PARTITIONING PROBLEMS IN DISCRETE AND COMPUTATIONAL GEOMETRY

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ABSTRACT OF THE DISSERTATION

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Many interesting problems in Discrete and Computational Geometry involve partitioning. A main question is whether a given set, or sets, may be separated into parts satisfying certain properties. Sometime we also need to find an efficient way to do it - in other words an algorithm. In this thesis, we discuss several of problems and results of this kind.

First we give a combinatorial proof of the existence and the uniqueness of the generalized ham-sandwich cut for well separated point sets in R^d, that have the weak general position property. The combinatorial proof allows us to derive an $O(n(\log n)^{d-3})$ running time algorithm to find a generalized cut for d given well separated point sets in R^d.

A second problem concerns the 6-way partition of a given convex set in R^2 using 3 lines. We reopen an old question and show that when the direction of one line is fixed, there is unique partition such that 6 regions have the same area.

In the Voronoi game two players A and B each play n points in a compact set S in R^d. They obtain scores equal to the total volume of the Voronoi cells of their points. There are one round and alternating versions of the game. We give the player A's best strategy to minimize the area of player B's one(first) Voronoi cell.

Acknowledgements

I would like to take this opportunity to thank my supervisor, Professor William Steiger, for his constant guidance, motivation and support during this research. He trained me how to approach a problem and how to establish formal proofs. He also guided me through the early years of difficulties and confusions. This thesis contains the joint work with him published in the papers [35, 36, 37, 38, 39].

Professor Mario Szegedy served on my committee. I thank him for his many penetrating questions, his interesting conjectures and helpful advices. The results in Chapter 3 is based on the joint work with him and Professor William Steiger presented in [39].

Professor Bahman Kalantari expressed his interest in discrete and computation geometry and had lots of helpful discussions with me. I also leaned convex geometry concepts in his course.

Dedication

To Qin, Nathan

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Chapter 1

Introduction

Computational Geometry is a young field of Computer Science. The research studies in this fields began in the early 1970's. Since its numerous and strong interactions with various fields such as Algorithms and Data Structures, Graphics, Combinatorial Mathematics, Euclidean Geometry, Convexity, and Optimization, Computational Geometry has attracted the interest of a large number of researchers.

On the other hand, Discrete Geometry is a relative old field which once stood for packing, covering, and tiling, but grew rapidly to include the problems in Combinatorial Geometry, Convex Polytopes, and Arrangements of geometric objects, such as points, lines, planes, circles.

Now the two fields have significant overlaps, and this overlap becomes one common practice as well: mathematicians and computer scientists are working on some of the same geometric problems and using some of the same techniques and method. Two great handbooks survey the well established results in these researching areas: one is *Handbook of Discrete and Computational Geometry* edited by Jacob E. Goodman and Joseph O'Rourke[22], another is *Handbook of Computational Geometry* edited by J.-R. Sack and J. Urrutia[32].

In Discrete and Computational Geometry, many problems involve partitioning. A main question is whether a given set, or sets, may be separated into parts satisfying some properties. Voronoi diagrams, polygon decompositions, point sets dissection are some problems of this type. For example, with Voronoi diagrams, we are given n points in R^d, the goal is a partition of R^d(or a bounded set in R^d) into n cells such that each cell is the set of points closer to one given point than all other given points. Sometime we also need to find an efficient algorithm to find the partitioning having desired properties.

In this thesis, we will discuss three main problems of the partitioning types. In this first chapter, we will give some background for these problems and describe the organization of this thesis.

We first address a problem related to the ham-sandwich theorem- itself an important partitioning result in Discrete and computational Geometry. Given d sets $S_1, S_2, ..., S_d \subset R^d$, a ham–sandwich cut is a hyperplane h that simultaneously bisects each S_i. "Bisect" means that $\mu(S_i \cap h^+) = \mu(S_i \cap h^-) < \infty$, h^+, h^- the closed halfspaces defined by h, and μ a suitable, "nice" measure on Borel sets in R^d, e.g., the volume. The well–known ham-sandwich theorem states that such a cut always exists. That is, given d measurable sets in d-dimensional space, it is always possible to find a $(d-1)$-dimensional flat that divides each set into two parts, each part having half of the total measure.

The ham-sandwich theorem takes its name from the case when $n = 3$ and the three sets are a chunk of ham and two chunks of bread. The theorem says that each can be bisected with a single guillotine cut (i.e., a plane). According to Beyer and Zardecki [8], Steinhaus first conjectured the ham-sandwich theorem in a published paper and Banach gave the first proof, specifically for $d = 3$. Stone and Tukey[40] later proved the ham-sandwich theorem in a more general setting for measures.

One way to prove this theorem - and many other partitioning theorems - is using a tool from algebraic topology: the Borsuk-Ulam theorem, one version of which is,

Theorem 1.1 ***[Borsuk-Ulam theorem]*** *If f is a continuous function from the unit sphere in R^{n+1} into R^n, there is a point x where $f(x) = f(-x)$; i.e., some pair of antipodal points has the same image..*

Borsuk-Ulam is an important theorem because it has several different equivalent versions, many different proofs, many extensions and generalizations, and many interesting applications. Matoušek's recent book[27] explains the Borsuk-Ulam theorem, its background, and some of its many consequences in algebraic topology, algebraic geometry, and combinatorics.

While the original ham-sandwich theorem holds for continuous mass functions, there is also interest in the problems in the discrete context. Here we can define the measures as the counting measure, that is, the mass functions that count the numbers of the points contained in the closed (or open) halfspaces of the cut. However, the theorem no longer applies directly since the mass functions are not continuous.

There is a general approach for extending the continuous version to the discrete case. We define a continuous family of well behaved d-dimensional mass distributions for each finite multiset of points (see Edelsbrunner's book [16]). For example, we could take the sum of k symmetric d-dimensional normal distributions, one centered at each of k data points. We decrease the variance of the distributions to zero, and argue about the limit of the cuts.

There are several examples where we can derive combinatorial statements that give discrete versions of these results and we in turn can find efficient algorithm to find the asserted combinatorial object. For example in the ham-sandwich case, Lo et. al. [25] gave a direct combinatorial proof and described algorithms to compute ham-sandwich cuts for point sets.

Another example of the extension of Borsuk-Ulam theorem is the recent result by Bárány and Matoušek[4], who combined Borsuk-Ulam with equivariant topology to show that three finite, continuous measures in R^2 can be equipartitioned by a 2-fan, the region spanned by two half-lines incident at a point. Bereg[6] strengthened this statement for finite set of points. He proved a discrete version and described a beautiful algorithm to find such a partitioning. Finally, Roy, and Steiger[31] followed a similar path to obtain complexity results for several other combinatorial consequences of the Borsuk-Ulam theorem.

Our first problem in this thesis is in the same spirit. We are trying to generalize the ham-sandwich theorem not only for bisecting, but for other fractions. We start from the recent result by Bárány etc [3], find a combinatorial proof in the discrete setting and then derive an efficient algorithm to find the unique generalized ham-sandwich cut. In Chapter 2, we will do the following:

- We first formulate the discrete version of the generalized ham-sandwich cut theorem: Given d well-separated point sets in R^d, $P_1, \ldots, P_d$, and $a_1, \ldots, a_d$ are positive integers, $a_i \leq |P_i|$, then

  (i) if an $(a_1, ..., a_d)$-cut exists, it is unique. Also

  (ii) if the points have weak general position, then a cut exists for each $(a_1, ..., a_d)$, $1 \leq a_i \leq |P_i|$.

- Then we give the new combinatorial proof of the existence and uniqueness of generalized ham-sandwich cut in for well-separated point sets in R^d, with the weak general position property.

- From the combinatorial proof, we derive an $O(n \log^{d-3} n)$ running time algorithm to find the unique generalized ham-sandwich cut.

- Furthermore, we prove the converse of the theorem for discrete case, that is, well-separation and weak general position are also necessary if all $(a_1, ..., a_d)$ cuts exist.
- Finally we extend the proof to the continuous case show the partial converse of the theorem.

This is the joint work with William Steiger and some results appeared in the papers [36, 37, 38].

Our second problem involves partitions of a bounded convex set in R^2 by lines. It is easily observed that any set in the plane can be cut by a pair of lines into four parts of equal area. And we have one degree of freedom left. Courant and Robbins [13] showed that (1) we can choose the first bisecting line of arbitrary direction, or, (2) the pair of lines can be taken to be perpendicular.

In 1949, Buck and Buck[9] showed that for any convex body K in the plane, there always exists three concurrent lines that equipartition K; that is, each of the six sectors they define cuts of the same area in K. This is called a concurrent six-way equipartition.

In general, three lines don't meet in one point: when the three intersection points are in K, these lines divide the set into seven regions. The central, bounded one is a triangle T, that degenerates into a point if the lines are concurrent. We assume area$(K) = 1$. In [9] Buck and Buck also showed that no convex set can be partitioned by three lines into seven regions, each with area $1/7$. They then asked whether there are partitions where six of the seven regions each has area $(1 - t)/6$, and the seventh has area t. The previous statement shows this is **impossible** for every K when $t = 1/7$, and their original result shows it is **always possible** for every K when $t = 0$.

Finally they showed that if such a six-way partitioning of K did exist, then the only region having different area t must be the central triangle. They conjectured that in every six-way equipartition with three lines, area$(T) = t$ is at most $t_0 = 1/49$. This is the value that does occur when K is, itself, a triangle. It is easily seen that this K may be six-way equipartitioned by lines parallel to its own sides. The fact that t_0 is the maximum central triangle area over any possible six-way equipartition of any convex body K was later proved by Sholander [34].

As the second problem in this thesis, we reopen an apparently unexplored aspect of the six-way equipartitioning. This is the topic of Chapter 3:

- We first show that given a convex body $K \subset R^2$ with area$(K) = 1$, and a unit vector $d \in R^2$, there exists a unique trio of lines that form a six-way equipartition of K, and where one of them has normal vector d.
- Let $\theta \in [0, \pi)$ and define function $f_K(\theta)$ as the area of the central triangle such that one cut has normal direction $(\cos\theta, \sin\theta)$. We study the behavior of the function $f_K(\theta)$ for some special cases.

This chapter is based on the joint work with William Steiger and Mario Szegedy. We presented some results in [39].

Our third partitioning topic addresses the Voronoi game. Given a finite point set $P \subset R^d$, the Voronoi region of a point $p \in P$ is defined as

Definition 1.2 $\mathrm{Vor}(p) := \{x \in R^d : \mathrm{dist}(x, p) < \mathrm{dist}(x, q) \textit{ for all } q \in P \backslash \{p\}\}.$

where dist(x, y) is the Euclidean distance between the points x and y. So the Voronoi region of p contains all the points in R^d closer to p than all the other points. And the Voronoi regions partition of the whole space. Its complement is Voronoi diagram. The whole region can also be a bounded set $S \subset R^d$. Its Voronoi diagram is clipped by the boundary of S.

The Voronoi game is a two-player game. Both players play n distinct points on a bounded set $S \subset R^d$. The score of each player is the total volume of the union of the Voronoi regions of his points. There are two versions of the games. In the *one round* version, the first player A places all his points and then the second player B places hers. In the *alternating version*, A plays and then B plays, in each of the n successive rounds.

Ahn et. al.[1] first studied the Voronoi game for $d = 1$. An illustrative case is $S = S^1$, the unit circle in R^2 (or any simple closed curve). In the one-round version B can achieve a tie by just placing a point in each of the intervals allocated by A's n points. In fact he can win unless A plays "correctly", and equally spaces his points.

If S is "cut" and opened into a finite interval, or segment, now A can always win the one-round version by playing the n "critical" (evenly spaced) points. Also, B can always win the alternating game (by getting as many critical points as possible and then playing cleverly).

Cheong et. al. [12] took S to be a square of area one, and studied the one-round version, where the behavior is now completely opposite to the $d = 1$ segment case. Now, A need not win because there are no "critical" points to guarantee victory. In fact, they proved that B always has a winning strategy. Using a beautiful probabilistic argument, they showed that $\exists \beta > 0$ and an n_0 so that when $n > n_0$, B can get $1/2 + \beta$ no matter what points A plays. They also showed this holds in higher dimensions.

We study the one-round Voronoi game for $d = 2$. We work on the torus and study player B's strategy when A plays on both a square grid and a hexagon grid (when A plays on equilateral triangular lattice). This allows us to bound B's payoff in the one-round game. Using the numerical optimization techniques, we show that the area of B's first point is minimized when A places his points on the equilateral triangular lattice.

Chapter 4 contains the work on the Voronoi games. This is the joint work with William Steiger. Some of the result was presented in [35].

In the final chapter, we discuss some interesting open problems and some possible directions for future work.

Chapter 2

Generalized Ham-sandwich Cut

We are given d sets $S_1, \ldots, S_d \subset R^d$, each set S_i measurable with respect to a suitable, "nice" measure μ_i. The ham-sandwich theorem states that there exists a hyperplane h with the property that $\mu_i(S_i \cap h^+) = \mu_i(S_i \cap h^-) = \frac{1}{2}\mu_i(S_i)$, $i = 1, ..., d$. The hyperplane h is called a ham-sandwich cut. It "slices off" the fraction $\alpha_i = 1/2$ from each set.

A natural question is whether we can ask for other fractions $\alpha_i \in [0, 1]$ to slice off by the cut: i.e. is there a hyperplane h such that $\mu_i(S_i \cap h^+) = \alpha_i \mu_i(S_i), i = 1, ..., d$. Such an h is called a *generalized ham-sandwich cut*. Bárány, Hubard, and Jerónimo [3] recently showed an interesting result about generalized ham–sandwich cuts for sets with finite, nice measures. They showed that generalized ham-sandwich cuts always exist when the sets are well-separated.

In this chapter, we formulate a discrete version of this problem and give a direct combinatorial proof for a discrete version of Bárány, Hubard, and Jerónimo's results. We did this because of the interest in the algorithmic problem where, given n points distributed among d well-separated sets in R^d, the object is to actually find the cut for given $a_1, \ldots, a_d$. It is interesting that in general the best-known algorithm for the discrete ham-sandwich cuts does not scale well with dimension d having running time $O(n^{d-1-\alpha_d})$, $0 < \alpha_d \to 0$ as $d \to \infty$. Our combinatorial proof of the discrete case leads directly to the formulation of an efficient, $O(n(\log n)^{d-3})$ algorithm to compute generalized cuts for dimension $d \geq 3$.

This algorithm may be practical because it scales well as the dimension increases. It is nearly linear and it is one of our main results.

There are two other useful consequences of an direct combinatorial proof. The ideas in the proof for the discrete case can be applied in the continuous case and we get an alternative, simpler proof for the original result of Bárány, Hubard, and Jerónimo [3]. Also, as a corollary to the existence theorem we observe that in the discrete case, the conditions for the existence and uniqueness of all cuts are also necessary. This enabled us to strengthen the original theorems by showing that something similar holds in the continuous context.

The topic is based on the joint work with William Steiger. And is based on the papers appeared in [36, 37, 38].

This chapter is organized as follows: In Section 2.1, we survey the previous results on the discrete version of ham-sandwich theorem and on the efficient algorithms to compute it in low dimensions ($d \leq 3$). These facts are part of the new algorithm for generalized ham-sandwich cuts. In Section 2.2, we give rigorous definitions for well-separation and for the generalized ham-sandwich theorem. We prove the discrete version of generalized ham-sandwich theorem in Section 2.3 and describe the algorithm in Section 2.4. In Section 2.5 we adapt the proof to the continuous case for general measures and give other results.

2.1 Previous Result on Discrete Ham-sandwich Cut

The discrete version of ham-sandwich cut is defined as follows.

Definition 2.1 *Let P be a set of n points in R^d. A hyperplane h is called a **bisector** of P if neither of the two open half-spaces defined by h contains more than $\frac{n}{2}$ points of P.*

In this case, we also say that h *bisects* P.

Throughout we assume that points are in general position. If not, there are standard perturbation techniques that enforce this[16]. Also we assume without loss of generality that the cardinality of P is odd, since otherwise we may delete any point q; any bisector h of $P\backslash\{q\}$ also bisects P. In this case n is odd and every bisector h must be incident with at least one point of P. The discrete ham-sandwich theorem is:

Theorem 2.2 *Let $P_1, \ldots, P_d$ be d non-empty finite sets of points in R^d, then there exists a hyperplane h that simultaneously bisects $P_1, \ldots, P_d$.*

This theorem automatically leads to the computational problem of finding a ham-sandwich cut given n points in R^d partitioned into sets $P_1, \ldots, P_d$.

In two dimensions, Megiddo [29] first described an $O(n)$ algorithm for the special, separated case. Here sets P_1, P_2 of $|P_1| + |P_2| = n$ points are linearly separated. The ham-sandwich theorem implies that there is a point $x \in P_1$ and a point $y \in P_2$: the line $\overline{xy}$ bisects both P_1 and P_2. Megiddo's algorithm finds these points in the dual setting. Since the two sets are separated, the points in first set, P_1 may be dualized to (red) lines having positive slopes and those in second set, P_2, to (blue) lines having negative slope. His algorithm works in stages. In each stage, a fraction $t \in (0, 1)$ of all points are pruned because they cannot be x or y. The running time is linear in the number of points entering the stage. Therefore the overall complexity is also linear.

Later, Edelsbrunner and Waupotitsch [17] showed how to adapt Megiddo's algorithm to the general, not necessarily separated case in R^2. It's running time is $O(n \log n)$. Finally, Lo and Steiger [24] found an optimal linear running time algorithm. This algorithm was extended to higher dimensions by Lo, Matoušek, and Steiger [25]. They showed that the ham-sandwich cut can be computed in time proportional to the (worst-case) time needed to construct a given level in the arrangement of n given hyperplanes in R^{d-1}. They also show that, for $d = 3$, if these sets have the separation property whereby there is no line transversal meeting all three convex hulls, and each pair is linearly separated, then a ham-sandwich cut may be found in linear time.

2.2 Well-Separated Sets and Generalized Ham-sandwich Theorem

The previous results are all for bisectors. The problem of slicing off factors $\alpha_i \in [0, 1]$ other than 1/2, however only can be done under extra conditions. Some examples are in [3]. We will show that well-separation is necessary for the discrete case if we want the generalized ham-sandwich cut to exist for all possible factors.

The well-separation property is defined as,

Definition 2.3 *(see [23]) Given $k \leq d+1$, a family $S_1, \ldots, S_k$ of connected sets in R^d is well-separated if, for every choice of $x_i \in S_i$, the affine hull of $x_1, \ldots, x_k$*[1] *is a $(k-1)$-dimensional flat in R^d.*

There are several equivalent forms of the well-separation property for connected sets [5], in particular the fact that such a family is well-separated if and only if the convex hulls are well-separated. These equivalent forms include

- C1 Sets $S_1, \ldots, S_k, k \leq d+1$ are well-separated if and only if, when I and J are disjoint subsets of $1, \ldots, d+1$, there is a hyperplane separating the sets $S_i, i \in I$ from the sets $S_j, j \in J$.

- C2 $S_1, \ldots, S_d$ are well-separated in R^d if and only if there is no $(d-2)$-dimensional flat that has non-empty intersection with all $\text{Conv}(S_i), i = 1, \ldots, d$.

We define the discrete version as follows,

[1] *The affine hull of $x_1, \ldots, x_k$ is defined as the set $\{\sum_{i=1}^{k} \lambda_i x_i | \sum_{i=1}^{k} \lambda_i = 1, \lambda_i \in R, i = 1, \ldots, k\}$.*

Definition 2.4 *Point sets $P_1, ..., P_d$ in R^d are well-separated if their convex hulls,* $\text{Conv}(P_1)$, $\ldots, \text{Conv}(P_d)$*, are well-separated.*

In view of this definition, properties (C1) and (C2) hold for the discrete context as well. We will use these equivalent forms in the proofs later.

The maximum number of affinely independent points in R^d is $d+1$. The maximum possible value of k in Definition 2.3 is $d+1$; that is, the maximum number of well-separated sets is $d+1$. But for $k = d+1$, the affine hull of $d+1$ points will be the whole of R^d. So from now on, we will assume that $k \leq d$.

In [3], Bárány, Hubard, and Jerónimo showed the existence of generalized ham–sandwich cuts for well–separated convex measurable sets. They proved the following:

Theorem 2.5 ***[Generalized ham-sandwich theorem]*** *Let $K_1, ..., K_d$ be well-separated convex bodies in $\mathbb{R}^d$ and $\beta_1, \ldots, \beta_d$ given constants with $0 \leq \beta_i \leq 1$. Then there is a unique hyperplane $h \subset R^d$ with the property that $Vol(K_i \cap h^+) = \beta_i \cdot Vol(K_i)$, $i = 1, \ldots, d$.*

Here h^+ denotes the closed, positive transversal halfspace defined by h: that is the halfspace where, if Q is an interior point of h^+ and $z_i \in K_i \cap h$, the d–simplex $\Delta(z_1, ..., z_d, Q)$ is *negatively* oriented [3]. Specifying one particular choice of halfspaces is what allows h to be uniquely determined. Bárány et. al. gave analogous results for such *generalized ham-sandwich cuts* for other kinds of well-separated sets that support suitable measures.

We are interested in the discrete version of Theorem 2.5 for n points partitioned into d sets in R^d; i.e., points in $S = P_1 \cup \cdots \cup P_d$, $P_i \cap P_j = \varnothing, i \neq j$, $|S| = n$. For this context we need some kind of general position condition, and will assume the following, weak form.

Definition 2.6 *Points in $S = P_1 \cup \ldots \cup P_d$ have weak general position if, for each $(x_1, \ldots, x_d)$, $x_i \in P_i$, the affine hull of $x_1, \ldots, x_d$ is a $(d-1)$-dimensional flat that contains no other point of S.*

Notice that this does not prohibit more than d data points from being in a hyperplane, e.g. if they are all in the same P_i. Thus it does not imply general position.

For the discrete analogue of a generalized cut we use

Definition 2.7 *Given positive integers $a_i \leq |P_i|$, an $(a_1, ..., a_d)$-cut is a hyperplane h for which $h \cap P_i \neq \phi$ and $|h^+ \cap P_i| = a_i$, $1 \leq i \leq d$.*

As in Theorem 2.5, a cut is a transversal hyperplane for the convex hulls of $P_1, \ldots, P_d$ (here incident with at least one data point in each P_i) and h^+ its positive closed halfspace. The discrete version of Theorem 2.5 is

Theorem 2.8 ***[Generalized ham-sandwich theorem, discrete version]*** *If $P_1, \ldots, P_d$ are well-separated point sets in R^d, and $a_1, \ldots, a_d$ are positive integers, $a_i \leq |P_i|$, then*

(i) if an $(a_1, ..., a_d)$-cut exists, it is unique. Also

(ii) if the points have weak general position, then a cut exists for every $(a_1, ..., a_d)$, $1 \leq a_i \leq |P_i|$.

It might be possible to prove this using the results of [3] along with the standard argument we described before (see [16]). Instead here we give a direct combinatorial proof in next section. We did this because of the interest in the algorithmic problem:

Problem 2.9 *Given a set S of n points partitioned into well-separated sets $P_1, ..., P_d$ in R^d, and in weak general position, along with positive integers $a_i \leq |P_i|$, find an $(a_1, ..., a_d)$-cut for S.*

2.3 Proof of Generalized Ham-Sandwich Theorem, Discrete Version

We are given d well-separated point sets, $P_1, \ldots, P_d$, in R^d, First of all, we define the positive transversal halfspace formally, closely following [3].

2.3.1 Positive Transversal Halfspace

We are given n points partitioned into d non-empty sets in R^d; i.e., points in $S = P_1 \cup \cdots \cup P_d$, $P_i \cap P_j = \varnothing, i \neq j$, $|S| = n$, and these d point sets are well-separated. For each point set $P_i, i = 1, \ldots, d$, we pick one point $p_i \in \mathrm{Conv}(P_i), i = 1, \ldots, d$ (not necessarily data points in S), then the hyperplane h determined by $\{p_1, \ldots, p_d\}$ is a transversal hyperplane of dimension $d-1$.

As in Bárány et. al. [3], if a unit vector c satisfies $\langle c, p_i \rangle = t$ for some fixed constant t and for all i, the unit normal vector v of h can be chosen as either c or $-c$. The *positive transversal hyperplane* arises when v is chosen so that,

$$\det \begin{vmatrix} p_1 & p_2 & \cdots & p_d & v \\ 1 & 1 & \cdots & 1 & 0 \end{vmatrix} > 0.$$

Writing h as $\{p \in R^d : \langle p, v \rangle = t\}$, h^+, the *positive transversal halfspace* is

$$h^+ = \{p \in R^d : \langle p, v \rangle \leq t\}.$$

The relation $p \in h^+$ is invariant under translation and rotation.

Fixing one point set, say P_1, we can find a hyperplane π which separates P_1 from the other sets. Picking a point $p \in \mathrm{Conv}(P_1)$, we connect p to each point, $q \in S$, $q \notin P_1$ by a ray $\overrightarrow{pq}$. We call $\overrightarrow{pq} \cap \pi$ the *ray projection image* of q on π from point p, denote as q_p^π.

It is easy to check that $h^+ \cap \pi$ is also the positive halfspace in π for all projection images, q_p^π's.

2.3.2 Proof of Main Theorem

The proof is by induction. The base case $d = 2$ is probably folklore (but see [29]). "Well-separated" implies that points in P_1 may be dualized to (red) lines having positive slopes and those in P_2, to (blue) lines having negative slope. If a red/blue intersection q has a_1 red lines and a_2 blue lines above it, vertex q is the dual of an (a_1, a_2)-cut. It must be the unique one because the red levels have positive slope and blue ones have negative slope, proving (2.8)i.

If P_1 and P_2 also have weak general position, every red/blue intersection in the dual is a distinct vertex, $|P_1| \cdot |P_2|$ of them in all, and each is incident with just those two lines. This implies that each level in the first arrangement has a unique intersection with every level of the second, proving (2.8)ii. In fact the unique intersection can be found in linear time by adapting Megiddo's prune-and-search algorithm [29] for the unique intersection of median levels.

Next, suppose the claims in Theorem 2.8 hold in every dimension $j < d$; we show they also hold in R^d. Let π be a hyperplane that separates P_1 from $\bigcup_{i=2}^{d} P_i$. Fix a point $x \in \mathrm{Conv}(P_1)$ and project each data point $q \in \bigcup_{i=2}^{d} P_i$ onto π via the mapping $M_x : q \to \overrightarrow{xq} \cap \pi$. Write P_i^π for the multiset of images in π of the points $q \in P_i, i \geq 2$.

Lemma 2.10 *$P_2^\pi, \ldots, P_d^\pi$ are $d-1$ well-separated sets in π.*

Proof: If not, there is a $(d-3)$-flat $\rho \subset \pi$ that meets all $\mathrm{Conv}(P_i^\pi), i \geq 2$. But the span of x and ρ is a $(d-2)$-flat that meets all $\mathrm{Conv}(P_1), \ldots, \mathrm{Conv}(P_d)$, a contradiction with the fact that $P_1, \ldots, P_d$ are well-separated. ■

Lemma 2.11 *If $P_1, \ldots, P_d$ have weak general position, and if we project from a point $x \in P_1$, then $P_2^\pi, \ldots, P_d^\pi$ have weak general position in π.*

Proof: This follows because each $q^\pi \in P_i^\pi$ is the image of a distinct point $q \in P_i$, $i \geq 2$. A transversal flat $\rho_x \subset \pi$ has dimension $d-2$ by Lemma 2.10. If it contains one point q_i^π

from each $P_i^\pi, i \geq 2$ *and any other* $z \in \bigcup_{i=2}^d P_i^\pi$, then x and ρ_x span a hyperplane that violates weak general position for $P_1, \ldots, P_d$. ■

These facts show that the induction hypotheses apply to the images $P_2^\pi, \ldots, P_d^\pi$ in π.

Given a point $x \in \mathrm{Conv}(P_1)$ and $(a_2, \ldots, a_d)$, a hyperplane h_x containing x is called *an* $(a_2, \ldots, a_d)$ *semi-cut (or just a semi-cut)* if, for each $i \geq 2$, it is incident with a point $p_i \in P_i$ and $|h_x^+ \cap P_i| = a_i$. The following useful fact is straightforward:

Lemma 2.12 *Given* $x \in \mathrm{Conv}(P_1)$ *and* $(a_2, \ldots, a_d)$*, if there is an* $(a_2, \ldots, a_d)$ *semi-cut* h_x *then it is unique.*

Proof: Suppose h_1 and h_2 are distinct $(a_2, \ldots, a_d)$ semi-cuts incident with $x \in \mathrm{Conv}(P_1)$. Then there are points $q_i = h_1 \cap P_i$ and $q_i' = h_2 \cap P_i$, $i = 2, \ldots, n$, and the images of these points in π would be distinct $(a_2, \ldots, a_d)$ cuts, in violation of the induction hypothesis. ■

To advance the induction, fix $(a_1, \ldots, a_d)$ and suppose h_x is a cut with these values, $x \in P_1$. By Lemma 2.12, it is the unique semi-cut cut containing x, so suppose there is an $(a_2, \ldots, a_d)$ semi-cut h_y through $y \in P_1$, $y \notin h_x$. Hyperplanes h_x and h_y cannot meet in $\mathrm{Conv}(P_1)$ since otherwise any point in the intersection would be in two different $(a_2, \ldots, a_d)$ semi-cuts, violating Lemma 2.12.

But this implies that $|P_1 \cap h_y^+| \neq a_1$: if $y \in h_x^+$, for every $z \in P_1 \cap h_y^+$, we have $z \in h_x^+$, then $|P_1 \cap h_y^+| < a_1$ since $x \in h_y^-$. Similar for $y \in h_x^-$, we have $|P_1 \cap h_y^+| > a_1$. Therefore h_x is unique cut with value a_1, which proves statement (i) in Theorem 2.8.

Now suppose $P_1, \ldots, P_d$ have weak general position and fix $p \in P_1$ and $(a_2, \ldots, a_d)$. Projecting from p, there is a unique $(a_2, \ldots, a_d)$-cut $\rho_p \subset \pi$ by the induction hypothesis and the fact that each $q^\pi \in \bigcup_{i=2}^d P_i^\pi$ is the image of a distinct $q \in \bigcup_{i=2}^d P_i$. The hyperplane

h_p determined by p and ρ_p is an $(m_p, a_2, \ldots, a_d)$-cut, m_p denoting $|P_1 \cap h_p^+|$. Lemma 2.12 implies that there is no other $(m_p, a_2, \ldots, a_d)$-cut. Also, repeating this procedure for every $p \in P_1$, existence and uniqueness imply that the integers $m_p, p \in P_1$ form a permutation of $1, \ldots, |P_1|$. So for some $p \in P_1$ we have the unique $(a_1, \ldots, a_d)$-cut, and this proves statement (2.8)(ii). ■

In fact the conditions of the Theorem 2.8 are also necessary.

Corollary 2.13 *Well separation and weak general position are necessary if every* $(a_1, \ldots, a_d)$-*cut exists and is unique.*

Weak general position is necessary for the existence and uniqueness of all $(a_1, \ldots, a_d)$-cuts by simple counting. There are $|P_1| \cdot |P_2| \cdots |P_d|$ different d-tuples $(a_1, \ldots, a_d)$ and there are this many *different* transversal hyperplanes through data points only if we have weak general position.

Now suppose $P_1, \ldots, P_d$ are not well-separated. By property 1 at the beginning of this section, there is a partition $I \cup J$ of $\{1, \ldots, d\}$, such that $A = \mathrm{Conv}(\bigcup_{i \in I} P_i) \cap \mathrm{Conv}(\bigcup_{j \in J} P_j) \neq \phi$. For points in A on the boundaries of the convex hulls, weak general position is violated. For points of A interior to both convex hulls, any half space containing $\bigcup_{i \in I} P_i$ also contains at least one point of $\bigcup_{j \in J} P_j$ in its interior. If we set $a_i = 1$ for $i \in I$, $a_i = |P_i|$ for $i \in J$, no $(a_1, \ldots, a_d)$-cut can exist. ■

2.4 An Algorithm for Generalized Cuts.

From now on we assume weak general position and well-separation. Theorem 2.8 implies that for every $1 \leq a_i \leq |P_i|$, $i = 1, \ldots, d$, there is a unique set of data points $p_1, \ldots, p_d$, $p_i \in P_i$, for which the affine hull of $(p_1, \ldots, p_d)$ is an $(a_1, \ldots, a_d)$-cut. So we could use a brute force enumeration and find it in $O(n^{d+1})$, $O(n)$ being the cost to test each d-tuple.

A small improvement can be obtained by resorting to the following algorithmic result in Proposition 2.14. In [25], Lo, Matoušek, and Steiger [25] described an algorithm to find a ham-sandwich cut in high dimensions. We state their main results here (slightly improved to reflect new upper bounds on k-sets [15], [33], [28]).

Proposition 2.14 *Given n points in R^d which are partitioned into d sets $P_1, \ldots, P_d$ in R^d, a ham-sandwich cut can be computed in time proportional to the (worst-case) time needed to construct a given level in the arrangement of n given hyperplanes in R^{d-1}. The latter problem can be solved within the following bounds:*

$$\begin{array}{ll} O(n^{4/3}\log^2 n/\log^* n) & \text{for } d = 3, \\ O(n^{5/2}\log^{1+\delta} n) & \text{for } d = 4, \\ O(n^{4-\frac{2}{45}}\log^{1+\delta} n) & \text{for } d = 5, \\ O(n^{d-1-a(d)}) & \text{for } d \geq 6. \end{array}$$

Here $\delta > 0$ is an appropriate constant and $a(d) > 0$ a small constant; also $a(d) \to 0$ as $d \to \infty$. They also show that, for $d = 3$, if these sets have the separation property whereby there is no line transversal meeting all 3 convex hulls and each pair is linearly separated, then a ham-sandwich cut may be found in linear time.

We can apply the ham-sandwich algorithms in [25] to the generalized cuts for well-separated points sets having weak general position - given that they exist - and in this way, the complexity of finding generalized cuts may be reduced to $O(n^{d-1-a(d)})$.

Here, we will describe a much more practical algorithm, applying ideas from the proof in Section 2.3. We showed there that for each data point $p \in P_1$ and $(a_2, \ldots, a_d)$, there is a unique $(m_p, a_2, \ldots, a_d)$-cut h_p that contains p. Furthermore, for each j, $1 \leq j \leq |P_1|$, there is a unique $p \in P_1$ for which $m_p = j$, where $m_p = |P_1 \cap h_p^+|$. Thus we could consider in turn all $x \in P_1$. For each we project onto π, find the unique $(a_2, \ldots, a_d)$ cut $\rho_p \subset \pi$, and compute $m_x = |P_1 \cap h_p^+|$ for h_p, the hyperplane spanned by p and ρ_p. At some stage we will discover the unique $z \in P_1$ for which $m_z = a_1$ and h_z is the $(a_1, \ldots, a_d)$-cut. The cost would be bounded by the cost to solve n problems in R^{d-1}.

In fact we will find the desired $z \in P_1$ by solving at most $O(\log n)$ problems in R^{d-1}. The key is the ability to prune a fixed fraction of remaining points in P_1 after a search step with $p \in P_1$ by using the fact that if $m_p < a_1$, no point $y \in h_p^+ \cap P_1$ has $m_y = a_1$. In order to do that, we use the idea of ϵ-approximation of the remaining points in P_1. Here is the entire algorithm. The details are explained after.

ALGORITHM GEN-CUT

1. **choose $c > 0$, a small, fixed integer (say 10)**
2. **Find a hyperplane π that separates P_1 from $P_2 \cup \cdots \cup P_d$**
3. $C \leftarrow P_1$
4. $a \leftarrow a_1$
5. **WHILE $|C| > c$ DO**
   (a) **Construct A, an ϵ-approximation to C with repect to halfspaces**
   (b) **FOR each $x \in A$ DO**
      i. **Project each $y \in P_2 \cup \cdots \cup P_d$ onto π; let P_i^π denote the projections of the points in P_i**
      ii. **Find the $(a_2, ..., a_d)$-cut $\rho_x \subset \pi$ for the projections $P_2^\pi, \ldots, P_d^\pi$ by solving a $(d-1)$-dimensional problem**
      iii. **Get h_x, the hyperplane that spans x and ρ_x.**
      iv. **Compute the number of points of C in the positive transversal halfspace h_x^+**
      v. **END FOR**

(c) **Prune from C points $x \in P_1$ whose n_x is too small or too large, and adjust C and a**

(d) **END WHILE**

6. **For each remaining data point in $x \in C$, project, find the $(a_2, ..., a_d)$-cut ρ_x in π for the projections by solving a $(d-1)$-dimensional problem, get h_x and compute $n_x = |P_1 \cap h_x^+|$, stopping when $n_x = a_1$.**

Now we explain the various steps. In Step 2, finding a separating hyperplane π can be formulated as a linear programming problem and can be solved in time $O(n)$, for fixed dimension d. In Step 3, C is the set of candidates for the sought point $z \in P_1$; initially $C = P_1$. The number of undeleted points in the positive transversal halfspace of z's semicut is denoted by a; initially $a = a_1$.

In the WHILE loop we construct an ϵ-approximation to C. The range space $(C, \mathcal{A})$, has VC dimension $d+1$, where $\mathcal{A}$ denotes the set of all halfspaces in R^d that contain some points in C. By [10], in $O(|C|)$ time [i.e., linear; in fact its $O((d+1)^{3(d+1)}(\frac{1}{\epsilon^2}\log\frac{d+1}{\epsilon})^{d+1}|C|)$] we can construct an ϵ-approximation $A \subset C$, having constant size [in fact, $|A| = k = O(\frac{d+1}{\epsilon^2}\log\frac{d+1}{\epsilon})$].

The FOR loop in 5b is traversed $k \equiv |A|$ times. The cost of each traversal is dominated by $O(B_{d-1})$, the cost of the $(d-1)$-dimensional problem in (ii); the cost of (i) is $O(n)$ and (iv) is $O(|C|)$.

At the end of the FOR we have for each $x \in A$, the value of $n_x = |h_x^+ \cap C|$. These distinct values order the elements $x \in A$, and our target value, a, is (1) less than the smallest n_x, (2) greater than the largest n_x, or (3) between a successive pair in the ordering. In the first case we delete all $y \in C$, $y \notin h_u^+$, where $n_u = \min(n_x, x \in A)$. In the second case we delete all $y \in C$, $y \in h_v^+$, where $n_v = \max(n_x, x \in A)$; here we also reduce a by $a \leftarrow a - n_v$.

Finally, for the middle case, we have $n_l < a < n_u$ where (n_l, n_u) is a successive pair for some points l, u, then we only keep the points of P_1 lying between hyperplanes h_l and h_u and change the value of a to $a - n_l$. Since A is an ϵ-approximation, only a constant fraction ($< 1/(k+1) + 2\epsilon$) of the points in C remains after pruning.

The geometric decrease in $|C|$ implies that the number of iterations of the WHILE loop is bounded by $O(\log |P_1|) = O(\log n)$. Therefore Step 5b contributes $O(B_{d-1} \log n)$ to the total cost of the loop, where B_k denotes the complexity of the present algorithm in dimension k. This dominates the total cost of the loop because all other steps have cost either $O(n)$ or $(O|C|)$ and contribute a total of $O(n \log n)$ to the loop.

When the loop terminates, each remaining point in C is treated in time $O(B_{d-1})$ by executing Steps (i) - (iv) in 5b. Then, instead of Step 5c, we test whether $|h^+ \cap P_1| = a_1$; exactly one point will have this property. Since the base case for dimension $d = 2$ has linear running time, the present algorithm will find a generalized cut in $O(n(\log n)^{d-2})$.

Finally, we can reduce the power of $\log n$ to $d - 3$. In R^3, Lo, et. al. [25] showed how to find a ham-sandwich cut for well-separated point sets in linear time. That algorithm is easily adapted to generalized cuts. Using this as the base case when $d > 2$, the algorithm just described will now have running time $O(n(\log n)^{d-3})$ for dimensions $d \geq 3$, and we have shown

Theorem 2.15 *Given n points partitioned into well-separated sets $P_1, \ldots, P_d$ and having weak general position, and $(a_1, \ldots, a_d) \in [0,1]^d$, an $(a_1, \ldots, a_d)-$cut can be found in time $O(n(\log n)^{d-3})$, $d \geq 3$, and in linear time if $d = 2$.*

The dependency of the running time on dimension d can also be calculated as follows: First of all, we do $O(\log n)$ ϵ-approximation problems in step 5(a). The running time for this step will be bounded by $O\left((d+1)^{3(d+1)}(\frac{1}{\epsilon^2} \log \frac{d+1}{\epsilon})^{d+1} n \log n\right)$. Then we will solve

$O\left(\log n\frac{d+1}{\epsilon^2}\log\frac{d+1}{\epsilon}\right)$ problems in dimension $d-1$. It's easy to check the total running time is bounded by,

$$O\left((d+1)^{3(d+1)}\left(\frac{1}{\epsilon^2}\log\frac{d+1}{\epsilon}\right)^{d+1} n(\log n)^{d-3}\right).$$

In other words, the running time is dominated by finding $O(\log n)^{d-3})$ ϵ-approximations.

Remark: The importance of the algorithm is that it scales well with the dimension and is nearly linear.

2.5 A Simple Proof for the Continuous Case

In this section, we apply the inductive approach we used in the proof of Theorem 2.8 to give a new, simple proof for the continuous case. We need to extend the approach to *nice measures* and, to be self-contained, we repeat notations and terminology from Bárány et. al. [3].

Writing $v \in S^{d-1}$ for the unit outer normal vector of a halfspace H, we denote the halfspace $\{x \in R^d : \langle x, v\rangle \le t\}$ by $H(v \le t)$. Analogously we write $H(v = t) = \{x \in R^d : \langle x, v\rangle = t\}$. Given a set $K \subset R^d$, a unit vector v and a scalar t , we denote the set $H(v = t) \cap K$ by $K(v = t)$; analogously $K(v \le t) = H(v \le t) \cap K$.

Let μ be a finite measure on the Borel subsets of R^d and let $v \in S^{d-1}$ be a unit vector. Define

$$t_0 = t_0(v) = \inf\{t \in R : \mu(H(v \le t)) > 0\},$$

$$t_1 = t_1(v) = \sup\{t \in R : \mu(H(v \le t)) < \mu(R^d)\}.$$

We write $H(s_0 \le v \le s_1)$ for the closed slab between the hyperplanes $H(v = s_0)$ and $H(v = s_1)$ and define the set K by

$$K = \bigcap_{v \in S^{d-1}} H(t_0(v) \le v \le t_1(v)).$$

K is called the support of μ. It is convex and $\mu(R^d \backslash K) = 0$.

Barany et al [3] used the following

Definition 2.16 *A measure μ on R^d is nice if:*

(1) $t_0(v)$ and $t_1(v)$ are finite for every $v \in S^{d-1}$.

(2) $\mu(H(v = t)) = 0$ for every $v \in S^{d-1}$ and $t \in R$.

(3) $\mu(H(s_0 \leq v \leq s_1)) > 0$ for every $v \in S^{d-1}$ and for every s_0, s_1 satisfying $t_0(v) \leq s_0 < s_1 \leq t_1(v)$.

We observe that

Lemma 2.17 *Condition (3) in the Definition 2.16 is equivalent to*

(3'). For any two hyperplanes h_1, h_2 with $h_1 \cap K \neq \phi$, and $h_2 \cap K \neq \phi$ but $h_1 \cap h_2 \cap K = \phi$, the measure of the closed slab of K between h_1 and h_2 is positive.

Proof: Condition (3) is a special case of (3'). On the other hand, given $h_1 \cap h_2 \cap K = \phi$, let $x \in h_1 \cap K$ be a point with the minimum distance from h_2. Then the hyperplane h' incident with x and parallel to h_2, together with h_2 form a slab with positive measure, by (3), and this is a subset of the slab defined by h_1 and h_2 since K is convex. ■

The generalized ham-sandwich theorem (Theorem 3 in [3]) is

Proposition 2.18 *Suppose μ_i is a nice measure on R^d with support K_i, $i \in \{1, \ldots, d\}$. Assume the family $\mathcal{F} = \{K_1, \ldots, K_d\}$ is well-separated and let $\alpha = (\alpha_1, \ldots, \alpha_d) \in [0, 1]^d$. Then there is a unique positive transversal halfspace H, such that $\mu_i(K_i \cap H) = \alpha_i \cdot \mu_i(K_i)$, $i = 1, \ldots, d$.*

Here is a simple proof along the lines we used for Theorem 2.8.

Proof: We will use induction on dimension d. First we normalize each measure so that $\mu_i(R^d) = 1$. As before, a hyperplane h is an $(\alpha_1, \ldots, \alpha_d)$-cut if its corresponding corresponding positive halfspace h^+ has measure $\mu_i(K_i \cap h^+) = \alpha_i \cdot \mu_i(K_i)$, $i = 1, \ldots, d$.

For the base case, take $d = 1$ under the nice measure μ_1. From Definition Definition 2.16(1), the support K_1 is a finite line segment $[l, u]$. From Definition 2.16 and Lemma 2.17, the function $f : x \mapsto \mu_1(v \le x)$ is easily seen to satisfy

(i) $f(x) = 0, x \le l$ and $f(x) = 1, x \ge u$ and

(ii) $f(x)$ is strictly increasing and continuous on $[l, u]$, properties that guarantee the existence and uniqueness of an α_1-cut for every $\alpha_1 \in [0, 1]$.

Now suppose the claim holds for every dimension $j < d$. Let π be a hyperplane that separates K_1 from $\bigcup_{i=2}^d K_i$. For a point $x \in K_1$ and $y \in \bigcup_{i=2}^d K_i$, define $M_x(y) = \overline{xy} \cap \pi$, a mapping that projects each K_i onto π, $i > 1$. We write

$$PK_i := M_x(K_i)$$

for the image of K_i in π and define the measure μ_i^π on π by

$$\mu_i^\pi(S) := \mu_i\{v | v \in R^d, M_x(v) \in S\}.$$

for all measurable $S \subseteq \pi$.

From the definition of nice measure, it easily follows that for each $i = 2, \ldots, d$, μ_i^π is also a nice measure on π. In addition, following the same argument as in the proof of the discrete case, we have,

Fact: $PK_2, \ldots, PK_d$ are $d - 1$ well-separated sets in π.

Therefore the induction hypotheses apply to $PK_2, \ldots, PK_d$ and an $(\alpha_2, \ldots, \alpha_d)$-cut, $\rho_x \subset \pi$ exists for $PK_2, \ldots, PK_d$ under measures $\mu_2^\pi, \ldots, \mu_d^\pi$, and it is unique. Here, as in Section

2.3, we call the hyperplane h_x determined by x and ρ_x an $(\alpha_2, \ldots, \alpha_d)$-semicut (or just a semi-cut). By the definition of μ_i^π we have $\mu_i(K_i \cap h_x^+) = \alpha_i$, $i = 2, \ldots, d$ where h_x^+ is the positive halfspace of h_x. Similar to Lemma 2.12 for the discrete case, we have

Lemma 2.19 *Fix $x \in K_1$ and $(\alpha_2, \ldots, \alpha_d)$, an $(\alpha_2, \ldots, \alpha_d)$ semi-cut h_x exists and is unique.*

This in turn implies that, for any $x \neq y \in K_1$, the semicuts h_x, h_y either are the same or they do not meet in K_1. Finally, fix $(\alpha_2, \ldots, \alpha_d)$ and define the function $f : x \in K_1 \mapsto \mu_1(K_1 \cap h_x^+)$, h_x^+ the positive halfspace of semicut h_x. Because μ_1 is a nice measure, and in view of (3'), $f(y) < f(x)$ if $y \in h_x^+$, and f is continuous. Therefore the existence and uniqueness of an $(\alpha_1, \ldots, \alpha_d)$-cut follows and the induction advances. ■

Now consider a partition $I \cup J$ of $\{1, ..., d\}$. A cut h with $\alpha_i = 0$ for $i \in I$ and $\alpha_j = 1$ for $j \in J$ exists only if $\bigcup_{i \in I} K_i$ and $\bigcup_{j \in J} K_j$ are in different closed halfspaces of h. Letting I and J range over all 2^d partitions of $\{1, \ldots, d\}$ we either have well-separation or, for some I, J, and corresponding h, there are points $P_i \in K_i \cap h$ and these d points do not span a $d-1$ flat. We call this *degenerate non-separation*, and we have the following a partial converse to Proposition 2.18.

Corollary 2.20 *If all possible cuts exist, either the K_i are well-separated or they are degenerately non-separated.*

2.6 Summary

Starting from the recent interesting extension of the ham-sandwich theorem of Bárány, Hubard, and Jerónimo, we formulated a discrete version for n points partitioned into d sets in R^d. We gave a simple direct proof for this case and showed that the converse is true. Most importantly, we found an $O(n \log^{d-3} n)$ algorithm for discrete generalized ham-sandwich cut. Finally, we applied the same proof technique to the continuous case of generalized ham-sandwich theorem and showed the partial converse.

Chapter 3

Six-way Equipartitioning by Three Lines in the Plane

3.1 Introduction

In this chapter we discuss the problem involving partitions of a bounded convex set in the plane by lines. It is easily observed that any Borel measurable set in the plane can be cut by a pair of lines into four parts of the same measure. And there is still some freedom left. Courant and Robbins [13] showed that (1) we can choose the first bisecting line of arbitrary direction: the ham-sandwich theorem then implies the existence of a unique line which with the first line forms the desired partition or (2) the pair of lines can be taken to be perpendicular.

This is not true if we want partition factors other than bisecting. For example, we want to cut the set into four regions with factors α, β, γ and $1-(\alpha+\beta+\gamma)$ of the total measure with $\alpha, \beta, \gamma, 1-(\alpha+\beta+\gamma) \in (0,1)$. The ham-sandwich theorem implies that we can always do it and we can also let the first cut have arbitrary direction. But we no longer can enforce the cutting lines to be perpendicular.

In [9], R. C. Buck and E. F. Buck considered the problem of partitioning a convex set $K \subset R^2$ using three lines. In general, three lines will divide K into seven regions if the intersections of the lines are all within K. The central, bounded triangle T degenerates into a point if the lines are concurrent. Throughout we will assume area$(K) = 1$. Buck and Buck showed the following:

- For any convex body K in the plane, there always exists three concurrent lines that equipartition K; that is, each of the six sectors they define has area $= 1/6$. This is called a *concurrent six-way equipartition*.

- No convex set can be partitioned by three lines into seven regions, each with area $1/7$. So 7-way equipartitions never exist for convex bodies in R^2.

- If six out of the seven regions do have the same area, then the central triangle T must be the one with the different area. And the area of T is $< 1/7$.

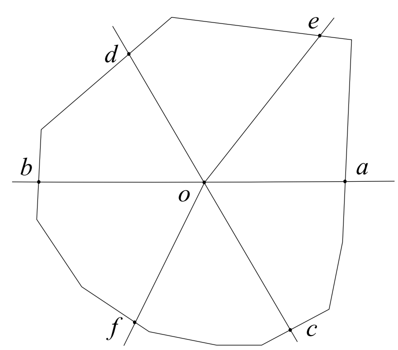

Figure 3.1: The existence of the six-way concurrent equipartition

The existence of the concurrent 6-way equipartition can be proved directly. See in Figure 3.1. We start with the unique horizontal line $\overline{ba}$ bisecting K. By the ham-sandwich theorem, there exists a unique halving line $\overline{dc}$ of K that meets $\overline{ba}$ at o such that area(bod) = area$(aoc) = 1/6$. There exists unique rays $\overline{oe}$ and $\overline{of}$ that separate K into 6 wedges with area $1/6$ each. If $\angle eof = 180°$, we have a concurrent 6-way equipartition. Otherwise, we rotate $\overline{ba}$ continuously, e.g. counter-clockwise $180°$, and preserve the above structure. Notice that

$$\angle eof_{begin} + \angle eof_{end} = 360°.$$

And the $\angle eof$ also changes continuously. So at some position of $\overline{ba}$, $\angle eof$ is $180°$, and this gives the concurrent 6-way equipartition.

Let the area of the central triangle T be t. To make things concrete we give,

Definition 3.1 *Given a convex body K in R^2 with* area$(K) = 1$, *lines l_1, l_2, l_3 form a* ***six-way equipartition*** *of K if*

(i) the points $P_{ij} = l_i \cap l_j, i < j$ are in K,

(ii) the triangle $T = \triangle P_{12}P_{13}P_{23}$ has area t, and

(iii) each of the other six regions of $K \backslash T$ has area $(1-t)/6$.

From Buck and Buck's results, we can see that it is always possible to find a six-way equipartition for every K when $t = 0$ and when $t = 1/7$ it is never possible. They stated the conjecture that in every six-way equipartition with three lines, area$(T) = t$ is at most $t_0 = 1/49$. This is the value that does occur when K is, itself, a triangle. It is easily seen that this K may be six-way equipartitioned by lines parallel to its own sides such that $t = t_0$.

The fact that t_0 is the maximum central triangle area over any possible six-way equipartition of any convex body K was later proved by Sholander[34], and he showed that the upper bound occurs when K is a triangle. See also [18, 19, 20] for another proof given by Eggleston.

Here we reopen an apparently unexplored aspect of the six-way equipartitioning. Surprisingly it is the crucial question concerning existence of 6-way equipartitions with $t > 0$. Our main result is

Theorem 3.2 *Given a convex body $K \subset R^2$ with* area$(K) = 1$ *and a unit vector $d \in R^2$, there exists a unique trio of lines that form a six-way partition of K, and where one of them has normal vector d.*

Before this result, the basic question of the existence was open except when $t = 0$. According to Theorem 3.2, for each convex body K and $\theta \in [0, 2\pi)$ there is a unique six-way equipartition of K where one of the lines has direction $d = (\cos\theta, \sin\theta)$. We write $f_K(\theta)$ as the area t of the central triangle in the partition. In Section 3.5, we will study this function and use it to characterize certain convex sets K.

3.2 Structure and Area Functions

Before we prove Theorem 3.2, we discuss our approach and introduce the notations and ideas that will be used..

3.2.1 Structure of Three Cutting Lines

Given a convex body $K \subset R^2$ with area$(K) = 1$, we introduce three cutting lines l_0, l_1 and l_2. If their intersection points are within K, they create a six-way equipartition if and only if they satisfy the following properties:

(C1) Each line l_i cuts K into two parts: K_i^- with area α for some $\alpha \leq \frac{1}{2}$. And the other part, K_i^+ has area $1 - \alpha$;

(C2) For each i, K_i^- is equipartitioned into three regions by two other lines $l_j, j \neq i$.

Clearly, when l_0, l_1 and l_2 satisfy both (C1) and (C2), the central triangle $T = \bigcap_{i=1}^{3} K_i^+$.

Given a fixed a unit vector $d \in R^2$, we let l_0 have the direction d. Without loss of generality, we assume that d is the vertical direction, so l_0 is a vertical line.

Let $\alpha \leq 1/2$ be the area of K_0^-, the smaller part cut by l_0. Let l_1^+ denote the halfplane above l_1, l_2^+ the halfplane above l_2. We choose lines l_1 and l_2 satisfying following invariants.

(I1) area$(l_1^+ \cap K_0^-) = \alpha/3$, area$(l_1^+ \cap K_0^+) = 2\alpha/3$;

(I2) area$(l_2^+ \cap K_0^-) = 2\alpha/3$, area$(l_2^+ \cap K_0^+) = 1 - 5\alpha/3$.

For example, as shown in Figure 3.2, the part to the right of l_0 has smaller area $\alpha \leq 1/2$, it is K_0^-. And A_1, the intersection of K_0^- and l_1^+, has area $\alpha/3$, and $A_5 \cup A_6$ $(= l_1^+ \cap K_0^+)$ have total area $2\alpha/3$. Similarly, $A_1 \cup A_2$ $(= l_2^+ \cap K_0^-)$ have total area $2\alpha/3$, and $A_6 \cup T$ $(= l_2^+ \cap K_0^+)$ has total area $1 - 5\alpha/3$, or $A_4 \cup A_5$ have total area $2\alpha/3$. So in this case, A_1, A_2 and A_3 have the same area $\alpha/3$. If A_5 also has area $\alpha/3$, l_0, l_1, and l_2 create a six-way equipartition.

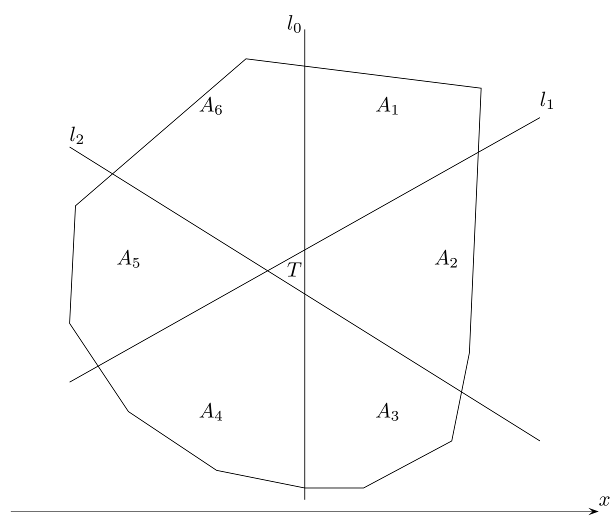

Figure 3.2: Three cutting lines and seven regions. l_0, l_1 and l_2 are bisectors of K when $x = 0$ and $\delta(x) = 0$.

Once the position of l_0 is fixed, the lines l_1 and l_2 satisfying (I1) and (I2) respectively are both unique since K_0^- and K_0^+ are separated convex bodies in the plane. And the seven regions are also determined.

Since l_0 is a vertical line, we describe the position of l_0 using its x-coordinates. We take $x = 0$ when l_0 is the vertical bisector of K. In this case, we have $\alpha = 1 - \alpha = 1/2$, we pick K_0^- as the right side. But in fact, no matter how we pick K_0^-, l_1 and l_2 are the same set of lines.

On the other hand, lines l_1, l_2 satisfying both (I1) and (I2) automatically satisfy (C1). If the intersection of l_1 and l_2 is within K_0^+, as in Figure 3.2, l_0, l_1, and l_2 satisfy (C2) when A_5 has the same area as A_1 and create a six-way equipartition. That is, we meet a six-way equipartition when A_5 and A_1 have the same area with invariants (I1) and (I2).

Fact: A six-way equipartition with l_0 of direction d must satisfy (I1) and (I2) and A_5 and A_1 have the same area. Furthermore, if the central triangle of six-way equipartition T has area t, then $\alpha = \frac{1-t}{2}$.

3.2.2 Area Functions

Let $\delta \equiv \frac{1}{2} - \alpha$. By the definition of α, we have $\delta \geq 0$. And δ is a function of x. We write is as $\delta(x)$. When $x = 0$, $\delta(x) = 0$ since $\alpha = 1/2$. If l_0, l_1 and l_2 are concurrent, we have a concurrent six-way equipartition. Otherwise, the central triangle, T, will have positive area. Without loss of generality, we assume that T is on the left side (the side where $x < 0$) of l_0. We label the other six regions as $A_1, ..., A_6$ in clockwise order as shown in Figure 3.2.

We denote the area of A_i as $A_i(x)$, $i = 1, ..., 6$ and denote the area of T as $T(x)$. When the value of x is given, all cutting lines l_0, l_1 and l_2 are uniquely determined. So are the regions of $A_1, ..., A_6$ and T. Thus $A_i(x)$'s and $T(x)$ are well defined functions of x. The first observation is that we can exclude the cases for $x < 0$.

Lemma 3.3 *Given the cuts as in Figure 3.2 when $x = 0$, that is, the central triangle is on the left of l_0, there is no six-way equipartition for $x < 0$.*

Proof: For a fixed $x < 0$, the vertical cuts moves to $l_0(x)$. In this case, K_0^- is on the left of l_0. If triangle T is on the left of l_0, we have $A_1(x) = A_3(x) = \frac{1}{6} - \frac{\delta(x)}{3}$. By invariants

(I1) and (I2), we have,

$$A_2(x) = \frac{1}{2} + \delta(x) - A_1(x) - A_3(x) = \frac{1}{6} + \frac{5}{3}\delta(x).$$

If triangle T moves to the right of $l_0(x)$, then $A_1(x) = A_3(x) = \frac{1}{6} - \frac{\delta(x)}{3} - T(x)$. And

$$A_2(x) = \frac{1}{2} + \delta(x) - A_1(x) - A_3(x) - T(x) > \frac{1}{6} + \frac{5}{3}\delta(x).$$

We always have $A_2(x) > \frac{1}{6}$, it is impossible to find a six-way equipartition. ■

So in what follows, we assume that $x \geq 0$. By invariants (I1) and (I2), we will have the following equalities:

Proposition 3.4

- *$A_1(x) = A_2(x) = A_3(x) = \frac{1}{6} - \frac{1}{3}\delta(x)$;*
- *$A_6(x) = A_4(x)$;*
- *$A_5(x) = \frac{1}{6} - \frac{2}{3}\delta(x) - A_4(x) = \frac{1}{6} - \frac{7}{3}\delta(x) + T(x)$.*

And $A_5(x) = A_1(x)$ if and only if l_0, l_1 and l_2 create a six-way equipartition of K. Especially when $x = 0$, $\delta(0) = 0$, we have,

- $A_1(0) = A_2(0) = A_3(0) = \frac{1}{6}$;
- $A_6(0) = A_4(0) = \frac{1}{6} - T(0)$;
- $A_5(0) = \frac{1}{6} + T(0) > A_1(0)$.

We will prove Theorem 3.2 in two steps:

1. First we move l_0 continuously, starting from where it bisects K to $x_{\max} > 0$ where it is the right vertical tangent to K. Throughout l_1 and l_2 satisfy invariants (I1) and (I2). We will show that during this process, all area functions change continuously.

In fact, they have continuous derivatives with respect to x. If a six-way equipartition exits, we will meet it at some $x \in [0, x_{\max}]$.

2. Then we establish the existence by showing that $A_1(x) = A_5(x)$ for some $x' \in [0, x_{\max}]$. From the invariants, area functions $A_1(), ...A(6)$ are equal at $x = x'$. We also show that once $x > x', A_5(x) < A_1(x)$. Since $A_5(0) \geq A_1(0)$, once we pass a six-way equipartition in the continuous motion of l_0, we never meet another one. Thus there is a unique 6-way equipartition with one line vertical.

3.3 Two Useful Lemmas

In order to prove Theorem 3.2, we need to study the area functions $A_i(x)$'s and $T(x)$ first. The following lemma is about the smoothness of these functions. This result may be well known, but we still give the proof.

Lemma 3.5 *$A_i(x)$'s and $T(x)$ are continuous functions on x. Furthermore, the first derivatives of $A_i(x)$'s exist and are continuous.*

Proof: Suppose x is changed to $x + \Delta x$ where $\Delta x > 0$ and small. And the chord on l_0 moves from $\overline{CD}$ to $\overline{C'D'}$ as shown in Figure 3.3. By the convexity of K, the differences between lengths $|CD|$ and $|C'D'|$ is $O(\Delta x)$ when $A_1(\cdot)$ keeps positive. So the area of K_0^- reduces with amount $|CD|\Delta x + O(\Delta x^2)$. This implies that $A_1(x), A_2(x)$ and $A_3(x)$ are continuous functions and the first derivatives of these functions is just the one third of $|CD|$. And the length $|CD|$ is a continuous function with respect to x by the convexity of K.

Now consider the movement of l_1, as shown in Figure 3.3. Let P_x be the x-coordinate value of P, the intersection of lines l_0 and l_1 and we write l_1 as the line $y = ax + b$ and new position l_1' as $y = (a + \Delta a)x + (b + \Delta b)$.

Notice that the part of K above l'_1, K_1^- will have the same area as the part of K on the right of l'_0, K_0^- by the invariants (I1) and (I2). So we have, (the area functions used here are with sign, for example, the area below EF will cancel out the area above EF.)

$$\begin{aligned} \text{area}(EE'F'F) &= \text{area}(CC'D'D) \\ \text{area}(CC'Q'P) + \text{area}(P'F'FQ') &= \frac{1}{3}\text{area}(CC'D'D) \end{aligned}$$

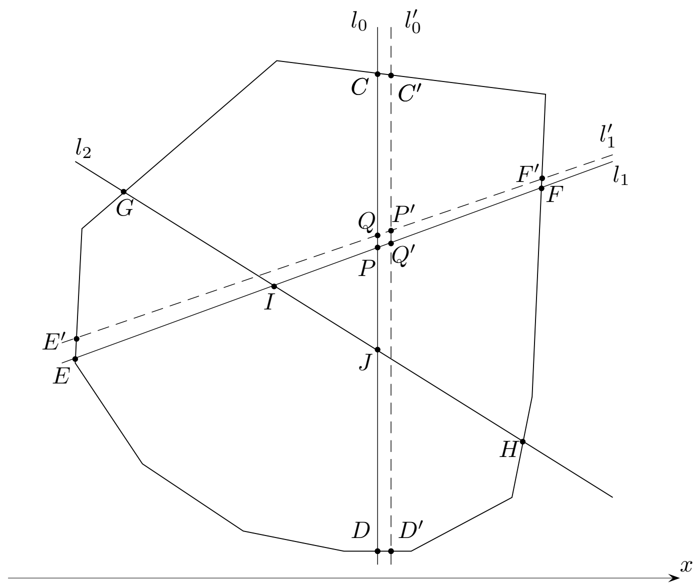

Figure 3.3: The position of l_0 moves from x to $x + \Delta x$. We only show the movements of l_0 and l_1. New positions are l'_0 and l'_1.

By the convexity of K, the x-coordinates difference between F' and F is $O(|\Delta a|+|\Delta b|)$. The same is true for E' and E, and we may rewrite the above equation as,

$$\int_{E_x}^{F_x} [\Delta a x + \Delta b] dx = |CD|\Delta x + O(|\Delta a| + |\Delta b| + \Delta x)^2;$$

$$\int_{P_x}^{F_x} \Delta a x + \Delta b dx = \left[\frac{1}{3}|CD| - |CQ|\right] \Delta x + O((|\Delta a| + |\Delta b| + \Delta x)^2.$$

One consistent solution of the above equations exists when $|\Delta a| = O(\Delta x), |\Delta b| = O(\Delta x)$. By the uniqueness of l_1', this is the only solution. In such a case, as $\Delta x \to 0$,

$$(F_x^2 - E_x^2)\Delta a + (F_x - E_x)\Delta b \approx |CD|\Delta x$$

$$(F_x^2 - P_x^2)\Delta a + (F_x - P_x)\Delta b \approx \left[\frac{1}{3}|CD| - |CQ|\right]\Delta x.$$

This shows that $\Delta a, \Delta b$ are changing linearly with respect to Δx. If we write l_1 as $y = a(x)z + b(x)$ where x is the current position of l_0, then $a(\cdot), b(\cdot)$ are both continuous functions. Furthermore, the first derivatives of $a(\cdot)$ and $b(\cdot)$ depend only on the shape of K and x and they are continuous functions of x.

Applying the same arguments on l_2, we can see that the "movement" of I, the intersection of l_1 and l_2 is $O(\Delta x)$. And the change of $A_5(\cdot)$ on segments EI and GI are both continuous with continuous first derivatives. So $A_5(\cdot)$ is also a continuous function with continuous first derivative.

Using the fact in Proposition 3.4, we can see that $A_4(\cdot), A_6(\cdot)$ and $T(\cdot)$ are also continuous functions with continuous first derivatives. ■

One implication of Lemma 3.3 is that triangle T is always on the left of l_0: given $T(\cdot)$ is continuous, if T moves to the right of l_0, then there exists some x such that $T(x) = 0$, we will have a concurrent six-way equipartition. But this contradicts the fact that $A_1(x) < \frac{1}{6}$ for all $x > 0$.

We also need to compare the first derivatives of the area functions, which in turn depend on the length of segments as shown in Figure 3.3. The following lemma is about the length of segments introduced by two cuts for a convex body K when each cut "slices off" K with one piece having area α.

Lemma 3.6 *Given a bounded convex body K in R^2, two cuts both "slice off" K with one piece having area α and intersects at point o as in Figure 3.4. So two cuts divides K into 4 regions R_1, R_2, R_3 and R_4 such that R_1 and R_3 have the same area. Using the labels in Figure3.4, we have the following inequalities,*

$$\frac{|ao|}{|ab|} + \frac{|co|}{|cd|} \geq \frac{\text{area}(R_4)}{\text{area}(R_1) + \text{area}(R_4)}, \frac{|bo|}{|ab|} + \frac{|do|}{|cd|} \geq \frac{\text{area}(R_2)}{\text{area}(R_1) + \text{area}(R_2)}. \quad (3.1)$$

And the larger of the sums in (3.1) is at least 1. They are both 1 if one is 1.

Proof: Draw the line $\overline{ad}$ passing both a and d and line $\overline{bc}$ passing both b and c. If $\overline{ad}$ and $\overline{bc}$ are parallels, then both sums are 1.

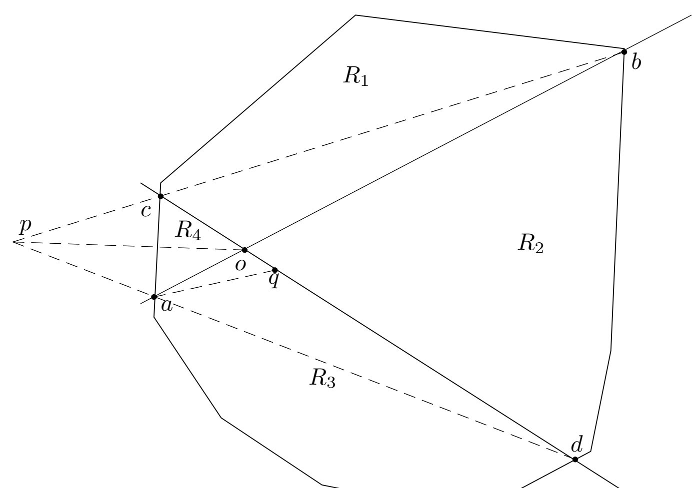

Figure 3.4: Lengths of segments of two cuts.

Otherwise, w.l.o.g, $\overline{ad}$ intersects with $\overline{bc}$ at point p as shown in Figure 3.4. We connect o and p, We also draw a line $\overline{aq}$ which is parallel to $\overline{bc}$ and interests with $\overline{od}$ at q. Then it is easy to check,

$$\frac{|bo|}{|ab|} + \frac{|do|}{|cd|} = \frac{|co|}{|cq|} + \frac{|do|}{|cd|} > \frac{|co|}{|cd|} + \frac{|do|}{|cd|} = 1.$$

And we have

$$\frac{|co|}{|cd|} = \frac{\text{area}(\triangle pco)}{\text{area}(\triangle pcd)}, \frac{|ao|}{|ab|} = \frac{\text{area}(\triangle pao)}{\text{area}(\triangle pab)}.$$

Say area($\triangle pab$) $\geq$ area($\triangle pcd$), we have,

$$\frac{|ao|}{|ab|} + \frac{|co|}{|cd|} \geq \frac{\text{area}(pcoa)}{\text{area}(pcoa) + \text{area}(\triangle ocb)}.$$

By the convexity of K, R_4 is contained in quadrilateral $pcoa$, and $\triangle ocb$ is contained in R_3, we have,

$$\frac{|ao|}{|ab|} + \frac{|co|}{|cd|} \geq \frac{\text{area}(R_4)}{\text{area}(R_4) + \text{area}(\triangle ocb)} \geq \frac{\text{area}(R_4)}{\text{area}(R_1) + \text{area}(R_4)}.$$

We can also see that for the smaller sum, the inequality becomes equality only when R_4 is exact the quadrilateral $pcoa$ (that is, $\triangle pac \subseteq R^4$), and R_1, R_3 are triangles $\triangle ocb$ and $\triangle oab$ which have the same area. ■

3.4 Existence and Uniqueness of Six-way Equipartition

Now we are ready to prove Theorem 3.2. In previous section, we showed that when $x = 0$, $A_5(0) > A_1(0) = \frac{1}{6}$. As in Figure 3.3, for small Δx, the area of A_1 is reduced on segment CQ and PF and the total reduced area is $\frac{1}{3}|CD|\Delta x$. And on l_1, the average movement(orthogonal to $\overline{EF}$) will be $\Delta_1 = \frac{|CD|}{|EF|}\Delta x$. Now by Lemma 3.6, we have

$$\frac{|CQ|}{|CD|} + \frac{|PF|}{|EF|} \geq \frac{1}{3}.$$

This implies the average movement on $\overline{PF}$ is no greater than Δ_1. So the average movement on $\overline{EP}$ is larger than Δ_1, so on segment $\overline{EI}$. By similar argument, the average movement (in absolute value) on $\overline{GI}$ is larger than $\frac{|CD|}{|GH|}\Delta x$. This shows that $A_5(x)$ is a decreasing function on x.

Keep moving line l_0, $A_1(x) = 0$ when l_0 is tangent to K, say for $x = x_{\max}$. But for some $x_0 < x_{\max}$, l_1 and l_2 intersect on the boundary of K and region A_5 vanishes. So $A_1(x_0) > A_5(x_0) = 0$. Since the $A_i(x)$'s are continuous functions, there exists at least one $x \in (0, x_0)$ such that $A_1(x) = A_5(x)$. And this is a six-way equipartition.

When $A_5(x) \geq A_1(x)$, by Lemma 3.6, we also have,

$$\frac{|JH|}{|GH|} + \frac{|JD|}{|CD|} \geq \frac{1}{3}, \frac{|GI|}{|GH|} + \frac{|EI|}{|EF|} \geq \frac{1}{3}.$$

As shown in the proof of Lemma 3.6, one inequality becomes equality only when, say K_0^- and K_1^- are both triangles and A_1 is the quadrilateral $K_0^- \cap K_1^-$. So for any convex $K \subset R^2$, at most one of three such inequalities can be equality. So $A_5(x)$ decreases faster than $A_1(x)$ when $A_5(x) > A_1(x)$. By the continuities of the derivatives, there exists a $\theta > 0$ such that for all x with $A_5(x) \geq A_1(x) - \theta$, we have

$$\frac{dA_5(x)}{dx} \leq \frac{dA_1(x)}{dx} < 0.$$

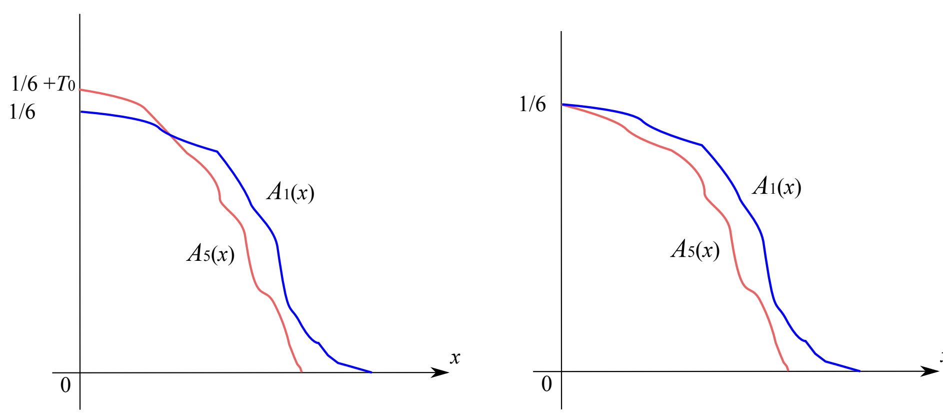

Figure 3.5: Area functions A_1 and A_5. The right one is for $T(0) = 0$, the concurrent six-way equipartition case.

The uniqueness of the intersection of $A_1(x)$ and $A_5(x)$ follows from the fact that once $A_5(x) < A_1(x)$, $A_5(x)$ cannot exceed $A_1(x) - \theta$ for larger x. So the six-way equipartition is unique. The argument also applies to the concurrent six-way equipartition case. Figure 3.5 shows the graph of functions $A_1(\cdot)$ and $A_5(\cdot)$.

3.5 The Area of Central Triangle of Six-Way Equipartition

Another interesting function is the area of the central triangle in the six-way equipartition as the function of the direction of the one line. We can define the direction of l_0 as $d = (\cos\theta, \sin\theta), \theta \in [0, \pi)$, and the area of the central triangle as $f_K(\theta)$. By Theorem 3.2, $f_K(\theta)$ is a well defined function.

Following the same idea as in the proof of Lemma 3.5, starting from a 6-way equipartition with fixed θ, once we change θ by a small angle δ, areas functions $A_1(\cdot)$ and $A_5(\cdot)$ are changed proportionally to δ, as are their differences. By Lemma 3.5, $A_1(\cdot)$ and $A_5(\cdot)$ have continuous first derivatives, so $f_K(\theta) - f_K(\theta + \delta) = O(\delta)$. Thus, $f_K(\theta)$ is a continuous function on θ.

We can characterize $f_K(\theta)$ for a convex set $K \subset R^2$,

- From the existence of concurrent six-way equipartition, $f_K(\theta)$ has at least 3 zeros since we restrict θ within $[0, \pi)$.
- Sholander's results[34] shows that $\max_K f_K(\theta) = 1/49$ for all $K \in R^2$ with area 1. And $f_K(\theta) = 1/49$ when K is a triangle and one side has direction $(\cos\theta, \sin\theta)$.

Furthermore, we have the following interesting properties as immediate corollaries of Theorem 3.2:

Lemma 3.7 *If convex region K is central symmetric, three lines in a 6-way equipartion must always be concurrent, i.e., $f_K(\theta) \equiv 0$.*

When K have an axis of symmetry, we can move the point p along the axis starting from the intersection of the boundary and the axis. We also make the two trisecting rays starting from p for each half on both sides of the axis. At some position, four trisecting rays become two lines by symmetry, that is a 6-way equipartition. So we have the following:

Lemma 3.8 *If K has a axis of symmetry, there exist concurrent 6-way equipartition with one of line being the axis.*

For example, when K is a regular n-gon, we have

- For even n, $f_K(\theta) \equiv 0$ since the regular n-gon has a center of symmetry.
- For odd n, regular n-gon has n axes of symmetry where each axis passes one vertex. When n is divisible by 3, the concurrent 6-way equipartition with one line passing a vertex must have all lines passing vertices. So we have n θ's with $f_K(\theta) = 0$. If n is odd and not divisible by 3, we have n concurrent 6-way equipartitions with one line passing a vertex. Using Theorem 3.2, it's easy to show that all n concurrent 6-way equipartitions are different. So we have at least $3n$ θ's in $[0, \pi]$ with $f_K(\theta) = 0$.

When n is even, clearly regular n-gon minimizes $\max f_K(\theta)$ for all polygons with n sides. This is not true for n odd and $n > 3$: we can always cut one angle of a regular $(n-1)$-gon and make the cut off small enough. By Lemma 3.5, $\max f_K(\theta)$ will also be smaller than that of regular n-gon. But we have the following conjecture:

Conjeture 3.9 *Regular n-gon maximizes the number of different concurrent six-way equipartitions over all polygons with n sides.*

This conjecture is true when n even and when $n = 3$. The first none trivial case is $n = 5$.

3.6 Summary

In this chapter we reopen an apparently unexplored aspect of the six-way equipartitioning by three lines in the plane. We prove the existence and uniqueness of the six-way equipartitioning when the direction of one line is given.

For a convex set K in the plane, we also introduce the function $f_K(\theta)$ as the area of the central triangle where θ is the given direction. We show that $f_K(\theta)$ is related to the symmetry of K.

Chapter 4

Voronoi Games

In this chapter, we discuss a simple geometric version of a location game, the Voronoi game. In the Voronoi game two players A and B each play n points in a bounded playing arena $S \subset R^d$. In this game, a point p "owns" the part of the playing arena that is closer to p than to any of the other $2n-1$ points; that is the Voronoi cell of p. There are two ways to play the game, one round and alternating. In the one round version, A places all his n points and then B places her n points. In the alternating version A plays and then B plays, in each of the n successive rounds. The score for a player in the game is the total volume of his/her Voronoi cells.

The one dimensional Voronoi game was recently studied by Ahn et. al.[1]. The game is played on a closed curve or open curve (e.g., a segment). When plays on a segment, A can always win the one round version by playing equally spaced points, the so called critical points. B can control this amount by which she loses. In the alternating version, B has a winning strategy.

Less is known for dimension $d \geq 2$. Cheong et. al. [12] took S to be a square of area one, and studied the one-round version, where the situation is opposite to the $d = 1$ segment game. Using a beautiful probabilistic argument they proved that for $n > n_0$, B has a winning strategy. They also extended these results to higher dimensions and showed that B still has a winning strategy for if n large enough.

An interesting followup paper by Fekete and Meijer [21] explored the continuum between a square and a segment by allowing S to be a rectangle of base x and height $1/x$. As $x \geq 1$ increases, S becomes more "linear", more like a segment. At a certain point, the behavior of the one round Voronoi game on $S \subset R^2$ switches to be that of the game on a segment: once $x \geq \sqrt{n/2}, n \geq 3$, there are again critical points which, if A takes them in his move, will guarantee victory. Almost nothing is known about the alternating version in dimension $d \geq 2$.

In this chapter, we amplify the results of Cheong et. al. [12]. We investigate A's best strategy to minimizing his disadvantage in several different ways. Some results were presented in [35].

4.1 Previous Results

In the one dimensional game, Ahn et. al.[1] first studied the one round game. On the a simple closed curve, e.g., the unit circle in R^2, B can achieve at least half of the total curve length by the simple strategy of placing one point within each interval created by A's n points thereby winning half of each interval. In fact unless A plays correctly (by evenly spacing his n points), B can win as follows:

1. Placing two points in the longest of A's intervals, arbitrarily close to the endpoints, (B gets nearly all, i.e. $\max_I -\delta$);

2. Placing one point arbitrarily in each of the remaining intervals except the smallest(B get 1/2 of each);

3. Avoiding the shortest of A's intervals with length $\min_I$.

In this way, B wins because A did not play evenly spaced points. B gets totally $\max_I - \delta$ from the longest and shortest intervals, combined, and half of the other $n-2$ intervals. With $\max_I - \min_I = \sigma > 0$ and as long as $\delta < \sigma/2$,

$$\max_I - \delta > \frac{\max_I + \min_I}{2}$$

greater than even share. So the A's best strategy is to place her points equally-spaced (these locations are called "critical" points), and in this case the best result of player B is a tie.

If S is "cut" and opened into a finite interval, or segment, these same observations can be used to show that now A can always win the one-round version by playing the "critical" points; $1/(2n), 3/(2n), ..., (2n-1)/(2n)$ along the length of S. There are $n+1$ intervals, two of length $L/2n$, the rest of length L/n. By A's placement, B must concede one interval and a positive part of another. She therefore loses, though by a margin she can control.

In the alternating version, B now always has the winning strategy by getting as many critical points as possible and then playing cleverly. Notice for the closed curve, the positions of the critical points depends on the position of first A's point. Player B needs to cover at least one "critical" point and has a winning strategy, but A can capture at least $\frac{1}{2} - \varepsilon$ of the curve for any $\varepsilon > 0$, in other words, she can make B's wins as small as possible.

For $d = 2$, Cheong et. al. [12] took S to be a square of area one, and studied the one-round version. First, they study the square $\mathcal{Q}$ under the topology of a torus by identifying the top edge with the bottom edge, and the left edge with the right edge. Let the area of $\mathcal{Q}$ be one. In the one round game, once player A plays all his n points in $\mathcal{Q}$, for a random point x, the expected area of the Voronoi region of x is

$$\mathbb{E}[\mathrm{vol}(\mathrm{Vor}(x, \mathcal{A}))] = \frac{1}{n} \int_{\mathcal{Q}} \mathrm{vol}(\{x \in \mathcal{Q} : y \in \mathrm{Vor}(x, \mathcal{A})\}) dy.$$

Here vol$(\cdot)$ is the measure(or area), $\mathcal{A}$ is the point set of player A, Vor$(x, \mathcal{A})$ is the Voronoi cell of x w.r.t $\mathcal{A}$ on $\mathcal{Q}$. We use dist(x, y) to denote the l_2-distance between two points $x, y \in R^2$. We also define dist$(x, \mathcal{S})$ for a point x and some points set S as

$$\mathrm{dist}(x, \mathcal{S}) := \min_{s \in \mathcal{S}} \ \mathrm{dist}(x, s).$$

A point $y \in \mathcal{Q}$ lies in Vor$(x, \mathcal{A})$ if and only if dist$(y, x) < r =$ dist$(y, \mathcal{A})$, so we have,

$$\{x \in \mathcal{Q} : y \in \mathrm{Vor}(x, \mathcal{A})\} = \{x \in \mathcal{Q} : \mathrm{dist}(x, y) < r\}.$$

By splitting $\mathcal{Q}$ by the Voronoi cells of $\mathcal{A}$ only, we have

$$\mathbb{E}[\mathrm{vol}(\mathrm{Vor}(x, \mathcal{A}))] = \frac{\pi}{n} \sum_{w \in \mathcal{A}} \int_{\mathrm{Vor}(w)} \mathrm{dist}(y, w)^2 dy. \tag{4.1}$$

where Vor(w) is the Voronoi cell of the point w.

On the other hand, by the following lemma by L. Fejes Tóth (see [30].),

Lemma 4.1 *Let $O = \{O_1, ..., O_n\}$ be n points in the plane, let H be a regular hexagon, and f be a monotone increasing function, then*

$$\int_H f(\mathrm{dist}(x, O)) dx \geq n \int_{H'} f(|x|) dx,$$

where H' is a regular hexagon with $A(H') = A(H)/n$, centered at $\boldsymbol{0}$.

Since the equilateral triangular lattice has regular hexagonal Voronoi cells, this implies that the equilateral triangular lattice minimizes the expected area of one point played at *random* by B, under the torus topology.

4.2 Equilateral Triangular Lattice

On the equilateral triangular lattice, each point p has 6 closest neighbors. The convex hull of these 6 closest neighbors is a regular hexagon with p as the center. The Voronoi region of p is also a regular hexagon.

We study B's strategy when A plays on equilateral triangular lattice. First, we show how to embed an equilateral triangular lattice in a square with torus topology. From now on, we will scale the $\mathcal{Q}$ to have area n to make this presentation clearer.

Let the distance between 2 lattice points on the equilateral triangular lattice be a. The hexagonal Voronoi cell for each point has area $\frac{\sqrt{3}}{2}a^2$. So when $a = \sqrt[4]{\frac{4}{3}}$, each point's Voronoi cell has area 1.

And the distance between two furthest layers on lattice is $\frac{\sqrt{3}}{2}a$, so we cannot embed the equilateral triangular lattice perfectly on $\mathcal{Q}$ when $\mathcal{Q}$ is a square. But by Minkowski's theorem (see [26][30]), we can always find two integers i_1 and i_2 such that,

$$\left|\frac{\sqrt{3}}{2} - \frac{i_1}{i_2}\right| \leq \frac{1}{i_2^2}.$$

Then we can set $n = i_1 i_2$, and let distance $a = \sqrt[4]{\frac{4}{3}}$ to embed the lattice on $\mathcal{Q}$. For i_2 (and so i_1) large enough, the embedding error is sufficiently small.

Now each point's hexagonal Voronoi cell will have area 1. And we consider the one round Voronoi game. Using Lemma 4.1, we already know that the equilateral triangular lattice minimizes the expected area of one random point played by B. In fact its value is easy to calculate. Two adjacent lattice point has distance $a = \sqrt[4]{4/3}$. Let one point at $\mathbf{0}$, its Voronoi cell H is the regular hexagon centered at $\mathbf{0}$ with area 1 and the length of each side H is $a/\sqrt{3}$. The minimum expected value as in Equation 4.1 is:

$$\pi \int_H |x|^2 dx = 12\pi \int_0^a \int_0^{\frac{x}{\sqrt{3}}} (x^2 + y^2) dy dx = \frac{5\sqrt{3}\pi}{54} = 0.5038....$$

So we have the following result:

Lemma 4.2 *Let $\mathcal{Q}$ be a square of torus topology with area n large enough, the expected area of the Voronoi region of a random point x on the torus is minimized when player A puts her n points on the equilateral triangular lattice, and the expectation is $\frac{5\sqrt{3}\pi}{54}$.*

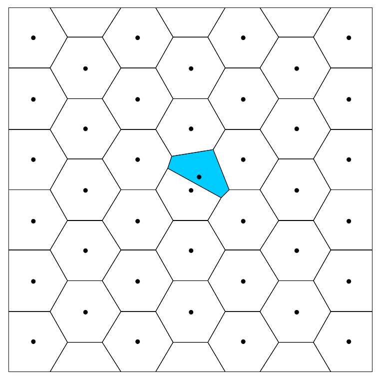

Figure 4.1: A random B's point on equilateral triangular lattice.

4.3 One Round Game on Lattices

We are interested in the question that what's B's best response when A plays on a lattice grid. We will study two lattice grid cases, the square grid and the hexagonal grid. For each grid, we will try to find the maximum area which B's first point can catch. Before we study the lattice grids, we show the following fact for any lattice.

Lemma 4.3 *Suppose A plays all her n points on some lattice on the torus $\mathcal{Q}$, and for each of A's points, B plays one point has the same offset $\vec{r}$. In other words, B places all his points on a shifted lattice. Then the result is a tie.*

Lemma 4.3 is easy to prove: let p be one bisecting point on the line segment of one pair of A's and B's point with offset $\vec{r}$. We do the $180°$ rotation centered at p. Then position of A's and B's lattices will swap. So the result must be a tie.

Now consider B's first move x. Let a_x be the closest point of A. If B places more points to make x on a shifted lattice from A's lattice, the area of the Voronoi cell of x is either reduced or not impacted. So Lemma 4.3 gives us the following corollary

Corollary 4.4 *Suppose A plays n lattice point on $\mathcal{Q}$, and B places a point x on $\mathcal{Q}$, then the area of the Voronoi region of x is at least* 0.5. *Furthermore, if we can place more points on A's neighbor point with the same offset as x from its closest point of A, and this does not impact the Voronoi cell of x, then the area of the Voronoi region of x is exactly* 0.5.

Square Grid

As the first step, we study the case in which player A places points on square grids. In this case, each of A's Voronoi cells is a unit square (before the player B moves).

For a particular point of A, it has 4 closest neighbors and 4 second closest neighbors. We call the line segment connecting the point to one closest neighbor "w-edge" and the line segment connecting the point to one second closest neighbor "w-diagonal".

Now we study the first move of player B. Call this point x. If x is placed on "w-diagonal"as shown in Figure 4.2, the Voronoi cell of x is an isosceles trapezoid. And if we place the other point like x in the neighbor cells, these trapezoids do not intersect. So we can see that the area of the Voronoi cell of x is minimized on these diagonals.

All w-diagonals split $\mathcal{Q}$ into small squares with area 0.5. So if x is placed on the boundary of these squares, $\text{area}(\text{Vor}(x, \mathcal{A})) = 0.5$. Another property of these square is that if x is placed as the interior point in one of these squares, the set of A's points which determine the Voronoi cell of x remains the same, and they are in convex position. By a

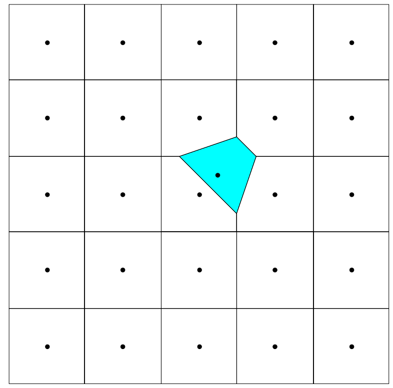

Figure 4.2: Square lattice: The area of B's Voronoi cell is 0.5.

result of Dehne et al, [14], there is a unique maximal $x_{\max}$ within one such square which maximizes the area of its Voronoi cell.

By symmetry of the square, the only possible $x_{\max}$ must be the center of the small square, which is also the mid point of one w-edge. And the maximal area is 0.5625.

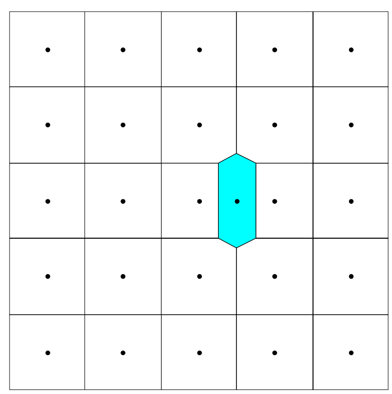

Figure 4.3: Square lattice: The maximal of B's Voronoi cell.

Player B may have two classes of strategies to win the game:

W1 She places some points on locations with the maximal Voronoi region. By Lemma 4.3, she cannot place all her points on such locations. Then she try to break player A's largest Voronoi regions;

W2 She places her points on some period "stripe" locations.

The following configurations are for W1. The configuration with win 0.5208 is the best strategy we could find.:

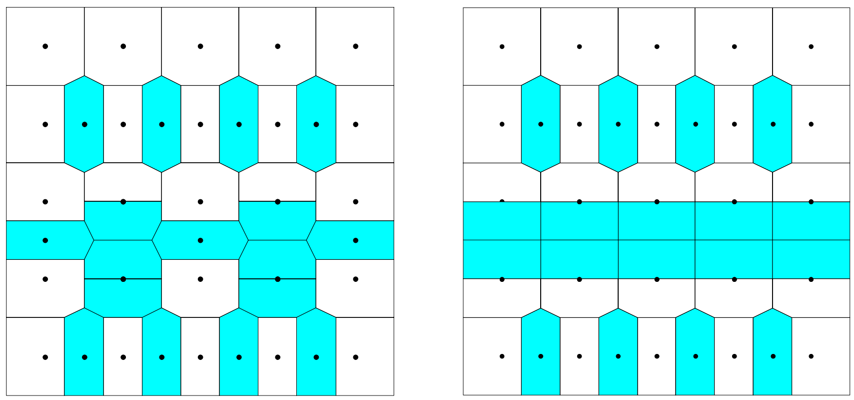

Figure 4.4: Two B's strategies with wins $0.5156, 0.5208$.

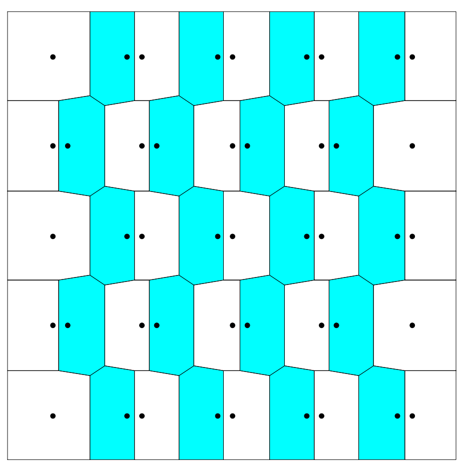

Figure 4.5: Stripe case of B, wins 0.5185.

For strategies of type W2, we have following lemma.

Lemma 4.5 *Using strategy of type W2, player B cannot have wins* > 0.5185.

Proof: We can see that period is 2. Let coordinates of one B's point be (a, b), the closest B's point are $(1-c, 1+d)$ and $(1-c, -1+d)$ with $\frac{1}{2} \geq a > b \geq 0$. It is easy to show that the optimal will need the $\frac{1}{2} \geq c > d \geq 0$.

We can show that the area will increase if we project the points to w-edges or let $b = 0, d = 0$. Then the wins of B will be 0.5 plus a factor times $(1-a-c)^2(a+c)$, it will be maximized when $a + c = \frac{1}{3}$. Figure 4.5 shows the corresponding Voronoi diagram when $a = c = \frac{1}{6}$ and B wins 0.5185. ■

Hexagonal Grid

If A plays on the equilateral triangular lattice, we have similar results to the square lattice case.

First of all, if B's first point x is on the boundary of one regular hexagonal Voronoi cell or she places a point one the segment connecting two neighbor lattice point, by the similar argument as in square grid which is based on Corollary 4.4, we must have $\text{area}(\text{Vor}(x, \mathcal{A})) = 0.5$.

One hexagonal cell is separated by these minimal lines into 6 similar quadrilaterals. Again, if x is placed in the same quadrilaterals, its Voronoi cell is given by 5 points of A in the convex position. Let these points be $(0,0), (0,2), (0,-2), (\sqrt{3}, 1), (\sqrt{3}, -1)$ (Here we scale the unit to make the calculation easier). And the quadrilateral will be the convex hull of $(0,0), (\frac{\sqrt{3}}{2}, \frac{1}{2}), (\frac{2}{\sqrt{3}}, 0), (\frac{\sqrt{3}}{2}, -\frac{1}{2})$.

Again, by the result in [14], there is only one point whose Voronoi region has local maximal area in the quadrilateral. And since the quadrilateral is symmetric about the line $y = 0$, the position of the local maximal must be on the line segment between $(0,0)$ and $(\frac{2}{\sqrt{3}}, 0)$.

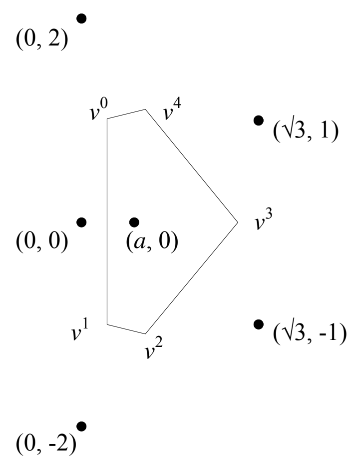

Figure 4.6: The maximal in hexagonal grid case.

Let $x = (a, 0), 0 < a < \frac{2}{\sqrt{3}}$ be a point on this line segment. Then we have,

$$f(a) = \text{area}(\text{Vor}(x, \mathcal{A})) = \left(\sqrt{3} + \frac{3a^3 - 4\sqrt{3}a^2 + 4a}{4(a^2 - 3\sqrt{3}a + 6)}\right).$$

The function $f(a)$ is maximized when $f'(a) = 0$. So $a_{\max}$ is a root of the following polynomial:

$$g(x) = \sqrt{3}x^3 - 16x^2 + 18\sqrt{3}x - 12.$$

The numerical result shows $a_{\max} = 0.511953...$ and area(Vor($a_{\max}$)) is 0.51273.... So we have the following result:

Lemma 4.6 *If A plays on the equilateral triangular lattice, the maximal area of the Voronoi region for a single point played by B is* 0.51273.

Clearly B cannot win 0.51273 in this case, we tried different configurations and cannot find the win of B exceed 0.5064.

4.4 Minimizing Maximal Area of Player B's First Move

From the results of the previous section, we have a strong belief that playing on the equilateral triangular lattice is A's best strategy to minimize the area of B's first Voronoi cell. In this section, we will use computer simulations and optimization techniques to support this claim.

Euler Formula for the Torus

First of all, the Voronoi diagram is a planer graph with vertices (without player's points), faces and edges. For a connected planer graph, the well-known Euler Formula is

$$f - e + v = 2,$$

where f is the number of cells(faces), e is the number of edges and v is the number of vertices. For the torus topology, the formula is changed to:

$$f - e + v = 0.$$

Since each vertex is adjacent to at least 3 edges and each edge is adjacent to exactly 2 vertices, we have,

$$2e \geq 3v.$$

But this implies that $e \leq 3f, v \leq 2f$. So the average number of Voronoi edges per cell on the torus is at most 6. Now we consider an arbitrary Voronoi diagram with n cells on the torus.

If some cell is triangle, its contribution to the Euler's formula is $c \leq 1 - 3/2 + 1 = 1/2$. For quadrilateral cell, the contribution is $c \leq 1 - 2 + 4/3 = 1/3$. Similarly, we have the following table for the upper bound of the c:

i	3	4	5	6	7	8	9	...
max(c)	1/2	1/3	1/6	0	-1/6	-1/3	-1/2	...

For example, if we have an enneagon(9-gon) cell in the Voronoi diagram, we also need to have at least one triangle cell, or one quadrilateral cell with one pentagon cell or 3 pentagon cells in the diagram.

Now consider the Voronoi diagram generated by A's points. We want to calculate the maximal area that one point can capture for different shapes of the Voronoi cells by simulation. We already showed that the maximal area of B's single point is 0.51273 if player A plays on the equilateral triangular lattice. So if any cell has area greater than 1.0255, B can always always find a position to capture at least half area of it, which means A has a worse result than equilateral triangular lattice case. So the first condition is that no cell can have area greater than 1.0255.

The Voronoi cell having more edges will decrease the factor of area that B's point can capture. But from the Euler formula, we must have some cell with less than 6 edges if we want to allow a cell with more than 6 edges. We want to study the triangle, quadrilateral and pentagon cases to show that the cost to have cells with less than 6 edges is just too high. That is, having these cells will increase the average area of other cells.

The easiest case is the triangle cell: since we can always use a point to capture at least 5/9 of the triangle area, we need the area of the triangle to be less than 0.922914. Now even with three heptagon(7-gon)s to offset its effect, at least one of them has area ≥ 1.025695, we can always find a position when the Voronoi area is greater than > 0.51273. So having a triangle cell will increasing the average area of other cells.

For convex quadrilateral and pentagon cells, we will do the computer simulation using the technique from numerical optimization. The idea of simulation is as follows: fix one point of A and one edge in direction of the x-axis. We treat the coordinates of the vertices of the Voronoi cell as variables. For example, a quadrilateral Voronoi cell will have 6 variables, a pentagonal Voronoi cell will have 8 unknowns.

Once the position of the vertices are given, we can have the edges and the neighbor points of the Voronoi cell. We will find the position of x for which the captured area is maximized. We do this by using Newton's method (with multiple variables). The maximal area is a function that depends only on the unknown variables. Then we can calculate the Hessian matrix and gradient vector by changing the coordinates by a small value. In the next step we apply Newton's method to move the solution to a new solution.

Using these numerical optimization techniques, we have the following results:

- for a quadrilateral Voronoi cell with area 1, we can always find a position in the cell which has Voronoi area > 0.54861.

- for a pentagon Voronoi cell with area 1, we can always find a position in the cell which has Voronoi area > 0.5277.

Both facts shows that the cost of having cells with less than 6 edges will increasing the average of the other cells. Therefore the optimal strategy of player A cannot have Voronoi cells that are of triangles, quadrilaterals and pentagons. Neither can it have cell with more than 6 edges, by Euler formula. The only shape of the cell is a hexagon.

Again, by computer simulation, we find that the regular hexagon minimized the area of the Voronoi region of a single point of player B. This result supports our belief.

4.5 Summary

In this chapter, we discuss several player strategy for player B when A plays on lattice grids when the Voronoi game is played on a torus, and when the number of points is large.

And we also gave the evidence that playing on the equilateral triangular lattice is A's best strategy to minimize the area of B's first Voronoi cell.

Chapter 5

Conclusions and Future Work

There are many interesting questions and problems in Discrete and Computational Geometry. We discussed three such problems in this thesis: the generalized ham-sandwich theorem, six-way equipartitions of convex bodies, and the Voronoi game.

Our results give rise to a variety of questions and open problems. We list some of these questions below,

Generalized ham-sandwich theorem

In Chapter 2, we gave an algorithm with running time $O(n(\log n)^{d-3})$ to find a generalized ham-sandwich cut. One question is the complexity of this problem:

Problem 5.1 *What is the complexity of finding the generalized ham-sandwich cut in dimension d?*

Lo et. al. [25] showed that for $d = 3$, the complexity is $O(n)$ by giving an optimal running time algorithm. It would be interesting to know if there is a linear time algorithm in higher dimensions, or whether there is a super linear lower bound.

Six-way equipartition

In Chapter 3 we showed that every convex body has a unique six-way equipartition by 3 lines, where one of the lines has a fixed orientation. And we state the conjecture 3.9 states that regular n-gons maximizes the number of different concurrent six-way equipartitions over all polygons with n sides.

There are also related algorithmic problems. Given n points on the plane, Roy and Steiger[31] gave an optimal $O(n \log n)$ algorithm to find a concurrent six-way equipartition. The running time is linear when points are in convex positions. There remains interesting algorithmic questions. One such question is related to Theorem 3.2:

Problem 5.2 *Given n points in R^2 and a vector d, find a six-way equipartition such that one line has direction d.*

And we can also formulate the problem for the number of the points in the central triangle.

Problem 5.3 *Given n points on the plan, find a six-way equipartition such that the number of points in the central triangle in the largest.*

In 6-way equipartition, we enforce that six of the seven pieces have the same areas. And the ratio of the minimum area (of which is the central triangle) over the maximum area is at most 1/8 when K is a triangle. Professor B. Kalantari asked the following question:

Problem 5.4 *For a convex set $K \subset R^2$ with three cutting lines, what is the maximum ratio between the smallest area and the larger area of the seven regions.*

It looks like that the triangle will still be the extremal body.

Voronoi game

The main problem is still open:

Problem 5.5 *What is B's best strategy for the Voronoi game in dimension* $d \geq 2$*, either one-round or alternating?*

And

Problem 5.6 *What is player A's best strategy in Voronoi game in dimension* $d \geq 2$*, either one-round or alternating?*

In fact, under the assumption that n is large and we can place the points arbitrarily close, the numerical results shown in Chapter 4 shows that for either the one-round or alternating Voronoi game, A can just play on the equilateral triangular lattice grid and force B's payoff to be lower than 0.51273. We conjecture that this is in turn A's best strategy for one-round game.

And we are also interested in a rigorous proof of

Problem 5.7 *The equilateral triangular lattice is the play for A that will minimize the maximum area of B's first point.*

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