SINGULAR PERTURBATION METHODS IN CREDIT DERIVATIVE MODELING

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ABSTRACT OF THE DISSERTATION

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This thesis introduces the dynamical pricing model and approximation method in pricing a "Collateralized Debt Obligation"(CDO). For this purpose we use a two-dimensional, self-affecting Markov process of discrete-valued aggregate loss process and stochastic factor process in its intensity. We review several models for pricing of multi-name credit derivative products and explain in detail a two-dimensional Markov intensity model proposed by Halperin and Arnsdorf.

Using the model by Halperin and Arnsdorf, we derive the Kolmogorov forward partial differential equation for the transition density function of the underlying two-dimensional Markov process. We use the singular perturbation method to obtain an approximate solution to this partial differential equation in the case of a fast mean reverting stochastic intensity model. We perform an error analysis to determine the accuracy of our approximate solution.

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Dedication

To my parents.

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Chapter 1

Introduction

1.1 Overview

Recent financial crises due to defaults in subprime mortgages have called attention to one of the most complex financial products involved in the crisis, namely the *Collateralized Debt Obligation* (CDO). A CDO works like an insurance product for a pool or portfolio of various financial assets. One party provides protection against losses in the pool while the other party pays a premium periodically for protection from losses in the underlying assets. Initially, such portfolios (called *cash CDOs*) were comprised of mortgages, auto and student loans, credit card debts, and so on. As the credit derivative market evolved, more asset securitization products were invented and the size of the global credit derivative market grew from $180 billion in 1996 to $20 trillion in 2006. One of the most important multi-name products is the *synthetic CDO*, whose constituents are traded *credit default swaps*, rather than the loans, etc., underlying a cash CDO. To produce a synthetic CDO, one has to diversify the pool and divide it into slices or *tranches*, with different levels of risk and return.

The central *pricing problem* for a synthetic CDO is to determine the correct "interest rate", or *credit spread*, at which each slice is offered to protection sellers in exchange for compensation of losses in the slice during the holding period. Methods of stochastic differential equations translate this practical question into one of the most challenging mathematical problems in financial engineering today. Through the credit spread formula of a CDO in §2.1.2

$$c = \frac{\mathbb{E}\left[\int_0^T D(0,t)dL_t^{a,b}\right]}{\sum_{i=1}^N \left((t_i - t_{i-1})\mathbb{E}\left[D(0,t_i)(1-L_{t_i})\right] + (T_i - t_i)\mathbb{E}\left[D(0,T_i)\Delta L_{T_i}^{a,b}\right]\right)},$$

it can be recognized that credit spread can be deduced from the knowledge of the

distribution of the default loss process.

The first approach to this problem was suggested by David Li [18], who proposed the "structural factor copula" model. (A copula is a method of combining marginal distribution functions to give a multivariate joint distribution function.) Li's approach gives joint default probabilities for the CDO portfolio provided that (a) the default probability of each firm can be obtained from each firm's value and (b) the correlation among firms is given exogenously (external to the model). However, several problems arise when implementing Li's model. First, the structural copula model cannot explain the short term credit spread observed in the market because the model assumes that each firm's value varies continuously. Second, this static model cannot be used to model the evolution of joint probability densities. Third, Li's model is not sophisticated or flexible enough to simultaneously match observed market prices for all traded credit derivative products.

The "intensity model" was proposed in order to explain the observed non-zero short term credit spreads. However, modeling the stochastic intensity of each firm (which captures when defaults occur) in the CDO is computationally too burdensome. It is unrealistic to model each of the firms individually because a CDO references more than a hundred firms.

The shortcomings of the two previous models led Giesecke and Goldberg to propose a "top-down" approach to pricing multi-name derivatives. The key idea in their approach is to model total portfolio loss with a single intensity process. This dramatic reduction in dimension allows us to apply more tractable mathematical methods in order to solve this problem. In addition, this transformation of the problem allows one to adapt the well-established Heath-Jarrow-Morton (HJM) forward rate curve model paradigm of interest rate theory. This HJM-style forward curve paradigm is incorporated in the work of Schönbucher [25], Bennani [2], and Sidenius-Piterbarg-Andersen [27] in their work on CDO modeling. The HJM style credit model is the most general form of credit derivative model and is very flexible because CDS market prices can be used as model input data, rather than an output the model must reproduce. Hence, the HJM-style credit model has fewer parameter calibration issues than other credit derivative models.

Unfortunately, the HJM-style curve credit model is still computationally too difficult to implement. Instead a top-down loss, spot intensity model has been proposed as a reasonable and tractable choice for credit derivative modeling. Giesecke and Goldberg [6], Lopatin and Misirpashaev [19], and Arnsdorf and Halperin [12], among others, have pursued this approach. In my dissertation, I rely on Halperin's work for modeling the loss process in multi-name credit products. Once the model for the loss process is set up, we use the *Kolmogorov forward equation* (a partial differential equation (PDE) for a transition probability density function) to compute the loss probability density function. The Kolmogorov forward equation — derived from the correlated Markov loss and intensity processes (defined by a two-dimensional system of stochastic ordinary differential equations) — is a system of partial differential equations (PDEs). We get a system of PDEs because our model has both a continuous random variable, namely the intensity process, and discrete random variable, namely the loss process. I obtain approximate solutions to the Kolmogorov forward equation for the loss and intensity processes using the singular perturbation methods introduced by Khasminskii and Yin in [15]. Recently the approximation method in pricing for financial products was used by Hagan and his collaborators [11] and by Fouque and his collaborators [9] in their work on the celebrated "SABR" and stochastic volatility models in interest rate and equity option pricing problems. The intensity process in my work is assumed to be a fast-varying diffusion process, so the corresponding Kolmogorov forward equation is a singularly perturbed one, meaning that the coefficients of the equation depend on a small parameter ε. Thus perturbation methods can be adapted to find approximate solutions to the problem.

1.2 Main Results

We begin with the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ equipped with a filtration $\{\mathcal{F}(t)\}_{t\geq 0}$. Let L_t be a continuous time Markov chain whose generating matrix A is upper bi-diagonal

and contains a stochastic factor Y_t.

$$A = a(Y_t)\begin{pmatrix} -F(t,0) & F(t,0) & 0 & \cdots & 0 \\ 0 & -F(t,1) & F(t,1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -F(t,N-1) & F(t,N-1) \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{1.1}$$

In our pricing model for multi-name credit derivatives, we adapt the process L_t as a loss process in the underlying portfolio. The random factor Y_t in the generating matrix A makes it possible to model the realistic evolution of the credit spread for dynamic credit products such as options because the random factor allows credit spread to change between credit events. Now the next-to-default intensity λ of L_t is stochastic and is given by

$$\lambda_t(N_t = k, Y_t, t) = a(Y_t) \cdot F(t,k).$$

In contrast with the usual Kolmogorov forward equation for one spatial variable, we derive a generalized Kolmogorov forward equation for a continuous variable Y_t and discrete variable L_t for the joint density function $P_{jk}(Y,T|y,t)$:

$$P_{jk}(Y,T|y,t) := \mathbb{Q}(N_T = k, Y_T = Y | N_t = j, Y_t = y).$$

We recognize that $P_{jk}(Y,T|y,t)$ can be approximated by a finite series of ε using the matching method of singular perturbation theory when Y_t follows

$$dY_t = \frac{1}{\varepsilon}\mu(t,Y_t)dt + \frac{1}{\sqrt{\varepsilon}}\sigma(t,Y_t)dW_t \qquad \varepsilon \ll 1, \tag{1.2}$$

where $\mu(t,y)$ and $\sigma(t,y)$ are real-valued smooth functions. We use the singular perturbation method to find an approximation of $P_{jk}(Y,T|y,t)$ and carry out an error analysis between our approximation and the exact solution.

Moreover assuming that Y_t evolves according to the formula (1.2) and the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is generated by the Brownian motion in (1.2), the pair (L_t, Y_t) makes a Markov process on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q}, \mathbb{F})$ meaning that

$$\mathbb{Q}(L_T, Y_T|\mathcal{F}_t) = \mathbb{Q}(L_T, Y_T|L_t, Y_t). \tag{1.3}$$

Theorem 1.2.1. *Let L_t be a continuous-time Markov chain on $(\Omega, \mathcal{F}, \mathbb{Q})$ equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ with generating matrix (1.1). If Y_t follows the stochastic differential equation*

$$dY_t = \frac{1}{\varepsilon}\mu(t, Y_t)dt + \frac{1}{\sqrt{\varepsilon}}\sigma(t, Y_t)dW_t \qquad \varepsilon \ll 1, \tag{1.4}$$

where $\mu(t, y)$ and $\sigma(t, y)$ are real-valued smooth functions representing the drift and diffusion respectively and W is a standard Brownian motion, then $P_{jk}(Y, T|y, t)$ satisfies the Kolmogorov forward equation

$$\begin{aligned} \frac{\partial P_{jk}}{\partial T} &= -\frac{\partial}{\partial Y}\left[\frac{1}{\varepsilon}\mu(T, Y)P_{jk}\right] + \frac{1}{2!}\frac{\partial^2}{\partial Y^2}\left[\frac{1}{\varepsilon}\sigma^2(T, Y)P_{jk}\right] \\ &\quad + \lambda(Y, k-1, T)P_{jk-1} - \lambda(Y, k, T)P_{jk}, \\ P_{jk}(Y, t|y, t) &= g_0 \cdot 1_{\{k=j\}}, \end{aligned} \tag{1.5}$$

where g_0 is the initial probability density function of Y_t and it satisfies $\int_{-\infty}^{\infty} g_0 dY = 1$ and $g_0 \geq 0$.

With backward variables fixed at $t = 0$, we simply denote P_{jk} as P_k. We plan to construct an asymptotic formula $\tilde{P}_k^{(n)}(T, Y)$ of the form

$$\begin{aligned} \tilde{P}_k^{(n)}(T, Y) &:= P_k^{(n)}(T, Y) + Q_k^{(n)}(T, Y), \\ &= \sum_{i=0}^{n} \varepsilon^i a_k^{(n)}(T, Y) + \sum_{i=0}^{n} \varepsilon^i b_k^{(n)}(\tau, Y), \end{aligned} \tag{1.6}$$

where $\tau = \frac{T}{\varepsilon}$.

Definition 1.2.2. *By substituting $\tilde{P}_k^{(n)}(T, Y)$ into (1.5), $P_k^{(n)}(T, Y)$ and $Q_k^{(n)}(T, Y)$ separate due to linearity of the differential operator in (1.5). Then we can define $a_k^{(i)}$*

in $P_k^{(n)}(T,Y)$ *and* $b_k^{(i)}$ *in* $Q_k^{(n)}(T,Y)$ *as solutions to the following differential equations:*

$$\begin{aligned}
&\mathcal{L}_0 a_k^{(0)} = 0, \quad \textit{with} \quad \int_{-\infty}^{\infty} a_k^{(0)}(Y,T)dY = 1_{\{k=0\}},\\
&\frac{\partial a_k^{(0)}}{\partial T} = \mathcal{L}_0 a_k^{(1)} + \mathcal{L}_1 a_k^{(0)} + \lambda(Y,k-1,T)P_{k-1}1_{\{k>0\}},\\
&\frac{\partial a_k^{(1)}}{\partial T} = \mathcal{L}_0 a_k^{(2)} + \mathcal{L}_1 a_k^{(1)},\\
&\frac{\partial a_k^{(2)}}{\partial T} = \mathcal{L}_0 a_k^{(3)} + \mathcal{L}_1 a_k^{(2)},\\
&\qquad\vdots\\
&\frac{\partial a_k^{(n-1)}}{\partial T} = \mathcal{L}_0 a_k^{(n)} + \mathcal{L}_1 a_k^{(n-1)}
\end{aligned} \tag{1.7}$$

and

$$\begin{aligned}
\frac{\partial b_k^{(0)}}{\partial \tau} &= \mathcal{L}_0^{(0)}(Y,0)b_k^{(0)},\\
\frac{\partial b_k^{(1)}}{\partial \tau} &= \mathcal{L}_0^{(0)}(Y,0)b_k^{(1)} + \tau\mathcal{L}_0^{(1)}(Y,0)b_k^{(0)} + \mathcal{L}_1^{(0)}(Y,0)b_k^{(0)},\\
\frac{\partial b_k^{(2)}}{\partial \tau} &= \mathcal{L}_0^{(0)}(Y,0)b_k^{(2)} + \tau\mathcal{L}_0^{(1)}(Y,0)b_k^{(1)} + \frac{\tau^2}{2!}\mathcal{L}_0^{(2)}(Y,0)b_k^{(0)} + \mathcal{L}_1^{(0)}(Y,0)b_k^{(1)}\\
&\quad + \tau\mathcal{L}_1^{(1)}(Y,0)b_k^{(0)},\\
&\vdots\\
\frac{\partial b_n}{\partial \tau} &= \sum_{i=0}^{n}\frac{\tau^i}{i!}\mathcal{L}_0^{(i)}b_k^{(n-i)} + \sum_{i=0}^{n-1}\frac{\tau^i}{i!}\mathcal{L}_1^{(i)}b_k^{(n-1-i)},
\end{aligned} \tag{1.8}$$

with initial conditions

$$\begin{cases} b_k^{(n)}(Y,0) &= -a_k^{(n)}(Y,0), \quad n \neq 0,\\ b_k^{(n)}(Y,0) &= g_0 \cdot 1_{\{k>0\}} - a_k^{(n)}(Y,0),\ n = 0, \end{cases}$$

where the operators are given by

$$\begin{aligned}
&\mathcal{L}_0 = 1 + (-\kappa(T)+Y)\frac{\partial}{\partial Y} + v\frac{\partial^2}{\partial Y^2} = -\frac{\partial}{\partial Y}\left((\kappa(T)-Y)\cdot\right) + v\frac{\partial^2}{\partial Y^2},\\
&\mathcal{L}_1 = -\lambda(Y,k,T),
\end{aligned} \tag{1.9}$$

and $\mathcal{L}_j^{(i)}$ *is the i-th derivative with respect to the time variable for* $j = 0, 1$

Thus finding the approximate formula $\tilde{P}_k^{(n)}(T,Y)$ depends on finding functions $a_k^{(i)}(T,Y)$ and $b_k^{(i)}(T,Y)$ in Equation (1.6).

Theorem 1.2.3. *For the operators* (1.9)*, we can solve the differential equation*

$$\mathcal{L}_0 a_k^{(0)} = 0$$

with auxiliary condition

$$\int a_k^{(1)}(Y,T)dY = 1_{\{k=0\}}$$

to find a continuous smooth function $a_k^{(0)}$*.*

Theorem 1.2.4. *For the operators* (1.9)*, we can solve the differential equation*

$$\frac{\partial a_k^{(0)}}{\partial T} = \mathcal{L}_0 a_k^{(1)} + \mathcal{L}_1 a_k^{(0)} + \lambda(Y, k-1, T) P_{k-1} 1_{\{k>0\}}$$

with auxiliary condition

$$\int a_k^{(1)}(Y,T)dY = 0$$

to find a continuous smooth function $a_k^{(1)}$*.*

Theorem 1.2.5. *The solution* $b_k^{(n)}$ *of the partial differential equation*

$$\frac{\partial b_k^{(n)}}{\partial \tau} = \sum_{i=0}^{n} \frac{\tau^i}{i!} \mathcal{L}_0^{(i)} b_k^{(n-i)} + \sum_{i=0}^{n-1} \frac{\tau^i}{i!} \mathcal{L}_0^{(i)} b_k^{(n-1-i)},$$

with initial condition

$$\begin{cases} b_k^{(n)}(Y,0) &= -a_k^{(n)}(Y,0), \quad n \neq 0, \\ b_k^{(n)}(Y,0) &= g_0 \cdot 1_{\{k>0\}} - a_k^{(n)}(Y,0),\ n = 0, \end{cases}$$

has the Green's function representation.

Since we only know the approximate value of P_{k-1} when we approximate the solution of next equation P_k, the assumption we made in Theorem 1.2.4 about knowing the exact solution of P_{k-1} can be removed by using vector-valued PDE instead of scalar-valued PDE so that we automatically have the approximate value of P_{k-1} when we calculate the approximate value of P_k. We make the system of PDEs into vector-valued equation,

$$\frac{\partial \vec{P}}{\partial T} = \frac{1}{\varepsilon} I \cdot \vec{P} - \frac{1}{\varepsilon}(\kappa(T) - Y) I \cdot \frac{\partial \vec{P}}{\partial Y} + \frac{v}{\varepsilon} I \cdot \frac{\partial^2 \vec{P}}{\partial Y^2} + \Lambda \cdot \vec{P}, \tag{1.10}$$

where

$$\vec{P}(T,Y) = \begin{pmatrix} P_0(T,Y) \\ \vdots \\ P_N(T,Y) \end{pmatrix}, \tag{1.11}$$

$$\Lambda(T,Y)=\begin{pmatrix} -\lambda(T,0) & 0 & 0 & \cdots & 0 \\ \lambda(T,0) & -\lambda(T,1) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \lambda(T,N-2) & -\lambda(T,N-1) & 0 \\ 0 & 0 & \cdots & \lambda(T,N-1) & -\lambda(T,N) \end{pmatrix},$$

and I is the $(N+1)\times(N+1)$ identity matrix.

We have regular and singular approximation vectors respectively,

$$\begin{aligned} \vec{P}(T,Y) &\approx \vec{P}^{(n)}+\vec{Q}^{(n)} \\ &= \begin{pmatrix} P_0^{(n)}+Q_0^{(n)} \\ \vdots \\ P_N^{(n)}+Q_N^{(n)} \end{pmatrix} \\ &= \sum_{i=0}^{n}\varepsilon^i\vec{A}^{(i)}(T,Y)+\sum_{i=0}^{n}\varepsilon^i\vec{B}^{(i)}(\tau,Y), \end{aligned} \tag{1.12}$$

where

$$\vec{A}^{(i)}(T,Y)=\begin{pmatrix} a_0^{(i)}(T,Y) \\ \vdots \\ a_N^{(i)}(T,Y) \end{pmatrix}, \vec{B}^{(i)}(\tau,Y)=\begin{pmatrix} b_0^{(i)}(\tau,Y) \\ \vdots \\ b_N^{(i)}(\tau,Y) \end{pmatrix}.$$

The regular part $\vec{P}^{(n)}$ satisfies

$$\frac{\partial\vec{P}^{(n)}}{\partial T}=\frac{1}{\varepsilon}I\cdot\vec{P}^{(n)}-\frac{1}{\varepsilon}(\kappa(T)-Y)I\cdot\frac{\partial\vec{P}^{(n)}}{\partial Y}+\frac{v}{\varepsilon}I\cdot\frac{\partial^2\vec{P}^{(n)}}{\partial Y^2}+\Lambda\cdot\vec{P}^{(n)}. \tag{1.13}$$

Equating the coefficients of like powers ε, we obtain equations for the zeroth order term $a_i^{(0)}$ and the first order term $a_i^{(1)}$ of approximation formula $P_i^{(1)}$ for P_i.

$$a_i^{(0)}-(\kappa(T)-Y)\frac{\partial a_i^{(0)}}{\partial Y}+v\frac{\partial^2 a_i^{(0)}}{\partial Y^2}=0, \tag{1.14}$$

$$a_i^{(1)}-(\kappa(T)-Y)\frac{\partial a_i^{(1)}}{\partial Y}+v\frac{\partial^2 a_i^{(1)}}{\partial Y^2}=\frac{\partial a_i^{(0)}}{\partial T}-\lambda(i-1)a_{i-1}^{(0)}+\lambda(i)a_i^{(0)}. \tag{1.15}$$

Similarly, the singular part of the approximation can be found:

$$\frac{1}{\varepsilon}\frac{\partial\vec{Q}^{(n)}}{\partial\tau}=\frac{1}{\varepsilon}I\cdot\vec{Q}^{(n)}-\frac{1}{\varepsilon}(\kappa(\varepsilon\tau)-Y)I\cdot\frac{\partial\vec{Q}^{(n)}}{\partial Y}+\frac{v}{\varepsilon}I\cdot\frac{\partial^2\vec{Q}^{(n)}}{\partial Y^2}+\Lambda\cdot\vec{Q}^{(n)}. \tag{1.16}$$

Substituting $\vec{Q}^{(n)}$ with $\sum_{i=0}^{n} \varepsilon^i \vec{B}^{(i)}(\tau, Y)$ and equating the coefficients of like powers of ε, for the zeroth order term we obtain

$$\frac{\partial b_i^{(0)}}{\partial \tau} = b_i^{(0)} - g(0,Y)\frac{\partial b_i^{(0)}}{\partial Y} + v\frac{\partial^2 b_i^{(0)}}{\partial Y^2}, \tag{1.17}$$

while for first order term, we have

$$\begin{aligned}\frac{\partial b_i^{(1)}}{\partial \tau} = b_i^{(1)} - g(0,Y)\frac{\partial b_i^{(1)}}{\partial Y} + v\frac{\partial^2 b_i^{(1)}}{\partial Y^2} - \tau\frac{\partial g}{\partial T}(0,Y)\frac{\partial b_i^{(0)}}{\partial Y} \\ + \lambda(0,Y,i)b_{i-1}^{(0)} - \lambda(0,Y,i)b_i^{(0)}.\end{aligned} \tag{1.18}$$

Theorem 1.2.5 is applied to equations (1.17) and (1.18).

Next we state the error analysis for the approximation formula $\tilde{P}_k^{(n)}(T,Y)$. The remainder $r_k^{(n)}$ is defined by

$$\mid r_k^{(n)} \mid := \mid P_k - (P^n + Q^n) \mid = \mid P_k - \sum_{i=0}^{n} \varepsilon^i a_k^{(i)}(Y,T) - \sum_{i=0}^{n} \varepsilon^i b_k^{(i)}(Y,\tau) \mid .$$

We only prove the case for $n = 1$ and $k = 0$.

Theorem 1.2.6. *With $a_k^{(i)}$ from Theorem 1.2.4 and $b_k^{(i)}$ from Theorem 1.2.5,*

$$\sup_{(T,Y)\in(0,T_M)\times(-\infty,\infty)} |r_0^{(1)}| = O(\varepsilon^1).$$

1.3 Outline of the Thesis

The thesis is organized in the following manner.

In Chapter 2 we provide the financial and mathematical background on which we rely in later chapters. Definitions of various multi-name credit products such as the credit default swap (CDS), collateralized debt obligation (CDO), forward starting CDO, option on CDO and leveraged super-senior tranche are given in §2.1. In the second section, we introduce the extension of the Kolmogorov forward equation and the matching method of singular perturbation.

In Chapter 3, we review several different modeling approaches for pricing credit derivatives. We start with the intuitive bottom-up model, of which the most celebrated is the Gaussian copula model in §3.1. In §3.2, the top-down model approach is reviewed. In particular we take a close look at the Markovian bivariate spread-loss model for portfolio credit derivatives proposed for later use in §5.2.

In Chapter 4, we review the work of Fouque, Papanicolaou, and Sirca [8] on use of the singular perturbation method in equity option pricing.

In Chapter 5, the singular perturbation method for the Gaussian copula model studied by Fouque and Zhou [10] is reviewed in the first section. In §5.2, we first look at the work of Yin and Khasminskii[15] and we present the Markov two-dimensional loss and intensity model and the singular perturbation method to find an approximate solution for the joint probability density function of loss and intensity. With the approximate solution, we perform an error analysis to determine the accuracy.

In Chapter 6, we summarize our results and briefly mention future research topics.

Chapter 2

Background

In this chapter we provide the financial and mathematical background on which we rely on in later chapters.

The first section is devoted to the financial background and in the second section we explain necessary mathematical tools which are used in chapter 5.

2.1 Financial Background

There are several important credit products in the current financial market. Here we explain credit derivative products and address the pricing problems for them.

2.1.1 Credit Default Swap (CDS)

The *credit default swap* is one of the most liquid single-name credit derivatives in the market and provides a good indication of the credit risk of the underlying entity. In a single-name credit default swap, the protection buyer pays a coupon c to another party, the protection seller, in return for protection against default on a firm. The coupon c is also called a CDS spread. At the time of default, the coupon payments are stopped.

The problem of CDS pricing reduces to a determination of the coupon rate c which makes the current price of the swap equal to zero.

We briefly illustrate the structure of a credit default swap to determine c. Let $t_0 < t_1 < \cdots < t_n$ be payment schedule dates. A random variable τ denotes the default time of the reference entity. A credit default swap *fixed leg* has the contingent claim paying $\alpha_i c$ at time t_i for all $i = 0, \ldots, n$, as long as default has not occurred by that time, where c is a constant coupon rate and α_i is the day-count fraction between t_{i-1}

and t_i. Thus a fixed leg is a portfolio of $n \geq 1$ defaultable zero coupon bond with maturity t_i with amounts $\alpha_i k$.

Then the value of fixed leg at $t = 0$ is given by

$$\pi_0^{\text{fixed}} = \sum_{i=1}^{n} \alpha_i c P_0^i V_0^i, \tag{2.1}$$

where each P_0^i is the current survival probability with maturity t_i, and V_0^i is the current value of a default-free zero with maturity t_i.

A credit default swap *default leg* consists of a single contingent claim C_i which has a maturity t_i and payoff $(1-R)1_{t_{i-1}<\tau\leq t_i}$, where R is the constant *recovery rate* of the default leg. It pays $1-R$ at time t_i if defaults occur in the interval $(t_{i-1}, t_i]$ under the assumption that notional is normalized to 1. Thus the default leg has a value at $t = 0$ of

$$\pi_0^{\text{default}} = (1-R)\sum_{i=1}^{n}(P_0^{i-1} - P_0^i)V_0^i. \tag{2.2}$$

Thus equating the value of fixed leg (2.1) and the value of default leg (2.2) gives the value of the coupon c

$$c = \frac{(1-R)\sum_{i=1}^{n}(P_0^{i-1} - P_0^i)V_0^i}{\sum_{i=1}^{n}\alpha_i P_0^i V_0^i}. \tag{2.3}$$

2.1.2 Synthetic Collateralized Debt Obligation (CDO)

The next most common portfolio credit derivative is the *synthetic collateralized debt obligation*. Portfolio credit derivatives are derivatives whose payoff depends on the losses due to defaults in the portfolio consisting of underlying reference such as bonds, loans or credit default swaps.

A synthetic CDO tranche is based on a portfolio of n single-name CDSs on n different reference entities. It is characterized by a maturity date T and a lower attachment point and upper attachment point (percentage) $0 \leq a \leq b \leq 1$. A synthetic CDO is quite similar to a CDS in terms of insuring underlying assets. But it can cover only a specified part of the loss known as a "tranche" — not all of it. Thereafter the initial notional amount which the first coupon is based on is $(b-a) \times$ (total notional). Gradually this

initial notional is decreased according to losses starting from $a \times$ (total notional) until maturity or $b \times$ (total notional), whatever comes first. If the cumulative losses reach $b \times$ (total notional) before maturity, then all payments are stopped.

Let L_t be the cumulated loss at time $t \in [0, T]$, rescaled between $[0, 1]$, and let N_t be also the rescaled number between $[0, 1]$ of defaults at time t. In order to find the tranche coupon, which is also called the spread, we need to calculate a default payment leg and a premium payment leg.

The cumulative loss of the tranche with upper and lower attachment points $a < b$ is

$$L_t^{a,b} := (L_t - a)^+ - (L_t - b)^+.$$

The premium payment leg at time $t = 0$ consists of regular premium payments made at fixed dates $0 = t_0 < t_1 < \cdots < t_N = T$ and accrued premium payments made at the default times $T_i \in (t_i, t_{i+1}]$ with $T_i \leq T$. This leads to

$$V_{[a,b]}^{\text{prem}} = c \sum_{i=1}^{N} \left((t_i - t_{i-1}) \mathbb{E}\left[D(0, t_i)(1 \times (b-1) - L_{t_i}^{a,b}) \right] + (T_i - t_i) \mathbb{E}\left[D(0, T_i) \Delta L_{T_i}^{a,b} \right] \right),$$

where $D(0, t)$ is a discount factor and c is the tranche spread. The notation $\Delta L_{T_i}^{a,b}$ is the loss occurring at time T_i. Sometimes the accrued premium payments are omitted when calculating c.

The default payment leg at time $t = 0$ is given by

$$V_{[a,b]}^{\text{def}} = \mathbb{E}\left[\int_0^T D(0, t) dL_t^{a,b} \right].$$

The integral with respect to L_t,

$$\mathbb{E}\left[\int_0^T D(0, t) dL_t^{a,b} \right]$$

is approximated as a Riemann sum

$$\mathbb{E}\left[\sum_{i=1}^{b} D(0, T_i)(L(T_i) - L(T_{i-1})) \right].$$

If there are no initial payments, the fair tranche spread can be computed by equating the values of the premium and default payment legs of the CDO tranche:

$$c = \frac{\mathbb{E}\left[\int_0^T D(0, t) dL_t^{a,b} \right]}{\sum_{i=1}^{N} \left((t_i - t_{i-1}) \mathbb{E}\left[D(0, t_i)(1 - L_{t_i}) \right] + (T_i - t_i) \mathbb{E}\left[D(0, T_i) \Delta L_{T_i}^{a,b} \right] \right)}. \tag{2.4}$$

2.1.3 Forward Starting CDO

A forward starting CDO is a forward contract which obligates the holder of contract to buy or sell the protection on a pre-specified tranche $[A, B]$ at a pre-specified rate over the period of future time $[T, T_b]$. A forward starting CDO begins at T, and its maturity is T_b. At time T, the contract becomes just a single tranche CDO with attachment point $A + L_T$ and detachment point $B + L_T$. The forward index spread S_T is defined so that the forward contract has a zero value at T. The formulas for the forward index spread with and without the knock-out feature are

$$S_T = \frac{1}{\Theta_T}\mathbb{E}_T\left[\int_T^{T_b} D(T,t)dL_t\right]$$
$$\bar{S}_T = \frac{1}{\Theta_T}\mathbb{E}_T\left[\int_T^{T_b} D(T,t)dL_t + L_T\right],$$

where

$$\Theta_T := \mathbb{E}_T\left[\sum_{i=1}^{b} \delta_i D(T,T_i)(1 - N_{T_i})\right]$$

and $D(T, T_i)$ is a discount factor and δ_i is a day counting factor.

Similarly, the spread of the forward CDO tranche with attachment point A and detachment point B can be defined as

$$S_T^{A,B} := \frac{\mathbb{E}_T\left[\int_T^{T_b} D(T,t)dL_t^{A,B} - U_T^{A,B}\delta_T\right]}{\mathbb{E}_T\left[\sum_{i=1}^{b} \delta_{T_i} D(T,T_i)\left((B-A) - L_{T_i}^{A,B}\right)\right]}.$$

The quantity $L_t^{A,B}$ is the cumulative loss adjusted to tranche $[L(T) + A, L(T) + B]$, which is

$$L_t^{A,B} = (L(t) - (L(T) + A))^+ - (L(t) - (L(T) + B))^+,$$

and $U_T^{A,B}$ is an up-front payment. If one wants to keep the tranche $[A, B]$ not adjusting to its loss level $L(T)$ at contract maturity time T, then the contract is considered to be canceled once cumulative losses $L(T)$ on the portfolio exceed the detachment point B before the forward contract starts. See [13] for more detail.

2.1.4 Option on Index CDO

An option on an index CDO is an option to enter a option contract at a given time (maturity) with a specified rate (strike value). The option payoff of the protection seller

at time T is

$$\begin{aligned} V_{\text{put}}(T,K) &= \left(K\Theta_T - \mathbb{E}_T \left[\int_T^{T_b} D(T,t) dL_t \right] \right)^+ \\ &= (K\Theta_T - S_T\Theta_T)^+ \\ &= \Theta_T (K - S_T)^+, \end{aligned}$$

where

$$\Theta_T := \mathbb{E}_T \left[\sum_{i=1}^{b} \delta_i D(T,T_i)(1 - N_{T_i}) \right]$$

and $D(T,T_i)$ is a discounting factor.

The option payoff of the protection buyer at time T is

$$V_{\text{call}}(T,K) = \left(\mathbb{E}_T \left[\int_T^{T_b} D(T,t) dL_t \right] + \bar{L}_T - K\Theta_T \right)^+.$$

Note that a put option depends only on the loss in $[T,T_b]$, whereas a call option includes the losses L_T occurred before the maturity T. For more detail, refer to [3].

Then the option value at time t is represented by the formulas

$$\begin{aligned} V_{\text{put}}(t,T,K) &= \mathbb{E}_t[D(t,T)V_{\text{put}}(T,K)] = \mathbb{E}_t[D(t,T)\Theta_T(K - S_T)^+] \\ V_{\text{call}}(t,T,K) &= \mathbb{E}_t[D(t,T)V_{\text{call}}(T,K)] = \mathbb{E}_t[D(t,T)\Theta_T(\bar{S}_T - K)^+]. \end{aligned}$$

2.1.5 Leveraged Super-Senior Tranche

In a *leveraged super-senior tranche*, the protection seller receives a spread for the entire principal just like a normal senior tranche, while its exposure to losses is capped by some portion (usually 10 or 20%) of its total notional. As a compensation for the limited coverage of loss, payment from the protection seller to the protection buyer is triggered by the underlying spread (spread trigger) or loss level (loss trigger) or a combination of both.

Thus assessing the risk coming from the additional trigger is important in valuing a leveraged super-senior tranche.

2.2 Mathematical Background

In §2.2.1, the necessary mathematical concepts for modeling credit derivatives are given. In §2.2.2, we derive the extension of a Kolmogorov forward equation for a discrete variable and a continuous variable. We present a simple example of the singular perturbation method used in approximating the solution of the ordinary differential equation in the section 5.

2.2.1 Mathematical Setup for Credit Derivatives

All random variables we discuss below are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$ where Ω represents the set of possible events and the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ gives us the information available up to time t. Source of generating filtration $\{\mathcal{F}_t\}_{t\geq 0}$ may vary in each model.

Definition 2.2.1. *A random variable $\tau : \Omega \to \mathbb{N} \cup \{\infty\}$ is called a stopping time if $\{\omega \in \Omega | \tau(\omega) \leq t\} \in \mathcal{F}_t$.*

Sometimes it is required that τ is finite almost surely. In the intensity-based model in the Chapter 3, the stopping time denotes the default time of an underlying name in a portfolio.

Now we can define a counting process N_t by

$$N_t := \sum_i 1_{\{\tau_i \leq t\}}$$

where τ_i is a stopping time. The process N_t is an increasing process as a function of t , thus a submartingale. By the Doob-Meyer decomposition theorem, there exists a unique increasing process A_t such that $A_0 = 0$ and $N_t - A_t$ is a martingale. Since A_t counteracts the upward trend in N_t, it is often called a *compensator*.

Using the martingale property of $N_t - A_t$, we have

$$\mathbb{E}[N_{t+\Delta t} - N_t | \mathcal{F}_t] = \mathbb{E}[A_{t+\Delta t} - A_t | \mathcal{F}_t].$$

Assuming that for sufficiently small time t, the process N_t only has one jump, we have

$$\begin{aligned}\mathbb{E}[A_{t+\Delta t} - A_t|\mathcal{F}_t] &= 1 \cdot Q(N_{t+\Delta t} - N_t = 1|\mathcal{F}_t) + 0 \cdot Q(N_{t+\Delta t} - N_t = 0|\mathcal{F}_t) \\ &= Q(N_{t+\Delta t} - N_t = 1|\mathcal{F}_t).\end{aligned}$$

This shows that knowledge of A_t can give some information about the distribution of counting process N_t.

Definition 2.2.2. *A non-negative predictable process λ_t is called the intensity of the counting process N_t if*

$$A_t := \int_0^t \lambda(s)ds$$

is a compensator for N_t.

From Aven's theorem below, we are able to derive the intensity if we can compute

$$\lim_{n\to\infty} \frac{1}{\epsilon_n}\mathbf{E}\left[N_{t+\epsilon_n} - N_t|\mathcal{F}_t\right].$$

Theorem 2.2.3. *(Aven, 1985) Let $\{\epsilon_n\}_{n=1}^{\infty}$ be a sequence which decreases to zero and let $Y_n(t)$, $t \in \mathbb{R}_+$ be a measureable version of the process*

$$Y_n(t) := \frac{1}{\epsilon_n}\mathbf{E}\left[N_{t+\epsilon_n} - N_t|\mathcal{F}_t\right].$$

Assume there are non-negative and measurable processes $g(t)$ and $y(t)$, $t \in \mathbb{R}_+$ such that

(i) for each t,

$$\lim_{n\to\infty} Y_n(t) = g(t) \qquad a.s.$$

(ii) for each t there exists ,for almost all $\omega \in \Omega$, an integer $n_0 = n_0(t,\omega)$ such that

$$|Y_n(s,\omega) - g(s,\omega)| \le y(s,\omega) \qquad \forall s \le t, n \ge n_0$$

(iii) if

$$\int_0^t y(s)ds < \infty \quad a.s. \quad t \in \mathbb{R}_+,$$

then $N_t - \int_0^t g(s)ds$ is a local martingale, and $\int_0^t g(s)ds$ is the compensator of N_t.

In credit modeling, we define the loss process where loss at default time τ_i is Δ_i.

$$L_t = \sum_i \Delta_i 1_{\{\tau_i \leq t\}}.$$

When the loss at default times is normalized to be 1, then L_t is simply a counting process N_t.

2.2.2 Extensions of the Kolmogorov Forward Equations

Here the generalization of the Kolmogorov forward equation to the case of a conditional joint probability density function of continuous and discontinuous random variables is derived, following to [23]. We begin with the transition density function $p(y, t|\tilde{y}, \tilde{t})$ of a continuous random variable y_t,

$$p(y, t + \Delta t|\tilde{y}, \tilde{t}) = \int_{-\infty}^{\infty} p(y, t + \Delta t|y', t)p(y', t|\tilde{y}, \tilde{t})dy'. \tag{2.5}$$

We express the first term in the integrand as the Fourier transform of the conditional characteristic function of $y - y'$, and expand this characteristic function in a Taylor series:

$$\begin{aligned} \psi(v, t + \Delta t|y', t) &= \mathbb{E}\left[e^{iv(y-y')}|y', t\right] \\ &= \int_{-\infty}^{\infty} e^{iv(y-y')}p(y, t + \Delta t|y', t)dy. \end{aligned} \tag{2.6}$$

Then by the inverse Fourier transform

$$\begin{aligned} p(y, t + \Delta t|y', t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iv(y-y')}\psi(v, t + \Delta t|y', t)dv \\ &= \sum_{n=0}^{\infty} \frac{a_n(y', t; \tilde{y}, \tilde{t})}{2\pi n!}\int_{-\infty}^{\infty} e^{iv(y-y')}(iv)^n dv \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} a_n(y', t; \tilde{y}, \tilde{t})\frac{\partial^n}{\partial y}\delta(y - y'), \end{aligned}$$

where

$$a_n(y', t; \tilde{y}, \tilde{t}) = \mathbb{E}[(y(t + \Delta t) - y(t))^n|y', t].$$

Substituting (2.6) into (2.5) and integrating gives

$$p(y, t + \Delta t|\tilde{y}, \tilde{t}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\frac{\partial^n}{\partial y^n}[a_n(y', t; \tilde{y}, \tilde{t})p(y, t|\tilde{y}, \tilde{t})].$$

From this, we can easily get the generalized infinite-order Kolmogorov forward equation,

$$\frac{\partial p}{\partial t} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial y^n}[A_n p],$$

where

$$A_n(y,t|\tilde{y},\tilde{t}) = \lim_{\Delta t \to 0} \mathbb{E}\left[(y(t+\Delta t) - y(t))^n | y,t;\tilde{y},\tilde{t}\right].$$

If $A_n < \infty$ for all n and if $A_n = 0$ for some even n, then we have the finite-order Kolmogorov forward equation,

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial y}(A_1 p) + \frac{\partial^2}{\partial y^2}(A_2 p). \tag{2.7}$$

For a discrete-valued random process y_t, we can similarly derive the Kolmogorov forward equation. Define

$$P_{ij}(t|\tilde{t}) := \Pr\left[y(t) = j | y(\tilde{t}) = i\right].$$

By the Chapman-Kolmogorov equation

$$P_{ij}(t+\Delta t|\tilde{t}) = \sum_{k \geq 0} \Pr[y(t+\Delta t) = j | y(t) = k, y(\tilde{t}) = i] P_{ik}(t|\tilde{t}).$$

We have

$$P_{ij}(t+\Delta t|\tilde{t}) - P_{ij}(t|\tilde{t}) = \left[\sum_k \Pr[y(t+\Delta t) = j | y(t) = k, y(\tilde{t}) = i] - \delta_{jk}\right] P_{ik}(t|\tilde{t}), \tag{2.8}$$

where

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$

Dividing Equation by Δt and taking the limit as $\Delta t \to 0$ yields

$$\frac{\partial}{\partial t} P_{ij}(t|\tilde{t}) = \sum_k a_{jki}(t|\tilde{t}) P_{ik}(t|\tilde{t}),$$

where

$$a_{jki}(t|\tilde{t}) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\Pr[y(t+\Delta t) = j | y(t) = k, y(\tilde{t}) = i] - \delta_{jk}\right].$$

By combining the ideas behind the continuous-valued process and the discrete-valued process, we can obtain the Kolmogorov forward equation of the joint density function of a discrete random process x_t and a continuous random process y_t. Define

$$p_{kj}(y,t|y_0,t_0) := p(y,t|x(t) = j; y(t_0), x(t_0) = k) \cdot P(x(t) = j | y(t_0), x(t_0) = k) \tag{2.9}$$

Taking the derivative of (2.9) with respect to t,

$$\frac{\partial}{\partial t}p_{kj}(y,t|y_0,t_0) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial y^n}[A_{nkj}p_{kj}(y,t|y_0,t_0)] + \sum_i a_{kij}p_{ki}(y,t|y_0,t_0), \quad (2.10)$$

where

$$A_{nkj} := \lim_{\Delta t\to 0}\mathbb{E}\left[(y(t+\Delta t)-y(t))^n|x(t+\Delta t)=j; y(t), x(t)=j; y(t_0), x(t_0)=k\right],$$

$$a_{kij} := \lim_{\Delta t\to 0}\left[\Pr[x(t+\Delta t)=j|y(t), x(t)=i; y(t_0), x(t_0)=k] - \delta_{ij}\right].$$

2.2.3 Singular Perturbation Method

Perturbation theory is a collection of iterative methods for obtaining approximate solutions to problems with a small parameter ε. This involves decomposing a big and complex problem into a number of relatively easy ones. Hence perturbation theory is useful when the first few steps reveal the important features of the solution and the remaining steps give small corrections.

In perturbation theory, approximation series occurs in two varieties. A *regular perturbation problem* is defined as one whose ε power series has a non-vanishing radius of convergence. A basic feature of all regular perturbation problems is that the exact solution for small nonzero ε approaches the unperturbed or zeroth-order solution as $\varepsilon \to 0$. A *singular perturbation problem* is defined as one whose perturbation series either does not take the form of a power series or, if it does, whose power series has a vanishing radius of convergence.

In a singular perturbation problem, the unperturbed solution sometimes does not exist, and when its solution exists, its qualitative features may be quite different from that of an exact solution with nonzero but small ε. When an abrupt change in the character of the solution occurs when ε becomes zero, we classify it as a singular perturbation problem.

Matching Method

Matching methods have been widely used in singular perturbation problems. Since we use this technique later for our model, we shall describe how this method works through a simple example.

When one tries to find an approximate solution involving ε, it is natural to seek a solution of the form

$$\sum_{j=0}^{\infty} a_j(x)\alpha_j(\varepsilon).$$

Often $\alpha_j(\varepsilon)$ is a power of ε^j and we expect the first few terms to capture important features of the solution and give us a good approximation to the exact solution. But such an expansion is not usually uniformly valid throughout the domain $\mathcal{D}$. Instead this expansion is satisfactory in a certain part of $\mathcal{D}$ called the *outer region*. In order to investigate the nonuniform convergent region of $\mathcal{D}$, a transformation from the variable x to the "stretched variable" $\xi = \psi(x, \varepsilon)$ is used. If we can transform the coordinate so $x = 0$ is the boundary where the nonuniformity occurs, then $\xi = \frac{x}{\varepsilon}$ might be used for nonuniform convergence at $x = 0$ (for fixed x and $\varepsilon \to 0$, $\xi \to \infty$). Selection of the "correct" stretching variable is an art and sometimes depends on consideration of a physical phenomenon. In terms of the stretched variable ξ, one can seek a solution of the form

$$\sum_{j=0}^{\infty} b_j(\xi)\beta_j(\varepsilon).$$

This is called an *inner expansion* and often accounts for boundary regions neglected by the outer expansion.

Since these two expansions represent the solution asymptotically in different regions, we want to match them in the overlapping domain of validity for two expansions through use of the variable ξ. Formally accomplishing the rule for this matching method can be very complicated and we shall not attempt this here. Instead we examine one simple example from [22] showing how this method works:

$$\varepsilon y'' + y' + y = 0, \qquad 0 \le x \le 1, \tag{2.11}$$

$$y(0) = \alpha, \qquad y(1) = \beta. \tag{2.12}$$

This ordinary differential equation problem has the solution

$$y_\varepsilon(x) = \frac{\beta - \alpha e^{\rho_2}}{e^{\rho_1} - e^{\rho_2}}\, e^{\rho_1 x} + \frac{\alpha e^{\rho_1} - \beta}{e^{\rho_1} - e^{\rho_2}}\, e^{\rho_2 x},$$

where

$$\rho_1(\varepsilon) = \frac{1}{2\varepsilon}(-1 + (1-4\varepsilon)^{\frac{1}{2}}) = -1 + O(\varepsilon),$$
$$\rho_2(\varepsilon) = \frac{1}{2\varepsilon}(-1 - (1-4\varepsilon)^{\frac{1}{2}}) = -\frac{1}{\varepsilon} + 1 + O(\varepsilon).$$

Since $\rho_1 \to -1$ and $\rho_2 \to -\infty$, we have

$$y_\varepsilon(x) = \beta e^{1-x} + (\alpha - \beta e)e^x e^{-\frac{x}{\varepsilon}} + O(\varepsilon)$$

throughout $0 \le x \le 1$ as ε goes to zero. For $x > 0$,

$$e^{-x/\varepsilon} = O(\varepsilon^N)$$

for every N as $\varepsilon \to 0$, while $e^0 = 1$. Thus $y_\varepsilon(x)$ converges nonuniformly on the interval $0 \le x \le 1$ as $\varepsilon \to 0$. We have

$$y_\varepsilon(x) = \beta e^{1-x} + O(\varepsilon) \qquad \text{for} \qquad x > 0.$$

Note that $y_\varepsilon(x)$ converges to the solution of the problem

$$y' + y = 0, \qquad y(1) = \beta(\varepsilon)$$

except at $x = 0$. This nonuniform convergence as $\varepsilon \to 0$ (unless $\alpha = \beta e$) implies that we have a singular perturbation problem.

We try to obtain an asymptotic approximation to the solution by the matching method. We shall seek an outer expansion as the asymptotic power series,

$$y^{\mathrm{o}} := \sum_j^\infty a_j(x)\varepsilon^j \tag{2.13}$$

valid for $0 < x \le 1$, and an inner expansion

$$y^{\mathrm{i}} := \sum_j^\infty b_j(\xi)\varepsilon^j \tag{2.14}$$

valid near $x = 0$ as $\varepsilon \to 0$ where $\xi = x/\varepsilon$.

The outer expansion satisfies the differential equation (2.11) and the terminal condition. We have

$$\begin{aligned} &\varepsilon(a_0'' + \varepsilon a_1'' + \cdots) + (a_0' + \varepsilon a_1' + \varepsilon^2 a_2' + \cdots) \\ &+ (a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \cdots) = 0, \end{aligned} \tag{2.15}$$

and

$$a_0(1) + \varepsilon a_1(1) + \varepsilon^2 a_2(1) + \cdots = \beta(\varepsilon).$$

Equating coefficients of ε in (2.15), we have

$$a_0' + a_0 = 0, \quad a_0(1) = \beta\varepsilon, \tag{2.16}$$

$$a_1' + a_1 + a_0'' = 0, \quad a_1(0) = 0. \tag{2.17}$$

Solving Equation (2.16), we obtain

$$\begin{aligned} a_0(x) &= \beta e^{1-x}, \\ a_1(x) &= (1-x)\beta e^{1-x}. \end{aligned} \tag{2.18}$$

The differential Equation (2.11) can be written in terms of variable ξ,

$$y_{\xi\xi} + y_\xi + \varepsilon y = 0 \tag{2.19}$$

Using the same reasoning as above, we obtain

$$\begin{aligned} b_{0\xi\xi} + b_{0\xi} = 0, \quad b_0(0) = \alpha, \\ b_{1\xi\xi} + b_{1\xi} + b_0 = 0, \quad b_1(0) = 0. \end{aligned} \tag{2.20}$$

Equation (2.20) has the solutions,

$$\begin{aligned} b_0(\xi) &= \gamma_0 + (\alpha - \gamma_0)e^{-\xi}, \\ b_1(\xi) &= -(\gamma_0\xi + \gamma_1) + \left[(\alpha - \gamma_0)\xi + \gamma_1\right] e^{-\xi}, \end{aligned} \tag{2.21}$$

where γ_0 and γ_1 are undetermined constants. We use matching of both the outer expansion and the inner expansion to determine γ_0 and γ_1. Here,

$$y^{\mathrm{o}}(x) = \beta e^{1-x} + (1-x)\beta e^{1-x} + O(\varepsilon^2), \tag{2.22}$$

$$y^{\mathrm{i}}(\xi) = \gamma_0 + (\alpha - \gamma_0)e^{-\xi} + -(\gamma_0\xi + \gamma_1) + \left[(\alpha - \gamma_0)\xi + \gamma_1\right] e^{-\xi} + O(\varepsilon^2). \tag{2.23}$$

Writing the outer expansion in terms of the variable ξ, we have

$$y^{\mathrm{o}} = \beta e e^{-\varepsilon\xi} + \varepsilon(1 - \varepsilon\xi)\beta e e^{-\varepsilon\xi} + O(\varepsilon^2). \tag{2.24}$$

This leads to the approximation up to $O(\varepsilon^2)$,

$$y^{\mathrm{oi}} = \beta e \left(1 + \varepsilon(-\xi + 1)\right). \tag{2.25}$$

Analogously, writing the inner expansion in terms of the outer variable x and neglecting e^{ξ} terms,

$$y^{\mathrm{i^o}} = \gamma_0(1-x) - \varepsilon\gamma_1. \tag{2.26}$$

Matching will be accomplished by choosing

$$\gamma_0 = \beta e = -\gamma_1$$

Chapter 3

Multi-Name Credit Pricing Models

In this chapter various models for pricing portfolio credit derivatives are reviewed. We begin with bottom-up approach in §3.1. Then top-down models are introduced in §3.2.

3.1 Bottom-up Model

In a bottom-up model, the problem of pricing multi-name derivatives begins with introducing either intensity or firm value of each name in the multi-name financial products. Bottom-up modeling has been popular since the early days of credit derivatives because of its instinctively clear structure and computational tractability.

3.1.1 Copula Model

A *copula* is a function that takes the marginal distributions of individual random variables and produces the dependence structure between them. For simplicity, we give a definition of a 2-dimensional copula. This can be easily generalized to an n-dimensional copula.

Definition 3.1.1. *[21] A two-dimensional copula C is a function from $I^2 = [0,1]\times[0,1]$ to $I = [0,1]$ with the following properties:*

1. *For every u, v in I,*

$$C(u,0) = C(0,v) = 0,$$

and

$$C(u,1) = u, \quad C(1,v) = v.$$

2. *For every u_1, u_2, v_1, v_2 in I such that $u_1 \leq u_2$ and $v_1 \leq v_2$,*

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_1) + C(u_1,v_1) \geq 0.$$

We can restate this technical definition of copula in a way that is more appropriate for financial modeling. In essence, an n-dimensional copula is a distribution function on $[0,1]^n$ with uniform marginal distributions. A copula is a tool which can be used to analyze the dependency structure between random variables. Sklar's theorem characterizes this feature of a copula.

Theorem 3.1.2 (Sklar [24])**.** *Let $X_1, \ldots, X_n$ be random variables with marginal distributions $F_1, \ldots, F_n$ and joint probability distribution F. Then there exists an n-dimensional copula C such that*

$$F(x_1, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)),$$
$$C(u_1, \ldots, u_n) = F(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)).$$

If $F_1, \ldots, F_n$ are continuous, then C is unique.

This theorem says that for every multivariate distribution function, there is a copula which contains all information on dependence.

The copula model is relatively simple and widely used in the industry. While computationally convenient, a calibrated copula model does not generate a consistent set of tranche spreads across different maturities and attachment/detachment points.

Gaussian Copula Model

The Gaussian copula model is by far the most popular copula model used in the financial industry in default dependency modeling. This is because it is easy to simulate and has the right number of parameters — equal to the number of correlation coefficients in the underlying names. A single name deterministic correlation is fit to the single name credit default swap market, and a fixed copula governs the dependence structure among names.

In this model, the value of each firm in multi-name credit product is

$$Z_i = \rho Y + \sqrt{1-\rho^2}\varepsilon_i, \ i = 1, \cdots, n$$

with common factor Y and idiosyncratic factor ε_i. We assume that each factor Y, ε_i follows the standard normal distribution and that they are independent of each other.

Consider for each obligor $i = 1, \cdots, n$ and its default stopping time τ_i and the default probability $p_i(t_i)$ which is derived from the credit market. We introduce the trigger level $z_i(t_i)$:

$$\mathbb{Q}(\tau_i < t_i) = \mathbb{Q}(Z_i < z_i(t)),$$
$$p_i(t_i) = \Phi(z_i(t_i)),$$
$$z_i(t_i) = \Phi^{-1}(p_i(t_i)),$$

where Φ is the cumulative probability density function of the standard normal variable. Conditioning on the common factor Y, we get

$$Q(\tau_i < t_i|Y) = \Phi\left(\frac{\Phi^{-1}(p_i(t_i)) - \rho Y}{\sqrt{1-\rho^2}}\right).$$

Since $Z_i, \cdots, Z_n$ are independent of each other conditioned on the common factor Y, we have the joint marginal distribution function F of $\tau_1, \cdots, \tau_n$,

$$F(t_1, \ldots, t_n) = \int \left(\prod_{i=1}^{n} Q(\tau_i < t_i|Y)\right) f(y)dy. \tag{3.1}$$

Equation (3.1) is the one-factor Gaussian copula function, in contrast to the standard Gaussian copula function,

$$C_{\sum}(\overrightarrow{u}) = \Phi_{\sum}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)),$$

where $\sum$ is the $n \times n$ correlation matrix $\sum$.

Empirical evidence [24] shows that the Gaussian distribution used here underestimates extreme events, so it cannot correctly price every tranche simultaneously. As an alternative way to overcome the shortcomings of the model, we can use a new dependency structure from different copulas. Here we only introduce the Archimedean copula as one example.

Definition 3.1.3. *An Archimedean copula function $C : [0,1]^I \longrightarrow [0,1]$ is a copula function which can be represented in the form*

$$C(X) = \phi\left(\sum_{i=1}^{I} \phi^{-1}(x_i)\right),$$

with a generator function $\phi : \mathbb{R}_+ \longrightarrow [0,1]$.

A generator function ϕ is the Laplace transform of a positive random variable Y:

$$\phi(s) = \mathbb{E}[e^{sY}].$$

Y is often called the *frailty* variable or *mixing* variable. In particular, the Clayton copula is an Archimedean copula function with $\phi^{-1}(x) = x^{-\theta} - 1$, corresponding to the mixing variable Y having a Gamma($\frac{1}{\theta}$) distribution, which results in

$$C_\theta^{\text{Clayton}}(X) = \left(1 - N + \sum_{i=1}^{I} x_i^{-\theta}\right)^{-\frac{1}{\theta}}.$$

Vasicek Homogeneous Large Portfolio Model

When the number of entities in portfolio is quite large, we can use the Law of Large Numbers to calculate the distribution of normalized total loss L_t. In a homogeneous pool, the firm's value is $Z_i = \rho Y + \sqrt{1-\rho^2}X_i$, where Y, X_i are independent identically distributed standard normal random variables. All participants in a homogeneous pool have the same loss at default and the same notional amount and the same default probability. Thus the number of defaults is sufficient to determine the loss of the portfolio. Define

$$L_t(y) := \frac{1}{N}\sum_{i}^{N} 1_i(t|y),$$

where the function $1_i(t|y)$ is the default indicator function of the firm i by the time t conditioned on Y. For the expectation of the default indication function,

$$\begin{aligned}\mathbb{E}[1_i(t|y)] &= 1 \cdot \beta_t(y) \\ &= \text{probability of firm's value is less than default barrier } d,\end{aligned}$$

where $\beta_t(y)$ is the probability of the firm defaulting by time t conditioned on the factor Y. The barrier is explicitly derived from the firm's default probability:

$$\mathbb{Q}[L_t(y) < x] = 1_{\beta_t(y)<x}.$$

This is because of the law of large numbers, which says that $L_t(y)$ converges to expectation of $1_i(t|y)$.

$$\begin{aligned}\mathbb{Q}[L_t < x] &= \int \mathbb{Q}[L_t(y) < x]dy \\ &= \int 1_{\beta_t(y)<x} f(y)dy \\ &= \mathbb{E}[1_{\beta_t(y)<x}] \\ &= \mathbb{Q}[\beta_t(y) < x].\end{aligned}$$

For $\mathbb{Q}[\beta_t(y) < x]$, we have

$$\begin{aligned}\mathbb{Q}[\beta_t(y) < x] &= \mathbb{Q}\left[\Phi\left(\frac{\Phi^{-1}(\rho_i(t)) - \rho_i y}{\sqrt{1-\rho_i^2}}\right) < x\right] \\ &= \mathbb{Q}\left[\frac{\Phi^{-1}(\rho_i(t)) - \rho_i y}{\sqrt{1-\rho_i^2}} < \Phi^{-1}(x)\right] \\ &= \mathbb{Q}\left[\Phi^{-1}(\rho_i(t)) - \rho_i y < \sqrt{1-\rho_i^2}\Phi^{-1}(x)\right] \\ &= \mathbb{Q}\left[\frac{\Phi^{-1}(\rho_i(t)) - \sqrt{1-\rho_i^2}\Phi^{-1}(x)}{\rho} < y\right] \\ &= \mathbb{Q}\left[\frac{-\Phi^{-1}(\rho_i(t)) + \sqrt{1-\rho_i^2}\Phi^{-1}(x)}{\rho} > -y\right] \\ &= \Phi\left(\frac{-\Phi^{-1}(\rho_i(t)) + \sqrt{1-\rho_i^2}\Phi^{-1}(x)}{\rho}\right).\end{aligned}$$

Because Y, X_i are independent variables,

$$\beta_t(y) = \mathbb{Q}\left[\rho Y + \sqrt{1-\rho^2}X_i < d|Y\right] = \Phi\left(\frac{d-\rho y}{\sqrt{1-\rho^2}}\right),$$

where

$$\Phi(d) = \rho_i(t).$$

Here $\rho_i(t)$ is the default probability of individual names in the portfolio. Note that this is a homogeneous group, so every name has same default probability.

3.1.2 Intensity-based Models

Due to its tractability and flexibility, factor copula modeling has become a standard tool in the valuation of basket credit derivatives. This approach is, however, problematic because the default time is predictable so credit spread in short maturity can not be captured in the copula model.

Given this problem, a natural alternative to the copula model is a multivariate version of the intensity-based model. In the intensity-based model the default time is constructed as an *inaccessible* stopping time and the parameters in the intensity have economic interpretations and can be estimated from available CDS market data.

Common Factor Model

Default time of single name in the portfolio is described by the first jump time of a Cox process. The intensity of an individual firm is constructed using a firm-specific idiosyncratic factor and the common market factor.

Given the common market factor, conditional default events of any two firms are independent. This is an important assumption of the factor model in terms of computational simplicity.

The default intensity $\lambda^{(i)}$ of firm i is decomposed into an idiosyncratic factor $X^{(i)}$ and a market factor Y:

$$\lambda_t^{(i)} = X_t^{(i)} + c_i Y_t.$$

The process N^i is called the *default process*, meaning that its first jump time represents the default of the firm i. So the default time τ_i is defined as

$$\tau_i := \inf\{t > 0 : N_t^i > 0\}.$$

We consider the loss process L_t in a pool of N names:

$$L_t = \sum_{i=1}^{N} 1_{\tau_i < t}.$$

The conditional survival probability of underlying entity i is

$$\mathbb{Q}(\tau_i > t|\mathcal{F}_s) = 1_{\tau_i > s}\mathbb{E}\left[-\exp\int_s^t \lambda_u^i du|\mathcal{F}_s\right].$$

Let Z_t be the integrated systematic process

$$Z_t = \int_0^t Y_s ds.$$

Under the assumption that $X^1, \ldots, X^n$ are independently identically distributed random variables, the law of the loss process is computed using binomial distribution,

$$\mathbb{Q}(L_t = m) = \mathbb{E}\left[\binom{n}{m}\left(1 - \mathbb{E}[-\exp\int_0^t \lambda_u^i du|Z_t]\right)^m \left(\mathbb{E}[-\exp\int_0^t \lambda_u^i du|Z_t]\right)^{n-m}\right].$$

In a homogeneous pool, the default probabilities are obtained by the following simple recursive algorithm due to [1].

Let $D^K(t)$ denote the number of defaults at time t in the pool consisting of K reference entities. Since defaults are conditionally independent, the conditional probability of observing j defaults in a pool is given by

$$\begin{aligned}\mathbb{Q}(D^K(t) = j|Z_t = z) = \mathbb{Q}(D^{K-1} = j|Z_t = z) \times (1 - p_K[t|z]) \\ + \mathbb{Q}(D^{K-1} = j-1|Z_t = z) \times p_K[t|z],\end{aligned}$$

where $p_i[t|z]$ is the default probability of entity i by time t given the common factor Z. Thus unconditional probabilities can be written as

$$\mathbb{Q}(L_t = m) = \int_{-\infty}^{\infty} \mathbb{Q}(D(t) = m|Z_t = z) f_{Z_t}(z) dz.$$

As a specific example of this approach, we look at the model proposed by Mortensen [20], which is an extension of the model of Duffie and Garleanu [4].

Let us go back to the intensity of each entity i in the portfolio:

$$\lambda_t^{(i)} = X_t^{(i)} + c_i Y_t.$$

For the common factor Y_t, we assume that

$$dY_t = \kappa_Y(\theta_Y - Y_t)dt + \sigma_Y\sqrt{Y_t}dW_t^Y + dJ_t^Y.$$

For the idiosyncratic factor $X_t^{(i)}$, we also assume that

$$dX_t^{(i)} = \kappa_i(\theta_i - X_t^{(i)})dt + \sigma_i\sqrt{X_t^{(i)}}dW_t^i + dJ_t^i,$$

where

- $W_Y, W_1, \ldots, W_N$ are independent Brownian motions,
- $J_Y, J_1, \ldots, J_N$ are independent pure jump processes with jump arrival intensities l_Y, l_i and exponentially distributed jump sizes with means μ_Y, μ_i, and
- the jump processes and Wiener processes are independent of each other.

Under these assumptions, the default probabilities are known to be

$$\begin{aligned}\mathbb{Q}(\tau_i \le t) &= 1 - \mathbb{E}[e^{-\int_0^t \lambda_s^{(i)} ds}] \\ &= 1 - \mathbb{E}[e^{-c_i \int_0^t Y_s ds}] \times \mathbb{E}[e^{\int_0^t X_s^{(i)} ds}] \\ &= 1 - \exp\left(A(t; \kappa_Y, c_i\theta_Y, \sqrt{c_i}\sigma_Y, l_Y, c_i\mu_Y) + B(t; \kappa_Y, \sqrt{c_i}\sigma_Y)c_i Y(0)\right) \\ &\quad \times \exp\left(A(t; \kappa_i, \theta_i, \sigma_i, l_i, \mu_i) + B(t; \kappa_i, \sigma_i)X^i(0)\right).\end{aligned} \tag{3.2}$$

To find the functions A, B above, we need to calculate the moment generating function of a CIR process with jumps.

Let us consider X_t the general CIR process with jump. We have

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t + dJ_t.$$

The process X_t has the same condition as X_t^i and Y_t in Mortensen's model [20]. That is, W is a Brownian motion and J is an independent pure jump process whose arrival times follow a Poisson process with intensity l and whose jump size is exponentially distributed with mean μ. We want to find

$$u(t, x) = \mathbb{E}\left[e^{-\int_0^t X_s ds} \mid X_0 = x\right].$$

As shown in [8], there is a Feynman-Kac partial differential integral equation satisfied by u, namely

$$u_t + \frac{1}{2}\sigma^2 x u_{xx} + \kappa(\theta - x)u_x + l\int_0^\infty (u(t, x+z) - u(t, x))ve^{-vz}dz - xu = 0, \tag{3.3}$$

with $u(T, x) = 1$. We construct a solution of the form

$$u(t, x) = A(T - t)e^{-B(T-t)x}. \tag{3.4}$$

After substitution of (3.4) into (3.3), we get the ordinary differential equations

$$-B' + \frac{1}{2}\sigma^2 B^2 + \kappa B - 1 = 0,$$
$$\frac{A'}{A} + \kappa\theta B - l\frac{B}{B + 1/\mu} = 0,$$

with $A(T) = 1$, $B(T) = 0$ and

$$A(T;\kappa,\theta,\sigma,l,\mu) = \frac{\kappa\theta\gamma}{bc_1d_1}\log\left(\frac{c_1 + d_1e^{bT}}{-\gamma}\right) + \frac{\kappa\theta}{c_1}T + \frac{l(\frac{c_2d_1}{c_1} - d_2)}{bc_2d_2}\log\left(\frac{c_2 + d_2e^{bT}}{c_2 + d_2}\right) + \frac{l - c_2l}{c_2}T,$$
$$B(T;\kappa,\sigma) = \frac{1 - e^{bT}}{c_1 + d_1e^{bT}},$$

where

$$\gamma = \sqrt{\kappa^2 + 2\sigma^2},$$
$$c_1 = \frac{-(\gamma + \kappa)}{2},$$
$$c_2 = 1 - \frac{\mu}{c_1},$$
$$d_1 = c_1 + \kappa,$$
$$d_2 = \frac{d_1 + \mu}{c_1},$$
$$b = d_1 + \frac{\kappa c_1 - \sigma^2}{\gamma}.$$

Thus probability of stopping time (3.2) can be calculated using the solution of $u(t, x)$.

3.2 Top-down Model

The bottom-up framework for multi-name credit modeling has very good features in terms of consistency with single-name constituents. However, it is difficult to calibrate the model with the available data of CDO indices and CDO tranches, because there are too many parameters unknown in bottom-up models.

Considering that a portfolio credit derivative is a contingent claim on the aggregate loss of a portfolio of credit sensitive securities, it is natural to model a loss process L_t directly rather than obtaining it from a default processes of the single names. In this approach, an important assumption is the homogeneous property of a pool of various names so that the model depends only on the number of defaulted names.

3.2.1 Intensity-Based Model

In intensity-based credit risk models, default is defined as the first jump of a counting process, and the counting process is characterized by its intensity. Constant intensity models were considered in [14], and stochastic intensities were introduced in [17].

Poisson Process

The loss process of a portfolio is constructed as a Poisson process with constant intensity parameter λ. The intensity λ can be generalized to a non-negative deterministic function of time $\lambda(t)$.

A Poisson process N_t is an integer-valued jump process such that

1. N_t has independent increments and has a probability distribution

$$\mathbb{Q}(N_t - N_s = k) = e^{-\lambda(t-s)} \frac{(\lambda(t-s))^k}{k!}.$$

2. N_t starts at zero, $N_0 = 0$

The distribution of the first jump time τ is given by

$$\mathbb{Q}(\tau > t) = \mathbb{Q}(N_t = 0) = e^{-\lambda t}.$$

The probability of a single jump in a small time interval can be seen to be

$$\mathbb{Q}(N_{t+\Delta t} - N_t = 1 | \mathcal{F}_t) = e^{-\lambda \Delta t} \lambda \Delta t = \lambda \Delta t + O(\Delta t^2).$$

This tells us that $\lambda \Delta t$ is the local jump probability over a small time period.

Generalized-Poisson Loss Dynamic Model

In the generalized Poisson loss dynamic model [3] , the number of defaulted names in a portfolio of names is modeled as a linear combination

$$Z_t = \sum_{j=1}^{n} \alpha_j N_j(t) \tag{3.5}$$

of independent Poisson processes with different intensities. The N_i for $i = 1, 2, \ldots, n$ are independent Poisson processes, and the α_i's are integers.

This allows multiple defaults to happen in a small time period, so we need to restrict the number of defaulted names to be bounded by the number of total names in the pool. If the pool contains M names, then

$$C_t = \min(Z_t, M).$$

Given the distribution of Z_t, the distribution of C_t is easily found by

$$\mathbb{Q}(C_t \leq x) = 1_{\{x<M\}}\mathbb{Q}(Z_t < x) + 1_{\{x \geq M\}}.$$

The distribution of Z_t can be obtained by:

$$\begin{aligned}
\varphi_{Z_t}(u) &= \mathbb{E}[\exp(-iuZ_t)] \\
&= \sum_n \exp(-iun)Q(Z_t = n) \\
&= \prod_{j=1}^{n} \mathbb{E}[\exp(-iu\alpha_j N_j(t))] = \prod_{j=1}^{n} \varphi_{N_j(t)}(\alpha_j u) \\
&= \exp\left[\sum_{j=1}^{n} \Lambda_j(t)(e^{i\alpha_j u} - 1)\right],
\end{aligned}$$

where $\Lambda_j(t) = \int_0^t \lambda_j(s)ds$.

Now we use the inverse Fourier transform to get the probability distribution of Z_t:

$$\begin{aligned}
\mathbb{Q}(Z_t = k) &= \sum_{m_1+\cdots+m_n=k} \mathbb{Q}(N_1(t) = m_1, \ldots, N_2(t) = m_n) \\
&= \sum_{m_1+\cdots+m_n=k} \prod_{j=1}^{n} \mathbb{Q}(N_j(t) = m_j).
\end{aligned}$$

Each $\mathbb{Q}(N_j(t) = m_j)$ is known already, and the independence of the $N_j(t)$'s is used here.

3.2.2 Markovian Self-Affecting Process

The loss process in the multi-name derivatives is, by its nature, a non-decreasing process. If the underlying portfolio has a discrete loss unit, a continuous time birth process of a Markov chain can be a very feasible candidate for the loss process.

To produce a more realistic model of the loss process, the transition intensities of the generating matrix of a Markov chain can be generalized from a simple linear birth

process. When we assume that at most one default occurs in an infinitesimal time interval, we get an upper bi-diagonal transition rate generating matrix. The simplest Markov non-increasing process with this feature is the well-known linear death process, and the transition rate generating matrix has the form

$$A_t = \lambda \begin{pmatrix} -N & N & 0 & \cdots & 0 \\ 0 & -(N-1) & (N-1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Here, λ is the average intensity of individual names in the portfolio of N names.

This can be modified by adding time-inhomogeneity, a nontrivial function $f(t, L_t)$ of time, and the default to the matrix to obtain

$$A_t = \lambda \begin{pmatrix} -f(t,0) & f(T,0) & 0 & \cdots & 0 \\ 0 & -f(t,1) & f(t,1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -f(t,N-1) & f(t,N-1) \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

We call the intensity of the generating matrix a *local intensity* since it depends only on the time and current level of loss. Note that the local volatility is a function of time and underlying equity value. The next step is to make the local intensity stochastic.

A completely general forward intensity modeling approach to this generating matrix is given by Schönbucher [25]. He allows each next-to-default intensity in the generating matrix for loss process to have its own stochastic dynamics. In contrast to this, the BSLP model discussed next introduces only one stochastic factor Y_t, applying to every transition intensity rate simultaneously.

BSLP: Markovian Bivariate Spread-Loss Model for Portfolio Credit Derivatives

A bivariate spread loss model [12] is a two-dimensional dynamic model with stochastic cumulative portfolio loss and loss intensity. The loss is considered to be a pure increasing

jump process with jump intensity λ_t determining how often it jumps. Here the loss unit is assumed to be one, so the counting process N_t and the loss process L_t become the same:

$$N_t = L_t.$$

As mentioned above, the generating matrix for BSLP has a stochastic factor Y_t instead of a deterministic value λ so the transition rate matrix is:

$$A = a(Y_t) \begin{pmatrix} -F(t,0) & F(t,0) & 0 & \cdots & 0 \\ 0 & -F(t,1) & F(t,1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -F(t,N-1) & F(t,N-1) \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where the function $a(\cdot)$ is a deterministic function which ensures that the intensity is positive. Thus, the next-to-default intensity is now stochastic and is given by

$$\lambda_t(N_t = k, Y_t, t) = a(Y_t) \cdot F(t,k).$$

The stochastic factor Y_t is assumed to satisfy the stochastic differential equation

$$dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t) \cdot dW_t + \gamma(t, Y_t)dN_t. \tag{3.6}$$

Several specifications of Y_t are possible. One example of a specification Y_t is an Ornstein-Ulenback process defined by setting

$$\begin{aligned} \mu(t, Y_{t-}) &:= a(\theta_t - Y_{t-}), \\ \sigma(t, Y_{t-}) &:= \sigma, \\ \gamma(t, Y_{t-}) &:= \gamma. \end{aligned} \tag{3.7}$$

Having a default counting process term dN_t in (3.6), The process N_t becomes a self-affecting process. The process N_t itself is not a Markov process anymore, but (N_t, Y_t) is a two-dimensional Markov process, and knowledge of their joint conditional probability distributions is important in pricing various credit derivative products.

Affine Point Processes

The aggregate loss is modeled as a counting process which increases due to default. The loss is determined by default times and default size. Default times are governed by an intensity driven by market-wide risk factors which follow an affine jump diffusion process. The size of each loss is described by another independent random variable. The loss process can be included as a risk factor in this model, so the model has the self-affecting property.

An affine point process provides a computationally tractable model for pricing credit derivatives. From the results of [4], the conditional transform of an affine point process is known to be an exponentially affine function of the risk factors with coefficients that satisfy ordinary differential equations. This transform determines the conditional distribution of future portfolio loss and the price of a contingent claim on the aggregate loss of portfolio, which is specified by an intensity λ and a random loss at default.

One example of such process is a *Hawkes process* [6]. The Hawkes process is a self-affecting affine point process whose intensity increases at an event as a function of the realized loss and reverts to a time-varying level between events:

$$\lambda_t = c(t) + \int_0^t f(t-s)dL_s. \tag{3.8}$$

The first-to-default intensity $c(t)$ is a nonnegative deterministic function of time. The impact of a loss on the intensity is determined by the exponential function

$$f(s) = \delta e^{ks}, \qquad s \geq 0,$$

with $k \geq 0$ and $\delta \geq 0$. When $c(t)$ has the form

$$c(t) = c(0)e^{kt} + k\int_0^t e^{k(t-s)}\varrho(s)ds.$$

We may write (3.8) as

$$d\lambda_t = k(\varrho(t) - \lambda_t) + \delta dL_t.$$

The intensity in the Hawkes process can be generalized to include additional diffusion and jump terms.

An *affine point process* has an intensity that is driven by an affine jump diffusion [6]. A Markov process X_t in a state space $D \subset \mathbb{R}^d \times \mathbb{R}_+$ is an affine jump diffusion in the sense of [5] if X is a strong solution of the stochastic differential equation

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t + \sum_{i=1}^{m} \zeta^i dZ_t^i,$$

where W_t is an $\mathbb{R}^d$-valued standard Brownian motion, $\mu : D \to \mathbb{R}^d$ is the drift, $\sigma : D \to \mathbb{R}^{d\times d}$ is the volatility, and each Z^i is a temporally consistent $\mathbb{R}_+$-valued point process. This means that the component processes of each vector Z^i have the same jump times and differ only in jump sizes. We denote the intensity of Z^i by $\lambda^i(X_t, t)$ for some $\lambda^i : D \to \mathbb{R}_+$. The jump sizes are drawn from a distribution ν^i on $\mathbb{R}_+$. Each parameter ζ^i is a d-dimensional diagonal matrix. We further assume that

$$\begin{aligned}
\mu(x,t) &= K_0(t) + K_1(t)x, \quad K_0(t) \in \mathbb{R}^d, \quad K_1(t) \in \mathbb{R}^{d\times d} \\
\left(\sigma(x,t)\sigma(x,t)^T\right)_{jk} &= (H_0)_{jk}(t) + (H_1)_{jk}(t) \cdot x, \quad H_0(t) \in \mathbb{R}^{d\times d}, \quad H_1(t) \in \mathbb{R}^{d\times d\times d} \\
\lambda^i(x,t) &= \Lambda_0^i(t) + \Lambda_1^i(t) \cdot x, \quad \Lambda_0^i(t) \in \mathbb{R}, \quad \Lambda_1^i(t) \in \mathbb{R}^d, \quad i = 1, 2, \ldots, m.
\end{aligned}$$

The affine point process J is said to have the *self-affecting property* if at least one of the component processes of $J = (L, N)^T$ is temporally consistent with one of the component processes of one of the jump terms Z^i of X.

One such example of affine point process is the Hawkes process we mentioned above. Suppose the coefficients of the affine diffusion jump process X are

$$\begin{aligned}
K_0(t) &= k\lambda_\infty, \\
K_1(t) &= -k, \\
H_0(t) &= \text{zero matrix}, \\
H_1(t) &= \text{a tensor of zeros}, \\
X_0 &= \lambda_\infty, \quad \zeta = \delta \geq 0, \\
\Lambda_0(t) &= 0, \quad \Lambda_1(t) = 1.
\end{aligned}$$

Then the intensity of J is $\lambda = X$, and the dynamics it satisfies are defined by

$$d\lambda_t = k(\lambda_\infty - \lambda_t)dt + \delta dL_t,$$

which is the exactly the intensity of the Hawkes process.

Two-Dimensional Markovian Model for Dynamics of Aggregate Credit Loss

The model proposed by Lopatin and Misirpashaev in [19] is Markovian with the state variables being the total accumulative loss L_t and the stochastic default intensity λ_t. The dynamics of the default intensity are governed by the equation

$$d\lambda_t = \kappa(\rho(L_t, t) - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t.$$

The calibration procedure for this model consists of two parts. In the first step, the intensity of an auxiliary one-step Markov chain, which is consistent with the current available CDO tranches, is found. The intensity of this Markov chain is a deterministic function of accumulative loss and time, so we call it a local intensity to distinguish it from the stochastic intensity of the original model. In the second calibration step, the stochastic intensity of the two-dimensional model is calibrated to the local intensity.

The forward Kolmogorov equation for the joint density $p(\lambda, L, t)$ for positive-valued intensity λ and non-negative multiples of h-valued loss unit loss process L is

$$\begin{aligned}\frac{\partial p(\lambda, L, t)}{\partial t} = &\left(-\kappa\frac{\partial}{\partial\lambda}(\rho(L, t) - \lambda + \frac{1}{2}\frac{\partial^2}{\partial\lambda^2}\sigma^2\lambda\right)p(\lambda, L, t) \\ &+ \lambda\left(1_{L\geq h}p(\lambda, L - h, t) - p(\lambda, L, t)\right),\end{aligned} \tag{3.9}$$

with boundary conditions

$$\begin{aligned}p(\lambda, L, 0) &= p_0(\lambda)\cdot 1_{\{L=0\}}, \\ p(0, L, t) &= 0,\end{aligned}$$

where p_0 is the initial distribution of λ_t. Integrating Equation (3.9) with respect to the intensity λ, we get the forward Kolmogorov equation for the probability density function $P(L, t)$ of the loss L:

$$\begin{aligned}P(L, t) &= \int_0^\infty p(\lambda, L, t)d\lambda, \\ \frac{\partial P(L, t)}{\partial t} &= 1_{\{L\geq h\}}\Lambda(L - h, t)P(L - h, t) - \Lambda(L, t)P(L, t),\end{aligned}$$

where

$$\Lambda(L,t)P(L,t) := \int_0^\infty \lambda p(\lambda, L, t) d\lambda,$$
$$\Lambda(L,t) := \frac{\int_0^\infty \lambda p(\lambda, L, t) d\lambda}{\int_0^\infty p(\lambda, L, t) d\lambda},$$
$$\Lambda(L,t) := \mathbb{E}[\lambda_t | L_t = L].$$

This can be seen to be a projection of the stochastic intensity-based model onto a one-dimensional Markov chain with local intensity $\Lambda(L,t)$.

Provided that the initial loss distribution surface is fully known for time t and the loss level L is known from the available market data, the forward Kolmogorov equation (3.9) leads to the calibration of the local intensity $\Lambda(L,t)$:

$$\Lambda(L=K,t) = -\frac{1}{P(K,t)} \frac{\partial}{\partial t} \sum_{L=0}^{K} P(L,t). \tag{3.10}$$

The next step in calibration is to find the function $\rho(L,t)$ consistent with the local intensity $\Lambda(L,t)$:

$$\begin{aligned} \rho(L,t) &= \Lambda(L,t) + \frac{1}{\kappa} \frac{\partial \Lambda(L,t)}{\partial t} \\ &= \frac{\Lambda(L,t)(1_{L \geq h} \Lambda(L-h,t) P(L-h,t) - \Lambda(L,t) P(L,t))}{\kappa P(L,t)} \\ &= \frac{M(L,t) - 1_{L \geq h} M(L-h,t)}{\kappa P(L,t)}, \end{aligned} \tag{3.11}$$

where

$$M(L,t) = \int_0^\infty \lambda^2 p(\lambda, L, t) d\lambda.$$

The calibration of the function ρ provides much more flexibility in terms of matching the initial market data than is afforded by the parameter models in other two-dimensional Markovian intensity-based top-down models. Substituting Equation (3.11) into Equation (3.9) yields the density function $P(L,t)$.

Chapter 4

Singular Perturbation Methods in Equity Option Pricing

In a stochastic volatility model for equity option pricing, Fouque and his collaborators show that asymptotic methods such as singular perturbation are very efficient for capturing the effects of stochastic volatility. Here we summarize their results in Chapter 4 in [8].

4.1 Fast Time Scale and Singular Perturbation

We begin with a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{Q})$. Filtration is generated by Brownian motions W_t and $W_t^{(1)}$. We consider two correlated stochastic processes, X_t and Y_t. Here X_t denotes the log-price process of an underlying asset, and Y_t is the driving random factor for stochastic volatility:

$$\begin{aligned} dX_t &= \sigma_t dW_t, \\ \sigma_t &= f(Y_t). \end{aligned} \tag{4.1}$$

where we assume that $f(z) : \mathbb{R} \to (0, \infty)$ is a smooth function bounded above and below by zero. We assume that Y_t obeys the stochastic differential equation

$$dY_t = \frac{1}{\varepsilon}\alpha(Y_t)dt + \frac{1}{\sqrt{\varepsilon}}\beta(Y_t)dW_t^{(1)}. \tag{4.2}$$

The Brownian motion processes W and $W^{(1)}$ are correlated in the sense that

$$W_t = \rho_1 W_t^{(1)} + \sqrt{1-{\rho_1}^2}W_t^{(2)},$$

where $W_t^{(1)}$ and $W_t^{(2)}$ are independent Brownian motions.

We consider a function $u(t,x,y)$ on $[0,T]\times\mathbb{R}^2$ defined by

$$\begin{aligned} u(t,x,y) &= \mathbb{E}[h(X_T)|\mathcal{F}_t] \\ &= \mathbb{E}[h(X_T)|X_t=x, Y_t=y], \\ u(T,x,y) &= h(x) \end{aligned} \tag{4.3}$$

for some payoff function h. Using Equations (4.2), (4.1), the infinitesimal generator[8] $\mathcal{L}_{(X,Y)}$ of the Markov process (X_t,Y_t) is

$$\mathcal{L}_{(X,Y)} = \frac{1}{2}f^2(y)\frac{\partial^2}{\partial x^2} + \frac{1}{\sqrt{\varepsilon}}\rho_1\beta(y)f(y)\frac{\partial^2}{\partial x\partial y} + \frac{1}{\varepsilon}\left(\frac{1}{2}\beta^2(y)\frac{\partial^2}{\partial y^2} + \alpha(y)\frac{\partial}{\partial y}\right). \tag{4.4}$$

Thus, the function u defined in (4.3) now satisfies Equation (4.5)

$$\frac{\partial u}{\partial t} + \frac{1}{2}f^2(y)\frac{\partial^2 u}{\partial x^2} + \frac{1}{\sqrt{\varepsilon}}\rho_1\beta(y)f(y)\frac{\partial^2 u}{\partial x\partial y} + \frac{1}{\varepsilon}\left(\frac{1}{2}\beta^2(y)\frac{\partial^2 u}{\partial y^2} + \alpha(y)\frac{\partial u}{\partial y}\right) = 0. \tag{4.5}$$

In order to find the approximate solution of u in (4.3), we expand the solution u in powers of $\sqrt{\varepsilon}$:

$$u = u_0 + \sqrt{\varepsilon}u_1 + \varepsilon u_2 + \varepsilon\sqrt{\varepsilon}u_3 + \cdots. \tag{4.6}$$

Substituting (4.6) into (4.5) and collecting powers of ε gives

$$\begin{aligned} &\frac{1}{\varepsilon}\left(\frac{1}{2}\beta^2(y)\frac{\partial^2 u_0}{\partial y^2} + \alpha(y)\frac{\partial u_0}{\partial y}\right) \\ &+ \frac{1}{\sqrt{\varepsilon}}\left(\frac{1}{2}\beta^2(y)\frac{\partial^2 u_1}{\partial y^2} + \alpha(y)\frac{\partial u_1}{\partial y} + \rho_1\beta(y)f(y)\frac{\partial^2 u_0}{\partial x\partial y}\right) \\ &+ \left(\frac{1}{2}\beta^2(y)\frac{\partial^2 u_2}{\partial y^2} + \alpha(y)\frac{\partial u_2}{\partial y} + \rho_1\beta(y)f(y)\frac{\partial^2 u_1}{\partial x\partial y} + \frac{\partial u_0}{\partial t} + \frac{1}{2}f^2(y)\frac{\partial^2 u_0}{\partial x^2}\right) \\ &+ \sqrt{\varepsilon}\left(\frac{1}{2}\beta^2(y)\frac{\partial^2 u_3}{\partial y^2} + \alpha(y)\frac{\partial u_3}{\partial y} + \rho_1\beta(y)f(y)\frac{\partial^2 u_2}{\partial x\partial y} + \frac{\partial u_1}{\partial t} + \frac{1}{2}f^2(y)\frac{\partial^2 u_1}{\partial x^2}\right) \\ &+ \cdots = 0. \end{aligned} \tag{4.7}$$

The coefficient of $1/\varepsilon$ in Equation (4.7) gives the partial differential equation

$$\frac{1}{2}\beta^2(y)\frac{\partial^2 u_0}{\partial y^2} + \alpha(y)\frac{\partial u_0}{\partial y} = 0. \tag{4.8}$$

Since the y-dependent solution of Equation (4.8) exhibits the unreasonable growth $\exp(y^2)$ at infinity, we look for a y-independent solution. Thus we assume that $u_0 = u_0(t,x)$ and subsequently we find u_0 from the existence condition of another partial

differential equation which we explain below. The coefficient of $1/\sqrt{\varepsilon}$ in Equation (4.7) leads to

$$\frac{1}{2}\beta^2(y)\frac{\partial^2 u_1}{\partial y^2} + \alpha(y)\frac{\partial u_1}{\partial y} + \rho_1\beta(y)f(y)\frac{\partial^2 u_0}{\partial x \partial y} = 0. \tag{4.9}$$

Since the function u_0 is independent of y, we see that (4.9) simplifies to

$$\frac{1}{2}\beta^2(y)\frac{\partial^2 u_1}{\partial y^2} + \alpha(y)\frac{\partial u_1}{\partial y} = 0.$$

For the same reason that the function u_0 is independent of y, we also know that u_1 is a function only of $(t,\, x)$. The coefficient of the zeroth-order term of ε gives the equation

$$\frac{1}{2}\beta^2(y)\frac{\partial^2 u_2}{\partial y^2} + \alpha(y)\frac{\partial u_2}{\partial y} = -\frac{\partial u_0}{\partial t} - \frac{1}{2}f^2(y)\frac{\partial^2 u_0}{\partial x^2}. \tag{4.10}$$

The operator on the left hand side,

$$\mathcal{L}_Y = \frac{1}{2}\beta^2(y)\frac{\partial^2}{\partial y^2} + \alpha(y)\frac{\partial}{\partial y},$$

only involves the variable y. Thus t and x are fixed parameters in Equation (4.10). This Equation (4.10) has solutions with reasonable polynomial growth at infinity only if the source term

$$-\frac{\partial u_0}{\partial t} - \frac{1}{2}f^2(y)\frac{\partial^2 u_0}{\partial x^2}$$

is in the orthogonal complement of the null space of the formal adjoint $\mathcal{L}_Y^*$ of $\mathcal{L}_Y$,

$$\mathcal{L}_Y^* = -\frac{\partial}{\partial y}(\alpha(y)\cdot) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{2}\beta^2(y)\cdot\right). \tag{4.11}$$

Let v be a solution of

$$\mathcal{L}_Y^* v(y) = 0.$$

The source function, right hand side of (4.10), should be orthogonal to v with respect to L^2 inner product. Using $\langle , \rangle$ to denote the L^2 inner product, we have

$$\begin{aligned} \left\langle -\frac{\partial u_0}{\partial t}, v \right\rangle - \left\langle \frac{1}{2}f^2\frac{\partial^2 u_0}{\partial x^2}, v \right\rangle = 0, \\ \frac{\partial u_0}{\partial t} + \left\langle \frac{1}{2}f^2, v \right\rangle \frac{\partial^2 u_0}{\partial x^2} = 0, \end{aligned} \tag{4.12}$$

where we use the property of the function v, which is the invariant distribution function,

$$\int_{-\infty}^{\infty} v(y)dy = 1.$$

Equation (4.12) finally reduces to the heat equation with constant coefficients and terminal condition:

$$\begin{aligned} \frac{\partial u_0}{\partial t} + \frac{1}{2}\left\langle f^2(y), v\right\rangle \frac{\partial^2 u_0}{\partial x^2} = 0, \\ u_0(T,x) = h(x). \end{aligned} \tag{4.13}$$

The coefficient of $\sqrt{\varepsilon}$ in Equation (4.7) gives a similar condition on u_1 by the same reasoning:

$$\begin{aligned} \frac{1}{2}\beta^2(y)\frac{\partial^2 u_3}{\partial y^2} + \alpha(y)\frac{\partial u_3}{\partial y} + \rho_1\beta(y)f(y)\frac{\partial^2 u_2}{\partial x \partial y} + \frac{\partial u_1}{\partial t} + \frac{1}{2}f^2(y)\frac{\partial^2 u_1}{\partial x^2} = 0, \\ \frac{1}{2}\beta^2(y)\frac{\partial^2 u_3}{\partial y^2} + \alpha(y)\frac{\partial u_3}{\partial y} = -\rho_1\beta(y)f(y)\frac{\partial^2 u_2}{\partial x \partial y} - \frac{\partial u_1}{\partial t} - \frac{1}{2}f^2(y)\frac{\partial^2 u_1}{\partial x^2}, \end{aligned}$$

and

$$\begin{aligned} \rho_1\left\langle \beta f \frac{\partial^2 u_2}{\partial x \partial y}\right\rangle + \frac{\partial u_1}{\partial t} + \frac{1}{2}\langle f^2, v\rangle \frac{\partial^2 u_1}{\partial x^2} = 0, \\ \frac{\partial u_1}{\partial t} + \frac{1}{2}\langle f^2, v\rangle \frac{\partial^2 u_1}{\partial x^2} = -\rho_1\left\langle \beta f \frac{\partial^2 u_2}{\partial x \partial y}\right\rangle. \end{aligned}$$

From Equations (4.10) and (4.13), we first find u_2 from

$$\frac{1}{2}\beta^2(y)\frac{\partial^2 u_2}{\partial y^2} + \alpha(y)\frac{\partial u_2}{\partial y} = \frac{1}{2}\left\langle f^2, v\right\rangle \frac{\partial^2 u_0}{\partial x^2} - \frac{1}{2}f^2(y)\frac{\partial^2 u_0}{\partial x^2}.$$

Let continuous function $\phi(y)$ be a solution of

$$\frac{1}{2}\beta^2(y)\frac{\partial^2 \phi}{\partial y^2} + \alpha(y)\frac{\partial \phi}{\partial y} = f^2(y) - \left\langle f^2, v\right\rangle.$$

Then

$$u_2 = -\frac{1}{2}\phi(y)\frac{\partial^2 u_0}{\partial x^2}(t,x) + C(t,x),$$

where for some function $C(t,x)$.

Finally, we get the following equation for u_1, which completes the first order y-independent approximation:

$$\begin{aligned} \frac{\partial u_1}{\partial t} + \frac{1}{2}\left\langle f^2, v\right\rangle \frac{\partial^2 u_1}{\partial x^2} = \frac{1}{2}\rho_1\left\langle \beta f \phi', v\right\rangle \frac{\partial^3 u_0}{\partial x^3}, \\ u_1(T,x) = 0. \end{aligned} \tag{4.14}$$

Here, u_1 is explicitly given in terms of u_0:

$$u_1 = -\frac{1}{2}(T-t)\rho_1\langle \beta f \phi', v\rangle \frac{\partial^3 u_0}{\partial x^3}.$$

Therefore, the approximation of u using the first two terms in the $\sqrt{\varepsilon}$ is given by

$$u \approx u_0 - \frac{1}{2}(T-t)\rho_1\langle \beta f \phi', v\rangle \frac{\partial^3 u_0}{\partial x^3}\sqrt{\varepsilon}.$$

This is a modification of the u_0 term with one of order $\sqrt{\varepsilon}$.

An error analysis of this approximation is undertaken in [9]. Define

$$\tilde{u}^{\varepsilon}(t,x,y) := \mathbb{E}[e^{-r(T-t)}h(X_T)|X_t = x, Y_t = y] \tag{4.15}$$

where r is the interest rate. Note that u defined previously does not have the discounting factor. Fouque, Sircar, Papanicolaou and Solna showed the error between the real solution $\tilde{u}^{\varepsilon}(t,x,y)$ and its approximation $w^{\varepsilon} := u_0 + \sqrt{\varepsilon}u_1$ in their setting [9] obeys

$$\lim_{\varepsilon \to 0} \frac{|\,\tilde{u}^{\varepsilon} - w^{\varepsilon}\,|}{\varepsilon|\log\varepsilon|^{1+p}} = 0, \tag{4.16}$$

for any $p > 0$.

Chapter 5

Singular Perturbation Methods in Credit Modeling

We examine the use of approximation methods in credit derivative modeling. In §5.1 we introduce the Gaussian copula with stochastic volatility in [10]. In §5.2 we model the loss in a portfolio with stochastic intensity and approximate the joint density function of loss and intensity using the singular perturbation.

5.1 Perturbed Gaussian Copula

In [10] Fouque and Zhou apply the approximation method used for equity option pricing [8] to the copula model because the Gaussian copula model, which is the most popular in the financial industry, does not show the tail dependence property. We here review their work briefly.

Tail dependence reflects the dependence structure when extreme events occur. Let $(Y_1, Y_2)^T$ be a vector of continuous random variables with marginal distribution functions F_1, F_2, then the coefficient of the *upper tail dependence* of $(Y_1, Y_2)^T$ is

$$\lambda_U = \lim_{u \to 1} P(Y_2 > F_2^{-1}(u) | Y_1 > F_1^{-1}(u)),$$

and the coefficient of the *lower tail dependence* of $(Y_1, Y_2)^T$ is

$$\lambda_L = \lim_{u \to 1} P(Y_2 < F_2^{-1}(u) | Y_1 < F_1^{-1}(u)).$$

If $\lambda_U > 0$, upper tail dependence exists and the positive extreme values can be observed simultaneously and if $\lambda_L > 0$, there exists lower tail dependence and negative extreme value can be observed simultaneously.

Some tail dependence can be restored by introducing stochastic volatility on a Gaussian copula. Using the perturbation method, an approximate copula called the *perturbed Gaussian copula* is derived here.

Unlike in the standard Gaussian copula setting, each firm's value variable X_i has a stochastic volatility in its dynamics.

We define the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$, $\mathcal{F} = \{\mathcal{F}\}_{t\geq 0}$. We only consider a portfolio with two underlying names below:

$$dX_t^i = f_i(Y_t)dB_t^i, \quad \text{for} \quad i = 1, 2,$$

where

$$dY_t = \frac{1}{\varepsilon}(m - Y_t)dt + \frac{v\sqrt{2}}{\sqrt{\varepsilon}}dB_t^Y,$$

and Brownian motions B_t^1, B_t^2 and B_t^Y are correlated with,

$$\begin{aligned} B_t^1 &= \rho B_t^2 + \sqrt{1-\rho^2}W_t, \\ B_t^1 &= \rho_{1Y} B_t^Y + \sqrt{1-\rho_{1Y}^2}W_t^1, \\ B_t^2 &= \rho_{2Y} B_t^Y + \sqrt{1-\rho_{2Y}^2}W_t^2. \end{aligned} \tag{5.1}$$

where Brownian motions W_t, W_t^1, W_t^2 are independent to Brownian motions B_t^1, B_t^2 and B_t^Y. We are interested in determining the three transition densities,

$$\begin{aligned} u^\varepsilon(t,x,y) &:= \mathbb{Q}(X_T^{(1)} \in d\xi_1, X_T^{(2)} \in d\xi_2 | X_t = x, Y_t = y), \\ v_1^\varepsilon(t,x,y) &:= \mathbb{Q}(X_T^{(1)} \in d\xi_1 | X_t = x, Y_t = y), \\ v_2^\varepsilon(t,x,y) &:= \mathbb{Q}(X_T^{(2)} \in d\xi_2 | X_t = x, Y_t = y), \end{aligned} \tag{5.2}$$

where $X_t = (X_t^{(1)}, X_t^{(2)})$ and $x = (x_1, x_2)$. The singular perturbation method is used to find the density functions in (5.2) in the following way. We only discuss the case of u^ε; the other cases can be treated in a similar way.

The Kolmogorov backward equation [8], [26] for u^ε is

$$\begin{aligned} \mathcal{L}^\varepsilon u^\varepsilon(t, x_1, x_2, y) &= 0, \\ u^\varepsilon(T, x_1, x_2, y) &= \delta(\xi_1 - x_1)\delta(\xi_2 - x_2), \end{aligned} \tag{5.3}$$

where $\delta(\xi_i - x_i)$ is the Dirac delta measure of x_i centered at ξ_i. The operator $\mathcal{L}^\varepsilon$ has the decomposition

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2,$$

where

$$\begin{aligned}
\mathcal{L}_0 &:= (m-y)\frac{\partial}{\partial y} + v^2\frac{\partial^2}{\partial y^2},\\
\mathcal{L}_1 &:= v\sqrt{2}\rho_{1Y} f_1(y)\frac{\partial^2}{\partial x_1 \partial y} + v\sqrt{2}\rho_{2Y} f_2(y)\frac{\partial^2}{\partial x_2 \partial y},\\
\mathcal{L}_2 &:= \frac{\partial}{\partial t} + \frac{1}{2}f_1^2(y)\frac{\partial^2}{\partial x_1^2} + \frac{1}{2}f_2^2(y)\frac{\partial^2}{\partial x_2^2} + \rho f_1(y) f_2(y)\frac{\partial^2}{\partial x_1 \partial x_2}.
\end{aligned} \tag{5.4}$$

Expanding the solution u^ε in powers of $\sqrt{\varepsilon}$ and substituting it into Equation (5.3), we obtain the first two terms in the same way as in [8]:

$$\mathcal{L}_0 u_0 = 0, \tag{5.5}$$

$$\mathcal{L}_0 u_1 + \mathcal{L}_1 u_0 = 0, \tag{5.6}$$

$$\mathcal{L}_0 u_2 + \mathcal{L}_1 u_1 + \mathcal{L}_2 u_0 = 0, \tag{5.7}$$

$$\mathcal{L}_0 u_3 + \mathcal{L}_1 u_2 + \mathcal{L}_2 u_1 = 0. \tag{5.8}$$

From the first two Equations (5.5) and (5.6), we know that u_0, u_1 are independent of y. From the Equation (5.7) for u_2, the source term becomes $\mathcal{L}_2 u_0$ since we know that u_1 is independent of y. To have at most polynomial growth of u_2 for large values of x_1, x_2(see chapter 5 in [8]), we must have

$$\begin{aligned}
\langle \mathcal{L}_2 \rangle u_0(t, x_1, x_2) &= 0,\\
u_0(T, x_1, x_2) &= \delta(\xi_1 - x_1)\delta(\xi_2 - x_2),
\end{aligned} \tag{5.9}$$

where $\langle \cdot \rangle$ denotes the average with respect to the invariant distribution $\mathcal{N}(m, v^2)$ of Y_t,

$$\langle g \rangle := \int_{-\infty}^{\infty} g(y)\frac{1}{v\sqrt{2\pi}} \exp -\frac{(y-m)^2}{2v^2} dy.$$

Note that $\langle \mathcal{L}_i \rangle$ means the coefficients functions of the operator $\mathcal{L}_i$ are averaged with respect to the normal distribution $\mathcal{N}(m, v^2)$. Let

$$\bar{\sigma}_1 := \sqrt{\langle f_1^2 \rangle}, \;\; \bar{\sigma}_2 := \sqrt{\langle f_2^2 \rangle}, \;\; \bar{\rho} := \frac{\langle \rho f_1 f_2 \rangle}{\bar{\sigma_1}\bar{\sigma_2}}. \tag{5.10}$$

Using the notation in Equation(5.10), Equation (5.9) becomes

$$\begin{aligned}
\frac{\partial u_0}{\partial t} + \frac{1}{2}\bar{\sigma_1}^2\frac{\partial^2 u_0}{\partial x_1^2} + \frac{1}{2}\bar{\sigma_2}^2\frac{\partial^2 u_0}{\partial x_2^2} + \bar{\rho}\bar{\sigma_1}\bar{\sigma_2}\frac{\partial^2 u_0}{\partial x_1 \partial x_2} &= 0,\\
u_0(T, x_1, x_2) &= \delta(\xi_1 - x_1)\delta(\xi_2 - x_2).
\end{aligned}$$

Here, u_0 is the joint transitional probability density function of two correlated scaled Brownian motions with instantaneous correlation $\bar{\rho}$ and scale factors $\bar{\sigma}_1$ and $\bar{\sigma}_2$:

$$\begin{aligned} u_0(t, x_1, x_2) &= \frac{1}{2\pi\bar{\sigma}_1\bar{\sigma}_2(T-t)\sqrt{1-\bar{\rho}^2}} \\ &= \exp -\frac{1}{2(1-\bar{\rho}^2)}\left(\frac{(\xi_1 - x_1)^2}{\bar{\sigma_1}^2(T-t)} - 2\bar{\rho}\frac{(\xi_1 - x_1)(\xi_2 - x_2)}{\bar{\sigma_1}\bar{\sigma_2}}\frac{(\xi_2 - x_2)^2}{\bar{\sigma_2}^2(T-t)}\right). \end{aligned}$$

We apply the same growth condition as in Equation (5.7) to Equation (5.8) and using the fact that $\langle\mathcal{L}_2\rangle u_0 = 0$ from Equation (5.7) , we have

$$\langle\mathcal{L}_2\rangle u_1 = \langle\mathcal{L}_1\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle)\rangle u_0,$$

with zero final condition,

$$u_1(T, x_1, x_2) = 0.$$

Let us denote by $\phi_1(y), \phi_2(y)$ and $\phi_{12}(y)$ the solutions of the Poisson equations,

$$\begin{aligned} \mathcal{L}_0\phi_1(y) &= f_1^2(y) - \langle f_1^2\rangle, \\ \mathcal{L}_0\phi_2(y) &= f_2^2(y) - \langle f_2^2\rangle, \\ \mathcal{L}_0\phi_{12}(y) &= f_1(y)f_2(y) - \langle f_1 f_2\rangle. \end{aligned}$$

Then $\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle)$ is defined by

$$\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle) = \frac{1}{2}\phi_1(y)\frac{\partial^2}{\partial x_1^2} + \frac{1}{2}\phi_2(y)\frac{\partial^2}{\partial x_2^2} + \rho\phi_{12}(y)\frac{\partial^2}{\partial x_1\partial x_2}.$$

By the definition of $\mathcal{L}_1$,

$$\begin{aligned} \mathcal{L}_1\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle) = {} & v\sqrt{2}\rho_{1Y}f_1(y)\left[\frac{1}{2}\phi_1'(y)\frac{\partial^3}{\partial x_1^3} + \frac{1}{2}\phi_2'(y)\frac{\partial^3}{\partial x_2^2\partial x_1} + \rho\phi_{12}'(y)\frac{\partial^3}{\partial x_1^2\partial x_2}\right] \\ & + v\sqrt{2}\rho_{2Y}f_2(y)\left[\frac{1}{2}\phi_1'(y)\frac{\partial^3}{\partial x_1^2\partial x_2} + \frac{1}{2}\phi_2'(y)\frac{\partial^3}{\partial x_2^3} + \rho\phi_{12}'(y)\frac{\partial^3}{\partial x_1\partial x_2^2}\right]. \end{aligned}$$

Then the operator $\langle\mathcal{L}_1\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle)\rangle$ is

$$R_1\frac{\partial^3}{\partial x_1^3} + R_2\frac{\partial^3}{\partial x_2^3} + R_{12}\frac{\partial^3}{\partial x_1\partial x_2^2} + R_{21}\frac{\partial^3}{\partial x_1^2\partial x_2},$$

with the constant parameters R_1, R_2, R_{12}, R_{21} defined as

$$\begin{aligned}
R_1 &= \frac{v\rho_{1Y}}{\sqrt{2}}\langle f_1\phi_1'\rangle,\\
R_2 &= \frac{v\rho_{2Y}}{\sqrt{2}}\langle f_2\phi_2'\rangle,\\
R_{12} &= \frac{v\rho_{1Y}}{\sqrt{2}}\langle f_1\phi_2'\rangle + v\sqrt{2}\rho\rho_{2Y}\langle f_2\phi_{12}'\rangle,\\
R_{21} &= \frac{v\rho_{2Y}}{\sqrt{2}}\langle f_2\phi_1'\rangle + v\sqrt{2}\rho\rho_{1Y}\langle f_1\phi_{12}'\rangle.
\end{aligned}$$

Then u_1 is given by

$$u_1 = -(T-t)\langle \mathcal{L}_1\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle)\rangle u_0.$$

Conditioning on $X_t = x, Y_t = y$, suppose $(X_T^{(1)}, X_T^{(2)})$ has a Gaussian copula C. Then

$$\mathbb{Q}(X_T^{(1)} \le \xi_1, X_T^{(2)} \le \xi_2 | X_t = x, Y_t = y) = \Psi(\xi_1, \xi_2) = C(z_1, z_2), \tag{5.11}$$

where

$$\begin{aligned}
z_1 &= \mathbb{Q}(X_T^{(1)} \le \xi_1 | X_t = x, Y_t = y),\\
z_2 &= \mathbb{Q}(X_T^{(2)} \le \xi_2 | X_t = x, Y_t = y).
\end{aligned}$$

Taking derivatives with respect to ξ_1, ξ_2 in (5.11) yields

$$\mathbb{Q}(X_T^{(1)} \in d\xi_1, X_T^{(2)} \in d\xi_2 | X_t = x, Y_t = y) = \frac{\partial^2 C(z_1, z_2)}{\partial z_1 \partial z_2}\frac{\partial z_1}{\partial \xi_1}\cdot\frac{\partial z_2}{\partial \xi_2}.$$

This gives the density function

$$\psi(\xi_1, \xi_2) = \frac{\partial^2 \Psi(\xi_1, \xi_2)}{\partial \xi_1 \partial \xi_2}$$

using equations (5.2) and

$$\psi(\xi_1, \xi_2) = \frac{u^\varepsilon}{v_1^\varepsilon v_2^\varepsilon}. \tag{5.12}$$

Even though the work in [10] is meaningful in the sense that it gives more interdependence among extreme events, when it comes to pricing multi-name products with many names, it still has the drawbacks of bottom-up modeling.

5.2 Markovian Stochastic Intensity Model

In this section we adapt the top-down intensity approach in Section 3.2.2 to model the aggregate loss process. We will investigate the use of the singular perturbation method for approximating the transition density function and analyze the error in the approximation. The loss process in our model together with stochastic factor in the intensity makes a two-dimensional Markov process. Thus our model can be seen to be similar to the models in [12] and [19]. In order to find the transition density function of Markov process, we apply the method of matched asymptotic expansion inspired by the work in [15] of Yin and Khasminskii for a certain type of Kolmogorov forward equation; see [23] for details. As the solution to the Cauchy problem, existence of the probability density function is studied in [16] and [7]. However, the perturbation method provides the opportunity to see the asymptotic properties of the solution as the coefficient ε in the Kolmogorov forward partial differential equation tends to zero.

5.2.1 Model Setting

We construct a loss process L_t in a pool of N names as a continuous-time Markov chain whose $(N+1) \times (N+1)$ generating matrix has the form:

$$A = a(Y_t) \begin{pmatrix} -F(t,0) & F(t,0) & 0 & \cdots & 0 \\ 0 & -F(t,1) & F(t,1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -F(t,N-1) & F(t,N-1) \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where $a(\cdot)$ is a continuous bounded deterministic function and the random factor Y_t can be assumed to be a diffusion process:

$$dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)dW_t. \tag{5.13}$$

Let λ be the intensity of the process L_t. The intensity is a deterministic function of the loss level L_t, the random factor Y_t, and the time t:

$$\lambda = \lambda(Y_t, N_t, t).$$

We solve the Kolmogorov forward equation to get the joint probability density function. Since Y_t is a continuous random variable and L_t is a discrete variable, we need a Kolmogorov forward equation which accommodates both types of random variables. Let $P_{jk}(Y,T|y,t)$ be a joint density function of L_t and Y_t with $L(T)=k$, $L(t)=j$ and $Y_T=Y$, $Y_t=y$ for $0\le t<T\le T_M$. Then P_{jk} satisfies the Kolmogorov forward equation; eee [23], [12], [19] for more details:

$$\frac{\partial P_{jk}}{\partial T}=-\frac{\partial}{\partial Y}[A_{1jk}P_{jk}]+\frac{1}{2!}\frac{\partial^2}{\partial Y^2}[A_{2jk}P_{jk}]+\sum_{i\in\{0,1,2,\dots,N\}}a_{kij}P_{ji}, \tag{5.14}$$

where

$$\begin{aligned}a_{kij}&=\lim_{\Delta T\downarrow 0}\frac{1}{\Delta T}\left[\mathbb{Q}\left(N_{T+\Delta T}=k|N_T=i,Y_T;N_t=j,Y_t\right)-\delta_{ki}\right],\\ A_{njk}&=\lim_{\Delta T\downarrow 0}\frac{1}{\Delta T}\mathbb{E}\left[(Y_{T+\Delta t}-Y_T)^n|N_{T+\Delta T}=k;Y_T,N_T=j;Y_t,N_t=j\right].\end{aligned}$$

With Equation (5.13), the Kolmogorov forward equation (5.14) becomes

$$\begin{aligned}\frac{\partial P_{jk}}{\partial T}=&-\frac{\partial}{\partial Y}[\mu(T,Y)P_{jk}]+\frac{1}{2!}\frac{\partial^2}{\partial Y^2}[\sigma^2(T,Y)P_{jk}]\\&+\lambda(Y,k-1,T)P_{jk-1}-\lambda(Y,k,T)P_{jk},\end{aligned} \tag{5.15}$$

where the intensity $\lambda(Y,k,T)$ of loss L_t can be specified in each model. The initial condition accompanying (5.15) is

$$P_{jk}(Y,t|y,t)=g_0\cdot 1_{\{j=k\}}, \tag{5.16}$$

where g_0 is the initial distribution of the process Y_t. Thus g_0 satisfies $\int_{-\infty}^{\infty}g_0dy=1$.

Once the backward variables j,y,t are fixed, we have a system of equations for $k\ge j$. We can iteratively solve this system of equations starting from $k=j$. The first equation for $k=j$ becomes

$$\begin{aligned}&\frac{\partial P_{jj}}{\partial T}=-\frac{\partial}{\partial Y}[\mu(T,Y)P_{jj}]+\frac{1}{2!}\frac{\partial^2}{\partial Y^2}[\sigma^2(T,Y)P_{jj}]-\lambda(Y,j,T)P_{jj},\\ &P_{jj}(Y,t|y,t)=g_0.\end{aligned} \tag{5.17}$$

For notational simplicity, we assume that the backward time t is zero and $L_t=0$. Then for forward loss level k, the joint density function P_{jk} becomes P_{0k}. We denote this by

P_k. Moreover, for the rapidly varying process Y_t, we rewrite Equation (5.13) for dY_t with slight notational abuse as

$$dY_t = \frac{1}{\varepsilon}\mu(t, Y_t)dt + \frac{1}{\sqrt{\varepsilon}}\sigma(t, Y_t)dW_t. \tag{5.18}$$

Then Equation (5.15) becomes

$$\begin{aligned} \frac{\partial P_k}{\partial T} &= -\frac{\partial}{\partial Y}[\frac{1}{\varepsilon}\mu(T,Y)P_k] + \frac{1}{2!}\frac{\partial^2}{\partial Y^2}[\frac{1}{\varepsilon}\sigma^2(T,Y)P_k] \\ &\quad + \lambda(Y, k-1, T)P_{k-1} - \lambda(Y,k,T)P_k, \\ P_k(0,Y) &= 0, \end{aligned} \tag{5.19}$$

and equation (5.17) becomes

$$\begin{aligned} \frac{\partial P_j}{\partial T} &= -\frac{\partial}{\partial Y}\left[\frac{1}{\varepsilon}\mu(T,Y)P_j\right] + \frac{1}{2!}\frac{\partial^2}{\partial Y^2}\left[\frac{1}{\varepsilon}\sigma^2(T,Y)P_j\right] - \lambda(Y,j,T)P_j \\ P_j(0,Y) &= g_0. \end{aligned} \tag{5.20}$$

In the simple case of periodic conditions imposed on the coefficients of dY_t, μ and σ, and also on the initial function g_0, we have the following results in [15] about approximation formula of

$$\varepsilon\frac{\partial p^\varepsilon}{\partial t} = \mathcal{L}^{*,\varepsilon}p^\varepsilon, \qquad t > 0, \tag{5.21}$$

with initial data $p^\varepsilon(0,y) = g(y) \geq 0$ such that

$$\int_{-\infty}^{\infty} g(y)dy = 1,$$

where ε is a small parameter and the operator $\mathcal{L}^{*,\varepsilon}$ is defined by

$$\begin{aligned} \mathcal{L}^{*,\varepsilon} &= \mathcal{L}_1^* + \varepsilon\mathcal{L}_2^* \\ \mathcal{L}_i^* &= \frac{1}{2}\frac{\partial^2}{\partial y^2}(a_i(t,y)\cdot) - \frac{\partial}{\partial y}(b_i(t,y)\cdot) \quad \text{for} \quad i = 1,2, \end{aligned}$$

and where additionally we have following assumptions.

(A1) There is an $n \in \mathbb{Z}_+$ such that for $i = 1, 2$, $a_i(\cdot), b_i(\cdot) \in C_b^{n+1,2(n+1)}([0,T] \times \mathbb{R})$. Moreover, $\frac{\partial^{n+1}}{\partial t^{n+1}}a_i(\cdot, y)$ and $\frac{\partial^{n+1}}{\partial t^{n+1}}b_i(\cdot, y)$ are Lipschitz continuous of $t \in [0,T]$ uniformly with respect to y.

(A2) For each $t \in [0,T]$ and each $y \in \mathbb{R}$,

$$a_i(t,y+1) = a_i(t,y), \quad b_i(t,y+1) = b_i(t,y),$$

$$g(y+1) = g(y), \quad a_1(t,y) > 0.$$

Khasminskii and Yin intend to show that the solution $p^\varepsilon(t,y)$ can be expanded as an asymptotic formula in terms of ε such that the leading term is the quasi-stationary density.

Definition 5.2.1. *A function $\bar{p}$ is said to be a quasi-stationary transition density for the periodic diffusion with period 1 corresponding to the operator $\mathcal{L}_i^*$ if $\bar{p}(t,\cdot)$ is a periodic function with period 1 and*

$$\mathcal{L}_i^* \bar{p}(t,y) = 0, \quad \int_0^1 \bar{p}(t,y)dy = 1, \quad \bar{p}(t,y) \geq 0.$$

A quasi-stationary density on the entire space can be defined similarly by deleting the periodic conditions.

We will construct an asymptotic formula to its solution p^ε of the form

$$\begin{aligned} \tilde{P}_n^\varepsilon(t,y) &= \sum_{i=0}^n \varepsilon^i p_i(t,y) + \sum_{i=0}^n \varepsilon^i q_i(t/\varepsilon, y) \\ &= P_n^\varepsilon(t,y) + Q_n^\varepsilon(t,y), \end{aligned} \tag{5.22}$$

and show that the asymptotic formula is valid in the sense that

$$|p^\varepsilon(t,y) - \tilde{P}_n^\varepsilon(t,y)| \leq K\varepsilon^n, \tag{5.23}$$

where K is a positive real number independent of ε. Substituting Equation (5.22) into Equation (5.21) and separating the part $P^\varepsilon(t,y)$ and $Q^\varepsilon(t,y)$ using linearity of the operator $\mathcal{L}^{*,\varepsilon}$, we get the following system of equations p_i by equating the coefficients of like powers of ε^i:

$$\begin{aligned} \mathcal{L}_1^* p_0 &= 0, \\ \mathcal{L}_1^* p_1 &= \frac{\partial p_0}{\partial t} - \mathcal{L}_2^* p_0, \\ &\vdots \\ \mathcal{L}_1^* p_i &= \frac{\partial p_{i-1}}{\partial t} - \mathcal{L}_2^* p_{i-1}. \end{aligned} \tag{5.24}$$

Solving Equation (5.24), we find that for $i = 0$

$$\tilde{p}_0(t,y) = C_{0,2}(t)\exp\left(2\int_0^y \frac{b_1(t,\xi)}{a_1(t,\xi)}d\xi\right) + C_{0,1}(t)\int_0^y \exp\left(2\int_u^y \frac{b_1(t,\xi)}{a_1(t,\xi)}\right)d\xi,$$

and for $i \geq 1$,

$$\begin{aligned}\tilde{p}_i(t,y) = C_{i,2}(t)&\exp\left(2\int_0^y \frac{b_1(t,\xi)}{a_1(t,\xi)}d\xi\right)\\ &+ C_{i,1}(t)\int_0^y \exp\left(2\int_u^y \frac{b_1(t,\xi)}{a_1(t,\xi)}d\xi\right)du\\ &+ \int_0^y \exp\left(2\int_u^y \frac{b_1(t,\xi)}{a_1(t,\xi)}\int_0^u p_{i-1}^{\{i\}}(t,\xi)d\xi\right)du,\end{aligned} \tag{5.25}$$

where

$$\begin{aligned}\tilde{p}_i(t,y) &= a_1(t,y)p_i(t,y),\\ p_{i-1}^{\{i\}}(t,y) &= \frac{\partial p_{i-1}}{\partial t} - \mathcal{L}_2^* p_{i-1}\end{aligned}$$

The Coefficients $C_{i,2}(t)$, and $C_{i,1}(t)$ can be found using the periodicity of p_i and the auxiliary condition.

We define a "stretched" time variable

$$\tau = \frac{t}{\varepsilon}.$$

Substituting Q_n^ε into Equation (5.21), we get

$$\frac{\partial}{\partial\tau}Q_n^\varepsilon(\tau,y) = \mathcal{L}_1^*(\varepsilon\tau,y)Q_n^\varepsilon(\tau,y) + \varepsilon\mathcal{L}_2^*(\varepsilon\tau,y)Q_n^\varepsilon(\tau,y). \tag{5.26}$$

Expanding the coefficient functions in the operator $\mathcal{L}_i^*$ as Taylor series in the time variable, we arrive at

$$\begin{aligned}\frac{\partial}{\partial\tau}\sum_{i=0}^n \varepsilon^i q_i(t/\varepsilon,y) = &\left(\sum_{j=1}^i \frac{(\varepsilon\tau)^j}{j!}\mathcal{L}_1^{*,(j)}(0,y)\right)\sum_{i=0}^n \varepsilon^i q_i(t/\varepsilon,y)\\ &+ \varepsilon\left(\sum_{j=1}^i \frac{(\varepsilon\tau)^j}{j!}\mathcal{L}_2^{*,(j)}(0,y)\right)\sum_{i=0}^n \varepsilon^i q_i(t/\varepsilon,y),\end{aligned}$$

where $\mathcal{L}_i^{*,(j)}(0,y)$ denotes the partial derivative of coefficient function of operator $\mathcal{L}_i^*$ of order j with respect to the time variable. Equating like powers of ε leads to the

equations,

$$\begin{aligned}
\frac{\partial}{\partial\tau}q_0(\tau,y) &= \mathcal{L}_1^*(0,y)q_0(\tau,y),\\
&\vdots\\
\frac{\partial}{\partial\tau}q_n(\tau,y) &= \mathcal{L}_1^*(0,y)q_n(\tau,y) + \sum_{j=1}^{n}\frac{s^j}{j!}\mathcal{L}_1^{*,(j)}(0,x)q_{n-j}(s,x)\\
&\quad + \sum_{j=0}^{n-1}\frac{s^j}{j!}\mathcal{L}_2^{*,(j)}(0,x)q_{n-j-1}(s,x),
\end{aligned} \tag{5.27}$$

with initial conditions

$$q_n(Y,0) = \begin{cases} -p_n(Y,0), & \text{for } n\neq 0\\ g_0 - p_n(Y,0), & \text{for } n=0. \end{cases} \tag{5.28}$$

In order to obtain the solutions q_i, we need the following lemma.

Lemma 5.2.2. *Consider the problem*

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \mathcal{L}^{*,\varepsilon}(t,y),\\
u(0,y) &= u_0,\\
u_0(y+1) &= u_0(y).
\end{aligned}$$

where

$$\mathcal{L}^{*,\varepsilon}(t,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}(a(t,y)\cdot) - \frac{\partial}{\partial y}(b(t,y)\cdot).$$

Suppose that $a(t,y)$ and $b(t,y)$ satisfy the assumptions (A1) and (A2). Then there exists a fundamental solution G for the problem.

For proof of this lemma, we refer to Lemma 4.1 in [15].

Using the lemma above, q_i can be represented in terms of the fundamental solution G of the corresponding parabolic equation in Lemma 5.2.2:

$$\begin{aligned}
q_0(\tau,y) &= \int_0^1 G(x,\tau,y)q_0(0,x)dx,\\
q_i(\tau,y) &= \int_0^1 G(x,\tau,y)q_i(0,x)dx + \int_0^\tau\int_0^1 G(x,\tau-s,y)f_i(s,x)dxds,
\end{aligned} \tag{5.29}$$

where f_i is the inhomogeneous term,

$$f_i(s,x) = \sum_{j=1}^{i}\frac{s^j}{j!}\mathcal{L}_1^{*,(j)}(0,x)q_{i-j}(s,x) + \sum_{j=0}^{i-1}\frac{s^j}{j!}\mathcal{L}_2^{*,(j)}(0,x)q_{i-j-1}(s,x)$$

for $i = 1, \ldots, n$.

Define a differential operator $\mathcal{F}^\varepsilon$ by

$$\mathcal{F}^\varepsilon f := \varepsilon \frac{\partial f}{\partial t} - \mathcal{L}^{*,\varepsilon} f.$$

Lemma 5.2.3. *Let $v^\varepsilon(\cdot, y)$ be continuous on $[0,T] \times [0,1]$ and Lipschitz with respect to $t \in [0,T]$ uniformly in y, and let f^ε be the solution to the problem*

$$\begin{aligned}
\mathcal{F}^\varepsilon f^\varepsilon &= v^\varepsilon(t,y), \\
f^\varepsilon(0,y) &= 0, \quad for\, y \in [0,1], \\
v^\varepsilon(t,y+1) &= v^\varepsilon(t,y),\ for\ each\, (t,y) \in [0,T] \times [0,1], \\
\int_0^1 v^\varepsilon(t,y)dy &= 0, \quad for\ each\, t \in [0,T], \\
\sup_{(t,y)\in[0,T]\times[0,1]} &| \, v^\varepsilon(t,y) \, | = O(\varepsilon^{n+1}).
\end{aligned}$$

Then for $\varepsilon > 0$ sufficiently small, under conditions (A1) and (A2),

$$\sup_{(t,y)\in[0,T]\times[0,1]} | \, f^\varepsilon(t,y) \, | = O(\varepsilon^n).$$

For the proof of this Lemma, we refer to Lemma 6.2 in [15].

With the results obtained about p_i in (5.25) and q_i in (5.29) above, we now turn to the error analysis of the approximate solution (5.22). Define

$$r_n^\varepsilon(t,y) := p^\varepsilon(t,y) - \left(P_n^\varepsilon(t,y) + Q_n^\varepsilon(t,y)\right).$$

Proposition 5.2.4. *Assume that conditions (A1) and (A2) are satisfied. Then*

$$\sup_{(t,y)\in[0,T]\times[0,1]} | \, r_n^\varepsilon(t,y) \, | = O(\varepsilon^n).$$

Proof. First consider the case $n = 1$. We have

$$\begin{aligned}
\mathcal{F}^\varepsilon r_1^\varepsilon = &-\varepsilon^2 \frac{\partial p_1(t,y)}{\partial t} - \varepsilon^2 \frac{1}{2} \frac{\partial b_2(t,y) p_1(t,y)}{\partial y} + \varepsilon^2 \frac{\partial^2 a_2(t,y) p_1(t,y)}{\partial y^2} \\
&+ \varepsilon^2 \frac{1}{2} \frac{\partial^2 a_2(\varepsilon\tau, y) q_1(t/\varepsilon, y)}{\partial y^2} - \varepsilon^2 \frac{\partial b_2(\varepsilon\tau, y) q_1(t/\varepsilon, y)}{\partial y} \\
&+ \frac{1}{2} \frac{\partial^2}{\partial y^2} \left(\left(a_1(\varepsilon\tau) - a_1(0,y) - \varepsilon\tau \frac{\partial a_1(0,y)}{\partial t} \right) q_0(\tau, y) \right) \\
&- \frac{\partial}{\partial y} \left(\left(b_1(\varepsilon\tau) - b_1(0,y) - \varepsilon\tau \frac{\partial b_1(0,y)}{\partial t} \right) q_0(\tau, y) \right) \\
&+ \frac{\varepsilon}{2} \frac{\partial^2}{\partial y^2} \left(\left(a_1(\varepsilon\tau) - a_1(0,y) \right) q_1(\tau, y) \right) \\
&- \varepsilon \frac{\partial}{\partial y} \left(\left(b_1(\varepsilon\tau) - b_1(0,y) \right) q_1(\tau, y) \right) \\
&+ \frac{\varepsilon}{2} \frac{\partial^2}{\partial y^2} \left(\left(a_2(\varepsilon\tau) - a_2(0,y) \right) q_0(\tau, y) \right) \\
&- \varepsilon \frac{\partial}{\partial y} \left(\left(b_2(\varepsilon\tau) - b_2(0,y) \right) q_0(\tau, y) \right).
\end{aligned} \tag{5.30}$$

By the condition (A1),

$$\left| \varepsilon^2 \frac{\partial p_1(t,y)}{\partial t} \right| = O(\varepsilon^2), \text{ uniformly in } t \in [0,T] \text{ and } y \in [0,1].$$

Similarly,

$$\begin{aligned}
\sup_{(t,y)\in[0,T]\times[0,1]} \left| \varepsilon^2 \frac{\partial^2 a_2(t,y) p_1(t,y)}{\partial y^2} \right| &= O(\varepsilon^2), \\
\sup_{(t,y)\in[0,T]\times[0,1]} \left| \varepsilon^2 \frac{\partial^2 b_2(t,y) p_1(t,y)}{\partial y^2} \right| &= O(\varepsilon^2).
\end{aligned}$$

Using the exponential decaying property of q_i and *a priori* estimates for $\frac{\partial q_i}{\partial y}$ and $\frac{\partial^2 q_i}{\partial y^2}$,

$$\left| \varepsilon^2 \frac{1}{2} \frac{\partial^2 a_2(\varepsilon\tau, y) q_1(t/\varepsilon, y)}{\partial y^2} - \varepsilon^2 \frac{\partial b_2(\varepsilon\tau, y) q_1(t/\varepsilon, y)}{\partial y} \right| = O(\varepsilon^2).$$

By virtue of the Taylor expansion and the decaying property of $\frac{\partial^n q_i}{\partial y^n}$ for $n = 0, 1, 2$, we have

$$\begin{aligned}
\left| \frac{\varepsilon}{2} \frac{\partial^2}{\partial y^2} \left(\left(a_1(\varepsilon\tau) - a_1(0,y) \right) q_1(\tau, y) \right) \right| &\le K\varepsilon^2\tau \left| \frac{\partial^3}{\partial y^2 \partial t} a_1(0,y) \right| |q_1(\tau, y)| \\
&\quad + K\varepsilon^2\tau \left| \frac{\partial^2}{\partial y \partial t} a_1(0,y) \right| \left| \frac{\partial}{\partial y} q_1(\tau, y) \right| \\
&\quad + K\varepsilon^2\tau \left| \frac{\partial}{\partial t} a_1(0,y) \right| \left| \frac{\partial^2}{\partial y^2} q_1(\tau, y) \right| + O(\varepsilon^3) \\
&\le K\varepsilon^2\tau \exp(-\gamma\tau) \\
&= O(\varepsilon^2).
\end{aligned}$$

We can get similar results for the other terms in Equation (5.30). Putting the estimates together gives

$$\sup_{(t,y)\in[0,T]\times[0,1]} |\, \mathcal{F}^{\varepsilon} r_1^{\varepsilon}(t,y)\,| = O(\varepsilon^2).$$

By Lemma 5.2.2,

$$\sup_{(t,y)\in[0,T]\times[0,1]} |\, r_1^{\varepsilon}(t,y)\,| = O(\varepsilon).$$

□

Next we apply the singular perturbation method to some special examples of dY_t.

5.2.2 Fast Mean Reverting Process

We assume the following dynamics of Y_t:

$$dY_t = \frac{1}{\varepsilon}(\kappa(t) - Y_t)dt + \frac{\sigma}{\sqrt{\varepsilon}}dW_t, \tag{5.31}$$

where $\varepsilon \ll 1$. This leads to the Kolmogorov forward equation,

$$\begin{aligned}\frac{\partial P_k}{\partial T} = \frac{1}{\varepsilon}\left(1 - (\kappa(T) - Y)\frac{\partial}{\partial Y} + v\frac{\partial^2}{\partial Y^2}\right)P_k \\ + \lambda(Y, k-1, T)P_{k-1} - \lambda(Y, k, T)P_k,\end{aligned} \tag{5.32}$$

where $v = \frac{\sigma^2}{2}$ and $k \geq 0$. At this stage, we assume that we already know P_{k-1}.

Define

$$\begin{aligned}\mathcal{L}_0 &:= 1 + (-\kappa(T) + Y)\frac{\partial}{\partial Y} + v\frac{\partial^2}{\partial Y^2} = -\frac{\partial}{\partial Y}\left((\kappa(T) - Y)\cdot\right) + v\frac{\partial^2}{\partial Y^2} \\ \mathcal{L}_1 &:= -\lambda(Y, k, T).\end{aligned} \tag{5.33}$$

Thus we can rewrite Equation (5.32) as

$$\frac{\partial P_k}{\partial T} = \frac{1}{\varepsilon}\mathcal{L}_0 P_k + \mathcal{L}_1 P_k + \lambda(Y, k-1, T)P_{k-1}. \tag{5.34}$$

We construct $n+1$ terms of finite power series of ε with two different time scale. One is called a regular expansion $P_k^{(n)}$ and the other is called a singular expansion $Q_k^{(n)}$.

$$P_k^{(n)} := \sum_{i=0}^{n} \varepsilon^i a_k^{(i)}(Y, T), \tag{5.35}$$

$$Q_k^{(n)} := \sum_{i=0}^{n} \varepsilon^i b_k^{(i)}(Y, \tau), \quad \text{where} \quad \tau = \frac{T}{\varepsilon}. \tag{5.36}$$

Substituting $P_k^{(n)} + Q_k^{(n)}$ into Equation (5.34) and separating terms into regular part with inhomogeneous term and singular part, we have

$$\frac{\partial P_k^{(n)}}{\partial T} = \frac{1}{\varepsilon}\mathcal{L}_0 P_k^{(n)} + \mathcal{L}_1 P_k^{(n)} + \lambda(Y, k-1, T)P_{k-1}, \tag{5.37}$$

$$\frac{\partial Q_k^{(n)}}{\partial T} = \frac{1}{\varepsilon}\mathcal{L}_0 Q_k^{(n)} + \mathcal{L}_1 Q_k^{(n)}. \tag{5.38}$$

First, let us look at the regular expansion. Following the approach in [15], we expand the solution of differential equation (5.34) in powers of ε as

$$P_k^{(n)} = a_k^{(0)} + \varepsilon a_k^{(1)} + \varepsilon^2 a_k^{(2)} + \cdots + \varepsilon^n a_k^{(n)}. \tag{5.39}$$

Now we substitute (5.39) into Equation (5.37) and collect powers of ε. In this way we obtain a series of simpler equations as the functions $a_k^{(0)}, a_k^{(1)}, a_k^{(2)}, \ldots$. Since ε is assumed to be very small, we may view the solution as a perturbation of $a_k^{(0)}$:

$$\begin{aligned}\frac{\partial(a_k^{(0)} + \varepsilon a_k^{(1)} + \varepsilon^2 a_k^{(2)} + \cdots + \varepsilon^n a_k^{(n)})}{\partial T} &= \frac{1}{\varepsilon}\mathcal{L}_0(Y,T)(a_k^{(0)} + \varepsilon a_k^{(1)} + \varepsilon^2 a_k^{(2)} + \cdots + \varepsilon^n a_k^{(n)}) \\ &\quad + \mathcal{L}_1(Y,j)(a_k^{(0)} + \varepsilon a_k^{(1)} + \varepsilon^2 a_k^{(2)} + \cdots + \varepsilon^n a_k^{(n)}) \\ &\quad + \lambda(Y, k-1, T)P_{k-1}(T,Y)1_{\{k>0\}}.\end{aligned} \tag{5.40}$$

By examining the coefficients of powers of ε in (5.40), we see that

$$\begin{aligned}\mathcal{L}_0 a_k^{(0)} &= 0, \\ \frac{\partial a_k^{(0)}}{\partial T} &= \mathcal{L}_0 a_k^{(1)} + \mathcal{L}_1 a_k^{(0)} + \lambda(Y, k-1, T)P_{k-1}(T,Y)1_{\{k>0\}}, \\ \frac{\partial a_k^{(1)}}{\partial T} &= \mathcal{L}_0 a_k^{(2)} + \mathcal{L}_1 a_k^{(1)}, \\ \frac{\partial a_k^{(2)}}{\partial T} &= \mathcal{L}_0 a_k^{(3)} + \mathcal{L}_1 a_k^{(2)}, \\ &\vdots \\ \frac{\partial a_k^{(n-1)}}{\partial T} &= \mathcal{L}_0 a_k^{(n)} + \mathcal{L}_1 a_k^{(n-1)},\end{aligned} \tag{5.41}$$

with conditions

$$\int_{-\infty}^{\infty} a_k^{(n)}(T,Y)dY = \begin{cases} 1_{\{k=0\}} & \text{for } n = 0. \\ 0 & \text{for } n > 0. \end{cases} \tag{5.42}$$

We look at the first equation in (5.41).

Proposition 5.2.5. *The equation*

$$-\frac{\partial}{\partial Y}\left((\kappa(T)-Y)a_k^{(0)}\right)+v\frac{\partial^2}{\partial Y^2}a_k^{(0)}=0 \tag{5.43}$$

with auxiliary condition $\int_{-\infty}^{\infty}a_k^{(0)}(T,Y)dY=1_{\{k=0\}}$ *has a smooth and bounded solution.*

Proof. Equation (5.43) can be rewritten as

$$\frac{\partial}{\partial Y}\left(\frac{\partial}{\partial Y}a_k^{(0)}-\frac{1}{v}(\kappa(T)-Y)a_k^{(0)}\right)=0. \tag{5.44}$$

Then we know that from (5.44)

$$\frac{\partial}{\partial Y}a_k^{(0)}-\frac{1}{v}(\kappa(T)-Y)a_k^{(0)}$$

is independent of Y, so

$$\frac{\partial}{\partial Y}a_k^{(0)}-\frac{1}{v}(\kappa(T)-Y)a_k^{(0)}=C_1(T). \tag{5.45}$$

Define

$$H(T,Y):=-\frac{1}{v}(\kappa(T)Y-\frac{1}{2}Y^2).$$

By the integrating factor method, we have

$$\begin{aligned}\exp\left(H(T,Y)\right)\frac{\partial}{\partial Y}a_k^{(0)}-\frac{1}{v}(\kappa(T)-Y)\exp\left(H(T,Y)\right)a_k^{(0)}&=\exp\left(H(T,Y)\right)C_1(T)\\ \frac{\partial}{\partial Y}\left(\exp\left(H(T,Y)\right)a_k^{(0)}\right)&=\exp\left(H(T,Y)\right)C_1(T).\end{aligned} \tag{5.46}$$

By (5.46), $a_k^{(0)}$ can be obtained as

$$a_k^{(0)}(Y,T)=\exp\left(-H(T,Y)\right)C_2(T)+\exp\left(-H(T,Y)\right)C_1(T)\int_{Y_0}^{Y}\exp\left(H(T,s)\right)ds. \tag{5.47}$$

To have boundedness at infinity, we must have the constant C_1 equal to zero. We determine the constant C_2 using the auxiliary condition. □

In next proposition, we find the $a_k^{(1)}$ from the second equation in (5.41).

Proposition 5.2.6. *The equation*

$$\frac{\partial a_k^{(0)}}{\partial T}=\mathcal{L}_0a_k^{(1)}+\mathcal{L}_1a_k^{(0)}+\lambda(Y,k-1,T)P_{k-1}1_{\{k>0\}} \tag{5.48}$$

with auxiliary condition $\int_{-\infty}^{\infty}a_k^{(1)}(Y,T)dY=0$ *has a smooth solution.*

Proof. Rearranging terms in (5.48), we have

$$\mathcal{L}_0 a_k^{(1)} = \frac{\partial a_k^{(0)}}{\partial T} - \mathcal{L}_1 a_k^{(0)} - \lambda(Y, k-1, T) P_{k-1} 1_{\{k>0\}}. \tag{5.49}$$

Let $G_1(T, Y)$ denote the right hand side of this equation.

Then (5.49) is equivalent to

$$\frac{\partial}{\partial Y}\left(\frac{\partial}{\partial Y} a_k^{(1)} - \frac{1}{v}(\kappa(T) - Y) a_k^{(1)}\right) = \frac{1}{v} G_1(T, Y). \tag{5.50}$$

Integrating both sides of (5.50) with respect to Y gives

$$\frac{\partial}{\partial Y} a_k^{(1)} - \frac{1}{v}(\kappa(T) - Y) a_k^{(1)} = \int_{Y_0}^{Y} \frac{1}{v} G_1(T, s) ds + C_1(T)$$

for some Y-independent function C_1. Thus,

$$\begin{aligned} &\exp\left(H(T, Y)\right) \frac{\partial}{\partial Y} a_k^{(1)} - \frac{1}{v}(\kappa(T) - Y) \exp\left(H(T, Y)\right) a_k^{(1)} \\ &\quad = \exp\left(H(T, Y)\right) \left(\int_{Y_0}^{Y} \frac{1}{v} G_1(T, s) ds + C_1(T)\right). \end{aligned} \tag{5.51}$$

The function $H(T, Y)$ is same as in Proposition 5.2.5. Then (5.51) becomes

$$\frac{\partial}{\partial Y}\left(\exp\left(H(T, Y)\right) a_k^{(1)}\right) = \exp\left(H(T, Y)\right) \left(\int_{Y_0}^{Y} \frac{1}{v} G_1(T, s) ds + C_1(T)\right). \tag{5.52}$$

By integrating both sides of (5.52) with respect to Y, we find $a_k^{(1)}$ to be

$$\begin{aligned} a_k^{(1)}(Y, T) = &\exp\left(-H(T, Y)\right) C_2(T) + \exp\left(-H(T, Y)\right) \times \\ &\left(\int_{Y_0}^{Y} \exp\left(H(T, u)\right) \left(\int_{Y_0}^{u} \frac{1}{v} G_1(T, s) ds + C_1(T)\right) du\right). \end{aligned}$$

Determination of C_1 and C_2 should ensure the integrability of $a_k^{(1)}$. □

Now we have found the first two terms $a_k^{(0)}, a_k^{(1)}$ in the regular expansion (5.39) of $P_k^{(n)}$.

Next we have $Q_k^{(n)}$ for the singular part of an approximation of P_k. As in the regular expansion, we assume that the backward time t is equal to zero and that $L_t = 0$. We have

$$Q_k^{(n)} = \sum_{i=0}^{n} \varepsilon^i b_k^{(i)}(Y, \tau), \quad \text{where} \quad \tau = \frac{T}{\varepsilon}. \tag{5.53}$$

We rewrite the partial differential equation (5.34) in terms of the stretched time variable τ:

$$\frac{1}{\varepsilon}\frac{\partial Q_k^{(n)}}{\partial \tau} = \frac{1}{\varepsilon}\mathcal{L}_0(Y,\varepsilon\tau)Q_k^{(n)} + \mathcal{L}_1(Y,\varepsilon\tau)Q_k^{(n)}. \tag{5.54}$$

Because the coefficient functions of the operators $\mathcal{L}_0$, $\mathcal{L}_1$ include ε, we want to use their formal Taylor series to collect terms according to the powers of ε. For the operators $\mathcal{L}_1$ and $\mathcal{L}_2$, we have

$$\mathcal{L}_m(\varepsilon\tau, Y) = \sum_{s=0}^{\infty} \frac{(\varepsilon\tau)^s}{s!}\mathcal{L}_m^{(s)}(Y,0) \qquad m = 0, 1. \tag{5.55}$$

Each of $\mathcal{L}_m^{(s)}(0,Y)$ denotes the partial derivative of order s with respect to the time variable calculated at $\tau = 0$. Using the Taylor series (5.55) above, we have

$$\begin{aligned}\mathcal{L}_0 &= -\frac{\partial}{\partial Y}\left(A(T,Y)\cdot\right) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial Y^2}, \quad \text{where} \quad A(T,Y) := \frac{1}{\varepsilon}(\kappa(T) - Y) \\ &= \frac{\sigma^2}{2}\frac{\partial^2}{\partial Y^2} - \frac{\partial}{\partial Y}\left(\left\{A(0,Y) + \frac{\varepsilon\tau}{1!}\frac{\partial A}{\partial T}(0,Y) + \frac{(\varepsilon\tau)^2}{2!}\frac{\partial^2 A}{\partial T^2}(0,Y) + \cdots\right\}\cdot\right).\end{aligned} \tag{5.56}$$

Substituting Equation (5.56) into Equation (5.54) gives

$$\frac{1}{\varepsilon}\frac{\partial Q_k^{(n)}}{\partial \tau} = \frac{1}{\varepsilon}\left(\sum_{s=0}^{\infty}\frac{(\varepsilon\tau)^s}{s!}\mathcal{L}_0^{(s)}(Y,0)\right)Q_k^{(n)} + \left(\sum_{s=0}^{\infty}\frac{(\varepsilon\tau)^s}{s!}\mathcal{L}_1^{(s)}(Y,0)\right)Q_k^{(n)}. \tag{5.57}$$

Equating the coefficients of ε^i in Equation (5.57), we obtain

$$\begin{aligned}\frac{\partial b_k^{(0)}}{\partial \tau} &= \mathcal{L}_0^{(0)}(0,Y)b_k^{(0)}, \\ \frac{\partial b_k^{(1)}}{\partial \tau} &= \mathcal{L}_0^{(0)}(0,Y)b_k^{(1)} + \tau\mathcal{L}_0^{(1)}(0,Y)b_k^{(0)} + \mathcal{L}_1^{(0)}(0,Y)b_k^{(0)}, \\ \frac{\partial b_k^{(2)}}{\partial \tau} &= \mathcal{L}_0^{(0)}(0,Y)b_k^{(2)} + \tau\mathcal{L}_0^{(1)}(0,Y)b_k^{(1)} + \frac{\tau^2}{2!}\mathcal{L}_0^{(2)}(0,Y)b_k^{(0)} + \mathcal{L}_1^{(0)}(0,Y)b_k^{(1)} \\ &\qquad + \tau\mathcal{L}_1^{(1)}(0,Y)b_k^{(0)}, \\ &\vdots \\ \frac{\partial b_k^{(n)}}{\partial \tau} &= \sum_{i=0}^{n}\frac{\tau^i}{i!}\mathcal{L}_0^{(i)}b_{n-i} + \sum_{i=0}^{n-1}\frac{\tau^i}{i!}\mathcal{L}_1^{(i)}b_{n-1-i},\end{aligned} \tag{5.58}$$

with initial condition

$$b_k^{(n)}(Y,0) = \begin{cases} -a_k^{(n)}(Y,0), & \text{for } n \neq 0, \\ g_0 - a_k^{(n)}(Y,0), & \text{for } n = 0. \end{cases} \tag{5.59}$$

We would like to solve the system of (5.58) to find the singular part of (5.53). In order to do this, we use the fundamental solution of the parabolic differential operator $\mathcal{L}_0^{(0)}$,

$$\mathcal{L}_0^{(0)}(0,Y) = -\frac{\partial}{\partial Y}\left(A(0,Y)\cdot\right) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial Y^2}.$$

We now discuss the fundamental solution of the general parabolic operator we are going to use here.

Define

$$\mathcal{L}\left(x,t,\frac{\partial}{\partial t}\right)u := \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x,t)\frac{\partial u}{\partial x_i} + a(x,t)u, \tag{5.60}$$

on the domain $D_{n+1}^{(T)} := \mathbb{R}^n \times (0,T)$ with uniformly parabolic and real smooth coefficients. In case of unbounded domain, we need to restrict the growth of the solution as $|x| \to \infty$.

For the construction of fundamental solution, we use the results of the fundamental solution $Z(x,\xi,t,\tau)$ in Chapter 4 of the book [16].

$$\begin{aligned} \mathcal{L}(x,t,\frac{\partial}{\partial t})Z(x,\xi,t,\tau) = 0 \text{ in}\, \mathbb{R}^n \times (\tau,T) \\ Z(x,\xi,t,\tau) = \delta(x-\xi) \text{ on}\, \mathbb{R}^n \times \{t=\tau\}. \end{aligned} \tag{5.61}$$

and $Z(x,\xi,t,\tau)$ is bounded for $|x| \to \infty$. They look for the fundamental solution Z in the form

$$Z(x,\xi,t,\tau) = Z_0(x,x-\xi,t,\tau) + \int_\tau^t d\lambda \int_{\mathbb{R}^n} Z_0(x-y,y,t,\lambda)Q(y,\xi,\lambda,\tau)dy,$$

For $t > \tau$, the function Z_0 has the form

$$\begin{aligned} Z_0(x-\xi,\xi,t,\tau) = &\frac{1}{[4\pi(t-\tau)]^{\frac{n}{2}}(\det A(\xi,\tau))^{\frac{1}{2}}} \\ &\times \exp\left(-\frac{1}{4(t-\tau)}\sum_{i,j=1}^{n} A^{(i,j)}(\xi,\tau)(x_i-\xi_i)(x_j-\xi_j)\right). \end{aligned}$$

Here, $A(\xi,\tau)$ is the matrix whose components are given by the leading coefficients $a_{ij}(\xi,\tau)$ of the operator, while the $A^{(i,j)}(\xi,\tau)$ are the components of the inverse matrix $A^{-1}(\xi,\tau)$. For $t < \tau$, we set $Z_0(x-\xi,\xi,t,\tau) = 0$. Here, the function Q is determined by a calculation to satisfy the equation (5.60). The function Q can be found by solving

the integral equation. For the detailed construction of function Q, we refer to §11 in Chapter 4 of the book [16],

$$Q(x,\xi,t,\tau)+\int_\tau^t d\lambda\int_{\mathbb{R}^n}K(x,y,t,\lambda)Q(y,\xi,\lambda,\tau)dy+K(x,\xi,t,\tau)=0, \tag{5.62}$$

where

$$K(x,y,t,\lambda)=\sum_{i,j}[a_{ij}(y,\lambda)-a_{ij}(x,t)]\frac{\partial^2 Z_0(x-y,y,t,\lambda)}{\partial x_i\partial x_j}$$
$$+\mathcal{L}_1 Z_0(x-y,y,t,\lambda),$$

and

$$\mathcal{L}_1=\sum_{i=1}^n a_i(x,t)\frac{\partial}{\partial x_i}+a_k^{(0)}(x,t).$$

We consider the Cauchy problem(see §14 in Chapter4 in [16] for more detailed discussion.) in the domain $D_{n+1}^{(T)}$:

$$\begin{cases}\mathcal{L}(x,t)v(x,t) &= f(x,t)\\ v|_{t=0} &= \varphi\end{cases} \tag{5.63}$$

Assume that f satisfies a Hölder condition in all of its arguments and φ is continuous and bounded. In this case, the solution of Cauchy problem (5.63) can be written in the form of a sum of two potentials with kernel Z:

$$v(x,t)=\int_0^t d\tau\int_{\mathbb{R}^n}Z(x,\xi,t,\tau)f(\xi,\tau)d\xi+\int_{\mathbb{R}^n}Z(x,\xi,t,0)\varphi(\xi)d\xi. \tag{5.64}$$

In order to ensure the convergence of the integrals in (5.64), we require that the functions f and φ do not increase too rapidly as $|x|\to\infty$. It is sufficient to require that they increase no faster than a function e^{ax^2}.

Going back to our problem of finding $b_k^{(n)}$ in equation (5.58), we can use the solution (5.64) of the Cauchy problem to obtain the following result.

Proposition 5.2.7. *Let $b_k^{(n)}$ be a solution to the parabolic partial differential equation*

$$\frac{\partial b_k^{(n)}}{\partial\tau}=\sum_{i=0}^n\frac{\tau^i}{i!}\mathcal{L}_0^{(i)}b_{n-i}+\sum_{i=0}^{n-1}\frac{\tau^i}{i!}\mathcal{L}_0^{(i)}b_k^{(n-1-i)}.$$

Then $b_k^{(n)}$ can be represented in a integral form using the fundamental solution.

Let $G(t,y;\tau,Y)$ denote the fundamental solution for the operator $\mathcal{L}_0^{(0)}(0,Y)$. We have

$$\begin{aligned} b_k^{(0)}(\tau,Y) &= \int_{-\infty}^{\infty} G(0,y;\tau,Y) b_k^{(0)}(0,y)dy \\ b_k^{(1)}(\tau,Y) &= \int_{-\infty}^{\infty} G(0,y;\tau,Y) b_k^{(1)}(0,y)dy + \int_0^{\tau} dt \int_{-\infty}^{\infty} G(t,y;\tau,Y) \\ &\quad \times \left(\tau \mathcal{L}_0^{(1)}(0,y) b_0(t,y) + \mathcal{L}_1^{(0)}(0,y) b_0(t,y)\right) dy. \end{aligned} \tag{5.65}$$

Similarly, we can construct $b_k^{(n)}(\tau,Y)$ for different n.

Vector-valued Fast Mean Reverting Process

When we have a system of PDE, each PDE of P_k has inhomogeneous term involved with P_{k-1} for $k = 1,\ldots,N$. We assume that exact solution of P_{k-1} is available when we approximate the solution of P_k. However this assumption is not quite right because we only have an approximate solution to each PDE for P_k and do not know the exact solution. Thus we use the vector-valued PDE instead of scalar-valued PDE so we automatically have the approximated value of P_{k-1} when we calculate the approximate value of P_k.

Remember that we approximated the solution P_k of the equation (5.32) with the regular formula $P_k^{(n)}$ and the singular term $Q_k^{(n)}$. Thus we have for $k = 0,\ldots,N$,

$$\begin{aligned} P_k &\approx P_k^{(n)} + Q_k^{(n)} \\ &= a_k^{(0)}(T,Y) + \varepsilon a_k^{(1)}(T,Y) + \cdots + \varepsilon^n a_k^{(n)}(T,Y) \\ &\quad + b_k^{(0)}(T,Y) + \varepsilon b_k^{(1)}(\tau,Y) + \cdots + \varepsilon^n b_k^{(n)}(\tau,Y). \end{aligned} \tag{5.66}$$

We rewrite the equation (5.32) using vector and matrix notation.

$$\frac{\partial \vec{P}}{\partial T} = \frac{1}{\varepsilon} I \cdot \vec{P} - \frac{1}{\varepsilon}(\kappa(T) - Y) I \cdot \frac{\partial \vec{P}}{\partial Y} + \frac{v}{\varepsilon} I \cdot \frac{\partial^2 \vec{P}}{\partial Y^2} + \Lambda \cdot \vec{P}, \tag{5.67}$$

where we have the vector $\vec{P}(T,Y)$ whose kth component is $P_k(T,Y)$,

$$\vec{P}(T,Y) := \begin{pmatrix} P_0(T,Y) \\ \vdots \\ P_N(T,Y) \end{pmatrix}, \tag{5.68}$$

and

$$\Lambda(T,Y) := \begin{pmatrix} -\lambda(T,0) & 0 & 0 & \cdots & 0 \\ \lambda(T,0) & -\lambda(T,1) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \lambda(T,N-2) & -\lambda(T,N-1) & 0 \\ 0 & 0 & \cdots & \lambda(T,N-1) & -\lambda(T,N) \end{pmatrix},$$

and I is the $(N+1)\times(N+1)$ identity matrix.

Similarly as in the (5.66), we want to approximate the vector $\vec{P}(T,Y)$ with a regular part vector $\vec{P}^{(n)}$ and a singular part vector $\vec{Q}^{(n)}$.

$$\begin{aligned} \vec{P}(T,Y) &\simeq \vec{P}^{(n)} + \vec{Q}^{(n)} \\ &= \sum_{i=0}^{n} \varepsilon^i \vec{A}^{(i)}(T,Y) + \sum_{i=0}^{n} \varepsilon^i \vec{B}^{(i)}(\tau,Y), \end{aligned} \tag{5.69}$$

where

$$\vec{P}^{(n)}(T,Y) = \begin{pmatrix} P_0^{(n)} \\ \vdots \\ P_N^{(n)} \end{pmatrix} = \sum_{i=0}^{n} \varepsilon^i \vec{A}^{(i)}(T,Y), \tag{5.70}$$

$$\vec{Q}^{(n)}(\tau,Y) = \begin{pmatrix} Q_0^{(n)} \\ \vdots \\ Q_N^{(n)} \end{pmatrix} = \sum_{i=0}^{n} \varepsilon^i \vec{B}^{(i)}(\tau,Y). \tag{5.71}$$

and

$$\vec{A}^{(i)}(T,Y) := \begin{pmatrix} a_0^{(i)}(T,Y) \\ \vdots \\ a_N^{(i)}(T,Y) \end{pmatrix}, \vec{B}^{(i)}(\tau,Y) := \begin{pmatrix} b_0^{(i)}(\tau,Y) \\ \vdots \\ b_N^{(i)}(\tau,Y) \end{pmatrix}.$$

We substitute approximation formula $\vec{P}^{(n)} + \vec{Q}^{(n)}$ into Equation (5.67).

$$\begin{aligned} \frac{\partial(\vec{P}^{(n)} + \vec{Q}^{(n)})}{\partial T} &= \frac{1}{\varepsilon} I \cdot (\vec{P}^{(n)} + \vec{Q}^{(n)}) - \frac{1}{\varepsilon}(\kappa(T) - Y) I \cdot \frac{\partial(\vec{P}^{(n)} + \vec{Q}^{(n)})}{\partial Y} \\ &\quad + \frac{v}{\varepsilon} I \cdot \frac{\partial^2(\vec{P}^{(n)} + \vec{Q}^{(n)})}{\partial Y^2} + \Lambda \cdot (\vec{P}^{(n)} + \vec{Q}^{(n)}) \end{aligned} \tag{5.72}$$

Once we separate Equation (5.72) into the regular and singular part, we get

$$\frac{\partial \vec{P}^{(n)}}{\partial T} = \frac{1}{\varepsilon} I \cdot \vec{P}^{(n)} - \frac{1}{\varepsilon}(\kappa(T) - Y) I \cdot \frac{\partial \vec{P}^{(n)}}{\partial Y} + \frac{v}{\varepsilon} I \cdot \frac{\partial^2 \vec{P}^{(n)}}{\partial Y^2} + \Lambda \cdot \vec{P}^{(n)} \tag{5.73}$$

and

$$\frac{1}{\varepsilon}\frac{\partial \vec{Q}^{(n)}}{\partial \tau} = \frac{1}{\varepsilon} I \cdot \vec{Q}^{(n)} - \frac{1}{\varepsilon}(\kappa(\varepsilon\tau) - Y) I \cdot \frac{\partial \vec{Q}^{(n)}}{\partial Y} + \frac{v}{\varepsilon} I \cdot \frac{\partial^2 \vec{Q}^{(n)}}{\partial Y^2} + \Lambda \cdot \vec{Q}^{(n)}. \tag{5.74}$$

We consider the case of $n = 1$ so that $\vec{P}^{(1)} = \vec{A}^{(0)} + \varepsilon \vec{A}^{(1)}$ and $\vec{Q}^{(1)} = \vec{B}^{(0)} + \varepsilon \vec{B}^{(1)}$. First we consider the regular part (5.73).

$$\begin{aligned}\frac{\partial(\vec{A}^{(0)} + \varepsilon \vec{A}^{(1)})}{\partial T} = \frac{1}{\varepsilon} I \cdot (\vec{A}^{(0)} + \varepsilon \vec{A}^{(1)}) - \frac{1}{\varepsilon}(\kappa(T) - Y) I \cdot \frac{\partial(\vec{A}^{(0)} + \varepsilon \vec{A}^{(1)})}{\partial Y} \\ + \frac{v}{\varepsilon} I \cdot \frac{\partial^2(\vec{A}^{(0)} + \varepsilon \vec{A}^{(1)})}{\partial Y^2} + \Lambda \cdot (\vec{A}^{(0)} + \varepsilon \vec{A}^{(1)})\end{aligned} \tag{5.75}$$

Equating the coefficients of like powers of ε^i, we have

$$\begin{aligned}&\vec{A}^{(0)} - (\kappa(T) - Y) I \cdot \frac{\partial \vec{A}^{(0)}}{\partial Y} + v I \cdot \frac{\partial^2 \vec{A}^{(0)}}{\partial Y^2} = 0 \\ &\vec{A}^{(1)} - (\kappa(T) - Y) I \cdot \frac{\partial \vec{A}^{(1)}}{\partial Y} + v I \cdot \frac{\partial^2 \vec{A}^{(1)}}{\partial Y^2} = \frac{\partial \vec{A}^{(0)}}{\partial T} - \Lambda \cdot \vec{A}^{(0)}.\end{aligned} \tag{5.76}$$

Each vector-valued equation in (5.76) corresponds to a set of scalar equations for $i = 0, \cdots, N$.

$$a_i^{(0)} - (\kappa(T) - Y)\frac{\partial a_i^{(0)}}{\partial Y} + v\frac{\partial^2 a_i^{(0)}}{\partial Y^2} = 0, \tag{5.77}$$

$$a_i^{(1)} - (\kappa(T) - Y)\frac{\partial a_i^{(1)}}{\partial Y} + v\frac{\partial^2 a_i^{(1)}}{\partial Y^2} = \frac{\partial a_i^{(0)}}{\partial T} - \lambda(i-1)a_{i-1}^{(0)} + \lambda(i)a_i^{(0)}. \tag{5.78}$$

Note that the $\lambda(i-1)a_{i-1}^{(0)}$ term is not included in Equation (5.78) when $i = 0$. We notice the difference between Equation (5.78) above and Equation (5.48) which contains whole $\lambda(Y, i-1, T)P_{i-1}1_{\{i>0\}}$ term while (5.78) has the only zeroth order approximation term $\lambda(i-1)a_{i-1}^{(0)}$ of P_{i-1}. We can solve equations (5.77), (5.78) using Proposition 5.2.5, Proposition 5.2.6. We summarize the results about regular expansion into the following Proposition.

Proposition 5.2.8. *Vector of regular expansion $\vec{P}^{(1)}$ consists of two vectors $\vec{A}^{(0)}$ and $\varepsilon \times \vec{A}^{(1)}$. Vector $\vec{A}^{(0)}$ has kth component $a_k^{(0)}$ which satisfies the equation,*

$$\begin{aligned}&a_k^{(0)} - (\kappa(T) - Y)\frac{\partial a_k^{(0)}}{\partial Y} + v\frac{\partial^2 a_k^{(0)}}{\partial Y^2} = 0 \\ &\int_{-\infty}^{\infty} a_k^{(0)}(T, Y) = \begin{cases} 1 & \text{if} \quad k = 0, \\ 0 & \text{if} \quad k \neq 0. \end{cases}\end{aligned} \tag{5.79}$$

Vector $\vec{A}^{(1)}$ has kth component $a_k^{(1)}$ which satisfies the equation,

$$\begin{aligned} a_k^{(1)} - (\kappa(T) - Y)\frac{\partial a_k^{(1)}}{\partial Y} + v\frac{\partial^2 a_k^{(1)}}{\partial Y^2} &= \frac{\partial a_k^{(0)}}{\partial T} - \lambda(k-1)a_{k-1}^{(0)} + \lambda(k)a_k^{(0)}, \\ \int_{-\infty}^{\infty} a_k^{(1)} &= 0. \end{aligned} \tag{5.80}$$

Proposition 5.2.5 and Proposition 5.2.6 are used to solve each equations.

Next we consider the singular part (5.74).

$$\begin{aligned} \frac{1}{\varepsilon}\frac{\partial(\vec{B}^{(0)} + \varepsilon\vec{B}^{(1)})}{\partial\tau} = \frac{1}{\varepsilon}I \cdot (\vec{B}^{(0)} + \varepsilon\vec{B}^{(1)}) - \frac{1}{\varepsilon}(\kappa(\varepsilon\tau) - Y)I \cdot \frac{\partial(\vec{B}^{(0)} + \varepsilon\vec{B}^{(1)})}{\partial Y} \\ + \frac{v}{\varepsilon}I \cdot \frac{\partial^2(\vec{B}^{(0)} + \varepsilon\vec{B}^{(1)})}{\partial Y^2} + \Lambda \cdot (\vec{B}^{(0)} + \varepsilon\vec{B}^{(1)}). \end{aligned} \tag{5.81}$$

We use the Taylor series expansion up to order ε^2 for the functions $\kappa(\varepsilon\tau)$ and $\Lambda(\varepsilon\tau)$. We obtain

$$\begin{aligned} \frac{\partial(\vec{B}^{(0)} + \varepsilon\vec{B}^{(1)})}{\partial\tau} &= (\vec{B}^{(0)} + \varepsilon\vec{B}^{(1)}) - g(0,Y)I \cdot \frac{\partial(\vec{B}^{(0)} + \varepsilon\vec{B}^{(1)})}{\partial Y} \\ &\quad - \varepsilon\tau\frac{\partial g}{\partial T}(0,Y)I \cdot \frac{\partial(\vec{B}^{(0)} + \varepsilon\vec{B}^{(1)})}{\partial Y} + vI \cdot \frac{\partial^2(\vec{B}^{(0)} + \varepsilon\vec{B}^{(1)})}{\partial Y^2} \\ &\quad + \varepsilon(\Lambda(0,Y) + \varepsilon\tau\Lambda'(0,Y)) \cdot (\vec{B}^{(0)} + \varepsilon\vec{B}^{(1)}). \end{aligned} \tag{5.82}$$

Equating the coefficients of like powers of ε^i, from the zeroth order of ε we have

$$\frac{\partial\vec{B}^{(0)}}{\partial\tau} = \vec{B}^{(0)} - g(0,Y)I \cdot \frac{\partial\vec{B}^{(0)}}{\partial Y} + vI \cdot \frac{\partial^2\vec{B}^{(0)}}{\partial Y^2}. \tag{5.83}$$

The corresponding scalar equation is

$$\frac{\partial b_i^{(0)}}{\partial\tau} = b_i^{(0)} - g(0,Y)\frac{\partial b_i^{(0)}}{\partial Y} + v\frac{\partial^2 b_i^{(0)}}{\partial Y^2}. \tag{5.84}$$

From the first order of ε, we obtain

$$\frac{\partial\vec{B}^{(1)}}{\partial\tau} = \vec{B}^{(1)} - g(0,Y)I \cdot \frac{\partial\vec{B}^{(1)}}{\partial Y} + vI \cdot \frac{\partial^2\vec{B}^{(1)}}{\partial Y^2} - \tau\frac{\partial g}{\partial T}(0,Y)I \cdot \frac{\partial\vec{B}^{(0)}}{\partial Y} + \Lambda(0,Y) \cdot \vec{B}^{(0)}, \tag{5.85}$$

and the corresponding scalar partial differential equation is,

$$\frac{\partial b_i^{(1)}}{\partial\tau} = b_i^{(1)} - g(0,Y)\frac{\partial b_i^{(1)}}{\partial Y} + v\frac{\partial^2 b_i^{(1)}}{\partial Y^2} - \tau\frac{\partial g}{\partial T}(0,Y)\frac{\partial b_i^{(0)}}{\partial Y} + \lambda(0,Y,i)b_{i-1}^{(0)} - \lambda(0,Y,i)b_i^{(0)}. \tag{5.86}$$

We use Proposition 5.2.7 to solve Equations (5.84) and (5.86).

Proposition 5.2.9. *Vector of singular expansion $\vec{Q}^{(1)}$ consists of two vectors $\vec{B}^{(0)}$ and $\varepsilon \times \vec{B}^{(1)}$. Vector $\vec{B}^{(0)}$ has kth component $b_k^{(0)}$ which satisfies the equation,*

$$\begin{aligned} \frac{\partial b_i^{(0)}}{\partial \tau} &= b_i^{(0)} - g(0,Y)\frac{\partial b_i^{(0)}}{\partial Y} + v\frac{\partial^2 b_i^{(0)}}{\partial Y^2} \\ b_k^{(0)}(Y,0) &= g_0 - a_k^{(0)}(Y,0), \end{aligned} \tag{5.87}$$

Vector $\vec{B}^{(1)}$ has kth component $b_k^{(1)}$ which satisfies the equation,

$$\begin{aligned} \frac{\partial b_k^{(1)}}{\partial \tau} &= b_k^{(1)} - g(0,Y)\frac{\partial b_k^{(1)}}{\partial Y} + v\frac{\partial^2 b_k^{(1)}}{\partial Y^2} - \tau\frac{\partial g}{\partial T}(0,Y)\frac{\partial b_k^{(0)}}{\partial Y} \\ &\quad + \lambda(0,Y,i)b_{k-1}^{(0)} - \lambda(0,Y,k)b_k^{(0)}, \\ b_k^{(1)}(Y,0) &= -a_k^{(1)}(Y,0). \end{aligned} \tag{5.88}$$

Proposition 5.2.7 can be used to solve each equations.

5.2.3 Error Analysis

We want to estimate the error occurring in the approximation from singular perturbation, namely

$$r_k^{(n)} := P_k - (P_k^{(n)} + Q_k^{(n)}) = P_k - \sum_{i=0}^{n} \varepsilon^i a_i(Y,T) - \sum_{i=0}^{n} \varepsilon^i b_i(Y,\tau).$$

We already constructed a_i, b_i in asymptotic expansion (5.39) and here want to analyze the error $r_k^{(n)}$, using the same method as [15]. We only consider the case $n = 1$ and $k = 0$.

Proposition 5.2.10. *With the results obtained in Propositions 5.2.5 and 5.2.6 and equation* (5.65)*, we have*

$$\sup_{(T,Y)\in(0,T_M)\times(-\infty,\infty)} |r_0^{(1)}| = O(\varepsilon^1).$$

Proof. In order to prove this, we use the following modified lemma from [15].

Lemma 5.2.11. *If $\mathcal{F}^\varepsilon f^\varepsilon(t,y) = v(t,y)$ and*

$$\sup_{(t,y)\in(0,T_M)\times(-\infty,\infty)} |v(t,y)| = O(\varepsilon^{m+1}),$$

where the operator $\mathcal{F}^\varepsilon$ *is* $\varepsilon\frac{\partial}{\partial t} - \mathcal{L}_0 - \varepsilon\mathcal{L}_1$ *and* $f^\varepsilon(0,y) = 0$*, then for* $\varepsilon > 0$ *sufficiently small,*

$$\sup_{(t,y)\in(0,T_M)\times(-\infty,\infty)} |f^\varepsilon(t,y)| = O(\varepsilon^m).$$

Let

$$\mathcal{F}^\varepsilon := \varepsilon\frac{\partial}{\partial T} - \mathcal{L}_0 - \varepsilon\mathcal{L}_1$$
$$f := P_k - (P_k^{(n)} + Q_k^{(n)}).$$

Since we have $n = 1$ and $k = 0$:

$$\begin{aligned}\mathcal{F}^\varepsilon r_1 = \varepsilon\frac{\partial}{\partial T}(P_k - (a_k^{(0)} + \varepsilon a_k^{(1)}) - (b_k^{(0)} + \varepsilon b_k^{(1)})) - \mathcal{L}_0(P_k - (a_k^{(0)} + \varepsilon a_k^{(1)}) - (b_k^{(0)} + \varepsilon b_k^{(1)}))\\ - \varepsilon\mathcal{L}_1(P_k - (a_k^{(0)} + \varepsilon a_k^{(1)}) - (b_k^{(0)} + \varepsilon b_k^{(1)}))\\ = -\left(\varepsilon\frac{\partial a_k^{(0)}}{\partial T} - \mathcal{L}_0 a_k^{(0)} - \varepsilon\mathcal{L}_1 a_k^{(0)}\right) - \varepsilon\left(\varepsilon\frac{\partial a_k^{(1)}}{\partial T} - \mathcal{L}_0 a_k^{(1)} - \varepsilon\mathcal{L}_1 a_k^{(1)}\right)\\ - \left(\varepsilon\frac{\partial b_k^{(0)}}{\partial T} - \mathcal{L}_0 b_k^{(0)} - \varepsilon\mathcal{L}_1 b_k^{(0)}\right) - \varepsilon\left(\varepsilon\frac{\partial b_k^{(1)}}{\partial T} - \mathcal{L}_0 b_k^{(1)} - \varepsilon\mathcal{L}_1 b_k^{(1)}\right).\end{aligned}$$

Using the definition of operators $\mathcal{L}_0$ and $\mathcal{L}_1$ in (5.33),

$$\begin{aligned}\mathcal{F}^\varepsilon r_k^{(1)} = &-\varepsilon\frac{\partial a_k^{(0)}}{\partial T} - \frac{\partial}{\partial Y}\left((\kappa(T) - Y)a_k^{(0)}\right) + \frac{\sigma^2}{2}\frac{\partial^2 a_k^{(0)}}{\partial Y^2} - \varepsilon\lambda a_k^{(0)}\\ &- \varepsilon^2\frac{\partial a_k^{(1)}}{\partial T} - \varepsilon\frac{\partial}{\partial Y}\left((\kappa(T) - Y)a_k^{(1)}\right) + \varepsilon\frac{\sigma^2}{2}\frac{\partial^2 a_k^{(1)}}{\partial Y^2} - \varepsilon^2\lambda a_k^{(1)}\\ &- \frac{\partial b_k^{(0)}}{\partial \tau} - \frac{\partial}{\partial Y}\left((\kappa(\tau\varepsilon) - Y)b_k^{(0)}\right) + \frac{\sigma^2}{2}\frac{\partial^2 b_k^{(0)}}{\partial Y^2} - \varepsilon\lambda b_k^{(0)}\\ &- \varepsilon\frac{\partial b_k^{(1)}}{\partial \tau} - \varepsilon\frac{\partial}{\partial Y}\left((\kappa(\tau\varepsilon) - Y)b_k^{(1)}\right) + \varepsilon\frac{\sigma^2}{2}\frac{\partial^2 b_k^{(1)}}{\partial Y^2} - \varepsilon^2\lambda b_k^{(1)}.\end{aligned}$$

Here we use the facts that $a_k^{(0)}, a_k^{(1)}$ satisfy the equations

$$\begin{aligned}\mathcal{L}_0 a_k^{(0)} &= -\frac{\partial}{\partial Y}\left((\kappa(T) - Y)a_k^{(0)}\right) + \frac{\sigma^2}{2}\frac{\partial^2 a_k^{(0)}}{\partial Y^2} = 0\\ \mathcal{L}_0 a_k^{(1)} &= \frac{\partial a_k^{(0)}}{\partial T} - \mathcal{L}_1 a_k^{(0)}.\end{aligned}$$

This reduces the regular part to

$$-\varepsilon^2\frac{\partial a_k^{(1)}}{\partial T} + \varepsilon^2\mathcal{L}_1 a_k^{(1)} = \varepsilon^2(-\mathcal{L}_0 a_k^{(2)}).$$

For the singular part, we use the stretched variable $\tau = \frac{T}{\varepsilon}$ and denote $\kappa(\varepsilon\tau) - Y$ by the function $g(\varepsilon\tau, Y)$. Then the singular part becomes

$$\begin{aligned}
&-\frac{\partial b_k^{(0)}}{\partial \tau} - \frac{\partial}{\partial Y}\left((\kappa(\varepsilon\tau) - Y)b_k^{(0)}\right) + \frac{\sigma^2}{2}\frac{\partial^2 b_k^{(0)}}{\partial Y^2} - \varepsilon\lambda b_k^{(0)} \\
&\quad - \varepsilon\frac{\partial b_k^{(1)}}{\partial \tau} - \varepsilon\frac{\partial}{\partial Y}\left((\kappa(\varepsilon\tau) - Y)b_k^{(1)}\right) + \varepsilon\frac{\sigma^2}{2}\frac{\partial^2 b_k^{(1)}}{\partial Y^2} - \varepsilon^2\lambda b_k^{(1)} \\
&= -\frac{\partial b_k^{(0)}}{\partial \tau} - \frac{\partial}{\partial Y}\left(g(\varepsilon\tau, Y)b_k^{(0)}\right) + \frac{\sigma^2}{2}\frac{\partial^2 b_k^{(0)}}{\partial Y^2} - \varepsilon\lambda b_k^{(0)} \\
&\quad - \varepsilon\frac{\partial b_k^{(1)}}{\partial \tau} - \varepsilon\frac{\partial}{\partial Y}\left(g(\varepsilon\tau, Y)b_k^{(1)}\right) + \varepsilon\frac{\sigma^2}{2}\frac{\partial^2 b_k^{(1)}}{\partial Y^2} - \varepsilon^2\lambda b_k^{(1)}.
\end{aligned}$$

We can replace $-\frac{\partial b_k^{(0)}}{\partial \tau}$ and $-\varepsilon\frac{\partial b_k^{(1)}}{\partial \tau}$ with terms involving the Y derivatives using equation (5.58),

$$\begin{aligned}
\frac{\partial b_k^{(0)}}{\partial \tau} &= \mathcal{L}_0^{(0)}(Y, 0)b_k^{(0)} \\
&= -\frac{\partial}{\partial Y}(g(0, Y)b_k^{(0)}) + \frac{\sigma^2}{2}\frac{\partial^2 b_k^{(0)}}{\partial Y^2} \\
\frac{\partial b_k^{(1)}}{\partial \tau} &= \mathcal{L}_0^{(0)}(Y, 0)b_k^{(1)} + \tau\mathcal{L}_0^{(1)}(Y, 0)b_k^{(0)} + \mathcal{L}_1^{(0)}(Y, 0)b_k^{(0)} \\
&= -\frac{\partial}{\partial Y}(g(0, Y)b_k^{(1)}) + \frac{\sigma^2}{2}\frac{\partial^2 b_k^{(1)}}{\partial Y^2} - \tau\frac{\partial}{\partial Y}\left(\frac{\partial g(0, Y)b_k^{(0)}}{\partial T}\right) - \lambda(Y, 0, 0)b_k^{(0)}.
\end{aligned}$$

Then the singular term becomes

$$
\begin{aligned}
&-\frac{\partial b_k^{(0)}}{\partial \tau}-\frac{\partial}{\partial Y}\left(g(\varepsilon\tau,Y)b_k^{(0)}\right)+\frac{\sigma^2}{2}\frac{\partial^2 b_k^{(0)}}{\partial Y^2}-\varepsilon\lambda b_k^{(0)}\\
&-\varepsilon\frac{\partial b_k^{(1)}}{\partial \tau}-\varepsilon\frac{\partial}{\partial Y}\left(g(\varepsilon\tau,Y)b_k^{(1)}\right)+\varepsilon\frac{\sigma^2}{2}\frac{\partial^2 b_k^{(1)}}{\partial Y^2}-\varepsilon^2\lambda b_k^{(1)}\\
=&\frac{\partial(g(0,Y)b_k^{(0)})}{\partial Y}-\frac{\sigma^2}{2}\frac{\partial^2 b_k^{(0)}}{\partial Y^2}+\frac{\sigma^2}{2}\frac{\partial^2 b_k^{(0)}}{\partial Y^2}-\frac{\partial g(\varepsilon\tau)b_k^{(0)}}{\partial Y}-\varepsilon\lambda b_k^{(0)}\\
&+\varepsilon\frac{\partial(g(0,Y)b_k^{(1)})}{\partial Y}-\varepsilon\frac{\sigma^2}{2}\frac{\partial^2 b_k^{(1)}}{\partial Y^2}+\varepsilon\tau\frac{\partial}{\partial Y}\left(\frac{\partial g}{\partial T}(0,Y)b_k^{(0)}\right)\\
&+\varepsilon\lambda(Y,L_T,0)b_k^{(0)}+-\varepsilon\frac{\partial(g(\varepsilon\tau,Y)b_k^{(1)})}{\partial Y}+\varepsilon\frac{\sigma^2}{2}\frac{\partial^2 b_k^{(1)}}{\partial Y^2}-\varepsilon^2\lambda b_k^{(1)}\\
=&-\frac{\partial}{\partial Y}\left(\left(g(\varepsilon\tau,Y)-g(0,Y)\right)b_k^{(0)}\right)-\varepsilon\left(\left(\lambda(Y,L_T,\varepsilon\tau)-\lambda(Y,L_T,0)\right)b_k^{(0)}\right)\\
&-\varepsilon\left(\frac{\partial}{\partial Y}\left(\left(g(\varepsilon\tau,Y)-g(0,Y)\right)b_k^{(1)}\right)\right)-\varepsilon^2\lambda b_k^{(1)}+\varepsilon\tau\frac{\partial}{\partial Y}\left(\frac{\partial g}{\partial T}(0,Y)b_k^{(0)}\right)\\
=&-\frac{\partial}{\partial Y}\left(\left(g(\varepsilon\tau,Y)-g(0,Y)-\varepsilon\tau\frac{\partial g}{\partial T}(0,Y)\right)b_k^{(0)}\right)\\
&-\varepsilon\left(\left(\lambda(Y,L_T,\varepsilon\tau)-\lambda(Y,L_T,0)\right)b_k^{(0)}\right)\\
&-\varepsilon\left(\frac{\partial}{\partial Y}\left(\left(g(\varepsilon\tau,Y)-g(0,Y)\right)b_k^{(1)}\right)\right)-\varepsilon^2\lambda b_k^{(1)}
\end{aligned}
$$

Using the Taylor series expansion,

$$
\begin{aligned}
=&-\varepsilon^2\frac{\partial}{\partial Y}\left(\tau^2\frac{\partial^2 g(\varepsilon'\tau,Y)}{\partial T^2}\right)-\varepsilon^2\tau\frac{\partial\lambda(Y,L_T,\varepsilon'\tau)}{\partial T}b_k^{(0)}\\
&-\varepsilon^2\tau\frac{\partial}{\partial Y}\left(\frac{\partial g(\varepsilon'\tau,Y)}{\partial T}b_k^{(1)}\right)-\varepsilon^2\lambda b_k^{(1)}.
\end{aligned}
$$

So $|\mathcal{F}^{\varepsilon}r_1| = O(\varepsilon^2)$ if coefficients of ε^2 are bounded. In order to have this property on coefficients of ε^2, we have the restriction when choosing the function κ in (5.31).

By adapting Lemma 5.2.3, we conclude

$$\sup_{(t,y)\in(0,T_M)\times(-\infty,\infty)}|r_1| = O(\varepsilon^1).$$

□

Error Analysis for Vector-valued PDE

Here, we estimate the error for vector-valued case for Equation (5.32). Error is defined by

$$\vec{r}_k^{\,(n)} := \vec{P} - (\vec{P}^{(n)} + \vec{Q}^{(n)}).$$

Let's consider the case $n = 1$.

$$\vec{r}^{\,(1)} = \vec{P} - (\vec{P}^{(1)} + \vec{Q}^{(1)}). \tag{5.89}$$

Define a operator $\mathcal{F}^\varepsilon$ on vector $\vec{v}$.

$$\mathcal{F}^\varepsilon \vec{v} = \varepsilon \frac{\partial \vec{v}}{\partial T} + \frac{\partial}{\partial Y}\left((\kappa(T) - Y) I \cdot \vec{v}\right) - vI \cdot \frac{\partial^2 \vec{v}}{\partial Y^2} - \varepsilon \Lambda \cdot \vec{v}.$$

As in the scalar-valued error analysis before, we also want to show the components of $\mathcal{F}^\varepsilon \vec{r}^{\,(1)}$ are $O(\varepsilon^2)$. Using Equation (5.67),

$$\begin{aligned} \mathcal{F}^\varepsilon \vec{r}^{\,(1)} &= \mathcal{F}^\varepsilon \vec{P} - \mathcal{F}^\varepsilon(\vec{P}^{(1)} + \vec{Q}^{(1)}) \\ &= -\mathcal{F}^\varepsilon(\vec{P}^{(1)} + \vec{Q}^{(1)}). \end{aligned} \tag{5.90}$$

We consider $\mathcal{F}^\varepsilon \vec{P}^{(1)}$ and $\mathcal{F}^\varepsilon \vec{Q}^{(1)}$ separatively. First $\mathcal{F}^\varepsilon \vec{P}^{(1)}$ is

$$\begin{aligned} \mathcal{F}^\varepsilon \vec{P}^{(1)} &= \mathcal{F}^\varepsilon \left(\vec{A}^{(0)} + \varepsilon \vec{A}^{(1)}\right) \\ &= \varepsilon \frac{\partial \vec{A}^0}{\partial T} + \frac{\partial}{\partial Y}\left((\kappa(T) - Y) I \cdot \vec{A}^0\right) - vI \cdot \frac{\partial^2 \vec{A}^0}{\partial Y^2} - \varepsilon \Lambda \cdot \vec{A}^0 \\ &\quad + \varepsilon \left(\varepsilon \frac{\partial \vec{A}^1}{\partial T} + \frac{\partial}{\partial Y}\left((\kappa(T) - Y) I \cdot \vec{A}^1\right) - vI \cdot \frac{\partial^2 \vec{A}^1}{\partial Y^2} - \varepsilon \Lambda \cdot \vec{A}^1\right). \end{aligned} \tag{5.91}$$

Simplifying $\mathcal{F}^\varepsilon \vec{P}^{(1)}$ using Equations (5.76),

$$-\vec{A}^{(0)} + (\kappa(T) - Y) I \cdot \frac{\partial \vec{A}^{(0)}}{\partial Y} - vI \cdot \frac{\partial^2 \vec{A}^{(0)}}{\partial Y^2}$$

are canceled to zero and

$$-\varepsilon \vec{A}^{(1)} + \varepsilon(\kappa(T) - Y) I \cdot \frac{\partial \vec{A}^{(1)}}{\partial Y} - \varepsilon v I \cdot \frac{\partial^2 \vec{A}^{(1)}}{\partial Y^2}$$

becomes

$$-\varepsilon \frac{\partial \vec{A}^{(0)}}{\partial T} + \varepsilon \Lambda \cdot \vec{A}^{(0)}.$$

Thus we obtain

$$\mathcal{F}^\varepsilon \vec{P}^{(1)} = \varepsilon^2 \frac{\partial \vec{A}^{(1)}}{\partial T} - \varepsilon^2 \Lambda \cdot \vec{A}^{(1)}. \tag{5.92}$$

Under our assumption that vector $\partial \vec{A}^{(1)}/\partial T$ and vector $\Lambda \cdot \vec{A}^{(1)}$ have bounded components, $\mathcal{F}^\varepsilon \vec{P}^{(1)}$ also has bounded components. This leads to the results that $\mathcal{F}^\varepsilon \vec{P}^{(1)}$ has corresponding $O(\varepsilon^2)$ scalar parts.

Next we look at $\mathcal{F}^\varepsilon \vec{Q}^{(1)}$,

$$\begin{aligned}\mathcal{F}^\varepsilon \vec{Q}^{(1)} = &\frac{\partial \vec{B}^0}{\partial \tau} + \frac{\partial}{\partial Y}\left((\kappa(\varepsilon\tau) - Y)I \cdot \vec{B}^0\right) - vI \cdot \frac{\partial^2 \vec{B}^0}{\partial Y^2} - \varepsilon\Lambda \cdot \vec{B}^0 \\ &+ \varepsilon\left(\frac{\partial \vec{B}^1}{\partial \tau} + \frac{\partial}{\partial Y}\left((\kappa(\varepsilon\tau) - Y)I \cdot \vec{B}^1\right) - vI \cdot \frac{\partial^2 \vec{B}^1}{\partial Y^2} - \varepsilon\Lambda \cdot \vec{B}^1\right).\end{aligned} \tag{5.93}$$

We use Equations (5.83) and (5.85) to simplify (5.93). We replace the terms $\partial\vec{B}^0/\partial\tau$ and $\partial\vec{B}^1/\partial\tau$ with the right side of (5.83) and (5.85) respectively. Note that function $g(\varepsilon\tau, Y)$ below denotes $\kappa(\varepsilon\tau) - Y$.

$$\begin{aligned}\mathcal{F}^\varepsilon \vec{Q}^{(1)} = &\vec{B}^0 - g(0,Y)I \cdot \frac{\partial \vec{B}^0}{\partial Y} + vI \cdot \frac{\partial^2 \vec{B}^0}{\partial Y^2} \\ &- \vec{B}^0 + g(\varepsilon\tau, Y)I \cdot \frac{\partial \vec{B}^0}{\partial Y} - vI \cdot \frac{\partial^2 \vec{B}^0}{\partial Y^2} - \varepsilon\Lambda \cdot \vec{B}^0 \\ &+ \varepsilon\vec{B}^1 - \varepsilon g(0,Y)I \cdot \frac{\partial \vec{B}^1}{\partial Y} + \varepsilon vI \cdot \frac{\partial^2 \vec{B}^1}{\partial Y^2} - \varepsilon\tau \frac{\partial g(0,Y)}{\partial T} I \cdot \frac{\partial \vec{B}^0}{\partial Y} + \varepsilon\Lambda(0,Y) \cdot \vec{B}^0 \\ &- \varepsilon\vec{B}^1 + \varepsilon g(\varepsilon\tau, Y)\frac{\partial \vec{B}^1}{\partial Y} - \varepsilon vI \cdot \frac{\partial^2 \vec{B}^1}{\partial Y^2} - \varepsilon^2\Lambda \cdot \vec{B}^1 \\ = &-\varepsilon(\Lambda(\varepsilon\tau, Y) - \Lambda(0,Y))\vec{B}^0 - \varepsilon^2\Lambda \cdot \vec{B}^1 \\ &+ \left(g(\varepsilon\tau, Y) - g(0,Y) - \varepsilon\tau\frac{\partial g(0,Y)}{\partial T}\right) I \cdot \frac{\partial \vec{B}^0}{\partial Y} + \varepsilon\left(g(\varepsilon\tau, Y) - g(0,Y)\right) I \cdot \frac{\partial \vec{B}^1}{\partial Y}.\end{aligned} \tag{5.94}$$

By virtue of a Taylor expansion, for some $\vec{\gamma}_1, \gamma_2, \gamma_3$,

$$\begin{aligned}\Lambda(\varepsilon\tau, Y) - \Lambda(0,Y) &= \varepsilon\tau\frac{\partial}{\partial T}\Lambda(\vec{\gamma}_1, Y) \\ g(\varepsilon\tau, Y) - g(0,Y) - \varepsilon\tau\frac{\partial g(0,Y)}{\partial T} &= \frac{\varepsilon^2\tau^2}{2}\frac{\partial^2 g(\gamma_2, Y)}{\partial T^2} \\ g(\varepsilon\tau, Y) - g(0,Y) &= \varepsilon\tau\frac{\partial g(\gamma_3, Y)}{\partial T}.\end{aligned} \tag{5.95}$$

We substitute the right hand side of equations in (5.95) into (5.94) and we can see that $\mathcal{F}^\varepsilon \vec{Q}^{(1)}$ is $O(\varepsilon^2)$ provided that coefficients terms of ε^2 are bounded.

Chapter 6

Summary and Future Work

We have seen explained the structure of credit-related multi-name products and have concluded that the distribution of the cumulative loss distribution L_t plays a crucial role in pricing the fair value of those financial products.

We started with the Kolmogorov forward equation of the joint density function of the cumulative loss process L_t and the stochastic factor Y_t of its intensity λ. Since the cumulative loss has the non-decreasing property and discrete values along time, our Kolmogorov forward equation accommodates both a discrete variable and a continuous variable. Under certain dynamics of a stochastic process Y_t, we implemented the perturbation method in order to find the joint density function. We computed the first few terms of the approximation formula for the joint density function and tried to analyze the error between the approximation and the joint density function.

For future work, we can consider error analysis for different n and k. The numerical implementation of this work using such as Mathlab can provide practical uses for computationally challenging credit derivatives pricing problems.

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