THE CLASSIFICATION PROBLEM FOR FINITE RANK DIMENSION GROUPS

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ABSTRACT OF THE DISSERTATION

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There has been much work done in the study of the Borel complexity of various naturally occurring classification problems. In particular, Hjorth and Thomas have shown that the Borel complexity of the classification problem for torsion-free abelian groups of finite rank increases strictly with rank.

In this thesis, we extend this result to dimension groups of finite rank. As these groups are naturally characterized by Bratteli diagrams, we obtain a similar theorem for Bratteli diagrams. We also obtain a similar result for a class of countable simple locally finite groups which are also characterized by Bratteli diagrams.

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Chapter 1
Countable Borel Equivalence Relations

1.1 Introduction

The isomorphism problem for countable simple locally finite groups has resisted satisfactory classification. As evidence that the classification problem is intractable, authors usually refer to the result of Kegel-Wehrfritz [27, 6.12] that there are $2^{\aleph_0}$ pairwise nonisomorphic groups that can be obtained as unions of chains of finite alternating groups. However, this is far from convincing, since the groups they constructed are of strongly diagonal type of rank 1, and the present Theorem 6.4 implies that there is an explicit function which associates to each such group a real number, so that two groups are assigned the same number if and only if they are isomorphic. On the other hand, we will show that the isomorphism problem for countable simple locally finite groups of strongly diagonal type of arbitrary finite rank is in fact intractable, in a sense which is made precise by the theory of Borel equivalence relations.

The study of Borel equivalence relations allows us to study the relative complexity of classification problems. We generally encode a classification problem as an equivalence relation E on a standard Borel space X. A *standard Borel space* is a Polish space equipped with its associated σ-algebra of Borel subsets. Then a *Borel equivalence relation* E on X is an equivalence relation $E \subseteq X^2$ which is a Borel subset of X^2. The first reason to work in this context is that there is a canonical method for encoding many naturally occurring classification problems into standard Borel spaces.

Suppose $\mathcal{L} = \{R_i \mid i \in I\}$ is a countable relational language, where R_i is an n_i-ary relation symbol. (If $\mathcal{L}$ contains n-ary function symbols or constant symbols, we consider them as $(n+1)$-ary or unary relation symbols.) Let $X_{\mathcal{L}} = \prod_{i \in I} 2^{\mathbb{N}^{n_i}}$. Then $X_{\mathcal{L}}$ is a Polish space whose elements represent countable $\mathcal{L}$-structures as follows. Given $x = (x_i) \in X_{\mathcal{L}}$, the structure $\mathcal{M}_x = \langle \mathbb{N}; R_i^x \rangle_{i \in I}$ is defined by

$$R_i^x(a_0, \ldots, a_{n_i - 1}) \iff x_i(a_0, \ldots, a_{n_i - 1}) = 1$$

Furthermore, the isomorphism relation on this space is precisely the orbit equivalence relation of the following natural action of the infinite symmetric group S_∞ on $X_{\mathcal{L}}$.

Definition 1.1. If $\sigma \in S_\infty$ is a permutation of $\mathbb{N}$ and $x = (x_i) \in X_\mathcal{L}$, then $\sigma.x$ is defined by

$$(\sigma.x)_i(a_0, \ldots, a_{n_i-1}) = x_i(\sigma^{-1}(a_0), \ldots, \sigma^{-1}(a_{n_i-1})).$$

Definition 1.2. If σ is an $\mathcal{L}_{\omega_1,\omega}$-sentence, then

$$\mathrm{Mod}(\sigma) = \{x \in X_\mathcal{L} \mid M_x \models \sigma\}.$$

Theorem 1.3. *[24] If σ is an $\mathcal{L}_{\omega_1,\omega}$-sentence, then* $\mathrm{Mod}(\sigma)$ *is a Borel subset of $X_\mathcal{L}$.*

Theorem 1.4. *(folklore) If X is a Polish space and $A \subseteq X$ is a Borel subset, then $(A, \mathcal{B}(X)\restriction_A)$ is a standard Borel space, where $\mathcal{B}(X)\restriction_A = \{Z \cap A \mid Z \in \mathcal{B}(X)\}$.*

Corollary 1.5. *If σ is an $\mathcal{L}_{\omega_1,\omega}$-sentence, then* $\mathrm{Mod}(\sigma)$ *is a standard Borel space.*

While this method works for any class of structures defined by an $\mathcal{L}_{\omega_1\omega}$-sentence, we note that it is rather unwieldy. Thus we usually try to find more natural representations for our classification problems. For example, Hjorth[23] and Thomas[36] studied the classification problem for torsion-free abelian groups of finite rank. By the *rank* of a torsion-free abelian group A, we mean size of the largest linearly independent subset of A. In this case, it is natural to identify the class of torsion-free abelian groups of rank n with the set of full-rank subgroups of $\mathbb{Q}^n$, denoted $R(\mathbb{Q}^n)$. Furthermore, there is a natural way to describe the isomorphism relation on this space, namely, that two groups $A, B \in R(\mathbb{Q}^n)$ are isomorphic if and only if there is a $\varphi \in \mathrm{GL}_n(\mathbb{Q})$ such that $A = \varphi(B)$. This illustrates the following definition.

Definition 1.6. Let X be a standard Borel space and let G be a locally compact second countable group acting on X. Then the *orbit equivalence* relation E_G^X of this action is given by

$$xEy \iff (\exists g \in G)\,(g(x) = y)\,.$$

There is a natural way to compare the relative complexity of two equivalence relations. If E, F are equivalence relations on standard Borel spaces X, Y, we say that E is *Borel reducible* to F and write $E \leq_B F$ if there exists a Borel map $f : X \to Y$ such that

$$xEy \iff f(x)Ff(y).$$

We call such a map f a *Borel reduction.* One standard interpretation of $E \leq_B F$ is that the problem of classifying elements of X up to E is effectively reduced to that of classifying elements of Y up to F. We say E and F are *Borel bireducible* and write $E \sim_B F$ if $E \leq_B F$ and $F \leq_B E$. Finally, we write $E <_B F$ if both $E \leq_B F$ and $F \not\leq_B E$. While we usually consider *Borel* equivalence relations, notice that the notion of Borel reducibility applies to arbitrary equivalence relations on standard Borel spaces. As an example of Borel (non)reducibility, let $\cong_n$ denote the isomorphism relation on the space of torsion-free abelian groups of rank n, then Hjorth and Thomas proved the following theorem. (A conjecture of Friedman and Stanley[14] implies that the isomorphism problem for infinite rank torsion-free abelian groups has maximal complexity, i.e., the isomorphism relation is Borel complete.)

Theorem 1.7. *For all $n \geq 1$, $(\cong_n) <_B (\cong_{n+1})$.*

Remarkably, the classification problem for a certain class of simple locally finite groups is Borel bireducible with a classification problem involving abelian groups. Specifically, we examine the isomorphism problem for simple dimension groups, a particular type of ordered abelian group. By an ordered abelian group, we mean an abelian group A together with a distinguished subset A^+, called the *positive cone*, such that

1. $A^+ + A^+ \subseteq A^+$;
2. $A^+ \cap (-A^+) = \{0\}$; and
3. $A^+ - A^+ = A$.

If $a, b \in A$, then we shall write $a \leq b$ if $b - a \in A^+$. If A and B are ordered abelian groups, then a homomorphism $\varphi : A \to B$ is an *order homomorphism* if $a \leq b$ implies $\varphi(a) \leq \varphi(b)$ for all $a, b \in A$. It follows that a homomorphism $\varphi : A \to B$ is an order homomorphism if and only if $\varphi[A^+] \subseteq B^+$.

An ordered abelian group is said to be *unperforated* if whenever $a \in A$ satisfies $na \in A^+$ for some $n \geq 1$, then $a \in A^+$. An element $u \in A^+$ is an *order unit* if for every $a \in A$, there exists an integer $n \in \omega$ such that $a \leq nu$.

Definition 1.8. An unperforated ordered abelian group A is a *dimension group* if A satisfies the *Riesz interpolation property*; ie. given elements $a_1, a_2, b_1, b_2 \in A$ with $a_i \leq b_j$ for $1 \leq i, j \leq 2$, then there exists $c \in A$ such that $a_i \leq c \leq b_j$ for $1 \leq i, j \leq 2$.

Notice that if $na = n(-a) = 0$ for some $n \in \mathbb{N}$, then unperforatedness gives $a \geq 0$ and $a \leq 0$ which implies that $a = 0$; hence we conclude that a dimension group A must be torsion free.

Example 1.9. The group $\mathbb{Z}^n$ is a dimension group with the following positive cone:

$$(\mathbb{Z}^n)^+ = \{(z_1, \ldots, z_n) \mid z_i \geq 0 \text{ for all } i\}.$$

To see that this group satisfies the Riesz interpolation property, let $a, b, c, d \in \mathbb{Z}^n$ such that $a, b \leq c, d$. Then, given $1 \leq i \leq n$, let $e_i = max\{a_i, b_i\}$ where a_i, b_i are the ith coordinates of a, b. Then $a, b \leq (e_1, \ldots, e_n) \leq c, d$.

In this thesis, we will prove an analogue of Theorem 1.7 for simple dimension groups equipped with a distinguished order unit. Here, as usual, simplicity means the lack of a nontrivial ideal, where an *ideal* in a dimension group (A, A^+) is a subgroup $J \leq A$ satisfying

1. $0 \leq a \leq b \in J \implies a \in J$, and
2. $J = J^+ - J^+$ where $J^+ = J \cap A^+$.

We define the standard Borel space of simple dimension groups of rank n as follows.

Definition 1.10. Let $n \geq 1$ and consider the standard Borel space $R(\mathbb{Q}^n) \times \mathcal{P}(\mathbb{Q}^n) \times \mathbb{Q}^n$ where $\mathcal{P}(\mathbb{Q}^n)$ denotes the power set of $\mathbb{Q}^n$. Then SDG_n denotes the Borel subset of $R(\mathbb{Q}^n) \times \mathcal{P}(\mathbb{Q}^n) \times \mathbb{Q}^n$ given by those (A, A^+, u) such that (A, A^+) is a simple dimension group (of rank n) and $u \in A^+ \setminus \{0\}$ is a distinguished order unit. Let $\cong_n^+$ denote the isomorphism relation on SDG_n.

Then our target theorem about dimension groups is the following.

Theorem 1.11. *For all* $n \geq 1$, $(\cong_n^+) <_B (\cong_{n+1}^+)$

Now what does any of this have to do with locally finite groups? In Chapter 2, we discuss locally finite groups of strongly diagonal type, which are defined as certain direct limits of finite groups. These direct limits are in turn described by a class of infinite graphs, known as Bratteli diagrams. A Bratteli diagram has as its vertex set a disjoint union of countably many finite levels. Each vertex corresponds to a finite alternating group, and each level to a (finite) product of finite alternating groups. The edges, which only lie between successive levels, define a sequence of group embeddings from one level to the next. The corresponding locally finite group is then the direct limit of this increasing sequence of finite groups.

However, each Bratteli diagram also defines a dimension group in a similar fashion. Here each vertex corresponds to a copy of the dimension group $\mathbb{Z}$, and each level to $\mathbb{Z}^m$, where m is the cardinality of the level. The edges now define a sequence of positive homomorphisms from one level to the next. Then the direct limit of this sequence is a dimension group, and Effros, Handelman, and Shen[11] have shown that every dimension group can be constructed in this fashion from some Bratteli diagram.

In each of these two constructions, we may compose two successive homomorphisms in the sequence without changing either the resulting dimension group or the resulting locally finite group. This "telescoping" operation, together with isomorphism, generates an equivalence relation $\sim$ on the class of Bratteli diagrams. In fact, two dimension groups (or locally finite groups of strongly diagonal type) are isomorphic if and only if the corresponding Bratteli diagrams are $\sim$-equivalent.

Now our goal is to prove a result analogous to Theorem 1.11 for simple locally finite groups of strongly diagonal type, and thus to show that the problem of classifying the countable simple locally finite groups is intractable. In order to do this, we will first show that the isomorphism problem for countable simple locally finite groups of strongly diagonal type is Borel bireducible with the isomorphism problem for simple dimension groups, by way of the classification problem for simple Bratteli diagrams. We will then prove Theorem 1.11, relying heavily on the work of Hjorth and Thomas. Finally, we define a notion of rank for countable simple locally finite groups of strongly diagonal type, and then show that it corresponds sufficiently well to the notion of rank for simple dimension groups.

This thesis is organized as follows. In Chapter 1, we will introduce all the relevant notions from the theory of Borel equivalence relations. In Chapter 2, we define countable simple locally finite groups of strongly diagonal type and simple Bratteli diagrams, and we examine the relationship between the two classification problems. In Chapter 3, we discuss simple dimension groups and show how simple dimension groups are characterized by simple Bratteli diagrams. In Chapter 4, we will examine the geometry of simple dimension groups of finite rank. In Chapter 5, we further examine the relationship between the classification problem for simple dimension groups and that for simple Bratteli diagrams. In Chapter 6, we prove Theorem 1.11 for two special cases. In Chapters 7 and 8, Theorem 1.11 is proved for the rest of the cases. Finally, in Chapter 9, we will show how Theorem 1.11 applies to simple Bratteli diagrams and simple locally finite groups of strongly diagonal type.

1.2 Countable Borel equivalence relations

Let X be a standard Borel space; i.e., a Polish space equipped with its associated σ-algebra of Borel subsets. Then a *Borel equivalence relation* E on X is an equivalence relation $E \subseteq X^2$ which is a Borel subset of X^2. We say that a Borel equivalence relation is *countable* if each equivalence class is countable. While we usually consider *Borel* equivalence relations, notice that the notion of Borel reducibility applies to arbitrary equivalence relations on standard Borel spaces. Now consider equivalence relations E and F on standard Borel spaces X and Y respectively, and a Borel reduction $f : X \to Y$ from E to F. Then the map $\hat{f} : X \times X \to Y \times Y$ given by $\hat{f}(x_1, x_2) = (f(x_1), f(x_2))$ is Borel. Thus if F is Borel, then E must also be Borel. Hence the class of Borel equivalence relations is closed downward under $\leq_B$. However, this closure property is not shared by the class of *countable* Borel equivalence relations. This is easily seen by the case of the trivial equivalence relation E on any standard Borel space X, where $x_1 E x_2$ for all $x_1, x_2 \in X$. Clearly, this equivalence relation is not countable, but it is

Borel reducible to *any* other equivalence relation on a standard Borel space. Thus we define the following class of equivalence relations which *is* closed downward under $\leq_B$:

Definition 1.12. An equivalence relation E on a standard Borel space X is *essentially countable* if there is a countable Borel equivalence relation F such that $E \leq_B F$.

Let G be a locally compact second countable group. Then a *standard Borel G-space* is a standard Borel space X equipped with a Borel action $(g, x) \mapsto g \cdot x$ of G on X. The Borel equivalence relation E_G^X on X, called the *G-orbit equivalence relation*, is then defined by

$$x_1 E_G^X x_2 \iff \text{there exists } g \in G \text{ such that } g \cdot x_1 = x_2.$$

By Kechris [26], E_G^X is Borel bireducible with a countable Borel equivalence relation. Conversely, we have the following theorem of Feldman and Moore:

Theorem 1.13. *[13] If E is an arbitrary countable Borel equivalence relation on the standard Borel space X, then there exists a countable group G and a Borel action of G on X such that $E = E_G^X$.*

The notion of Borel reducibility gives a partial pre-order on the collection of Borel equivalence relations. While much of the structure of this hierarchy is unknown, there are some benchmarks, especially within the realm of *countable* Borel equivalence relations. The first step to understanding Borel equivalence relations is the following result of Kuratowski.

Theorem 1.14. *[28] There exists a unique uncountable standard Borel space up to isomorphism.*

Thus we may naturally think of the identity equivalence relation on $\mathbb{R}$, denoted $\mathrm{id}_{\mathbb{R}}$, as the identity relation on whichever standard Borel space we happen to be working with. Silver has shown that $\mathrm{id}_{\mathbb{R}}$ is $\leq_B$-minimal.

Theorem 1.15. *[31] If E is a Borel equivalence relation with uncountable many classes, then* $\mathrm{id}_{\mathbb{R}} \leq_B E$.

Definition 1.16. A Borel equivalence relation E is *smooth* if $E \leq_B \mathrm{id}_X$ for some (and thus every) uncountable standard Borel space X.

One example of a countable Borel equivalence relation which is *not* smooth is the following:

Definition 1.17. E_0 is the Borel equivalence relation defined on $2^{\mathbb{N}}$ by

$$x E_0 y \iff x(n) = y(n) \text{ for all but finitely many } n \in \mathbb{N}.$$

Furthermore, we have the following remarkable result, which says that E_0 is an immediate $\leq_B$-successor of $\mathrm{id}_{\mathbb{R}}$:

Theorem 1.18. *[21] If E is a nonsmooth Borel equivalence relation, then $E_0 \leq_B E$.*

We also have a nice characterization of the countable Borel equivalence relations E such that $E \leq_B E_0$.

Definition 1.19. A Borel equivalence relation F is said to be *finite* if all of the equivalence classes of F are finite. A Borel equivalence relation E is *hyperfinite* if $E = \bigcup_{n \in \mathbb{N}} F_n$, where each F_n is a finite Borel equivalence relation, and for each $n \in \mathbb{N}$, $F_n \subseteq F_{n+1}$.

For example, E_0 is hyperfinite. To see this, note that $E_0 = \bigcup_{n \in \mathbb{N}} F_n$, where we define the sequence of equivalence relations F_n on $2^{\mathbb{N}}$ by

$$xF_ny \iff x(i) = y(i) \text{ for all } i > n.$$

In fact, every nonsmooth hyperfinite Borel equivalence relation is Borel bireducible with E_0. Furthermore, if F is hyperfinite and $E \leq_B F$, then E is also hyperfinite. Also, by a result of Dougherty, Jackson, and Kechris[7], if E is a countable Borel equivalence relation, then E is hyperfinite if and only if E can be realized as the orbit equivalence relation of a Borel $\mathbb{Z}$-action. A recent result of Gao and Jackson[16] shows that the orbit equivalence relations of arbitrary countable abelian groups are hyperfinite, and it is conjectured that the orbit equivalence relations of arbitrary countable amenable groups are hyperfinite.

At the other end of the spectrum, the following Borel equivalence relation turns out to be $\leq_B$-universal for the class of countable Borel equivalence relations:

Definition 1.20. Let $\mathbb{F}_\omega$ be the free group on infinitely many generators and define a Borel action of $\mathbb{F}_\omega$ on

$$(2^{\mathbb{N}})^{\mathbb{F}_\omega} = \{p \mid p : \mathbb{F}_\omega \to 2^{\mathbb{N}}\}$$

by setting

$$(g \cdot p)(h) = p(g^{-1}h), \qquad p \in (2^{\mathbb{N}})^{\mathbb{F}_\omega}.$$

Let E_ω be the resulting orbit equivalence relation.

Lemma 1.21. *[25] If E is a countable Borel equivalence relation, then $E \leq_B E_\omega$.*

Proof. Let X be a standard Borel space, and let E be a countable Borel equivalence relation on X. Since every countable group is a homomorphic image of $\mathbb{F}_\omega$, Theorem 1.13 implies that E is the orbit equivalence relation of a Borel action of $\mathbb{F}_\omega$. Let $\{U_i\}_{i \in \mathbb{N}}$

be a sequence of Borel subsets of X which separates points and define $f : X \to (2^{\mathbb{N}})^{\mathbb{F}_\omega}$ by $x \mapsto f_x$ where

$$f_x(h)(i) = 1 \text{ iff } x \in h(U_i).$$

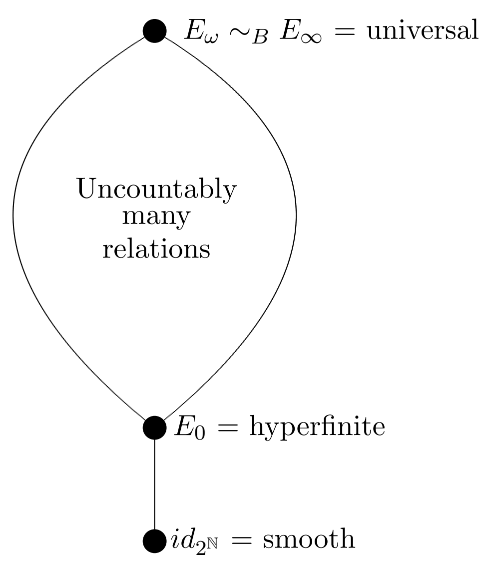
Then f is injective and

$$\begin{aligned}(g \cdot f_x)(h)(i) = 1 &\text{ iff } f_x(g^{-1}h)(i) = 1\\ &\text{ iff } x \in g^{-1}h(U_i)\\ &\text{ iff } g \cdot x \in h(U_i)\\ &\text{ iff } f_{g \cdot x}(h)(i) = 1\end{aligned}$$

Thus there is some $g \in \mathbb{F}_\omega$ such that $x = g \cdot y$ if and only if there is some $g \in \mathbb{F}_\omega$ such that $f_x = g \cdot f_y$. Hence f is a Borel reduction from E to E_ω. $\square$

Let $\mathbb{F}_2$ denote the free group on 2 generators and consider the action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$ given by setting $(g \cdot f)(h) = f(g^{-1}h)$. Then by [7], $E_\infty = E_{\mathbb{F}_2}^{2^{\mathbb{F}_2}}$ is also a universal countable Borel equivalence relation.

Finally, we note that Adams and Kechris[1] have shown that there exist uncountably many countable Borel equivalence relations up to Borel bireducibility. Thus, we have the following picture for countable Borel equivalence relations.



Let $R(\mathbb{Q}^n)$ be the standard Borel space consisting of the additive subgroups of $\mathbb{Q}^n$ of rank n, and let $\cong_n$ be the isomorphism relation on $R(\mathbb{Q}^n)$. It is easy to check that $\cong_n$ is the orbit equivalence relation given by the natural action of $\mathrm{GL}_n(\mathbb{Q})$ on $R(\mathbb{Q}^n)$. In 1937, Baer essentially showed that $\cong_1 \sim_B E_0$. In 1998, Hjorth [23] proved that $(\cong_1) <_B (\cong_2)$. Two years later, making essential use of earlier work of Adams-Kechris [1], Thomas [37] proved that $(\cong_n) <_B (\cong_{n+1})$ for all $n \geq 2$. Thus the equivalence relations $\cong_n$ form an increasing chain in the middle of the above picture, anchored at $\cong_1 \sim_B E_0$. Thomas [34] has also shown that the union of this chain does not reach the top, i.e., that the isomorphism relation on the space of all torsion-free abelian groups of finite rank is not countable universal.

1.3 Ergodicity

Given a standard Borel G-space X and a standard Borel H-space Y, and letting $E = E_G^X$ and $F = F_H^Y$, it is generally more difficult to prove results of the form $E \not\leq_B F$ rather than results of the form $E \leq_B F$. A significant tool in this endeavor is an invariant ergodic probability measure on X.

Definition 1.22. Let X be a standard Borel G-space. Throughout this thesis, a probability measure on X will always mean a Borel probability measure, i.e., a countably-additive measure which is defined on the collection of Borel subsets of X. Then a probability measure μ on X is said to be *nonatomic* if $\mu(\{x\}) = 0$ for all $x \in X$, and μ is said to be *G-invariant* if $\mu(g(A)) = \mu(A)$ for each $g \in G$ and Borel subset $A \subseteq X$. A G-invariant probability measure μ is *ergodic* if for every G-invariant Borel subset $A \subseteq X$, either $\mu(A) = 0$ or $\mu(A) = 1$. In this case, we say that G *acts ergodically* on (X, μ).

Lemma 1.23 (folklore)**.** *Suppose G is a countable group and X is a standard Borel G-space with G-invariant probability measure μ. Then the following are equivalent:*

(a) μ is ergodic.

(b) If Y is a standard Borel space and $f : X \to Y$ is a G-invariant Borel function (i.e., $f(g.x) = f(x)$ for every $x \in X$ and every $g \in G$), then there exists a G-invariant Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that $f \restriction_M$ is a constant function.

Notice that this says that the existence of an ergodic invariant probability measure on the standard Borel G-space X implies that E_G^X is not smooth.

Proof. $(b) \implies (a)$. Let $A \subseteq X$ be a G-invariant Borel subset. Consider the characteristic function $\chi_A : X \to \{0, 1\}$, where $\{0, 1\}$ is the standard Borel space that arises from the discrete topology on the set of 2 elements. Then χ_A is clearly Borel and G-invariant. Thus there is a G-invariant Borel subset $M \subseteq X$ such that $\mu(M) = 1$ and $\chi_A \restriction_M$ is constant. If $\chi_A(M) = 1$, then $\mu(A) = 1$, and if $\chi(M) = 0$ then $\mu(A) = 0$.

$(a) \implies (b)$. Let $f : X \to Y$ be a G-invariant Borel function. Let $\{U_n \mid n \in \omega\}$ enumerate a countable basis for a topology τ on Y such that $\mathcal{B}(Y) = \mathcal{B}(\tau)$. Then

$\{U_n \mid n \in \omega\}$ separates the points of Y, i.e., for all $a, b \in Y$, if $a \neq b$ then there is some $n \in \omega$ with $a \in U_n$ and $b \notin U_n$. Next, for each $n \in \omega$, define

$$A_n^1 = f^{-1}(U_n)$$
$$A_n^0 = X \backslash f^{-1}(U_n) = f^{-1}(Y \backslash U_n)$$

Notice that A_n^1 and A_n^0 are Borel since f is Borel. Furthermore, they are both G-invariant subsets of X. Thus, for each $n \in \omega$, either $\mu(A_n^1) = 1$ or $\mu(A_n^0) = 1$. If $\mu(A_n^1) = 1$ set $A'_n = A_n^1$, otherwise set $A'_n = A_n^0$. Finally, set $M = \bigcap A'_n$. Then $\mu(M) = 1$, and since $\{U_n \mid n \in \omega\}$ separates the points of Y, $f(M)$ must be a single point. $\square$

We will make use of the following Theorem in Chapter 7. However we must first define the notion of a Borel homomorphism.

Definition 1.24. If E, F are Borel equivalence relations on standard Borel spaces X, Y, then we say the Borel map $f : X \to Y$ is a *Borel homomorphism from E to F* if xEy implies $f(x)Ff(y)$.

In order to prove the nonexistence of a Borel reduction, it is often useful to examine Borel homomorphisms and show that none can be a Borel reduction. The following result is a special case of Hjorth-Kechris [24, 10.5].

Theorem 1.25. *Let $n \geq 3$ and let X be a standard Borel $SL_n(\mathbb{Z})$-space with an invariant ergodic probability measure μ. Suppose that Y is a standard Borel space and that F is a hyperfinite equivalence relation on Y. If $f \colon X \to Y$ is a Borel homomorphism from $E^X_{\mathrm{SL}_n \mathbb{Z}}$ to F, then there exists an $SL_n(\mathbb{Z})$-invariant Borel subset M with $\mu(M) = 1$ such that f maps M into a single F-class.*

1.4 Fréchet-amenable equivalence relations

As mentioned above, it is known that every orbit equivalence relation arising from a Borel action of an abelian group is hyperfinite. On the other hand, while the same is conjectured for arbitrary amenable groups, we currently only have some partial results involving the notion of Fréchet-amenable equivalence relations, which were introduced by Jackson, Kechris, and Louveau. The following account of Fréchet-amenability is based upon Section 2.4 of [25].

A countable group G is *amenable* if there exists a finitely additive G-invariant probability measure $\nu : \mathcal{P}(G) \to [0, 1]$ defined on every subset of G. In Chapter 6, we shall make use of the fact that solvable groups are amenable, and we shall also make use of the fact that if a countable group contains an isomorphic copy of $\mathbb{F}_2$, the free group on two generators, then it is not amenable.

Example 1.26. To see that $\mathbb{F}_2$ is not amenable, suppose that μ is a finitely additive $\mathbb{F}_2$-invariant probability measure on $\mathbb{F}_2 = \langle a, b\rangle$. First notice that the measure of a single element must be 0. Then given a reduced word σ in the alphabet $\{a, b, a^{-1}, b^{-1}\}$, let X_σ be the set of all reduced words of $\mathbb{F}_2$ for which σ is an initial segment. Then

$$\begin{aligned} 1 &= \mu(\mathbb{F}_2) \\ &= \mu(X_a) + \mu(X_b) + \mu(X_{a^{-1}}) + \mu(X_{b^{-1}}) \\ &= \mu(bX_a) + \mu(bX_b) + \mu(bX_{a^{-1}}) + \mu(X_{b^{-1}}) \\ &= \mu(X_{ba}) + \mu(X_{bb}) + \mu(X_{ba^{-1}}) + \mu(X_{b^{-1}}) \\ &= \mu(X_b) + \mu(X_{b^{-1}}). \end{aligned}$$

Similarly, $1 = \mu(X_a) + \mu(X_{a^{-1}})$. But then

$$\mu(\mathbb{F}_2) = \mu(X_a) + \mu(X_{a^{-1}}) + \mu(X_b) + \mu(X_{b^{-1}}) = 2,$$

a contradiction. It is also easy to see that if a group G contains a subgroup isomorphic to $\mathbb{F}_2$, then the same argument shows that G is not amenable.

The definition of Fréchet-amenable equivalence relations is motivated by the following characterization of the amenability of countable groups due to Day.

Theorem 1.27. *[6] Let G be a countable group. Then G is amenable iff there is a sequence (f_n), $f_n \in l_1(G)$, $f_n \geq 0$, $||f_n||_1 = 1$, such that for all $g \in G$, $||f_n - f_n^g||_1 \to 0$, where $f_n^g(h) = f_n(hg)$.*

A *free* filter on $\mathbb{N}$ is a filter containing the *Fréchet filter*

$$Fr = \{A \subseteq \mathbb{N} : A \text{ is cofinite}\}.$$

A filter $\mathcal{F}$ on $\mathbb{N}$ is said to be Borel if it is Borel when viewed as a subset of $2^{\mathbb{N}}$.

Definition 1.28. Let E be a countable Borel equivalence relation on a standard Borel space X. Let $\mathcal{F}$ be a free filter on $\mathbb{N}$. We say that E is *$\mathcal{F}$-amenable* if there is a sequence $f_n : E \to \mathbb{R}^+$ of nonnegative Borel functions such that letting $f_n^x(y) = f_n(x, y)$ we have:

1. For all $x \in X$, $f_n^x \in l_1([x]_E)$ and $||f_n^x||_1 = 1$
2. xEy implies $||f_n^x - f_n^y||_1 \to_{\mathcal{F}} 0$

If Y is a topological space and $y_n, y \in Y$, then $y_n \to_{\mathcal{F}} y$ means that for every neighborhood U of y, there is $A \in \mathcal{F}$ such that $n \in A \Rightarrow y_n \in U$. Note that $y_n \to_{Fr} y$ if and only if $y_n \to y$.

Define the partial order $\leq$ on filters on $\mathbb{N}$ by

$$\mathcal{F} \leq \mathcal{G} \Leftrightarrow \text{ there exists } h : \mathbb{N} \to \mathbb{N} \text{ such that } h^{-1}(\mathcal{F}) \subseteq \mathcal{G}$$

and the corresponding equivalence relation by

$$\mathcal{F} \equiv \mathcal{G} \Leftrightarrow \mathcal{F} \leq \mathcal{G} \text{ and } \mathcal{G} \leq \mathcal{F}.$$

Lemma 1.29. *If E is $\mathcal{F}$-amenable and if $\mathcal{F}$ and $\mathcal{G}$ are filters such that $\mathcal{F} \leq \mathcal{G}$, then E is $\mathcal{G}$-amenable.*

Proof. Let E be $\mathcal{F}$-amenable, $\mathcal{F} \leq \mathcal{G}$, and let $f_n : E \to \mathbb{R}^+$ be a sequence of Borel functions satisfying

(1) For all $x \in X$, $f_n^x \in l_1([x]_E)$ and $||f_n^x||_1 = 1$

(2) xEy implies $||f_n^x - f_n^y||_1 \to_{\mathcal{F}} 0$.

Then let $h : \mathbb{N} \to \mathbb{N}$ be a function satisfying $h^{-1}(\mathcal{F}) \subseteq \mathcal{G}$, and for each $n \in \mathbb{N}$, let $g_n = f_{h(n)}$. Certainly each of the functions g_n satisfies (1). Since $h^{-1}(\mathcal{F}) \subseteq \mathcal{G}$, we have that $y_n \to_{\mathcal{F}} y$ implies $y_{h(n)} \to_{\mathcal{G}} y$ for any $y_n, y \in \mathbb{R}$. Thus $||f_n^x - f_n^y||_1 \to_{\mathcal{F}} 0$ implies $||g_n^x - g_n^y||_1 \to_{\mathcal{G}} 0$, and so xEy implies $||g_n^x - g_n^y||_1 \to_{\mathcal{G}} 0$. Hence E is $\mathcal{G}$-amenable. □

Thus, the notion of $\mathcal{F}$-amenability only depends on the $\equiv$-equivalence class of $\mathcal{F}$. We will now define a canonical transfinite iteration of the Frèchet filter. Fix a bijection $\varphi : \mathbb{N} \to \mathbb{N}^2$. Also, for $m \in \mathbb{N}$, $A \subseteq \mathbb{N}^2$, let $A_m = \{n \in \mathbb{N} : (m, n) \in A\}$. For two filters $\mathcal{F}, \mathcal{G}$ on $\mathbb{N}$, their (Fubini) product $\mathcal{F} \otimes \mathcal{G}$ is defined by

$$\varphi(\mathcal{F} \otimes \mathcal{G}) = \{A \subseteq \mathbb{N}^2 : \{m \in \mathbb{N} : A_m \in \mathcal{G}\} \in \mathcal{F}\}.$$

It is easily checked that $\mathcal{F} \otimes \mathcal{G}$ is a filter on $\mathbb{N}$. We also define for each sequence $(\mathcal{F}_n)$ of filters, the filter $\mathcal{F} \otimes (\mathcal{F}_n)$ by

$$\varphi(\mathcal{F} \otimes (\mathcal{F}_n)) = \{A \subseteq \mathbb{N}^2 : \{m \in \mathbb{N} : A_m \in \mathcal{F}_m\} \in \mathcal{F}\}.$$

Now fix, for each countable limit ordinal λ, an increasing sequence $\alpha_0 < \alpha_1 < \ldots < \lambda$ whose limit is λ and inductively define the αth *iterated Frèchet filter* Fr_α as follows:

$$\begin{aligned} Fr_1 &= Fr \\ Fr_{\alpha+1} &= Fr \otimes Fr_\alpha \\ Fr_\lambda &= Fr \otimes (Fr_{\alpha_n}). \end{aligned}$$

It is clear that this definition depends on the choice of φ and the choice of the sequences (α_n), but it is a simple exercise to check that it is independent up to $\equiv$ and therefore the following definition makes sense.

Definition 1.30. Let E be a countable Borel equivalence relation. We say that E is *α-amenable* if E is Fr_α-amenable and *Frèchet-amenable* if it is α-amenable for some $\alpha < \omega_1$.

To see how this notion approximates hyperfiniteness, we note the following results, all from Section 2.4 of [25], which we will employ here in Chapter 6.

Theorem 1.31. *Suppose G is a countable amenable group, and E_G is the orbit equivalence relation induced by some Borel action of G. Then E_G is 1-amenable.*

Thus, since each hyperfinite equivalence relation can be realized as the orbit equivalence relation of a Borel $\mathbb{Z}$-action, we note

Corollary 1.32. *Hyperfinite equivalence relations are 1-amenable.*

Theorem 1.33. *Suppose G is a countable group acting in a Borel way on X, μ is a G-invariant probability measure on X, and the action is free on an invariant Borel set of measure 1. If E_G^X is Frèchet-amenable, then G is amenable.*

Theorem 1.34. *Let E, F be countable Borel equivalence relations. If $E \leq_B F$ and F is α-amenable, then E is also α-amenable.*

1.5 Constructing the measure space (X, μ)

A key ingredient in Thomas' proof that $(\cong_{n-1}) <_B (\cong_n)$, for $n \geq 3$ was that given a Borel homomorphism $f : R(\mathbb{Q}^n) \to R(\mathbb{Q}^{n-1})$ from $\cong_n$ to $\cong_{n-1}$, he was able to reduce the analysis to the situation where the domain of f was a standard Borel $\mathrm{SL}_n(\mathbb{Z})$-space X with an invariant ergodic probability measure μ. The proof of our main theorem will require the same thing. The following construction of an appropriate standard Borel $\mathrm{SL}_n(\mathbb{Z})$-space X and invariant ergodic probability measure μ is condensed from Sections 3 and 4 of [35].

Definition 1.35. Let $\mathbb{P}$ denote the set of primes. If $p \in \mathbb{P}$, then a group $A \in R(\mathbb{Q}^n)$ is said to be *p-local* iff $A = qA$ for every prime $q \neq p$; i.e., A is a $\mathbb{Z}_{(p)}$-module, where

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \in \mathbb{Q} \mid a, b \in \mathbb{Z} \text{ and } b \text{ is relatively prime to } p \right\}.$$

Let $R^{(p)}(\mathbb{Q}^n)$ denote the p-local subgroups of $\mathbb{Q}^n$ of rank n.

Suppose that K is a compact second countable group and that L is a closed subgroup. Then there exists a unique K-invariant probability measure μ on the standard Borel K-space K/L. The measure μ is called the Haar probability measure on K/L and can be described explicitly as follows. Suppose ν is the Haar probability measure on K and let $\pi : K \to K/L$ be the canonical surjection. Then $\mu = \pi_*\nu$, that is, given $A \subseteq K/L$, then $\mu(A) = \nu(\pi^{-1}(A))$. Below, we shall make use of the following observation.

Lemma 1.36. *[37, 2.2(a)] Let K be a compact second countable group, let $L \leqslant K$ be a closed subgroup and let μ be the Haar probability measure on K/L. If Γ is a countable dense subgroup of K, then μ is an ergodic Γ-invariant probability measure on K/L.*

For the rest of this section, fix an integer $n \geq 3$, and let p be any prime number. To obtain our measure space, we will examine the action of $\mathrm{PSL}_n(\mathbb{Z})$ on the standard Borel space $S_n(\mathbb{Q}_p)$ of nontrivial proper subspaces of the n-dimensional vector space $\mathbb{Q}_p^n$ over the p-adic field.

Definition 1.37. If $0 \leq k \leq n$, then $V^{(k)}(n, \mathbb{Q}_p)$ denotes the standard Borel space consisting of the k-dimensional vector subspaces of $\mathbb{Q}_p^n$.

It is easily checked that the compact group $\mathrm{PSL}_n(\mathbb{Z}_p)$ acts transitively on each $V^{(k)}(n, \mathbb{Q}_p)$. (For example, see [37, 6.1]) Thus we can identify $V^{(k)}(n, \mathbb{Q}_p)$ with the coset space $\mathrm{PSL}_n(\mathbb{Z}_p)/L$, where L is a suitably chosen closed subgroup of $\mathrm{PSL}_n(\mathbb{Z}_p)$. Let $\mu_{n,k}$ be the corresponding Haar probability measure on $V^{(k)}(n, \mathbb{Q}_p)$. Since $\mathrm{PSL}_n(\mathbb{Z})$ is a dense subgroup of $\mathrm{PSL}_n(\mathbb{Z}_p)$, the above lemma shows that $\mathrm{PSL}_n(\mathbb{Z})$ acts ergodically on $(V^{(k)}(n, \mathbb{Q}_p), \mu_{n,k})$. Thus, since a set $X \subseteq V^{(k)}(n, \mathbb{Q}_p)$ is $\mathrm{PSL}_n(\mathbb{Z})$-invariant if and only if it is $\mathrm{SL}_n(\mathbb{Z})$-invariant, it follows that $\mathrm{SL}_n(\mathbb{Z})$ also acts ergodically on $(V^{(k)}(n, \mathbb{Q}_p), \mu_{n,k})$.

Now we will discuss how this space relates to the isomorphism relation on $R^{(p)}(\mathbb{Q}^n)$.

Definition 1.38. For each $A \in R^{(p)}(\mathbb{Q}^n)$, let $\hat{A} = \mathbb{Z}_p \otimes A$

We shall regard each $\hat{A}$ as a subgroup of $\mathbb{Q}_p^n$ in the usual way; i.e., $\hat{A}$ is the subgroup consisting of all finite sums

$$\gamma_1 a_1 + \gamma_2 a_2 + \ldots + \gamma_t a_t,$$

where $\gamma_i \in \mathbb{Z}_p$ and $a_i \in A$ for $1 \leq i \leq t$. By Lemma 93.3 [15], there exist integers $0 \leq k, l \leq n$ with $k + l = n$ and elements $v_i, w_j \in \hat{A}$ such that

$$\hat{A} = \bigoplus_{i=1}^{k} \mathbb{Q}_p v_i \oplus \bigoplus_{j=1}^{l} \mathbb{Z}_p w_j.$$

Definition 1.39. For each $A \in R^{(p)}(\mathbb{Q}^n)$, let $V_A = \bigoplus_{i=1}^{k} \mathbb{Q}_p v_i$.

Theorem 1.40. *[35, 4.3 and 4.4] Suppose $A \in R^{(p)}(\mathbb{Q}^n)$ and that $\dim V_A = n - 1$. Then for each $B \in R^{(p)}(\mathbb{Q}^n)$, we have that $A \cong B$ if and only if there exists $\pi \in \mathrm{GL}_n(\mathbb{Q})$ such that $\pi(V_A) = V_B$.*

Definition 1.41. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{Q}_p^n$. Suppose that S is a $\mathbb{Q}_p$-subspace of $\mathbb{Q}_p^n$ of dimension $0 \leq k \leq n$. Then

$$\sigma(S) = (S \oplus \mathbb{Z}_p \mathbf{e}_{i_1} \oplus \ldots \oplus \mathbb{Z}_p \mathbf{e}_{i_{n-k}}) \cap \mathbb{Q}^n$$

where $i_1 < \ldots < i_{n-k}$ is the lexicographically least sequence such that

$$\mathbb{Q}_p^n = \langle S, \mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{n-k}} \rangle.$$

Theorem 1.42. *[35, 4.6] If S is a $\mathbb{Q}_p$-subspace of $\mathbb{Q}_p^n$ of dimension $0 \leq k \leq n$, then*

(a) $\sigma(S) \in R^{(p)}(\mathbb{Q}^n)$;

(b) $V_{\sigma(S)} = S$.

Definition 1.43. For $n \geq 3$, set

- $X_n = V^{(n-1)}(n, \mathbb{Q}_p)$,
- $\mu_n = \mu_{n,n-1}$, and
- $\sigma_n : X_n \to R(\mathbb{Q}^n)$ by $S \mapsto \sigma(S)$.

Then Theorems 1.40 and 1.42 imply that σ_n is a countable-to-one Borel homomorphism from $E_{\mathrm{SL}_n(\mathbb{Z})}^{X_n}$ to $\cong_n$ (In particular, σ_n does not map a measure one subset of X to a single $\cong_n$-class). We will use this as well as the fact that $\mathrm{SL}_n(\mathbb{Z})$ acts ergodically on (X_n, μ_n) in Chapters 7 and 8, while the fact that all the abelian groups in the range of this map are p-local will be used in Chapter 9. (Notice that our construction did not depend on the choice of the prime p.)

1.6 Cocycles

Let G be a locally compact second countable (lcsc) group, and let X be a standard Borel G-space with invariant probability measure μ. Let H be an lcsc group. A *cocycle* of the G-space X into H is a Borel map $\alpha : G \times X \to H$ such that for all $g, h \in G$,

$$\alpha(hg, x) = \alpha(h, g \cdot x)\alpha(g, x) \;\; \mu\text{-a.e.}(x).$$

If this equation holds for all x, then we say that α is a *strict cocycle.* If $\beta : G \times X \to H$ is also a cocycle, we say that α *is equivalent to* β, written $\alpha \sim \beta$, if there is a Borel map $A : X \to H$ such that for all $g \in G$,

$$\alpha(g, x) = A(g \cdot x)\beta(g, x)A(x)^{-1} \;\; \mu\text{-a.e.}(x).$$

In addition to being the only type of cocycle which we will encounter in this thesis, the following canonical example motivates the above definitions. Suppose $E = E_G^X$ and

$F = E_H^Z$, where H acts freely on Z. Let $f : X \to Z$ be a Borel function such that xEy implies $f(x)Ff(y)$, i.e., f is a Borel homomorphism from E to F. Then the function $\alpha : G \times X \to H$ defined by $f(g \cdot x) = \alpha(g, x) \cdot f(x)$ is a strict cocycle. (There exists a *unique* such element $\alpha(g, x) \in H$ since the action of H on Z is free.)

Some of our proofs will proceed by considering the cocycles associated with various Borel homomorphisms. There are various cocycle reduction results which say that, under certain hypotheses, cocycles α are equivalent to cocycles β, whose range $\beta(G\times X)$ is contained in a "small" subgroup of H. In Chapters 7 and 8 we shall make essential use of the following such theorem.

Theorem 1.44. *[36, 2.3] Let $n \geq 3$ and let X be a standard Borel $SL_n(\mathbb{Z})$-space with an invariant ergodic probability measure. Suppose that G is an algebraic $\mathbb{Q}$-group such that $\dim G < n^2 - 1$ and that $H \leq G(\mathbb{Q})$. Then for every Borel cocycle $\alpha\colon SL_n(\mathbb{Z}) \times X \to H$, there exists an equivalent cocycle β such that $\beta(SL_n(\mathbb{Z}) \times X)$ is contained in a finite subgroup of H.*

1.7 Relative ergodicity of equivalence relations

Recall that in order to construct the cocycle associated with a Borel homomorphism, we required that the action of H on Y is free. In the case of Thomas' proof that $(\cong_n) <_B (\cong_{n+1})$, this corresponds to the action of $GL_n(\mathbb{Q})$ on $R(\mathbb{Q}^n)$. However, this action is far from free. Thomas' innovation was to work with the coarser equivalence relation of quasi-isomorphism, defined here in Chapter 7, which enabled him to obtain a free action of a quotient of a suitable subgroup of $GL_n(\mathbb{Q})$ on a suitable quotient of $R(\mathbb{Q}^n)$. While working with this coarser equivalence relation, Thomas implicitly proved the following lemma.

Definition 1.45. Suppose that X is a standard Borel G-space with invariant ergodic probability measure μ, and that F is a countable Borel equivalence relations on a standard Borel space Y. Then E *is F-ergodic* if for any Borel homomorphism $f : X \to Y$ from E to F, there is a Borel subset $X_1 \subseteq X$ with $\mu(X_1) = 1$ such that f maps X_1 into a single F-class.

Lemma 1.46. *[33] Let G be a countable group. Suppose X is a standard Borel G-space with invariant ergodic probability measure μ, and that F and F' are countable Borel equivalence relations on a standard Borel space Y such that $F \subseteq F'$. Suppose that $E = E_G^X$ is F'-ergodic. Then E is F-ergodic.*

Proof. Suppose that $E = E_G^X$ is F'-ergodic. Let $f : X \to Y$ be a Borel homomorphism from E to F. Then since $F \subseteq F'$, it follows that f is also a Borel homomorphism from E to F'. Hence there is a Borel subset $X' \subseteq X$ such that $\mu(X') = 1$ and $f(X')$ is contained in a single F'-class, say C. Since C is countable, there exists a Borel subset $Z \subseteq X'$ with $\mu(Z) > 0$ and a fixed element $y \in C$ such that $f(x) = y$ for all $x \in Z$. Since μ is ergodic, the G-invariant Borel subset $M = G.Z$ satisfies $\mu(M) = 1$, and clearly f maps M into the F-class containing y. Hence E is F-ergodic. $\square$

Chapter 2
Groups of strongly diagonal type and Bratteli diagrams

In this chapter, we shall define the countable locally finite groups of strongly diagonal type, as well as the notion of a Bratteli diagram. We show that every Bratteli diagram gives rise to a countable locally finite group of strongly diagonal type via a canonical construction, and also that every countable simple locally finite group of strongly diagonal type can be constructed from a Bratteli diagram in this way. We determine when two Bratteli diagrams yield isomorphic groups, and show that a Bratteli diagram is simple if and only if the corresponding group is simple.

2.1 Countable locally finite groups of strongly diagonal type

Definition 2.1. Let G be a countable locally finite group and let

$$G_0 \leq G_1 \leq ... \leq G_n \leq ...$$

be an increasing chain of finite groups such that $G = \bigcup_{n \in \omega} G_n$. Suppose further that for each $n \geq 1$,

$$G_n = A_{n,1} \times ... \times A_{n,d_n}$$

where each $A_{n,i}$ is an alternating group on a finite set $\Omega_{n,i}$. For each $1 \leq i \leq d_n$, let

$$B_{n,i} = A_{n,1} \times ... \times \widehat{A_{n,i}} \times ... \times A_{n,d_n}$$

where $\widehat{A_{n,i}}$ indicates that $A_{n,i}$ has been omitted from the product.

(a) The above chain is said to be of *diagonal type* if whenever $n < m$ and Σ is a nontrivial orbit of G_n on some $\Omega_{m,k}$, then there exists $1 \leq i \leq d_n$ such that

  (1) $|\Sigma| = |\Omega_{n,i}|$;

  (2) $A_{n,i}$ acts naturally on Σ; and

  (3) $B_{n,i}$ acts trivially on Σ.

(b) The above chain is said to be of *strongly diagonal type* if it is of diagonal type and whenever $n < m$ and $1 \leq k \leq d_m$, then each element of $\Omega_{m,k}$ lies in some nontrivial G_n-orbit.

(c) The above chain is said to be of *regular type* if whenever $n < m$, then there exists $1 \leq k \leq d_m$ such that G_n has at least one regular orbit on $\Omega_{m,k}$.

Whenever we have an embedding of two finite products of alternating groups which satisfies (a) above, we say that the embedding is *diagonal*. To understand (c), recall that a permutation group $H \leq Alt(\Omega)$ is said to act *regularly* if H acts transitively on Ω, and

- If $h \in H$, $x \in \Omega$, and $hx = x$, then $h = 1$.

In particular, given a diagonal embedding of finite products of alternating groups

$$Alt(\Omega_{i,1}) \times Alt(\Omega_{i,2}) \times \ldots \times Alt(\Omega_{i,d_i}) \hookrightarrow Alt(\Omega_{j,1}) \times Alt(\Omega_{j,2}) \times \ldots \times Alt(\Omega_{j,d_j}),$$

then $Alt(\Omega_{i,1}) \times Alt(\Omega_{i,2}) \times \ldots \times Alt(\Omega_{i,d_i})$ cannot have any regular orbits on $\bigsqcup_{k=1}^{d_j} \Omega_{j,k}$. The following theorem is much stronger. It shows that a simple locally finite group cannot be expressed as both the union of a chain of diagonal type and the union of a chain of regular type.

Theorem 2.2. *[22] Let G be a countably infinite simple locally finite group, and suppose that G is the union of an increasing chain*

$$G_0 \leq G_1 \leq \ldots \leq G_n \leq \ldots$$

of finite groups, each of which is a direct product of alternating groups. Let K be an algebraically closed field of characteristic 0.

(a) If the above chain is of diagonal type, then the group algebra KG has at least four ideals.

(b) If the above chain is of regular type, the group algebra KG has exactly three ideals.

The next theorem shows that when studying simple locally finite groups which can be expressed as the unions of chains of finite groups, each of which is the direct product of alternating groups, we may restrict our attention to chains of either diagonal type or regular type.

Theorem 2.3. *[22] Let G be a countably infinite simple locally finite group, and suppose that G is the union of an increasing chain*

$$G_0 \leq G_1 \leq \ldots \leq G_n \leq \ldots$$

of finite groups, each of which is a direct product of alternating groups. Then there exists a subsequence $\{i_n \mid n \in \omega\}$ such that the chain

$$G_{i_0} \leq G_{i_1} \leq \ldots \leq G_{i_n} \leq \ldots$$

is either of diagonal type or of regular type.

Thus it is natural to define a countable locally finite group G to be of *(strongly) diagonal* type if G is isomorphic to the union of a chain of (strongly) diagonal type.

2.2 Bratteli Diagrams

Definition 2.4. A *Bratteli diagram* (V, E) consists of a vertex set V and an edge set E, where V and E can be written as countable disjoint unions of nonempty finite sets $V = \bigsqcup_{n \geq 0} V_n$ and $E = \bigsqcup_{n \geq 1} E_n$ such that the following conditions hold.

1. $V_0 = \{v\}$ is a singleton set.

2. There exist range and source maps r, s from E to V such that $r[E_n] \subseteq V_n$ and $s[E_n] \subseteq V_{n-1}$. Furthermore, $s^{-1}(v) \neq \emptyset$ for all $v \in V$ and $r^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$.

For each $n \geq 1$, the edge set E_n determines a corresponding *incidence matrix* $M_n = (m_{u,v})$, with rows indexed by V_n and columns indexed by V_{n-1}, such that

$$m_{u,v} = |\{e \in E_n \mid r(e) = u \text{ and } s(e) = v\}|$$

is the number of edges joining v to u.

Definition 2.5. If $0 \leq k \leq l$, then

$$E_{k+1} \circ ... \circ E_l = \{(e_{k+1}, ..., e_l) \mid e_i \in E_i, r(e_i) = s(e_{i+1})\}$$

denotes the set of all paths from V_k to V_l. We define range and source maps on $E_{k+1} \circ \ldots \circ E_l$ by $r((e_{k+1}, ..., e_l)) = r(e_l)$ and $s((e_{k+1}), ..., e_l)) = s(e_{k+1})$.

For each Bratteli diagram (V, E), we shall now define a countable locally finite group $G(V, E) = \bigcup_{n \geq 0} G_n$ of strongly diagonal type in such a way that

1. the factors of the direct product $G_n = Alt(\Omega_{n,1}) \times ... \times Alt(\Omega_{n,d_n})$ are indexed by the set of vertices $V_n = \{v_{n,i} \mid 1 \leq i \leq d_n\}$; and

2. the subgroup $Alt(\Omega_{n,i})$ has exactly

$$m_{v_{n+1,j}, v_{n,i}} = |\{e \in E_{n+1} \mid r(e) = v_{n+1,j} \text{ and } s(e) = v_{n,i}\}|$$

   nontrivial orbits on each $\Omega_{n+1,j}$.

Definition 2.6. If (V, E) is a Bratteli diagram, then we define the locally finite group $G(V, E)$ as follows. First, let $\Omega_0 = \{1, 2, 3, 4, 5\}$ and let $G_0 = Alt(\Omega_0)$. Then, for each $n \geq 1$, let $V_n = \{v_{n,i} \mid 1 \leq i \leq d_n\}$. Let

$$P_{n,i} = \{(e_1, ...e_n) \in E_1 \circ ... \circ E_n \mid r(e_n) = v_{n,i}\}$$

be the set of all paths from $V_0 = \{v_0\}$ to $v_{n,i}$ and let

$$\Omega_{n,i} = \{(e_0, e_1, ..., e_n) \mid 1 \leq e_0 \leq 5 \text{ and } (e_1, ..., e_n) \in P_{n,i}\}.$$

Then let

$$G_n = Alt(\Omega_{n,1}) \times ... \times Alt(\Omega_{n,d_n}).$$

We regard G_n as a subgroup of G_{n+1} by identifying each element $\pi \in G_n$ with the permutation $\tilde{\pi} \in G_{n+1}$ defined by

$$\tilde{\pi}(e_0, e_1, ..., e_n, e_{n+1}) = (f_0, f_1, ..., f_n, e_{n+1})$$

where $\pi(e_0, e_1, ..., e_n) = (f_0, f_1, ..., f_n)$. Then $G(V, E)$ is the union of the strongly diagonal chain of finite groups:

$$G_0 \leq G_1 \leq ... \leq G_n \leq ...$$

Next we shall consider the question of when two Bratteli diagrams yield isomorphic groups. Firstly, there is an obvious notion of isomorphism between two Bratteli diagrams (V, E) and (V', E'); namely there exists a bijection $\varphi : V \to V'$ such that for all $n \in \omega$,

1. $\varphi[V_n] = V'_n$; and
2. $m_{\varphi(u),\varphi(v)} = m_{u,v}$ for all $v \in V_n$ and $u \in V_{n+1}$. (This implies that φ can be extended to include a bijection $\varphi : E \to E'$ which preserves the range and source maps.)

And clearly if (V, E) and (V', E') are isomorphic, then $G(V, E) \cong G(V', E')$. Secondly, let $G(V, E) = \bigcup_{n \geq 0} G_n$ be as in Definition 2.6; and let

$$0 = m_0 < m_1 < ... < m_n < ...$$

be an increasing sequence of natural numbers. Then the strongly diagonal chain

$$G_{m_0} \leq G_{m_1} \leq ... \leq G_{m_n} \leq ...$$

corresponds to the Bratteli diagram (V', E') which is obtained from (V, E) by the "telescoping" operation of the following definition.

Definition 2.7. Given a Bratteli diagram and an increasing sequence

$$0 = m_0 < m_1 < ... < m_n < ...$$

of natural numbers, we define the corresponding *telescoping* (V', E') of (V, E) to the sequence $(m_n \mid n \in \omega)$ by $V'_n = V_{m_n}$ and $E'_n = E_{m_{n-1}+1} \circ ... \circ E_{m_n}$, together with the range and source maps as defined in Definition 2.5.

Definition 2.8. We define $\sim$ to be the equivalence relation on Bratteli diagrams generated by the isomorphism and telescoping relations.

Theorem 2.9. *If (V,E), (V',E') are Bratteli diagrams, then $G(V,E) \cong G(V',E')$ if and only if $(V,E) \sim (V',E')$.*

Proof. By the discussion preceding Definition 2.7, it is clear that if (V',E') is a telescoping of (V,E), then $G(V',E') \cong G(V,E)$. Thus, $(V',E') \sim (V,E)$ implies that $G(V',E') \cong G(V,E)$.

Conversely, let us suppose that (V,E), (W,F) are Bratteli diagrams, and that $f : G(W,F) \cong G(V,E)$ is a group isomorphism. Let $G_0 \leq G_1 \leq ... \leq G_n \leq ...$ be the strongly diagonal chain of groups defined from (V,E) as in definition 2.6. Similarly, let $H_0 \leq H_1 \leq ... \leq H_n \leq ...$ be the strongly diagonal chain of groups defined from (W,F). Then since $G(V,E)$ is the union of the increasing sequence of the subgroups $G_0 \leq G_1 \leq ... \leq G_n \leq ...$, for each $p \in \omega$, there is some $p' > p$ such that $f(H_p) \subseteq G_{p'}$. Similarly, for each $m \in \omega$, there is some $m' > m$ such that $f^{-1}(G_m) \subseteq H_{m'}$. Thus there are telescopings (V',E') of (V,E) to $(m_n \mid n \in \omega)$ and (W',F') of (W,F) to $(p_n \mid n \in \omega)$ such that the following chain of embeddings exists

$$G_0 \overset{\theta_0}{\to} H_{p_0} \overset{\psi_0}{\to} G_{m_1} \overset{\theta_1}{\to} H_{p_1} \overset{\psi_1}{\to} ... \overset{\psi_{n-1}}{\to} G_{m_n} \overset{\theta_n}{\to} H_{p_n} \overset{\psi_n}{\to} G_{m_{n+1}} \overset{\theta_{n+1}}{\to} ...$$

where $\psi_n = f \restriction H_{p_n}$ and $\theta_n = f^{-1} \restriction G_{m_n}$.

Claim 1. *Each of the maps θ_n and ψ_n are diagonal embeddings.*

Claim 2. *If $n \geq 1$, then both θ_n and ψ_n satisfy part (b) of definition 2.1.*

Assuming these two claims, we have that the chain of embeddings

$$G_0 \hookrightarrow G_{m_1} \overset{\theta_1}{\to} H_{p_1} \overset{\psi_1}{\to} ... \overset{\psi_{n-1}}{\to} G_{m_n} \overset{\theta_n}{\to} H_{p_n} \overset{\psi_n}{\to} G_{m_{n+1}} \overset{\theta_{n+1}}{\to} ...$$

is of strongly diagonal type. Furthermore, it naturally defines a Bratteli diagram (U,H), such that (V',E') and (W',F') are each isomorphic to telescopings of (U,H). Then it is clear that $(V,E) \sim (V',E') \sim (U,H) \sim (W',F') \sim (W,F)$, and we are done.

Proof of claim 1. In order to prove this, we will make use of the following theorem of Zalesskii [39, Lemma 10].

Lemma 2.10. *Suppose $m > l > k > 4$. Let $\tau_1 : A(k) \to A(l)$ and $\tau_2 : A(l) \to A(m)$ be embeddings of alternating groups, and let $\tau = \tau_2 \circ \tau_1$. If τ is diagonal, then both τ_1 and τ_2 are diagonal.*

Let us suppose that there is some $\theta_n : G_{m_n} \to H_{p_n}$ which is not a diagonal embedding. (The argument for ψ_n is similar.) To simplify notation, let

$$G_{m_n} = Alt(\Omega_{a,1}) \times ... \times Alt(\Omega_{a,d_a}),$$

$$H_{p_n} = Alt(\Omega_{b,1}) \times ... \times Alt(\Omega_{b,d_b}),$$

$$G_{m_{n+1}} = Alt(\Omega_{c,1}) \times ... \times Alt(\Omega_{c,d_c}).$$

Then for any $1 \leq i \leq d_a$, consider $\theta_n \restriction Alt(\Omega_{a,i})$.

We shall first show that this map is a diagonal embedding. For any $1 \leq j \leq d_b$, let $\pi_{b,j}$ be the projection map of H_{p_n} onto the j-th factor, and for any $1 \leq k \leq d_c$, let $\pi_{c,k}$ be the projection map of $G_{m_{n+1}}$ onto the k-th factor. Then, for any $1 \leq j \leq d_b$, $\phi_k = (\pi_{c,k} \circ \psi_n) \circ (\pi_{b,j} \circ \theta_n) \restriction Alt(\Omega_{a,i})$ is a homomorphism from $Alt(\Omega_{a,i})$ to $Alt(\Omega_{c,k})$ passing through $Alt(\Omega_{b,j})$.

$$\phi_k : Alt(\Omega_{a,i}) \stackrel{\pi_{b,j}\circ\theta_n}{\longrightarrow} Alt(\Omega_{b,j}) \stackrel{\pi_{c,k}\circ\psi_n}{\longrightarrow} Alt(\Omega_{c,k})$$

Note that every orbit of $Alt(\Omega_{a,i})$ on $\Omega_{c,k}$ given by ϕ_k must be included in some orbit of $Alt(\Omega_{a,i})$ on $\Omega_{c,k}$ given by $\psi_n \circ \theta_n$. Since ψ_n is one-to-one, there must be some $1 \leq k \leq d_c$ such that some orbit of $Alt(\Omega_{a,i})$ on $\Omega_{c,k}$ given by ϕ_k is nontrivial. Fix such a value of k. Then, since the action of $Alt(\Omega_{a,i})$ on each of its orbits in $\Omega_{c,k}$ given by $\psi_n \circ \theta_n$ is either natural or trivial, then the action of $Alt(\Omega_{a,i})$ on each of its orbits in $\Omega_{c,k}$ given by ϕ_k is either natural or trivial, i.e., ϕ_k is diagonal. Since we chose k so that at least one of these orbits is non-trivial, we have that ϕ_k is an embedding. Thus by Lemma 2.10, the map $\pi_{b,j} \circ \theta_n \restriction Alt(\Omega_{a,i})$ is a diagonal embedding. Since our choice of j was arbitrary, we have that $\pi_{b,j} \circ \theta_n \restriction Alt(\Omega_{a,i})$ is a diagonal embedding for every $1 \leq j \leq d_b$, and thus $\theta_n \restriction Alt(\Omega_{a,i})$ is a diagonal embedding.

Now assume that θ_n is not a diagonal embedding. Since for each $1 \leq i \leq d_a$, $\theta_n \restriction Alt(\Omega_{a,i})$ is a diagonal embedding, there exists $1 \leq i < j \leq d_a$ and $1 \leq k \leq d_b$ such that the orbits of $Alt(\Omega_{a,i})$ and $Alt(\Omega_{a,j})$ on $\Omega_{b,k}$ are not disjoint. However, since the above argument applies also to $\psi_n \restriction Alt(\Omega_{b,k})$, we have that $\psi_n \restriction Alt(\Omega_{b,k})$ is a diagonal embedding. Thus after applying ψ_n, the orbits of $Alt(\Omega_{a,i})$ and $Alt(\Omega_{a,j})$ on $\Omega_{c,1} \sqcup \Omega_{c,2} \sqcup \ldots \sqcup \Omega_{c,d_c}$ are not disjoint. However, this violates the diagonality of the embedding $\psi_n \circ \theta_n : G_{m_n} \to G_{m_{n+1}}$. □

Proof of claim 2. Fix some $n \geq 1$. Notice that the map $\theta_n \circ \psi_{n-1} : H_{p_{n-1}} \to H_{p_n}$ is the inclusion map given from the definition of H. Thus, given $1 \leq i \leq d_{p_n}$ and some $x \in \Omega_{p_n,i}$, there is some $\pi \in H_{p_{n-1}}$ which moves x. Thus $\psi_{n-1}(\pi) \in G_{m_n}$ also moves x. The proof for ψ_n is similar. □

□

Definition 2.11. Given a Bratteli diagram (V, E), an *ideal* is a subset $V^* \subseteq V$ such that whenever $e \in E_n$ and $s(e) \in V^*$, then $r(e) \in V^*$. An ideal V^* is then said to be *proper* if for every $n < \omega$, $V^* \cap V_n \neq V_n$. A Bratteli Diagram is said to be *simple* if it has no nonempty proper ideals.

Notice that if (V, E) is a Bratteli diagram, $V^* \subseteq V$ is an ideal, and if there exists $n \in \omega$ for which $V^* \cap V_n = V_n$, then since $r^{-1}(v) \neq \emptyset$ for all $v \in V \backslash V_0$, we have that $V^* \cap V_m = V_m$ for every $m \geq n$. Thus, an ideal $V^* \subseteq V$ is proper if and only if $V \backslash V^*$ is infinite.

Theorem 2.12. *Let (V, E) be a Bratteli diagram, and let $G(V, E)$ be the corresponding locally finite group of strongly diagonal type. Then (V, E) is simple if and only if $G(V, E)$ is simple.*

Proof. We will show that (V, E) has a nonempty proper ideal if and only if $G(V, E)$ has a nontrivial normal subgroup. First assume that V^* is a nonempty proper ideal of (V, E). Then we can define a normal subgroup $H \lhd G(V, E)$ as follows. For each $n \in \omega$, let

$$G_n = Alt(\Omega_{n,1}) \times ... \times Alt(\Omega_{n,d_n})$$

be as in the definition of $G(V, E)$. Then let $H_n = \prod_{i \in V^* \cap V_n} Alt(\Omega_{n,i})$. Since V^* is an ideal, we have that $H_n < H_{n+1}$ for every $n \in \omega$. Then let $H = \bigcup_{n \in \omega} H_n$.

First, let us show that $H \lhd G(V, E)$. Let $g \in G(V, E)$ and $h \in H$ be arbitrary. Then there is some $n \in \omega$, such that $g \in G_n$ and $h \in H_n$. Then it is clear that $ghg^{-1} \in H_n$, so $ghg^{-1} \in H$.

Now we shall show that $H \neq G(V, E)$. Since V^* is a proper ideal, there is a sequence of vertices $\{v_{i_n,n} \mid n \in \omega\}$ such that $v_{i_n,n} \in V_n \backslash V^*$ for each $n \in \omega$. Now let $g_0 \in Alt(\Omega_0)$ be any nontrivial permutation. Then g_0 acts nontrivially on each $\Omega_{i_n,n}$. So for each $n \in \omega$, $g_0 \notin H_n$. Hence, $g \notin H$.

Conversely, suppose $\{1\} \neq H \lhd G(V, E)$. We will first show that for any $n \in \omega$, $H \cap G_n$ is a subproduct of

$$G_n = Alt(\Omega_{n,1}) \times ... \times Alt(\Omega_{n,d_n}).$$

Suppose that there is some $n \in \omega$ and some $1 \neq g \in H \cap G_n$, and that we can write $g = (g_1, ..., g_{d_n}) \in \prod_{1 \leq i \leq d_n} Alt(\Omega_{n,i})$. Choose $1 \leq i \leq d_n$ such that $g_i \neq 1$. Then since $Alt(\Omega_{n,i})$ is simple, for any $a \in Alt(\Omega_{n,i})$, there is $h \in H \cap G_n$ such that $h = (h_1, ..., h_{d_n}) \in \prod_{1 \leq i \leq d_n} Alt(\Omega_{n,i})$ and $h_i = a$. Since $|\Omega_{n,i}| \geq 5$, we can choose a so that $a \neq a^{-1}$ and such that a is conjugate to a^{-1} (For example, let $a = (123)$). Now since H is a subgroup of $G(V, E)$, $h^{-1} \in H$. Then since $H \lhd G(V, E)$, there is some $b \in Alt(\Omega_{n,i})$ such that $bh^{-1}b^{-1} = h^* = (h_1^{-1}, \ldots, a, \ldots, h_{d_n}^{-1}) \in H$.

Then $h^*h = (1, \ldots, 1, aa, 1, \ldots, 1) \in H$, and $h^*h \neq 1$. Now, since $Alt(\Omega_{n,i})$ is simple, $Alt(\Omega_{n,i}) \leq H$. This implies that for every n, there is some $I \subseteq \{1, ..., d_n\}$ such that $\{g \in \prod_{1 \leq i \leq d_n} Alt(\Omega_{n,i}) \mid g \in H\} = \prod_I Alt(\Omega_{n,i})$.

Now, let $\{v_{n,i}, v_{n+1,j}\} \in E$. Assume that $Alt(\Omega_{n,i}) \leq H$. Then, given some element $1 \neq g \in Alt(\Omega_{n,i})$, we have that the image of g in G_{n+1} can be expressed as $(g_1, \ldots, g_{d_{n+1}})$ where $g_j \neq 1$. Thus, by the above argument, $Alt(\Omega_{n+1,j}) \leq H$. Hence, $V^* = \{v_{n,i} \mid Alt(\Omega_{n,i}) \leq H\}$ is an ideal of (V, E).

Finally, $\{1\} \neq H$ implies $\emptyset \neq V^*$, and if V^* were not a proper ideal, then there would be some $n \in \omega$ such that $H \cap G_m = G_m$ for every $m > n$, and then we would have $H = G(V, E)$. $\square$

Notice that we have also shown that the following type of subgroup of a locally finite group of diagonal type is always normal.

Definition 2.13. Let G be a countable locally finite group which is the union of the diagonal chain

$$G_0 \leq G_1 \leq \ldots \leq G_n \leq \ldots$$

where each G_n is the product of finite alternating groups

$$G_n = Alt(\Omega_{n,1}) \times ... \times Alt(\Omega_{n,d_n}).$$

Then for each $n < m$ and $1 \leq k \leq d_n$, let

$$I_{n,k,m} = \{1 \leq i \leq d_m \mid Alt(\Omega_{n,k}) \text{ has a nontrivial orbit on } \Omega_{m,i}\},$$

and define

$$\lceil Alt(\Omega_{n,k}) \rceil = \bigcup_{m>n} \left(\prod_{i \in I_{n,k,m}} Alt(\Omega_{m,i}) \right).$$

2.3 The corresponding standard Borel spaces

Definition 2.14. Let $\mathcal{L} = (1, \cdot)$ be the language of group theory, and consider $X_{\mathcal{L}}$ as in the paragraph preceeding Definition 1.1. We will let $\mathcal{SDT}$ be the subspace of countable simple locally finite groups of strongly diagonal type. Then denote the isomorphism relation on $\mathcal{SDT}$ by $\cong_{\mathcal{SDT}}$.

The next two results show that the class of countably infinite simple locally finite groups of (strongly) diagonal type can be axiomatized by an $\mathcal{L}_{\omega_1 \omega}$-sentence, and thus by Corollary 1.5 is a standard Borel space.

Theorem 2.15. *A countably infinite simple locally finite group G is of diagonal type if and only if the following conditions are satisfied.*

(a) *Every finite subset X of G is contained in a finite subgroup of G which is a direct product of alternating groups.*

(b) *There exists a finite subgroup F of G such that whenever*

$$F \leq A_1 \times \ldots \times A_n < G$$

where each A_i is an alternating group on a finite set Ω_i, then F has no regular orbits on any of the Ω_i.

Proof. Assume that G satisfies conditions (a) and (b). Then condition (a) allows us to express G as the increasing union of finite subgroups, each of which is the direct product of alternating groups, say

$$G_0 \leq G_1 \leq \ldots \leq G_n \leq \ldots$$

Theorem 2.3 implies that we may then select a subchain which is either of diagonal type or of regular type. However, condition (b) implies that G is not expressible as a union of a chain of regular type. Thus G must be of diagonal type.

Conversely, let G be the union of the diagonal chain of finite subgroups, each of which is the direct product of alternating groups,

$$G_0 \leq G_1 \leq \ldots \leq G_n \leq \ldots$$

Then G clearly satisfies condition (a). Moreover, there exists an $m \geq 0$ such that G_m has a factor of the form $Alt(\Omega)$ for some finite set $|\Omega| \geq 5$. Otherwise, G would be locally solvable, and there are no infinite locally solvable simple groups. (For example, this follows directly from [27, Corollary 1.B.5]). So let $Alt(\Omega)$ be such a factor. Since $\lceil Alt(\Omega) \rceil \lhd G$, we actually have that $\lceil Alt(\Omega) \rceil = G$. Thus we may assume that $G_0 = Alt(\Omega)$. Now suppose that $G_0 \leq A_1 \times \ldots \times A_n < G$, where each A_i is a finite alternating group. Then there is some $l > 0$ so that $G_0 \leq A_1 \times \ldots \times A_n < G_l$. Then arguing as in the proof of Claim 1 of Theorem 2.9, we see that the embedding $G_0 \leq A_1 \times \ldots \times A_n$ is diagonal. Thus G satisfies condition (b) with respect to $F = G_0$. □

Corollary 2.16. *A countably infinite simple locally finite group G is of strongly diagonal type iff the following conditions are satisfied.*

(a) *There exists a finite subgroup G_0 such that every finite subset X of G is contained in a finite subgroup*

$$G_0 \cup X \subseteq A_1 \times \ldots \times A_n < G,$$

where each A_i is an alternating group on a finite set Ω_i and each element of $\sqcup \Omega_i$ lies in some nontrivial G_0-orbit.

(b) There exists a finite subgroup F of G such that whenever

$$F \leq A_1 \times \ldots \times A_n < G,$$

where each A_i is an alternating group on a finite set Ω_i, then F has no regular orbits on any of the Ω_i.

Proof. As before, condition (a) allows us to build an appropriate chain of subgroups, and condition (b) together with Theorem 2.2 ensure that it is of diagonal type. The new clause in condition (a) ensures that the chain is *strongly* diagonal. On the other hand, if G is the union of the strongly diagonal chain of finite subgroups

$$G_0 \leq G_1 \leq \ldots \leq G_n \leq \ldots$$

then G clearly satisfies condition (a). Arguing as in the previous theorem, we see that G must also satisfy condition (b). $\square$

We now work to encode the class of simple Bratteli diagrams into an appropriate standard Borel space, which we call $\mathcal{BD}$. We then show that $\cong_{\mathcal{SDT}}$ and the relation $\sim$ on $\mathcal{BD}$ are Borel bireducible.

Definition 2.17. We encode $\mathcal{BD}$, the standard Borel space of simple Bratteli diagrams, as follows. First we encode each Bratteli diagram as a member of the standard Borel space $(\mathbb{N} \times \mathbb{N})^{\mathbb{N}}$. Fix a particular Bratteli diagram (V, E). We will associate to it a function $f \in (\mathbb{N} \times \mathbb{N})^{\mathbb{N}}$. We may assume that $V = \{n \in \mathbb{N} \mid n \text{ even}\}$ with $V_0 = \{0\}$ and that $E = \{n \in \mathbb{N} \mid n \text{ odd}\}$. Then encode the source and range maps by setting, for each edge $e \in E$, $f(e) = (s(e), r(e))$. Next encode the levels of V by setting, for each $v \in V_n$, $f(v) = (0, n)$. Finally, we let $\sim$ denote the equivalence relation on $\mathcal{BD}$ given by Definition 2.8.

Lemma 2.18. *$\mathcal{BD}$ is a standard Borel space.*

Proof. Given a function $f \in (\mathbb{N} \times \mathbb{N})^{\mathbb{N}}$ and $f(n) = (i, j)$, use $l_f(n) = i$ and $r_f(n) = j$ to denote the corresponding projections. Then notice that $\mathcal{BD}$ is the set of functions $f \in (\mathbb{N} \times \mathbb{N})^{\mathbb{N}}$ which satisfy the following conditions:

- If n is even, then $l_f(n) = 0$.
- $f(n) = (0, 0)$ if and only if $n = 0$.
- For each odd $n \in \mathbb{N}$, if $f(n) = (i, j)$, then i, j are even and $r_f(j) = r_f(i) + 1$.
- For each $j \in \mathbb{N}$, $\{n \in \mathbb{N} \mid n \text{ is even and } r_f(n) = j\}$ is nonempty and finite.
- For each even $i \in \mathbb{N}$, $\{n \in \mathbb{N} \mid n \text{ is odd and } l_f(n) = i\}$ is nonempty and finite.

- For each even $j \in \mathbb{N}\backslash\{0\}$, $\{n \in \mathbb{N} \mid n \text{ is odd and } r_f(n) = j\}$ is nonempty and finite.

  It is now evident that we may similarly express the following condition, which defines simplicity for Bratteli diagrams.

- $(\forall v \in V_n)(\exists m > n)(\forall w \in V_m)$(there is a path in $E_{n+1} \circ \ldots \circ E_m$ from v to w).

Clearly $\mathcal{BD}$ is a Borel subset of the Polish space $(\mathbb{N} \times \mathbb{N})^{\mathbb{N}}$, and so by Theorem 1.4 is a standard Borel space. □

Lemma 2.19. $(\sim) \leq_B (\cong_{\mathcal{SDT}})$

Proof. Theorems 2.9 and 2.12 show that the map $(V, E) \mapsto G(V, E)$ is a Borel reduction from $\sim$ to $\cong_{\mathcal{SDT}}$ □

Lemma 2.20. $(\cong_{\mathcal{SDT}}) \leq_B (\sim)$

Proof. Given a group $G \in \mathcal{SDT}$ together with a chain of strongly diagonal type

$$G_0 \leq G_1 \leq \ldots \leq G_n \leq \ldots \leq G$$

such that G is the limit of the G_n, we may naturally choose a Bratteli diagram (V, E) so that $G \cong G(V, E)$. Thus if we show how, given a group $G \in \mathcal{SDT}$, to explicitly choose an appropriate chain of subgroups, then Theorems 2.9 and 2.12 imply that this assignment would give us a Borel reduction from $\cong_{\mathcal{SDT}}$ to $\sim$.

Since each $G \in \mathcal{SDT}$ has $\mathbb{N}$ as its underlying set we may, after fixing a well order on the finite subsets of $\mathbb{N}$, refer to the least finite subset of G satisfying a given property. So we begin by choosing as G_0 the least subset of G which satisfies part (b) of Corollary 2.16. Then given G_n we choose as G_{n+1} the least subset of G which satisfies part (a) of Corollary 2.16 with respect to $G_n \cup \{0, 1, \ldots, n\}$. Then Theorem 2.3 together with part (b) of Corollary 2.16 tell us that we may choose a subchain of diagonal type. Finally, the second clause of part (a) of Corollary 2.16 assures us that this subchain is *strongly* diagonal. □

Chapter 3
Dimension groups

In this chapter, for each Bratteli diagram (V, E), we shall define an associated dimension group $K_0(V, E)$. We note that, for any Bratteli diagrams (V, E) and (W, F), $(V, E) \sim (W, F)$ if and only if $K_0(V, E)$ and $K_0(W, F)$ are isomorphic. We then show that (V, E) is simple if and only if $K_0(V, E)$ is simple.

Definition 3.1. If (V, E) is a Bratteli diagram, then we can explicitly define an associated dimension group $K_0(V, E)$, equipped with a distinguished order unit, as follows. For each integer $n \in \omega$, let $\mathbb{Z}^{V_n}$ be the free abelian group on the set of vertices $V_n = \{v_{n,i} \mid 1 \leq i \leq d_n\}$. We regard $\mathbb{Z}^{V_n}$ as an ordered abelian group with positive cone

$$\left(\mathbb{Z}^{V_n}\right)^+ = \left\{\sum_{i=1}^{d_n} z_i v_{n,i} \mid z_i \geq 0 \text{ for all } 1 \leq i \leq d_n\right\}.$$

For each $n \geq 1$, let $\varphi_n : \mathbb{Z}^{V_{n-1}} \to \mathbb{Z}^{V_n}$ be the homomorphism given by matrix multiplication by the incidence matrix M_n from Definition 2.4. Since all of the entries of M_n are nonnegative, $\varphi_n[(\mathbb{Z}^{V_{n-1}})^+] \subseteq (\mathbb{Z}^{V_n})^+$. Then we define $K_0(V, E)$ to be the direct limit of the system of ordered groups

$$\mathbb{Z}^{V_0} \xrightarrow{\varphi_1} \mathbb{Z}^{V_1} \xrightarrow{\varphi_2} \mathbb{Z}^{V_2} \xrightarrow{\varphi_3} \dots \xrightarrow{\varphi_n} \mathbb{Z}^{V_n} \xrightarrow{\varphi_{n+1}} \dots$$

endowed with the induced order. Given a group element $a \in \mathbb{Z}^{V_n}$ for some $n \in \mathbb{N}$, we define $[a] = \lim_{m \geq n} (\varphi_m \circ \dots \circ \varphi_{n+1})(a) \in K_0(V, E)$. Thus, if there are $n, m \in \mathbb{N}$ such that $a \in \mathbb{Z}^{V_n}$ and $b \in \mathbb{Z}^{V_m}$, then $[a] = [b]$ if and only if there is some $l > n, m$ such that $(\varphi_l \circ \dots \circ \varphi_{n+1})(a) = (\varphi_l \circ \dots \circ \varphi_{m+1})(b)$. Notice also that $x \in (K_0(V, E))^+$ if and only if there is some $n \in \omega$ and $a \in \left(\mathbb{Z}^{V_n}\right)^+$ such that $x = [a]$.

Lemma 3.2. *[9] $K_0(V, E)$ is a dimension group.*

Proof. It is routine to check that $K_0(V, E)$ is an unperforated ordered abelian group, so we will only verify the Riesz interpolation property. Let $a_i, b_j \in K_0(V, E)$ where $a_i \leq b_j$ $(1 \leq i, j \leq 2)$. Then since $b_j - a_i \in K_0(V, E)^+$ for $1 \leq i, j \leq 2$, there is some $n \in \mathbb{N}$ and some $a'_i, b'_j \in \mathbb{Z}^{V_n}$ such that $[a'_i] = a_i$, $[b'_j] = b_j$, and $b'_j - a'_i \in \left(\mathbb{Z}^{V_n}\right)^+$ for $1 \leq i, j \leq 2$. That is, $a'_i \leq b'_j$ for $1 \leq i, j \leq 2$. Since $\mathbb{Z}^{V_n}$ is a dimension group, there is some $c' \in \mathbb{Z}^{V_n}$ such that $a'_i \leq c' \leq b'_j$ for $1 \leq i, j \leq 2$. Thus, letting $c = [c']$, we have that $a_i \leq c \leq b_j$ for $1 \leq i, j \leq 2$. $\square$

Finally, the *distinguished order unit* is the element of $K_0(V,E)^+$ corresponding to the element $v_0 \in \mathbb{Z}^{V_0}$.

Theorem 3.3. *[8] If (V,E) and (V',E') are Bratteli diagrams, then $(V,E) \sim (V',E')$ if and only if the ordered groups $K_0(V,E)$ and $K_0(V',E')$ are isomorphic via a map sending the distinguished order unit v_0 of $K_0(V,E)$ to the distinguished order unit v_0' of $K_0(V',E')$.*

Theorem 3.4. *Let (V,E) be a Bratteli diagram, and let $K_0(V,E)$ be the corresponding dimension group. Then (V,E) is simple if and only if $K_0(V,E)$ is simple.*

Proof. First, let $V^* \subset V$ be a nonempty proper ideal of (V,E), and let

$$X = \bigcup_{n\in\omega} \left\{ \sum_{i=1}^{d_n} z_i v_{n,i} \in \mathbb{Z}^{V_n} \mid z_i \neq 0 \Rightarrow v_{i,n} \in V^* \right\}.$$

Note that X is "upward closed" in the sense that if $a \in X \cap \mathbb{Z}^{V_n}$, $b \in \mathbb{Z}^{V_{n+1}}$, and $b = \varphi_{n+1}(a)$, then $b \in X$. We show that $J = \{[a] \mid a \in X\}$ is an ideal of A.

We first prove that J is in fact a subgroup of $K_0(V,E)$. Clearly, $a \in X$ implies $a^{-1} \in X$, and so $[a] \in J$ implies $[a^{-1}] \in J$. Now let $[a],[b] \in J$, where $a \in X \cap V_m$ and $b \in X \cap V_l$. Then there is some $n \geq m,l$ and some $c,d \in \mathbb{Z}^{V_n}$ such that $[a]=[c]$ and $[b]=[d]$. Then $c,d \in X$, and so $c+d \in X \cap \mathbb{Z}^{V_n}$, and so $[a]+[b]=[c]+[d]=[c+d] \in J$.

Now suppose $0 \leq a \leq b \in J$. Then $a=[x]$ where $x \in \mathbb{Z}^{V_m}$ for some m, and $b=[y]$ where $y \in X \cap \mathbb{Z}^{V_n}$ for some n. Then there is some $l \geq m,n$ and $x',y' \in \mathbb{Z}^{V_l}$ such that $a=[x']$, and $b=[y']$ and $0 \leq x' \leq y'$. Then $y' \in X$ implies $x' \in X$. Thus $a \in J$.

If we let

$$X^+ = \bigcup_{n<\omega} \{\sum_{i=1}^{d_n} z_i v_{n,i} \in \left(\mathbb{Z}^{V_n}\right)^+ \mid z_i \neq 0 \Rightarrow v_{i,n} \in V^*\},$$

then $J^+ = \{[a] \mid a \in X^+\}$, and clearly $J = J^+ - J^+$. Thus J is an ideal of $K_0(V,E)$.

Note that $\emptyset \neq V^*$ implies $\{0\} \neq J$. To show that $J \neq K_0(V,E)$, consider the element of $K_0(V,E)$ corresponding to $v_0 \in \mathbb{Z}^{V_0}$. We claim that $[v_0] \notin J$. Otherwise, there is some $a \in X$ such that $[a]=[v_0]$. Suppose $n \in \mathbb{N}$ and $a = \sum_{i=1}^{d_n} z_i v_{n,i} \in \mathbb{Z}^{V_n}$. Then there must be some $l > n$ such that $(\varphi_l \circ \ldots \circ \varphi_{n+1})(a) = (\varphi_l \circ \ldots \circ \varphi_1)(v_0)$. However, since all of the source maps associated with (V,E) are nonempty, $(\varphi_l \circ \ldots \circ \varphi_1)(v_0)$ has all positive coordinates. But since $(\varphi_l \circ \ldots \circ \varphi_{n+1})(a) \in X$, this implies that $V^* \cap V_l = V_l$ and this contradicts the assumption that V^* is a proper ideal.

Conversely, let $J \neq \{0\}$ be an ideal of $K_0(V,E)$ and let

$$V^* = \{v_{n,i} \mid \exists a = \sum_{i=1}^{d_n} a_i v_{n,i} \in \left(\mathbb{Z}^{V_n}\right)^+ \text{ such that } [a] \in J^+ \text{ and } a_i > 0\}.$$

Then V^* has the property that whenever $e \in E_n$ and $s(e) \in V^*$, then $r(e) \in V^*$. Since $J = J^+ - J^+$ and $J \neq \{0\}$, then $J^+ \neq \{0\}$, and so $V^* \neq \emptyset$.

Now assume that V^* is not a proper ideal. Then there is some $n \in \mathbb{N}$ such that $V^* \cap V_m = V_m$ for every $m \geq n$. It is clear that if $V^* \cap V_m = V_m$, then $\mathbb{Z}^{V_m} \subseteq J$. Hence $\mathbb{Z}^{V_m} \subseteq J$ for all $m \geq n$, and thus $J = K_0(V, E)$. $\square$

Chapter 4
The positive cone of a simple dimension group

We now turn our attention to the structure of dimension groups. The beginning of this chapter follows Chapter 4 of [10]. First we observe that if $a \leq b$ and $c \leq d$, then $a + c \leq b + d$, and if $na \leq nb$ for some $n \in \mathbb{N}$, then $a \leq b$. The following lemma gives a useful characterization of the Riesz interpolation property. The proof follows that of Effros [10, Lemma A3.1].

Lemma 4.1. *[29, pp. 175-6][3, Theorem 49] Let A be an unperforated ordered group. Then the following are equivalent properties for A:*

(1) Given $a_i \leq b_j$ $(1 \leq i, j \leq 2)$ there exists some $c \in A$ with $a_i \leq c \leq b_j$ for all i, j.

(2) Given $a_i \leq b_j$ $(i = 1, ..., r; j = 1, ..., s)$ there is some $c \in A$ with $a_i \leq c \leq b_j$ for all i, j.

(3) If $0 \leq a \leq b_1 + \cdots + b_s$, and $0 \leq b_i$ $(1 \leq i \leq s)$, then there exist $a_i \in A$ $(1 \leq i \leq s)$ with $0 \leq a_i \leq b_j$ $(1 \leq i, j \leq s)$ and $a = a_1 + \cdots + a_s$.

(4) If $\sum_{i=1}^{r} a_i = \sum_{j=1}^{s} b_j, a_i, b_j \geq 0$, then there exist $c_{ij} \in A^+$ with $a_i = \sum_j c_{ij}$ and $b_j = \sum_i c_{ij}$.

Proof. First let

$$[0, b] = \{a \in A \mid 0 \leq a \leq b\},$$

rewrite (3) as

$$[0, b_1 + \ldots + b_s] = [0, b_1] + \ldots + [0, b_s],$$

and represent (4) as a table as follows:

	b_1	$\ldots$	b_s
a_1	c_{11}	$\ldots$	c_{1s}
a_2	c_{21}	$\ldots$	c_{2s}
$\vdots$	$\vdots$	$\ldots$	$\vdots$
a_r	c_{r1}	$\ldots$	c_{rs}

In (1) or (2), we say that the element $c \in A$ *interpolates*, or that we can *interpolate with c*. We refer to either (3) or (4) as the *Riesz decomposition property.*

(1) $\Rightarrow$ (2). We assume (1) and prove (2) by induction on $r+s$. If $r+s \leq 4$, then either a_1 interpolates (if $r=1$), b_1 interpolates (if $s=1$), or (2) follows immediately from (1) (if $r=s=2$). Assume then that $r+s \geq 5$. Assume $r \geq 3$ (otherwise $s \geq 3$ and the proof is similar). Then by induction there is some $c' \in A$ such that $a_1, a_2, \ldots, a_{r-1} \leq c' \leq b_1, b_2, \ldots, b_s$. Then again by induction there is some $c \in A$ such that $c', a_r \leq c \leq b_1, b_2, \ldots, b_s$. Then $a_1, a_2, \ldots, a_r \leq c \leq b_1, b_2, \ldots, b_s$.

(2) $\Rightarrow$ (3). We assume (2) and prove (3) by induction on s. If $s=2$ then $0, a-b_1 \leq a, b_2$. Thus there is some $c \in A$ such that $0, a-b_1 \leq c \leq a, b_2$. Then $0 \leq c \leq b_2$ and $0 \leq a-c \leq b_1$ and clearly $a+(a-c)=a$. Thus we have shown that $[0, b_1+b_2] = [0, b_1] + [0, b_2]$. Then $[0, b_1+\ldots+b_s] = [0, b_1]+\ldots+[0, b_s]$ follows by induction.

(3) $\Rightarrow$ (4). Once again, we use induction on $r+s$. If $r+s=2$, then (4) is trivial, so assume that $r+s \geq 3$. Assume $r \geq 2$ (otherwise $s \geq 2$ and the proof is similar). Then $0 \leq a_r \leq \sum_{j=1}^{s} b_j$, and so by (3) there are $c_{r1}, c_{r2}, \ldots, c_{rs}$ such that $0 \leq c_{rj} \leq b_j$ for each $1 \leq j \leq s$ and that $c_{r1}+c_{r2}+\ldots+c_{rs} = a_r$. Then by induction we can decompose $\sum_{i=1}^{r-1} a_i = \sum_{j=1}^{s}(b_j - c_{sj})$ as

$$\begin{array}{c|ccc} & b_1 - c_{r1} & \ldots & b_s - c_{rs} \\ \hline a_1 & c_{11} & \ldots & c_{1s} \\ a_2 & c_{21} & \ldots & c_{2s} \\ \vdots & \vdots & \ldots & \vdots \\ a_{r-1} & c_{(r-1)1} & \ldots & c_{(r-1)s} \end{array}$$

Then we obtain (4) by adding the row

$$\begin{array}{c|ccc} a_r & c_{r1} & \ldots & c_{rs} \end{array}$$

to the bottom of the above matrix.

(4) $\Rightarrow$ (1). Assume $a_1, a_2 \leq b_1, b_2$. Then $(b_1-a_1), (b_1-a_2), (b_2-a_1), (b_2-a_2) \geq 0$ and $(b_1-a_1)+(b_2-a_2) = (b_2-a_1)+(b_1-a_2)$. Then by (4) we have the following decomposition:

$$\begin{array}{c|cc} & b_1 - a_2 & b_2 - a_1 \\ \hline b_1 - a_1 & c_{11} & c_{12} \\ b_2 - a_2 & c_{21} & c_{22} \end{array}$$

where $c_{ij} \geq 0$. We claim that $c = b_1 - c_{11}$ interpolates. $c \leq b_1$ is immediate. Then $c_{11} \leq b_1 - a_1$ implies $c \geq a_1$, and $c_{11} \leq b_1 - a_2$ implies $c \geq a_2$. Finally, $b_1 - a_1 = c_{11} + c_{12}$ implies that $c = b_1 - c_{11} = a_1 + c_{12} \leq a_1 + (b_2 - a_1) = b_2$. $\square$

Recall that if A is a dimension group, then a subgroup J is an *ideal* if $J = J^+ - J^+$ (where $J^+ = J \cap A^+$) and $0 \leq a \leq b \in J$ implies $a \in J$. Now a *face* F in A^+ is defined to be a subset $F \subseteq A^+$ satisfying $F + F \subseteq F$ and $0 \leq a \leq b \in F$ implies $a \in F$. It is easy to check that $J \mapsto F = J^+$ provides a one-to-one correspondence between the ideals in A and the faces in A^+. In particular if $b \in A^+$ and we let

$$[b] = \{a \in A \mid 0 \leq a \leq nb \text{ for some } n \in \mathbb{N}\}$$

then $[b]$ is a face, and $J = [b] - [b]$ is the smallest ideal containing b. Recall that given a dimension group A, an element $u \in A^+$ is an order unit if $[u] = A^+$. In particular, if A is a simple dimension group, then $a \in A^+ \setminus \{0\}$ implies the ideal $[a] - [a]$ is the whole of A, i.e., every element of $A^+ \setminus \{0\}$ is an order unit. Similarly, if every $a \in A^+ \setminus \{0\}$ is an order unit, then A must be simple. (There are, in fact, non-simple dimension groups which have no order units.)

If A is a dimension group, we say an element $b \in A$ is *minimal* if $b > 0$ and $0 \leq a \leq b$ implies $a = 0$ or $a = b$. If $b \in A^+$ is minimal then $[b] = \mathbb{Z}^+ b$. To see this, note that if $0 \leq a \leq nb$, then $a = a_1 + \cdots + a_n$ where $0 \leq a_i \leq b$ (see Lemma 4.1(3)), and thus $a_i = 0$ or $a_i = b$ for each i. Thus if A is a simple dimension group with a minimal positive element, then A is isomorphic to $\mathbb{Z}$ as a group. Since there are exactly two orderings on $\mathbb{Z}$, we see that $(A, A^+) \cong (\mathbb{Z}, \mathbb{Z}^+)$ as a dimension group. In other words, if $(A, A^+) \not\cong (\mathbb{Z}, \mathbb{Z}^+)$ is simple, then A contains no minimal positive elements. The following is included in [11, Corollary 1.2].

Lemma 4.2. *If $(A, A^+) \not\cong (\mathbb{Z}, \mathbb{Z}^+)$ is a simple dimension group, then (A, A^+) satisfies the strong Riesz interpolation property: given elements $a, b, c, d \in A$, if $a, b < c, d$, then there is some $e \in A$ so that $a, b < e < c, d$.*

Proof. By the usual Riesz interpolation property, we know that there is some $f \in A$ with $a, b \leq f \leq c, d$. If $a, b < f < c, d$, then we are done, so assume for example that $f = b$ (the other cases are similar). Then $a \leq b = f < c, d$, and $0 < (c - f), (d - f)$. But then since (A, A^+) is simple, $(c - f)$ is an order unit, and so there is some positive $n \in \mathbb{N}$ such that $0 < (d - f) < n(c - f)$. Now, since (A, A^+) contains no minimal positive element, there is some $\epsilon \in A$ with $0 < \epsilon < (d - f) < n(c - f)$. Then by the Riesz Decomposition Property, there are $\epsilon_1, \ldots, \epsilon_n \in A^+$ so that $\epsilon = \epsilon_1 + \ldots + \epsilon_n$, and $\epsilon_i \leq (c - f)$ for each $1 \leq i \leq n$. Next fix i so that $\epsilon_i > 0$. Again since (A, A^+) has no minimal positive elements, there is some $\delta \in A^+$ such that $0 < \delta < \epsilon_i$. Then we interpolate with $e = f + \delta$. Clearly $a, b < e$, since $\delta > 0$. Furthermore $e < c, d$, since $(d - f), (c - f) > \delta$. $\square$

We will soon see that the existence of an order unit allows much insight into the structure of dimension groups. So from now on, we will work with the standard Borel space of simple dimension groups *with a distinguished order unit*, whose definition we restate here.

Definition 4.3. Let $n \geq 1$ and consider the standard Borel space $R(\mathbb{Q}^n) \times \mathcal{P}(\mathbb{Q}^n) \times \mathbb{Q}^n$ where $\mathcal{P}(\mathbb{Q}^n)$ denotes the power set of $\mathbb{Q}^n$. Let SDG_n denote the Borel subset of

$R(\mathbb{Q}^n) \times \mathcal{P}(\mathbb{Q}^n) \times \mathbb{Q}^n$ given by those (A, A^+, u) such that (A, A^+) is a simple dimension group (of rank n) and $u \in A^+ \setminus \{0\}$. (Here we see that simplicity may be encoded by an $\mathcal{L}_{\omega_1\omega}$-sentence, since it is equivalent to asserting that every non-zero element of A^+ is an order unit.) Let $\cong_n^+$ denote the isomorphism relation on SDG_n. Since this is the orbit equivalence relation given by the diagonal action of $GL_n(\mathbb{Q})$ on $SDG_n \subseteq R(\mathbb{Q}^n) \times P(\mathbb{Q}^n) \times \mathbb{Q}^n$, we see that $\cong_n^+$ is a countable Borel equivalence relation.

Our focus for the remainder of this thesis is the following theorem. We prove it for $n = 1, 2$ in Chapter 6, and for the rest of the cases in Chapters 7 and 8. In Chapter 9, we use it to prove similar theorems about simple Bratteli diagrams and simple locally finite groups of strongly diagonal type.

Theorem 4.4. *For all $n \geq 1$, $(\cong_n^+) <_B (\cong_{n+1}^+)$*

In order to prove this for $n \geq 3$, we will first need to study the space of states of a simple dimension group of finite rank. Fix $n \geq 3$ and some $(A, A^+, u) \in SDG_n$. We say that a homomorphism $p : A \to \mathbb{R}$ is a *state* if p is positive (i.e., $p(A^+) \geq 0$), and $p(u) = 1$. We let $S_u(A, A^+)$ be the set of all states on (A, A^+, u), and we give it the weakest topology for which each of the functions $\hat{a} : f \mapsto f(a)$ $(a \in A)$ is continuous.

It is clear that $S_u(A, A^+)$ is convex. Since A has finite rank, it follows that $S_u(A, A^+)$ is compact. To see this, let e_i $(1 \leq i \leq k)$ be a maximally linearly independent set of elements of A^+. Then since $A = A^+ - A^+$, any state p is determined by $p(e_i)$, $(1 \leq i \leq k)$. Also there is some integer $z_i \in \mathbb{N}$ such that $0 \leq e_i \leq z_i u$. Thus $0 \leq p(e_i) \leq z_i$, regardless of the choice of p. Hence any sequence of states must have a convergent subsequence, and it follows that $S_u(A, A^+)$ is compact.

Now since $S_u(A, A^+)$ is a convex compact subset of the locally convex space $\mathbb{R}^A$ of all functions $f : A \to \mathbb{R}$ equipped with the product topology, the Krein-Milman theorem says that $S_u(A, A^+)$ is the convex hull of its extreme points. Let $E(S_u(A, A^+))$ be this set of extreme points. Our goals for the remainder of this chapter are

1. to show that $E(S_u(A, A^+))$ is finite, and

2. to explore the manner in which $E(S_u(A, A^+))$ determines the positive cone A^+.

Understanding the nature of $E(S_u(A, A^+))$ will be crucial in Chapter 8.

4.1 $E(S_u(A, A^+))$ is finite.

Fix $n \geq 3$ and some $(A, A^+, u) \in SDG_n$. Following Chapter 10 of Goodearl [18], we define a *classical simplex* to be the convex hull of finitely many affinely independent points in a real vector space. We define below an infinite-dimensional analogue of a classical simplex which we will call a *simplex*. In particular, a classical simplex will be a simplex, and the extreme points of a simplex will be affinely independent. Then we will show that $S_u(A, A^+)$ is a simplex, and thus the extreme points of $S_u(A, A^+)$ are affinely independent.

Assume for a moment that the extreme points of a $S_u(A, A^+)$ are affinely independent. Let e_i, $(1 \le i \le n)$ be linearly independent elements of A, and define homomorphisms $f_i : A \to \mathbb{R}$ by $f_i(e_j) = 1$ if $i = j$ and $f_i(e_j) = 0$ if $i \ne j$. Since the elements of $S_u(A, A^+)$ are linear combinations of the functions f_i, $E(S_u(A, A^+))$ must be finite. (Notice that this actually means that $S_u(A, A^+)$ is a classical simplex.) In order to generalize the definition of a classical simplex, we need some notions from convexity theory.

Definition 4.5. (a) A *convex cone* in a real vector space E is any subset C such that $C + C \subseteq C$ and $aC \subseteq C$ for all $a \in \mathbb{R}^+$. A convex cone C is *strict* if $C \cap (-C) = \{0\}$.

(b) A partially ordered set is a *lattice* if any two elements have a least upper bound and greatest lower bound.

(c) A *lattice cone* of a real vector space is any strict cone C that is a lattice under the order defined by $a \le b \iff b - a \in C$.

(d) A *base* of a convex cone C is any subset K so that every nonzero element of C can be uniquely expressed as αx for some $\alpha \in \mathbb{R}^+$ and $x \in K$.

(e) A *simplex* in a real vector space E is a compact subset of E which is affinely isomorphic to a base for a lattice cone in some real vector space.

First, note that if K is an l-dimensional classical simplex, then it is affinely isomorphic to the convex hull (in $\mathbb{R}^{l+1}$) of the basis vectors

$$(1, 0, \ldots, 0) \quad (0, 1, \ldots, 0) \quad \ldots \quad (0, 0, \ldots, 1)$$

and this is the base for the usual positive cone of $\mathbb{R}^{l+1}$, which is lattice-ordered. Thus any classical simplex is a simplex. The next proposition is folklore for convexity theorists. (See, for example, Proposition 10.7 and Corollary 10.8 in Goodearl [18].)

Proposition 4.6. *If K is a simplex in a real vector space E, then the set of extreme points of K is an affinely independent subset of E.*

In the remainder of this section, we show that $S_u(A, A^+)$ is a simplex. We say that a homomorphism f from A to $\mathbb{R}$ is *relatively bounded* if, given any bounded subset $K \subset A$, then $f(K)$ is bounded as well. Notice that the set of relatively bounded homomorphisms from A to $\mathbb{R}$ forms a real vector space.

Claim. *The set of positive homomorphisms from A to $\mathbb{R}$ forms a strict cone in the space of relatively bounded homomorphisms $f : A \to \mathbb{R}$.*

Proof. The set of positive homomorphisms is clearly closed under addition and scalar multiplication by $r \in \mathbb{R}^+$. So we need to show that if f is a positive homomorphism from A to $\mathbb{R}$, then f is relatively bounded. Let $X \subseteq A$ be a bounded set, and let $a \in A^+$ be a bound for X. That is, $x \in X$ implies $-a \leq x \leq a$. Then there is some natural number n such that $a \leq nu$, and thus $f(a) \leq n$. Thus $f(X)$ is bounded by $-n$ and n. $\square$

Claim. *The state space $S_u(A, A^+)$ forms a base of this cone.*

Proof. If f is a positive homomorphism from A to $\mathbb{R}$ such that $f \neq 0$, then $f(u) > 0$. Thus $f(u)^{-1}f \in S_u(A, A^+)$, and $f = f(u)[f(u)^{-1}f]$ is the unique way to represent f as a scalar multiple of an element of $S_u(A, A^+)$. $\square$

Thus we are only left to show:

Lemma 4.7. *[12] Let (A, A^+) be a dimension group, and let $\hom(A, \mathbb{R})^+$ be the space of positive homomorphisms from A to $\mathbb{R}$ partially ordered by*

$$f \leq g \iff f(a) \leq g(a) \text{ for all } a \in A^+.$$

Then $\hom(A, \mathbb{R})^+$ is lattice ordered.

Proof of Lemma 4.7. Let $f_1, f_2 \in \hom(A, \mathbb{R})^+$ be any two positive homomorphisms. We construct a least upper bound by setting, for each $a \in A^+$

$$(f_1 \vee f_2)(a) = \sup\{f_1(a_1) + f_2(a_2) \mid a_1 + a_2 = a \text{ and } a_1, a_2 \in A^+\}.$$

Then, $(f_1 \vee f_2)$ has a unique extension to A, since $A = A^+ - A^+$. We then claim that if $a + b = c$ $(a, b \in A^+)$, then $(f_1 \vee f_2)(a) + (f_1 \vee f_2)(b) = (f_1 \vee f_2)(c)$. Certainly we have that $(f_1 \vee f_2)(a) + (f_1 \vee f_2)(b) \leq (f_1 \vee f_2)(c)$, since for any decomposition $a = a_1 + a_2$ and $b = b_1 + b_2$ $(a_1, a_2, b_1, b_2 \in A^+)$, we have

$$(f_1 \vee f_2)(c) \geq f_1(a_1 + b_1) + f_2(a_2 + b_2) = [f_1(a_1) + f_2(a_2)] + [f_1(b_1) + f_2(b_2)].$$

On the other hand, to show that $(f_1 \vee f_2)(a) + (f_1 \vee f_2)(b) \geq (f_1 \vee f_2)(c)$, we need to find, given some decomposition $c = c_1 + c_2$ $(c_1, c_2 \in A^+)$, decompositions $a_1 + a_2 = a$ and $b_1 + b_2 = b$ $(a_1, a_2, b_1, b_2 \in A^+)$ such that

$$f_1(c_1) + f_2(c_2) \leq [f_1(a_1) + f_2(a_2)] + [f_1(b_1) + f_2(b_2)].$$

However, the Riesz decomposition property tells us that since $c_1 + c_2 = a + b$, then there are $a_1, a_2, b_1, b_2 \in A^+$ such that $a_1 + a_2 = a$, $b_1 + b_2 = b$, $a_1 + b_1 = c_1$, and

$a_2 + b_2 = c_2$, and the above inequality (actually, equality) holds. It is then easily checked that $(f_1 \vee f_2)(a) + (f_1 \vee f_2)(b) = (f_1 \vee f_2)(c)$ for arbitrary $a, b, c \in A$ such that $a + b = c$.

Since $(f_1 \vee f_2)(a) \geq f_1(a) + f_2(0) = f_1(a)$ for all $a \in A^+$, it is clear that $(f_1 \vee f_2) \geq f_1$. Similarly, $(f_1 \vee f_2) \geq f_2$. On the other hand, if $f_1, f_2 \leq h \in \hom(A, \mathbb{R})^+$, then for any $a_1 + a_2 = a$ where $a_1, a_2 \in A^+$, we have

$$f_1(a_1) + f_2(a_2) \leq h(a_1) + h(a_2) = h(a)$$

and so $(f_1 \vee f_2)(a) \leq h(a)$. Hence $(f_1 \vee f_2) \leq h$, and in fact $(f_1 \vee f_2) = \sup\{f_1, f_2\}$.

It then follows that $(f_1 \wedge f_2) := (f_1 + f_2) - (f_1 \vee f_2)$ is also an element of $\hom(A, \mathbb{R})^+$, and in fact it is the greatest lower bound of f_1 and f_2. To see this, let $a \in A^+$. Then

$$(f_1 \wedge f_2)(a) = f_1(a) + f_2(a) - (f_1 \vee f_2)(a) \leq f_1(a) + f_2(a) - [f_1(0) + f_2(a)] = f_1(a)$$

and so $(f_1 \wedge f_2) \leq f_1$. Similarly $(f_1 \wedge f_2) \leq f_2$. Also, if $f_1, f_2 \geq h \in \hom(A, \mathbb{R})^+$, then for any $a_1 + a_2 = a$ where $a_1, a_2 \in A^+$, we have

$$f_1(a) + f_2(a) - [f_1(a_1) + f_2(a_2)] = f_1(a - a_1) + f_2(a - a_2) \geq h(a - a_1) + h(a - a_2) = h(a)$$

and so $(f_1 \wedge f_2)(a) \geq h(a)$. $\square$

4.2 $E(S_u(A, A^+))$ determines the positive cone.

If X is a compact Hausdorff space, we let $C_{\mathbb{R}(\mathbb{X})}$ denote the Banach space of continuous functions $h : X \to \mathbb{R}$ with the norm $\|h\|_\infty = \sup\{|h(x)| : x \in X\}$, and we define the *ordinary* and *strict* orderings on $C_{\mathbb{R}(X)}$ by

$$C_{\mathbb{R}}(X)^+ = \{h \in C_{\mathbb{R}}(X) : h(x) \geq 0 \text{ for all } x \in X\},$$

$$C_{\mathbb{R}}(X)^{++} = \{h \in C_{\mathbb{R}}(X) : h(x) > 0 \text{ for all } x \in X\} \cup \{0\},$$

using the notation $\ll$ for the latter cone.

If K is a compact convex subset of a locally convex space, we let $\operatorname{Aff} K$ be the *affine* functions in $C_{\mathbb{R}}(K)$, i.e., the continuous functions $h \in C_{\mathbb{R}}(K)$ such that

$$h(\alpha p + (1 - \alpha)q) = \alpha h(p) + (1 - \alpha)h(q) \text{ for all } p, q \in K \text{ and } 0 \leq \alpha \leq 1.$$

$\operatorname{Aff} K$ is a closed subspace of $C_{\mathbb{R}}(K)$ and we let $(\operatorname{Aff} K)^+ = \operatorname{Aff} K \cap C_{\mathbb{R}}(X)^+$, and $(\operatorname{Aff} K)^{++} = \operatorname{Aff} K \cap C_{\mathbb{R}}(X)^{++}$. In particular, if K is a classical simplex, $\operatorname{Aff} K$ may be identified with $C_{\mathbb{R}}(E(K))$ where $E(K)$ is the set of extreme points as follows. Let $E(K) = \{k_1, \ldots, k_d\}$ be the extreme points of K. Then for any $h \in \operatorname{Aff} K$ and any element $\alpha_1 k_1 + \ldots + \alpha_d k_d \in K$ such that $\alpha_1 + \ldots + \alpha_d = 1$, we have that

$h(\alpha_1 k_1 + \ldots + \alpha_d k_d) = \alpha_1 h(k_1) + \ldots + \alpha_d h(k_d)$. Thus we may identify h with the function $f_h \in C_{\mathbb{R}}(E(K))$ defined by $f_h(k_i) = h(k_i)$ for all $1 \leq i \leq d$.

Thus we have an ordinary and strict order, norm, and linear isomorphism

$$\mathrm{Aff}\, K = C_{\mathbb{R}}(E(K)) \cong \mathbb{R}^d.$$

Now fix some $n \in \mathbb{N}$ and $(A, A^+, u) \in SDG_n$. Then we define a positive homomorphism $\theta : A \to \mathrm{Aff}\, S_u(A, A^+) : a \mapsto \hat{a}$ by letting $\hat{a}(p) = p(a)$. In particular, we have that $\hat{u} = 1$. To see that $\hat{a}$ is an affine function, note that

$$\hat{a}(\alpha p + (1-\alpha)q) = (\alpha p + (1-\alpha)q)(a) = \alpha p(a) + (1-\alpha)q(a) = \alpha\hat{a}(p) + (1-\alpha)\hat{a}(q).$$

The range of the function $\hat{a}$ may be calculated by comparing a with u. More precisely, we have the following result of Goodearl and Handelman, based on their ordered group analogue of the Hahn-Banach Theorem.

Theorem 4.8. *[19] Suppose that A is an ordered group with order unit u. Then for any $a \in A^+$,*

$$\inf\{\hat{a}(p) : p \in S_u(A, A^+)\} = \sup\{\alpha \in \mathbb{Q}^+ : \alpha u \leq a\}.$$

The inequality on the right is interpreted as follows. If $\alpha = p/q$ with $p, q \in \mathbb{N}$, then $\alpha u \leq a$ means that $pu \leq qa$.

Definition 4.9. Suppose that u is an order unit in an unperforated order group A. We say that $a \in A$ is *infinitesimal* if $-u \leq na \leq u$ for all $n \in \mathbb{N}$.

Notice that the infinitesimal elements do not depend on the choice of u, and that the set of infinitesimals forms a subgroup of A. For example, if $a, b \in A$ are infinitesimal, then $2n(a+b) = 2na + 2nb \leq u + u = 2u$, and so $n(a+b) \leq u$, for all $0 < n \in \mathbb{N}$.

Corollary 4.10. *[11, Corollary 1.5] If (A, A^+, u) is a simple dimension group, then the map*

$$\theta : A \to \mathrm{Aff}\, S_u(A, A^+)$$

determines the order on A in the sense that $A^+ = \{a \in A : \hat{a} \gg 0\} \cup \{0\}$. Furthermore, we have $a \in \ker\theta$ (i.e., $\hat{a} = 0$) if and only if a is infinitesimal.

Proof. If $a \in A^+ \backslash \{0\}$, then since (A, A^+, u) is simple, a is an order unit and $na \geq u$ for some n. But then $n\hat{a} \geq \hat{u} = 1$, i.e., $\hat{a} \gg 0$. Conversely if $\hat{a} \gg 0$, then since $S_u(A, A^+)$ is compact, $\hat{a} \geq \varepsilon\hat{u}$ for some $\varepsilon > 0$. Thus by Theorem 4.8, there exists an $p/q \in \mathbb{Q}^+$ such that $qa \geq pu$. Thus $qa \in A^+$, and since (A, A^+, u) is unperforated, $a \in A^+$.

If $-u \leq na \leq u$ for all $n \in \mathbb{N}$, then $-1 \leq np(a) \leq 1$ for all $p \in S_u(A, A^+)$ and $n \in \mathbb{N}$. Thus $\hat{a} = 0$. Conversely, if $\hat{a} = 0$, then given $\varepsilon > 0$, $\varepsilon \in \mathbb{Q}$, $\widehat{(a + \varepsilon u)} = \hat{a} + \widehat{\varepsilon u} \gg 0$ implies that $a + \varepsilon u \geq 0$, i.e., $a \geq -\varepsilon u$. Since the same applies to $-a$, $-\varepsilon u \leq a \leq \varepsilon u$. $\square$

Corollary 4.11. *Let $n \geq 1$ and let $(A, A^+, u), (B, B^+, v) \in SDG_n$. Suppose that $A = B$ and $u = v$. Then the following are equivalent:*

1. $A^+ = B^+$
2. $S_u(A, A^+) = S_v(B, B^+)$
3. $E(S_u(A, A^+)) = E(S_v(B, B^+))$

Proof. Clearly (1) $\implies$ (2) $\implies$ (3). So assume that $E(S_u(A, A^+)) = E(S_v(B, B^+))$. Then since each of $S_u(A, A^+)$ and $S_v(B, B^+)$ is the set of affine combinations of elements of $E(S_u(A, A^+))$, they must be equal as well. Now assume that there is some $a \in A^+ \backslash B^+$. Then Corollary 4.10 implies that $p(a) > 0$ for every $p \in S_u(A, A^+)$, but that there is some $q \in S_v(B, B^+)$ such that $q(a) \leq 0$. This contradicts $S_u(A, A^+) = S_v(B, B^+)$. $\square$

Chapter 5

Countable dimension groups of finite rank are characterized by Bratteli diagrams.

In this chapter, we shall present Effros' proof [10] of the surprising theorem of Effros-Handelman-Shen [11] that every countable dimension group arises from a suitably chosen Bratteli diagram via the construction in Definition 3.1. Using Theorems 3.3 and 3.4, this will give, for each $n \geq 1$, a Borel reduction from the isomorphism relation $\cong_n^+$ on SDG_n to the relation $\sim$ on $\mathcal{BD}$.

Theorem 5.1. *[10] Fix $n \geq 1$. For every dimension group $(A, A^+, u) \in SDG_n$, there is a Bratteli diagram (V, E) such that $(A, A^+) \cong K_0(V, E)$. Furthermore, (V, E) can be chosen in a Borel way.*

The first step in proving this theorem is to note that the Riesz interpolation property can be applied in a Borel fashion. That is, we fix a well-ordering on $\mathbb{Q}^n$, so that given a dimension group (A, A^+) such that $A \leq \mathbb{Q}^n$, and $a, b, c, d \in A$ such that $a, b \leq c, d$, we can pick the *least* element x of $A \subseteq \mathbb{Q}^n$ such that $a, b \leq x \leq c, d$. We also note that since in the proof of Theorem 4.1, all of our constructions were explicit, each of the reformulations the Riesz interpolation property can be applied in a Borel fashion.

Next we shall need the following lemma, which gives a necessary and sufficient condition (Shen's condition) for an unperforated ordered group to be isomorphic to $K_0(V, E)$ for some Bratteli diagram (V, E).

Lemma 5.2. *[30] Suppose that (A, A^+) is a countable unperforated ordered group. Then there is a Bratteli diagram (V, E) such that $(A, A^+) \cong K_0(V, E)$ if and only if for every positive homomorphism $\varphi : \mathbb{Z}^r \to A$, there exists an $s \in \mathbb{N}$, and positive homomorphisms $\sigma : \mathbb{Z}^r \to \mathbb{Z}^s$, $\varphi' : \mathbb{Z}^s \to A$ such that the following diagram commutes*

$$\begin{tikzcd} & \mathbb{Z}^s \arrow[d, "\varphi'"] \\ \mathbb{Z}^r \arrow[r, "\varphi\ \ \sigma"] & A \arrow[r] & {} \end{tikzcd} \tag{5.1}$$

and $\ker \sigma = \ker \varphi$. *Furthermore, in the case that (A, A^+) satisfies Shen's condition, there exists a Borel choice of a corresponding Bratteli diagram (V, E).*

We will only use the harder 'if' direction of this lemma in the proof of Theorem 5.1, but we include the easier direction to help the reader understand the meaning of Shen's condition.

Proof of Lemma 5.2. Let (V, E) be a Bratteli Diagram and $(A, A^+) = K_0(V, E)$ the associated dimension group. Let $r \geq 1$ and let $\varphi : \mathbb{Z}^r \to A$ be any positive homomorphism. Then since $\mathbb{Z}^r$ is finitely generated, so is $\varphi(\mathbb{Z}^r)$. Choose the least $n \in \mathbb{N}$ so that $\varphi(\mathbb{Z}^r) \subseteq \mathbb{Z}^{V_n}$, where V_n is the n-th level of V. Then we obtain the following commuting diagram of positive maps:

$$\begin{tikzcd} & \mathbb{Z}^{V_n} \arrow[d] & \\ \mathbb{Z}^r \arrow[r, "\varphi\ \ \varphi'"] & A \arrow[r] & {} \end{tikzcd} \tag{5.2}$$

It is clear that $\ker \varphi' \subseteq \ker \varphi$. However, $\ker \varphi$ is again finitely generated, so we can just increase n until all of the generators of $\ker \varphi$ are mapped to 0 by φ'. Then $\ker \varphi = \ker \varphi'$.

Conversely, let (A, A^+) be an unperforated ordered group which satisfies Shen's condition. We construct a Bratteli diagram (V, E) such that $(A, A^+) \cong K_0(V, E)$ as follows. Enumerate $A^+ = \{a_1, a_2, \ldots\}$. Then we actually construct a commuting diagram of positive maps

$$\begin{tikzcd} \mathbb{Z}^{r(1)} \arrow[r, "\theta_1"] \arrow[d, "\varphi_1"'] & A \arrow[d, "id"] \\ \mathbb{Z}^{r(2)} \arrow[r, "\theta_2"] \arrow[d, "\varphi_2"'] & A \arrow[d, "id"] \\ \vdots & \vdots \end{tikzcd} \tag{5.3}$$

so that for each $n \geq 1$, $\ker \theta_n = \ker \varphi_n$ and

$$\{a_1, a_2, \ldots, a_n\} \subseteq \theta_n\left((\mathbb{Z}^{r(n)})^+\right). \tag{5.4}$$

Then (V, E) will be determined by letting $|V_n| = r(n)$ for each $n \in \mathbb{N}$, and for each $v \in V_n$ the edges between v and V_{n+1} will be determined by reading off the coordinates of $\varphi_n(v)$. But before we construct this diagram, we show that

Claim. *(A, A^+) is isomorphic to the direct limit $K_0(V, E)$ of the $\mathbb{Z}^{r(n)}$.*

Proof of claim. First, define $A_n = \theta_n(\mathbb{Z}^{r(n)})$ for each $n \geq 1$ Then since $\ker \theta_n = \ker \varphi_n$ for each $n \geq 1$, we obtain a sequence of (not necessarily positive) maps $\eta_n : A_n \to \mathbb{Z}^{(r(n+1))}$ so that the following diagram commutes.

$$
\begin{tikzcd}
\mathbb{Z}^{r(1)} \arrow[r, "\theta_1"] \arrow[d, "\varphi_1"'] & A_1 \arrow[d] \arrow[dl, phantom, ""] \\
\mathbb{Z}^{r(2)} \arrow[r, shift left, "\theta_2"] \arrow[d, "\varphi_2"'] & A_2 \arrow[l, shift left, "\eta_1"] \arrow[d] \\
\mathbb{Z}^{r(3)} \arrow[r, shift left, "\theta_3"] \arrow[d] & A_3 \arrow[l, shift left, "\eta_2"] \arrow[d] \\
\vdots & \vdots
\end{tikzcd}
\tag{5.5}
$$

Since this diagram commutes, and since $K_0(V, E)$, (A, A^+) are the direct limits implicit in this diagram (the latter because of Formula (5.4)), we naturally obtain homomorphisms $f : K_0(V, E) \to (A, A^+)$ and $g : (A, A^+) \to K_0(V, E)$. Since all the θ_n are positive, so is f. On the other hand, let $a \in A^+$. Then by Formula (5.4) there is some $n \geq 1$ where $a \in A_n$ and $a = \theta_n(\alpha)$ for some $\alpha \in (\mathbb{Z}^{r(n)})^+$. However, $\eta_n(a) = \varphi_n(\alpha)$ and φ_n is positive, so $\eta_n(a) = \varphi_n(\alpha) \in (\mathbb{Z}^{r(n+1)})^+$. Thus g is also positive.

Finally, we show that f and g are inverse maps. Let $b \in K_0(V, E)$. Then there is some $n \geq 1$ such that $b = \varphi_{n\infty}(b_n)$ for some $b_n \in \mathbb{Z}^{r(n)}$. (Here $\varphi_{n\infty}$ denotes the natural map from $\mathbb{Z}^{r(n)}$ to $K_0(V, E)$.) Then

$$(gf)(b) = (g\theta_n)(b_n) = (\varphi_{(n+1)\infty}\eta_n\theta_n)(b_n) = (\varphi_{(n+1)\infty}\varphi_n)(b_n) = (\varphi_{n\infty})(b_n) = b.$$

Next let $a \in A$. Then there is some $n \geq 1$ and some $\alpha \in \mathbb{Z}^{r(n)}$ such that $a = \varphi_n(\alpha)$. Then

$$g(a) = (\varphi_{(n+1)\infty}\eta_n)(a) = (\varphi_{(n+1)\infty}\eta_n\theta_n)(\alpha),$$

and we calculate $f(g(a))$ by applying θ_{n+1} to an element of $\mathbb{Z}^{r(n+1)}$ which has $g(a)$ as a limit:

$$(fg)(a) = (\theta_{n+1}\eta_n\theta_n)(\alpha) = (\theta_n)(\alpha) = a.$$

$\square$

Finally, we construct Diagram 5.3 by induction on n. First, let $r(1) = 1$ and define $\theta_1 : \mathbb{Z} \to A$ by setting $\theta_1(1) = a_1$. Next assume we have defined $\theta_1, \ldots, \theta_n$ and

$\varphi_1, \ldots, \varphi_{n-1}$. Then set $\varphi'_n : \mathbb{Z}^{r(n)} \to \mathbb{Z}^{r(n)+1}$ as the natural inclusion map, and define $\theta'_{n+1} : \mathbb{Z}^{r(n)+1} \to A$ by defining $\theta'_{n+1}(e_0), \ldots, \theta'_{n+1}(e_{r(n)})$ according to θ_n and setting $\theta'_{n+1}(e_{r(n)+1}) = a_{n+1}$. So we have the following commutating diagram, but we still desire the appropriate kernel condition.

$$\begin{tikzcd}
\mathbb{Z}^{r(n)} \arrow[r, "\theta_n"] \arrow[d, "\varphi'_n"] & A \arrow[d, "id"] \\
\mathbb{Z}^{r(n)+1} \arrow[r, "\theta'_{n+1}"] & A
\end{tikzcd} \tag{5.6}$$

However Shen's condition gives us maps σ and θ_{n+1} so that $\ker \sigma = \ker \theta'_{n+1}$ and so that the following diagram commutes

$$\begin{tikzcd}
\mathbb{Z}^{r(n)} \arrow[r, "\theta_n"] \arrow[d, "\varphi'_n"'] & A \arrow[d, "id"] \\
\mathbb{Z}^{r(n)+1} \arrow[r, "\theta'_{n+1}"] \arrow[d, "\sigma"'] & A \arrow[d, "id"] \\
\mathbb{Z}^{r(n+1)} \arrow[r, "\theta_{n+1}"] & A
\end{tikzcd} \tag{5.7}$$

We then let $\varphi_n = \sigma \circ \varphi'_n$ and (noting that $\ker \varphi'_n = \{0\}$) finally obtain

$$\ker \varphi_n = \ker \sigma = \ker \theta'_{n+1} = \ker(\theta'_{n+1} \circ \varphi'_n) = \ker(id \circ \theta_n) = \ker \theta_n.$$

$\square$

Now, to prove Theorem 5.1, we only need to verify that dimension groups satisfy Shen's condition.

Proof of Theorem 5.1. Let (A, A^+) be a dimension group and let $\varphi : \mathbb{Z}^r \to A$ be a positive homomorphism. We begin by noting that diagrams such as (5.1) may be composed, i.e., given two commuting diagrams of positive homomorphism

$$\begin{tikzcd}
 & \mathbb{Z}^s \arrow[d, "\varphi'"] & & & \mathbb{Z}^t \arrow[d, "\psi"] & \\
\mathbb{Z}^r \arrow[r, "\varphi \quad \sigma"] & A \arrow[r] & {} & \mathbb{Z}^s \arrow[r, "\varphi' \quad \tau"] & A \arrow[r] & {}
\end{tikzcd} \tag{5.8}$$

then composing the ascending maps we obtain another commuting diagram of positive homomorphisms

$$\begin{tikzcd} & \mathbb{Z}^t \arrow[d, "\psi"] \\ \mathbb{Z}^r \arrow[r, "\varphi\tau \circ \sigma"] & A \end{tikzcd} \tag{5.9}$$

It follows that if we wish to verify Shen's condition, it suffices to construct a diagram in which $\sigma(\alpha) = 0$ for a single $\alpha \in \ker\varphi$. This is the case since $\ker\varphi$ is finitely generated, and we may annihilate its generators (and their images) one at a time.

If $\alpha = e_i$ for some $0 \leq i \leq r$, then we may accomplish this by defining $\sigma : \mathbb{Z}^r \to \mathbb{Z}^{r-1}$ via $\sigma(e_j) = e_j$ for $j < i$, $\sigma(e_i) = 0$, and $\sigma(e_j) = e_{j-1}$ for $j > i$, and defining $\psi : \mathbb{Z}^{r-1} \to A$ so that the following diagram commutes

$$\begin{tikzcd} & \mathbb{Z}^{r-1} \arrow[d, "\psi"] \\ \mathbb{Z}^r \arrow[r, "\varphi \quad \sigma"] & A \end{tikzcd} \tag{5.10}$$

Otherwise, we work inductively on $\deg(\alpha)$, which is defined below. First express α as $\alpha = \Sigma m_i f_i - \Sigma n_j g_j$, where $m_i, n_j \in \mathbb{N}^+$ and $f_1, \ldots, f_s, g_1, \ldots, g_t, h_1, \ldots, h_u$ is a rearrangement of the basis elements of $\mathbb{Z}^r$. We may assume that $m_1 \geq m_2 \geq \ldots \geq m_s$ and (by considering $-\alpha$ instead of α if necessary) that $m_1 \geq n_1 \geq n_2 \geq \ldots \geq n_t$. Then the *degree* of α is $\deg(\alpha) = (m, d)$, where $m = m_1$ and d is the number of times that m_1 appears among the m_i and n_j. We order the degrees lexicographically, and note that $\deg(\alpha) = (1, 1)$ corresponds to the case that $\alpha = e_i$ for some $0 \leq i \leq r$. Thus, since we can compose diagrams, we only need to construct a commuting diagram of positive homomorphisms

$$\begin{tikzcd} & \mathbb{Z}^{q} \arrow[d, "\psi"] \\ \mathbb{Z}^r \arrow[r, "\varphi \quad \sigma"] & A \end{tikzcd} \tag{5.11}$$

so that the minimum degree of $\sigma(\alpha) \in \ker\psi$ is less than that of $\alpha \in \ker\varphi$.

So let $\alpha \in \ker\varphi$ have minimal degree, and again let $\alpha = \Sigma m_i f_i - \Sigma n_j g_j$ as above. Then let $a_i = \varphi(f_i)$ for $1 \leq i \leq s$, $b_j = \varphi(g_j)$ for $1 \leq j \leq t$, and $c_k = \varphi(h_k)$ for $1 \leq k \leq u$. Then since $\varphi(\alpha) = 0$, we obtain

$$m_1 a_1 + \ldots + m_s a_s = n_1 b_1 + \ldots + n_t b_t,$$

and since $m_1 \geq n_1 \geq n_2 \geq \ldots \geq n_t$,

$$m_1 a_1 \leq n_1 b_1 + \ldots + n_t b_t \leq m_1 b_1 + \ldots + m_1 b_t.$$

Thus $a_1 \leq b_1 + \ldots + b_t$, and $a_1, b_1, \ldots, b_t \geq 0$ since φ is positive. So we can apply the Riesz Decomposition Property to obtain $a_{11}, \ldots, a_{1t}$ so that $0 \leq a_{1j} \leq b_j$ for all $1 \leq j \leq t$ and $a_{11} + \ldots + a_{1t} = a_1$.

Now let $\mathbb{Z}^q$ have basis elements $f'_{11}, \ldots, f'_{1t}, f'_2, \ldots, f'_s, g'_1, \ldots, g'_t, h'_1, \ldots, h'_u$, and define $\sigma : \mathbb{Z}^r \to \mathbb{Z}^q$ and $\psi : \mathbb{Z}^q \to A$ by

$$\begin{array}{lll} \sigma(f_1) = f'_{11} + \ldots + f'_{1t} & \psi(f'_{1j}) = \varphi(a_{1j}) & 1 \leq j \leq t \\ \sigma(f_i) = f'_i & \psi(f'_i) = a_i & 2 \leq i \leq s \\ \sigma(g_j) = f'_{1j} + g'_j & \psi(g'_j) = b_j - a_{1j} & 1 \leq j \leq t \\ \sigma(h_k) = h'_k & \psi(h'_k) = c_k & 1 \leq k \leq u \end{array}$$

Then it is clear that σ and ψ are positive and that the above diagram commutes. Also,

$$\begin{aligned} \sigma(\alpha) &= m_1\sigma(f_1) + m_2\sigma(f_2) + \ldots + m_s\sigma(f_s) - [n_1\sigma(g_1) + \ldots + n_t\sigma(g_t)] \\ &= m_1(f'_{11} + \ldots + f'_{1t}) + m_2 f'_2 + \ldots + m_s f'_s \\ &\quad - [n_1 f'_{11} + \ldots + n_t f'_{1t} + n_1 g_1 + \ldots + n_t g_t] \\ &= (m_1 - n_1) f'_{11} + \ldots + (m_1 - n_t) f'_{1t} + m_2 f'_2 + \ldots + m_s f'_s - [n_1 g_1 + \ldots + n_t g_t]. \end{aligned}$$

and since $(m_1 - n_1), \ldots, (m_1 - n_t)$ are all nonnegative and less than m_1, we have $\deg(\sigma(\alpha)) < \deg(\alpha)$ and we are done. $\square$

In order to be able to relate SDG_n to $\mathcal{BD}$, we will need to incorporate the extra structure of the order unit into the above theorem. The following is a condensed version of [17, 21.9 and 21.10].

Theorem 5.3. *[17] Fix some $n \geq 1$. For every dimension group with order unit $(A, A^+, u) \in SDG_n$, there is a Bratteli diagram (V, E) such that $(A, A^+, u) \cong K_0(V, E)$. Furthermore, (V, E) can be chosen in a Borel way.*

Proof. Theorem 5.1 gives us a Borel choice of a Bratteli diagram (V, E), such that $(A, A^+) \cong K_0(V, E)$. Letting $H_m = \mathbb{Z}^{V_m}$, we have a sequence of positive maps

$$H_0 \xrightarrow{\varphi_0} H_1 \xrightarrow{\varphi_1} H_2 \xrightarrow{\varphi_2} \ldots \xrightarrow{\varphi_{m-1}} H_m \xrightarrow{\varphi_m} \ldots$$

so that there is an isomorphism $g : (A, A^+) \cong \varinjlim(H_m, H_m^+)$. Set $v = g(u)$. Then v is an order unit of $\varinjlim(H_m, H_m^+)$. Thus, there is some $k \in \mathbb{N}$ and some $v' \in H_k^+$ such that $[v'] = v$. Contracting levels 1 through k of (V, E), we may assume that $v' \in H_1$. Next set $u_1 = v'$, and for $m > 1$, $u_m = \varphi_{m-1}(u_{m-1})$. Then for each $m \geq 1$, let

$$G_m = \{x \in H_m \mid -tu_m \leq x \leq tu_m \text{ for some } t \in \mathbb{N}\}.$$

Notice that $G_m \leq H_m$ and $u_m \in H_m^+$. Thus G_m is precisely $\mathbb{Z}^{W_m}$, where $W_m \subseteq V_m$ is the set of non-zero coordinates of u_m.

Claim. $\varinjlim(H_m, H_m^+) = \varinjlim(G_m, G_m^+)$.

Proof. Given $x \in \varinjlim(H_m, H_m^+)$, we have that there is some $m \in \mathbb{N}$ and some $y \in H_m$ such that $x = [y]$. Since v is an order unit for $\varinjlim(H_m, H_m^+)$, there is some $t \in \mathbb{N}$ such that $-tv \leq x \leq tv$, and thus $[-tu_m] \leq [y] \leq [tu_m]$. After increasing m if necessary, we may assume that $-tu_m \leq y \leq tu_m$, and so $y \in G_m$. Thus $x \in \varinjlim(G_m, G_m^+)$. □

Finally, set $|W_0| = 1$, $G_0 = \mathbb{Z}^{W_0}$, and place edges between W_0 and W_1 so that $\varphi_0 : G_0 \to G_1$ is the unique positive map with $\varphi_0(1) = u_1$. Then the Bratteli diagram induced from the W_m fulfills our purpose. □

Theorem 5.4. *For each $n \geq 1$, $(\cong_n^+) \leq_B (\sim)$.*

Proof. Theorem 5.3 gives a Borel map from $f : SDG_n \to \mathcal{BD}$ so that for each dimension group $(A, A^+, u_A) \in SDG_n$, $(A, A^+, u_A) \cong K_0(f(A, A^+, u_A))$. Theorem 3.3 implies that f is a Borel reduction. □

Chapter 6

$(\cong_1^+) <_B (\cong_2^+) <_B (\cong_3^+)$

Proposition 6.1. *Let $n \geq 2$. The map $f_n : SDG_n \to SDG_{n+1}$ given by $f_n((A, A^+, u_A)) = (B, B^+, u_B)$ where*

1. $B = A \oplus \mathbb{Q}$
2. $B^+ = \{(a, q) \in A \oplus \mathbb{Q} : a \in A^+ \backslash \{0\}$ *and* $q > 0\} \cup \{(0, 0)\}$
3. $u_B = (u_A, 1)$

is a Borel reduction from $\cong_n^+$ to $\cong_{n+1}^+$.

Proof. We first need to check that $(B, B^+, u_B) \in SDG_{n+1}$. It is easy to check that (B, B^+, u_B) is an unperforated ordered group. For example, to see that $B^+ - B^+ = B$, let (a, q) be any element of B. Then there are $a_1, a_2 \in A^+$ so that $a = a_1 - a_2$. If either a_1 or a_2 are 0, then replace them with $a_1 + u_A$ and $a_2 + u_A$ so that they both lie in $A^+ \backslash \{0\}$. Next let q_1, q_2 be any two positive rational numbers so that $q_1 - q_2 = q$. Then $(a, q) = (a_1, q_1) - (a_2, q_2)$ and $(a_1, q_1), (a_2, q_2) \in B^+$.

To see that (B, B^+, u_B) satisfies the Riesz interpolation property, consider elements $(a_i, q_i), (b_j, p_j) \in B$ $(1 \leq i, j \leq 2)$ such that $(a_i, q_i) \leq (b_j, p_j)$. First note that if $q_i = p_j$ for some $i, j \in \{1, 2\}$, then it must be the case that $a_i = b_j$ and then we can choose (a_i, q_i) to interpolate. Thus we can assume that $q_1, q_2 < p_1, p_2$, and so $a_1, a_2 < b_1, b_2$. Then let q be some rational number such that $q_1, q_2 < q < p_1, p_2$. Now applying Lemma 4.2 to (A, A^+, u_A), there exists $c \in A$ with $a_i < c < b_j$ for $1 \leq i, j \leq 2$, and so we can choose (c, q) to interpolate.

To see that (B, B^+, u_B) is simple, let J be a nontrivial ideal of (B, B^+, u_B), and let $(a, q) \in J^+ \backslash \{0\}$. Now let (b, r) be any other element of B^+. Since (A, A^+, u_A) is simple and $a \in A^+ \backslash \{0\}$, we know that a is an order unit in (A, A^+). That is, there is some natural number $n \in \mathbb{N}$ such that $na - b \in A^+$. Since $q > 0$, there is some natural number $n' \in \mathbb{N}$ such that $n'q - r > 0$. Then letting $m = \max\{n, n'\}$, $m(a, q) - (b, r) \in B^+$, and so $(b, r) \in J^+$. Thus $J^+ = B^+$ and so $J = B$.

We now need to check that f_n is a Borel reduction. If $(A, A^+, u_A) \cong (C, C^+, u_C)$, then there exists some $\varphi \in \mathrm{GL}_n(\mathbb{Q})$ so that $\varphi(A, A^+, u_A) = (C, C^+, u_C)$. Then $(\varphi \oplus 1)(f_n(A, A^+, u_A)) = f_n(C, C^+, u_C)$, and so $f_n(A, A^+, u_A) \cong f_n(C, C^+, u_C)$

On the other hand, suppose that $(A, A^+, u_A), (C, C^+, u_C) \in SDG_n$ and also that $f_n(A, A^+, u_A) \cong f_n(C, C^+, u_C)$. Let $(B, B^+, u_B) := f_n(A, A^+, u_A)$ and $(D, D^+, u_D) := f_n(C, C^+, u_C)$. Let $\varphi : (B, B^+, u_B) \to (D, D^+, u_D)$ be an isomorphism, identify A, $\mathbb{Q}$ with the corresponding factors of B, and identify C, $\mathbb{Q}$ with the corresponding factors of D. Consider the set

$$\begin{aligned} B^\circ &= \{b \in B \mid b \notin B^+ \text{ and for every } b' \in B^+\backslash\{0\}, b + b' \in B^+\} \cup \{(0,0)\} \\ &= \{(0,q) \in B \mid q \in \mathbb{Q}^+\} \cup \{(a,0) \in B \mid a \in A^+\}. \end{aligned}$$

Then the first equality above shows that $\varphi(B^\circ) = D^\circ$. Notice that $u_A + 1_\mathbb{Q}$ is the unique way to express u_B as a sum of two elements of B°. Thus $\varphi(\{u_A, 1_\mathbb{Q}\}) = \{u_C, 1_\mathbb{Q}\}$. Notice also that if $g \in B^\circ$, then

- $g \in A^+$ iff $(\exists n \geq 1)(nu_A - g \in B^0)$
- $g \in \mathbb{Q}^+$ iff $(\exists n \geq 1)(n1_\mathbb{Q} - g \in B^0)$

Thus we either have that

(1) $\varphi(u_A) = u_C$ and $\varphi(A^+) = C^+$, and φ extends linearly to an isomorphism $\varphi : (A, A^+, u_A) \cong (C, C^+, u_C)$; or

(2) $\varphi(u_A) = 1_\mathbb{Q}$ and $\varphi(A^+) = \mathbb{Q}^+$, and φ extends linearly to an isomorphism $\varphi : (A, A^+, u_A) \cong (\mathbb{Q}, \mathbb{Q}^+, 1_\mathbb{Q})$. However, this is impossible since $n \geq 2$.

□

Proposition 6.2. *Let $n \geq 1$. The map $g_n : R(\mathbb{Q}^n) \to SDG_{n+1}$ given by $g_n(G) = (A, A^+, u)$ where*

1. $A = G \oplus \mathbb{Q}$
2. $A^+ = \{(h,q) \in G \oplus \mathbb{Q} : q > 0\} \cup \{(0,0)\}$
3. $u = (0,1)$

is a Borel reduction from $\cong_n$ to $\cong^+_{n+1}$.

Proof. It is easily seen that g_n does map each group G to a dimension group. To see that (A, A^+, u_A) is simple, let $J \subseteq A$ be a nontrivial ideal, fix some $(g,q) \in J^+\backslash\{0\}$ and choose any $(h,r) \in A^+$. Now choose $n \in \mathbb{N}$ so that $nq > r$. Then since we have $n(g,q) \geq (h,r) \geq 0$, it must be the case that $(h,r) \in J^+$. Since our choice of (h,r) was arbitrary, $J^+ = A^+$, and so $J = A$.

It is clear that $G \cong H$ implies $g_n(G) \cong g_n(H)$. Conversely, if $g_n(G) \cong g_n(H)$, then

$$G \cong G \oplus \{0\} = \text{infin}(g_n(G)) \cong \text{infin}(g_n(H)) = H \oplus \{0\} \cong H,$$

where $\text{infin}((A, A^+, u))$ is the group of infinitesimals of (A, A^+, u). $\square$

We will begin our analyses of $\cong_1^+$ and $\cong_2^+$ by giving Thomas' [33] description of Baer's [2] classification of the rank 1 torsion-free abelian groups. Let $\mathbb{P}$ be the set of primes. If G is a torsion-free abelian group and $0 \neq x \in G$, then the *p-height* of x is defined to be

$$h_x(p) = \sup\{n \in \mathbb{N} \mid \text{ There exists } y \in G \text{ such that } p^n y = x\} \in \mathbb{N} \cup \{\infty\},$$

and the *characteristic* $\chi(x)$ of x is defined to be the function

$$\langle h_x(p) \mid p \in \mathbb{P}\rangle \in (\mathbb{N} \cup \{\infty\})^{\mathbb{P}}.$$

Two functions $\chi_1, \chi_2 \in (\mathbb{N} \cup \{\infty\})^{\mathbb{P}}$ are said to be *similar* or *belong to the same type*, written $\chi_1 \equiv \chi_2$, iff

(a) $\chi_1(p) = \chi_2(p)$ for all but finitely many primes p; and

(b) if $\chi_1(p) \neq \chi_2(p)$, then both $\chi_1(p)$ and $\chi_2(p)$ are finite.

Clearly $\equiv$ is an equivalence relation on $(\mathbb{N} \cup \{\infty\})^{\mathbb{P}}$. If G is a torsion-free abelian group and $0 \neq x \in G$, then the *type* $\tau(x)$ is defined to be the $\equiv$-equivalence class containing the characteristic $\chi(x)$.

Now suppose that $G \in R(\mathbb{Q})$ is a rank 1 group. Then it is easily checked that $\tau(x) = \tau(y)$ for all $0 \neq x, y \in G$. Hence we can define the *type* $\tau(G)$ of G to be $\tau(x)$ where x is any non-zero element of G. In [2], Baer proved the following:

Theorem 6.3. *If $G, H \in R(\mathbb{Q})$, then $G \cong H$ iff $\tau(G) = \tau(H)$.*

In other words, $(\cong_1) \leq_B ((\mathbb{N} \cup \{\infty\})^{\mathbb{P}}, \equiv)$. We also have $((\mathbb{N} \cup \{\infty\})^{\mathbb{P}}, \equiv) \leq_B (\cong_1)$, via $\chi \mapsto G_\chi$, where G_χ is the group generated by $\{1/(p^n) \mid n \in \mathbb{N}, p \in \mathbb{P}, n \leq \chi(p)\}$. Thus $(\cong_1) \sim_B ((\mathbb{N} \cup \{\infty\})^{\mathbb{P}}, \equiv)$. Hence, since $((\mathbb{N} \cup \{\infty\})^{\mathbb{P}}, \equiv) \sim_B E_0$ and $(\cong_1) \leq_B (\cong_2^+)$, we know that $\cong_2^+$ is not smooth.

On the other hand, $\cong_1^+$ is smooth, since for any rank 1 simple dimension groups $(A, A^+, u_A), (B, B^+, u_B) \in SDG_1$, $(A, u_A) \cong (B, u_B)$ if and only if $\chi(u_A) = \chi(u_B)$. In fact, since

$$A^+ \backslash \{0\} = \{a \in A \mid (\exists q \in \mathbb{Q}^+)\ a = q u_A\},$$

it follows that $(A, A^+, u_A) \cong (B, B^+, u_B)$ if and only if $\chi(u_A) = \chi(u_B)$. Thus, we have shown:

Theorem 6.4. $(\mathrm{id}_{2^{\mathbb{N}}}) \sim_B (\cong_1^+) <_B (\cong_2^+)$.

Next, we will show that $(\cong_2^+) <_B (\cong_3^+)$ by examining the group actions which give rise to these equivalence relations. Instead of analyzing $\cong_2^+$, we first note that it is enough to analyze the Borel equivalence relation obtained by restricting $\cong_2^+$ to those dimension groups (A, A^+, u_A) for which $u_A = (1,0) \in \mathbb{Q}^2$. Denote this space of rank 2 dimension groups by $SDG_2^{(e_0)}$ and the resulting equivalence relation by $(\cong_2^+)^{<e_0>}$. Then we have a Borel reduction from $\cong_2^+$ to $(\cong_2^+)^{<e_0>}$ via $(A, A^+, u_A) \mapsto g(A, A^+, u_A)$, where g is some element of $\mathrm{GL}_2(\mathbb{Q})$ such that $g(u_A) = (1,0)$. Now we have that $(\cong_2^+)^{<e_0>}$ is the orbit equivalence relation of the natural action of the group

$$H = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_n(\mathbb{Q}) \mid a = 1, c = 0\}$$

on $SDG_2^{(e_0)}$. Since H is solvable and hence amenable, we have by Theorem 1.31 that $(\cong_2^+)^{<e_0>}$ is Frèchet-amenable.

Now suppose that $(\cong_3^+) \leq_B (\cong_2^+)$, and thus $(\cong_2) \leq_B (\cong_3^+) \leq_B (\cong_2^+) \leq_B (\cong_2^+)^{<e_0>}$. Then by Theorem 1.34, we would have that $\cong_2$ is Frèchet-amenable. However, in [23, Section 5], Hjorth has constructed a $\mathrm{PSL}_2(\mathbb{Z})$-invariant measure μ on $R(\mathbb{Q}^2)$, and a Borel subset $X \subset R(\mathbb{Q}^2)$ with $\mu(X) = 1$ such that $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{1,-1\}$ acts freely on X. Thus Theorem 1.33 would imply that $\mathrm{PSL}_2(\mathbb{Z})$ is amenable. However, this is not the case, since $\mathrm{PSL}_2(\mathbb{Z})$ contains an isomorphic copy of $\mathbb{F}_2$, namely, $\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle / \{-1, 1\}$. Thus we have shown:

Theorem 6.5. $(\cong_2^+) <_B (\cong_3^+)$.

Chapter 7

The relationship between dimension groups and torsion-free abelian groups.

In order to prove Theorem 1.11 for the case $n \geq 3$, we will first reduce the analysis to the case of a Borel homomorphism $f : SDG_{n+1} \to SDG_n$ whose image is a single isomorphism class *of the underlying torsion-free abelian group.* Let $g_n : R(\mathbb{Q}^n) \to SDG_{n+1}$ be the Borel reduction from $\cong_n$ to $\cong^+_{n+1}$ defined in Proposition 6.2, and let $\pi'_n : SDG_n \to R(\mathbb{Q}^n)$ be the forgetful map $\pi'_n(A, A^+, u) = A$. Consider initially a Borel homomorphism $f : SDG_{n+1} \to SDG_{n-1}$ (recall $n \geq 3$). Then composing these maps, we obtain a Borel homomorphism $h = \pi'_{n-1} \circ f \circ g_n$ from $\cong_n$ to $\cong_{n-1}$; and examining Thomas' proof that $(\cong_n) <_B (\cong_{n+1})$, we see that, with respect to a suitable invariant ergodic probability measure, h maps a measure one subset of an $SL_n(\mathbb{Z})$-invariant Borel subset of $R(\mathbb{Q}^n)$ to a single isomorphism class of $R(\mathbb{Q}^{n-1})$. This implies that f maps this subset to a collection of dimension groups with isomorphic underlying torsion-free abelian groups.

This is how we *would* start to prove Theorem 1.11 for $n \geq 3$, except that our Borel homomorphism f should be a map from SDG_{n+1} to SDG_n. To fix this, we observe that by first adjusting by an appropriate element of $\mathrm{GL}_n(\mathbb{Q})$, we can assume that the order unit of every dimension group in the image of f is $u = e_0$, thus shrinking the group which acts on SDG_n. It will also turn out to be useful to reduce the analysis to the case when the group of infinitesimals of every dimension group in the image of f is identical. Notice that the group of infinitesimals is always a torsion-free abelian group of rank less than that of the associated dimension group. To realize these two goals simultaneously, we proceed as follows:

Given $n \geq 1$, let $S(\mathbb{Q}^n)$ be the space of all subgroups of $\mathbb{Q}^n$. Then the isomorphism relation on this space is the orbit equivalence relation of the action of the group $\mathrm{GL}_n(\mathbb{Q})$. Next let $\mathrm{Mat}_n^{\langle e_0 \rangle}(\mathbb{Q}) \subset \mathrm{Mat}_n(\mathbb{Q})$ be the subset of all $n \times n$ matrices which fix the one-dimensional subspace $\langle e_0 \rangle$. Let $\mathrm{GL}_n^{\langle e_0 \rangle}(\mathbb{Q}) = \mathrm{GL}_n(\mathbb{Q}) \cap \mathrm{Mat}_n^{\langle e_0 \rangle}(\mathbb{Q})$. Let $\cong_{n^*}^{\langle e_0 \rangle}$ be the orbit equivalence relation of the diagonal action of $\mathrm{GL}_n^{\langle e_0 \rangle}(\mathbb{Q})$ on $R(\mathbb{Q}^n) \times S(\mathbb{Q}^n)$.

Theorem 7.1. *Let $n \geq 3$ and let X be a standard Borel $\mathrm{SL}_n(\mathbb{Z})$-space with an invariant ergodic probability measure μ. Suppose that $f \colon X \to R(\mathbb{Q}^n) \times S(\mathbb{Q}^n)$ is a Borel function such that $xEy \Rightarrow f(x) \cong_{n^*}^{\langle e_0 \rangle} f(y)$. Then there exists an $\mathrm{SL}_n(\mathbb{Z})$-invariant Borel subset*

$M \subset X$ with $\mu(M) = 1$ such that f maps M into a single $\cong_{n^*}^{\langle e_0 \rangle}$ -class.

We will use this theorem in the next chapter in the case when f is the composition of the following Borel homomorphisms:

1. The Borel homomorphism σ_n from $E_{\mathrm{SL}_n(\mathbb{Z})}^{X_n}$ to $\cong_n$, where (X_n, μ_n) is the measure space given by Definition 1.43,
2. The Borel reduction $g_n : R(\mathbb{Q}^n) \to SDG_{n+1}$ from $\cong_n$ to $\cong_{n+1}^+$, defined in Proposition 6.2,
3. An arbitrary Borel homomorphism $h : SDG_{n+1} \to SDG_n$ from $\cong_{n+1}^+$ to $\cong_n^+$,
4. A Borel function which replaces each dimension group $(A, A^+, u) \in SDG_n$ by $\varphi(A, A^+, u)$ for some $\varphi \in \mathrm{GL}_n(\mathbb{Q})$ so that $\varphi(u) = e_0$.
5. The function which takes a dimension group $(A, A^+, u) \in SDG_n$ and gives the element of $R(\mathbb{Q}^n) \times S(\mathbb{Q}^n)$ corresponding to the underlying torsion-free abelian group and the group of infinitesimals of (A, A^+, u).

Our goal for the rest of this chapter is to prove Theorem 7.1. In order to accomplish this, we will first prove an analogous theorem for the quasi-isomorphism relation.

Definition 7.2. Suppose that $A, B \in S(\mathbb{Q}^n)$. Then A is said to be *quasi-contained* in B, written $A \prec_n B$, if there exists an integer $m > 0$ such that $mA \leq B$. If $A \prec_n B$ and $B \prec_n A$, then A and B are said to be *quasi-equal* and we write $A \approx_n B$.

It is well-known that if $A \in S(\mathbb{Q}^n)$ and $m > 0$, then $[A : mA] < \infty$. This implies that if $A, B \in S(\mathbb{Q}^n)$, then $A \approx_n B$ if and only if $A \cap B$ has finite index in both A and B.

Theorem 7.3. *[36, Lemma 3.2] For all $n \geq 1$, $\approx_n$ is a countable Borel equivalence relation on $S(\mathbb{Q}^n)$.*

Proof. It is clear that $\approx_n$ is a Borel equivalence relation. We must check that for each group $A \in S(\mathbb{Q}^n)$, there are only countably many groups $B \in S(\mathbb{Q}^n)$ such that $A \approx_n B$. Fix some $A \in S(\mathbb{Q}^n)$ and assume that $A \approx_n B$. Then there are finite positive integers m, l so that $mB \leq A$ and $lA \leq B$. However, this implies that $mlA \leq mB \leq A$. Thus since $[A : mlA] < \infty$, there are only finitely many choices for mB, and thus only countably many choices for B. □

We will rely on the following highly nontrivial result of Thomas.

Theorem 7.4. *[36, Theorem 3.8] For each $n \geq 1$, the relation $\approx_n$ is hyperfinite.*

Definition 7.5. Suppose that $A, B \in S(\mathbb{Q}^n)$. Then A and B are said to be *quasi-isomorphic*, written $A \sim_n B$, if there exists $\varphi \in \mathrm{GL}_n(\mathbb{Q})$ such that $\varphi(A) \approx_n B$. We shall write $A \sim_n^{\langle e_0 \rangle} B$ iff there exists some $\varphi \in \mathrm{GL}_n^{\langle e_0 \rangle}(\mathbb{Q})$ such that $\varphi(A) \approx_n B$. Given $(A, A'), (B, B') \in R(\mathbb{Q}^n) \times S(\mathbb{Q}^n)$, we write $(A, A') \sim_{n^*}^{\langle e_0 \rangle} (B, B')$ if there exists $\varphi \in \mathrm{GL}_n^{\langle e_0 \rangle}(\mathbb{Q})$ such that $\varphi(A) \approx_n B$ and $\varphi(A') \approx_n B'$.

Since $\sim_{n^*}^{\langle e_0 \rangle}$ is the smallest equivalence relation on $R(\mathbb{Q}^n) \times R^k(\mathbb{Q}^n)$ containing both $\approx_n \restriction_{R(\mathbb{Q}^n)} \times \approx_n$ and $\cong_{n^*}^{\langle e_0 \rangle}$, then it is also a countable Borel equivalence relation. In particular, since $\left(\cong_{n^*}^{\langle e_0 \rangle}\right) \subseteq \left(\sim_{n^*}^{\langle e_0 \rangle}\right)$, Theorem 7.1 is an immediate consequence of Lemma 1.46 and the following result.

Theorem 7.6. *Let $n \geq 3$ and let X be a standard Borel $\mathrm{SL}_n(\mathbb{Z})$-space with an invariant ergodic probability measure μ. Suppose that $f \colon X \to R(\mathbb{Q}^n) \times S(\mathbb{Q}^n)$ is a Borel function such that $xEy \Rightarrow f(x) \sim_{n^*}^{\langle e_0 \rangle} f(y)$. Then there exists an $\mathrm{SL}_n(\mathbb{Z})$-invariant Borel subset $M \subset X$ with $\mu(M) = 1$ such that f maps M into a single $\sim_{n^*}^{\langle e_0 \rangle}$-class.*

To prove this, we first need a few definitions. For each $A \in S(\mathbb{Q}^n)$, let [A] be the $\approx_n$-class containing A. If $A \in S(\mathbb{Q}^n)$, then a linear transformation $\varphi \in \mathrm{Mat}_n(\mathbb{Q})$ is said to be a *quasi-endomorphism* of A if $\varphi(A) \prec_n A$. Equivalently, φ is a quasi-endomorphism of A if and only if there exists an integer $m > 0$ such that $m\varphi \restriction_A \in \mathrm{End}(A)$. It is easily checked that the collection $\mathrm{QE}(A)$ of quasi-endomorphisms of A is a $\mathbb{Q}$-subalgebra of $\mathrm{Mat}_n(\mathbb{Q})$ and that if $A \approx_n B$, then $\mathrm{QE}(A) = \mathrm{QE}(B)$. Let

$$\mathrm{RQE}(A) = \mathrm{QE}(A) \cap \mathrm{Mat}_n^{\langle e_0 \rangle}(\mathbb{Q}).$$

Note that if $A \approx_n B$, then $\mathrm{RQE}(A) = \mathrm{RQE}(B)$, and that $\mathrm{RQE}(A)$ is also a $\mathbb{Q}$-subalgebra of $\mathrm{Mat}_n(\mathbb{Q})$. A linear transformation $\varphi \in \mathrm{Mat}_n(\mathbb{Q})$ is said to be a *quasi-automorphism* of A if φ is a unit of the $\mathbb{Q}$-algebra $\mathrm{QE}(A)$. The group of quasi-automorphisms of A is denoted by $\mathrm{QAut}(A)$. Let $\mathrm{RQAut}(A) = \mathrm{QAut}(A) \cap \mathrm{Mat}_n^{\langle e_0 \rangle}(\mathbb{Q}) =$ the group of units of $\mathrm{RQE}(A)$

Lemma 7.7. *If $A \in S(\mathbb{Q}^n)$, then $\mathrm{RQAut}(A)$ is the setwise stabilizer of $[A]$ in $\mathrm{GL}_n^{\langle e_0 \rangle}(\mathbb{Q})$.*

Proof. First, suppose that $\varphi \in \mathrm{RQAut}(A) \subseteq \mathrm{QAut}(A)$. Then there exists an integer $m > 0$ such that $\psi = m\varphi \in \mathrm{End}(A)$. Clearly, ψ is also a unit of $\mathrm{QE}(A)$ and so ψ is a monomorphism. Hence, by Exercise 92.5 [15], $\psi(A)$ has finite index in A and so $\psi(A) \approx_n A$. Since $\psi(A) = m\varphi(A)$, it follows that $\psi(A) \approx_n \varphi(A)$. Thus, $\varphi(A) \approx_n A$ and so φ stabilizes $[A]$.

Conversely, suppose that $\varphi \in GL_n^{\langle e_0 \rangle}(\mathbb{Q})$ stabilizes $[A]$. Then $\varphi(A) \approx_n A$ and so there exists an integer $m > 0$ such that $m\varphi(A) \leq A$. Since $m\varphi \in \mathrm{End}(A)$ is a monomorphism, it follows that $m\varphi \in \mathrm{QAut}(A)$ and so $\varphi \in \mathrm{QAut}(A)$. Thus $\varphi \in \mathrm{RQAut}(A)$. $\square$

Now to prove Theorem 7.6, we will follow closely the proof of Theorem 3.5 of Thomas [33]. So let $n \geq 3$ and let X be a standard Borel $SL_n(\mathbb{Z})$-space with an invariant ergodic probability measure μ. Suppose that $f\colon X \to R(\mathbb{Q}^n) \times S(\mathbb{Q}^n)$ is a Borel function such that $xE^X_{SL_n(\mathbb{Z})}y$ implies $f(x) \sim^{\langle e_0 \rangle}_{n^*} f(y)$. Let $E = E^X_{SL_n(\mathbb{Z})}$ and for each $x \in X$, let $(A_x, I_x) = f(x) \in R(\mathbb{Q}^n) \times S(\mathbb{Q}^n)$. First, notice that there are only countably many possibilities for the $\mathbb{Q}$-algebra $\mathrm{QE}(A_x)$. Thus, since $\mathrm{RQE}(A_x) = \mathrm{QE}(A_x) \cap \mathrm{Mat}_n^{<e_0>}(\mathbb{Q})$, there are only countably many possibilities for $\mathrm{RQE}(A_x)$. Hence, there exists a Borel subset $X_1 \subseteq X$ with $\mu(X_1) > 0$ and a fixed $\mathbb{Q}$-subalgebra S' of $\mathrm{Mat}_n(\mathbb{Q})$ such that $\mathrm{RQE}(A_x) = S'$ for all $x \in X_1$. By the ergodicity of μ, we have that $\mu(SL_n(\mathbb{Z}).X_1) = 1$. In order to simplify notation, we shall assume that $SL_n(\mathbb{Z}).X_1 = X$. After slightly adjusting f if necessary, we can also assume that $\mathrm{RQE}(A_x) = S'$ for all $x \in X$. (That is, let $c\colon X \to X$ be a Borel function such that $c(x)Ex$ and $c(x) \in X_1$ for all $x \in X$. Then we can replace f with $f' = f \circ c$.) By a similar argument, we can assume that there is a fixed $\mathbb{Q}$-subalgebra S'' of $\mathrm{Mat}_n(\mathbb{Q})$ such that $\mathrm{RQE}(I_x) = S''$ for all $x \in X$. Finally, let $S = S' \cap S''$. Then, letting S^* denote the group of units of S, we have $S^* = \mathrm{RQAut}(A_x) \cap \mathrm{RQAut}(I_x)$ for each $x \in X$.

Now suppose that $x, y \in X$ and that xEy. Then $(A_x, I_x) \sim^{\langle e_0 \rangle}_{n^*} (A_y, I_y)$ and so there exists $\varphi \in \mathrm{GL}_n^{\langle e_0 \rangle}(\mathbb{Q})$ such that $\varphi(A_x) \approx_n A_y$ and $\varphi(I_x) \approx_n I_y$. Notice that

$$\begin{aligned} \varphi S \varphi^{-1} &= \varphi(\mathrm{RQE}(A_x) \cap \mathrm{RQE}(I_x))\varphi^{-1} \\ &= \mathrm{RQE}(\varphi(A_x)) \cap \mathrm{RQE}(\varphi(I_x)) \\ &= \mathrm{RQE}(A_y) \cap \mathrm{RQE}(I_y) \\ &= S \end{aligned}$$

and so $\varphi \in N = N_{GL_n^{\langle e_0 \rangle}(\mathbb{Q})}(S)$. Clearly we also have $\varphi([A_x]) = [A_y]$ and $\varphi([I_x]) = [I_y]$. By Lemma 7.7, for each $x \in X$, the stabilizer of $[A_x]$ (resp. $[I_x]$) in $GL_n^{\langle e_0 \rangle}(\mathbb{Q})$ is $\mathrm{RQAut}(A_x)$ (resp. $\mathrm{RQAut}(I_x)$). Thus for each $x \in X$, the stabilizer of $[A_x] \times [I_x]$ in $GL_n^{\langle e_0 \rangle}(\mathbb{Q})$ is $\mathrm{RQAut}(A_x) \cap \mathrm{RQAut}(I_x) = S^*$

Let $H = N/S^*$ and for each $\varphi \in N$, let $\overline{\varphi} = \varphi S^*$. Then we can define a cocycle $\alpha\colon \mathrm{SL}_n(\mathbb{Z}) \times X \to H$ by

$$\alpha(g, x) = \text{ the unique element } \overline{\varphi} \in H \text{ such that } \varphi([A_x] \times [I_x]) = [A_{g.x}] \times [I_{g.x}]$$

Lemma 7.8. *There exists an algebraic $\mathbb{Q}$-group G with $\dim G < n^2 - 1$ such that $H \leq G(\mathbb{Q})$.*

Proof. Let Ω be a fixed algebraically closed field containing $\mathbb{R}$ and all of the p-adic fields $\mathbb{Q}_p$. Let

$$\Lambda = \Omega \otimes S \subseteq \mathrm{Mat}_n(\Omega)$$

be the associated Ω-algebra. Then Λ is an affine $\mathbb{Q}$-variety; and the Cayley-Hamilton Theorem implies that the group of units of Λ is given by

$$\Lambda^* = \{\varphi \in \Lambda \mid \det(\varphi) \neq 0\}.$$

Thus Λ^* is an algebraic $\mathbb{Q}$-group and $\Lambda^*(\mathbb{Q}) = S^*$. Furthermore, by Proposition 1.7 [4], $\Gamma = N_{GL_n^{\langle e_0 \rangle}(\Omega)}(\Lambda)$ is also an algebraic $\mathbb{Q}$-group and clearly $\Gamma(\mathbb{Q}) = N$. By Theorem 6.8 [4], $G = \Gamma/\Lambda^*$ is an algebraic $\mathbb{Q}$-group and

$$H = \Gamma(\mathbb{Q})/\Lambda^*(\mathbb{Q}) \leq G(\mathbb{Q}).$$

Finally note the following, where the last inequality holds because $n \geq 3$.

$$\dim G \leq \dim \Gamma \leq \dim GL_n^{\langle e_0 \rangle}(\Omega) = n^2 - (n-1) < n^2 - 1$$

$\square$

By Theorem 1.44, α is equivalent to a cocycle β such that $\beta(\mathrm{SL}_n(\mathbb{Z}) \times X)$ is contained in a finite subgroup K of H. To simplify notation, we shall assume that $\beta = \alpha$. Then for each $x \in X$,

$$\begin{aligned} \Phi(x) &= \{\varphi([A_x] \times [I_x]) \mid \overline{\varphi} = \alpha(g, x) \text{ for some } g \in \mathrm{SL}_n(\mathbb{Z})\} \\ &= \{[A_z] \times [I_z] \mid zEx\} \end{aligned}$$

is a nonempty finite set of $\approx_n \times \approx_n$-classes; and clearly if xEy, then $\Phi(x) = \Phi(y)$. By the ergodicity of μ, we can assume that there exists an integer $1 \leq l \leq |K|$ such that $|\Phi| = l$ for all $x \in X$. Now let $x \mapsto (x_1, \dots, x_l)$ be a Borel function from X to X^l such that for each $x \in X$,

(a) $x_i E x$ for all $1 \leq i \leq l$; and

(b) $\Phi(x) = \{[A_{x_1}] \times [I_{x_1}], \dots, [A_{x_l}] \times [I_{x_l}]\}$.

Finally, let $\tilde{f} \colon X \to (R(\mathbb{Q}^n) \times S(\mathbb{Q}^n))^l$ be the Borel function defined by

$$\tilde{f}(x) = (A_{x_1} \times I_{x_1}, \dots, A_{x_l} \times I_{x_l});$$

and let F be the countable Borel equivalence relation on $(R(\mathbb{Q}) \times S(\mathbb{Q}^n))^l$ defined by

$$\begin{gathered} (A_1 \times I_1, \dots, A_l \times I_l) F (B_1 \times J_1, \dots, B_l \times J_l) \\ \iff \{[A_1] \times [I_1], \dots, [A_l] \times [I_l]\} = \{[B_1] \times [J_1], \dots, [B_l] \times [J_l]\}. \end{gathered}$$

Since the relation $\approx_n$ on $R(\mathbb{Q}^n)$ and the relation $\approx_n$ on $S(\mathbb{Q}^n)$ are both hyperfinite, it follows that F is also hyperfinite. (For example, see [25, Section 1].) Notice that if xEy, then $\Phi(x) = \Phi(y)$ and so $\tilde{f}(x) F \tilde{f}(y)$. By Theorem 1.25, there exists an $SL_n(\mathbb{Z})$-invariant Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that $\tilde{f}$ maps M into a single F-class; and this implies that f maps M into a single $\sim_{n^*}^{\langle e_0 \rangle}$-class. This completes the proof of Theorem 7.6.

Chapter 8
Proof of Theorem 1.11

Let $n \geq 3$ and assume toward a contradiction that $f : SDG_{n+1} \to SDG_n$ is a Borel reduction from $\cong^+_{n+1}$ to $\cong^+_n$. Let $g_n : R(\mathbb{Q}^n) \to SDG_{n+1}$ be the Borel reduction defined in Proposition 6.2. Then $h = f \circ g_n$ is a Borel reduction from $\cong_n$ to $\cong^+_n$. Letting $X = X_n$, $\mu = \mu_n$, and $\sigma = \sigma_n$ as in Definition 1.43, we have that

(a) X is a standard Borel $\mathrm{SL}_n(\mathbb{Z})$-space with $\mathrm{SL}_n(\mathbb{Z})$-invariant ergodic probability measure μ,

(b) σ is a Borel homomorphism from $E^{X_n}_{\mathrm{SL}_n(\mathbb{Z})}$ to $\cong_n$, and

(c) σ is countable-to-one and hence does not map a measure one subset of X to a single $\cong_n$-class.

Adjusting by the appropriate elements of $GL_n(\mathbb{Q})$, we may assume that the order unit of every element in the range of h is $u = e_0$. Now let $\pi : SDG_n \to R(\mathbb{Q}^n) \times S(\mathbb{Q}^n)$ be the map $\pi(A, A^+, u) = (A, \mathrm{infin}(A, A^+, u))$. Then Theorem 7.1 implies that we may assume that $\pi \circ h \circ \sigma$ maps X into a single $\cong^{\langle e_0 \rangle}_{n^*}$ -class. Hence, after adjusting by the appropriate elements of $\mathrm{GL}^{\langle e_0 \rangle}_n(\mathbb{Q})$, we can assume that $\pi \circ h \circ \sigma$ maps X to a single pair, say (A, I). So we have reduced our analysis to the case when all the dimension groups in the image of $h \circ \sigma$ have the same underlying torsion-free abelian group A, the same group of infinitesimals I, and the same distinguished order unit $u = e_0$.

In both of the following cases, we will use (a) and (b) above to show that $h \circ \sigma$ maps a measure-one subset of X to a single $\cong^+_n$-class. However, this violates (c), and thus completes the proof of Theorem 1.11.

8.1 Case I: $I = \{0\}$

Fix some $x \in X$. Let $(A, A^+_x, u) = (h \circ \sigma)(x)$, and let S_x be the stabilizer of (A, A^+_x, u) in $GL^{\langle e_0 \rangle}_n(\mathbb{Q})$.

Claim. *S_x is finite.*

Proof. We examine the action of S_x on the state space $S_u(A, A^+_x)$ defined by

$$\varphi.p(a) = p(\varphi^{-1}(a)) \text{ for } p \in S_u(A, A^+_x) \text{ and } \varphi \in S_x.$$

Notice that $\varphi \in S_x$ implies that $\varphi^{-1}(a) \in A$ for each $a \in A$, and so the above is well-defined. Notice also that

1. $\varphi.p(u) = p(\varphi^{-1}(u)) = p(u) = 1$; and

2. for any $a \in A^+$, $\varphi^{-1}(a) \in A^+$, and so $\varphi.p(a) = p(\varphi^{-1}(a)) \in \mathbb{R}^+$.

Thus $\varphi.p \in S_u(A, A_x^+)$. Notice also that, for any $\varphi \in S_x$, $p, q \in S_u(A, A_x^+)$, and $0 \leq \alpha \leq 1$,

$$\varphi.(\alpha p + (1-\alpha)q) = \alpha\varphi.p + (1-\alpha)\varphi.q.$$

Thus since any $\varphi \in S_x$ is an affine permutation of the classical simplex $S_u(A, A_x^+)$, it must permute the elements of the finite set $E(S_u(A, A_x^+))$. Hence the following statement implies that S_x is finite.

Subclaim. *If $\varphi \in S_x$, and φ acts as the identity on $E(S_u(A, A_x^+))$, then $\varphi = \mathrm{id}$.*

Proof. In this case, since each $p \in S_u(A, A_x^+)$ is an affine combination of elements of $E(S_u(A, A_x^+))$, φ fixes *every* state $p \in S_u(A, A_x^+)$. Now given any $a \in A_x$, recall that $\hat{a} \in \mathrm{Aff}(S_u(A, A_x^+))$ is defined by $\hat{a}(p) = p(a)$. So choose any state $p \in S_u(A, A_x^+)$ and any $a \in A$. Then $p(a) = \varphi^{-1}.p(a)$. Thus $p(a) = p(\varphi(a))$, and so $p(a - \varphi(a)) = 0$. This implies that $\widehat{a - \varphi(a)}(p) = 0$, and since our choice of p was arbitrary, $\widehat{a - \varphi(a)} = 0$. But since $I = \{0\}$, Corollary 4.10 implies $a - \varphi(a) = 0$. Thus $a = \varphi(a)$ for every $a \in A$, and so $\varphi = \mathrm{id}$. □

□

Thus there are only countably many possibilities for the stabilizer of $(h \circ \sigma)(x) = (A, A_x, u)$ in $\mathrm{GL}_n^{\langle e_0 \rangle}(\mathbb{Q})$ and we can proceed as in the proof of Theorem 7.6. For the rest of this case, let $E = E^X_{\mathrm{SL}_n(\mathbb{Z})}$. Since μ is countably-additive, there exists a Borel subset $X_1 \subseteq X$ with $\mu(X_1) > 0$ and a fixed finite subgroup S of $\mathrm{GL}_n^{\langle e_0 \rangle}(\mathbb{Q})$ such that $S_x = S$ for all $x \in X_1$. By the ergodicity of μ, we have that $\mu(\mathrm{SL}_n(\mathbb{Z}).X_1) = 1$. In order to simplify notation, we shall assume that $\mathrm{SL}_n(\mathbb{Z}).X_1 = X$. After slightly adjusting $h \circ \sigma$ if necessary, we can also assume that $S_x = S$ for all $x \in X$. (That is, let $c\colon X \to X$ be a Borel function such that $c(x)Ex$ and $c(x) \in X_1$ for all $x \in X$. Then we can replace $h \circ \sigma$ with $h \circ \sigma \circ c$.)

Now suppose that $x, y \in X$ and that xEy. Then $(A, A_x^+, u) \cong (A, A_y^+, u)$ and so there exists $\varphi \in \mathrm{GL}_n^{\langle e_0 \rangle}(\mathbb{Q})$ such that $\varphi(A, A_x^+, u) = (A, A_y^+, u)$. Notice that

$$\varphi S \varphi^{-1} = \varphi S_x \varphi^{-1} = S_y = S$$

and so $\varphi \in N = N_{\mathrm{GL}_n^{\langle e_0 \rangle}(\mathbb{Q})}(S)$. Let $H = N/S$ and for each $\varphi \in N$, let $\overline{\varphi} = \varphi S$. Then we can define a cocycle $\alpha\colon \mathrm{SL}_n(\mathbb{Z}) \times X \to H$ by

$$\alpha(g, x) = \text{ the unique element } \overline{\varphi} \in H \text{ such that } \varphi(A, A_x^+, u) = (A, A_{g \cdot x}^+, u).$$

Now since S is finite, it is a closed subgroup of N, and so H is a algebraic $\mathbb{Q}$-group (See for example [32, 5.5.10]). Furthermore we have the following, where the last inequality holds because $n \geq 3$,

$$\dim H \leq \dim GL_n^{\langle e_0 \rangle}(\Omega) = n^2 - (n-1) < n^2 - 1.$$

Thus, by Theorem 1.44, α is equivalent to a cocycle β such that $\beta(SL_n(\mathbb{Z}) \times X)$ is contained in a finite subgroup K of H. To simplify notation, we shall assume that $\beta = \alpha$. Then for each $x \in X$,

$$\begin{aligned} \Phi(x) &= \{\varphi(A, A_x^+, u) \mid \overline{\varphi} = \alpha(g, x) \text{ for some } g \in \mathrm{SL}_n(\mathbb{Z})\} \\ &= \{(A, A_z^+, u) \mid zEx\} \end{aligned}$$

is finite; and clearly if xEy, then $\Phi(x) = \Phi(y)$. But this means that Φ is a Borel homomorphism from E to the identity relation on the standard Borel space of finite subsets of SDG_n. Hence, by Theorem 1.23, there exists a Borel subset $X_2 \subseteq X$ with $\mu(X_2) = 1$ such that $\Phi(x) = \Phi(y)$ for all $x, y \in X_2$; and this means that $h \circ \sigma$ maps X_2 into a single $\cong_n^+$-class, as desired.

Of course, after a suitable adjustment of $h \circ \sigma$, we can assume that $h \circ \sigma$ maps X_2 to a single dimension group. This observation will be helpful in our analysis of Case II.

8.2 Case II: $I \neq \{0\}$

Consider some $x \in X$ and the dimension group $(A, A_x^+, u) = (h \circ \sigma)(x)$. Consider the quotient group A/I. Theorem 4.10 implies

$$a \in A_x^+ \backslash \{0\} \text{ and } b \in I \implies a + b \in A_x^+ \backslash \{0\},$$

since in this case $\widehat{(a+b)} = \widehat{a} + \widehat{b} = \widehat{a} \gg 0$. It is easy to see that $(A/I, C_x^+, v)$ is a simple dimension group, where $C_x^+ = \{a + I \mid a \in A_x^+\}$ and $v = u + I$. We check the Riesz Interpolation Property. Consider $a, b, c, d \in A$ such that $a + I, b + I \leq c + I, d + I$. Then $c - a + I, c - b + I, d - a + I, d - b + I \in C_x^+$. This implies that $c - a, c - b, d - a, d - b \in A_x^+$, and so we may apply the Riesz Interpolation Property to obtain some $e \in A$ such that $a, b \leq e \leq c, d$. Then $a + I, b + I \leq e + I \leq c + I, d + I$, and we are done.

So by Case I, we may assume that there is a subset $X_1 \subseteq X$ with $\mu(X_1) = 1$ such that for every $x, y \in X_1$, $(A/I, C_x^+, v) = (A/I, C_y^+, v)$. Notice that if $p \in S_u(A, A_x^+)$, then $p^*(a + I) = p(a)$ defines a state $p^* \in S_v(A/I, C_x^+)$. In fact, this defines a one-to-one correspondence between $S_u(A, A_x^+)$ and $S_v(A/I, C_x^+)$. Thus for $x, y \in X_1$, we have the following, where the last implication is due to Corollary 4.11:

$$\begin{aligned}(A/I, C_x^+, v) = (A/I, C_y^+, v) &\implies S_v((A/I, C_x^+)) = S_v((A/I, C_y^+)) \\ &\implies S_u(A, A_x^+) = S_u(A, A_y^+) \\ &\implies (A, A_x^+, u) = (A, A_y^+, u).\end{aligned}$$

And so $h \circ \sigma$ maps X_1 to a single dimension group.

Chapter 9

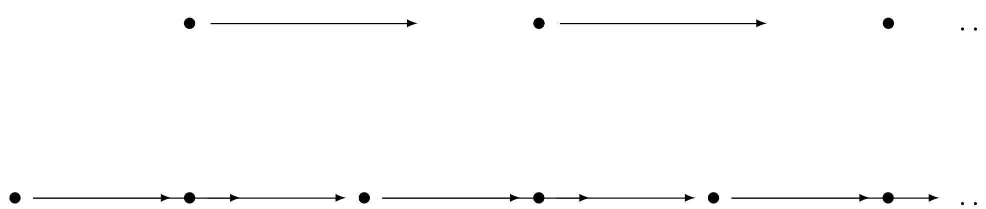
The rank of a Bratteli diagram

In Chapter 5, we gave an explicit construction which assigned to each countable dimension group a Bratteli diagram. What we have *not* done, however, is to understand what kind of Bratteli diagrams correspond to dimension groups of a given finite rank. A first guess would be that the following notion of rank in a Bratteli diagram corresponds to the rank of the resulting dimension group.

Definition 9.1. Given a Bratteli diagram (V, E) where $V = \bigsqcup_{n \in \mathbb{N}} V_n$, we define

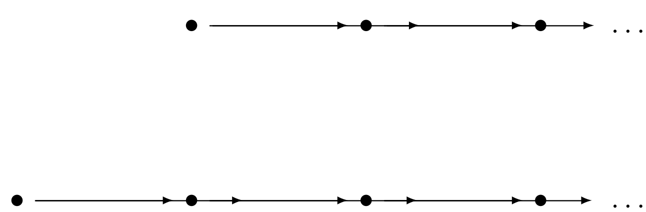
$$\operatorname{rank}(V, E) = \liminf_{n \to \infty} |V_n|.$$

However, there are $\sim$-equivalent Bratteli diagrams with different ranks. For example the rank 1 diagram:



(9.1)

telescopes to the rank 2 diagram:



(9.2)

Thus we define:

Definition 9.2. Let $\mathcal{BD}_n$ denote the standard Borel space of simple Bratteli diagrams which are $\sim$-equivalent to a Bratteli diagram of rank at most n. That is, we let

$$\mathcal{BD}_n = \{(V, E) \in \mathcal{BD} \mid \exists (W, F) \sim (V, E) \text{ with } \operatorname{rank}(W, F) \leq n\}.$$

Let BD_n be the equivalence relation obtained by restricting $\sim$ to $\mathcal{BD}_n$.

Notice that for any Bratteli diagram (V, E), $\operatorname{rank}(V, E) \geq \operatorname{rank}(K_0(V, E))$. Thus Theorems 3.3 and 3.4 imply that Definition 3.1 gives us a Borel function $\Phi : \mathcal{BD}_n \to \bigsqcup_{i=0}^{n} SDG_i$ such that $(V, E) \sim (W, F)$ if and only if $\Phi((V, E)) \cong^+ \Phi((W, F))$. Hence we have that BD_n is an essentially countable Borel equivalence relation, and clearly $BD_n \leq_B BD_{n+1}$.

While it is true that $\operatorname{rank}(V, E) \geq \operatorname{rank}(K_0(V, E))$, there are simple dimension groups whose rank is strictly less than that of each of the Bratteli diagrams which generate them.

Example 9.3. [9, 2.7] Consider the simple dimension group $A = \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}$ (here $\mathbb{Z}[\frac{1}{3}]$ denotes the triadic rationals) with positive cone $A^+ = \{(a, b) \in A \mid a > 0\} \cup \{(0, 0)\}$. We will show that $(A, A^+) \neq K_0(V, E)$ for every Bratteli diagram (V, E) such that $\operatorname{rank}(V, E) = 2$. Assume otherwise, and let $(A, A^+) = K_0(V, E)$ where $\operatorname{rank}(V, E) = 2$. Then there must be some $n \in \omega$, such that $|V_n| = 2$, and $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in (\mathbb{Z}^{V_n})^+$ such that $[(a_1, b_1)] = (1, -1)$, $[(a_2, b_2)] = (1, 0)$, and $[(a_3, b_3)] = (1, 1)$.

Then let $(c_1, d_1) = [(1, 0)]$ and $(c_2, d_2) = [(0, 1)]$ where $(1, 0)$ and $(0, 1)$ are the basis elements of $\mathbb{Z}^{V_n}$. Since $(1, 0), (0, 1) \in (\mathbb{Z}^{V_n})^+$, we have $(c_1, d_1), (c_2, d_2) \in A^+$. Now, for $i = 1, 2, 3$, we have

$$(a_i, b_i) = m_i(1, 0) + n_i(0, 1) \text{ for some } m_i, n_i \in \mathbb{Z}^+,$$

which gives us the following set of equations:

$$m_1(c_1, d_1) + n_1(c_2, d_2) = (1, -1) \qquad \text{where } m_1, n_1 \in \mathbb{Z}^+; \tag{9.3}$$

$$m_2(c_1, d_1) + n_2(c_2, d_2) = (1, 0) \qquad \text{where } m_2, n_2 \in \mathbb{Z}^+; \tag{9.4}$$

$$m_3(c_1, d_1) + n_3(c_2, d_2) = (1, 1) \qquad \text{where } m_3, n_3 \in \mathbb{Z}^+. \tag{9.5}$$

(9.3) and (9.5) imply that d_1 and d_2 are nonzero and have opposite signs. Then since $(c_1, d_1), (c_2, d_2) \in A^+ \backslash (0, 0)$, it follows that $c_1, c_2 > 0$. Subtracting (9.3) from (9.4), we obtain

$$(m_2 - m_1)(c_1, d_1) + (n_2 - n_1)(c_2, d_2) = (0, 1),$$

and so $(m_2 - m_1)c_1 + (n_2 - n_1)c_2 = 0$. But then since $c_1, c_2 > 0$ either (i) $(m_2 - m_1)$ and $(n_2 - n_1)$ are both zero, or (ii) neither are zero and they have opposite signs. If they are both zero, then $(m_2 - m_1)d_1 + (n_2 - n_1)d_2 = 0$, a contradiction. If they have opposite signs, then $(m_2 - m_1)d_1$ and $(n_2 - n_1)d_2$ are non-zero integers with the same sign, and then $|(m_2 - m_1)d_1 + (n_2 - n_1)d_2| \geq 2$, a contradiction.

It can be calculated that this dimension group is generated by the Bratteli diagram whose incidence matrices are all

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

Note that this matrix is singular, and that the corresponding map φ_n is not one-to-one. For example,

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}.$$

However, we will find it convenient to ignore these types of dimension groups:

Definition 9.4. If a dimension group (A, A^+, u) may be written as $K_0(V, E)$ for some Bratteli diagram (V, E) where all the maps φ_n are one-to-one, then (A, A^+, u) is said to be *ultrasimplicial*.

Lemma 9.5. *If (V, E) is a Bratteli diagram such that all the maps φ_n are one-to-one, and $K_0(V, E)$ is a finite rank dimension group, then* $\operatorname{rank}(V, E) = \operatorname{rank}(K_0(V, E))$*, and there exists $N \geq 1$ such that $|V_n| = \operatorname{rank}(K_0(V, E))$, for all $n > N$.*

Proof. Set $r = \operatorname{rank}(V, E)$. Let $N \geq 1$ be the least natural number such that $|V_N| = \liminf_{n\to\infty} |V_N| = r$. We claim that if $n > N$, then $|V_n| = |V_N|$. Otherwise, either $|V_n| < |V_N|$ and then $\varphi_n \circ \ldots \circ \varphi_{N+1}$ is not injective, or else $|V_n| > |V_N|$ and then there is some $m > n$ such that $|V_m| = |V_N| < |V_n|$ and $\varphi_m \circ \ldots \circ \varphi_{n+1}$ is not injective.

Finally, if $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_r \in \mathbb{Z}^{V_N}$ are the natural basis elements, and $\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \ldots + \alpha_r\mathbf{e}_r \neq 0$ ($\alpha_i \in \mathbb{Z}, 1 \leq i \leq r$) is any nontrivial linear combination, then $\alpha_1[\mathbf{e}_1] + \alpha_2[\mathbf{e}_2] + \ldots + \alpha_r[\mathbf{e}_r] \neq [0]$. Otherwise, we would have $[\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \ldots + \alpha_r\mathbf{e}_r] = [0]$ which would violate the injectivity of the maps φ_n. Thus $\operatorname{rank}(K_0(V, E)) \geq r$. ☐

We shall show that the dimension groups involved in the proof of Theorem 1.11 are all ultrasimplicial.

Theorem 9.6. *Suppose G is a p-local torsion-free abelian group of rank n, where $p > n$. Then the dimension group $g_n(G)$ given by Lemma 6.2 is ultrasimplicial.*

Before we prove this, we show how this gives the analogue of Theorem 1.11 for simple Bratteli diagrams.

Corollary 9.7. *For $n \geq 3$,* $\mathrm{BD}_n <_B \mathrm{BD}_{n+1}$

Proof. Suppose that $f : \mathcal{BD}_{n+1} \to \mathcal{BD}_n$ is a Borel reduction from BD_{n+1} to BD_n. Let $g_n : R(\mathbb{Q}^n) \to SDG_{n+1}$ be the Borel reduction from $\cong_n$ to $\cong_{n+1}^+$ defined in Lemma 6.2.

As in Chapter 8, we consider $X = X_n$, $\mu = \mu_n$, and $\sigma = \sigma_n$ from Definition 1.43. If we pick $p > n$ when defining X, μ, and σ, then Theorem 9.6 says that every group in the image of $g_n \circ \sigma$ is ultrasimplicial.

Hence combining Lemma 9.5 and Theorem 5.4, we obtain a Borel reduction

$$j : (g_n \circ \sigma)(X) \to \mathcal{BD}_{n+1}$$

from $\cong^+_{n+1}\restriction_{(g_n\circ\sigma)(X)}$ to BD_{n+1}. Next, Definition 3.1 gives a Borel reduction

$$h : \mathcal{BD}_n \to \bigsqcup_{i\leq n} SDG_i$$

from BD_n to $\bigsqcup_{i\leq n} \cong^+_i$. Then the following composition is a Borel homomorphism from $E^X_{\mathrm{SL}_n(\mathbb{Z})}$ to $\bigsqcup_{i\leq n} \cong^+_i$:

$$X \xrightarrow{\sigma} R(\mathbb{Q}^n)\restriction_{\sigma(X)} \xrightarrow{g_n} SDG_{n+1}\restriction_{(g_n\circ\sigma)(X)} \xrightarrow{j} \mathcal{BD}_{n+1} \xrightarrow{f} \mathcal{BD}_n \xrightarrow{h} \bigsqcup_{i\leq n} SDG_i.$$

Clearly there exists a subset $X_1 \subseteq X$ with $\mu(X_1) > 0$ such that the above maps X_1 to SDG_k for some $k \leq n$. Then by the ergodicity of μ, $\mu(\mathrm{SL}_n(\mathbb{Z}).X_1) = 1$. Replacing X by $\mathrm{SL}_n(\mathbb{Z}).X_1$, the analysis of Chapter 8 again shows that there is a measure one subset of X which maps to a single $\cong^+_k$-class. This implies that σ maps a measure one subset of X to a single $\cong_n$-class, which is a contradiction. $\square$

Proof of Theorem 9.6. We will prove that $g_n(G)$ satisfies the following criteria for ultrasimpliciality:

Lemma 9.8. *[20] Let (A, A^+, u) be a countable dimension group. Then (A, A^+, u) is ultrasimplicial if and only if for all finite subsets $\{x_i\}_{i=1}^n$ of A^+,*

(∗) *there exists a finite subset $\{s_j\}_{j=1}^m$ of A^+ such that*

1. *$\{s_j\}_{j=1}^m$ is rationally independent;*
2. *there exist m_{ij} in $\mathbb{N} \cup \{0\}$ with $x_i = \sum m_{ij}s_j$, for all i.*

Below we will use the following extension of the notion of gcd to the rationals.

Definition 9.9. Given a finite set of positive rational numbers $\{q_1, q_2, \ldots, q_n\}$, define $\gcd\{q_1, q_2, \ldots, q_n\}$ to be the greatest positive rational number q such that for every $1 \leq i \leq n$, $q_i = m_i q$ for some $m_i \in \mathbb{N}$.

Let G be a p-local torsion-free abelian group of rank n, where $p > n$. (The condition $p > n$ will allow us to divide any element of G by n.) Let $(G \oplus \mathbb{Q}, (G \oplus \mathbb{Q})^+, (0, 1))$ be the dimension group defined by $(G \oplus \mathbb{Q})^+ = \{(h, q) \in G \oplus \mathbb{Q} : q > 0\} \cup \{(0, 0)\}$. Recall that $G \leq \mathbb{Q}^n$, and let $\{x^i = (x^i_0, x^i_1, x^i_2, \ldots, x^i_{n-1}) \oplus (x^i_n)\}_{i\leq m}$ be a finite set of elements of $(G \oplus \mathbb{Q})^+$. Let $y_k = \frac{1}{n} \gcd_{i \leq m}\{|x^i_k|\}$ for all $0 \leq k \leq n$. Next, for $0 \leq j \leq n-1$, let $s_j = (0, 0, \ldots, y_j, \ldots, 0) \oplus (\frac{y_n}{n^N})$, where y_j is in the j-th slot, and $N \in \mathbb{N}$ is some constant determined below. Finally, let $s_n = (-y_0, -y_1, \ldots, -y_j, \ldots, -y_{n-1}) \oplus (\frac{y_n}{n^N})$.

We claim that if N is large enough, then $\{s_j\}_{i=0}^n$ fulfills $(*)$. Clearly, the $\{s_j\}$ are rationally independent. Given $i \leq m$, we want to express x^i as a sum of nonnegative integer multiples of the s_j. First, note that the sum $\sum_{k=0}^{n-1} \left(\frac{x^i_k}{y_k}\right) s_k$ does the trick, but only on the first n coordinates. We can then add some multiple M^i of $\sum_{j=0}^{n} s_j$ to this sum without changing the first n coordinates. So we just solve for M^i. We have that

$$x^i = M^i \sum_{j=0}^{n} s_j + \sum_k \left(\frac{x^i_k}{y_k}\right) s_k$$

thus,

$$x^i_n = M^i n \frac{y_n}{n^N} + \sum_k \frac{x^i_k}{y_k} \frac{y_n}{n^N}$$

Then,

$$M^i = \frac{x^i_n - \sum_k \frac{x^i_k}{y_k} \frac{y_n}{n^N}}{n \frac{y_n}{n^N}} = \frac{x^i_n}{y_n} n^{N-1} - \left(\sum_k \frac{x^i_k}{y_k}\right) \frac{1}{n}$$

Now, by the definition of the y_k, $(\sum_k \frac{x^i_k}{y_k})\frac{1}{n}$ and $\frac{x^i_n}{y_n}$ are positive integers. Finally, if we choose N large enough, then M^i is positive for all $0 \leq i \leq m$. □

9.1 Simple groups of strongly diagonal type

We will now see how our analysis can be applied to the classification problem for simple countable locally finite groups of strongly diagonal type.

Definition 9.10. Given a countable locally finite group of strongly diagonal type G, define $\text{rank}(G) = \min\{\text{rank}(V, E) \mid G(V, E) \cong G\}$.

Definition 9.11. For each $n \geq 1$, let $\mathcal{SDT}_n \subseteq \mathcal{SDT}$ be the standard Borel space of countable simple locally finite groups of strongly diagonal type of rank at most n. That is, let

$$\mathcal{SDT}_n = \{G \in \mathcal{SDT} \mid \exists (V, E) \in \mathcal{BD}_n \text{ such that } G \cong G(V, E)\}.$$

Then let $\cong^s_n$ be the equivalence relation obtained by restricting $\cong_{\mathcal{SDT}}$ to $\mathcal{SDT}_n$.

Then Theorems 2.9 and 2.12 imply that the assignment $(V, E) \mapsto G(V, E)$ gives a Borel reduction from BD_n to SDT_n, and that the map defined in the proof of Theorem 2.20 gives a Borel reduction from SDT_n to BD_n. Thus we have shown:

Theorem 9.12. *For each $n \geq 1$, $\cong_n^s \sim_B BD_n$.*

Corollary 9.13. *For each $n \geq 3$, $(\cong_n^s) <_B (\cong_{n+1}^s)$.*

9.2 Questions

Bratteli diagrams also characterize other naturally occurring structures, such as approximately finite-dimensional (AF) C^*-algebras and AF-relations on Cantor sets.

Question 9.14. *For which other classes of structures that are described by Bratteli diagrams can we obtain a result similar to Theorem 1.11?*

Recall that $\cong_n$ is the isomorphism relation on the space of torsion free abelian groups of rank n, and that E_∞ is the universal countable Borel equivalence relation. In [34], Thomas showed that $\left(\bigsqcup_{n\geq 1} \cong_n\right) <_B E_\infty$.

Conjecture 9.15. $\left(\bigsqcup_{n\geq 1} \cong_n^+\right) <_B E_\infty$.

It is natural to define the class of Bratteli diagrams of rank exactly n as

$$\mathcal{BD}_n^* = \{(V,E) \in \mathcal{BD} \mid \exists (W,F) \sim (V,E) \text{ with } \operatorname{rank}(W,F) = n\},$$

and then to define BD_n^* as $\sim\restriction_{\mathcal{BD}_n^*}$. It is easy to rewrite the proof of Corollary 9.7 to show that, for $n \geq 3$, $BD_{n+1}^* \nleq_B BD_n^*$. However, the intuitively "easy" fact below is not currently known.

Conjecture 9.16. *For $n \geq 1$, $BD_n^* \leq_B BD_{n+1}^*$. Thus, for $n \geq 3$, $BD_n^* <_B BD_{n+1}^*$.*

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