INVARIANT THEORY IN CAUCHY-RIEMANN GEOMETRY AND APPLICATIONS TO THE STUDY OF HOLOMORPHIC MAPPINGS

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ABSTRACT OF THE DISSERTATION

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In this dissertation, proper holomorphic maps between some types of CR manifolds have been studied. For non-degenerate holomorphic Segre maps between $\mathcal{H}^n$ and $\mathcal{H}^N$, the complexifications of Heisenberg hypersurfaces, we show that they possess a partial rigidity property when $N \leq 2n-2$. As an application under the same assumption, we prove that the holomorphic Segre non-transversality for these maps propagates along Segre varieties. this propagation property fails when $N > 2n-2$. For any proper rational holomorphic map between complex balls, we derive a simple and explicit criterion when it is equivalent to a holomorphic polynomial map. This criterion is used to show that proper rational holomorphic maps from $\mathbb{B}^2$ into $\mathbb{B}^N$ of degree two are equivalent to polynomial maps. For general smooth CR embeddings from a Levi non-degenerate hypersurface into another one with the same signature, a monotonicity property of the Chern-Moser-Weyl curvature along directions in the null space of the Levi-form has been obtained.

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Chapter 1

Introduction to CR geometry

1.1 Preliminaries

Let M be a real submanifold of $\mathbb{C}^n$ of codimension d. Namely, for any point $p \in M$, there exists a neighborhood U of p so that $M \cap U = \{z \in \mathbb{C}^n : \rho(z, \bar{z}) = 0\}$ for some neighborhood U of p, where $\rho = (\rho_1, \ldots, \rho_d)$ is a real vector-valued functions in U and $d\rho \neq 0$ on M. ρ is called the local defining function of M. We say M is a $\mathcal{C}^k$ (real analytic) submanifold if $\rho \in \mathcal{C}^k$ (real analytic).

For $p \in \mathbb{C}^n$, write $T_p\mathbb{C}^n = T_p\mathbb{R}^n = \big\{ \sum_{j=1}^n a_j \frac{\partial}{\partial x_j}|_p + \sum_{j=1}^n b_j \frac{\partial}{\partial y_j}|_p : a_j, b_j \in \mathbb{R}\big\}$. We say $X \in T_p\mathbb{C}^n$ is tangent to M at p if

$$X\rho_k(p) = \sum_{j=1}^n a_j \frac{\partial \rho_k}{\partial x_j}(p, \bar{p}) + \sum_{j=1}^n b_j \frac{\partial \rho_k}{\partial y_j}(p, \bar{p}) = 0 \quad \text{for} \quad 1 \leq k \leq d.$$

Denote T_pM to be the subspace of all real vectors tangent to M at p and $T(M) = \sqcup_{p\in M} T_p(M)$ the corresponding vector bundle over M. The space of all smooth sections in $T(M)$ is denoted by $\Gamma^\infty(T(M))$. By allowing the coefficients a_j and b_j in the above expressions to be complex numbers we can define the complexified tangent spaces $\mathbb{C}T\mathbb{C}^n$ and $\mathbb{C}T_p(M) = T_p(M) \otimes \mathbb{C}$. Specifically, if we write

$$\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}),$$

Then

$$\mathbb{C}T_p\mathbb{C}^n = \left\{ \sum_{j=1}^n a_j \frac{\partial}{\partial z_j}|_p + \sum_{j=1}^n b_j \frac{\partial}{\partial \bar{z}_j}|_p : a_j, b_j \in \mathbb{C} \right\},$$

$$\mathbb{C}T_p(M) = \left\{ X = \sum_{j=1}^n a_j \frac{\partial}{\partial z_j}|_p + \sum_{j=1}^n b_j \frac{\partial}{\partial \bar{z}_j}|_p : a_j, b_j \in \mathbb{C}, X\rho_k(p) = 0 \ \text{ for } \ 1 \leq k \leq d \right\}.$$

We can also denote the smooth sections on $\mathbb{C}T(M)$ by $\Gamma^\infty(\mathbb{C}T(M))$.

We are about to introduce the concept of *CR manifold* in the general sense.

Definition 1.1.1 *Let M be a real smooth manifold. M is called a CR manifold if there exists a complex subbundle $T^{(1,0)}M$ of the complexified tangent bundle $\mathbb{C}TM = TM \otimes \mathbb{C}$ such that*

1. $T^{(1,0)}M \cap \overline{T^{(1,0)}M} = \{0\}$,
2. $[X, Y] := XY - YX \in \Gamma^\infty(T^{(1,0)}M) \quad X, Y \in \Gamma^\infty(T^{(1,0)}M)$.

Here $\Gamma^\infty(T^{(1,0)}M)$ is the collections of sections on $T^{(1,0)}M$. The bundle $T^{(1,0)}M$ is called the CR structure of M.

We also call the above $T^{(1,0)}M$ the CR tangent bundle of M. Notice in the sense of the above definition, the CR structure for a general CR manifold can be highly abstract. If we restricts merely on submanifold of some complex manifold, say $\mathbb{C}^n$, the CR structure of the CR manifold will be naturally induced by the complex structure of the ambient manifold and hence, the definition of CR manifolds in those cases may be simplified. In detail, let M be a real smooth submanifold in $\mathbb{C}^n$. Denote $T^{(1,0)}\mathbb{C}^n$ to be the subbundle of $\mathbb{C}T\mathbb{C}^n$ whose sections are complex linear combinations of the $\frac{\partial}{\partial z_j}$'s and $T^{(0,1)}\mathbb{C}^n$ to be its conjugate. Then $\mathbb{C}T\mathbb{C}^n$ has a natural decomposition:

$$\mathbb{C}T\mathbb{C}^n = T^{(1,0)}\mathbb{C}^n \oplus T^{(0,1)}\mathbb{C}^n.$$

We denote $T^{(1,0)}M$, the bundle of $(1,0)$ vectors, to be the intersection

$$T^{(1,0)}M = \mathbb{C}TM \cap T^{(1,0)}\mathbb{C}^n.$$

Respectively one can define its conjugate $T^{(0,1)}M$, the bundle of $(0,1)$ vectors. Furthermore, it may be easily verified $T^{(1,0)}M$ satisfies the following three properties:

1. $T^{(1,0)}M \cap T^{(0,1)}M = \{0\}$.
2. $[X, Y] := XY - YX \in T^{(1,0)}M \quad X, Y \in \Gamma^\infty(T^{(1,0)}M)$.
3. $T^{(1,0)}M = \overline{T^{(0,1)}M}$.

Therefore, the definition of CR manifolds for manifolds in $\mathbb{C}^n$ can be rephrased as follows:

Definition 1.1.2 *A real submanifold* $M \subset \mathbb{C}^n$ *is a CR submanifold if* $\dim_{\mathbb{C}} T_p^{(1,0)}M$ *is constant for* $p \in M$. *The constant is called the CR dimension of* M.

On those CR manifolds we introduce a special type of functions, which are called *CR functions*:

Definition 1.1.3 *A continuous function* $f : M \to \mathbb{C}$ *is said to be a CR function on CR manifold* M *if* $\bar{X}f = 0$ *(in the sense of distribution) for all* $X \in T^{(1,0)}M$.

Note the notion of CR functions on CR manifolds in $\mathbb{C}^n$ is a generalization of holomorphic functions in $\mathbb{C}^n$. Precisely speaking, given a holomorphic function f on $U \subset \mathbb{C}^n$. Since $\bar{\partial} f \equiv 0$ on U, the restriction of f on $M \cap U$ naturally induces a CR function. However on the other hand, not all CR functions on M come from the restriction of a holomorphic function on M. See the following example:

Example 1.1.4 *Let* $M = \{(z_1, z_2) \in \mathbb{C}^2 : \Im z_2 = 0\}$. *It is obvious that* M *is a CR manifold and* $\{\frac{\partial}{\partial z_1}\}$ *forms a global basis for* $\Gamma^\infty(T^{(1,0)}M)$. *Consider any non real-analytic function which depends only on* $\Re z_2$*, say* $f(\Re z_2)$. *By definition, it is always a CR function. However it cannot be written as a restriction of a holomorphic function on* M*, which is always real analytic by Schwarz reflection principle.*

It has been discussed in [BER] that 'minimality' of M is a necessary and sufficient condition for holomorphic extension of CR functions to one side of M. In other words, if M does not contain a proper CR submanifold of the same CR dimension, then any CR function can be extended as a holomorphic function on at least one side of M.

Mappings between two CR manifolds can also be defined.

Definition 1.1.5 *Let* $(M, T^{(1,0)}M)$ *and* $(N, T^{(1,0)}N)$ *be two CR manifolds. A CR mapping* $F : (M, T^{(1,0)}M) \to (N, T^{(1,0)}N)$ *(of class* $\mathcal{C}^k$*) is a* $\mathcal{C}^k$ *mapping* $F : M \to N$ *such that for each* $p \in M$*,* $F_*(T_p^{(1,0)}M) \subseteq T_{F(p)}^{(1,0)}N$. *If in addition the mapping is an embedding, namely,* F *is a one-to-one mapping and the Jacobian of* F *is of full rank on* M*, then we call* F *is a CR embedding.*

Simply speaking, a CR map from a CR manifold to another is a map preserving the CR structures. A straight forward conclusion of the definition is $F : (T^{(1,0)}M) \to (\mathbb{C}^N, T^{(1,0)}\mathbb{C}^N)$ is a CR mapping if and only if each component $F_j, j = 1, \ldots, N$ is a CR function.

Throughout this thesis, since we are interested in the smooth boundaries of domains in $\mathbb{C}^{n+1}$, from now on we will focus the attention only on smooth hypersurfaces, submanifolds with real codimension one in $\mathbb{C}^{n+1}$. Later we will show smooth hypersurfaces always admit CR structures and hence are called the CR manifolds of hypersurface type or CR hypersurfaces. The structure of CR hypersurfaces can even be explicitly written down under certain well chosen local coordinate. Let us first describe such a local coordinate for a given hypersurface.

Lemma 1.1.6 *Let M be a real smooth hypersurface in $\mathbb{C}^{n+1}$ through p. Then there exists a holomorphic change of local coordinates near p such that under the new coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$, M near p is given by*

$$0 = \Im w - \phi(z, \bar{z}, \Re w) \tag{1.1}$$

for some smooth function $\phi(z, \bar{z}, s)$ defined near 0 with $\phi(0) = 0$ and $d\phi(0) = 0$.

Proof of Lemma 1.1.6: Let $\rho(\tilde{z}, \bar{\tilde{z}})$ be a local defining function for M near p. After making a translation, we can assume p is the origin. Then the real smooth function ρ can be written as:

$$\rho(\tilde{z}, \bar{\tilde{z}}) = \Im \sum_{j=1}^{n+1} a_j \tilde{z}_j + O(2).$$

Here $O(k)$ denotes terms of vanishing order at least k. Since $d\rho(0) \neq 0$, without loss of generality we can assume $a_{n+1} = 2i\frac{\partial \rho}{\partial \tilde{z}_{n+1}}(0) \neq 0$. Take the linear transformation:

$$\begin{aligned} z_j &= \tilde{z}_j, \quad j = 1, \ldots, n. \\ w &= \sum_{j=1}^{n+1} a_j z_j. \end{aligned}$$

Then under the new coordinates $(z_1, \ldots, z_n, w)$, the defining equation of M near p can be written as: $\rho(z, w) = \Im w + O(2) = 0$. Applying Implicit function theorem one can solve for $\Im w$ in terms of $z, \bar{z}, \Re w$ by $\Im w = \phi(z, \bar{z}, \Re w)$ near 0. $\phi(0) = 0$ is obvious. Moreover, since $\rho(z, w) = \Im w + O(2) = 0$ on M, $X(\rho(z, w)) = X(\Im w) + X(O(2)) = 0$ for any $X \in \mathbb{C}T_0M$. On the other hand, $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$ and $\frac{\partial}{\partial \Re w}$ form a basis for $\mathbb{C}T_0M$. Therefore $d\phi(0) = 0$. $\square$

Under the above local coordinates of M, its CR vector bundle can be explicitly formulated as follows:

Lemma 1.1.7 *Suppose a real smooth hypersurface M in $\mathbb{C}^{n+1}$ is given by (1.1) near 0. Consider smooth sections in $\mathbb{C}TM$:*

$$L_j = \frac{\partial}{\partial z_j} + 2i\frac{\phi_{z_j}}{1 + i\phi_s}\frac{\partial}{\partial w}; \quad j = 1, \ldots, n. \tag{1.2}$$

$$T = \frac{2}{1 + i\phi_s}\frac{\partial}{\partial w} + \frac{2}{1 - i\phi_s}\frac{\partial}{\partial \bar{w}}. \tag{1.3}$$

Then $\mathbb{C}TM = Span_{\mathbb{C}}\{T, L_j, \overline{L_j}, j = 1, \ldots, n\}$. Moreover, the following holds:

1. $[L_j, L_k] \in Span_{\mathbb{C}}\{L_j, j = 1, \ldots, n\}$. (1.4)

2. $[L_j, \overline{L_k}] = \lambda_{j\bar{k}}T$. (1.5)

3. $\dim_{\mathbb{C}} Span_{\mathbb{C}}\{L_j, j = 1, \ldots, n\} = n$. (1.6)

Therefore M is a CR manifold of hypersurface type and its CR bundle $T^{(1,0)}M$ is given by $Span_{\mathbb{C}}\{L_j, j = 1, \ldots, n\}$.

Proof of Lemma 1.1.7: Straight forward computation shows that $L_j\rho = \overline{L_j}\rho = T\rho = 0$ and $L_j, \overline{L_j}, T$ are $\mathbb{C}$-linear independent. Therefore $T, L_j, \overline{L_j}$ consists of a local basis for $\mathbb{C}TM$ and $[L_j, L_k] \in \mathbb{C}TM$. Furthermore, note $L_j \in T^{(1,0)}\mathbb{C}^{n+1}$. Then $T^{(1,0)}M = Span_{\mathbb{C}}\{L_j, j = 1, \ldots, n\}$ and its CR dimension is equal to n. On the other hand, since $L_j \in T^{(1,0)}\mathbb{C}^{n+1}$, we also have $[L_j, L_k] \in T^{(1,0)}\mathbb{C}^{n+1}$. Therefore $[L_j, L_k] \in T^{(1,0)}\mathbb{C}^{n+1} \cap \mathbb{C}TM = T^{(1,0)}M = Span_{\mathbb{C}}\{L_j, j = 1, \ldots, n\}$. The proof for the rest of the results is obvious. $\square$

Denote $H(M) := T^{(1,0)}M \cup T^{(0,1)}M$. It is a subbundle of $\mathbb{C}TM$ of codimension one. Along M there exists a globally defined nonvanishing real one form θ, which vanishes on $H(M)$. We call θ to be *contact form.* If M is defined by (1.1), then one may check

$\theta_0 = i\partial r$ is a contact form and $\theta_0(T) = 1$. Given any contact form θ, there exists a nonvanishing function c on M such that $\theta = c\theta_0$.

Definition 1.1.8 *The Levi form with respect to a given contact form θ at $p \in M$ is the Hermitian form on $T_p^{(1,0)}M$ defined by*

$$\mathcal{L}_\theta|_p(L, K) = -d\theta|_p(L, K), \quad \text{for } \; L \in \Gamma^\infty(T^{(1,0)}M), K \in \Gamma^\infty(T^{(0,1)}M) \text{ near } p.$$

We call the real hypersurface M to be Levi nondegenerate at a point $p \in M$ if $\mathcal{L}_\theta|_p$ is nondegenerate.

Since $d\theta(L, K) = L(\theta(K)) - K(\theta(L)) - \theta([L, K])$ and $\theta(K) = \theta(L) = 0$, the definition of Levi form is also given by

$$\mathcal{L}_\theta(L, K) = \langle \theta, [L, K] \rangle.$$

Using Lemma 1.1.7, the matrix of the levi form with respect to θ under the basis (1.2) can be written to be $(c\lambda_{j\bar{k}})$ for some nonzero function c. Therefore M is Levi nondegenerate at $p \in M$ if and only if $rank(\lambda_{j\bar{k}}) = n$ at p. Denote ν^+ and ν^- to be the number of positive and negative eigenvalues of $\lambda_{j\bar{k}}$, respectively, then

Definition 1.1.9 *A Levi nondegenerate hypersurface is called of signature ℓ if*

$$\min(\nu^+, \nu^-) = \ell.$$

A Levi non-degenerate smooth hypersurface M_ℓ in $\mathbb{C}^{n+1}$ is of signature ℓ near the origin if it locally is defined by an equation of the form: $r(z, w) = \Im w - \sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n} |z_j|^2 + \circ(|z|^2 + |z \cdot \Re w|) = 0$ for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. Another remark mentioned here is by multiplying -1 to the contact form θ if necessary, we can always assume $\min(\nu^+, \nu^-) = \nu^-$.

Example 1.1.10 *Consider the boundary of the unit ball in $\mathbb{C}^{n+1}$: $\mathbb{S}^n = \{z \in \mathbb{C}^{n+1} : |z|^2 = 1\}$. Here we use the notation $\langle a, \bar{b} \rangle = \sum_j a_j \bar{b}_j$ and $|z|^2 = \langle z, \bar{z} \rangle$. In order to write the defining function near the origin to be of the form in Lemma 1.1.6, we introduce the Cayley Cayley transformation:*

$$\rho(z, w) = \left(\frac{2z}{1 - iw}, \frac{1 + iw}{1 - iw} \right).$$

One can check ρ^{-1} biholomorphically maps $\mathbb{S}^n \backslash \{(0,1)\}$ onto $\mathbb{H}^{n+1}$, where $\mathbb{H}^{n+1} = \{(z,w) \in \mathbb{C}^n \times \mathbb{C} : 0 = \Im w - |z|^2\}$. We call $\mathbb{H}^{n+1}$ to be the standard Heisenberg Hypersurface in $\mathbb{C}^{n+1}$. Apparently it is a Levi nondegenerate hyperquadrics of signature 0 in $\mathbb{C}^{n+1}$. Applying Lemma 1.1.7 to $\mathbb{H}^{n+1}$, We have a global basis for $\mathbb{C}T\mathbb{H}^{n+1}$:

$$L_j = \frac{\partial}{\partial z_j} + 2i\bar{z}_j \frac{\partial}{\partial w}, j = 1, \ldots, n. \tag{1.7}$$

$$\overline{L_j} = \frac{\partial}{\partial \bar{z}_j} - 2iz_j \frac{\partial}{\partial \bar{w}}, j = 1, \ldots, n. \tag{1.8}$$

$$T = 2(\frac{\partial}{\partial w} + \frac{\partial}{\partial \bar{w}}). \tag{1.9}$$

and a special contact form

$$\theta = \frac{1}{2}\partial w - i\bar{z}_j \partial z_j.$$

Moreover,

$$[L_j, \overline{L_k}] = -2i\delta_{jk},$$

where $\delta_{jk} = 1$ if $j = k$ and 0 otherwise.

On $\mathbb{H}^{n+1}$, there are three special self maps of interests:

1. Given any point $p_0 = (z_0, w_0) \in \mathbb{H}^{n+1}$, define:

$$\sigma^0_{p_0}(z,w) = (z + z_0, w + w_0 + 2i\langle z, \bar{z}_0\rangle).$$

2. Given an $n \times n$ unitary matrix U and a nonzereo scalar $\lambda \in \mathbb{R}$, define:

$$(z,w) \to (\lambda U z, \lambda^2 w)$$

3. Given a vector $\mathbf{a} \in \mathbb{C}^n$, $r \in \mathbb{R}$ define:

$$(z,w) \to (\frac{z - \mathbf{a}w}{1 + 2i\langle z, \bar{\mathbf{a}}\rangle + (r - i|\mathbf{a}|^2)w}, \frac{w}{1 + 2i\langle z, \bar{\mathbf{a}}\rangle + (r - i|\mathbf{a}|^2)w})$$

It is not hard to verify the above three self maps of $\mathbb{H}^{n+1}$ are also one-to-one. We denote the group consisting of all bimeromorphic self maps of $\mathbb{H}^{n+1}$ by $Aut(\mathbb{H}^{n+1})$ and denote by $Aut_0(\mathbb{H}^{n+1})$ the subgroup of $Aut(\mathbb{H}^{n+1})$ consisting of those who preserve the origin. It's worth mentioning that any element in $Aut(\mathbb{H}^{n+1})$ can always be written as a combination of the above three mappings.

We will continue to discuss the application of the properties of $\mathbb{H}^n$ in the next section. Similar results may also hold on the generalized Heisenberg hypersurfaces $\mathbb{H}_\ell^{n+1} = \{(z,w) \in \mathbb{C}^{n+1} : 0 = \Im w - |z|_\ell^2\}$ - hyperquadrics of signature ℓ. Here our notation is $\langle a, \bar{b}\rangle_\ell = -\sum_{j\le\ell} a_j\bar{b}_j + \sum_{j>\ell} a_j\bar{b}_j$ and $|z|_\ell^2 = \langle z, \bar{z}\rangle_\ell$.

Definition 1.1.11 *Let D_1 and D_2 be two domains in $\mathbb{C}^n$ and $\mathbb{C}^n$, respectively. A holomorphic map $F : D_1 \to D_2$ is said to be a proper holomorphic map if for any compact subset $K \in D_2$, the inverse image $F^{-1}(K)$ is compact in D_2.*

Remark 1.1.12 *Given a holomorphic map $F : D_1 \to D_2$. Suppose F extend continuously to the boundary of D_1. Then F is proper if and only if F maps ∂D_1 into ∂D_2.*

The next lemma is about polarization (or complexification). This technique allows us to replace the dependent conjugate variable to another independent variable in certain identities. It has been shown quite useful when dealing with proper holomorphic maps between CR hypersurfaces.

Lemma 1.1.13 *Let D be a domain in $\mathbb{C}^n$ and let $\bar{D} = \{z : \bar{z} \in D\}$ be its conjugate domain. Suppose that $H : D \times \bar{D} \to \mathbb{C}$ is a holomorphic mapping and $H(z,\bar{z}) = 0$ for any $z \in D$. Then $H(z,w) \equiv 0$ on $D \times \bar{D}$.*

Proof of Lemma 1.1.13: Without loss of generality, we can assume D is the unit disk in $\mathbb{C}^n$. For $z \in \mathbb{C}^n$, we can write $z = (r_1e^{i\theta_1}, \ldots, r_ne^{i\theta_n}) := re^{i\theta}$ for some vectors r, θ and write the Taylor expansion of $H(z,w)$ to be $H(z,w) = \sum_{\alpha,\beta} a_{\alpha\beta}z^\alpha w^\beta$. Plug the expression of z into the above Taylor expansion, we get

$$\sum_{\alpha,\beta} a_{\alpha\beta} r^{\alpha+\beta} e^{i\theta(\alpha-\beta)} = 0$$

for any r, θ. Collect terms containing r^k, we have

$$\sum_{\alpha} a_{\alpha(k-\alpha)} e^{i\theta(2\alpha-k)} = 0.$$

Since θ is arbitrary, we immediately have $a_{\alpha\beta} = 0$. $\square$

We call $H(z,w)=0$ to be the complexification of $H(z,\bar{z})=0$. Another approach of the proof is to show $M=\{(z,\bar{z}) : z\in D\}$ is a totally real submanifold of maximal dimension, i.e. $\dim T^{(1,0)}M=0$ and $\dim M=n$. Therefore M is a uniqueness set for holomorphic functions in $D\times\bar{D}$.

Suppose $M=\{z\in D : r(z,\bar{z})=0\}$ and $\mathrm{d}r\neq 0$. We also call the complexification of M to be $\mathcal{M}=\{(z,\xi)\in D\times \mathrm{Conj}(D) :\ r(z,\xi)=0\}$.

Corollary 1.1.14 *Let M and $\tilde{M}$ be two real real-analytic hypersurafaces in $\mathbb{C}^n$ and $\mathbb{C}^N$, respectively locally defined by real analytic functions $r,\tilde{r}$. $F : M\to\tilde{M}$. Then $\tilde{r}(f(z),\bar{f}(w))=0$ on $r(z,w)=0$.*

Proof of Corollary 1.1.14: By the assumption, we already have

$$\tilde{r}(f(z),\overline{f(z)})=0 \quad \text{on} \quad r(z,\bar{z})=0$$

and $dr, d\tilde{r}\neq 0$ along M. This implies there exists some holomorphic function $h(z,w)$ such that

$$\tilde{r}(f(z),\overline{f(z)})-h(z,\bar{z})r(z,\bar{z})=0. \tag{1.10}$$

Now apply Lemma 1.1.13 to holomorphic function $\tilde{r}(f(z),\bar{f}(w))-h(z,w)r(z,\bar{w})$. The Corollary is then straight forward by (1.10). □

Suppose M and $\tilde{M}$ are two CR hypersurfaces in $\mathbb{C}^{n+1}$ and in $\mathbb{C}^{N+1}$, respectively. $F:\mathbb{C}^{n+1}\to\mathbb{C}^{N+1}$ is holomorphic and $F(M)\subset\tilde{M}$. We have the definition of CR transversality of F to $\tilde{M}$ as follows:

Definition 1.1.15 *A holomorphic map $F : M(\in\mathbb{C}^{n+1})\to\tilde{M}(\in\mathbb{C}^{N+1})$ is said to be CR transversal at $p\in M$ if*

$$T^{(1,0)}_{F(p)}\tilde{M}+dF(T^{(1,0)}_p\mathbb{C}^{n+1})=T^{(1,0)}_{F(p)}\mathbb{C}^{N+1}$$

In the following lemma, we will show CR transversality of the mapping is equivalent to the fact that the derivatives of normal components along normal direction is nonvanishing.

Lemma 1.1.16 *Consider a holomorphic map* $F : M(\in \mathbb{H}^{n+1}) \to \tilde{M}(\in \mathbb{H}^{N+1})$ *and* $F(0) = 0$. *M and $\tilde{M}$ are given under the local coordinates as in (1.1). Write* $F = (\tilde{f}, g)$. *Then F is CR transversal to $\tilde{M}$ at 0 if and only if* $\frac{\partial g}{\partial w}(0) = 0$.

Proof of Lemma 1.1.16: Denote (z, w) and $(\tilde{z}, \tilde{w})$ to be the local coordinates for M and $\tilde{M}$. Obviously $T_0^{(1,0)}\tilde{M} = \{\frac{\partial}{\partial \tilde{z}_j}|_0, j = 1, \ldots, n\}$. In order that F is CR transversal to $\tilde{M}$ at 0, we just need to show $\frac{\partial}{\partial \tilde{w}} \in dF(T_p^{(1,0)}\mathbb{C}^{n+1})$ by definition. On the other hand, recall

$$dF(\frac{\partial}{\partial w})|_0 = \sum_{j \le N} \frac{\partial \tilde{f}_j}{\partial w}|_0 \frac{\partial}{\partial \tilde{z}_j} + \frac{\partial g}{\partial w}|_0 \frac{\partial}{\partial \tilde{w}} \equiv \frac{\partial g}{\partial w}|_0 \frac{\partial}{\partial \tilde{w}} \quad mod \ (T_0^{(1,0)}\tilde{M}).$$

The Lemma thus holds. □

1.2 Normalization on proper holomorphic maps between balls

In this section, we will introduce a useful normalization process given by Huang [Hu]. Let $M \in \mathbb{H}^{n+1}$ and $\tilde{M} \in \mathbb{H}^{N+1}$ be two germs of $\mathbb{H}^{n+1}$ and $\mathbb{H}^{N+1}$ near the origin. As is shown in Lemma 1.1.7, a choice of a global basis for $T^{(1,0)}M$ is $\{L_j = \frac{\partial}{\partial z_j} + 2i\bar{z}_j \frac{\partial}{\partial w}, j = 1, \ldots, n\}$. Let

$$F = (\tilde{f}, g) = (f, \phi, g) = (f_1, \ldots, f_n, \phi_1, \ldots, \phi_{N-n}, g)$$

be a smooth nontrivial CR map from M to $\tilde{M}$ with $F(0) = 0$. It's known by Lewy extension theorem F can be extended holomorphically to one side of M denoted by $D \in \mathbb{C}^{n+1}$.

Next, we assign the weight of z and w to be 1 and 2, respectively. For a non-negative integer m, We say a function $h(z, w)$ defined in a small neighborhood U of 0 to be in $o_{wt}(m)$, if $h(tz, t^2w)/|t|^m \to 0$ as $t \to 0$. For a smooth function $h(z, w)$ defined in U, we also denote by $h^{(k)}(z, w)$ terms of weighted degree k in the Taylor expansion of h at 0.

Since $F(M) \subset \tilde{M}$, we have

$$\frac{g - \bar{g}}{2i} = |f|^2 + |\phi|^2 \tag{1.11}$$

Write $\tilde{f} = zA + w\mathbf{a} + O((z,w)^2)$ where A is an $n \times N$ matrix and $\mathbf{a} \in \mathbb{C}^n$. Applying $L^\alpha = L_1^{\alpha_1} L_2^{\alpha_2} \cdots L_n^{\alpha_n}$ to (1.11), we have

$$\frac{L^\alpha g}{2i} = \langle L^\alpha \tilde{f}, \bar{\tilde{f}} \rangle,$$

Letting $(z,w) = 0$, it follows $L^\alpha g|_0 = 0$ and therefore

$$g(z,0) \equiv 0.$$

Write $g(z,w) = \lambda w + O((z,w)^2)$. We claim $\lambda \neq 0$. Indeed, consider subharmonic function $u = \Im g - |\tilde{f}|^2$ on $\{(z,w) \in \mathbb{C}^{n+1} : \Im w - |z|^2 > 0\}$. By Hopf lemma, $0 \neq \frac{\partial u}{\partial \mathbf{n}}(0) = \frac{\partial(\Im g)}{\partial(\Im w)}(0)$, where $\mathbf{n}$ is the normal direction of M at 0. Therefore $\lambda = \frac{\partial g}{\partial w}(0) \neq 0$.

Apply $\overline{L_k} L_j$ to (1.11),

$$\frac{\overline{L_k} L_j g}{2i} = \langle \overline{L_k} L_j \tilde{f}, \bar{\tilde{f}} \rangle + \langle L_j \tilde{f}, \overline{L_k \tilde{f}} \rangle.$$

Let $(z,w) = 0$ and notice $\overline{L_k} L_j g|_0 = \delta_{jk} \lambda$. This then leads to

$$\lambda I_n = A \bar{A}^t$$

and hence $\lambda > 0$. Extend $A/\sqrt{\lambda}$ to a Unitary matrix of size $N \times N$, denoted by $\tilde{A}/\sqrt{\lambda}$ and consider a new map F^* given by:

$$F^* = (\tilde{f}^*, g^*) = (f^*, \phi^*, g^*) = \frac{1}{\sqrt{\lambda}} F \cdot \begin{pmatrix} \overline{\tilde{A}^t} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix}.$$

Then $F^*(M_1) \in \mathbb{H}^{N+1}$ and

$$\tilde{f}^*(z,w) = (z,0) + \mathbf{a}w + \tilde{f}^{*(2)}(z) + o_{wt}(2)$$
$$g^*(z,w) = w + dw^2 + O(|zw|, |z^2 w|) + o_{wt}(4),$$

Here $\mathbf{a} \in \mathbb{C}^N$ and $d \in \mathbb{C}$. Now write $r = \Re d$ and define $G \in Aut_0(\mathbb{H}^{N+1})$ by:

$$G(z,w) = \left(\frac{z^* - \mathbf{a}w^*}{1 + 2i\langle z^*, \bar{\mathbf{a}} \rangle + (r - i|\mathbf{a}|^2)w^*}, \frac{w^*}{1 + 2i\langle z^*, \bar{\mathbf{a}} \rangle + (r - i|\mathbf{a}|^2)w^*}\right).$$

We then define the second normalization F^{**} by

$$F^{**} = (\tilde{f}^{**}, g^{**}) = (f^{**}, \phi^{**}, g^{**}) := G \circ F^*.$$

Still we have $F^{**}(M) \in \mathbb{H}^{N+1}$ and

$$\tilde{f}^{**}(z,w) = (z,0) + \tilde{f}^{**(2)}(z) + o_{wt}(2), \tag{1.12}$$

$$g^{**}(z,w) = w + iew^2 + O(|zw|, |z^2w|) + o_{wt}(4). \tag{1.13}$$

Here $e \in \mathbb{R}$. Moreover, we have the following Lemma:

Lemma 1.2.1 *[Hu] After composing F with certain $Aut(\mathbb{H}^{N+1})$, the map $F = (f, \phi, g)$ can be assumed to take the following normal form:*

$$f(z,w) = z + a^{(1)}(z)w + o_{wt}(3),$$

$$\phi(z,w) = \phi^{(2)}(z) + o_{wt}(2),$$

$$g(z,w) = w + o_{wt}(4)$$

with

$$-2i\langle \bar{z}, a^{(1)}(z)\rangle |z|^2 = |\phi^{(2)}(z)|^2. \tag{1.14}$$

Proof of Lemma 1.2.1: First according to the above normalization process we already have

$$f(z,w) = z + a^{(2)}(z) + a^{(1)}(z)w + a^{(3)}(z) + o_{wt}(3),$$

$$\phi(z,w) = \phi^{(2)}(z) + o_{wt}(2),$$

$$g(z,w) = w + iew^2 + c^{(1)}(z)w + c^{(2)}(z)w + o_{wt}(4)$$

and (f, ϕ, g) satisfy (1.11) on $w = u + i|z|^2$. Plugging the above into (1.11) and replacing $w = u + i|z|^2$ we have:

$$\begin{aligned} & v + e(u^2 - |z|^4) + u\Im(c^{(1)}(z) + c^{(2)}(z)) + |z|^2\Re(c^{(1)}(z) + c^{(2)}(z)) \\ =& \langle z + a^{(2)}(z), \overline{z + a^{(2)}(z)}\rangle + 2\Re\langle z, \overline{a^{(1)}(z)}(u - i|z|^2) + a^{(3)}(z)\rangle \\ & + \langle \phi^{(2)}(z), \overline{\phi^{(2)}(z)}\rangle + o_{wt}(4). \end{aligned}$$

Collecting terms containing u^2, we have $e = 0$. Collect terms of uz^2 and uz, then $c^{(2)}(z) = c^{(1)}(z) = 0$. Collect terms of the form $z^3\bar{z}$ and $z^2\bar{z}$, we get $a^{(3)}(z) = a^{(2)}(z) = 0$. Finally, the above identity becomes:

$$0 = 2\Re\langle z, \overline{a^{(1)}(z)}(u - i|z|^2)\rangle + \langle \phi^{(2)}(z), \overline{\phi^{(2)}(z)}\rangle + o_{wt}(4).$$

Since u is an independent variable, we then get

$$\Re\langle z, \overline{a^{(1)}(z)}\rangle = 0$$

and therefore

$$-2i\langle z, \overline{a^{(1)}(z)}\rangle|z|^2 = \langle \phi^{(2)}(z), \overline{\phi^{(2)}(z)}\rangle.$$

The proof of Lemma 1.14 is thus complete. □

The above argument may also be used to proper holomorphic maps between hyperquadrics of the same signature $\ell > 0$ ([BH]).

In order to obtain the results of Chapter 2, the next lemma will be used. It was proved in [Hu], Lemma 3.2 for $\ell = 0$. It turns out the same proof also applies with no change to $\ell > 0$ case.

Lemma 1.2.2 *[Hu] Let k, ℓ, n be nonnegative integers such that $1 \le k < n$. Assume that $g_1, \ldots, g_k, f_1, \ldots, f_k$ are germs at $0 \in \mathbb{C}^n$ of holomorphic functions such that*

$$\sum_{i=1}^{k} g_i(z)\overline{f_i(z)} = A(z, \bar{z})|z|^2_\ell,$$

where $A(z, \xi)$ is a germ at $0 \in \mathbb{C}^n \times \mathbb{C}^n$ of a holomorphic function. Then $A(z, \bar{z}) \equiv 0$.

Applying Lemma 1.1.13 by taking complexification, we can rephrase the above lemma as follows.

Remark 1.2.3 *Let $1 \le k < n$. Assume $g_1, \ldots, g_k, f_1, \ldots, f_k$ are germs at $0 \in \mathbb{C}^n$ of holomorphic functions and $A(z, \xi)$ is a germ at $0 \in \mathbb{C}^n \times \mathbb{C}^n$ of a holomorphic function. Then*

$$\sum_{i=1}^{k} g_i(z) f_i(\xi) = A(z, \xi)\langle z, \xi\rangle_\ell$$

implies $A(z, \xi) \equiv 0$.

1.3 Chern-Moser normal form and Chern-Moser-Weyl curvature tensor

Before stating the Chern-Moser's normal coordinates, some notations will be explained in advance. For any analytic function $F(z, \bar{z}, u)$, we decompose F to be of the form

$$F = \sum_{k,j \geq 0} F_{kl},$$

where $F_{kj}(\lambda z, \mu \bar{z}, u) = \lambda^k \mu^j F_{kj}(z, \bar{z}, u)$ for all complex numbers λ, μ, and call (k, j) the "type" of F_{kj}. For any analytic function of type (k, j), we can introduce a contraction operation 'tr' as follow:

$$tr(F_{kj}) = -\sum_{i=1}^{\ell} \frac{\partial^2 F_{kj}}{\partial z_i \partial \bar{z}_i} + \sum_{i=\ell+1}^{n} \frac{\partial^2 F_{kj}}{\partial z_i \partial \bar{z}_i}.$$

$tr(F_{kj})$ is a function of type $(k-1, j-1)$.

Theorem 1.3.1 *[CM]: Given a real real analytic Levi-nondegenerate hypersurface M in $\mathbb{C}^{n+1}$ of signature ℓ, there exists a unique holomorphic transformation of local coordinates such that under the new coordinates M can be written as:*

$$\Im w = \langle z, z \rangle_\ell + \sum_{i \geq 2, j \geq 2} F_{ij}(z, \bar{z}, \Re w).$$

What's more,

$$tr F_{22} = (tr)^3 F_{33} = 0, \quad (tr)^2 F_{32} = 0.$$

It's worth noting that the equations $tr F_{22} = (tr)^3 F_{33} = 0$ and $(tr)^2 F_{32} = 0$ remain meaningful for smooth manifolds. Indeed, if M is $\mathcal{C}^6$-smooth, one can achieve the above normal forms up to terms of order 6.

In the same paper when studying the equivalence problems of real analytic CR hypersurfaces, Chern-Moser introduced a 4-th order tensor $\mathcal{S}$ on $T^{(1,0)}M \otimes T^{(0,1)}M \otimes T^{(1,0)}M \otimes T^{(0,1)}M$. $\mathcal{S}$ is what we called Chern-Moser-Weyl curvature tensor. Its coefficients at a given $p \in M$ with respect to some well-chosen 1-forms ω_α, θ satisfies the following properties:

$$(S_{\theta|_p})_{\alpha\bar{\beta}\gamma\bar{\delta}} = (S_{\theta|_p})_{\gamma\bar{\beta}\alpha\bar{\delta}} = (S_{\theta|_p})_{\gamma\bar{\delta}\alpha\bar{\beta}},$$

$$\overline{(S_{\theta|_p})_{\alpha\bar{\beta}\gamma\bar{\delta}}} = (S_{\theta|_p})_{\beta\bar{\alpha}\delta\bar{\gamma}},$$

and the following trace-free condition:

$$\sum_{\beta,\alpha=1}^{n} g^{\bar{\beta}\alpha}(S_{\theta|_p})_{\alpha\bar{\beta}\gamma\bar{\delta}} = 0.$$

Here $(g^{\bar{\beta}\alpha})$ is the diagonal matrix whose first ℓ entries are -1 and rest 1. Chern-Moser-Weyl curvature tensor $\mathcal{S}$ in local coordinates realizes as the 4-th degree power series of the defining function. The above trace-free condition is equivalent to $trF_{22} = 0$ in Theoerem 1.3.1. Later Webster [We2] developed the pseudo Hermitian connection with respect to an admissible coframe $\theta, \theta_\alpha, \theta_{\bar{\alpha}}$ and induced the corresponding curvature tensor $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$. He showed that the $\mathcal{S}$ given in [CM] indeed is the traceless component of the curvature tensor in the sense of pseudo Hermitian connection. It is pseudo conformally invariant in the sense that if given any two contact forms $\tilde{\theta}$ and θ, then

$$(\mathcal{S}_{\tilde{\theta}})^{\bar{\alpha}}{}_{\bar{\beta}\gamma\bar{\delta}} = (\mathcal{S}_{\theta})^{\bar{\alpha}}{}_{\bar{\beta}\gamma\bar{\delta}}.$$

He also gave the explicit formula for the conformally invariant Chern-Moser-Weyl curvature tensor:

$$\begin{aligned}\mathcal{S}_{\alpha\bar{\beta}\gamma\bar{\delta}} =& R_{\alpha\bar{\beta}\gamma\bar{\delta}} - \frac{1}{n+2}(R_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + R_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}} + \delta_{\alpha\bar{\beta}}R_{\gamma\bar{\delta}} + \delta_{\gamma\bar{\delta}}R_{\alpha\bar{\beta}}) \\ &+ \frac{R}{(n+1)(n+2)}(\delta_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + \delta_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}})\end{aligned}$$

Here $(g_{\alpha\bar{\beta}})$ is the pseudo Hermitian metric induced by the admissible coframe, $R_{\alpha\bar{\beta}}$ and R are Ricci and Scalar tensors. $\mathcal{S}$ vanishes identically when $n = 1$.

Another stunning result in [CM] is that locally spherical hypersurfaces are the only strongly pseudoconvex hypersurfaces with vanishing Chern-Moser-Weyl tensors.

Theorem 1.3.2 *[CM] When $n > 1$, $\mathcal{S}$ vanishes if and only if M is locally equivalent to the generalized Heisenberg hypersurface in $\mathbb{C}^{n+1}$.*

Chapter 2

Rigidity and holomorphic Segre transversality for holomorphic Segre mappings

2.1 Background

In the category of several complex variables, the following problems are fundamental and of great importance.

Question 2.1.1 *: Given two complex manifolds M_1 and M_2, classify proper holomorphic mappings from M_1 into M_2, modulo automorphisms of the source and the target manifolds.*

Question 2.1.2 *: (CR Version of Problem 2.1.1): Given two CR manifolds M_1 and M_2, classify smooth CR mappings from M_1 into M_2, modulo CR automorphisms of the source and the target manifolds.*

There have been extensive studies in the past for Problem 2.1.1 especially when M_1 and M_2 are bounded symmetric domains in complex Euclidean spaces of different dimensions or when M_1 and M_2 are of the same complex dimension. (See [Mok], [For], [Mir], [BER], [DA2], [EL], [LM], [LM2] etc). When the complex manifolds in Problem 2.1.1 have smooth boundary and when the mappings also extend smoothly up to the boundary, Problem 2.1.1 can be reduced to the study of Problem 2.1.2.

In the context of Problem 2.1.2, due to the existence of CR invariants, for two generic M_1 and M_2, there is no non-trivial smooth CR map between M_1 and M_2. Hence, one often focuses on mappings between CR manifolds with vanishing invariants– namely, the boundary of generalized spheres or their unbounded realization - hyperquadrics (through Cayley transformation). Denote the unbounded realization - Heisenberg hypersurface - of the unit sphere in $\mathbb{C}^n$ by $\mathbb{H}^n := \{(z_1, \ldots, z_{n-1}, w) \in \mathbb{C}^n : \Im w =$

$\sum_{j=1}^{n-1} |z_j|^2\}$. A general result of Forstnerič states that all $\mathcal{C}^{N-n+1}$-smooth CR mappings from $\mathbb{H}^n$ into $\mathbb{H}^N$ are rational mappings, the quotients of polynomials.

An important but special case of Problem 2.1.1 and Problem 2.1.2 can then be phrased as follows:

Question 2.1.3 *: Classify proper (rational) holomorphic maps f from the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ into the unit ball $\mathbb{B}^N \subset \mathbb{C}^N$ up to the automorphisms of the target and source domains.*

The study of the above problem dates back to Poincaré. He proved that rational proper holomorphic maps from $\mathbb{B}^2$ into itself must be automorphisms of $\mathbb{B}^2$. Since then, much attention has been paid to this study. Alexander [Al] showed any proper holomorphic self-map of $\mathbb{B}^n$ $(n > 1)$ is an automorphism, hence completing with the equal dimensional case. Webster [We] was the first one studying proper holomorphic maps between balls of different dimension. He showed any proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^{n+1}$ with $\mathcal{C}^3$-smoothness up to the boundary is indeed a linear embedding modulo automorphisms of balls. Cima-Suffridge [CS] later reduced the boundary regularity to $\mathcal{C}^2$ smoothness in Webster's result. For larger codimensional case, Faran [Fa] proved any proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^N$ with $N < 2n-1$, that is analytic up to the boundary, is a linear embedding. Later Huang [Hu] relaxed the boundary assumption and proved the same conclusion holds provided the map is $\mathcal{C}^2$-smooth up to the boundary. When $N \geq 2n-1$, the proper mapping is no longer rigid. When $N = 2n-1, n > 2$ for instance, if the map is $\mathcal{C}^2$-smooth up to the boundary, Huang-Ji [HJ] shows there are only two possibilities: linear embedding or Whitney map $W : z = (z_1, \ldots, z_n) \to (z_1, \ldots, z_{n-1}, z_n z)$. Interesting readers may find more about the classifications of proper holomorphic maps between balls of higher codimensions in [HJX], [DA] etc.

From a slightly different point of view, consider the complexifications of balls and some appropriate type of mappings between them. The appropriate mapping over those complex analyitic varieties of dimension $2n-1$ is the so called *holomorphic Segre mapping.* For these types of mappings, the rigidity properties no longer holds. However, one may still expect a semi-rigidity property and transversality property. The basic setting is given in the next section.

2.2 Setup and Main Theorem

Let D be an open subset in $\mathbb{C}^n$ and M be a real analytic hypersurface in D with a real analytic defining function r. Namely, $M = \{z \in D : r(z, \bar{z}) = 0\}$ and $\mathrm{d}r \neq 0$. The complexification of M is defined as: $\mathcal{M} = \{(z, \xi) \in D \times \mathrm{Conj}(D) : \ r(z, \xi) = 0\}$. Here for a set $E \subset \mathbf{C}^n$, $\mathrm{Conj}(E) := \{\bar{z} : \ z \in E\}$. Assume that $\mathrm{d}r \neq 0$ over $\mathcal{M}$. Then $\mathcal{M}$ is a complex submanifold of complex codimension one in $D \times \mathrm{Conj}(D)$ which contains $\{(z, \bar{z}) : \ z \in M\}$ as a maximally totally real submanifold. Define complex analytic varieties $Q_\xi := \{z \in D : \ r(z, \xi) = 0\}$ for $\xi \in \mathrm{Conj}(D)$ and $\hat{Q}_z := \{\xi \in \mathrm{Conj}(D) : r(z, \xi) = 0\}$ for $z \in D$. We call Q_ξ and $\hat{Q}_z$ the *Segre variety* of M with respect to ξ and z, respectively. (See [We]). Notice that $\mathcal{M}$ is then holomorphically foliated by $\{Q_\xi \times \{\xi\}\}$ and also by $\{\{z\} \times \hat{Q}_z\}$ for $\xi \in \mathrm{Conj}(D)$ and $z \in D$. As in the literature (see, for instance, [Ch] [Fa2] [HJ2]), we call $\mathcal{M}$ the Segre family associated with M.

A fundamental fact for the Segre family is its invariant property by holomorphic maps. (See the famous paper of S. Webster [We]). More precisely, let $\widetilde{M}$ be another real analytic hypersurface in $\widetilde{D} \subset \mathbf{C}^N$ and let $\widetilde{\mathcal{M}}$ be its Segre family. f is a holomorphic map from D into $\widetilde{D}$ sending M into $\widetilde{M}$. Then $f(Q_\xi) \subset \widetilde{Q}_{\overline{f}(\xi)}$ and $\overline{f}(\hat{Q}_z) \subset \hat{\widetilde{Q}}_{f(z)}$ for $\xi \in \mathrm{Conj}(D)$ and $z \in D$. Here, for instance, we write $\widetilde{Q}_{\overline{f}(\xi)}$ for the Segre variety of $\widetilde{M}$ with respect to $\overline{f}(\xi)$. In particular, f induces a holomorphic map $\mathcal{F} := (f(z), \overline{f}(\xi))$ from $\mathcal{M}$ into $\widetilde{\mathcal{M}}$. Here, as usual, we write $\bar{f}(z)$ for $\overline{f(\bar{z})}$. More generally, we introduce the following notion as in [HJ2]:

Definition 2.2.1 *: Φ is called a holomorphic Segre map from $\mathcal{M}$ into $\widetilde{\mathcal{M}}$ if Φ is a holomorphic map from $\mathcal{M}$ to $\widetilde{\mathcal{M}}$ such that Φ sends each $Q_\xi \times \{\xi\}$ of $\mathcal{M}$ into a certain $Q_{\widetilde{\xi}} \times \{\widetilde{\xi}\}$ of $\widetilde{\mathcal{M}}$ and sends each $\{z\} \times \hat{Q}_z$ into a certain $\{\widetilde{z}\} \times \hat{\widetilde{Q}}_{\widetilde{z}}$ of $\widetilde{\mathcal{M}}$ for $\xi \in$ Conj(D) and $z \in D$.*

We remark that there is another important but very different class of real-analytic maps closely related to the Segre families introduced by Baouendi-Ebenfelt-Rothschild

(See [BER1]), which are called the Segre maps in many references. A holomorphic Segre map Φ is called a *holomorphic Segre embedding* if it is also a holomorphic embedding. Holomorphic Segre maps from $\mathcal{M}$ into $\widetilde{\mathcal{M}}$ are the natural generalizations of holomorphic mappings from M into $\widetilde{M}$. As already demonstrated by E. Cartan, holomorphic Segre maps play a very role in the study of holomorphic equivalence problems and many other related fields. (See [Car] [Ch] [BER3] [Hu2] and the references therein, for instance).

In this chapter, we focus our attention on holomorphic Segre maps between the Segre family of the model manifolds: Heisenberg hypersurfaces. Holomorphic Segre maps are much less restricted than holomorphic maps induced from CR maps between Heisenberg hypersurfaces. It is thus not surprising that many properties for the latter are no longer true for holomorphic Segre maps. For instance, it is an easy consequence of the classical Hopf lemma that any non-constant holomorphic map between Heisenberg hypersurfaces must have non-vanishing normal derivative in its normal component. (This property is called the Hopf Lemma property by Baouendi-Rothschild [BR2] or the CR transversality by Ebenfelt-Rothschild [ER] and Ebenfelt-Huang-Zaitsev [EHZ]). However, such a property does not hold anymore for general holomorphic Segre maps. Also, easy examples show that there are many non-rational holomorphic Segre embeddings from $\mathcal{H}^n$ into $\mathcal{H}^N$ for $N \geq n+1$, which can not occur for holomorphic maps between the Heisenberg hypersurfaces as is well known from the work of Forstneric [For]. (See below for the precise definition of $\mathcal{H}^n$ and Remark 5.3 for related examples.)

Making use of the recent work of Baouendi-Huang [BH] and Ebenfelt-Huang-Zaitsev [EHZ], we will provide, in this chapter, two theorems for holomorphic Segre maps from $\mathcal{H}^n$ into $\mathcal{H}^N$ with $N \leq 2n-2$. As was done in those papers, we first normalize the holomorphic Segre maps by composing the maps with automorphisms in the source and in the target. For those normalized holomorphic Segre maps, we then prove that although full linearity fails, linearity for certain components still holds. Afterwards, we use this semi-linearity property to prove a propagation theorem concerning the failure of a holomorphic notion of the Hopf lemma. Namely, we will show that if the transversality breaks down at one point, then it breaks down along one of the two Segre varieties through this point. One may compare this with the work of Baouendi-Huang

[BH], where the transversality breaks down at one point if and only if it does for all points.

Theorem 2.2.2 *[BH] Let M be a small neighborhood of 0 in $\mathbb{H}^n_\ell$ with $0<\ell<n$. Suppose that $F=(f_1,\ldots,f_{N-1},g)$ is a holomorphic map from a neighborhood U of M in $\mathbb{C}^n$ into $\mathbb{C}^N$ with $F(M)\subset\mathbb{H}^N_\ell$, $N\geq n$, and $F(0)=0$. Suppose $\ell\leq(n-1)/2$. Then, the following hold:*
(i) If $\frac{\partial g}{\partial w}(0)\neq 0$, then F is linear fractional. Moreover, there exists $\tau\in Aut_0(\mathbb{H}^N_\ell)$ such that $\tau\circ F(z_1,...,z_{n-1},w)=(z_1,...,z_{n-1},0,...,0,w)$.
(ii) If $\frac{\partial g}{\partial w}(0)=0$, then $F(U)\subset\mathbb{H}^N_\ell$. More precisely, there is a constant $(N-\ell-1,\ell-1)$ complex matrix V, with $VV^t=Id_{N-\ell-1}$, such that $g\equiv 0,(f_1,...,f_\ell)\equiv(f_{\ell+1},...,f_{N-1})V$.

We next give more notation to state our main theorems.

Recall the Heisenberg hypersurface in $\mathbf{C}^n$ is given by

$$\mathbb{H}^n := \{(z_1,\ldots,z_{n-1},w)\in\mathbf{C}^n:\ \Im w=\sum_{j=1}^{n-1}z_j\bar{z}_j\}.$$

Then its complexification is

$$\mathcal{H}^n := \{(z_1,\ldots,z_{n-1},w,\xi_1,\ldots,\xi_{n-1},\eta)\in\mathbf{C}^{2n}:\ w-\eta=2i\sum_{j=1}^{n-1}z_j\xi_j\}.$$

$\mathcal{H}^n$ is the Segre family associated with $\mathbb{H}^n$. $\mathcal{H}^n$ is holomorphically foliated by $\{Q_{(\xi,\eta)}\times\{(\xi,\eta)\}\}$ and also by $\{\{(z,w)\}\times\hat{Q}_{(z,w)}\}$ where $Q_{(\xi,\eta)}=\{(z,w)\in\mathbf{C}^n:\ w-\eta=2i\sum_{j=1}^{n-1}z_j\xi_j\}$ for any $(\xi,\eta)\in\mathbf{C}^n$ and $\hat{Q}_{(z,w)}=\{(\xi,\eta)\in\mathbf{C}^n:\ w-\eta=2i\sum_{j=1}^{n-1}z_j\xi_j\}$ for any $(z,w)\in\mathbf{C}^n$.

Next let Φ be a holomorphic Segre map from an open piece $\mathcal{M}$ of $\mathcal{H}^n$ into $\mathcal{H}^N$. It is known that Φ takes the following form: $\Phi(z,w,\xi,\eta)=(\Phi_1(z,w),\Phi_2(\xi,\eta))$ for certain holomorphic maps Φ_1,Φ_2 defined in a neighborhood of $\mathcal{M}$ in $\mathbf{C}^{2n}$ into $\mathbf{C}^{2N}$. (See [Fa2] [Hu2] [HJ2]). In the following, we always assume that $N\geq n$.

We say $\sigma\in Aut(\mathcal{H}^n)$ if σ is a bimeromorphic self-map of $\mathcal{H}^n$ which is also a holomorphic Segre map away from its pole. Further, we write $\sigma\in Aut_0(\mathcal{H}^n)$ if $\sigma\in Aut(\mathcal{H}^n)$, σ is holomorphic near 0 and $\sigma(0)=0$. For any $p_0=(z_0,w_0,\xi_0,\eta_0)\in\mathcal{H}^n$, define $\sigma^0_{p_0}$ by sending $(z,w,\xi,\eta)\in\mathbf{C}^{n-1}\times\mathbf{C}^1\times\mathbf{C}^{n-1}\times\mathbf{C}^1$ to $(z+z_0,w+w_0+2iz\cdot\xi_0,\xi+\xi_0,\eta+\eta_0-2i\xi\cdot z_0)$. Then $\sigma^0_{p_0}\in Aut(\mathcal{H}^n)$ with $\sigma^0_{p_0}(0)=p_0$. (See Theorem 6.3 of [HJ2] for the explicit formula for elements in $Aut(\mathcal{H}^n)$)

Theorem 2.2.3 *Let $\mathcal{M}$ be a connected neighborhood of p_0 in $\mathcal{H}^n$. Let Φ be a holomorphic Segre map from $\mathcal{M}$ into $\mathcal{H}^N$ and write $\Phi(z,w,\xi,\eta)=(\Phi_1(z,w),\Phi_2(\xi,\eta))=(\tilde{f}_1(z,w),\ldots,\tilde{f}_{N-1}(z,w),g(z,w),\tilde{h}_1(\xi,\eta),\ldots,\tilde{h}_{N-1}(\xi,\eta),e(\xi,\eta))$ for $(z,w,\xi,\eta)\in\mathcal{M}$. Assume that Φ is holomorphic in a neighborhood $\mathcal{U}$ of $\mathcal{M}$ in $\mathbf{C}^{2n}$, $p_0\in\mathcal{M}$ with $\Phi(p_0)=\widetilde{p_0}$ and $N\le 2n-2$. For $p\in\mathcal{M}$, write $\Phi_p=((\widetilde{f}_1)_p,\ldots,(\widetilde{f}_{N-1})_p,g_p,(\widetilde{h}_1)_p,\ldots,(\widetilde{h}_{N-1})_p,e_p)$ $:=(\widetilde{\sigma}^0_{\Phi(p)})^{-1}\circ\Phi\circ\sigma^0_p$. Then the following holds:*
(1). If $\frac{\partial g_{p_0}}{\partial w}(0)\neq 0$, then there exists a $\tau\in Aut_0(\mathcal{H}^N)$ such that

$$\tau\circ\Phi_{p_0}(z,w,\xi,\eta)=(z,\phi(z,w),w,\xi,\psi(\xi,\eta),\eta)$$

with $\phi_j(z,w)\psi_j(\xi,\eta)\equiv 0$ for $(z,w,\xi,\eta)\in\mathcal{U}$ for each $j=1,\cdots,N-n$.
(2). If $\frac{\partial g_p}{\partial w}(0)=0$ for all $p\in\mathcal{M}$ near p_0, then $g_{p_0}\equiv e_{p_0}\equiv 0$ and $\sum_{j=1}^{N-1}(\widetilde{f}_j)_{p_0}(z,w)(\widetilde{h}_j)_{p_0}$ $(\xi,\eta)\equiv 0$ for $(z,w,\xi,\eta)\in\mathcal{U}$.

Motivated by the concept of CR transversality, we introduce the following notion:

Definition 2.2.4 *A holomorphic Segre map $\mathcal{F}:(\mathcal{M},p)(\subset(\mathbf{C}^{2n},p))\to(\tilde{\mathcal{M}},\tilde{p})(\subset(\mathbf{C}^{2N},\widetilde{p}))$ with $\mathcal{F}(p)=\widetilde{p}=(q_1,q_2)\in\mathbf{C}^N\times\mathbf{C}^N$ is called to be holomorphic Segre transversal to $\mathcal{M}$ at p if:*

$$d\mathcal{F}(T^{(1,0)}_p\mathcal{M})+T^{(1,0)}_{\widetilde{p}}\widetilde{Q}_{q_2}+T^{(1,0)}_{\widetilde{p}}\hat{\widetilde{Q}}_{q_1}=T^{(1,0)}_{\widetilde{p}}\mathcal{H}^N,$$

where $T^{(1,0)}_p\mathcal{M}$ is the holomorphic tangent space of $\mathcal{M}$ at p.

In §5, we shall show that the assumption in Theorem 2.2.3 (1) is equivalent to the statement that Φ is holomorphic Segre transveral to $\mathcal{H}^N$ at p_0.

Definition 2.2.5 *Let $\mathcal{M}$ be an open subset of $\mathcal{H}^n$. A connected non-empty complex analytic variety $\mathcal{E} \subset \mathcal{M}$ is called a holomorphic Segre-related set of $\mathcal{M}$ if either for any $(z_0, w_0, \xi_0, \eta_0) \in \mathcal{E}$, the connected component of $\{(z_0, w_0, \xi, \eta) \in \mathcal{M} : (\xi, \eta) \in \hat{Q}_{(z_0,w_0)}\}$ containing $(z_0, w_0, \xi_0, \eta_0)$ is a subset of $\mathcal{E}$, or for any $(z_0, w_0, \xi_0, \eta_0) \in \mathcal{E}$, the connected component of $\{(z, w, \xi_0, \eta_0) \in \mathcal{M} : (z, w) \in Q_{(\xi_0,\eta_0)}\}$ containing $(z_0, w_0, \xi_0, \eta_0)$ is a subset of $\mathcal{E}$.*

Write $\mathrm{Hol}(\mathcal{M}, \mathbf{C})$ for the collection of holomorphic functions from $\mathcal{M}$ into $\mathbf{C}$. Then as an immediate application of Theorem 2.2.3, we have the following characterization of holomorphic non-transversal points of a holomorphic Segre map:

Theorem 2.2.6 *Let $\mathcal{M}$ be a connected open piece in $\mathcal{H}^n$. Let Φ be a holomorphic Segre map from $\mathcal{M}$ into $\mathcal{H}^N$ and write $\Phi(z, w, \xi, \eta) = (\Phi_1(z, w), \Phi_2(\xi, \eta)) = (\tilde{f}_1(z, w), \ldots, \tilde{f}_{N-1}(z, w), g(z, w), \tilde{h}_1(\xi, \eta), \ldots, \tilde{h}_{N-1}(\xi, \eta), e(\xi, \eta))$ for $(z, w, \xi, \eta) \in \mathcal{M}$. Assume that Φ is holomorphic in a neighborhood $\mathcal{U}$ of $\mathcal{M}$ in $\mathbf{C}^{2n}$, $p_0 \in \mathcal{M}$ with $\Phi(p_0) = \widetilde{p_0}$ and $N \leq 2n-2$. Let $\mathcal{E}_\Phi$ be the collection of points, where Φ fails to be holomorphically Segre transversal. Then the following holds:*

(1) $\mathcal{E}_\Phi$, if not empty nor the whole space $\mathcal{M}$, must be a complex analytic variety of codimension one, whose irreducible components are holomorphic Segre-related sets of codimension one in $\mathcal{M}$. Moreover for each irreducible component $\mathcal{E}_j$ of $\mathcal{E}_\Phi$, there is a point $(z_0, w_0, \xi_0, \eta_0) \in \mathcal{M}$ such that $\Phi(\mathcal{E}_j) \subset \widetilde{Q}_{\Phi_2(\xi_0,\eta_0)} \times \{\Phi_2(\xi_0, \eta_0)\}$ or $\Phi(\mathcal{E}_j) \subset \{\Phi_1(z_0, w_0)\} \times \hat{\widetilde{Q}}_{\Phi_1(z_0,w_0)}$.

(2) When $N = n+1$, $n \geq 3$, either there is a $\chi_1 \in Hol(\mathcal{M}, \mathbf{C})$ depending only on (z, w)-variables such that $\mathcal{E}_\Phi$ is precisely the zero set of χ_1, or there is a $\chi_2 \in Hol(\mathcal{M}, \mathbf{C})$ depending only on (ξ, η)-variables such that $\mathcal{E}_\Phi$ is precisely the zero set of χ_2.

(3) When $2n - 2 \geq N > n + 1$, there are a $\chi_1 \in Hol(\mathcal{M}, \mathbf{C})$ depending only on (z, w)-variables and a $\chi_2 \in Hol(\mathcal{M}, \mathbf{C})$ depending only on (ξ, η)-variables such that $\mathcal{E}_\Phi$ is precisely the union of the zero sets of χ_1 and χ_2.

Proposition 2.2.7 *Let $\mathcal{E} \subset \mathcal{M}$ be a holomorphic Segre-related set of $\mathcal{M}$ of codimension one, where $\mathcal{M}$ is a connected open subset of $\mathcal{H}^n$ $(n \geq 2)$. Suppose that either*

$\mathcal{E} = \{(z,w,\xi,\eta) \in \mathcal{M} :\ \chi_1(z,w) = 0,\ \chi_1 \in Hol(\mathcal{M},\mathbf{C})\}$ or $\mathcal{E} = \{(z,w,\xi,\eta) \in \mathcal{M} :\ \chi_2(\xi,\eta) = 0,\ \chi_2 \in Hol(\mathcal{M},\mathbf{C})\}$. Assume that $\chi_1(\mathcal{M}) \neq \mathbf{C}$ in the first case and $\chi_2(\mathcal{M}) \neq \mathbf{C}$ in the latter. Then there is a holomorphic Segre map Φ from $\mathcal{M}$ into $\mathcal{H}^N$ with $N = n+1$ such that $\mathcal{E}$ is precisely the collection of points, where Φ fails to be holomorphic Segre transversal.

Proposition 2.2.8 *Let $\mathcal{M}$ is a connected open subset of $\mathcal{H}^n$ ($n \geq 2$). Suppose that $\mathcal{E} = \{(z,w,\xi,\eta) \in \mathcal{M} :\ \chi_1(z,w) = 0,\ \chi_1 \in Hol(\mathcal{M},\mathbf{C})\} \cup \{(z,w,\xi,\eta) \in \mathcal{M} :\ \chi_2(\xi,\eta) = 0,\ \chi_2 \in Hol(\mathcal{M},\mathbf{C})\}$, where $\chi_1(\mathcal{M}) \neq \mathbf{C}$ and $\chi_2(\mathcal{M}) \neq \mathbf{C}$. Then there is a holomorphic Segre map Φ from $\mathcal{M}$ into $\mathcal{H}^N$ with $N = n+2$ such that $\mathcal{E}$ is precisely the collection of points, where Φ fails to be holomorphic Segre transversal.*

The new phenomenon in Theorem 2.2.6 is the propagation property for the holomorphic Segre non-transversality along the Segre varieties for the case of $N \leq 2n-2$. Interestingly, the following example shows that this property does not hold for $N > 2n-2$:

Example 2.2.9 *Let Φ be the holomorphic Segre embedding $\Phi : \mathcal{H}^n \to \mathcal{H}^N$ with $N = 2n-1$, where*

$$\Phi(z_1,\cdots,z_{n-1},w,\xi_1,\cdots,\xi_{n-1},\eta) =$$

$$(z_1w,\cdots,z_{n-1}w,z_1,\cdots,z_{n-1},w^2,\xi_1,\cdots,\xi_{n-1},\xi_1\eta,\cdots,\xi_{n-1}\eta,\eta^2).$$

We will see at the end of §5 that Φ fails to be holomorphic Segre transversal at $(z,w,\xi,\eta) \in \mathcal{H}^n$ if and only if $w+\eta = 0$. We also shows there, however, the connected complex submanifold of codimension one defined by $w+\eta = 0$ is not a holomorphic Segre-related set of $\mathcal{H}^n$.

2.3 Normalization and a curvature equation

In this section, we shall use the strategy of Huang([Hu]) and Huang-Ji([HJ]). One can also refer Chapter 1 for related material. Let $n \geq 2$ and $\mathcal{M}$ be a connected open piece of $\mathcal{H}^n$ containing the origin. Let

$$\Phi = (\Phi_1(z,w), \Phi_2(\xi,\eta)) : \mathcal{M}(\subset \mathcal{H}^n) \to \mathcal{H}^N$$

be a holomorphic Segre map, where

$$\Phi := (\widetilde{f}, g, \widetilde{h}, e) = (\tilde{f}_1, \ldots, \tilde{f}_{N-1}, g, \tilde{h}_1, \ldots, \tilde{h}_{N-1}, e)$$

which is also written as $\Phi = (f, \phi, g, h, \psi, e) = (f_1, \ldots, f_{n-1}, \phi_1, \ldots, \phi_{N-n}, h_1, \ldots, h_{n-1},$ $\psi_1, \ldots, \psi_{N-n}, e)$. Write

$$\mathcal{L}_j = 2i\xi_j \frac{\partial}{\partial w} + \frac{\partial}{\partial z_j},\ \mathcal{K}_j = -2iz_j \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi_j},\ \mathcal{T} = \frac{\partial}{\partial w} + \frac{\partial}{\partial \eta}.$$

Then $\{\mathcal{L}_j, \mathcal{K}_j, \mathcal{T}\}_{j=1}^n$ forms a global basis for the space of sections of the complex tangent bundle $T^{(1,0)}\mathcal{H}^n$ of $\mathcal{H}^n$.

Notice that $\Phi(\mathcal{M}) \subset \mathcal{H}^N$ gives the following equation:

$$g(z,w) - e(\xi,\eta) = 2i\tilde{f}(z,w)\cdot\tilde{h}(\xi,\eta) \quad \text{over} \quad w - \eta = 2iz\cdot\xi \tag{2.1}$$

where $a \cdot b := \sum_{j=1}^m a_j b_j$ for $a, b \in \mathbb{C}^m$.

Let $\mathbf{Z}_+$ be the set of non-negative integers. Then applying $\mathcal{L}^\alpha,\ \mathcal{K}^\alpha,\ \mathcal{T}$ and $\mathcal{K}_j\mathcal{L}_k$ to (), respectively, where $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbf{Z}_+^{n-1}$ and $\mathcal{L}^\alpha = \mathcal{L}_1^{\alpha_1} \cdots \mathcal{L}_{n-1}^{\alpha_{n-1}}$, we have

$$\mathcal{L}^\alpha g = 2i\mathcal{L}^\alpha \widetilde{f} \cdot \tilde{h}, \qquad \mathcal{K}^\alpha e = 2i\widetilde{f}\cdot\mathcal{K}^\alpha\tilde{h}, \tag{2.2}$$

$$\frac{\partial g}{\partial w} - \frac{\partial e}{\partial \eta} = 2i\frac{\partial \tilde{f}}{\partial w}\cdot\tilde{h} + 2i\tilde{f}\cdot\frac{\partial \tilde{h}}{\partial \eta}, \qquad \delta_j^k \frac{\partial g}{\partial w} = 2i\delta_j^k \frac{\partial \tilde{f}}{\partial w}\cdot\tilde{h} + \mathcal{L}_k\tilde{f}\cdot\mathcal{K}_j\tilde{h} \tag{2.3}$$

where δ_j^k is the standard Kronecker function. Assuming that $\Phi(0) = 0$ and letting $(z,w,\xi,\eta) = (0,0,0,0)$ in (2.2),(2.3), we have

$$e(\xi,0) \equiv 0,\ \frac{\partial g}{\partial w}(0) = \frac{\partial e}{\partial \eta}(0) = \mathcal{L}_j\tilde{f}(0)\cdot\mathcal{K}_j\tilde{h}(0), \quad \mathcal{L}_j\tilde{f}\cdot\mathcal{K}_k\tilde{h} = 0\ (j \neq k). \tag{2.4}$$

Let $M_{n\times N}(n \le N)$ denote the set of n by N matrixes with all entries in $\mathbb{C}$. We then have the following elementary lemma:

Lemma 2.3.1 *Let A, $B \in M_{n\times N}$ and $A\cdot B^t = Id_{n\times n}$. Then there exist $\tilde{A}$, $\tilde{B} \in M_{N\times N}$ whose first n rows are A and B, respectively, such that $\tilde{A}\cdot\tilde{B}^t = Id_{N\times N}$.*

Proof of Lemma 2.3.1: Consider the linear equation $A \cdot y^t = 0$ with $y \in \mathbb{C}^N$. Then its solution space has dimension $N - n$. Choose a basis $\{y^1, \cdots, y^{N-n}\}$ and define D to be the matrix whose k^{th}-row is precisely y^k. Then $A \cdot D^t = 0$. Considering the new matrix $\begin{pmatrix} B \\ D \end{pmatrix}$, it has full rank. In fact, suppose that $\begin{pmatrix} B \\ D \end{pmatrix}^t \cdot y^t = 0$ with $y = (y_1, \cdots, y_N) \in \mathbb{C}^N$. Then multiplying from the left by A, we obtain $y_j = 0$ for $j \leq n$. Hence, we get $(y_{n+1}, \cdots, y_N) \cdot D = 0$. Since D has rank $N - n$, we conclude that $y = 0$. Similarly, we can construct $C \in M_{(N-n)\times N}$ with rank $N - n$ such that $C \cdot B^t = 0$ and Rank $\begin{pmatrix} A \\ C \end{pmatrix} = N$. Since $\begin{pmatrix} A \\ C \end{pmatrix} \cdot \begin{pmatrix} B \\ D \end{pmatrix}^t = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & C \cdot D^t \end{pmatrix}$, $C \cdot D^t$ is invertible. Let $\tilde{A} = \begin{pmatrix} A \\ (C \cdot D^t)^{-1} \cdot C \end{pmatrix}$, $\tilde{B} = \begin{pmatrix} B \\ D \end{pmatrix}$. Then we see the proof of the Lemma. $\square$

Now assume $\lambda := \frac{\partial g}{\partial w}(0) \neq 0$. Write

$$A = \begin{pmatrix} \mathcal{K}_1(\tilde{h})(0)/\sqrt{\lambda} \\ \vdots \\ \mathcal{K}_{n-1}(\tilde{h})(0)/\sqrt{\lambda} \end{pmatrix}, \qquad B = \begin{pmatrix} \mathcal{L}_1(\tilde{f})(0)/\sqrt{\lambda} \\ \vdots \\ \mathcal{L}_{n-1}(\tilde{f})(0)/\sqrt{\lambda} \end{pmatrix}.$$

Then (2.3) immediately gives that $A \cdot B^t = Id_{(n-1)\times(n-1)}$. Applying Lemma 2.3.1 to A and B, we obtain $\tilde{A}$ and $\tilde{B}$ with $\tilde{A} \cdot \tilde{B}^t = Id_{(N-1)\times(N-1)}$.

Next let $\tau(z^*, w^*, \xi^*, \eta^*) = (\frac{1}{\sqrt{\lambda}} z^* \tilde{A}^t, \frac{1}{\lambda} w^*, \frac{1}{\sqrt{\lambda}} \xi^* \tilde{B}^t, \frac{1}{\lambda} \eta^*)$. Obviously $\tau \in Aut_0(\mathcal{H}^N)$. By composing Φ with τ, we obtain the *first normalization* of Φ as follows:

$$\begin{aligned} \Phi^* &= (\tilde{f}^*, g^*, \tilde{h}^*, e^*) : \\ &= \tau \circ \Phi = \frac{1}{\sqrt{\lambda}} \Phi \cdot \begin{pmatrix} \tilde{A}^t & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} & 0 & 0 \\ 0 & 0 & \tilde{B}^t & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix}, \text{ with} \end{aligned}$$

$$\frac{\partial \widetilde{f}_j^*}{\partial z_k}(0) = \frac{\partial \widetilde{h}_j^*}{\partial \xi_k}(0) = \delta_j^k \qquad \text{for } 1 \leq k \leq n-1 \text{ , } 1 \leq j \leq N-1, \tag{2.5}$$

$$\frac{\partial g^*}{\partial w}(0) = \frac{\partial e^*}{\partial \eta}(0) = 1. \tag{2.6}$$

Write

$$u = \frac{1}{2}\frac{\partial^2 e^*}{\partial \eta^2}\bigg|_0,$$

$$\vec{a} = (\frac{\partial \tilde{f}_1^*}{\partial w}, \ldots, \frac{\partial \tilde{f}_{N-1}^*}{\partial w})\bigg|_0,$$

$$\vec{s} = 2i(\frac{\partial \tilde{h}_1^*}{\partial \eta}, \ldots, \frac{\partial \tilde{h}_{N-1}^*}{\partial \eta})\bigg|_0$$

and let $G \in Aut_0(\mathcal{H}^N)$ be defined by $G(z^*, w^*, \xi^*, \eta^*) :=$

$$\left(\frac{z^* - \vec{a}w^*}{1 + z^* \cdot \vec{s} + uw^*}, \frac{w^*}{1 + z^* \cdot \vec{s} + uw^*}, \frac{\xi^* + \frac{i}{2}\vec{s}\eta^*}{1 - 2i\xi^* \cdot \vec{a} + (u + \vec{a}\cdot\vec{s})\eta^*}, \frac{\eta^*}{1 - 2i\xi^* \cdot \vec{a} + (u + \vec{a}\cdot\vec{s})\eta^*}\right).$$

(See Theorem 6.3 of [HJ2]).

We then get the *second normalization* Φ^{**} as follows:

$$\Phi^{**} = (\tilde{f}^{**}, g^{**}, \tilde{h}^{**}, e^{**}) = (f^{**}, \phi^{**}, g^{**}, h^{**}, \psi^{**}, e^{**}) := G \circ \Phi^* = G \circ \tau \circ \Phi.$$

Simple computation shows that

$$\frac{\partial \tilde{f}^{**}}{\partial w}(0) = \frac{\partial \tilde{h}^{**}}{\partial \eta}(0) = 0, \qquad \frac{\partial f^{**}}{\partial z}(0) = \frac{\partial h^{**}}{\partial \xi}(0) = Id, \tag{2.7}$$

$$\frac{\partial \phi^{**}}{\partial z}(0) = \frac{\partial \psi^{**}}{\partial \xi}(0) = 0, \qquad \frac{\partial g^{**}}{\partial w}(0) = \frac{\partial e^{**}}{\partial \eta}(0) = 1, \qquad \frac{\partial^2 e^{**}}{\partial \eta^2}(0) = 0. \tag{2.8}$$

Now we assign the weight of (z, ξ) to be 1 and of (w, η) to be 2. In the following, we use $(\cdot)^{(l)}$ for a homogeneous polynomial of weighted degree l. For a function f on $\mathcal{H}^n$ we say that $f \in o_{wt}(k)$ if $\lim_{t\to 0} \frac{f(tz, t^2w, t\xi, t^2\eta)}{t^k} = 0$ uniformly for (z, w, ξ, η) in any compact subset of $\mathcal{H}^n$.

Next we do the weighted Taylor series expansion of Φ^{**}:

Lemma 2.3.2 *Write Φ for the Φ^{**} above. Then Φ has the following form:*

$$f_j^{**} = z_j + J_j^{(1)}(z)w + o_{wt}(3),$$

$$\phi_j^{**} = B_j^{(2)}(z) + o_{wt}(2),$$

$$g^{**} = w + o_{wt}(4),$$

$$h_j^{**} = \xi_j + M_j^{(1)}(\xi)\eta + o_{wt}(3),$$

$$\psi_j^{**} = F_j^{(2)}(\xi) + o_{wt}(2),$$

$$e^{**} = \eta + o_{wt}(4), \ \textit{with}$$

$$2i(J^{(1)}(z)\cdot\xi)(z\cdot\xi) = -B^{(2)}(z)\cdot F^{(2)}(\xi).$$

Proof of Lemma 2.3.2: First applying (2.7), (2.8) to Φ^{**}, we get the following weighted expansion:

$$\begin{aligned}
f_j^{**} &= z_j + A_j^{(2)}(z) + I_j^{(3)}(z) + J_j^{(1)}(z)w + o_{wt}(3),\\
\phi_j^{**} &= B_j^{(2)}(z) + o_{wt}(2),\\
g^{**} &= w + Cw^2 + D^{(1)}(z)w + N^{(2)}(z)w + o_{wt}(4),\\
h_j^{**} &= \xi_j + E_j^{(2)}(\xi) + L_j^{(3)}(\xi) + M_j^{(1)}(\xi)\eta + o_{wt}(3),\\
\psi_j^{**} &= F_j^{(2)}(\xi) + o_{wt}(2),\\
e^{**} &= \eta + H^{(1)}(\xi)\eta + P^{(2)}(\xi)\eta + o_{wt}(4)
\end{aligned}$$

and

$$g^{**} - e^{**} = 2i(f^{**}\cdot h^{**} + \phi^{**}\cdot\psi^{**}). \tag{2.9}$$

Collecting terms of weighted degree 3 in (2.9), we have:

$$D^{(1)}(z)w - H^{(1)}(\xi)\eta = 2i(z\cdot E^{(2)}(\xi) + A^{(2)}(z)\cdot\xi) \quad \text{over} \quad w = \eta + 2iz\cdot\xi.$$

Hence, $(D^{(1)}(z) - H^{(1)}(\xi))\eta + 2iz\cdot\xi^{(1)}(z) = 2i(z\cdot E^{(2)}(\xi) + A^{(2)}(z)\cdot\xi)$. Collecting coefficients of terms of the form: η, $z^2\xi$ and $z\xi^2$ in the above, we have:

$$\begin{aligned}
&D^{(1)}(z) = H^{(1)}(\xi) = 0,\\
&A^{(2)}(z) = 0,\\
&E^{(2)}(\xi) = 0.
\end{aligned}$$

Collecting terms of weighted degree 4 in (2.9), we have:

$$Cw^2 + N^{(2)}(z)w - P^{(2)}(\xi)\eta = 2i(z\cdot L^{(3)}(\xi) + z\cdot M^{(1)}(\xi)\eta + I^{(3)}(z)\cdot\xi + J^{(1)}(z)w\cdot\xi + B^{(2)}(z)\cdot F^{(2)}(\xi))$$

where $w = \eta + 2iz\cdot\xi$.

Similar arguments then show that the following holds:

$$C = 0, \tag{2.10}$$

$$N^{(2)}(z) = P^{(2)}(\xi) = 0, \tag{2.11}$$

$$L^{(3)}(\xi) = I^{(3)}(z) = 0, \tag{2.12}$$

$$2i(J^{(1)}(z) \cdot \xi)(z \cdot \xi) = -B^{(2)}(z) \cdot F^{(2)}(\xi), \tag{2.13}$$

$$z \cdot M^{(1)}(\xi) + J^{(1)}(z) \cdot \xi = 0. \tag{2.14}$$

This completes the proof of Lemma 2.3.2. □

If we assume $N - n \leq n - 2$, i.e. $N \leq 2n - 2$, then applying [Lemma 3.2, [Hu]] to (2.13), we immediately get:

$$J^{(1)}(z) = 0,$$

$$B^{(2)}(z) \cdot F^{(2)}(\xi) = 0.$$

Hence, we have the following:

Lemma 2.3.3 *With the same assumption as above, if we further assume that $N \leq 2n-2$, then for the Φ in Lemma 2.3.2, we have the following weighted Taylor expansion:*

$$f_j^{**} = z_j + o_{wt}(3), \tag{2.15}$$

$$\phi_j^{**} = B_j^{(2)}(z) + o_{wt}(2), \tag{2.16}$$

$$g^{**} = w + o_{wt}(4), \tag{2.17}$$

$$h_j^{**} = \xi_j + o_{wt}(3), \tag{2.18}$$

$$\psi_j^{**} = F_j^{(2)}(\eta) + o_{wt}(2), \tag{2.19}$$

$$e^{**} = \eta + o_{wt}(4) \tag{2.20}$$

with

$$B^{(2)}(z) \cdot F^{(2)}(\xi) = 0. \tag{2.21}$$

2.4 A partial linearity for Φ

We assume in this section that $N \leq 2n-2$. Let Φ satisfies (2.15) through (2.21) in Lemma 2.3.3. Write

$$\Phi = (z + \hat{f}(z,w), \hat{\phi}(z,w), w + \hat{g}(z,w), \xi + \hat{h}(\xi,\eta), \hat{\psi}(\xi,\eta), \eta + \hat{e}(\xi,\eta)). \tag{2.22}$$

Theorem 2.4.1 *: With the above notation, we have $\hat{f} = \hat{h} = 0$, $\hat{g} = 0$ on $\mathcal{M}$ and $\hat{\phi} \cdot \hat{\psi} \equiv 0$ over $\mathcal{U}$. Moreover, after composing an element $\tau_U \in Aut_0(\mathcal{H}^N)$ from the left onto Φ, if necessary, there is a non-negative integer k such that $\phi_j \equiv 0$ for $j > k$ and $\psi_j \equiv 0$ for $j \leq k$.*

Proof of Theorem 2.4.1: We will follow the approach used in [EHZ]. Since $\Phi(\mathcal{M}) \subset \mathcal{H}^N$, from (2.22), we have:

$$w+\hat{g}(z,w)-\eta-\hat{e}(\xi,\eta) = 2i(z+\hat{f}(z,w))\cdot(\xi+\hat{h}(\xi,\eta))+2i\hat{\phi}(z,w)\cdot\hat{\psi}(\xi,\eta),\ w-\eta = 2iz\cdot\xi.$$

Therefore,

$$\begin{aligned}\hat{g}(z,w) - \hat{e}(\xi,\eta) - 2i\xi\cdot\hat{f}(z,w) - 2iz\cdot\hat{h}(\xi,\eta) &= 2i\hat{\phi}(z,w)\cdot\hat{\psi}(\xi,\eta) + 2i\hat{f}(z,w)\cdot\hat{h}(\xi,\eta)\\ &:= 2iA(z,w,\xi,\eta) \qquad \text{over} \quad w-\eta = 2iz\cdot\xi.\end{aligned} \tag{2.23}$$

In view of the normalization obtained in Lemma 2.3.3, we have the following expansions:

$$\begin{aligned}
\hat{f}(z,w) &= \sum_{\mu+\nu\geq 2} f_{\mu\nu}(z)w^\nu, & \hat{h}(\xi,\eta) &= \sum_{\mu+\nu\geq 2} h_{\mu\nu}(\xi)\eta^\nu,\\
\hat{\phi}(z,w) &= \sum_{\mu+\nu\geq 2} \phi_{\mu\nu}(z)w^\nu, & \hat{\psi}(\xi,\eta) &= \sum_{\mu+\nu\geq 2} \psi_{\mu\nu}(\xi)\eta^\nu,\\
\hat{g}(z,w) &= \sum_{\mu+\nu\geq 2} g_{\mu\nu}(z)w^\nu, & \hat{e}(\xi,\eta) &= \sum_{\mu+\nu\geq 2} e_{\mu\nu}(\xi)\eta^\nu,\\
A(z,w,\xi,\eta) &= \sum_{\alpha+\beta+\mu+\nu\geq 4} A_{\alpha\mu\beta\nu}(z,\xi)w^\mu\eta^\nu
\end{aligned}$$

where $(\cdot)_{\mu\nu}(z)w^\nu$ and $(\cdot)_{\mu\nu}(\xi)\eta^\nu$ are homogeneous polynomials of degree (μ,ν) with respect to (z,w) and (ξ,η), respectively, and $(\cdot)_{\alpha\mu\beta\nu}(z,\xi)w^\mu\eta^\nu$ is a homogeneous polynomial of degree (α,μ,β,ν) with respect to (z,w,ξ,η). Letting $w=0$, i.e. $\eta = -2iz\cdot\xi$

in (2.23) and collecting terms of a fixed bi-degree (α, β) with respect to (z, ξ), we obtain:

$$\beta = 1 \Rightarrow \hat{f}(z, 0) = 0, \tag{2.24}$$

$$\beta = 0 \Rightarrow \hat{g}(z, 0) = 0. \tag{2.25}$$

Similarly, we also have

$$\hat{h}(\xi, 0) = 0, \ \hat{e}(\xi, 0) = 0. \tag{2.26}$$

We will use an induction argument to prove that $\hat{f}(z, w) = \hat{h}(\xi, \eta) = 0$ and $\hat{g}(z, w) = \hat{e}(\xi, \eta) = 0$. First by Lemma 2.3.3, we have $\hat{f}(z, w) = o_{wt}(3)$, $\hat{h}(\xi, \eta) = o_{wt}(3)$, $\hat{g}(z, w) = o_{wt}(4)$, $\hat{e}(\xi, \eta) = o_{wt}(4)$. Now, suppose that for l with $k - 1 \geq l \geq 3$, $\hat{f}^{(l)}(z, w) = \hat{h}^{(l)}(\xi, \eta) = 0$ and $\hat{g}^{(l+1)}(z, w) = \hat{e}^{(l+1)}(\xi, \eta) = 0$. We then need to show that $\hat{f}^{(k)}(z, w) = \hat{h}^{(k)}(\xi, \eta) = 0$ and $\hat{g}^{(k+1)}(z, w) = \hat{e}^{(k+1)}(\xi, \eta) = 0$. Collecting terms of a fixed bi-degree (α, β) with respect to (z, ξ) with $\beta \geq 2$ and $\alpha + \beta \leq k + 1$ and letting $w = 0$ in (2.23), we get

$$\begin{aligned} -e_{\beta-\alpha,\alpha}(\xi)\eta^{\alpha} - 2iz \cdot h_{\beta-\alpha+1,\alpha-1}(\xi)\eta^{\alpha-1} \\ = 2i \sum_{p=0}^{\alpha-2} A_{\alpha-p,0,\beta-p,p}(z, \xi)\eta^{p} \\ = 2i \sum_{p=0}^{\alpha-2} \phi_{\alpha-p,0}(z)\psi_{\beta-p,p}(\xi)\eta^{p} + 2i \sum_{p=0}^{\alpha-2} f_{\alpha-p,0}(z)h_{\beta-p,p}(\xi)\eta^{p}. \end{aligned} \tag{2.27}$$

By the induction assumption, we have for $k - 1 \geq l \geq 3$, $\hat{f}^{(l)}(z, w) = \hat{h}^{(l)}(\xi, \eta) = 0$. We thus obtain $\sum_{p=0}^{\alpha-2} f_{\alpha-p,0}(z)h_{\beta-p,p}(\xi)\eta^{p} = 0$ for $\alpha + \beta \leq k + 1$. Then (2.27) becomes:

$$\begin{aligned} -e_{\beta-\alpha,\alpha}(\xi)\eta^{\alpha} - 2i\ z \cdot h_{\beta-\alpha+1,\alpha-1}(\xi)\eta^{\alpha-1} \\ = 2i \sum_{p=0}^{\alpha-2} A_{\alpha-p,0,\beta-p,p}(z, \xi)\eta^{p} \\ = 2i \sum_{p=0}^{\alpha-2} \phi_{\alpha-p,0}(z)\psi_{\beta-p,p}(\xi)\eta^{p} \end{aligned} \tag{2.28}$$

where $\eta = -2iz \cdot \xi$.

Since we assumed that $N \leq 2n - 2$, i.e. $N - n \leq n - 2$. Applying Lemma 3.2, [EHZ] to (2.28), we have

$$A_{\mu 0 \nu \delta} = 0 \ \text{ for } \ \mu + \gamma + 2\delta \leq k + 1. \tag{2.29}$$

Hence

$$-e_{\beta-\alpha,\alpha}(\xi)\eta^{\alpha}-2iz\cdot h_{\beta-\alpha+1,\alpha-1}(\xi)\eta^{\alpha-1}=0 \quad \text{where} \quad \eta=-2iz\cdot\xi. \tag{2.30}$$

Similarly

$$A_{\mu\delta\nu 0}=0 \ \text{ for } \ \mu+\gamma+2\delta\leq k+1 \tag{2.31}$$

and

$$g_{\beta-\alpha,\alpha}(z)w^{\alpha}-2i\xi\cdot f_{\beta-\alpha+1,\alpha-1}(z)w^{\alpha-1}=0 \quad \text{where} \ \ w=2iz\cdot\xi. \tag{2.32}$$

Applying $\mathcal{L}_j$ and $\mathcal{K}_j$ to (2.23), we have

$$\mathcal{L}_j\hat{g}(z,w)-2i\xi\cdot\mathcal{L}_j\hat{f}(z,w)-2i\hat{h}_j(\xi,\eta)=2i\mathcal{L}_jA(z,w,\xi,\eta), \tag{2.33}$$

$$-\mathcal{K}_j\hat{e}(\xi,\eta)-2iz\cdot\mathcal{K}_j\hat{h}(\xi,\eta)-2i\hat{f}_j(z,w)=2i\mathcal{K}_jA(z,w,\xi,\eta) \tag{2.34}$$

over $w-\eta=2iz\cdot\xi$.

By (2.24),(2.25),(2.26),(2.29) and (2.31), we have

$$\mathcal{L}_j(\hat{f},\hat{g})(z,0)=2i\xi_j\sum_{\mu\geq 2}(f_{\mu 1},g_{\mu 1})(z), \tag{2.35}$$

$$(\mathcal{L}_j(A))^{(l)}(z,0,\xi,\eta)=2i\xi_j\sum_{\mu+\nu+2\delta=l-1}A_{\mu 1\nu\delta}(z,\xi)\eta^{\delta} \ \text{ for } \ 0\leq l\leq k, \tag{2.36}$$

$$\mathcal{K}_j(\hat{h},\hat{e})(\xi,0)=-2iz_j\sum_{\mu\geq 2}(h_{\mu 1},e_{\mu 1})(\xi), \tag{2.37}$$

$$(\mathcal{K}_j(A))^{(l)}(z,w,\xi,0)=-2iz_j\sum_{\mu+\nu+2\delta=l-1}A_{\mu\delta\nu 1}(z,\xi)w^{\delta} \ \text{ for } \ 0\leq l\leq k. \tag{2.38}$$

where in (2.36), $\eta=-2iz\cdot\xi$; in (2.38), $w=2iz\cdot\xi$.

Substituting (2.35),(2.36),(2.37) and (2.38) to (2.33) and (2.34), we have

$$\beta=2\Rightarrow 2i\xi_j\xi\cdot f_{\alpha 1}(z)+h_{j;2-\alpha,\alpha}(\xi)\eta^{\alpha}=0 \ \text{ for } \ \alpha\leq 2 \ \text{ over } \ \eta=-2iz\cdot\xi, \tag{2.39}$$

$$\beta=1\Rightarrow g_{\mu 1}(z)=0 \ \text{ for } \ \mu\leq k-1, \tag{2.40}$$

$\beta\geq 3$ and $\alpha+\beta=k\Rightarrow$

$$-h_{j;\beta-\alpha,\alpha}(\xi)\eta^{\alpha}=\xi_j\sum_{p=0}^{\alpha-1}A_{\alpha-p,1,\beta-p-1,p}(z,\xi)\eta^{p}$$

$$= \xi_j \sum_{p=0}^{\alpha-1} \phi_{\alpha-p,1}(z)\psi_{\beta-p-1,p}(\xi)\eta^p \quad \text{where} \quad \eta = -2iz\cdot\xi. \tag{2.41}$$

Applying Lemma 3.2, [EHZ] again, we conclude:

$$h_{\mu\nu}(\xi) = 0 \qquad \text{for} \quad \mu+\nu \geq 3 \ \text{ and } \quad \mu+2\nu = k.$$

Back to (2.30), we get

$$e_{\mu\nu}(\xi) = 0 \qquad \text{for} \quad \mu+\nu \geq 3 \ \text{ and } \quad \mu+2\nu = k+1.$$

Similarly, we have

$$f_{\mu\nu}(z) = 0 \qquad \text{for} \quad \mu+\nu \geq 3 \ \text{ and } \quad \mu+2\nu = k,$$

$$g_{\mu\nu}(z) = 0 \qquad \text{for} \quad \mu+\nu \geq 3 \ \text{ and } \quad \mu+2\nu = k+1.$$

Notice that for $k \geq 4$, $\{(\mu,\nu) : \ \mu+\nu \geq 3 \text{ and } \mu+2\nu = k\} = \{(\mu,\nu) : \ \mu + 2\nu = k\} - \{(0,2)\}$. On the other hand, substituting $f_{21}(z) = 0$ into (2.39) yields $h_{02} = 0$. Similarly we have $f_{02} = 0$. Hence, we proved that $\hat{f}^{(k)}(z,w) = \hat{h}^{(k)}(\xi,\eta) = 0$, $\hat{g}^{(k+1)}(z,w) = \hat{e}^{(k+1)}(\xi,\eta) = 0$.

By induction, we conclude that

$$\tau\circ\Phi(z,w,\xi,\eta) = (z,\hat{\phi}(z,w),w,\xi,\hat{\psi}(\xi,\eta),\eta)$$

with

$$\sum_{j=1}^{N-n} \hat{\phi}_j(z,w)\hat{\psi}_j(\xi,\eta) = 0 \ \text{ over } \mathcal{M}.$$

Next we prove that $\sum_{j=1}^{N-n} \hat{\phi}_j(z,w)\hat{\psi}_j(\xi,\eta) \equiv 0$. To this aim, we need only to prove the following lemma, whose proof follows the same line as in Lemma 4.2, [EHZ]:

Lemma 2.4.2 *If $A(z,w,\xi,\eta) := \sum_{j=1}^{k_0} \phi_j(z,w)\psi_j(\xi,\eta) = 0$ over $w-\eta = 2iz\cdot\xi$, $k_0 \leq n-2$. then $A(z,w,\xi,\eta) \equiv 0$ as a formal power series in (z,w,ξ,η).*

Proof of Lemma 2.4.2: Write

$$\phi_j(z,w) = \sum_{\mu,\nu} \phi_{j;\mu\nu}(z)w^\nu, \quad \psi_j(\xi,\eta) = \sum_{\mu,\nu} \psi_{j;\mu\nu}(\xi)\eta^\nu$$

where $(\cdot)_{\mu\nu}(z)w^{\nu}$, $(\cdot)_{\mu\nu}(\xi)\eta^{\nu}$ are homogeneous polynomials of degree (μ,ν) with respect to (z,w) and (ξ,η). Denote $A_{\alpha\nu_1\beta\nu_2}(z,\xi)w^{\nu_1}\eta^{\nu_2}$ to be terms of degree $(\alpha,\nu_1,\beta,\nu_2)$ with respect to (z,w,ξ,η), respectively, for A. Assume $A(z,w,\xi,\eta)\neq 0$. Then there is a smallest nonnegative l_0 such that $A_{\alpha\nu_1\beta l_0}(z,\xi)\neq 0$ for some α,β,ν_1. We are going to reach a contradiction. In fact, since $A(z,w,\xi,\eta)=0$ on $w-\eta=2iz\cdot\xi$ or equivalently $\sum_{\alpha,\beta,\nu_1,\nu_2}A_{\alpha\nu_1\beta\nu_2}(z,\xi)(\eta+2iz\cdot\xi)^{\nu_1}\eta^{\nu_2}=0$. By factoring out η^{l_0} and setting $\eta=0$, we have

$$\sum_{\alpha,\beta,\nu_1}A_{\alpha\nu_1\beta l_0}(z,\xi)(2iz\cdot\xi)^{\nu_1}\equiv 0. \tag{2.42}$$

Collecting terms of bi-degree (μ_1,μ_2) with respect to (z,ξ) in (2.42), we obtain

$$\sum_{\nu_1}A_{\mu_1-\nu_1,\nu_1,\mu_2-\nu_1,l_0}(z,\xi)(2iz\cdot\xi)^{\nu_1}\equiv 0.$$

On the other hand

$$A_{\mu_1-\nu_1,\nu_1,\mu_2-\nu_1,l_0}(z,\xi)=\sum_{j=1}^{k_0}\phi_{j;\mu_1-\nu_1,\nu_1}(z)\psi_{j;\mu_2-\nu_1,l_0}(\xi)$$

where $k_0\leq n-2$. It now follows from Lemma 3.2, [EHZ] that for any α,β,ν_1,

$$A_{\alpha\nu_1\beta l_0}(z,\xi)=0,$$

which contradicts the choice of l_0. This completes the proof of the lemma. $\Box$

Now, for simplicity of notation, assume that $\{\phi_1,\cdots,\phi_k\}$ is basis for the vector space spanned by $\{\phi_j\}_{j=1}^{N-n}$ over $\mathbb{C}$ with $k>0$. Then there is an $(N-n)\times(N-n)$ invertible matrix U such that

$$(\phi_1,\cdots,\phi_k,0,\cdots,0)=(\phi_1,\cdots,\phi_{N-n})\cdot U.$$

Now, define $\tau_U\in Aut_0(\mathcal{H}^N)$ by mapping $(z_1,\cdots,z_{N-1},w,\xi_1,\cdots,\xi_{N-1},\eta)$ to

$$\left(z_1,\cdots,z_{n-1},(z_n,\cdots,z_{N-1})\cdot U,\xi_1,\cdots,\xi_{n-1},(\xi_n,\cdots,\xi_{N-1})\cdot(U^{-1})^t\right).$$

Then $\tau_U\circ\Phi$ has the same form as above but with the extra property: $\sum_{j=1}^{k}\phi_j\psi_j\equiv 0$. Since $\{\phi_j\}_{j=1}^{k}$ is a linearly independent system, we get that $\psi_j\equiv 0$ for $j=1,\cdots,k$. The proof of Theorem 2.4.1 is complete. $\Box$

2.5 Holomorphic Segre Transversality

For any $p \in \mathcal{H}^n$ close to the origin, recall in the introduction:

$$\Phi_p = (\tilde{f}_p, g_p, \tilde{h}_p, e_p) = (f_p, \phi_p, g_p, h_p, \psi_p, e_p) := (\widetilde{\sigma}^0_{\Phi(p)})^{-1} \circ \Phi \circ \sigma^0_p,$$

where for each $p = (z_0, w_0, \xi_0, \eta_0) \in \mathcal{H}^n$, $\sigma^0_p \in Aut(\mathcal{H}^n)$ and

$$\sigma^0_p(z, w, \xi, \eta) = (z + z_0, w + w_0 + 2iz \cdot \xi_0, \xi + \xi_0, \eta + \eta_0 - 2i\xi \cdot z_0)$$

for any $(z, w, \xi, \eta) \in \mathcal{H}^n$. Easy computation tells that for any $(z^*, w^*, \xi^*, \eta^*) \in \mathcal{H}^N$,

$$\begin{aligned}(\widetilde{\sigma}^0_{\Phi(p)})^{-1}(z^*, w^*, \xi^*, \eta^*) :&= (z^* - \tilde{f}(z_0, w_0), w^* - e(\xi_0, \eta_0) - 2iz^* \cdot \tilde{h}(\xi_0, \eta_0), \\ &\qquad \xi^* - \tilde{h}(\xi_0, \eta_0), \eta^* - g(z_0, w_0) + 2i\xi^* \cdot \tilde{f}(z_0, w_0)).\end{aligned}$$

Obviously $\Phi_p(0) = 0$. Without loss of generality, we may assume in this section $p_0 = 0, \widetilde{p}_0 = 0$ in Theorem 2.2.3.

Lemma 2.5.1 *Φ is defined as in Theorem 2.2.3. Then Φ is holomorphic Segre transversal at the origin if and only if $\frac{\partial g}{\partial w}(0) \neq 0$.*

Proof of Lemma 2.5.1: Write the coordinate of $\mathcal{H}^N$ to be $(\widetilde{z}, \widetilde{w}, \widetilde{\xi}, \widetilde{\eta})$. It is straightforward then $\{\frac{\partial}{\partial \widetilde{z}_j}\Big|_0\}_{j=1}^{N-1}$ and $\{\frac{\partial}{\partial \widetilde{\xi}_j}\Big|_0\}_{j=1}^{N-1}$ span $T_0^{(1,0)}\widetilde{Q}_0$ and $T_0^{(1,0)}\hat{\widetilde{Q}}_0$, respectively. On the other hand, $d\Phi((\frac{\partial}{\partial w} + \frac{\partial}{\partial \eta})\Big|_0) = \frac{\partial g}{\partial w}(0) \cdot (\frac{\partial}{\partial \widetilde{w}} + \frac{\partial}{\partial \widetilde{\eta}})\Big|_0 \bmod(\frac{\partial}{\partial \widetilde{\xi}_j}\Big|_0, \frac{\partial}{\partial \widetilde{z}_j}\Big|_0, j = 1, \ldots, N-1)$ by the second equality of (2.4). The proof is thus complete by the definition of holomorphic Segre transversality. $\square$

Since holomorphic Segre transversality is invariant under the composition of holomorphic automorphisms, we see that Φ is holomorphic Segre transversal at p if and only if Φ_p is holomorphic Segre transversal at 0. This is equivalent to $(g_p)_w(0) = g_w(z_0, w_0) - 2i\widetilde{f}_w(z_0, w_0) \cdot \tilde{h}(\xi_0, \eta_0) \neq 0$ where $(\cdot)_w := \frac{\partial(\cdot)}{\partial w}$ by the above Lemma. Hence, write $\mathcal{E}_\Phi$ for the set of points where Φ fails to be holomorphic Segre transversal. Then we have

$$\mathcal{E}_\Phi = \{(z, w, \xi, \eta) \in \mathcal{M} : g_w(z, w) - 2i\widetilde{f}_w(z, w) \cdot \tilde{h}(\xi, \eta) = 0\}.$$

In particular if $\mathcal{E}_\Phi \neq \mathcal{M}$, then we conclude $\mathcal{E}_\Phi$ is either empty or a complex analytic variety of codimension one in $\mathcal{M}$.

Proof of Proposition 2.2.7: We keep the same notation as in Proposition 2.2.7. Without loss of generality, we assume that $\mathcal{E}$ is defined by $\chi_1(z,w)=0$ with $\chi_1(z,w)$ holomorphic over $\mathcal{M}$ and $\chi_1(z,w)\neq -K_0$ for any $(z,w,\xi,\eta)\in\mathcal{M}$ for some constant $K_0\in\mathbf{C}$. Define

$$\Phi(z,w,\xi,\eta)=\left(\frac{\chi_1(z,w)z}{K_0+\chi_1(z,w)},\frac{K_0}{K_0+\chi_1(z,w)},\frac{\chi_1(z,w)w}{K_0+\chi_1(z,w)},\xi,\frac{i}{2}\eta,\eta\right).$$

Then, one can verify that Φ is a holomorphic Segre map from $\mathcal{M}$ into $\mathcal{H}^N$ with $N=n+1$. Also, we can verify that $\mathcal{E}_\Phi$ is precisely the complex analytic variety defined by $\chi_1(z,w)=0$.

Notice that when $\mathcal{E}$ is defined by $\chi_2(\xi,\eta)=0$ where $\chi_2(\xi,\eta)$ is holomorphic over $\mathcal{M}$ and $\chi_2(\xi,\eta)\neq -K_0$ for any $(z,w,\xi,\eta)\in\mathcal{M}$ for some constant $K_0\in\mathbf{C}$, then the holomorphic Segre map Φ with $\mathcal{E}_\Phi=\mathcal{E}$ is given as follows:

$$\Phi(z,w,\xi,\eta)=\left(z,-\frac{i}{2}w,w,\frac{\chi_2(\xi,\eta)\xi}{K_0+\chi_2(\xi,\eta)},\frac{K_0}{K_0+\chi_2(\xi,\eta)},\frac{\chi_2(\xi,\eta)\eta}{K_0+\chi_2(\xi,\eta)}\right).$$

This proves Proposition 2.2.7. $\square$

Proof of Proposition 2.2.8: If we assume $\chi_1(z,w)\neq K_1$ and $\chi_2(\xi,\eta)\neq K_2$ for any $(z,w,\xi,\eta)\in\mathcal{M}$ for some constants $K_1,K_2\in\mathbb{C}$, then the holomorphic Segre map defined below meets the requirement:

$$\begin{aligned}\Phi(z,w,\xi,\eta)=\Bigg(&\frac{\chi_1(z,w)z}{K_1+\chi_1(z,w)},\frac{K_1}{K_1+\chi_1(z,w)},-\frac{i}{2}\frac{\chi_1(z,w)w}{(K_1+\chi_1(z,w))},\frac{\chi_1(z,w)w}{K_1+\chi_1(z,w)},\\ &\frac{\chi_2(\xi,\eta)\xi}{K_2+\chi_2(\xi,\eta)},\frac{i}{2}\frac{\chi_2(\xi,\eta)\eta}{(K_2+\chi_2(\xi,\eta))},\frac{K_2}{K_2+\chi_2(\xi,\eta)},\frac{\chi_2(\xi,\eta)\eta}{K_2+\chi_2(\xi,\eta)}\Bigg)\end{aligned}$$

This proves Proposition 2.2.8. $\square$

Proof of Theorem 2.2.6: Notice that a holomorphic Segre-related set is mapped to a holomorphic Segre-related set by a holomorphic automorphism of the complexification of the Heisenberg hypersurface. Without loss of generality, we can assume that $0\in\mathcal{M}$ and Φ is holomorphic Segre transversal at 0 and Φ satisfies first normalization condition

(2.5) and (2.6). Consider the Φ_p defined above. Assume that $\mathcal{E}_\Phi \neq \emptyset$. Then it is of codimension one. By Theorem 2.4.1, there exists $\tau \in Aut_0(\mathcal{H}^N)$ such that $\Phi^{**} = \tau \circ \Phi$ and

$$\Phi^{**}(z,w,\xi,\eta) = (z,\phi^{**}(z,w),w,\xi,\psi^{**}(\xi,\eta),\eta) \text{ with } \phi^{**}(z,w)\cdot\psi^{**}(\xi,\eta) \equiv 0 \text{ over } \mathcal{U}. \tag{2.43}$$

Notice that for any $(N-n)\times(N-n)$ invertible matrix U, $\Phi = (f,\phi,g,h,\psi,e)$, $\hat{\Phi} = (f,\phi,0,g,h,\psi,0,e)$ and $\hat{\hat{\Phi}} := (f,\phi\cdot U,g,h,\psi\cdot(U^{-1})^t,e)$ all have the same set of non-holomorphic Segre transversal points. Making use of the same argument as in the last paragraph in the proof of Theorem 2.4.1, we can assume, without loss of generality, that both $\{\phi\}$ and $\{\psi\}$ are linearly independent over $\mathbb{C}$ and $\phi_j\psi_j \equiv 0$ for $1 \le j \le N-n$.

Write E for the set of points in $\mathcal{M}$, whose image under Φ is contained in the pole of τ. Suppose that $p = (z_0,w_0,\xi_0,\eta_0) \notin E$. We have $(g^{**})_p = g^{**}\circ\sigma_p^0 - e^{**}(\xi_0,\eta_0) - 2i(\tilde{f}^{**}\circ\sigma_p^0)\cdot\tilde{h}^{**}(\xi_0,\eta_0)$. Hence

$$((g^{**})_p)_w(0) = (g^{**}_w)(z_0,w_0) - 2i\tilde{f}^{**}_w(z_0,w_0)\cdot\tilde{h}^{**}(\xi_0,\eta_0). \tag{2.44}$$

Since $\phi_j{}^{**}_w(z,w)\cdot\psi_j^{**}(\xi,\eta) \equiv 0$ on $\mathcal{M}$, we thus get:

$$((g^{**})_p)_w(0) = 1 \neq 0 \quad \text{for any} \quad p \in \mathcal{M}.$$

This shows that Φ^{**} and thus Φ are holomorphic Segre transversal at p. Therefore, we get that $\mathcal{E}_\Phi \subset E$.

From the construction in §3, we notice that $\Phi^{**} = \tau\circ\Phi$ is given by the following expression:

$$\left(\frac{\tilde{f}-\vec{a}g}{1+\tilde{f}\cdot\vec{s}+ug}, \frac{g}{1+\tilde{f}\cdot\vec{s}+ug}, \frac{\tilde{h}+\frac{i}{2}\vec{s}e}{1-2i\tilde{h}\cdot\vec{a}+(u+\vec{a}\cdot\vec{s})e}, \frac{e}{1-2i\tilde{h}\cdot\vec{a}+(u+\vec{a}\cdot\vec{s})e}\right) \tag{2.45}$$

for certain $\vec{a},\vec{s} \in \mathbb{C}^{N-1}$.

Now write $\chi_1 := 1+\tilde{f}\cdot\vec{s}+ug$, $\chi_2 := 1-2i\tilde{h}\cdot\vec{a}+(u+\vec{a}\cdot\vec{s})e$, $A := \phi-\vec{a}g$ and $B := \psi-\frac{i}{2}\vec{s}e$ in (4.3). Then $\phi^{**}(z,w) = \frac{A(z,w)}{\chi_1(z,w)}$, $\psi^{**}(\xi,\eta) = \frac{B(\xi,\eta)}{\chi_2(\xi,\eta)}$ for any $(z,w,\xi,\eta) \in \mathcal{M}$. Write $E_1 := \{(z,w,\xi,\eta) \in \mathcal{M} : \ \chi_1(z,w) = 0\}$, $E_2 := \{(z,w,\xi,\eta) \in \mathcal{M} : \ \chi_2(\xi,\eta) = 0\}$. Then $E = E_1 \cup E_2$.

claim 2.5.2 $\{E_1 - E_2\} \cup \{E_2 - E_1\} \subset \mathcal{E}_\Phi$.

Since $E_1 \cap E_2$ is of codimension 2 in $\mathcal{M}$, together with what we obtained above, Claim 4.2 leads to the following:

Corollary 2.5.3 $E = \mathcal{E}_\Phi$.

Assuming Claim 2.5.2 for the moment, we next complete the proof of Theorem 2.2.6. Since each irreducible component of E is obviously a holomorphic Segre-related set of codimension one, we see that $\mathcal{E}_\Phi$ must be a locally finite union of holomorphic Segre-related sets of codimension one.

Back to (2.45), for any $p \in \mathcal{E}_\Phi$, we have either $p \in E_1$ or $p \in E_2$. Without loss of generality, assume the first one. Then $g(p) = 0$. Now replacing Φ by Φ_{p_0} for any $p_0 \in \mathcal{M}$ near the origin, then the holomorphic Segre transversality breaks down for Φ_{p_0} over $(\sigma_{p_0}^0)^{-1}(E_1)$. By a similar argument as above, we may also assume, without loss of generality, that $g_{p_0}(p) = 0$ for $p \in (\sigma_{p_0}^0)^{-1}(E_1)$ and p_0 is in a certain connected open subset U of $\mathcal{M}$ with 0 in its closure.

Since $g_{p_0} = g \circ \sigma_{p_0}^0 - e(p_0) - 2i\widetilde{f} \circ \sigma_{p_0}^0 \cdot \widetilde{h}(p_0)$, we have $g_{p_0}((\sigma_{p_0}^0)^{-1}(p)) = g(p) - e(p_0) - 2i\widetilde{f}(p) \cdot \widetilde{h}(p_0)$. Let $p \in E_1$, then we have

$$e(p_0) = -2i\widetilde{f}(p) \cdot \widetilde{h}(p_0) \quad \text{for any } p \in E_1,\ p_0 \in U. \tag{2.46}$$

Notice that Φ satisfies the first normalization condition. Hence we have

$$h_j = \xi_j + o_{wt}(1), \quad \psi_j = o_{wt}(1), \quad e = o_{wt}(1).$$

Substituting the above into (2.46), we get

$$f_j(p) = 0 \quad \text{for } j = 1, \ldots, n-1,\ p \in E_1.$$

Therefore (2.46) becomes

$$e(p_0) = -2i \sum_{j=1}^{N-n} \phi_j(p)\psi_j(p_0) \quad \text{for } \ p \in E_1,\ p_0 \in U.$$

Since $\{\psi_k\}_{l=1}^{N-n}$ is a linearly independent system and U is a uniqueness set for holomorphic functions over $\mathcal{M}$, we immediately get $\phi_k \equiv$ constant over E_1. This proves that $\Phi(E_1) \subset \{q\} \times \hat{\tilde{Q}}_q$ with $\Phi_1(E_1) \equiv q$. Similarly we can prove that $\Phi(E_2) \subset \widetilde{Q}_{\Phi_2(E_2)} \times \{\Phi_2(E_2)\}$, where $\Phi_2 \equiv$ constant on E_2. Hence, to complete the proof of Theorem 2.2.6, we need only to give a proof for Claim 2.5.2.

Proof of Claim 2.5.2: We write $\mathcal{H}^N_{proj}$ for the compactification of $\mathcal{H}^N$ in $\mathbb{CP}^N \times \mathbb{CP}^N$ with the following homogeneous coordinates and defining equation:

$$[z_1, \cdots, z_N, t; \xi_1, \cdots, \xi_N, \gamma], \quad \gamma z_N - t\xi_N = 2i \sum_{j=1}^{N-1} z_j \xi_j.$$

Then the τ in (2.43) extends naturally to a new mapping, denoted by $\hat{\tau}$, from $\mathcal{H}^N$ to $\mathcal{H}^N_{proj}$. Write $\hat{\Phi} = \hat{\tau} \circ \Phi$. Then $\hat{\Phi}$ is a holomorphic map from $\mathcal{M}$ into $\mathcal{H}^N_{proj}$. Apparently, Φ is holomorphic Segre transversal at $p \in \mathcal{M}$ if and only if $\hat{\Phi}$ is holomorphic Segre transversal in a similar way.

For a fixed $p_0 \in E_1 - E_2$, there exists a $j \in \{1, \ldots, N-n\}$ such that $A_j(p) \neq 0$ by (2.45) and the definition of E_1. Without loss of generality, assume that $j = 1$. We next use the following local holomorphic coordinates chart of $\mathbb{CP}^N \times \mathbb{CP}^N$ near $\hat{\Phi}(p_0)$:

$$\sigma^n_N([z_1, \ldots, z_{N-1}, w, t; \xi_1, \cdots, \xi_{N-1}, \eta, 1]) :=$$
$$\left(\frac{z_1}{z_n}, \frac{z_2}{z_n}, \ldots, \frac{z_{n-1}}{z_n}, \frac{z_{n+1}}{z_n}, \frac{z_{N-1}}{z_n}, \frac{w}{z_n}, \frac{t}{z_n}, \xi_1, \ldots, \xi_{N-1}, \eta\right).$$

Then, with the new coordinates, the image of $\mathcal{H}^N_{proj}$ under this coordinate transformation is locally defined by the following equation:

$$\hat{\mathcal{H}}^N = \{(\hat{z}, \hat{w}, \hat{t}, \hat{\xi}, \hat{\eta}) \in \mathbb{C}^{N-2} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{N-1} \times \mathbb{C} : \ \hat{w} - \hat{t}\hat{\eta} = 2i(\sum_{j=1}^{n-1} \hat{z}_j \hat{\xi}_j + \hat{\xi}_n + \sum_{j=n+1}^{N-1} \hat{z}_{j-1} \hat{\xi}_j)\}.$$

An easy computation shows that $\{\hat{\mathcal{L}}_j, \hat{\mathcal{K}}_j, \hat{\mathcal{T}}\}_{j=1}^{N-1}$ forms a global basis of the sections of holomorphic tangent bundle $T^{(1,0)}\hat{\mathcal{H}}^N$ of $\hat{\mathcal{H}}^N$ and that

$$\langle \hat{\theta}, \hat{\mathcal{T}} \rangle = 1, \quad \langle \hat{\theta}, \hat{\mathcal{L}}_j \rangle = \langle \hat{\theta}, \hat{\mathcal{K}}_j \rangle = 0,$$

where

$$\begin{aligned}
&\hat{\mathcal{L}}_j = \frac{\partial}{\partial \hat{z}_j} + 2i\hat{\xi}_j \frac{\partial}{\partial \hat{w}}, && \hat{\mathcal{K}}_j = -\frac{\partial}{\partial \hat{\xi}_j} + \hat{z}_j \frac{\partial}{\partial \hat{\xi}_n}, && j = 1, \ldots, n-1, \\
&\hat{\mathcal{L}}_n = \frac{\partial}{\partial \hat{t}} + \hat{\eta} \frac{\partial}{\partial \hat{w}}, && \hat{\mathcal{K}}_n = -\hat{t} \frac{\partial}{\partial \hat{\xi}_n} + 2i \frac{\partial}{\partial \hat{\eta}}, && \\
&\hat{\mathcal{L}}_j = \frac{\partial}{\partial \hat{z}_{j-1}} + 2i\hat{\xi}_j \frac{\partial}{\partial \hat{w}}, && \hat{\mathcal{K}}_j = -\frac{\partial}{\partial \hat{\xi}_j} + \hat{z}_{j-1} \frac{\partial}{\partial \hat{\xi}_n}, && j = n+1, \ldots, N-1, \\
&\hat{\mathcal{T}} = 2i \frac{\partial}{\partial \hat{w}} + \frac{\partial}{\partial \hat{\xi}_n}, && \hat{\theta} = \sum_{j=1}^{n-1} \hat{z}_j d\hat{\xi}_j + d\hat{\xi}_n + \sum_{j=n+1}^{N-1} \hat{z}_{j-1} d\hat{\xi}_j + \frac{\hat{t}}{2i} d\hat{\eta}. &&
\end{aligned}$$

Write $\mathcal{F}(z, w, \xi, \eta) := \sigma_N^n \circ \hat{\Phi}(z, w, \xi, \eta) = \sigma_N^n \circ \hat{\tau} \circ \Phi(z, w, \xi, \eta) = (\frac{\chi_1(z,w)z}{A_1(z,w)}, \frac{A_2(z,w)}{A_1(z,w)}, \ldots, \frac{A_{N-n}(z,w)}{A_1(z,w)}, \frac{\chi_1(z,w)w}{A_1(z,w)}, \frac{\chi_1(z,w)}{A_1(z,w)}, \xi, \psi^{**}(\xi, \eta), \eta)$ for $(z, w, \xi, \eta) \in \mathcal{M}$. Notice that σ_N^n is biholomorphic near $\hat{\Phi}(p_0)$. Hence Φ is holomorphic Segre transversal at p_0 if and only if $\mathcal{F}$ is holomorphic Segre transversal at p_0. However,

$$\langle \hat{\theta}, d\mathcal{F}(\mathcal{T}) \rangle \Big|_{p_0} = \left(\frac{\partial \psi_1^{**}}{\partial \eta} + \sum_{j=2}^{N-n} \frac{A_j}{A_1} \frac{\partial \psi_j^{**}}{\partial \eta} \right) \Bigg|_{p_0} = 0,$$

where we used the facts that $\chi_1(p_0) = 0, \psi_1^{**} \equiv 0$ and $A_j \cdot \psi_j^{**} \equiv 0$ for $j \geq 2$. This yields that $\mathcal{F}$ and thus Φ are not holomorphic Segre transversal at p_0. Hence, $p_0 \in \mathcal{E}_\Phi$.

Similarly we can prove that $E_2 - E_1 \in \mathcal{E}_\Phi$. this completes the proof of Claim 2.5.2 and thus the proof of Theorem 2.2.6. $\square$

As we pointed out before, Φ is holomorphic Segre transversal at p if Φ_p is holomorphic Segre transversal at 0 or equivalently $(g_p)_w(0) \neq 0$. It then follows in Example 2.2.9 that Φ is not holomorphic Segre transversal at $p = (z, w, \xi, \eta)$ iff $w + \eta = 0$ by using (2.44). Note that the submanifold $\mathcal{G} = \{(z, w, \xi, \eta) \in \mathcal{H}^n : \ w + \eta = 0\}$ is not a holomorphic Segre-related set of $\mathcal{H}^n$. In fact, for any point $(z_0, w_0, \xi_0, \eta_0) \in \mathcal{G}$, $\{(z_0, w_0)\} \times \hat{Q}_{(z_0, w_0)} = \{(z_0, w_0, \xi, w_0 - 2iz_0 \cdot \xi) : \ \xi \in \mathbb{C}^{n-1}\}$, which is not contained in $\mathcal{G}$. Similarly one can show that $Q_{(\xi_0, \eta_0)} \times \{(\xi_0, \eta_0)\}$ is not totally contained in $\mathcal{G}$. Thus $\mathcal{G}$ is not a holomorphic Segre-related set by definition. This example shows that the condition $N \leq 2n - 2$ is critical for Theorem 2.2.6.

2.6 Proof of Theorem 2.2.3

We keep the same notation set up before.

Lemma 2.6.1 *Let $\mathcal{M}$ be a connected neighborhood of 0 in $\mathcal{H}^n$. Suppose that the holomorphic Segre map Φ maps a neighborhood $\mathcal{U}$ of $\mathcal{M}$ in $\mathbf{C}^{2n}$ into $\mathbf{C}^{2N}$ with $\Phi(\mathcal{M}) \subset \mathcal{H}^N, \Phi(0) = 0$ and write $\Phi(z,w,\xi,\eta) = (\Phi_1(z,w), \Phi_2(\xi,\eta)) := (\tilde{f}_1(z,w), \ldots, \tilde{f}_{N-1}(z,w), g(z,w), \tilde{h}_1(\xi,\eta), \ldots, \tilde{h}_{N-1}(\xi,\eta), e(\xi,\eta))$ for $(z,w,\xi,\eta) \in \mathcal{M}$. Assume that $N \leq 2n-2$. If there exists a neighborhood V of 0 in $\mathcal{M}$, such that for every $p \in V$, $(g_p)_w(0) = 0$, then $g \equiv 0, e \equiv 0$ and $\widetilde{f} \cdot \widetilde{h} \equiv 0$ over $\mathcal{U}$.*

To prove Lemma 2.6.1, we need the following:

Lemma 2.6.2 *: Suppose that $A, B \in M_{(n-1)\times(N-1)}$ where $N \leq 2n-2$ satisfy that $A \cdot \bar{B}^t = 0$. Then either A or B has rank less than $n-1$.*

Proof of Lemma 2.6.2: Suppose A has rank $n-1$. Then the linear equation $A \cdot y^t = 0$ has at most $N-1-(n-1) = N-n$ linearly independent solutions, which implies that $rank(B) \leq N-n < n-1$. $\square$

Proof of Lemma 2.6.1: We follows the same approach in [BH] for the proof of the Lemma. By the definition of Φ_p, we have:

$$g_p = g \circ \sigma_p^0 - e(\xi_0,\eta_0) - 2i\tilde{f} \circ \sigma_p^0 \cdot \tilde{h}(\xi_0,\eta_0). \tag{2.47}$$

Hence it follows that

$$(g_p)_w(0) = g_w(z_0,w_0) - 2i\tilde{f}_w(z_0,w_0) \cdot \tilde{h}(\xi_0,\eta_0),$$

where $p = (z_0,w_0,\xi_0,\eta_0) \in \mathcal{M}$. By the assumption, $g_w(z,w) = 2i\tilde{f}_w(z,w) \cdot \tilde{h}(\xi,\eta)$ on V, i.e.,

$$g_w(z, \eta + 2iz \cdot \xi) = 2i\tilde{f}_w(z, \eta + 2iz \cdot \xi) \cdot \tilde{h}(\xi,\eta). \tag{2.48}$$

Let $\xi = 0,\ \eta = 0$, we have

$$g_w(z,0) = 0.$$

Applying $\frac{\partial}{\partial \xi_j},\ j = 1, \ldots, n-1$ to (2.48) and letting $\xi = 0,\ \eta = 0$, we get

$$z_j g_{w^2}(z,0) = \tilde{f}_w(z,0) \cdot \tilde{h}_{\xi_j}(0). \tag{2.49}$$

On the other hand, applying $\mathcal{K}_j\mathcal{L}_k$ to $g_p - e_p = 2i\tilde{f}_p \cdot \tilde{h}_p$ and letting $(z, w, \xi, \eta) = 0$, we have

$$\delta_j^k (g_p)_w(0) = (\tilde{f}_p)_{z_k}(0) \cdot (\tilde{h}_p)_{\xi_j}(0) \qquad \text{for any } j, k = 1, \ldots, n-1. \tag{2.50}$$

Applying Lemma 2.6.2 and making use of the assumption $(g_p)_w(0) = 0$ for $p \in V$ on (2.50), we get that

$$V = A \cup B,$$

where $$A = \{p : \{(\tilde{h}_p)_{\xi_j}(0)\}_{j=1}^{n-1} \text{ are linearly dependent}\};$$

$$B = \{p : \{(\tilde{f}_p)_{z_j}(0)\}_{j=1}^{n-1} \text{ are linearly dependent}\}.$$

Then either A or B contains an open neighborhood of V. Without loss of generality, assume that $V_1(\ni p_0) \subset V$ is an open piece of $\mathcal{M}$ that is contained in A. By considering Φ_{p_0} instead of Φ, we assume, without loss of generality, that $p_0 = 0$.

Therefore $\{(\tilde{h}_p)_{\xi_j}(0)\}_{j=1}^{n-1}$ are linearly dependent for p in some small neighborhood of $\mathcal{M}$ near 0. Take a non-zero $(n-1)$-tuple $(a_1, \ldots, a_{n-1})$ such that $\sum_{j=1}^{n-1} a_j \tilde{h}_{\xi_j}(0) = 0$. It thus follows from (2.49) that $\sum_{j=1}^{n-1} a_j z_j g_{w^2}(z, 0) = 0$. Since $\sum_{j=1}^{n-1} a_j z_j \neq 0$, we conclude $g_{w^2}(z, 0) = 0$. Now applying the previous argument to Φ_p, we then get $(g_p)_{w^2}(z, 0) = 0$ for p in some small neighborhood in $\mathcal{M}$. An induction argument then shows that $g_{w^k}(z, 0) = 0$ for $k \geq 0$. Since we also have $g_{z^k}(0, 0) = 0$ for $k \geq 0$, we then proved that $g \equiv 0$. Similarly we have $g_p \equiv 0$. Substituting $g \equiv 0$ and $g_p \equiv 0$ into (2.47), we have

$$e(\xi_0, \eta_0) = -2i\tilde{f} \circ \sigma_p^0(z, w) \cdot \tilde{h}(\xi_0, \eta_0)$$

for any $p = (z_0, w_0, \xi_0, \eta_0)$ in $\mathcal{M}$. Choose $(z, w) = (\sigma_p^0)^{-1}(0)$ in the above, then we have $e \equiv 0$. Again by (2.47) we see $\widetilde{f} \cdot \widetilde{h} \equiv 0$ in $\mathcal{U}$. The proof is complete. $\square$

Proof of Theorem 2.2.3: Theorem 2.2.3 (1) follows from Theorem 2.4.1 and Theorem 2.2.3 (2) is an easy consequence of Lemma 2.6.1. $\square$

Remark 2.6.3 *When $\Phi_2(z, w) = \bar{\Phi}_1(z, w)$ in Theorem 2.2.3, a super-rigidity can be deduced. Namely, one further concludes that $\phi(z, w)$ and $\psi(\xi, \eta)$ must be identically* *zero and Theorem 2.2.3 (2) can not occur. Theorem 2.2.3 in this case then reduces to a Theorem of Faran and Huang (See [Fa] [Hu]). When* $\Phi_2(z,w) \neq \bar{\Phi}_1(z,w)$*, one can easily write down the following example, showing that* Φ *does not have to be linear nor rational:* $\Phi : \mathcal{H}^3 \to \mathcal{H}^4$*, where*

$$\Phi(z_1, z_2, w, \xi_1, \xi_2, \eta) = (z_1, z_2, \cos z_1, w, \xi_1, \xi_2, 0, \eta).$$

Chapter 3

Polynomial and rational maps between balls

3.1 Introduction

Let $\mathbb{B}^n$ be the unit ball in the complex space $\mathbb{C}^n$. Write $Rat(\mathbb{B}^n, \mathbb{B}^N)$ for the space of proper rational holomorphic maps from $\mathbb{B}^n$ into $\mathbb{B}^N$ and $Poly(\mathbb{B}^n, \mathbb{B}^N)$ for the set of proper holomorphic polynomial maps from $\mathbb{B}^n$ into $\mathbb{B}^N$. We say that F and $G \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ are *equivalent* if there are automorphisms $\sigma \in Aut(\mathbb{B}^n)$ and $\tau \in Aut(\mathbb{B}^N)$ such that $F = \tau \circ G \circ \sigma$.

Proper holomorphic maps from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N \le 2n-2$, that are sufficiently smooth up to the boundary, are equivalent to the identity map ([Fa] [For3] [Hu]). In [HJX], it is shown that $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ with $N \le 3n-4$ is equivalent to a quadratic monomial map, called the D'Angelo map. However, when the codimension is sufficiently large, there is plenty of room to construct rational holomorphic maps with certain arbitrariness by the work in Catlin-D'Angelo [CD]. Hence, it is reasonable to believe that after lifting the codimension restriction, many proper rational holomorphic maps are not equivalent to proper holomorphic polynomial maps. In the last paragraph of the paper [DA], D'Angelo gave a philosophic discussion on this matter.

However, the problem of determining if an explicit proper rational holomorphic map is equivalent to a polynomial map does not seem to have been studied so far.

This chapter is concerned with such a problem. We will first give a simple and explicit criterion when a rational holomorphic map is equivalent to a holomorphic polynomial map. With the help of the classification result in [CJX], this criterion is used in §3 to show that proper rational holomorphic maps from $\mathbb{B}^2$ into $\mathbb{B}^N$ of degree two are equivalent to polynomial maps. On the other hand, making use of the criterion, we construct in §4 rational holomorphic maps of degree 3 that are 'almost' linear but are not equivalent to holomorphic polynomial maps.

3.2 A criterion

Let $F = \frac{P}{q} = \frac{(P_1,...,P_N)}{q}$ be a non-constant rational holomorphic map from the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ into the unit ball $\mathbb{B}^N \subset \mathbb{C}^N$, where $(P_j)_{j=1}^N$, q are holomorphic polynomial functions and $(P_1, ..., P_N, q) = 1$. We define $\deg(F) = \max\{deg(P_j)_{j=1}^N, deg(q)\}$. Then F induces a rational map from $\mathbb{CP}^n$ into $\mathbb{CP}^N$ given by

$$\hat{F}([z_1 : \cdots : z_n : t]) = \left[t^k P(\frac{z}{t}) : t^k q(\frac{z}{t}) \right]$$

where $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $\deg(F) = k > 0$.

$\hat{F}$ may not be holomorphic in general. Denote by $Sing(\hat{F})$ the singular set of $\hat{F}$, namely, the collection of points where $\hat{F}$ fails to be (or fails to extend to be) holomorphic. Then $Sing(\hat{F})$ is an algebraic subvariety of codimension two or more in $\mathbb{CP}^n$. For instance, we have the following:

Example 3.2.1 *I. Let $F_\theta(z, w) = (z, cos\theta\ w, sin\theta\ zw, sin\theta\ w^2)$ be the proper monomial map from $\mathbb{B}^2$ into $\mathbb{B}^4$ (called the D'Angelo map), where $0 < \theta < \frac{\pi}{2}$. Then $Sing(\hat{F_\theta})$ of $\hat{F_\theta}$ consists of one point: $\{[z : w : t] \in \mathbb{CP}^2 :\ w = 0,\ t = 0\} = \{[1 : 0 : 0]\}$.*

II. Let $G_\alpha = (z^2, \sqrt{1 + cos^2\alpha}\ zw, cos\alpha\ w^2, sin\alpha\ w)$ be the proper monomial map from $\mathbb{B}^2$ into $\mathbb{B}^4$ where $0 \leq \alpha < \frac{\pi}{2}$. Then G_α induces

$$\hat{G}_\alpha = [z^2 : \sqrt{1 + cos^2\alpha}\ zw : cos\alpha\ w^2 : sin\alpha\ wt : t^2].$$

There are no singular points for $\hat{G}_\alpha$. Hence $\hat{G}_\alpha$ is holomorphic.

Write $\mathbb{B}^n_1 = \{[z_1 : \cdots : z_n : t] \in \mathbb{CP}^n :\ \sum_{j=1}^n |z_j|^2 < |t|^2\}$, which is the projective realization of $\mathbb{B}^n$. Write $U(n+1, 1)$ for the collection of the linear transforms $\mathcal{A} :\ [Z](\in \mathbb{CP}^n) \mapsto [ZA](\in \mathbb{CP}^n)$ such that

$$AE_{n+1,1}\overline{A}^t = E_{n+1,1}$$

where

$$E_{n+1,1} = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $U(n+1,1)/\{\pm Id\} = Aut(\mathbb{B}_1^n) \approx Aut(\mathbb{B}^n)$.

Lemma 3.2.2 *For any hyperplane $H \subset \mathbb{CP}^n$ with $H \cap \overline{\mathbb{B}_1^n} = \emptyset$, there is a $\sigma \in U(n+1,1)$ such that $\sigma(H) = H_\infty = \{[z_1 : \cdots : z_n : 0] \in \mathbb{CP}^n\}$.*

Proof of Lemma 3.2.2: Assume that $H : \sum_{j=1}^n a_j z_j - a_{n+1} t = 0$ with $\vec{a} = (a_1, ..., a_{n+1}) \neq 0$. Under the assumption that $H \cap \overline{\mathbb{B}_1^n} = \emptyset$, we have $a_{n+1} \neq 0$. Without loss of generality, we can assume that $a_{n+1} = 1$. Let U be an $n \times n$ unitary matrix such that

$$(a_1, ..., a_n)\overline{U} = (\lambda, 0, ..., 0),$$

for some $\lambda \in \mathbb{C}$. Let $\sigma = \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}$. Then $\sigma(H) = \{[z : t] \in \mathbb{CP}^n \mid \lambda z_1 - t = 0\}$ with $|\lambda| < 1$. Let $T \in Aut(\mathbb{B}^n)$ be defined by

$$T(z_1, z') = \left(\frac{z_1 - \overline{\lambda}}{1 - \lambda z_1}, \frac{\sqrt{1 - |\lambda|^2} z'}{1 - \lambda z_1} \right)$$

with $z' = (z_2, ..., z_n)$. Then $\hat{T} \in U(n+1,1)$ is defined by

$$\hat{T}([z_1 : z' : t]) = [z_1 - \overline{\lambda} t : \sqrt{1 - |\lambda|^2} z' : t - \lambda z_1].$$

Now, it is easy to see that $\hat{T} \circ \sigma$ maps H to H_∞. $\square$

Our criterion can be stated as follows:

Theorem 3.2.3 *Let F be a non-constant rational holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N, n \geq 1$. Then F is equivalent to a holomorphic polynomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$, namely, there are $\sigma \in Aut(\mathbb{B}^n)$ and $\tau \in Aut(\mathbb{B}^N)$ such that $\tau \circ F \circ \sigma$ is a holomorphic polynomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$, if and only if there exist (complex) hyperplanes $H \subset \mathbb{CP}^n$ and $H' \subset \mathbb{CP}^N$ such that $H \cap \overline{\mathbb{B}_1^n} = \emptyset$, $H' \cap \overline{\mathbb{B}_1^N} = \emptyset$ and*

$$\hat{F}(H \setminus Sing(\hat{F})) \subset H', \quad \hat{F}\left(\mathbb{CP}^n \setminus (H \cup Sing(\hat{F}))\right) \subset \mathbb{CP}^N \setminus H'.$$

Proof of Theorem 3.2.3: If F is a non-constant holomorphic polynomial map, then $\hat{F} = [t^k F(\frac{z}{t}), t^k]$ with $deg(F) = k > 0$. Let $H = H_\infty$ and $H' = H'_\infty$. Then they satisfy the property described in the theorem.

If F is equivalent to a holomorphic polynomial map G, then there exist $\hat{\sigma} \in U(n+1,1), \hat{\tau} \in U(n+1,1)$ such that $\hat{F} = \hat{\tau} \circ \hat{G} \circ \hat{\sigma}$. Let $H = \hat{\sigma}^{-1}(H_\infty)$ and $H' = \hat{\tau}(H'_\infty)$. Then they are the desired ones.

Conversely, suppose that $\hat{F}$, H and H' are as in the theorem. By Lemma 3.2.2, we can find $\hat{\sigma} \in U(n+1,1)$ and $\hat{\tau} \in U(n+1,1)$ such that $\hat{\sigma}(H) = H_\infty$ and $\hat{\tau}(H') = H'_\infty$. Let $\hat{Q} = \hat{\tau} \circ \hat{F} \circ \hat{\sigma}^{-1}$. Then $\hat{Q}$ induces a rational holomorphic map Q from $\mathbb{B}^n$ into $\mathbb{B}^N$. If $Q = \frac{P}{q}$ where $(P,q) = 1$ and $deg(Q) = k > 0$, then

$$\hat{Q} = [t^k P(\frac{z}{t}) : t^k q(\frac{z}{t})].$$

Suppose that $q \not\equiv constant$. Let $z_0 \in \mathbb{C}^n$ be such that $q(z_0) = 0$ but $P(z_0) \neq 0$. Then $[z_0 : 1] \notin Sing(\hat{Q}) \cup H_\infty$ and $\hat{Q}([z_0 : 1]) \in H'_\infty$. Notice that $\hat{Q}(H_\infty \setminus Sing(\hat{Q})) \subset H'_\infty$ and $\hat{Q}\left(\mathbb{CP}^n \backslash (H_\infty \cup Sing(\hat{Q}))\right) \subset \mathbb{CP}^N \backslash H'_\infty$. This is a contradiction. Thus, we showed that Q is a polynomial. $\square$

Remark 3.2.4 (A) *Suppose that* $\hat{F} = [F_1 : \cdots : F_N : F_0]$ *is a non-constant rational map from* $\mathbb{CP}^n$ *into* $\mathbb{CP}^N$*, where* $F_0, \ldots, F_N$ *are homogeneous polynomials in* (z,t) *of degree* $k > 0$ *with*

$$(F_1, \ldots, F_N, F_0) = 1.$$

Assume that $H := \{[z_1 : \cdots : z_n : t] \in \mathbb{CP}^n : \sum_{j=1}^n a_j z_j + a_0 t = 0,\ a_j \in \mathbb{C},\ (a_0, \ldots, a_n, a_0) \neq 0\}$, $H' := \{[z'_1 : \cdots : z'_N : t'] \in \mathbb{CP}^N : \sum_{j=1}^N A_j z'_j + A_0 t' = 0,\ A_j \in \mathbb{C},\ (A_0, \ldots, A_N, A_0) \neq 0\}$ *are (complex) hyperplanes. Also assume that* $H \cap \mathbb{B}_1^n = \emptyset$ *and* $H' \cap \mathbb{B}_1^N = \emptyset$*. We easily see that* $a_0, A_0 \neq 0$ *(thus we can always make* $a_0,\ A_0 = 1$*). Under such a set-up, by the basic division property for polynomials, we can easily conclude that*

$$\hat{F}(H \setminus Sing(\hat{F})) \subset H', \quad \hat{F}\left(\mathbb{CP}^n \backslash (H \cup Sing(\hat{F}))\right) \subset \mathbb{CP}^N \backslash H'$$

if and only if

$$\sum_{j=1}^N A_j F_j + A_0 F_0 \equiv C \cdot (\sum_{j=1}^n a_j z_j + a_0 t)^k,$$

where $C \neq 0$ is a constant and $k(>0)$ is the degree of F. This observation will be used for our later application of Theorem 3.2.3.

(B): *From the argument of Theorem 3.2.3, it is clear that a similar result can also be proved for non-constant rational maps from $\mathbb{CP}^n$ into $\mathbb{CP}^N$.*

Write the Cayley transformation

$$\rho_n(z', z_n) = \left(\frac{2z'}{1 - iz_n}, \frac{1 + iz_n}{1 - iz_n}\right).$$

Then ρ_n biholomorphically maps $\partial\mathbb{H}^n$ to $\partial\mathbb{B}^n \backslash \{(0, 1)\}$, where $\mathbb{H}^n = \{(z', z_n) \in \mathbb{C}^n : \Im(z_n) > |z'|^2\}$. ρ_n induces a linear transformation of $\mathbb{CP}^n$:

$$\hat{\rho}_n = [2z' : t + iz_n : t - iz_n].$$

$\hat{\rho}_n$ maps $\mathbf{S}_1^n = \{[z' : z_n : t] \in \mathbb{CP}^n : \frac{z_n\bar{t} - t\bar{z}_n}{2i} > |z'|^2\}$ to $\mathbb{B}_1^n$.

Now let G be a non-constant rational holomorphic map from an open piece of $\partial\mathbb{H}^n$ into $\partial\mathbb{H}^N$. Then $\rho_N \circ G \circ \rho_n^{-1}$ extends to a proper rational holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^N$. By Theorem 3.2.3, we have the following:

Theorem 3.2.5 *$\rho_N \circ G \circ \rho_n^{-1}$ is equivalent to a proper holomorphic polynomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$ if and only if there are (complex) hyperplanes $H \subset \mathbb{CP}^n$, $H' \subset \mathbb{CP}^N$ such that $\hat{G}(H \setminus Sing(\hat{G})) \subset H'$ and $\hat{G}(\mathbb{CP}^n \setminus (H \cup Sing(\hat{G})) \subset \mathbb{CP}^N \setminus H'$ with*

$$H \cap \overline{\mathbf{S}_1^n} = \emptyset, \quad H' \cap \overline{\mathbf{S}_1^N} = \emptyset.$$

3.3 Proper rational holomorphic maps from $\mathbb{B}^2$ into $\mathbb{B}^N$ of degree two

As a first application of Theorem 3.2.3, we prove the following:

Theorem 3.3.1 *A map $F \in Rat(\mathbb{B}^2, \mathbb{B}^N)$ of degree two is equivalent to a polynomial proper holomorphic map in $Poly(\mathbb{B}^2, \mathbb{B}^N)$.*

Proof of Theorem 3.3.1: By [HJX], we know that any rational holomorphic map of degree 2 from $\mathbb{B}^2$ into $\mathbb{B}^N$ is equivalent to a map of the form $(G, 0)$, where the map G is from $\mathbb{B}^2$ into $\mathbb{B}^5$. Hence, to prove Theorem 3.3.1, we need only to assume that $N = 5$. After applying Cayley transformations and using the result in [CJX], we can assume that $F = (f, \phi_1, \phi_2, \phi_3, g)$ is from $\mathbb{H}^2$ into $\mathbb{H}^5$ with either

(I)

$$f = \frac{z + \frac{i}{2}zw}{1 + e_2w^2},\ \phi_1 = \frac{z^2}{1 + e_2w^2},\ \phi_2 = \frac{c_1zw}{1 + e_2w^2},\ \phi_3 = 0,\ g = \frac{w}{1 + e_2w^2}$$

where $-e_2 = \frac{1}{4} + c_1^2$ and $c_1 > 0$ or

(II)

$$f = \frac{z + (\frac{i}{2} + ie_1)zw}{1 + ie_1w + e_2w^2},\ \phi_1 = \frac{z^2}{1 + ie_1w + e_2w^2},$$

$$\phi_2 = \frac{c_1zw}{1 + ie_1w + e_2w^2},\ \phi_3 = \frac{c_3w^2}{1 + ie_1w + e_2w^2},\ g = \frac{w + ie_1w^2}{1 + ie_1w + e_2w^2}$$

where $-e_1, -e_2 > 0$, $c_1, c_3 > 0$ and

$$e_1e_2 = c_3^2, \quad -e_1 - e_2 = \frac{1}{4} + c_1^2.$$

Write $[z : w : t]$ for the homogeneous coordinates of $\mathbb{CP}^2$. In Case (I) the map F induces a rational map $\hat{F} : \mathbb{CP}^2 \to \mathbb{CP}^5$ given by

$$\hat{F}([z : w : t]) = [tz + \frac{i}{2}zw : z^2 : c_1zw : 0 : tw : t^2 + e_2w^2] \quad \forall [z : w : t] \in \mathbb{CP}^2.$$

In Case (II), F induces a rational map $\hat{F} : \mathbb{CP}^2 \to \mathbb{CP}^5$ given by

$$\hat{F}([z : w : t]) = [tz + (\frac{i}{2} + ie_1)zw : z^2 : c_1zw : c_3w^2 : tw + ie_1w^2 : t^2 + ie_1wt + e_2w^2]$$

$\forall [z : w : t] \in \mathbb{CP}^2$. In terms of Theorem 3.2.5, we will look for $H = \{-t = \mu_1z_1 + \mu_2z_2\} \subset \mathbb{CP}^2$ and $H' = \{-t' = \sum_{j=1}^5 \lambda_j z_j'\} \subset \mathbb{CP}^5$ such that $H \cap \overline{\mathbf{S}_1^2} = \emptyset, \quad H' \cap \overline{\mathbf{S}_1^5} = \emptyset$ with

$$\hat{F}(H \setminus Sing(\hat{F})) \subset H' \ \ and \ \ \hat{F}\left(\mathbb{CP}^2 \setminus (H \cup Sing(\hat{F}))\right) \subset \mathbb{CP}^5 \setminus H'.$$

We next prove the following lemma:

Lemma 3.3.2 *Let $H = \{-t = \sum_{j=1}^n K_jz_j\} \subset \mathbb{CP}^n$. Then $H \cap \overline{\mathbf{S}_1^n} = \emptyset$ if and only if*

$$4\Im(K_n) + \sum_{j=1}^{n-1} |K_j|^2 < 0.$$

Proof of Lemma 3.3.2: Suppose for z_j and $t = -\sum_{j=1}^{n} K_j z_j$, we have

$$\frac{w\bar{t} - t\overline{w}}{2i} < \sum_{j=1}^{n-1} |z_j|^2.$$

Here we identify $z_n = w$. We then get

$$\frac{-\overline{K_n}|w|^2 + K_n|w|^2}{2i} + \sum_{j=1}^{n-1} \frac{-\overline{K_j}\overline{z_j}w + K_j z_j \overline{w}}{2i} < \sum_{j=1}^{n-1} |z_j|^2.$$

Hence

$$|w|^2 \Im(K_n) < \sum_{j=1}^{n-1} \{|z_j|^2 - 2\Re(\frac{K_j}{2i} z_j \overline{w})\},$$

or

$$|w|^2 \left(\Im(K_n) + \sum_{j=1}^{n-1} \frac{|K_j|^2}{4} \right) < \sum_{j=1}^{n-1} |z_j - \frac{i}{2}\overline{K_j} w|^2.$$

Since $\{z_j, w\}$ are independent variables, this can only happen if and only if

$$\Im(K_n) + \sum_{j=1}^{n-1} \frac{|K_j|^2}{4} < 0.$$

This proves the lemma. $\square$

We first consider Case (I). Here, we need only to find out $\mu_1, \mu_2, \lambda_1, ..., \lambda_5 \in \mathbb{C}$ such that

$$4\Im(\mu_2) + |\mu_1|^2 < 0, \qquad 4\Im(\lambda_5) + \sum_{j=1}^{4} |\lambda_j|^2 < 0$$

and

$$\lambda_1(tz + \frac{i}{2}zw) + \lambda_2 z^2 + \lambda_3 c_1 zw + \lambda_5 tw + (t^2 + e_2 w^2) = (t + \mu_1 z + \mu_2 w)^2,$$

$\forall [z : w : t] \in \mathbb{CP}^2$. It is easy to verify that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \mu_1 = 0$, $\lambda_5 = -2\sqrt{|e_2|}i$ and $\mu_2 = -\sqrt{|e_2|}i$ satisfy the above conditions. Hence in Case (I), the map is always equivalent to a holomorphic polynomial map in $Poly(\mathbb{B}^2, \mathbb{B}^5)$.

We next consider the second case. Similar to Case (I), it suffices for us to find

$$\mu_1, \mu_2, \lambda_1, ..., \lambda_5 \in \mathbb{C}$$

such that

$$4\Im(\mu_2) + |\mu_1|^2 < 0, \qquad 4\Im(\lambda_5) + \sum_{j=1}^{4} |\lambda_j|^2 < 0$$

and

$$\begin{aligned}&\lambda_1(tz + i(\frac{1}{2} + e_1)zw) + \lambda_2 z^2 + \lambda_3 c_1 zw + \lambda_4 c_3 w^2 + \lambda_5(tw + ie_1 w^2)\\ &+(t^2 + ie_1 tw + e_2 w^2) = (t + \mu_1 z + \mu_2 w)^2, \quad \forall [z : w : t] \in \mathbb{CP}^2.\end{aligned}$$

Comparing the coefficients, we get

$$\begin{aligned}&\lambda_1 = 2\mu_1,\ \lambda_2 = \mu_1^2,\ \lambda_3 = \frac{1}{c_1}[-i(1 + 2e_1)\mu_1 + 2\mu_1\mu_2],\\ &\lambda_4 = \frac{1}{c_3}(\mu_2^2 - e_2 - 2ie_1\mu_2 - e_1^2),\ \lambda_5 = 2\mu_2 - ie_1.\end{aligned}$$

By Theorem 3.2.5 and Remark 3.2.4, we thus obtain the following statement:

$\rho_N \circ F \circ \rho_n^{-1}$ is equivalent to a holomorphic polynomial map if and only if there are $\mu_1, \mu_2 \in \mathbb{C}$ such that $4\Im(\mu_2) + |\mu_1|^2 < 0$ and that

$$-4e_1+8\Im(\mu_2)+4|\mu_1|^2+|\mu_1|^4+\frac{1}{c_1^2}|2\mu_1\mu_2-i(1+2e_1)\mu_1|^2+\frac{1}{c_3^2}|\mu_2^2-e_2-e_1^2-2ie_1\mu_2|^2 < 0.$$

We will look for μ_1 and μ_2 with $\mu_1 = 0$ and $\mu_2 = iy$ $(y < 0)$.

To prove that $\rho_N \circ F \circ \rho_n^{-1}$ is equivalent to a polynomial map, it suffices for us to show that there exists $y < 0$ such that

$$-4e_1 + 8y + \frac{1}{c_3^2}(-y^2 - e_2 - e_1^2 + 2e_1 y)^2 < 0,$$

or

$$J(y) := (-4e_1+8y)e_1e_2+(y^2-2e_1y+e_1^2+e_2)^2 = (8y-4e_1)e_1e_2+((y-e_1)^2+e_2)^2 < 0.$$

Notice that as a function in $y < 0$,

$$\lim_{y\to-\infty} J(y) = +\infty,\ J(0) = (e_1^2 - e_2)^2 > 0.$$

We need to show that

$$\min_{y\le 0} J(y) < 0.$$

Notice that $J'(y) = 8e_1e_2 + 4((y-e_1)^2 + e_2)(y-e_1)$. Setting $J'(y) = 0$, we get

$$(y-e_1)^3 + e_2(y-e_1) + 2e_1e_2 = 0.$$

$J'(y) = 0$ thus has a root $y_0 \in (-\infty, 0)$; for

$$\lim_{y\to-\infty} J'(y) = -\infty, \quad J'(0) = 4(-e_1^3 + e_1e_2) > 0.$$

Let $\zeta_0, \zeta_1, \zeta_2$ be the solution of

$$\zeta^3 + e_2\zeta + 2e_1e_2 = 0 \ \ \textit{with}\ \zeta_0 = y_0 - e_1.$$

Then $\zeta_0+\zeta_1+\zeta_2 = 0$, $\zeta_0\zeta_1+\zeta_0\zeta_2+\zeta_1\zeta_2 = e_2$ and $\zeta_0\zeta_1\zeta_2 = -2e_1e_2$. Hence $\zeta_0 = -\zeta_1-\zeta_2$. We get

$$-\zeta_0^2 + \zeta_1\zeta_2 = e_2,$$

or $\zeta_1\zeta_2 = e_2 + \zeta_0^2$, and

$$\frac{1}{\zeta_1\zeta_2} = -\frac{\zeta_0}{2e_1e_2}.$$

In particular, $\frac{1}{\zeta_1\zeta_2} \in \mathbb{R}\backslash\{0\}$.

Now $J(y_0) = (-4e_1 + 8\zeta_0 + 8e_1)e_1e_2 + (\zeta_0^2 + e_2)^2 = 2e_1e_2(4\zeta_0 + 2e_1) + (\zeta_1\zeta_2)^2 = -\zeta_0\zeta_1\zeta_2(4\zeta_0 + 2e_1) + (\zeta_1\zeta_2)^2$.

Notice that $4\zeta_0^3 = -8e_1e_2 - 4e_2\zeta_0$. We see that

$$\begin{aligned} 2e_1e_2\frac{J(y_0)}{(\zeta_1\zeta_2)^2} &= 2e_1e_2 + \zeta_0^2(4\zeta_0 + 2e_1) = 2e_1e_2 - 8e_1e_2 - 4e_2\zeta_0 + 2e_1\zeta_0^2 \\ &= -6e_1e_2 - 4e_2\zeta_0 + 2e_1\zeta_0^2 = -2e_2(3e_1 + 2\zeta_0) + 2e_1\zeta_0^2. \end{aligned}$$

Since $\zeta_0 = y_0 - e_1 < -e_1$, $3e_1 + 2\zeta_0 < e_1 < 0$. Therefore $\frac{J(y_0)}{(\zeta_1\zeta_2)^2}2e_1e_2 < 0$. Hence we showed that $J(y_0) < 0$. This also completes the proof of Theorem 3.3.1. $\square$

Our proof of Theorem 3.3.1 is, in fact, a constructive proof, which can be used to find precisely polynomial maps equivalent to the original ones. In the following, we demonstrate this by giving an explicit example:

Proposition 3.3.3 *Let* $F = (f, \phi_1, \phi_2, \phi_3, g) : \mathbb{H}^2 \to \mathbb{H}^5$ *be defined as follows:*

$$f(z,w) = \frac{z - \frac{i}{2}zw}{1 - iw - \frac{1}{3}w^2}, \ \phi_1(z,w) = \frac{z^2}{1 - iw - \frac{1}{3}w^2},$$

$$\phi_2(z,w) = \frac{\sqrt{\frac{13}{12}}zw}{1 - iw - \frac{1}{3}w^2}, \ \phi_3(z,w) = \frac{\frac{\sqrt{3}}{3}w^2}{1 - iw - \frac{1}{3}w^2}, \ g(z,w) = \frac{w - iw^2}{1 - iw - \frac{1}{3}w^2}.$$

It is equivalent to the proper polynomial holomorphic map G *from* $\mathbb{B}^2$ *into* $\mathbb{B}^5$*:*

$$G(z,w) = \left(\frac{\sqrt{3}}{9}(-2+4z+z^2), -\frac{\sqrt{6}}{9}(1+z+z^2), \frac{\sqrt{3}}{12}(5+3z)w, \frac{\sqrt{6}}{6}w^2, \frac{\sqrt{13}}{12}i(1-z)w\right).$$

Proof of Proposition 3.3.3: In fact, for the map F given above, $e_1 = -1$, $e_2 = -\frac{1}{3}$, $c_1 = \sqrt{\frac{13}{12}}$, $c_3 = \frac{\sqrt{3}}{3}$. From the proof of Theorem 3.3.1, the hyperplanes $H \subset \mathbb{CP}^2$, $H' \subset \mathbb{CP}^5$ are defined by

$$H : t = -y_0 iw, \ \textit{or} \ \frac{w}{t} = \frac{i}{y_0},$$
$$H' : t' = -\lambda_4 z_4' - \lambda_5 w', \ \textit{or} \ -\lambda_4 \frac{z_4'}{t'} - \lambda_5 \frac{w'}{t'} = 1.$$

Here $y_0 < 0$ is a solution for $(y_0+1)^3 - \frac{1}{3}(y_0+1) + \frac{2}{3} = 0$, $\lambda_4 = \frac{1}{c_3}[-(y_0-e_1)^2 - e_2] = -\frac{(y_0-e_1)^2+e_2}{\sqrt{e_1e_2}}$ and $\lambda_5 = 2iy_0 - e_1 i$. Therefore $y_0 = -2$, $\lambda_4 = -\frac{2}{\sqrt{3}}$ and $\lambda_5 = -3i$. Thus we see that

$$H : t = 2iw, \ \textit{or} \ \frac{w}{t} = \frac{1}{2i},$$
$$H' : t' = \frac{2}{\sqrt{3}}z_4' + 3iw', \ \textit{or} \ \frac{2}{\sqrt{3}}\frac{z_4'}{t'} + \frac{3iw'}{t'} = 1.$$

Consider $\tilde{F} := \rho_5 \circ F \circ \rho_2^{-1} : \mathbb{B}^2 \to \mathbb{B}^5$ where ρ_i are the corresponding Cayley transformations. An easy computation shows that the projectivization of $\tilde{F}$, denoted by $\hat{\tilde{F}}$, is as follows:

$$\hat{\tilde{F}}([z : w : t]) = \Big[z(3t+w) : 2z^2 : 2i\sqrt{\frac{13}{12}}z(t-w) : -\frac{2\sqrt{3}}{3}(t-w)^2$$
$$: \frac{1}{3}(t^2 + 10tw + w^2) : \frac{1}{3}(13t^2 - 2tw + w^2)\Big]$$

and

$$\hat{\tilde{H}} := \hat{\rho}_2(H) : t = \frac{1}{3}w,$$
$$\hat{\tilde{H}}' := \hat{\rho}_5(H') : t' = \frac{\sqrt{3}}{6}z_4' + \frac{1}{2}w'.$$

We have $\hat{\hat{H}} \subset \mathbb{CP}^2$ and $\hat{\hat{H}}' \subset \mathbb{CP}^5$, that satisfy the property:

$$\hat{\hat{H}} \cap \overline{\mathbb{B}_1^2} = \emptyset, \;\; \hat{\hat{H}}' \cap \overline{\mathbb{B}_1^5} = \emptyset \text{ and}$$

$$\hat{\hat{F}}(\hat{\hat{H}}) \subset \hat{\hat{H}}', \quad \hat{\hat{F}}(\mathbb{CP}^2 \backslash \hat{\hat{H}}) \subset \mathbb{CP}^5 \backslash \hat{\hat{H}}'.$$

According to Lemma 3.2.2, let

$$\begin{aligned} \hat{\sigma}_1([z:w:t]) &= \left[\frac{2\sqrt{2}}{3} w : z + \frac{t}{3} : t + \frac{z}{3}\right] \\ \hat{\sigma}_2([z_1':z_2':z_3':z_4':w':t']) &= \left[\frac{1}{2}(z_4' + \sqrt{3}w') - \frac{\sqrt{3}}{3}t' : \frac{\sqrt{6}}{6}(w' - \sqrt{3}z_4') \right. \\ &\quad \left. : \frac{\sqrt{6}}{3}z_1' : \frac{\sqrt{6}}{3}z_2' : \frac{\sqrt{6}}{3}z_3' : t' - \frac{\sqrt{3}}{6}(z_4' + \sqrt{3}w')\right], \end{aligned}$$

then $\hat{\sigma}_1 \in U(3,1)$ and $\hat{\sigma}_2 \in U(6,1)$ with $\hat{\sigma}_1(\hat{\hat{H}}_\infty) = \hat{\hat{H}}$ and $\hat{\sigma}_2(\hat{\hat{H}}') = \hat{\hat{H}}'_\infty$. The desired proper polynomial holomorphic map G is thus induced by $\hat{\sigma}_2 \circ \hat{\hat{F}} \circ \hat{\sigma}_1$, which has the expression given in Proposition 3.3.3. $\square$

Remark 3.3.4 *: It may be interesting to notice that the map G in Proposition 3.3.3 does not preserve the origin and is not equivalent to a map of the form $(G', 0)$. We do not know other examples of proper polynomial maps between balls of this type.*

3.4 Examples of proper rational holomorphic maps that are not equivalent to proper polynomial maps

In this section, we apply Theorem 3.2.3 to construct examples of rational holomorphic maps which are not equivalent to proper holomorphic polynomial maps.

Example 3.4.1 *: Let $G(z,w) = \left(z^2, \sqrt{2}zw, w^2(\frac{z-a}{1-\overline{a}z}, \frac{\sqrt{1-|a|^2}w}{1-\overline{a}z})\right)$, $|a| < 1$, be a map in $Rat(\mathbb{B}^2, \mathbb{B}^4)$. Then G is equivalent to a proper holomorphic polynomial map in $Poly(\mathbb{B}^2, \mathbb{B}^4)$ if and only if $a = 0$.*

Proof of Example 3.4.1: Indeed, we have

$$\hat{G} = \left[(t - \overline{a}z)z^2 : (t - \overline{a}z)\sqrt{2}zw : w^2(z - at) : w^2\sqrt{1 - |a|^2}w : (t^3 - \overline{a}t^2 z)\right].$$

Suppose there exist hyperplanes $H = \{\mu_1 z_1 + \mu_2 w + \mu_0 t = 0\} \subset \mathbb{CP}^2$ and $H' = \{\sum_{j=1}^4 \lambda_j z'_j + \lambda_0 t' = 0\} \subset \mathbb{CP}^4$ such that

$$H \cap \overline{\mathbb{B}_1^2} = \emptyset,\ H' \cap \overline{\mathbb{B}_1^4} = \emptyset,\ \hat{G}(H \backslash Sing(\hat{G})) \subset H',\ \ \hat{G}\left(\mathbb{CP}^2 \backslash (H \cup Sing(\hat{G}))\right) \subset \mathbb{CP}^4 \backslash H'.$$

Then

$$\lambda_1(t - \overline{a}z)z^2 + \lambda_2(t - \overline{a}z)\sqrt{2}zw + \lambda_3 w^2(z - at) + \lambda_4 w^2\sqrt{1 - |a|^2}w$$
$$+\lambda_0(t^3 - \overline{a}t^2 z) = (\mu_1 z + \mu_2 w + \mu_0 t)^3 \quad \forall [z : w : t] \in \mathbb{CP}^2.$$

Apparently $\lambda_0 \neq 0$. Hence we can assume that $\lambda_0 = 1, \mu_0 = 1$. By comparing the coefficient of $z^3, w^3, wt^2, zt^2, z^2t, zwt, z^2w, zw^2, w^2t$, respectively, in the above equation, we get

$$\mu_1^3 = -\overline{a}\lambda_1,\ \mu_2^3 = \lambda_4\sqrt{1 - |a|^2},\ 3\mu_2 = 0,\ 3\mu_1 = -\overline{a},\ 3\mu_1^2 = \lambda_1,$$
$$6\mu_1\mu_2 = \sqrt{2}\lambda_2,\ 3\mu_1^2\mu_2 = -\sqrt{2}\lambda_2\overline{a},\ 3\mu_1\mu_2^2 = \lambda_3,\ 3\mu_2^2 = -a\lambda_3.$$

We then have $\lambda_2 = \lambda_3 = \lambda_4 = \mu_2 = 0$. If $a \neq 0$, then $\mu_1,\ \lambda_1 \neq 0$. From $\mu_1^3 = -\overline{a}\lambda_1$ and $3\mu_1^2 = \lambda_1$, we get $\mu_1 = -3\overline{a}$. Since $3\mu_1 = -\overline{a}$, we get $\overline{a} = 0$. This is a contradiction. Notice that when $a = 0$, F is a polynomial. By Theorem 3.2.3, we see the conclusion. $\square$

Example 3.4.2 *: Let* $F(z', w) = \left(z', wz', w^2(\frac{\sqrt{1-|a|^2}z'}{1-\overline{a}w}, \frac{w-a}{1-\overline{a}w})\right)$ *with* $|a| < 1$ *be a map in* $Rat(\mathbb{B}^n, \mathbb{B}^{3n-2})$*. Then* F *has geometric rank* 1 *and is linear along each hyperplane defined by* $w = constant$*.* F *is equivalent to a proper polynomial map in* $Poly(\mathbb{B}^n, \mathbb{B}^{3n-2})$ *if and only if* $a = 0$.

Proof of Example 3.4.2: The projectivization of F is

$$\hat{F} = \left[tz'(t - \overline{a}w) : twz' : w^2\sqrt{1 - |a|^2}z' : w^2(w - at) : t^2(t - \overline{a}w)\right].$$

Assume $a \neq 0$ and suppose there exist hyperplanes $H \subset \mathbb{CP}^n$ and $H' \subset \mathbb{CP}^{3n-2}$ such that

$$H \cap \overline{\mathbb{B}_1^n} = \emptyset,\ H' \cap \overline{\mathbb{B}_1^{3n-2}} = \emptyset,\ \hat{F}(H \backslash Sing(\hat{F})) \subset H',\ \ \hat{F}\left(\mathbb{CP}^n \backslash (H \cup Sing(\hat{F}))\right) \subset \mathbb{CP}^{3n-2} \backslash H'.$$

Then

$$\lambda_1' tz'(t-\overline{a}w)+\lambda_2' twz'+\lambda_3' w^2\sqrt{1-|a|^2}z'+\lambda_n w^2(w-at)+\lambda_0 t^2(t-\overline{a}w) = (\mu_0 t+\mu' z'+\mu_n w)^3$$

for some $\lambda_1', \lambda_2', \lambda_3', \mu' \in \mathbb{C}^{n-1}$ and $\lambda_n, \lambda_0, \mu_0, \mu_n \in \mathbb{C}$.

Then $\lambda_0 = \mu_0^3 \neq 0$. We thus can assume at the beginning that $\lambda_0 = \mu_0 = 1$.

Since there are no terms like $z_j^3 (j < n)$ on the left hand side, we conclude that $\mu' = 0$.

Thus we get

$$\lambda_n w^2(w - at) + t^2(t - \overline{a}w) = (t + \mu_n w)^3.$$

Therefore $-\overline{a} = 3\mu_n,\ -\lambda_n a = 3\mu_n^2,\ \lambda_n = \mu_n^3$ or $\mu_n = -\frac{\overline{a}}{3}$ and $\mu_n = -\frac{3}{a}$. This contradicts the assumption that $0 < |a|^2 < 1$. $\square$

Chapter 4

Monotonicity for the Chern-Moser-Weyl curvature tensor and CR embeddings

4.1 Introduction

Our research in this direction is motivated by the following problems:

Question 4.1.1 *(Embedding problem): Let M be a strongly pseudoconvex hypersurface in $\mathbb{C}^{n+1}$ with $n \geq 1$ defined by a real polynomial. For any $p \in M$, does there exist a sufficiently large positive integer N, which may depend on p, such that a small piece of M near p can be holomorphically embedded into the Heisenberg hypersurface $\mathbb{H}^{N+1}$ (with signature 0)?*

Question 4.1.2 *(CR transversality problem): Let M and $\widetilde{M}$ be smooth Levi non-degenerate hypersurfaces in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{N+1}$, respectively, with $N > n$. Assume that both M and $\widetilde{M}$ have the same signature ℓ with $0 < \ell \leq [\frac{n}{2}]$. Let U be a (connected) neighborhood of M in $\mathbb{C}^{n+1}$. Suppose that F is not a totally degenerate holomorphic map from U into $\mathbb{C}^{N+1}$ with $F(M) \subset \widetilde{M}$. Is then F CR transversal (or equivalently a local holomorphic embedding) along M?*

Here recall a Levi non-degenerate smooth hypersurface M_ℓ in $\mathbb{C}^{n+1}$ is of signature ℓ near the origin if it locally is defined by an equation of the form: $r(z,w) = \Im w - \sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n} |z_j|^2 + \circ(|z|^2 + |z \cdot \Re w|) = 0$ for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.

Question 4.1.1 is more along the lines of the general embedding problem, which asks when a Levi non-degenerate hypersurface M_ℓ in $\mathbb{C}^{n+1}$ of signature ℓ with $0 \leq \ell \leq n/2$ can be embedded into a hyperqradric $\mathbb{H}_\ell^{N+1}$ in $\mathbb{C}^{N+1}$ of the same signature for $N >> n$.

By the general invariant theory and a Baire category argument, Forstnerič [For2] showed that there exist real analytic strictly pseudoconvex hypersurfaces in $\mathbb{C}^{n+1}$ which cannot be smoothly embedded into $\mathbb{H}^{N+1}$ for any N. (See also a recent paper of Zaitsev [Zait] on the related issue). On the other hand, 30 years ago, Webster in [We] showed that a Levi non-degenerate hypersurface in $\mathbb{C}^{n+1}$ of signature ℓ, defined by a real polynomial, can always be embedded into the hyperquadric $\mathbb{H}^{n+2}_{\ell+1}$ of signature $\ell+1$ but in the $(n+2)$-complex space. This has then led to an interesting open problem to understand whether any algebraic Levi non-degenerate hypersurface in $\mathbb{C}^{n+1}$ can be embedded into a hyperquadric of the same signature but in a much higher dimensional complex space.

In this chapter, jointly with Huang, we gave a checkable necessary condition whether M_ℓ can be embedded into $\mathbb{H}^{N+1}_\ell$ when $\ell \in (0, [n/2]]$. Our criterion is based on a monotonicity property for the Chern-Moser-Weyl tensor along the cone defined by tangent vectors of type (1,0) in the null space of the Levi form. Roughly speaking, our monotonicity property says that a CR embedding from a Levi non-degenerate hypersurface into another one with the same signature decreases the Chern-Moser-Weyl curvature. This phenomenon may be compared with various monotonicity properties for (some type of) curvatures under the application of holomorphic maps, initiated from the classical Ahlfors-Pick-Schwarz lemma (see [GH] and [Yau], for instance). In the CR setting, the natural curvature tensor to be considered is the Chern-Moser-Weyl curvature tensor and the mappings to be involved are CR mappings. Unfortunately, there is no monotonicity phenomenon in general. Our crucial observation is that the monotonicity exists along directions in the null space of the Levi-form. Since the null space of the Levi-form may be regarded as the 'largest' holomorphic subspace inside $T^{(1,0)}M$, our result may be considered as a generalization of those results on complex manifolds. In our investigation, we have to exclude the important strongly pseudoconvex case: $\ell = 0$; for the null space of the Levi-form in this setting is the 0-space.

Since the hyperquadrics have vanishing Chern-Moser-Weyl tensor, our criterion makes it possible to construct many algebraic Levi non-degenerate hypersurfaces which can not be embedded into a hyperquadric of the same signature $\ell > 0$ in a complex space of higher dimension. However, Question 4.1.1 still remains open.

Question 4.1.2 is to ask when a holomorphic map between Levi non-degenerate real analytic hypersurfaces of complex spaces of different dimensions is CR transversal. The problem follows trivially from the classical Hopf lemma when the signature is 0. However, it appears to be a quite difficult problem in the general signature setting, even in the hyperquadrics case (See [BH], [BR], [BR2], [BER], [BER2] etc). In the following sections, we proposed a geometric method to approach this problem. We showed that the Hopf lemma can fail at most at those points with a pseudo semi-negative Chern-Moser-Weyl curvature tensor.

4.2 Chern-Moser-Weyl curvature tensor on a Levi non-degenerate hypersurface

We use $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ for the coordinates of $\mathbb{C}^{n+1}$. We always assume that $n \geq 2$ and $\ell \leq n/2$. Let M be a smooth real hypersurface. Recall that M is Levi non-degenerate at $p \in M$ with signature $\ell \leq n/2$ if there is a local holomorphic change of coordinates, that maps p to the origin, such that in the new coordinates, M is defined near 0 by an equation of the form:

$$r = v - |z|_\ell^2 + o(|z|^2 + |zu|) = 0 \tag{4.1}$$

Here, we write $u = \Re w, v = \Im w$ and $\langle a, \bar{b}\rangle_\ell = -\sum_{j\leq \ell} a_j \bar{b}_j + \sum_{j=\ell+1}^n a_j \bar{b}_j, |z|_\ell^2 =< z, \bar{z} >_\ell$. When $\ell = 0$, we regard $\sum_{j\leq\ell} a_j = 0$.

Assume that M is Levi non-degenerate with the same signature ℓ at any point. A contact form θ over M is said to be appropriate if the Levi form $L_{\theta|_p}$ associated with θ at any point $p \in M$ has ℓ negative eigenvalues and $n - \ell$ positive eigenvalues. (See (4.2) for our definition of the Levi form.) Since our consideration in this chapter is local, we only focus on a small piece of M with $0 \in M$ and M is defined by an equation as in (4.1). In particular, $\theta_0 = i\partial r$ is appropriate near 0. When $\ell < n/2$, a contact form θ is appropriate if and only if $\theta = k_0\theta_0$ with $k_0 > 0$.

Let θ be an appropriate contact form over M. Then from the Chern-Moser Theory, there is a unique 4th order curvature tensor $\mathcal{S}_\theta$ associated with θ ([CM], [We]), which we call the Chern-Moser-Weyl tensor with respect to the contact form θ over M or along the contact form θ. (One can also use (4.3) and (4.4) as the definition of $\mathcal{S}|_{\theta|_0}$. The invariant property or the transformation law is given in (4.18).) $\mathcal{S}_\theta$ can be regarded as a section over $T^{*(1,0)}M \otimes T^{*(0,1)}M \otimes T^{*(1,0)}M \otimes T^{*(0,1)}M$. We write $\mathcal{S}_{\theta|_p}$ for the restriction of $\mathcal{S}_\theta$ at $p \in M$. For a basis $\{X_\alpha\}_{\alpha=1}^n$ of $T_p^{(1,0)}M$ with $p \in M$, write $(S_{\theta|_p})_{\alpha\bar\beta\gamma\bar\delta} = \mathcal{S}_{\theta|_p}(X_\alpha, \overline{X}_\beta, X_\gamma, \overline{X}_\delta)$. We then have the following symmetric properties:

$$(S_{\theta|_p})_{\alpha\bar\beta\gamma\bar\delta} = (S_{\theta|_p})_{\gamma\bar\beta\alpha\bar\delta} = (S_{\theta|_p})_{\gamma\bar\delta\alpha\bar\beta}$$
$$\overline{(S_{\theta|_p})_{\alpha\bar\beta\gamma\bar\delta}} = (S_{\theta|_p})_{\beta\bar\alpha\delta\bar\gamma},$$

and the following trace-free condition:

$$\sum_{\beta,\alpha=1}^{n} g^{\bar\beta\alpha}(S_{\theta|_p})_{\alpha\bar\beta\gamma\bar\delta} = 0.$$

Here

$$g_{\alpha\bar\beta} = L_{\theta|_p}(X_\alpha, X_\beta) := -i < d\theta|_p, X_\alpha \wedge \overline{X}_\beta >= - < \partial\bar\partial r|_p, X_\alpha \wedge \overline{X}_\beta > \qquad (4.2)$$

is the Levi form of M associated with θ at $p \in M$ and $(g^{\bar\beta\alpha})$ is the inverse matrix of $(g_{\alpha\bar\beta})$. For a different contact form $\tilde\theta = \tilde k\theta$ smooth along M with $\tilde k > 0$, we have the following transformation formula:

$$\mathcal{S}_{\tilde\theta|_p}(X_\alpha, \overline{X}_\beta, X_\gamma, \overline{X}_\delta) = \tilde k \mathcal{S}_{\theta|_p}(X_\alpha, \overline{X}_\beta, X_\gamma, \overline{X}_\delta)$$

This implies $\mathcal{S}$ is conformal invariant with respect the metric $g_{\alpha\bar\beta}$. For a smooth vector field X, Y, Z, W of type $(1,0)$ and a smooth contact form along M, $\mathcal{S}_\theta(X, \bar Y, Z, \bar W)$ is also a smooth function along M. One easy way to see this is to use the Webster-Chern-Moser-Weyl formula obtained in [We] through the curvature tensor of the Webster pseudo-Hermitian metric, whose constructions are done by only applying the algebraic and differentiation operations on the defining function of M.

$\mathcal{S}_\theta$ is described in terms of the normal coordinates for M as follows: First, by the Chern-Moser normal form theory [CM], we can find a coordinate in which M is defined near 0 by an equation of the following form (see (6.25), (6.30), [CM]):

$$r = v - |z|_\ell^2 + \frac{1}{4}s(z,\bar z) + o(|z|^4) = v - |z|_\ell^2 + \frac{1}{4}\sum s_{\alpha\bar\beta\gamma\bar\delta} z_\alpha \bar z_\beta z_\gamma \bar z_\delta + o(|z|^4) = 0. \quad (4.3)$$

Here $s(z,\bar z)=\sum s_{\alpha\bar\beta\gamma\bar\delta}z_\alpha\bar z_\beta z_\gamma\bar z_\delta$, $i\partial r|_0=\theta|_0$, $s_{\alpha\bar\beta\gamma\bar\delta}=s_{\gamma\bar\beta\alpha\bar\delta}=s_{\gamma\bar\delta\alpha\bar\beta}$, $\overline{s_{\alpha\bar\beta\gamma\bar\delta}}=s_{\beta\bar\alpha\delta\bar\gamma}$ and $\sum_{\alpha,\beta=1}^n s_{\alpha\bar\beta\gamma\bar\delta}g^{\bar\beta\alpha}=0$ where $g^{\bar\beta\alpha}=0$ for $\beta\neq\alpha$, $g^{\bar\beta\beta}=1$ for $\beta>\ell, g^{\bar\beta\beta}=-1$ for $\beta\le\ell$. Then

$$\mathcal{S}_{\theta|_0}(\frac{\partial}{\partial z_\alpha}|_0,\frac{\partial}{\partial\bar z_\beta}|_0,\frac{\partial}{\partial z_\gamma}|_0,\frac{\partial}{\partial\bar z_\delta}|_0)=s_{\alpha\bar\beta\gamma\bar\delta}. \tag{4.4}$$

Write $\triangle_\ell=-\sum_{j\le\ell}\frac{\partial^2}{\partial z_j\partial\bar z_j}+\sum_{j=\ell+1}^n\frac{\partial^2}{\partial z_j\partial\bar z_j}$ and also write $s_{\theta|_0}(z,\bar z)$ for $s(z,\bar z)$. Then the trace-free condition above is equivalent to

$$\triangle_\ell s_{\theta|_0}(z,\bar z)\equiv 0.$$

Indeed, this follows from the following fact: Let $\Delta_H=\sum_{l,k=1}^n h^{l\bar k}\partial_l\bar\partial_k$ with $\overline{h^{l\bar k}}=h^{k\bar l}$ for any l,k. Then

$$\Delta_H s_{\theta|_0}(z,\bar z)=4\sum_{\gamma,\delta=1}^n\sum_{\alpha,\beta=1}^n h^{\alpha\bar\beta}s_{\alpha\bar\beta\gamma\bar\delta}z_\gamma\bar z_\delta. \tag{4.5}$$

For the rest of this section, we assume that $\ell>0$ and define

$$\mathcal{C}_\ell=\{z\in\mathbb{C}^n:|z|_\ell=0\}.$$

Then $\mathcal{C}_\ell$ is a real algebraic variety of real codimension 1 in $\mathbb{C}^n$ with the only singularity at 0. For each $p\in M$, write $\mathcal{C}_\ell T_p^{(1,0)}M=\{v_p\in T_p^{(1,0)}M:\ \langle d\theta_p,v_p\wedge\bar v_p\rangle=0\}$. Apparently, $\mathcal{C}_l T_p^{(1,0)}M$ is independent of the choice of θ. Let F be a CR diffeomorphism from M to M'. We also have $F_*(\mathcal{C}_\ell T_p^{(1,0)}M)=\mathcal{C}_\ell T_{F(p)}^{(1,0)}M'$. (We will explain this in details in the later discussion). Write $\mathcal{C}_\ell T^{(1,0)}M=\coprod_{p\in M}\mathcal{C}_\ell T_p^{(1,0)}M$ with the natural projection π to M. We say that X is a smooth section of $\mathcal{C}_\ell T^{(1,0)}M$ if X is a smooth vector field of type $(1,0)$ along M such that $X|_p\in\mathcal{C}_\ell T_p^{(1,0)}M$ for each $p\in M$. Later, we will see that $\mathcal{C}_\ell T^{(1,0)}M$ is a kind of smooth bundle with each fiber isomorphic to $\mathcal{C}_\ell$. (See Remark 4.3.5.)

We say that the Chern-Moser-Weyl curvature tensor $\mathcal{S}_\theta$ is pseudo semi-positive definite (or pseudo semi-negative definite) at $p\in M$ if $\mathcal{S}_{\theta|_p}(X,\bar X,X,\bar X)\ge 0$ for any $X\in\mathcal{C}_\ell T_p^{(1,0)}M$ (or $\mathcal{S}_{\theta|_p}(X,\bar X,X,\bar X)\le 0$, respectively, for all $X\in\mathcal{C}_\ell T_p^{(1,0)}M$). We say that $\mathcal{S}_\theta$ is pseudo positive-definite (or pseudo negative-definite) at $p\in M$ if $\mathcal{S}_{\theta|_p}(X,\bar X,X,\bar X)>0$ for all $X\in\mathcal{C}_\ell T_p^{(1,0)}M\setminus 0$ (or $\mathcal{S}_{\theta|_p}(X,\bar X,X,\bar X)<0$, respectively, for all $X \in \mathcal{C}_\ell T_p^{(1,0)} M \setminus 0)$. We use the terminology pseudo semi-definite to mean either pseudo semi-positive definite or pseudo semi-negative definite. We can similarly define the notion of pseudo definiteness.

$\mathcal{C}_\ell$ is obviously a uniqueness set for holomorphic functions. The following lemma shows that it is also a uniqueness set for the Chern-Moser-Weyl curvature tensor.

Lemma 4.2.1 *(I). Suppose that $H(z,\bar{z})$ is a real real-analytic function in $(z,\bar{z})$ near 0. Assume that $\triangle_\ell H(z,\bar{z}) \equiv 0$ and $H(z,\bar{z})|_{\mathcal{C}_\ell} = 0$. Then $H(z,\bar{z}) \equiv 0$ near 0.*

II). Assume the above notation. If $\mathcal{S}_{\theta|_p}(X,\bar{X},X,\bar{X}) = 0$ for any $X \in \mathcal{C}_\ell T_p^{(1,0)} M$, then $\mathcal{S}_{\theta|_p} \equiv 0$.

Proof of Lemma 4.2.1: *(I)* Write $H(z,\bar{z}) = \sum_{m=1}^{\infty} H^{(m)}(z,\bar{z})$ with $H^{(m)}(z,\bar{z})$ homogeneous polynomials in $(z,\bar{z})$ of degree m. Then we easily see that $\triangle_\ell H(z,\bar{z}) \equiv 0$ if and only if $\triangle_\ell H^{(m)}(z,\bar{z}) \equiv 0$ for each m. For $q \in \mathcal{C}_\ell$, since $tq \in \mathcal{C}_\ell$ for $t \in \mathbb{R}^+$, we see that $H(tp,\overline{tp}) = \sum_{m=1}^{\infty} t^m H^{(m)}(p,\bar{p})$ and $H(tp,\overline{tp}) = 0$ for each $t \in \mathbb{R}$ if and only if $H^{(m)}(p,\bar{p}) = 0$ for each m. Hence we see that $H(z,\bar{z})|_{\mathcal{C}_\ell} = 0$ if and only if $H^{(m)}(z,\bar{z}) = 0$ along $\mathcal{C}_\ell$ for each m. Therefore, to prove Lemma 2.1, we can assume that $H(z,\bar{z})$ is already a homogeneous polynomial of degree m in $(z,\bar{z})$. Next, notice that

$$\mathrm{V} = \{(z,\xi) \in \mathbb{C}^n \times \mathbb{C}^n : \ \langle z,\xi\rangle_\ell = -\sum_{j=1}^{\ell} z_j\xi_j + \sum_{j=\ell+1}^{n} z_j\xi_j = 0\}$$

is a complex analytic variety defined by $\langle z,\xi\rangle_\ell = 0$ with $\langle z,\xi\rangle_\ell$ irreducible as an element in $\mathscr{O}_{(p,q)}$ for each $(p,q) \in V$. Hence, we easily see that $H(z,\xi) = h(z,\xi)\langle z,\xi\rangle_\ell$ for a certain holomorphic function $h(z,\xi)$ in $(z,\xi) \in \mathbb{C}^n \times \mathbb{C}^n$. Then it follows that $h(z,\xi)$ is a homogeneous polynomial of degree $m-2$. Now by a well-known argument in harmonic analysis (see [SW], pp140), we can prove $H \equiv 0$ as follows:

First, write $H(z,\bar{z}) = \sum_{\alpha+\beta=m} a_{\alpha\bar{\beta}} z^\alpha \bar{z}^\beta$. Then

$$\begin{aligned}\sum_{\alpha+\beta=m} |a_{\alpha\bar{\beta}}|^2 \alpha!\beta! &= H(\partial_z,\partial_{\bar{z}})(H(z,\bar{z})) \\ &= h(\partial_z,\partial_{\bar{z}})(\triangle_\ell(H(z,\bar{z}))) \\ &= 0\end{aligned}$$

Thus $H(z,\bar{z}) \equiv 0$.

(II): By the transformation law for the Chern-Moser-Weyl curvature tensor, we can assume that $p = 0$ and M near 0 is given in normal coordinates as in (4.3) with $\theta|_0 = i\partial r$. Write $X = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}|_0$. Then $X \in \mathcal{C}_\ell T_0^{(1,0)} M$ if and only if $|z|_\ell = 0$. Moreover $\mathcal{S}_{\theta|_0}(X,\bar{X},X,\bar{X}) = s_{\theta|_0}(z,\bar{z})$ with $\Delta_\ell s_{\theta|_0}(z,\bar{z}) \equiv 0$. Now, since $s_{\theta|_0}(z,\bar{z}) = 0$ for $|z|_\ell = 0$, we have, by Part I of the Lemma, $s_{\theta|_0}(z,\bar{z}) = 0$ for any z. Namely, $\mathcal{S}_{\theta|_0}(X,\bar{X},X,\bar{X}) \equiv 0$. This then immediately shows that $\mathcal{S}_{\theta|_0} \equiv 0$. $\Box$

Write $\mathbb{H}_\ell^{n+1} := \{(z,w) \in \mathbb{C}^n \times \mathbb{C} : \Im w = \langle z,\bar{z}\rangle_\ell\}$ for the Levi non-degenerate real hyperquadric with signature $\ell > 0$. By the Chern-Moser theory, M is locally CR equivalent to $\mathbb{H}_\ell^{n+1}$ if and only if $\mathcal{S}_\theta \equiv 0$. Together with the above lemma, we have the following:

Theorem 4.2.2 *Let M be a Levi non-degenerate hypersurface of signature ℓ with $0 < \ell \le \frac{n}{2}$. Then M is locally CR equivalent to the hyperquadric $\mathbb{H}_\ell^{n+1}$ of signature ℓ if and only if for any contact form θ and any vector $X_p \in \mathcal{C}_\ell T_p^{(1,0)} M$ with $p \in M$, it holds that $\mathcal{S}_{\theta|_p}(X_p,\bar{X}_p,X_p,\bar{X}_p) = 0$.*

4.3 Monotonicity for the Chern-Moser-Weyl tensor and CR embeddings

Next, we let $\widetilde{M} \subset \mathbb{C}^{N+1} = \{(z,w) \in \mathbb{C}^N \times \mathbb{C}\}$ be also a Levi non-degenerate smooth real hypersurface near 0 of signature $\ell \ge 0$ defined by an equation of the form:

$$\widetilde{r} = \Im\widetilde{w} - |\widetilde{z}|_\ell^2 + o(|\widetilde{z}|^2 + |\widetilde{z}\widetilde{u}|) = 0. \tag{4.6}$$

Assume that $N \ge n$ and let $F := (\tilde{f}, g) = (f_1, \ldots, f_N, g) : M \to \widetilde{M}$ be a smooth CR map. We say that F is CR transversal at a point $p \in M$, if the normal component of F has a non-vanishing normal derivative at p. Assume that $F(0) = 0$. Then F is CR transversal at 0 if and only if $\frac{\partial g}{\partial w}|_0 \ne 0$.

In the consideration of this chapter, namely, when M and $\widetilde{M}$ are both Levi non-degenerate hypersurfaces with the same signature, the CR transversality of F is equivalent to the local embeddability. More precisely, F is CR transversal at p if and only if F is a CR embedding from a small neighborhood of p in M into $\widetilde{M}$. When F extends to a holomorphic map to a neighborhood of p in $\mathbb{C}^{n+1}$, which is automatically the case when $0 < \ell \leq n/2$ by the Lewy extension theorem, this is further equivalent to the property that F is a local holomorphic embedding from a neighborhood of p in $\mathbb{C}^{n+1}$ into $\mathbb{C}^{N+1}$. To see this, we can assume, without loss of generality, that $p = 0$. Since by the classical Hopf lemma, when $\ell = 0$, either F is a constant map or F is a local CR embedding at any point in M, we thus assume that $0 < \ell \leq n/2$. When F is CR transversal at $p = 0$, by the following (4.7), we easily see that F is a local embedding from a neighborhood of 0 in $\mathbb{C}^{n+1}$. Conversely, if F is not CR transversal at 0, then near 0, we have $g = O(|(z,w)|^2)$ and $\tilde{f} = zU + \vec{a}w + O(|(z,w)|^2)$, where U is an $n \times N$ matrix and $\vec{a} \in \mathbb{C}^N$. Since $F(M) \subset \widetilde{M}$, we have

$$\Im g = |\tilde{f}|_\ell^2 + O(3), \ (z,w) \in M.$$

We easily see that $U \cdot E_\ell \cdot \bar{U}^t = 0$. Here E_ℓ is the diagonal matrix with the first ℓ diagonal elements -1 and the rest diagonal elements 1. Hence, by Lemma 4.2 in [BH], the rank of U is strictly less than n. Thus the Jacobian matrix of F at 0 can at most have rank $n < n+1$. Namely, F can not be a holomorphic embedding near 0 in $\mathbb{C}^{n+1}$.

Since the set of points where a holomorphic map fails to be local embedding is a complex analytic variety in a neighborhood of M where F is holomorphic, the above observation has an immediate consequence: When $0 < \ell < n/2$, either F fails to be CR transversal at any point in M or the set of CR non-transversal points of F in M is an intersection of a certain proper holomorphic variety with M and thus is a thin set in M. In particular, when M is real analytic, it has codimension at least 2 in M. Hence, in this situation, the complement of the set of the CR non-transversal points of F is an open dense and connected subset of M. (We always assume $M, \widetilde{M}$ to be connected.)

Now, assume that F is CR transversal at 0. Then, as in §2, [BH], we can write

$$\begin{aligned} \tilde{z} &= \tilde{f}(z,w) = (f_1(z,w),\dots,f_N(z,w)) = \lambda z U + \vec{a} w + O(|(z,w)|^2) \\ \tilde{w} &= g(z,w) = \sigma\lambda^2 w + O(|(z,w)|^2). \end{aligned} \tag{4.7}$$

Here U can be extended to an $N\times N$ matrix $\widetilde{U}\in SU(N,\ell)$ (namely $\langle X\widetilde{U}, Y\overline{\widetilde{U}}\rangle_\ell = \langle X,Y\rangle_\ell$ for any $X,Y\in\mathbb{C}^N$). Moreover, $\vec{a}\in\mathbb{C}^N$, $\lambda>0$ and $\sigma=\pm 1$ with $\sigma=1$ for $\ell<\frac{n}{2}$. When $\sigma=-1$, by considering $F\circ\tau_{n/2}$ instead of F, where $\tau_{\frac{n}{2}}(z_1,\dots,z_{\frac{n}{2}},z_{\frac{n}{2}+1},\dots,z_n,w) = (z_{\frac{n}{2}+1},\dots,z_n,z_1,\dots,z_{\frac{n}{2}},-w)$, we can make $\sigma=1$. Hence, we will assume in what follows that $\sigma=1$.

Write $r_0=\frac{1}{2}\Re\{g''_{ww}(0)\}$, $q(\tilde{z},\tilde{w}) = 1+2i\langle\tilde{z},\lambda^{-2}\overline{\vec{a}}\rangle_\ell + \lambda^{-4}(r_0 - i|\vec{a}|^2_\ell)\tilde{w}$,

$$T(\tilde{z},\tilde{w}) = \frac{(\lambda^{-1}(\tilde{z}-\lambda^{-2}\vec{a}\tilde{w})\widetilde{U}^{-1}, \lambda^{-2}\tilde{w})}{q(\tilde{z},\tilde{w})}. \tag{4.8}$$

Then

$$F^\sharp(z,w) = (\tilde{f}^\sharp, g^\sharp)(z,w) := T\circ F(z,w) = (z,0,w) + O(|(z,w)|^2) \tag{4.9}$$

with $\Re\{g^{\sharp''}_{ww}(0)\}=0$.

Assume that $\widetilde{M}$ is also defined in the Chern-Moser normal form up to the 4th order:

$$\tilde{r} = \Im\tilde{w} - |\tilde{z}|^2_\ell + \frac{1}{4}\tilde{s}(\tilde{z},\bar{\tilde{z}}) + o(|\tilde{z}|^4) = 0. \tag{4.10}$$

Then $M^\sharp = T(\widetilde{M})$ is defined by

$$r^\sharp = \Im w^\sharp - |z^\sharp|^2_\ell + \frac{1}{4}s^\sharp(z^\sharp,\bar{z^\sharp}) + o(|z^\sharp|^4) = 0 \tag{4.11}$$

with $s^\sharp(z^\sharp,\bar{z^\sharp}) = \lambda^{-2}\tilde{s}(\lambda z^\sharp\tilde{U}, \lambda\overline{z^\sharp\tilde{U}}) = \lambda^2\tilde{s}(z^\sharp\tilde{U},\overline{z^\sharp\tilde{U}})$.

One can verify that

$$\Big(-\sum_{j=1}^{\ell}\frac{\partial^2}{\partial z_j^\sharp\partial\bar{z}_j^\sharp} + \sum_{j=\ell+1}^{N}\frac{\partial^2}{\partial z_j^\sharp\partial\bar{z}_j^\sharp}\Big)s^\sharp(z^\sharp,\overline{z^\sharp}) = 0. \tag{4.12}$$

Therefore (4.11) is also in the Chern-Moser normal form up to the 4th order. Now we assign the weight of $z,\bar{z}$ to be 1, and that of w to be 2. We use the standard notation $h^{(k)}$ and $o_{wt}(k)$ to denote terms in the Taylor expansion for the function h of weighted degree k and terms vanishing to the weighted degree higher than k, respectively. Write $F^\sharp(z,w) = \sum_{k=1}^{\infty} F^{\sharp(k)}(z,w)$. Since $F^\sharp$ maps M into $M^\sharp = T(\widetilde{M})$, we get the following

$$\begin{aligned} &\Im\{\sum_{k\geq 2} g^{\sharp(k)}(z,w) - 2i\sum_{k\geq 2}\langle f^{\sharp(k)}(z,w), \bar{z}\rangle_\ell\} \\ &\quad = \sum_{k_1,\ k_2\geq 2} \langle f^{\sharp(k_1)}(z,w), \overline{f^{\sharp(k_2)}(z,w)}\rangle_\ell + \frac{1}{4}(s(z,\bar{z}) - s^\sharp((z,0),\overline{(z,0)})) + o_{wt}(4) \end{aligned} \tag{4.13}$$

over $\Im w = |z|_\ell^2$.

Here, we write $F^\sharp(z,w) = (\tilde{f}^\sharp(z,w), g^\sharp(z,w)) = (f^\sharp(z,w), \phi^\sharp(z,w), g^\sharp(z,w))$.

Collecting terms of weighted degree 3 in (4.13), we get

$$\Im\{g^{\sharp(3)}(z,w) - 2i\langle f^{\sharp(2)}(z,w), \bar{z}\rangle_\ell\} = 0 \ \text{ on } \ \Im w = |z|_\ell^2.$$

By [Hu], we get $g^{\sharp(3)} \equiv 0, f^{\sharp(2)} \equiv 0$.

Collecting terms of weighted degree 4 in (4.13), we get

$$\Im\{g^{\sharp(4)}(z,w) - 2i\langle f^{\sharp(3)}(z,w), \bar{z}\rangle_\ell\} = |\phi^{\sharp(2)}(z)|^2 + \frac{1}{4}(s(z,\bar{z}) - s^\sharp((z,0),\overline{(z,0)})).$$

Similar to the argument in [Hu] and making use of the fact that $\Re\{\frac{\partial^2 g^{\sharp(4)}}{\partial w^2}(0)\} = 0$, we get the following:

$$\begin{aligned} & g^{\sharp(4)} \equiv 0,\ f^{\sharp(3)}(z,w) = \frac{i}{2}a^{(1)}(z)w, \\ & \langle a^{(1)}(z), \bar{z}\rangle_\ell |z|_\ell^2 = |\phi^{\sharp(2)}(z)|^2 + \frac{1}{4}(s(z,\bar{z}) - s^\sharp((z,0),\overline{(z,0)})), \text{ or} \\ & \langle a^{(1)}(z), \bar{z}\rangle_\ell |z|_\ell^2 = |\phi^{\sharp(2)}(z)|^2 + \frac{1}{4}(s(z,\bar{z}) - \lambda^2\widetilde{s}((z,0)\widetilde{U}, \overline{(z,0)\widetilde{U}})). \end{aligned} \tag{4.14}$$

We assume in the following (except in Proposition 4.3.1 and Remark 4.3.2) that $\ell > 0$. Letting $z \in \mathcal{C}_\ell$, we get

$$\begin{aligned} 4|\phi^{\sharp(2)}(z)|^2 &= s^\sharp((z,0),\overline{(z,0)}) - s(z,\bar{z}) \\ &= \lambda^{-2}\widetilde{s}((\lambda z,0)\widetilde{U}, \overline{(\lambda z,0)\widetilde{U}}) - s(z,\bar{z}) \\ &= \lambda^2\widetilde{s}((z,0)\widetilde{U}, \overline{(z,0)\widetilde{U}}) - s(z,\bar{z}). \end{aligned} \tag{4.15}$$

We claim that, for $v_p \in \mathcal{C}_\ell T_p^{(1,0)}M$, $F_*(v_p) \in \mathcal{C}_\ell T_{F(p)}^{(1,0)}\widetilde{M}$ and $F_*^\sharp(v_p) \in \mathcal{C}_\ell T_{F^\sharp(p)}^{(1,0)}M^\sharp$. Indeed, to see this, we need only to notice that for any contact form $\tilde{\theta}$ along $\widetilde{M}$, $F^*(\tilde{\theta})$ is also a contact form of M and

$$\langle d(F^*(\tilde{\theta}))|_p, v_p \wedge \bar{v}_p\rangle = \langle d\tilde{\theta}_{F(p)}, F_*(v_p) \wedge \overline{F_*(v_p)}\rangle.$$

Thus, if $v_p \in \mathcal{C}_\ell T_p^{(1,0)}M$, then $\langle d\tilde{\theta}_{F(p)}, F_*(v_p) \wedge \overline{F_*(v_p)}\rangle = 0$ and hence $F_*(v_p) \in \mathcal{C}_\ell T_{F(p)}^{(1,0)}\widetilde{M}$. Next, if we identify z with the $(1,0)$ vector $v = \sum z_j(\frac{\partial}{\partial z_j}|_0)$, then $(\lambda z, 0)\widetilde{U}$ is identified with the vector $F_*(v)$. Moreover, $z \in \mathcal{C}_\ell$ if and only if $v \in \mathcal{C}_\ell T_0^{(1,0)}M$.
Set $\theta = i\partial r$ and $\tilde{\theta} = i\partial\tilde{r}$. Then

$$F^*(\tilde{\theta})|_0 = \frac{1}{2}dg|_0 = \lambda^2\theta|_0.$$

Write $F^*(\tilde{\theta}) = k\theta$, then $k(0) = \lambda^2$. Hence (4.15) can now be written as:

$$\tilde{\mathcal{S}}_{\tilde{\theta}|_0}(F_*(v), \overline{F_*(v)}, F_*(v), \overline{F_*(v)}) = \lambda^2\mathcal{S}_{\theta|_0}(v, \bar{v}, v, \bar{v}) + 4\lambda^2|\phi^{\sharp(2)}(z)|^2,$$

$$\text{or} \;\; \tilde{\mathcal{S}}_{\tilde{\theta}|_0}(F_*(v), \overline{F_*(v)}, F_*(v), \overline{F_*(v)}) = \mathcal{S}_{F^*(\tilde{\theta})|_0}(v, \bar{v}, v, \bar{v}) + 4\lambda^2|\phi^{\sharp(2)}(z)|^2, \tag{4.16}$$

where $v = \sum_{j=1}^n z_j\frac{\partial}{\partial z_j} \in T_0^{(1,0)}M$. Summarizing the above, we have the following: (In Proposition 4.3.1 and Remark 4.3.2, ℓ can be 0.)

Proposition 4.3.1 *Let M and $\widetilde{M}$ be defined by (4.3) and (4.10), respectively. Let*

$$F(\widetilde{z}, \widetilde{w}) = (\widetilde{f}(z,w), g(z,w)) = (f_1(z,w), \cdots, f_{N-1}(z,w), g(z,w))$$

be a smooth CR map sending M into $\widetilde{M}$, satisfying the normalization in (4.7) with $\sigma = 1$. Let T be given as in (4.8) and write $F^{\#} = T\circ F = (\widetilde{f}^{\#}, g^{\#})$ as in (4.9). Then, for any $v = \sum_{j=1}^n z_j\frac{\partial}{\partial z_j} \in T_0^{(1,0)}M$, the following holds:

$$g^{\sharp(2)} - w = g^{\sharp(3)} = g^{\sharp(4)} \equiv 0, \;\; f^{\sharp(2)} = 0, f^{\sharp(3)}(z,w) = \frac{i}{2}a^{(1)}(z)w, \quad \text{and}$$

$$\begin{aligned} 4\langle a^{(1)}(z), \bar{z}\rangle_\ell |z|_\ell^2 =& 4|\phi^{\sharp(2)}(z)|^2 - \lambda^{-2}\tilde{\mathcal{S}}_{\tilde{\theta}|_0}(F_*(v), \overline{F_*(v)}, F_*(v), \overline{F_*(v)}) \\ &+ \mathcal{S}_{\theta|_0}(v, \bar{v}, v, \bar{v}). \end{aligned} \tag{4.17}$$

Remark 4.3.2 *(1). We notice that when $N = n$, $\phi^{\sharp(2)}(z) \equiv 0$. Since the left hand side of the second equation in (4.17) is divisible by $|z|_\ell^2$ and the right hand side of the second equation in (4.17) is annihilated by Δ_ℓ, we conclude that both sides have to be identically zero and thus we have:*

$$\tilde{\mathcal{S}}_{\tilde{\theta}|_0}(F_*(v), \overline{F_*(v)}, F_*(v), \overline{F_*(v)}) = \mathcal{S}_{F^*(\tilde{\theta}|_0)}(v, \bar{v}, v, \bar{v}) \text{ for any } v \in T_0^{(1,0)}M. \tag{4.18}$$

This is the Chern-Moser invariant property (or the biholomoprhic transformation law) of the Chern-Moser Weyl tensor in the case of $N = n$.

(2). Our proof of the above proposition uses the same argument as that first appeared in [Hu], where a certain version of Proposition 4.3.1 was obtained. We repeated it here due to the reason that we have to trace precisely how the tangent vectors of type (1,0) and others are transformed when we normalize the map, which will be crucial for our later application. Indeed, as in [Hu], in the case of $\ell = 0$*, we can just assume that the map* F *is only a* C^2*-smooth CR map. We should mention that some other versions of the second equation in Proposition 3.1 were also obtained in the later work (see [EHZ] [BH], for instance), where this type of the results was called (the CR version of) the Gauss equation.*

Notice that when $\tilde{\theta}$ is an appropriate contact form along $\widetilde{M}$, then $F^*(\tilde{\theta})$ is also an appropriate contact form. From (4.16), we get the following monotonicity property for the Chern-Moser-Weyl curvature tensor under a CR embedding:

Theorem 4.3.3 *Let* $M \subset \mathbb{C}^{n+1}$ *and* $\widetilde{M} \subset \mathbb{C}^{N+1}$ *be two Levi non-degenerate smooth real hypersurfaces with the same signature* $0 < \ell < \frac{n}{2}$*. Suppose that* $F : M \to \widetilde{M}$ *is a CR transversal mapping (or, equivalently, a local holomorphic embedding). For an appropriate contact form* $\tilde{\theta}$ *along* $\widetilde{M}, p \in M$ *and* $v_p \in \mathcal{C}_\ell T_p^{(1,0)}M$*, we have*

$$\mathcal{S}_{F^*(\tilde{\theta})|_p}(v_p, \bar{v}_p, v_p, \bar{v}_p) \leq \tilde{\mathcal{S}}_{\tilde{\theta}|_{F(p)}}(F_*(v_p), \overline{F_*(v_p)}, F_*(v_p), \overline{F_*(v_p)}).$$

When $\ell = \frac{n}{2}$*, after replacing* M *by* $\tau_{\frac{n}{2}}(M)$ *and* F *by* $F \circ \tau_{\frac{n}{2}}$ *(to make* $F^*(\tilde{\theta}) = \tilde{k}\theta$ *with* $\tilde{k} > 0$*) if necessary, we also have the same statement as above. Here* $\tau_{\frac{n}{2}}(z_1, \ldots, z_{\frac{n}{2}}, z_{\frac{n}{2}+1}, \ldots, z_n, w) = (z_{\frac{n}{2}+1}, \ldots, z_n, z_1, \ldots, z_{\frac{n}{2}}, -w)$*.*

Now, assume that F is a holomorphic mapping from a domain $U \subset \mathbb{C}^{n+1}$ into $\mathbb{C}^{N+1}$. F is said to be totally degenerate if F fails to be a local holomorphic embedding at any point inside U, namely, if the rank of the Jacobian matrix of F is less than $n+1$ at any point $p \in U$. Hence, F is not totally degenerate over U if and only if it is a local holomorphic embedding away from a proper holomorphic variety. Now, let $M, \widetilde{M}$ be as above with $M \subset U$, $F \in Hol(U, \mathbb{C}^{N+1})$ and $F(M) \subset \widetilde{M}$. If F is not totally degenerate, then we apparently have $F(U) \not\subset \widetilde{M}$. Conversely, in case $M, \widetilde{M}$ are real analytic, if

$F(U) \not\subset \widetilde{M}$, by a result of Baouendi-Ebenfelt-Rothschild [BER] (see already the paper of Baouendi-Huang [BH] for a related investigation), F is not totally degenerate over U and thus is CR transversal over a connected dense open subset of M.

As the first application of Theorem 4.3.3, we have the following:

Corollary 4.3.4 *Let $M \subset \mathbb{C}^{n+1}$ be a smooth Levi non-degenerate hypersurface of signature ℓ. Suppose that F is not a totally degenerate holomorphic mapping defined in a neighborhood U of M in $\mathbb{C}^{n+1}$ that sends M into $\mathbb{H}_\ell^{N+1} \subset \mathbb{C}^{N+1}$. Then when $\ell < \frac{n}{2}$, the Chern-Moser-Weyl curvature tensor with respect to any appropriate contact form θ is pseudo semi-negative. When $\ell = \frac{n}{2}$, along any contact form θ, $\mathcal{S}_\theta$ is pseudo semi-definite.*

Proof of Corollary 4.3.4: By the observation above, since F is not totally non-degenerate, F is CR transversal over an open dense subset E_F of M. Without loss of generality, we assume that $\ell < \frac{n}{2}$. Since the Chern-Moser-Weyl pseudo-conformal curvature tensor for the hyperquadric $\mathbb{H}_\ell^{N+1}$ vanishes, by the previous theorem, we have for $p \in E_F$,

$$\mathcal{S}_{F^*(\tilde{\theta})|_p}(v_p, \bar{v}_p, v_p, \bar{v}_p) \leq 0$$

when $v_p \in \mathcal{C}_\ell T_p^{(1,0)}M$ and $\tilde{\theta}$ is an appropriate contact form of $\mathbb{H}_\ell^{N+1}$ near $F(p)$. This implies that $\mathcal{S}$ is pseudo semi-negative definite at each point $p \in E_F$.

When $p \notin E_F$, let θ be an appropriate contact form at p and $X_1, \ldots, X_n$ an orthonormal basis of $T^{(1,0)}M$ with respect to L_θ on some neighborhood of p, say U_p. Indeed, $\forall p \in M$, choose $X_1(p), \ldots, X_n(p)$ to be an orthonormal basis of $T_p^{(1,0)}M$ with respect to $L_{\theta|_p}$, i.e.,

$$\langle X_j(p), X_k(p)\rangle_{L_{\theta|_p}} = \begin{cases} -1 & \text{if } j = k \leq \ell; \\ 1 & \text{if } j = k > \ell; \\ 0 & \text{otherwise.} \end{cases}$$

Applying the Gram-Schmidt process if necessary, one can always extend $\{X_j(p)\}_{j=1}^n$ to an orthonormal basis $\{X_j\}_{j=1}^n$ (with respect to the Levi form L_θ) of $T^{(1,0)}M$ on some small neighborhood U_p of p. Moreover, a straightforward computation shows that for any vector-valued smooth function $\vec{a}(q) = (a_1(q), \ldots, a_n(q))$ along M near p,

$$\sum_{j=1}^{n} a_j X_j \text{ is a smooth section of } \mathcal{C}_\ell T^{(1,0)} U_p \Leftrightarrow |\vec{a}(q)|_\ell^2 = 0 \text{ for all } q \in U_p.$$

Now for the above $p \notin E_F$ and any $v_p = \sum_{j=1}^{n} a_j X_j|_p \in \mathcal{C}_\ell T_p^{(1,0)} M$ with $a_j \in \mathbb{C}$, take a sequence $\{q_k\}_{k=1}^{\infty} \in E_F$ converging to p. By the previous argument, $\sum_{j=1}^{n} a_j X_j|_{q_k} \in \mathcal{C}_\ell T_{q_k}^{(1,0)} M$ and $\mathcal{S}_{\theta|_{q_k}}(v_{q_k}, \bar{v}_{q_k}, v_{q_k}, \bar{v}_{q_k}) \leq 0$ for any k. Moreover, $\mathcal{S}_{\theta|_q}$ depends smoothly on q as we mentioned before. Letting $k \to \infty$, we then obtain the desired inequality at p. $\square$

Remark 4.3.5 *From the above, we see the following fact: For any point $p \in M$, there is an open neighborhood U_p of p in M and a smooth frame $\{X_1, \cdots, X_n\}$ of $T^{(1,0)} U_p$ such that the diffeomorphism Ψ from $T^{(1,0)} U_p$ to $U_p \times \mathbf{C}^n$ defined by $\Psi(\sum_{j=1}^{n} a_j X_j|_q) = (q, (a_1, \cdots, a_n))$ maps $\mathcal{C}_\ell T_q^{(1,0)} U_p$ to $\{q\} \times \mathcal{C}_\ell$ for each $q \in U_p$.*

In Theorem 4.3.3, suppose we only assume that F is not a totally degenerate holomorphic map in a neighborhood U of M. Then F is CR transversal along an open dense subset of M, as observed at the beginning of this section. Assume that F fails to be CR transversal at $p \in M$. Choose a sequence of points $\{q_j\} \subset M$ with $q_j \to p$, where the CR transversality holds. Apply a standard procedure to normalize M and $\widetilde{M}$ at $q \in M$ and $F(q)$ up to 4th order, respectively, for any $q \approx p$. Notice that we can make the normalizations to depend continuously (or even smoothly) on q and $F(q)$, respectively. Now, we can similarly define $\lambda(q)$ as in (4.7). Then $\lambda(q)$ depends continuously on q and thus converges to 0 as $q \to p$, by the assumption that F is not CR transversal at p. Now, applying (4.15) with $q = q_j$ and then letting $q_j \to p$, we see the following:

$$\mathcal{S}_{\theta|_p}(v_p, \overline{v_p}, v_p, \overline{v_p}) \leq 0, \quad \text{for } v_p \in \mathcal{C}_\ell T_p^{(1,0)} M. \tag{4.19}$$

Here when $\ell < n/2$, we have assumed that θ is appropriate and when $\ell = n/2$, we have assumed that $F^*(\tilde{\theta}|_{F(q_j)}) = \tilde{k}(q_j)\theta|_{q_j}$ with $\tilde{k}(q_j) > 0$ for a certain choice of the sequence $q_j \to p$. Hence, we get another application of Theorem 4.3.3:

Corollary 4.3.6 *Let $M \subset \mathbb{C}^{n+1}$ and $\widetilde{M} \subset \mathbb{C}^{N+1}$ be two smooth Levi non-degenerate hypersurfaces with the same signature $0 < \ell \leq \frac{n}{2}$. Suppose that F is not a totally degenerate holomorphic map defined over a neighborhood U of M in $\mathbb{C}^{n+1}$ with $F(M) \subset \widetilde{M}$. Let $p \in M$. If F fails to be CR transversal at p (or, equivalently, if F fails to be a local holomorphic embedding near p), then the following holds:*

(I) If $0 < \ell < n/2$, then the Chern-Moser-Weyl tensor at p with respect to any appropriate contact form is pseudo semi-negative definite.

(II) If $\ell = n/2$, then the Chern-Moser-Weyl tensor of M (with respect to any contact form) at p is pseudo semi-definite.

Corollary 4.3.4 can be used to construct many examples which fail to be embeddable into hyperquadrics. Here we provide one example as follows.

Example 4.3.7 *(1). Suppose that $P(z, \bar{z})$ is a real-valued homogeneous polynomial of bidegree $(2, 2)$ for $z \in \mathbf{C}^n$ $(n \geq 3)$and $P(z, \bar{z}) > 0$ for $z \neq 0$. Let $0 < \ell < n/2$. Let $M \subseteq \mathbb{C}^{n+1}$ be defined by*

$$\Im w = |z|_\ell^2 - N_4(z, \bar{z}) \tag{4.20}$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$, where N_4 is obtained from the following decomposition

$$P(z, \bar{z}) = N_4(z, \bar{z}) + N_2(z, \bar{z})|z|_\ell^2$$

with $\Delta_\ell N_4(z, \bar{z}) = 0$. Then M cannot be CR embedded into $\mathbf{H}_\ell^N$ for any N.

(2). Suppose that $P(z, \bar{z})$ is a real-valued homogeneous polynomial of bidegree $(2, 2)$ for $z \in \mathbb{C}^n$ $(n = 2k \geq 4)$ and $P(z, \bar{z})$ does not have a fixed sign for $|z|_\ell = 0$. (Namely, neither $P \geq 0$ for all $|z|_\ell = 0$ nor $P \leq 0$ for all $|z|_\ell = 0$.) Let $0 < \ell = k$. Let $M \subseteq \mathbb{C}^{n+1}$ be defined by

$$\Im w = |z|_\ell^2 - N_4(z, \bar{z}) \tag{4.21}$$

as above. Then M cannot be CR embedded into $\mathbf{H}_\ell^N$ for any N.

Indeed, (4.20) and (4.21) are already of the Chern-Moser normal form near the origin and their corresponding Chern-Moser-Weyl curvature tensor $\mathcal{S}_{\theta|_0}(z, \bar{z}) = 4N_4(z, \bar{z})$.

Moreover, by the construction of N_4, it is pseudo positive-definite in (4.20) and not pseudo semi-definite in (4.21). Corollary 4.3.4 then directly implies that M cannot be CR embedded into $\mathbb{H}_\ell^N$. In particular, the following two real hypersurfaces M_1 and M_2 can not be CR embedded into real hyperquadrics of the same signature in any $\mathbb{C}^N$:

$$\begin{aligned} &M_1 \subset \mathbb{C}^4: \ \Im w = |z|_\ell^2 - \tfrac{1}{2}(|z_1|^4 + |z_2|^4 + |z_3|^4 + 2|z_1 z_2|^2 + 2|z_1 z_3|^2 - 2|z_2 z_3|^2), \quad \ell = 1; \\ &M_2 \subset \mathbb{C}^5: \ \Im w = |z|_\ell^2 - \tfrac{1}{3}(|z_1|^4 - |z_3|^4 - 2|z_1 z_2|^2 + 2|z_1 z_4|^2 - 2|z_2 z_3|^2 + 2|z_3 z_4|^2), \quad \ell = 2. \end{aligned} \tag{4.22}$$

One may verify that, for M_1, the corresponding $P(z,\bar{z}) = |z_1|^4 + |z_2|^4 + |z_3|^4$ and $N_4(z,\bar{z}) = P(z,\bar{z}) - \frac{1}{2}|z|_\ell^4$, which falls into Case (1); while for M_2, the corresponding $P(z,\bar{z}) = |z_1|^4 - |z_3|^4$ and $N_4(z,\bar{z}) = P(z,\bar{z}) + \frac{2}{3}(|z_1|^2 + |z_3|^2)|z|_\ell^2$, which falls into Case (2).

We conclude this chapter with the following two open problems related to our Corollary 4.3.4, Example 4.3.7 and Crollary 4.3.6:

Question 4.3.8 *Let M be a strongly pseudoconvex hypersurface in $\mathbb{C}^{n+1}$ with $n \geq 1$ defined by a real polynomial. For any $p \in M$, does there exist a sufficiently large positive integer N, which may depend on p, such that a small piece of M near p can be holomorphically embedded into the Heisenberg hypersurface $\mathbb{H}_0^{N+1}$ (with signature 0)?*

Question 4.3.9 *Let M and $\widetilde{M}$ be smooth Levi non-degenerate hypersurfaces in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{N+1}$, respectively, with $N > n$. Assume that both M and $\widetilde{M}$ have the same signature ℓ with $0 < \ell \leq [\frac{n}{2}]$. Let U be a (connected) neighborhood of M in $\mathbb{C}^{n+1}$. Suppose that F is not a totally degenerate holomorphic map from U into $\mathbb{C}^{N+1}$ with $F(M) \subset \widetilde{M}$. Is then F a local holomorphic embedding along M?*

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