int_{\partial D_2^*} \frac{\partial (v_3^*)_{\varphi^*}}{\partial \nu} = \int_{\partial D_2^*} \frac{\partial (v_3^*)_H}{\partial \nu} + \int_{\partial D_2^*} \frac{\partial (v_3^*)_{s^*}}{\partial \nu} \geq \frac{1}{C} \neq 0.$$

It is also clear that $\int_{\partial\Omega} \frac{\partial v_1^*}{\partial \nu} < 0$, Thus,

$$Q^*[\varphi^*] = -2 \int_{\partial\Omega} \frac{\partial v_1^*}{\partial \nu} \int_{\partial D_2^*} \frac{\partial (v_3^*)_{\varphi^*}}{\partial \nu} \neq 0.$$

This proof is completed. □

Remark 2.3.2. *In $\mathbb{R}^2$, when $D_{1\varepsilon}$ and $D_{2\varepsilon}$ are identical balls of radius 1, the estimate (2.44) was established in [2] under a weaker assumption $\partial_{x_1} H(0) \neq 0$.*

2.4 Proof of Theorem 2.1.3 and 2.1.4

In the introduction, similar to the harmonic case, we still decompose $u = C_1 V_1 + C_2 V_2 + V_3$ as in (2.13).

Proposition 2.2.1 holds since Lemma 2.2.1−2.2.3 hold for V_1, V_2, V_3 defined by (2.14)−(2.16) and $\rho \in C^2(\widetilde{\Omega})$ which is the solution to:

$$\begin{cases} \partial_{x_j}\Big(a_2^{ij}(x)\ \partial_{x_i}\rho\Big) = 0 & in\ \widetilde{\Omega}, \\ \rho = 0 \ \ on\ \partial D_1 \cup \partial D_2, \quad \rho = 1 \ \ on\ \partial\Omega. \end{cases}$$

The proofs are essentially the same.

Now we start to estimate $|C_1 - C_2|$. By the decomposition formula (2.13), instead of (2.20), we denote

$$\begin{aligned} a_{lm} &= \int_{\partial D_l} a_2^{ij}(x)\ \partial_{x_i} V_m\ \nu_j \quad (l, m = 1, 2), \\ b_l &= \int_{\partial D_l} a_2^{ij}(x)\ \partial_{x_i} V_3\ \nu_j \quad (l = 1, 2). \end{aligned} \tag{2.50}$$

Then Lemma 2.2.4 and (2.21)−(2.23) still hold for a_{lm} and b_l defined above.

In fact, to prove Lemma 2.2.4 with general coefficients, we only need to change $\frac{\partial *}{\partial \nu}$ to $a_2^{ij}(x)\ \partial_{x_i} *\ \nu_j$, change $\Delta *$ in $\partial_{x_j}\left(a_2^{ij}(x)\ \partial_{x_i} *\right)$ and change v_1, v_2, v_3 in V_1, V_2, V_3, respectively, in the original proof of Lemma 2.2.4. For instance, (2.24) is changed to

$$\begin{aligned} 0 &= \int_{\widetilde{\Omega}} \partial_{x_j}\left(a_2^{ij}(x)\ \partial_{x_i} V_1\right) \cdot V_2 - \int_{\widetilde{\Omega}} \partial_{x_j}\left(a_2^{ij}(x)\ \partial_{x_i} V_2\right) \cdot V_1 \\ &= -\int_{\partial D_2} a_2^{ij}(x)\ \partial_{x_i} V_1\ \nu_j \cdot 1 + \int_{\partial D_1} a_2^{ij}(x)\ \partial_{x_i} V_2\ \nu_j \cdot 1 \\ &= -a_{21} + a_{12}. \end{aligned} \tag{2.51}$$

Therefore, to estimate $|C_1 - C_2|$, it is equivalent to estimating $|a_{11} - \alpha a_{12}|$ and $|b_1 - \alpha b_2|$.

For $|a_{11} - \alpha a_{12}|$, Lemma 2.2.5–2.2.7 still hold for $a_{ll}(l = 1, 2)$ defined by (2.50). The proof is quite similar and the only thing which needs to be shown is the following:

$$\begin{aligned} 0 &= \int_{\widetilde{\Omega}} \partial_{x_j}\left(a_2^{ij}(x)\ \partial_{x_i} V_1\right) \cdot V_1 \\ &= -\int_{\widetilde{\Omega}} a_2^{ij}(x)\ \partial_{x_i} V_1 \partial_{x_j} V_1 - \int_{\partial D_1} a_2^{ij}(x)\ \partial_{x_i} V_1\ \nu_j \cdot 1 \\ &= -\int_{\widetilde{\Omega}} a_2^{ij}(x)\ \partial_{x_i} V_1 \partial_{x_j} V_1 - a_{11}, \end{aligned}$$

i.e.

$$a_{11} = -\int_{\widetilde{\Omega}} a_2^{ij}(x)\ \partial_{x_i} V_1 \partial_{x_j} V_1.$$

Then by the uniform ellipticity of $a_2^{ij}(x)$ and the harmonicity of v_1,

$$|a_{11}| \geq \lambda \int_{\widetilde{\Omega}} |\nabla V_1|^2 \geq \lambda \int_{\widetilde{\Omega}} |\nabla v_1|^2,$$

and

$$|a_{11}| \leq \int_{\widetilde{\Omega}} a_2^{ij}(x)\ \partial_{x_i} w \partial_{x_j} w \leq \Lambda \int_{\widetilde{\Omega}} |\nabla w|^2 \leq \Lambda \int_{\widetilde{\Omega} \cap O_{r/2}} |\nabla \overline{w}|^2 + C,$$

where w is defined in the proof of Lemma 2.2.5 with the same boundary data of V_1 and $\overline{w}$ is defined by (2.26) and (2.31).

Thus, Lemma 2.2.5–2.2.7 follow by the same computations. Then Lemma 2.2.8 and Proposition 2.2.2 hold with the same proofs.

For $|b_1 - \alpha b_2|$, Proposition 2.2.3 also holds for $b_l (l = 1, 2)$ defined by (2.50) and $Q_\varepsilon[\varphi]$ defined by (2.17). The proof is the same after changing $\frac{\partial *}{\partial \nu}$ to $a_2^{ij}(x)\ \partial_{x_i} *\ \nu_j$.

Combining the above propositions, we obtain our theorems.

2.5 Appendix

Some elementary results for the conductivity problem

Assume that in $\mathbb{R}^n$, Ω and ω are bounded open sets with $C^{2,\alpha}$ boundaries, $0 < \alpha < 1$, satisfying

$$\overline{\omega} = \bigcup_{s=1}^{m} \overline{\omega}_s \subset \Omega,$$

where $\{\omega_s\}$ are connected components of ω. Clearly, $m < \infty$ and ω_s is open for all $1 \leq s \leq \omega$. Given $\varphi \in C^2(\partial\Omega)$, the conductivity problem we consider is the following transmission problem with Dirichlet boundary condition:

$$\begin{cases} \partial_{x_j}\Big\{\Big[\big(ka_1^{ij}(x) - a_2^{ij}(x)\big)\chi_\omega + a_2^{ij}(x)\Big]\partial_{x_i} u_k\Big\} = 0 & in\ \Omega, \\ u_k = \varphi & on\ \partial\Omega, \end{cases} \tag{2.52}$$

where $k = 1, 2, 3, \cdots$, and χ_ω is the characteristic function of ω.

The $n \times n$ matrixes $A_1(x) := \big(a_1^{ij}(x)\big)$ in ω, $A_2(x) := \big(a_2^{ij}(x)\big)$ in $\Omega\backslash\overline{\omega}$ are symmetric and $\exists$ a constant $\Lambda \geq \lambda > 0$ such that

$$\lambda|\xi|^2 \leq a_1^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2\ (\forall x \in \omega), \qquad \lambda|\xi|^2 \leq a_2^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2\ (\forall x \in \Omega\backslash\omega)$$

for all $\xi \in \mathbb{R}^n$ and $a_1^{ij}(x) \in C^2(\overline{\omega})$, $a_2^{ij}(x) \in C^2(\overline{\Omega}\backslash\omega)$.

Equation (2.52) can be rewritten in the following form to emphasize the transmission condition on $\partial\omega$:

$$\begin{cases} \partial_{x_j}\Big(a_1^{ij}(x)\ \partial_{x_i}u_k\Big) = 0 & in\ \omega, \\ \partial_{x_j}\Big(a_2^{ij}(x)\ \partial_{x_i}u_k\Big) = 0 & in\ \Omega\backslash\overline{\omega}, \\ u_k|_+ = u_k|_-, & on\ \partial\omega, \\ a_2^{ij}(x)\partial_{x_i}u_k\nu_j\big|_+ = ka_1^{ij}(x)\partial_{x_i}u_k\nu_j\big|_- & on\ \partial\omega, \\ u_k = \varphi & on\ \partial\Omega. \end{cases} \tag{2.53}$$

Here and throughout this paper ν is the outward unit normal and the subscript $\pm$ indicates the limit from outside and inside the domain, respectively.

We list the following results which are well known and omit the proofs.

Theorem 2.5.1. *If $u_k \in H^1(\Omega)$ is a solution of equation (2.52), then $u_k \in C^1(\overline{\Omega\backslash\omega}) \cap C^1(\overline{\omega})$ and satisfies equation (2.53).*

If $u_k \in C^1(\overline{\Omega\backslash\omega}) \cap C^1(\overline{\omega})$ is a solution of equation (2.53), then $u_k \in H^1(\Omega)$ and satisfies equation (2.52).

Theorem 2.5.2. *There exists at most one solution $u_k \in H^1(\Omega)$ to equation (2.52).*

The existence of the solution can be obtained by using the variational method. For every k, we define the energy functional

$$I_k[v] := \frac{k}{2}\int_\omega a_1^{ij}(x)\partial_{x_i}v\partial_{x_j}v + \frac{1}{2}\int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}v\partial_{x_j}v, \tag{2.54}$$

where v belongs to the set

$$H^1_\varphi(\Omega) := \{v \in H^1(\Omega)|\ v = \varphi\ \ on\ \partial\Omega\}.$$

Theorem 2.5.3. *For every k, there exists a minimizer $u_k \in H^1(\Omega)$ satisfying*

$$I_k[u_k] = \min_{v\in H^1_\varphi(\Omega)} I_k[v].$$

Moreover, $u_k \in H^1(\Omega)$ is a solution of equation (2.52).

Comparing equation (2.53), when $k = +\infty$, the perfectly conducting problem turns out to be:

$$\begin{cases} \partial_{x_j}\Big(a_2^{ij}(x)\ \partial_{x_i} u\Big) = 0 & in\ \Omega\backslash\overline{\omega}, \\ u|_+ = u|_- & on\ \partial\omega, \\ \nabla u = 0 & in\ \omega, \\ \displaystyle\int_{\partial\omega_s} a_2^{ij}(x)\partial_{x_i} u\nu_j\big|_+ = 0 & (s = 1, 2, \cdots, m), \\ u = \varphi & on\ \partial\Omega. \end{cases} \tag{2.55}$$

We also have similar results:

Theorem 2.5.4. *If $u \in H^1(\Omega)$ satisfies equation (2.55) except for the fourth line, then $u \in C^1(\overline{\Omega\backslash\omega}) \cap C^1(\overline{\omega})$.*

Proof: By the third line of equation (2.55), we have $u \equiv$ const on each component of ω, so $u \equiv$ const on each component of $\partial\omega$. Thus $u \equiv$ const on each component of $\partial(\Omega\backslash\overline{\omega})$.

Since $u \in H^1(\Omega)$ satisfies $\partial_{x_i}\Big(a_2^{ij}(x)\ \partial_{x_i} u_k\Big) = 0$ in $\Omega\backslash\overline{\omega}$, $u|_{\partial\Omega} = \varphi \in C^2(\partial\Omega)$ and $u \equiv$ const on each component of $\partial(\Omega\backslash\overline{\omega})$, by the elliptic regularity theory, we have $u \in C^1(\overline{\Omega\backslash\omega}) \cap C^1(\overline{\omega})$. $\square$

Theorem 2.5.5. *There exists at most one solution $u \in H^1(\Omega) \cap C^1(\overline{\Omega\backslash\omega}) \cap C^1(\overline{\omega})$ of equation (2.55).*

Proof: It is equivalent to showing that if $\varphi = 0$, equation (2.55) only has the solution $u \equiv 0$. Integrating by parts in the first line of equation (2.55), we have

$$\begin{aligned} 0 &= -\int_{\Omega\backslash\overline{\omega}} \partial_{x_j}\Big(a_2^{ij}(x)\ \partial_{x_i} u_k\Big) \cdot u \\ &= \int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i} u \partial_{x_j} u - \int_{\partial\Omega} u \cdot a_2^{ij}(x)\partial_{x_i} u\nu_j\big|_- + \int_{\partial\omega} u \cdot a_2^{ij}(x)\partial_{x_i} u\nu_j\big|_+ \\ &\geq \lambda \int_{\Omega\backslash\overline{\omega}} |\nabla u|^2 - \int_{\partial\Omega} \varphi \cdot a_2^{ij}(x)\partial_{x_i} u\nu_j\big|_- + C_s \int_{\partial\omega_s} a_2^{ij}(x)\partial_{x_i} u\nu_j\big|_+ \\ &= \lambda \int_{\Omega\backslash\overline{\omega}} |\nabla u|^2. \end{aligned}$$

Thus $\nabla u = 0$ in $\Omega\backslash\overline{\omega}$. And since $u = \varphi = 0$ on $\partial\Omega$, we have $u \equiv 0$ in $\Omega\backslash\overline{\omega}$. Since $u|_+ = u|_-$ on $\partial\omega$ and $u \equiv C$ on $\overline{\omega}$, we get $u = 0$ on $\overline{\omega}$. Hence $u \equiv 0$ in Ω, i.e. $u \equiv 0$ is the only solution of (2.55) when $\varphi = 0$. $\square$

Define the energy functional

$$I_\infty[v] := \frac{1}{2}\int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}v\partial_{x_j}v, \tag{2.56}$$

where v belongs to the set

$$\mathcal{A} := \left\{v \in H^1_\varphi(\Omega)\middle|\nabla v \equiv 0 \ in\ \omega\right\}.$$

Theorem 2.5.6. *There exists a minimizer $u \in \mathcal{A}$ satisfying*

$$I_\infty[u] = \min_{v\in\mathcal{A}} I_\infty[v].$$

Moreover, $u \in H^1(\Omega) \cap C^1(\overline{\Omega\backslash\omega}) \cap C^1(\overline{\omega})$ is a solution of equation (2.55).

Proof: By the lower-semi continuity of I_∞ and the weakly closed property of $\mathcal{A}$, it is easy to see that the minimizer $u \in \mathcal{A}$ exists and satisfies $\partial_{x_j}\left(a_2^{ij}(x)\partial_{x_i}u\right) = 0$ in $\Omega\backslash\overline{\omega}$. The only thing which needs to be shown is the fourth line in equation (2.55), i.e.

$$\int_{\partial\omega_s} a_2^{ij}(x)\partial_{x_i}u\nu_j\big|_+ = 0, \quad s = 1, 2, \cdots, m.$$

In fact, since u is a minimizer, for any $\phi \in C_c^\infty(\Omega)$ satisfying $\phi \equiv 1$ on $\overline{\omega}_s$ and $\phi \equiv 0$ on $\overline{\omega}_t (t \neq s)$, let

$$i(t) := I_\infty[u + t\phi] \quad (t \in \mathbb{R}),$$

we have

$$i'(0) := \frac{di}{dt}\Big|_{t=0} = \int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u\phi_{x_j} = 0.$$

Therefore

$$\begin{aligned} 0 &= -\int_{\Omega\backslash\overline{\omega}} \partial_{x_j}\left(a_2^{ij}(x)\ \partial_{x_i}u_k\right)\phi = \int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u\phi_{x_j} + \int_{\partial\omega_s} \phi\cdot a_2^{ij}(x)\partial_{x_i}u\nu_j\big|_+ \\ &= \int_{\partial\omega_s} a_2^{ij}(x)\partial_{x_i}u\nu_j\big|_+, \end{aligned}$$

for all $s = 1, 2, \cdots, m$. $\square$

Finally, we give the relationship between u_k and u.

Theorem 2.5.7. *Let u_k and u in $H^1(\Omega)$ be the solutions of equations (2.53) and (2.55), respectively. Then*

$$u_k \rightharpoonup u \quad in\ H^1(\Omega), \qquad as\ k \to +\infty,$$

and

$$\lim_{k\to+\infty} I_k[u_k] = I_\infty[u],$$

where I_k and I_∞ are defined as (2.54) and (2.56).

Proof: Step 1. By the uniqueness of the solution to equation (2.55), we only need to show that there exists a weak limit u of a subsequence of $\{u_k\}$ in $H^1(\Omega)$ and u is the solution of equation (2.55).

(1) To show that after passing to a subsequence, u_k weakly converges in $H^1(\Omega)$ to some u.

Let $\eta \in H^1_\varphi(\Omega)$ be fixed and satisfy $\eta \equiv 0$ on $\overline{\omega}$, then since u_k is the minimizer of I_k in $H^1_\varphi(\Omega)$, we have

$$\frac{\lambda}{2}\|\nabla u_k\|^2_{L^2(\Omega)} \le I_k[u_k] \le I_k[\eta] = \frac{1}{2}\int_{\Omega\setminus\overline{\omega}} a_2^{ij}(x)\eta_{x_i}\eta_{x_j} \le \frac{\Lambda}{2}\|\eta\|^2_{H^1(\Omega)},$$

i.e.

$$\|\nabla u_k\|_{L^2(\Omega)} \le \|\eta\|_{H^1(\Omega)} \doteq \overline{M},$$

where $\overline{M}$ is independent of k.

Since $u_k = \varphi$ on $\partial\Omega$ and $\sup_k \|u_k\|_{H^1(\Omega)} < \infty$, we have $u_k \rightharpoonup u$ *in* $H^1_\varphi(\Omega)$.

(2) To show that u is a solution of equation (2.55).

In fact, we only need to prove the following three conditions:

$$\partial_{x_j}\Big(a_2^{ij}(x)\ \partial_{x_i} u\Big) = 0 \qquad in\ \Omega\setminus\overline{\omega}, \tag{2.57}$$

$$\nabla u = 0 \qquad in\ \omega, \tag{2.58}$$

$$\int_{\partial\omega_s} a_2^{ij}(x)\partial_{x_i} u_k \nu_j\big|_+ = 0, \qquad s = 1, 2, \cdots, m. \tag{2.59}$$

(i) For every k, since $u_k \in H^1(\Omega)$ is the solution of equation (2.52), then

$\forall\, \phi \in C_c^\infty(\Omega)$, we have

$$k\int_\omega a_1^{ij}(x)\partial_{x_i} u_k \phi_{x_j} + \int_{\Omega\setminus\overline{\omega}} a_2^{ij}(x)\partial_{x_i} u_k \phi_{x_j} = 0.$$

Thus, $\forall\ \phi \in C_c^\infty(\Omega\backslash\overline{\omega}) \subset C_c^\infty(\Omega)$,

$$0 = \int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u_k\phi_{x_j} \longrightarrow \int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u\phi_{x_j},$$

since $u_k \rightharpoonup u\ \ in\ \ H^1_\varphi(\Omega) \subset H^1(\Omega)$.

Therefore,

$$\int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u\phi_{x_j} = 0, \quad \forall\ \phi \in C_c^\infty(\Omega\backslash\overline{\omega}),$$

i.e. (2.57).

(ii) Let $\eta \in H^1_\varphi(\Omega)$ be fixed and satisfy $\eta \equiv 0$ on $\overline{\omega}$, then since u_k is the minimizer of I_k in $H^1_\varphi(\Omega)$, we have

$$\frac{k\lambda}{2}\|\nabla u_k\|^2_{L^2(\omega)} \le I_k[u_k] \le I_k[\eta] = \frac{1}{2}\int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}\eta\partial_{x_j}\eta \le \frac{\Lambda}{2}\|\eta\|^2_{H^1(\Omega)},$$

which implies

$$\|\nabla u_k\|^2_{L^2(\omega)} \to 0, \qquad as\ k \to \infty.$$

By (1), since $u_k \rightharpoonup u\ in\ H^1(\Omega)$, then $u_k \rightharpoonup u\ in\ H^1(\omega)$. Therefore, by the lower-semi continuity, we get

$$\begin{aligned} 0 \le \lambda \int_\omega |\nabla u|^2 \le \int_\omega a_1^{ij}(x)\partial_{x_i}u\partial_{x_j}u \le \int_\omega a_1^{ij}(x)\partial_{x_i}u_k\partial_{x_j}u_k \\ \le \Lambda\|\nabla u_k\|^2_{L^2(\omega)} \longrightarrow 0, \qquad as\ \ k \longrightarrow \infty. \end{aligned}$$

Hence, $\int_\omega |\nabla u|^2 = 0 \implies \nabla u \equiv 0$ in ω, which is just (2.58).

(iii) By (i) and (ii), u satisfies (2.57) and is either constant or φ on each component of $\partial(\Omega\backslash\overline{\omega})$. Thus, $u \in C^2(\overline{\Omega\backslash\omega})$. For each $s = 1, 2, \cdots, m$, we construct a function $\varrho \in C^2(\overline{\Omega\backslash\omega})$, such that $\varrho = 1$ on $\partial\omega_s$, $\varrho = 0$ on $\partial\omega_t (t \neq s)$, and $\varrho = 0$ on $\partial\Omega$.

By Green's Identity, we have the following:

$$\begin{aligned} 0 &= -\int_{\Omega\backslash\overline{\omega}} \partial_{x_j}\Big(a_2^{ij}(x)\ \partial_{x_i}u_k\Big)\varrho \\ &= \int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u_k\partial_{x_j}\varrho - \int_{\partial\Omega} \varrho \cdot a_2^{ij}(x)\partial_{x_i}u_k\nu_j\big|_- + \int_{\partial\omega} \varrho \cdot a_2^{ij}(x)\partial_{x_i}u_k\nu_j\big|_+ \\ &= \int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u_k\partial_{x_j}\varrho + k\int_{\partial\omega_s} a_1^{ij}(x)\partial_{x_i}u_k\nu_j\big|_- \\ &= \int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u_k\partial_{x_j}\varrho. \end{aligned}$$

Similarly,

$$0 = -\int_{\Omega\setminus\overline{\omega}} \partial_{x_j}\left(a_2^{ij}(x)\ \partial_{x_i}u\right)\varrho = \int_{\Omega\setminus\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u\partial_{x_j}\varrho + \int_{\partial\omega_s} a_2^{ij}(x)\partial_{x_i}u\nu_j\big|_+.$$

Since $u_k \rightharpoonup u$ in $H^1(\Omega)$, it follows

$$0 = \int_{\Omega\setminus\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u_k\partial_{x_j}\varrho \longrightarrow \int_{\Omega\setminus\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u\partial_{x_j}\varrho.$$

Thus,

$$\int_{\partial\omega_s} a_2^{ij}(x)\partial_{x_i}u\nu_j\big|_+ = 0,$$

for any $s = 1, 2, \cdots, m$. Therefore, we finish the proof of the first part.

Step 2. Since u_k is a minimizer of I_k and $\nabla u = 0$ in ω, for any $k \in \mathbb{N}$,

$$I_k[u_k] \leq I_k[u] = I_\infty[u].$$

Then $\limsup_{k\to+\infty} I_k[u_k] \leq I_\infty[u]$.

On the other hand, by Theorem 2.5.7, since u is the weak limit of $\{u_k\}$ in $H^1(\Omega)$, we obtain

$$I_\infty[u] = \int_\Omega a_2^{ij}(x)\partial_{x_i}u\partial_{x_j}u \leq \liminf_{k\to+\infty}\int_\Omega a_2^{ij}(x)\partial_{x_i}u_k\partial_{x_j}u_k \leq \liminf_{k\to+\infty} I_k[u_k].$$

Therefore,

$$\lim_{k\to+\infty} I_k[u_k] = I_\infty[u].$$

$\square$

Chapter 3

The perfect and insulated conductivity problems with multiple inclusions

In this chapter, we investigate the two extreme cases of the conductivity problems, i.e. the perfect and insulated conductivity problems, in the general sense that multiple inclusions with extreme conductivity are imbedded in the surrounding matrix.

3.1 Mathematical set-up and the main results

let Ω be a domain in $\mathbb{R}^n$ with $C^{2,\alpha}$ boundary, $n \geq 2$, $0 < \alpha < 1$. Let $\{D_i\}$ $(1 \leq i \leq m)$ be m strictly convex open subsets in Ω with $C^{2,\alpha}$ boundaries, $m \geq 2$, satisfying (1.1)

Given $\varphi \in C^{1,\alpha}(\partial\Omega)$, the conductivity problem can be modelled by the following equation:

$$\begin{cases} div(a_k(x)\nabla u_k) = 0 & \text{in } \Omega, \\ u_k = \varphi & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $k = (k_1, \ldots, k_m)$ and

$$a_k(x) = \begin{cases} k_i \in (0, \infty) & \text{in } D_i, \\ 1 & \text{in } \widetilde{\Omega}. \end{cases} \tag{3.2}$$

The existence and uniqueness of solutions to the above equation is well known. Moreover, we have $\|u_k\|_{H^1(\Omega)} \leq C\|\varphi\|_{C^{1,\alpha}(\partial\Omega)}$ for some constant C independent of k. Therefore, by passing to a subsequence, we have $u_k \rightharpoonup u_\infty$ in $H^1(\Omega)$ as $k \to \infty$, where $u_\infty \in H^1(\Omega)$ is the solution to the following perfect conductivity problem, for details, see e.g. the Appendix of [6],

$$\begin{cases} \Delta u = 0 & in \ \ \widetilde{\Omega}, \\ u|_{+} = u|_{-} & on \ \ \partial D_i, \ (i = 1, 2, \ldots, m), \\ \nabla u \equiv 0 & in \ \ D_i \ (i = 1, 2, \ldots, m), \\ \displaystyle\int_{\partial D_i} \frac{\partial u}{\partial \nu}\Big|_{+} = 0 & (i = 1, 2, \ldots, m), \\ u = \varphi & on \ \ \partial\Omega, \end{cases} \tag{3.3}$$

where

$$\frac{\partial u}{\partial \nu}\Big|_{+} := \lim_{t \to 0^+} \frac{u(x + t\nu) - u(x)}{t}.$$

Here and throughout this paper ν is the outward unit normal to the domain and the subscript $\pm$ indicates the limit from outside and inside the domain, respectively. Since the high stress concentration only occurs in the narrow regions between the fibers, we only need to focus on those narrow regions.

For $i \neq j$, denote

$$\mathrm{dist}(x_{ij}^i, x_{ij}^j) = \mathrm{dist}(D_i, D_j) = \varepsilon_{ij} > 0, \ x_{ij}^i \in \partial D_i, \ x_{ij}^j \in \partial D_j,$$

and

$$x_{ij}^0 := \frac{1}{2}(x_{ij}^i + x_{ij}^j).$$

It is easy to see that there exists some positive constant $\delta < \frac{1}{4}$ which depends only on κ_0, r_0 and $\{\|\partial D_i\|_{C^{2,\alpha}}\}$, but is independent of $\{\varepsilon_{ij}\}$ such that

$$\text{if } \varepsilon_{ij} < 2\delta, \ B(x_{ij}^0, 2\delta) \text{ only intersects with } D_i \text{ and } D_j. \tag{3.4}$$

Denote

$$\rho_n(\varepsilon) = \begin{cases} \dfrac{1}{\sqrt{\varepsilon}} & for \ \ n = 2, \\ \dfrac{1}{\varepsilon|\ln \varepsilon|} & for \ \ n = 3, \\ \dfrac{1}{\varepsilon} & for \ \ n \geq 4. \end{cases} \tag{3.5}$$

Then we have the following gradient estimates for the perfect conductivity problem

Theorem 3.1.1. *Let $\Omega, \{D_i\} \subset \mathbb{R}^n$, $\{\varepsilon_{ij}\}$ be defined as in (1.1), $n \geq 2$, $\varphi \in L^\infty(\partial\Omega)$, δ be the universal constant satisfying* (3.4). *Suppose $u_\infty \in H^1(\Omega)$ is the solution to equation (3.3), then for any $\varepsilon_{ij} < \delta$, we have*

$$\|\nabla u_\infty\|_{L^\infty(\widetilde{\Omega}\cap B(x_{ij}^0,\delta))} \leq C\rho_n(\varepsilon_{ij})\|\varphi\|_{L^\infty(\partial\Omega)}$$

where C is a constant depending only on n, m, κ_0, r_0, $\{\|\partial D_i\|_{C^{2,\alpha}}\}$, but independent of ε_{ij}.

Note that if $\varepsilon_{ij} \geq \delta$, by boundary estimates of harmonic functions, we immediately get $\|\nabla u_\infty\|_{L^\infty(\widetilde{\Omega}\cap B(x_{ij}^0,\delta))} \leq C\|\varphi\|_{L^\infty(\partial\Omega)}$. Then by Theorem 3.1.1 and standard boundary Schauder estimates, see e.g. Theorem 8.33 in [12], we have the global gradient estimates of u_∞ in $\widetilde{\Omega}$.

Corollary 3.1.1. *Let $\Omega, \{D_i\} \subset \mathbb{R}^n$, $\{\varepsilon_{ij}\}$ be defined as in (1.1), $\varepsilon := \min\limits_{i\neq j} \varepsilon_{ij} > 0$, and $\varphi \in C^{1,\alpha}(\partial\Omega)$, $0 < \alpha < 1$, let $u_\infty \in H^1(\Omega)$ be the solution to equation (3.3). Then*

$$\|\nabla u_\infty\|_{L^\infty(\widetilde{\Omega})} \leq C\rho_n(\varepsilon)\|\varphi\|_{C^{1,\alpha}(\partial\Omega)}.$$

where C is a constant depending only on n, m, κ_0, r_0, $\|\partial\Omega\|_{C^{2,\alpha}}$, $\{\|\partial D_i\|_{C^{2,\alpha}}\}$, but independent of ε.

Remark 3.1.1. *Theorem 3.1.1 and Corollary 3.1.1 do not really need D_i and D_j to be strictly convex, the strict convexity is only needed in a fixed neighborhood (the size of the neighborhood is independent of ε) of a pair of points on ∂D_i and ∂D_j which realize minimal distance ε. In fact, our proofs of Theorem 3.1.1 and Corollary 3.1.1 also apply, with minor modification, to more general situations where two closely spaced inclusions, D_i and D_j, are not necessarily convex near points on the boundaries where minimal distance ε is realized; see discussions after the proof of Theorem 3.1.1 in Section 2.*

Next, we study the insulated conductivity problem. Similar to the perfect conductivity problem, the solution to the insulated conductivity problem can also be treated as the weak limit of u_k in $H^1(\widetilde{\Omega})$ as k approaches to 0. Here we consider the insulated conductivity problem with anisotropic conductivity.

Let $\Omega, D_i \subset \mathbb{R}^n$, ε_{ij} be defined as in (1.1), $\varphi \in C^{1,\alpha}(\partial\Omega)$, suppose $A(x) := \big(a^{ij}(x)\big)$ is a symmetric matrix function in $\widetilde{\Omega}$, where $a^{ij}(x) \in C^{\alpha}(\overline{\widetilde{\Omega}})$ and and for constants $\Lambda \geq \lambda > 0$,

$$\|a^{ij}\|_{C^{\alpha}(\overline{\widetilde{\Omega}})} < \Lambda, \qquad a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \ (\forall \xi \in \mathbb{R}^n, \forall x \in \widetilde{\Omega}).$$

Then the anisotropic insulated conductivity problem can be described by the following equation,

$$\begin{cases} \partial_i(a^{ij}\partial_j u) = 0 & \text{in } \widetilde{\Omega}, \\ a^{ij}\partial_j u \nu_i = 0 & \text{on } \partial D_i (i = 1, 2, \ldots, m), \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

The existence and uniqueness of solutions to equation (3.6) are elementary, see the Appendix.

As we mentioned before, the blow-up only occurs in the narrow regions between two closely spaced inclusions. Therefore, we only derive gradient estimates for the solution to (3.6) in those regions. Without loss of generality, we consider the insulated conductivity problem in the narrow region between D_1 and D_2. Assume

$$\varepsilon = \mathrm{dist}(D_1, D_2)$$

After a possible translation and rotation, we may assume

$$(-\varepsilon/2, 0') \in \partial D_1, \ (\varepsilon/2, 0') \in \partial D_2.$$

Here and throughout this paper by writing $x = (x_1, x')$, we mean x' is the last $n-1$ coordinates of x.

We denote the narrow region between D_1 and D_2 and its boundary on ∂D_1 and ∂D_2 as follows

$$\begin{aligned} &\mathcal{O}(r) := \widetilde{\Omega} \cap \{x \in \mathbb{R}^n \big| |x'| < r\} \\ &\Gamma_+ := \partial D_1 \cap \{x \in \mathbb{R}^n \big| |x'| < r\} \\ &\Gamma_- := \partial D_2 \cap \{x \in \mathbb{R}^n \big| |x'| < r\} \end{aligned} \tag{3.7}$$

where r is some universal constant depending only on $\{\|\partial D_i\|_{C^{2,\alpha}}\}$.

Under the above notations, we consider the following problem,

$$\begin{cases} \partial_i(a^{ij}\partial_j u) = 0 & \text{in } \mathcal{O}(r), \\ a^{ij}\partial_j u \nu_i = 0 & \text{on } \Gamma_+ \cup \Gamma_-. \end{cases} \tag{3.8}$$

Then we have:

Theorem 3.1.2. *If $u_0 \in H^1(\mathcal{O}(r))$ is a weak solution of (3.8), then*

$$|\nabla u_0(x)| \leq \frac{C\|u_0\|_{L^\infty(\mathcal{O}(r))}}{\sqrt{\varepsilon + |x'|^2}}, \quad \textit{for any} \;\; x \in \mathcal{O}(\frac{r}{2}). \tag{3.9}$$

where C is a constant depending only on n, r, Λ, λ and $\|\partial D_i\|_{C^{2,\alpha}} (i = 1, 2)$, but independent of ε.

Remark 3.1.2. *It is possible that $\|u_0\|_{L^\infty(\mathcal{O}(r))}$ is infinity, in that case, the above theorem is automatically true. Theorem 3.1.2 also remains true for the general second order elliptic systems, its proof is essentially the same as for the equations.*

As an application of Theorem 3.1.2, we have the global gradient estimates for the insulated conductivity problem

Corollary 3.1.2. *Let $\Omega, \{D_i\} \subset \mathbb{R}^n$, $\{\varepsilon_{ij}\}$ be defined as in (1.1), $\varepsilon := \min\limits_{i \neq j} \varepsilon_{ij} > 0$, and $\varphi \in C^{1,\alpha}(\partial\Omega)$, let $u_0 \in H^1(\widetilde{\Omega})$ be the weak solution to equation (3.6), then*

$$\|\nabla u_0\|_{L^\infty(\widetilde{\Omega})} \leq \frac{C}{\sqrt{\varepsilon}} \|\varphi\|_{C^{1,\alpha}(\partial\Omega)}. \tag{3.10}$$

where C is a constant depending only on n, m, κ_0, r_0, $\|\partial\Omega\|_{C^{2,\alpha}}$, $\{\|\partial D_i\|_{C^{2,\alpha}}\}$, but independent of ε.

Note that through this paper we often use C to denote different constants, but all these constants are independent of ε, in this sense, we will not distinguish them.

The paper is organized as follows. In Section 2 we consider the perfect conductivity problem and prove Theorem 3.1.1. In Section 3 we show Theorem 3.1.2 for the insulated case. Finally in the Appendix we present some elementary results for the insulated conductivity problem.

3.2 The perfect conductivity problem with multiple inclusions

In this section, we consider the perfect conductivity problem (3.3). Note that from equation (3.3), we know that $u \equiv C_i$ on $\overline{D}_i,\ 1 \le i \le m$, where $\{C_i\}$ are some unknown constants. In order to prove Theorem 3.1.1, we first estimate $|C_i - C_j|$ for $1 \le i \ne j \le m$, which later will allow us to control the gradient of u in the narrow region between D_i and D_j.

3.2.1 A Matrix Result

To estimate $|C_i - C_j|$, the following proposition plays a crucial role.

Let m be a positive integer, $P = (p_{ij})$ an $m \times m$ real symmetric matrix satisfying,

$$(A1).\ p_{ij} = p_{ji} \le 0\ (i \ne j);$$
$$(A2).\ r_1 \le \bar{p}_i := \sum_{j=1}^{m} p_{ij} \le r_2,$$

where r_1 and r_2 are some positive constants.

Then we have

Proposition 3.2.1. *Given an integer $m \ge 1$, let $P = (p_{ij})$ be an $m \times m$ real symmetric matrix satisfying $(A1)$ and $(A2)$. for $\beta \in \mathbb{R}^m$, let α be the solution of*

$$P\alpha = \beta, \tag{3.11}$$

then

$$|\alpha_i - \alpha_j| \le m(m-1)\frac{r_2}{r_1}\frac{|\beta|}{|p_{ij}| + r_1}, \tag{3.12}$$

where $|\beta| = \max_i |\beta_i|$.

Remark 3.2.1. *An $m \times m$ matrix P satisfying $|p_{ii}| > \sum_{j \ne i} |p_{ij}|$ is called a diagonally dominant matrix. Such a matrix is nonsingular, see [13]. $(A1)$ and $(A2)$ implies that the matrix P is diagonally dominant, therefore (3.11) has a unique solution.*

Before proving the above theorem we introduce the following lemmas.

Denote

$$\mathcal{I}(l) = \{\text{all } l \times l \text{ diagonal matrices whose diagonal entries are 1 or } -1\},$$

$$\mathcal{I}_e(l) = \{\overline{I} \in \mathcal{I}(l) \big| \overline{I} \text{ has even numbers of } -1 \text{ in its diagonal}\},$$

$$\mathcal{I}_o(l) = \{\overline{I} \in \mathcal{I}(l) \big| \overline{I} \text{ has odd numbers of } -1 \text{ in its diagonal}\}.$$

Then we have

Lemma 3.2.1. *Given a positive integer l, suppose $\mathcal{I}$, $\mathcal{I}_e(l)$, $\mathcal{I}_o(l)$ are defined as above, then for any $x \in \mathbb{R}$ and any $l \times l$ matrix A,*

$$\sum_{\overline{I} \in \mathcal{I}_e(l)} \det(xI + \overline{I}A) \equiv 2^{l-1}(x^l + \det A)$$

$$\sum_{\overline{I} \in \mathcal{I}_o(l)} \det(xI + \overline{I}A) \equiv 2^{l-1}(x^l - \det A)$$

Proof: We prove it by induction. The above identities can be easily checked for $l = 1$. Suppose that the above identities stand for $l = k-1$, we will prove them for $l = k$. Observe that if $x = 0$ then the above identities are true, to prove they are true for any x, it suffices to show that the derivatives with respect to x in both sides of the identities coincide. Since for any $\overline{I} \in \mathcal{I}(k)$,

$$(\det(xI + \overline{I}A))' = \sum_{i=1}^{k} \det(xI + \overline{I}_i A_i)$$

where A_i and $\overline{I}_i$ are the submatrices obtained by eliminating the ith row and the ith column of A and $\overline{I}$ respectively.

Notice that if $\overline{I}$ runs through all the elements of $\mathcal{I}_e(k)$, $\overline{I}_i$ will achieve all the elements of $\mathcal{I}(k-1)$ for any $i \in \{1, 2, \ldots, k\}$, so we have

$$\begin{aligned}
&\sum_{\overline{I} \in \mathcal{I}_e(k)} (\det(xI + \overline{I}A))' \\
&= \sum_{i=1}^{k} \big(\sum_{\overline{I} \in \mathcal{I}_e(k-1)} \det(xI + \overline{I}A_i) + \sum_{\overline{I} \in \mathcal{I}_o(k-1)} \det(xI + \overline{I}A_i) \big) \\
&= \sum_{i=1}^{k} \big(2^{k-2}(x^{k-1} + \det A_i) + 2^{k-2}(x^{k-1} - \det A_i) \big) \qquad \text{(By induction)} \\
&= k2^{k-1}x^{k-1} = 2^{k-1}(x^k + \det A)'.
\end{aligned}$$

Therefore, we have proved the first identity. The second one follows from the first one by changing the sign of one row of A.

As a consequence of Lemma 3.2.1, we have

Corollary 3.2.1. *Let A be an $l \times l$ matrix, if $\det(I + \overline{I}A) \geq 0$ for any $\overline{I} \in \mathcal{I}(l)$, then $|\det A| \leq 1$.*

Lemma 3.2.2. *Given integers $m > l \geq 1$, let $Q = (q_{ij})$ be an $m \times l$ real matrix which satisfies, for $j = 1, 2, \ldots, l$,*

$$q_{jj} > \sum_{i \neq j} |q_{ij}| \tag{3.13}$$

Let $\mathcal{A}$ be the set of all $l \times l$ submatrices of the above matrix Q and $S_1 \in \mathcal{A}$ the matrix obtained from the first s rows of Q. then we have

$$\det S_1 = \max_{S \in \mathcal{A}} |\det S|.$$

Proof: For any $S \in \mathcal{A}$, by rearranging the order of its rows we do not change $|\det S|$. Thus we can treat S as a matrix obtained by replacing some rows of S_1 by some other rows of Q, note that S and S_1 could have no rows in common, which means S is obtained by replacing all the rows of S_1 by some other rows of Q.

Given any $\overline{I} \in \mathcal{I}(l)$, we claim:

$$\det(S_1 + \overline{I}S) \geq 0$$

Proof of the claim: There are two cases between S_1 and S:

Case 1. S_1 and S have no rows in common. Then by (3.13), we know that $S_1 + \overline{I}S$ is diagonally dominant, therefore $\det(S_1 + \overline{I}S) > 0$.

Case 2. S_1 and S have some common rows, denote the order of these rows by $1 \leq i_1 < \cdots < i_s \leq l, 1 \leq s \leq l$. If row i_{s_0} of $\overline{I}S$ is opposite to row i_{s_0} of S for some $1 \leq s_0 \leq s$, then row i_{s_0} of $S_1 + \overline{I}S$ is 0, therefore $\det(S_1 + \overline{I}S) = 0$. Otherwise row i_t of $\overline{I}S$ is the same as that of S and S_1 for any $1 \leq t \leq s$, then we take out the common factors 2 in these rows when we compute the $\det(S_1 + \overline{I}S)$, thus we have

$$\det(S_1 + \overline{I}S) = 2^s \det(S_1 + \overline{I}\hat{S}),$$

where $\hat{S}$ is the matrix obtained by replacing row i_t of S by 0 for any $1 \leq t \leq s$. We know that $S_1 + \overline{I}\hat{S}$ is diagonally dominant according to (3.13), then $\det(S_1 + \overline{I}\hat{S}) > 0$, therefore, $\det(S_1 + \overline{I}S) > 0$.

Since $\det S_1 > 0$ and

$$\det(S_1 + \overline{I}S) = \det(I + \overline{I}SS_1^{-1})\det S_1$$

we have, by the claim, that for any $\overline{I} \in \mathcal{I}(l)$,

$$\det(I + \overline{I}SS_1^{-1}) \geq 0$$

By Corollary 3.2.1, we have

$$|\det(SS_1^{-1})| \leq 1$$

therefore

$$\det S_1 \geq |\det S|.$$

Now we are ready to prove Proposition 3.2.1.

Proof of Proposition 3.2.1: For $m = 1$ the inequality is automatically true. For $m = 2$, we have, by Cramer's rule,

$$\alpha_1 - \alpha_2 = \frac{\begin{vmatrix} \beta_1 & p_{12} \\ \beta_2 & p_{22} \end{vmatrix}}{\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix}} - \frac{\begin{vmatrix} p_{11} & \beta_1 \\ p_{21} & \beta_2 \end{vmatrix}}{\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix}} = \frac{\begin{vmatrix} \beta_1 & \bar{p}_1 \\ \beta_2 & \bar{p}_2 \end{vmatrix}}{\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix}}$$

Since $r_1 \leq \bar{p}_i \leq r_2$ by Condition (A2),

$$\begin{vmatrix} \beta_1 & \bar{p}_1 \\ \beta_2 & \bar{p}_2 \end{vmatrix} = \beta_1\bar{p}_2 - \beta_2\bar{p}_1 \leq 2r_2|\beta|$$

On the other hand, by Condition (A1) and (A2)

$$\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix} = \begin{vmatrix} \bar{p}_1 & p_{12} \\ \bar{p}_2 & p_{22} \end{vmatrix} = \bar{p}_1 p_{22} - \bar{p}_2 p_{12} \geq \bar{p}_1 p_{22} \geq r_1(r_1 + |p_{12}|)$$

Therefore, Proposition 3.2.1 for $m = 2$ follows from the above.

For $m \geq 3$, we only estimate $|\alpha_1 - \alpha_2|$ since the other estimates can be obtained by switching columns of P.

Since α satisfies (3.11), by Cramer's rule, we have:

$$\alpha_1 - \alpha_2 = \frac{\begin{vmatrix} \beta_1 & p_{12} & \cdots & p_{1m} \\ \beta_2 & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_m & p_{m2} & \cdots & p_{mm} \end{vmatrix}}{\begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{vmatrix}} - \frac{\begin{vmatrix} p_{11} & \beta_1 & \cdots & p_{1m} \\ p_{21} & \beta_2 & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & \beta_m & \cdots & p_{mm} \end{vmatrix}}{\begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} \beta_1 & p_{11}+p_{12} & p_{13} & \cdots & p_{1m} \\ \beta_2 & p_{21}+p_{22} & p_{23} & \cdots & p_{2m} \\ \beta_3 & p_{31}+p_{32} & p_{33} & \cdots & p_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_m & p_{m1}+p_{m2} & p_{m3} & \cdots & p_{mm} \end{vmatrix}}{\begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{vmatrix}}$$

By adding the last $(m-2)$ columns of the matrix in the numerator to its second column, we have

$$\alpha_1 - \alpha_2 = \frac{\begin{vmatrix} \beta_1 & \bar{p}_1 & p_{13} & \cdots & p_{1s} \\ \beta_2 & \bar{p}_2 & p_{23} & \cdots & p_{2s} \\ \beta_3 & \bar{p}_3 & p_{33} & \cdots & p_{3s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_m & \bar{p}_m & p_{m3} & \cdots & p_{mm} \end{vmatrix}}{\begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{vmatrix}} := \frac{\det \widetilde{P}}{\det P}.$$

Next we estimate the determinants of the above two matrices separately.

Expanding $\det P$ with respect to the first column, we have

$$\det P = \sum_{j=1}^{m} p_{j1} P_{j1}$$

where P_{ji} is the cofactor of p_{j1}.

Applying Lemma 3.2.2 to the $m \times (m-1)$ matrix obtained by eliminating the first column of P, we know that, among the cofactors P_{j1}, $P_{11} > 0$ has the largest absolute value. Since $p_{j1} = p_{1j} \leq 0$ $(j \neq 1)$ and $p_{11} > 0$ by condition $(A1)$ and $(A2)$, we have

$$\det P \geq \sum_{j=1}^{m} p_{j1} P_{11} = \bar{p}_1 P_{11}.$$

For the same reason, we have

$$P_{11} = \begin{vmatrix} p_{22} & \cdots & p_{2m} \\ \vdots & \ddots & \vdots \\ p_{m2} & \cdots & p_{mm} \end{vmatrix} \geq \Big(\sum_{j=2}^{m} p_{2j}\Big) \begin{vmatrix} p_{33} & \cdots & p_{3m} \\ \vdots & \ddots & \vdots \\ p_{m3} & \cdots & p_{mm} \end{vmatrix}.$$

Combining the above two inequalities and using condition $(A1)$ and $(A2)$, we have

$$\begin{aligned} \det P &\geq \bar{p}_1 \sum_{j=2}^{m} p_{2j} \begin{vmatrix} p_{33} & \cdots & p_{3m} \\ \vdots & \ddots & \vdots \\ p_{m3} & \cdots & p_{mm} \end{vmatrix} = \bar{p}_1(\bar{p}_2 - p_{21}) \begin{vmatrix} p_{33} & \cdots & p_{3m} \\ \vdots & \ddots & \vdots \\ p_{m3} & \cdots & p_{mm} \end{vmatrix} \\ &\geq r_1(|p_{12}| + r_1) \begin{vmatrix} p_{33} & \cdots & p_{3m} \\ \vdots & \ddots & \vdots \\ p_{m3} & \cdots & p_{mm} \end{vmatrix}. \end{aligned} \tag{3.14}$$

By Laplace expansion, see e.g. page 130 of [23], we can expand $\det \widetilde{P}$ with respect to the first two columns of P, namely,

$$\det \widetilde{P} = \sum_{i_1, i_2} \begin{vmatrix} \beta_{i_1} & \bar{p}_{i_1} \\ \beta_{i_2} & \bar{p}_{i_2} \end{vmatrix} \widetilde{P}_{i_1 i_2 12}, \tag{3.15}$$

where $1 \leq i_1 < i_2 \leq m$ and $\widetilde{P}_{i_1 i_2 12}$ is the cofactor of the 2nd-order minor in row i_1, i_2 and column $1, 2$ of $\widetilde{P}$.

Applying Lemma 3.2.2 to the $m \times (m-2)$ matrix obtained by eliminating the first 2 columns of $\widetilde{P}$, we know that, among all those cofactors,

$$\begin{vmatrix} p_{33} & \cdots & p_{3m} \\ \vdots & \ddots & \vdots \\ p_{m3} & \cdots & p_{mm} \end{vmatrix}$$

has the largest absolute value. Since $0 < \bar{p}_i \leq r_2$ by condition $(A2)$,

$$\begin{vmatrix} \beta_{i_1} & \bar{p}_{i_1} \\ \beta_{i_2} & \bar{p}_{i_2} \end{vmatrix} \leq 2r_2|\beta|,$$

then by (3.15), we have

$$\left|\det \widetilde{P}\right| \leq m(m-1)r_2|\beta| \begin{vmatrix} p_{33} & \cdots & p_{3m} \\ \vdots & \ddots & \vdots \\ p_{m3} & \cdots & p_{mm} \end{vmatrix}. \tag{3.16}$$

By (3.14) and (3.16), we have

$$|\alpha_1 - \alpha_2| = \frac{|\det \widetilde{P}|}{|\det P|} \leq m(m-1)\frac{r_2}{r_1}\frac{|\beta|}{|p_{12}| + r_1}.$$

3.2.2 Proof of Theorem 3.1.1

We decompose u_∞ into $m+1$ parts:

$$u_\infty = v_0 + \sum_{i=1}^{m} C_i v_i, \tag{3.17}$$

where $v_i \in H^1(\widetilde{\Omega})$ $(i = 0, 1, 2, \ldots, m)$ are determined by the following equations:

for $i = 0$,

$$\begin{cases} \Delta v_0 = 0 & in \quad \widetilde{\Omega}, \\ v_0 = 0 & on \;\; \partial D_1, \; \partial D_2, \ldots \; \partial D_m, \\ v_0 = \varphi & on \;\; \partial \Omega. \end{cases} \tag{3.18}$$

for $i = 1, 2, \ldots, m$,

$$\begin{cases} \Delta v_i = 0 & in \quad \widetilde{\Omega}, \\ v_i = 1 & on \;\; \partial D_i, \\ v_i = 0 & on \;\; \partial D_j, \text{ for } j \neq i, \\ v_i = 0 & on \;\; \partial \Omega. \end{cases} \tag{3.19}$$

Since u_∞ satisfies the integral conditions in equation (3.3), using the decomposition formula (3.17), we know that the vector $(C_1, C_2, \ldots, C_m)$ satisfies the following system of linear equations

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_m \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \tag{3.20}$$

where

$$a_{ij} := \int_{\partial D_j} \frac{\partial v_i}{\partial \nu}, \quad (i, j = 1, 2, \ldots, m), \tag{3.21}$$

$$b_i := -\int_{\partial D_i} \frac{\partial v_0}{\partial \nu}, \quad (i = 1, 2, \ldots, m). \tag{3.22}$$

Similar to the two inclusions case in [6], we first investigate the properties of v_i $(i = 0, 1, \cdots, m)$, the matrix A and the vector b in equation (3.20). Here we state several lemmas, for their proofs, readers may refer to Lemma 2.1, Lemma 2.3, and Lemma 2.4 in [6].

Lemma 3.2.3. *Let $v_0, v_i (i = 1, \ldots, m)$ be the solutions of equations (3.18) and (3.19) respectively, then there exists a universal constant C depending only on n, m, r_0, k_0, ∂D_i and $\partial\Omega$, but independent of ε_{ij} such that,*

$$\|\nabla v_0\|_{L^\infty(\widetilde{\Omega})} \le C, \quad \|\nabla v_i\|_{L^\infty(\widetilde{\Omega})} \le \frac{C}{\varepsilon}, \quad \|\frac{\partial v_i}{\partial \nu}\|_{L^\infty(\partial\Omega)} \le C.$$

Lemma 3.2.4. *For $1 \le i,\ j \le m$, let a_{ij} and b_i be defined by (3.21) and (3.22), then they satisfy the following:*

1) $a_{ii} < 0, \quad a_{ij} = a_{ji} > 0 \ (i \ne j)$,

2) $-C \le \sum_{1 \le j \le m} a_{ij} \le -\frac{1}{C}$,

3) $|b_i| \le C \|\varphi\|_{L^\infty(\partial\Omega)}$,

where $C > 0$ is a universal constant depending only on n, r_0, $\partial\Omega$, but independent of ε_{ij}.

Remark 3.2.2. *From property (1) and (2) in Lemma 3.2.4, we know that A is diagonally dominant, therefore it is nonsingular.*

Next, we derive some further estimates of a_{ij}.

Lemma 3.2.5. *Let a_{ij} be defined as in (3.21), if $\varepsilon_{ij} < 1/2$, then there exists a universal constant $C > 0$, depending only on n, m, r_0, κ_0, $\{\|\partial D_i\|_{C^{2,\alpha}}\}$ and $\|\partial\Omega\|_{C^{2,\alpha}}$, but independent of ε_{ij}, such that for any $1 \le i \neq j \le m$,*

$$\begin{aligned}
-\frac{C}{\sqrt{\min\limits_{j\neq i}\varepsilon_{ij}}} < a_{ii} < -\frac{1}{C\sqrt{\min\limits_{j\neq i}\varepsilon_{ij}}}, &\qquad \frac{1}{C\sqrt{\varepsilon_{ij}}} < a_{ij} < \frac{C}{\sqrt{\varepsilon_{ij}}}, &\qquad \textit{for } n = 2,\\
-C|\ln(\min_{j\neq i}\varepsilon_{ij})| < a_{ii} < -\frac{1}{C}|\ln(\min_{j\neq i}\varepsilon_{ij})|, &\qquad \frac{1}{C}|\ln\varepsilon_{ij}| < a_{ij} < C|\ln\varepsilon_{ij}|, &\qquad \textit{for } n = 3,\\
-C < a_{ii} < -\frac{1}{C}, &\qquad \frac{1}{C} < a_{ij} < C, &\qquad \textit{for } n \ge 4.
\end{aligned}$$

Proof: Without loss of generality, we assume $i = 1, j = 2$. The proof of the estimates for a_{11} and a_{22} is the same as that in Lemma 2.5, Lemma 2.6, and Lemma 2.7 in [6]. Here we prove the estimate for a_{12}. In this following, we use C to represent the universal constants depending only on n, m, r_0, κ_0, $\{\|\partial D_i\|_{C^{2,\alpha}}\}$ and $\|\partial\Omega\|_{C^{2,\alpha}}$, but independent of $\{\varepsilon_{ij}\}$.

Notice that if ε_{12} is larger than some universal constant, then the proof is trivial. Therefore, we can assume $\varepsilon_{12} < \delta$, where $\delta < 1/4$ is the universal constant satisfying (3.4). By (3.4), we know that $B(x_{12}^0, \delta)$ only intersects with D_1 and D_2.

Denote

$$\Gamma_i := \partial D_i \cap B(x_{12}^0, \delta)\ (i = 1, 2), \quad \Gamma_3 := \partial B(x_{12}^0, \delta) \setminus (D_1 \cup D_2)$$

Since $B(x_{12}^0, 2\delta)$ does not intersect with $D_i (i \ge 3)$ or $\partial\Omega$ by (3.4), then

$$dist(\Gamma_3, \cup_{i=3}^m \partial D_i) > \delta, \quad dist(\Gamma_3, \partial\Omega) > \delta,$$

by the gradient estimates and boundary estimates, we have

$$\|\nabla v_1\|_{L^\infty(\Gamma_3)} < C \tag{3.23}$$

Next we show $\|\nabla v_1\|_{L^\infty(\partial D_2\setminus\Gamma_2)} < C$. Since the tangential derivatives of v_1 on ∂D_2 is 0, we only need to consider its normal derivative.

Let $\widetilde{v}_1 \in H^1(\Omega \setminus (\overline{D_1 \cup D_2}))$ be the solution to the following equation

$$\begin{cases} \Delta \widetilde{v}_1 = 0, \text{ in } \Omega \setminus (\overline{D_1 \cup D_2}) \\ \widetilde{v}_1 = 1 \text{ on } \partial D_1 \\ \widetilde{v}_1 = 0 \text{ on } \partial D_2 \cup \partial\Omega \end{cases}$$

Then $\widetilde{v}_1 - v_1 \geq 0$ in $\widetilde{\Omega}$ by the maximum principle. Since $\widetilde{v}_1 - v_1 = 0$ on ∂D_2, by the Hopf Lemma, we have

$$\frac{\partial \widetilde{v}_1}{\partial \nu} > \frac{\partial v_1}{\partial \nu} > 0 \text{ on } \partial D_2.$$

But by boundary estimates of harmonic functions, we have

$$\|\nabla v_1\|_{L^\infty(\partial D_2 \setminus \Gamma_2)} \leq C \tag{3.24}$$

By (3.24), we have

$$\begin{aligned} a_{12} = \int_{\partial D_2} \frac{\partial v_1}{\partial \nu} &= \int_{\Gamma_2} \frac{\partial v_1}{\partial \nu} + \int_{\partial D_2 \setminus \Gamma_2} \frac{\partial v_1}{\partial \nu} \\ &= \int_{\Gamma_2} \frac{\partial v_1}{\partial \nu} + O(1). \end{aligned} \tag{3.25}$$

By the harmonicity of v_1 on $B(x_{12}^0, \delta) \cap \widetilde{\Omega}$ and (3.23), we have

$$\begin{aligned} 0 &= \int_{\Gamma_1} \frac{\partial v_1}{\partial \nu} + \int_{\Gamma_2} \frac{\partial v_1}{\partial \nu} + \int_{\Gamma_3} \frac{\partial v_1}{\partial \nu} \\ &= \int_{\Gamma_1} \frac{\partial v_1}{\partial \nu} + \int_{\Gamma_2} \frac{\partial v_1}{\partial \nu} + O(1). \end{aligned} \tag{3.26}$$

Meanwhile, by Green's formula and (3.23), we have

$$\begin{aligned} -\int_{B(x_{12}^0,\delta)\cap\widetilde{\Omega}} |\nabla v_1|^2 &= \int_{\Gamma_1} v_1 \frac{\partial v_1}{\partial \nu} + \int_{\Gamma_2} v_1 \frac{\partial v_1}{\partial \nu} + \int_{\Gamma_3} v_1 \frac{\partial v_1}{\partial \nu} \\ &= \int_{\Gamma_1} \frac{\partial v_1}{\partial \nu} + \int_{\Gamma_3} v_1 \frac{\partial v_1}{\partial \nu} \\ &= \int_{\Gamma_1} \frac{\partial v_1}{\partial \nu} + O(1) \end{aligned} \tag{3.27}$$

Therefore, by combining (3.25), (3.26) and (3.27), we have

$$a_{12} = \int_{B(x_{12}^0,\delta)\cap\widetilde{\Omega}} |\nabla v_1|^2 + O(1).$$

Similar to the energy estimates given in Lemma 1.5, Lemma 1.6, and Lemma 1.7 in [6], we have

$$\begin{aligned}
\frac{1}{C\sqrt{\varepsilon_{12}}} &< \int_{B(x_{12}^0,\delta)\cap\widetilde{\Omega}} |\nabla v_1|^2 < \frac{C}{\sqrt{\varepsilon_{12}}}, && \text{for } n=2\\
\frac{1}{C}|\ln\varepsilon_{12}| &< \int_{B(x_{12}^0,\delta)\cap\widetilde{\Omega}} |\nabla v_1|^2 < C|\ln\varepsilon_{12}|, && \text{for } n=3\\
\frac{1}{C} &< \int_{B(x_{12}^0,\delta)\cap\widetilde{\Omega}} |\nabla v_1|^2 < C, && \text{for } n\geq 4.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{C\sqrt{\varepsilon_{12}}} &< a_{12} < \frac{C}{\sqrt{\varepsilon_{12}}}, && \text{for } n=2,\\
\frac{1}{C}|\ln\varepsilon_{12}| &< a_{12} < C|\ln\varepsilon_{12}|, && \text{for } n=3,\\
\frac{1}{C} &< a_{12} < C, && \text{for } n\geq 4.
\end{aligned}$$

□

Knowing enough properties of the system of linear equations (3.20) from Lemma 3.2.4 and Lemma 3.2.5 , we have

Proposition 3.2.2. *Let $u_\infty \in H^1(\Omega)$ be the weak solution to equation (3.3) and C_i the value of u_∞ on D_i, then for any $1 \leq i \neq j \leq m$ with $\varepsilon_{ij} < \delta$, there exists a universal constant $C > 0$ depending only on n, m, κ_0, r_0, $\|\partial\Omega\|_{C^{2,\alpha}}$, $\{\|\partial D_i\|_{C^{2,\alpha}}\}$, but independent of $\{\varepsilon_{ij}\}$ such that*

$$\begin{aligned}
|C_i - C_j| &\leq C\sqrt{\varepsilon_{ij}}\|\varphi\|_{L^\infty(\partial\Omega)} && for\;\; n=2,\\
|C_i - C_j| &\leq C\frac{1}{|\ln\varepsilon_{ij}|}\|\varphi\|_{L^\infty(\partial\Omega)} && for\;\; n=3, \\
|C_i - C_j| &\leq C\|\varphi\|_{L^\infty(\partial\Omega)} && for\;\; n\geq 4.
\end{aligned} \tag{3.28}$$

Proof: By Lemma 3.2.4, we know that the matrix $-A$ satisfies condition $(A1)$ and $(A2)$, then applying Proposition 3.2.1 on (3.20), we have, for any $1 \leq i \neq j \leq m$,

$$|C_i - C_j| \leq \frac{C}{a_{ij}}\|\varphi\|_{L^\infty(\partial\Omega)}$$

where C is some constant depending on n, m, κ_0, r_0, $\|\partial\Omega\|_{C^{2,\alpha}}$, $\{\|\partial D_i\|_{C^{2,\alpha}}\}$, but independent of $\{\varepsilon_{ij}\}$.

By Lemma 3.2.5, we immediately finish the proof. □

Now we are ready to complete the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1: We prove the estimates in dimension 2, the higher dimension cases follow from the same idea. Without loss of generality, we assume $\varepsilon_{12} < \delta$ and prove the gradient estimates for u_∞ in the narrow region between D_1 and D_2. For simplicity, we assume $\|\varphi\|_{L^\infty(\partial\Omega)} = 1$.

By the decomposition formula (3.17), we have

$$\nabla u_\infty = (C_1 - C_2)\nabla v_1 + C_2(\nabla(v_1 + v_2)) + \sum_{i=3}^{m} C_i \nabla v_i + \nabla v_0$$

By Lemma (3.2.3), we have

$$\|\nabla v_1\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))} < \frac{C}{\varepsilon_{12}}, \quad \|\nabla v_0\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))} < C \tag{3.29}$$

where C is some universal constant.

Next we show that, for $i = 3, \dots, m$,

$$\|\nabla v_i\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))} < C, \quad \|\nabla(v_1 + v_2)\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))} < C. \tag{3.30}$$

Let $\widetilde{v}_3$ be the solution of the following equation,

$$\begin{cases} \Delta \widetilde{v}_3 = 0 & \text{in} \quad \Omega \setminus \overline{D_1 \cup D_3}, \\ \widetilde{v}_3 = 0 & \text{on} \ \ \partial D_1, \\ \widetilde{v}_3 = 1 & \text{on} \ \ \partial D_3, \\ \widetilde{v}_3 = 0 & \text{on} \ \ \partial\Omega. \end{cases}$$

Then we have $\widetilde{v}_3 \geq v_3$ on $\partial\widetilde{\Omega}$, by the maximum principle, $\widetilde{v}_3 \geq v_3$ in $\widetilde{\Omega}$. Since $\widetilde{v}_3 = v_3 = 0$ on ∂D_1, by the Hopf lemma, we have

$$\frac{\partial \widetilde{v}_3}{\partial \nu} > \frac{\partial v_3}{\partial \nu} > 0$$

But $|\nabla \widetilde{v}_3| < C$ on $\partial D_1 \cap B(x_{12}^0, \delta)$ by the boundary estimates of harmonic functions, then we have

$$\|\nabla v_3\|_{L^\infty(\partial D_1 \cap B(x_{12}^0,\delta))} = \|\frac{\partial v_3}{\partial \nu}\|_{L^\infty(\partial D_1 \cap B(x_{12}^0,\delta))} < C \tag{3.31}$$

Similarly, we have

$$\|\nabla v_3\|_{L^\infty(\partial D_2\cap B(x_{12}^0,\delta))} = \|\frac{\partial v_3}{\partial \nu}\|_{L^\infty(\partial D_2\cap B(x_{12}^0,\delta))} < C \tag{3.32}$$

Furthermore, by gradient estimates and boundary estimates of harmonic functions, we have

$$\|\nabla v_3\|_{\partial B(x_{12}^0,\delta)\cap\widetilde{\Omega}} < C. \tag{3.33}$$

Since ∇v_3 is still harmonic function on $B(x_{12}^0,\delta)\cap\widetilde{\Omega}$, by (3.31), (3.32) and (3.33) and the maximum principle, we have

$$\|\nabla v_3\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))} < C.$$

Similarly, we get, for $i = 3,\dots,m$,

$$\|\nabla v_i\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))} < C.$$

Since $v_1 + v_2 = 1$ on both ∂D_1 and ∂D_2, similar to the proof in the above, we can show that

$$\|\nabla(v_1+v_2)\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))} < C.$$

By Proposition 3.2.2, (3.29) and (3.30), we have

$$\begin{aligned}\|\nabla u_\infty\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))} &\le |C_1 - C_2|\|\nabla v_1\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))} + |C_2|\|\nabla(v_1+v_2)\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))}\\ &\quad + \sum_{i=3}^m |C_i|\|\nabla v_i\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))} + \|\nabla v_0\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))}\\ &\le C\sqrt{\varepsilon_{12}}\frac{1}{\varepsilon_{12}} + C\\ &\le \frac{C}{\sqrt{\varepsilon_{12}}}.\end{aligned}$$

As we mentioned in Remark 3.1.1, the strict convexity assumption of the two inclusions can be weakened. In fact, our proof of Theorem 3.1.1 applies, with minor modification, to more general inclusions.

In $\mathbb{R}^n$, $n \ge 2$, for two closely spaced inclusions D_i and D_j which are not necessarily strictly convex, assume $\partial D_i \cap B(0,r)$ and $\partial D_j \cap B(0,r)$ can be represented by the graph of $x_1 = f(x') + \frac{\varepsilon_{ij}}{2}$ and $x_1 = -g(x') - \frac{\varepsilon_{ij}}{2}$, then $f(0') = g(0') = 0$, $\nabla(g+f)(0') = 0$. Assume further that

$$\lambda_1|x'|^{2l} \le g(x') + f(x') \le \lambda_2|x'|^{2l}, \quad \forall |x'| \le r/2, \tag{3.34}$$

where $\lambda_2 \ge \lambda_1 > 0, l \in \mathbb{Z}^+$.

Under the above assumption, let $u_\infty \in H^1(\Omega)$ be the solution to equation (3.3). Then, for ε_{ij} sufficiently small, we have

$$\begin{aligned}
&\|\nabla u_\infty\|_{L^\infty(\widetilde{\Omega}\cap B(x^0_{ij},\delta))} \le C\|\varphi\|_{L^\infty(\partial\Omega)}\varepsilon_{ij}^{-\frac{n-1}{2l}} && \textit{if } n-1<2l,\\
&\|\nabla u_\infty\|_{L^\infty(\widetilde{\Omega}\cap B(x^0_{ij},\delta))} \le C\|\varphi\|_{L^\infty(\partial\Omega)}\frac{1}{\varepsilon_{ij}|\ln\varepsilon_{ij}|} && \textit{if } n-1=2l,\\
&\|\nabla u_\infty\|_{L^\infty(\widetilde{\Omega}\cap B(x^0_{ij},\delta))} \le C\|\varphi\|_{L^\infty(\partial\Omega)}\frac{1}{\varepsilon_{ij}} && \textit{if } n-1>2l.
\end{aligned} \tag{3.35}$$

where C is a constant depending on n, λ_1, λ_2, r_0, $\|\partial D_i\|_{C^{2,\alpha}}$ and $\|\partial D_j\|_{C^{2,\alpha}}$, but independent of ε_{ij}.

For the proof, please refer to the corresponding discussion after the proof of Theorem 0.1-0.2 in [6].

3.3 The insulated conductivity problem

In this section, we consider the anisotropic insulated conductivity problem, which is described by Equation (3.6). As we mentioned in the introduction, the gradient only blows up when two inclusions are close to each other. In order to establish the gradient estimates for this problem, we first consider the local version of the problem, namely Equation (3.8).

To make the problem easier, we first consider the equation in a strip. In this case, by using a "flipping" technique, we manage to derive the gradient estimates in the strip. Denote, for any integer l

$$\begin{aligned}
&\mathcal{Q}_l := \{z \in \mathbb{R}^n \big| (2l-1)\delta < z_1 < (2l+1)\delta, |z'| \le 1\},\\
&\Gamma_l^+ := \{z \in \mathbb{R}^n \big| z_1 = (2l+1)\delta \text{ and } |z'| \le 1\},\\
&\Gamma_l^- := \{z \in \mathbb{R}^n \big| z_1 = (2l-1)\delta \text{ and } |z'| \le 1\},
\end{aligned}$$

and

$$\mathcal{Q} = \{z \in \mathbb{R}^n \big| |z_1| \le 1 \text{ and } |z'| \le 1\}.$$

We consider the following equation in $\mathcal{Q}_0$

$$\begin{cases} \partial_{z_i}\Big(b^{ij}(z)\ \partial_{z_j} w\Big) = 0 & \text{in } \mathcal{Q}_0, \\ b_{1j}\partial_{z_j} w = 0 & \text{on } \Gamma_0^{\pm}. \end{cases} \tag{3.36}$$

where $(b^{ij}) \in C^{\alpha}(\overline{\mathcal{Q}}_0)(0 < \alpha < 1)$ is a symmetric matrix function in $\mathcal{Q}_0$, and there exist constants $\Lambda_2 \ge \lambda_1 > 0$ such that, for all $\xi \in \mathbb{R}^n$,

$$\|b^{ij}(z)\|_{C^{\alpha}(\overline{\mathcal{Q}}_0)} \le \Lambda_2, \qquad \lambda_2|\xi|^2 \le b^{ij}(z)\xi_i\xi_j\ (\forall z \in \mathcal{Q}_0).$$

Then we have

Lemma 3.3.1. *Suppose $w \in H^1(\mathcal{Q}_0) \cap L^{\infty}(\mathcal{Q}_0)$ is a weak solution of (3.36), then there exists a constant $C > 0$ depending only on $n,\ \lambda_2, \Lambda_2$, but independent of δ, such that*

$$\|\nabla w\|_{L^{\infty}(\mathcal{Q}_0(\frac{1}{2}))} \le C\|w\|_{L^{\infty}(\mathcal{Q}_0)},$$

where $\mathcal{Q}_0(\frac{1}{2}) := \{z \in \mathbb{R}^n \big| |z_1| \le \delta$ and $|z'| \le \frac{1}{2}\}$.

Proof: For any integer l, We construct a new function $\widetilde{w}$ by "flipping" w evenly in each $\mathcal{Q}_l$. $\widetilde{w}$ is defined by induction on l. We first define $\widetilde{w} = w$ in $\mathcal{Q}_0$. Suppose $\widetilde{w}$ is defined in $\mathcal{Q}_{\pm(l-1)}$ for some $l \ge 1$, we define $\widetilde{w}$ in $\mathcal{Q}_l$ and $\mathcal{Q}_{-l}$ in the following

$$\widetilde{w}(z) = \begin{cases} \widetilde{w}((4l-2)\delta - z_1, z') & \text{if } z \in \mathcal{Q}_l, \\ \widetilde{w}(-(4l-2)\delta - z_1, z') & \text{if } z \in \mathcal{Q}_{-l}. \end{cases} \tag{3.37}$$

In this way, we can define $\widetilde{w}$ in $\mathcal{Q}$ by the above flipping process.

Similarly we define the elliptic coefficients by induction on l in the following

Let $\widetilde{b}^{ij} = b^{ij}$ in $\mathcal{Q}_0$. Suppose $\widetilde{b}^{ij}$ is defined in $\mathcal{Q}_{\pm(l-1)}$ for $l \ge 1$, then $\widetilde{b}^{ij}$ on $\mathcal{Q}_l$ and $\mathcal{Q}_{-l}$ is defined as follows,

for $\alpha = 2, 3, \ldots, n$,

$$\widetilde{b}^{\alpha 1}(z) = \widetilde{b}^{1\alpha}(z) = \begin{cases} -\widetilde{b}^{1\alpha}((4l-2)\delta - z_1, z') & \text{if } z \in \mathcal{Q}_l; \\ -\widetilde{b}^{1\alpha}((4l+2)\delta - z_1, z') & \text{if } z \in \mathcal{Q}_{-l}. \end{cases} \tag{3.38}$$

for all other indices

$$\widetilde{b}^{ij}(z) = \begin{cases} \widetilde{b}^{ij}((4l-2)\delta - z_1, z') & \text{if } z \in \mathcal{Q}_l; \\ \widetilde{b}^{ij}(-(4l-2)\delta - z_1, z') & \text{if } z \in \mathcal{Q}_{-l}. \end{cases} \tag{3.39}$$

Under the above definition of $\widetilde{w}$ and $\widetilde{b}^{ij}$, we can easily check that, for any integer l,

$$\begin{cases} \partial_{z_i}\Big(\widetilde{b}^{ij}(z)\ \partial_{z_j}\widetilde{w}\Big) = 0 & \text{in } \mathcal{Q}_l, \\ \widetilde{b}_{1j}\partial_{z_j}\widetilde{w} = 0 & \text{on } \Gamma_l^{\pm}, \end{cases} \tag{3.40}$$

Then for any test function $\psi \in C_0^\infty(\mathcal{Q})$, we have

$$\begin{aligned} \int_{\mathcal{Q}} \widetilde{b}^{ij}(z)\ \partial_{z_j}\widetilde{w}\partial_{z_i}\psi &= \sum_l \int_{\mathcal{Q}_l} \widetilde{b}^{ij}(z)\ \partial_{z_j}\widetilde{w}\partial_{z_i}\psi \\ &= 0 \quad \big(\text{by the definition of weak solution}\big) \end{aligned}$$

Therefore $\widetilde{w} \in H^1(\mathcal{Q})$ satisfies

$$\partial_{z_j}\big(\widetilde{b}^{ij}(z)\ \partial_{z_i}\widetilde{w}\big) = 0 \qquad \text{in } \mathcal{Q}. \tag{3.41}$$

Following exactly from [17], we first introduce a new equation

$$\partial_{z_i}\big(\widetilde{B}^{ij}(z)\ \partial_{z_j}u\big) = 0 \qquad \text{in } \mathcal{Q}$$

where

$$\widetilde{B}^{ij}(z) = \begin{cases} \lim_{z\in\mathcal{Q}_l,\ z\to((2l-1)\delta,\ 0')} \widetilde{b}^{ij}(z) & z \in \mathcal{Q}_l, l > 0; \\ \widetilde{b}^{ij}(0) & z \in \mathcal{Q}_0 \\ \lim_{z\in\mathcal{Q}_l,\ z\to((2l+1)\delta,\ 0')} \widetilde{b}^{ij}(z) & z \in \mathcal{Q}_l, l < 0; \end{cases}$$

then we define the norm

$$\|F\|_{Y^{s,p}} = \sup_{0<r<1} r^{1-s}(⨍_{r\mathcal{Q}}|F|^p)^{\frac{1}{p}}$$

Since $b^{ij}(z) \in C^\alpha(\overline{\mathcal{Q}}_0)$, $\widetilde{b}^{ij}(z)$ is piecewise C^α continuous in $\mathcal{Q}$, then we can immediately check that

$$\|\widetilde{b}^{ij} - \widetilde{B}^{ij}\|_{Y^{1+\alpha,2}} < C$$

where C is some constant only depending on Λ_2. Using Proposition 4.1 in [17], we have

$$\|\nabla\widetilde{w}\|_{L^\infty(\frac{1}{2}\mathcal{Q})} \le C\|\widetilde{w}\|_{L^2(\mathcal{Q})} \le C\|\widetilde{w}\|_{L^\infty(\mathcal{Q})},$$

therefore

$$\|\nabla w\|_{L^\infty(\mathcal{Q}_0(\frac{1}{2}))} \le C\|w\|_{L^\infty(\mathcal{Q}_0)}$$

where $C > 0$ depends on n, λ_2, Λ_2, but is independent of δ. $\square$

Since D_1 and D_2 are strictly convex, we can write $\mathcal{O}(r)$, which is defined by (3.7), as follows

$$\mathcal{O}(r) = \{x \in \mathbb{R}^n \big| -g(x') - \varepsilon/2 < x_1 < f(x') + \varepsilon/2,\ |x'| < r\}$$

With the side boundary Γ_+ and Γ_- as

$$\Gamma_+ = \{x \in \mathbb{R}^n \big| x_1 = f(x') + \varepsilon/2, |x'| < r\},\ \Gamma_- = \{x \in \mathbb{R}^n \big| x_1 = -g(x') - \varepsilon/2, |x'| < r\}$$

where $f(x')$ and $g(x')$ are strictly convex functions, moreover they satisfy

$$f(0') = g(0') = 0,\ \nabla f(0') = \nabla g(0') = 0.$$

Under the above notation, we prove Theorem 3.1.2:

Proof of Theorem 3.1.2: Fix one point $(0, x_0') \in \mathcal{O}(\frac{r}{2})$ and let $\delta = \sqrt{f(x_0') + g(x_0') + \varepsilon}$, since $f(x')$ and $g(x')$ are strictly convex, then there exists a universal constant C depending only on $\|\partial D_1\|_{C^{2,\alpha}}$ and $\|\partial D_2\|_{C^{2,\alpha}}$ such that

$$\frac{1}{C}\sqrt{|x_0'|^2 + \varepsilon} < \delta < C\sqrt{|x_0'|^2 + \varepsilon}. \tag{3.42}$$

We shift the origin to $(0, x_0')$ and rescale the coordinates with δ, then the new coordinates $y = (y_1, y')$ can be written as follows

$$\begin{cases} y_1 = x_1/\delta, \\ y' = (x' - x_0')/\delta. \end{cases} \tag{3.43}$$

Let

$$v(y) = u_0(\delta y_1, x_0' + \delta y'), \qquad \widetilde{a}^{ij}(y) = a^{ij}(\delta y_1, x_0' + \delta y').$$

Denote

$$\widetilde{\mathcal{O}}(\widetilde{r}) := \{y \in \mathbb{R}^n \big| -\frac{\varepsilon}{2} - g(x_0' + \delta y') < \delta y_1 < \frac{\varepsilon}{2} + f(x_0' + \delta y'),\ |y'| < \widetilde{r}\}$$

With its side boundary

$$\widetilde{\Gamma}_+ := \{y \in \mathbb{R}^n \big| \delta y_1 = \frac{\varepsilon}{2} + f(y_0' + \delta y'),\ |y'| < \widetilde{r}\}$$
$$\widetilde{\Gamma}_- := \{y \in \mathbb{R}^n \big| \delta y_1 = -\frac{\varepsilon}{2} - g(y_0' + \delta y'),\ |y'| < \widetilde{r}\}.$$

By (3.42), we can find some universal constant $\widetilde{r}$ depending only on ∂D_1 and ∂D_2, such that $\widetilde{\mathcal{O}}(\widetilde{r})$ is in the image of $\mathcal{O}(r)$ under the above transform. Thus we have

$$\begin{cases} \partial_{y_i}(\widetilde{a}^{ij}\partial_{y_j} v(y)) = 0 & \text{in} \quad \widetilde{\mathcal{O}}(\widetilde{r}), \\ \widetilde{a}^{ij}\partial_{y_j} v \nu_i = 0 & \text{on} \quad \widetilde{\Gamma}_+ \cup \widetilde{\Gamma}_-. \end{cases} \tag{3.44}$$

where the coefficients $\widetilde{a}^{ij}$ satisfy, for some universal constant C,

$$\|\widetilde{a}^{ij}\|_{C^\alpha(\widetilde{\mathcal{O}}(\widetilde{r}))} \le C\|a^{ij}\|_{C^\alpha(\mathcal{O}(r))} \le C\Lambda_1,\ \lambda_1|\xi|^2 \le \widetilde{a}^{ij}(y)\xi_i\xi_j\ (\forall y \in \widetilde{\mathcal{O}}(\widetilde{r}),\ \forall \xi \in \mathbb{R}^n).$$

Next we can construct a map $\Phi : \widetilde{\mathcal{O}}(\widetilde{r}) \longmapsto \mathcal{Q}_0$, $\Phi(y) = z$ with

$$\begin{cases} z_1 = \delta \dfrac{\delta y_1 + g(x_0' + \delta y') + \varepsilon/2}{f(x_0' + \delta y') + g(x_0' + \delta y') + \varepsilon}, \\ z' = \dfrac{y'}{\widetilde{r}}. \end{cases} \tag{3.45}$$

It can be verified directly that this map is a diffeomorphism from $\widetilde{\mathcal{O}}(\widetilde{r})$ to $\mathcal{Q}_0$.

Let

$$w(z) = v(\Phi^{-1}(z))$$

Then from the definition of weak solution, we know that $w(z)$ satisfies the following equation

$$\begin{cases} \partial_{z_j}\big(b^{ij}(z)\partial_{z_i} w(z)\big) = 0 & \text{in } \mathcal{Q}_0, \\ b^{1i}(z)\partial_{z_i} w(z) = 0 & \text{on } \Gamma_0^+ \cup \Gamma_0^-. \end{cases} \tag{3.46}$$

where

$$\big(b^{ij}(z)\big) = \frac{(\partial_y z)\big(\widetilde{a}^{ij}(y)\big)(\partial_y z)^t}{|\det(\partial_y z)|}$$

Therefore, we have transferred the original problem into Equation (3.36).

In order to use Lemma 3.3.1, we have to check that $b^{ij}(z)$ is strictly elliptic and $\|b^{ij}\|_{C^\alpha(\overline{\mathcal{Q}}_0)}$ is bounded by some universal constant. First we show that there exists a universal constant λ_2 such that

$$\xi^t\big(b^{ij}(z)\big)\xi \ge \lambda_2|\xi|^2(\forall \xi \in \mathbb{R}^n,\ \forall z \in \mathcal{Q}_0) \tag{3.47}$$

Notice that the eigenvalues of $(\partial_y z)$ are $\frac{1}{\widetilde{r}}$ with multiplicity $n-1$ and $\partial_{y_1} z_1$. By (3.42), we can prove that

$$\frac{1}{C} < |\partial_{y_1} z_1| = \partial_{y_1} z_1 = \frac{\delta^2}{f(x_0' + \delta y') + g(x_0' + \delta y') + \varepsilon} < C \tag{3.48}$$

where C is some universal constant.

Based on (3.48),we have

$$\xi^t\big(b^{ij}(z)\big)\xi = \xi^t(\partial_y z)\frac{\big(\widetilde{a}^{ij}(y)\big)}{|\det(\partial_y z)|}(\partial_y z)^t\xi > \lambda_2|\xi|^2$$

where $\lambda_2 > 0$ is some universal constant

The boundedness of $\|b^{ij}\|_{C^\alpha(\overline{\mathcal{Q}}_0)}$ can be checked similarly.

Now applying Lemma 3.3.1, we have

$$\|\nabla w\|_{L^\infty(\mathcal{Q}_0(\frac{1}{2}))} \le C\|w\|_{L^\infty(\mathcal{Q}_0)}$$

Tracing back to u_0 through the transforms, we have, for any point $x \in \mathcal{O}(\frac{r}{2})$,

$$|\nabla u_0(x)| \le \frac{C\|u_0\|_{L^\infty(\mathcal{O}(r))}}{\delta} \le \frac{C\|u_0\|_{L^\infty(\mathcal{O}(r))}}{\sqrt{|x'|^2 + \varepsilon}}.$$

$\square$

3.4 Appendix

Some elementary results for the insulated conductivity problem

Assume that in $\mathbb{R}^n$, Ω and ω are bounded open sets with $C^{2,\alpha}$ boundaries, $0 < \alpha < 1$, satisfying, for some $m < \infty$,

$$\overline{\omega} = \bigcup_{s=1}^{m} \overline{\omega}_s \subset \Omega,$$

where $\{\omega_s\}$ are connected components of ω. Clearly ω_s is open for all $1 \le s \le m$. Given $\varphi \in C^2(\partial\Omega)$, the conductivity problem we consider is the following transmission problem with Dirichlet boundary condition:

$$\begin{cases} \partial_{x_j}\Big\{\Big[\big(ka_1^{ij}(x) - a_2^{ij}(x)\big)\chi_\omega + a_2^{ij}(x)\Big]\partial_{x_i} u_k\Big\} = 0 & in\ \Omega, \\ u_k = \varphi & on\ \partial\Omega, \end{cases} \tag{3.49}$$

where $0 < k < 1$, and χ_ω is the characteristic function of ω.

The $n \times n$ matrixes $A_1(x) := \big(a_1^{ij}(x)\big)$ in ω, $A_2(x) := \big(a_2^{ij}(x)\big)$ in $\Omega\backslash\overline{\omega}$ are symmetric and $\exists$ a constant $\Lambda \geq \lambda > 0$ such that

$$\lambda|\xi|^2 \leq a_1^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \ (\forall x \in \omega), \qquad \lambda|\xi|^2 \leq a_2^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \ (\forall x \in \Omega\backslash\omega)$$

for all $\xi \in \mathbb{R}^n$ and $a_1^{ij}(x) \in C^2(\overline{\omega})$, $a_2^{ij}(x) \in C^2(\overline{\Omega}\backslash\omega)$.

Equation (3.49) can be rewritten in the following form to emphasize the transmission condition on $\partial\omega$:

$$\begin{cases} \partial_{x_j}\Big(a_1^{ij}(x)\ \partial_{x_i}u_k\Big) = 0 & \textit{in } \omega, \\ \partial_{x_j}\Big(a_2^{ij}(x)\ \partial_{x_i}u_k\Big) = 0 & \textit{in } \Omega\backslash\overline{\omega}, \\ u_k|_+ = u_k|_-, & \textit{on } \partial\omega, \\ a_2^{ij}(x)\partial_{x_i}u_k\nu_j\big|_+ = ka_1^{ij}(x)\partial_{x_i}u_k\nu_j\big|_- & \textit{on } \partial\omega, \\ u_k = \varphi & \textit{on } \partial\Omega. \end{cases} \tag{3.50}$$

It is well known that equation (3.49) has a unique solution u_k in $H^1(\Omega)$, and the solution u_k is in $C^1(\overline{\Omega\backslash\omega}) \cap C^1(\overline{\omega})$ and satisfies equation (3.50). On the other hand, if $u_k \in C^1(\overline{\Omega\backslash\omega}) \cap C^1(\overline{\omega})$ is a solution of equation (3.50), then $u_k \in H^1(\Omega)$ satisfies equation (3.49).

For $k \in (0,1)$, consider the energy functional

$$I_k[v] := \frac{k}{2}\int_\omega a_1^{ij}(x)\partial_{x_i}v\partial_{x_j}v + \frac{1}{2}\int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}v\partial_{x_j}v, \tag{3.51}$$

defined on

$$H^1_\varphi(\Omega) := \{v \in H^1(\Omega)|\ v = \varphi \ \ \textit{on } \partial\Omega\}.$$

It is well known that for $k \in (0,1)$, the solution u_k of (3.49) is the minimizer of the minimization problem:

$$I_k[u_k] = \min_{v \in H^1_\varphi(\Omega)} I_k[v].$$

For $k = 0$, the insulated conducting problem is:

$$\begin{cases} \partial_{x_j}\left(a_2^{ij}(x)\ \partial_{x_i}u_0\right) = 0 & in\ \Omega\backslash\overline{\omega}, \\ a_2^{ij}(x)\partial_{x_i}u_0\nu_j\big|_+ = 0 & on\ \partial\omega, \\ u_0 = \varphi & on\ \partial\Omega, \\ \partial_{x_j}\left(a_1^{ij}(x)\ \partial_{x_i}u_0\right) = 0 & in\ \omega, \\ u_0|_+ = u_0|_-, & on\ \partial\omega. \end{cases} \tag{3.52}$$

Equation (3.52) has a unique solution $u_0 \in H^1(\Omega)$, which can be solved in $\Omega \setminus \overline{\omega}$ by the first three lines in (3.52), and then, with $u_0|_{\partial\omega}$, be solved in ω using the fourth line in (3.52). It is well known that $u_0 \in C^1(\overline{\Omega} \setminus \omega) \cap C^1(\overline{\omega})$.

Define the energy functional

$$I_0[v] := \frac{1}{2}\int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}v\partial_{x_j}v, \tag{3.53}$$

where v belongs to the set

$$\mathcal{A}_0 := \left\{v \in H^1(\Omega \setminus \overline{\omega})\big|\ v = \varphi \text{ on } \partial\Omega\right\}.$$

It is well known that there is a unique $v_0 \in \mathcal{A}_0$ which is the minimizer to the minimization problem:

$$I_0[v_0] = \min_{v\in\mathcal{A}_0} I_0[v].$$

Moreover, $v_0 = u_0$ a.e. in $\Omega \setminus \overline{\omega}$, where u_0 is the solution of (3.52).

Now, we give the relationship between u_k and u_0.

Theorem 3.4.1. *For $0 < k < 1$, let u_k and u_0 in $H^1(\Omega)$ be the solutions of equations (3.50) and (3.52), respectively. Then*

$$u_k \rightharpoonup u_0 \quad in\ H^1(\Omega), \quad as\ k \to 0, \tag{3.54}$$

and, consequently,

$$\lim_{k\to 0} I_k[u_k] = I_0[u_0]. \tag{3.55}$$

Proof: We will first show that

$$\sup_{0<k<1} \|\nabla u_k\|_{L^2(\Omega)} < \infty. \tag{3.56}$$

Since u_k is the minimizer of I_k in $H^1_\varphi(\Omega)$ and $v_0 := u_0|_{\Omega\setminus\overline{\omega}}$ is the minimizer of I_0 in $\mathcal{A}_0$, we have

$$\begin{aligned}\frac{\lambda k}{2}\|\nabla u_k\|_{L^2(\omega)} + I_0[v_0] &\le \frac{k}{2}\int_\omega a_1^{ij}(x)\partial_{x_i}u_k\partial_{x_j}u_k + I_0[v_0]\\ &\le \frac{k}{2}\int_\omega a_1^{ij}(x)\partial_{x_i}u_k\partial_{x_j}u_k + I_0[u_k|_{\Omega\setminus\overline{\omega}}] = I_k[u_k]\\ &\le I_k[u_0] = \frac{k}{2}\int_\omega a_1^{ij}(x)\partial_{x_i}u_0\partial_{x_j}u_0 + I_0[v_0],\\ &\le \frac{\Lambda k}{2}\|\nabla u_0\|_{L^2(\omega)} + I_0[v_0].\end{aligned}$$

Thus

$$\sup_{0<k<1}\|\nabla u_k\|_{L^2(\omega)} < \infty.$$

On the other hand,

$$\frac{\lambda}{2}\|\nabla u_k\|_{L^2(\Omega\setminus\overline{\omega})} \le I_k[u_k] \le I_k[u] \le \frac{\Lambda}{2}\|\nabla u_0\|_{L^2(\Omega)}.$$

Estimate (3.56) follows from the above.

Since $u_k = \varphi$ on $\partial\Omega$, we derive from (3.56) that $\sup_{0<k<1}\|u_k\|_{H^1(\Omega)} < \infty$. Let $u_k \rightharpoonup u_0^*$ *in* $H^1_\varphi(\Omega)$ along a subsequence of $k\to 0$ (still denoted as $k \to 0$).

We will show that u_0^* is a solution of equation (3.52). Therefore, $u_0^* = u_0$.

We only need to establish the following three properties:

$$\partial_{x_j}\left(a_2^{ij}(x)\ \partial_{x_i}u_0^*\right) = 0 \qquad \textit{in } \Omega\setminus\overline{\omega}, \tag{3.57}$$

$$\partial_{x_j}\left(a_1^{ij}(x)\ \partial_{x_i}u_0^*\right) = 0 \qquad \textit{in } \omega, \tag{3.58}$$

$$u_0^* \in C^1(\Omega\setminus\omega), \qquad a_2^{ij}(x)\partial_{x_i}u_0^*\nu_j\big|_+ = 0 \qquad \textit{on } \partial\omega. \tag{3.59}$$

(i) For $k \in (0,1)$, we see from equation (3.49) that

$$\partial_{x_j}\left(a_2^{ij}(x)\ \partial_{x_i}u_k\right) = 0, \qquad \text{in } \Omega\setminus\overline{\omega},$$

$$\partial_{x_j}\left(a_1^{ij}(x)\ \partial_{x_i}u_k\right) = 0, \qquad \text{in } \omega.$$

Since u_k converges to u_0^* weakly in $H^1(\Omega)$, (3.57) and (3.58) follow from the above.

(ii) For any $w \in \mathcal{A}_0$, we extend it to $\tilde{w} \in H^1_\varphi(\Omega)$ (i.e. $\tilde{w} = w$ in $\Omega\setminus\overline{\omega}$). By the minimality of u_k,

$$I_k(u_k) \le I_k(\tilde{w}).$$

Sending k to 0 leads to

$$I_0(u_0^*|_{\Omega \setminus \omega}) \leq I_0(w).$$

Thus $u_0^* = u_0$ a.e. in $\Omega \setminus \omega$. (3.59) follows.

We have proved (3.54). Theorem 3.4.1 is established.

Chapter 4

Gradient estimates for elliptic systems

In this chapter, we study the elliptic systems in a narrow domain bounded by two quadratic hypersurfaces. By utilizing the well known L^2 estimates and $W^{2,p}$ estimates, we are able to establish the gradient estimates in this kind the special domains. Next, we apply our estimates into the systems of linear elasticity .

4.1 Elliptic systems and main results

Before stating results we first describe more precisely the domain with which this paper is concerned. For $r \leq 1$, we consider the domain

$$\Omega_r := \{x \in \mathbb{R}^n \big| -\varepsilon - g(x') < x_n < \varepsilon + h(x'),\ |x'| < r\},$$

with top and bottom boundary denoted, respectively, as

$$\Gamma_r^+ = \{x \in \mathbb{R}^n \big|\ x_n = \varepsilon + h(x'),\ |x'| < r\},$$

and

$$\Gamma_r^- = \{x \in \mathbb{R}^n \big|\ x_n = -\varepsilon - g(x'),\ |x'| < r\},$$

where g, h are C^2 convex functions and satisfy

$$g(0) = h(0) = 0, \quad g'(0) = h'(0) = 0,$$

$$\frac{1}{\kappa}|x'|^2 \leq g(x'), h(x') \leq \kappa |x'|^2, \quad \text{and} \quad |\nabla g(x')|, |\nabla h(x')| \leq \kappa.$$

Now suppose $u = (u_1, \cdots, u_N) \in H^1(\Omega_1, \mathbb{R}^N)$ satisfies the following problem

$$\begin{cases} \partial_\alpha (A_{ij}^{\alpha\beta}(x) \partial_\beta u_j) = 0 \quad \text{in } \Omega_1, \\ u = \mathbb{a} \quad \text{on } \Gamma_1^+, \quad u = \mathbb{b} \quad \text{on } \Gamma_1^-, \\ u = \varphi \quad \text{on } \partial\Omega_1 \backslash (\Gamma_1^+ \cup \Gamma_1^-), \end{cases} \tag{4.1}$$

where $0 \le \alpha, \beta \le n$, $0 \le i, j \le N$, the coefficients $A_{ij}^{\alpha\beta}(x) \in C^\infty(\Omega_1)$ satisfy the weak ellipticity condition

$$\lambda \int_{\Omega_1} |\nabla \psi|^2 \le \int_{\Omega_1} A_{ij}^{\alpha\beta} \partial_\alpha \psi^i \partial_\beta \psi^j \le \Lambda \int_{\Omega_1} |\nabla \psi|^2, \quad \text{for any} \quad \psi \in H_0^1(\Omega_1, \mathbb{R}^N),$$

$\mathbb{a}, \mathbb{b}$ are constant vectors and the value of u is equal to the vector-valued function $\varphi \in H^1(\Omega_1)$ on the lateral boundary of Ω_1.

Now we state our main results. When $\mathbb{a} = \mathbb{b}$, we can obtain the C^k estimate of the potential.

Theorem 4.1.1. *When $\mathbb{a} = \mathbb{b}$, suppose $u \in H^1(\Omega_1, \mathbb{R}^N)$ satisfies* (4.1)*, then for any positive integer k,*

$$\|u\|_{C^k(\Omega_{1/2})} \le C \Big(\int_{\Omega_1} |\nabla u|^2 \Big)^{1/2}$$

where C depends on $k, n, N, \lambda, \Lambda$, κ, A, but does not depend on ε.

Remark 4.1.1. *In fact, our proof yields a more explicit dependence. For any positive integer k,*

$$|\nabla^k u(z)| \le C \varepsilon^{1-k-\frac{n}{2}} \mu^{\frac{9}{\sqrt{\varepsilon}} - \frac{4|z'|}{\varepsilon}} \Big(\int_{\Omega_1} |\nabla u|^2 \Big)^{1/2}, \quad \text{for } 0 \le |z'| \le \sqrt{\varepsilon},$$

and

$$|\nabla^k u(z)| \le C |z'|^{2-2k-n} \mu^{\frac{5}{|z'|}} \Big(\int_{\Omega_1} |\nabla u|^2 \Big)^{1/2}, \quad \text{for } \sqrt{\varepsilon} \le |z'| < \frac{1}{2},$$

where μ is some constant less than 1, and C depends on n, N, λ, Λ, but does not depend on ε.

When $\mathbb{a} \neq \mathbb{b}$, we can obtain the gradient estimates of u.

Theorem 4.1.2. *When $\mathbb{a} \neq \mathbb{b}$, if $u \in H^1(\Omega_1, \mathbb{R}^N)$ satisfy* (4.1) *and $\int_{\Omega_1} |\nabla u|^2 \le c\rho_n(\varepsilon)$ for some constant c, where ρ_n is defined in (2.2), then we have*

$$\|\nabla u(x)\|_{L^\infty(\Omega_{1/2})} \le \frac{C}{\varepsilon},$$

where C depends only on $\lambda, \Lambda, \kappa, n, N, \mathbb{a}, \mathbb{b}$, c, A, but does not depend on ε.

4.2 Proof of Theorem 4.1.1

In this section, we intend to derive the C^k estimates for the solutions of elliptic system (4.1) when $\mathbb{a} = \mathbb{b}$. Without loss of generality, we suppose that $\mathbb{a} = \mathbb{b} = 0$ and $\int_{\Omega_1} |\nabla u|^2 = 1$. Before proving Theorem 4.1.1, we first show that the energy decays exponentially.

Lemma 4.2.1. *Suppose $u \in H^1(\Omega_1, \mathbb{R}^N)$ satisfies (4.1), then for any $\sqrt{\varepsilon} < t < 1$,*

$$\int_{\Omega_t} |\nabla u|^2 \leq C(\frac{1}{4})^{\frac{1}{4t}}. \tag{4.2}$$

where C depends on N, λ, Λ, but does not depend on ε.

Proof. For any $0 \leq t < s \leq 1$, we introduce a cutoff function $\eta \in C^\infty(\Omega_1)$ satisfying $0 \leq \eta \leq 1$, $\eta = 1$ in Ω_t, $\eta = 0$ in $\Omega_1 \backslash \Omega_s$, and $|\nabla \eta| \leq \frac{C}{s-t}$. By multiplying the test function $u\eta^2$ on both sides of the equation in (4.1), and in virtue of the weak ellipticity condition, we have the inequality

$$\begin{aligned}\lambda \int_{\Omega_s} |\nabla u|^2 \eta^2 \leq \int_{\Omega_s} A_{ij}^{\alpha\beta}(\eta\partial_\beta u_j)(\eta \partial_\alpha u_i) = -2 \int_{\Omega_s} \eta\, A_{ij}^{\alpha\beta} \partial_\beta u_j \partial_\alpha \eta\, u_i \\ \leq \frac{\lambda}{2} \int_{\Omega_s} |\nabla u|^2 \eta^2 + C \int_{\Omega_s} u^2 |\nabla \eta|^2,\end{aligned}$$

By Hölder inequality, we have

$$\int_{\Omega_s} u^2 dx \leq 4(\varepsilon + s^2)^2 \int_{\Omega_s} |\nabla u|^2 dx.$$

So that

$$\int_{\Omega_s} |\nabla u|^2 \eta^2 \leq C \int_{\Omega_s} u^2 |\nabla \eta|^2 \leq C \left(\frac{\varepsilon + s^2}{s - t}\right)^2 \int_{\Omega_s} |\nabla u|^2,$$

where C depends only on λ, Λ. For simplicity of notation, in the following we denote $F(t) = \int_{\Omega_t} |\nabla u|^2$ and take $C = 1$, then we have the iterative formula,

$$F(t) \leq \left(\frac{\varepsilon + s^2}{s - t}\right)^2 F(s). \tag{4.3}$$

For $\sqrt{\varepsilon} \leq t < s \leq 1$, take $t_0 = t < 1/8$, and $t_{i+1} = \frac{1}{4}(1 - \sqrt{1 - 8t_i})$ if $t_i \leq 1/8$. It is clear that $\{t_i\}$ is an increasing sequence. Let k be the integer such that $t_k \leq 1/8$ and $t_{k+1} > 1/8$, then $t_{k+1} \leq 1/4$. For $0 \leq i \leq k$,

$$F(t_i) \leq \left(\frac{t_{i+1}^2}{t_{i+1} - t_i}\right)^2 F(t_{i+1}) = \frac{1}{4} F(t_{i+1}),$$

Iterating the above inequality k times, we have

$$F(t_0) \leq (\frac{1}{4})^{k+1} F(t_{k+1}) \leq (\frac{1}{4})^{k+1} F(\frac{1}{4}) \leq (\frac{1}{4})^{k+1}.$$

Now we estimate how large k should be. From the iterating formula, for $0 \leq i \leq k$,

$$\frac{1}{2t_i} = \frac{1}{2t_{i+1}} + \frac{1}{1-2t_{i+1}},$$

then

$$\frac{1}{2t_0} - \frac{1}{2t_{k+1}} = \sum_{i=1}^{k+1} \frac{1}{1-2t_i}.$$

Since $0 < t_i < \frac{1}{4}$ for $1 \leq i \leq k+1$, and $\frac{1}{8} < t_{k+1} < \frac{1}{4}$, we have

$$\frac{1}{4t} - 3 < k < \frac{1}{2t} - 3.$$

Therefore, we obtain that

$$\int_{\Omega_t} |\nabla u|^2 \leq (\frac{1}{4})^{k+1} \leq C(\frac{1}{4})^{\frac{1}{4t}}.$$

So the energy decays exponentially. □

Proof of Theorem 4.1.1. **Step 1.** Given a point $(y', y_n) \in \Omega_1$ with $|y'| = a < \frac{1}{2}$, by rotation of coordinates, we can assume $y' = (a, 0, \cdots, 0,)$. Define

$$\widehat{\Omega}_s := \{x \in \Omega_1 \big| -\varepsilon - g(x') < x_n < \varepsilon + h(x'),\ |x' - (a, 0, \cdots, 0)| < s\}.$$

For $0 < t < s < 1$, we choose another cutoff function $\zeta \in C^\infty(\Omega_1)$ satisfying $0 \leq \zeta \leq 1$, $\zeta = 1$ in $\widehat{\Omega}_t$, $\zeta = 0$ in $\Omega_1 \setminus \widehat{\Omega}_s$, and $|\nabla \zeta| \leq \frac{C}{s-t}$. By the same way, multiplying the test function $u\zeta^2$ on both sides of the equation in (4.1), by Cauchy inequality and Hölder inequality, we have

$$\int_{\widehat{\Omega}_s} |\nabla u|^2 \zeta^2 \leq C \int_{\widehat{\Omega}_s} u^2 |\nabla \zeta|^2 \leq C\left(\frac{\varepsilon + (s+a)^2}{s-t}\right)^2 \int_{\widehat{\Omega}_s} |\nabla u|^2.$$

denote $\widehat{F}(t) = \int_{\widehat{\Omega}_t} |\nabla u|^2$, then we have another iterative formula,

$$\widehat{F}(t) \leq C\left(\frac{\varepsilon + (s+a)^2}{s-t}\right)^2 \widehat{F}(s). \tag{4.4}$$

Step 2. For $0 \leq a < \sqrt{\varepsilon}$ and $0 < s < t < 2\sqrt{\varepsilon} - a$, we have

$$\widehat{F}(t) \leq C \frac{\varepsilon^2}{(s-t)^2} \widehat{F}(s).$$

For the purpose of simplicity, we assume $C = 1$.

Let $t_0 = \varepsilon$, $t_{i+1} - t_i = 2\varepsilon$, then

$$\widehat{F}(t_i) \leq \frac{1}{4}\widehat{F}(t_{i+1}).$$

By iteration and Lemma 4.2.1, we obtain

$$\widehat{F}(\varepsilon) \leq C(\frac{1}{4})^{\frac{2\sqrt{\varepsilon}-a}{2\varepsilon}}\widehat{F}(2\sqrt{\varepsilon}-a) \leq C(\frac{1}{4})^{\frac{2\sqrt{\varepsilon}-a}{2\varepsilon}}F(2\sqrt{\varepsilon}) < C(\frac{1}{4})^{\frac{9}{8\sqrt{\varepsilon}}-\frac{a}{2\varepsilon}}.$$

Then by Poincarè inequality and Lemma 4.2.1 , we have

$$\int_{\widehat{\Omega}_\varepsilon} u^2 \leq C\varepsilon^2 \int_{\widehat{\Omega}_\varepsilon} |\nabla u|^2 \leq C\varepsilon^2(\frac{1}{4})^{\frac{9}{8\sqrt{\varepsilon}}-\frac{a}{2\varepsilon}}.$$

Now taking a point z with $|z'| = a$, we do the following scaling

$$\begin{cases} \varepsilon y' = x' - z', \\ \varepsilon y_n = x_n. \end{cases}$$

Let $\hat{u}(y) = u(\varepsilon y' + z', \varepsilon y_n)$, then $\hat{u}(y)$ satisfies

$$\partial_\alpha\big(A_{ij}^{\alpha\beta}(y)\partial_\beta \hat{u}^j(y)\big) = 0 \quad \text{in} \quad Q_1,$$

where $Q_1 := \{y\big| -1 - g(\varepsilon y' + z')/\varepsilon < y_n < 1 + f(\varepsilon y' + z')/\varepsilon,\ |y'| < 1\}.$, namely, Q_1 is the image of $\widehat{\Omega}_\varepsilon$ under the above rescaling so that Q_1 is of size 1.

Using L^2 estimates on the new equation on Q_1 and Sobolev Imbedding Theorems, we have

$$\|\nabla^k \hat{u}\|_{L^\infty(Q_{1/2})} \leq C\|\hat{u}\|_{L^2(Q_1)} \leq C\varepsilon^{-n/2}\|u\|_{L^2(\widehat{\Omega}_\varepsilon)} \leq C\varepsilon^{1-n/2}(\frac{1}{2})^{\frac{9}{8\sqrt{\varepsilon}}-\frac{a}{2\varepsilon}}.$$

In particular, for $z \in \Omega_1$ with $0 \leq |z'| < \sqrt{\varepsilon}$,

$$|\nabla^k u(z)| \leq C\varepsilon^{1-k-\frac{n}{2}}(\frac{1}{2})^{\frac{9}{8\sqrt{\varepsilon}}-\frac{|z'|}{2\varepsilon}}.$$

Step 3. For $\sqrt{\varepsilon} \leq a < \frac{1}{2}$ and $0 < s < t < a$, we have

$$\widehat{F}(t) \leq C\left(\frac{a^2}{s-t}\right)^2\widehat{F}(s).$$

As we did in Step 2, we assume $C = 1$ for simplicity. Let $t_0 = a^2$ and $t_{i+1} - t_i = 2a^2$, then we have the iterative formula

$$\widehat{F}(t_i) \leq \frac{1}{4}\widehat{F}(t_{i+1}).$$

Therefore, by iteration and (4.2), we have

$$\widehat{F}(a^2) = \widehat{F}(t_0) \leq C(\frac{1}{4})^{\frac{a}{2a^2}} \widehat{F}(a) \leq C(\frac{1}{4})^{\frac{1}{2a}} F(2a) \leq C(\frac{1}{4})^{\frac{5}{8a}},$$

By Poincarè inequality, we know that

$$\int_{\widehat{\Omega}_{a^2}} u^2 \leq C(\varepsilon + a^2)^2 \int_{\widehat{\Omega}_{a^2}} |\nabla u|^2 \leq C(\varepsilon + a^2)^2 (\frac{1}{4})^{\frac{5}{8a}}.$$

Now taking a point z with $|z'| = a$, we do the following scaling

$$\begin{cases} a^2 y' = x' - z', \\ a^2 y_n = x_n. \end{cases}$$

Let $\hat{u}(y) = u(a^2 y' + z', a^2 y_n)$, then $\hat{u}(y)$ satisfies

$$\partial_\alpha \big(A_{ij}^{\alpha\beta}(y) \partial_\beta \hat{u}^j(y) \big) = 0, \quad \text{in} \quad Q_1,$$

where $Q_1 := \{y \big| -\frac{\varepsilon}{a^2} - g(a^2 y' + z')/a^2 < y_n < \frac{\varepsilon}{a^2} + f(a^2 y' + z')/a^2,\ |y'| < 1\}$.
Using L^2 estimates and Sobolev Imbedding Theorems, we have

$$\|\nabla^k \hat{u}\|_{L^\infty(Q_{1/2})} \leq C \|\hat{u}\|_{L^2(Q_1)} < \frac{C(\varepsilon + a^2)}{a^n} (\frac{1}{2})^{\frac{5}{8a}}.$$

In particular, for $z \in \Omega_{1/2}$ with $\sqrt{\varepsilon} \leq |z'| < \frac{1}{2}$,

$$|\nabla^k u(z)| < \frac{C(\varepsilon + |z'|^2)}{|z'|^{2k+n}} (\frac{1}{2})^{\frac{5}{8|z'|}} \leq C|z'|^{2-2k-n} (\frac{1}{2})^{\frac{5}{8|z'|}}.$$

Taking $\mu = (\frac{1}{2})^{1/8}$, the proof is completed. □

4.3 Proof of Theorem 4.1.2

Let Ω_r, $\Gamma_r^\pm$ defined as in Section 4.1, we consider the gradient estimates for the case $\mathbb{a} \neq \mathbb{b}$. Without loss of generality, we take $\mathbb{a} = (1, 0, \cdots, 0)$ and $\mathbb{b} = (0, 0, \cdots, 0)$. The proof of Theorem 4.1.2 consists the following steps.

Proof of Theorem 4.1.2: **Step 1.** we first construct

$$\bar{u}^1 = \frac{x_n + g(x') + \epsilon}{g(x') + h(x') + 2\epsilon} + \frac{A_{i1}^{n\gamma}(x)(g_\gamma(x') + h_\gamma(x'))}{8A_{i1}^{nn}(x)} \left(\left(\frac{2x_n + g(x') - h(x')}{g(x') + h(x') + 2\epsilon} \right)^2 - 1 \right),$$

for $2 \le j \le n$,

$$\bar{u}^j = \frac{A_{ij}^{n\gamma}(x)(g_\gamma(x') + h_\gamma(x'))}{8A_{ij}^{nn}(x)}\left(\left(\frac{2x_n + g(x') - h(x')}{g(x') + h(x') + 2\epsilon}\right)^2 - 1\right),$$

where $1 \le \gamma \le n-1$. $g_\gamma(x'), h_\gamma(x')$ are denoted as the partial derivatives of g and h.

Denote

$$\bar{u} = (\bar{u}^1, \cdots, \bar{u}^N). \tag{4.5}$$

It is clear that $\bar{u} = (1, 0, \cdots, 0)$ on Γ_1^+, $\bar{u} = (0, \cdots, 0)$ on Γ_1^-.

For $1 \le \alpha, \beta \le n-1$, by direct computation, we have

$$\begin{aligned}
\frac{\partial \bar{u}^j}{\partial x_\alpha} &= \frac{g_\alpha}{g+h+2\epsilon} - \frac{(x_n+g+\epsilon)(g_\alpha+h_\alpha)}{(g+h+2\epsilon)^2} \\
&+ \frac{A_{ij}^{n\gamma}(x)(g_\gamma+h_\gamma)}{4A_{ij}^{nn}(x)}\left(\frac{(2x_n+g-h)(g_\alpha-h_\alpha)}{(g+h+2\epsilon)^2} - \frac{(2x_n+g-h)^2(g_\alpha+h_\alpha)}{(g+h+2\epsilon)^3}\right) \\
&+ \left(\frac{(A_{ij}^{n\gamma})_\alpha(g_\gamma+h_\gamma) + A_{ij}^{n\gamma}(g_{\alpha\gamma}+h_{\alpha\gamma})}{8A_{ij}^{nn}(x)} - \frac{A_{ij}^{n\gamma}(g_\gamma+h_\gamma)(A_{ij}^{nn})_\alpha}{8(A_{ij}^{nn})^2}\right)\left(\left(\frac{2x_n+g-h}{g+h+2\epsilon}\right)^2 - 1\right),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \bar{u}^j}{\partial x_n} &= \frac{1}{g+h+2\epsilon} + \frac{A_{ij}^{n\gamma}(x)(g_\gamma+h_\gamma)}{2A_{ij}^{nn}(x)}\left(\frac{(2x_n+g-h)}{(g+h+2\epsilon)^2}\right) \\
&+ \left(\frac{(A_{ij}^{n\gamma})_n(g_\gamma+h_\gamma)}{8A_{ij}^{nn}(x)} - \frac{A_{ij}^{n\gamma}(g_\gamma+h_\gamma)(A_{ij}^{nn})_n}{8(A_{ij}^{nn})^2}\right)\left(\left(\frac{2x_n+g-h}{g+h+2\epsilon}\right)^2 - 1\right),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \bar{u}^j}{\partial x_\alpha \partial x_n} &= -\frac{(g_\alpha+h_\alpha)}{(g+h+2\epsilon)^2} + \frac{A_{ij}^{n\gamma}(x)(g_\gamma+h_\gamma)}{2A_{ij}^{nn}(x)}\left(\frac{(g_\alpha-h_\alpha)}{(g+h+2\epsilon)^2} - \frac{2(2x_n+g-h)(g_\alpha+h_\alpha)}{(g+h+2\epsilon)^3}\right) \\
&+ \left(\frac{(A_{ij}^{n\gamma})_\alpha(g_\gamma+h_\gamma) + A_{ij}^{n\gamma}(g_{\alpha\gamma}+h_{\alpha\gamma})}{2A_{ij}^{nn}(x)} - \frac{A_{ij}^{n\gamma}(g_\gamma+h_\gamma)(A_{ij}^{nn})_\alpha}{2(A_{ij}^{nn})^2}\right)\left(\frac{(2x_n+g-h)}{(g+h+2\epsilon)^2}\right) \\
&+ \left(\frac{(A_{ij}^{n\gamma})_n(g_\gamma+h_\gamma)}{4A_{ij}^{nn}(x)} - \frac{A_{ij}^{n\gamma}(g_\gamma+h_\gamma)(A_{ij}^{nn})_n}{4(A_{ij}^{nn})^2}\right) \\
&\quad \times \left(\frac{(2x_n+g-h)(g_\alpha-h_\alpha)}{(g+h+2\epsilon)^2} - \frac{(2x_n+g-h)^2(g_\alpha+h_\alpha)}{(g+h+2\epsilon)^3}\right) \\
&+ \left(\left(\frac{2x_n+g-h}{g+h+2\epsilon}\right)^2 - 1\right) \times \left\{\frac{(A_{ij}^{n\gamma})_{n\alpha}(g_\gamma+h_\gamma) + (A_{ij}^{n\gamma})_n(g_{\alpha\gamma}+h_{\alpha\gamma})}{8A_{ij}^{nn}(x)}\right. \\
&\qquad - \frac{(A_{ij}^{n\gamma})_n(g_\gamma+h_\gamma)(A_{ij}^{nn})_\alpha + A_{ij}^{n\gamma}(g_\gamma+h_\gamma)(A_{ij}^{nn})_{n\alpha}}{8(A_{ij}^{nn})^2} \\
&\quad \left. - \frac{((A_{ij}^{n\gamma})_\alpha(g_\gamma+h_\gamma) + A_{ij}^{n\gamma}(g_{\alpha\gamma}+h_{\alpha\gamma}))(A_{ij}^{nn})_n}{8(A_{ij}^{nn})^2} + \frac{2A_{ij}^{n\gamma}(g_\gamma+h_\gamma)(A_{ij}^{nn})_\alpha(A_{ij}^{nn})_n}{8(A_{ij}^{nn})^3}\right\},
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 \bar{u}^j}{\partial x_n^2} &= \frac{A_{ij}^{n\gamma}(x)}{A_{ij}^{nn}(x)}\Big(\frac{g_\gamma + h_\gamma}{(g+h+2\epsilon)^2}\Big)\\
&+\Big(\frac{(A_{ij}^{n\gamma})_n(g_\gamma+h_\gamma)}{A_{ij}^{nn}(x)} - \frac{A_{ij}^{n\gamma}(g_\gamma+h_\gamma)(A_{ij}^{nn})_n}{(A_{ij}^{nn})^2}\Big)\Big(\frac{(2x_n+g-h)}{(g+h+2\epsilon)^2}\Big)\\
&+\Big(\frac{(A_{ij}^{n\gamma})_{nn}(g_\gamma+h_\gamma)}{8A_{ij}^{nn}(x)} - \frac{2(A_{ij}^{n\gamma})_n(g_\gamma+h_\gamma)(A_{ij}^{nn})_n + A_{ij}^{n\gamma}(g_\gamma+h_\gamma)(A_{ij}^{nn})_{nn}}{8(A_{ij}^{nn})^2}\\
&\qquad + \frac{2A_{ij}^{n\gamma}(g_\gamma+h_\gamma)(A_{ij}^{nn})_{nn}(A_{ij}^{nn})_n^2}{8(A_{ij}^{nn})^3}\Big) \times \Big((\frac{2x_n+g-h}{g+h+2\epsilon})^2 - 1\Big),
\end{aligned}
$$

We notice that in

$$A_{ij}^{\alpha\beta}\partial_{\alpha\beta}\bar{u}^j + \partial_\alpha(A_{ij}^{\alpha\beta})\partial_\beta \bar{u}^j,$$

the term

$$\frac{g_\alpha + h_\alpha}{(g+h+2\epsilon)^2},$$

will be cancelled, while all the other terms can be controlled by

$$\frac{C}{\epsilon + |x'|^2},$$

where C depending only on $A_{ij}^{\alpha\beta}$ and κ. By direct computation, we have

$$\int_{\Omega_1} |\nabla \bar{u}|^2 dx \le C\rho_n(\varepsilon).$$

Heuristically, u should be very close to the function $\bar{u}$. In the following we give gradient estimate of u by the estimate for their difference $u - \bar{u}$.

Step 2. Let $w := \bar{u} - u$, then w will satisfy the following equation

$$
\begin{cases}
\partial_\alpha(A_{ij}^{\alpha\beta}(x)\partial_\beta w^j) = f_i(x) \quad \text{in} \quad \Omega_1,\\
\qquad\qquad w = 0 \qquad \text{on} \quad \Gamma_1^+ \cup \Gamma_1^-,\\
\displaystyle\int_{\Omega_1} |\nabla w|^2\, dx \le C\rho_n(\varepsilon),
\end{cases}
$$

where $f_i(x) = \partial_\alpha(A_{ij}^{\alpha\beta}\partial_\beta \bar{u}^j)$, and for any $x \in \Omega_1$,

$$|f(x)| \le \frac{C}{\varepsilon + |x'|^2}.$$

Given a point $(y', 0) \in \Omega_1$ with $|y'| = a < \frac{1}{2}$, by rotation of coordinates, we can assume $y' = (a, 0, \cdots, 0,)$. Define

$$\widehat{\Omega}_s := \{x \in \Omega_1 \big| -\varepsilon - g(x') < x_n < \varepsilon + h(x'),\ |x' - (a, 0, \cdots, 0)| < s\}.$$

We first give energy estimates of w in dimension $n = 2$.

For $0 < t < s < 1/2$, we pick the cutoff function $\eta \in C^\infty(\Omega_i)$ satisfying $0 \le \eta \le 1$, $\eta = 1$ in $\widehat{\Omega}_t$, $\eta = 0$ in $\Omega_1 \backslash \widehat{\Omega}_s$, and $|\nabla \eta| \le \frac{C}{s-t}$. Since $w = 0$ on Γ_1^-, and by Hölder inequality, we have

$$\begin{aligned}\int_{\widehat{\Omega}_s} w^2 dx &= \int_{\widehat{\Omega}_s} \left(\int_{-\varepsilon - g(x_1)}^{x_2} \frac{\partial w}{\partial x_2} dx_2 \right)^2 dx \\ &\le (2\varepsilon + g + h)^2 \int_{\widehat{\Omega}_s} |\nabla w|^2 dx \\ &\le C(\varepsilon + (a+s)^2)^2 \int_{\widehat{\Omega}_s} |\nabla w|^2 dx.\end{aligned}$$

Multiplying the equation by $w\eta^2$ and integrating by parts, we have

$$\int_{\Omega_s} |\nabla w|^2 \eta^2 + 2 \int_{\Omega_s} w \nabla w \eta \nabla \eta = \int_{\Omega_s} f w \eta^2.$$

By Cauchy inequality and the properties of η, we immediately have

$$\begin{aligned}\int_{\widehat{\Omega}_t} |\nabla w|^2 &\le C \int_{\widehat{\Omega}_s} w^2 |\nabla \eta|^2 + \int_{\widehat{\Omega}_s} f w \eta^2 \\ &\le C \left(\frac{\varepsilon + (a+s)^2}{s-t} \right)^2 \int_{\widehat{\Omega}_s} |\nabla w|^2 + (s-t)^2 \int_{\widehat{\Omega}_s} f^2.\end{aligned}$$

Defining $\widehat{F}(t) = \int_{\widehat{\Omega}_t} |\nabla w|^2$, we have

$$\widehat{F}(t) \le C \left(\frac{\varepsilon + (a+s)^2}{s-t} \right)^2 \widehat{F}(s) + (s-t)^2 \int_{\widehat{\Omega}_s} f^2. \tag{4.6}$$

For simplicity, we assume that $C = 1$ in the following.

Step 2.1. For $\sqrt{\varepsilon} \le a < \frac{1}{8}$, and $0 < s < t < a$, by assuming the constant $C = 1$ for simplicity, we can write the above formula as follows,

$$\widehat{F}(t) \le \left(\frac{a^2}{s-t} \right)^2 \widehat{F}(s) + (s-t)^2 \int_{\widehat{\Omega}_s} f^2.$$

Since $|f(x)| \le \frac{1}{\varepsilon + |x'|^2}$, let $t_0 = 2a^2$ and $t_{i+1} - t_i = 2a^2$, then we have the iterative formula

$$\widehat{F}(t_i) \le \frac{1}{4} \widehat{F}(t_{i+1}) + 4a^4 \cdot \frac{1}{a^4} \cdot 2[\varepsilon + (a + t_{i+1})^2] t_{i+1} \le \frac{1}{4} \widehat{F}(t_{i+1}) + 40 a^2 t_{i+1}.$$

Claim: we can find a constant c such that

$$\widehat{F}(t_i) - c a^2 t_i \le \frac{1}{4} \Big(\widehat{F}(t_{i+1}) - c a^2 t_{i+1} \Big).$$

Indeed, we need only to find c such that

$$(40 + \frac{c}{4})t_{i+1} \leq \, ct_i,$$

that is,

$$(\frac{40}{c} + \frac{1}{4})t_{i+1} \leq \, t_i.$$

Since $t_{i+1} - t_i = 2a^2$ and $t_0 = 6a^2$, it follows that $t_i > 6a^2$ for $i = 1, 2, \cdots$. Then

$$t_{i+1} - t_i = 2a^2 < \frac{1}{3}t_i.$$

Therefore, we can choose $c = 80$.

Now using the above inequality recursively, we have

$$\widehat{F}(t_0) - ca^4 \leq (\frac{1}{4})^i(\widehat{F}(t_i) - ca^2 t_i) \leq C(\frac{1}{4})^{\frac{1}{2a}}\widehat{F}(a).$$

Therefore, we have

$$\widehat{F}(t_0) \leq C(\frac{1}{4})^{\frac{1}{2a}}\frac{1}{\sqrt{\varepsilon}} + ca^4,$$

that is,

$$\int_{\widehat{\Omega}_{6a^2}} |\nabla w|^2 \leq \, C\biggl((\frac{1}{4})^{\frac{1}{2a}}\frac{1}{\sqrt{\varepsilon}} + a^4\biggr), \tag{4.7}$$

Step 2.2. For $0 \leq \, a < \sqrt{\varepsilon}$, $0 < s < t < \sqrt{\varepsilon}$, we have for simplicity,

$$\widehat{F}(t) \leq \frac{\varepsilon^2}{(s-t)^2}\widehat{F}(s) + (s-t)^2\int_{\widehat{\Omega}_s} f^2.$$

Let $t_0 = 4\varepsilon$, $t_{i+1} - t_i = 2\varepsilon$, then

$$\widehat{F}(t_i) \leq \frac{1}{4}\widehat{F}(t_{i+1}) + 4\varepsilon^2 \cdot \frac{1}{\varepsilon^2} \cdot 4\varepsilon \, t_{i+1}.$$

By the same way as in Step 2.1, we have, for some constant c,

$$\widehat{F}(t_0) \leq C(\frac{1}{4})^{\frac{1}{\sqrt{\varepsilon}}}\widehat{F}(\sqrt{\varepsilon}) + c\varepsilon \, t_0.$$

Therefore,

$$\widehat{F}(4\varepsilon) \leq \, C\varepsilon \, t_0 = C\varepsilon^2. \tag{4.8}$$

Similarly, in dimension $n = 3$ we have, for $\sqrt{\varepsilon} \leq \, a < 1/2$,

$$\widehat{F}(6a^2) \leq C(\frac{1}{4})^{\frac{1}{a}}|\ln\varepsilon| + ca^6, \tag{4.9}$$

for $0 \leq a < \sqrt{\varepsilon}$,

$$\widehat{F}(4\varepsilon) \leq C\varepsilon\, t_0 = C\varepsilon^3. \tag{4.10}$$

In dimension $n \geq 4$, we have, for $\sqrt{\varepsilon} \leq a < 1/2$,

$$\widehat{F}(6a^2) \leq C(\frac{1}{4})^{\frac{1}{a}} + Ca^{2n} \leq C\, a^{2n}, \tag{4.11}$$

for $0 \leq a < \sqrt{\varepsilon}$,

$$\widehat{F}(4\varepsilon) \leq C\varepsilon\, t_0 = C\varepsilon^n. \tag{4.12}$$

Step 3. We next derive the gradient estimates of w based on the above energy estimates and the $W^{1,p}$ estimates.

Step 3.1. We still discuss in dimension $n = 2$. For $\sqrt{\varepsilon} \leq a < 1/2$, we do the following change of variables

$$\begin{cases} x_1 - a = a^2 y_1, \\ x_2 = a^2 y_2, \end{cases}$$

Under this change of variables, $\widehat{\Omega}_{a^2}$ is mapped into a domain of size 1, Denote this domain as

$$Q_r = \{y \in \mathbb{R}^2 \big| \ -\frac{\varepsilon}{a^2} - \frac{g(a + a^2 y_1)}{a^2} < y_2 < \frac{\varepsilon}{a^2} + \frac{h(a + a^2 y_1)}{a^2}, |y_1| < r\}.$$

For any $(y_1, y_2) \in Q_1$, define

$$\widetilde{w}(y_1, y_2) = \frac{1}{B} w(x_1, x_2),$$

where $B = \big(a^4 + (\frac{1}{4})^{\frac{1}{2a}} \frac{1}{\sqrt{\varepsilon}}\big)^{\frac{1}{2}}$, then by (4.7), we have

$$\int_{Q_1} |\nabla \widetilde{w}|^2 \leq B^{-2} \int_{\widehat{\Omega}_{a^2}} |\nabla w|^2 \leq C.$$

Since $\widetilde{w} = 0$ on the top and bottom boundary of Q_1, by Sobolev inequality, we have $\|\widetilde{w}\|_{H^1(Q_1)} \leq C$, then by Sobolev Imbedding Theorem, we have $\|\widetilde{w}\|_{L^p(Q_1)} \leq C\|\widetilde{w}\|_{H^1(Q_1)} \leq C$, for $1 < p < \infty$. On the other hand,

$$\partial_\alpha\Big(A_{ij}^{\alpha\beta}\partial_\beta \widetilde{w}^j(y)\Big) = \partial_\alpha\Big(A_{ij}^{\alpha\beta}\partial_\beta \bar{u}^j(y)\Big), \tag{4.13}$$

Since the coefficients are in C^2, so we can differentiate equation. Apply ∂_l to (4.13), we obtain

$$\begin{aligned}&\partial_\alpha\Big(A_{ij}^{\alpha\beta}(y)\partial_\beta\big(\partial_l\widetilde{w}^j(y_1,y_2)\big)\Big)\\&=-\partial_\alpha\Big(\big(\partial_l A_{ij}^{\alpha\beta}(x)\big)\partial_\beta w^j(x)\Big)+\partial_\alpha\Big(\partial_l(A_{ij}^{\alpha\beta}(x)\partial_\beta\bar{u}^j(x))\Big),\end{aligned}$$

and

$$\begin{aligned}|f_i^\alpha|&=\Big|\Big(-\big(\partial_l A_{ij}^{\alpha\beta}\big)\partial_\beta w^j+\partial_l(A_{ij}^{\alpha\beta}\partial_\beta\bar{u}^j)\Big)\Big|\\&\le C(|\nabla w|+|\nabla\bar{u}|+|\nabla\partial_l\bar{u}|)\\&\le C|\nabla w|+C,\end{aligned}$$

where C is independent ε. We notice that

$$\partial_\alpha(A_{ij}^{\alpha\beta}\partial_\beta w^j)=\partial_\alpha(A_{ij}^{\alpha\beta}\partial_\beta\bar{u}^j).$$

By the $W^{1,p}$ estimate for systems, we have

$$\|\nabla w\|_{L^p}\le\|\nabla\bar{u}\|_{L^p}\le\frac{C}{\varepsilon^{1-\frac{1}{p}}}.$$

Then using $W^{1,p}$ estimates([10], Theorem 2.2 in Chapter 10) and L^2 estimates, we have

$$\|\nabla\partial_l\widetilde{w}\|_{L^p(\frac{1}{2}\widetilde{\Omega},\mathbb{R}^N)}\le\frac{C}{\varepsilon^{1-\frac{1}{p}}}.$$

So that

$$\|D^2\widetilde{w}\|_{L^p(\frac{1}{2}\widetilde{\Omega},\mathbb{R}^N)}\le\frac{C}{\varepsilon^{1-\frac{1}{p}}}.$$

By Sobolev Imbedding theorem, for $p>2$, we have

$$\|\widetilde{w}\|_{C^{1,\alpha}(\frac{1}{2}\widetilde{\Omega})}\le C\|\widetilde{w}\|_{W^{2,p}(\frac{1}{2}\widetilde{\Omega})}\le\frac{C}{\varepsilon^{1-\frac{1}{p}}}.$$

where $\alpha=1-2/p$. So that, for any $(y_1,y_2)\in\frac{1}{2}\widetilde{\Omega}$,

$$|\nabla\widetilde{w}(y_1,y_2)|=a^2\bigg(a^4+(\frac{1}{4})^{\frac{1}{a}}\frac{1}{\sqrt{\varepsilon}}\bigg)^{-\frac{1}{2}}|\nabla w(x_1,x_2)|\le\frac{C}{\varepsilon^{1-\frac{1}{p}}}.$$

In particular, for any point $(x_1,x_2)\in\Omega_1$ with $\sqrt{\varepsilon}\le|x_1|=a<\frac{1}{2}$, we have

$$|\nabla w(x_1,x_2)|\le\frac{C}{\varepsilon^{1-\frac{1}{p}}}\bigg(a^4+(\frac{1}{4})^{\frac{1}{a}}\frac{1}{\sqrt{\varepsilon}}\bigg)^{\frac{1}{2}}\frac{1}{a^2}<\frac{C}{\varepsilon}.\tag{4.14}$$

Step 3.2. For $0 \le a < \sqrt{\varepsilon}$, similarly as the above, we do the following change of variables

$$\begin{cases} x_1 - a = \varepsilon y_1, \\ x_2 = \varepsilon y_2, \end{cases}$$

and denote

$$\widetilde{\Omega} = \{y \in \mathbb{R}^2 \big| \ |y_2| < 1 + (\frac{a}{\sqrt{\varepsilon}} + \sqrt{\varepsilon} y_1)^2, |y_1| < 1\}.$$

For any $(y_1, y_2) \in \widetilde{\Omega}$, define

$$\widetilde{w}(y_1, y_2) = \frac{1}{\varepsilon} w(x_1, x_2)$$

then by (4.8), we have

$$\int_{\widetilde{\Omega}} |\nabla \widetilde{w}|^2 < \frac{1}{\varepsilon^2} \int_{\widehat{\Omega}_\varepsilon(a)} |\nabla w|^2 \le C.$$

Since $w = 0$ on the top and below boundary of $\widetilde{\Omega}$, by Sobolev inequality, $\|\widetilde{w}\|_{H^1(\widetilde{\Omega})} \le C$, which implies by Sobolev Imbedding Theorem that $\|\widetilde{w}\|_{L^p(\widetilde{\Omega})} \le C \|\widetilde{w}\|_{H^1(\widetilde{\Omega})} \le C$, for $1 < p < \infty$. We now have

$$\partial_\alpha \left(A_{ij}^{\alpha\beta} \partial_\beta \widetilde{w}^j(y_1, y_2) \right) = \partial_\alpha \left(A_{ij}^{\alpha\beta} \partial_\beta \bar{u}^j(a + a^2 y_1, a^2 y_2) \right),$$

Then Similarly as the above, by $W^{1,p}$ estimates and Sobolev imbedding argument as above, we have, for $p > 2$,

$$\|\widetilde{w}\|_{C^{1,\alpha}(\frac{1}{2}\widetilde{\Omega})} \le C$$

where $\alpha = 1 - \frac{2}{p}$. Therefore for any $(y_1, y_2) \in \frac{1}{2}\widetilde{\Omega}$

$$|\nabla \widetilde{w}(y_1, y_2)| = |\nabla w(x_1, x_2)| \le C$$

In particular, for any point $(x_1, x_2) \in \Omega_1$ with $0 < |x_1| = a < \sqrt{\varepsilon}$, we have

$$|\nabla w(x_1, x_2)| \le C \tag{4.15}$$

Step 3.3. In dimension $n = 3$, using the same argument as above, for $(x', x_3) \in \Omega_1$, we have

$$\begin{aligned} |\nabla w(x', x_3)| &\le C\left(1 + \frac{1}{a^2}(\frac{1}{4})^{\frac{1}{2a}} \sqrt{|\ln \varepsilon|}\right) \\ &\le C\sqrt{|\ln \varepsilon|}, \quad \text{for } \sqrt{\varepsilon} \le |x'| < 1/2, \end{aligned}$$

$$|\nabla w(x', x_3)| \leq C, \quad \text{for } 0 \leq |x'| < \sqrt{\varepsilon}.$$

For $n = 4$, based on the energy estimates (4.11) (4.12) and the above argument, we have

$$|\nabla w(x', x_n)| \leq C, \quad \text{for } 0 \leq |x'| < 1/2. \tag{4.16}$$

Now we consider in all dimensions. Since

$$|\nabla \bar{u}(x)| \leq \frac{C}{\varepsilon},$$

combining with all the above estimates, we have for any $x \in \Omega_{1/2}$,

$$\|\nabla w(x)\|_{L^\infty(\Omega_{1/2})} \leq \frac{C}{\varepsilon}.$$

$\square$

4.4 Some applications in the systems of linear elasticity

Let Ω be a bounded open set in $\mathbb{R}^n$ with $C^{2,\alpha}$ boundary, $n \geq 2$, $0 < \alpha < 1$, D_1 and D_2 be two bounded strictly convex open subsets in Ω with smooth boundaries which are ε apart and far away from $\partial\Omega$, i.e.

$$\begin{gathered} \overline{D}_1, \overline{D}_2 \subset \Omega, \quad \text{the principal curvature of } \partial D_1, \partial D_2 \geq \kappa_0 \\ \varepsilon := \operatorname{dist}(D_1, D_2) > 0, \quad \operatorname{dist}(D_1 \cup D_2, \partial\Omega) > r_0, \quad \operatorname{diam}(\Omega) < \frac{1}{r_0}, \end{gathered} \tag{4.17}$$

where $\kappa_0, r_0 > 0$ are universal constants independent of ε.

In two dimensions, we can treat Ω as the cross section of one composite material, where D_1 and D_2 are the cross sections of the fibers. If this composite material is homogeneous and isotropic and suppose that the Lamé pair of the surrounding matrix is (λ, μ) and the Lamé pair of the fibers D_1 and D_2 is $(\widetilde{\lambda}, \widetilde{\mu})$, then we know from the introduction that the displacement u satisfies the system of equations (1.8).

Denote $\mathcal{R}$ be the linear space of rigid displacements of $\mathbb{R}^n$, i.e. the set of all vector valued functions $\eta = (\eta_1, \eta_2, \ldots, \eta_n)^T$ such that $\eta = a + Ax$, where $a = (a_1, a_2, \ldots, a_n)^T$ is a constant vector, A is a skew-symmetric $n \times n$ matrix, it is easy to see that $\mathcal{R}$ is a linear space of dimension $m := n(n+1)/2$.

Here we consider the extreme case in which the shear modulus of the fibers $\widetilde{\mu} = \infty$. Given the boundary condition $\varphi \in H^1(\Omega)$, then the displacement $u = (u_1, u_2, \cdots, u_n)$ satisfies the following equation

$$\begin{cases} \mathcal{L}_{\lambda,\mu} u = 0 & \text{in } \widetilde{\Omega} \\ \nabla u + \nabla u^T = 0 & \text{in } D_1 \cup D_2 \\ u|_+ = u|_- & \text{on } \partial D_1 \cup \partial D_2, \\ u = \varphi & \text{on } \partial\Omega, \\ \int_{\partial D_i} \eta \frac{\partial u}{\partial \nu}|_+ = 0 & \forall \eta \in \mathcal{R}. \end{cases} \tag{4.18}$$

where $\mathcal{L}_{\lambda,\mu} u := \mu \Delta u + (\lambda + \mu) \nabla(\nabla \cdot u)$.

Proposition 4.4.1. *The solution of (4.18) exists and is unique.*

Proof. We first prove the uniqueness of solutions to (4.18). in fact, if u is a solution of (4.18) with $\varphi = 0$, then using u as a test function in Ω, we have,

$$\int_\Omega \lambda |\nabla \cdot u|^2 + \frac{\mu}{2} |\nabla u + \nabla u^T|^2 = 0$$

Therefore, $\nabla u + \nabla u^T = 0$, i.e. $u \in \mathcal{R}$ in Ω, since $u = 0$ on $\partial\Omega$, $u = 0$ in Ω.

Next we prove the existence of the solution of (4.18). Actually its solution can be viewed as the minimizer of the functional

$$I[u] := \int_\Omega \lambda |\nabla \cdot u|^2 + \frac{\mu}{2} |\nabla u + \nabla u^T|^2$$

in the Hilbert space

$$\mathcal{A} := \{v \in H^1_\varphi(\Omega) | \nabla v + \nabla v^T = 0 \text{ in } D_1 \cup D_2\}.$$

Now we prove the existence of the minimizer. Let $\{u^i\}$ be a minimizing sequence in H^1_φ, then we have $\|\nabla u^i + (\nabla u^i)^T\|_{L^2(\Omega)} \le C$ for some constant C independent of i. By Korn's Inequality, see page 13-14 in [22] for example, we have,

$$\|\nabla u^i\|^2_{L^2(\Omega)} \le \frac{1}{2} \|\nabla u^i + \nabla u^{iT}\|^2_{L^2(\Omega)} + \|\varphi\|^2_{H^1(\Omega)} \le C$$

Therefore $\{u^i\}$ is bounded in $H^1(\Omega)$, let $u^i \rightharpoonup u$ in $H^1(\Omega)$. Then $u \in \mathcal{A}$ since $\mathcal{A}$ is H^1 weakly closed. Moreover, u is exactly the minimizer of $I[u]$ since $I[u]$ is a convex function of the components of ∇u. $\square$

From the second equation in (4.18), we know that $u \in \mathcal{R}$ in $D_i (i = 1, 2)$. Given $\{e^i\}$ as a basis of $\mathcal{R}$, let v^i be the solution of the following system

$$\begin{cases} \mathcal{L}_{\lambda,\mu} u = 0 & \text{in } \widetilde{\Omega}, \\ u = e^i & \text{on } \partial D_1, \\ u = 0 & \text{on } \partial D_2 \cup \partial\Omega. \end{cases} \tag{4.19}$$

$\widetilde{v}^i$ be the solution of the following system

$$\begin{cases} \mathcal{L}_{\lambda,\mu} u = 0 & \text{in } \widetilde{\Omega}, \\ u = e^i & \text{on } \partial D_2, \\ u = 0 & \text{on } \partial D_1 \cup \partial\Omega. \end{cases} \tag{4.20}$$

v^0 be the solution of the following system

$$\begin{cases} \mathcal{L}_{\lambda,\mu} u = 0 & \text{in } \widetilde{\Omega}, \\ u = 0 & \text{on } \partial D_1 \cup \partial D_2 \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{4.21}$$

Then we can decompose the solution u of (4.18) as follows

$$u = \sum_{i=1}^{m} C_i v^i + \sum_{i=1}^{m} \widetilde{C}_i \widetilde{v}^i + v^0$$

Denote

$$\alpha_{ij} = \int_{\partial D_1} \frac{\partial v^i}{\partial \nu} e^j, \quad \widetilde{\alpha}_{ij} = \int_{\partial D_1} \frac{\partial \widetilde{v}^i}{\partial \nu} e^j; \quad \beta_{ij} = \int_{\partial D_2} \frac{\partial \widetilde{v}^i}{\partial \nu} e^j, \quad \widetilde{\beta}_{ij} = \int_{\partial D_2} \frac{\partial v^i}{\partial \nu} e^j$$

and

$$\gamma_i = \int_{\partial D_1} \frac{\partial v^0}{\partial \nu} e^i, \qquad \widetilde{\gamma}_i = \int_{\partial D_2} \frac{\partial v^0}{\partial \nu} e^i$$

Then by the last equation of (4.18), we have

$$\begin{pmatrix} \alpha_{ij} & \widetilde{\alpha}_{ij} \\ \widetilde{\beta}_{ij} & \beta_{ij} \end{pmatrix} \begin{pmatrix} C_j \\ \widetilde{C}_j \end{pmatrix} = \begin{pmatrix} \gamma_i \\ \widetilde{\gamma}_i \end{pmatrix} \tag{4.22}$$

To establish the gradient estimate for u, as we did on the perfect conductivity problems, we first estimate $|\nabla v^i|$. Next we try to estimate $|C_j - \widetilde{C}_j|$. As an application of Theorem 4.1.1 and Theorem 4.1.2, we have,

Corollary 4.4.1. $\|\nabla v^0\|_{L^\infty(\widetilde{\Omega})} < C, \quad \|\nabla v^i\|_{L^\infty(\widetilde{\Omega})} < \frac{C}{\varepsilon} \quad (i = 1, 2, \ldots, m)$, *where* C *is some constant depending on* n, λ,μ, κ_0, r_0, *but independent of* ε.

The difficulty here is to estimate $|C_j - \widetilde{C}_j|$, we conjecture that $|C_j - \widetilde{C}_j| < C\varepsilon\rho_n(\varepsilon)$.

References

[1] H. Ammari, G. Dassios, H. Kang and M. Lim, *Estimates for the electric field in the presence of adjacent perfectly conducting spheres*, Quat. J. Appl. Math., 65 (2007), 339-355.

[2] H. Ammari, H. Kang, H. Lee, J. Lee and M. Lim, *Optimal estimates for the Electrical Field in Two Dimensions*, J. Math. Pures Appl. 88 (2007), 307-324.

[3] H. Ammari, H. Kang and M. Lim, *Gradient estimates to the conductivity problem*, Math. Ann. 322 (2005), 277-286.

[4] I. Babuska, B. Anderson, P.J. Smith, and K. Levin, *Damage analysis of fiber composites. I. Statistical analysis on fiber scale*, Comput. Methods Appl. Mech. Engrg. 172 (1999), 27-77.

[5] E.S. Bao, H. Li, Y.Y. Li and B. Yin, *Estimates for elliptic systems in narrow regions*, in preparation.

[6] E.S. Bao, Y.Y. Li and B. Yin, *Gradient estimates for the perfect conductivity problem*, Arch. Rational Mech. and Anal., 193 (2009), 195-226.

[7] E.S. Bao, Y.Y. Li and B. Yin, *Gradient estimates for the perfect and insulated conductivity problems with multiple inclusions*, in preparation.

[8] E. Bonnetier and M. Vogelius, *An elliptic regularity result for a composite medium with "touching" fibers of circular cross-section*, SIAM J. Math. Anal. 31 (2000), 651-677.

[9] B. Budiansky and G.F. Carrier, *High shear stresses in stiff fiber composites*, J. App. Mech. 51 (1984), 733-735.

[10] Y.Z. Chen and L.C. Wu: *Second order elliptic equations and elliptic systems*, American Mathematical Society (1997)

[11] G.B. Folland, *Introduction to Partial Differential Equations*, Princeton University Press, Princeton, NJ, 1976.

[12] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[13] R. Horn and C. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[14] J.B. Keller, *Stresses in narrow regions*, Trans. ASME J. Appl. Mech. 60 (1993), 1054-1056.

[15] D. Kapanadze and B.W. Schulze, *Boundary-contact problems for domains with edge singularities*, J. Differential Equations 234 (2007), 26-53.

[16] L.D. Landau and E.M. Lifshitz, *Theory of Elasticity (3rd ed.)* Oxford, England (1986).

[17] Y.Y. Li and L. Nirenberg, *Estimates for elliptic system from composite material*, Comm. Pure Appl. Math. 56 (2003), 892-925.

[18] Y.Y. Li and M. Vogelius, *Gradient estimates for solution to divergence form elliptic equation with discontinuous coefficients*, Arch. Rational Mech. Anal. 153 (2000), 91-151.

[19] M. Lim and K. Yun, *Blow-up of Electric Fields between Closely Spaced Spherical Perfect Conductors*, Communications in PDE, to appear.

[20] X. Markenscoff, *Stress amplification in vanishingly small geometries*, Computational Mechanics 19 (1996), 77-83.

[21] J. Mateu, J. Oorobitg and J. Verdera, *Extra cancellation of even Calderon-Zygmund operators and quasiconformal mappings*, arXiv:math.CA/0802.1185v2 1 Oct 2008.

[22] O.A. Oleinik, A.S. Shamaev and G.A. Yosifian *Mathematical problems in elasticity and homogenization*, North-Holland (1992).

[23] V.V. Voyevodin, *Linear Algebra*, Mir Publishers, Moscow, 1983.

[24] K. Yun, *Estimates for electric fields blown up between closely adjacent conductors with arbitrary shape*, SIAM J. on Applied Math. 67 (2007), 714-730.

[25] K. Yun, *Optimal bound on high stresses occurring between stiff fibers with arbitrary shaped cross-sections*, J. Math. Anal. Appl. 350 (2009), no. 1, 306–312.

[26] K. Yun, *Blow-up of Electric Fields between Closely Spaced Spherical Perfect Conductors*, preprint.

Vita

Biao Yin

2009	Ph. D. in Mathematics, Rutgers University
1999-2003	B. Sc. in Mathematics, University of Science and Technology of China
1999	Graduated from Huanggang Middle School in Hubei Province of China.

2004-2009	Teaching assistant, Department of Mathematics, Rutgers University