TRACING BELIEFS AND BEHAVIORS OF A PARTICIPANT IN A LONGITUDINAL STUDY FOR THE DEVELOPMENT OF MATHEMATICAL IDEAS AND REASONING: A CASE STUDY

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A dissertation submitted to

The Graduate School-New Brunswick

Rutgers - the State University of New Jersey,

in partial fulfillment of the requirements

for the degree of

Doctor of Education

Graduate Program in Mathematics Education

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April, 2010

ABSTRACT OF THE DISSERTATION:

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This research provides an analysis of the relationship between a student's beliefs and mathematical behaviors over a seventeen-year period. Romina, the student of focus in this case study, was among the original participants in a longitudinal study which explored how students build mathematical ideas when working collaboratively on problem-solving tasks with as little outside intervention as possible (Maher, 2005). A qualitative, phenomenological approach was taken in analyzing videotape recordings from the Rutgers-Kenilworth longitudinal study between February 6, 1992 and July 15, 2009 in the Robert B. Davis Institute of Learning archive, along with student work, questionnaires, and researcher field notes. To better understand the development of math ideas by tracing her *knowing* and *sense-making*, the research examined four sessions of Romina's problem-solving behavior in terms of justification, representation, and collaboration from fourth through twelfth grades. In addition, this study explored her mathematical beliefs based upon five interviews from high school, college, and her post-graduate career concerning her views about the knowledge, conditions, and processes of mathematical learning. Addressing a documented need in the literature for investigation of the interplay between personal epistemology and mathematical reasoning over time, this study contributes to a larger body of work considering how social interaction, teacher questioning, and task design affect a student's cognitive growth.

The research suggests that Romina constructed mathematical ideas by building relationships among concepts and produced justifications through continuously evolving personal representations that promoted mathematical understanding. Further, the findings provide evidence that Romina engaged in a range of collaborative behaviors in which she questioned others' ideas, found teacher-researcher interaction a catalyst to her thinking, worked through frustration, and moved fluidly among many roles within the group – facilitator, manager, communicator, and secretary. Simultaneously, the data suggest she developed three very "healthy" mathematical beliefs involving the active construction of conceptual knowledge, learning environments that built "comfortable" collaborative relationships while engaging in complex tasks over long periods of time, and, finally, a learning process of "group thinking" where personally relevant problems were shared, questioned, and argued. Through systematic examination of the relationship between Romina's beliefs and problem-solving behaviors, the results of this study imply specific instructional interventions that support the development of mathematical ideas and reasoning from elementary grades through college and into the workplace.

ACKNOWLEDGEMENTS

Words are things, and a small drop of ink,
Falling like dew upon a thought produces
That which makes thousands, perhaps millions,
Think.
~ Lord Byron

One of my favorite photographs shows me with my mom many years ago: I'm about five years old, in my ruffled pink Strawberry Shortcake pajamas, and I'm twirling a piece of hair around my finger while I sit in my mom's lap. She reads me a bed-time story - *Where the Wild Things Are*, *The Little Engine That Could*, or perhaps *Frog and Toad Are Friends* - the cover is not clear. It must be a good book though - neither one of us even notices when my father comes to the door with the camera. Looking every day at that picture on my dresser, I see a four by six inch window into my everlasting love for language and the new worlds that "a small drop of ink" can create. I can almost hear myself, "Read it to me again, Mom, please." I thank my mother for reading me stories when I was little, inspiring a little girl's passion for literature, and helping a young woman believe there was beauty and meaning in her own voice.

Just as the subject of my research often resists the first person singular, I too must stress that my "I" is the product of many lives. Any success I ever found was a result of the very supportive, creative, and loving "we" made up of my family, friends, and teachers. My father has been my continued champion, guide, and counselor. Over bowls of mixed vegetables, bean curd, aphorism, and advice, our talks have sustained me through difficult times. Thank you, Dad, for listening to me, supporting me, and being my constant advocate. To my two sisters, Karla and Laura, you are my heroes and best friends. Beautiful, talented, and brilliant – you two are amazing and I am forever in awe of you both. An incredibly gifted artist and graphic designer, Karla also was very kind to provide invaluable help and advice with the images in this document. David – your love and support have been such an amazing gift to me each and every day – thank you. Mike, Wayne, Aunt Pat, Christen, and Tricia – I am so happy to have you all in my life. Ella and Molly – you make me the happiest and proudest aunt ever! I also remember my grandparents now passed away. I know how all of you dreamed of higher education for your families: Grandma Jo, who kept a small red book of poetry stashed away with her recipes – Shakespearean sonnets among Swiss chard and stewed tomatoes; my Grandpa Bill whose fantastic stories of dinosaurs and far away places kept me clamoring for more; Grandpa Jim whose encouragement made me feel like an adult with valuable ideas when I was only in elementary school; and my Grandma Maggie whom I wished I could have met. I hope you would be proud.

I am so grateful to my dear friends and colleagues at Applegarth Middle School. Thank you to my wonderful "Team Alpha" - Fran Cundari, John Devine, Kim Garavente, Kim Lowden, Michelle Murphy, Jillian Palmieri, Claire Rascoe, Candace Silverman, and Gail Snipp - as well as sincere gratitude to my mentors, Irene Curran and Nina Hirsch. I am blessed with truly supportive administrators: Jeff Gorman, Robert O'Donnell, and Chari Chanley. Jeff, your vision has been transforming; Chari, your kindness and caring have been sustaining. Bob, you are one in a googol – your encouragement and support have meant the world to me. My continued thanks go to my favorite collaborator and constant friend, Erin Muschla.

There are so many people to thank at Rutgers University. My unending thanks to Marjory Palius and Robert Sigley – I don't know what I would have done without you! Thank you, Marjory, for always being there to help me navigate the often confusing waters of Rutgers through your tireless support. Robert, you never cease to amaze me with the wealth of your knowledge and incredible generosity of your time. I am also very grateful to fellow graduate students in my dissertation group for their continued encouragement and thought-provoking discussion: Manjit K. Sran, Andrea de Paolo, WeiWei Lo, Anna Brophy, Pam Brett, and Eman. I am indebted to John Zengerle and Melissa Lieberman, who provided crucial sections of transcription and verification.

My sincerest appreciation goes to my teachers through the years – I have been blessed to be the student of truly gifted educators in my life like Mr. Robert Darton, Dr. Stephen J. Greenfield, Dr. Amy Cohen-Corwin, and Dr. Ann K. Stehney. Dr. Greenfield – I can never thank you enough - if it were not for your encouragement and luminous teaching, I never would have had the courage to become a math major.

My committee members rank highest among my teachers. I reserve my most heartfelt thanks for Dr. Carolyn A. Maher, Dr. Alice Alston, Dr. Robert Speiser, and Dr. Elena P. Steencken. From my first class in the masters program, Dr. Alston has encouraged me to seek justification for my mathematical assertions and critically reflect on my educational beliefs. Since the Problem Solving course I was lucky enough to have with him, Dr. Speiser has taught me to seek the poetry of mathematics – his research proves that pedagogy can be analyzed with a brilliant lyric sensibility and sensitivity. With her caring manner and keen scholarship, Dr. Steencken has provided me the opportunity to find both intellectual challenge and comfort in graduate study. And, Dr. Maher - Carolyn, you are my educational superhero. Without your constant support, insightful critiques, and unwavering belief, I would not have finished this research. Indeed, you are the reason I began this journey. The inspiration of your research and brilliance of your vision has been my guide - I offer my unending gratitude.

For my final acknowledgement, I thank Romina, whose participation in the longitudinal program allowed me to learn so much about mathematical thinking.

DEDICATION

For my family – my love always

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Chapter 1 INTRODUCTION

The mind is an enchanting thing
is an enchanted thing
    like the glaze on a
katydid-wing
    subdivided by the sun
    till the nettings are legion.

- Marianne Moore, *The Mind is an Enchanting Thing* (1944)

1.1 Statement of the Problem

This research examines the relationships between a student's mathematical beliefs and behaviors in problem-solving tasks through experiences and reflections from a longitudinal study[1]. Based on careful analysis of video data involving clinical and semi-structured interviews as well as problem-solving task sessions, this qualitative study employs a phenomenological approach. It is argued that this research addresses a call in the literature for tracing the development of both views and behaviors that concern a student's *knowing* and *sense-making* in problem-solving over time.

In the *Principles and Standards for School Mathematics* (2000), the National Council of Teachers of Mathematics (NCTM) identifies "mathematics for a changing world" and "continued improvement of mathematics education" as the major needs facing classroom teachers, administrators, curriculum developers, researchers, and policymakers today (pp. 4 – 5). Indeed, the NCTM (2000) asserts that "the need to understand and be able to use mathematics in everyday life and in the workplace has never been greater" (p. 4). To meet these needs, NCTM stresses an emphasis on discourse, rich mathematical tasks, and learning through problem-solving. Specifically, the "learning principle" is put forth such that "students must learn mathematics with

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[1] Two grants from the National Science Foundation supported the longitudinal study: MDR-9053597 (directed by R. B. Davis and C. A. Maher) and REC-9814846 (directed by C. A. Maher).

understanding, actively building new knowledge from experience and prior knowledge" (NCTM, 2000, p. 20). But what do we mean by "understanding" and "knowledge"? In order to better understand "understanding" and know "knowledge," epistemological research can help us begin to define and investigate the beliefs that underpin these broad concepts. Our beliefs inform our knowledge and understanding of the world. Schoenfeld (1985) argues that people's mathematical beliefs define their mathematical problem-solving approach. In reflecting on the implications of the longitudinal study, Maher (2005) observes that through the importance the students themselves place on creating a "culture of sense-making," such carefully reasoned arguments and justifications can emerge in their problem-solving behavior (p. 12). Thus, to address the necessity for mathematical understanding that translates to our ever "changing world," we can simultaneously explore both mathematical behaviors and beliefs. Through examination of her problem-solving behavior from fourth to twelfth grades along with interviews concerning her beliefs about the knowledge, conditions, and processes of mathematical learning, this research seeks to better understand the development of math ideas by tracing the *knowing* and *sense-making* of a student named Romina.

1.2 Background of the Longitudinal Study

Romina, the student of focus in this study, was among the original participants in a longitudinal study resulting from a research partnership between the Rutgers University Graduate School of Education and the Kenilworth School District. Initiated in 1989 with a class of eighteen first graders at Harding Elementary School, a public school in a working class community, the study has continued for over 20 years with a smaller subset of the original class of students in addition to a few other students who joined later in the program (Martino, 1992; Maher & Martino, 1996a; Tarlow, 2004).

As Maher (2005) recounts, the main goal of the longitudinal research was to explore how students built mathematical ideas when working collaboratively on problem-solving tasks with as little outside intervention as possible (p. 1). The researchers would meet with the students about four times a year, for two to three days at a time. In problem sessions that could last up to three hours in length, the researchers invited the students to work together on well-defined and open-ended tasks from five content strands (number operations, algebra, counting/combinatorics, probability, and precalculus/calculus) and then present convincing justifications for their solutions. During high school, the smaller subset of students continued to meet in group after-school sessions and follow-up interviews. Members of the high school cohort met individually and in groups during college and then after as they transitioned to the workplace. The students would often revisit problems from earlier sessions, spanning months or even years. Evidence of sophisticated proof-like arguments and generalizations emerged from the "culture of sense-making" in which the students were brought up throughout the study (Maher, 2005, p. 12). Many considerations of how social interaction, teacher questioning, and task design affected students' cognitive growth in terms of mathematical justification, proof, and generalization have been extensively documented (Maher, 2002, 2005; Maher & Martino, 1996a, 1996b, 1999, 2000; Martino, 1992; Muter, 1999; Powell, 2003; Uptegrove, 2005; Uptegrove & Maher, 2004a, 2004b). To investigate students' views, interview data from students at the high school and university levels have also been analyzed (Francisco, 2004).

1.3 Research Questions

> I know it in my own way, not in their way. Everything I explain is in my own words, not in anyone else's words. It's not from some mathematician from a thousand years ago, because I don't know that. I didn't know what the pyramid [Pascal's Triangle] was called. I just know everything in my own way. Everything has Romina's definition to it. *– Romina, 11th grade, May 1999* (Francisco, 2004, p. 34)

> Everything has to make sense in my terms. Someone else may have done it already in a book, but I just don't understand it unless I try it myself and put it in my own terms.
> *– Romina, college sophomore, March 2002* (Maher, 2005, p. 12)

What does it mean when Romina says she has to "know everything in my own way" and "make sense in my own terms"? Furthermore, how did she come to develop such views about herself as a learner of mathematics? The goal of this research is to better understand the growth of math ideas by analyzing Romina's *knowing* and *sense-making* in the context of various problem-solving situations over time. This study traces the development of Romina's problem-solving through video data from a longitudinal study and examines her behaviors in select examples from fourth to twelfth grades in conjunction with her adult beliefs about the knowledge, conditions, and processes of mathematical learning. More specifically, the following questions guide the research:

1. Within the context of problem-solving situations, how do Romina's representations and justifications for her ideas develop over time?
2. To what extent, if at all, does Romina collaborate and incorporate the ideas of others into her own ideas?
3. How do Romina's later adult views about learning relate to evidence of her earlier mathematical behavior in terms of her descriptions of knowledge, the conditions for learning environments, and the learning process?

Chapter 2 : LITERATURE REVIEW AND THEORETICAL FRAMEWORK

A motley collection of shelves and cabinets contained boxes with labels such as “Cemetery Soil Samples” and “Marc Kelley's Ribs.” There were countless books on medicine and on the ancient world, including the works of Diodorus Siculus and Herodotus. “All knowledge is connected to all other knowledge,” Aufderheide said. “The fun is in making the connections.”

- Excerpt from “The Mummy Doctor (Arthur Aufderheide)” by Kevin Krajick in *The New Yorker*, March 16, 2005

2.1 Introduction

Studying the connections between a student’s mathematical beliefs and problem-solving behaviors with problem-solving tasks across various strands of mathematics necessitates a review of the literature concerning both personal epistemology and students’ mathematical reasoning. In addition, since this research is based on data from a longitudinal study of children’s mathematical thinking that spans over twenty years now at Rutgers University, it is also necessary to review and trace specifically the previous work of students in Rutgers’ strands of tasks.

The chapter is organized into two main sections. The first section explicates the theoretical framework in which this study is situated. The second section reviews the literature of relevant studies. Since the nature of this research concerns both behavior and belief within the context of the Rutgers longitudinal study, the second section divides further into sections that explore the literature of: a) mathematical behavior in terms of justification, representation, and collaboration; b) personal epistemological beliefs; and c) student reasoning in representative strands of the longitudinal study.

2.2 Theoretical Framework

The theoretical framework that guided this study was provided by the research of Davis and Maher whereby to “do mathematics” is to build a collection of individual mental representations that can be applied, revisited, and modified as new experiences are encountered (Davis, 1984; Davis & Maher, 1990, 1997). When faced with a mathematical task, the learner first builds mental representations for both the input data and previous relevant knowledge. Then one must construct, check, and possibly modify a mapping between those two mental representations - the input data representation and the existing knowledge representation. Though the teacher or researcher does not have direct access to the individual's internal representations, features of those mental representations are made public and can be evaluated when shared with others. The teacher's role should be to provide experiences that allow the student to further develop and revise those mental representations. A "constructivist" teacher for Davis (1984) would design isomorphic task situations related to relevant mathematics in as a "paradigm teaching strategy." The goal of the teacher's carefully planned experience would be to provide the student with a mathematical metaphor that could be employed as an "assimilation paradigm" given its specific correspondence to a mathematical concept.

Davis's concept of "assimilation paradigm" has its foundations in Jean Piaget's genetic epistemology and developmental theories of assimilation and accommodation. In his description of the various forms of knowledge, Piaget (1967) describes a "sui generis equilibrium situation" in the relationship between assimilation and accommodation during the development of a child's "logico-mathematical structures," (p. 849). "Assimilation" involves the integration of previous knowledge and the "experimental datum" with which the child is currently presented. "Accommodation" happens when new learning requires some modification of the child's operational structures. As a result of his developmental theories, Piaget recommends in his *Comments on Mathematical*

Education (1972) rather explicit "psycho-pedagogical principles" for math learning involving reinvention, understanding in actions, and intuition before axiomatization. Specifically, Piaget describes his precepts:

- *Reinvention*: "The first is that real comprehension of a notion of a theory implies the reinvention of this theory by the subject."
- *Understanding in Actions*: "A second consideration should constantly be present in the teacher's mind: that is, at all levels, including adolescence and in a systematic manner at the more elementary levels, the pupil will be far more capable of "doing" and "understanding in actions" than of expressing himself verbally."
- *Intuition before Axiomatization*: ".. the representations or models used should correspond to the natural logic of the levels of the pupils in question, and formalization should be kept for a later moment as a type of systematization of the notions already acquired. This certainly means the use of intuition before axiomatization." (Piaget, 1972, pp. 731 – 732)

One can see a direct correspondence between Piaget's recommendations to the theory of individual representational mapping by Davis and Maher which requires the student to <u>reinvent</u> mental representations by frequently revisiting problems in problem-solving experiences that necessitate Piaget's understanding in action and intuition before outside intervention presents axiomatization (Davis, 1984; Davis & Maher, 1990, 1997).

If one were to ask a constructivist within this conceptual framework the origin of new mathematical ideas, the most direct answer would be: "new ideas come from old ideas" (Davis & Maher, 1990). One must not lose sight then of the isomorphic forest for the task-trees, so to speak, when considering all of the separate problems implemented within the Rutgers-Kenilworth longitudinal study. The overall focus remains on the multi-faceted ideas evoked because, as Davis once reflected, "Mathematics is about ideas, not about symbols written on paper" (1992, p. 732).

Mathematics education aims to increase students' power of representation for their ideas. Isomorphism, defined as discovering and then making use of structural relationships, fuels the strength of students' representations. As Greer and Harel (1998) suggest, laboratory-based research using "artificial problems" and research about mathematical cognition within "highly circumscribed contexts" have very little relevance to exploring how students recognize isomorphisms (p. 22). Instead, they argue that more investigations need to be undertaken in the type of environment described by Maher, Martino, and Alston (1993) in which a fourth-grade student named Brandon was able to recognize the isomorphism between two combinatorial tasks called "Towers" and "Pizza." In a later analysis, Maher and Martino (1998) remark on the circumstances that made Brandon's insight into isomorphic structure possible: "his active building and rebuilding of representations, over an extended period of time, in situations that encouraged communication and thoughtfulness" (p. 91). Time to build, revisit, and communicate the representation of a rich mathematical task thus set the stage for student understanding.

Maher and Martino (2000) further specify a set of conditions that facilitate conceptual change in learners: a student's choice to become "cognitively involved" with a meaningful task, a classroom environment that provides "sufficient time for exploration and reinvention," and finally a teacher who can "seize the opportunity to provoke thought and to support reconsideration and reinvention of the mathematics" (pp. 268-269). When these conditions are in place, what emerges is a "culture of sense-making" which Maher (2005) describes can exist among a community of learners like those in the Rutgers-Kenilworth longitudinal study where:

> Their discourse, naturally, involved arguing about ideas and providing convincing evidence to each other. This in turn, led to their proof making and generalizing. The processes developed in students through their activity doing mathematics in

> the context of coherent strands of investigations that they were invited to explore. (p. 12)

This "culture of sense-making" encourages collaboration among learners given "coherent strands" of tasks which necessitate they argue, convince, proof-make, and generalize with their mathematical ideas.

Note the prominence of the "coherent strands of investigations" in the foundation of a culture of sense-making. Indeed, Francisco and Maher (2005) elaborate upon the connection between the strands of tasks and the resultant culture created among the students: "task design is crucial for sustained engagement of students in problem solving and for promoting sense-making and mathematical reasoning" (p. 365). In contrast to the traditional atomistic approach where complex mathematics problems are divided into simpler bite-size pieces for the students to quickly swallow, Francisco and Maher emphasize that a rich, complex task should be presented *first*. Though the complex task will take longer for the students to digest, it will succeed in "promoting meaningful and thoughtful mathematical activity" that otherwise would have been lost (p. 365).

Given the conditions that support a culture of sense-making, the educator would hope that the learners truly *internalize* mathematical concepts. Within this theoretical framework, then, is the view that the development of math ideas necessitates collaboration. The pioneer of sociocultural theory in psychology, Vygotsky (1978) defines "internalization" as the "internal reconstruction of an external operation" (p. 56). He argues that all higher-order cognition results from a person's internalization of interaction with others. For Vygotsky the "process of internalization" requires that three "transformations" occur:

> (a) *An operation that initially represents an external activity is reconstructed and begins to occur internally.*

(b) *An interpersonal process is transformed into an intrapersonal one*. Every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first, *between* people (*interpsychological*), and then *inside* the child (*intrapsychological*). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All the higher functions originate as actual relations between human individuals.

(c) *The transformation of an interpersonal process into an intrapersonal one is the result of a long series of developmental events…* (1978, pp. 56-57)

Conceptual development thus works from the outside in as external interactions on the social level are gradually incorporated internally for the learner. Vygotsky's theory provides strong support for the use of collaborative small groups and the presence of a knowledgeable teacher/researcher in problem-solving environments. He hypothesizes that a person's potential for cognitive development is limited by "the zone of proximal development" (ZPD) defined as "the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (1978, p. 86). Vygotsky theorizes that one gains much more cognitively by problem solving in the zone of proximal development with others and then by working alone. On her own, a student could reach an "actual developmental level," but through activity in the ZPD in the company of "capable peers" or the guiding adult, that same student can function at a higher developmental level.

Schoenfeld (1987) argues for the prominent role of social context in metacognitive development and comments that, "almost by definition, small-group discussions (when they work well) result in the individual's working in his ZPD" (p. 211). Schoenfeld goes on to observe, however, that referencing Vygotsky as the primary theoretical support for the necessity of collaboration in problem-solving fails to do justice to the entire phenomenon. The issue is larger and culturally-engrained. He provides individual accounts from the field of professional mathematicians with Paul Erdos, Peter Hilton, and Persi Diaconis as detailed in *Mathematical People: Profiles and Interviews* (1985) by Donald Albers and Gerald Alexanderson. A quote from Persi Diaconis, the famous magician-turned-mathematician and statistics professor at Stanford University, best captures the collaborative nature of real life mathematicians, which is anything but a solitary pursuit:

> Mathematical people enjoy talking to each other… Collaboration forces you to work beyond your normal level. Ron Graham has a nice way to put it. He says when you've done a joint paper, both co-authors do 75% of the work, and that's about right… Collaboration for me means enjoying talking and explaining, false starts, and the interaction of personalities. It's a great, great joy to me. (Albers & Alexanderson, 1985, pp. 74-75).

The quote suggests that the natural state of "mathematical people" is that of collaboration. Diaconis summarizes what it means to be part of a community of problem-solvers where there is talking, explaining, false starts, and interactions – all of which contribute to potential "great, great joy." Here too we hear echoed an application of Vygotsky's hypothesis that work within the ZPD pushes learners further then they would have gone otherwise. Indeed, the definition of a good "joint paper" according to Ron Graham would be one where the collaborators each contribute "75% of the work" leading to a sum that yields a much greater cognitive return! Schoenfeld reflects that in his own problem-solving courses he has established a "microcosm of mathematical culture" in which "mathematics was the medium of exchange" (1987, p. 213).

Inherent within this research's theoretical framework is that students can make mathematics the "medium of exchange" and build mathematical ideas when given the conditions for a "culture of sense-making." These conditions have been specifically addressed and discussed in the literature (Alston & Maher, 1993; Davis, 1984; Davis & Maher, 1990, 1997; Francisco & Maher, 2005; Maher & Martino, 1998, 2000; Maher, 2005). Thus, to summarize, the environment that supports sense-making includes: *complex and coherent tasks* – inviting students to explore mathematically rich problems; *sufficient time* – providing extended time for investigation and reinvention, *stimulating teacher/researcher interactions* – the educator carefully listening to and questioning student reasoning to stimulate reexamination, justification, and generalization; and *collaboration* – promoting the exchange of ideas in groups where students share their representations and make convincing arguments.

2.3 Literature Review

2.3.1 MATHEMATICAL BEHAVIOR

2.3.1.1 Justification

Justification and proof have been documented across the grade levels through longitudinal study (Alston & Maher, 1993, Maher & Martino, 1996a, 1996b, 1999, 2000; Maher, 2002, 2005; Francisco & Maher, 2005; Powell, 2003). We may well ask then, what precisely does it mean for a student to "justify" or "prove" a mathematical idea? And how do we distinguish between justification and proof? The definition of "mathematical proof" given by the National Council of Teachers of Mathematics (NCTM) has evolved over the years. In the late eighties, mathematical proof was given as "a careful sequence of steps with each step following logically from an assumed or previously proved statement and from previous steps" (NCTM, 1989, p. 144). Eleven years later, a revised section on reasoning and proof explains that "by the end of secondary school, students should be able to understand and produce mathematical proofs

- arguments consisting of logically rigorous deductions of conclusions from hypotheses – and should appreciate the value of such arguments" (NCTM, 2000, p. 56). Notice how the linear "sequence of steps" definition has be replaced by a description that emphasizes "arguments" of logical rigor. Yackel and Hanna go further to define a "good proof" as "one that also helps one understand the meaning of what is being proved: to see not only that it is true but also why it is true" (2003, p. 228). More recently, Harel and Sowder (2007) attempted to put forward a "comprehensive perspective on proof" for the *Second Handbook of Research on Mathematics Teaching and Learning* for NCTM. In their chapter, Harel and Sowder stress the subjective character of proof and that for them, "a proof is what established truth for a person or a community" (p. 806). They assert that proving combines two processes: *ascertaining* where an individual or community attempts to remove its own doubts about the truth of an assertion and *persuading* where an individual or community works to remove others' doubts about the truth of an assertion. Proving then requires convincing oneself and others and can thereby be viewed as a necessarily social practice. Balacheff (1991) comments that, "there is a long way between this [1989] definition and the students' concept-image of mathematical proof as a result of teaching interactions" (p. 177). He emphasizes the importance of social interaction and distinguishes between different types of "proving processes" like that of providing justification versus constructing a rigorous mathematical proof. Balacheff goes on to discuss the social dimension of proof as an argument whose validity must be accepted by a mathematical community (p. 178). He quotes the Russian logician Yuri Manin, "a proof becomes a proof after the social act of 'accepting it as a proof' –

this is true of mathematics as it is of physics, linguistics, and biology" (Manin, 1977, p. 48).

To illustrate the meaning of its "Reasoning and Proof Standard," NCTM (2000) uses the example of a longitudinal research case study by Maher and Martino (1996a) tracing the development of justification for a student named Stephanie. NCTM includes an example of Stephanie's "proof by cases" produced in grade 5 for the Towers 3-high from two color selection. Stephanie was a member of the class of 18 first grade children in Harding School who became subjects for the longitudinal study in 1989. The research uses videotapes, individual clinical interviews, and small group evaluations for Stephanie from grade 1 to grade 5. The videotapes were transcribed, verified, described, and coded to trace the development of proof by specifically following use of heuristics, "local organization" description, and "global organization" argument. The data were organized into 11 critical events which document a progression of Stephanie's justifications for the Towers Problem over the five years: using early trial and error heuristics, pairing a tower with its "opposite" or "cousin" in a "local organization," employing a more sophisticated organization of "upside-down and opposites," next accounting for all possibilities with a "proof by contradiction," transitioning from physically building towers to recording towers with symbolic notation, controlling for variables by holding a certain color fixed, developing a new "letter-grid" notation to more simply record all her combinations, discovering a "proof by cases" based on the number of white cubes per tower to create five categories, and then writing a modified proof by cases of only four categories that incorporated her classmates' suggestions. NCTM uses illustrations such as this research study to support its call to educators: "by developing ideas, exploring phenomena, justifying results, and using mathematical conjectures in all content area and – with different expectations of sophistication – at all grade levels, students should see and expect that mathematics makes sense" (2000, p. 56). The suggestion then is that proof and justification under the environmental conditions of proving processes within a supportive and stimulating mathematical community as described by Maher and Martino can encourage and promote sense-making.

Francisco and Maher (2005) propose an "epistemological distinction" between justification and proof as well as suggest a connection to mathematical reasoning:

> Justification refers to how students explain their mathematical actions and decisions. Proof is the formal and rigorous argument, which helps mathematicians explain their ideas. The present study highlights the importance of emphasizing justification over rigorous proofs as a way to promote students' mathematical reasoning. (p. 371)

Contrasting the need for students' explanations of their mathematical decisions to more rigorous arguments, the authors suggest that "explanatory proofs" should be emphasized over formalism in order to achieve mathematical understanding. Through video recordings of classroom sessions and interviews in addition to written questionnaires, Francisco and Maher (2005) conducted a qualitative longitudinal/cross-sectional study of three New Jersey school districts. Approximately 80 students from the districts were videotaped doing mathematics over the course of 3 to 18 years, depending on the district. A phenomenological approach was taken to identify, describe, code, and interpret the "critical events or episodes" which the authors defined as "the students' different forms of mathematical reasoning and the research conditions associated with them" (p. 363). The behavioral problem-solving data and verbal interview and questionnaire data were first analyzed independently of each other and then cross-analyzed for consistency of results. Results were reported based on examination of a smaller group of four high school students from the 18-year study as they worked on and then later reflected upon a probability task called the "World Series Problem." In addition to proposing a distinction between proof and justification, the authors reported results that highlighted key "conditions for promoting mathematical reasoning" which include "the role of basic ideas, complex tasks, strands of problems, students' ownership of their mathematical activity, justification of ideas, and student collaborative work" (p. 371). Francisco and Maher further note that the students' association between "the building of proofs with collaborating" and the consistency between the students' reflections about beliefs and behaviors. Having earlier observed that "few studies have systematically examined problem-solving from the interplay of the students' behavior and their mathematical beliefs," they suggest a "parallel development of students' mathematical behavior and their views about mathematics and learning" (pp. 362, 371). Given the implications of this study, the reader wonders whether and to what extent this "parallel development" would be suggested if the students' mathematical behavior and views were traced even farther back to incorporate tasks that preceded the World Series Problem in the longitudinal strand.

Maher & Martino (1998) describe a case study tracing a student's mental representations as they affected his justifications over two years with the Towers Problem and the Pizza Problem. The account of the fourth grader Brandon occurred in between 1992 and 1993 but in Colts Neck, NJ. The researchers analyzed videotape data from a class session about the Towers 4-tall Problem in November 1992, a classroom written assessment about the Towers Problem in December 1992, video of a class session on the

Pizza Problem with 4-Toppings in March 1993, and a follow-up clinical interview in April 1993. Maher and Martino detail how he established the isomorphism between the Towers and Pizza problems and "physically mapped each tower onto the appropriate row of his zeros and ones" (p. 28). For each problem task, the students worked for about 90 minutes over two consecutive days. Brandon and his partner, Justin, first encountered the Towers 4-tall problem and then four months later in March of 1993, he worked with a new partner, Colin to solve the 4-toppings Pizza problem. With the Towers problem, Brandon and his partner followed a path of first trial and error, then simple relationship names like "partner" and "opposite," and finally more sophisticated local organization strategies for "upside-down" pairs to generate all eight pairs or 16 total towers 4-cubes high (Maher & Martino, 1998, p. 77).

When given the 4-Topping Pizza problem, Brandon created a chart and developed numerical notation which represented the absence of a topping as the digit "0" and the presence of a topping as the digit "1." Using this notation, he began to generate pizzas with a guess and check strategy – writing under the column headings peppers, sausage, mushrooms, and pepperoni a numerical sequence like "1 0 1 0" to represent a pizza with peppers and sausage only. Colin also had a chart with the same column headings but used check marks instead of Brandon's two-digit number notation. Later, Brandon began to reorganize his solution and eventually created a new chart that grouped the pizzas by cases – pizzas with no toppings, exactly one topping, and then exactly two toppings (by pairing the topping in the first column with each of the toppings in the three remaining columns, then pairing the topping in the second column with each of the toppings in the two remaining columns, and so on). Next he added combinations for the cases of three toppings and four toppings. As Brandon explained his work to Colin, he further refined his representation and justified his solution of sixteen pizzas with a proof by cases.

Maher & Martino (1999) describe how teacher questioning "opened the way" for Brandon to recognize and build a justification for the isomorphic connection between the pizza and tower problems (p. 67). In an interview on April 5, 1993, Brandon revisited and articulated his global organization strategy for the Pizza Problem. Under the four columns – "P" for pepper, "M" for mushroom, "S" for sausage, and "peponi" for pepperoni – he had listed the sixteen possible pizza combinations as numerical sequences of zeros and ones indicating the absence or presence of toppings. He further grouped the list into the five cases of no toppings, one topping, two toppings, three toppings, and four toppings. When the interviewer asked Brandon if this problem reminded him of any other problems they had done, Brandon recalled the towers problem. Given red and yellow cubes, he reassembled the 4-tall towers in the same "opposite" pairs he had built back in November. When the interviewer asked him to justify that he had all of the towers based on his "opposite" organization, Brandon rethought and then reorganized his towers from opposite pairs into three groups. He explained, "It's kind of like the pizza problem… like this would be the one's group" (Maher & Martino, 1998, p. 86). In order to probe Brandon's understanding of isomorphism, the interviewer then asked Brandon to focus on one color within the towers. He turned his attention to the yellow cube and then changed his organization again to now five groups based on the number of yellow. Then, as the interviewer asked for clarification, Brandon began to map each tower to a particular pizza in his chart. He explained "since we're looking at yellow, the yellow cube would be 1 and the red would be 0" (p. 89) and thus the sequence "1 1 1 1" on his chart would be represented by an all yellow tower or a “pizza with everything.” Maher and Martino (1998) note the purposeful conditions of Brandon’s learning environment: building and rebuilding representations, working over extended time, and collaborating. They suggest that his “cycling through” process of representing and revisiting his representation under stimulating teacher/researcher questioning, enabled him to construct convincing proof-like arguments for Towers and Pizza as well as the isomorphic connection between the two.

2.3.1.2 Representation

Describing the place of representation in student problem solving, Davis (1984) wrote that “representations are fundamental to mathematical thought” (p. 78). Students’ use of representations to build, interpret, justify, and communicate their mathematical ideas as well as the teacher/researcher’s role in supporting and probing the learners’ representations have been extensively documented in longitudinal studies (Bellisio, 1999; Davis & Maher, 1997; Davis, Maher, & Martino, 1992; Francisco & Maher, 2005; Kiczek, 2000; Kiczek, Maher & Speiser, 2001; Maher & Martino 1996a, 1996b; Maher & Speiser, 1997; Uptegrove, 2005; Uptegrove & Maher, 2004a, 2004b).

Ironically and appropriately, the word “representation” defies easy interpretation. How best can we represent what we mean by student representations? Goldin and Janvier (1998) describe the evolution that the terms “representation” and “system of representations” have undergone for math education researchers as evidenced by the “Working Group on Representations” at the Annual Meeting of the International Group for the Psychology of Mathematics Education (PME). They explain how that group math education researchers have discussed and debated “representation” in connection with four ideas: 1) "an external structured physical situation" of environmental enactment for mathematical ideas; 2) "a linguistic embodiment, or a system of language" during problem-posing or mathematical discussion; 3) "a formal mathematical construct, or system of constructs" like symbolic or graphical notation; and 4) "an internal, individual cognitive configuration" as in the mental representations (p. 1 – 2). Notice that "representation" encompasses both *dynamic processes* and *static products* – physical situations, linguistic exchanges, mathematical constructs, and mental configurations.

NCTM (2000) incorporates the both the process and product aspects of the term by defining "representation" as referring "to the act of capturing a mathematical concept or relationship in some form and to the form itself" (p. 67). Anointing "Representation" as a standard for math education, NCTM (2000) calls for instructional programs from prekindergarten to grade 12 to enable students to create and use representations that communicate mathematical ideas, solve problems, and model and interpret "physical, social, and mathematical phenomena" (p. 67).

Goldin (1998) explores how to create a "unified psychological model" for mathematical learning and problem solving based on multi-layered "representational systems" (p. 137). Examples of "spoken symbols, written symbols, static figural models or pictures, manipulative models, and real world situations" would fall under the umbrella of "external systems of representation" for Goldin (1998, p. 143). He cites von Glaserfeld's (1987) distinction between *Darstellung* and *Vorstellung* which both translate to "representation" from the German, but connote external for the former and internal for the latter. Goldin writes that processes within external representational systems are "mediated" by internal systems of cognitive and affective representation (p. 147). His model for internal representational systems has five components: “verbal/syntactic systems” like the word-to-definition association a student has reading a task; “imagistic systems” where a learner imagines the problem situation; “formal notational systems of mathematics” as in mentally linking a symbolic code to a pattern; “a system of planning, monitoring, and executive control” when the student chooses heuristic approaches or metacognitively decides a next step; and a “system of affective representation” that includes the problem-solver’s attitudes and states of feeling (p. 148). For Goldin, belief systems are “broad constructs cutting across systems of representation” from the mental image construction to metacognitive choices and affective attitudes (p. 158). Regardless of the definition of representation, however, Goldin stresses that the overall purpose of math education should be “to foster in students the construction of powerful, internal systems of representation” (p. 159).

Davis, Maher, and Martino (1992) illustrated the variety of representations students use when problem solving. Drawing on video data and written student work, the research reports on six children – three girls and three boys – over second and third grades as they worked on a combinatorial task called “Shirts and Pants”:

> Stephen has a white shirt, a blue shirt, and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make? (p. 178)

On May 30, 1990, a group of three second graders, Dana, Stephanie, and Michael, were given the Shirts and Jeans problem. The students used a variety of representation strategies. After Michael suggested there were only two possible outfits, both Stephanie and Dana observed that they had to find all “different” outfits. Stephanie proceeded to record each distinct outfit with a pair of letters, the first for a shirt and the second for the jeans – so, for instance, “W W” would represent a white shirt with white jeans. Dana drew three shirts labeled W, B, Y and two jeans labeled B and W. She proceeded to draw lines to match each shirt with a pair of jeans. While she drew two lines from the white and blue shirts, she only drew only line from the yellow shirt. Dana chose not to include the combination of a yellow shirt with the white jeans. Dana, in fact, observed, "It can't… yellow can't go with white" though Stephanie later argued, "It doesn't matter if it doesn't match as long as it can make outfits. It doesn't have to go with each other, Dana!" (Davis, Maher, & Martino, 1992, p. 181). The researchers noted Dana's behavior in limiting her set of possible outfit solutions to only five, instead of six which would require the yellow shirt and white jeans combination:

> For Dana, an outfit is the kind of combination of clothing items that her experience has taught her to consider appropriate. She appears to ignore Stephanie's remark that the outfit doesn't have to match. Dana has not moved to the stage of thinking about abstract outfits which are to include every possible combination of one shirt with one pair of jeans, however unsightly the result. (p. 181)

Dana's previous experience and mental representation of "outfit" informed her written representation and thus her solution to this problem. Though all three students worked together, each approached the problem with a slightly different representation. None of the students at this time arrived at the answer of six either. Dana had five total outfits because she excluded the yellow shirt and white jeans outfit. Stephanie arrived at five because her coding neglected to include the white shirt and blue pants outfit, though she intended to have all outfits whether they "matched" or not. Finally, Michael, who began with the answer of only two outfits, drew a situation with three shirts and, mistakenly, three choices of pants. The researchers observe that "what is particularly interesting about this classroom episode is that each child produced an independent solution, and each seemed to be satisfied with his or her own solution" (p. 182).

The researchers had the opportunity to revisit this problem with the children five months later in third grade. This time the results were different – the students now concluded there were six possible outfits. Dana and Stephanie worked together as partners and Michael worked with another student, Jaime. No mention was made about whether certain color combinations would match, though both Stephanie and Michael adopted the connecting lines representation that Dana had introduced in grade two. The researchers explored and questioned how representation and meaning evolved in the children's minds from second grade to third grade:

> There is further question of the premathematical building blocks from which the representations are constructed, and the distinction between metaphors based on experience (which probably have an essential role to play, and probably must not be bypassed) vs. subsequent abstract ideas that are constructed after one has used metaphoric assimilation paradigms for an adequate length of time. For some of these children, a true outfit had to represent a harmonious match of colors; only later did they come to the idea of putting things together in every possible way, no matter how unsightly the result. (p. 188)

The researchers suggest that understanding students' "premathematical building blocks" would be an important first step for any educator tracing the children's mathematical ideas in these episodes. "Abstract ideas" emerged after students built, rebuilt, revisited, and discussed their personal representations for an extended length of time.

The connection between student representations and increasing conceptual abstraction is explored further by Cifarelli (1998) who interviewed fourteen first-year college calculus students as they solved algebraic word problems. Videotapes of the students' problem solving interviews were transcribed and then analyzed for "significant solution activity" (p. 244). Detailed case studies were prepared that included a written summary of the student's specific problem-solving methods and a "macroscopic summary" of the performance that included both a "general overview of the conceptual knowledge" and a "characterization" of the student's work coded for levels of conceptual abstraction (p. 245). Eight of the fourteen cases were chosen for further analysis based on their "high levels" of task involvement and verbal response. Cifarelli inferred from the eight students' representations while solving the problems a 3-tiered system of increasing abstraction called the "levels of conceptual structure." Two students demonstrated the lowest "Recognition" level, two students performed at the "Re-presentation" level, and four students worked at the highest "Structural Abstraction" level. At the first "Recognition" level, students recognize the utility of current and prior activity but reflect only on the actual solution and not any potential future ones (p. 260). "Re-presentation" solvers are able to present again prior solution activity, recognize the appropriateness of prior representations in a new context, and anticipate potential problems with a prior representation in a new context (p. 261). Students at the "Structural Abstraction" level not only "re-present" representations in new contexts in which they can operate, but also anticipate the results of potential activity without actually carrying out the activity. As a result of the study, Cifarelli suggests that researchers acknowledge and re-evaluate the "constructive function of representation in the development of conceptual knowledge" (p. 261). He calls for future studies that will analyze "situations where the solvers' representations do not work for them and need to be modified through novel solution activity" (p. 262). A possible place to heed this call for future research of representation would be in the context of a longitudinal study.

In a teaching experiment, Maher and Speiser (1997) examine how a student from a longitudinal study related more concrete representations to abstract symbolic notation. Using two video cameras, one focused on the discussion and the other on the written work, the researchers analyzed two of eight individual task-based interviews with Stephanie, an eighth grader at the time, conducted over a six month period 11/8/95 to 5/1/96. Data included videotape, transcripts, Stephanie's written work, and observer/researcher notes. During the March 13, 1996 interview, Stephanie explored how 3-high towers with a choice of two colors, a physical representation she had used frequently in grades 4 and 5 of the longitudinal study, could be related now to a monomial of degree three in two variables (p. 128). During a March 27, 1996 interview, Stephanie is able to connect the binomial coefficient notation $C(n,r)$ to towers where n is the height of the tower. Later, Stephanie explains that "she can use Pascal's triangle to predict the terms of $(a+b)^n$ for new, and hence larger, exponents" (p. 129). The researchers suggest that Stephanie's earlier mental representations for Towers and Pascal's Triangle enable her to construct more abstract symbolic notations for the binomial expansion.

Likewise in the context of a longitudinal study of students' development of mathematical ideas, Kiczek, Maher, and Speiser (2001) report on a student's use of binary number representation to relate two different combinatorial problems. They trace the origin, use, and extension of representation with a case study of a student named Michael. Analyzing transcripts of video data from small group task sessions and student written work in the form an unedited e-mail sent to the researchers, they record how Michael employed a binary scheme of 1s and 0s in the tenth grade to keep track of the number of combinations in the 4-topping Pizza Problem and the Towers problem.

Kiczek, Maher, and Speiser (2001) compared Michael's high school binary representation to his strategy in archival video footage from fifth grade when he was first presented with the Pizza Problem. In fifth grade, Michael drew circles to represent each pizza and used a notational code of letters like "pl = plain" and "c = cheese" for the toppings. Whereas the other students in tenth grade used a similar notational code to their fifth-grade selves when solving this problem again, Michael now developed and applied his "binary coding scheme" (p. 207). Later, Michael extended this representation to demonstrate the addition rule in Pascal's triangle and systematically organize his solutions for the Pizza and Towers problems. The researchers note that "Michael's representation, triggered by the need to find and justify a particular solution, served as a tool for him and others to connect mathematical situations that he and his classmates explored for a number of years" (p. 212). Representation here served as "tool" for mathematical justification and connection. Michael's representation evolved further as it came to be incorporated into the representations of his peers – a phenomenon documented by Muter (1999) in counting towers when three colors are available and by Kiczek (2000) in modeling sample spaces for probability problems.

2.3.1.3 Collaboration

Cole and Engestrom (1993) provide a detailed review of the established tradition of educational research into the fields of socio-cultural understanding and "distributed cognition." Drawing from this tradition, there is a growing body of research into collective mathematical learning and reasoning (Bowers & Nickerson, 2001; Cobb & Yackel, 1996; Cobb, Yackel, & Wood, 1992; Lave & Wenger, 1991; Martin, Towers, & Pirie, 2006; Mueller, 2007; Schoenfeld, 1987; Steencken, 2001; Yackel & Cobb, 1996). Most of this research assumes, as Bowers and Nickerson do, that "individual learning can be seen as an inherently social process" (2001, p. 2). The importance of collaborative experiences in a longitudinal study has been extensively discussed in the literature and mentioned here in an earlier section (Alston & Maher, 1993; Davis, 1984; Davis & Maher, 1990, 1997; Francisco & Maher, 2005; Maher & Martino, 1998, 2000; Maher, 2005).

Martin, Towers, and Pirie (2006) seek to explore classroom examples of “collective mathematical understanding” with data drawn from two studies. Participants in the first study were students and teachers from kindergarten, first grade, fourth grade, and sixth grade classes in a large, urban elementary school in Canada. The researchers collected classroom and interview data over a two year period. Participants in the second study were enrolled in a year-long teacher education program for secondary mathematics student teachers. Data for the second study came from two days of video taping the pre-service teachers’ problem solving sessions as part of the course. Using the Pirie-Kieren (1994) theoretical model, the researchers viewed and reviewed videotapes as well as transcripts of the task sessions. They interpreted the data through the “lens of improvisational theory” whereby a growth in mathematical understanding was observed at the collective level. They echo Sawyer’s (2000) assertion that in an “ensemble improvisation” the individual creativity of each performer is not identified, but rather the performance makes sense only at the collaborative level when we experience the overall effect of the melody. The students and student-teachers were given challenging open-ended tasks for their age group (like quadrilateral area for the sixth graders or taxicab geometry for the pre-service teachers). They were encouraged to work together by being provided with only a single sheet of paper to record the group solution. With both groups, the researchers found it significant that the students did not seek to involve the interviewer/researcher in any way and demonstrated a "powerful sense of collective purpose" (p. 172). Also they found evidence of the students "deferring to a group mind" by being willing to abandon their own personal strategies in favor of contributing to ideas other group members offered that appeared "better" based on their justification (p. 174). Martin, Towers, and Pirie (2006) suggest a framework for viewing students' mathematical behavior as having the "power and potential of improvisational co-action for occasioning the growth of understanding" (p. 175).

Yackel and Cobb (1996) advance the idea of "sociomathematical norms" in their research conducted in teaching experiments with a second-grade class. They define "sociomathematical norms" as the "normative aspects of mathematical discussions that are specific to students' mathematical activity" (p. 459). The data collected included: video recordings for all mathematics lessons for the entire school year of a second grade class, individual interviews conducted with each student three times during the year (beginning, middle, and end), researcher field notes, and copies of students' written work. In analysis, the researchers suggested that the increasingly sophisticated mathematical behavior they observed coincided with the students' opportunities to make sense of others' explanations as well as their own justifications. They traced the significance of the teacher's role in setting and modeling the norms for mathematical sense making in the classroom to include an inquiry approach. Yackel and Cobb (1996) further propose that "in the process of negotiating sociomathematical norms, students in these classrooms actively constructed personal beliefs and values that enabled them to be increasingly autonomous in mathematics" (p. 474).

Powell (2006) probed the socio-cultural relationship of student discourse to cognition. Further elaborating on the data set previously analyzed (Powell, 2003; Powell & Maher, 2004), Powell (2006) analyzed the conversational exchanges that occurred among four high school students for a "Taxicab Problem" task session within the counting/combinatorics strand of a longitudinal study. Powell defines discourse as "language (natural or symbolic, oral or gestic) used to carry out tasks – for example, social or intellectual – of a community" (2006, p. 34). Using video data that was transcribed and student work collected over two after school sessions of the students working in a group of four, Powell coded for four categories of interlocution: evaluative, informative, interpretive, and negotiatory. In his results, Powell suggests that "socially emergent cognition is possible when interlocutors are engaged in negotiatory interlocution" (p. 38). Tracing the students' discourse during their problem solving, Powell asserts that the students build an isomorphism between two combinatorics problems as a result of their "negotiatory discursive interaction" – specifically, "not one student presents the isomorphism fully formed, but rather their discursive interactions constitute a co-construction of the isomorphism" (p. 40). This suggests that collaboration on a task influences the development of the students' mathematical ideas.

Further investigating the role of "co-construction" in mathematical reasoning, Mueller (2007) studied a group of sixth grade students engaged in fraction tasks with Cuisenaire rods as part of an informal, after school program. Video tape data from five after school sessions was transcribed and analyzed, student written work and researcher/observer field notes were collected, and interviews were conducted. Unlike Powell's (2003) original study which included a group of students who had been exposed to a “culture of sense making” since at least fourth grade, the students in Mueller’s research had not grown up in the supportive learning environment of a longitudinal study. Mueller wondered how the sixth graders would reason about the fraction tasks given that this would be the first time the students were placed in a collaborative problem-solving setting and that they had previously been exposed to the standard fraction operation algorithms. Mueller concludes that, “the students developed their own mathematical microculture, co-constructed ideas, and questioned and challenged each other in such a way as to promote the collective growth of understanding” (p. 303). Again, collaboration on a task seemed to influence the students’ mathematical reasoning and understanding. Based on her findings, Mueller (2007) calls for a mathematical learning environment that promotes exploration, reinvention, and collaboration.

2.3.2 EPISTEMOLOGICAL BELIEFS

2.3.2.1 Introduction

From the Greek *epistḗmē*, “knowledge,” the field of ***epistemology*** investigates the theory of knowledge and justification. How can we begin to describe what it means to know? To discover what fills such a daunting conceptual chalice as knowledge, it would seem we need to embark upon an Arthurian journey of legendary proportions with Gawain and Perceval in attendance. Fortunately, the rich body of research into epistemology can guide the careful reader. Audi (2002) presents a thorough historical background to epistemological inquiry as well as a discussion of the concepts, problems, and methods in the field of epistemology as a philosophical pursuit. Before one can understand the development and structure of beliefs, knowledge, and justification, one must first identify and explore the source of such constructions. Audi delineates the six major "sources": perceptual, memorial, introspective, a priori, inductive, and testimony-based beliefs (p. 6). A belief based on the senses is *perceptual*, like believing a glass is cold after touching it. A belief based on an experience stored in memory is *memorial*. Looking within oneself, as in the act of imagining, conceives an *introspective* belief. An *a priori* belief arises not from direct observational experience, but rather an intuition of what was "former" to any perceptual evidence. For example, we could use the example of Descartes. His belief that he exists ("Cogito ergo sum") is *a priori*. However, our belief that Descartes existed is not a priori, as we need evidence of his existence while he did not. Or consider the belief, "the rose is red," which would be *a posteriori* or perceptual as it derives <u>after</u> an experience, or specifically here from visual evidence. Generalizing from something more basic develops an *inductive* belief. One's general belief that a rose needs to be frequently watered could arise inductively from many experiences where a dry rose wilted. Finally, many beliefs occur based on the *testimony* of others. If you did not have experience with watering roses, a belief about the proper care for roses could have also arisen from hearing the testimony of other more botanically-inclined acquaintances. Reason intertwines these sources of belief: "reason yields no knowledge or justified belief until experience, whether perceptual, reflective, or introspective, acquaints us with (or develops in us) concepts sufficient for grasping a priori propositions" (p. 119).

Whereas epistemology from the philosophical perspective concerns the nature, source, and methods of human knowledge, the psychological and educational lens focuses rather on ***<u>personal</u> epistemology*** to concentrate on how an individual develops and applies conceptions of knowledge and knowing. Numerous studies demonstrate the divergent definitions, conceptual frameworks, and methodologies through which research into personal epistemology has been undertaken by educational psychologists (Hofer & Pintrich, 1997). In their review of epistemological research in educational psychology, Hofer and Pintrich (1997) noted the general lines of inquiry: (a) extension of developmental sequences, (b) creation of measurement tools, (c) exploration of gender-related patterns in knowing, (d) examination of the relationship between epistemological awareness and reasoning, (e) identification of epistemological belief systems, and (f) assessment of connections among beliefs, cognition, and motivation (p. 89). The authors state that many theoretical and methodological issues have arisen from the various models of knowledge and knowing. Among the issues they note are that "the general definition of the construct varies across the field" and "there has been a lack of conceptual clarity about the elements or dimensions that constitute individual theories or beliefs" (p. 111). The authors address the need for better clarification of definition and dimensions in personal epistemology research in educational psychology for the future.

Not only do divergent definitions and frameworks of epistemology exist within the field of educational psychology, but there also exist questions as to what branch of inquiry epistemology should even be properly undertaken in the first place. Toulmin (2003) highlights what he calls the "ambiguous" status of epistemology when considered as a branch of educational psychology. He proposes that epistemology would be "more properly thought of as a branch of comparative applied logic" so that questions of "innate" abilities and physiological cognitive development can be put aside in favor of studying the structure of argument to determine knowledge:

> Considered as psychology, the subject is concerned with intellectual or 'cognitive' *processes*, with our intellectual equipments and endowments, with

> 'cognition' and its mechanism; considered as a branch of general logic, it is concerned with intellectual or rational *procedure*s, with methods of argument, and with the rational justification of claims of knowledge. (Toulmin, 2003, p. 196)

For Toulmin, the language itself of epistemology needs to be changed from posing questions like "How do we know that?" to asking instead "What adequate ground do we ever have for the claims of knowledge we make?" (p. 201). He acknowledges that when inquiries are undertaken into how children come to "know" certain concepts, one will have to employ inductive *a posteriori* methods from a psychological perspective. However, if one were to instead consider whether the grounds the child has to believe something is "up to standard," the issue is now in the court of a logician:

> A man who puts forward some proposition, with a claim to know that it is true, implies that the grounds which he could produce in support of the proposition are of the highest relevance and cogency: without the assurance of such grounds, he has no right to make any claim to knowledge (p. 201).

The problem of comparative applied logic then is to determine what is meant by the "highest relevance and cogency" of standards to be applied to any field of argument. Toulmin advocates ideological cooperation of "*rapprochement* between logic and epistemology" so that investigations can proceed in a single direction toward the "merits and defects" of argument and subject to analysis of such elements of argument as claims, data, warrants, qualifiers, and rebuttals (p. 234).

However one ultimately defines the field, Hofer (2002) observes that the investigation of personal epistemology crosses many more disciplines than simply philosophy, logic, or educational psychology. Indeed, within the various dimensions of our own daily lives, we constantly engage and enact epistemological beliefs: at home, judging the credibility of a newspaper's claim about a presidential candidate; at work, determining the function of a new computer-auditing process; or at school, encountering a problem-solving task. At such moments, individuals must make decisions like "whether they view knowledge as a set of accumulated facts or an integrated set of constructs, or whether they view themselves as passive receptors or active constructors of knowledge" (Hofer, 2002, p. 3). One's personal epistemological framework gives form to how one makes sense and meaning from new experiences. As Hofer explains, given the wide-ranging application, the study of personal epistemology incorporates research from diverse disciplines including not only educational, developmental, and instructional psychology, but also higher education, counseling studies, science and math education, reading and literacy investigations, and teacher education (p. 4).

2.3.2.2 Theoretical Models of Personal Epistemology

Certain notable studies contributed influential theoretical models to the field of personal epistemology. The first significant steps in the field were taken by Piaget (1950) who used the phrase *épistémologie génétique* or "genetic epistemology" to describe his intellectual development theory and bridged the fields of philosophy and psychology to spur interest by researchers in the same pursuits. Specifically, the notable modern theoretical models include: the "scheme" of nine positions for epistemological development among college students (Perry, 1970), a five perspective categorization on "Women's Ways of Knowing" (Belenky, Clinchy, Goldberger, & Tarule, 1986), an "Epistemological Reflection Model" exploring the role of gender differences (Baxter Magdola, 1992), the "Reflective Judgment Model" of seven stages for epistemic cognition (King & Kitchener, 1994), and a theoretical framework for describing epistemological belief systems (Schommer, 1994).

In a reflection of the primary theoretical models mentioned above, Hofer (2002) suggests that all the models offer certain common themes:

> Regardless of the number of stages, positions, or perspectives, the sequence invariably suggests movement from a dualistic, objectivist view of knowledge to a more subjective, relativistic stance and ultimately to a contextual, constructivist perspective of knowing. (p. 7)

Not only do the models share a similar developmental path from objectivist to constructivist perspectives, but most also concerned research with college-level students. The uniformity of the subjects for most of these influential models raises questions about how these models translate across a broader range of grade levels. In addition, none of these major studies concerned specifically the domain of mathematics and thus, it is important to investigate how these theories would operate in mathematical problem-solving contexts.

2.3.2.3 Students' Epistemological Beliefs about Mathematics

In a review of 33 studies, Muis (2004) synthesized and summarized empirical research concerning students' epistemological beliefs about mathematics in order to help develop "a more cohesive theoretical framework across and within disciplines" as well as offer direction for future work in the field (p. 318). Including only those studies which focused on *students'* beliefs about mathematics, Muis's methodology identified "relevant literature" as empirical research falling within the time period 1980 to February 2004 and involving at least one component of the definition she selected for epistemological beliefs: defining features, conditions/sources, and limits of personal knowledge and justification. From an initial list of 355 articles, books, or book chapters, she employed her criteria to winnow down the list to thirty-three items that she coded on the basis of their applicability to one of five categories: (i) students' epistemological beliefs about mathematics, (ii) development of epistemological beliefs, (iii) effects of epistemological beliefs on behavior, (iv) domain differences in epistemological beliefs, and (v) changing epistemological beliefs (p. 325). She offered critical analysis of the research across and within each of the five categories.

Based on her review, Muis (2004) writes that one common theme across the majority of the math education research suggests that "students at all levels hold nonavailing beliefs" (p. 330). Other common beliefs she found among the studies included: the math learner being a *passive* recipient of knowledge, mathematical knowledge consisting of *unrelated and isolated* pieces of information, and mathematical thought occurring in a *quick* amount of time. She writes that the body of empirical evidence correlates with the claim that "students' classroom experiences greatly influence their beliefs" (Muis, 2004, p. 338). Moreover, the research suggests that students' beliefs can become more availing over time. Muis remarked on a possible methodological issue in the line of environmentally-influenced beliefs as inadequately measuring students' beliefs. Specifically, she suggested that future researchers should "directly measure their sample's beliefs about mathematics and compare them with the classroom environment and activities" (p. 338).

Within the category of belief and behavior, Muis (2004) reported that qualitative studies observed that students' beliefs seemed to impact the time spent on a problem, strategies employed, and justifications made (p. 345). Most of the research reviewed focused on how beliefs would influence behaviors and, in turn, achievement. Muis warned that although the evidence she surveyed supported a relationship among beliefs, behaviors, and achievement, the reader should not conclude a "cause-and-effect" relationship. Finally, the majority of studies Muis included found support for a "domain-specific hypothesis" – particularly that student beliefs about mathematics were less availing then other subjects. Investigations of domain-specificity pose methodological challenges and raise questions like how to compare mean scores across domains in a statistically valid way.

2.3.2.4 Relationship between Beliefs and Behaviors in Mathematics

In early studies of discipline-specific beliefs, Schoenfeld (1992) has examined and identified "typical" beliefs students hold about the nature of mathematics which he asserts have a very strong relationship to students' behavior. He summarizes a review of research on students' beliefs and problem solving, his own included, with two observations:

1. Students abstract their beliefs about formal mathematics – their sense of their discipline – in a large measure from their experiences in the classroom.
2. Students' beliefs shape their behavior in ways that have extraordinarily powerful (and often negative) consequences. (Schoenfeld, 1992, p. 359)

Given these possible "extraordinarily powerful" consequences, he calls for more research in math education to examine the development of students' epistemological beliefs together with their problem-solving behaviors. Specifically, he calls the field as it stands now "under-conceptualized" and in need of "new methodologies and new explanatory frames" (p. 364).

In observations of problem-solving protocols he implemented with college students, Schoenfeld (1983) argues that the students' behaviors must be interpreted in the light of their beliefs. He advances his thesis about the relationship between problem-solving behaviors and mathematical beliefs:

> …Cognitive behaviors we customarily study in experimental fashion take place within, and are shaped by, a broad social-cognitive and meta-cognitive matrix. That is, the tangible cognitive actions produced by our experimental subjects are often the result of consciously or unconsciously held beliefs about (a) the task at hand, (b) the social environment within which the task takes place, and (c) the individual problem-solver's perception of self and his or her relation to the task and the environment. (Schoenfeld, 1983, p. 330)

Notice the multi-faceted framework of *social-cognitive* and *meta-cognitive* beliefs in which Schoenfeld situates all "cognitive behaviors" – beliefs about the task itself, beliefs about the surrounding social environment, and the individual's personal beliefs of self-perception in regard to the present task and environment. In order to characterize students' problem-solving, Schoenfeld considers three distinct categories of analysis: resources, control, and belief systems. "Category 1 -Resources" involves the individual's mathematical knowledge as it relates specifically to the task at hand in terms of facts, algorithms, and "local" heuristics like trial and error. "Category 2 – Control" refers to the problem solver's "selection and implementation of tactical resources: monitoring, assessment, decision-making, conscious meta-cognitive acts" (p. 331). In other words, the strategic global decision to use the heuristic "try a simpler problem" would be a Control, whereas once that decision is made and the problem solver starts computing within the simpler problem the level is now back at Resources. Finally, "Category 3 – Belief Systems" regards the individual's understandings about self, environment, topic/task, and mathematics in general.

Exploring problem-solving through the lens of socio-cognitive and meta-cognitive beliefs, Schoenfeld (1983) provides vignettes of two college freshmen working on a straightedge-and-compass construction problem from geometry. Both students had already completed a semester of calculus and were currently enrolled in his problem solving course. Schoenfeld analyzes and contrasts audio transcripts of the pair of students solving the same problem as a professional mathematician who had not "done" geometry for some time. Schoenfeld classifies the students' behavior as "purely empiricist" in that hypothetical solutions derive from features of the drawings, hypotheses are tested "seriatum," sequentially, until accepted or rejected, and "mathematical proof is irrelevant" since verification is made by whether the construction "appears" to work (p. 338). The students based their justifications on this empirical approach and did not pursue more logical, time-saving proofs in their problem-solving. The mathematician, in contrast, exhibits a more rationalist approach with "better control behavior, more reliable recall of relevant facts, and (not to be underestimated) more confidence" (p. 344). Schoenfeld concludes that there is a "dynamic interplay" among the three levels of problem-solving beliefs and behaviors. He acknowledges that a "complete explanation" of such interactions will not be a simple matter for future research.

To more intensively examine the development of beliefs and their implications for student behavior, Schoenfeld (1988) conducted a year-long case study of a 10th grade geometry class. The setting for the study was a suburban school district in upstate New York during the school year of 1983 - 1984. Twelve mathematics classes were periodically observed with interviews conducted for the students and teachers as well as an 80-item questionnaire given to all 230 students to assess their perspectives. A target geometry class of 20 students was chosen from among the twelve for weekly observations and two weeks of having every lesson videotaped in its entirety. Most of the analysis made was qualitative in nature, though a quantitative analysis of the questionnaires was also provided.

At the outset, Schoenfeld (1988) sought to explore the students' subject matter understanding and the influence of "classroom practice" on that understanding's development. The target class is described as "well run" and successful from an outside perspective, given that the class scored in the top 15% on the New York State Regents geometry exam. He cautions that the results of his data analysis indicate however that "as a direct result of their experience in the course, the students developed (or, at least, were reinforced in)" a series of four "unhealthy" beliefs (p. 152). The four beliefs that developed were:

> *Belief 1*: The processes of formal mathematics (e.g., "proof") have little or nothing to do with discovery or invention.
> *Belief 2*: Students who understand the subject matter can solve assigned mathematics problems in five minutes or less.
> *Belief 3*: Only geniuses are capable of discovering, creating, or really understanding mathematics.
> *Belief 4*: One succeeds in school by performing the tasks, to the letter, as described by the teacher. (Schoenfeld, 1988, p. 151)

Schoenfeld argues that the first belief, that formal mathematics has little to do with discovery or invention, correlates with the district-wide stress on instructional preparation for the state tests. He comments that on the large amount of class time spent in practicing and checking for speed and accuracy in the precise sequence of steps needed to solve exercises. The second belief that all problems can be solved in just a few minutes could be traced to the implementation and type of tasks given to the students in the classroom. In a "typical" class, the students were given homework assignments of between 18 and 45 "problems" - Schoenfeld notes these "problems" were limited recognition or procedure application exercises with obvious and immediate answers. In a 54-minute class period students were given usually about 25 problems, giving them an average of 2 minutes and 10 seconds per problem. The instructor in the target class would make comments like, "You'll have to know all your constructions cold so you don't spend a lot of time thinking about them" (p. 159).

Regarding the third belief that only geniuses are capable of really discovering or understanding mathematics, Schoenfeld (1988) directed attention to the amount of class time spent discussing the two-column "form" of the students' work, rather than the substance. The resultant belief that the format of an argument is just as, if not more, important than the argument itself was the unintended lesson Schoenfeld found the students learning. Finally, the students came to view themselves as "passive consumers of others' mathematics" (p. 160). Schoenfeld points to how often in the classroom observations and video transcripts the problems were introduced and practiced with step-by-step procedures and memorization. He concludes that the students in the target geometry class mastered much more than proof and construction procedures. Indeed, Schoenfeld suggests that the students learned new beliefs about what it means to do mathematics and those beliefs impacted their problem solving: "their views about mathematical form, 'problems,' and their role as passive consumers to others' mathematics, all shaped their mathematical behavior" (p. 165).

Using the questionnaire data from his 1988 study, Schoenfeld (1989) further explored aspects of students' beliefs and their mathematical performance. A questionnaire with 70 closed and 11 open-ended items was administered to 230 "college-bound" mathematics students, 112 female and 118 male, from tenth to twelfth grades in "highly regarded" suburban high schools. The students were enrolled in geometry, trigonometry/pre-calculus, or calculus. The questionnaire items included six categories: attributions of success or failure; students' perceptions of mathematics and school practice; students' views of school content areas; students' views on the nature of geometric proof and reasoning; students' motivation; and students' personal and scholastic performance (p. 342). The article's appendix provides the 81 items in full. In all multiple choice items students were asked to rate their agreement with a 4-point Likert-type scale. For example, an item in the section about students' perceptions of mathematics and school practice is item #11, "The math that I learn is school is mostly facts and procedures that have to be memorized." Finding sex differences in the answers to be statistically negligible, the data analysis is reported for the population as a whole as opposed to by gender. Within the category on classroom practice items, Schoenfeld observes that of the 206 responses to "how long should it take to solve a typical homework problem," the mean time given was just less than 2 minutes with not a single student writing a time exceeding 5 minutes. When asked "what is a reasonable amount of time to work on a problem before you know it's impossible," the answers from the 215 responses ranged from 2 minutes to 20 minutes. The mean time given for when one would "know it's impossible" is 12 minutes. Schoenfeld also found that "the students' overall academic performance, their expected mathematical performance, and their sense of their own mathematical ability all correlate strongly with each other" (p. 347). He also reports that students with higher mathematical performance gave the empirical non-availing belief questions a lower rating. In other words, "the better the student is, the less likely he or she is to believe that mathematics is mostly memorizing (item 11), that success depends on memorization (item 38), or that problems get worked from the top down in step-by-step procedures (item 41)" (pp. 347-348). Schoenfeld concludes that the most "troubling" aspect of the study is the suggestion that of two separate mathematics for students: school mathematics of empirical memorization and 2-minute exercises versus abstract mathematics of problem solving and discovery.

Sharing Schoenfeld's view about the interaction of belief and behavior, Lampert (1990) conducted a teaching experiment with a fifth grade class over a year and specifically focused on the case of one lesson about exponents. She claims at the outset of her paper that "mathematics is associated with certainty" in popular culture. More specifically, she elaborates that:

> These cultural assumptions are shapes by school experience, in which doing mathematics means following the rules laid down by the teacher; knowing mathematics means remembering and applying the correct rule when the teacher asks a question; and mathematical truth is determined when the answer is ratified by the teacher. Beliefs about how to do mathematics and what it means to know it in school are acquired through years of watching, listening and practicing. (Lampert, 1990, p. 32)

Lampert describes how she considered the transcripts of lessons that occurred throughout a year she was teaching a fifth-grade mathematics class. While acknowledging the methodological issue of the researcher's position as both subject and author, she does not mention how the transcripts were acquired – the reader is left to wonder whether there were video or audio recordings made. Two stages of analysis were used with her data: detailed daily field notes and reflections on lessons and then comparisons of lessons across the year. In the focus lesson described, after looking for patterns in the sequence of square numbers, the students were asked to figure out the last digit in 5^4, 6^4, and 7^4 without multiplying. Through limited but probing questioning of students' representations and extended time for the students to discuss and justify their responses, Lampert tested her hypothesis that "changing students' ideas about what it means to know and do mathematics was in part a matter of creating a social situation that worked according to rules different from those that ordinarily pertain in classrooms" (p. 58). Lampert suggests that it is possible for "knowing mathematics" can be "taught and learned," because her students acted "differently" about "mathematical knowledge" by the end of the year than at the beginning - though how this was ascertained is left vague and unresolved.

Diaz-Obando, Plasencia-Cruz, and Solano-Alvarado (2003) found similar results when they conducted a study of the mathematics beliefs for two students from two different countries. The authors took a case study approach with the purpose of observing, analyzing, and determining the "interpretation that the participants gave to their mathematics knowledge as well as their actions" (Diaz-Obando et al., 2003, p. 163). Fifteen year old Kevin and seventeen year old Sam were from public schools in Spain and Costa Rica, respectively. The study's methodology included field notes of classroom observations and audio-taped and video-taped interviews that varied in format – semi-structured, clinical, and short follow-up. Full transcriptions were made of all classroom observations and interviews and "coded" based on categories that emerged. The authors examine beliefs about the role of teacher and student in mathematics by highlighting comments made by Kevin, "at school you are not given time to think" and "the classroom problems are explained" (p. 167) and by Sam, "at the beginning of a topic the teacher usually explains how to formulate the problem and solve it" and "at the end, what really counts for me is to use what I understand best" (p. 170). After a very limited number of cited examples, the authors suggest that Kevin and Sam believe that school mathematics is rule-bound and procedure-based. The reader wonders how the authors' conclusions would generalize given the very small scope of their study. Given also that the students were only studied within the confines of a single year, would the students' professed beliefs have remained the same over a wider range of time?

In a larger study, Kloosterman and Cougan (1994) considered the mathematical beliefs and performance of 62 elementary students in grades 1 to 6 at a single school from a working-class neighborhood. Three to five students from each of the two or three classrooms at each grade level were interviewed. Classroom teachers chose the students after being instructed to provide children "with a range of abilities in mathematics" (p. 377). Half of the students qualified for free and reduced lunch, roughly equal numbers of male and female students were included, and, although the majority of the students were white, several minority students were represented. The open-ended interview protocol had eight categories of beliefs and seven mathematics problems to be solved during the interview. Students were asked to think aloud and explain their reasoning. While the belief-interview questions were the same across grade levels, the mathematics problems became progressively harder for each grade. The final part of the interview then asked the students to categorize a set of ten story problems as "mathematical" or "nonmathematical." All interviews were audio-taped, though it is not indicated whether they were fully transcribed. Rather, the authors say that most analyses were based on observational field notes written during and after the interview by the researchers. To supplement the interview data, achievement data were collected from the California Achievement Test taken two months before the interviews.

Of the eight categories in the original interview protocol, five were chosen for further analysis: liking school and liking math; parental support for school and math; perceived usefulness of math; self-confidence in math; and existence of a "math mind" (as in the item "Are there any students who just aren't smart enough to be good at math or can every student learn math if they try hard enough?"). Kloosterman and Cougan (1994) coded each student as "high," "medium," or "low" in each of the data sets collected – achievement test scores, problem-solving test scores, and mathematical belief comments. For the achievement test data, "high" indicated a score in the 70^{th} percentile or above, "medium" referred to between the 30^{th} and 70^{th} percentiles, and "low" meant a score below the 30^{th} percentile. With the problem-solving test scores, the students were ranked based on grade-level z-scores: "high" for z-scores greater than +1 on each item, "medium" for z-scores between +1 and -1, and "low" for all z-scores less than -1. Finally, the students' comments about beliefs were ranked as "high," "medium," or "low" in each of the five belief categories.

Kloosterman and Cougan (1994) report five results. First, they describe the responses of the older students as more expansive and "easier to understand" than the younger students – not surprising given the verbal skills of a first grader versus a sixth grader. Second, most to all of the students' responses to the question "Do you do well in math?" mentioned teacher feedback and grades. The researchers conclude that "the children we interviewed indicated that grades and teacher feedback about the correctness of their assignments rather than conceptual understanding or mathematical power were the basis of their self-confidence in mathematics" (p. 381). Third, the majority of student responses suggested they "liked" both school and mathematics. Fourth, "almost every" student answered affirmatively to the question “do you think it is important to learn mathematics” and the older students provided more detailed reasons. The researchers found “no consistent relationship” between the students’ reports of parental involvement and the corresponding students’ achievement scores. Finally, the students’ responses varied by age with regard to the question about whether all students can learn mathematics – their comments often linked effort with ability. Kloosterman and Cougan acknowledge the constraints for how their results would generalize given their data were from a single school with a majority of white, lower- to middle- socioeconomic families. They raise the question of how their protocol would translate within other settings. The report left the reader wanting to know more about how the students’ professed beliefs related to their problem-solving behaviors which, though part of the data collection, did not have a prominent place in any of the researchers’ discussion.

Though focusing on only a single student, Francisco (2008) provides a close examination of the relationship between the beliefs and mathematical behaviors of a student named Mike while in the 12th year of a longitudinal study. Data were collected from video recordings of a three-session problem solving task called the “World Series Problem” and a 1-hour semi-structured interviewed focused on Mike’s reflections about mathematics and the longitudinal study. Francisco comments that the methodology for the interview data follows Perry’s (1970) approach of “inferring epistemological beliefs from individual’s reflections on their educational experiences” (Francisco, 2008, p. 5). Two separate data analyses were undertaken – one phenomenological analysis for the interview data and one problem-solving analysis of the behavior data. Next, Francisco looked for any relationship between the mathematical beliefs Mike expressed in his interview and the behavior he displayed while solving the World Series Problem. Francisco suggests that Mike engaged in problem-solving behavior that was consistent with his views. Specifically, Francisco concludes that “[Mike’s] quest for justification, flexibility and timing in working with others, and ability to engage effectively in collective mathematical activity with a peer and an expert reflect not only useful problem-solving skills, but also forms of behavior that support his views of mathematics as a sense making and discursive activity” (pp. 13 – 14). The researcher argues that these results indicate that secondary students can hold epistemologically sophisticated views and display behavior consistent with those beliefs. He also suggests that Mike’s beliefs about mathematics as a “sense making and discursive activity” were linked and perhaps the result of the collaborative and supportive learning conditions which Mike experienced in the longitudinal study. Calling for further research to be done, Francisco highlights the advantages of a “simultaneous analysis of mathematical beliefs and behaviors” as providing a methodology that will “unveil important aspects” of student problem solving (p. 15).

Mason (2003) also examined students’ beliefs and behaviors, but on a much larger scale and in a very different setting. In a study of 599 students, 302 girls and 297 boys, from two different high schools in southeastern Italy, Mason attempted to address four different research questions involving: the application of an American questionnaire to an Italian audience, the differences of beliefs by grade and gender, the relationship between beliefs and achievement, and the reasons for mature versus naïve epistemological views. Mason administered her own Italian translation and adaptation of Kloosterman and Stage’s (1992) five-point Likert-type scale called the Indiana

Mathematics Beliefs Scales and the Fennema-Sherman Usefulness of Mathematics Scale. Students completed the questionnaires in their respective classrooms and then their individual grades in mathematics were collected, which in Italy are expressed on a scale of 1 – 10. A group of 24 students were selected for follow-up individual interviews based on their lowest or highest scores in the beliefs questionnaires and were asked questions like "Could you tell me why you rated this item 1 (or 2, 4, 5)?" All interviews were tape-recorded and transcribed in their entirety. Mason first analyzed the reliability coefficients for the six scales of the questionnaire and found only one – scale 4 "Word problems are important in mathematics" – to be significantly different. Discarding scale 4 from subsequent analysis, Mason proceeded with a multivariate analysis of variance (MANOVA) using a 5 (grades) x 2 (gender) design with grade and gender as the between-subject variables and scores from the questionnaire's scales as the dependent variables (p. 77). She found a main effect by grade and gender but no interaction between the two. A post hoc Tukey's Honestly Significant Difference test revealed that the students' beliefs that they could solve time-consuming math problems decreased over the years while their beliefs that not all problems could be solved with step-by-step procedures increased over the five years. She reports that "belief in the usefulness of mathematics decreased fairly linearly" over the five years as well (p. 78). A stepwise regression analysis indicated that all mathematical beliefs but one ("effort can increase mathematical ability") predicted achievement in mathematics. Mason concludes that the decreasing trends in both students' beliefs in the usefulness of mathematics and their ability to solve difficult problems over the five years of high school are quite "worrying." Given that the students' beliefs predicted their achievement in all but one area, she suggests that her findings support the importance of measuring students' views in addition to their behaviors. She calls for further studies to investigate the underlying reasons for students' belief systems.

To summarize the major ideas underlying the studies already examined here, it is useful to consider the review of mathematics-related belief in research literature conducted by De Corte, Op't Eynde, and Verschaffel (2002). They acknowledge the "general agreement" among researchers today that students' beliefs have a significant impact on their mathematical behavior, but argue that there is "still a lack of clarity on the specific nature of beliefs and even more with respect to the different beliefs that are studied in relation to mathematical learning and problem solving" (p. 298). Drawing upon previous research in the field, De Corte, Op't Eynde, and Verschaffel suggest that a clearer categorization system could be the first step towards a more "comprehensive approach" and suggest a more heuristic model like that of Power and Dagleish (1997): beliefs about math education, beliefs about self in mathematical learning, and beliefs about social context in mathematics. Finally, they highlight how little knowledge and understanding exists in the present literature concerning "how positive beliefs about mathematics can be stimulated in students and how mis-beliefs that many students hold can be remedied" (p. 316). As a result, De Corte, Op't Eynde, and Verschaffel challenge future researchers to systematically study the "interplay among students' beliefs and instructional interventions" in order to better understand the influence of learning environments on mathematical views and behaviors (p. 317). It is a hope that this present study will begin to address the need evidenced by the literature for further examination of the "interplay" that exists among students' mathematical beliefs, behaviors, and learning environment over time.

2.3.3 STRANDS OF TASKS IN THE LONGITUDINAL STUDY

2.3.3.1 Overview of Tasks in the Longitudinal Study

The strands of tasks in number operations, counting/combinatorics, probability, algebra and pre-calculus/calculus implemented during the Rutgers-Kenilworth longitudinal study illustrate an example of mathematical concepts being addressed throughout the grade levels as called for by the NCTM *Principles and Standards* (2000). The task design of the longitudinal study reflects the theoretical framework of Davis and Maher. From his time with the Madison Project through his research at Rutgers, Robert B. Davis made a career of thinking about and acting upon math education ideas. As a result of his experience and reflection, some of Davis's writing examines mathematics curriculum in schools. In 1972 he outlined a "Piaget-based curriculum" on which future mathematics learning could be founded. He identified the "major task of schools" in teaching mathematics to be "not to tell the adult version, but to work with a child on describing, elucidating, and improving <u>his</u> ideas" (Davis, 1972, p. 8). He advocated a "developmental" approach that allowed students to "schemata" as they encountered carefully designed "assimilation paradigm"-building experiences. Lamenting that most curricula he witnessed in classrooms were dangerously "severed" from the real world and unfortunately consisted of "meaningless bits and pieces," Davis (1992) surveyed new approaches being taken across the country. He highlighted the "particularly important series of studies" that Carolyn Maher had undertaken with the inception of the Kenilworth longitudinal study at Rutgers (p. 731). He looked "hopefully" toward how the small-group work with mathematically rich tasks given plenty of time and manipulative materials could be incorporated into schools.

Francisco and Maher (2005) elaborate on the definition, implementation, and advantage of a "task design model based on a strand of problems." For example, the Tower problems, Pizza problems, and Taxicab problems, to be described in full later, were part of the counting/combinatorics "strand" of the Rutgers-Kenilworth longitudinal study. To be precise, a "strand" by definition of Francisco and Maher is "a series of related tasks designed around identified mathematical concepts with comparable levels of difficulty and similar problem-solving structure" (2005, p. 366). Some tasks within a strand are more obvious extensions of previous problems like the "Pizza with 4-Toppings" problem after the "Pizza with Halves" problem while others have more subtle isomorphic connections like the tasks "Towers 5-high from a choice of two colors" and "Taxicab Geometry." The strand approach enables the students to move forward, laterally, and backward among different investigations. This is an advantage for the researcher because the task strand approach "provides an opportunity for the students to revisit the same issues or concepts in a different context, which may be more cognitively appealing or familiar and increase the potential for a breakthrough" (Francisco & Maher, 2005, p. 366). The final benefit the authors suggest for the strand approach is the possibility for students to use the problems as "metaphors," as Davis would say, for building meaning as they discover underlying structural isomorphisms that exist among the tasks in the given research strand.

2.3.3.2 Content in the School Curriculum

The current NCTM *Principles and Standards for School Mathematics* (2000) asserts that "all students should learn algebra" and includes algebra among the five major content strands, along with number and operations, geometry, probability and data analysis, and measurement. One may well ask, what *is* "algebra" for math educators?

Believing a child's algebraic development to be a recapitulation of history, the much cited Anna Sfard (1991) argues that Boyer's (1965) three stages in the history of algebra – rhetorical, syncopated, and symbolic – correspond to a three-part schema of children's algebraic development in school. She proposes that a child's concept transition from computational operations to abstract objects develops in three "steps": (1) *interiorization* – where the student can perform operations on lower-level mathematical objects like substituting for a variable to find the value of a dependent variable in a function; (2) *condensation* – when the student can think about a given process as a whole like alternating among graphical, tabular, and mapping diagram representations of a function; and finally (3) *reification* – in which an "instantaneous quantum leap" occurs whereby the student sees a mathematical process anew as a static entity whose properties can be investigated like describing composition or inversion on functions (pp. 18-20).

Within this three-stage framework, Sfard also proposes that mathematical concepts can only be regarded as "fully developed" if they can be understood both *operationally* and *structurally* or, in other words, as both objects and processes (p. 23). Arguing that achieving "reification" was very difficult because of the requirement to think both operationally and structurally, it might, in fact, be out of reach for certain students. Sfard suggests that students might need to "put up with a certain amount of 'mechanical' drill accompanied by doubts about meaning" (p. 32). Sfard wrote several widely cited papers (1991, 1994, 1995) that advance her historical framework theory using interviews to illustrate her points.

An alternate theory of students' early algebraic reasoning and algebra's use in the school curriculum was advanced by Robert B. Davis. In the Madison Project, so named for the Madison School in Syracuse, NY where the Project's earliest experiments took place, Davis (1965) developed a series of classroom materials to introduce fifth to ninth graders – though some were used even at the second grade level – to concepts like variable, open sentence, truth set, function, mapping, number line, Cartesian coordinates, matrices, and quadratic formula. Over seven years, he and his collaborators collected video data from grades 2 to 9 of what he termed the students' "creative learning experiences," as opposed to lessons, since often times there was no "teaching" in the traditional sense in these videos. Davis explained that "what we do instead is to suggest to the children one or more mathematical tasks, and then work with them, unobtrusively, as they devise their own methods for tackling the tasks" (p. 3). He likened the way he sequenced the curricular materials in the Madison Project for the student to how an astronaut would explore the moon: first seeing broadly that the moon was there in the night sky – a very rough idea; then getting a slightly clearer vision of the moon's surface through a telescope – a clearer idea; then a more detailed picture through a satellite – a more refined idea; and then finally a very detailed immediate understanding through a moon landing – a minutely conceived idea (p. 6).

Considering the research of his colleagues, Davis (1985) took on the question of algebra's meaning in modern times. In his ICME-5 Report, Davis suggested that there were two different views about "algebra" in the math education community. The first view consisted of the "typical U.S. 'ninth-grade algebra' course" based on tasks involving the factoring of an algebraic expression and expectations of students being "shown what to do" so that then they could practice the instructor's method (1985, p. 199). In contrast to the show and tell pedagogy inherent within the "typical ninth-grade algebra course," the second view Davis outlined incorporated a focus on *experience*:

> In general, using the sequence: a) first *have* an appropriate *experience*; b) second, be able to talk about it accurately in simple language; c) third, learn to write about it in a nonmisleading notation (p. 199).

To lend better understanding to what he meant by an "appropriate experience," Davis gave examples of tasks like "Guessing Functions" which was first introduced by Warwick Sawyer in the 1950s and "U.V. (use of variables) and the Rule for Substitution." For the Guessing Functions game, the teacher might use $\square$ to represent the number the students said and $\triangle$ to represent the response after the rule was applied, generating the table:

$\square$	$\triangle$
0	7
10	27
1	9
100	207

(pp. 201 – 202)

A representation of a "rule" for the table above then would be $(\square \times 2) + 7 = \triangle$. According to Davis, such a task as "Guessing Functions" provided students with an experience from which a mental representation of the algebraic concept of function could be *built* along with a developing notation to clearly communicate that representation. Davis referenced a remark by Diane Resek to provide further clarification of algebraic meaning. There were three guises of algebra: first, going *from* reality *to* algebraic expression; second, going *from* algebraic expression *to* reality; and third, going from one algebraic expression to another. Davis applied Resek's thinking to the question of how to define algebra in schools: a traditional "ninth-grade algebra" course would focus only on the third aspect of algebra, whereas the first two aspects would be fundamental for algebraic reasoning in the earlier grades (pp. 204 – 205).

Whereas algebra has served long as a prominent content standard for the NCTM , combinatorics has been the lesser known and celebrated sibling in the academic family. Until recently, discrete mathematics, which integrates the three fields of combinatorics, iteration and recursion, and vertex-edge graphs, was not usually taught before the college level. NCTM introduced a Discrete Mathematics Standard for grades 9 – 12 in 1989 and then with the publication of *Principles and Standards* (2000), called for discrete mathematics to be a topic that should be "distributed across the Standards, instead of receiving separate treatment" throughout prekindergarten to grade 12 (p. 31). More recently, *Navigating through Discrete Mathematics in Grades 6 – 12* (Hart, Kenney, DeBellis, & Rosenstein, 2008) and the forthcoming companion book for grades K – 5 from NCTM address what discrete mathematical processes and content students should learn throughout the K-12 levels.

The very recent *Navigations Series* book for discrete mathematics (2008) offers many suggestions for how to incorporate the combinatorial ideas of systematic listing and counting in classrooms. Hart, Kenney, DeBellis, and Rosenstein (2008) recommend that students in K – 5 should become familiar with counting representations like lists, tables, arrays, tree diagrams, and Venn diagrams and then in grades 6 – 8 analyze counting more closely and in a wider field of situations (p. 15). They further suggest that teachers in grades 6 – 8 informally introduce permutations and combinations without emphasizing "the technical formulas and formal terminology" (p. 16). Then the authors argue that the "major focus" of discrete mathematics in grades 9 – 12 should be on applying algebraic notation and more formal reasoning to "extend, explain, and connect" the topics from earlier grades (p. 35). The NCTM *Navigations* recommendations share a great deal in common with the implementation work of the combinatorial tasks through the Rutgers-Kenilworth longitudinal study. There is a similar emphasis on providing mathematically rich experiences from which various representations can arise and connections can be made among them.

2.3.3.3 Student Reasoning in Longitudinal Study Tasks

Problems from the combinatorics and probability strands of tasks in the Rutgers-Kenilworth longitudinal study that are relevant to this research include Towers 4 and 5 tall selected from two or three colors, Pizza with Halves, Pizza with 4-toppings, "Ankur's Challenge," Binomial Expansion, and Taxicab. What follows below is an account of research analyzing the Kenilworth students' mathematical reasoning with regard to these specific tasks using video data of problem-solving sessions. The precise phrasing as posed to the students for each task is given as a footnote for each sub-section.

2.3.3.3.1 Towers Problem 3, 4, 5, n-tall (Grades 3, 4, and 5)[2]

Many illustrations exist of the progression of students' thinking with respect to the Towers problem in grades 3, 4, and 5 within the context of the Rutgers-Kenilworth longitudinal study (Alston & Maher, 1993; Maher & Martino, 1996a, 1996b, 1999, 2000;

[2] TOWERS PROBLEM 3, 4, 5, n-tall: *Your group has two colors of Unifix cubes. Work together and make as many different towers, say 4 cubes tall, as is possible when selecting from two colors. Convince us that you have found them all. What about 3-tall towers? What about n-tall?*

Martino, 1992). A specific case that was traced through grades 3, 4, and 5 follows the students Milin and Stephanie as they progressed from random guess and check methods to more systematic "local organization" to finally advanced justifications involving proof by cases and/or induction. A compilation of the task tables included from three different research reports (Alston & Maher, 1993, p. 2; Maher & Martino, 1996b, p. 433-434; Maher & Martino, 2000, p. 253) appear in *Figure 2-1* below to situate the reader within the timeframe of the studies to be discussed about these students in grades three, four, and five.

Grade	*Date Recorded*	*Activity/Description*
3	October 11, 1990	**Tower Problem 1, Class Activity**: Students find all possible towers that are four cubes tall when selecting from Unifix cubes in two colors.
3	October 12, 1990	**Tower Problem 1, Interview 1**: Individual children talk about the combinations they found and discuss their organization strategies.
4	February 6, 1992	**Tower Problem 2, Class Activity**: Students find all possible towers that are five cubes tall when selecting from Unifix cubes in two colors.
4	February 7, 1992	**Tower Problem 2, Interview 1**: Students reconstruct and explain their solutions from the previous day. "In this interview, random guess and check methods are replaced by local organizations as students monitor their production of combinations." (Maher & Martino, 1996b, p. 434)
4	February 21, 1992	**Tower Problem 2, Interview 2**: Stephanie, Milin, and Michelle further discuss organization of towers – "extension to towers with 3 colors; building from towers of 1 to towers of 2 cubes tall" (Alston & Maher, 1993, p. 2)
4	March 6, 1992	**Tower Problem 2, Interview 3**: Stephanie and Milin describe their reasoning. Milin uses "organization in 'families' for towers of 1-5 cubes tall" (Alston & Maher, 1993, p. 2) and "Stephanie discovered that the number of towers (selecting from two colors) doubles each time the height is increased by one cube. She used this pattern to calculate towers of heights 6 and 10" (Maher & Martino, 2000, p. 253).
4	March 10, 1992	**"The Gang of Four" Interview/Assessment**: Stephanie, Milin, Michelle and Jeff are asked to find all possible towers of three cubes high when selecting from two colors. Here, Stephanie justifies her answer with a "proof by cases" and Milin employs a "proof by mathematical induction" (Maher & Martino, 1996b, p. 434). Stephanie mentions a "doubling pattern" but does not explain how it works.
4	May 15, 1992	**Interview**: Researcher introduces Stephanie to a tree

		diagram. Stephanie does not adopt it as her own representation.
4	June 15, 1992	**Written assessment, partners**: Stephanie and Milin write an explanation for the benefit of a hypothetical absent classmate to account for all possible towers three cubes tall given a selection of two colors.
5	October 25, 1992	**Written assessment, individual**: Stephanie accounts for all towers three cubes tall using a "proof by cases." She checks her work with a "doubling rule" (Maher & Martino, 2000, p. 253).
5	February 26, 1993	**Tower Problem 1, Class Activity**: Students find all possible towers that are four cubes tall when selecting from Unifix cubes in two colors. Stephanie works with Matt to find all towers. After some disconnect between her "doubling rule" and then number of towers found so far, Matt employs Milin's "tree organization." Stephanie shares a "tree of towers" with the entire class that explains her doubling rule about the towers problem.

Table 2-1. **Towers in grades 3, 4, and 5 (Alston & Maher, 1993, p. 2; Maher & Martino, 1996b, p. 433-434; Maher & Martino, 2000, p. 253)**

In the fall of third grade when first exposed to the Towers Problem, many of the students use trail and error and guess and check and then reason that they had all possible tower combinations because they could not find any more (Martino, 1992). Still this reasoning persists into grade four when Milin justified his solution by arguing that "if you go about 4 minutes without finding one, you're probably done" (Alston & Maher, 1993). Maher and Martino (2000) summarize Stephanie's various strategies in grade 3: building a new tower and comparing it with the others to check for duplication, classifying certain individual towers as "red in the middle" or "patchwork," and relating pairs of towers together using the terms "opposite" or "cousin," though no explicit mention was made of particular groupings by cases (p. 438).

In grade 4, there is more evidence of the students grouping towers according to certain characteristics and using those groupings as sets to solve the problem. For example, in fourth grade, Stephanie demonstrates an "upside down and opposite procedure" to generate towers five cubes tall: "she and her partner would build a tower (call it A), build the 'opposite' of A, build the 'cousin' of A, and build the 'opposite of the cousin of A'" (Maher & Martino, 1996b, p. 439).

Maher and Martino (1996a) define Stephanie's "upside-down and opposites" pattern as an example of "local organization" strategy whereby relationships that she simply identified in third grade now become the foundation for her more sophisticated generating rule for sets. Likewise in fourth grade, Milin offers explanation based on groupings based around "staircase" patterns. He even constructs twenty of his total for the towers five-high using six cases that incorporate staircases as noted by Alston and Maher (1993) below. He generates the final twelve towers needed (for a total of thirty-two) by employing an "opposites" strategy after this staircase grouping work.

By the time of his third interview on March 6, 1992, Milin begins using the term "family" to describe the relationship between shorter and taller towers. In such a way, he progresses to reasoning about how a simpler case, like towers 3-tall, relates to towers 4 tall or 5-tall. He explains to the interviewer that to go from the towers 3-tall to the towers 4-tall, there would be 16 total because "2 for this, 2 for this, 2 for this, 2 for this, 2 for this, 2 for this, 2 for this, and 2 for this" pointing to the eight 3-tall towers that already exist that could get either of the two new colored cubes when it grows in height to 4-tall (Alston & Maher, 1993).

In fourth grade on March 10, 1992, Stephanie and Milin also participate in what has become known as the "Gang of Four" discussion where they both present their most sophisticated arguments yet. Stephanie tries to convince Jeff there were only eight possible towers that were three cubes tall by reasoning by cases. She organizes her work into five cases: towers with no blue cubes, one blue cube, two blue cubes "stuck together," three blue cubes, and two blue cubes "stuck apart" (Maher & Martino, 1996b, p. 437). During Stephanie's explanation both Milin and Jeff interrupt her and argue that the tower "blue, red, blue" should be included with her other towers of two blues and a red. Michelle also argues for only four cases that would combine Stephanie's blues "stuck together" and blues "stuck apart" into a single case of two blues. Stephanie stays with her original five cases, however, despite the others' protests. After a discussion of the importance of patterns, Milin introduces his exhaustive "building up strategy by multiples of two" which is essentially induction as it relates the previous height to the next consecutive height (Maher & Martino, 1996b, p. 442).

During the "Gang of Four" discussion, Stephanie also notes a doubling pattern – that there are twice as many towers with each additional cube of height - though she does not at the time show evidence of understanding the underlying reason for the pattern despite listening to Milin's induction argument. Maher and Martino (2000) trace Stephanie's pattern recognition in fourth grade to her understanding and justification of the pattern in fifth grade, when, during the February 26, 1993 session after listening to her classmates, she integrates a new scheme into her previous one and shares how all the towers could be generated at each height with a "tree of towers" finally justifying her doubling pattern.

Careful consideration must be given to *how* Stephanie and Milin were able to progress from methods of guess and check and "local organization" to the sophistication of proof by cases and induction. Maher and Martino (2000) note the conditions of the classroom environment facilitated conceptual change. They never give artificial closure to the Towers Problem, but rather allowed the students the necessary time to justify, reinvent, and further explore their oral and written responses in various flexible groupings (individual, pairs, small groups, and whole class).

While the focus here has been on examination of student approaches to the Towers Problem in grades 3, 4, and 5, the Towers Problem was revisited again in later grades of the Kenilworth longitudinal study. Muter (1999) provides an analysis of a problem-solving session that occurred when the students were in tenth grade and the binary coding system was used by the students to solve the 5-tall towers problem. Then an extension that involves towers four-tall choosing from three colors with each color used at least once was offered by Ankur.

2.3.3.3.2 Towers Problem 4-Tall selecting from Three Colors (Grades 4, 5, 10)[3]

Muter (1999) considered a cohort of five students (Ankur, Jeff, Brian, Michael, and Romina) from the Rutgers-Kenilworth longitudinal study and compared their work on several of the combinatorics tasks when they first encountered them in fourth and fifth grades and then when they later revisited the same problems in tenth grade. Muter asserts that the students' insight into the Towers 4-tall with a choice of Three Colors came only *after* they had spent time on the Towers 4-tall selecting from Two Colors and the Pizza problems. She notes that while the students developed the formula 2^n to predict the number of pizzas that could be created with *n* toppings, they were not initially able to explain *why* the base would be 2. When revisiting the towers and pizza problems in tenth grade, they discussed the possible connections between them. The "realization" that the

[3]TOWERS 4-Tall with Three Colors: *Your group has three colors of Unifix cubes. Work together and make as many different towers four cubes tall as is possible when selecting from three colors. See if you and your partner can plan a good way to find all the towers four cubes tall. Convince us that you have found them all.*

base 2 resulted from the two choices of presence or absence for a topping came when they solved the Towers 4-tall selecting from Three Colors problem (Muter, 1999, p. 127). During the December 1997 session, Michael explained his idea about a simpler version of the problem that considered only towers 2-tall and referred to the three colors as "numbers":

> I'm a hundred percent sure. But here is what I'm thinking. You have two different places to put the colors. For the first place you have the possibility of three numbers. In the second place you have another possibility of three numbers. And if you times them, it comes up to nine. (p. 114, Line 384)

Eventually they reasoned that if there were four positions on the tower into which any of the three colors could be placed, the result would be $3 \times 3 \times 3 \times 3 = 81$ possible towers. Muter suggests that revisiting the meaning of base 3 in this problem gave them insight into the earlier Towers problem with the use of base 2.

2.3.3.3.3 The Pizza Problem with Halves (Grade 5)[4]

From 1992 to 1993, different pizza problems were administered by Rutgers in three school districts in New Jersey: Kenilworth, New Brunswick, and Colts Neck. The Four-Topping Pizza Problem was given to fourth and fifth grade classes at all three sites and the Pizza with Halves problem was given only to the fifth grade classes at Kenilworth and New Brunswick. For the fifth grade classes who received both pizza problems, the students tackled the Pizza with Halves problem first. Bellisio (1999) provides a thorough description of distinct case studies about students from all three sites

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[4] PIZZA PROBLEM WITH HALVES: *A local pizza shop has asked us to help them design a form to keep track of certain pizza sales. Their standard "plain" pizza contains cheese. On this cheese pizza, one or two toppings could be added to either half of the plain pizza or the whole pie. How many choices do customers have if they could choose from two different toppings (sausage and pepperoni) that could be placed on either the whole pizza or half of a cheese pizza? List all the possibilities. Show your plan for determining these choices. Convince us that you have accounted for all possibilities and that there could be no more.*

as they developed ideas about the pizza problems. Attention here will be given to the Kenilworth students (the first group of Michelle I., Jeff, Matt, Stephanie, and Milin and the second group of Romina, Ankur, Bobby, Amy Lynn, Michael, Michelle R., and Brian) who first encountered the Pizza with Halves problem as fifth graders on March 1, 1993. Bellisio (1999) notes that the first hurdle the students had to overcome was one of notation. Michelle I. wanted "P" to represent a "cheese pizza," Matt said "P" should mean "plain," and Jeff insisted "P" denote "pepperoni." That Jeff used the letter "C" to represent a plain pizza caused some confusion because, as Stephanie later pointed out, cheese was understood to be on all pizzas. Thus Matt duplicated pizzas when he listed both "S" (sausage pizza) and "CS" (cheese with sausage pizza). Likewise Milin's notation created duplicates when he wrote both "CPS" and "PS." The students also seemed to struggle with the notational organization for half pizzas – Jeff used "SP" for a mix of sausage and pepperoni and "S|P" for sausage on one half and pepperoni on the other. The first group tested different organizational strategies like categorically listing the pizzas using full words instead of abbreviations but didn't agree on a solution by the end of the first day. On the next day, the students in the first group continued to discuss notation. Matt then had the idea to group the pizza types as C, P, S for the first group, C/P, C/S, P/S, SP for a second group and S/SP, P/SP, C/SP for the third group (Bellisio, 1999, Fig 46, page 122). While the students eventually agreed that there were 10 possible pizzas, Stephanie continued to feel uncomfortable with the use of "C" for the plain pizza and thus different charts were created and discussed.

The second group of Kenilworth students used two different strategies: Michael drew pictures (with circles for pepperoni and ovals for sausage) which Amy Lynn later helped label while Ankur created a list. The group did not work together until Romina observed that "we have to start talking, communicating" (Bellisio transcript K-2, line 21). By the end of the first day of work, Michael had drawn 12 pizzas that the rest of the group suggested. The next day Brian observed that he thought there were 13 pizzas and suggested that someone should double-check their list. Romina read out the list as the others checked it against the drawings. Similar to the other group, most of the discussion focused on the use of "cheese" as a separate topping – eventually they decided to eliminate such entries as "half pepperoni-cheese, half sausage" since it would duplicate "half pepperoni, half sausage." They created a new list without pictures using written out labels and categorized into three groups: wholes, single toppings on one half, and the half pizzas with mixed toppings. They agreed that 10 pizzas would meet the criteria of the problem.

A month later, all twelve of the Kenilworth students met together on April 2, 1993 and were again given the Pizza with Halves problem. Bellisio (1999) notes that this time "most of the students organized their pizzas in the categories of whole, half, and mixed" (p. 137). She suggests that the two days of work and a third day of writing the month prior had an effect on the students' now "confident" and "systematic" presentations of the problem. Bellisio also observes that this new session offered the students an opportunity to revisit *why* they chose the categories they did and reason about their problem solving at a higher meta-level.

2.3.3.3.4 The Four-Topping Pizza Problem (Grade 5)[5]

[5] FOUR-TOPPING PIZZA PROBLEM: *A local pizza shop has asked us to help design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many different choices for pizza does a customer have? List all the possible choices. Convince us that you have them all.*

The Four-Topping Pizza Problem is a question that invites students to make connections. When implemented in what Davis termed a "paradigm teaching strategy" as it has on numerous occasions by members of the Rutgers math education research community, the Pizza Problem is an isomorphic task situation to the 4-cube high Towers Problem.

The group of Kenilworth fifth graders, who had encountered the Pizza with Halves problem over a period of three days in March 1993, was given the four-topping pizza problem next. When the twelve students reconvened the next month in April, they first reflected on the Pizza with Halves problem for about forty minutes. Then the researcher facilitating the discussion presented the Four-Topping Pizza Problem. Bellisio (1999) describes how Jeff, Ankur, Brian, and Romina started to work together. As the others suggested choices, Romina recorded an initial list of possible pizzas because she had the "best handwriting" (Bellisio, 1999, p. 141, figure 53).

As a result of the researcher's questioning, the group realized another confusion of notation resulting from the use of "Pl" for "plain" since "P/Pl" (a pepper with plain pizza) would duplicate "P" (a pepper pizza). Ankur then described a new way to organize:

> Okay, you start with the first, P, and you mix it with the second one, that's P slash S. And then you start with the first one again, skip the second and mix it with the third, that's M, P slash M. Then you start with the P again and mix it with the fourth one, PE. And then you start with the S because that's the, you can't use plain. We start with S and mix it with M. (Bellisio, 1999, Transcript K-6, line 145)

Based on Ankur's new organization strategy, Romina created a new chart that listed all sixteen pizzas. Meanwhile, Stephanie and Matt worked together as a separate pair. They went back to the "whole" and "half" categories from the Pizza with Halves problem and, after discarding the “half” option, generated a list of pizza combinations beyond the single topping wholes by pairing one topping with the others. Stephanie and Matt eventually reorganized their list into cases of one topping, two toppings, three toppings, and four toppings after discussing their work with the researcher. Bellisio notes that “in presenting the more difficult, pizza with halves problem first, researchers from Rutgers were exploring the idea that once the students had solved the first problem, they would solve the simpler problem very easily and quickly” (p. 106). She suggests that the Kenilworth students did indeed seem to solve the “simpler” problem of the 4-Topping Pizzas with relative ease and quickness as it took under an hour for all the students to arrive at the correct answer of 16 pizzas.

Muter (1999) recounts the sessions the Kenilworth students had with the various pizza problems as fifth graders in 1993 and then extends the story to December 12, 1997 when a group of five from the original twelve students revisited the pizza problem as tenth graders. As the students ate actual pizza around a table, the researcher asked them to consider the pizza problem given a choice of five toppings (rather than the four originally in fifth grade). The initial notations differed among the students. Jeff, and Romina employed letters for their possible toppings, Ankur and Jeff decided to use a numerical notation of 1, 2, 3, and so on to represent the first, second, third toppings, and finally Michael worked alone with a binary code of 0s and 1s. Though their notations varied and after getting slightly side-tracked by attempting to impose some kind of factorial solution, they all eventually set about listing the possible pizza combinations in an organizational strategy not unlike what they had done as fifth graders. They concluded there were thirty-one possible pizzas though this did not fit with the doubling rule they felt to exist (8 pizzas with three toppings, 16 pizzas with four toppings). At that point, Michael interrupted and presented his binary number representation. Drawing on previous knowledge, he explained to the rest of the group the place value meaning in the binary number system: for four toppings, they would use four digits for each binary number, five digits for five toppings, and therefore an additional digit for each extra topping. He convinced the group that there would be thirty-two 5-topping pizzas.

2.3.3.3.5 "Ankur's Challenge" (Grade 10)[6]

Muter (1999) describes the session that occurred on January 9, 1998 when Ankur posed his own tower extension problem that later came to be known as "Ankur's Challenge." Just as one month earlier (described previously when the students revisited the pizza problem), the researcher began by asking the students (Michael, Ankur, Brian, Jeff, and Romina) to revisit a corollary to the problem done in fourth grade: when selecting from red and yellow, find all towers five-tall that contain exactly two reds. The students quickly answer that there are 10 towers. When asked to justify this response, Michael and Ankur produce a representation using the binary coding scheme Michael had shared in the December 1997 last session for pizzas. They share their reasoning: fix a red cube "1" in a position and generate the other possibilities that contain one other "1" in the sequence.

Meanwhile, Romina Brian, and Jeff recall that when they had previously done the problem there were thirty-two total towers five-high. Combining written work and building with actual Unifix cubes, they begin to organize their work by cases. Jeff then

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[6] "ANKUR'S CHALLENGE": *Find all possible towers that are 4 cubes tall, selecting from cubes available in three different colors, so that the resulting towers contain at least one of each color. Convince us that you have found them all*.

suggests a multiplication scheme: since one times five equals the five towers with one red, two times five equals ten is the "reason" there are ten towers with two reds. Romina points out the error in Jeff's reasoning when she indicates it would not work for a tower with no reds: "But then this one wouldn't be times zero. 'Cause five times zero is zero" (Line 222, page 91). They then move forward, disregarding the multiplication scheme, to justify why there would be ten towers that meet the criteria.

While the others are finishing, Ankur proposes his extension problem to find all the towers four-tall when selecting from three colors so the towers contain at least one of each color. He and Michael begin by determining the total number of towers four-tall when selecting from three colors. They recall the relationship of base 2 to the towers selected from two colors and that there would be 2^4 towers four-tall selected from two colors. They conclude there would be 3^4 towers four-tall selected from three colors. Next, using the notation 1, 2, and 3 to represent the three colors red, yellow, and blue, Ankur and Michael try to list the combinations that would form the complement of Ankur's problem. However, they do not come to an accurate resolution (finding 39 towers by subtracting 81 – 42); they believe they have accidently listed duplicates.

By this time, Jeff, Romina, and Brian join the discussion. Romina questions the others about why there are an odd number of total towers 4-high selected from three colors: "don't they have pairs?" and "how did you get 81?" After Jeff and Ankur explain the reasoning of three to the fourth powers, she asks them to restate the problem: "Oh, so that's not the problem. So what is the problem?" (Muter, 1999, p. 95) After Ankur restates his problem, Jeff conjectures that the answer is 36 since he has already listed 37 and believes there is a duplicate. Jeff joins Ankur and Michael to work. Commenting that "it might be thirty-six 'cause I'm working with sixes now" and requesting that they "let me think first, organize my thoughts first," Romina develops her own proof (p. 96). She uses the notation X, 0, and 1 to represent the three colors and the fact that each tower will have exactly two of the same color. In sets of six, she creates pattern groups in which a color is used twice. Then, working with Brian, she goes through several drafts of a generalization. She draws a line with the duplicated cubes (using the number 1 to denote the duplicated color) in the first two positions of the tower. On the next line she draws the duplicated cubes in the third and fourth positions. Then in line three she put them in the second and third positions; for line four, in the first and fourth; for line five, in the second and fourth; and then for line six, in the first and third.

Romina explains her proof to Jeff. Each of the six possibilities could be multiplied by 2 because the two non-repeating colors in each row could take either of the two remaining spaces available. Then you would multiply the twelve by 3 because the duplicated color could be any one of three possible colors, resulting in the answer 36. To better convince the group, she revised her representation yet again to include horizontal rectangles divided into four subsections to indicate a tower 4-tall and notational code of X, 0, and 1. She presents her proof two more times during the session, both at the chalkboard and then in a written description, each time her argument becoming more and more refined. Muter (1999) remarks that the iterative process through which Romina developed her proof demonstrates that "students need to have occasion not only to explain and write about heir ideas, but that repeating the opportunities for the students to present thoughts are essential for erudition" (p. 126).

2.3.3.3.6 Exploring Binomial Expansions (Grades 8 and 11)[7]

As mentioned earlier, Maher and Speiser (1997) examined how Stephanie, a student from a longitudinal study, was able to build a connection among the Towers Problem, Pizzas Problem, and Pascal' Triangle. During an early interview of the teaching experiment, Stephanie explored how 3-high towers with a choice of two colors could be related to a monomial of degree three in two variables (p. 128). During a later interview, Stephanie was able to connect the binomial coefficient notation $C(n,r)$ to towers where n is the height of the tower. The researchers suggest Stephanie's earlier mental representations for Towers and Pascal's Triangle enable her to connect to the abstract representation of binomial expansion.

Muter (1999) mentions how the Kenilworth cohort of Romina, Ankur, Brian, Jeff, and Michael related the towers and pizza problems to binomial expansion and Pascal's triangle in sessions from 1998 when they were in high school. In February 1998, after reviewing work from previous sections, the interviewer asked the students if there was a relationship between combinatorial notation in Pascal's triangle and the Pizza problem. Uptegrove (2005) more explicitly explores the students' ideas about isomorphisms among the combinatorics tasks and presents a timeline for their work with binomial coefficients: "a connection between binomial coefficients and Pascal's Triangle was first discussed by this group on January 9, 1998 and reiterated on March 6, 1998 and June 12, 1998" (p. 93). During the March session the students explored the connections among binomial coefficients, pizzas, and towers when they represented a blue block as "a" and a

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[7] *Expand binomial expressions to different powers. Represent and describe what you observe about the results, particularly the coefficients. How would you predict the coefficients for* $(a+b)^n$ *? Justify your prediction and explain why it works.*

white block as "b" in the expression $(a+b)$ as well as two different toppings for a pizza. When prompted to rewrite Pascal's triangle in "choose" notation, the students made reference to towers and binomial expansion when they expressed the n^{th} row: Romina said "make *n* your height"; Ankur remarked, "write a plus b to whatever it is" next to row *n*; and Jeff observed row *n* was "*a* plus *b* to the n^{th}" (Uptegrove, 2005, p. 132). In the "Night Session" on May 12, 1999 when the students derived Pascal's identity, they drew upon these earlier isomorphisms.

2.3.3.3.7 The Taxicab Problem (Grade 12)[8]

The Taxicab Problem as explored in the Kenilworth longitudinal study is analyzed by Powell (2003). Romina, Michael, Brian, and Jeff attended the Taxicab session on May 5, 2000. Uptegrove (2005) suggests that Romina took more of a "lead role" in the session: she made two suggestions that the group consider the Towers Problem and then observed that Pascal's Triangle could be connected to the Taxicab routes. Later when Romina and Brian described their solution, they used the terminology "x" and "y" to refer to "across" and "down" (perhaps linking to binomial expansions as well). Powell notes that, underscoring her observation of the isomorphisms present, Romina mentions that they could just as easily have used topping or color instead of direction.

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[8] *A taxi driver is given a specific territory of a town, represented by the grid provided. All trips originate at the taxi stand, the point in the top left corner of the grid. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated by the other points on the grid. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route. What is the shortest route from the taxi stand to each point? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many?*

Chapter 3 METHODOLOGY

> To define a word, then, the dictionary editor places before him the stack of cards illustrating that word; each of the cards represents an actual use of the word by a writer of some literary or historical importance. He reads the cards carefully, discards some, rereads the rest, and divides up the stack according to what he thinks are the several senses of the word. Finally, he writes his definitions, following the hard-and-fast rule that each definition *must* be based on what the quotations in front of him reveal about the meaning of the word. The editor cannot be influenced by what *he* thinks a given word *ought* to mean. He must work according to the cards, or not at all.
>
> - Excerpt from "How Dictionaries are Made" by S. I. Hayakawa in *Language in Thought and Action* (pp. 55, 1949)

3.1 Design of the Study

This research employs a qualitative, phenomenological approach using videotaped data selected from recordings by the Rutgers-Kenilworth longitudinal study[9] between February 6, 1992 and May 22, 2006, along with student work, questionnaires, and researcher field notes. The design addresses two methodological issues: Romina's mathematical behavior as a participant in problem-solving sessions during the longitudinal study and her beliefs about those experiences as expressed during later clinical and semi-structured interviews.

To investigate Romina's behavior and beliefs within the framework of her own lived-in experiences of the longitudinal study, a phenomenological approach seemed most appropriate. According to Moustakas (1994), empirical phenomenology requires the researcher to "return to experience in order to obtain comprehensive descriptions that provide the basis for a reflective structural analysis that portrays the essences of experience" (p. 13). Thus, this approach works to understand the underlying framework or "essences" for the phenomenon of human behavior by interpreting it in the original situation of the experience's occurrence. Moustakas explains that the dual aim is to both

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[9] Two grants from the National Science Foundation supported the longitudinal study: MDR-9053597 (directed by R. B. Davis and C. A. Maher) and REC-9814846 (directed by C. A. Maher).

determine the meaning of a person's experience and provide a "comprehensive description" of it (p. 13). Giorgi (1985) advocates this type of qualitative research as one in which, after first carefully describing the phenomena in a more "naïve" way, a direction towards further investigation and more "general" significance can be found.

3.2 Data Collection

In addition to student work, questionnaires, and researcher field notes, this study utilizes two types of video data: clinical and semi-structured *interviews* and small and large-group *problem-solving task sessions*. The interviews ranged in length from about a half hour to approximately two hours. The clinical interviews made during May 1999, when Romina was in eleventh grade, were conducted in collaboration with Annenberg Media and the Science Media Group of the Harvard Astrophysics Observatory for *The Private Universe Project in Mathematics* or *"PUP-Math"* (2000). These recordings used separate digital cameras focused on student faces and, when applicable, student work. The semi-structured interview on May 22, 2006 took place approximately two hours in a large-group format and was filmed by a staff member of the Robert B. Davis Institute for Learning (RBDIL) in the Rutgers Graduate School of Education. The semi-structured interview conducted at the Graduate School of Education at Rutgers University on July 15, 2009 occurred after Romina's graduation earlier in May with her M.B.A. from the Kellogg School of Management at Northwestern University. The interview lasted approximately 90 minutes and followed a loosely structured format. There was a single video camera and microphone. All of Romina's written work during the interview was subsequently scanned. The interview from 2009 is the only one included in this study in which the researcher was a participant – in this case, the main questioner.

The problem-solving behavioral data were selected from an archive maintained by the RBDIL that includes approximately 3500 video recordings of whole classroom, small-group, and individual task sessions, session descriptions with corresponding observer and researcher notes, student written work, and video transcripts. The interview data and problem-solving data were analyzed separately according to the plans described below.

3.3 Analysis of Interview Data

Analysis for the interviews follows an adaptation of an analytical model (Powell, A. B., Francisco, J. M., & Maher, C. A., 2003) that incorporates transcription, coding, and narrative and builds upon the phenomenological qualitative data analysis methods by Moustakas (1994), Giorgi (1985), and Belenky, Clinchy, Goldberger, and Tarule (1986). Video of the interviews are first viewed repeatedly to get a sense of the overall structure. If not already in existence as part of the RBDIL archive, transcripts of the entire interview sessions are produced and then verified. Analysis then follows Francisco's (2004) interview model and suggestions by Moustakas (1994) and Belenky, Clinchy, Goldberger, and Tarule (1986) to identify "significant" statements and then "cluster" those significant statements to identify larger thematic categories according to "epistemological position." Next, one creates a table to delineate the following researcher observations: issue, significant statement, summary, and interpretation. What then follows is a narrative constructed as the last step of the video data contextual analysis to address the question: how do Romina's later adult views about learning relate to evidence of her earlier mathematical behavior in terms of her descriptions of knowledge, the conditions for learning environments, and the learning process? To summarize, the following five steps are incorporated into the interview data analysis:

1. Viewing the video
2. Transcribing and verifying the interviews
3. Determining "significant statements"
4. Clustering into general thematic categories
5. Writing a structural descriptive narrative with a coding scheme

3.3.1 Viewing the video

The researcher repeatedly watches and listens to the video data set in order to be familiar with the over-arching scope and sequence of events. Powell, Francisco, and Maher (2003) emphasize that the researchers watch and listen "attentively" and "without intentionally imposing a specific analytic lens on their viewing" (pp. 415-416).

3.3.2 Transcribing and verifying the interviews

Powell, Francisco, and Maher (2003) offer several reasons to transcribe a videotape: first, to allow researchers to implement coding procedures on a "static rendering" of an inherently dynamic problem-solving session; second, to afford researchers the opportunity to analyze "participants' discursive practices"; third, to create a "permanent record" that can yield detailed categories not always captured during a visual and auditory inspection of the original video; and fourth, to provide "evidence of findings in the participants' own words" in research reports (p. 422). Here, the motivations for transcription involve all the reasons stated above except perhaps the second because, although a permanent and "static rendering" of Romina's comments are needed for analysis, this study does not focus on Romina's discursive practices. Transcripts for the interviews are produced and verified by at least one other viewer for accuracy. The transcripts include line numbers, time codes (hour:minute:second), speaker codes (T/R1 = "teacher/researcher 1"), and verbatim transcription of the speakers' utterances.

3.3.3 Determining the significant statements

Following Moustakas (1994) and Francisco (2004), "significant statements" are tagged in the transcripts. Using Francisco's model, "the significant statements are determined on the basis of their ability to summarize the students' response to the issue and, in particular, perceived relevance of the statement to the phenomenon under study" (p. 12, 2004). Here "the phenomenon under study" indicates Romina's views on mathematical learning.

ISSUE	*SIGNIFICANT STATEMENT*	*SUMMARY*	*INTERPRETATION*
KNOWING - Definition *Interviewer Question: "But for yourself as a learner, when do you feel you know something really well?" (line A-158)*	"When you're able to explain it to someone else and they ask you every question under the sun and you can still answer it. I think then you know it." (line A-160)	Romina compares being able "to know" with being "able to explain" and address "every question under the sun."	For Romina, knowledge and the learning experience are situated in dialogue (the knower must be able to *explain* to a questioner). She seems to contextualize knowing in a broad sphere. To be able to answer "every question under the sun" about a topic implies a deep and very well-rounded understanding. It might also indicate being able to connect among many disciplines at once.

Figure 3-1 **Sample analysis of an interview**

3.3.4 Clustering into general thematic categories

Next, the significant statements identified in the transcript are reorganized or "clustered," to use Francisco's (2004) terminology, into broader thematic categories. To aid in this process, a table delineating the following researcher observations - issue, significant statement, summary, and interpretation – is produced. *Figure 3-1* below provides an example of this process as applied to Romina's May 2006 interview. Reading from left to right on Figure 3-1, notice that the first column identifies an issue within a broader thematic category, "knowing." The second column gives a significant statement Romina made that falls into that category. In the third column, a summary of strict observation is given - as free of personal analysis or bias as possible. The last column allows for the researcher's analytical lens or perspective to intrude as an "interpretation" is given of the statement.

The significant statements are purposely re-copied from the original transcripts into this analytical table in a manner similar to Belenky, Clinchy, Goldberger, and Tarule (1986). They describe their analysis of the interviews conducted with 135 women for *Women's Ways of Knowing* as "labor intensive method" whereby they would underline significant text in a transcribed interview, copy verbatim the "most salient quotes," and then group the quotes by "epistemological position" (p. 17). The authors comment:

> The very process of recopying the women's words, reading them with our eyes, typing them with our fingers, remembering the sounds of the voices when the words were first spoken helped us hear meanings in the words that had previously gone unattended. We moved back and forth between these excerpts and the unabridged interviews. This enabled us to maintain a dual perspective, hearing the statements as exemplars of a particular epistemological position but hearing them also in the context of the woman's whole story. (p. 17)

By moving back and forth between the full original transcript and the selected excerpts of significant statements, the researcher can thereby find a deeper way of "hearing" the subject. Belenky, et al (1986) likens their manner of analysis to the description Hayakawa (1964) gives of how dictionary editors define new words as they enter the common discourse. A new word and an example sentence of its use are written on a card. Editors take all the cards that have accumulated for a particular word and, after re-reading and sorting them according to different contextual uses, write a general definition for the word as it will appear in the dictionary. Similarly here as the researcher records separate significant statements of Romina, through the process of re-reading, sorting, and categorizing her statements into broader themes, a larger definition of her process of knowing or sense-making will hopefully emerge.

3.3.5 Writing a structural descriptive narrative with a coding scheme

The final step of the analysis involves writing a narrative that describes the general structure of the student's statements. The goal at this step is to explicate the relationships between Romina's statements about her experiences in order to gain insight into the phenomenon being studied here, namely, her beliefs about mathematical learning. Contextual analysis of the interview video data employs the following scheme.

Code Name	*Code Description*
Knowledge and Knowing	Ontological and epistemological descriptions of mathematical learning *Example*: "*When you're able to explain it to someone else and they ask you every question under the sun and you can still answer it. I think then you know it." (5/12/06, line A-160)*
Conditions of Learning Environment	Description of the conditions for learning environments *Example*: "*We'd have tables, no desks, tables. I don't know, we'd all sit in groups of 4 or 5 and we'd rotate periodically so we could work with different people all the time so we'd have to re-learn how to work with people." (5/12/06, line A- 277)*
Process of Learning	Description about the process of activities that contributed to knowing in mathematics Example: "*I hate learning things that don't, like I feel like if you learn one concept that doesn't connect to other concepts, like you're learning something almost useless." (5/12/06, line B-242)*

Figure 3-2. **Coding scheme for the analysis of the interviews.**

The coding scheme is an organic construct based on the most frequent emergent categories that occur during the analysis of significant epistemological statements. Notice there are three main categories of focus: knowledge and knowing, conditions of the learning environment, and the process of learning. As further interviews were included within the scope of this study, the codes adapted accordingly, but the categories remained the same.

Below is a summary of the specific interview video data analyzed for the research:

DATE	*FORMAT/TOPIC*	*PARTICIPANTS*	*GRADE*
1999-05-18	*Clinical Interview*: Reflections on longitudinal study for Private Universe Project in Mathematics (for PUPMath)	Romina	*11*
1999-07-21	*Clinical Interview*: Reflections on longitudinal study for Private Universe Project in Mathematics (for PUPMath)	Romina	*12*
2002-03-11	*Semi-structured*: Romina and Jeff review and comment upon video of their own mathematical behavior	Romina, Jeff, T/R1, math education seminar	*College Sophomore*
2006-08-22	*Semi-structured*: Romina, Angela, and Magda reflect on the longitudinal study and discuss learning in the context of their current career paths	Romina, Angela, Magda, T/R1, math education seminar	*Business Analyst*
2009-07-15	*Semi-structured*: Romina reflects on the longitudinal study and discusses learning in the context of her graduate work in an M.B.A. program. Romina also revisits the Towers 5-High task.	Romina	*Post-Graduate*

Figure 3-3. **Summary of interview videos for analysis**

3.4 Analysis of Problem-Solving Data

Analysis for the behavioral problem-solving data follows similar procedures - transcription, coding, and narrative - as that outlined above for the analysis of interviews. An adaptation of the analytical model (Powell, A. B., Francisco, J. M., & Maher, C. A.,

2003) is again followed. Video of the problem-solving sessions are first viewed repeatedly to get a sense of the overall structure. If not already in existence as part of the RBDIL archive, transcripts of the entire interview sessions are produced and then verified. Next, significant verbal and non-verbal behaviors are noted and summarized. "Critical events" are identified. Powell et al. (2003) clarify that "an event is called critical when it demonstrates a significant or contrasting change from previous understanding, a conceptual leap from earlier understanding" (p. 416). The use of critical events in video analysis has been extensively documented (Kiczek, 2000; Maher, 2002; Maher & Martino, 1996a; Steencken, 2001). A coding scheme is developed to aid the researcher in generalizing the themes that emerged in the data. Then, by applying the codes to the data, a "storyline" is constructed. Powell, et al. (2003) explain that this step involves coming up with "insightful and coherent organizations of the critical events" and discerning "traces" or the "collection of events, first coded and then interpreted, to provide insight into a students' cognitive development" (p. 430). Using "traces" to illuminate a student's personal cognitive growth as with as the collective collaborative growth of the community of learners has been documented (Maher & Speiser, 1997). Finally, a narrative is constructed as the last step of the video data behavioral analysis to address the research questions about: 1) how Romina's representations and justifications for her ideas develop over time and 2) to what extent, if at all, does Romina collaborates and incorporates the ideas of others into her own ideas. To summarize, the following six steps will be incorporated:

1. Viewing the video
2. Transcribing and verifying the interviews

3. Determining "critical events"
4. Developing a coding scheme
5. Writing a structural descriptive narrative

3.4.1 Developing a Coding Scheme

While the viewing, transcribing, and verifying of the videodata do not require further explication (as the process mirrors exactly the process for the interview data previously described), the coding scheme and narrative for the behavioral data did follow a slightly different course. Coding allows the researcher to annotate transcripts in such a way that underlying themes may begin to emerge. A result of many different drafts, my final behavior coding scheme sought to highlight the major themes of the research questions – namely, how Romina's representations and justifications of mathematical ideas developed over time, to what extent, if at all, she collaborated and incorporated the ideas of others, and how Romina's beliefs about learning and knowledge relate to her behavior. Below is a figure summarizing the behavioral coding scheme:

Coding Scheme for Analysis of Behavior Data *Note* – [] denotes that the researcher would leave a "blank" space in the column				
Code Name	*Code Description*			
EVALUATE Previous Statement	How the speaker assesses the previous speaker's statement or action.			
	+ (agree)	- (disagree)	[] (neutral)	* (change topic)
INVITE another's participation	The method through which the speaker provokes further group collaboration through choices of verbal discourse			
	[] (declarative statement)	! (imperative command)	? (question) -**letter** - *sub-code for questions*	
	i (Inform)	!-i (Inform)	?-i (Inform)	Asks the addressee to provide a piece of information like, "Which one did we just make?"
	v (Verify)	!-v (Verify)	?-v (Verify)	Asks the addressee to verify or confirm the speaker's assumption as in, "Aren't we supposed to go down to number two?"
	r (Reiterate/ repeat)	!-r (Reiterate/ repeat)	?-r (Reiterate/ repeat)	Elicits or puts forward a repetition of a previous statement like, "Zero times two plus one?"
	e (Explain/ clarify)	!-e (Explain/ clarify)	?-e (Explain/ clarify)	Invites the addressee to further explain or clarify a preceding action or statement like, "Can you run that past me?"
	s (Suggest)	!-s (Suggest)	?-s (Suggest)	Invites the addressee to consider a suggested course of action as in, "How about we try one on the bottom?"
JUSTIFY an Idea	Whether the student provides an informal explanation or more formal logically rigorous argument for his/her mathematical action(s) or decisions			
	+ (yes, informal explanation)	++ (yes, more formal argument)	-(no justification)	[] (not applicable)
Express AFFECT	Whether the student verbally or nonverbally expresses an emotional state or perceptual belief that registers as a change from a previous state.			
	+ (positive)	- (negative)	[] (neutral)	
Express KNOWLEDGE	Whether the student verbally expresses knowledge or lack of knowledge about a particular math topic			
	+ (I know)	- (I do NOT know)	[] (no mention)	*P(Reference to previous knowledge) *S(Reference to shared knowledge in group)

Figure 3-4. **Coding Scheme for the Analysis of Behavior Data.**

Behavior data analysis incorporated and modified suggestions from Chiu (2008) for a five dimensional coding of problem-solving discourse: evaluation of previous action, invitation for participation, justification, affective expression, and knowledge expression. The detailed five-dimensional coding was employed after "critical events" were highlighted in the transcripts. Within each of these five dimensions, Romina's statements in each behavioral transcript at "critical event" junctures were considered and then summarized – thus, the entire transcript was not so coded. Often, the classification within each dimension became binomial in nature as a statement such as "Yeah, we have that" would be an evaluation of a previous statement that is in agreement – it would thus be coded with the binomial "E +" (evaluate and agree). A statement like, "Aren't we supposed to go down to number two?" would be flagged under the "Invite" dimension as "?-v" as it is a question seeking verification. When applicable to a "critical event," each line in a behavior transcript was coded under the five dimensions of evaluation, invitation, justification, affect, and knowledge. Consider an excerpt:

Line	Time	Speaker	Transcript	**E**valuate	**I**nvite	**J**ustify	**A**ffect	**K**now
125		Romina	Yeah, think!	+	!			
126		Brian	Yo, white-white-blue?		?s	-		
127		Romina	White-white-blue? Two whites. [Hands Brian two white cubes].		?v			
128		Brian	White-white-blue. [Puts two white cubes on top of one blue cube]. Another two whites. [Romina hands him another two white cubes]. Do we have that?		?i			
129		Romina	Yeah, we have that.	+		-		
130		Brian	Where? [Romina points] No. White. Ahhh!		?i		-	

Figure 3-5. Example of coded transcript for "Towers 5-high with a choice of two colors" task

Notice here that Romina's statement, "Yeah, think!" is coded under both the "evaluate" and "invite" dimensions with the "+" to represent agreement and the "!" to represent an imperative command. Indeed, each statement could be either declarative, imperative, or interrogative in nature and was coded as [blank], !, or ?, in turn respectively. As Romina's statements seemed so multi-faceted, the coding scheme grew to include even more sub-categories like whether Romina sought to inform, verify, reiterate, explain, or suggest an idea by a particular statement. The researcher allows that this determination could sometimes be subjective and attempted to alleviate the possible limitations of this procedure by independently verifying her codes with other graduate students analyzing the same behavioral transcripts.

3.4.2 Writing a Structural Descriptive Narrative

The final step of the behavior analysis involves writing a narrative that describes the general structure of the student's behavior. The phase of the research resulted from first constructing a skeleton outline of critical events in each problem-solving session and then composing a narrative that fleshed out the themes that seemed to emerge from the coding. During the process, I continually revisited my original research questions and the original video datasets in order to examine Romina's behavior under the lenses of representation, justification, and collaboration. A sample of Romina's behavior in math was taken from elementary school, middle school, high school, college, and post-graduate school. The choice of problem-solving sessions also sought to include both large classroom examples as well as small voluntary after school meetings. Below is a summary of the specific data included:

DATE	*TASK*	*ENVIRONMENT*	*PARTICIPANTS*	*GRADE*
1992-02-06	Towers 5-high	Classroom	Romina, Brian, T/R1, other fourth graders	*4*
1993-10-01	Guess My Rule		Romina, Brian, Stephanie, Jeff, Michelle I., Milin, Michael, Bobby, Amy-Lynn, Ankur, Michelle R., & Matt	*6*
1998-01-09	"Ankur's Challenge" for Towers 4-tall selecting from 3 colors with each color represented at least once	Informal small-group after school	Ankur, Michael, Romina, Jeff, Brian, and T/R1	*11*
2000-05-05	Taxicab Geometry		Romina, Michael, Brian, Jeff, T/R1, T/R2, T/R3	*12*
2009-07-15	Towers 5-Tall selecting from 2 colors	Rutgers GSE	Romina, T/R4	*21*

***Figure 3-6*. Summary of problem-solving behavior videos for analysis**

Chapter 4 BEHAVIOR RESULTS – Elementary & Middle School

Think left and think right and think low and think high.
Oh, the thinks you can think up if only you try!
~ Dr. Seuss, *Oh, The Thinks You Can Think!* (1975)

4.1 Introduction

Through analysis of video data selected from the archives of the Rutgers-Kenilworth longitudinal study, this study focuses on the relationships between a student's mathematical beliefs and behaviors over a period of seventeen years – specifically, from 1992 to 2009. The research examines Romina's mathematical behavior as a participant in five problem-solving sessions from 4th, 6th, 10th, 12th grades, and post-graduate level and her beliefs about problem-solving experiences as expressed during clinical and semi-structured interviews conducted during high school through her post-graduate career as a business analyst at Deloitte Consulting. More specifically, the problem solving and interview data selections are summarized the table below.

Grade	*Date*	*Task or Interview Topic*
Mathematical Behavior Data – Problem Solving Tasks		
4	1992-02-06	Towers 5-Tall selecting from 2 colors
6	1993-10-01	Guess My Rule
10	1998-01-09	"Ankur's Challenge" for Towers 4-tall selecting from 3 colors with each color represented at least once
12	2000-05-05	Taxicab Geometry
21	2009-07-15	Towers 5-Tall selecting from 2 colors
Mathematical Beliefs Data – Clinical or Semi-structured Interviews		
11	1999-05-18	Reflections I - PUPMath
12	1999-07-21	Reflections II - PUPMath
14	2002-03-11	Reflections III – Math Education Seminar
18	2006-05-12	Reflections IV – Longitudinal Study and Career
21	2009-07-15	Reflections V – Longitudinal Study and Graduate Work

Table 4-1. **Problem-solving and Interview Data Selections**

The research questions that guide this research are: 1) Within the context of problem-solving situations, how do Romina's representations and justifications for her ideas develop over time? 2) To what extent, if at all, does Romina collaborate and incorporate the ideas of others into her own ideas? 3) How do Romina's later adult views about learning relate to evidence of her earlier mathematical behavior in terms of her descriptions of knowledge, the conditions for learning environments, and the learning process?

Directed by the research questions, the study's main objective was to examine the "interplay" of Romina's beliefs, behaviors, and learning environment over an extended period of time. Behavior data and interview data were analyzed separately and are presented in different chapters that follow a chronological order: behavior results in 4th and 6th grade (elementary and middle school episodes), behavior results in 10th and 12th grade (high school episodes), interview results from 11th and 12th grade (high school), interview results from undergraduate and graduate school, and finally interview results from post-graduate study. Analysis for the behavioral problem-solving data followed similar procedures - transcription, coding, and narrative - as that for the analysis of interviews. Behavior data analysis incorporated and modified suggestions from Chiu (2008) for a five dimensional coding of problem-solving discourse: evaluation of previous action, invitation for participation, justification, affective expression, and knowledge expression. Romina's statements in each behavioral transcript were considered under this coding scheme and then summarized. What follows in the behavior data chapters is a narrative constructed to address the first two research questions concerning how Romina's representations and justifications for mathematical ideas developed over time as well as to what extent, if at all, Romina collaborated and incorporated other's ideas.

Interview data analysis followed Moustakas' (1994) suggestion and Francisco's (2004) interview model to identify "significant" statements and then "cluster" those significant statements to identify larger themes. A table was produced with the following categories as columns: issue, significant statement, summary, and interpretation. What follows in the interview data chapters is a narrative constructed as the last step of the video data analysis that begins to address the question: How do Romina's descriptions of knowledge, conditions, and process, as they relate to doing mathematics, inform her views about mathematical learning?

This chapter considers two episodes from elementary and middle school. On each occasion, a problem-solving task was introduced during an extended period of the regular school day to a large group of students who were working together in smaller partnerships. Romina's first problem-solving task within the Rutgers-Kenilworth longitudinal study was, in fact, when she and Brian worked on the combinatorial "Towers 5-High" on February 6, 1992 in 4th grade. We see Romina and Brian working again in 6th grade on the algebraic "Guess My Rule" task on October 1, 1993.

4.2 Towers 5-High: February 6, 1992 (4th Grade)

4.2.1 Setting

On February 6, 1992 two classes of 4th grade students at Harding Public School in Kenilworth, NJ were given the "Towers 5-high" task involving a choice of two colors of Unifix® cubes. One class used blue and white cubes and the other class used yellow and red. Romina and her partner, Brian, were in the class using blue and white cubes. They sat in the back of the classroom, furthest from the chalkboard. Two camera views were combined and transcribed for analysis: "People View" with the camera focused on their faces and "Work View" with the camera focused on their desktops. A transcript of the 40 minute episode combining audio and visual information from both camera views is included in the research here (see Appendix A). This was Romina's first interaction with the Rutgers research team and Dr. Carolyn Maher (coded in the transcripts as T/R1).

4.2.2 Background and Exploration

During the first three minutes of videotaping, Brian and Romina look at the video cameras and remark on the presence of cameras and microphones. When Brian comments, "look at these cameras – they've got like them TV cameras," Romina replies, "that's cause we're one of the best" (14-15). Brian points at the different cameras and remarks how "huge" they are. Mrs. Barnes, one of the teachers in the room, notices their attention to the cameras and says, "Don't worry about it" (22). Romina and Brian continue to discuss the "microphone thingie" on their desk. When Brian again brings attention to the camera by saying, "Look at the TV" and "You're on *Candid Camera!*" Romina and Brian cover their faces with their hands and laugh (31-33).

Starting the problem-solving session, T/R1 introduces herself to the whole class and comments upon the fact that this is her first time with them:

> This is a new group. Okay, I'm Dr. Maher. I'm from Rutgers and I've been in some of your classes and I'm very happy to be with you today because your teacher tells me that you like to solve problems. Is this that true? And that you're very good at it. Is that correct? (36)

When T/R1 asks if they "like to solve problems" and if "you're very good at it," both Brian and Romina nod their heads. T/R1 proceeds to ask the students about Unifix®

cubes and if they have used them before. Romina is one of the students who responds and says that “we used them to make patterns” when their teacher “gave us a lot of different colors and we had to put them in patterns” (42-44). T/R1 directs the students’ attention to a task called “building a tower.” By working with a partner and two different color cubes, she explains the task:

> … using the blue and white cubes, you’re going to work together to build as many different towers that you can that either use white, blue, or blue and white together. But it has to be five cubes tall. (50)

Taking a minute to explore what a tower looks like, T/R1 demonstrates that towers have “sort of like a bump on the top – a chimney” and that you cannot “count it upside down” (52). She asks the students to build an example tower that meets the requirements she has explained: five cubes tall, a chimney on top, and blue, white, or blue and white together. Romina makes a tower using five blue cubes and Brian makes a tower of five white cubes. After the students have shown her these example towers, T/R1 explains that “we want to find all possible towers that are five cubes high” and asks if there are more and how many more there would be. When students in the classes offer different guesses like eight, nine, or twelve possible towers, Romina whispers to Brian that she thinks there are ten total. T/R1 then repeats the task to the students: “find all possible [towers] and try to be able to convince us and each other and Mrs. Barnes that you have found all possible towers and that you haven’t missed any” (72).

After T/R1 has introduced and discussed the Towers 5-high task, eight minutes have elapsed and the students begin working on the problem. Brian and Romina spend the next twenty minutes working with each other on the task. During that time, they interact with a teacher/researcher four times – twice with a graduate student and twice with Dr. Alston. During the last ten minutes of class, T/R1 conducts a whole class discussion of how many towers students have discovered so far. By this time, Brian and Romina have found twenty-six towers. Finally, a graduate student helps the students tape their towers together and label a sheet with their names.

Many illustrations exist of the progression of students' thinking with respect to the Towers problem in grades 3, 4, and 5 within the context of the Rutgers-Kenilworth longitudinal study (Alston & Maher, 1993; Maher & Martino, 1996a, 1996b, 1999, 2000; Martino, 1992). Consideration how social interaction, teacher questioning, and task design affected students' cognitive growth in terms of mathematical justification, proof, and generalization has been extensively documented (Alston & Maher, 1993; Francisco, 2004; Glass, 2001; Kiczek, 2000; Maher, 2002; Maher & Martino, 1996a, 1996b, 1999, 2000; Maher & Speiser, 1997; Martino, 1992; Muter, 1999; Powell, 2003; Uptegrove, 2005; Uptegrove & Maher, 2004a, 2004b). In a case study of a student named Stephanie over the period of grade 1 through grade 5, Maher and Martino (1996a) describe evidence of three types of thinking about the Towers problem: "(a) the spontaneous use of heuristics (guess and check, looking for patterns, etc.), (b) the development of an argument to support a component of a solution (local organization), and (c) the extension of an argument to build a full solution (global organization)" (p. 199). Here, we will consider specifically how Romina developed her mathematical ideas through collaboration, representation, and justification, as she and Brian progressed from "spontaneous" guess and check methods to a more systematic "local organization" in this fourth grade session.

4.2.3 Spontaneous Heuristics: "How about…"

For about the first eight minutes of their work on the problem, Romina and Brian generate towers through a guess and check strategy. Before she has built any new towers, Romina immediately asserts her belief about the task's solution. She repeats five times that there will be exactly ten towers: "ten" (66); "I got ten" (71); "Yo, Brian, there has to be ten" (73); "There has to be ten" (75); and "There's gonna be ten, Brian" (79). In a seeming effort to keep their efforts private, she also pulls the cubes closer to her at the beginning and says that they "can't let anybody see these" (77). Notice that Romina's initial problem-solving behavior here consists of repeated insistence on an answer with no evidence of justification and a desire to not publicize or share any physical models with other students in the classroom.

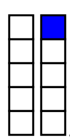
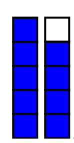
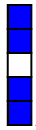
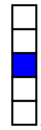
Romina and Brian then begin to generate new towers. They mirror the color pattern of each other's towers. For instance, Brian begins by taking one blue cube and putting it on top of a stack of four white cubes. He puts this new tower next to his original tower of five whites: . Romina looks at the pair he has just produced and then takes a white cube and puts in on top of a stack of four blue cubes. She puts this new tower next to her original tower of five blue cubes: . At this point, Romina takes the pair that Brian has created and the one that she has made and pulls it in closer saying that they "can't let anybody see these" (77). She also lays all of the towers down flat on the table in front of her. Brian suggests that they make the "same thing as Alex did." He proceeds to put one white cube in the middle position with two blues below and two blues above: . Without comment, Romina builds a tower with one blue cube in the middle position with two whites below and two whites above: . Figure 4-1 below depicts the six towers they have built in this early stage.

Figure 4-1. **After about one minute of building, Romina and Brian have six towers.**

Next, Brian builds a tower with a white cube in the second position from the top and blue cubes in all the other positions. When he places this tower next to the previous one, he observes, "Oh! Move it up" (82). His comment indicates that he has noticed a pattern of the white cube first being in the middle (third from top) position and then in the second from top position. He encourages Romina to build a tower that uses this pattern and tells her to "do the same thing with blues" (84). While Brian was working, however, Romina had created a different pattern of alternating cubes: white-blue-white-blue-white. Romina and Brian then adopt the other's pattern strategy to build a tower which switches the colors in that pattern. Romina says, "I'm doing this one" (85) and builds a tower with a blue cube in the second position from the top and whites in all the other positions. Referring to Romina's alternating pattern, Brian comments, "I'm doing it – I'm doing it like that" (86) and builds a tower of alternating color cubes: blue-white-blue-white-blue. At this time, Brian counts the towers they have built so far - "two, four, six, eight, ten" (see Figure 4-2). There are ten total towers.

Figure 4-2. **Romina and Brian build a total of ten towers by their alternating colors strategy.**

Romina smiles at the observation that there are "ten" towers and comments, "See, I told you!" (91) - referring to her earlier prediction that there would be ten towers. However, at the same time, she asks if there might be another possible: "How 'bout one with one on the bottom?" and "How about one on the bottom?" (89 and 93).

Whereas their initial collaboration consisted of mirroring the other person's pattern with opposite colors in each position, now they begin to offer suggestions of new towers. Romina asks many questions that can be classified based on what they seek from the addressee (Brian in this situation): a piece of information, verification/confirmation of an assumption, or suggestions for a future course of action. Later, Romina will also adopt the role of checking new towers that Brian builds to see whether or not they are duplicates. Consider the following example of how Romina asks questions during the students' collaboration.

Building Towers 5-high from two colors (lines 87-170)		
	COLLABORATION –Asking Questions	
Line	*Romina's Statement*	*Description*
87	How does that look?	Romina asks Brian how the group of towers "look" when she adds a tower with a white cube in the second from top position mirroring the tower he had made with a blue cube in the second from top position.
89	How 'bout one with one on the bottom? One on the bottom.	Romina suggests another possible tower with "one" different color

		at the bottom (like white-blue-blue-blue-blue or blue-white-white-white-white). In the towers they've built so far they have used a different color in the second and third positions, but not at the bottom position.
93	How about one on the bottom?	Although Brian observes that they have ten total towers at this point, Romina repeats her suggestion for a new tower with the pattern of "one on the bottom." Brian comments that "we have one like that" and then corrects himself and realizes that "no, we don't" (96). Then Brian suggests his own idea for a new tower with "a white one right there – a blue, white?" (103) where the white cube is in the second position from the bottom of all blue cubes.
106	Do we have them like this, but only with white on top?	Next, Romina suggests another possible tower "like this" (pointing to a tower of all whites with a blue on top) where all the other cube positions are blue but with a white cube "on top." Brian repeats back Romina's suggestion, "four blues and a white" (107) and then builds the tower she has suggested. When Brian puts the new tower down among the others, however, he realizes it is a duplicate – "four blues and a white there" (113).
114	Which ones are those that we did… this one and this one. How about we put one there? No, there!	In response to Brian's observation that the new tower is in fact a duplicate, Romina asks about identifying towers they've already constructed – "which ones are those that we did." She pulls two pairs of towers away from the long row in a new arrangement: BBBWB, BBWBB, WWBWW, WWWBW. She points to the second position from the bottom

		of these towers. She suggests that a new tower could be included among those in this new grouping: "how about we put one there?" Brian ignores Romina's suggestion and says, "no, take the kinds that are like this" (115). He puts the BBBBB and WWWWW together. Romina insists on her tower, "No, there!" and constructs her tower BWBBB. She also exclaims, "Well, I thought of it first!"
118	Which one did we just make?	Brian says that the tower she has just made is a duplicate: "We have that" (117). Romina asks "which one" was just made. Meanwhile, she makes a similarly pattern tower of WBWWW.
120	That one? Well, if we go…	When Brian points out the tower she just built, Romina repeats, "that one?" and then begins to check the other towers for a duplicate ("if we go…"). Brian also looks to see if it is a duplicate and cannot decide: "We don't have that. No, yes, we do" (121). They push all the towers back together without removing the duplicate tower.
127	White-white-blue? Two whites.	Brian wonders if there are "any others" and Romina affirms that there will be and encourages, "yeah, think" (124-125). When Brian then suggests the combination, "white-white-blue," Romina repeats his suggestion as a question. She gives him two white cubes to build the tower. Brian asks her for two more white cubes and builds the tower WWBWW. Romina observes that they already have that tower. She disassembles Brian's tower and builds BBWWW in its place saying, "I thought you meant like

		this" (131). Brian asks if they should "do that with the blues" and Romina takes the cubes out of Brian's hands and builds WWBBB while saying, "Yeah, that's what I thought" (135).
139	Wait a minute, didn't we just do that?	Next, Brian builds what he calls "a three and two" (136): BBBWW. Romina asks him to "wait a minute" and wonders if they didn't already "just do that" tower. Brian disagrees and Romina observes that the previous tower of WWBBB was "upside-down" (141).
143 – 145	How about… three whites and two blues.	Romina begins another suggestion that Brian helps complete: "three whites and two blues" WWWBB.
149	How about…	Romina begins another suggestion but never finishes it with anything specific.
165	Here. Sure we don't have it?	Brian then begins to suggest different combinations: "a blue on the top and four whites?" (154); "four whites and one blue?" (156); "a white and then three blues and then a white?" (162). As Brian makes these suggestions, Romina drags her finger down the row of towers they have to check if a duplicate exists. When Brian insists that his last suggestion of WBBBW will work, she asks if he is "sure we don't have it."
170	Hmm. Did we… do you think we have all of them?	Brian asks Romina, "How many do we have now?" Romina replies that they have "twenty-one" because "I counted." When Brian begins to speak about how he sees a "word" spelled in the row of towers, Romina interjects and asks if he thinks they have built "all of them" [the towers].

***Table 4-2.* Examples of Romina's collaboration by asking questions in the task Towers 5-high.**

In the span of about six minutes (00:09:40:29 – 00:15:49:00), Romina asks thirteen questions. About half of the questions are informative in nature – they involve asking Brian to provide a piece of information to her like "Which one did we just make?" (118). The other questions allow Romina the opportunity to offer suggestions "How about we put one there?" (114) or seek verification of one of her assumptions like, "Wait a minute, didn't we just do that?" (139). Notice the high frequency of the phrase "how about" in Romina's questions as she suggests new towers. Also notice that the students do not yet have a strategy for systematically checking for duplicate towers. Only now is Romina beginning to question "which ones are those that we did" (114) and if they have built "all of them" yet. There is some small evidence of a local organization when the students reorder some of the towers so that some of the towers with four blues and one white are next to each other and a couple of towers that Romina called "upside-down" of each other are side by side like BBWWW & WWBBB and WWWBB & BBBWW. Their heuristics seem haphazardly determined however. They guess a tower suggestion and then sometimes, but not always, check. The other heuristic they employ is based on color patterns like all blue with a single white in the second position or three blue cubes and two white cubes. Pretty consistently, though, whenever they determine a color pattern, they reverse colors to build a second tower like BBWWW and WWBBB. During this six minute episode, the Romina and Brian have generated eleven more towers, so now they have a total of twenty-one towers on the desk (See Figure 4-3).

Figure 4-3. **Examples of spontaneous heuristics as Romina and Brian build twenty-one towers.**

4.2.4 Local Organization: "Opposites," "Matches," and the "Husband & Wife" take "Strolls in the Park"

For the second half of the problem-solving session, the students develop more local organizations for generating and categorizing towers. The students employ a particular strategy and assumption: for any tower, one can find another tower in which each position's color has been reversed. For instance, for the tower of five blue cubes, one can find a tower of five white cubes. Romina introduces and uses several different names with metaphoric underpinnings to towers that fulfill this condition: "opposites," "matches," "husband and wife," and pairs that take "strolls in the park."

4.2.4.1 Romina describes the "opposite" strategy

When T/R2 asks the students about their twenty-one towers, she questions whether they think they have them all. Brian observes that was the same question Romina just asked, "Yeah, that's what she just said, but we're still working it" (175). T/R2 asks if they had begun to see "any pattern" with the towers. Romina then proceeds to describe how they could always find what she calls "the opposite":

T/R2	Did you begin to see any kind of pattern with them?
ROMINA	We can always find. Well, this and – wait, where's the other one? The one – white, blue, blue?

BRIAN	Right there.
ROMINA	[*Holds up the towers BWWWB and WBBBW*] We can always do the opposite.
T/R2	Okay. You can do the opposite. Do you have any pairs that are opposites of each other?
BRIAN	I had one. I just had one.
ROMINA	This is opposite. [*Puts down the towers she was holding up BWWWB and WBBBW*]
T/R2	Okay, that's opposite. (178 – 186 and Figure 4-4)

***Figure 4-4*. Romina: "This is opposite." (185)**

When T/R2 asks about a "pattern," Romina describes that you can "always" find what she calls an "opposite." She uses the example of WBBBW and BWWWB to illustrate "opposite" and then reiterates that "this is opposite" when T/R2 asks if they have any "pairs" of opposites. Notice that Romina does not justify her statement that a person can "always" find opposites, nor does she further define or explain her strategy of opposites. At this point she has applied the term, "opposite," to their strategy and provided a single example to demonstrate their color patterning work thus far.

4.2.4.2 Romina and Brian look for "sames"

Next, T/R2 questions Brian and Romina as to whether there is "another kind of pattern you could have besides having opposites" (200). T/R2 suggests that the students "find some more" towers and "see if there's a pattern." As Romina has already answered T/R2's earlier question about a "pattern" with her description of their "opposite" strategy, Romina tells Brian that they could see if they have "some of the same" and they begin a strategy that checks for duplicates by driving a tower over the other existing ones:

T/R2	Why don't you work on seeing if you can find some more and then you maybe you can see if there's a pattern.
ROMINA	Let's see if we have some of the same.
BRIAN	Let's put the sames in an order.
ROMINA	[*Takes a tower and passes it along the top of an existing row. She makes a driving sound*]. Vroooooom.
BRIAN	No, wait. Put the pairs with the opposites.
ROMINA	Some of them could be the same.
BRIAN	Oh, good idea. [*Brian repeats the action Romina was doing by passing the tower along the top of the row and making a driving sound*].
ROMINA	Neeeeeeerr. [*Passes a tower along the top of the row*]. Nothing matches with this.
BRIAN	Wait, where is this?
ROMINA	Wait. One almost matched. (205 – 214)

Notice that two ideas have emerged here: a more systematic strategy to check for duplicate towers and the thought to pair all the opposites together. Romina suggests that she and Brian check to see if they have "some of the same" where "same" indicates a duplicate tower. It is unclear what Brian means when he comments that they should put the "sames in an order" – Romina does not ask for nor does Brian volunteer a clarification. Since "same" for Romina indicates a duplicate tower, one wonders how the duplicates would be put "in an order." Romina's check strategy consists of her taking a questionable tower and passing it along top the long row of towers they currently have. She makes the driving sound "vroooooom" or "neeeeeerr" as she moves the tower. At first, Brian tells her to "wait" and begin to group "the pairs with opposites," but Romina argues that they need to check for any that are the "same" first. Brian agrees this is a good idea and mirrors the same driving action Romina initiated.

4.2.4.3 Romina and Brian refine and define "opposites"

Romina and Brian continue to check for any duplicate "same" towers. At T/R2's suggestion, they also transition to standing up the towers on the desk as opposed to laying them flat. T/R2 comments to Brian, "why don't you stand them up – you could see them better" (230). After they have stood all of the towers up, the students begin searching for new towers by guessing different combinations: "How about three and two?" (239); "How about three blues and two whites or four blues and one white?" (244); "How about one blue on the bottom and four whites up?" (246); "One white and four blues up?" (250). Brian suggests each combination and then Romina checks the row of towers and then indicates whether they have that particular tower already. At this point, Dr. A. kneels down by the desk and asks the students what they're "thinking about" with their towers. She probes their definition of "opposites" more closely when it's mentioned.

Dr. A.	Can you tell me what you're thinking about?
BRIAN	Well, once when we find one, we just do the opposite.
Dr. A.	What do you mean 'the opposite'?
BRIAN	Like, when we found this one out [*holds up WBBBW*]
Dr. A.	Yeah?
BRIAN	We just put two blues on top and three whites in the middle.
	[*Brian holds up the BWWWB*]
Dr. A.	Oh. Do they always have an opposite?
ROMINA	Yeah.
BRIAN	Yes. Well, not. Yeah. [*Romina nods her head*]. Well not like ones that have two in the middle.
Dr. A.	Hmm. So it works sometimes?
BRIAN	Like if you have two here. You can't do that. Switch it around. [*Brian points to the tower BBWWW*]
ROMINA	What? You can switch this around. You can put two whites and three blues. [*Romina picks up the tower BBWWW*]
Dr. A.	That would be the opposite to that one?

ROMINA	Which one? That one?
BRIAN	Do we have it? Yes. [*Brian picks up WWBBB and hands it to Romina*]
ROMINA	This one. [*Romina holds the BBWWW and BBWWW together*]
Dr. A.	Oh, so that's what you mean by 'opposite.'
ROMINA	Yeah.
Dr. A.	So that's the way you've been working?
BRIAN	Yeah. Let's see if we have any one without an opposite.
ROMINA	Yeah, that's a good idea. (256-276)

Notice that Dr. A's questions like "Can you tell me what you're thinking about?" and "What do you mean 'the opposite'?" provide the first examples of questions that serve to elicit further explanations and clarifications of mathematical thinking. This is also the first time that the students have been asked to justify a more general and abstract question: "Do they always have an opposite?" Following this questioning, Brian demonstrates an example of "opposite" using the same towers that Romina used earlier for T/R2: the BWWWB and WBBBW towers. Romina affirms that towers will "always" have an opposite, but does not provide justification. When Brian expresses doubt as to whether all the towers will have opposites, Romina disagrees at Brian's example of BBWWW as a tower without an opposite. Romina argues that "you can switch this around" and Brian helps her find the opposite to the tower. Brian then suggests, and Romina agrees, that they check over all of their towers to see if there exists a tower "without an opposite." Since the students have just established that every tower will have an "opposite," searching for towers without opposites among their existing constructions would allow them the opportunity to build more towers.

4.2.4.4 Sorting the "opposites"

After their conversation with Dr. A., the students begin sorting the towers into two groups: "opposites" and not opposites. Romina directs Brian to "take the ones that are opposites and put them over here" indicating a separate area of the desk. Brian later uses the term "sames" for the group of towers without opposites.

ROMINA	They're opposites. Take the ones that are opposites and put them over here.
BRIAN	[*Separates a group of towers*] All of them are opposites.
ROMINA	Maybe you can find one that's not. Well these are [*holds up WWWWW and BBBBB*]
BRIAN	Wait, I found one, I found one! This one. Oh no, I found it.
ROMINA	Da, da, da, da. [*Hums and moves the tower*]
BRIAN	With two whites on top? Right there.
ROMINA	Oh, I guess I have bad eyesight.
BRIAN	We don't have it, we don't have it, we don't have it! Ah! This one we don't have.
ROMINA	What do you mean, 'this one we don't have'?
BRIAN	We don't have one blue and four whites?
ROMINA	[*Holds up a tower*] One blue and four whites. [*Brian takes two towers out of her hand.*] Whoa, whoa, whoa, whoa.
BRIAN	They're the same, look. Yeah, no opposites.
ROMINA	All opposites you mean.
BRIAN	Yeah, no sames. (284 – 296 and Figure 4-5)

Figure 4-5. **A pile of "opposites" versus the "sames"**

When they begin sorting the towers, Brian asserts that “all of them are opposites.” Romina does not ask for clarification, but it seems that he assumes without justification that all of the towers they have built thus far would pair up in the “opposite” pattern he described to Dr. A. Romina tells Brian that they might find a tower “that’s not” with its opposite, however. Romina proceeds to pick up a tower, look for its “opposite,” and then hand the pair to Brian, at which he puts the pair into a pile in front of himself.

At a certain point, Brian gets excited that there might be a tower for which they have not already built an opposite. He calls out, “we don’t have it!” over and over again and then says, “this one we don’t have.” It is unclear to which tower exactly he is referring. Romina asks for clarification by questioning, “What do you mean, ‘this one we don’t have’?” Brian clarifies and tells her that he meant the tower with “one blue and four whites” because he is holding the tower of one white and four blue cubes. Romina then finds the tower of one blue and four white cubes and hands it to Brian to put in the pile of opposites. During the time that Brian was calling out, Romina placed another pair of towers in the pile that were not, in fact, opposites by their definition: BBBBW and WWBBB. Neither student notices, however, that this pair does not create an opposite. Brian grabs more of the towers to put into the pile of opposites and Romina attempts to slow him down by saying, “whoa, whoa, whoa, whoa.” There are now no towers standing; all of the towers are in the pile of opposites. Brian concludes there are “no opposites.” When Romina corrects him by saying, “all opposites, you mean,” Brian agrees and replies that there are “no sames.”

4.2.4.5 The "matches" go for "strolls in the park"

The students survey the pile of towers now on the desk. Brian observes that, "you shouldn't have done that because now we can't see which one we did" (299) and he picks the towers up to stand them upright on the table again. He begins to line them up in the same long row formation they had previously, but Romina stops him. She directs him to instead "get the matches together" and place the towers in groups of two. Doing this also allows them to revisit the tower pairings and discover which did not satisfy their original "opposite" definition.

ROMINA	Well, get the matches together. Which one – is this right? [*Romina holds up WWWWB and BWWWW together*]
BRIAN	Yeah.
ROMINA	No it's not.
BRIAN	Wait.
ROMINA	No, we had to find four blues and one white.
BRIAN	Four blues and one white? Did you find it?
ROMINA	No.
BRIAN	Oh! We might not have it.
ROMINA	But we do. Four blues – [*Romina leans over the pile of towers*]
BRIAN	We don't have it. We don't have it. We don't have it! [*Romina picks up a tower BBBBW and shows it to Brian*]. Oh, we have it! [*Holds his hands up to his face*]. Oh! So close!
ROMINA	We have this one. We have this one. [*Romina stands pairs of towers up on the desk in front of her: WBBBB and BWWWW, BBBBW and WWWWB*]
BRIAN	Two whites, three blues. [*Brian stands up WWBBB and BBWWW next to the two pairs Romina put up*].
ROMINA	Don't put them together. [*She separates the six towers into pairs*].
BRIAN	No, we already know they're watch-a-macall.
ROMINA	They're going for strolls in the park.
BRIAN	[*Laughs and leans back*] It's like playing with two Barbie dolls.
	Here – match. [*Brian hands Romina more towers to stand up: BWBBB and WBWWW*]

BRIAN [*Looks at the pair WWBWW and WWWBW that Romina stood up*]. That ain't no match.

ROMINA I know that. Hmm. (300 – 317 and Figure 4-6)

Figure 4-6. Brian: "That ain't no match" (316)

Notice that throughout this excerpt above Romina takes on more of an authoritative role as she directs Brian in how to reorganize the tower grouping on the desk. By telling Brian to "get the matches together" and then "don't put them together" when Brian tries to push towers into a row, she ensures that the towers are now paired off and separated by enough space to make the pairs visually distinct. By the end of the excerpt, Brian is no longer even placing the towers himself but rather handing them to Romina to place on the desk. When Brian tries to argue the regrouping by saying that "we already know they're whatcha-macall," he implies that he does not see the necessity to pair them off since they already checked they were opposites. Romina supports her reasoning by saying that the towers are "going for strolls in the park." Brian then likens her metaphor to doll-playing by commenting that "it's like playing with Barbie dolls."

As a result of the regrouping, the students have the opportunity to argue and refine what an "opposite" pair looks like. For instance, as soon as Romina directs the match-making, she questions her own pairing and asks Brian, "is this right?" Indeed, the pair she is holding at this time is not an "opposite"-match, but rather an "upside-down" as she described much earlier in the task session: WWWWB and BWWWW. Even though

Brian agrees, “yeah,” it is a match, Romina disagrees with herself and says “no, it’s not.” She realizes that the opposite-match would require that they find a tower with “four blues and one white” for the white-white-white-white-blue tower. Later, Brian questions a different pair that Romina puts together: WWBWW and WBWWW. He observes, “That ain’t no match.” Romina replies, “I know that” and they correct the pairing to be WWBWW with BBWBB. This is the first time that Romina has used the word “know” or referred to knowledge in reference to this task. It is interesting that the first instance of this word occurs after about twenty minutes of the students exploring, defining, and refining their definition of “opposites” and “matches” for towers.

4.2.4.6 Checking for “Husband and Wife” pairs

Very soon after Romina directs the regrouping of the towers into “matches” that are “going for strolls in the park,” she uses another analogy for the pairing strategy. As she moves each pair of towers closer together, she describes their tower pairs as “husband and wife.” Their check strategy has become looking for any tower “without a pair,” or a tower without a spouse of opposing color pattern, so to speak.

ROMINA	Do we have this one?
BRIAN	What? Is there any without a pair? Any without a pair? [*Romina holds the tower BBBWB against a duplicate already standing up*]. Yeah, same thing.
ROMINA	[*She moves two of the opposite pair towers closer to each other*]. Husband and wife. (319 – 321 and Figure 4-7)

Figure 4-7. **Romina: "Husband and wife" (321)**

4.2.4.7 "Go strolling" to check

After they have all their existing towers paired up in "opposite" matches of "husband and wife," the students try to generate more tower combinations. They return to their spontaneous heuristic of guess and check where one student calls out a suggested tower combination, constructs it, and the other checks to see if it already exists on the table. Their check strategy has changed, however, from earlier in the task session. Romina calls this strategy "strolling" and Brian describes it as "match it up." That is, they hold the suggested tower against each of the standing towers to see if it is a duplicate.

ROMINA	Two whites and two blues?
BRIAN	Yes. We don't have that, I don't think. We don't have that! Ow.
	Wait. Match them up. You gotta match it up. [*Romina holds the tower up against the first pair*].
ROMINA	Go strolling again. [*Holds the tower BBWWB against each of the other existing pairs of towers*]. Wait a minute, isn't this? No. Opposite?
BRIAN	Yes, we don't have it! We don't! [*Pumps hands in the air*].

(346 – 349)

Although Romina does not ask him to clarify, Brian uses the word "match" for a different purpose here then previously. Before, a "match" was a pair of opposites. Here, Brian intends "match" to be a pair of duplicate towers. Romina continues her metaphor from before of the tower as a person strolling in the park. Brian becomes very animated when they find a tower that they “don’t have” in their tower park which indicates it will be a new addition.

4.2.4.8 There has to be an even number

Soon after they “go strolling again” with the towers, Brian observes that they have twenty-five towers now on the desk. Dr. A. approaches the students and asks about how many towers they have. Brian counts the towers to check the total and corrects the amount to actually twenty-four. This leads to an observation about whether the total number of towers in this problem could ever be an odd number.

BRIAN	Oh yes! We have twenty-five.	
Dr. A.	You have twenty-five?	
BRIAN	Yeah, two, four, six, eight, ten, twelve, fourteen, sixteen, eighteen, twenty… twenty-four.	
ROMINA	You can’t have twenty-five. Twenty-four.	
Dr. A.	Why can’t you have twenty-five?	
BRIAN	Cause there’s even numbers.	
ROMINA	Yeah.	(353 – 359)

After Brian corrects himself that there are actually twenty-four towers on their table so far, Romina observes that “you can’t have twenty-five.” Dr. A. asks why not. Brian explains and Romina agrees that “there’s even numbers” in reference to the pairs of towers on the table. The students do not explore or justify this idea further, however, as Brian suddenly suggests another tower combination.

4.2.4.9 Romina insists on an exhaustive check

In this last excerpt before the end of the session, Romina and Brian try to generate more towers. The students question each other with suggested tower combinations. The first couple suggestions prove to be duplicates. When Brian suggests a tower combination that seems new and not among their already existing total, Brian becomes excited and likens the new tower to a rocket ship that has achieved "lift-off." Romina cautions him to wait, however, until she checks the new tower against each of their existing pairs.

ROMINA	Yeah, we found it. What did you say? Brian – what did you say last time?
BRIAN	Three blues, a white, and a white. Do we have that?
ROMINA	That's what I just said. A white, and a white.
BRIAN	No, we had that. Ah! No, we don't.
ROMINA	No, unless you want two girls and two boys. That would be odd.
BRIAN	Do we have that? Oh, yes we do.
ROMINA	What?
BRIAN	Do we have a white, two blues, and two whites?
ROMINA	A white?
BRIAN	Oh, yes we do. We have it right there.
BRIAN	Do we have a blue, a white, two blues, and a white?
ROMINA	A blue? A blue, white
BRIAN	A blue, a white, two blues, and a white. [*Builds tower*].
	Do we have it, no! We have another lift-off.
ROMINA	Will you wait on. [*Picks up the tower and holds it against the existing towers on the the desk*]. Let's check.
BRIAN	[*Leans down and watches Romina*]. We have lift-off. We have ignition. [*A couple of towers fall over*].
ROMINA	We got a strike.
BRIAN	[*Laughs*] No, where are the pairs? Hey, we're missing a pair, dude.
ROMINA	We're not missing a pair.
BRIAN	Okay, I got the one white.
ROMINA	Whoa, whoa. Dun. Dun. [*Holds the new tower against each existing pair*] (377 – 396 and Figure 4-8)

Figure 4-8. Romina: “Will you wait on. Let’s check” (390).

Romina checks each one of Brian’s suggestions. Whereas the “white, two blues, and two whites” proves to be already among their towers, his suggestion of “a blue, a white, two blues, and a white” does not. Romina tells him to “wait” and says, “let’s check.” Brian is already building the “opposite”-match to his new tower while Romina is still completing the check. She cautions him to slow down by saying, “whoa, whoa.” She moves the new tower exhaustively by each existing tower. Eventually Romina agrees that, “I don’t think we have this” (398). They add the new pair and recount the towers. T/R1 closes the session by asking the whole class how many towers they’ve generated so far. By this time, Brian and Romina have generated twenty-six towers. T/R1 asks students to raise their hands based on how many towers they think exist. Brian and Romina express surprise when other students in the class claim to have thirty-nine or forty towers. Brian exclaims, “Oh!” and Romina comments, “You have to be kidding” (418). However, as T/R1 asks the other students about their larger groups of towers, Romina whispers to Brian that “I’m going to start thinking” (420). She constructs another tower, hands it to Brian, and tells him to “check if we have this” (427). Brian and Romina begin to whisper back and forth about other possible towers while T/R1 addresses the whole class. T/R1 tells the class:

> Okay, I think you might want some more time to check what you found. And you might want some more time to find some more. So we're going to save them and would you like to finish this tomorrow? And then maybe share with each other what you found and maybe think about how many there are? (446 – 448)

Brian nods in agreement at T/R1's questions. He comments that "there's gotta be one more" (450). Romina qualifies his statement by referring back to their earlier observation about the towers coming in even numbers and says, "There's gotta be one more – no, if we find one, there's got to be two" (450). A graduate student then interrupts and asks the students to save and label their cubes for another day.

4.3 Guess My Rule: October 1, 1993 (6th Grade)

4.3.1 Setting and Introduction of Task

Over a period of several days, a class of sixth grade students from the Harding Public School in Kenilworth, NJ was given algebra tasks. On October 1, 1993 the students used a worksheet containing ten "Guess My Rule" problems. Drawing from his *Madison Project* materials for the development of early algebra ideas, Robert B. Davis employed the game "Guess My Rule" to introduce the concept of function. The students had begun the worksheet the day before. Each problem had a table with two columns headed by the symbols: □ and△. The objective of the activity was to create a "rule" for each problem which would take the given input values of the box column and result in the corresponding output values provided in the triangle column. Including Romina, twelve sixth grade students were present: Stephanie, Jeff, Michelle I., Milin, Michael, Bobby, Amy-Lynn, Brian, Romina, Ankur, Michelle R., and Matt. Four cameras captured the 90 minute session. The "RC" camera angle focused on the area in which

Romina and her partner, Brian, were sitting and thus transcripts from the RC videodata are included in this research (see Appendix B).

RBD began the session by asking the students about scientists – what sorts of things scientists do, what problems they hoped scientists would solve, and what famous scientists they could identify. After a student mentioned Einstein as a famous scientist, RBD segued into a discussion about the idea of "secrets." He asked about when it would be appropriate for a scientist to keep a secret versus when one should share information. He summarized that the "main thing" is to both "find secrets" and then "share them too" and encouraged:

> Maybe the first time you find a secret you keep it a secret for a little bit so other people can think about it too and see if they can find it. And then at some point, probably, we want to share it. (RC66-72).

Next, RBD reviewed the Guess My Rule tasks from the previous day and reminded students that the numbers that replaced the empty box or triangle had to make a "true statement." Michelle R. wrote the equation $(\square \times 2) + 1 = \Delta$ on the board and the class discussed how they would substitute a value for the box and then get a value for the triangle, like 0 for box and 1 for triangle. RBD mentioned that "we started turning the problem around" so that instead of him giving the equation and the students constructing the table, he began to give them the table first. He asked, "I gave you the table and what are you supposed to do?" and Romina answered, "Find the equation" (RC151-153). At that point, RBD handed back the worksheets and instructed:

> Why don't you talk to your neighbors and see what you can do with problem two. We know about problem one. So, problem two, you've got the table and you're trying to find the equation, just what Romina told us. (RC179-182).

Over the remaining eighty minutes, the students worked on problems two through nine. Problems two through five involved a linear function rule – the box times a constant that was then increased or decreased by a number. Problems six through nine involved non-linear function rules. At various times throughout the session, students would explain their "secret" to the camera. After about twenty-five minutes of working on problems two through five, RBD encouraged the students to share ideas at the board. Much interaction and discussion about problem six, the first quadratic on the sheet, ensued among Romina, Brian, Ankur, Michelle I., Bobby, AmyLynn, Stephanie, and Jeff during the second half of the session. The students began investigating the "code" of multiplying a box times another box. After approximately an hour of work during the session, RBD mentioned that though the students were finding "interesting secrets," he wanted to redirect them to find a "formula" where the following would hold true:

> All you need is to put in the number in the box and it will tell you what the number in the triangle is. (RC1015-1016)

Brian, Romina, Michelle I., and Ankur worked on problems seven, eight, and nine. RBD closed the session by returning to the idea of secrets and asking, "Can we take one minute to talk about this question about keeping secrets and so forth?" (RC1123-1124). RBD said that there were "two sides" – good and bad – to telling secrets. While the students argued that it's important to share ideas, RBD also provided an analogy of weightlifting to support the case of not telling too soon. Michelle I. reiterated his analogy by explaining that to get stronger, you would need to weight lift yourself, not just watch someone else weight lift (RC1143-1144). Thus, one would need to figure an idea out first before just being told the "secret" of that idea.

4.3.2 Problem Number Two

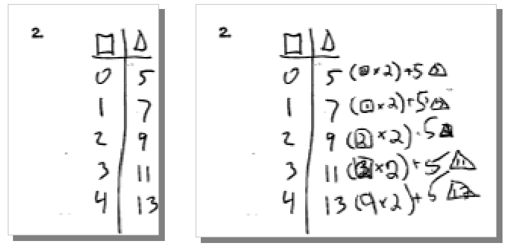


Figure 4-9. Problem Number Two for Guess My Rule: before and after Romina's work.

When Romina and Brian began work on problem two (see Figure 4-9), there was initial disagreement about the appropriate equation to use.

BRIAN	Zero times two plus one.
ROMINA	What?
BRIAN	We have to get the equation.
ROMINA	Zero times two plus one?
BRIAN	Yeah, like her. Look at that.
ROMINA	Yeah, but she used the next number, and the next number is five.
BRIAN	I know, but the next number shouldn't have been five. She used zero and one.
ROMINA	But zero and one so zero times two plus one.
ROMINA	Don't we have to use the zero and the five? Aren't we supposed to go down to number two? (191 – 201)

Brian first suggests that the equation is "zero times two plus one" and urges Romina that they have to be "like her" - Michelle I. - who, earlier, had put the equation $(\square \times 2) + 1 = \Delta$ on the board. Romina notices the "1" in the box column of the first problem and the "+1" of Michelle I.'s equation and remarks that "she used the next number and the next number is five" (RC196) since 5 is the first entry in the box column for the second problem. Romina corrects Brian that he should be considering the second problem as opposed to the first when she says, "Don't we have to use the zero and the five? Aren't we supposed to go down to number two?" (RC200-201). Romina then remarks that the rule will include "zero times something" (RC205). Brian follows, "we got the 'x' in the zero, the square in the zero" (RC211). Romina reasons aloud, "plus five, equals five" and then wonders, "would it work with other problems?" (RC216,218). The reader is left with some question as to what the "it" refers, but Romina's subsequent statements imply that she is referring to the use of five as the y-intercept (the "plus number"). When Brian states that the method to get the next entry in the table is "one times one plus six," she challenges him by questioning, "Doesn't it have to be all the same equation?" (RC222, 225).

Brian and Romina then engage Bobby in the conversation about whether the "same equation" must be used to generate all the entries in the table:

BRIAN	[*To Bobby*] Did you get it?
ROMINA	Yeah, he got it.
BRIAN	Bobby's got the big, giant, enormous egghead brain.
ROMINA	[*laughs*] No you, duh, you just don't use it.
BRIAN	Do we have to have the same plus there too? The same plus number?
ROMINA	[*To Bobby*] Is it all the same equation?
BOBBY	What do you mean by that?
BRIAN	See this? This has got to be the same for every one except you change the numbers in there?
ROMINA	[*Turns and holds up her paper to Bobby and Amy-Lynn*] Okay, look, you guys, did you only change the square and the triangle or did you change the whole entire equation? (RC235 – RC248)

Brian asks Bobby if he used the "same plus number" for his equation – essentially questioning whether the y-intercept must remain constant. Romina questions Bobby as well as to whether the "same equation" must apply to all values in the table. Romina continues to question Bobby and AmyLynn. She shows them her worksheet and inquires whether they changed "only change the square and the triangle" or rather, the "whole entire equation" (RC245-247). As Bobby answers, Romina continues to question him about what stays the same in an equation and what changes. Bobby explains that they only changed "just the ones in the squares and triangle" (RC252). Romina then shares with Brian that they need to use the rule, "times two plus five" (RC261).

Notice that in the space of ten minutes, Romina uses questions fourteen times. By contrast, Brian asks three questions in the same time span and Bobby asks only one. In addition to her frequent questioning, Romina collaborates with her peers by expanding on their ideas and redefining or reiterating their statements. The table below summarizes instances of Romina's collaboration for Problem Number Two.

Problem Number Two (lines 184-261)		
	COLLABORATION - ASKING QUESTIONS	
Line	*Romina's Statement*	*Description*
192	What?	Here, Romina's question is in response to Brian's statement of "zero times two plus one" as the answer for problem two. She may also be expressing some confusion because $(\square \times 2) + 1 = \Delta$ was the equation Michelle I. gave for problem number one and they are supposed to be working on problem two now.
194	Zero times two plus one?	In her question, Romina reiterates Brian's statement of his proposed rule.
200	Don't we have to use the zero and the five?	Romina redirects Brian to the first entries in problem number two (zero in the box column and five in the triangle column)
201	Aren't we supposed to go down to number two?	Romina again redirects Brian to problem number two
207	Can you run that past me?	She seeks an explanation and possible clarification when Brian states that "zero times one plus four equals five," which is not a true statement mathematically.
218	Would it work with the other problems?	Romina questions whether "it" – most likely the "plus five" she noted earlier - would work as the

		y-intercept in the function rule for this table.
223	One times one times six?	Romina attempts to reiterate Brian’s statement about “one time one plus six”
225	Doesn’t it have to be all the same equation?	Romina questions Brian about whether the rule they are using should remain consistent. For the previous entry they used “times zero” and “plus five” and now they are using “times one” and “plus six” (effectively changing both the slope and the y-intercept)
233	What?	Brian makes a sudden exclamation of “Ohh!” and Romina seeks information.
240	Is it all the same equation?	Romina follows up on Brian’s question to Bobby about whether they need the “same plus number” in their function rule or not.
242	What do you mean, what do I mean by that?	Bobby asks what she “mean[s]” by “that” and Romina is unsure what “that” refers to (most likely, Bobby is questioning Romina’s phrase “same equation”).
246-7	Okay, look, you guys, did you only change the square and the triangle or did you change the whole entire equation?	Romina shows her paper to AmyLynn and Bobby and attempts to redefine her previous question about the “same equation” by offering a more detailed query, contrasting a change to the box and triangle elements of the equation versus a change to every element of the equation (slope, box, y-intercept, and triangle).
249	What did you do?	When Bobby responds that “no” they did not change every element of the equation each time, Romina asks for more information.
251	The whole equation or just the ones in the square and triangle?	Bobby’s response that they “changed numbers” prompts her to again ask about what specifically they changed.
253	And you got them?	Since Romina looks at Bobby’s paper at this point, it seems

		Romina is seeking verification of the validity of his approach – "them" being the appropriate elements in each triangle.

Table 4-3. **Examples of Romina's collaboration by asking questions in Guess My Rule.**

4.3.3 Problem Number Three

Very soon after her exchange with Bobby about problem number two (exactly twenty-one minutes into the problem-solving session), Romina makes the statement, "Oh duh. Zero times two is in between so this one has three in it" (RC272). The "in between" number to which she refers here seems to be the slope of the linear function rule since the first finite differences for the triangle column in problem number two was the constant 2 and the first finite differences for problem number three is 3. At this point then, Romina's comments indicate that she has already recognized the slope for problem number three and the rule will have a 3 in it. Before she and Brian continue with problem number three however, Romina becomes involved with a conversation about perceptions of herself and relationships to the task and the learning environment. This interaction is summarized in Table 4-4 below.

Problem Number Three (lines 272 - 296)		
	PERCEPTIONS of Self and Relationships to Task & Learning Environment	
Speaker	*Statement*	*Description*
Romina	Oh, duh. Zero times two is in between so this one has three in it. …Three, yeah. Oh now I figure it out. (RC272 – 275)	Romina recognizes that 2 was the number "in between" the triangle column entries – i.e. the first order finite difference and thus, the slope. "This one" is problem number three which will have 3 as the slope in the function rule.
RBD	Ok we need to talk about that. Is it okay to give away secrets or is it too early to do that? (RC276-277)	When RBD asks whether it's okay to "give away secrets" or still "too early" to do so, Romina answers that it's "too early" to share secrets. Romina is not yet ready then to
Romina	Too early. (278)	

		publicize her results.
Brian	Ankur, Ankur. She got it. I can't believe it. (279)	Brian draws Ankur's attention to the fact that Romina "got it" and that he "can't believe it." Presumably, the "it" here refers to the function rule she expressed earlier for problem number two. Romina acknowledges Brian's surprised comment about her ability to have "got[ten] it" with a laugh.
Romina	[Laughs] Oh thanks. (280)	
Brian	Let's see if it works for every one. (281)	Brian is still working on problem number two and is in the process of checking if "it" – Romina's rule of "times two plus five" works for the problem.
Romina	It does. Bobby has it. (282)	Romina tells Brian that "it" – the function rule – does work. Her reason for why the rule works is the fact that "Bobby has it."
Brian	Zero times two plus five equals. (283)	Brian reiterates the rule for the first entry in the table.
Romina	Hold on, we have to write this down. (284)	Romina urges a written representation.
Brian	Does it? It doesn't. Oh it does. One times two plus seven, it does. Oh my God it does. It does. (285-286)	When the rule works for the second entry in the problem, Brian expresses excitement and repeats "it does… oh my God it does. It does."
Romina	No duh, that's why I'm writing it down. (287)	Romina does not express surprise that the rule works by saying "No duh." She uses the fact that she's writing down the rule as evidence that it works.
Brian	Zero times two. (288)	
Romina	Don't say it out loud. (289)	Romina again expresses a desire not to publicize their results when she tells Brian not to use the rule "out loud."
Brian	[looks up across the table] We got some. Oh my, I can't believe we got that!... [addresses another student] No, do you know the secret? It's easy. [Gestures to Romina] Romina got it. (292 – 295)	Unlike Romina, Brian does share across the table that "we got some." He says the "secret" is "easy" and qualifies how easy by adding that "Romina got it."
Romina	Oh, that makes me feel real good. Oh come on. (296)	Romina again counters a statement by Brian where he equates a problem's ease with her ability to solve it. She comments "that makes

		me feel real good.”

Table 4-4. Examples of Romina’s perception of self and relationships in Guess My Rule.

Whereas at the beginning of this exchange Romina had already moved on to problem number three (see Figure 4-10) and had expressed the slope for the new function rule, by the end of this exchange, she and Brian are back again to solving problem number two. What Romina writes on her paper during this time must contain a mistake, because Bobby interrupts soon after Romina counters Brian’s statement about how “easy” the problem must be because she got it. Bobby points to Romina’s paper and tells her that “you can’t change this – it has to stay the same” (RC299). The “it” to which Bobby refers is not entirely clear, but it seems that though she orally stated the function rule for problem number two as “times two plus five” – that is not what she has written because Bobby continues that “this has to stay the same” (RC301) referring either to the slope or y-intercept she was using.

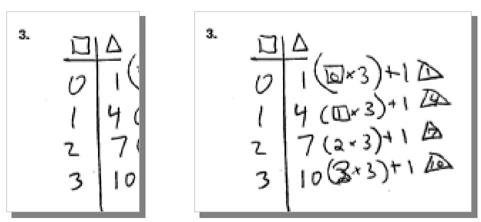


Figure 4-10. **Problem Number Three before and after Romina’s work.**

The next sequence of interchanges in Table 4-5 illustrates a variety of at times strong, powerful, and variable affect in which Romina acknowledges that she “messed up,” expresses “I didn’t care,” describes that she “copied off you guys,” clenches her fists to her head, argues with Brian about why her function rule works and his does not, and finally hums a tune “dum, dum, dum, dum, dum – you’re so slow” to him.

Problem Number Three (lines 302-348)
VARIABLE AFFECT

Line	*Romina's Statement*	*Description*
302	Oh der, I messed up.	Bobby points out to Romina that "this has to stay the same" indicating that the equation she has written on her paper for problem number two is not yet correct. She immediately states that she "messed up" and laughs with Brian.
305	Hey, I was just making up numbers. I didn't care.	Brian leans over as Romina erases her work and asks her what she did. She now characterizes her earlier actions as "just making up numbers" and says she "didn't care."
315-316	Mm, I got the answer. [Sticks out her tongue and makes a face].	Brian repeats again that "we got it" and "I can't believe that." Romina laughs at his observation. Then, Romina brings to the attention of Stephanie, Jeff, and Michelle I. across the table that she "got the answer." She punctuates her statement by sticking out her tongue and making a face at them. She takes a primarily non-verbal and offensive approach in this public display of taunting at least four people at the table.
319	Yeah, but the first time I copied off of you guys I didn't even get it.	When Bobby hears Romina say that she got the answer, he interjects that happened "cause you copied off us." AmyLynn agrees, "Yeah, you copied off us." Romina now takes a more defensive posture. She acknowledges what they're saying with the affirmative "yeah" but qualifies that when she "copied off you guys I didn't even get it." Romina draws a distinction between literally getting the answer by copying Bobby and AmyLynn's written representation of the

		function rule versus figuratively getting the knowledge to understand the function rule.
326	We just have to write it down.	RBD asks the students how many problems they have done and Brian replies that "we can just fly through this – we just have to write it down" (RC325). Romina then reiterates Brian's statement that "we just have to write it down" implying the confidence in their work that they merely have to record it.
333	Hold on what did you write? It's not two this time, it's one this time, isn't it?	Romina questions what Brian is writing on his paper for problem number three. Here, the "one this time" to which she refers is the y-intercept they need for their function rule.
336-337	Oh. [Clenches her fists and puts them to her head]. Zero times three, zero how come it doesn't work?	Brian states that what Romina is suggesting for problem number three "doesn't work" (RC335). Romina expresses frustration nonverbally by clenching her fists and putting them to her head. She states the beginning of her rule out loud, "zero times three." She then persists with her rule by questioning Brian, "how come it doesn't work?"
341	Zero times three is zero, plus one is one.	Brian tells Romina that "it still doesn't work." Romina then repeats the application of her function rule (times three, plus one) for the first entry in the table: "zero times three is zero, plus one is one."
343	I don't know where you got this one – it works!	Romina expresses confidence in "it" - her rule - more forcefully by stating she doesn't know "where" Brian got his answer because "it works!"
348	[Leans on her elbow and hums] Dum, dum, dum, dum. You're so slow.	Brian agrees, "okay, okay." Romina finishes filling in her table faster than Brian. He asks her to "wait up" for him.

		Romina leans on her elbow and hums, "dum, dum, dum, dum. You're so slow."

Table 4-5. Examples of Romina's variable affect in Guess My Rule – Problem Three

This research's consideration of the affective domain is guided by DeBellis and Goldin (2006) and takes "affect" as a representational structure that includes both the "local affect" of "changing states of emotional feeling during mathematical problem solving" and the "global affect" of the longer-term constructs established for local affect (p. 133). DeBellis and Goldin describe "affective pathways" as the sequences of local states of emotion as they interact with cognitive configurations. When faced with a problem, for example, one might feel curiosity which then leads to the self-motivation to better understand the problem. Or one might first feel bewilderment and then fall into frustration. Strategic thinking would hopefully lead to feelings of pleasure and satisfaction.

How would we characterize Romina's affective pathway in this particular episode? When Bobby points out an error in her work, Romina characterizes herself as not caring - "Hey, I was just making up numbers. I didn't care." (RC305). She then taunts students across the table by sticking out her tongue and saying, "I got the answer." When Bobby challenges that she copied off of their ideas, she defends herself by drawing a distinction between having initially copied their rule without understanding of the meaning but now having gaining her own knowledge of the rule. When Brian challenges her conjecture for the new rule, she initially expresses frustration by clenching her fists to her head but then perseveres and defends her function rule by demonstrating that it works for the first entry. She then lightly mocks him by humming "dum, dum, dum" when she finishes faster than he does. Perseverance seems to characterize Romina's affective pathway in this episode. While being criticized alternately by both Bobby and Brian, she persists in repeating and arguing for her function rule. She then criticizes Brian.

4.3.4 Problem Number Four

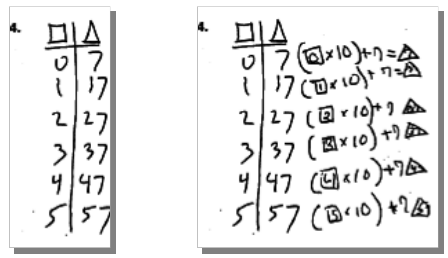


Figure 4-11. **Problem Number Four: before and after Romina's work**

Whereas problems two and three took the students longer, they quickly (within a matter of minutes) resolve what linear rule applies to problem number four:

ROMINA	This one's ten.
BRIAN	We're on four. Ten?
BOBBY	It's the first number, it's the plus number.
ROMINA	[*Turns toward Bobby and AmyLynn*] Der, you didn't know that?
BOBBY	No, we heard you guys.
ROMINA	No, I got that one by myself.
BRIAN	Oh, I know how, I know what the multiple is [turns to the previous page]
ROMINA	How? [*Romina leans over and points to his paper*] Der, all you have to do, Brian, is take the first number and add it.
BRIAN	Okay, okay.
ROMINA	That's what I told you in the beginning, but no.
BRAIN	This is zero. I'm just writing this out. This is blank times four, right? [*Looks over at Romina's paper*] No, times seven.
ROMINA	[*Laughs and points a Brian's table for number four.*] Whatever is between seven and seventeen. (RC353 – 368)

When Brian turns his page to problem number four, Romina immediately observes, "This one's ten" (RC352). Here, she refers to the slope for the new linear function rule (see Figure 4-11). Bobby interjects with an observation about the y-intercept by saying that, "it's the first number – it's the plus number" (RC354). Romina replies, "Der, you didn't know that?" (RC355) and Bobby acknowledges, "we heard you guys." Romina takes personal credit for the observation that the "plus number" in the function rule is the first entry in the triangle column when the square is zero – thus making it the y-intercept. She tells Bobby, "No, I got that one by myself" (RC358). Romina soon leans over and corrects the rule Brian is writing on his paper. She explains how to identify the y-intercept for the function rule: "All you have to do, Brian, is take the first number and add it" (RC361-2). After Brian agrees, she reminds him that she had made this observation to him before about how to find the y-intercept: "That's what I told you in the beginning, but no" (RC364). Brian then asks her if the rest of the rule is "blank times four" or "times seven." Romina laughs and corrects him by explaining that to find the slope he needs to look for "whatever is between seven and seventeen" – two of the entries in the triangle column. Then Brian writes the correct rule for problem number four as a linear function with a slope of ten and y-intercept seven.

In this episode we see Romina asserting more personal authority and collaborating with Brian by sharing her observations about how to identify both the slope and y-intercept. As opposed to previous problems, after she has guided Brian through the problem, she expresses a desire to make their ideas public by saying to the group at the table, "we've had a secret" (RC378) and then repeating a minute later, "we've had the secret" (RC387).

4.3.5 Problem Number Five

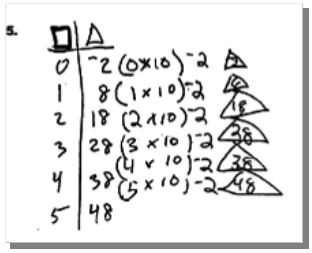


Figure 4-12. Problem Number Five – Romina's work.

By the time they reach the last linear function on the worksheet – problem number five - Brian and Romina generate and record a rule quickly (under five minutes) and without as much discussion as previously (see Figure 4-12):

ROMINA Hey, wait, we're doing the [*looks over at Brian's paper*]. This time you beat me.

BRIAN [*Puts down his pen and sticks out his tongue at Romina*] Okay, this goes up by –

ROMINA Is that a minus two? [Indicating the first entry in the triangle column for problem number five]

JEFF No, it's a plus two.

BRIAN Yeah, it's minus two so it goes up by ten.

ROMINA Ten again. Okay, negative two. Brian?

BOBBY Did you get this one?

ROMINA Hey! You guys can't look at ours. (RC408 – 418)

By this point Brian and Romina are finishing each other's sentences. Brian begins, "Okay, this goes up by…" (RC409-410) and Romina finishes, "Is that a minus two?" (RC411). Brian observes that the sequence of dependent values "goes up by ten" and Romina remarks that it is "ten again" – making the connection that both this problem and the previous had slopes of ten. When Bobby asks if they have gotten the problem, Romina tells him that "you guys can't look at ours" (RC417) expressing a desire to keep their rule a secret here. A minute later, Bobby remarks that he gets it and says that "this is a lot easier way" (RC429) presumably commenting on the pattern of identifying the y-intercept from the first triangle entry and slope from the “between” finite differences. Romina agrees about the ease of this strategy by beginning, “Yeah, I know – when you know the answer it’s like” and Brian finishing, “boom, boom, boom, boom, - you get done with the answer” (RC430-431). Bobby agrees, “you multiply that and minus two” (RC432). Brian and Romina notice Michelle I. explaining her pattern to the camera. Brian implies that their strategy generalizes to all of the problems by commenting, “Our one goes with everything” (RC447). Romina agrees that “this is easy once you get the hang of it” and then comments, “you’re so slow” (RC448).

4.3.6 Problem Number Six

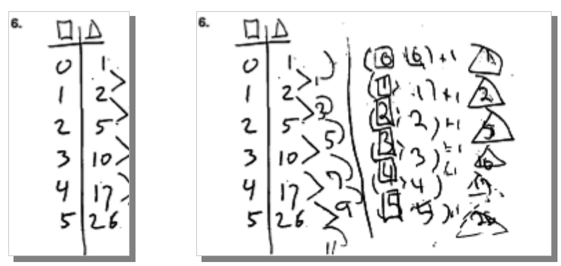


Figure 4-13. Problem Number Six: before and after Romina’s work.

Problem number six is the first quadratic function the students have encountered thus far in the worksheet (see the “before” of Figure 4-13 above). Brian and Romina approach the problem in the same way they approached the others by looking for the finite differences in the triangle column. Brian asks, “Okay, what does this go by?” and Romina asserts, “I get it” (RC451-2). A few seconds later, however, Romina makes a crying sound and asks, “Why do they do this to us?” (RC453). As they begin to guess function rules, Romina asks, “Wait – what’s between each one?” As she and Brian list the first order finite differences as one, three, and five, they realize that it is not a constant number as it was in the previous linear problems. Romina observes a pattern between the "between" numbers:

> Five, seven, nine, it goes up by two. No. Yeah, what's in between it, it goes up by two. This is not fair. (RC465-467)

Romina notes that the second order finite differences are constant since the first order differences like 5, 7, and 9 increase by the constant two (see Figure 4-13 in the "after" image). Brian comments, "you don't have the same number there – oh, you can't do that" (RC470-471). For Brian, the fact that the first order differences are not constant means you "can't do" the problem. Romina laughs and says "it's not fair." While Brian begins to work on his paper, Romina calls out across the table to engage another group, "Guys, did you get number six?" (RC474). At this time, RBD comes over. Romina says that they "know the secret" but are "stuck on six" – their secret applies to the previous linear functions but not to the current quadratic problem. When Brian explains to RBD that "it keeps on going up by two," Romina interrupts and corrects, "No, what's in between goes" (RC487-488). Romina rephrases and tries to explain the second order differences to RBD, "No, he means like this doesn't go up by two, but what's in between this goes up by two" (RC490-491). RBD encourages them to share their observations with the camera.

When they return to their seats to continue work on the problem, Romina's problem-solving strategy is similar to that which she employed when beginning work on problem two earlier: frequent questioning of her partner and other co-construction collaborative techniques. In addition to asking eight questions in under four minutes, Romina collaborates with Brian by expanding on his ideas and redefining, reiterating, or correcting his statements. Table 4-6 below summarizes instances of Romina's collaboration for one episode of Problem Number Six.

Problem Number Six (lines 518-573)		
	COLLABORATION	
Line	*Romina's Statement*	*Description*
518-9	One's the first number. Then wouldn't it be plus one? We did that with all the other ones.	Since 1 is the "first number" in the triangle column, Romina suggests that 1 should be the y-intercept for the function rule. She uses as evidence the fact that they used the first entry in the triangle column as the y-intercept for all of the previous function rules up to this point.
523	Why are you 'oh yeah'?	When Brian replies, "one – oh yeah" to Romina's observation, she asks him to elaborate.
525	Okay, the answer has to be one so.	Romina summarizes what they know so far about the function rule "answer": that there has to be a "one" in it. She prompts Brian to continue by saying, "so."
543-4	But, what? These are the answers you're supposed to be getting right now.	Brian develops a function rule for finding the first order differences instead of the entries in the triangle column. He describes to Romina his rule for multiplying by two and adding one. The numbers that result are one, three, five, and seven (the first order differences). Romina asks for clarification and redirects Brian to the entries in the triangle column and tells him "these are the answers you're supposed to be getting right now" (1, 2, 5, 10, …)
546	But did you get five for this one?	Brian insists that his method works. Romina asks if his rule would "get five" when the square is two. Brian's rule does happen to work for this entry.
548	Yeah, but those aren't the answers, Brian. Those aren't the ones we're supposed to be getting right now.	Brian gives another example of his rule when the square is four. He uses "four times two plus one"

		and gets nine (which is the first order difference) – the triangle entry should be 17 when the square is 4. Because Brian's rule gives nine and not seventeen, Romina argues, "those aren't the answers." She tells him they are "supposed" to be developing a rule to get the numbers in the triangle column.
551	Why do you think?	Brian tells Romina it is "impossible" to get the numbers in the triangle column, but they can use the numbers in between. Romina asks him "why" he thinks this.
553-4	Yeah, but – but one, three, five, seven, and nine are not the ones we're supposed to get in the triangle. What is that?	Brian tells Romina that his rule "goes" and does not elaborate further. Romina again reminds Brian that the first order differences ("one, three, five, seven, and nine") are not the numbers they should be arriving at as solutions to their rule. Brian continues working and Romina asks him what he's doing.
559	But these are the numbers which are supposed to be in the triangle.	Brian applies his same rule of times two plus one to the five to get the first order difference of 11. He shows her again how his rule will give them each of the first order differences. Romina again redirects Brian to the numbers "which are supposed to be in the triangle."
565	Each one's gonna have to be a different number.	Romina predicts that they are going to have to use different numbers in their rule.
570	Okay, and where is eleven?	Brian again argues for his rule by saying, "look, this is exactly what it is, two times five plus one is eleven." Romina challenges him to show him "where" the eleven is in the triangle column for problem number six.
573	Yeah, but aren't we supposed to get these	When Brian shows Romina the

	numbers in the triangle, not these?	eleven in the first order differences "between" the 26 and 37, Romina again asks, "aren't we supposed to get these numbers in the triangle" and "not these" numbers between them.

Table 4-6. Examples of Romina's collaboration in Guess My Rule

Notice that although Brian argues for his rule throughout this episode, Romina remains persistent in redirecting his attention back to the entries in the triangle column through her questioning, reiterating, and rephrasing of his statements. She asks him questions that probe "what," "where," and "why" he is applying his rule of multiplying by two and adding one.

Directly after the exchange detailed above, Romina calls RBD over by saying that "I think we've got something for six, but we're not totally sure" (RC579-580). Ankur and Michelle interject that they have "figured out how to write it" for problem six. RBD encourages Ankur and Michelle and then Brian and Romina to share their findings with the camera. Bobby and AmyLynn then volunteer that they have "finished six" so they also go to the camera. Out of the three pairs who go to the camera to share their secrets, two pairs - Ankur and Michelle and Bobby and AmyLynn - have found the quadratic function rule. Ankur and Michelle join Romina and Brian. Brian explains how he used a rule to generate the first order differences, but Ankur interrupts and says "there's a different way to write it." He tells Romina and Brian that they have to "write it all in code" (RC642). Ankur says that when they write it in "code" with "like squares and triangles," then he'll tell them the secret and "share."

For the next ten minutes Romina and Brian try to develop the "code." Romina goes through a similar cycle of variable affect as she did in problem three and the process of questioning she has used before. Initially she professes to not care about the problem:

> Three, five, seven. Switch them all, Bri. Who cares? We'll just get a different answer. Different problem. (RC654-655)

Next, she tries to engage other students and RBD by asking if the equation has to "stay the same through" (RC672). Bobby suggests that Romina should write an equation that "develops a pattern that you notice," but Romina rebuffs him with, "Oh, leave me alone" (RC687-8). At this point RBD invites Ankur and Michelle I. to the board in order to "publish" part of their "secret" about the function rule to the class. Michelle I. mentions how to find the y-intercept and then that you can multiply the zero times itself. Romina and Brian remain unsure how to "make a code" for this problem. Michelle I. and Stephanie join Brian and Romina. Michelle I. gives them "one hint" by indicating that they focus on the box. Stephanie asks, "does this number always multiply by itself or something?" (RC778) and Romina points out that "yeah, we have that much" (RC781) since they know by this point to multiply zero times zero plus one, one times one plus one, two times two plus one. They are just unsure how to write it in "code." Michelle I. then tells them, "if it's going to be the same number, it's gonna be a square" – so the code will be square times square plus one. Romina remarks, "that's cheap." She does not express any desire to publicize this information because she tells Brian, "this time you're explaining it Brian 'cause I'm not saying anything" (RC795-6). Brian shares the correct quadratic rule with the camera: square times square plus one equals triangle.

4.3.7 Problem Number Seven

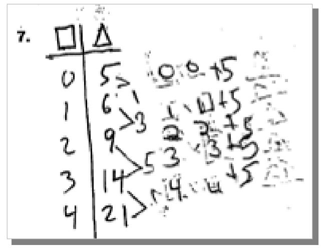


Figure 4-14. **Problem Number Seven – Romina's work.**

After the more than twenty minutes spent on problem number six, Romina and Brian solve problem number seven rather quickly (less than three minutes) and with limited discussion. First, Romina observes "not again" because the first order differences are the same as in problem six: 1, 3, 5, 7, and 9 (see Figure 4-14). Brian comments, "one, three, five, seven – it's the same thing as it was last time" and Romina responds that, "Yeah, I know" (RC805-807). Problem number seven is also a quadratic function. The only other comment Romina makes about this problem is, "this is five, right?" (RC867) in reference to the y-intercept. They record the correct quadratic rule: square times square plus five equals triangle.

4.3.8 Problem Number Eight

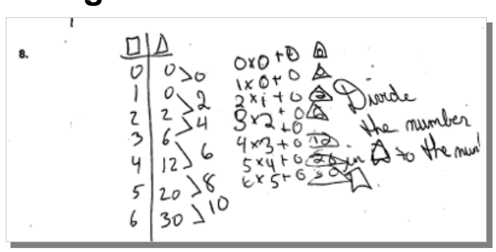


Figure 4-15. Problem Number Eight – Romina's work.

Whereas the quadratic rule followed very quickly for Romina and Brian in problem number seven after the long time spent with problem number six, they encounter difficulty again with problem number eight since the first order differences do not follow the same pattern. First, Romina and Brian list the first order differences: 0, 2, 4, 6, 8, and

10 (see Figure 4-15). Brian suggests that "we do the same thing we've been doing" (RC874) implying perhaps further use of the first order differences. We again see examples of variable affect – specifically here Romina exhibits frustration and confusion. Romina remarks that "this is really ticking me off" (RC875) because the pattern is not like the previous two quadratic rules. When Brian claims to "know what it is" but it "ain't two times two," Romina expresses confusion and says, "Brian, you are confusing me so much" (RC886). Brian's suggests a new rule using the term in the box times the previous term in the box plus zero. Romina asks, "Aren't we supposed to be using a code?" (RC891). At that point Brian responds, "I hate number eight – number eight stinks" (RC892). Brian then returns to his earlier idea. Brian explains his pattern to both Romina and Michelle I.:

> Hey look at this one, how many times does two go into two? Once. How many times does three go into six? Twice. How many times does four go into twelve? Three times. How many times does five go into twenty? Four times. How many times – (RC916-919)

Using a sequence of questions accompanied by his own answers, Brian builds a case for his the pattern he has noticed in the sequence of terms in the triangle column. Romina at first cautions him, "don't talk so loud" but as Brian continues to argue for his pattern, Romina soon calls out to the group and RBD that "we have another code" (RC935). Brian explains their code as "divide the number in the square by the number in the triangle" (RC939) and Romina corrects the division sentence "in the triangle to the number in the square" (RC940). While they have not yet fully expressed the function rule for problem number eight, indeed the quotient of the triangle term divided by the square term equals the previous square term. By this time, Ankur, Michelle I. Romina, and Brian are all working together. Though the group claims to have found "the code" for this problem, by the end of this session Romina has recorded “divide the number Δ to the number” (see Figure 4-15) and Ankur has recorded the expression “(Δ÷□) +” on his paper.

4.3.9 Problem Number Nine

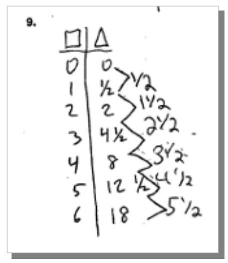


Figure 4-16. **Problem Number Nine – Romina’s work.**

As Romina begins work on problem number nine (see Figure 4-16), she displays another example of strong and variable affect. She first asks for time “to think,” expresses frustration, claims to have “messed up,” and then brings attention to the fact that she was actually “right.”

Problem Number Nine (lines 956-967)		
VARIABLE AFFECT		
Speaker	*Statement*	*Description*
Romina	I got to think. Ahhh! This is hard. Brian is this right? (RC956)	When she first looks at the new problem, Romina tells the others that, “I got to think.” Her next vocalizations of “Ahhh!” and “this is hard” indicate an amount of frustration.
Brian	Ok. That goes in that four. That goes in that five. Six times eight, ahhh! (RC957-958)	Brian tries his previous pattern of division by looking at what “goes in” each term. When this does not work, he displays frustration with the cry of “ahhh!” Michelle then directs their attention to the first order finite differences: 1 and 1 ½
Michelle I.	Look, look. This is one. That’s one and a half. (RC959)	
Brian	Let me see, let me see, let me see. (RC961)	Brian asks to see what they are doing. Romina tells him that she “erased” the work with the common
Romina	I erased it now. I messed up. (RC962)	

		differences because she "messed up."
Brian	Half, two and a half, five and a half? (RC963)	Brian notices the finite differences Romina had written and not completely erased: ½, 2 ½, and 5 ½
Romina	I messed up, I know. (RC962)	Romina seems to believe Brian's comment implies a criticism of her work because she says, "I messed up, I know."
Brian	Ok, that goes one and a half, two and a half, three and a half, four and a half… you were right, you were right. (RC965-966)	Brian then lists the finite differences out loud: 1 ½, 2 ½, 3 ½, 4 ½. He tells Romina that "you were right, you were right." The numbers she had written earlier are indeed part of the finite differences he now observes in the triangle column.
Romina	Whoa, whoa, whoa. Thank you – I was right. (RC967)	Romina now brings attention to Brian's comment by exclaiming, "whoa, whoa, whoa" and then, "thank you – I was right."

Table 4-7. Examples of Romina's variable affect in Guess My Rule – Problem Number Nine

When Romina was ready to abandon her problem-solving approach – in fact, she was already in the process of erasing her work and claiming that she "messed up," it took Brian's renewed interest in the sequence of finite differences and his supportive comment that "you were right" to change Romina's affect from one of frustration to one of confidence, "thank you – I was right."

Chapter 5 BEHAVIOR RESULTS – High School

Never be afraid to sit awhile and think.

~ Lorraine Hansberry, *A Raisin in the Sun* (1961)

5.1 Introduction

The two problem-solving sessions, "Ankur's Challenge" and "Taxicab Geometry," that follow in this chapter both occurred when Romina was in high school – 10^{th} and 12^{th} grades, respectively. What merited a separate chapter for these behavioral analyses was not only the fact that they are situated in a high school setting, but also that they represent a different path the longitudinal study took with the students after middle school. While in the elementary and middle grades the interventions took place with the whole classroom during an extended period of the normal school day, what distinguishes the high school years of the longitudinal study is that the students met on a *voluntary* basis *after school*, often on Fridays, and in much *smaller groups* – only four or five students at a time. In almost all these high school sessions from the RBDIL video archive, Romina was also usually the only female participant. While different graduate students and researchers would be present at different times over the high school years, the same lead teacher-researcher, coded here as T/R1, was present in all of the sessions. In what became known as the "Ankur's Challenge" session from 10^{th} grade on Friday, January 9, 1998, the participants were Ankur, Michael, Jeff, Romina, and Brian and the session lasted 90 minutes. In the Taxicab Geometry session from 12^{th} grade on Friday, May 5, 2000, the participants were Michael, Romina, Jeff, and Brian and the session lasted 100 minutes. What follows are analyses of Romina's behavior in both of these sessions.

5.2 "Ankur's Challenge": January 9, 1998 (10th Grade)

5.2.1 Setting

On Friday afternoon, January 9, 1998, five tenth-grade students met with T/R1 for a problem-solving session of approximately 90 minutes in duration. This session was one of several after school problem-solving sessions held at David Brearly High School in Kenilworth, New Jersey that were videotaped as part of the Rutgers-Kenilworth longitudinal study. For this session, five students were present: Ankur, Michael, Jeff, Romina, and Brian. They sat around a conference table with Michael sitting adjacent to Ankur at one end and Jeff, Romina, and Brian sitting at the other end. One video camera and microphone were used. Two disks of video data (Disk I – 59 minutes and Disk II – 34 minutes) were transcribed and verified (see Appendix C). During several sections of Disk I both groups of students were discussing problems simultaneously and thus a separate transcript for those times which directly overlapped was developed called the "Addendum" – line numbers from this transcript are indicated by the prefix "Ad." For the majority of the time, the students wrote their work on paper but occasionally they would make use of the chalkboard located on the wall behind the conference table. For the most part, the students were left by themselves at the table to work. At certain points, however, T/R1 would sit with the students and ask questions.

5.2.2 Revisiting Towers 5-Tall

Two problems were considered during the session. The first problem was posed by T/R1. She asked the students to reconsider a problem they had encountered in fourth grade involving towers five tall. Specifically, she asked how students could have gotten an answer of ten for the number of 5-tall towers that could be built from a choice of two colors (red or yellow) and would have *exactly* two red cubes. She requested that the students "convince" her that they had found all of them:

> So now you have to convince me that you found them - that there couldn't be any others. So why don't you think about that for a minute. (4)

The students broke into two sub-groups: Ankur working with Michael and Jeff, Romina, and Brian working together.

5.2.2.1 Ankur and Michael's Solution

Within four minutes, Ankur and Michael had developed an argument which involved representing each of the ten towers with a code of 1 and 0 (for the red and yellow color choices, respectively): "the ones are like red and those are like yellow" (12). While Jeff, Romina, and Brian continued to work on their solution, Ankur and Michael presented their solution strategy (see Figures 5-1 and 5-2) to T/R1. A tower written as 11000 was their "first tower" where each 1 indicated a red cube and the 0 a yellow cube. Thus, their first tower recorded was Red-Red-Yellow-Yellow-Yellow. They proceeded to move the 1 digit to different locations among the 0 digits.

***Figure 5-1*. Ankur and Michael's written work – first version of their solution**

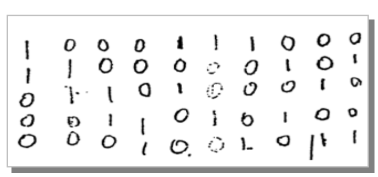


***Figure 5-2*. Ankur and Michael's written work – second version of their solution**

They rewrote their first version into a second version (Figure 5-2) so that the red “1” cube would be held constant in the top position while the other red cube moved among the second, third, fourth, and fifth positions. Then the constant top red “1” cube was kept stationary in the second position while the other red cube moved among the third, fourth, and fifth positions. Then, the top-most red “1” cube was held stationary in the third position while the other red cube moved among the fourth and fifth positions. Finally the top-most red “1” cube was held stationary in the fourth position with the other red cube in the bottom red position. This solution gave a total of 4 + 3 + 2 + 1 = 10 towers. When R1 commented that “you have a very powerful strategy here” (67), Michael responded that he could “apply that to anything you give me” (68).

5.2.2.2 Romina, Jeff, and Brian – First Approach

Meanwhile at the other end of the table and for about twenty minutes, Romina, Jeff, and Brian worked on the same problem of towers. Although T/R1 had posed the problem as five-high with a choice of two colors that would have exactly two red cubes, Romina, Jeff, and Brian began listing towers that would have exactly two *yellow* cubes and three reds. Their first approach was to volunteer different combinations of towers that had exactly two yellow cubes while Romina recorded the suggestions on paper.

ROMINA	So it is that it? I don’t know.
BRIAN	Y, Y, R, R, R? Der. Do you have Y, R, R. No, that ain’t working.
BRIAN	You have Y, R, R, Y, R?
BRIAN	You have R, R, Y, Y, R?
JEFF	What did you say?
ROMINA	That would be four reds.
BRIAN	You got them. R, R, Y, Y, R. [*Holds out a finger of one of his hands as he repeats each letter until all five of his fingers are extended*].

BRIAN	How many is that? Ten. Boom.
JEFF	Twenty. It's twenty total.
ROMINA	Come on. That doesn't make an equation. Zero, One, One. You have the same output for two inputs. Why is it you guys? We know this. It's Friday… Don't panic. You guys, two-fifths.
BRIAN	It's not two-fifths.
ROMINA	I know it's not. I'm just saying. Why is it ten? [*To Ankur and Michael at the other end of the table*] You guys, why is it ten? [*To Brian and Jeff*] Are they on a different problem already?
BRIAN	Can we just say we got it, and then go to another one?
JEFF	How are we supposed to get the next one, if we can't get the first one?
T/R1	You have to convince me that you have them. You can't just say, "I have 10." You have to be able to prove to me that there can't be more. Sixteen, Eighteen?
ROMINA	This is so frickin… We don't even have, hold on, we don't even have it, cause they did it… (Ad32 –Ad47)

Romina begins by asking the two boys "is that it" for the list they have generated so far. She comments that "I don't know." Brian calls out different sequences of letters to form towers with two yellow cubes and three red like YYRRR, YRRYR, and RRYYR. Jeff and Romina check his suggestions against the written list. Notice the list of ten tower combinations Romina had recorded in Figure 5-3 below.

Figure 5-3. **Romina's written list of the towers: "So is that it? I don't know." (Ad32)**

After his suggestions have been checked against Romina's list, Brian observes that there are now "ten" total and exclaims, "Boom." Although he does not provide justification, Jeff disagrees and says that there are "twenty total" because he is including the ten towers of two yellow and three red plus the ten towers of three yellow and two red. Romina observes, "that doesn't make an equation." She writes on her paper an x-y table where the two x-entries are {2, 3} and the two y-entries are {10, 10} (see Figure 5-4). Indeed the total number of towers with exactly 2 yellow cubes is 10 and the total number of towers with exactly 3 yellow cubes is 10.

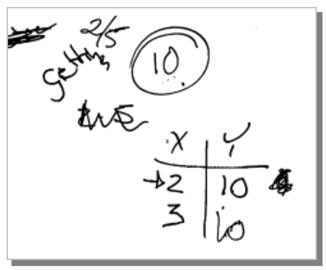


***Figure 5-4*. Romina's table - "…You have the same output for two inputs" (Ad41)**

Perhaps investigating a function relationship between the number of yellow cubes and the total number of towers, Romina comments that "you have the same output for two inputs." Romina then asks a string of several questions. She asks Brian and Jeff for an explanation: "Why is this?" She coaxes, "We know this. It's Friday – don't panic." Without justification, she says that the answer is "two-fifths" and when Brian disagrees, she responds that she does "know" that it's not that answer. Romina questions Brian and Jeff, "Why is it ten?" Then she asks Ankur and Michael at the other end of the table, "Why is it ten?" Instead of answering, Brian wonders aloud if they can "just say we got it" and continue to the next problem. Jeff observes that they will have difficulty with the next problem if they "can't get the first" and T/R1 interjects that "you have to convince me that you have them." She reminds the students that "you have to prove to me that there can't be more." Implying some negative affect, Romina responds that "this is so frickin'" and notices that Ankur and Mike have answered the problem but they have not – "we don't even have it, but they did it."

5.2.2.3 Romina, Jeff, and Brian – <u>Second Approach</u> – "Backwards" & "Opposites"

Romina, Jeff, and Brian continue to work on the Towers problem. Their notation shifts from letters to numbers. As opposed to listing the sequences with letters "Y" and "R," their second approach involves Jeff generating a list of sequences using the digits 1 and 0. Again, they began by suggesting different sequences. Although Jeff starts the list, Romina soon takes the paper and pen from Jeff and continues writing it herself (see Figure 5-5). Throughout, Brian continues to make suggestions for sequences. Soon, a discussion of "opposite" sequences within their list ensues and leads to further sequence generation.

BRIAN	Would 0, 0, 0, 1, 1 be the same thing as 1, 1, 0, 0, 0? (*Jeff and Romina point to paper*) I didn't see that. Never mind.
JEFF	Oh, I know. No, I don't know. 1, 1, 0, 0
ROMINA	[*Takes the paper from Jeff and points with the pen*]. Hold on, here we go then. Where is the opposite to this? [*Taking the paper and pen from Jeff*] This. This. What would be the opposite to this?
BRIAN	0, 1, 1, 1, 0
JEFF	The difference of that would be because these two are opposites.
BRIAN	It's like read them backwards
JEFF	This one doesn't have a backwards 'cause it's the same thing.
ROMINA	They proved it already. They could be like that, and then this, and this and this, and then you would have one. This one. This one you would have. (Ad68 – 75)

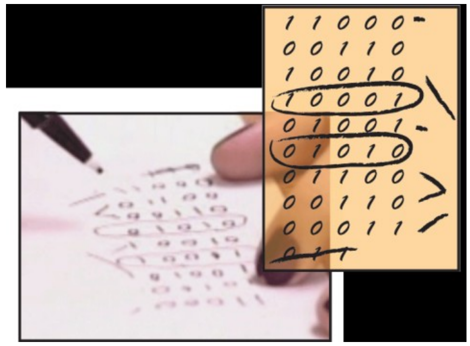


***Figure 5-5*. Romina's binary list: "What would be the opposite to this?" (Ad70)**

First, Brian asks if the sequence 00011 would be the "same" as 11000. When Jeff begins to suggest another sequence, Romina interrupts with "hold on." Taking the paper from him, Romina questions Jeff and Brian. Pointing to the sequence 10001, she asks, "What is the opposite to this?" Romina directs the boys' attention to the paper by pointing and saying "this, this" (see Figure 5-5). She repeats her question seeking information and clarification about the meaning of "opposite." There is some confusion in the definition because Brian first says that 01110 would be the opposite to the sequence 10001. However, after Jeff indicates a different pair on the list as opposites 11000 and 00011, Brian changes his definition to be that "opposites" occur when you "read them backwards." Jeff indicates a palindrome sequence 10001 as one that "doesn't have a backwards" because it would read as the "same thing." Narrating with a series of "this and this" comments, Romina then begins to note which pairs of sequences on the list are "opposites" and which are palindromes. She circles the palindromes 10001 and 01010. She draws an arrow to match 11000 and 00011 as "opposites." She writes 00101 on the list as the "opposite" to 10100. She connects 01100 and 00110. With a dash mark by 01001, she includes the opposite 10010 as the opposite. Their final list of towers five high with a choice of two colors and exactly two red are as follows in Figure 5-6.

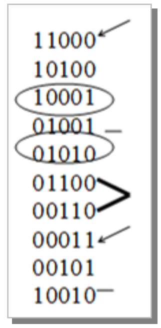


***Figure 5-6*. Jeff and Romina's list of the ten towers adapted to show Romina's notation for linking the "opposites" on the list.**

Although they have again generated a list of ten towers as the solution to the problem, the students continue to wonder about the answer. When they question the solution of *ten* towers further, the students seem to express some negative affect as well.

JEFF	Yeah, but you're saying if you had two yellows it would be ten.
BRIAN	As long as it's five high, it's still three of one color and two of another.
ROMINA	Why? Cause that's what she wants to know.
BRIAN	You're gonna have ten. You could have this and this *[holding up the blocks WW next to YYY and then WWW next to YY*] . You would have ten, and then if you just got.
ROMINA	Oh, that, yeah.
BRIAN	It don't matter if you have three. The height is going to make the amount itself.
ROMINA	Oh, I know that. I'm just saying I thought you had like –
BRIAN	I don't have no breakthrough. I don't have breakthroughs in my life.
JEFF	I'm getting a little frustrated.
ROMINA	Oh, okay then.
BRIAN	School's just gonna go on.
ROMINA	[*Puts head down on desk. Jeff has the pen and paper*]. I have no clue.
ROMINA	[*Picks up her head and looks on Jeff's paper*]. What's the total? What's it doing? Alright?
ROMINA	I don't know what I'm doing. (Ad87 – Ad99)

Figure 5-7. Romina's affect: "I have no clue." (Ad96)

After finishing the list of ten tower sequences for exactly two red cubes, Jeff questions why the solution would be ten for "two yellows" as well. Brian explains that for any tower that is "five high," it is "still three of one color and two of another" – three reds and two yellows in their first list and three yellows and two reds in their second list. Romina interjects with a question, "Why?" and reminds Brian and Jeff that "why" is what T/R1 "wants to know." Brian holds up towers and replies that "you're gonna have ten" – the only justification he offers is that "the height is going to make the amount itself." When Romina presses further and says that she knows that but "thought you had" further reasoning, Brian comments that he doesn't have a "breakthrough," nor does he have "breakthroughs in my life." Jeff observes that he is also "getting a little frustrated." Romina puts her head on the table and responds that "I have no clue" (see Figure 5-7). Breaking the sequence of negative affect comments however, Romina then lifts her head and asks a couple new questions. She probes, "What's the total?" and "What's it doing?" perhaps attempting a new consideration of patterns. A minute later, when Jeff and Brian have not responded, Romina again seems to express negative affect when she says that "I don't know what I'm doing." Below is a table (Table 5-1) summarizing the variable affect displayed within this section of the problem solving for Romina.

Second Approach for Towers (lines Ad96 – Ad117)		
	VARIABLE AFFECT & SELF-PERCEPTIONS	
Line	*Romina's Statement*	*Description*
Ad96	I have no clue.	Having earlier asked "why" there would be ten total towers with exactly two red cubes and this number would be the same as having exactly two yellow cubes, Romina puts her head down on the table and says she has "no clue." She says this directly after Brian has said he had "no breakthrough" and Jeff mentioned he was "getting a little frustrated."
Ad97	What's the total? What's it doing? All right?	Romina picks her head off the table and looks over at what Jeff is writing. She asks him what the "total" is and 'what's it doing?"
Ad99	I don't know what I'm doing.	When Jeff does not respond, Romina returns to her earlier comments of negative self-perception. Expressing a negative perception of her current actions, she states that she does not know what she's doing.
Ad101	Maybe we should just do the binary system. This could have been easier.	Romina proposes that it could have been "easier" if they had used the "binary system" for this tower problem. Jeff refers to Mike as a "mad scientist" who uses the binary system.
Ad110	I don't know what the binary system is. I don't know how to do this. I think I was also absent when you did this.	Jeff continues to re-work the towers problem and creates new groupings. He records the towers with "one" cube of another color. Romina proposes that "would only make four" towers (Ad104). Brian and Jeff disagree with her and argue there would be five towers with one cube of another color. Romina expresses this in her own words by saying that if we

		had "four red and one yellow then that would be five" (Ad108). After this observation however, she also claims that "I don't know" the binary system and "I don't know how to do this."
Ad113-115	What are you doing? … You're getting mad.	Romina continues to ask Jeff about his new approach to the problem and questions, "What are you doing?" Jeff responds by saying, "Can I think for one second?" Then Romina claims that he is "getting mad."
Ad117	Okay, go!	In reference to Romina's repeated questions about "what" he's doing, Jeff wonders aloud, "How many times are you going to ask me that?" (Ad116). Romina then gives the imperative, "Okay, go!"

Table 5-1. Examples of Romina's variable affect and self perceptions in second approach

After five minutes during which there is evidence of variable affect as detailed above, Jeff shares with Romina the new list he has created to answer the Towers problem. In contrast to their earlier representations that included on the 10 sequences, Jeff includes all 32 total towers five-high from a choice of the two colors yellow and red. Recall that, using the letters Y and R to represent the two colors, the group originally had recorded 10 tower sequences with exactly two yellow cubes through a process of guess and check. Brian or Jeff would call out a sequence while Romina would check or write it on this list. Then, the group switched to recording the tower sequences using the digits 1 and 0 for red and yellow, respectively. They included a patterning strategy that employed the use of "opposites" to pair up sequences that would read "backwards" to each other. They found ten towers that would have exactly two red cubes. In the final iteration of problem solving representation and as a result of Romina's earlier questioning about "why" there would be ten towers with exactly two red cubes, Jeff decided to create a new list that included all thirty-two towers five high with a choice of two colors. After Romina's repeated questions about "what are you doing?" and her perception that he was getting "mad" at her, Jeff explained what he his list to Romina.

> JEFF [*Shows Romina from his paper*] Ten of these. Five of these. Two of these. Oh, no. One of these. And that's sixteen. And then everyone has the opposite colors and so forth. Five, five, one color (inaudible) I thought it was one deal. [*Brian is building towers*]. Five high and zero of the other color. Five high and two of the other colors. And just half. Do you understand? Five high… (Ad125)

Jeff groups the towers as "ten" with two red, "five" with one red, and then "one" of no reds. He explains that would be a total of "sixteen" towers. Next he argues to Romina that "everyone has the opposite colors." Therefore, there would be 10 with two yellow, 5 with one yellow and 1 with no yellows. Jeff describes the group names as "five high and zero of the other color" and "five high and two of the other color." This creates another 16 or "half" of the total 32 towers five-tall with a choice of two colors. The ten towers with exactly two reds are now a subset of this whole set of 32 towers. Notice that in Jeff's argument, a new definition of "opposite" emerges. Whereas for the previous list, "opposite" meant towers read "backwards" like 11000 and 00011, now Jeff is using the word opposite in the context of "opposite colors." Thus, 11000 and 00111 would now be opposites in his new list.

After having explained his groupings to Romina, Jeff next makes an argument to T/R1. Here, Romina joins him in justifying where the ten towers emerge in the set.

> JEFF And then, you have the opposite colors so you can go say one's red and zero is yellow, then you can go yellow, red, red, red, so there would be ten for that, right? And if there's ten, we did there's five high times the two of one color in it, and that gave us ten. Flipped over the other way would give you twenty. Twenty plus ten is thirty. Excuse me. And then there's the zeroes or like all reds or all yellows which makes thirty-two

	which is the total that you can get. And that's how they divide up into and that's the number of ones that have…
ROMINA	Just simple multiplication. It's just simple multiplication. (*pointing to paper*) This one with one would be five times one, and this would be five times two, like how many with two colors. Five times two.
T/R1	Okay. So, you answered even more than I asked, right? You didn't just tell me how many with just two reds. (Ad203–Ad205)

Notice that as Jeff begins his justification to T/R1, he uses "opposite colors" instead of the "backwards" opposite definition the group used earlier for generating towers. He explains to T/R1 that to get the solution of ten towers, he multiplied the height of the tower by the number red cubes: "there's five high times the two of one color in it and that gave us ten." He does not justify this multiplication strategy further. Continuing, Jeff indicates that they switched colors or directions next since "flipped over the other way would give you twenty." He calls the "zeroes" group the towers with "all reds or all yellows." Jeff summarizes that "thirty-two which is the total you can get" for the towers five high with a choice of two colors. Romina interjects that their strategy is "just simple multiplication." The group with one red and four yellows is the "one" group and is generated by "five times one." The group with two reds and three yellows (or two yellows and three reds) is the "two colors" group and is generated by "five times two." Following this reasoning, however, the "zero" group would be five times zero which would be no towers – Romina and Jeff do not mention this flaw in their logic. T/R1 comments that "you answered even more than I asked." Romina, Jeff, and Brian then learn of the new task that Ankur began working on with Michael while they were waiting for Romina's side to finish with the original towers problem.

5.2.3 Ankur's Challenge

While waiting for Jeff, Romina, and Brian to complete the original tower problem of five-tall with a choice of two colors, T/R1 sits at the other end of the table and begins to ask Ankur and Michael "another question" about towers four tall with a selection of three colors. She directs the students to think of the question themselves: "I want to know how many, how many there – you raise the question. What would be a reasonable question?" (72). Ankur wonders aloud, "How many with at least one of each color?" (73). T/R1 responds that that is a "good question" and reiterates the new problem. This problem became known in later math education research at Rutgers as "Ankur's Challenge": how many towers can you build four tall, selecting from cubes available in three different colors, so that the resulting towers contain at least one of each color?

Ankur and Michael work on the new problem for the next fifteen minutes. After Romina, Jeff, and Brian have presented their solution to the original towers problem to T/R1, Ankur shares his new problem with them.

ANKUR	You have four high and three colors and you have to use at least one of each color in each tower.
JEFF	And... what's the answer?
MICHAEL	We have that.
ANKUR	We have that.
MICHAEL	But it's not like working.
T/R1	They think it's now... They have a conjecture but they can't prove it.
ANKUR	But that answer is right. That answer is right.
JEFF	What did you… what did you come up with?
ANKUR	Um, seventy-two.
JEFF	That's a lot. That's a lot.
ROMINA	Seventy-two? With four high?
ANKUR	Do you want to try it? Four high, you have to use one of each color.
ROMINA	And you have seventy-two?
ANKUR	Yes. Trust us. (276 – 289)

Ankur describes his new problem as one in which you consider towers “four high” with a choice of “three colors” and “you have to use a least one of each color in the tower.” As soon as Ankur has shared the problem, Jeff immediately asks “what’s the answer?” Michael and Ankur both say that they “have” the answer but “it’s not like working.” T/R1 qualifies that “they have a conjecture but they can’t prove it.” Ankur then insists that the “answer is right.” When Jeff asks for what specific answer they found, Ankur tells them that their solution is “seventy-two.” Jeff comments that 72 is “a lot” and Romina questions whether they could get seventy-two with towers “four-high.” Ankur challenges them, “do you want to try it?” Romina questions again, “and you have seventy-two?” Ankur affirms, “yes, trust us.” Ankur goes on to say a minute later that “if you start to do it, then you will realize” (295). Ankur seems to imply confidence that his answer of seventy-two towers is correct and that the groups merely need to provide evidence to support and “realize” this solution.

5.2.3.1 Developing Notation: “ones, zeroes, and X’s”

Romina, Jeff, and Brian begin working on Ankur’s Challenge problem. While Brian builds with the actual cubes on a physical model, both Jeff and Romina start writing on separate pieces of paper. Their first consideration is what notation to use.

ROMINA	Okay, I’m going to use ones, zeroes, and X’s.
JEFF	Ones, zeroes, and X’s? I want to use hearts, squares, and O’s.
ROMINA	Fine. You do it. [*Puts down pen and pushes paper away*]
JEFF	It was a joke – that’s a great idea.
ROMINA	Shut up. [*Picks up the pen again. Writes.*]
JEFF	How ‘bout we just use three letters though? Or three numbers?
ROMINA	I don’t want to -

(Ad210 – Ad216)

Romina informs Brian that she is going to use "ones, zeroes, and X's" to represent the three colors in the problem. When Jeff replies that he will use "hearts, squares, and O's" instead, Romina pushes away her paper. Jeff tells her that he was teasing her by replying that "it was a joke" and he thinks her idea for notation is "a great idea." Romina tells him to "shut up" and picks up her pen again to write. Jeff then suggests that they use "three letters" or "three numbers" for the three colors, but Romina continues to use her notation of ones, zeroes, and X's. Her reason for not using another notation is that "I don't want to." Jeff adopts Romina's notation and they both begin to write possible tower sequences that meet Ankur's requirements of four-high with each color represented.

Soon Jeff attempts to create a number sentence that will give Ankur's answer of 72. Not fully explaining, Jeff observes that from the sequences of three colors "X, O, 1" or "O, 1, X" that they "could have three" (Ad222) – he implies perhaps that if one begins a sequence with X, 0, 1, there could be three possible color choices for the fourth spot. He does not clarify his meaning at this time however. Instead, Jeff continues with a number sentence, "nine times three is seventy-two" (Ad224). Romina questions him and wonders aloud, "nine times three?" and then says that nine times three would be "twenty-seven" (Ad225 – 227). Jeff, Romina, and Brian laugh as Jeff acknowledges the calculation error. Romina then criticizes her own calculation ability.

ROMINA	I was like oh my god. [*laughing*] I have such trouble with simple stuff.
JEFF	My god, I just got them totally like mixed up. Okay.
BRIAN	I'm happy. I'm the only one who hasn't screwed up yet. First time in my whole life.
ROMINA	Okay, come on. Two times three; I'm an idiot. Twenty-four.

(Ad230 – Ad233)

Twice Romina uses language of self-criticism in this one minute segment. First, she says that she has “such trouble with simple stuff” and then commenting that “I’m an idiot.” Her comments her in the context of the three students laughing at the calculation of nine times three. Jeff observes that he got the numbers totally “mixed up” and Brian says that he is the “only one who hasn’t screwed up yet.”

The students continue to work on writing sequences of towers with the notation of ones, zeros, and X’s. Romina’s interaction with Brian and Jeff reveals several instances of asking questions.

Romina begins Ankur’s Challenge (lines 245 - 258)		
COLLABORATION – Asking Questions		
Line	*Romina’s Statement*	*Description*
Ad245	All right, we put it in every space, right? And that would give us like four times three. That would give us twelve.	Romina asks for verification on her strategy. She suggests that since they need to put a cube (“it”) “in every space” and there are four spaces on a tower four-high and three color choices, then they should calculate “four times three.” She suggests the answer 12.
Ad248	For what?	Jeff turns to look on with Ankur and Michael. Romina continues to work with Brian. Brian says that they should calculate “fifteen times three” (Ad247) instead. Romina questions him for information and clarification by asking “for what?” Brian responds that they should look at what Jeff did.
Ad252	It is seventy-two?	Brian wonders aloud how they would get an answer of 72 like Ankur. Brian wonders, “I don’t know where you gonna get that” (Ad251). Romina then seems to question whether the answer really is 72 at all when she says, “It is seventy-two?”

Ad256	Five, six. So he got? I don't know what he did.	Next, Romina says without justification that the answer is "fifty-four" (Ad254). She looks at Jeff's paper and wonders what "he got" and admits that "I don't know what he did."
Ad258	…Can you explain what you did there?	Romina then tries to get Jeff's attention back by asking him for explanation. She requests that he "explain what you did there" on his paper that she's been reading.

***Table 5-2*. Examples of Romina's collaboration by asking questions in Ankur's Challenge**

In the space of about a minute and a half, Romina asks five questions. Her questions alternately seek information ("we put it in every space, right?" or "for what?") or clarification/explanation ("can you explain what you did here?"). Following Romina's questions, Brian and Jeff share their approaches to the problem.

ROMINA	Can you explain what you did there?
BRIAN	You changed the first space on each one.
JEFF	No, I changed the last space. I just did. All right. X you could only have. You could have, X, O, 1 and you could get three for each of them. X, O, 1, X; X, O, 1, 1; X, O, 1, O.
BRIAN	And you multiply that by three.
JEFF	Yeah, so this would be like one, two, three, four, five, six. Now, you could say take out the middle one, so we have X, blank, O, X.

(Ad258 – 262)

Romina's question seeking an explanation from Jeff about his work leads Brian to speculate that Jeff's method "changed the first space on each one." Jeff disagrees and explains that he "changed the last space." Clarifying further, he says that with the sequence "X, O, 1," there could be "three for each of them." He lists the possibilities as X01X, X011, and X010. Consider the four positions on a tower that is four-cubes high. Notice that Jeff kept the first three positions constant with the cubes "X,0, 1" and varied the last position only. In the last position he substituted the three possible colors and therefore got "three" possible tower sequences. When Jeff counts "one, two, three, four, five, six," it is not clear to what he is referring. It seems that he means to include another set of three tower sequences because he discusses that "you could take out the middle one." Jeff elaborates that there could be a sequence built from "X, blank, 0, X" – indicating that the first, third, and fourth positions would be held constant with a color while the second position varied.

5.2.3.2 Experimenting with a "Blank"

Brian picks up on Jeff's use of the word "blank" and suggests that they use it to help generate new tower sequences. The group discusses how to use a "blank" and Romina experiments with and refines a new approach.

BRIAN	So why don't we just put like X, something, blank, whatever?
ROMINA	You have the right idea.
JEFF	Because X, O, blank, 1. (*inaudible*) Blank, O, X. Can't do that. Then you go X, O, O, 1. We didn't write these up here, because these are the only ones you can get three out of. You can't get three out of these because if you put X, O, O, you can't put another X. It's not going to work, because you can't put X, O, O, O, so there's only one of these. X, O, O, O.
ROMINA	It was easy. Give him a number and he multiplies. Why don't I. We'll do the first …
BRIAN	(*inaudible*)
ROMINA	Okay, four times three (*inaudible*). I'm not sure where you get the four.
BRIAN	'Cause you could move that blank into four different spots. Like you could come up with a certain amount for this one…
ROMINA	Make it double.
BRIAN	A certain amount for that.
JEFF	So, X…
BRIAN	So, come up with a formula. Which will be… it can't be that, doubles.
ROMINA	That's what we're trying to do. All right. (Ad270 – Ad281)

Brian suggests that they use a sequence like “X, something, blank, whatever.” Brian’s use of “something” and “whatever” allows for the second and fourth positions to vary between 0 and 1 (since the X has already been taken for the first position). Brian’s “blank” in the third position could be any of the three colors X, 0, and 1. Notice that Brian’s “X, something, blank, whatever” generalizes Jeff’s sequence of X01X further. Romina agrees that Brian’s suggestion is the “right idea.” Jeff disagrees, however, and says that there are certain sequences where simply filling in the “blank” with one of the three colors will lead to a sequence that does not meet the criteria. For instance, if they begin by holding the first three positions constant at “X, 0, 0” then the fourth position could not be any of the three colors. As Jeff argues, “you can’t put another X” in the last position because then not all three colors would be represented. Another “0” cannot be placed in the fourth position either. Jeff observes that “there’s only one of these” that would work: the tower X, 0, 0, 1 – if there were a “blank” in the fourth position it would have to be filled by the digit 1. Romina then probes Brian’s reasoning further and comments that perhaps it was “easy.” Since Brian is using four times three, she tells him that she is “not sure where you get the four.” Brian explains that she could “move the blank into four different spots” on the tower four cubes high. Without explaining, Romina says that he should “make it double.” Brian encourages them to “come up with a formula.” Romina agrees that “we’re trying” to do that.

Romina continues to make an initial list of towers that blends the suggestions of both Jeff and Brian. Consider Figure 5-8 below of the list Romina generates at this time. Notice the sets of three and the underlined variable position.

Figure 5-8. **Romina's list of possible towers for Ankur's Challenge indicating a "blank"**

Notice that like Jeff's idea to begin a sequence with X01, Romina begins her first set of three towers with 10X. She then varies only the fourth position with the three possible colors. In the next set of three towers, she keeps the second, third, and fourth positions constant with 10X – she varies the first position among the three color choices. Emphasizing the three positions held constant in each set, Romina has boxed in the 10X in the first set of three and the 10X in the second set. In the last set, the first, third, and fourth positions are held constant while the second position varies. Romina's underlining of the variable position perhaps draws from Brian's "blank" suggestion. Of course, whereas the other sets of towers created possible solutions, two of the sequences in the last set of towers would not meet the criteria of including all three colors: 1X1X and 111X. Romina does not comment directly on this discrepancy at the moment – however, Romina does start to express some affective language as detailed in the next section.

5.2.3.3 Initial Exploration of 36: "It's confusing me"

For about a minute, the two sub-groups interact. Michael wonders, "What do you have?" to Brian, Romina, and Jeff (Ad283). Brian replies that they got "seventy-two" and then “fifty-four” as answers. Ankur says that he and Michael went from “seventy-two to fifty-four” and then now “to forty-five” (Ad286-288). Ankur asks again what Romina’s sub-group has as a solution and Brian responds “forty-eight” (Ad294). Jeff admits that “we don’t” have 48 as a solution, but rather Brian “just picked a number out of his head” (Ad295). The students all laugh together and then they continue working as sub-groups. Romina observes that “this is getting really confusing” (Ad300). She and Brian continue to write sequences of 1’s, 0’s and X’s, but Romina seems to express some frustration. She comments that “I keep on writing the same thing over and over again.” (Ad306). Romina wonders aloud, “Did we cancel out fifty-four as a possibility?” (Ad313). Ankur overhears her question from the other side of the table and says that it is not fifty-four. Brian asks Ankur and Michael if they “have a formula” and if they do not, how they are “going to prove that” (Ad317 – Ad320).

Brian, Romina, and Jeff continue to work together as a separate group from Michael and Ankur. When Brian tells Romina that he is “on thirty right now” (Ad322), Romina’s response implies some confusion as well as her new approach.

> “I don’t know how I got it. You have three or two of them put together…” (Ad323)

Notice that Romina again focuses on the reasoning behind her solution. She finds it necessary to mention “I don’t know how I got it.” The knowledge of “how” seems to be a priority. Romina comments on possible groupings in her current list – specifically, there are “three or two of them put together.” In Figure 5-9 below, one can see the lines Romina draws separating her tower sequences into pairs or triplets.

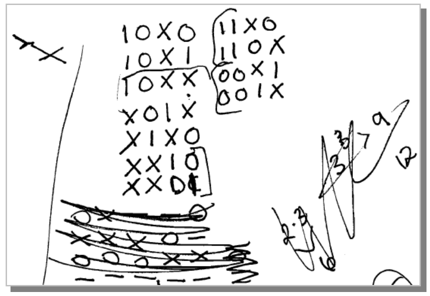


Figure 5-9. **Romina's list: "…You have three or two of them put together…" (Ad323)**

Romina pairs the following together: XX10 and XX01, 11X0 and 110X, and 00X1 and 001X. For all of these pairings, she has kept the first two tower positions constant and interchanged the color in the third and forth positions. For instance, "11" in the first two positions and then X in the third or X in the forth position to give 11X0 and 110X. The other five tower sequences are not as clear for her grouping strategy, however. Though there is a line beneath 10X0 and 10X1, here she has kept the first three positions constant and only the forth position varies among 0 and 1. The reasoning behind her triplet of 10XX, X01X and X1X0 is not apparent. Romina does not justify this categorization at all during her dialogue with the boys. One would thing that the 10XX would fit more with the 10X0 and 10X1 as the first three positions are constant and in common.

As they continue to write possible tower sequences, Romina, Brian, and Jeff discuss what the solution might be. Romina implies possible negative affect several times during this interchange.

BRIAN	That's thirty so far.
JEFF	Yeah, thirty.
ROMINA	And what is that one on the end? Double at the end?
JEFF	I don't get it now.
ROMINA	Are you boys done? I just don't want to do this.
JEFF	Um, X, X.

ROMINA	It's it's. I don't know why, it's just that I'm looking at it and it's confusing me.
JEFF	0, 0,…
BRIAN	Aren't we doing four high?
ROMINA	Yes.
JEFF	Wait, wait. I can't believe –
ROMINA	Dude, I've written the same thing ten times now.

(Ad329 – Ad342)

Brian and Jeff both state that the solution is "thirty" so far. Romina questions this, however. She asks about a specific tower on the list "that one on the end" and then asks if they "double at the end" – perhaps alluding to the doubles formed when two of the positions are held constant like her 11X0 and 110X earlier. Notice Romina makes three statements that refer to her emotional state. She asks if the boys are "done" and then says "I just don't want to do this." She continues that "I don't know why" and "it's confusing me." Then she comments that she has "written the same thing ten times now."

Next, while Jeff and Brian debate whether the answer would be 13 or 31, Romina writes a new list. Though she is mostly silent during this time as she records new sequences, one comment she makes hints at her new strategy. When she wonders aloud, "What happens when I change the 1's around?" (Ad350), one might conclude that her focus is now on the placement of the digit "1" in her sequences. Soon she announces to the boys, "All right. I came up with all the combinations" (Ad353). Notice the list of twenty-four tower sequences Romina has written in Figure 5-10 below.

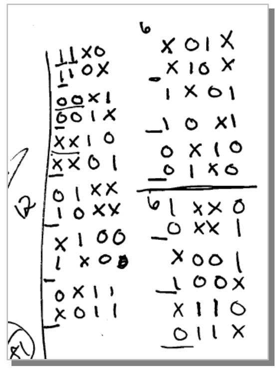


Figure 5-10. **Romina's expanded list: "All right. I came up with all the combinations." (Ad353)**

Though she has written two 6's and a 12 in the margins of her list, she seems to have employed a pairing strategy for the most part as evidenced by her small line marks after every two towers. Indeed 11X0 and 110X form a pair where the first two positions are held constant and the third and fourth positions vary among X and 0. Her next two pairings hold the first two positions constant with 00 and then XX. Her next three pairings involve holding the last two positions constant with a color and varying the first and second positions with the remaining two colors. Thus the pairs 01XX/10XX, X100/1X00, and 0X11/X011 are generated. In the next column, Romina seems to hold the first and forth positions constant and interchanges the second and third positions with the remaining colors: X01X/X10X, 1X01/10X1, and 0X10/01X0. Finally, the last six towers seem to be grouped by holding the middle two positions constant and varying the outer first and fourth positions: 1XX0/0XX1, X001/100X, and X110/011X.

5.2.3.4 Even versus Odd: Brian and Romina argue with Ankur and Jeff

Another intersection of dialogue between the two sub-groups occurs when Jeff asks Ankur and Brian about their progress. At the time, Jeff has 37 towers as a solution. He is surprised to learn that Ankur and Michael have 45 towers – Jeff comments that "there's no way it's forty-five" (463). Then Ankur explains that he and Michael took a different approach and found the "other ones" that do not meet the criteria. Romina and Brian begin a discussion of whether the solution should be even or odd when they overhear the last part of Jeff's conversation with Ankur and Michael.

JEFF	There's no way it's forty-five, there's too many.	
MICHAEL	That's what we are trying to figure out.	
JEFF	Cause now, I have thirty-seven right now. I have the same thing somewhere but I don't know where it is...	
ANKUR	If it's not forty-five, then it's probably forty-two. But, either one of those two…	
BRIAN	How could you have an odd though?	
ANKUR	We found thirty-nine other ones.	
ROMINA	But don't they have like pairs?	
JEFF	You found thirty-nine of these?	
BRIAN	Doesn't each one have a pair?	
ANKUR	No, like *[pause*].	
ROMINA	Yeah, doesn't, don't they have pairs?	(482 – 492)

Jeff repeats that there is "no way it's forty-five" because that would be "too many." He adds that currently he has "thirty-seven" as the answer. Ankur qualifies that "if it's not forty-five, then it's probably forty-two." At this time it is not entirely clear whether Jeff and Ankur realize that they are speaking about different problems. Jeff refers to the number of towers that meet Ankur's Challenge criteria – towers four-high with each of three colors represented. Ankur on the other hand has been trying to find the number of towers that do *not* meet these criteria – the complement to his problem. Brian and Romina overhear the last part of Jeff and Ankur's conversation. Brian asks how there could be an "odd" solution to Ankur's challenge problem. Ankur says that they found "thirty-nine other ones" – here the "other ones" are the towers that meet the criteria (not the complement). Romina questions his answer as well and wonders, "don't they have like pairs?" Here, the "pairs" perhaps refers to the pairing strategy she employed in her most recent list of towers where she arrived at 36 towers. Brian, who has been sitting next to and working with Romina, is familiar with the list of paired towers Romina has. Questioning whether he is talking about the complement to the problem or not, Jeff asks if Ankur found "thirty-nine of these." Brian follows up on Romina's question and asks whether each tower "has a pair." When Ankur replies that they do not, Romina asks again, "Don't they have pairs?" Romina and Brian seem to be questioning whether the solution to Ankur's Challenge could be an odd number. The fact that they heard both Jeff and Ankur mention odd numbers (39 and 45) seems to have caused their questioning.

Rather than directly addressing the issue of whether the solution should be an even or odd number, Ankur argues for why there should 81 total towers four-tall with a choice of three colors. The ensuing conversation reveals group dynamics.

JEFF	There's eighty-one total of these. You weren't listening.
ANKUR	And we found…
ROMINA	You weren't talking to me.
ANKUR	You butted in our conversation and then…
ROMINA	You have a conversation between yourselves.
ANKUR	I think they are calling you, Jeff.
ROMINA	Hold on. What, okay. Could you run the conversation by me one more time then?
JEFF	There's eighty-one total things you could have.
ROMINA	How did you get eighty-one?
ANKUR	Do it and you'll figure it out.
JEFF	The x times the y deal.
ROMINA	No, Ankur
JEFF	All right. X times the y. What is it?
ANKUR	The x times the y deal. Remember when we?

JEFF	Wait, wait. X is three? X was three.
ANKUR	It's three to the fourth. (503 – 518)

***Figure 5-11*. Romina and Ankur argue. Ankur: "You butted in our conversation…" (506)**

Jeff charges that Romina and Brian "weren't listening" and the "eighty-one" refers to the "total" of all the towers four-high. Romina replies that they "weren't talking to me." When Ankur then remarks that Romina "butted in our conversation" (see Figure 5-11) Romina observes that Ankur and Jeff were just having a "conversation between yourselves." After the more emotionally-charged back and forth, Romina then asks a question that returns to the problem. Seeking clarification, she asks Ankur and Jeff if they could "run the conversation by me one more time." Jeff reiterates that there are "eighty-one total" towers." When Romina asks "how did you get eighty-one," Ankur makes a quick retort that she should just "do it." Jeff, however, suggests she use the "x to the y deal" referring most likely to the exponential expression x^y. Agreeing with Jeff, Ankur continues that they use the "x times the y deal" and in this case "it's three to the forth" in order to get eighty-one total towers.

Ankur further clarifies that his strategy with Michael has been to find the complement of his original problem. Romina asks them to restate the original problem.

ANKUR	So instead of finding all the ones that we can use one of each color we found the other ones.
JEFF	Because if she just said that just find all of the ones you can do four towers, three high, we would have been done three hours ago.
ROMINA	Okay, so that's not the problem. So what is the problem?
JEFF	The problem is how many using one in each slot, using, well you have to use all three colors.
ANKUR	We just used eighty-one to try to help us to find the other one, the other side. Do you know what I mean?
ANKUR	Instead of one of each color, we just found, we were just trying to figure out like not ones with each other.
ROMINA	See that is where I misunderstood you. I thought you meant eighty-one of these things. (542 – 549)

Ankur describes his strategy of finding the complement to the set of towers four-high with each of three colors represented as finding the "other ones" – that is, finding the set of towers four-high *without* all three colors represented. Romina asks for clarification then as to "what is the problem?" Jeff reiterates Ankur's original problem as to find "how many using one in each slot" when "you have to use all three colors." Ankur explains that he and Michael used the eighty-one total towers to "help us to find the other one, the other side" – here, Ankur again tries to define his strategy of the complementary set. Ankur elaborates on his definition of the complement as "trying to figure out like not ones with each color." Romina then expresses that she "misunderstood you" before because she thought they meant "eighty-one of these things" – the solution to the challenge problem to which she has found thirty-six.

5.2.3.5 Four Iterations of Romina's Solution to Ankur's Challenge

Over the course of the next forty minutes, Romina presents arguments for why thirty-six would be the solution for Ankur's Challenge. First, she puts forward her reasoning of "sixes" to T/R1 and Brian. Then she tries to convince Jeff and Brian. Next, she works to convince Ankur. Finally, she presents her solution at the board for Michael. In the four iterations of her argument, her representations and justifications evolve in detail and structure. Each of Romina's four presentations will be detailed below under the heading of "solution version" – while Romina's numerical answer remains constant at thirty-six, the means through which she convinces her audience varies.

5.2.3.5.1 SOLUTION VERSION 1 - Romina Presents to Brian and T/R1

Before Romina's first explanation, T/R1 asks all of the students what "ideas" they are pursuing and if there are any they would like to "share with each other" (594). Jeff mentions that he is "at thirty-seven" but he thinks he has the "same one somewhere" in which case "it will be thirty-six"(595). Jeff then turns away from Jeff and Romina and begins to discuss his work with Ankur. Romina says to T/R1 and Brian that she thinks it "might be thirty-six" because she is "working with sixes now":

ROMINA	It might be thirty-six, cause I'm working with sixes now. I mean. Okay. You put them. You pair them up. 'Cause you're only going to have. Okay. [*Holds up her hands*] Let me organize my thoughts a little. You can have 'em together, together, like here these are together, these are together, these are together. Like two of the same color like in a pattern and then you put them somewhere and you like switch them around. So, I'm up to twenty-four now and I'm going to put them the same way here and here. (*Referring to her paper*). So that's thirty. I'm going to put the same one here, here, um.
BRIAN	Can you do that maybe right there? (*pointing to paper*)
ROMINA	Here, here, that's one. Here, here. I didn't put them yet. And there's your thirty-six 'cause one, two, thirty-six. Right? (*See figure* **) That's four and that's six, and then (*makes more marks on the paper and counts lines*) How did I just get thirty-six again? Okay, we know that this is… Six, six, six. Okay, that's thirty. Oh, no, you guys for thirty-six?
BRIAN	You had the way right there…
ROMINA	Oh, that's not it then. [*Crosses off her 36 answer on the paper*] Hold on.

(Ad359 – Ad363)

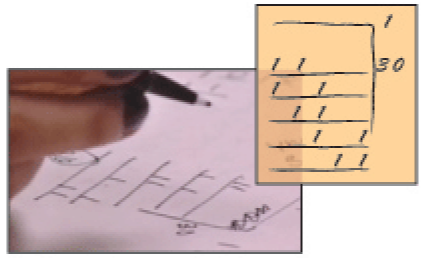


Figure 5-12. **Romina's written solution representation for Brian and T/R1 – Romina: "…there's your thirty-six" (Ad361)**

Here, when Romina first presents her reasoning for why the solution "might be thirty-six," she references how "you pair them up" – linking her work here with what we saw earlier in Romina's private work of tower pairs. Romina holds up her hands in the middle of her explanation however and requests that they "let me organize my thoughts a little." Then Romina starts again and attempts to explain the reasoning behind her pairs. Her language is general and vague as she points to her paper: "these are together, these are together, these are together." Romina's definition of how she grouped tower "together" lacks specific detail: "together" seems to mean when you have "two of the same color like in a pattern" and then "you put them somewhere and like switch them around." Romina does not articulate what her "pattern" is, but does continue to reference her earlier list of paired towers. Then Romina makes lines on her paper and puts a pair of the digits "1 1" in different positions. For instance, on the first line, she puts 1 1 in the first and second positions. On the second line, she places 1 1 in the first and third positions with a blank space in the remaining positions. Romina writes this on her paper but does not explain her method at this time other than to say "I'm going to put them the same way here and here." Romina repeatedly refers to what she is writing as she says "here, here" and again "here, here." At one point, Brian points to her paper and seems to a new combination she could include "that one." As she finishes writing, Romina states, "there's your thirty-six" (see Figure 5-12), but then seems to lose track of her counting. She recounts in groups of six. At this time she has only five lines on her paper and each line she seems to refer to as a "six." She recounts, "six, six, six – Okay, that's thirty" and then wonders where her "thirty-six" is. She crosses out thirty-six on her paper and states "that's not it." Brian observes that she had the "right way" however. Romina and Brian continue to look at her paper as T/R1 turns to speak to Ankur and Michael.

5.2.3.5.2 SOLUTION VERSION 2 - Romina presents to Jeff and Brian

When T/R1 turns to speak to Ankur and Michael at their end of the table, Jeff returns to join Romina and Brian. Meanwhile, Romina takes another clean sheet of paper and redraws her lines. Romina explains to Jeff and Brian as she writes on the new paper:

ROMINA	[*Gets a new sheet of paper. Explains to Jeff and Brian*] First and third. Okay. Thirty-six. Ready? The way we did it. You've got two of the same color, right? Two of the same color which stands for where I'm putting these and these. You're going to have 'em in. And the rest you fill up, right? And you going to have 'em in. And there's only two other ones that you could have. So you have this one which you're going to multiply by two. Hold on. One, two, three, four, five, six.
JEFF	So that's only twelve.
ROMINA	Okay. No, you multiply this by two, and this by two. Multiply this by two and this by two. By two, and by two. Then, how much is that? One, two, three, four, five, six. (Ad365 – Ad367)

Romina draws six horizontal lines on the paper and places two one-digits "1 1" on each of the horizontal lines. Earlier when she was describing this representation to Brian and T/R1 she only had five horizontal lines and thus became confused when she only had a total of thirty towers (five lines time six combinations each). Notice in Figure 5-13 she has now included six horizontal lines.

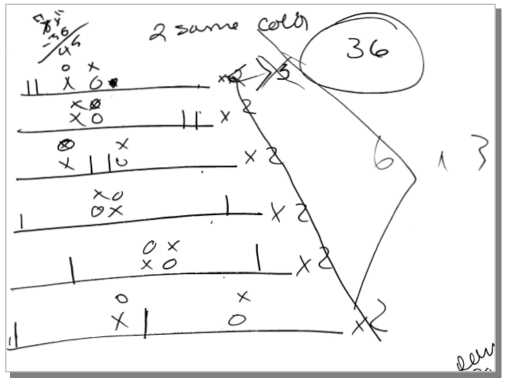


***Figure 5-13*. Romina's written solution representation for Jeff and Brian**

She finishes including the line she was missing in the earlier representation – where the 1s digits are in the "first and third" position – and says out loud that there are "thirty-six." She asks Jeff and Brian if they are "ready" for her to explain. She explains that, in order to meet Ankur's criteria, each tower much have "two of the same color." The 1's digits represent the repeated color in her tower. As Romina explains it, "two of the same color stand for where I'm putting these and these" – the 1's digits on each horizontal line. Then "you fill up" the other horizontal lines with the other ways you could arrange two 1's digits in four positions. As Romina has represented it, the six combinations are: 1's in the first and second positions, 1's in the third and fourth positions, 1's in the second and third positions, 1's in the first and fourth positions, 1's in the second and fourth positions, and 1's in the first and third positions. Romina recounts that she has six horizontal lines for each of the six positional arrangements: "one, two, three, four, five, six." When Jeff says that would make a total of "twelve" towers; Romina states that, for each horizontal line, they would need to "multiply this by two." She asks "how much" that would be as she records a "x 2" by each of her horizontal lines. Jeff continues to question her representation, however, and asks her to "hold on."

As a result of Jeff's continued comments and questions about multiplying by two, Romina refines her argument and includes more calculation on her paper:

ROMINA	Okay, guys. One, two, three, four, five, six, right? (*inaudible*) For each one here, you have six other combinations. You have two for this one
JEFF	That's why we multiply by two.
ROMINA	You multiply by two, and then you multiply this by two.
BRIAN	Multiply by two. Multiply all of them by two not the whole thing by three.
JEFF	The whole thing by three by three you're saying?
ROMINA	Yeah.
BRIAN	'Cause each one has three.
ROMINA	You want to make a neater one? (Ad373 – Ad380)

Romina recounts the lines on her paper – "one, two, three, four, five, six." Then she explains that "for each one here, you have six other combinations." It seems that she indicates that she is multiplying the six horizontal line towers by six. For each line like the two 1s in the first and second positions, there would be "six other combinations" if the 1's could represent any of the three colors and the other blank positions were filled with the remaining two colors. For the "two" blank spots in each horizontal line, Romina writes an X0 or 0X to represent the two remaining colors. Romina and Jeff both remark that this is "why we multiply by two." Brian clarifies that they should "multiply all of them by two" as opposed to each one individually. Then Jeff asks if they should multiply "the whole thing by three." Romina agrees and on her paper she writes "x3" (see Figure 5-13). The reasoning for why they should multiply by three is given by Brian as "cause each one has three." One can infer that he means there is a choice of three colors for each tower arrangement, but neither Brian nor Romina have yet fully articulated their meaning behind the calculation 6 x 2 x 3 = 36 on the paper. Romina asks if they should make a "neater" written representation of the solution. When the boys do not directly answer her, Romina states that "I'm making a neater copy for her" (Ad387). As Romina begins writing, Jeff mentions that "you multiply by three 'cause there are three different colors" (Ad388).

A minute elapses as Romina writes a new copy. Then Jeff asks how they could "justify" their answer even more and a discussion of the complement ensues:

JEFF	So how do we justify this even more? Um, this, we have thirty-six of these, right? That means that there's fifty…	
BRIAN	Forty-six	
JEFF	Forty-eight. No, forty-one. It's forty-one.	
BRIAN	Four colors?	
ROMINA	I don't know if it's going to be –	
JEFF	It's eighty-one.	
ROMINA	It has to be eighty-one, but –	
JEFF	But that means that forty –	
ROMINA	What are you doing? Eighty-one minus thirty-six?	
BRIAN	Of what?	
JEFF	Of not with, with no requirements.	(Ad403 – Ad414)

Given that they have "thirty-six" towers for Ankur's problem, Jeff asks "how do we just this even more?" Jeff and Brian calculate the difference between 81 and 36 to find the complement of the problem. Though the difference should be 45, both Jeff and Brian make calculation errors as Brian says "forty-six" and Jeff arrives at "forty-eight" and then "forty-one." As the boys mention numbers, Romina says that "I don't know." Then Jeff states that the total number of towers is "eighty-one." When he does not explain his numbers, Romina asks directly, "what are you doing?" She questions if they are calculating the difference, "eighty-one minus thirty-six?" Jeff says this difference would give the number of towers "with no requirements." Romina records the difference 81 –

36 = 45 in the top right corner of her paper (see Figure 5-13) and mentions to the boys that there are “forty-five” and it is an “odd number” (Ad424). Before the students can explore the complement further however, Romina tries to get the attention of Ankur and Michael at the other end of the table. She tells them that “we have an explanation for you… we figured out thirty-six” (Ad430).

5.2.3.5.3 SOLUTION VERSION 3 – Romina presents to Ankur and Michael

After Romina has asked for Ankur’s attention, it takes another two minutes until she can begin to explain. Ankur and Michael continue their own discussion until Jeff interrupts them and says, “Pay attention… ‘cause you’re gonna think it’s thirty-six when we’re done” (703). Again Jeff calls to them to “just pay attention” (707). Romina then begins her explanation from a new paper on which she has written a representation with boxed-in towers as opposed to lines (see Figure 5-14). She narrates how she organized the towers on her paper:

ROMINA	So you have to organize them so they... so that you don't have any doubles. So either you can have them next to each other. You can have them separated by one. You have them on the ends, in the middle, two and fourth spot, and third and fourth spot. Right?
ANKUR	Yes.
ROMINA	So that’s six.
ANKUR	Yes.
ROMINA	Okay. Now you, in the other spots you can have an 0 and an X. Those are colors. Like these are three different colors – an 0 and an X and an X and an 0. Right?
ANKUR	Mhm.
ROMINA	So you have to multiply each of these six by two. (718 – 724)

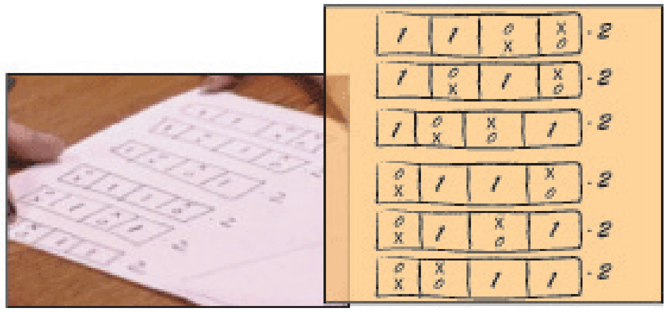


***Figure 5-14*. Romina's representation for Ankur and Michael**

Romina first refers to the placement of the 1's digits in each of the six arrangements given on her paper. She explains that are six combinations for where to place the two 1's given four possible positions: "next to each other," "separated by one," "on the ends," "in the middle," in the "two and fourth spot," or in the "third and fourth spot." Notice here Romina has become much more explicit about how she is arranging her towers. She goes on to describe how "in the other spots you can have an 0 and an X." She refers to the "0 and an X and an X and an 0" that she has written in the boxes. She defines the 0 and X as "colors" and mentions again that the 1, 0, and X represent the "three different colors." Finally she instructs that then they would have to "multiply each of these by two."

The students probe the reasoning further behind the multiplication by two and a more refined argument for a solution of 36 emerges through discussion:

ANKUR	Okay. Hold up. I just want to think about it for a second.
ROMINA	Six times two, twelve; six times two, twelve; six times two, twelve; six times two, twelve; six times two, twelve; six times two, twelve.
ANKUR	Yeah, now when you add them...
JEFF	Why do you keep saying six times two?
MICHAEL	You get thirty-six for the ones without...
JEFF	Why do you keep crossing that out?
ROMINA	'Cause it's wrong! No, you multiply all this by two. Right? And then you multiply all that by three, because of the three different colors.
JEFF	Yeah, yeah, no.
ROMINA	So that is what we were trying to say but we wrote it bad.

JEFF	We were saying that, but she wrote it funny.
ANKUR	Okay.
ROMINA	Okay, so you can multiply these all by two, right? Because you have one color or the other.
ANKUR	An O or an X or an X or an O.
ROMINA	Right? Then you have to multiply all by three because the ones can be any colors.
JEFF	And then you could switch the numbers around, x's and then you could bring...
ROMINA	It could be the three colors.
JEFF	There's like... there's twelve this way. And there would be twelve if you took the x's put them here. And took the one's and put them there, that's twelve more. And there's twelves more if you took the zeros and put them here and put the x's back over there with the ones.
ROMINA	So it's thirty-six. (730 – 747)

While Ankur asks if she can "hold up" so he can "think about it for a second," Romina continues to count multiply each of her horizontal tower bars by two. She repeats "six times two, twelve" six times as she writes "*2" next to each horizontal tower - see Figure 5-15 below. Jeff interrupts and asks Romina why she keeps saying "six times two."

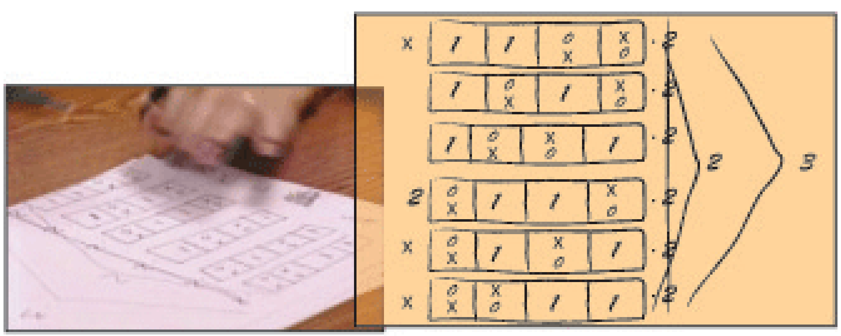


Figure 5-15. Romin'as revised solution representation - Romina: "'Cause it's wrong! No, you multiply all this by two. Right?" (736)

Romina crosses out the six separate "*2" on her paper and Jeff wonders why she is "crossing that out." Romina explains that "it's wrong" and they have to "multiply all this by two" and then multiply that result "by three because of the three different colors." See

Figure 5-15 above where Romina has drawn lines through the six separate written multiplications by 2 – now she has one large "*2" in a bracket for all the towers nested within a larger bracket with "*3." Romina indicates the initial error on her paper as "that is what we were trying to say but we wrote it bad." Continuing, Romina reiterates that "you have to multiply all by three." She more carefully articulates the use of the 1's digits in her representation as "the ones can be any colors." Jeff interjects that there would be twelve possibilities when you multiply the six tower sequences by the two possibilities for the other colors represented by X and 0. He describes how the X's and 0's can be switched "back over there." Romina summarizes, "so it's thirty-six."

5.2.3.5.4 Michael's Challenge: "Proof the Other Way Around"

Rather than accepting Romina's argument, Michael insists on a "proof the other way around" (see Figure 5-16). Since Michael and Ankur have been working on a systematic list of all the tower sequences that do *not* meet the requirements, Michael challenges Romina, Jeff, and Brian to justify what the complement would be to their set of thirty-six.

MICHAEL	No, I want proof the other way around. For that there's, 'cause that's what we did.
JEFF	They're forty-five.
ANKUR	We proved the other side.
MICHAEL	We came up with seventy-two. Okay, then we just, if you were right, then eighty-one minus seventy-two that is only nine.
JEFF	Yeah, that's what I'm saying. So we could be wrong. That's what I was starting to do there.
MICHAEL	I want you to prove
JEFF	The other one.
MICHAEL	The other one.

ANKUR The only way you could prove that you were right is to prove the other side.

MICHAEL We proved the other one. But I don't. That's not enough for me. I want to prove the other. (794 – 803)

***Figure 5-16*. Michael's Challenge: "No, I want proof the other way around." (794)**

Throughout this discussion, Michael and Ankur refer to the complement of the problem as the "other way around" and the "other side," respectively. Michael initially challenges the others to "prove the other way around" because "that's what we did." When Jeff replies that the complementary set would have "forty-five" towers, Ankur says that he and Michael "proved" this. Michael explains that he and Ankur got "seventy-two" towers in the complement and thus he believes there could be a flaw in Romina's answer of thirty-six because "eighty-one minus seventy-two is only nine." Notice that Michael uses "eighty-one" towers as the union of both the set that meets Ankur's requirements and the complementary set. Jeff remarks that "we could be wrong." Michael reiterates his challenge: "I want you to prove… the other one." Ankur elaborates that the "only way" Romina's group could "prove that you were right is to prove the other side." For Ankur and Michael then, justification for this problem involves establishing the existence of both the set and its complement. Again, Michael challenges them and says that it is "not enough for me" just to "prove" the existence of one set – instead, he explains, "I want to prove the other."

5.2.3.5.5 Romina Argues with Ankur for 84 Total Towers

After Michael challenges the students to "prove the other" complementary set, they begin to discuss the possible numbers of elements in each set. Meanwhile, Romina writes on her sheet and then announces to the others that she has 84 total towers four tall with a choice of three colors:

ROMINA	You guys, I got eighty-four.
MICHAEL	Eighty-four what, total?
ROMINA	Total.
JEFF	And wait, what was your number?
ROMINA	Hold on. But I got, you guys, it makes sense.
JEFF	What was your number?
ANKUR	We had eighty-one total. (829 – 835)

Romina tells them that she "got eighty-four" and when asked to be more specific as to what the 84 refers, she clarifies that it is the "total.' She tells the others to "hold on" and that her number "makes sense." Ankur disagrees however, and says that they "had eighty-one total" instead. Whether the total number of towers four-high with a choice of three colors is 81 or 84 then becomes an issue as both groups were using this total in their calculations of differences between sets and the complementary sets.

An argument follows between Ankur and Romina as they each try to present their reasoning for the total number of towers. Ankur ends up explaining his reasoning first as to why there are 81 total towers four high with a choice of three colors.

ROMINA	You guys, you guys you know how we have our x to the y system? Oh, I'm just talking to myself.

JEFF	No, we were –
ANKUR	Can I tell you right now why it's not eighty-four?
ROMINA	Hold on. Can I tell you why it could be eighty-four?
ANKUR	Can I tell you first?
ROMINA	No, I don't want to. No. Okay, go ahead.
ANKUR	Cause look –
BRIAN	We don't have to have a brawl like we do in history.
ANKUR	There's four spots, right? So for the first one, there's three colors...
ROMINA	But Ankur, do you agree, hold on, do you agree that your other thing works?
ANKUR	Just cut me off.
ROMINA	Ankur, I'm just doing your other thing.
BRIAN	Where do you think you've been for the last sixteen years of our life?
ROMINA	I know.
ANKUR	She's like, "Okay, I'll let you explain." I start to explain.
ROMINA	Okay, go, go, go.
JEFF	Go.
ANKUR	There could be three colors for the first one, three colors for the second one, three colors for the third one, three colors for the fourth one. Right?
ROMINA	Yes.
ANKUR	Multiply them and you get eighty-one. Now there's no way there can be eighty-four now. (843 – 862)

Notice the dynamic between Romina and Ankur as they both press for their point of view to be heard. Romina begins by asking "you guys" if they remember "our x to the y system" – with the use of the pronoun "our" Romina defines the exponential model for towers as shared, group knowledge. When no one responds, Romina comments, "Oh, I'm just talking to myself." Ankur and Romina go back and forth for a minute. Ankur asks, "Can I tell you right now why it's not eighty-four?" Romina immediately tells him to "hold on" and asks, "Can I tell you why it could be eighty-four?" Ankur requests that he get to "tell you first." Brian observes the quick verbal exchange and cautions that "we don't want to have a brawl like we do in history" – perhaps implying that debates are commonplace between this group of students in history class. Ankur starts his explanation by saying there are “four spots” on the tower and for the first spot “there’s three colors.” Before Ankur can finish, however, Romina interrupts with a question about whether he agrees that “your other thing works.” Ankur characterizes Romina’s interruption as “just cut me off.” He muses out loud about Romina’s questioning and how he was starting to explain. Romina tells him to “go, go, go” and then remains silent as Ankur proceeds. Ankur explains that there could be “three colors” for each position on the tower: “for the first one… for the second one, … for the third one, … for the fourth one.” He instructs her to “multiply them and you get eighty-one” since three times three times three times three is eighty-one. Ankur concludes that “now there’s no way it could be eighty-four.”

After Ankur presents his justification for why there would be 81 total towers four-tall with a choice of three colors, Romina counters with her reasoning for 84 total:

ROMINA	But it works. It just works. I don’t know why.
ANKUR	What do you mean, it just works?
ROMINA	Hold on. Look at. I am not saying that I am right. I'm not saying that I'm right.
JEFF	You prove what you thought. Prove what you think.
ROMINA	Okay, you have the thirty-six.
ANKUR	Thirty-six what? What do you have?
JEFF	Uh-hum.
ROMINA	[*Writing on paper*]. And then you're going... yeah, and then you are going with the x, y deal, right? And say you can't work 'em in all at the same time so you figure one of them might be dropped. Cause that's what we did we worked them in all at the same time and one of them has got to be dropped the other ways we do it. So then it would be two to the fourth because there's two colors, right? And for each one you have to multiply that...
ANKUR	What’s the fourth one?
ROMINA	That's how high it is. That is like your x to the y system. And that equals sixteen. And then there's colors. (866 – 877)

As for 84 total towers, Romina begins by insisting that “it works” but that “I don’t know why.” Romina then qualifies her earlier assertion by twice repeating that she is “not saying that I am right.” When Jeff tells her to “prove what you thought,” Romina begins with the “thirty-six” towers she found earlier that meet the requirements of four-tall with each of three colors represented. After alluding to their shared vocabulary of “the x, y deal” with exponential notation, she defines this set of thirty-six as having “worked them in all at the same time” where “them” refers to the three colors. The complementary set, in contrast, “can’t work ‘em in all at the same time” so one color “might be dropped.” She continues with her description of the complement as the set in which one of the colors “has got to be dropped the other ways we do it.” Then, Romina concludes that “it would be two to the fourth because there’s two colors.” Ankur asks why she would take the “fourth one” and Romina responds that she is taking the fourth power because “that’s how high” the tower is. Again she references their shared vocabulary for exponential notation for the different numbers of towers by saying it is “like your x to the y system” (see Figure 5-17) – however, notice now she has switched from the pronoun “our” to “your.” She writes $2^4 = 16$ on her paper and tells Ankur “that equals sixteen.”

Figure 5-17. Romina argues for 84 Total: “…That is like your x to the y system” (877)

Ankur agrees that two to the fourth would be sixteen, but wonders why she is multiplying this power by three. Romina continues her argument for 84 towers, but a paradox emerges as the students wonder how there could be both 81 and 84 total towers at the same time for their problem:

ROMINA	Okay and then you multiply that by three. I was getting, okay.
ANKUR	Why by three?
ROMINA	Because three different colors. Right? So one of them is going to be dropped out one time, and then the other one and then the other one. So that's three. So what's that? I didn't do this...
ANKUR	Sixty-four.
ROMINA	Sixty-four
ANKUR	No.
JEFF	Six times three is eighteen –
ANKUR	Forty-eight.
JEFF	Carry the one
ROMINA	And add that to the thirty-six. Eighty-four.
ANKUR	But we've used this method all the time.
ROMINA	Well, I'm just saying that could be –
BRIAN	So things are subject to change over a lifetime.
ROMINA	But I could, I could
JEFF	So we are saying that we have to go back and reprove all of the other problems that we did because we did this wrong?
ROMINA	I could be completely wrong, you guys. Chill out, I could be completely wrong. I'm just saying, couldn't work like that?

(884 – 899)

When Romina describes multiplying sixteen by three, Ankur asks Romina "why by three." Romina explains that it is "because three different colors." Then she goes on to say that "one of them is going to be dropped out one time." Given the "colors" 1, 0, X (using Romina's notation), her explanation implies that she is finding the total number of towers four high with a choice of two colors at a time – three choose two would be three possible combinations 1 with 0, 1 with X, and 0 with X. Romina asks what the product would be of sixteen times three. At first, Ankur replies “sixty-four,” but then he corrects himself by saying “no” and the true product of “forty-eight.” Romina states that if they find the sum of forty-eight and thirty-six, then the result will be 84: “add that to the thirty-six. Eighty-four.” Ankur expresses doubt and refers back to his 81 where he employed the “method” they had used “all the time” in the longitudinal study. Brian remarks that “things are subject to change over a lifetime.” Jeff reacts more forcefully and wonders if Romina’s 84 answer means that “we have to go back and reprove all of the other problems” – implying that the three to the fourth argument that Ankur gave has been an accepted method of substantiating the total number of towers in the past. Romina tells the others to “chill out” – twice she repeats that she “could be completely wrong.” She again asks them to consider whether her reasoning could “work” however.

As Romina continues writing, she makes an observation that she could subtract the duplicate towers (that she terms “doubles”) and resolves the seeming paradox of how her answer of 84 could coexist with Ankur’s 81.

ROMINA	There’s three doubles in there.
JEFF	Yeah, and those are the three of each one. That double, that double, and that double.
ROMINA	Out of this? So there's three doubles in there. So then there's eighty-one, there.
JEFF	What are you saying?
ROMINA	And then there's these doubles, because those go over there?
JEFF	No, we are counting these as the three doubles that you just subtracted? Not just any –
ANKUR	What number does that leave us with? [*laughing*]
ROMINA	I’m just saying.
JEFF	Eighty-one. (902 – 910)

Romina writes down the sequences 1111, 0000, and XXXX of all the same color (see Figure 5-18 below) and then observes that within her earlier calculation of 16 x 3 there are duplicates.

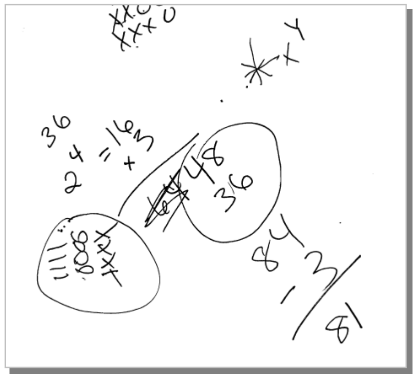


Figure 5-18. Romina's work for the total number of towers 4-high with a choice of 3 colors

In reference to the product 48, she remarks that "there's three doubles in there." Here, "doubles" means a tower that was double-counted or duplicated inadvertently. Indeed, the 24 towers four-high using the colors 1 with 0 would include the sequence 0000 as would the 24 towers four-high using the colors 0 with X. Thus, each sequence of all the same color (0000, 1111, and XXXX) would be duplicated when Romina multiplied 24 towers by three. Jeff agrees and points to each of the single color towers "that double, that double, and that double." Romina continues that if "there's three doubles in there," then the final answer would be "eight-one." On her paper, Romina subtracts 84 – 3 and gets 81 (see Figure 5-18 above). Jeff asks her to specify what she is "saying." Ankur asks Romina "what number" that would "leave us with" – laughing as Jeff remarks it is now down to the "eighty-one" for which Ankur had originally argued.

Once Romina resolves the issue of 84 versus 81 for herself, Ankur asks Romina to re-explain why she originally used two to the fourth in her reasoning. Romina explains that whereas she, Jeff, and Brian "did the one where we have all three" of the colors included in the set, for the complement, you would "drop" one or two colors to get sequences like "X, X, 0, 0" or "X, X, X, 0" (940). She reiterates that when you multiply by three, "that's probably where the doubles would be" (948) – i.e., the duplication of the single color towers. Romina points out that she has now corrected for the double-counting because "that's why I subtracted three" (953). Ankur agrees "you get eighty-one which is the same" as his answer of eighty-one – he characterizes Romina's work as having made a "mistake" (954).

5.2.3.5.6 SOLUTION VERSION 4- Romina presents at the chalkboard

Alternately, the students share their strategies at the board for the original towers problem and then Ankur's Challenge problem. Michael then asks Romina to re-explain where she got the answer of 36 for Ankur'r problem. He admits that hew was not really paying attention when she explained her solution earlier to Ankur:

> "How about this – explain the thirty-six one more time because I was not paying attention." (1137)

Romina goes up to the chalkboard to explain her solution of thirty-six for Michael. Her explanation takes a little under two minutes.

ROMINA	Okay. So what we did, we, well, let's say these are your different ones. *[Someone sneezes*] And we came up with six different like possibilities for like the, the match it could be. It would be here and here, the same. Here and here. Here and here. Come on. Which one am I missing?
ANKUR	The second.
ROMINA	Okay.
ANKUR	And the last.
ROMINA	Yeah, the second one and the last. Okay. Do you agree with me? And then each one, this is either going to be an O or an X.
BRIAN	Or an X.

ANKUR	Or an X or an O. So each one, there's two of each one. You can't have X and X.
MICHAEL	Yeah. I get that.
ROMINA	You get that?
MICHAEL	Yeah.
ROMINA	So should I…?
MICHAEL	What are you doing?
ROMINA	I'm writing.
MICHAEL	No. I was talking to Brian.
ROMINA	Oh. Okay. So so far we have six. And then we have to multiply the six by the two for all of these so you get twelve. Right? And multiply the twelve times the three to get thirty-six. You multiply it because it's three different colors.
MICHAEL	Yeah. The one's can be any color.
ROMINA	So each one here can be three.
BRIAN	Yeah.
ROMINA	Yeah. So you multiply that to get thirty-six.
MICHAEL	Okay. (1159 – 1178)

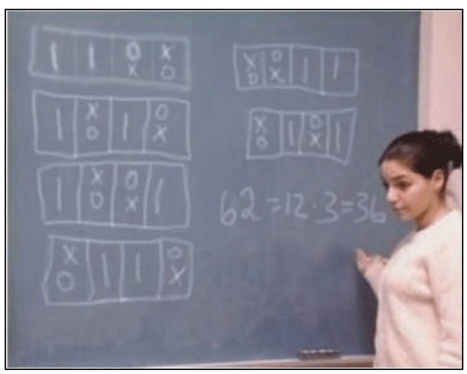


Figure 5-19. Romina at the board: "So you multiply that to get thirty-six." (1177)

Romina begins her justification by drawing out the "six different like possibilities" by varying the position of the two 1's digits among the four boxes she creates for each horizontally represented tower. She begins by placing the 1's in the same way she did on paper in her third solution version, however at the last two towers she makes a change and ends up missing a sequence. It is interesting to note that Romina does not place the 1's digits in exactly the same system for any of her solution representations. Here, her fifth tower has the 1's in the third and fourth positions (in earlier representations she had this particular sequence alternately as her second tower, third tower, or last tower). As Romina writes the 1's in the positions, she narrates with a repeated "here and here." Then she stops when at five towers and asks, "Which one am I missing?" Ankur observes that she is missing the sequence in which the 1's digits are in the "second" and "last" positions. Romina agrees and writes 1's in "the second and the last" positions for a tower possibility. She asks Michael if he "agrees" with her thus far. When he does not answer, she continues by writing X over 0 or 0 over X in each of the remaining blank position boxes. She explains that "there's going to be two of each one" and cautions that they must be different because "you can't have X and X." Michael says that he gets her and Romina asks again, "You get that?" Romina then summarizes with an algorithmic calculation whereby you "multiply the six by two for all these" to get twelve and then "multiply the twelve times the three to get thirty-six." On the board, Romina writes $6*2=12*3=36$ (see Figure 5-19). She explains that the reason for the last step of multiplying by three is "because it's three different colors." Michael asks for verification that the "ones" digits represent a cube of "any color." Romina clarifies that the 1's digits can take on any of the three colors – "each one here can be three." She repeats that you multiply to get the solution of thirty-six. Michael agrees by saying "okay," and Romina returns to her seat.

5.2.3.6 Romina Records for Michael and Ankur

After Romina presents her argument for thirty-six, T/R1 requests that Michael and Ankur now shares their reasoning for why the complement to the problem would be forty-five. With reference to what Romina has written on the board, Ankur laughs that “it is not as simple as that” (1180). Neither Ankur nor Michael volunteers to go to the board. Michael continues to write on his paper and Ankur announces that “I don’t like writing” (1191). T/R1 suggests then that perhaps someone else might write it for them. Romina volunteers to be the recorder for Ankur and Michael: “I’ll write it for you. So you don’t have to write” (1194). At first Ankur says that “I decline” Romina’s offer, but then a minute later he takes her up on the offer:

ANKUR	All right, Romina?
ROMINA	Yes?
ANKUR	Could you write something for me please?
ROMINA	Yes. That exactly without changing a thing? [She *goes to the board*] (1213 – 1216)

Figure 5-20. Romina records for Ankur: “Write whatever I read. Write going down.” (1226)

Romina spends the rest of the student presentation section of this problem-solving session as the recorder for Ankur and Michael. Ankur requests that she write “exactly without changing a thing” of what he dictates (see Figure 5-20). He and Michael proceed to share their representation of the towers using the three colors red, blue, and yellow as the digits 1, 2, and 3, respectively. As Ankur defines it, they use the digit 0 to represent “any one of the three except the one that is present” (1281). Thus, the first set of three towers that

Ankur requests Romina record on the board is: 1110, 2220, and 3330. Then the digit 0 is moved to the next position as 1101, 2202, and 3303. Romina continues writing tower sequences as dictated by Ankur and Michael (see Figure 5-21).

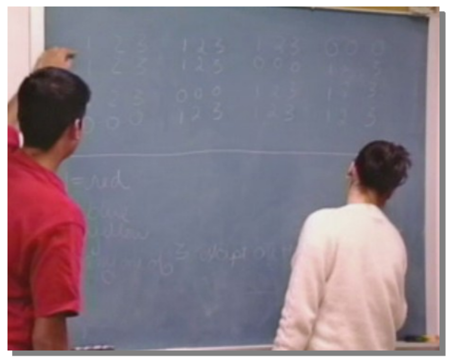


Figure 5-21. Romina continues to record for Ankur: "Any one of the three..." (1281)

Finally Romina has recorded four sets of three on the board. Ankur states that this would represent "twenty-four" towers (1290). Romina asks him "why" it would be twenty-four and Ankur explains that you would multiply each of these twelve by two because the 0 digit will represent one of two colors for each tower. Then he adds the single color towers "red, red, red, red; yellow, yellow, yellow, yellow; and blue, blue, blue, blue" to get "twenty-seven" (1295). He tells Romina to erase those twenty-seven towers and then write another group of towers (see Figure 5-22). There are eighteen towers he dictates for her to write – he describes these towers as "two of one color and two of another color" (1306). Ankur summarizes that "twenty-seven plus eighteen equals forty-five" (1303).

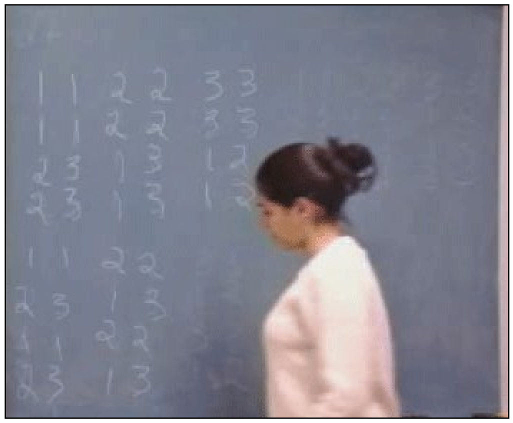


Figure 5-22. Romina finishes following Ankur's directions at the board: "These are two of one color and two of another color." (1306)

After Ankur finishes dictating and Romina finishes writing, T/R1 asks if the students are "convinced" that they have "all possibilities" for the complement to this problem. Romina agrees that she is convinced by commenting that this argument is "an extended version of what we did" (1311). T/R1 asks her to explain what she means. Romina elaborates on her observation of Ankur and Michael's argument:

> Because we didn't actually go through, we just went on the math. Which, that's why I said we could have been wrong because we didn't actually go through them. This one showed us every single possibility. Very good. So we're sure. (1312)

Romina contrasts her group's justification with Ankur and Michael's. Notice that she does not characterize her previous solution presentation as singularly her own – she defines it as what "we" did. She describes her group as having "just went on the math" as opposed to going through "every single possibility" as Ankur and Michael did in their systematic list. Romina remarks that her group "could have been wrong because we didn't actually go through them." She calls what Ankur and Michael did "very good" and "we're sure" that they are right. Observe that even at the end of this session after she has presented at least four different ways, Romina still seems to believe that her justification could have been "wrong" because it did not involve exhaustively listing each of the thirty-six possibilities. T/R1 brings to the students' attention that Ankur and Michael did use a variable in their notation (and did not, in fact, exhaustively list every possibility). She asks Ankur and Michael about where "your notion of variable got you in trouble" (1319). The session concludes with the students reflecting on the use of the digit 0 as a variable in Ankur and Michael's justification. Michael takes the opportunity to rewrite some of the towers that Ankur had earlier dictated to Romina back on the board. Michael explains that he and Ankur had trouble using their notation of 1, 2, 3, and 0 – in fact, he admits to Romina that "we used your explanation – that was best" for how to arrive at the thirty-six towers (1356). Ankur agrees that the justification Romina presented for thirty-six was "a lot easier" (1357). The session then ends after a total of 92 minutes on these two tower problems have elapsed.

5.2.3.7 SOLUTION VERSION 5- Romina writes up her solution a month later

Though the group session ended on January 9, 1998, Romina actually revisited Ankur's Challenge problem one final time on her own. T/R1 requested that she write up her justification to the problem. About month later on February 6, 1998, Romina handed in her final hand-written version of the problem (see Figure 5-23 below). Notice that here Romina has abandoned her notation of 1's, X's, and 0's. Now instead she uses the letters R, Y, and B to represent the colors red, yellow, and blue, respectively. She places the two R's in the same locations as the 1's but now the R's mean only red (as opposed to the 1's representing any of the three colors). This gives "six possibilities." She writes that in the "remaining blocks" there must be "a combination of two possibilities: yellow and blue or blue and yellow." Here then, she uses the Y and R as she used X and 0 previously. In a similar argument to her calculation earlier, she explains that the reader should "multiply the 6 by 2 leaving 12." Then she continues that "since there are 3 colors and each can be the doubled color, one must multiply the 12 by 3." She arrives at the answer of thirty-six towers that are four-cubes high with a choice of three colors where all three colors are represented in each tower.

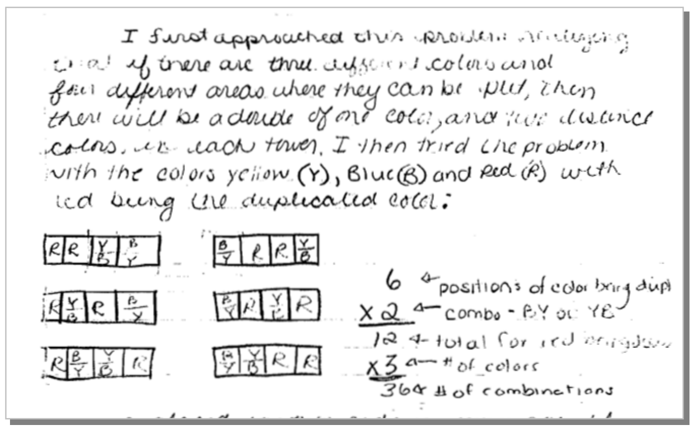
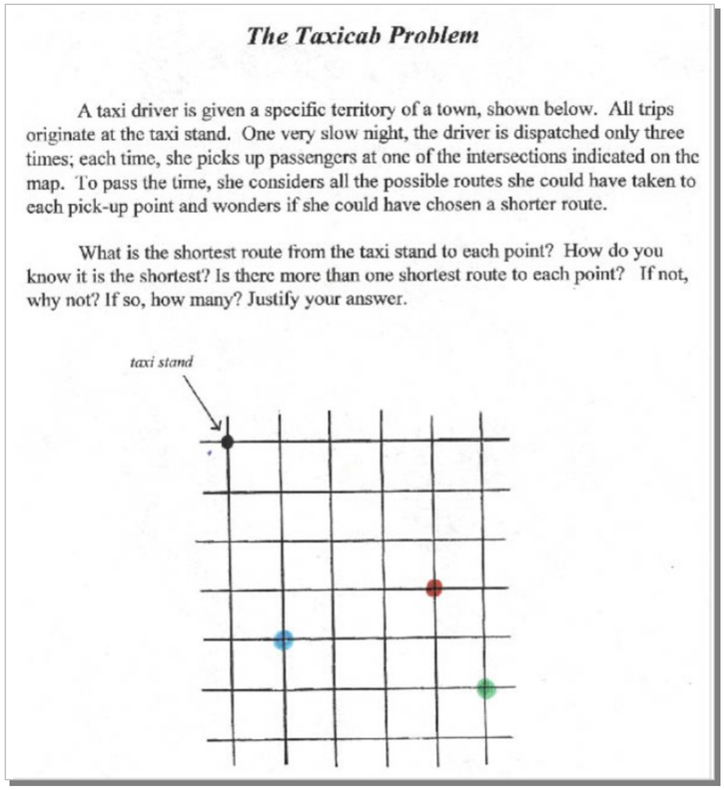


Figure 5-23. Excerpt of Romina's written solution turned in a month later February 6, 1998

5.3 Taxicab Geometry: May 5, 2000 (12th Grade)

5.3.1 Setting

On Friday, May 5, 2000, when the students were in their senior year, a problem-solving session that was part of the Rutgers-Kenilworth longitudinal study took place after regular school hours at the David Brearley High School. Sitting around a small table, four students were present as participants – Michael, Romina, Jeff, and Brian (from left to right at the table from the video camera's perspective). Two teacher-researchers were also in the room a various times. The students worked on a task of the combinatorics research strand – The Taxicab Problem shown below in Figure 5-24.



The Taxicab Problem

A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route.

What is the shortest route from the taxi stand to each point? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.

***Figure 5-24*. The Taxicab Problem as given to the students on May 5, 2000**

Powell (2003) provides a detailed analysis of the mathematical ideas and forms of reasoning employed during the Taxicab Problem session. Specifically, Powell investigated how the students' discursive propositions and inscriptions illustrated their justifications, heuristic development, and articulations of isomorphisms.

Lasting for about 1 hour and 40 minutes, the problem-solving session was recorded with two cameras by the videographers, Lynda Smith and Sergey Kornienko. The videotapes were digitized, compressed, and stored on five compact disks – three for Lynda Smith's video and two for Sergey Kornienko's. Using Lynda Smith's video as the primary source because its audio was the most complete, Powell and others transcribed and verified the session. Referencing the transcript provided in Appendix C of Powell's dissertation (Powell, 2003, pp. 196 – 283) and reformatted here, with permission, as

Appendix D, here we specifically trace Romina's problem-solving behavior through the session with a new lens that particularly focuses on her representations, justifications, and collaborations.

5.3.2 Taxicab Conjectures – "Can't We Do Towers on This?"

After being given the problem, the students took a minute to read through the task statement. T/R1 had them restate problem in their own words. Jeff explains that the problem is asking them to find "how many different shortest routes" there are to each of the three colored dots on the given grid: blue, red, and green. Brian suggests that they "do the blue" dot first. T/R1 reminds the students that they will "have to convince us" as well when they do find the number of shortest routes.

5.3.2.1 Exploring and Making Conjectures

The students begin by tracing paths one-by-one on the paper to determine the number of shortest routes to the blue dot. After initially making a mistake and getting 7 routes, they agree on a total of 5 shortest routes to the blue dot. They then employ different terminology to denote sub-sections of the grid. Romina suggests that the blue dot was part of a "four by one" subsection because it was four units down and one unit to the right of the starting point. When Romina asks how they could "devise an area" for the red dot, both Jeff and Michael correct her by saying that they are not considering area but rather what they call "perimeter" since they are counting the number of units to a particular point on the grid. Since they are in agreement for the blue dot, Romina directs that the students split the work for the remaining two points on the grid. She suggests that she and Michael "do greens" and Brian and Jeff "do red." At this time, Romina's suggestion for their strategy is that they "count how many ways" to each terminal point (126) on the given grid.

After trying to count each individual route, the students observe that they are losing track of their running totals. Jeff comments that "this is hard" and "I can't keep track of what I'm doing" (147). Romina suggests that they switch strategies. Thus, after about 7 minutes since beginning, Romina introduces the idea that the Taxicab problem might be related to the Towers problem. Specifically, she wonders if they could "do towers" to solve the problem:

ROMINA	Okay, we can't count. Like we need a – can't we – can't we do towers on this?
JEFF	That's what I'm saying. Look, all right, you go here.
ROMINA	And they're like blocks.
JEFF	All right, you go to here and you got a choice of going there or there. Right? [*Indicating a choice of across or down at an intersection point of the grid on his problem sheet*.] So then you pick one of those and then you got a choice of there or there. When you get to you know what I'm saying? Maybe we can add all those up or something and get like a whole- [*Explaining routes on grid paper.]*
ROMINA	All right.
MICHAEL	There's a lot.
ROMINA	Okay, for ours there's ten //
MICHAEL	There's more than ten.
ROMINA	No. I mean there's ten blocks. Like ten lines to that thing, right?
MICHAEL	Yeah, six by five.
ROMINA	So if there's ten, ten could be like the number of blocks we have in the tower.
MICHAEL	This is one –
ROMINA	How do we do that? Two to the n? [*Moving her pen cap on and off of her pen*]
MICHAEL	How- how many? This was five they said? [*Pointing to the blue pick-up point on his problem sheet.*]
ROMINA	Yeah. [*Looking back to her problem sheet*] (159 – 173)

Figure 5-25. Romina suggests towers: "… like the number of blocks we have in the tower" (169)

Notice that, in this conversation lasting about a minute, Romina asks four questions that make a suggestion about linking this problem to the Towers problem. First she suggests that since "we can't count" each route through simply tracing on the grid, they need another strategy. She asks, "Can't we do towers on this?" and follows with the observation that "they're like blocks." Jeff observes that at each intersection "you got a choice" of two directions: right or down. He does not seem to acknowledge her suggestion of Towers. Soon after, Romina wonders if there are ten line segments in any path to the green dot, then "ten could be like the number of blocks we have in the tower" (see Figure 5-25). Thus, Romina proposes that the distance between the starting point and terminal point is related to the height of a tower. Michael continues counting. Romina asks again about relating this problem to Towers and proposes an algebraic rule, "How do we do that? Two to the n?" Michael seems to ignore Romina's suggestion because instead of responding directly to her question, he asks her what answer Jeff and Brian had given for their red dot.

As the students continue to count the number of paths to the red and green dots, Romina's interaction with Michael reveals more examples of her asking questions, proposing strategies that incorporate Towers, and then trying to adopt Michael's suggestions for a strategy that counts the number of interior line segments of sub-grids. Her statements that suggest collaboration by asking questions are summarized here in Table 5-3 below:

Romina explores the Taxicab Problem (lines 181 -)		
	COLLABORATION – Asking Questions	
Line	*Romina's Statement*	*Description*
181	Well how many – okay, there's ten. How many lines// end up in the thing?	Directly after Romina has proposed that they look at towers ten high or use the rule "two to the n," Michael seems to dismiss her idea and redirect her work. He tells Romina to "think of the possibilities of doing this and then doing that" (180). He shows her on the grid to count the number of routes. Romina then asks him for information - how many lines are "in the thing" (the number of line segments making up the sub-section of the grid).
189	So – it couldn't be like a block ten high in six different colors type deal? That would be -	Asking for verification, Romina wonders again if Michael is sure that "it couldn't be a block ten high" tower.
191	So maybe it's thirty?	Instead of acknowledging her question, Michael suggests that because there were ten "lines" in the sub-grid that led to an answer of 5 routes, the rule for predicting the number of routes might be to take "half" (190). Romina asks if there thirty paths to the red dot.
193	How many are there in here? One, two, three, four, twelve, twenty – You guys got at least twenty-four yet?	Romina uses Michael's suggestion to count the interior line segments in order to predict the number of routes. She asks "how many are in here" and invites participation from Brian and Jeff by asking them if they "got at least twenty-four."
213	Yeah, but, how you going to know when we –	After Michael tells her to "try

	how are you keeping track though?	doing the red one" (212), Romina asks him for an explanation as for how he is "keeping track" of the number of routes.

Table 5-3. Examples of Romina's collaboration by asking questions in the Taxicab problem

After questioning Michael about his strategy of counting the line segments and dividing by two, Romina asks Jeff, "What are you doing?" (236). Jeff describes how he is focused on the fact that there are two choices at every intersection – "you can either go over or up again" (237). Romina reiterates his observation in her own words when she says that at any point, "you have one of two choices" (242). By this time, Brian has counted more routes to the red dot. Based on his totals, Romina shares her and Michael's "theory" that you count the number of line segments in the interior of a rectangle surrounding the initial and terminal points and then divide by two to get the number of shortest routes to the terminal point:

ROMINA	But then there's more. [*Brian counting with his pen on the grid*]
JEFF	There's more than fourteen?
ROMINA	No, I don't know how many there are.
BRIAN	Are you sure you got –
ROMINA	No, I was just saying that if – that wouldn't work with our theory.
JEFF	What theory is that?
MICHAEL	Divide // it by two.
ROMINA	//Divide it by two. It's like a highly – it was like a –
JEFF	Was it – like what divided by two? All the – add them all up. // [*Inaudible. Pointing at paper.*]
ROMINA	//Because there's ten lines – ten lines like that are all within this rectangle. [*Pointing at paper with pen*]
JEFF	All right.
ROMINA	There's five ways to get to it. So if there are twenty-four lines there would be twelve different lines to get to it. But, it's hard to prove. [*Pointing to her grid with a pen*].

BRIAN	Actually, this whole thing, if you count the middle lines there's thirteen. [*Referring to the rectangular region between the blue pick-up point and the taxi stand*]

(268 – 280)

Under Jeff's questioning, Romina shares the "theory" she and Michael were using to "divide it by two." She explains that there are "ten lines like that are all within this rectangle" around the blue dot and there are "five ways" to get to the blue dot. Romina does not elaborate on why this rule would make sense. She predicts that, with this theory however, if there were twenty-four lines within a rectangle, then there would be twelve ways to get to the terminal point. When Brian counters that there are actually 13 "middle lines" for the rectangular region between the starting point and the blue dot to which Romina has been referring, she dismisses the theory.

5.3.2.2 A New Approach - "Make it Simple" and Label Intersections

After Brian mentions that there are thirteen "middle lines," the students briefly discuss whether there exists a connection between prime numbers and the number of shortest routes in the taxicab problem. Dismissing prime numbers, Romina and Jeff then suggest that they modify their problem-solving approach:

JEFF	There's like no way it could work with a prime number – like you can't even make something up.
BRIAN	All right.
ROMINA	I think we're going to have to break it apart and draw as many as possible.
BRIAN	Yeah, // that's what I'm going to do.
JEFF	//And then have that lead us to something? What if we do – why don't we do easier ones? You know what I'm saying? What if the – the thing – do you have another one of these papers? *[Speaking to T/R2*]
ROMINA	Here, to make it simple, just draw on here.
JEFF	All right. Well, yeah. We're just going to make a grid.

(291 – 297)

After Jeff observes that there is "no way" the prime number theory is going to work, Romina suggests they "break it apart" and draw as many cases as possible. Brian agrees. Jeff suggests that they "do easier ones" rather than the original large 6 by 5 grid. Romina also asserts that they should "make it simple". On a new grid paper Romina records the number of paths to shorter intersections. As she writes the number of paths at each intersection point she observes that, "It looks like a multiplication table" (314 and see Figure 5-26 below).

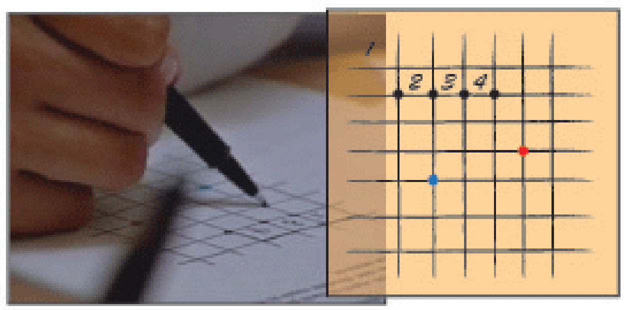


Figure 5-26. Romina's labeling scheme: "It looks like a multiplication table." (314)

Romina encourages the other group members not to count out loud so that they can "see if we get the same thing" (326). Over the next several minutes, Romina counts and re-counts to each intersection point starting with a 1x1 grid, then 1x2, 2x1, 3x1, and so on. Notice in Figure 5-27 that Romina has 2 paths for the 1x1 grid and 3 paths for the 2x1 grid. She wonders aloud about the accuracy of her answers by saying that she is "not sure if I'm counting right" (334). Indeed there are inaccuracies in her records at this time. Romina and Jeff disagree about number of routes to record for certain intersections like the 2x2 grid because Romina has written 5 routes instead of the correct 6 routes.

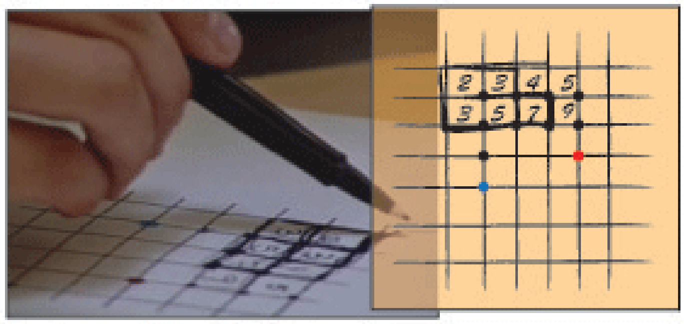


Figure 5-27. Romina records the number of paths at each intersection point

After Jeff finds the mistake in Romina's grid, he questions the other intersection point labels and asks, "How do you know we did five, right?" (382) Romina and Jeff try to confirm number of routes independently and they show each other their results thus far.

5.3.2.4 Systematically Numbering Smaller Sub-grids and Looking for Patterns - "Couples," "Towers," and "Pascal"?

By the time twenty-two minutes have elapsed of the problem-solving session, Romina stops the others and redirects them to search more systematically for a "pattern":

ROMINA	You already did that one.
BRIAN	//I don't remember if I did that.
JEFF	Which one?
BRIAN	//There's definitely twenty-three.
ROMINA	All right guys. This is what we're trying to do. Why don't we try to do this – [*Taking a blank piece of 1-centimeter grid paper*]
JEFF	All right, what's –
ROMINA	We're getting all confused. You see how we're like going to like we're drawing like we're going to here. How many it takes to get to that point and then we're going to here and it's like a- this is just going up like one, two, three- two, three, four, five and then we go down to here and there's the same thing and then like how much we'll get to this point and how much we'll get to that point. [*Pointing to intersection points on a blank 1-centimeter grid paper.*] Why don't we all try to do that because we're getting confused and we're-
JEFF	Yeah.

ROMINA	We're doing the same mistakes.
JEFF	And it's real hard. My brain –
ROMINA	If we do that and we see a pattern I'm sure we'll be able to-
JEFF	Hey, you know what we could even do, we could, uh where are those transparencies? We could exploit the fact that we have those. You know what I'm saying? Like- [*Michael silently writes.*] (398 – 409)

Observing that "we're getting confused," Romina articulates a strategy that they should all "try to do." Specifically, she asks that they systematically record their results for the smaller sub-grids as she and Jeff have been doing. She says that this may allow them to "see a pattern." Adding to Romina's directive, Jeff suggests that they also "exploit" the transparencies to help them keep track of the number of paths.

As Romina re-counts the number of paths to each intersection of the smaller sub-grids, she continues to check. She asks if they "definitely know that's two" at the top of their grid and suggests that they "make sure" for each new recorded label. When Jeff demonstrates how he is using duplicate 2x2 subgrids to record each path independently, he convinces her that there are actually 6 paths as opposed to the earlier 5 Romina had written. When they then move on to a 3x2 subgrid, Romina introduces a local organization and the language of "couples" to describe the relationship between certain paths. Romina asks Jeff, "You want to do them in couples?" (463 and Figure 5-28). For instance, a path going down two units and then across three units would be a "couple" to going across three and then down two units. Then Jeff uses the name "opposite" (467) for this relationship. However, when they try to find a partner for the path of one-down and three-over, they question whether couples will work since one-across and three-down would not result at the desired end point. Romina states, "We can't go in couples" (492). Having never explicitly defined the "couples" relationship for each other, they discuss whether all paths can be paired up in "couples." Romina observes that the path "going all the way across in the middle is never going to have a couple" (495). Jeff wonders if certain intersection points will then have to "always be odd." Romina speculates that "maybe any one with an odd length or width" will not have a couple. They conclude (incorrectly) that there are 9 paths on a 3x2 sub-grid.

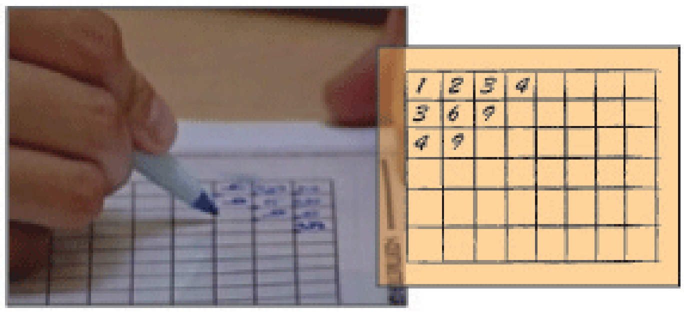


Figure 5-28. Romina's new labeling strategy: "You want to do them in couples?" (463)

Romina and Jeff next attempt to record all the paths on a 3x3 sub-grid. After arguing whether to go "down first" or "across first," Jeff tells Romina not to "blow it" (544). Romina adopts Jeff's earlier strategy to systematically list all the paths on duplicate 3x3 grids. She suggests to Jeff that, "shouldn't we draw them just to make sure though?"(562). In Figure 5-29 below, notice how Romina has partitioned the grid paper into smaller 3x3 rectangular grids. In red marker, Romina draws one path per sub-grid. They go back and forth checking on each other with the repeated, "you got that?"

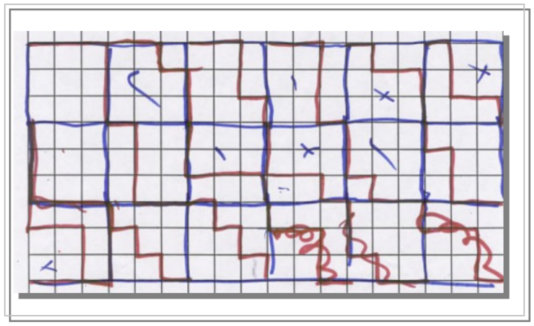


Figure 5-29. Romina partitions into 3x3 sub-grids and records paths: "Over, down, across. Down one – down one over two" (560)

Over the next few minutes, Romina, Jeff, Brian, and Michael continue to enumerate the number of paths to certain intersection points for small sub-grids. Romina and Jeff collaborate on a 4x3 grid. Brian redoes the number of paths for a 3x2 grid and shares his solution of ten paths. Since Romina had earlier counted only nine paths, Brian works to convince her at the chalkboard. Romina then wonders aloud if they could associate the number of paths with the "Towers" problem:

ROMINA	Couldn't we just do something like in towers where like lines over are like the color and the lines down are the, um, number of blocks?
JEFF	All right. And that would?
ROMINA	Because, okay, lines over – because what is it – the number of blocks to the number of colors?
JEFF	I don't know what you're – what – what's that?
ROMINA	Two to the n. Two is the amount of blocks or the colors?
MICHAEL	For what? Like towers on them?
ROMINA	Yeah.
JEFF	Colors. N is the number of blocks. I think. I don't know. I'm not sure.
MICHAEL	Well you figure a block has this – you got two – two ten over like this. Or two colors actually. I thinks it's, uh, the colors and n is the blocks.
ROMINA	Color two - //right. [*Writing the words "color" and "blocks" on a piece of paper*]
JEFF	//Same thing.

ROMINA All right, here we have one color – nah; it doesn't work for the first one. Scratch that idea. [*Crossing out the words on her paper*]. (738 – 750)

Romina asks the others if they could do something "like in towers" where the two different directions: across ("lines over") and down ("lines down") would be associated with the two different color choices. Jeff says that he does not know what she means. Romina suggests that they use "two to the n" but she cannot recall whether the base "two" represented the "amount of blocks or the colors." Michael asks her for what this rule would be used. Jeff observes that the base number represents "colors" and the exponent "n" represents the "number of blocks" but he also is "not sure." Michael agrees that the two is the "colors" and "n is the blocks." However, when Romina tries to apply her rule, she says it "doesn't work for the first one" and they should "scratch that idea."

The students go back to listing routes from sub-grids. Brian draws routes on the chalkboard for the 2x3 grid while Romina reads the possibilities from her paper. When she realizes that she was missing one route, she corrects her paper to read "10" instead of "9" routes for the 2x3 grid. Looking now at the corrected numerical array on her centimeter grid transparency paper, she makes an observation: "All right - it's, um, - it's Pascal's Triangle" (778 and Figure 5-30 below).

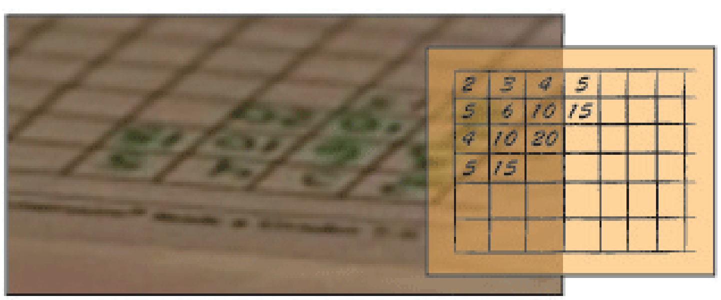


***Figure 5-30*. Romina's transparency representation where she sees Pascal's Triangle**

When Jeff asks to "see it," Romina repeats that "it's Pascal's Triangle," but then retracts her statement by saying that "No, it's not – it doesn't work out" (787). She begins pointing to certain entries on their numerical array that do not match Pascal's Triangle. For instance, they have recorded 12 paths to a point where Pascal's Triangle would have predicted 15 paths. At another point they have recorded 15 paths where Pascal's Triangle would have 20. Romina worries that "if it's Pascal's triangle, it'll just give us problems" (804). Jeff counters that "it's nice" because they "start from nothing" and now have an a connection to Pascal's Triangle (805) – he suggests that it would not have been hard to "miss" some paths when they were counting each individually. Romina agrees that they "got a few wrong" (815). She also suggests that they reorient the transparency paper ("just turn it around") so they can see Pascal's Triangle. While Romina leaves briefly to go to the restroom, Jeff summarizes a strategy to "figure out all the ways to get to the beginning parts" first (847). Then Michael and Brian recheck one of the questionable entries and both "get fifteen" as the number of paths. When Michael asks, "What does that mean?" (858). Jeff responds that this "means that it is the triangle" – thereby indicating that he takes the corrected entry that corresponds to Pascal's Triangle to be justification. When Romina returns to the group, they tell her of the corrected entries. Romina observes that "now it's working" so if they redo one of the other questionable entries and it also corresponds to Pascal's predicted entry of 20 paths, "then we're done" (890). They recount and find that the number of paths through a 3x3 sub-grid is indeed 20. Romina gets new transparency paper and rewrites their new entries for the intersection points that correspond to Pascal's triangle.

5.3.3 Romina's Justifications - "Relate this back to the blocks"

5.3.3.1 Why is it Pascal's Triangle? Blocks and Pizza Reasoning

After the students have corrected their entries on the grids so that the number of paths corresponds to elements of Pascal's Triangle, Jeff predicts that the next question the researchers will ask is "why" Pascal's Triangle works for the Taxicab problem:

JEFF	All right well then – I mean can't we explain why we think – well - all right.
MICHAEL	//They're going to ask us –
JEFF	//All right then the next question is why - //why
ROMINA	//Now –
MICHAEL	//How do you know –
ROMINA	//Just relate this back to the //blocks [*Pointing to the 1-centimeter grid on the transparency*]
JEFF	//Wait – Why is this? Why does the Pascal's Triangle work for this is the question.
ROMINA	//Exactly. Relate it back to the blocks.
MICHAEL	//Just think first how do you know it's twenty? You know, how do you know it's not nothing else? (908 – 916)

Both Jeff and Michael raise the issue of justification. Jeff comments that the "next question" they need to address is "why" Pascal's Triangle works and Michael asks "How do you know." To both of the boys' questions, Romina responds that they need to "just relate this back to the blocks." Though not elaborating on her answer at this point, Romina seems to indicate that they use their knowledge of the Towers problem to build a justification for the isomorphic relationship between Pascal's triangle and the Taxicab problem. Jeff again asks "why" Pascal's triangle would "work" for this problem. Again, without further elaboration, Romina reiterates that they should "relate it back to the blocks." Michael seems to press for more explanation as he counters with the question of "how" she would know that particular entries would not be something else.

After about a minute, Romina directs the boys' attention back to the numerical array on her transparency paper and again makes reference to an exponential rule:

ROMINA	Two colors - It's, it's two to the *x* like that? [*Pointing to the second diagonal "row" of the array of numbers on the 1-centimeter-square transparency, containing the numerals 1, 2, 1*.]
MICHAEL	Yeah, it's two.
ROMINA	So it's two colors –
MICHAEL	Think of it as zero, one, two – you only have two colors of choices – zero, one , two, three.
ROMINA	Huh
MICHAEL	Three toppings on a pizza
ROMINA	Yeah, like- so then how could this- this is two what? Two? Two different ways- like- [*Pointing to the top numbers on the transparency with her marker.]*
MICHAEL	Two- Uh- it's the total. One, two, three- That's, that's the total length that you can get, have to get there- to get there. [*Pointing at numbers on transparency with marker.*]

(947 – 954)

Romina recalls that the Towers problem had "two colors" and wonders if "two to the *x*" refers to the entries 1 2 1 on her transparency. Michael agrees that "it's two," but there seems to be some confusion as to which "two" each student in referring. Michael comments that there were "two colors" and then, without further explanation, refers also to "toppings on a pizza." Romina points to the numerical array and seems to probe for the meaning of the two – "this is two what?" She muses that there are "two different ways" to move on the grid (across or down). Meanwhile, Michael refers to the entry "two" on the grid and says that it the "total" number of paths to "get there." Romina and Michael do not seem to be referencing the same "two." Here, Romina seems to be asking for the meaning behind the base two in the algebraic rule and Michael seems to be indicating the entry of two paths on their recorded transparency paper. She indicates a possible relationship between the base two in the exponential rule and the two directions in which to move through the grid (across and down).

Romina tells the others that she is going to rewrite the entries so that she can "see it." When Romina accidentally gets the transparency marker on her white sweater, the boys tease her that she "could buy little Shout wipes." Romina tells them that she is "very upset right now" because of the apparent stain on her sweater (1004). After a side conversation about the stain, the students return to the problem when the researcher returns to the room. T/R1 asks the students to tell her what they have done and whether they "like" the problem:

ROMINA	No. Nah, it was okay.
JEFF	It's just-, doing all this kind of stuff really hurts your brain, but other than that //it was all right.
ROMINA	//It hurts your eyes. All right. What we did is we- we took it
JEFF	We broke it down.
ROMINA	Yeah, we just went from point to point on the thing.
JEFF	Yeah. Like we even- we'll just say we started making the box like that. How many different ways can you get from this point to this point? You know, make an easier problem. Like the basic math- deal. [*Romina draws in points with her marker and points to the numbers on the transparency grid*.]
ROMINA	So we did like up to this point there's two. Up to this point there's three, four, six, three- So that- those are our numbers. Those are up to the points like down and diagonal. And what we got is Pascal's triangle. [*Jeff points to the numbers on the transparency grid with his marker*.]
JEFF	Yeah. We started, you know, and then as we started, you know, like it takes two to get to there. Three to you know, to get there as Romina just went through and did. And then as we started filling it out we noticed that if you tilt it like that *[Rotating the transparency.*] and throw ones on the outside and a one on top, I mean you're looking at Pascal's triangle. And so we stopped at this point [*Jeff points to a point on the transparency grid with his marker*.] because I mean making, you know, like thirty plus different things like this it gets- it just gets confusing you know. [*Drawing a curve on his paper*.] (1075 – 1082)

In response to T/R1's question about whether they "like" the problem, Romina replies that it was "okay" and Jeff explains that it "really hurts your brain, but other than that it was all right." Romina and Jeff proceed to explain how they solved the problem. Jeff describes their process as that they "broke it down." Romina says that they "went from point to point" on the grid paper. Jeff continues that they then made an "easier problem" by considering smaller sub-grids and counting the number of paths for simpler cases. Indicating the transparency grid, Romina demonstrates how they annotated the number of paths to each intersection point at the top of the grid. She comments that they "got" Pascal's triangle. Jeff adds that they realized the numerical array they were recording was Pascal's triangle when they went to "tilt it."

After Romina and Jeff have offered their narrative of how they solved the problem, T/R1 asks the students for more explanation. She comments: "You're showing me that's Pascal's Triangle, but I don't see it - help me see it" (1120). Jeff reads the entries in their numerical array as they would correspond to Pascal's triangle. For this time, Romina remains silent. After Jeff again indicates certain numerical elements on their grid as corresponding to entries on Pascal's triangle (like the entry of 35 from the 7^{th} row), T/R1 asks, "So can you explain, for instance, to me why that works?" (1130). Jeff repeats that a particular element would be "thirty-five" and T/R1 rephrases her question as, "How-where would the thirty-five come from?" (1134). Jeff explains that they did not count out all of the paths to the entry of 35, so their only justification at the moment is that they are "following the pattern" of Pascal's triangle (1136). Romina and Michael begin counting the number of paths to the intersection point that they have predicted to have 35 routes. Then Michael offers an analogy to the Pizza problem when he observes that "this is like one topping – you know on the pizza?" (1147). Romina agrees with his allusion and comments, "yeah, one topping, two toppings" (1148).

5.3.3.2 "Why do those numbers seem to work?" Direction is color

T/R1 continues to question the students about how they would justify a connection between the Taxicab Problem and Pascal's Triangle. Specifically, T/R1 asks, "Why do those numbers seem to work? How could you explain those numbers?" (1166). Romina then comments that she is "having trouble seeing Pascal's Triangle" (1168). She proceeds to rewrite the first five rows of Pascal's Triangle on a separate sheet of paper and puts it next to the transparency grid with the Taxicab entries (see Figure 5-31 below).

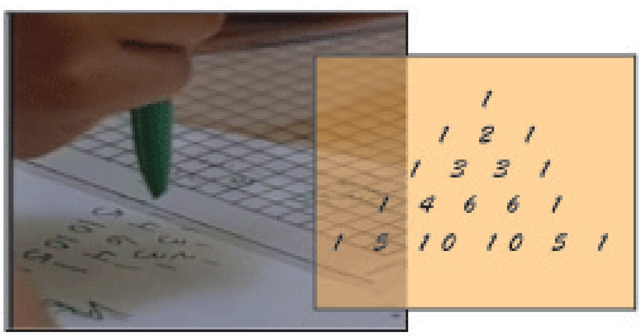


Figure 5-31. Romina rewrites Pascal's Triangle on a separate sheet next to the Taxicab transparency

After rewriting Pascal's triangle on a separate sheet, Romina begins to wonder aloud about where "the two comes from." She suggests that they "go back to Towers" (1178). She asks Michael a series of questions about Pascal's triangle and how it relates to the Towers problem. She uses questions like: "Hold on – how's this go? Just tell me where this comes from" (1180); "This is with just one block?" (1184); "This is zero block, one block, two block?" (1188). Her repeated use of the word "block" as she references the numerical entries of Pascal's triangle seems to indicate that she seeking information about the Unifix "block" cubes used in the Towers problem. Romina's continued discussion with Michael illustrates further examples of her questions:

ROMINA	That's what it goes one, two, three, four? Because then okay for this one for the three. If we name all the ones going horizontal- *A*s and ones going down same with *B*. And this would be with two *A*s and one *B* there's three and then there's two *B*s with one *A*, three. [*Pointing with a green marker at the intersections Points † (3,2) and † (3,1) on the transparency grid*.] And for this one remember like two *A*s two *B*s- //six. [*Now pointing to the intersections point † (4,2) and on the transparency grid*.]
MICHAEL	//You could say, um -
ROMINA	Do you understand what I'm saying?
MICHAEL	Like yeah, these are like this row is everything with perimeter two. I mean one half the perimeter, like.
ROMINA	//Well no, I'm saying that –
MICHAEL	//In order to get to that point you have to go over one and down, uh, one or down one and over one. Just like that row. Everything is this row, over two and down two and over one.
ROMINA	Yeah but like I'm just saying like if she were to pick anything like right there we could say it's like eight *A*s and like six *B*s. [*Tracing a rectangle on the transparency grid*.] You know like- and then we could tell you where you it is in this one. [*Pointing to the redrawn Pascal's triangle on the piece of paper*.] (1196 – 1202)

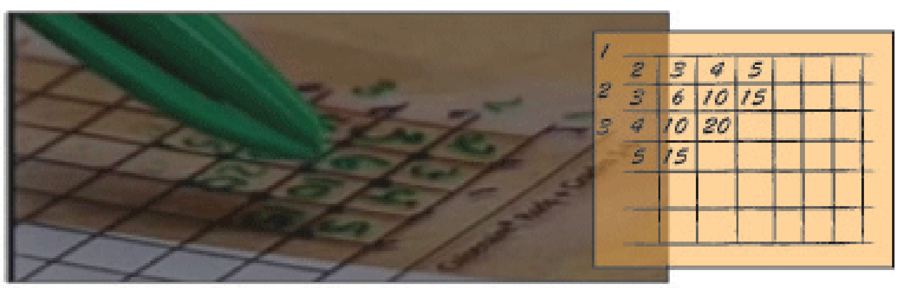


Figure 5-32. Romina names directions right and down as A and B: "…If we name all the ones going horizontal As…" (1196)

After seeking verification from Michael about the numerical order of the array, Romina suggests that they introduce a new notation. Using the letters "A" and "B" to represent the two directions of horizontal or vertical direction, respectively, she states that they should "name all the ones going horizontal As and ones going down same with B" (see

Figure 5-32). Thus, she indicates that moving two units to the right and one unit down from the starting point in the top corner of the grid would be represented as “two As and one B.” Romina provides other examples of how she would name paths to certain intersection points with a sequence of As and Bs. She asks Michael, “Do you understand what I’m saying?” Michael then restates Romina’s new representation in his own words. He describes the paths with such language as “over one and down one.” Romina clarifies that she is saying that if the researcher were to pick any point on the Taxicab grid, they could name the point using a particular sequence of As and Bs (“like eight As and like six Bs”). Gesturing from the Taxicab transparency grid to the Pascal’s triangle on a separate sheet, Romina indicates that they could use the “A” and “B” notation find corresponding elements between the two arrays.

5.3.3.3 Justification Version 1: Romina explains to Michael and T/R1

Soon after introducing the “A” and “B” notation for horizontal and vertical movement on the grid, Romina offers a connection between the two directions in the Taxicab problem and the two colors in the Towers problem:

ROMINA	//Because isn’t that how- isn’t that how we get like these? Like doesn’t the two- there’s- that I mean, that’s one- that means it’s one of *A* color, one of *B* color. [*Pointing to the 2 on the redrawn triangle on paper*.] Here’s one- it’s either one- either way you go. It’s one of across and one down. [*Pointing to a number on the transparency grid and motions with her pen to go across and down*.] And for three that means there’s two *A* color and one *B* color [*pointing to the 3 on the redrawn triangle*], so here it’s two across, one down or the other way [*tracing across and down on the transparency grid*] you can get three is //two down- [Pointing to the grid.]
MICHAEL	//You mean like one A color and two –
ROMINA	Yeah.
ROMINA	Like two blues, one red. Two across, one down or this is two reds, one blue, two down, one across. And that’s how we would get the Pascal’s

> Triangle. [*Pointing to numbers on the redrawn grid and transparency grid*]. (1210 – 1214)

Indicating the entry 2 in the second row of the Pascal's Triangle that she wrote on a separate page, Romina asks Michael if that "means it's one of A color, one of B color." She seems to refer to the fact that there would be two towers that could be built two-cubes tall from a choice of two colors: AB or BA (taking the letters to represent the two different colors). Romina then points to an entry on her Taxicab transparency grid and describes it as having two shortest paths: "one of across and one down" – AB or BA (taking the letters now to represent the two different directions across-down and down-across). Next, Romina points to the entry 3 in the third row of Pascal's triangle and says that it "means there's two A color and one B color" for the Towers problem, or analogously, "two across and one down" for the Taxicab problem. Romina clarifies that two As and one B could be "like two blues, one red" or "two across, one down." By linking the two colors from the Towers problem to the two directions one can move in the Taxicab problem, Romina claims that is "how we would get the Pascal's Triangle."

After Michael and T/R1 question Romina further on her explanation that color in the Towers problem is related to direction in the Taxicab problem, T/R1 asks Romina to reiterate her argument. Specifically, T/R1 asks, "Why don't you try saying that again" because "I'm not so sure Brian and Michael followed what you said" (1232). Romina then offers a reiterated argument for why Towers relate to Taxicab:

> //When we look- whenever we do this we always- we always talk about towers and how this is like a tower of two high with two different colors and there's one- one tower you can make that makes one color and one and one and then all the other color. And- and then for this one it's three high and this is all one color. There's two of one color and one of the other, whatever. And for this it's basically the same thing because this is- let's see. This is- this is two but usually you go one across and one down so there's two different ways to get to that one. And for this

> one there's going to be two across and one down. Or to go down here it's two down and one across which is basically the same thing and it just goes on. Do you understand? Understand? Was that good? Or, do you want more? [*Connecting the data from the grid and the triangle drawings by pointing to the numbers on each back and forth*.] (1239)

Notice that Romina provides more specific details and even phrases her argument more precisely as relating "a tower two high with two different colors" to the Taxicab problem. First, she describes towers with different color combinations for two-tall and then towers "three high" like one that is "all one color" or "two of one color and one of the other." Romina states that the two color choice is "basically the same thing" as the two direction choice in the Taxicab problem. She indicates a point on the grid that would have two shortest paths (consisting of "one across and one down" in either order) and then a point with three shortest paths (consisting of "two across and one down"). As Romina explains, she continually gestures from the numerical array data on the grid to the Pascal's triangle numbers. She asks if they "understand" or if they "want more" explanation.

Brian asks Michael if he understands what Romina just explained. Michael says that he does, but he suggests that "we'll think of it as pizza because that's the thing I like" (1246). Romina disagrees and directs Michael to "think of towers" because "the tower is easier" (1247 and 1251). Michael then begins to describe an entry in terms of towers with "color x and two of color y" that would be analogous to "direction x and two directions of y" (1252). T/R1 then asks the students to clarify their notation. Romina suggests that they use the "x and y" notation for colors and direction rather than her previous A and B notation.

5.3.3.4 Justification Version 2: Romina explains to Michael

After Romina has explained her reasoning to T/R1, Brian, and Michael, T/R3 tells the students that “it’s still not clear to me how they know that to get to any particular corner corresponds to one of the numbers in Pascal’s triangle” (1272). Romina acknowledges that she has not “done” that yet. T/R1 says the researchers will “leave you be while you think” (1276).

The students determine that some particular corners they will investigate further are the points at which they are predicting 21 and 35 shortest paths. Romina records more rows for her Pascal’s triangle and then refers to her augmented triangle in her explanations to Brian and Michael (see Figure 5-33 below).

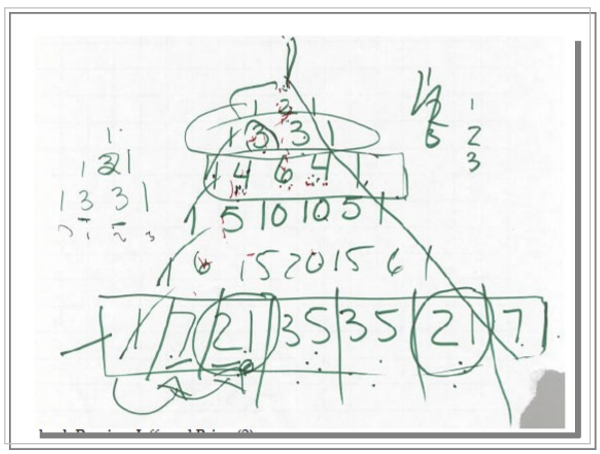


***Figure 5-33*. Romina’s augmented Pascal’s Triangle representation.**

Under Michael’s questioning, Romina explains again why the number of “blocks” in a tower would be equivalent to the number of “spaces” they move on the taxicab grid:

MICHAEL //But why- you know, why is it thirty-five? If you go-

Or why is it- let’s go- go a little easier. Why is it, you know, four if- of, um

ROMINA All right. Four, right? Four is- all right, why don’t we do six? Six is a little harder. Six is one two- the one with six. There’s one, two, three, four- you know there’s four- [*Pointing to triangle*.]

MICHAEL	It's two and two. All right. Two, four –
ROMINA	This one.
MICHAEL	One, two, three, four.
ROMINA	It's because it's four blocks. No matter how you go there you had to take four spaces. And any direction you take has to be four spaces, right? So that means it's four- it's four blocks high. So you go to the fourth one. So you know it's in here. [*Circling the 4th row of the triangle*.] And it's- to get here it's two across and two down. So whatever, like you know- Do you understand? Whatever route you take you'll end up two across two down. So it means there's-

(1323 – 1328)

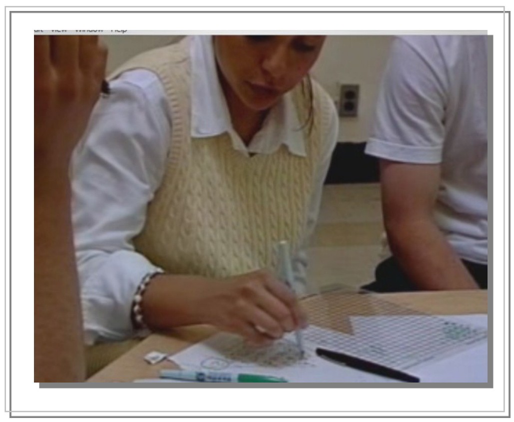


***Figure 5-34*. Romina explains to Michael that the number of "blocks" on her Pascal's Triangle relates to the number of "spaces" on her Taxicab grid**

Michael first asks Romina "why" there would be thirty-five paths to a particular point on the taxicab grid, but then amends his question to be "a little easier." He directs his question to the fourth row of Pascal's triangle. Romina points to her augmented Pascal's triangle and circles particular elements with her marker (see Figure 5-34 above). She explains that the fourth row of Pascal's triangle would correspond to a tower "four blocks" high or a point that takes "four spaces" to reach on the taxicab grid. She says that to "get here" on the grid (indicating a particular point) it would take "two across and two down." Michael continues to question Romina until she makes a correspondence between each element of the fourth row of Pascal's triangle with each element of their taxicab grid's fourth row:

ROMINA	That's two across two down. That's four so you're in the four blocks. And then it's this- to get to here the only way to get to here is somewhere you got to go two across and two down. So there's two of one color and two of another. This is all one color. This is one and three. Two and two. Three and one.// [*Pointing to grid and redrawn triangle*]
MICHAEL	//All right. Yeah – that makes sense//
ROMINA	//All one color. And the- the four is still three and one but then it's three across and one down so it means it's three of one color and one of the other color. [*Pointing to triangle*]
MICHAEL	That – that's a pretty good explanation.
BRIAN	It's cool.
ROMINA	Who's calling them in?

(1342 – 1345)

Romina indicates the point on the taxicab grid with 6 shortest routes. She explains that it is "two across two down" which is a total of four spaces in distance so it corresponds to the "four blocks" row of Pascal's triangle. Relating "two across two down" to "two of one color and two of another," Romina indicates the corresponding 6 entry on Pascal's triangle in the fourth row since there are 6 towers that can be built from a choice of two colors that are four-cubes high. Romina points to other elements in the fourth row of the numerical array which she names "all one color," "one and three," "two and two," "three and one," and "all one color." Romina also distinguishes between the four paths which are "three across and one down" from the four paths which are one across and three down which she had mentioned earlier. Michael then acknowledges,"that's a pretty good explanation" and Brian says that "it's cool." Romina says that they should call in the researchers. The students announce they are "ready" for T/R3's question now. Brian tells the researchers that "Romina's got something good" (1354).

5.3.3.5 Justification Version 3: Romina explains to T/R3

After inviting the researchers back into the room, Romina presents her reasoning to T/R3. Romina's explanation now includes the language of "moves" as in, "four moves equals four blocks." She also equates going in the "same direction" on the taxicab grid with having a tower built with "all one color." To illustrate her justification, Romina again uses two elements 6 and 4 from the fourth row of Pascal's triangle as she did earlier with Michael:

ROMINA	Well we'll do the six and the four.
MICHAEL	All right.
ROMINA	Okay, to this point you know you need to take at least you have to take four moves. That's the shortest amount of moves because just like a simple one, two, three, four. So that means it's- let's say you we're relating back to this four moves equals four blocks. So I'd have to go down to the four block area. So that's one, two, three, four. [*Pointing to the fourth row of her Pascal's triangle*.] And now here you're going three across and one down. Or- so- [*Illustrating the moves on the taxi grid and pointing to the numbers on the grid and redrawn triangle*.]
MICHAEL	There's no possible way you could –
ROMINA	//Do anything else.
MICHAEL	//You have to – no matter how or which way you go you have to go three and then one.
ROMINA	Right. In any move you're going one down and three across no matter- in any direction you take. So the three across and one down, that relates to three colors and then-
MICHAEL	Of one –
ROMINA	Three of one color and one of another. So you go and
	you look in here. Say- Okay, here's with all one color.
MICHAEL	That's – that's nothing.
ROMINA	No that's all one color but we're not using that because you can't all go all in the same direction. That's all one color. That's with one of one color and three of the other. So that's four and that's what we have and if you go down to here this is two and two and this is three and one which is the same thing. So there's your other four. And if you go to the sixth, the only way you can get there again is by four moves. It goes one- one, two, three, four. So you're in the four block again but this time you have to take, no matter what you do, you go two across and two down anyway

> you do it. So that would be two and two which is your six but you're still in like the four block area. [*Relating the taxi grid to Pascal's triangle*.]
> (1371 – 1379)

Figure 5-35. Romina explains to T/R3: "…Four moves equals four blocks" (1371)

In her explanation to T/R3, Romina maps the elements 6 and 4 from the fourth row of Pascal's Triangle to numbers of shortest routes in what she calls the "four block area." Since it takes "four moves" to get to a particular point on the taxicab grid, she says "we're relating back to this four moves equals four blocks." She points to the fourth row of Pascal's Triangle (see Figure 5-35 above). Romina discusses each entry from the fourth row of Pascal's triangle: 1 4 6 4 1. She says the 1 is "all one color," but qualifies that they are "not using that because you can't go all in the same direction." Romina goes back and forth between the language of color and moves as she describes each entry in turn. For instance, the entry 4 is "one of one color and three of the other" using a block tower, while the entry 6 is "two across and two down" using moves on the taxicab grid.

5.3.3.6 Justification Version 4: Romina explains to both T/R1 and T/R3

After Romina finishes her latest explanation of how towers, taxicab grids, and Pascal's triangle are related, Michael observes that they could also reason with pizza toppings. T/R3 asks the students to "help me see how you're relating the number of toppings and the number of blocks" (1387). Romina responds by directing Michael to answer. She says, "Mike, if we were to use pizza, could you explain this 'cause I don't know how to do this" with pizzas." (1393). Michael proceeds to explain to Romina how the fourth row of Pascal's triangle would relate to a pizza with a choice of four toppings. For instance, he indicates that there would be six pizzas that could be made with three of the four available toppings. Romina then announces that this is "just the same way I just did with the blocks – it's the same thing" (1405). Michael agrees and he equates a choice of topping with a choice of direction. He says that getting a topping would be like "being able to go across" on the taxicab grid. Michael then continues to describe how each number in the fourth row of Pascal's triangle would relate to the Taxicab grid.

T/R1 then asks how Michael's description of toppings "would work with the A's and the B's" from Romina's earlier comments. Brian recalls that Romina had used the A and B representation for color and direction as well as "x" and "y":

BRIAN	Romina was bringing it up.
ROMINA	Um, I'm sorry. What am I trying?
BRIAN	x's and y's like -
T/R1	I think it was Romina who did it, yes. She used x's and y's for across and downs.
ROMINA	Okay, so if we're doing the same one with, um, with no- no *x*'s then you'd have to go four down- four *y*'s down and that would be this one. [*Motioning across and down on grid*] But you're not going to get there. Whatever. But if you're trying to get there it's one *x* and then you go three *y*'s. So that's your four. If you're trying to get to this one over here it's two *x*'s, two *y*'s then three *x*'s, one *y* and they all- they all equal four but they all have different amounts of *x*'s and *y*'s and that's how we get this. Yes? No? [*Referring to the drawing of Pascal's triangle*.]
T/R3	And the x's and y's – What does the x correspond to again?
ROMINA	//x is across.

BRIAN	//Going across. And y is //down.
ROMINA	//Or a topping or a color. All the same thing. And all our y's are down, toppings, color. (1440 – 1448)

Figure 5-36. Romina explains that toppings, colors, and directions are "all the same thing" (1448)

After both Brian and T/R1 prompt her to use her earlier "x" and "y" representation, Romina indicates the fourth row of Pascal's triangle. She then names each element of the fourth row in terms of x and y directions: 1 path with "no x's" ("four down"), 4 paths with "one x and then you go three y's" (one across, three down), 6 paths with "two x's, two y's" (two across, two down), 4 paths with "three x's, one y" (three across, one down), and finally 1 path with all x's. She explains the flexibility of her notation: when she uses the letter "x" she means the direction "across" or "a topping or a color." She summarizes that the choice of direction, topping, or color is "all the same thing" (see Figure 5-36). Similarly, she defines "all our y's are down, toppings, color."

T/R1 follows Romina's comment by asksing if they could use "zeros and ones" too as well to describe the paths on the taxicab grid (1449). Romina agrees "sure," but directs the researcher to Michael by saying "that's his area" (1452). Brian encourages Michael to "break out the binary" (1457) although Michael claims "I really don't remember" (1459). Instead, Romina defines how binary could be applied to the Taxicab problem. She defines the one and zero in turn as: "one would be every time across" (1462) and "zero would be every time down" (1464). After Romina defines the binary digits one and zero as the directions across and down, respectively, Michael begins to write binary sequences like 110 and 011 on his paper. As Michael records binary sequences, Romina claims that "I can't work like that – I work in, um, towers" (1476) whereas she Michael "works in pizzas and binary" (1478).

5.3.4 Romina's Generalization to an Algebraic Rule

T/R1 then asks, "How would you talk about some general numbers?" (1498). Michael states that "we've proved to you that you understand why it relates to the Pascal's triangle" (1501). Michael and Romina then challenge the researcher to give them any "general number" on Pascal's triangle and they will explain how to relate a particular number of shortest paths on the taxicab grid to that number. Romina uses an example from the tenth row of Pascal's triangle. She says that she would use the tenth row for any path on the taxicab grid that took a total of "ten moves." T/R1 then asks for a generalization about the *r*th row of Pascal's triangle as it relates to the Taxicab problem:

T/R1	So, what about the *r*th row?
MICHAEL	Would be -
ROMINA	The *r* th row would be- r moves
MICHAEL	Yeah, *r* moves *r* shortest distance. Whatever-
ROMINA	Yeah.
T/R3	Uh hum.
MICHAEL	*r* half the perimeter whichever, you know-
T/R3	Okay.
T/R1	Are you convinced?
T/R3	Yeah.

T/R1	It's really very interesting. Interesting problem. Did you ever do anything like this before?
MICHAEL	No, no I've never seen it before in my life.
ROMINA	We just discovered Pascal's triangle. (1590 – 1602)

Immediately after T/R1 asks about "the *r*th row," Romina's answers that "the r^{th} row would be *r* moves." Michael agrees that the *r*th row of Pascal's triangle would correspond to intersection points on the taxicab grid that would take "*r* moves *r* shortest distance." When T/R1 asks if they ever did a problem like this before, Michael observes that he had not. Romina announces that, in doing the problem, "we just discovered Pascal's Triangle." T/R1 then makes some final remarks to the students and thanks them for staying and working on the problem. The session concludes.

Chapter 6 INTERVIEW RESULTS – High School

I think there's [sic] two big different areas of math: one of them is like the thinking involved and one of them is just like spitting out numbers. I know I was never good at the spitting out the numbers thing, but I was decent at the thinking about it.

~ Romina, 1999

6.1 Introduction

Chapters six through eight summarize the video data analysis of interviews in which Romina participated from 1999 to 2009. What set the interviews from 1999 apart from the rest, and why they are presented separately here in their own chapter, is that they provide insight into Romina's beliefs about her own mathematical ideas and learning in the longitudinal study while she was still actively participating in the high school problem-solving sessions. The two interviews included here were videotaped relatively close together in time – the first, during Romina's 11th grade year on May 18, 1999 and the second, after a problem-solving summer program on July 21, 1999. The filming of both interviews was also made possible by a collaboration between the researchers of the Rutgers-Kenilworth longitudinal study with the assistance of the Science Media Group of the Harvard Astrophysics Observatory.

6.2 Reflections I - PUPMath: May 18, 1999 (11th Grade)

6.2.1 Setting

On May 18, 1999, toward the end of her 11th grade junior year of high school, Romina participated in an interview conducted with the assistance of the Science Media Group of the Harvard Astrophysics Observatory in preparation for the Private Universe Project in Mathematics (PUPMath). Romina discussed her reflections on the longitudinal study. Excerpts of the interview were included in Workshop 1 of the series of six K-12 teacher video workshops, support materials, and the companion interactive website (http://www.learner.org/workshops/pupmath/). The interview lasted about 22 minutes and followed a loosely structured format in which the researcher questioned Romina about the following topics: her memories of the longitudinal study, her associations with particular years of classroom mathematics, her thoughts about learning, and her self-perception as a problem-solver. After the interview was transcribed and verified, "significant statements" were tagged and analyzed. For the full interview transcript, see Appendix E. Her significant statements clustered into three thematic categories: knowledge, conditions of her learning environment, and her learning process in general.

Reflections I Interview – May 18, 1999 - SIGNIFICANT STATEMENT SUMMARY	
Issue	***Significant Statement(s)***
Knowledge & Knowing	***Arguing – "The Only Reason I Know Math":*** "And we just argued with him and he explained it to me, and he explained it to Jeff and Brian. And now we understand it." (24); "I've always had to argue to get somewhere, because they never actually told me where we were heading with anything. So, through arguing, that's how I discovered… that's the only reason I know math. Because I did it myself, all these years." (38); "Because we weren't afraid to come out with our ideas, because that's how we were taught. If we were sitting there in math, I would argue with her, I would ask her a question, and she was so surprised, she didn't know what to do with us." (42)
	"Everything has Romina's Definition to It" – Constructing versus Receiving Knowledge: "…you can't live your whole life being told what to do. You're going to eventually have to do it yourself. And they're going to have more knowledge about everything… you have to go deeper, you have to, if you understand something from the beginning, you're going to always understand it." (52); "I taught myself, basically, that year, from what I knew. The rest of my class did really bad, because they weren't used to that. They looked to the book for answers. …They were lost. And they couldn't do anything for themselves." (56); "…But I know it in my way, not in their way. And everything I explain is in my words, not in anyone else's words. It's not from some mathematician from thousands of years ago, because I don't know that. Like I didn't know what the pyramid – Pascal's - was called. I just know everything in my own way. Everything has Romina's definition to it." (58)
Conditions for the Learning Environment	***Collaboration – "Asking Why" in a "Socialized Class"***: "Like the why thing - everyone wanted to know why we did everything." (12); "I have problems that have to do with me… And then having, like, a socialized class. If you all sit there in neat rows and have to look at the teacher, and listen to them, you're going to be bored, you're not going to pay attention… But if you come into my math class, we're all in a big circle, and our teacher is in the middle sometimes, and sometimes he just kind of sits down and let's us do our own thing. He gives us problems that we want to know the answer to - that we're interested in, and then he doesn't have to give us an equation. We all just kind of talk about it, and then come to a point. And you're kind of socializing while you do your math." (68);
	"Comfortable" Relationships with Teachers & Researchers: "In fourth grade, I didn't know anything. I didn't know who you were. Now, we're comfortable with you." (16); "They don't like the idea that I'm a friend with my math teacher, and I can talk to him, like not only about math. And he's got a comfortable relationship with me..." (70); "And I've been doing it for ten years, and I think it

	really pays off. I've had amazing teachers who have gotten so involved in what we're doing. And they're not the regular book teachers." (78)
Learning Process	***"Arguing," "Disagreeing," & "Group Talking"***: "Like we could never do any of the things, well I don't think I could ever do any of the things we do alone. Like they just help you bring out things you didn't know were there. And we have a relationship where we argue a lot, so, like through arguing is where we come up with most of our answers." (20); "But if I disagree with someone, they'll have to explain it to me, and if you're explaining it, they're either going to find something right, or they're going to find something more. So, if I don't agree with it, they're going to explain it to me, but if they find something wrong, maybe I can help, and then someone else may disagree with me. And that's how we get through everything. We just disagree." (22); "Well math is where the most arguing is. Like, you can't do this in other classes. It's not like, in English, you read. You don't argue; it's there. It's written." (38); "If people learned the way I did with, like, group talking, I think people would learn more and be able to do more because if someone that was taught with just a teacher teaching them, if you're given something in, like, the real world, you're not going to know how to handle it. Whereas I would probably question it, and like, throw different ideas in the air. Other people, they get intimidated, and they don't know how to do that." (52); "…But once I explain it to them, and I say, "Maybe you could do it a different way, some way you understand it." Then, we've had arguments but I've helped them, so it's been okay." (60); "I like the socializing …" (62)
	Affective Dimension – "Scared" & "We Amazed Ourselves": "My first memories of Rutgers were, I got pulled out of class one day, and I didn't know why, and I got put into a special class, which is kind of scary, because you don't want to be different back then…" (10); "We called ourselves your guinea pigs…We thought you thought that we were smart, and we didn't think we were all that smart, and we were kind of scared. … none of us had any confidence." (18); "Yeah, we thought we were real weird. Like, fourth graders interested in math and arguing with their own friends about it? ...I don't know what happened there that we started arguing. But it just like got us so far. And what happened is we amazed ourselves with the things we got…" (36); "I was not interested in geometry…." (46)

Table 6-1**. Reflections I Interview – May 18, 1999 - Summary of Significant Statements**

6.2.2 Knowledge and Knowing

6.2.2.1 Arguing – "the only reason I know math"

Romina identifies "arguing" as the source of her understanding and knowledge in mathematics. She gives an example of how "arguing" with Michael led her, Jeff, and Brian to "understand" the binary number system:

> And now I think I understand what the binary system is, and so does everyone else. But we didn't know what that was when we started. Only Michael did, and I was arguing with him, because I thought it was wrong the whole time. And we just argued with him and he explained it to me, and he explained it to Jeff and Brian. And now we understand it. (24)

Romina describes how "we didn't know" about the binary system and then, through a process of "arguing" where "we just argued" and "he explained," they came to an understanding of the binary system. Notice that she does not speak only of her own understanding, but rather of a collective knowledge where "I understand" and "so does everyone else." It was not until she, Jeff, and Brian had all participating in the arguing and explaining that the understanding was complete.

Asserting that arguing was not limited to helping her understanding in just the instance of binary, Romina generalizes the usefulness of arguing to her knowledge:

> I've always had to argue to get somewhere, because they never actually told me where we were heading with anything. So, through arguing, that's how I discovered… that's the only reason I know math. Because I did it myself, all these years. (38)

Romina says that she "always had to argue to get somewhere." Notice that she does not speak of a particular solution, answer, or topic when referring to her knowledge but rather an abstract location, "somewhere." Implying that tasks were open-ended and complex, she explains that "never" told "where we were heading with anything. She reiterates that "through arguing," she "discovered" and arguing is the "only reason I know math." For Romina, arguing seems to have served a purpose of helping her construct her own knowledge. Arguing allowed her to discover and know. She summarizes with personal authority and another action verb – "I did it myself all these years."

Unlike the fields of combinatorics, probability, algebra, and calculus, Romina claims to not "know" anything about geometry. Indeed, during the middle portion of the interview when she is reviewing past courses, she states, "Ask me one question about geometry - because I won't know it." (40) Notice that she locates her knowledge in the context of questioning. She challenges the interviewer to "ask her one question" about geometry and implies that her failure to answer a question about the course indicates that she does not "know" geometry. She goes on to explain why she thinks she does not "know" geometry:

> Because my geometry teacher wasn't aware of this, and she, it was a completely different town. And when we went there, she was amazed at how much math knowledge we knew. And that was just through what we thought. Because we weren't afraid to come out with our ideas, because that's how we were taught. If we were sitting there in math, I would argue with her, I would ask her a question, and she was so surprised, she didn't know what to do with us. (42)

She describes her geometry teacher as someone from a "completely different town" (not Kenilworth) who was not "aware" of the philosophy of the longitudinal study. From Romina's perspective, her geometry teacher was "amazed" at "how much math knowledge" the students from the longitudinal study already had. Romina justifies the source of this "math knowledge" as "what we thought" and the fact that they "weren't afraid to come out with our ideas." The geometry teacher "didn't know what to do" when Romina would "argue with her" or "ask her a question." Knowing for Romina seems linked to being able to argue, ask questions, and share ideas. Since she was not able to argue, ask questions, or share ideas as she was used to doing in the longitudinal study, Romina concludes that she does not "know" geometry.

6.2.2.1 Constructing Knowledge - "Everything has Romina's Definition"

Romina draws a distinction between receiving knowledge versus constructing one's own knowledge. She also explains into which category she would put herself:

> …you can't live your whole life being told what to do. You're going to eventually have to do it yourself. And they're going to have more knowledge about everything. Because everything I do I understand, because it's more than just numbers to me. It's like you have to go deeper, you have to, if you understand something from the beginning, you're going to always understand it. You can't forget something like that. (52)

Asserting that one cannot "live your whole life being told what to do," Romina makes a strong argument against received knowledge. Indeed, she argues that when you "do it yourself," one will gain "more knowledge about everything." She explains that by doing, understanding follows – "everything I do, I understand." Doing mathematics seems to be an act of constructing "deeper" understanding for Romina. She says that "it's more than just numbers for me." She also claims that such constructed knowledge would lead to more lasting learning. She describes the way she goes "deeper" as working to understand a topic "from the beginning." If one knows something is this "deeper" and "beginning" way, that Romina asserts that a person will "always understand it" and "can't forget" it. The sense of knowing something "from the beginning" also implies rebuilding and reinventing for oneself on a problem task.

Regarding her freshmen year when she went to a different school and was not with students from the longitudinal study, Romina explains her success in terms of her ability to construct her own knowledge:

> I got through most of my tests, because I went back to sixth, seventh, and eighth grade, and what I learned then, and what I could put together. I taught myself, basically, that year, from what I knew. The rest of my class did really bad, because they weren't used to that. They looked to the book for answers. And they didn't understand the book, and the teacher wouldn't help them. They were lost. And they couldn't do anything for themselves. (56)

Romina contrasts "the rest of her class," who were not participants in the longitudinal study, with herself. These other students "looked to the book for answers," whereas Romina asserts that she "taught herself" from what she already "knew" in sixth, seventh, and eighth grades when she had participated in the longitudinal study. She states that the other students "couldn't do anything for themselves," but she was able to "put together" her learning. By looking to the book or the teacher for answers, Romina implies that these other students were relying on received knowledge. In contrast, she describes herself as someone who could construct knowledge by doing and putting together ideas on her own. There seems to be an implicit value judgment as well that Romina is making – the students who relied on received book knowledge were "lost" and "did really bad." She remarks that the others in her class "weren't used" to being able to "put together" their own knowledge as she claims she did in the course.

Specifically reflecting on her own knowledge, Romina explains that it is necessary for her to know in her "own way" and not someone else's:

> I'm not confident because, I know I can do a lot, and I can do it. But when I try to explain to a person what I know, I can't explain to you what I know. They might throw out, "Oh, do you know this rule, and this guy and all this stuff?" and I'm like, "No, but if you sit me down, maybe I know do know it." But I know it in my way, not in their way. And everything I explain is in my words, not in anyone else's words. It's not from some mathematician from thousands of years ago, because I don't know that. Like I didn't know what the pyramid – Pascal's - was called. I just know everything in my own way. Everything has Romina's definition to it. (58)

Romina says she is "not confident" because, although she knows she can "do a lot," she is not always able to explain what she knows to others particularly if they want the explanation in terms of a particular "rule" or established authority "guy." She describes how she can "know it in my way, but not their way." Knowledge then becomes something deeply personal and uniquely located within "my words." Notice the necessity of language here – to know is to be able to use one's own voice, not someone else's voice and "not in anyone else's words." Her knowledge is not that of "some mathematician from thousands of years ago." She provides an example of how she knows how to use Pascal's triangle, but she does not know what it is "called" formally. She insists that her knowledge has to originate in her own constructed understanding. Reiterating that "I just know everything in my own way" with her own terms and through her own action, she concludes that "everything has Romina's definition to it."

6.2.3 Conditions for the Learning Environment

6.2.3.1 Collaboration – "Asking why" in a "socialized class"

When asked about her first memories of the longitudinal study in fourth grade, Romina recalls that "everyone kept asking why" (10). Asking questions within a group emerged as common theme through her experience in the study. From her perspective, Romina remembers the researchers being very interested in "why" they did things:

> Like the why thing - everyone wanted to know why we did everything. We didn't know why we did everything. And we didn't know why things worked. (12)

Romina's phrase "the why thing" summarized her recollection that the researchers' would regularly question the students' reasoning in the longitudinal study. In fact, Romina remembers the questioning as pervasive – the researchers "wanted to know why we did everything." She remembers that in fourth grade they "didn't know why did everything" or "why things worked."

The questioning continued throughout Romina's time in the study and by high school, Romina had developed a clear definition of what "math class" should be:

> In my math class, I don't have problems about going to the market and buying apples. I have problems that have to do with me. Like with having enough money to buy clothes, and things like that. And then having, like, a socialized class. If you all sit there in neat rows and have to look at the teacher, and listen to them, you're going to be bored, you're not going to pay attention. You're going to hate school. But if you come into my math class, we're all in a big circle, and our teacher is in the middle sometimes, and sometimes he just kind of sits down and let's us do our own thing. He gives us problems that we want to know the answer to - that we're interested in, and then he doesn't have to give us an equation. We all just kind of talk about it, and then come to a point. And you're kind of socializing while you do your math. And you get an answer and you weren't that bored. (68)

Romina's definition of "my math class" includes two major elements: relevant tasks and collaborative problem solving settings. First, the problems in Romina's math class "have to do with me" and are thus relevant to her life. She dismisses problems like "going to the market and buying apples" as irrelevant. Instead, she would be interested in a problem involving "clothes" and her wardrobe. These tasks should be interesting to the students. She describes how her teacher now gives them problems "we want to know the answer to – that we're interested in." Implying that these are more complex, open-ended tasks, Romina explains that her teacher "doesn't have to give us an equation" to solve these problems. A great deal of discussion and consensus-building among a group is required. She emphasizes as her second point that the math class should be a "socialized class." Elaborating on this idea, she explains that rather than "neat rows" where you have to passively "look" and "listen" to a teacher, her math class has all the students in "a big circle" with the teacher sometimes in the middle and other times on the side so that they can "do our own thing." Solving a problem in this setting means "socializing" with others. The group will "talk and then come to a point." Romina predicts a student will not be "bored" in such a learning environment where the tasks are of personal interest and the problem-solving is done collaboratively.

6.2.3.2 "Comfortable" Relationships with Teachers and Researchers

Another key component that Romina includes about her learning environment involves her relationship with the teachers and researchers involved. She comments on her comfort level in the longitudinal study:

> In fourth grade, I didn't know anything. I didn't know who you were. Now, we're comfortable with you. (16)

Romina links that "I didn't know anything" in fourth grade with the fact that "I didn't know who you were." Knowledge gained from personal relationships becomes related then to knowledge of mathematics for Romina in the longitudinal study. She remarks that "now, we're comfortable with you." Having a "comfortable" relationship with the researchers seems a prerequisite for her further learning.

Continuing this idea of the importance of the relationship with the researchers, Romina describes how her family felt her relationship with teachers was unconventional:

> They don't like the idea that I'm a friend with my math teacher, and I can talk to him, like not only about math. And he's got a comfortable relationship with me. And they think that's very odd. And they think that my teacher should give me homework every night, in the book, and I should bring a nice big thick math book, with a whole bunch of numbers in it, and a notebook. (70)

Just as she described being "comfortable" with the researchers, she labels her math teacher as her "friend." She uses the same phrase as she did previously and says that they have a "comfortable relationship." Her family, on the other hand, perceives this as "very odd." They have expressed to Romina that a teacher's role should be to give her "homework every night in the book" and that she should have a "nice big thick math book, with a whole bunch of numbers in it." Romina indicates that she values the "comfortable relationship" of a teacher who is a "friend," over textbook work.

Romina explains that the "comfortable relationships" she has developed over the years with the teachers and researchers through the longitudinal study have had a positive effect in her estimation:

> And I've been doing it for ten years, and I think it really pays off. I've had amazing teachers who have gotten so involved in what we're doing. And they're not the regular book teachers. (78)

Having been part of this study for "ten years," Romina says that her teachers' investment "really pays off." She describes "amazing teachers" who get "so involved" with the students' work. She calls these teachers "not the regular book teachers." Again, Romina expresses that she values the personal relationship over the book in her learning.

6.2.4 Learning Process

6.2.4.1 "Arguing," "Disagreeing," and "Group Talking"

"Arguing" is the word Romina uses most frequently when recalling her interaction with her peers in the longitudinal study. She remembers the arguments as not only helpful, but also necessary to solving problems:

> Like we could never do any of the things, well I don't think I could ever do any of the things we do alone. Like they just help you bring out things you didn't know were there. And we have a relationship where we argue a lot, so, like through arguing is where we come up with most of our answers. (20)

Romina explains that neither "we" nor "I" could "ever do any of the things we do alone" – problem solving necessitates group work for her. Crediting collaboration for their success, she explains that the others in the group "help you bring out things you didn't know were there." If one were to solve the problem alone then, one might conclude that knowledge would stay hidden and unexpressed "there." She characterizes the "relationship we have" in the group as one in which "we argue a lot." This collaboration takes the form of argumentation. Romina summarizes that "through arguing is where we come up with most of our answers."

Romina elaborated on the specific nature of how "disagreeing" within the group would catalyze further exploration and justification:

> But if I disagree with someone, they'll have to explain it to me, and if you're explaining it, they're either going to find something right, or they're going to find something more. So, if I don't agree with it, they're going to explain it to me, but if they find something wrong, maybe I can help, and then someone else may disagree with me. And that's how we get through everything. We just disagree. (22)

Romina explains that if she disagrees with someone, they "have to explain it to me." Through the other person explaining, one of two things will happen: "they're either going to find something right or they're going to find something more." Notice that Romina does not say right or wrong - the two possible outcomes of arguing would be to find something "right" or something "more." Indeed, Romina implies that through the process of arguing, students can find something "more" than simply right. Arguing is also an iterative process. She describes how she could disagree with the person explaining and then that person will go back again "to explain it to me" again. If something is found to be "wrong," she or someone else can always "help." She also allows that "someone else may disagree with me" causing her to now take the role of the explainer. Romina claims that this back-and-forth of arguing and explaining is "how we get through everything." She reiterates and labels her collaboration as simply, "we just disagree."

Interestingly enough, Romina identifies "arguing" as localized to math classes as opposed to her other core curriculum courses:

> Well math is where the most arguing is. Like, you can't do this in other classes. It's not like, in English, you read. You don't argue; it's there. It's written. (38)

Romina states that mathematics is "where the most arguing is." Saying that a person "can't do this in other classes," she goes on to position argument squarely in a mathematics classroom. She provides an example of an English class where she says "you read" and "you don't argue" because "it's there" and "written" on the page. Thus Romina implies a belief of something uniquely unwritten about the nature of mathematics.

Romina names the type of learning where actions like arguing, questioning, and explaining take place as “group talking.” She explains what she views as its benefits:

> If people learned the way I did with, like, group talking, I think people would learn more and be able to do more because if someone that was taught with just a teacher teaching them, if you’re given something in, like, the real world, you’re not going to know how to handle it. Whereas I would probably question it, and like, throw different ideas in the air. Other people, they get intimidated, and they don’t know how to do that. (52)

Romina claims that if other people “learned” as she did through “group talking,” then they would be able to “learn more” and “be able to do more.” She asserts that this would transfer to real world applicability. A person “taught with just a teacher teaching” would not “know” how to handle a problem from the real world. Romina argues that she, on the other hand, would “question it” and “throw different ideas in the air.” She would actively work to construct a solution in a group context. People who are unfamiliar with “group talking,” might “get intimidated” and would not “know” to do the same questioning and sharing of ideas that Romina believes she would do in that situation.

Later, Romina again offers what she describes as the helpful nature of argument in mathematical problem solving contexts:

> …But once I explain it to them, and I say, “Maybe you could do it a different way, some way you understand it.” Then, we’ve had arguments but I’ve helped them, so it’s been okay. (60)

Romina gives the example of how she will “explain” to others in a group that there are other possible methods for solving and understanding a problem. She recalls how she would recommend to others that they try a “different way” and a “way you understand it” (recalling perhaps her admission that she needs ideas to be in her own terms and thus others would need problems to be solve in their own way as well). She characterizes these “arguments” she has had as having “helped them” and therefore provided a benefit.

Romina describes herself as someone who likes to socialize with other people, but she qualifies that the tendency to enjoy the company of others over the company of textbooks and homework as normal:

> I like the socializing and, I don't know, I like being involved in things, not, I don't come to school to do homework and work and all that. I'm a normal person. (62)

The process of "group talking" complements someone who likes "socializing" well. However, Romina says that the fact that she likes socializing and being involved in a more interpersonal way is not unique. She describes the desire to want to be social in school as opposed to do homework as being a "normal person."

6.2.4.2 Affective Dimension – "Scared" but "we amazed ourselves"

When asked about her first memories of the Rutgers longitudinal study, Romina recalls many initial feelings as opposed to specific tasks:

> My first memories of Rutgers were, I got pulled out of class one day, and I didn't know why, and I got put into a special class, which is kind of scary, because you don't want to be different back then… We were scared, but then you made it fine. It wasn't that bad. (10)

Being "pulled out of class" and "put into a special class," Romina remembers as "scary." She recalls that she and the other students were "scared" at first because they did not want to be labeled as "different." Despite their initial anxiety however, Romina remembers that eventually "you made it fine" and "it wasn't that bad."

When asked about what she was thinking back in fourth grade, Romina responds in terms of affect instead of cognition:

> We called ourselves your guinea pigs. Because we were never sure. We thought you thought that we were smart, and we didn't think we were all that smart, and we were kind of scared. … none of us had any confidence. (18)

Romina remembers labeling themselves as “guinea pigs.” As she continues she uses language to categorize herself and the others as having had low affect. Specifically, she says that “we” were “never sure,” “scared,” and without “confidence.” She explains that although they thought that the researchers considered them to be smart, “we didn’t think we were all that smart.”

Later, she elaborates on how in fourth grade they thought they were “weird” to be so “interested in math and arguing with their friends about it.” She explains the evolution of how they perceived themselves from fourth grade until high school:

> Yeah, we thought we were real weird. Like, fourth graders interested in math and arguing with their own friends about it? And we still think this today, like, why do we sit here and argue about math? It’s math. It’s not going to, it’s weird for us, and like, in fourth grade, we – I don’t know what happened there that we started arguing. But it just like got us so far. And what happened is we amazed ourselves with the things we got, and where it led us to now, that I guess we’re not that weird anymore. (36)

In fourth grade, Romina said that they considered themselves “real weird” because of their interest in mathematics and arguing. She states that they “still think this today” and wonder “why” they still “sit and argue about math.” Repeating that arguing about math might have been considered “weird,” she says that she is unsure “what happened” in fourth grade to cause them to start arguing about math. Whatever the reason, however, she explains that it had a positive outcome. Arguing “got us so far” and it ended up that “we amazed ourselves with the things we got.” Given the success, she concludes that “we’re not that weird anymore.”

Romina contrasts the positive affect she had as a result of combinatorics tasks in the longitudinal study with the negative affect she experienced during the geometry class in the other school that was not part of the longitudinal study:

> I was not interested in geometry. When I went to school the first day, we were just talking, it was a regular day, and she brought up something about a line. And I was telling her about ***y*** equals ***mx*** plus ***b***. And every kid in the class turned around and looked at me going, 'What are you talking about? I have no clue what you're talking about.' So, I took it upon myself, just like we do in Rutgers, I started explaining it to them, and I got up, I was like, "This is what you do, and this is what M equals, and how it equals that," and the teacher was kind of upset with me, because she didn't want me teaching them. She wanted her to be teaching them. And whenever there was a question to be asked, I'd raise my hand. Or when I had a question, I'd raise my hand, and she wouldn't answer. …Like, she wouldn't answer any of my questions, or she wouldn't call on me. So, it was weird, and it turned me off from math completely. (46)

After first stating that she was "not interested in geometry," Romina tells a story of what happened one day in this geometry class when the issue came up of linear equations. Romina recalls telling the teacher about "***y*** equals ***mx*** plus ***b***." She remembers all of the other students expressing surprise and expressing "no clue" as to what she was talking about. Romina states that she then "took it upon myself, just like we do in Rutgers" to explain slope-intercept form to the other students. For Romina, the "Rutgers" way would be for one student to have to present a justification to convince group members. However, in this case, Romina remembers the geometry teacher getting "upset" because "she didn't want me teaching them" but rather the teacher to teach and Romina to sit and raise her hand with her comments or questions. Romina continues that later in the course when she would raise her hand with a question, the teacher "wouldn't answer any of my questions" or "call on" her. She terms this experience of not being about to ask questions and explain ideas to her classmates as "weird" and says "it turned me off from math completely."

6.3 Reflections II - PUPMath: July 21, 1999 (12^{th} Grade)

6.3.1 Setting

In an interview conducted with the assistance of the Science Media Group of the Harvard Astrophysics Observatory on July 21, 1999 after an NSF-funded two-week Summer Institute held at Rutgers University, Romina discussed her reflections on the longitudinal study as she got ready for her 12^{th} grade senior year that coming September. The interview lasted 38 minutes and followed a loosely structured format whereby the researcher questioned Romina about the following topics: her memories of the longitudinal study, her reactions to the summer institute of which she had just been a part, her thoughts about mathematics and learning, and her self-perception as a problem-solver. After the interview was transcribed and verified, "significant statements" were tagged and analyzed. For the full transcript, see Appendix F. Her significant statements clustered into three thematic categories: knowledge, conditions of her learning environment, and her learning process in general.

Reflections II Interview – July 21, 1999 - SIGNIFICANT STATEMENT SUMMARY	
Issue	***Significant Statement(s)***
Knowledge & Knowing	***Two Types of Math***: "I think there's two big different areas of math: one of them is like the thinking involved and one of them is just like spitting out numbers." (85)
	Intelligence based on "Ideas" versus "Memorization": "Maybe they don't think we are as smart as we can like as we are but if you give real problems like problems that actually matter not just spitting back numbers and then memorization, we can apply… we can come up with pretty interesting things like point of views and ideas that no one else really looks at" (101)
	Constructing Knowledge versus Receiving Knowledge: "I don't like being reassured in like the problem. Like I look for reassurance but if they gave it to me, it's almost like they're treating me like a little child… when we do come up with something it's so much better because we came up by ourselves without someone holding our hand and walking us through it…" (83); "We kind of had to like invent anything… we had to choose what path we were going to take" (69)
Conditions for the Learning	***Tasks that "Interest Us":*** "If it is something that seems too easy or something that we can get done in a matter of two minutes, then we will do it real quick… the problem really has to interest us" (16); "it was typical Rutgers. They give us something – they give us like very little information about something and see what we take it to" (49)

Environment	***Sufficient time to "Talk about it Forever"***: "… we will talk about it forever – we will argue about it forever. We will do anything that's required. We'll come up with anything, like we will come up with weird things too. We will keep going as far as we can with the problem if we are interested in it." (18); "I can be using my time for more like thinking – like thinking up my own ideas" (104)
	Collaboration for "Sharing our Ideas and Working in Groups": "…it usually works best when we disagree with each other cause that way we will be like it – it helps so much. We are the type of people that if we disagree with each other just cause we are disagreeing we will work on any problem you'll give us." (16); "We did a lot of thinking – like we just sat and thought for hours a day and we came up with a lot of interesting things and we were able to go in front of a large audience and just talk about our ideas and then argue our points and prove our points" (81); "Like if you gave us like this big long test with all these problems that seems like a lot for us cause it's either right or wrong, but like when we come in here we are just sharing our ideas and like working in groups to come out with an answer" (93)
Learning Process	***"Group Thinking" & "Asking Questions":*** I like pushed him along a lot throughout the thing cause when he'd go up there and presenting, I would ask him questions and he hated that so much but by the end he was like, 'It's all right – I expect questions from me [Romina].' (34); "… we came up with so much – many different like point of views and areas and methods and like we had hour conversations about our math which I didn't think was possible. Not a lot of people think you can talk about math but we – it was just surprising what you can do and what like how controversial it could get." (67); "… first of all, I wouldn't be like finding the solution for a big problem by myself. I would – a lot of other peoples they'd be like – we would have to have some sort of arguing like to bring up points that maybe I don't see that could help the solution. And people arguing will help and people would just keep talking about it and we have to find as many solutions as possible and go from there to see which one's the best solution." (97); "Like I am a more verbal person. I can speak well and I can communicate my ideas where other people might like my same age level can't because they never had to – they don't know. They're intimidated where I was kind of put on the spot and had to and it just develops your idea and maybe when we are like running the world, we can come up with better solutions cause we know more and we can like we've practiced and we have been able to have like group thinking and solutions." (99)
	Connecting to "Real Life": "… we were able to put like real life things into it and like what can it affect it like not math like real things." (61); "There's like real life situations that math doesn't account for." (63); "… math is just so vague and in so many areas of something – it's everywhere… you can't get away from it – it's everywhere you can find and every situation you could possibly think of" (87)
	Affective Dimension – "People Underestimate Us": "professors we were to have in years to come were like kinda impressed by what we were doing and how we where thinking, maybe we can do it so maybe it changed us a lot." (91); "I think we could do it if we really we would have to work at it, but from what we know, I think we can handle it." (95); "People underestimate what we can do and if you can just give us problems and keep working at it – it like builds us up. Makes us more." (99)

Table 6-2. **Reflections II Interview – July 21, 1999 – summary of significant statements**

6.3.2 Knowledge and Knowing

6.3.2.1 Two Types of Math: <u>Thinking</u> versus "spitting out numbers"

What is "math" and how does one come to know something in mathematics? Toward the middle of the interview, Romina explained that she thought there were "two big different areas of math." She defined these two areas of math as follows:

> I think there's two big different areas of math: one of them is like the thinking involved and one of them is just like spitting out numbers. I know I was never

> good at the spitting out number thing and everything but I was decent at the thinking about it. (85)

Romina divides math into two distinct, non-overlapping regions: “thinking” and “spitting out numbers.” She claims that she was “never good” at the “spitting out numbers” part of mathematics, but that she was “decent” at the “thinking about it.” She seems to attribute a negative connotation to the type of mathematics she terms “spitting out numbers.” Notice that even the language used locates this mathematics in the mouth (not the brain) – you are only expected to “spit” back a number as you would a distasteful bit of food. In contrast, “thinking” would require personal authority and cognition.

6.3.2.2 Intelligence based on “ideas” versus “memorization”

Following her two-part mathematics definition of “thinking” and “spitting back,” Romina splits intelligence into two categories: “ideas” and “memorization.” Speaking in the first person plural throughout, she describes that “we” can do “interesting things” when we do not just “memorize”:

> …Maybe they don’t think we are as smart as we can like as we are but if you give us real problems like problems that actually matter not just spitting back numbers and then memorization, we can apply. It’s almost like higher level thinking like real life situations… we are pretty rational people when we can come with interesting things like point of views and ideas that no one else really looks at especially for our age level. They just think that we can memorize things – that’s it and if we can’t memorize things, we are not that intelligent. (101)

Notice that Romina distinguishes between problems that are “real” and “actually matter” versus ones that require just “spitting back numbers” and “memorization.” She asserts that they use “higher level thinking” in “real life situations.” Romina dismisses intelligence based solely on memorization: an unidentified “they” think that “if we can’t memorize things, we are not that intelligent." For Romina, the "interesting things" that indicate intelligence include "point of views and ideas."

6.3.2.3 Constructing knowledge versus receiving knowledge

In reference to constructing knowledge versus receiving knowledge, Romina says that "when we do come up with something it's so much better" (83). She likens received knowledge to "someone holding our hand." Specifically, she says that:

> I don't like being reassured in like the problem. Like I look for reassurance but if they gave it to me, it's almost like they're treating me like a little child… when we do come up with something it's so much better because we came up by ourselves without someone holding our hand and walking us through it… we are going to get the right answer, but if they do this they don't know what's going to happen, what direction we are going to take, so it makes it all the better. (83)

Romina asserts that, "I don't like being reassured" in a problem. She defines "reassurance" as when "they're treating me like a little child," "holding our hand," and "walking us through." She says this process of instruction allows the students to "get the right answer." If "they" – the instructors – had not "reassured" them, then the students could go in an unknown "direction" that "makes it all the better."

Given her repeated use of "better" in reference to when she and her peers "come up with something" themselves, one might infer that Romina believes that personally constructing a concept is more valuable and lasting than being given direct instruction on the topic. Here, Romina places constructed knowledge above received knowledge. By comparing received knowledge with someone "treating me like a little child" and "holding our hand," she equates such an experience with cognitive immaturity. She seems to makes a value judgment that their thinking is "better" when "they" do not directly instruct her, but rather allow her to construct her own knowledge.

Romina also recalls how they had to "invent" their knowledge in the longitudinal study and make choices about what "path" they would take to solve problems.

> We kind of had to like invent anything… We had to choose what path we were going to take and what we were going to do. With other things we had things given to us, with this we kind of had to make up our own – we all had to agree with the other group…. So it was a lot more compromising. (69)

Romina's language contrasts the active role she and her peers took in their problem solving during the longitudinal study versus the more passive role normally taken when "we had things given to us." In their problem-solving situations, she recalls that "we" would "invent," "choose," "make up," "agree," and engage in "compromising." Again, Romina places constructed knowledge over received knowledge. This type of knowledge is also defined in terms of the collective. She speaks only in the first person plural "we" again and describes how "we" would have to negotiate with "the other group."

6.3.3 Conditions for the Learning Environment

During the interview, Romina repeatedly refers to three conditions that should be present in her learning environment: *tasks that "interest us"* – inviting her and her group members to solve a complex, open-ended problem; *sufficient time that allows students to "talk about it forever"* – providing extended time for investigation and reinvention; and *collaboration* – promoting what Romina terms "group thinking." One should also note the frequency with which Romina uses the first person plural "we" when describing her experiences in the longitudinal study – she very rarely employs a singular pronoun when referring to her own learning and problem-solving. One might then infer that a final underlying condition of Romina's learning environment is the presence of a plural lens of "we" through which to consider problems as opposed to only the single "I."

6.3.3.1 Tasks that "interest us"

Romina explains that a problem cannot seem "too easy" or too "quick" to solve, but rather it "really has to interest us":

> If it is something that seems too easy or something we can get done in a matter of two minutes then we will do it real quick. We will like we will throw an answer on a sheet and we will stop, I mean the problem really has to interest us… (16)

With tasks that seem "too easy" or too "quick," Romina claims that she and her group will "throw an answer on a sheet" and "stop." The negative connotation of "throw[ing]" an answer recalls her earlier association of "spitting" back numbers in certain math problems requiring only memorization. Notice also that the problem does not just have to interest her alone, but instead has to interest "us." Throughout the interview, Romina returns to how tasks must engage her entire group – she refers to her own mathematical thinking almost exclusively within a first person plural context of what "we" did or how things would occur to "us." Her discussion of problems that "really interest us" also recalls her description of "real problems… problems that actually matter" and links these types of complex, open-ended tasks with "higher level thinking" (101).

Romina defines the complex mathematical tasks she experienced in the Summer Institute along with the others from the longitudinal study as "typical Rutgers":

> … it wasn't structured – they didn't give us all, 'This is what it is' and 'This is what we want you to figure out.' It was just typical Rutgers. They give us something – they give us like very little information about something and see what we take it to… (49)

According to Romina, "typical Rutgers" tasks in the longitudinal study are not "structured" – the students are given "very little information" at the outset. She provides two examples of what the researchers would NOT say: 'this is what it is' and 'this is what we want you to figure out.' She implies that there was an open-ended, complex nature to the tasks and that the researchers were looking for more than a particular answer. Instead of a task requiring a quick application of an algorithm, these tasks for Romina seem to lead them to new locations of thought. She observes that the researchers wanted to "see what we take it [the problem] to."

6.3.3.2 Sufficient time – "We will talk about it forever"

Given that "we are interested in it [the problem]," Romina claims there are no bounds on the amount of time that she and her group will invest in problem-solving:

> … we will talk about it forever – we will argue about it forever. We will do anything that's required. We'll come up with anything, like we will come up with weird things too. We will keep going as far as we can with the problem if we are interested in it. (18)

Notice the language of limitlessness that Romina uses to explain how much time "we" will use to work on a problem. They "talk" and "argue" about problems "forever." Romina does not mention any that there was ever a time by which they needed a problem to be completed. Rather, a problem for Romina seems to be something that one could spend "forever" talking about. It is significant to note that Romina never says that they finished a problem here or elsewhere in the interview, but instead implies a continuation to their tasks – "we will keep going as far as we can" and "we'll come up with anything... with weird things too." Instead of closure, there is continuing exploration and discussion as the group discovers new and unexpected "weird things" about a problem.

Sufficient time seems to be a condition then for Romina's learning environment - time to "talk" and "argue" mathematics. Later, she explains that she believes her time should be spent in thought: "I can be using my time for more like thinking – like thinking up my own ideas" (104). Romina's time should be used for "thinking" because that thought leads to her own "ideas."

6.3.3.3 Collaboration – "Sharing our ideas and working in groups"

In addition to being given enough time and a problem that "interests" her and her group, Romina also invokes a collaborative atmosphere based on "disagreeing" as a necessary condition to their work:

> …it usually works best when we disagree with each other cause that way we will be like it – it helps so much. We are the type of people that if we disagree with each other just cause we are disagreeing we will work on any problem you'll give us. We will go on and on until we figure which one of us wins kind of. (16)

It "helps so much" and "works best" for Romina in a problem-solving situation when "we disagree." She says that just by "disagreeing," they will continue to "work on any problem" and "go on and on."

Romina also comments on her familiarity and comfort with both her group members and the researchers. When the interviewer asks if seeing some of the other Kenilworth students in the Summer Institute was "a bit of a reunion," Romina agrees that, "I remember working a lot with those kids" and " we all got along now" (20). She observes that "if you're from Kenilworth, you're from Kenilworth forever" (26). She also describes the researchers from the longitudinal study as her "acquaintances" and says that "we get along well" (36).

She summarizes her time during the Rutgers Summer Institute as one of "problem solving," "thinking," talking, arguing, and proving:

> … we did a lot of problem solving. We did a lot of thinking – like we just sat and thought for hours a day and we came up with a lot of interesting things and we were able to go in front of a large audience and just talk about our ideas and then argue our points and prove our points- I think it was a very good experience. (81)

Her elaboration on what it means to engage in "problem solving" involves spending a great deal of time "thinking" where they "thought for hours a day" and came up with "a lot of interesting things." The thinking over extended periods of time would then lead to presentations "in front of a large audience" where they would "just talk about our ideas" and then have to "argue" and "prove our points." She summarizes the it as a "very good experience."

She explains that the expectations in the Rutgers longitudinal study are different from their regular schooling.

> Like if you gave us like this big long test with all these problems that seems like a lot for us cause it's either right or wrong, but like when we come in here we are just sharing our ideas and like working in groups to come out with an answer like this – it's not easier for us, it's completely different, but we usually we don't think people expect that. (93)

Here she contrasts the "big long test" with answers that are "either right or wrong" with what they do "here" in the longitudinal study. She describes their work as "just sharing our ideas" and "working in groups." Romina recognizes this environment as "completely different" and something other people would not "expect."

6.3.4 Learning Process

The conditions that Romina includes in describing her learning environment correlate directly with what she identifies as her learning process as well – thinking about meaningful tasks over an extended period of time in collaboration with others. She discusses how she engages in "group thinking" and "asking questions" during problems that she is able to connect to her "real life." She expresses an affective dimension of this learning as well. She explains "people underestimate us" and "don't think we are as smart," but that problem solving as they do in the longitudinal study "builds us up."

6.3.4.1 "Group thinking" and "asking questions"

Romina explains that during the Summer Institute she "pushed" a fellow student named Victor who was not originally part of the longitudinal study by consistently seeking to "ask him questions" during his presentations to the group:

> I like pushed him along a lot throughout the thing cause when he'd go up there and presenting, I would ask him questions and he hated that so much but by the end he was like, 'It's all right – I expect questions from me [Romina].' (34)

Romina views asking questions as something that will encourage and help another student's presentation and general problem solving. She associates asking questions as a very positive give-and-take of ideas. The learning environment she describes is one that is dynamic (the "pushing" back and forth of questions does not seem to end) and contributes to learning growth. She indicates that asking questions as part of problem solving is a learned behavior – at first this other student Victor "hated" her questions and then came to "expect questions from me" and judge it as "all right."

Reflecting on her experience in the Summer Institute when they worked on the *Placenticeras* and Catwalk problem, Romina attempts to summarize their learning:

> … we came up with so much – many different like point of views and areas and methods and like we had hour conversations about our math which I didn't think was possible. Not a lot of people think you can talk about math but we – it was just surprising what you can do and what like how controversial it could get. Like how many different opinions and ideas and like we all had we all knew we were working with the same things, but we had so many different ideas. (67)

She makes certain observations about the time-consuming, collaborative nature of their work: "hour long conversations about our math" where "different" points of views and methods were shared. Romina remarks that math can be “controversial” and that “not a lot of people think you can talk about math.” She emphasizes and reiterates that it can be “surprising” to people “how many different opinions and ideas” they can have when solving a problem. She also returns to the idea that knowledge is constructed as opposed to received as she describes learning as a process of personal authority where they “came up” with their own “points of views,” “areas,” “methods,” “different opinions,” and “different ideas.” Notice also that, according to Romina, they were working on something personal – “our math” – and mathematics necessitates that people “talk about” it. Romina marvels that they could all be working on the “same things” and yet have “so many different ideas.”

When asked more specifically about how she solves problems, Romina emphasizes first and foremost that mathematics for her is not a solitary pursuit:

> … first of all, I wouldn’t be like finding the solution for a big problem by myself. I would – a lot of other peoples they’d be like – we would have to have some sort of arguing like to bring up points that maybe I don’t see that could help the solution. And people arguing will help and people would just keep talking about it and we have to find as many solutions as possible and go from there to see which one’s the best solution. (97)

Romina explains that she would not be “finding the solution for a big problem” by herself, but rather be engaged in a collaborative effort with “a lot of other peoples.” She describes this process of problem solving as one where there a great deal of dynamic vocalization by repeating that there is “some sort of arguing,” “people arguing,” and people “talking.” Through this “arguing” and “talking,” then “we” would be able to “bring up points” and find “as many solutions as possible.” Finally, the group would then evaluate which of these suggestions is the “best solution.”

Romina describes herself as a "verbal" learner who needs to "communicate my ideas" in order to solve a problem. She asserts that it is through communication and "group thinking" that "better solutions" can be found:

> Like I am a more verbal person. I can speak well and I can communicate my ideas where other people might like my same age level can't because they never had to – they don't know. They're intimidated where I was kind of put on the spot and had to and it just develops your idea and maybe when we are like running the world, we can come up with better solutions cause we know more and we can like we've practiced and we have been able to have like group thinking and solutions. (99)

Throughout this excerpt, Romina emphasizes what she identifies as her communication skills; she is a self-described "verbal person" who can "communicate my ideas." She recalls being "put on the spot" to present ideas in front of a group through the longitudinal study, but claims that the process of convincing others "develops your idea." Being able to "communicate" and engage in "group thinking" constitute a type of knowledge for Romina. She claims that other students at her "same age level" do not "know" how to do this possibly because they are "intimidated," whereas she does "know" how. She implies a belief that this will have a positive outcome when "we are running the world" because they will "come up with better solutions cause we know more."

6.3.4.1 Connecting to "real life"

Just as she contrasted "thinking" and "spitting back numbers," Romina makes a distinction between "math" and "real things." She explains what she and her group did with the Catwalk Problem during the Summer Institute:

> … we were able to put like real life things into it and like what can it affect it like not math like real things. (61)

Romina expresses that they brought “real life things” to their problem solving. She reiterates and says it was “not math,” but rather “real things.”

Later she continues discussing her distinction between mathematics they learn in school and what she terms as “real life”:

> … when you do math, we do math two plus two equals four – there is nothing involved and when we do like word problems we never take anything else into consideration like you take when in real life like little things like a person like just running, that just doesn’t happen all of a sudden. You have to be kind of gradual to it or just things like air resistance and things just friction. There’s like real life situations that math doesn’t account for. (63)

Romina argues that there are many “real life situations” that the “math” of school “doesn’t account for.” When a person is running, she provides examples of “air resistance” and “friction” that one might not consider. She defines school “math” as “two plus two equals four” and “word problems” where “we never take anything else into consideration.” She labels this type of math as “nothing involved” and seems to express a belief that school mathematics does not usually connect to her life.

When the interviewer asks, “What is mathematics?” Romina responds that mathematics is “problem solving.” She then elaborates:

> I think it’s problem solving. It’s taking up a lot of things into consideration, coming up with a reasonable answer to something… math is just so vague and in so many areas of something – it’s everywhere… you can’t get away from it – it’s everywhere you can find and every situation you could possibly think of (87)

The far-reaching and all-encompassing mathematics of “problem solving” and “taking up a lot of things into consideration” that Romina defines here is in sharp contrast to the “math” of “two plus two equals four” she described before that was disconnected from her life. She seems to indicate here that mathematics should be “everywhere.” Connection is the theme to her definition: repeating “everywhere” twice, it is “in so many areas" and "every situation" so much so that "you can't get away from it." Just as we heard Romina defining "two areas of math" earlier in the interview, again we hear her divide mathematics into two categories: "two plus two equals four" that does not connect to her life and "problem solving" that is "everywhere" and connected to "every situation you could possibly think of."

6.3.4.3 Affective dimension of learning – "people underestimate"

When Romina reflects back on her feelings before coming to the Summer Institute, she describes "low self-esteem":

> … we all have very low self esteem about everything and we didn't think we were capable. We were very scared coming to this two weeks cause we thought a lot was expected from us and we were not going to be able to perform under all the pressure… (81)

Romina says that "we" all have "very low self esteem" and were "very scared" coming to the two-week Rutgers Summer Institute. She explains that they thought "a lot was expected" and that they would not be able to "perform under all the pressure."

Given that many of the students in the Summer Institute were veterans of the longitudinal study, Romina said she was not sure what had happened and why they had become "turned off by math," but that the problem solving over the summer had renewed their feelings of confidence:

> …we were already turned off by math and we already thought we couldn't do it and that was it – our math career was over when a lot of us had hoped to pursue math in the future but this changes it around a little because if we were able to go in there like professors we were to have in years to come were like kinda impressed by what we were doing and how we where thinking, maybe we can do it so maybe it changed us a lot. (91)

Romina asserts before the summer problem solving, for her and many of her peers, "our math career was over" even though "a lot of us" had planned on pursuing mathematics in the future. They had become “turned off by math” and thought “we couldn’t do it.” What made them feel more confident was the fact that “professors” they might have in the future were “kinda impressed” by what they were “doing” and “thinking.” An affective dimension thus emerges as important to Romina’s problem solving. She seems to express that having an authority like “professors” be impressed with her thinking helps reinforce that “we can do it.”

The interviewer asks Romina what she would do if she would “know what to do” if she got a “real world problem” on the “job.” Romina responds that “we” would be able to work on the problem even though she expects it would take a long time:

> …if they gave us a problem and we work at it until we get somewhere until we start right off in the right direction, it might take us a long time, but pretty much anything – I think we could do it if we really we would have to work at it, but from what we know, I think we can handle it. (95)

Notice that Romina immediately rephrases the question into what “we” would do as opposed to what she individually would do. She says that “we would work at it” and it might “take us a long time.” Based on what “we know,” she says that they would be able to “handle” a problem on the job. She seems to express a belief that her ability to solve a real world problem is predicated on a collaborative effort taken over a period of time.

Later she explains that outside people think of students her age and how problem-solving like she engages in with the longitudinal study helps them:

> People underestimate what we can do and if you can just give us problems and keep working at it – it like builds us up. Makes us more. (99)

Romina says that people “underestimate what we can do.” Later, she repeats this sentiment later about the judgment of an outside “they” on the students’ intelligence when she observes that “maybe they don’t think we are as smart” (101). She explains that, given “problems” and the time to work on them, the problem-solving itself will “build us up” and make them “more.”

Chapter 7 INTERVIEW RESULTS – Undergraduate and Career

> It makes a difference where and when we grew up. The culture we belong to and the legacies passed down by our forebears shape the patterns of our achievement in ways we cannot begin to imagine. It's not enough to ask what successful people are like, in other words. It is only by asking where they are from that we can unravel the logic behind who succeeds and who doesn't.
>
> - from *Outliers* by Malcolm Gladwell (p. 19, 2008)

7.1 Introduction

Chapter 7 summarizes data analysis based on interviews videotaped when Romina was a college undergraduate in 2002 and then a business analyst at Deloitte Consulting in 2006. What sets the interviews over this time span apart from the rest, and why they are presented separately here in their own chapter, is that they offer a new lens on Romina's beliefs about her own problem solving and learning in the longitudinal study as seen in contrast to the two new learning environments in which she found herself during the interviews: the college classroom and the business workplace. The two interviews included here were also set in larger group settings. The 2002 filming took place in a math seminar meeting where both Romina and Jeff joined a collection of math education graduate students and professors. The 2006 interview had three women from the longitudinal study - Romina, Magda, and Angela – join math education researchers at another seminar meeting of Rutgers students and professors.

7.2 Reflections III – Math Seminar: March 11, 2002 (College Sophomore)

7.2.1 Narrative

An interview of both Jeff and Romina was conducted during a seminar meeting of math education researchers on March 11, 2002 at the Graduate School of Education on the Rutgers-New Brunswick campus. Eight researchers were present in addition to the two longitudinal study participants, Jeff and Romina, who were college sophomores at the time. In addition to questioning from the researchers, the students were asked to view and comment upon video clips from the PUPMath-Rutgers collaboration. Romina saw herself in 4th grade working with Brian on the Towers problem. She also viewed the videodata of her engaged with Jeff, Brian, Michael, and Ankur in 10th grade on the "Ankur's Challenge" problem. Through the course of the interview, Romina discussed her reflections on the longitudinal study as it related to the videodata clips she was viewing and the college courses, specifically calculus, which she was taking at the University of Pennsylvania. The interview lasted 134 minutes and followed a loosely structured format whereby the researcher questioned Romina about the following topics: her memories of the longitudinal study, her reactions to the video clips from 4th and 10th grades she watched, her thoughts about mathematics and learning, her self-perception as a problem-solver, and her comments on college coursework. See Appendix G for the full transcript. After the interview was transcribed and verified, "significant statements" were tagged and analyzed. Her significant statements clustered into three thematic categories: knowledge, conditions of her learning environment, and her learning process in general.

Reflections III Interview –March 11, 2002- SIGNIFICANT STATEMENT SUMMARY	
Issue	***Significant Statement(s)***
Knowledge & Knowing	***We Needed to Know from the "Beginnings":*** "We each needed to know from the absolute, like, beginnings, because if we didn't, you would ask…"(88); "You would ask me, and I would be like, 'I really don't know,' and then I'd try to ask Michael." (90); "everything has to make sense in my terms other than I can't like I someone else might have done it already in a book but I just don't understand it unless I do try it myself and put it in my own terms." (428)
	Knowing the "Background" versus Using the Formula: "I didn't know a lot of, like, the simple notation, and I would work with a friend, and she could spit out all the formulas, and she didn't understand it, and I only knew the background behind every formula…" (121); "So I brought out towers and I was like, "Say you have towers four high, and you have two colors" '…" (123); "…They say it's a very professional-oriented school so they don't deal with a lot. They just give you an

	answer, and it's like that in all my classes..." (159); "… I didn't even know how to add, like, exponents 'cause I just never thought of it like that." (163)
Conditions for the Learning Environment	***Teacher-Researcher's Role: Introduce Formalization AFTER Understanding***: "That's why I remember it, 'cause then when you taught us the, uh, how to write it actually what we were doing for years…" (70); "…we never, we never formalized … we had a way of thinking about this and we always pretty much tried this same way, but we needed to end it… We came up with that formula, and then we actually use that formula now." (76)
	Collaboration - "Discussions" and "Group Work": "I'm better at learning if like thinking about things, discussions, group work, and I've always been, and now when I, now I'm not, I'm not doing as well as I think I could be doing in college because we're just not taught like that anymore…" (226); "I think I, I deal with groups. I work very well with groups…" (232)
	"Horrible" College Calculus – Learning in a "Completely Different Way" from the Longitudinal Study: "I don't, I don't really have a chance to apply it much in college. …In my college, I'm learning in a completely different way…" (246); "That's why I, I did so horribly and they and it was a ten-page exam…they only want the answer." (132); "And they did evil things, like… It was like 2.5e to the -3 and .25e to the… It was horrible, like they made 'em all really close so if you were off even one little…" (134); "Well, the way we did it, we were taught, each of us were taught by, um, there were two hundred lecture…" (149); "I don't learn well. Like if you give me a book …" (226); "…That's horrible to say, but, after a while, I just gave up. It just wasn't worth my time." (228)
Learning Process	***"Argue" Mathematics:*** "…we'd argue it out and then probably take ideas from each other and then worked from there and come back." (16); "We used to call each other and we'd just discuss ideas and what happened and details and things. And that's how I learn – and it's a group setting. It's - I learn in groups." (259); "'…And I don't talk to anyone. I just keep reading over and I just don't retain the information. As soon as I – if I do sit up all night, and try to memorize my sixty pages of notes and I write down what I know on the exam. As soon as I walk out of there it's done. I don't remember anything." (261); "I think a big way we learn is we tend to argue a lot..." (275)
	Concept before Formula - We Learned a "Thought Process": "If I didn't understand, if I didn't understand a problem, or if I didn't work enough through it by myself to understand where like… I guess Michael didn't know where I was heading with what I was doing, and if I didn't understand where the other person was heading, I liked to work on it before I form a couple options and see which one he takes." (30); "I came up with this is how I would do it, now what formulas would I use to get the answers if I were to do it like this?" (179); "I think we learned more of a thought process and how we deal when we were first given questions…" (181)
	Affective Dimension – This is an "Accomplishment of Mine": "And I feel that this is now an accomplishment of mine..." (474); "It just seems like so many times they were impressed by what we would do and we would just sit there and be like we are doing anything of any value." (497); "I don't know if this was a direct correlation but we were the ones who did better in school in general…" (515); "Like our group of kids. We were in the top in the class. Out of everyone involved in this program we were all pretty much 1 through 10." (517)

Table 7-1. **Reflections III Interview – March 11, 2002 – summary of significant statements**

7.2.2 Knowledge and Knowing

7.2.2.1 We needed to know from the "beginnings"

After viewing the "Night Session" video clip of their younger selves working on May 12, 1999, Romina and Jeff discussed how they made sense of more abstract combinatorics like the addition rule for Pascal's Triangle (Pascal's Identity) at that time. Uptegrove (2005) provides a full background and analysis of the Night Session as the students built Pascal's Identity by recognizing and using isomorphic relationships between combinatorial tasks like Pizzas and Towers with which they were already familiar. When asked about how they would develop their justifications for the researchers, Jeff remarked that "if we tried to just present a final thing, and really didn't know it from the beginning, we couldn't explain it in a way that you would accept from us" (85). T/R1 commented that it seemed that Jeff and Romina seemed to "demand the same thing of each other" when justifying and supporting arguments. Jeff and Romina agreed with this statement. Romina elaborated:

> We each needed to know from the absolute, like, beginnings, because if we didn't, you would ask… (88)

The knowledge that Romina discusses here is one of active construction. She explains that they "needed to know" from the "beginnings" of a task – indeed, the "absolute" beginning. She implies if she and her peers did not provide a complete and thorough justification of each step in their work (all the way back to the "beginnings" of their reasoning), they expected the researchers would request it – "you would ask." Romina remembers when the researcher would ask her for more information:

> You would ask me, and I would be like, 'I really don't know,' and then I'd try to ask Michael. (90)

The process of deeply examining justifications and representations was iterative: the researcher would ask the students and the students would ask each other and then the researchers would ask again. In Romina's recollection, she would be the student who would have to reply that, 'I don't know." Michael was a student she remembered turning to for help in understanding.

With the expectation that their arguments would have to be justified, Romina later comments that "we had to be prepared" (96) to explain their reasoning. She explains what happened when their high school teacher, Mr. Pantozzi, had tried to explain to them combinatorial notation earlier in class before the Night Session and how the knowledge they gained at that time contrasted with their knowledge from the Night Session itself:

> And I think, earlier in class, Mr. Pantozzi had written that and we all, he's like "you should know this." And we all looked at him like "I don't know what you're talking about." And he was like this, and he tried to relate it back for us, and we just didn't see how we reached from what we, from the work we had done to that formula. So we had to start at the very bottom and then she showed us. She showed us that extra step that we were missing. (105)

Romina remembers her experience when their teacher, Mr. Pantozzi, wrote about the additive rule in combinatorial notation ("that formula") on the board and then even "tried to relate it back for us." Despite his efforts, Romina says that they "didn't see how we reached" that formula from his explanation. She says that they were left thinking, "I don't know what you're talking about." Thus Romina implies that her experience of receiving the knowledge of the additive rule was unsuccessful – as a passive recipient to her teacher's explanations, she still "didn't see" and "didn't know." Romina explains that it was necessary for the students to actively rebuild meaning for themselves – "we had to start at the very bottom." At the Night Session as part of the longitudinal study that evening, they reconstructed the isomorphisms among the towers, pizzas, and Pascal's Triangle. They developed their own notation but it was not yet formal. Romina observes that the researcher needed to provide them with the formal notation – "she showed us." After having constructed the rest of the meaning themselves, that formal notation then became the only "extra step that we were missing."

Romina emphasizes the necessity of constructing her knowledge versus receiving it not only in the specific example of the Night Session, but also more generally. One of the researchers asked her to comment on her earlier statement in the May 18, 1999 interview in which she said that "everything has Romina's definition to it." Although Romina did not recall her specific words from 1999, she observed:

> …I understand it cause everything has to make sense in my terms other than I can't like I someone else might have done it already in a book but I just don't understand it unless I do try it myself and put it in my own terms. (428)

When she says that "everything has to make sense in my terms," Romina reiterates the same idea from her 1999 interview. Knowledge must be personally constructed – "I do try it myself" – and then uniquely voiced "in my own terms." Implied also is Romina's belief in personal ownership of knowledge. Her fingerprint must be on both the action (the doing and trying) as well as the oral or written representation (saying or writing in her "own terms"). Here again, Romina places constructed knowledge above received knowledge as her preferred mode of learning.

Romina expresses the belief that all of her learning comes through active construction. When asked about gaining understanding through a textbook, Romina asserts that textbooks are unsuccessful for her:

> Like that I have learned successfully through a textbook? I can't think of anything right now. (432)

Romina wonders aloud if she has ever "learned successfully through a textbook." She claims that she cannot think of "anything right now." Indeed she supports her earlier comments about her difficult understanding when only being told through a verbal explanation or written one on the blackboard.

7.2.2.2 Knowing the "background" versus using the formula

Romina describes her experience in Calculus I and Calculus II in her freshman year of college as very "difficult" and attributes this difficulty to her lack of knowledge about notation and formulas:

> I took Calculus all last year, Calc I and Calc II, and it was very difficult for me, and a major part of that was, um, I didn't know a lot of, like, the simple notation, and I would work with a friend, and she could spit out all the formulas, and she didn't understand it, and I only knew the background behind every formula… (121)

According to Romina, the "major part" of why Calculus was so difficult was that she "didn't know" the "simple notation" and "formulas" necessary for her classes. She professes to have known the "background behind every formula" however. Romina contrasts her friend's procedural knowledge with her own conceptual knowledge. Her friend could "spit out all the formulas" though she "didn't understand" them, whereas Romina "only knew the background behind ever formula."

She expresses a clear concept orientation as opposed to a rule orientation in her description of knowledge in the content area of calculus. It is interesting that she seems to imply that knowing concept and knowing formula are mutually exclusive for her. Romina describes her ability to know "background" but "not know" the notations and formulas. To explain a topic she would try to help others understand the concept *behind* it – the "background." She uses the Towers Problem, an abiding learning metaphor for her, as a building block for conceptual knowledge. Romina describes how she would study with her friend in this college calculus class:

> So I brought out towers and I was like, "Say you have towers four high, and you have two colors" 'cause we had four choose two or something. I'm like, and then, relating it to… Cause I knew how to do it, like I understood, like, "say this one is four choose zero, so you have none of this color, and now you have one of this color, four choose one," and I went through this whole explanation. She's just

> looking at me. She's like "you claim you can't even…" I'm like, "no, 'cause I don't know the formulas.' I don't know that that means that, but this is if we were to think of it like that, this is the reasoning… (123)

Romina recalls how she "brought out towers" four-high with a choice of two colors to explain to her friend how to make sense of the combinatorial concept of "four choose zero." She says she "understood" when "relating it" to the Towers problem. Her experiences in the longitudinal study translated to how she would approach problems in college. Romina professes to have been able to go through "this whole explanation" of combinatorics but that "I don't know the formulas." If they were able to use "reasoning" and "think of it" as with the Towers in the longitudinal study, then Romina thinks she would do better in class. The assimilation paradigm of the Towers Problem emerged as a powerful tool for Romina's learning.

Knowing for Romina is an active pursuit of "reasoning" through a concept and "relating" her conceptual knowledge with generalized formulas. She describes it as difficult to memorize a formula that does not relate to a previous concept she has built. In her explanations there exist two different types of knowing: from her time with the longitudinal study and from her experience at her university. She describes the focus of the university classes:

> …They say it's a very professional-oriented school so they don't deal with a lot. They just give you an answer, and it's like that in all my classes. They don't… They never have to explain anything; they just… that's how they were taught. So it was easier for them, but I struggled through Calculus and they didn't 'cause they just knew the formula, they just put the numbers in and they got an answer. (159)

Romina provides the reason that the university is "very professional-oriented" for why "they don't deal with a lot." For Romina, this means that "they just give you an answer" and students "never have to explain anything." She perceives that it is "easier" for the other students who were “taught” this way. On the other hand, she “struggled” through calculus whereas the other students “didn’t.” Romina implies that other students possessed a different type of knowledge. The other students “just knew the formulas.” Romina’s language makes this “formula” knowledge sound like a machine in which students could “just put the numbers in” and get out an answer. Romina contrasts the two actions as a tension between conceptual and procedural knowledge: to “explain” versus to “put the numbers in.”

Romina further elaborates on her experience with the focus on procedural knowledge in her university-level calculus classes:

> Like, I didn’t know basic things: how to manipulate log. Like you know how… I don’t know kinda like if you multiply two different logs and you get… I didn’t know how to do that. I had to learn that to take my exams, or the things with *e*. I didn’t know how to… I didn’t even know how to add, like, exponents ‘cause I just never thought of it like that. (163)

Romina repeats five times that she “didn’t know” what she terms “basic things” in calculus. All of the specific examples Romina provides of what she “didn’t know” involve numerical computation or algebraic manipulation: “manipulate log,” “multiply two different logs,” “things with e,” and “add exponents.” She explains that she “just never thought” of topics like that. She reiterates her struggle with knowing procedures – “I didn’t know how to do that.” Again, Romina provides a contrast between conceptual and procedural knowledge – she describes herself as someone who does not “know” procedures like exponential or logarithmic manipulation.

7.2.3 Conditions for the Learning Environment

7.2.3.1 Teacher-Researcher’s role

During the review of how developed the additive rule for Pascal's Triangle in the Night Session of 1999, Romina offered some specific comments about what she saw as the teacher-researcher role during the longitudinal study at that time:

> That's why I remember it, 'cause then when you taught us the, uh, how to write it actually what we were doing for years… (70)

Romina asserts that the reason "why I remember" Pascal's Identity is that the teacher-researcher had "taught" them "how to write" the formal notation for an idea "we were doing for years." Give her previous statements about how difficult it is for her to remember and apply formulas, one might conclude that it is significant that this is the only instance during the entire interview that Romina claims to have been about to "remember" a formula. This particular rule was acquired through the process of the teacher-researcher introducing formalization *after* the students had already built intuitive understanding – according to Romina, this was a topic "we were doing for years," they just had not previously known how to formally "write it" in standard combinatorial notation. The teacher-researcher in the longitudinal study is thus someone who would allow the students to work on constructing personal, informal, and intuitive meaning "for years" and then introduce the formal mathematical notation at the end.

With regard to the additive rule of Pascal's Triangle, Romina elaborated on her observations about how and when the teacher-researchers within the longitudinal study introduced the standard combinatorial notation:

> I think you tied it in for us 'cause, I mean, that equation, I've seen that now. I see that in my calc classes, and we, I mean we worked on this what, since we were in first grade, and we worked on a lot of the same problems and we never, we never formalized like we never had 'cause we didn't have this every day so we never had a set equation or we just, we had a way of thinking about this and we always pretty much tried this same way, but we needed to end it almost, and that's how

> we ended it. We came up with that formula, and then we actually use that formula now. (76)

Although they have "worked on a lot of the same problems" since "first grade," Romina observes that "we never formalized" the notation until that instance of the Night Session in eleventh grade. Romina explains the teacher-researcher role as having been one where "you tied it in for us" by introducing the formal notation. In their own work "we never had a set equation" but rather "a way of thinking." Notice to whom she attributes Pascal's Identity – implying ownership, she states that "we came up with that formula." She implies that by having a "way of thinking" first and continually through the years, the "formula" was an appropriate way to "end" it. She states that they "actually use that formula now" as opposed to other formulas they've encountered in their classes. According to Romina, the environment of the longitudinal study encouraged the students have a "way of thinking" first and a "formula" after the reasoning was established.

7.2.3.2 Collaboration by "discussions" and "group work"

Except for the very "end" in high school when the teacher-researchers introduced formal notation, Romina remembers that they would be given a task and then the teacher-researcher would "leave the room" (92). For the most part, Romina remembers that they would work in groups – their problem solving would consist of collaborating and asking questions of each other. Romina summarized her perception of her own learning style as it related to collaborative environments:

> I'm better at learning if like thinking about things, discussions, group work, and I've always been, and now when I, now I'm not, I'm not doing as well as I think I could be doing in college because we're just not taught like that anymore… (226)

Romina claims that she is “better at learning” when she is engaged with “discussions” and “group work.” For Romina, discussions and group work are synonymous with “thinking about things.” Thought, then, is implied to be a collaborative enterprise. Contrasting the group work she has been doing “always” in the longitudinal study with the work of her college classes, Romina observes that now “I’m not doing as well.” She explains that she is not doing as well as she “could be doing,” because “we’re not taught” with an emphasis on collaboration as they were in the longitudinal study “anymore.”

Soon after the observation above in the interview, Romina again implied that being with a group would be an optimal condition for her learning environment:

> I think I, I deal with groups. I work very well with groups. Um, I do some of my best work with other people so that’s helped me because I’m assuming that in the long run, I’m going to have to be, I hope to be in some sort of leadership role where I’m going to have deal with people and delegate, and I do that very well… I’ve dealt with professors and I have no problem walking into a room and sitting down and just discussing things, and I don’t get, I get nervous, because it was odd, someone new, but I didn’t, I performed well, performed well under pressure especially when, like, when people are older. (232)

Romina describes a self-perception that she can “deal with groups” and can “work very well with groups.” In fact, she qualifies the statement to be that she does “some of my best work with other people.” Romina states that working with groups has “helped” her in preparing for the future. Explaining that she hopes to be “in some sort of leadership role” where she would need to “deal with people and delegate,” Romina says that the environment of collaboration encouraged in the longitudinal study works well with that goal. Indeed, Romina expresses that she can “deal” with people “very well” – she has “dealt with professors” through the longitudinal study and now has “no problem walking into a room” and “discussing things.” She also states she does well “when people are older" as the teacher-researchers in the longitudinal study would have been in relation to her own age.

7.2.3.3 "Horrible" College Calculus

As opposed to the "discussions" and "group work" in which she remembers being engaged through the longitudinal study, college courses presented Romina with what she considered a very different learning environment. In fact she expressed that the environment to which she was accustomed was no longer applicable:

> I don't, I don't really have a chance to apply it much in college. …In my college, I'm learning in a completely different way… (246)

Romina states that she does not have much of a "chance to apply" her style of working in the longitudinal study to college. Indeed, she concludes that the environment she is in now is completely dissimilar and she is "learning in a completely different way."

As a result of the "different way" she is learning now, Romina claims that she is not performing well:

> That's why I, I did so horribly and they and it was a ten-page exam so you hand in your exam, and they had the question on top of the page, gave you all the room to work on it, but at the end, you take the exam home with you. They only want the answer. (132)

From Romina's perspective, one reason she "did so horribly" in college calculus was that "they only want the answer." She describes the types of assessment she was given in the calculus courses. The students were given a "ten-page exam" where the students were only the final answers were graded. In fact, the exam booklets with all of the students' written work could be taken "home with you" – the professors did not collect or assess the students' written work, but rather only the final answer.

Romina characterizes the calculus classes as doing “evil things” to the students and seems to express a consistently negative opinion about the assessments:

> And they did evil things, like… It was like 2.5e to the -3 and .25e to the… It was horrible, like they made ‘em all really close so if you were off even one little… like you didn’t get any credit for it. It was all or none. (134)

The “evil things” of calculus included that the final answers would be very specific numerical answers like “2.5e to the -3.” Since they were assessed on their final answers, she says it was “horrible” because the final numerical answers were very close to each other making it difficult for the student to distinguish among them. Romina claims that there was no room for error – “if you were off even one little,” the students would not get any credit. She summarizes the college calculus assessments as “all or none.”

As Romina describes it, the learning environment in college calculus included not only “horrible” assessments that required just an answer, but also large lectures with time restrictions:

> Well, the way we did it, we were taught, each of us were taught by, um, there were two hundred lecture, like two-hundred people in a lecture, and then our exams were at night, and then everyone, everyone in class, one-fifty say it was, took the same exact, the same exam at the same time. (149)

Romina says that the way they were “taught” calculus consisted of a “two hundred people in a lecture” with night-time exams. Implying a strict uniformity she was unused to, Romina states that everyone would take “the same exam at the same time.” Later she is more specific about the time constraints when she judges that “it averaged out to about three to four minutes per question” (187) on the calculus exams.

In addition to the direct instruction of a large lecture and carefully timed assessments requiring a single specific answer, the learning environment in Romina’s college calculus classes also included a great deal of focus on textbook material:

> I don't learn well. Like if you give me a book. I didn't even really use textbooks in high school 'cause I mean for math I never really had a textbook ever, and I don't learn well like that and that's… I'm having a lot of trouble in college now with that because I don't even know who my teacher is. Like if I saw, if they saw me on the street, they wouldn't recognize me, and most of 'em it's like you have to read a book and then you're tested from what's in the book… (226)

Romina defines herself as someone who "learn well" and then qualifies this to be only when "you give me a book." She does not remember using textbooks that often in high school. In fact, she says for math "I never really had a textbook ever" and that "I don't learn well like that." She attributes the "trouble in college now" to the focus on textbooks and that she does not "know who my teacher is." She asserts that if her professors saw her on the street, then "they wouldn't recognize me." The personal connection and comfort level that existed for her in her elementary and high school classes as well as the longitudinal study no longer seems to apply in her college courses. Likewise she is troubled by the issue that now "you have to read a book" and then be "tested" by the contents of that book.

Direct instruction lectures, strict time limits, single answer assessments, unfamiliarity with the professors, and textbook-driven content - the accumulation of the conditions present in her college calculus learning environment seem to have taken a toll on Romina. She expresses negative affect in relation her entire college calculus experience:

> In the beginning of my first year, I put a lot of effort into some of my classes, and you get your grades back and you're graded on a curve. So about this many people (puts fingers together) get a good grade and the rest of us all get B minuses. So it really doesn't matter. It's a huge range —I give up. That's horrible to say, but, after a while, I just gave up. It just wasn't worth my time. (288)

While in the first year at the university she "put in a lot of effort," now she says she decided to just "give up." After realizing that she would always be "graded on a curve," she argues that it “wasn’t worth my time.” She concludes that her effort “doesn’t really matter” and recognizes that this is “horrible to say” but what she feels as a result of her year in college calculus.

7.2.4 Learning Process

7.2.4.1 “Argue” Mathematics

When recalling the problem solving within the context of the longitudinal study, verbs that Romina uses frequently in her descriptions are active verbal ones like “argue,” “talk,” and “discuss.” Indeed, Romina describes problem solving as a dynamic process:

> …we’d argue it out and then probably take ideas from each other and then worked from there and come back. (16)

Notice the active give-and-take Romina remembers as “we’d argue it out,” “take ideas,” and then work “there and come back.” In her description, “ideas” are to be shared – they can be transferred from one person to the next. Discussions move fluidly “there” and “back” among the different ideas and different group members.

Romina classifies herself as someone whose learning involves discussions and groups. She remembers instances of this from the longitudinal study:

> We used to call each other and we’d just discuss ideas and what happened and details and things. And that’s how I learn – and it’s a group setting. It’s - I learn in groups. I don’t know, it’s just – I think this has a big part to do with it. (259)

Notice that the singular “I” for Romina learns when part of a plural “we.” She recalls how the students would “call each other” and “discuss ideas.” Explaining that “how” she learns is in a “group setting,” she summarizes her learning process by stating, “I learn in groups.”

Romina draws a distinction between how they "always did it" in the longitudinal study versus how things are done at the college level:

> 'Cause that's how we always did it. And, um, now in college, I try to do that. Now that I'm getting – and I don't know anyone in my classes, so it's harder and I have to sit there and instead I have to memorize everything myself. And I don't talk to anyone. I just keep reading over and I just don't retain the information. As soon as I – if I do sit up all night, and try to memorize my sixty pages of notes and I write down what I know on the exam. As soon as I walk out of there it's done. I don't remember anything. (261)

In her college classes, Romina claims that since she doesn't "know anyone," it is "harder" for her to learn. She describes how she will sit and "have to memorize everything myself." She expresses the concern that "I don't talk to anyone." As a result of not being able to talk and discuss, she observes that she cannot "retain the information." More specifically, she says that after staying up all night to "memorize my sixty pages of notes," she takes her exam and then as soon as she leaves, "I don't remember anything." She implies that if she were able to learn in a group setting again where she could "talk" to others about her reasoning, she would be able to retain more.

After illustrating how not talking and arguing seemed to affect her learning in college calculus, she returned to the importance of argument in problem solving:

> I think a big way we learn is we tend to argue a lot. So that's how we get places because we argue. And then we have to take their argument into consideration. When it's just me, I don't have much to argue about with myself because I think I'm right. Jeff doesn't have the same ideas as me. (275)

Romina remarks that "a big way we learn" is "to argue a lot." She goes on to say that "we get places because we argue" – argument thus becomes a literal and figurative vehicle for her reasoning. By having to "take their argument into consideration," a student goes to another location in learning (new "places") as opposed to staying stagnant within one's own opinions. Romina implies there is a danger in working individually.

When she is alone, she doesn't have "much to argue about with myself because I think I'm right." However, argument allows a student to encounter and engage with the reasoning of someone who "doesn't have the same ideas as me" like Jeff.

7.2.4.2 We learned a "thought process"

Gaining conceptual understanding of a problem emerged as a theme in Romina's interview as to what she prioritized when problem solving. Romina remembers that when she sometimes needed to think through a problem by herself first when working with her group in the longitudinal study:

> If I didn't understand, if I didn't understand a problem, or if I didn't work enough through it by myself to understand where like… I guess Michael didn't know where I was heading with what I was doing, and if I didn't understand where the other person was heading, I liked to work on it before I form a couple options and see which one he takes. (30)

There were several different instances Romina remembers where she might have needed to take some time for her own private thinking about a task. Romina would work on the problem alone if she "didn't understand," "didn't work enough through", another student like Michael "didn't know where I was heading," or she "didn't understand where the other person was heading." Thus there were four cases in which Romina would take personal time to "work it on" the problem and "form a couple of options" involving her assessment of either the level of her own understanding or level of her group members' understanding.

When asked by if there was "anything from the things that you used" in the Rutgers longitudinal study that "apply" to her current reasoning, Romina gives an account of her problem-solving process as one in which she would think about a plan first and leave specific formula choice until last:

> I mean that's how I arrived at most of my answers. I thought of them like that, and I came up with this is how I would do it, now what formulas would I use to get the answers if I were to do it like this? (179)

She describes "how I arrived" at answers as a process in which she would first ask herself "how would I do it" and then decides "now what formulas would I use." The reasoning of "how" would precede the application or "use" of formulas.

Instead of learning specific formulas, Romina expresses that they "learned more of a thought process" from the longitudinal study:

> I think we learned more of a thought process and how we deal when we were first given questions, which is how I always deal with how I'm given questions now. And that's how we do it; we talked it out, like, between my friend and I and then we came up with the how are we going to do this. (181)

Romina proposes that the she and the other students in the longitudinal study learned a "thought process" with which to apply to new questions. Indeed, she says that it is through this process that she "always" deals with "given questions now." She implies a three-step process for problem-solving that seems to necessitate at least one other person's collaboration: first discussion among group members ("we talked it out"), then development of a plan of action ("how are we going to do this"), and finally associate any necessary formula as she mentioned previously.

7.2.4.3 Affective Dimension: This is an "accomplishment of mine"

As opposed to negative affect in other academic areas, Romina seems to express positive affect about what she perceives as her success in the longitudinal study and presents her participation as an "accomplishment":

> And I feel that this is now an accomplishment of mine. I never viewed it like that before until I went away to college and I got shot down in every other area. This is the one thing that makes me feel sometimes all right about myself. (474)

Romina explains that the longitudinal study is “an accomplishment of mine” and that it is the “one thing” that makes her “feel sometimes all right about myself.” Romina’s qualifiers of “one thing” and “sometimes” may indicate some negative affect about other areas of academic study. She becomes more pointed when she says that in college she “got shot down in every other area.”

Elaborating on why she describes the longitudinal study as an accomplishment, Romina says:

> It just seems like so many times they were impressed by what we would do and we would just sit there and be like we are doing anything of any value. (497)

Though she remembers that the students would question whether they were doing “anything of value,” the fact that others seemed to be “impressed by what we do” made an impression on her. That their math work was valued by others was a frequent occurrence in Romina’s memory - Romina recalls that “so many times” teachers and researchers would be “impressed.”

Romina wonders if there was any relationship between the students’ participation in the longitudinal study and their success in high school:

> I don’t know if this was a direct correlation but we were the ones who did better in school in general… (515)

She seems to express a belief that she and her peers in the longitudinal study were more academically successful. Although she is not sure if it would be a “direct correlation,” Romina observes that the Kenilworth longitudinal students “did better in school in general.” She elaborates on this success factor and comments:

> Like our group of kids. We were in the top in the class. Out of everyone involved in this program we were all pretty much 1 through 10. That was us and then everybody else and if anyone out of our group. We were like the ones who did more math-oriented college-y things whereas I know my friend she was not in this

> program; she went to college. She found out the first day that she could not use a calculator in her chemistry class and she dropped pre-med. And she didn't even you know so scared. (517)

Romina claims that "our group of kids" who participated in the longitudinal study ended up being "in the top" of the class. She states that they ranked "1 through 10" out of their high school graduating class and became the "ones" doing "more math-oriented college-y things" after graduation. She compares their relative success to her "friend" who was not in the longitudinal program and "dropped pre-med" as her major after learning that she could "not use a calculator" in her college chemistry class. Romina describes this friend as being "so scared" by work that did not allow students to use calculators. Romina seems to indicate that, unlike her friend, she and her other longitudinal study group members would have been able to persevere and not be "scared" or intimidated by mathematics without a calculator. She goes on to comment that she thinks her friend could have been successful in the class:

> I know she could. She was so intimidated. She was like that something you could do but it's not true. I never used a calculator. I rarely used it during all this. (519)

Romina recognizes that her friend was "so intimidated" by work that did not allow the use of a calculator although Romina knows that "she could" have done it. Saying "it's not true" and "I never" or "rarely used" a calculator through the longitudinal study, Romina seems to imply that she would have stuck with the course and been able to do math reasoning without the aid of a calculator.

7.3 Reflections IV - Longitudinal Study and Career: May 12, 2006

7.3.1 Setting

On May 12, 2006, a semi-structured interview took place with Romina, Angela, and Magda. Video of the interview lasts a total of 114 minutes (see Appendix H for full transcripts – Disk I recorded the first 50 minutes and Disk II recorded the next 64 minutes). At the time, the three women, all of whom had been participants when they were younger in the Rutgers-Kenilworth longitudinal study, were finished with college and currently in the workplace. The main researcher/interviewer was joined by seven other graduate students and researchers in a large seminar-style group held in her home. The first five minutes of the session consisted of introductions of the graduate students and researchers. David, Marjory, Charlene, and Frances were introduced as graduate students in the math education seminar. Kelly introduced herself as a math educator with a particular interest in ethnomathematics – she described working in South Africa, Oregon, and now in New York for three years at Bard College. Kate was a former student of Kelly's through Bard and now was a teacher at the Fannie Lou Hamer High School in the South Bronx. Finally, Liz Uptegrove was introduced as a current professor at Felcian College.

> T/R1 then turned the focus to Romina, Angela, and Magda by saying,
> So we want to hear about you guys. We have some questions we want to ask that we really want your opinion about. We're not going to ask you about yourself, we want your opinions about things. (Disk I -38)

During the initial part of the interview, the women's respective job descriptions were established: Romina was now a business analyst at Deloitte Consulting; Angela worked as a marketing assistant at an IT company; and Magda was employed as an auditor at Deloitte Consulting.

Reflections IV Interview – May 12, 2006 - SIGNIFICANT STATEMENT SUMMARY	
Issue	**Significant Statement(s)**
Knowledge & Knowing	***Definition of knowing***: "When you're able to explain it to someone else and they ask you every question under the sun and you can still answer it. I think then you know it." (160)
	Two Types of Problem Solving: "Original Ideas" versus "Forced Frameworks" – someone should "just think about a problem" (201)
	Two Types of Intelligence: "Genuine Thinking" versus "Business Knowledge"– "it's just genuine thinking impresses me" (221)
	Understanding the concept versus Using a formula – "I walked them through the concept, but I don't know how to solve it through the formula" (209)
	Explaining the Concept Yourself versus Being Told – "We learned so much from just getting up in front and explaining what we thought of concepts versus someone just telling me what it was" (291)
	Building, Associating, Aligning the Concept versus Memorizing a Formula – "we built that whole concept… it's not like I can memorize a formula… I'm completely confusing it because it doesn't align with my conceptual knowledge" (B-311)
	Concept First and Formula Second – "…The formula will hit me much later after I thought about the problem and thought about the picture… It's not an automatic association if I have just the formula – I have to have a concept in mind" (B-318)
Conditions for the Learning Environment	***No "Formalized" Classroom*** – "I'd also get rid of the formalized classroom" (A-269); "We'd have tables, no desks… we'd all sit in groups of 4 or 5 and we'd rotate periodically so we could work with different people all the time" (A-277)
	Teacher: Research-oriented and dedicated – "…We had someone who really dedicated a lot of time to his own education and learning about how people think" (B-37); "…he'd customize every lesson" (B-39)
	Convincing & "Pushing Their Thought"– "… it's really interesting to se how different people think through problems and just them talking to you about it and depending on how well, how much they convince you" (A-179); "… make sure everyone's working together, make sure everyone's contributing ideas and pushing each other to you know, pushing their thought" (A-193)
	Collaborating – "Talk" Math – "We used to talk about math" (B-85); "…being able to interact with a group and kind of assessing someone's strengths and capitalizing them and then delegating work well" (A-273)
	Supportive and "Comfortable" – "we were so comfortable with each other, so I think I was fine not knowing something and being like, I don't know this, you guys, we have to go back and explain something to me for the tenth time" (B-136); "we were never embarrassed with each other" (B-141); "I wasn't afraid to ask you guys anything" (B-151); "I'm like so comfortable with being uncomfortable" (B-163); "I feel very comfortable asking questions" (B-184)
	Asking Questions – "you're constantly pushing them, and they push me back and it's great" (B-186); "… I know that we used to drive our teachers crazy because we'd always be like, why?" (B-348)
	Alternative Assessment – "I'd get rid of standardized tests, that'd be my first thing" (A-267)
Learning Process	***Willing to Learn*** – "… the most unsuccessful people that come into my class are those people that just want to get by on very little and not invest in the time, invest in the time upfront to learn, invest in the time to produce quality deliverables" (A-258)
	Learning to Learn – "……I know what questions to ask and I put the effort in and I know how to learn and how to absorb information, the right information, and weed through it – I mean, that's all we have to learn, to know how to do." (A-262)
	Making a Discovery & Connecting – "… we thought we'd discovered Pascal's Triangle" (B-8); "… I feel like if you learn one concept that doesn't connect to other concepts, you're learning something almost useless" (B-242); "…it's all interconnected as it is in the real world" (B-244); "I will never forget towers" (B-284)

Table 7-2. **Reflections IV Interview – May 12, 2006 – summary of significant statements**

7.3.2 Knowledge and Knowing

7.3.2.1 A Definition

In response to T/R1's question toward the beginning of the interview, Romina provides a definition of what it means for her as a learner to "know" something well:

T/R1	But for yourself as a learner, when do you feel you know something really well?
MAGDA	I guess when you can explain something to someone else.
ROMINA	Kind of. When you're able to explain it to someone else and they ask you every question under the sun and you can still answer it. I think then you know it. (Disk I -158 – 160)

Here, Romina compares being able "to know" with being "able to explain" and address "every question under the sun." For Romina, knowledge and the learning experience are situated in dialogue (the knower must be able to *explain* to a questioner). She seems to contextualize knowing in a broad sphere. To be able to answer "every question under the sun" about a topic implies a deep and very well-rounded understanding. It might also indicate being able to connect among many disciplines at once.

7.3.2.2 Two Types of Problem-Solving

When asked about types of problem-solving she has observed in others, Romina compares knowledge and approaches to problem-solving in her workplace between those people who have <u>not</u> been to business school (like her) to "those kids that went to business school":

> …And it's so funny, because you have those kids that went to business school, and like they've been doing it since day one. And you give them a problem and they're starting to talk about all these frameworks and all these Porters Five Forces and SWOT analysis and they're doing it. And I am working for two years have still not done this, so they're to impress you, and you're like you don't actually use that every day. Like it's a framework, you don't actually use it to solve a problem. And then you have those kids that are so intimidated because

> they've never done business before. But they get in there and they just think about a problem, and they come up with a better solution, because they're not just trying to force all this knowledge that they already have onto this problem and they're just looking at it as if they've, they've never seen anything like that before. So it's interesting to hear what they would come up with, because sometimes they're really original ideas that you wouldn't have thought of because you're so constrained by this business mentality. (Disk I - 201)

She observes that the business school students usually begin a problem by looking for a relevant "framework" like Porter's Five Forces or "SWOT" analysis. She then speaks of those people who "just think" and then "come up with a better solution" than the business students. She also classifies the non-business students' answers as "interesting to hear" and "really original ideas." Romina remarks that business students "force all this knowledge" and become "constrained by this business mentality."

Romina draws a sharp distinction between what she classifies as business and non-business knowledge when applied to problem-solving. She groups herself with the non-business thinkers and seems to value that kind of thinking, calling it "original," "better," and "interesting." She reserves harsh criticism for "them" (the people with business school backgrounds). She describes a type of negative knowledge that they "force" on problems and causes their thinking to become "constrained." Her language for the business mentality refers to a cage-like "framework" for their ideas. Romina seems to believe that problem-solving should be free of outside intervention at the outset (like the SWOT analysis that the business students try to impose when they begin a problem).

7.3.2.3 Two Types of Knowledge

When asked to share her perspective of her learning, Romina describes two types of knowledge and how she sees evidence of them in her career:

> I'm just so frustrated that we are, and I feel like in high school I was much more confident about my abilities than I am now. Because they always they're testing me with things that aren't relevant to how successful I'm going to be in the business world. Like, for example, to get my job I had to go through this big case interview. What they'd like to see, it depends on who, like for example, when I look at people's thinking, I don't know the business models because I still haven't had a chance to work with them, so that doesn't interest me, that doesn't impress me, because I don't understand them anyway. It's just genuine thinking impresses me but when I was interviewed, one of the people interviewing me was very upset that I didn't have all this business knowledge and they would give me all these things, throw numbers at me to see how fast I could spit out spit back numbers, and that is NOT how I grew up. I never had to do that, ever. So I'm not used to that, and they would get really upset, and that by no means indicates how I've done. Because I perform very well at work and I get like top ratings all the time, and because I can't shoot out answers within two seconds that you throw at me, they think I'm not, like my intelligence is underrated. (Disk I - 221)

Romina says that she does not "know the business models" and they do not "impress" her. What does impress her is "genuine thinking." She describes the interview process at her current job where people with "all this business knowledge" threw numbers at her "to see how fast I could spit out, spit back numbers." She observes that this business model example was "not how I grew up" and she is not used to it at all. In her opinion, her performance is not connected to an ability to get answers "within two seconds." As a result, she asserts that her "intelligence is underrated" and she is "frustrated."

Romina contrasts "genuine thinking" with "business knowledge." For Romina, "business knowledge" could be interpreted to be a stagnant, didactic thing whereas the type of knowledge she possesses and values in others (which she terms "genuine thinking") is a dynamic construct that has contributed to her "top ratings" at work. She dismisses superficial timed exercises like being able to "spit back numbers" or "shoot out answers." Notice that even the language she uses here about business modeled knowledge sets up an adversarial as opposed collaborative relationship where someone *throws* a number or problem at you and expects you to immediately spit or shoot back.

7.3.2.4 Knowing: Concept versus Formula

After Angela mentions that several of her friends would often use formulas to solve problems, Romina details a seeming paradox of knowing "probability stuff inside and out" and yet not knowing any of the formulas:

> And that's what would drive me crazy! In my calc class, because I took my math courses in my preliminary economics courses with all my Wharton counterparts, and we would get to Calculus, and we would all be studying because these exams were just impossible, and they could not tell you, and I loved the probability stuff, because I knew the probability stuff inside and outside, but I didn't know any of the formulas. And I couldn't - they were like computation, permutation formulas, which one do we apply? And I was like I don't know! You just think about it, and they couldn't understand the concept behind it. They never thought about it. And I'm like, well say you had I don't know, 4 colors, and you had to make – [Laughter] Towers, and I'm like, god, and I just assumed, because I've probably told you this before. I walked in, it was just funny to see how much of a different learning experience we had, because I walked them through the concept, but I don't know how to solve it through the formula, I'm like I don't know. And we'd get that far, and I'd be like, I don't know what that means, but I could solve it using just… (Disk I - 208 – 209)

When asked by others in her calculus classes which formulas to apply to problems, Romina recalls that she would reply "I don't know." However, she encountered people like her "Wharton counterparts" who could apply formulas but "couldn't understand the concept behind it." She describes how she would explain the concept to people by using the example of towers. She also remarks that she had such a "different learning experience" since she could describe topics in terms of concept as opposed to formula. Romina professes to have a deep and thorough knowledge of probability and combinatorics though she did not "know" the formulas. She "loved" probability because she possessed conceptual knowledge. She expresses a clear concept orientation as opposed to a rule orientation in her description of knowledge of a content area. It is interesting that she seems to imply that knowing concept and knowing formula are mutually exclusive for her. Romina attributes her ability to "know" concept but "not know" formula to her "different learning experience" in the longitudinal study. To explain a topic she would try to help others understand the concept *behind* it. She uses the Towers Problem, an abiding learning metaphor for her, as a building block for conceptual knowledge.

Romina recalls that in high school so much was learned from "getting up in front and explaining what we thought of the concepts." She says as a result of explaining as opposed to being told a definition, she will "always" remember "what an integral is":

> You're always prepared for those classes where you have to contribute something and discuss things you're actually taught. I remember, we learned so much from just getting up, and this is back in high school, not in college. We learned so much from just getting up in front and explaining what we thought of concepts versus someone just telling me what it was, and I'm always going to remember what an integral is – it's the area of a – (Disk I - 291)

Romina's description of having "learned so much" from "getting up" to justify arguments in high school indicates that she believes personally explaining a concept is more valuable and lasting than being told what a concept is. She directly contrasts personally "explaining" her own thinking with someone else "telling" her. Melding collaboration with conceptual knowledge, a large component of learning is the active process of explaining to others. Here, Romina seems to place constructed knowledge above received knowledge.

Romina describes the way she learned combinatorial topics in the longitudinal study as "very conceptual." She explains how her experiences in the longitudinal study translated to how she would approach problems in college:

> See and for me, like for me, like we built that whole concept and then we were introduced with this formula, so like that formula I, when I look at that because I remember I had to do it my first year in college and I remember looking at that

> and it's not like I can memorize a formula but I would look at that formula and I was like okay, so this means that I have my options for this could be a tower 5 tall and I have 3 blues and 2 whites, and that's how I remembered it and where the numbers went [gestures]. So for me I really took probability and combinations a lot very conceptual. And even now I'm trying to relearn it, and the way, I haven't done this in years, but the way they're teaching us now, it's similar and I think they taught us a whole new concept that I'm trying to learn now in my class, and I am completely confusing it because it doesn't align with my conceptual knowledge of like, okay, how many spaces are there, which is how high it is [more gesturing with hands]. How many different colors do I have, and it's really mixing me up because they're trying to teach me in a different, with a whole formula that's different, I can't associate conceptually so I'm having so much trouble just memorizing this one formula, I can't do it. Like it's very simple, it's like yeses and nos, and I can't do it cause it doesn't, cause I can't associate it conceptually. (Disk II - 311)

When she would look at a formula in college she recalls not memorizing it, but rather trying to figure out what the formula "means" in terms of a "tower." She describes a topic that they are currently learning in her GMAT class as "completely confusing" because it "doesn't align with my conceptual knowledge" and she "can't associate conceptually" with towers. She describes having so much trouble memorizing this new formula as a result of not having a conceptual link.

Romina seems to indicate that knowing is an active pursuit of building a concept, associating her conceptual knowledge with generalized formulas, and aligning new concepts with previous ones. She finds it very difficult and troublesome to memorize a formula that does not align or associate to a previous concept she has built. Here again, Romina places constructed knowledge above received knowledge as her preferred mode of learning. She mentions the Towers Problem twice in her discussion here of the importance of building, associating, and aligning with conceptual knowledge. The assimilation paradigm of the Towers Problem truly was powerful for Romina's learning. She describes making sense of a formula her first year of college by specifically thinking of a tower 5-tall with 3 blue cubes and 2 cubes. The amount of gesturing she does during her discussion also indicates the physicality of literally and figuratively building knowledge. Lasting knowledge for Romina is what she defines as "conceptual."

Romina gives a detailed account of her problem-solving process in which she describes how application of a formula is usually the last step:

> Well and especially because I find it like in a lot of problems what I would have trouble with is I don't automatically associate a formula when I read a problem. I think about a problem. So if I'm thinking about a problem and I kind of understand what it's asking me first and I have to draw some sort of picture, and like, I still do this a lot, I still draw some sort of picture. Then, the formula will hit me much later after I thought about the problem and thought about the picture. I'm like, oh, so this is that formula where we used this. But it doesn't, it's not an automatic association if I have just the formula. I have to have a concept in mind, and it's not good because it takes me a long time to do stuff and I don't learn it right away. (Disk II – 318)

Romina first observes that she does not "automatically associate a formula" with a given problem. Rather she needs to "think" about the problem and "understand what's it's asking me first." Then she usually needs to "draw some sort of picture." She says that the formula will "hit" her last. Romina asserts that she has "a concept in mind" before she can proceed to the formula application.

For Romina, she must have "a concept in mind" before she uses a formula. Romina asserts that her problem-solving process has four parts: think about the problem, understand what it's really asking, draw some kind of picture, reflect on the problem and picture further, and finally associate a formula. Here is an excellent illustration of Piaget's math education precept, "intuition before axiomatization." An intuitive grasp of the concept must precede an axiomatic formula. It is interesting to note however that Romina seems ambivalent about whether it's a "good" think that she is unable to apply a formula right away because building a conceptual model first takes a longer amount of time.

7.3.3 Conditions for the Learning Environment

7.3.3.1 The un-Formalized Classroom

Romina describes the physical nature of "our learning environment" in the longitudinal study as well as her ideal learning environment:

> Through my first experiences I'm very biased against business because I didn't have that structured – my, our learning environment wasn't that structured, we didn't have this homework and problems sets and all that. (Disk I – 219)

She recalls that her learning environment "wasn't that structured" and elaborates to define structure in terms of "homework and problem sets." Later, she describes how in an idealized learning environment, she would "get rid of the formalized classroom" (Disk I – 269). She also describes a specific physical environment:

> We'd have tables, no desks, tables. I don't know, we'd all sit in groups of 4 or 5 and we'd rotate periodically so we could work with different people all the time so we'd have to re-learn how to work with people. (Disk I – 277)

In an idealized learning environment, it would not be a formalized classroom, but rather have tables instead of desks so people could sit in groups of four or five. Rotations of group members would be in place so "we'd have to re-learn how to work with people." She also recalls all the time she had, like, for instance, the "weeks and hours" (Disk II – 20) just trying to figure out Pascal's Triangle.

It is sometimes difficult to distinguish descriptions of Romina's idealized learning environment from her recollections of scenes from being a participant in the longitudinal study. The fact that the longitudinal study conditions blend so seamlessly with her conceptions of the ideal seems to indicate her extremely high regard for her experience with Rutgers. She describes a physical environment of tables rather than desks which would promote group work and no arbitrary time limits to allow for deep investigations. She defines the "formalized classroom" as opposite to what she encountered. In her recollection, her learning environment "wasn't that structured" in terms of homework and problem sets and classrooms with separate rows and columns of desks.

7.3.3.2 The Teacher's Role

Romina identifies a "huge component" of learning environments as "teachers who are genuinely invested" in their students' learning (Disk II – 35). She goes on to describe one of her math teachers from high school, Mr. Pantozzi:

> Yeah, I think it's, I mean, we had a math teacher right who was getting, still getting his PhD, is still getting his PhD, and he'll get it you know, but we had someone who really dedicated a lot of time into his own education and learning about how people think and learning about how people learn and like he just spent all these years learning and applied them all on us and tested then out. And I know he, you couldn't, you wouldn't know, but I know he spent hours thinking up our lessons, and then we went to other classes –And he'd think, he'd customize every lesson because he'd be like okay, Bobby's going to say this and no one's going to understand him, so then Angela's going to ask, and Romina, he's going to have to explain it to Romina, and then Mike is going to get it, you know? And he went through all these different scenarios about how people learn and then you go to another class where your teacher gives you the same thing that she's been using or he's been using for the last ten years. And it's like this same paper, years and years before. So I think it's a lot about how invested your teachers are going to be too, that's probably one of the first things you have to change before your – (Disk II -37 – 39)

She remarks on the fact that her teacher Mr. Pantozzi was pursuing his doctorate while teaching them and engaged in research. She says he "dedicated a lot of time into his own education" and was interested in "learning how people learn."   In her recollection of the time, Mr. Pantozzi would "customize every lesson" according to the anticipations of the students' questions and actions as he would go through "all these different scenarios about how people learn." She illustrates with her classmates' names and actions: Bobby, Angela, and Mike. Later, she asserts that her class found him to be "very inspiring" and that they had "this unspoken commitment" (Disk II – 46) to perform their best for him.

Invested, dedicated, research-oriented, and inspiring were among the terms Romina used to describe her teacher, Mr. Pantozzi, from Kenilworth high school. Incidentally, Ralph Pantozzi was also pursuing his doctorate in math education at Rutgers during the time he was Romina's teacher. He participated as a researcher in the longitudinal study as well. Romina's description seems to express a belief that a teacher's role is paramount in shaping the optimal learning environment. The pedagogical traits she particularly lauds include a teacher's dedication to both his students and his own personal growth through research. The fact that he "applied" and "tested" his own learning on them is something she finds admirable. She seems to value his differentiated approach to lessons that anticipated individual learning styles. From her perspective, this teacher planned according to each of his students' unique voices.

7.3.3.3 Collaboration – "Talk about Math"

Romina mentions problem-solving immediately as a skill for young adults in the job market. She links problem-solving with the ability to convince:

> … it's really interesting to see how different people think through problems and just them talking to you about it and depending on how well, how much they convince you even though they have no idea what they're doing – (Disk I – 179)

Romina says that she finds it "interesting" to watch people "think through" and "convince you" about problems. She links the act of "think through" with "talking to you" in the same sentence as if the first would necessitate the latter.

Romina goes on to illustrate the type of problem solving by collaboration and talking through ideas occurs in her work place:

> One thing we do, we have these case competitions. We put all these college kids in a room, we break them up into teams of four and five. And we give them a problem, and they have all day to just sit and break this problem out and we're observing the whole time. So it's groups of five and, kinda like we used to do, they just work on it all day. At the end they have to present their findings so a board of partners. And you know, just one day is probably all you need you can tell how people work in groups and who contributes… So you observe them to make sure everyone's working together, make sure everyone's contributing ideas and pushing each other to you know, pushing their thought… (Disk I – 191 – 193)

In the case competitions she describes, the college students applying for jobs break into "teams of four and five," get a problem, and then "break this problem out." She compares the experience at her job with the Rutgers experience because "it's groups of five," they "just work on it all day," and they have to "present their findings." She asserts that observers of the problem-solving teams need to look for a multi-dimensional collaboration: contributing personal ideas as well as "pushing each other to know."

Being able to make a convincing argument is an important skill for the learning environment of a workplace for Romina. Her description emphasizes instances of voice and dialogue – the action verbs she chooses to use are "talk," "convince," and "present." Romina also finds her workplace experience similar to what she encountered as a student in the Rutgers longitudinal study in terms of being given a problem for a long time, working with a group, and having to present a convincing argument to others. Her description of group problem-solving indicates limited outside intervention. Just as the Rutgers teacher/researchers would "observe," so does she when she watches the case competitions for her company's job application process. There is also a sense of shared responsibility in knowledge and learning for Romina. She values "pushing each other to know" as much as knowing for oneself. Romina's belief about knowledge requires responsibility for others' knowing. Her concern for others is an issue in educational decision-making.

Romina recounts a story of describing the "Rutgers program" as a place where they would "talk about math" to a co-worker who doubted her experience as "too fluffy." Romina recalls the criticism of a co-worker who commented that "you can't talk about math, you just do math" (Disk II – 85) when Romina described the longitudinal program to him. Romina remembers emphasizing to him that the way she would solve problems with her peers in the longitudinal study was to "talk about math."

In another part of the interview, Romina contrasts the collaborative work in the longitudinal study with the individual work she would do in college:

> …And I think a lot of the skills that I learned growing up are very, they're people skills, being able to interact with a group and kind of assessing someone's strengths and capitalizing them and then delegating work well. And that's what we have to do all the time in the workplace, so I'd get rid of all those classrooms that are set up like with the rows of chairs, and I'd, yeah, I mean, that's we loved – Yeah, I think that's what we loved, and that's why college was so hard for me. I went from always leaning on Magda to explain something I wasn't going to get or when I couldn't write something very well I'd turn to Angela and be like, could you rewrite this for me. I mean we all had our strengths and we taught each other and we learned from each other and we got to college and it was completely different. No one worked in groups, and we all just sat there and we listened with three hundred other people and I think I lost a lot of the leadership capabilities I had and the speaking, I mean, I'm the most not confident, I hate speaking in front of crowds now, and I used to do it all the time. Like I did every day in school, I could do it. And it's just so different, and it's not like that in the workplace. So why even do that? (Disk I – 273 – 275)

She recalls skills from her school days that transferred to her work now as "people skills." She elaborates that having "people skills" means interacting in groups, delegating to others, assessing colleague's strengths, and capitalizing on group strengths. She asserts that "what we have to do all the time in the work place" is use these people skills. She also states this is what she "loved" growing up. Using her interaction with Magda and Angela in high school as illustrations, she further defines what collaboration means to her. She was “always leaning on Magda” or would “turn to Angela.” They would share their respective strengths, learn from each other, and teach each other. Her early experiences involving collaboration were unlike the solitary environment she found in college where “no one worked in groups” and “we all just sat there and listened.”

Romina’s description seems to indicate a belief in a learning environment that involves being with others in such a way that is collaborative and cooperative. The process of sharing in each others’ strengths and teaching each other is necessary to learn. She defines the Rutgers longitudinal study as a place where this collaborative atmosphere flourished – there they would “talk about math.” She relates the collaborative learning she “loved” to the “people skills” she values in at work. She describes working together as a shared experience.

7.3.3.4 “Comfortable with Being Uncomfortable”

In her recollections of the longitudinal study, Romina defines her relationship with both her peers and the actual problem tasks as “comfortable”:

> … we were so comfortable with each other, so I think I was fine not knowing something and being like, I don’t know this, you guys, we have to go back and explain something to me for the tenth time because I don’t understand this. And you don’t have that comfort in bigger classes. Until you get older, and you gain that confidence, and that ability to accept that you just aren’t going to know everything, which I know is hard for us. Like, then you can get the class sizes bigger. But I’m going to argue that they should be smaller when you’re younger to kind of instill these habits and this, I think a lot of us have trouble learning because we won’t ever fess up when we don’t understand something or we don’t feel comfortable actually voicing our opinions. (Disk II – 136)

Romina explains that not only were they “so comfortable with each other,” but also that she was comfortable with being unsure about a solution – specifically, she was “fine not knowing something." Romina remembers asking the others to "go back and explain something to me the tenth time." She says that she would not have that "comfort" in bigger classes. Romina asserts that class sizes "should be smaller when you're younger" so others could experience the "habits" she did. She says that many people have "trouble learning" because they do not "fess up" when they have a question or "feel comfortable actually voicing our opinions."

Romina characterizes the dynamic within her longitudinal study group as being one in which they were "never embarrassed with each other" (Disk II- 141). Later, she asserts that she "wasn't afraid" to ask "anything" in high school. She explains how her school experience has translated to her current workplace:

> I've had very different experiences, because on the one hand, like I'm comfortable going into something I don't know, like when I get on a new project, at first I was uncomfortable because I'm too nervous about everything, but now I'm like so comfortable with being uncomfortable, not knowing what I'm going to do because I know I'll figure it out… (Disk II – 163)

Romina describes that now in the work place she is "comfortable with being uncomfortable" when faced with new projects. She elaborates that even though she does not initially know what she will do to solve a problem, she is comfortable that "I know I'll figure it out."

When on project teams at her job, she also states that she feels "comfortable asking questions" of work colleagues when she doesn't agree with their "logic":

> No, I'm very comfortable, and one thing I'm comfortable with is I'm one of the youngest people at all, almost all the time on my project teams, and like, I feel very comfortable asking questions. I mean, I think we've always worked, we've always worked kinda facing older peers, so I mean, I feel very comfortable asking questions or taking the lead or questioning people when I don't think their logic is right. And, I mean, maybe I shouldn't feel as comfortable questioning my superiors, but I am. But they seem to like it. And I also work with people that think and work like I, like we've grown up with… (Disk II – 184)

Romina explains that she is "very comfortable" in the workplace. She explains that, although she is "one of the youngest people" on her project teams, she feels "very comfortable asking questions," "taking the lead," and "questioning people."

Notice that the word Romina uses again and again in her description of her remembered experiences in the longitudinal study is "comfortable." That comfort she experienced in the longitudinal study involved being able to ask questions with embarrassment. She connects the "comfort" that was cultivated in her early learning experiences to her present workplace ability to be "comfortable with being uncomfortable." She seems to express confidence that she will be able to figure out a new project task. Her ability to ask questions of her project team members is also evidence of how her Rutgers longitudinal experience translated to her job as she continues to "feel comfortable asking questions." Notice also that all of Romina's descriptions of comfort involve being with others in a very vocal way. Significantly, *her <u>voice</u>* and *her group members' voices* are what she remembers hearing. Her descriptive language seems to be auditory in nature rather than visual or tactile.

7.3.3.5 Asking Questions

Romina describes situations involving asking questions frequently in both her current job and previously as a student in the longitudinal study. She explains that her coworkers and superiors at work "like it" when she asks questions:

> Yeah, it's smaller groups, and they like it. They love it, they're like thank you because you're constantly pushing them, and they push me back and it's great. So it's a good learning environment, because you just learn more. (Disk II-186)

At work, Romina says that they not only "like it" when she asks questions in her "smaller groups," but they "love it" and "thank you" for constantly "pushing them" with questions. She asserts that they "push me back" with questions. She defines an environment where such smaller groups continually question each other as "a good learning environment" where "you just learn more."

Romina also remembers her experience with the longitudinal study in high school as being "conditioned" to ask "why":

> But I think of, because we were all in the same classes so maybe it was that we were conditioned to it, but I know that we used to drive our teachers crazy because we'd always be like why? And they would have to go to the next level, like in our chemistry class we'd be like, but we don't understand exactly why that happened. She's like you just do this and we're like no, why? And that used to drive them insane. (Disk II – 348)

Romina remarks that "we used to drive our teachers crazy" with constantly asking "why." She describes that questioning in such a way caused her teachers to "go to the next level" like in her chemistry class where they would ask when they did not "understand exactly why" something happened. Recalling that teachers would attempt to answer at the procedural level ("you just do this"), Romina recalls that they would "drive them insane" by asking instead for the conceptual "why."

Romina remembers asking questions a great deal both at the time of the longitudinal study in high school and now in her workplace. She associates asking questions in small groups as a very positive give-and-take of ideas. The learning environment she describes is dynamic (the "pushing" back and forth of questions would never need to end) and contributes to learning growth. She indicates that the question "why," though positive and necessary for her learning, was not always valued by her teachers in high school. In fact, it would drive them "crazy" and "insane." Her idea of conditioning indicates an almost behaviorist sense of the collaborative conditions the longitudinal study put in place which encouraged inquisitiveness.

7.3.3.6 Alternative Assessment

Romina questions whether doing "every single problem in under a minute," as standardized tests like the GMAT require, really assesses her knowledge (Disk I-225). In an idealized learning environment, Romina states that she would "get rid of standardized tests" as assessment measures (Disk I – 267). Romina remembers tests in Mr. Pantozzi's class. She recalls being able to "do our tests over" and verbally "explain things" to him for test items (Disk II – 48).

Romina articulates a belief that time should not be a criterion in assessment. She questions the value of timed tests like the GMAT. She asserts that she would get rid of standardized tests entirely. She seems to prefer the alternative assessments that Mr. Pantozzi would administer that would allow for extended time and explanation.

7.3.4 Learning Process

7.3.4.1 Willing to Learn

Since a part of Romina's job is to work with prospective employees, T/R1 asks Romina to discuss what qualities she looks for in candidates. Romina discusses learning:

> Just a desire to learn, like we, the most unsuccessful people that come into my class are those people that just want to get by on very little and not invest in the time, invest in the time upfront to learn, invest in the time to produce quality deliverables. Investing the time to just, they want to be in and out in three hours, and it doesn't work like that. So if you have a strong work ethic, you can teach those people, I mean, most of the people that couldn't do it are weeded out during an interview process, or weeded out in the first year. Now we're all in our second or third year. I mean, we can all do it, it's just a matter of how much time we're going to put into it. (Disk I – 258)

In describing what she would look for in a candidate for employment at her workplace, Romina includes two attributes: a "strong work ethic" and "a desire to learn." She employs very business-oriented diction when she describes what she looks for in prospective job candidates when she says they should "invest the time upfront to learn" and "invest the time to produce quality deliverables." Though business language infuses her description, the two traits that supposedly impress her – strong work ethic and a desire to learn – are essentially just a willingness to learn. Being willing to learn requires patience (time investment).

7.3.4.2 Learning to Learn

T/R1 asks for clarification about what Romina means by learning. T/R1 suggests that Romina is saying "something about learning to learn." Romina agrees that the phrase, "learning to learn," summarizes her earlier comments and then she considers further how it applies to her workplace:

> I feel, like that's completely applicable to what I do, because even at this stage, we don't specialize in anything, and we're in industry, and if you go in thinking that you know what you're going to do every time I've changed a client, or every time I've gotten a new project or a new task on my project, it's completely new. And it's just being able to pick things up quickly and ask the right questions to get an answer, it's going to take me a long time to get to my solution or my answer, but just in the fact that I know what questions to ask and I put the effort in and I know how to learn and how to absorb information, the right information, and weed through it. I mean, that's all we have to learn, to know how to do. I mean, I don't know about you guys with your jobs. (Disk I – 262)

Romina elaborates on the researcher's question about knowing how to learn to learn by observing that "all we have to learn" is to "ask the right questions," "absorb the information," and "weed through it." She acknowledges that this method requires a "long time" and "effort." Rather than focus on discrete topics that someone needs to learn, Romina asserts that a learner just needs to know *how* to learn. In her very meta-cognitive perspective, learning to learn requires a process of *asking* the "right" questions, *absorbing* the "right" information, and then *sorting through* the information to get to a solution. This process is admittedly more time and labor intensive, but Romina seems to indicate that it is more worthwhile.

7.3.4.3 Making Discoveries and Connections

Romina recalls how she told someone at work the "story" of how her group believed "we'd discovered Pascal's Triangle" (Disk II-8). Through her example of her group's discovery of Pascal's Triangle, Romina seems to translate the act of becoming a knower in terms of discovery which involves *constructed knowledge* as opposed to didacticism which would involve *received knowledge*. After offering the specific instance of making a connection to Pascal's Triangle during problems in high school, Romina later comments on the importance of making connections in general:

T/R1	If you have one idea in mathematics, can you imagine a mathematical idea where it would connect to another mathematical idea?
ROMINA	I hate learning things that don't, like I feel like if you learn one concept that doesn't connect to other concepts, like you're learning something almost useless. Because it's never ever going to be presented to you –
T/R1	That's a yes answer.
ROMINA	Yes, well it's never going to be presented to you. Nothing is ever going to be that simple. Nothing's going to be presented to you as just one little issue that if you figure that out it's all done. It's all interconnected as it is in the real world. (Disk II – 238 – 244)

Romina claims that concepts without connections to each other are "almost useless." She expresses strong negative emotion for learning situations that do not involve connections among concepts when she describes such a situation as one that "I hate." Romina asserts that "nothing is ever going to be that simple" where connections to other topics do not exist. For Romina, concepts in "the real world" are "interconnected."

Later, Romina spontaneously brings up another connection she often made during problem solving tasks in the longitudinal study – towers. During the interview in the context of "connections" and says she will "never forget towers" (Disk II-284). Throughout the latter part of the interview, Romina seems to profess a belief that learning without a sense of connection among concepts is tantamount to an "almost useless" experience. The Towers Problem has become an assimilation paradigm for Romina. It remains a lasting metaphor for Romina's understanding of conceptual knowledge. The relationship between learning and life is that of relation and synthesis rather than compartmentalization of discrete topics.

Chapter 8 INTERVIEW RESULTS – M.B.A. Graduate

When he needs reassurance about his life and his profession, Graham thinks about Godel's theorem, which states, roughly, that there is no end to mathematics, that the adventure of mathematical discovery will continue forever. 'Mathematics is to me, and to a lot of mathematicians, a very exciting thing,' Graham says. 'It's an open-ended challenge. No one's good enough to do even a small fraction of what there is to be done. The problems are more than adequate to challenge anyone, and as far as I can tell, that's always going to be the case. It's like juggling. When have you become the absolute juggler? When you can do all the tricks? Well, there's always one more ball.'
~ Ronald Graham, *Mathematical People (*Albers & Alexanderson, 1985, p. 117)

8.1 Introduction

This chapter focuses on a single session videotaped on July 15, 2009 in which Romina reflected back over the seventeen years she had been part of the Rutgers longitudinal study. The 90-minute session provided a unique opportunity for Romina both to discuss her beliefs about learning and to demonstrate her problem-solving by revisiting the Towers 5-tall task she has first considered in 1992. She looked back at her mathematical ideas through eyes that had seen close to two decades of participation in the longitudinal study as well as the inside of school classrooms yet again when she recently completed her masters in business administration.

8.2 Reflections V – Looking Back over Seventeen Years: July 15, 2009 (MBA Graduate)

8.2.1 Setting

An interview conducted at the Graduate School of Education at Rutgers University on July 15, 2009 occurred after Romina's graduation earlier in May with her M.B.A. from the Kellogg School of Management at Northwestern University. Romina discussed her reflections on the longitudinal study as she got ready to return to work at Deloitte in September after a trip to Asia. The interview lasted approximately 90 minutes and followed a loosely structured format. There was a single video camera and microphone. For the first 60 minutes the researcher questioned Romina about the following topics: her memories of the longitudinal study, her reactions to her graduate studies of which she had just completed, her thoughts about mathematics and learning, her definitions of knowledge, and her self-perception and beliefs about being a problem-solver. In the last 30 minutes, the researcher provided Romina with Unifix cubes and asked her about the Tower Problem (5-tall with a choice of two colors) she had first done in 4th grade in 1992 – seventeen years ago. Romina developed a solution which she then mapped to Pascal's Triangle. An analysis of this 30-minute behavior portion is included in the next section of this chapter. After the interview was transcribed and verified, "significant statements" were tagged and analyzed. Her significant statements clustered into three thematic categories: knowledge, conditions of her learning environment, and her learning process in general. What follows is an analysis of the significant statements made during the first 60 minutes of her interview.

Reflections V Interview –July 15, 2009 - SIGNIFICANT STATEMENT SUMMARY	
Issue	***Significant Statement(s)***
Knowledge & Knowing	***Understanding the Concept versus Memorizing a Formula –*** ***"So I understand kind of the basic idea behind it" –*** "I also just taught someone how to do a derivative and what it means based on my little graph…"(9); "Yes, because we did it so much and I don't think other people did…They just kind of memorize a formula and when you forget the formula, it was kind of hard to figure out how to do a derivative" (11); "Just to understand where it comes from…to be able to not have thought about it or even talked about it for five years, then still recall something about it…."
	Two Types of Mathematical Knowledge – "I think I'm pretty good at this point just getting a lot of information and being able to - organizing it to see what the problem is and then working to find the solution. It's more of like that process that I'm good at, not necessarily all the little details that go along with it" (137); "…I had the entire problem figured out; I knew how to analyze it. I knew what to do, but then, when it came to actually doing it, I'm a little confused with these little parts…" (141);
	"Right" Resides Only with the Group – Collective Knowledge when "Everything is Up for Debate": "I don't think I'm an expert at anything yet. And that's with always meeting new people and finding where they're at with things." (159); "I just figured out that no one is—you can be right in many different ways…" (163); "So it's kind of just assessing everything around you and just being able to kind of take everything into consideration - this is the best decision we can make. From the study perspective, I think what we – I mean, we just kept testing it." (165)

Conditions for the Learning Environment	***Business School is "this [the Longitudinal Study] every day in every single class… we always work in groups": "***So it's kind of funny because business school was pretty much this every day in every single class. So we always work in groups but it's a little difficult to get five people looking at the same numbers thinking the same thing" (5); "I've learned a lot more about group dynamics and stuff, because we had to work in groups for everything and Kellogg is just---I mean, every school is different-- but that's what their schtick is—teamwork. That's why I wanted to go there. You learn a lot about people, and people's priorities and how to manage that…" (23)
	The Collaboration of Socializing, Communicating, & Asking Questions – "You're in the Center of this Type of Wheel": "But to promote the group dynamic and getting a better output, you should disagree or just ask enticing questions." (45); "…And it's a lot of just socializing…communicating with a lot of different groups. You're kind of the center of this type of wheel..." (96)
	Being "Comfortable" with Groups and New Problems: "But, I really like it. I really like working like that; I always have. It might have been from this program; I'm not sure. But it's how I'm comfortable working." (53); "It was a long time; we built a lot of relationships with them…" (78); "Then, when you're in high school, I've known these people for ten years, so it's not that…it's a level of comfort…" (90); "They know my abilities at that point. They know me pretty well. It was just a lot easier to ask questions… (106); "But, within that specific group I was, like, the most comfortable I'll ever be..." (108); "To this day, if you put the group of us in a room, I think we'd still come up with something pretty good..." (123); "I think that whole problem-solving aspect of it and that whole kind of being comfortable which I think is really very important. Being comfortable being put in a situation where I have no idea how to do this…" (191)
	Sufficient Time: "I mean, hours for us. But, even with us, I think our sessions were like a few hours at a time—maybe 3 or 4 hours—we'd come up with an answer. But, we'd always go back and refine it. So I think that was what—that's why we'd get to right answers eventually, because we weren't scared, even after 4 hours, to say, "You know what? We need to go back to this…" (175); "You work on something for a month..." (177)
Learning Process	***Fighting, Disagreeing, and Asking Questions – "We were more like siblings than anything else": "*** I think we all fought a little bit. I think that's why we worked well together. Because we were more like siblings than anything else…" (39); "We'd fight a lot. I don't know if that was always visible on camera. We could get a little snippy. But I think we always had good intentions. And I think we still, to this day, always try to help each other out." (65); "We always came it from very different perspectives and different ways…"(41); "Most days I sit in a small room with six other people and you just argue your point. And that's what I'm fine with…"(104)
	Making Math "My Own" by Digesting, Visualizing, & Talking: "I think I'm really quick to jump to something, and then can explain it really quick. I've always talked really fast and I'm really animated. I'm very visual, too…" (57); "…So, I'm still a very visual, hands on , and I'm also—I think I can learn—I can read something and go ok. ..but I'm all about doing it myself. That's the only way I can really learn something is once I do it myself." (181); "I think I'm very—I need a little bit of quiet time, digesting time, at the beginning. I need to really understand something. Have some alone time to really think through my own thoughts…Then, it's like, I only get to a certain point by myself by kind of organizing the problem. I like to talk about it with other people…"(151);
	Making Connections – From Towers to Theorems: "Yeah, those towers, two colors, four high. I don't know if that is my actual first memory, but that's the first thing I remember." (29); "Yeah, I think the way we built on ideas—I think it was more interesting as we got older and we were able to figure out, go from towers to kind of an equation to kind of like a standard theorem, you know. That kind of stuff was a little bit—when we started connecting that…" (117);
	Gender Roles at work and in the study – "I was Always the Secretary": "Yes, I was always the secretary. I was always the one - to this day, I'm still the one who has to get Brian, Ankur. No, they worked and they went to Rutgers. So I think now. But I used to have get dragged them into every after school program like I was their personal secretary…" (70); "…they tended to be more talkers than I was and take the spotlight when people came into the room. Anything that is kind of a little bit gender, a little bit how we always interacted." (72); "I work with mostly men - I was the one always ordering dinner every night and doing all our grunt work. But I don't know if that was low level or it's a little bit of both." (74); "… And, talk to some of the girls and, "did you ever notice that you're the one always taking notes and setting up the meetings?…"(76)

	***Affective Dimension:* "**…So, I don't think we ever felt that confident to walk into one of these sessions and say, "We're about to amaze people right now." (115); "I think my group, our little group, I was really impressed by us sometimes - how did we do that?" (117)

***Table 8-1*. Reflections V Interview – July 15, 2009 – summary of significant statements**

8.2.2 Knowledge and Knowing

8.2.2.1 "So I understand kind of the basic idea behind it"

Throughout the interview Romina alluded to and contrasted two different types of knowledge – conceptual and procedural. Romina asserted that her knowledge had been conceptually-based throughout her time in the longitudinal study. She expressed concern about other students she encountered in business school who did not seem to know things in the same way that she did. Very early, during general questions about her experience in graduate school at the Kellogg School, Romina volunteered a story about how she had "just taught someone" about the concept of derivative:

> I also just taught someone how to do a derivative and what it means based on my little graph… the graph in the shaded area that Mr. Pantozzi taught us…he taught us that. I just taught it to someone which is sad because at the age of thirty we're business school students and we should know how to do that. (9)

Romina described how she had taught someone bother "how to do a derivative" and "what it means" based on a graph she had learned in high school with her teacher Mr. Pantozzi. She seems to imply that her knowledge of the concept of "derivative" has been more durable than some of the other business school students. In addition her knowledge of derivative allows her to understand both the procedure ("how to do") and the concept ("what it means"). Romina says it is "sad" that her peers lack an understanding of derivative as she does – she asserts that at the "age of thirty" she and her business peers "should know how" to deal with derivative calculus.

When asked if teaching someone else a concept like derivative was something with which she is "comfortable," Romina responded affirmatively and elaborated:

> Yes, because we did it so much and I don't think other people did…They just kind of memorize a formula and when you forget the formula, it was kind of hard to figure out how to do a derivative (11)

Here Romina provides what she believes to be the reason that she can teach a concept of derivative and her business school peers cannot. She explains that "we" (the students in the longitudinal study) "did it so much" whereas her peers probably did not. She says that the other business school students "just kind of memorize a formula" and so she was not surprised when they were unable to relearn derivatives. Implying that she learned in a way that was not just formula-based because she was able to relearn the concept, Romina asserts that "when you forget a formula" it is "hard to figure out" how to work with a concept like derivative again.

Later in the interview, the researcher returned the idea of knowledge and asked Romina to define more explicitly her own knowledge. Specifically, the interviewer asked, "What does it mean to you to know something really well?" (130). Romina explained that her definition of knowing something would involve a multi-faceted understanding of the "idea behind it":

> Just to understand where it comes from…to be able to not have thought about it or even talked about it for five years, then still recall something about it. I mean, I think that's what we did with a lot of these—the way we learned... With that, I mean I'm not really good at instant recall, crunching numbers type of—the normal thing. But, to this day, I've still—I've talked about this before—in college, when everyone was failing calculus, I could talk to all of them and explain. I don't know if I could do it now. It's been nine years. But, I could probably explain to them the fundamental theorem of calculus and kind of explain to them how all these things happened and worked. Visually, how everything was represented… So, being able to explain that to people—for them to be able to understand it—to explain the mechanics behind, just moving numbers around…I

> think that's very—back then I was very frustrated that I couldn't do the mechanical part of it. But, now, as I'm getting older, I don't have to do that. No one really does all that, really. Like logging things. We don't do that. So I understand kind of the basic idea behind it. I'm going to get through life just fine with that. (131)

Notice that throughout Romina's definition of what it means *to know* something well, the verbs "understand" and "explain" occur again and again. Romina says that to know something is "to understand where it comes from," to be able to "recall something" after years of not thinking about it, and "to explain" the concept to people and "for them to be able to understand it." For Romina, knowledge requires more than just to be able to explain the mechanical procedure associated with a concept (as with derivative or logarithm). Indeed, a knowledgeable person for her should be able to explain and understand the "basic idea behind" a concept regardless of how many years have passed since the concept was first introduced. In addition, knowledge must be such that one's audience must also "understand" the concept after an explanation is given. Thus knowledge should be durable over time and deep with association in both personal and group contexts. For Romina, someone with knowledge should be able to reconstruct that knowledge. She states that even today she would be able to "explain" for others both background and computation of "the fundamental theorem of calculus" – specifically, "how all these things happened and worked." Romina implies a lack of appreciation for knowing only a procedure – what she calls the "mechanics – just moving numbers around." She admits frustration in the past that she "couldn't do the mechanical part" of mathematics like computing logarithms ("logging"), but that "no one really does that" in her career and since she does "understand" the ideas behind concepts, she will "get through life just fine." For Romina, conceptual knowledge trumps procedural knowledge in real world application and currency.

Like her earlier discussion of derivative, Romina further elaborates on her definition of knowledge by providing another example of how she would contrast conceptual "understanding" with procedural "figuring" in the mathematics of slope:

> It's a little quantitative thing to me, but it's more—understanding how slope works versus actually figuring out the slope. It's much more higher level—I have tools which help me do like the basic, the number crunching—I have Excel, I don't need - it's much more understanding and setting up a problem in more of a quantitative in an easy to see, easy to calculate type of way. (135)

Romina discusses the difference between conceptual and procedural knowledge as "understanding how slope works" versus "actually figuring out the slope." She describes the "understanding"-type of knowledge to be "much more higher level." The "figuring out"-type of knowledge is "basic" and "number crunching" with which "tools" like "Excel" help. In Romina's estimation, understanding is higher order because computational tools will not be much help. "Setting up a problem" requires personal cognitive decisions, whereas calculating can be done by readily available tools.

The researcher probed Romina's definition of what would make someone an "expert." The interviewer asked, "How do you know if someone is an expert at something?" (154). Again, Romina referenced the ideas of long-lasting conceptual durability over time and deeply understanding fundamentals:

> Probably it's someone who worked with something for a very long time. I think you obtain expertise through just a lot of hours. And understanding the fundamental aspect— like understanding every point of the way versus certain aspects. (155)

Romina defines expertise as a result of time and multi-faceted understanding. An expert works with something "for a very long time" and over "a lot of hours." In addition, an expert understands what Romina calls "the fundamental aspect" of a concept which entails "understanding every point of the way" as opposed to just "certain aspects of a subject. This knowledge of the "fundamental aspect" seems to imply an ability to make multiple connections over time and subject area. Instead of discrete compartmentalization, an expert possesses understanding that involves relations among subject matter and synthesis of ideas through long-lasting study.

8.2.2.2 Two types of mathematical knowledge

What is knowledge for Romina? Throughout the interview, Romina offered a dichotomy whenever discussing knowledge and mathematics. She would mention problem analyzing versus numerical computing; planning versus carrying out a plan. Romina asserts that she is good at the former (analyzing and planning the larger picture), but not necessarily the latter (computing and carrying out the small details of a plan). Notice how she responds to a question about what would be something she "knows well":

> I think I'm pretty good at this point just getting a lot of information and being able to - organizing it to see what the problem is and then working to find the solution. It's more of like that process that I'm good at, not necessarily all the little details that go along with it (137)

As a problem-solver, Romina states she is "pretty good" at a three things: "getting a lot of information," "organizing" the information to "see" the problem, and finally "working to find the solution." She defines getting the information, organizing the problem, and working on a final solution as the "process" at which she excels. She asserts that while she is "good" at the analysis and organization process, she is "not necessarily" good at "all the little details" that are part of carrying out plans she devises.

In considering Romina's description of mathematical knowledge, one might recall Polya's classic, *How To Solve It* (1945) in which problem solving is given four phases:

> First, we have to *understand* the problem; we have to see clearly what is required. Second, we have to see how the various items are connected, how the unknown is linked to the data, in order to obtain the idea of the solution, to make a *plan*. Third, we *carry out* our plan. Fourth, we *look back* at the completed solution, we review and discuss it. (Polya, 1945, pp. 5 – 6)

Polya recommends four steps to the student when solving a problem: first to "understand" what the problem is asking; second to "see" the connections among the various givens in order to "make a plan," third to "carry out" the plan; and finally to "look back" and reflect on the solution preferably in a manner that allows discussion with others. One could contrast Polya's four phases of problem solving with the elements of Romina's "process" description of mathematics. We have already heard Romina's repeated use of the word "understand" in relation to knowledge and mathematics. For both Romina and Polya, understanding is the fundamental touchstone. Romina's process then involves "getting" the information, "organizing" the problem, and then "working to find the solution" – one could see parallels to making and carrying out a plan.

When asked about her quote from 1999 when she stated there were two types of math – "thinking" and "spitting out numbers" (138), Romina says she still stands by that definition now. She describes a class from the spring of 2009 in which she again had trouble reconciling the "thinking" with the "numbers part" of logarithms:

> This last class I took, I think the professor got really upset with me because I couldn't figure out how to do the algebraic logging part of the equation, because I don't remember what log is. …I had the entire problem figured out; I knew how to analyze it. I knew what to do, but then, when it came to actually doing it, I'm a little confused with these little parts. But, I think that's more important, because you can always find someone to help you with—how do I log both sides of this equation versus

> thinking about this whole problem. So, I still stand by that. I still think I'm not that great with the numbers part. (141)

Just as in 1999 when Romina divided math into two distinct, non-overlapping regions of "thinking" and "spitting out numbers," she still splits mathematics into knowing how to "analyze" a problem versus "actually doing" the calculations. In 1999 she had claimed that she was "never good" at the "spitting out numbers" part of mathematics, but that she was "decent" at the "thinking about it." Now in this passage she says she can "still stand by" that statement because she thinks she is "not that great with the numbers part" of mathematics. She recalls how in this "last class" from the spring semester of 2009, the professor "got upset" because she was unable to "do the algebraic logging part" although she "had the entire problem figured out." Specifying her quandary further, she explains that she "knew how to analyze" the problem and "knew what to do" to solve it, but could got "confused" with the "actually doing it" part. Romina classifies knowing how to "analyze" a problem "more important" than being able to do the associated calculations.

8.2.2.3 Shared Knowledge - "Right" resides with the group

When asked if she considered herself an "expert" in anything, Romina explained that she was not an expert yet because she had "a little way to go with everything":

> I don't think I'm an expert at anything yet. And that's with always meeting new people and finding where they're at with things. No, I still have a little way to go with everything. I haven't really chosen what I want to become an expert. (159)

Notice that Romina's description associates "expert" with a location toward which she moves. Not allowing for a single area of expertise, she says that she has "a little way to go with everything." Asserting that she does not consider herself "an expert at anything yet," she seems to use language of a journey as she looks forward. Connecting and communicating with others seems to be a part of this path - she explains that she is "always meeting new people and finding where they're at with things." Choice is also a factor in expertise for Romina – she explains that she has not yet "chosen" in what she would want to "become" an expert.

Probing the idea of "expert" further, the interviewer asked how Romina would know someone was "right" even if he or she were called an expert. Romina stated that no individual is right, but rather the group must come to agreement on what is finally "right." She made a distinction between individual and collective "right":

> I just figured out that no one is—you can be right in many different ways. Especially at work, I mean. Even at school, we'd come up with so many different answers to the problem that we're all right—no one is wrong. It's kind of all coming to an agreement and just eventually it's the group saying this is right. It's not one person knows the right or wrong answer. I think everything is up for debate. (163)

Alluding to relative rather than absolute truth in correctness, Romina states that "no one" is right because a person can "be right in many different ways." She says that while this holds "especially" true for work, she noticed the same phenomenon at school where her group would "come up with some many different answers" so that, in the end, "we're all right." To be "right" then, would involve the group "all coming to an agreement." So who decides if something is right? Romina identifies no individual figure – authority or otherwise. Rather, Romina asserts that "it's the group saying this is right." She reiterates her position that "right" does not reside with the individual but with the collective – "not one person knows the right or wrong answer." For Romina, there is a knowledge then that exists beyond individual capability. She implies this applies to all areas of life and requires discourse. To attain the knowledge of right or wrong, a group must converse – after all, "everything is up for debate."

When asked to provide an example of what Romina called the "group right," she described a general process of intense and broad reflection that assesses "everything":

> So it's kind of just assessing everything around you and just being able to kind of take everything into consideration - this is the best decision we can make. From the study perspective, I think what we – I mean, we just kept testing it. (165)

When given a problem as a consultant for her company, Romina explains that they have to go through a process of very broad terms whereby the group is "assessing everything around you" and taking "everything into consideration." The more wide-ranging their analysis and attention, the better their decision will be. In fact if they have been able to consider "everything," this would lead to the "best decision." She then refers back to the longitudinal study and says that her group would use this same process of reflection and assessment for a given problem – "we just kept testing it."

8.2.3 Conditions for the Learning Environment

8.2.3.1 Business School is "this [the Longitudinal Study] every day in every single class…"

Throughout the interview, Romina made references to how conditions in the longitudinal study paralleled those she experienced as a graduate student in her MBA program at Northwestern University. Specifically, she emphasized the use of working in groups and collaborative learning during both her time in the longitudinal study and then most recently in business school:

> So it's kind of funny because business school was pretty much this every day in every single class. So we always work in groups but it's a little difficult to get five people looking at the same numbers thinking the same thing (5)

Romina says that business school was like the longitudinal study “every day in every single class.” The condition of the learning environment she highlights is that “we always work in groups.” She qualifies that the constant group work can be “difficult” however, since it is challenging to get people looking at the same numerical data to be “thinking the same thing.”

Later, when asked about what she liked “best” at business school, Romina elaborates further being in groups and engaging in “teamwork”:

> I’ve learned a lot more about group dynamics and stuff, because we had to work in groups for everything and Kellogg is just---I mean, every school is different-- but that’s what their schtick is—teamwork. That’s why I wanted to go there. You learn a lot about people, and people’s priorities and how to manage that. We had to hand everything in, in groups. It’s a group project for…even papers, I’ve had to write twenty page papers with people… (23)

Romina identifies what she liked the most about business school as the fact that she “learned a lot more about group dynamics.” Explaining again that as business students they worked in groups “for everything,” Romina states that the reason she “wanted to go” to the Kellogg School of Management was because of this very emphasis on group work. Romina summarizes Kellogg as “their schtick is teamwork.” Not only did they have to work as groups on problems, but they also had to “hand everything in” as a group. She notes that even “twenty page papers” would be the result of collaboration among her group. As a result of all of the teamwork, Romina says that she had the opportunity to “learn a lot about people” – particularly how to gauge “people’s priorities” and how to “manage” them.

When asked to identify more specifically insight she gained about “group dynamics” through business school that she did not know before (26), Romina explained that she already had "a lot of experience" with groups as a result of the longitudinal study. Rather than identifying something new she had learned about groups, she described peoples' motivation as something she had always found interesting about group work throughout the years of the longitudinal study and then in graduate school:

> I mean, I had a lot of experience. I did that a lot; through this program, growing up with the same twelve people all the time. And, at my job, that's what we did; we worked in small rooms with each other all the time. I just think it's always surprising what motivates people and figuring out what motivates people is kind of like a new thing every single time. … And figuring out how to divide and conquer sometimes; or you have to sit in the room for three hours and just work together on coming up with theories… (27)

Through the longitudinal study to which she refers as "this program," Romina declares that she had "a lot of experience" with groups because she was "growing up with the same twelve people." In her workplace, she experienced similar conditions to which she was accustomed from the longitudinal study - working "in small rooms with each other all the time." Whether in school or at work, Romina explains that she finds it "always surprising" to learn what "motivates people." Notice that Romina applies the phrase "figuring out" not to objects like numbers or problems, but rather people. She says that people's motivation can be "a new thing every single time" and she also is always figuring out "how to divide and conquer" within the group. The conditions have remained constant for her – sitting in a room "for three hours" and working on "coming up with theories."

Romina further elaborates on how business conditions have paralleled what she remembers of the longitudinal study conditions – being left with a group to work out a problem over a period of many hours. She recalls what it was like to work in the longitudinal study when it was "just us":

> …But, to us, it was like people would walk out—everyone was silent. No one would talk to us for hours; so it would just be us sitting there. There were points when you were like, 'We're not ever going to get this. It's never going to happen for us.' And then, we'd always manage to get something… (117)

Romina recalls that the researchers would "walk out" and be "silent." The students would be left as "just us sitting there" for a period of "hours." She remembers although her group's initial concern was that they were "not ever going to get" the problem or make something "happen," they would be successful in the end and "always manage to gets something." Romina's recollection of both business school and the longitudinal study thus share long periods of work together without the presence of the authority figures where they would be presented with initially daunting tasks.

8.2.3.2 The collaboration of socializing, communicating, & asking questions – "You're in the center of this type of wheel"

In Romina's descriptions, collaboration involves socializing, communicating, and asking questions. In fact, Romina argues that disagreement and "enticing questions" are vital components of collaboration as they create better group results:

> But to promote the group dynamic and getting a better output, you should disagree or just ask enticing questions. (45)

For Romina, when members of a group "disagree" and "ask enticing questions," positive results occur. Questioning and disagreeing "promote" two things: the "group dynamic" and the "group output." Thus, Romina does not separate the ongoing functioning of a group from its eventual results – both seem to be held in equal esteem.

Romina provides a recent example of how she defines and engages in business collaboration when she participated in an internship with Target the previous summer:

> But it's very similar to what I do in consulting. I kind of…I did a competitor kind of research and analysis and came up with a…I did launch a juniors type of line. Which I did – it's out in the stores. It's really exciting. And it's a lot of just socializing…communicating with a lot of different groups. You're kind of the center of this type of wheel, they call it, The buyer—you're the person who defines the strategy but you don't necessarily do anything. You don't make the clothes; you don't create the marketing or the advertising. So you have to kind of work with everyone to convince them. (96)

Remarking that it was "really exciting," Romina explains how collaboration was integral to helping her "launch a juniors type of line" of clothing for Target that is actually "out in the stores" now. She likens the experience to being in the "center of this type of wheel" which necessitated a great deal of "socializing" and "communicating with a lot of different groups." She worked as a buyer. A buyer's role "defines strategy" but does not make the clothes or create marketing and advertising. Instead, her job came down to her ability to *convince* others – she had to "work with everyone to convince them."

8.2.3.3 Being "comfortable" with groups and new problems

Romina used the word "comfort" and "comfortable" many times in her descriptions of specific personal relationships or more general group experiences over the years. When asked how she felt about working in groups, she characterized herself as someone who was "comfortable" working in groups:

> But, I really like it. I really like working like that; I always have. It might have been from this program; I'm not sure. But it's how I'm comfortable working. (53)

Romina repeats that "I really like" working in groups and that she "always" has. While she is unsure of the origin of this affinity – she is "not sure" whether it was a result of "this program," she summarizes that she is "comfortable" working in groups.

Long-lasting personal relationships contributed to Romina's sense of comfort with groups. When asked about which relationships were "important" to her during the longitudinal study, Romina says that while her group developed many relationships with the researchers, their most significant relationship was with their high school teacher and Rutgers researcher, Mr. Pantozzi:

> It was a long time; we built a lot of relationships with them. I think Mr. Pantozzi kind of indirectly came out of that; we always saw him as a kind of Rutgers person, so I don't know if they thought of him like that. He was pretty significant to all of us. We had him for three years in math and he was very just invested in our learning. He's the reason I went to Penn. He said, "No, you're going to do this." And he wrote all my recommendations to college too. I still talk to him too. (78)

Describing Mr. Pantozzi as "indirectly" coming out of the longitudinal study since he was both a Rutgers graduate student at the time as well as a high school teacher at Kenilworth, Romina says that he was "pretty significant" to all of the students. The long time ("three years") he taught them and his involvement in their lives (being so "invested in our learning") created his significance. Romina states that Mr. Pantozzi was "the reason" she attended the University of Pennsylvania as an undergraduate. He wrote her recommendations and continues to talk to her now over ten years later.

Romina goes on to explain that the fact that she has "known" participants in the longitudinal study for such an extended time as given her a "level of comfort":

> Then, when you're in high school, I've known these people for ten years, so it's not that…it's a level of comfort… (90)

Romina explains that she had "known these people for ten years" and thus it led to a certain "level of comfort." Long-lasting relationships seem to be associated with the definition of comfort for Romina.

When asked specifically about her use of the words "comfort" and "comfortable" in past interviews, Romina puts the words in a context of how the others would make it feel permissible to ask questions.. Romina attributes the fact that the other longitudinal study participants "know" her for why they were all able to work "so well" together:

> They know my abilities at that point. They know me pretty well. It was just a lot easier to ask questions. You're like, "Should I ask this?" Whatever. What are they going to say? I think all of us were like that; we had no problem…sometimes, when I really didn't get it, I didn't mind, being like, "I don't get that. You'll have to explain that again and again and again." So, I think that comes with comfort and new environments you don't necessarily do that as much, because you're everyone else seems to get it, so I'm going to get it, too. So, I think that's why we worked so well together. Once they brought me along and I got it, I could probably add something later on. So, that's how we came to a better end product, I would say. (106)

By high school, the other longitudinal study participants had come to "know my abilities" and "know me pretty well." Knowing each other made it "easier to ask questions." She recalls an atmosphere where "we had no problem" asking questions when someone "didn't get it" or needed a topic re-explained. Romina says that she could tell the others when she did not understand and ask them "to explain that again and again and again." Romina attributes "comfort" with the ability to ask for help and explanations. She muses that in "new environments" that sort of questioning does not happen as frequently. Indeed she concludes that their "comfort" with each other in asking questions was "why we worked so well together." The other group members "brought her along" and then she could "add something later on" to their problem solving so, in the end, they "came to a better product."

The researcher asked if Romina would characterize herself now as "comfortable" with groups. Continuing with her reminiscences of the longitudinal study, Romina uses the language of “comfort” and “comfortable” again and again in her description. While she says she was the “most comfortable” with the students from the longitudinal study, she judges herself to still be quite comfortable in group settings:

> But, within that specific group I was, like, the most comfortable I’ll ever be. But, now I’m…I think I am comfortable…I don’t mind asking questions, especially after you establish yourself at work or at school or anything. But, I don’t think it’ll ever be at that level of comfort where I will just keep asking over... At one point, you’re just, “ok, move on without me. If I don’t get it, just go. I’m stalling the group.” But I still am pretty comfortable within groups. (108)

Contrasting being in a group during the longitudinal study to being in a group now, Romina asserts that she was the “most comfortable” with “that specific group” but she still considers herself “comfortable.” She explains that she continues to not mind “asking questions” but thinks that she will never achieve the same “level of comfort” where she could ask questions over and over again. Romina summarizes that she is still “pretty comfortable within groups.”

Later in the interview, Romina predicts that if her group from the longitudinal study were asked to do another problem, they would be successful. She equates knowing each other well with doing well:

> To this day, if you put the group of us in a room, I think we’d still come up with something pretty good. I think we’re all very good working with each other; we know each other so well. We all have very different strengths, I think. (123)

For Romina, knowledge of each other translates into a continually productive group. She asserts that “to this day” if they were to be “put in a room,” her group from the longitudinal study would “come up with something pretty good.” She claims that they are “all very good at working with each other” because they “know each other so well.” She also observes that they all have “different strengths.

When asked how she would compare her experience with the students she had known since 4th grade in the longitudinal study with other groups of students she had encountered, Romina explained that her longitudinal group had a shared common "basis" and were thus more comfortable with each other:

> …Because we all started off at the same point, and we always remembered towers in the fourth grade, you know. "Remember that; it's kind of just like the towers but with four colors?" So we always had that basis—we also, coming into it I knew who was good at what. That's really important when you're working in groups. Like I knew to expect certain things from certain people, so that made it go a little bit faster. Versus, I think, when it's a whole different group of people—we still did an ok job, but, if it was just the six of us, it would have been a lot more comfortable with each other. (125)

Since they all "started off at the same point" with "towers in the fourth grade," Romina characterizes her group as having a shared "basis" from which they would know "who was good at what." Remarking that is it "really important" to know group members' strengths, Romina says that she "knew to expect certain things from certain people" – this knowledge allowed her group to work "faster." In a different group, Romina reflects that although it would be "an ok job," the original "six of us" were "a lot more comfortable."

Following up on her comment about knowing what to "expect" from others, the researcher asked Romina to clarify what she meant by group expectations:

> Like Bobby or Mike coming up with some binary code; we would expect that from them. I think, like I asked a lot of questions, so prying that way. I think Brian and Jeff were like the our presenters to the outside world and they were very good at communicating once our ideas to everybody. Bobby and Mike got the real intense math, like they thought in a different…A lot of times we'd ask them questions, they'd start with an idea. And based on that idea, we could really take it far. (127)

Romina identifies expectations as roles she attributed to her longitudinal group members. For instance, Bobby and Mike were associated with "binary code" or getting the "real intense math." She was the one who "asked a lot of questions." Brian and Jeff were the "presenters to the outside world" because they were both "very good at communicating" the group's ideas. She explains that Bobby and Mike would usually "start with an idea," but then through the rest of them asking questions, the entire group was able to "really take it far." For Romina, each group member's role (getting the idea, questioning, presenting) was necessary to their final success.

Later in the interview, Romina added another dimension to her definition of "comfortable." Not only did Romina describe problem solving as requiring being comfortable with other people in a group, but also being comfortable with complex task situations where you initially have "no idea how to do this." When asked what she thinks has been the most long-lasting effect of the longitudinal study, Romina immediately identifies problem-solving and this multi-layered idea of "being comfortable" as what she has taken from the entire experience:

> I think that whole problem-solving aspect of it and that whole kind of being comfortable which I think is really very important. Being comfortable being put in a situation where I have no idea how to do this. Like, "I don't even know what you're talking about." Being able to break it into smaller parts and organize yourself and get the information you need for each part. Then, work in a group is the other big thing. Work in a group to kind of figure it out—something that is a daunting task, but if you work in a group, you figure it out together. Those two things. (191)

Romina points to two things which she feels have stayed with her the most from the longitudinal study: the "problem solving aspect" and the "whole kind of being comfortable." She expands on her definition of "comfortable" by saying that it includes being comfortable in a situation where "I have no idea how to do this." The type of complex task situation where someone does not immediately understand the problem and needs to be able to "break it into smaller parts," "organize yourself," and "get the information you need." Finally, Romina says that one needs to "work in a group" because if something is a "daunting task," you can "figure it out together" rather than alone. By the end of the interview, the word "comfortable" had taken on this multi-layered definition of comfort with complex, intimidating situations and comfort with other people. For Romina, problem solving seems to require this "comfortable" group effort.

8.2.3.4 Sufficient time

Drawing on some of her previous allusions which she makes here more explicit, a final element of the conditions for the learning environment is having *sufficient time* to complete a task with a group. When asked how long she thinks it takes to "typically" solve a problem, Romina recalls the hours and hours of time they were given during problem solving sessions in the longitudinal study:

> I mean, hours for us. But, even with us, I think our sessions were like a few hours at a time—maybe 3 or 4 hours—we'd come up with an answer. But, we'd always go back and refine it. So I think that was what—that's why we'd get to right answers eventually, because we weren't scared, even after 4 hours, to say, "You know what? We need to go back to this…we need to go back a few steps and start this from step 5. Not all the way to the beginning because we had some basis, but we started over a lot. (175)

Romina remembers having "hours," sometimes "3 or 4 hours" at a time," to come up with a solution. But even after they would have "an answer," they would "always go back and refine it." She observes that they were able to "get to right answers" because they "weren't scared" even after a long period of time (like "4 hours") to go back, reflect upon, refine, and possibly reconstruct a solution. She recalls that "we started over a lot" though not necessarily at the very beginning of a task because they would already have "some basis" from which to work. During the hours they would work, they had the opportunity to reflect in such a way that they would "go back a few steps and start this from step 5."

When asked if her experience of long amounts of time in the longitudinal study problems would be typical for other problem situations she has encountered in life, Romina observed the long amount of time it takes her to complete a project for work:

> You work on something for a month and it's not perfect, you think, "Well, we're going to go with not perfect right now." (177)

Romina remarks that she can work on something "for a month" and it will still not be "perfect." Sometimes they have to be satisfied with a "not perfect" solution. She continued to describe the typical amount of time for a project:

> Our standard project is probably six to eight weeks. And a lot of other stuff comes up and usually we get extended to another six to eight weeks for everything (179)

With a project lasting "probably six to eight weeks," Romina has found that "usually" they need an extension of another six to eight weeks. Considering problems over a long period of time is thus nothing new for Romina. Whereas for others doing a problem for hours and then returning to it again weeks or years later might not be the norm, for her, sufficient time was a necessary element to productive problem solving.

8.2.4 Learning Process

8.2.4.1 "We were more like siblings than anything else"

In recounting her learning process over the years, Romina uses the dynamic language of fighting, disagreeing, arguing, and questioning. She likens her relationship with the others in the longitudinal study as to that which would exist among siblings:

> I think we all fought a little bit. I think that's why we worked well together. Because we were more like siblings than anything else. I think

> we all fought a little at the beginning until, then, ok, someone would come up with a good point and we'd work towards it. But we'd always started off a little bit rocky… (39)

Like family members who "fought a little bit" and start off "a little bit rocky" with each other, Romina and her longitudinal group members "were more like siblings than anything else." In fact, Romina attributes this close sibling-bond to the reason "why we worked well together" because they could fight "at the beginning" but still be able to work together in the end. She recalls that, after the initial fighting, "someone would come up with a good point" and then together they would all "work towards it."

Romina returned to her characterization of fighting for her group's interactions, but qualified it more as *well-intentioned* and *helpful* fighting:

> We'd fight a lot. I don't know if that was always visible on camera. We could get a little snippy. But I think we always had good intentions. And I think we still, to this day, always try to help each other out. I mean if they need anything, like…I still talk to Brian and Jeff pretty regularly. (65)

Although they would "fight a lot" and get "snippy" with each other on and off camera, Romina contends that she and her group "always had good intentions." She says their good intentions have continued "to this day" as they still "always" make an effort to "help each other out." Thus Romina identifies the good intention of fighting as helping each other. She says that even now, after almost twenty years, they still talk "pretty regularly." Romina seems to give the impression of a close-knit, supportive group who used a "fight" as a helpful catalyst in problem solving.

In addition to the words "fight," Romina also often uses the words "disagree" and "question" when discussing how they would solve problems together:

> We always came it from very different perspectives and different ways. We always had a very different way of thinking; we disagreed a lot and

> then came to a conclusion together. Which is better; it was never, "Oh, ok, that's how you do it." We always questioned each other a lot. (41)

Noticing that the members of her group approached problems "from very different perspectives and different ways," Romina explains that the fact that they "disagreed a lot" was almost a foregone conclusion. But Romina emphasizes that the combination of their "different way of thinking" and disagreements in which they "always questioned each other," allowed for "better" results. They were able to reach conclusions "together" and were never satisfied with a single approach on "how" to do a problem. The diversity of their perspectives, backgrounds, and learning styles created for Romina productive sessions of intense questioning.

When asked to explain what she means by her continued use of the words disagree and question in her descriptions of group activity, Romina explains that disagreements and interrogations allow for people to "probe" and "dissect" their thinking:

> I think it's a probe—I think that's what we did. I don't think we ever thought someone was completely wrong. It's just that not everyone may have understood it. So, I just keep asking them questions so that they can dissect their whole thought process. (47)

Romina defines disagreement as "a probe" which allows for continued analysis. Since "not everyone may have understood" a problem or approach, the act of disagreeing or questioning causes people to "dissect their whole thinking process." Romina says that these questions were not about right or wrong – in fact, she thinks they never considered someone "completely wrong." For Romina, right or wrong was not the goal – understanding was the desired target. She remembers that specifically she would "just keep asking" questions to help others achieve this end. Asking others to reflect on their own thinking seems to have been part of Romina's learning process. She seems to use the words argue and question as synonyms for *convince*.

Romina points out that her problem solving in the longitudinal study was not an "individual thing," but rather a constant collaboration:

> I pretty much carried Jeff through most of grammar school and high school, so…I'm assuming I had to have helped him if he needed it. But, I think we were all pretty…I don't think it was an individual thing, so especially Ankur, Brian, Mike, Bobby, Jeff and I worked a lot together . We had after school. We definitely; I think we did. At least, we tried to. I don't think we were ever, "Oh you don't get this; that's it." We would always try to be on the same page. (61)

Although she "carried" Jeff through school, Romina asserts that problem solving was not an "individual thing." Rather, Romina remembers that she, Ankur, Brian, Mike, Bobby, and Jeff "worked a lot together." She recalls that they continually tried to "be on the same page" instead of stopping discussion with "that's it" if someone appeared not to "get" it. The group – "we" – wanted their understanding to be together.

Asked about her learning today, Romina explains that it still involves sitting in a room with a small group of people and arguing:

> Most days I sit in a small room with six other people and you just argue your point. And that's what I'm fine with. It's the more formal presentations. I don't think I liked it that much in the study either. Usually I let Jeff take that or Brian. When they made us stand…I really tried to stay away from the board unless they made me. (104)

Romina's description of the process through which she solves problems in her job parallels closely the process during in the longitudinal study. On "most days" of work, she sits "in a small room with six other people" in order to "just argue your point." While she asserts she is "fine" with the small group arguments, she does not care for "more formal presentations." She says that she thinks she didn't like formal presentations in the longitudinal study either and would “let Jeff take that or Brian.” Romina recalls that she “really tried to stay away from the board” unless required.

8.2.4.2 Making math “my own”

How does Romina describe her own learning? In addition to her consistent reference to working with a group of her peers, Romina identifies first organizing a problem on her own and then trying to come up with a visual representation as part of her learning process. When asked to characterize what makes her “uniquely you” when solving a problem (56), Romina talks about her animated explanations and visualizations:

> I think I’m really quick to jump to something, and then can explain it really quick. I’ve always talked really fast and I’m really animated. I’m very visual, too. So I draw these charts. So, I’m like, “You put this on the y axis and you put this on the x axis. (57)

Romina describes herself as someone who is “really quick to jump” into a problem and “explain it” in an “animated” manner. She calls herself “very visual” because she says she will make visual representations for a problem like “charts” or graphs on the coordinate plane with the x- and y-axis.

When asked to compare herself as a learner now to a learner in the longitudinal study, Romina explains that she still considers herself “very visual” and still “hands on” since she needs to construct her own understanding:

> I’m very visual. I have a lot—I can’t just hear something which made college real hard in lectures. So, I’m still a very visual, hands on , and I’m also—I think I can learn—I can read something and go ok. I can see someone else doing it but I’m all about doing it myself. That’s the only way I can really learn something is once I do it myself. (181)

Romina describes herself as “very visual” and that she has continued to be “very visual, hands on.” While she “can’t just hear something,” she is able to “read something” or "see someone else doing it" in order to learn. However, what she considers her most effective mode of learning – what she is "all about" - is constructed knowledge ("doing it myself"). So although she would prefer seeing something over hearing it in a direct instruction setting, she asserts that the "only way" she can "really learn" is to actually "do" the work herself. What she asserts then is a preference for constructed learning.

When asked to be more specific about how she would describe herself as a problem-solver, Romina explains certain phases through which she sees herself going:

> I think I'm very—I need a little bit of quiet time, digesting time, at the beginning. I need to really understand something. Have some alone time to really think through my own thoughts…Then, it's like, I only get to a certain point by myself by kind of organizing the problem. I like to talk about it with other people. Kind of be like, "Is this what you think? The issues? How are we going to tackle this?" Then, work on it together. And then, come up with some kind of plan. And I think it works out well, because then we get it a little bit further. I don't like going down and doing a problem all by myself, because the chances of me getting to the right answer that everyone else gets to, is gonna be—it's probably not going to happen. So, it's just bringing people along and then solving it together. (151)

During the problem solving process, Romina identifies four phases: understanding the problem and organizing her thoughts during "alone time," talking with and questioning others, working as a group to form a plan, and then solving it together. Romina emphasizes that at the beginning of a problem she likes to have "a little bit of quiet time, digesting time" that she also names "alone time." She says it is during this quiet-alone-digesting time that she gets to "really think through" her thoughts. She admits that she can only get to a "certain point" however on her own. After her initial alone time, she then finds it necessary to include a collaborative component. Saying that "I like to talk about it with other people," Romina gives examples of questions she might ask others. Her example questions involve asking what the others think and what strategies they might have. After this talking phase, then, as a group, they can form "some kind of plan." She says this process of problem solving – first alone and then with others - "works out well" because together "we get it a little bit further" then they were able to go individually. Again, she mentions that she does not like doing a problem "all by myself" because then she does not think she will get a solution – "it's probably not going to happen." Through the group effort and by "bringing people along," she finds that then they are sucessful "solving it together."

After being asked how she would define herself as a problem solver, the interviewer wondered how Romina thinks others would describe her. Romina returned to the idea of her initial alone time: "I definitely think a lot of people would say I like my alone time, my quiet time at the beginning" (153). Later, she returned again to what she considered this crucial first step in her problem-solving: "I'd have to walk through it a few times myself. It's my own time type of thing…" (189). During that beginning "quiet time" Romina needs to "walk through" the problem herself and make it her "own time" for understanding the problem.

8.2.4.3 Making connections from Towers to theorems

As one of the initial questions concerning the longitudinal study, the researcher asked, "What are your first memories?" (28). Romina immediately referred to "towers":

> Yeah, those towers, two colors, four high. I don't know if that is my actual first memory, but that's the first thing I remember. (29)

Gesturing to the Unifix cubes on one of the tables in the room, Romina identified "towers" as her first memory. Specifically, she seems to indicate a problem with towers – "two colors, four high." In fourth grade the students were given the Towers Problem in which they were to build all of the towers five high with a choice of two colors. Romina explains that while she is not sure if it is her "actual first memory," the towers problem is the "first thing" she remembers.

Later, when asked about what she was proud of during the longitudinal study, Romina referred again to the towers but in the context of how her group was able to make connections between towers and theorems:

> Yeah, I think the way we built on ideas—I think it was more interesting as we got older and we were able to figure out, go from towers to kind of an equation to kind of like a standard theorem, you know. That kind of stuff was a little bit—when we started connecting that… (117)

Romina points to "the way we built on ideas" as something of which to be proud. Notice the use of action verbs throughout her answer here. She uses the language of construction ("built"), cognition ("figure out"), and association ("connecting"). She found things "more interesting" when they were about to connection towers to "an equation" to "a standard theorem."

8.2.4.4 "I was always the secretary"

Asked about her role in the group during the longitudinal study, Romina brought up her gender forming part of her identity within the group, "I think I would be the most compassionate, considering they were all men and it was just me" (68). She called herself "the most compassionate" when in a group of all males. Indeed, it would often happen in the high school years of the longitudinal study that Romina would be the only female in the group - Romina, Ankur, Brian, Michael, and Jeff. The researcher followed up on Romina's self-characterization and asked if gender ever played a role in their group. Romina answered affirmatively and then elaborated by naming herself the "secretary" of the group:

> Yes, I was always the secretary. I was always the one - to this day, I'm still the one who has to get Brian, Ankur. No, they worked and they went to Rutgers. So I think now. But I used to have get dragged them into every after school program like I was their personal secretary. In the thing, I was always the one writing. That came up. I don't know if you guys caught that on camera or it was after camera—we had a discussion one day. (70)

When asked if gender played any role, Romina introduces the label "secretary" to name herself in the context of the group. She says that she was "always the one" who "dragged them into every after school program." Calling herself the boys' "personal secretary," she goes on to explain that she was "always the one writing." She wonders aloud if the researchers ("you guys") ever noticed on videotape that Romina was in this role. She recalls that the group had a "discussion one day" about her secretary role observations.

Asked what the others felt about Romina's assertion that she was the secretary, Romina explained that the boys were often the "talkers" and would "take the spotlight":

> No, they thought I was probably crazy. No, sometimes, I think they—they tended to be more talkers than I was and take the spotlight when people came into the room. Anything that is kind of a little bit gender, a little bit how we always interacted. (72)

Romina indicates that her view that she was the "secretary" was not necessarily shared by the boys. Replying that "no" the boys thought she was "probably crazy" in her observation, Romina explained that nevertheless "they tended to be more talkers" and "take the spotlight" when others entered the room. She acknowledges that this might not be completely the result of gender but rather a combination of "a little bit gender" and a little bit of "how we always interacted."

Following Romina’s discussion of her view of gender in the longitudinal study, the researcher questioned whether gender ever played a role at work for her now. Romina explained that she was still in an environment of “mostly men”:

> I work with mostly men - I was the one always ordering dinner every night and doing all our grunt work. But I don’t know if that was low level or it’s a little bit of both. (74)

At work, Romina points out that she has been the person “always ordering dinner” and “doing all our grunt work.” Again she acknowledges, however, that this might not entirely be a result of her gender but rather be because of her “low level” in the company.

Finally the researcher asked about gender during her business school experience. Romina explained that the girls in her school had actually gotten together to address this very topic of gender:

> There’s a lot less girls at school, but we always get together and talk about it. We kind of made a conscious effort, you know. We’re not going to be the ones who - we have to set up meetings on Outlook; it’s a very kind of business atmosphere. We always try to - I think it happened the first year—you’re trying not to stir the water; everyone get along. And, talk to some of the girls and, “did you ever notice that you’re the one always taking notes and setting up the meetings?” Yep, so we tried to be more conscious of not doing it. A lot of guys don’t think like that. I was really surprised, especially at business school when we’re supposed to be on the same level, they’re like, “Can’t you just do it?” (76)

With fewer girls in her business school, Romina says that they would “always get together” and talk about gender. Romina explains that the female graduate students “made a conscious effort” to not fall into certain stereotypical roles. During the “first year,” she says that they were trying “not to stir the water” so that everyone would “get along.” But then as she talked to more of her female peers, they would observe that, as the woman in a group, “you’re the one always taking notes and setting up the meetings.” Thus, they attempted to be “more conscious of not doing it” – taking notes and setting up meetings. Romina observes that many of the guys "don't think like that" and it "really surprised" her how often the men would ask the women to "just do it" when it came to notes and meetings.

8.2.4.5 Affective dimension

Reading a quote from Romina's interview in 1999 about feeling scared and having low self-esteem, the researcher asked if Romina remembered feeling low self-esteem in the longitudinal study. Romina explained herself:

> I think that is something I've kind of suffered with a lot. Even now, still…. But, to all of us, we're not capable of this high-level —once the Rutgers group left, we'd kind of talk amongst ourselves and our teachers would talk to us: "Oh, you guys are doing such high level math and you don't even know and we're like, " Yeah, right, we had no idea. So, I don't think we ever felt that confident to walk into one of these sessions and say, "We're about to amaze people right now." (115)

Saying that low self-esteem is something she "suffered with a lot" then and "even now still," Romina goes on to say that as a group they often did not realize they were doing something significant. She recalls that her group never felt "capable" of doing "high-level" mathematics. When they were told that they were doing "high level math," she remembers that they "had no idea." She does not remember feeling "confident" to walk into a problem solving session with the intention to "amaze people." Despite her impression that they lacked confidence, she expresses a pride in what her longitudinal group ended up accomplishing. Asked if she felt "proud," Romina replies, "I think my group, our little group, I was really impressed by us sometimes - how did we do that?"(117). Of "our little group," Romina summarizes that she was "really impressed" but still wonders "how" they did it.

8.3 Revisiting Towers 5-High: July 15, 2009 (MBA Graduate)

8.3.1 Setting

During the same July 15, 2009 interview described above and after discussed her reflections on the longitudinal study, Romina engaged in a problem solving session by revisiting the Towers 5-high task from fourth grade. Recall that this 2009 interview lasted approximately ninety minutes and followed a loosely structured format that split into two parts: I) a 60-minute epistemologically-oriented discussion focused on perceptions, beliefs, and reflections and then II) a 30-minute problem solving session on the Towers Problem 5-tall with a choice of two colors. An analysis of this 60-minute beliefs portion preceded this section. Here, we consider the second part of the interview. In the last 30 minutes, the researcher provided Romina with Unifix cubes and asked her about the Tower Problem (5-tall with a choice of two colors) she had first done in 4th grade in 1992 – seventeen years ago. Romina developed a solution which she then mapped to Pascal's Triangle. An analysis of Romina's problem solving behavior during this last 30 minutes follows in this section.

8.3.2 Local Organization: "Inverses," "Opposites," and "Couples"

While Romina's initial response to the Towers Problem is to use an algebraic rule of "n to the x," she pretty quickly abandons the rule in favor of listing the tower sequences on her paper when she cannot remember what the variables "n" and "x" represent. Next, Romina explains her local organizations for generating and categorizing towers. She employs a particular strategy and assumption: for any tower, one can find another tower in which each position's color has been reversed. For instance, for the tower of four blue cubes, one can find a tower of four yellow cubes. Romina introduces and uses several different names with metaphoric underpinnings to towers that fulfill this condition: "inverses," "opposites," and "couples." Notice below how Romina begins work through the Towers problem as an adult as she moves away from an algebraic rule she cannot fully remember in favor of a systematic list created with her "couples" organization. After five minutes of such exploration, she then returns to a more general abstract strategy as she introduces Pascal's Triangle and works to map cases of the Towers problem to its different rows. This mapping lasts for ten minutes. Finally, she returns to the algebraic rule, determines the variables, and explains the exponential nature of the growth pattern. As a last argument, she justifies the additive rule for Pascal's Triangle by using the binomial nature of the Towers problem with varying height as support of her generalization when given a choice of two colors.

8.3.2.1 Building "from scratch"

In the first minutes of the problem-solving discussion, when asked what she remembers about "Towers," Romina immediately responds with a question to the researcher to verify that the algebraic rule for Towers is "n to the x":

T/R	…So what do you remember about Towers?
ROMINA	Isn't it n to the x? The whole - I hope it is. I guess I just remembered that this was how they showed us combinations and permutations and all that. Right? This was the basis we learned in fourth grade and then we carried it on over and over until we figured out if you have, like. We started out: There's towers 4 high and there are two colors, so how many different towers can you make? That eventually led us into (*writes with her finger on the desk*) 4 squared or 2 to the 4^{th}. I don't remember. It's one of those. I can't remember which one n is: the number of blocks? You don't have to tell me.
T/R	I bet you could rebuild it. [*T/R pushes the bag of Unifix cubes toward Romina*]
ROMINA	Help me - I can't do this alone! No, I'm kidding. (225 – 228)

Romina initial memories of the "Towers" problem include several points: an algebraic rule she recalls as "n to the x," an association with "combinations and permutations," and her first exposure in fourth grade with "towers 4 high and there are two colors." Specifically, Romina defines the Towers problem as, given towers 4 high and a choice of two colors, "how many different towers can you make?" She says that her group found the answer to be "4 squared or 2 to the 4th" but she "can't remember" because she is not sure whether the variable "n" is the "number of blocks" or something else. Romina remembers using the Towers problem "over and over" after fourth grade in different contexts. Notice that throughout Romina's recollection, she phrases many of her key memories in the form of a question to the researcher – "isn't it n to the x?"; permutations and combinations, "right?"; and "the number of blocks?" When the researcher suggests that she could "rebuild" the problem, Romina turns to Robert, who is videotaping the interview, and says "help me – I can't do this alone!" Then she says she is "kidding."

In the next two minutes, Romina works to define the problem more specifically. She claims to only remember towers "4 high" at first, but then discusses how she would "recreate" and think of Towers with variable height "from scratch":

ROMINA	Do you want me to recreate the original towers problem? Wasn't that 4 high, 2 different colors?
T/R	Do you want to do that problem?
ROMINA	Sorry, do you have a different problem? That's the only one I remember.
T/R	In fourth grade, you guys were doing one with 5 high. You were looking at 5 high.
ROMINA	That's one of the variables that changes.
T/R	So you can look at either one. You want to do 4 high or 5 high?
ROMINA	Do you actually want me to build all the towers?
T/R	What would be the way you would think of this?
ROMINA	The way I would think of it from scratch? I would probably build the towers—we got pretty quick at it—we just got pretty quick [*uses a pen to*

> *write lists*]. Didn't we do that? Are we seriously doing this? You told me we didn't have to do math. Did you find that I was the anal one about patterns? (230 – 238)

Romina asks the researcher for verification that the "original towers problem" was with "4 high, 2 different colors" – she goes on to say that it is the "only one I remember." The researcher replies that in fourth grade, she worked on the Towers Problem with cubes "5 high." Romina observes that height is "one of the variables that changes." Wondering aloud if she should "build all the towers," Romina says that the way she would think about Towers "from scratch" would be to "build the towers." Beginning to write a list of tower sequences with the letters B and Y for the blue and yellow Unifix cubes on the table, Romina mentions that her group "got pretty quick" at this. She again asks for verification that this was indeed the method her group used – "didn't we do that?" She seems to express surprise that she is "seriously" doing the Towers problem again. She questions the researcher again if anyone noticed that she was "the anal one about patterns." Notice that, in this single episode of under a minute, Romina asks the researcher seven questions seeking, for the most part, verification.

8.3.2.2 "I work in little couples"

For a little over five minutes, Romina employs written representations of letter sequences for different towers. She explains that she like to "work in little couples" in order to organize them. Choosing to write out sequences for towers four-high with a choice of two colors, yellow (Y) and blue (B), Romina describes her grouping strategy:

ROMINA	This is how I'd go about it and add like (*writes more*). Then, do that again.
T/R	What are these groups? (*Points at paper*).

ROMINA I think this is how I used to do it—I'm not sure but. I take…this is 4 high, 2 colors –yellow and blue, obviously (*points at paper*). I work in little couples, I guess you could say. I start with 3 yellow—3Y,1B group This is the 3B, 1Y group.(*T/R gives Romina a new marker*).3Y, 1B group. Then, 3B, 1Y group. This is going to be, eventually, the same thing. So, I think what we used to do-- we double checked ourselves on everything. We used to kind of write them all out and even though this we knew… We'd just cross out the ones that are visually like the same thing. I would just keep going…Then, you'd have the 4B. We'd have 4 blue and 4 yellow and then we'd work on them like that. …introduce one new color until you get to the inverse. (242 – 244)

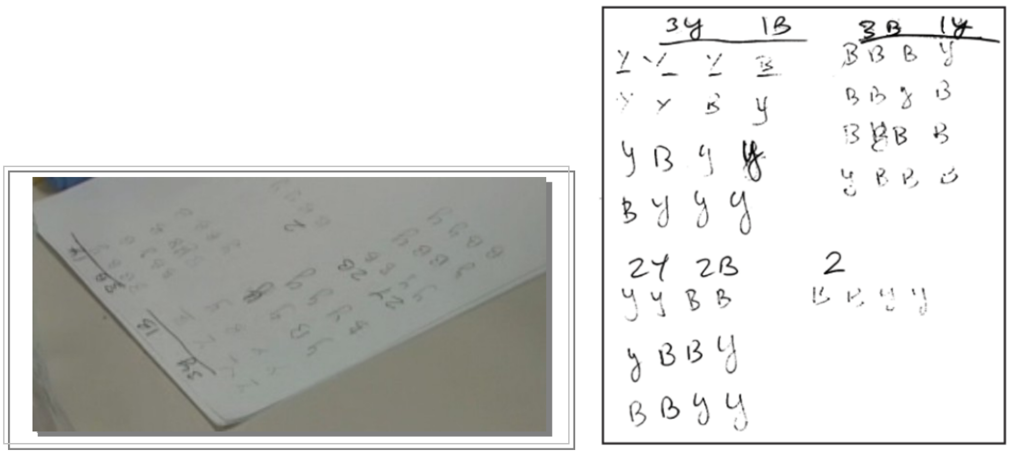


***Figure 8-1*. Romina's Tower sequence list: "I think is how I used to do it…" (244)**

Romina begins by writing the sequences YYYB, YYBY, YBYY, and BYYY horizontally on her paper. Next to this set of four towers, she writes BBBY, BBYB, BYBB, and YBBB. Romina explains that this is "how I used to do it." Prefacing that she is "not sure," Romina goes on to narrate how they would group the tower sequences as she simultaneously records them on her paper (see Figure 8-1). Calling them "little couples," she describes the sequences she wrote as having the headings of the "3Y, 1B group" and the "3B, 1Y group." She observes that "we double checked ourselves on everything." She says that the strategy would be to "write them all out" and "just cross out the ones that are visually the same thing" thereby eliminating any duplicate tower sequences. She explains that you "keep going" with different groups like "the 4 blue and 4 yellow."

Romina mentions that her grouping strategy involves finding the “inverse” of a tower sequence. The researcher asks her to clarify what she means by “inverse.” Referring to each tower as a “person,” Romina then defines the names she has for how towers are “related” together in pairs – “inverse,” “couple,” and “opposite”:

T/R	The inverse – so what do you mean by the inverse?
ROMINA	The inverse—wherever this was yellow, you’d turn to blues—to get to the opposite of that little tower.
T/R	So the opposite – so inverse and opposite, is that –
ROMINA	Um, I don’t know if it’s mathematically is the same thing, but, like the opposing couple to this person (*indicating with her marker*) would be this guy. These are related—this is what I’d call the inverse.
T/R	So, when you said “a couple” before, is this a couple?
ROMINA	Yes, this is a couple. This is a group. (*indicating on paper with her marker*) So, that’s the way I would think about it. Let me see, so this would be (*writing on paper*)—is that how we do it?—this high(*crosses out on paper*); sorry. (*Begins writing again*) So we would be—is this how it goes?—I don’t remember how this one. Bobby, can you tell me if this is how it goes? (245 – 250)

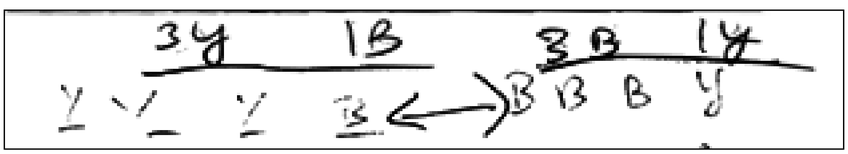


***Figure 8-2*. Romina’s representation for “inverse” (248)**

Romina defines “inverse” by saying that whenever a yellow cube appeared in a sequence, “you’d turn to blues” and thereby get the “opposite” of that tower. When asked about the relationship between “inverse” and “opposite,” Romina explains that although she is unsure whether they are “mathematically” the same thing, here she is using them as synonyms. Anthropomorphizing the towers, Romina elaborates that “the opposing couple to this person would be this guy.” She draws an arrow between two lettered sequences on her paper YYYB and BBBY (see Figure 8-2) and explains that they illustrate what she means by “a couple” with this “inverse” and “opposite” relationship.

Considering each letter as a position, YYYB would be a tower of four cube positions with yellow cubes in the first three positions and a blue cube in the last. BBBY, on the other hand, would have the color in each position switched – blue cubes now occupy the first three positions and a yellow cube occupies the last. Romina explains that a “group” would be a set of sequences YYYB, YYBY, YBYY, and BYYY. Each “person” in this group would create a “couple” with its “inverse”/ “opposite” in the group of BBBY, BBYB, BYBB, and YBBB. She summarizes that this how she would “think” about towers 4-high with a choice of two colors. Almost musing to herself, Romina then begins writing something else on her paper underneath her systematic list. She wonders, “is this how it goes” and asks “Bobby” (the videographer) for help. Romina continues to write on her own and the researcher asks her what she is writing now. It is at this point that Romina introduces how she remembers relating Towers to Pascal’s Triangle.

8.3.3 Grouping by Cases within Pascal’s Triangle

8.3.3.1 Pascal’s Triangle – First Version – “I love Pascal’s Triangle”

After her initial approach to systematically list the tower sequences using her “couple” grouping strategy, Romina takes the next ten minutes to reconstruct and explore how she could use Pascal’s Triangle for the Towers problem. When asked what she is writing under her list of tower sequences, Romina explains what she is “building” now:

ROMINA I’m building Pascal’s Triangle, to figure out the fourth row of it. To figure how many combinations there’d be; then, I’d go back to figure out which was this little…But, see my problem is—this is where the technical stuff comes in—so, 4 squared is 4 times 4, right? Two to the fourth - 2 times 2. So, that would be the same thing anyway. So that’s the way I—Sometimes I have to build it up from scratch. But, I’m building—I love Pascal’s Triangle. We thought we discovered it. 1 goes on the outside; you add these 2 to get this (*she is pointing and writing with the marker*). And, then, the 1 goes—is that how…I feel like…

T/R So where did the (*points with finger*) So is the 1; So where did the 2 come from?

ROMINA I just know it goes there. Is that…This is just one color; so this would be 2; so this would be all blue; this would be all yellow. There would be two: One's blue-yellow; one's yellow-blue. That's how I remember that line. Is this line wrong; could you tell me that? (253 – 255)

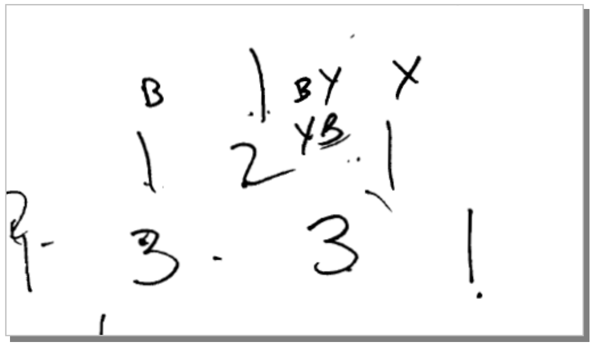


***Figure 8-3*. Romina's first representation for Pascal's Triangle (253)**

Romina explains that what she has written above in Figure 8-3 constitutes "building Pascal's Triangle." She describes how she wants to map entries in Pascal's Triangle to groups in her Tower problem. Her goal, Romina says, is to find the "fourth row" to "figure out how many combinations there'd be" and then "go back" to the towers. Not yet explaining how she has linked numerical expressions to Pascal's Triangle, she observes that "her problem" is the "technical stuff" because "four squared" and "two to the fourth" would be the "same thing anyway." Romina explains to the researcher that "sometimes I have to build it up from scratch" but that "I love Pascal's Triangle." She remembers that she and the others in the longitudinal study believed they had "discovered" Pascal's Triangle. When asked her reasoning behind the number placement in the triangular array, Romina says, without further justification, "I just know." Notice that her first version of Pascal's Triangle is in fact incorrect as it is missing the first row (see Figure 8-3) – Romina has written the 0^{th} row followed by the 2^{nd} row. At this time, Romina does not seem to be aware of this omission. She explains that the row with entries 1 2 1 refers to the towers with two colors. The "1" on either end would represent the “all blue” and “all yellow” towers. She says the entry “2” refers to the “blue-yellow” and “yellow-blue” towers. Seeming to express some doubt, Romina acknowledges “that’s how I remember that line” but questions aloud if the line is “wrong.” She asks the researcher to “tell” her.

Rather than answering, the researcher asks Romina to “explain” Pascal’s Triangle further. Romina then rewrites Pascal’s Triangle by combining the entries with tower sequences beneath each numeral. She explains her reasoning for this new representation:

ROMINA	The way I remember this, and I could be thinking about this wrong, is: this one would be my blue/blue; this one would be my yellow/yellow; this would be blue and yellow. So, that’s why I was thinking it would be…the one is the…Then, we’d have three of (*keeps writing*)
T/R	So, can you explain – you said you’d have three?
ROMINA	This is the 1, 3, 1
T/R	3 – So, what do these have in common?
ROMINA	1 yellow—is that 2 yellows? (*Draws boxes around letter sequences on paper*) My—I may be grouping them incorrectly though from what the original theorem is supposed to be. (263 – 267)

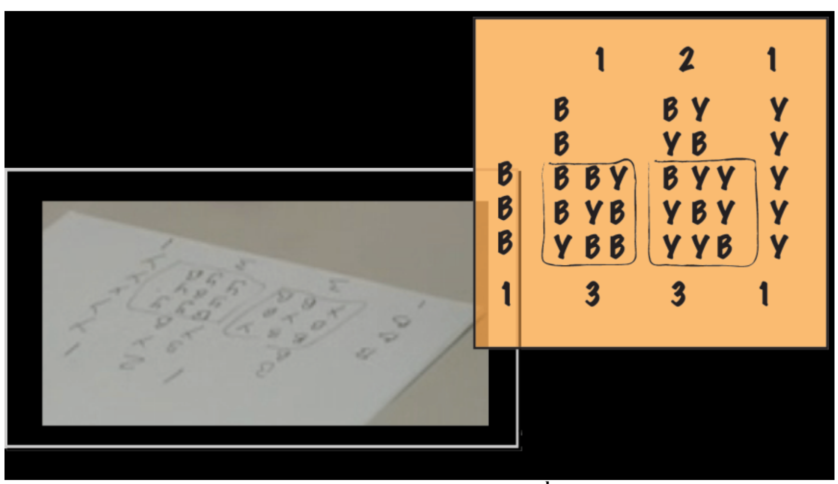


***Figure 8-4*. Romina’s Tower sequence groups for the 3rd row of Pascal’s Triangle**

Notice that Romina now further refines her previous explanation. In her new representation (see Figure 8-4 above), Romina writes and explains the 1 2 1 row of Pascal’s Triangle as summarizing towers 2-high with a choice of two colors, yellow and blue. She says "this one" in Pascal's Triangle refers to "my blue-blue" tower – she writes BB vertically below the entry – while the other "one" refers to "my yellow-yellow" which she records as YY. The 2 would then relate to the number of towers with a single "blue and yellow." Indeed, underneath the 1 she has written BB, beneath the 2 she has BY and YB, and underneath the final 1 she has YY. She goes on to list towers three-high with a choice of two colors into four groups which she places above the entries 1 3 3 1 in Pascal's Triangle. Notice in Figure 8-4 she draws boxes around the groups she calls "1 yellow" (BBY, BYB, YBB) and "2 yellows" (BYY, YBY, YYB). Again she seems to express doubt however when she observes that "I may be grouping them incorrectly" from what she calls the "original theorem."

Asked how she would proceed next in her reasoning, Romina takes another sheet of paper and writes the entries of a row from Pascal's Triangle 1 4 6 4 1 on the top. She rewrites the lettered sequences of towers underneath the row and explains what her categorizations mean:

> This would be part of the 6. Then you do the little inverse guys…get the other 3 of these to get the 6…Then, you do the inverse of these; switch the blues and yellows to get the other 4…these would be all my yellows. (275)

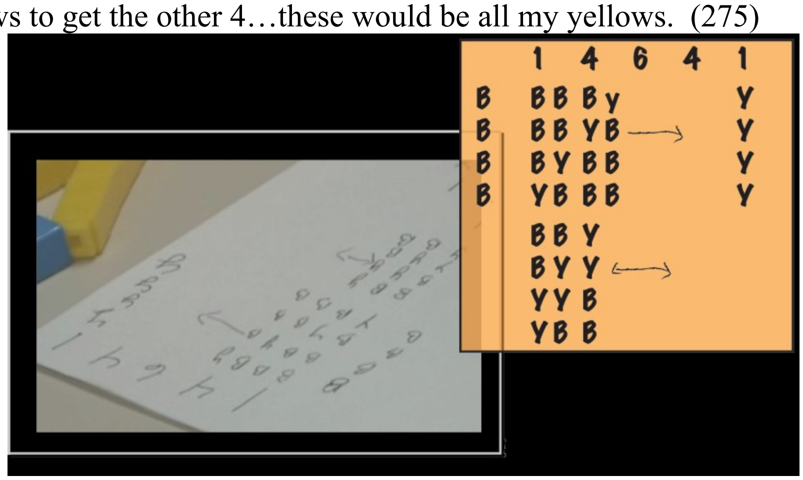


Figure 8-5. Romina demonstrates "inverses" within Pascal's Triangle

Indicating the group of two blues and two yellows - BBYY, BYYB, YYBB (see Figure 8-5 above), Romina draws an arrow and says this would be "part of the 6" in Pascal's Triangle. In order to get the "other three" from the three towers she has already listed, Romina explains that "you do the little inverse guys" with each one. Similarly, she indicates the group of three blue cubes and one yellow (BBBY, BBYB, BYBB, YBBB) that she has recorded under the number 4. She observes that "you do the inverse of these" or, in other words, "switch the blues and yellow to the other 4" in the row of Pascal's Triangle. Thus, she has associated each of her original groups of towers four-tall with a choice of two colors to one of the numerical entries in the fourth row of Pascal's Triangle. She writes YYYY under the last 1 in the row and explains that would be the group of "all my yellows."

8.3.3.2 Making "Ideal Couples" and "Carrying Through" the Pattern

Continuing with her strategy of associating each numerical entry of Pascal's Triangle with a group of towers, Romina records more tower sequences. In order to generate towers she employs her "couple" strategy of reversing the color of sequences she has already written. She worries, however, as she completes the group of towers with two blues and two yellows that perhaps "I didn't do these couples right":

> I don't know - This would be…let me see if this one doesn't work exactly like... This isn't working; maybe that's the 4. I didn't do these couples right, but… (277)

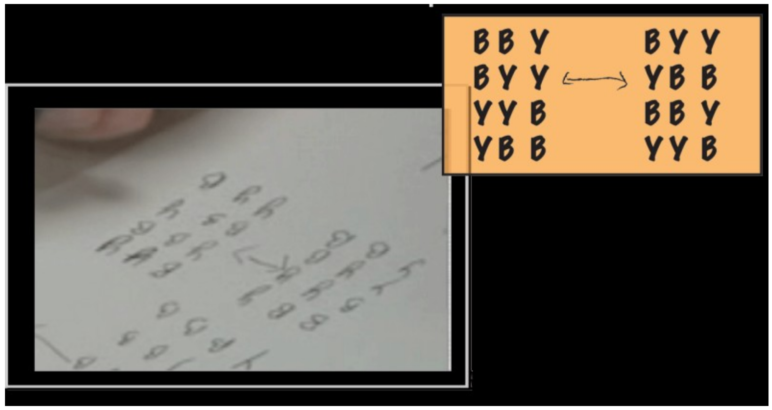


Figure 8-6. Romina tries to record Tower couples: "…I didn't do these couples right…"

Notice that in Figure 8-6 above, Romina has written the sequences BYBY, YBBY, and YBYB next to the sequences BBYY, BYYB, and YYBB. Romina observes that "this isn't working" and that "I didn't do these couples right." Though she has drawn an arrow connecting YYBB and BYBY, Romina indicates that they do not follow her original "couple" definition as the color in each position has not been reversed.

The researcher asks Romina what the "couple" would be in her recorded list. Romina then clarifies and gives an example of her "ideal couple":

T/R	What is the couple?
ROMINA	I know; I know; I messed up the couples….My ideal couple to this person would be this person (*writes* – see Figure 8-7) (280 – 281)

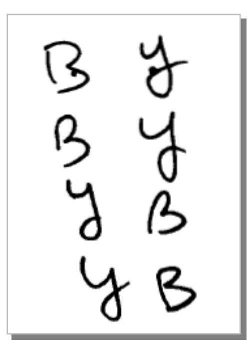


Figure 8-7. Romina's representation for an "ideal couple" (281)

Asked for the meaning of "couple," Romina repeats that she "messed up the couples" when she recorded sequences of two blues and two yellows. Writing in Figure 8-7, she asserts that “my ideal couple” relationship would be the sequence BBYY paired with YYBB. Notice that again, instead of calling them towers, Romina refers to each of her lettered sequences as a “person” who couples with another “person.”

Asked to elaborate why BBYY and YYBB would make an “ideal” couple, Romina refines her definition of “opposite” and “inverse.” She also describes the heuristic she uses when generating initial tower sequences as “carrying through” a color through the different positions:

T/R	How come? Why is this an ideal couple?
ROMINA	This is like the exact opposite—the inverse type of relationship.
T/R	When you said “the exact opposite,” what did you mean?
ROMINA	Wherever there’s a blue here, there’s a yellow here. So they switch off. But, I didn’t do it right. I was dragging the yellows through. So that would be the…that’s what I tried to do…This one and this one are little couples, and this one and this one are little couples. (*Draws a line from one sequence to another*) And, these two are couples. That would make up my six, because the whole idea that you have, like, repeats—that’s why you have 6 and not 8 Um - You can talk to me about how poorly I communicate my ideas throughout the twelve years. The reason that was 6 and not 8 and instead of adding the 4, because as you move the couples, they become the same thing eventually. The way I used to do it very systematically with carrying the Y’s through; if you do that, and do that the opposite with the B’s, eventually you get to the issue where you carried it through and then it’s the same pattern. It’s the same tower. So, then we started doing them in couples. I think that’s how I –
T/R	You said, “carrying through” – what’s the carrying through?
ROMINA	It’s just a pattern; and, then, I lift the two y’s up to make sure—just to be systematic about the different—So you can just see it and build it up systematically. So you don’t have repeats instead of just building it all out. Whatever comes to your mind. You start with one yellow; and then you bring the yellow through and make all the combinations: one yellow and 3 blue. You keep going like that. So this would be (*writes – see Figure 8-8*); then - (282 – 287)

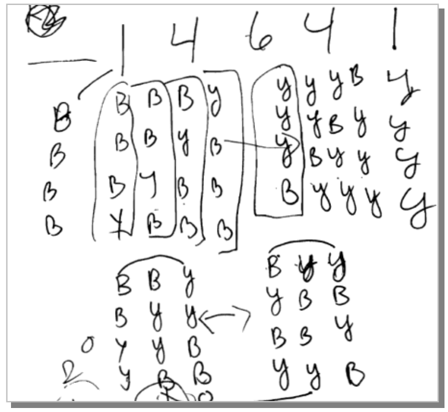


Figure 8-8. Romina's representation for "systematically" linking "little couples" 4-tall

For an "ideal couple," Romina says that whenever there is a blue in the original sequence, it is replaced with a yellow – "they switch off." Saying she "didn't do it right" originally, she now draws lines linking what she would consider "ideal" couples. Notice in Figure 8-8 above she has related BBYY with YYBB, BYBY with YBYB, and BYYB with YBBY. As she draws the lines, she narrates that she is making "little couples" and the three "little couples" together "make up my six" in Pascal's Triangle. She seems to express concern about the way she is describing her method – she observes "how poorly I communicate my ideas throughout the twelve years" of the longitudinal study. She remembers that this process allowed her to generate all the towers "very systematically." The researcher asks her about the use of the phrase "carrying through." Romina explains that phrase describes a "pattern" she uses to "systematically" generate towers. For instance, she could "start with one yellow" and then put the yellow block in each of the four positions. She indicates how "all the combinations" of "one yellow and three blue" blocks would be generated in this way. In the grouping BBBY, BBYB, BYBB, and YBBB, she is "carrying through" the yellow from the last position, to the second from last, to the third from last, and finally to the first position. She fills the other positions with blue cubes. To get the group of three yellow and one blue, she would then find the "couple" to each of the sequences she just wrote. Thus, with the combination strategy of "carrying through" and then finding the "ideal couples," Romina concludes that all of the towers can be generated four-high with a choice of two colors. Indeed, in Figure 8-8, one observes that all sixteen towers have been recorded accurately.

8.3.4 Refining an Algebraic Rule and Justifying Pascal's Identity

8.3.4.1 "So, this is my issue: 4 to 2, 2 to 4, - it's the same"

After Romina explained her strategy for generating towers 4-high, the researcher asked if there were any more than the ones she had recorded. Offering two answers, Romina replied that "there's 12 – no, there should be 16 there" (191). Questioned further about where she got her final total, Romina corrected herself but also criticized her ability to do "actual math" in the addition:

> Don't you add them up? Is that how you do that thing? Ah, that wouldn't be twelve. That's sixteen. Good. That makes - whew. See, it's the actual math I'm not so good. (294)

Romina wonders aloud that she is supposed to "add them up" to get the final total which would yield "sixteen" towers. She seems to express relief that the answer is not twelve. With a "whew," she says that is "good." She directs the researcher to notice that "I'm not so good" at the "actual math" since she had offered 12 as the final total before instead of the 16 total which she in fact had written on her paper.

Why there "should" be sixteen total towers 4-high with a choice of two colors develops into a new conversation about powers:

ROMINA	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16…yeah, I think so. Each one of these is one tower. So there are 16 groupings.
T/R	So, when you said, "n to the x" before –
ROMINA	So, this is my issue: 4 to 2, 2 to 4,--it's the same…

T/R	They both go to 16.
ROMINA	So that's why I count –
T/R	So, how would you resolve that then?
ROMINA	Umm (*laughter*) – I think it's this one (*boxes the 2 to the fourth power on her paper – see Figure 8-9*) – I think it's that one because it would be changing 2 to the n; n would be the row you're on – and 2, because there are always 2 colors. This one was – did I do this one? Yeah because this one would be 2 to the 3. (298 – 304)

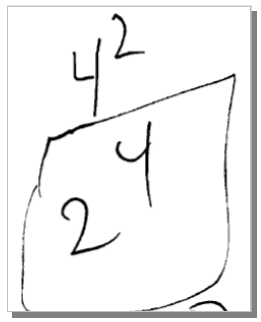


Figure 8-9. Romina's two possible exponential representations for Towers 4-tall

Romina explains that her "issue" is whether the sixteen total towers is a result of "4 to 2" or "2 to 4." Both four to the second power and two the fourth power would give the solution sixteen. When asked how she would "resolve" that fact, she boxes the fourth power of two on her paper (see Figure 8-9) and explains that she thinks the general rule is "2 to the n." She offers more details about her reasoning for the base and the exponent. She says the exponent "n" represents the "row you're on" and the base "2" is a result of the "two colors." For justification, she turns to an example on her paper of towers three-high with a choice of two colors. She points out that this rule would work for towers three high because "this one would be 2 to the 3" or the third power of two. Asked to elaborate here, Romina further refines her argument for the exponential rule by offering the case of the third row of Pascal's Triangle as evidence:

> So this equals…There are 8 combinations here on the third row. 3 times 3 would be 9…I'm hoping it's 2, yeah. (306)

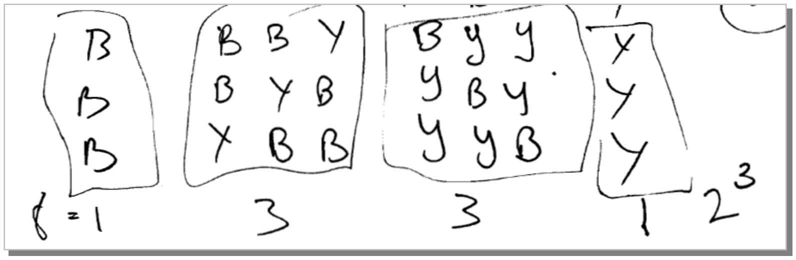


Figure 8-10. Romina's representation for the "8 combinations" in Pascal's third row

Before Romina was unsure if her 16 total towers was the result of two to the fourth power or four to the second power, so now she turns to towers 3-high for resolution. She has already written out the total of "8 combinations" of towers 3-high with a choice of two colors and mapped them to the "third row" of Pascal's Triangle (see Figure **). She writes 2^3 to the right of the row numbers 1 3 3 1. She notes that "3 times 3 would be 9" and so if the row number were the base, she would not get the correct number of towers three high. In other words, given a choice of 2^3 or 3^2 for how many towers exist that are 3-high with a choice of two colors, 3^2 gives an incorrect answer of nine. She is "hoping" that the base number is "2" as it would be in this case because 2^3 does indeed yield eight.

8.3.4.2 Pascal's Triangle – Second Version

When asked if her rule of "2 to the n" would "always" predict the number of towers "n"-high with a choice of two colors, Romina tests her rule for another case of towers. Working backwards to towers two-high and then one-high, she realizes that something is "missing" in her Pascal's Triangle and so she refines her representation:

T/R So does that always work then?

ROMINA So, this would be 2 to the 2 (*writes 2^2 next to the 1 2 1 line and 2^1 next to the 1 – See Figure 8-11*) That doesn't work, does it? What is 2 to the - There's a strong possibility… I might be missing something here *(draws line in her Pascal's Triangle-Figure 8-11*)

T/R Hold on. These are - when you wrote these out, these are ones that are –

ROMINA	So, this is this row (*points on paper*)
T/R	That row: those are the two high.
ROMINA	This is the three high.
T/R	What would come before that then?
ROMINA	Maybe, two 1's, can I do that? (l*aughter*) This would be a yellow, then the blue. I don't know what this one would be—the beginning of all towers? (*Writes in a 1 1 line and then circles the single 1 at the top*)
T/R2	But what if you – you've got things moving up, right?
ROMINA	So, this would be zero. This would be 2 to the one. This would be 2 to the zero, which is one, isn't it? (307 – 316)

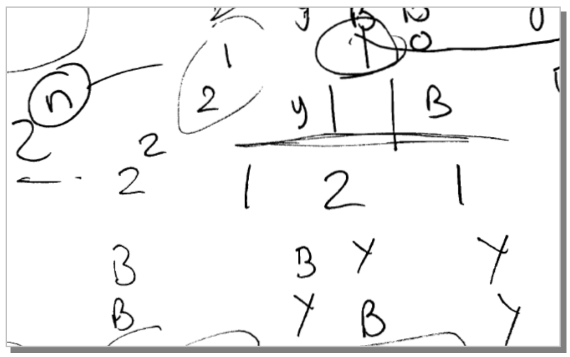


***Figure 8-11*. Romina's second version of Pascal's Triangle with the "beginning of all towers" (314)**

To support her generalized rule of 2^n for the total number of towers *n*-high with a choice of two colors, Romina begins to work backwards from the cases of towers 4-high and 3-high which she previously demonstrated to have 2^4 and 2^3 towers, respectively. Next to the line 1 2 1 in her Pascal's Triangle, Romina writes 2^2 (see Figure 8-11) and says that would be "2 to the 2." Following her reasoning the line above 1 2 1 would represent towers one-high, but Romina stops herself. Recall that in her original incorrect version of Pascal's Triangle she had skipped the first row 1 1. She notices "that doesn't work" and observes that perhaps she is "missing something here." She asks if she can include "two 1s" as a row. She then draws a line in her Pascal's Triangle and adds the proper first row of 1 1 (see Figure 8-11). Explaining that the first 1 would represent the number of towers one-high with "yellow" and the second 1 would represent the "blue," Romina wonders aloud what the row above would then represent. She admits that "I don't know what this one would be" and suggests it might be "the beginning of all towers." She then gives the row a name based on how she has named the other rows. She calls it the "zero" because if the row below it is "two to the one," then the row above it would be "two to the zero." She asks for verification that 2^0 is indeed equal to one. Through this reasoning then, Romina has successfully associated the sum of the entries in rows four, three, two, one, and zero of Pascal's Triangle with a power of two that will also predict the number of towers that row number high given a choice of two colors.

The researcher then asked Romina the original question of the problem solving session - how she would find the number of towers five tall with a choice of two colors. Romina replied that she would use the fifth row of Pascal's Triangle:

> I'd just---(writes 2^5) eventually, you'd get to that. I would draw out the whole (begins to write on a new sheet of paper – see Figure 8-12) So, this would be my 5 row. 0, 1, 2, 3, 4—yeah. (320)

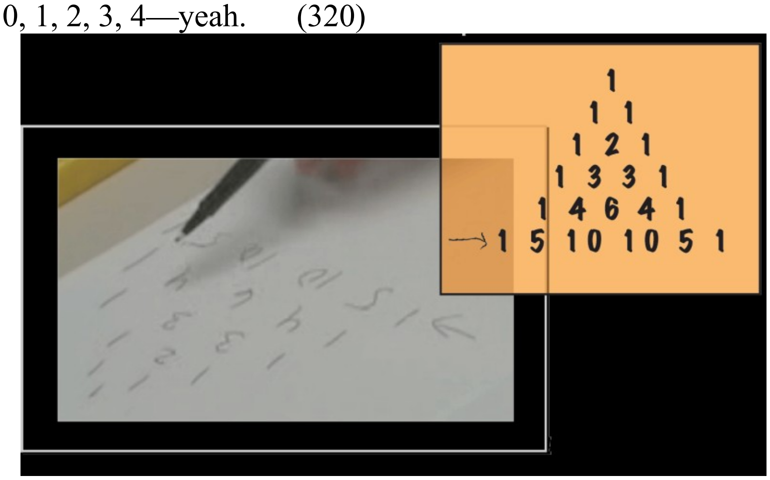


***Figure 8-12*. Romina's fifth row: "…So this would be my 5 row." (320)**

Notice that, to answer the question of Towers 5-high, Romina immediately answers that she "would draw out the whole." She proceeds to redraw Pascal's Triangle and indicates "my 5 row" with an arrow (see Figure 8-12 above). She names the other previous rows consecutively "0, 1, 2, 3, 4."

The researcher asked Romina how she knew which parts of Pascal would correspond to the yellow cubes and which to the blue cubes, Romina provided a more detailed explanation of how elements of the triangle related to the towers problem:

ROMINA	I don't think it—I don't think it matters—whatever this is, this is the opposite. So if I started with blue, this would be my yellow. So, I would work with…this would be my all blue…this would be 4 blue, one yellow…these would be 3 blue, 2 yellow…Then, we'd switch here to 3 yellow, 2 blue…this would be my – oh, I should write this out.
T/R	Let's go through it one more time to be sure I understand.
ROMINA	This would be my all blue towers. (*writes below the fifth row of Pascal's Triangle – see Figure 8-13*) This would be—this is 5 high, right? So this is 5 blues. This would be 4 blues, 1 yellow. This would be 3 blues, 2 yellows. This would be 2 blues, 3 yellow. This would be 1 blue, 4 yellow. And this would be 5 yellow. So, I just kind of gradually (*motions across the top of the row*). (324 – 326)

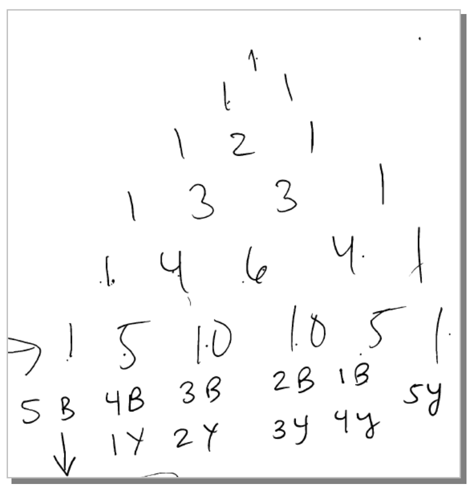


Figure 8-13. Romina's Tower cases within Pascal's fifth row

When mapping the Towers problem to the fifth row of Pascal's Triangle, Romina observes that "I don't think it matters" with which color you choose to start. She seems to indicate an awareness of symmetrical patterns, when she observes that whatever "this is, this is the opposite." Under each entry in the fifth row of Pascal's Triangle she writes a corresponding case of the Towers problem 5-high. For instance, notice in Figure 8-13 how, under the first 1, she writes "5B" and calls it her "5 blues" - for the entry 5, she writes “4B, 1Y” and names it “4 blues, 1 yellow.” She then finishes mapping each entry of the fifth row to Towers: 5B, 4B/1Y, 3B/2Y, 2B/3Y, 1B/4Y, 5Y (see Figure 8-13).

8.3.4.3 Justifying Pascal’s Identity

After mapping each entry in the fifth row to a case of Towers 5-high, Romina goes on to discuss “why” each entry is the sum of the two entries above it:

> Because you’re just adding. You’re adding an extra block, so that adds that many more combinations to that set. I also just know that this is how you do it. (laughter) I’m sure I had a good reason at one point, but now…I didn’t have any thought behind why I’m going…I just know that is how to do it and that is what it means. I think when we did it a long time ago, it was…this up here would be 4…so what you do is you add—you’re adding on—your 4 all-blue here, when you add an extra block, this becomes…you’re adding an extra block and then the 3 blue, one yellow to each one of these to become…you assume it’s one blue box. These two become the same grouping. (*Writes on her paper – Figure 8-14*) (332)

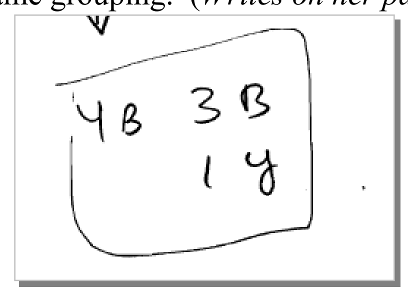


Figure 8-14**. Romina’s classification of “the same grouping” for Pascal’s Identity**

When Romina begins to discuss Pascal’s Identity she speaks in more general terms. She tells the researcher that “you’re just adding” and more combinations are included by “adding an extra block.” She laughs that “I also just know” and admits she “didn’t have any thought behind why.” Romina recalls that she believes her group did have a justification when they used Pascal’s Triangle “a long time ago.” At this point, she begins writing on her paper and tries to reconstruct a justification. She indicates that the group of towers with four blue cubes and three blue with 1 yellow cube would be the “same grouping” when “you’re adding an extra block.” She puts a box around 4B and 3B1Y (see Figure 8-14) on her paper but does not elaborate yet on how these are related.

The researcher asks Romina for more information about how the groups of one row of Towers are "added" to become the next. Romina reaches for the Unifix cubes for the first time in the interview and starts to illustrate her justification of Pascal's Identity with a physical model of how to build the third row from the second row:

ROMINA	Yeah, so, each time, if you're just building this, it would be like…We'll start with *(picks up blocks*) the two high, right? Then, you're just essentially, you have that and like what happens is this duplicates (*building with blocks*).
T/R	So, this is the two?
ROMINA	No, now, what happens, then, is you get…you have all of this. It's been awhile. So, you're gonna kind of have these guys again (*illustrates with blocks*). And, then, you're just adding.
T/R	So, what is the – that's the - (*points to the blocks*)
ROMINA	This is your – this guy. (*Places four towers over the row 1 2 1 previously written on her paper of Pascal's Triangle – see Figure 8-15*)

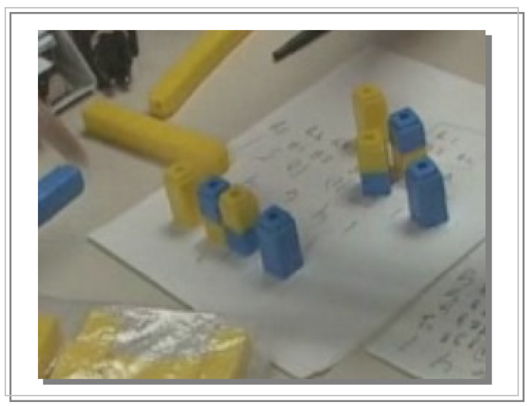


***Figure 8-15*. Romina builds with Unifix cubes for Towers 2-high**

Romina explains that, when "building" with the Unifix cubes, the row of towers corresponding to "two high" Towers "duplicates." Though it has "been awhile" since she has worked with the towers in this way, she reconstructs an argument by bringing the researcher's attention to "these guys" (the towers). She builds the four towers that are two-high and places them over the entries of the second row of Pascal's Triangle 1 2 1 (see Figure 8-15). She continues her argument for the researcher:

T/R	Oh, that row? I see.

ROMINA	Then, you're gonna have to add. It's just as if you were systematically –
T/R	Now you're adding a blue block to –
ROMINA	- get the next. A blue block. You can also add a yellow block. What should happen is – this should cancel out, I hope. Let me see if I have this.
T/R	You're saying this will make the third row?
ROMINA	Yeah. I hope – hope this works out. (*Rearranges the towers – see Figure 8-16 below*). Did I make the third row? (340 – 348)

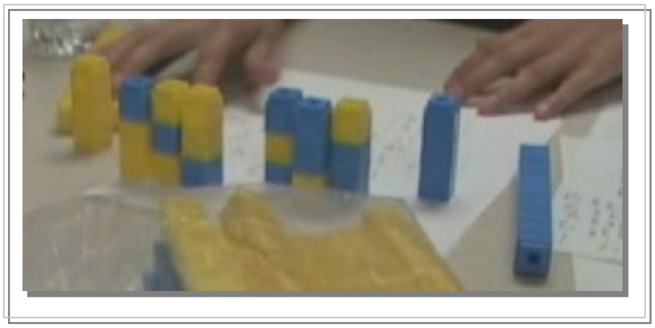


***Figure 8-16*. Romina builds the "third row" of Towers over Pascal's Triangle**

Romina tells the researcher that to generate the third row from the second row entries "you're gonna have to add" and it must be done "systematically." She adds "a blue block" and "a yellow block" to each existing tower from the second row. Not explaining what she means, she says "this should cancel out" and that she "hope(s) this works out." Placing a set of eight towers three-high in front of the researcher, Romina asks, "Did I make the third row?" Indeed, Romina has constructed the third row of Pascal's Triangle (see Figure 8-16): 1 tower with three blue cubes, 3 towers with one yellow and two blues, 3 towers with two yellows and one blue, and 1 tower with three yellow cubes.

The researcher asks Romina to explain her "process" again. Romina rebuilds the tower model for both the second row and third row. She refines her argument:

> So, this is the row we started with. And, then, initially, I made a duplicating row that looked exactly like this. And, then, to this one I added a blue block on top. To this one, I added a yellow block on top. (354)

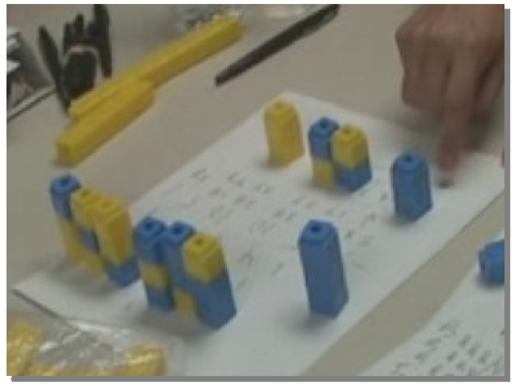


***Figure 8-17*. Romina refines her argument for the Additive Rule. (354)**

Indicating the four towers of the second row, Romina says that “this is the row we started with” of 1 with two blues, 2 with one yellow, one blue, and 1 with two yellows (see Figure 8-17 above). Her next step was to make a “duplicating row that looked exactly like this” row. To the original four towers she “added a blue block on top” and to the “duplicating row” she “added a yellow block on top.” In such a way, she built eight towers three-high with the two colors yellow and blue. She rearranged the eight towers and indicated she had the third row of Pascal’s Triangle (see Figure 8-17) of 1 with three blues, 3 with one yellow, two blues, 3 with two yellows, one blue, and 1 with three yellow cubes. The time of the interview at this point was coming to an end since her parking meter had just about run out, Romina concluded her interview with the researcher by saying that this method she had explained for the second and third rows of Pascal’s Triangle would work “any time” because she could always “add a blue or yellow cube to jump down to the next number of high towers” (358). Thus, Romina seemed to indicate a belief that her justification would generalize for any row of Pascal’s Triangle. Also, until the very end of this session, Romina continued to refer to entries of Pascal’s Triangle as a “number of high towers” – for instance, a tower 3-high or 4-high.

Chapter 9 CONCLUSIONS

"The time has come," the Walrus said, "To talk of many things…"

~ Lewis Carroll, *Through the Looking Glass* (1871)

9.1 Introduction

While mathematics itself may date back thousands of years to the development of early civilizations, the history of ideas within mathematics education has a much shorter lineage. In *Learning Mathematics: The Cognitive Science Approach to Mathematics Education*, Davis (1984) notes the contrast of the relatively "young" field of math education, dating back only to the early twentieth century with David E. Smith and J. W. A. Young, to the "old" study of mathematics:

> Now this confronts us with something of a paradox; the study of mathematics itself dates back at least several thousand years, as we know from research on Ishango, not to mention Euclid and Archimedes, and has been of considerable importance for centuries. How can the study of mathematics be so old and the study of mathematical thinking be so new? (pp. 2-3)

Davis resolves the seeming "paradox" by likening the emergence of math education to the development of modern medicine. For centuries, health, whether good or poor, was ascribed to ungovernable forces like gods or fates; only more recently did a scientific field of medicine emerge that sought out and defined causes, diagnoses, treatments, and cures for human ailments. Likewise, mathematical knowledge is no longer always viewed as a result of natural forces beyond our control – math education researchers endeavor "to get a more precise and detailed description of how human beings think about mathematical problems, and this can move us towards far more control, and far less need for fatalistic acceptance of everyday obstacles to learning" (Davis, 1984, p.4). The precise and detailed descriptions to which Davis refers is evidenced in the wealth of research that exists today that examines not only how human beings build mathematical ideas when they think about problems, but also how they retrieve and refine those ideas to solve new problems.

If it has taken thousands of years for mathematics to develop and more than a hundred years for the theories of math education to grow, how much time would one need to understand the development of a single student's mathematical ideas? Is such a feat even possible? And why should anyone care? Perhaps research might never construct a complete picture of each individual student's mathematical identity, but such epistemological portraits are incumbent upon the educational community. How else can we move toward the control and alleviation of obstacles to students' learning as Davis (1984) suggests? Since the demands of everyday life are increasingly and fundamentally mathematical, the National Council of Teachers of Mathematics (NCTM) argues in *Principles and Standards for School Mathematics* (2000) that "all students should have the opportunity and support necessary to learn significant mathematics with depth and understanding" (p. 5). More recently, NCTM (2009) emphasizes that "reasoning and sense making are the foundations of the NCTM Process Standards" (p. 5). Calling for reasoning and sense making to occur "in every mathematics classroom every day," NCTM encourages educators to constantly reflect on how students are drawing conclusions based on assertions ("reasoning") and developing understanding of situations and concepts by connecting it with existing knowledge ("sense making").

Though our goal is to reach "all students" in every classroom, we begin with one student. The story of Romina offers a rare glimpse into a student's mathematical thinking over seventeen years. Here we encounter a student who, by her own description, learned complex mathematical tasks with “understanding” from the fields of algebra, discrete mathematics, and calculus. Analysis of Romina’s understanding suggests instructional interventions that would support the development of students’ mathematical ideas over time.

The *American Heritage Dictionary* (1985) defines “success” as the “achievement of something desired, planned, or attempted.” By most measures of what it means to be successful, one could call Romina a success story. Educationally, she attained favorable and desired outcomes: after graduating as the valedictorian of her class in high school, she attended the Ivy League University of Pennsylvania as an undergraduate, began a career in consulting and corporate auditing with Deloitte, one of the largest professional services organizations, and then received a master’s degree in business administration from Northwestern University. What characterizes Romina’s apparent success and what might it mean for mathematics education? Often, we associate success with an individual’s personal efforts. In *Outliers: the Story of Success* (2008), Malcolm Gladwell asserts that there is something “profoundly wrong” with the way society usually defines success. He states that personal explanations of success are not enough:

> It makes a difference where and when we grew up. The culture we belong to and the legacies passed down by our forebears shape the patterns of our achievement in ways we cannot begin to imagine. It’s not enough to ask what successful people are like, in other words. It is only by asking where they are from that we can unravel the logic behind who succeeds and who doesn’t. (p. 19)

Gladwell argues that “the culture we belong to” helps to shape the “patterns of our achievement.” In order to better decode who succeeds and who does not, one must probe the conditions that cultivated that person’s growth. In other words, one must refocus attention away from individual personality and ask, instead, what was the “culture” in which Romina's mathematical ideas developed? What shaped Romina's particular "patterns" of mathematical achievement?

Let us approach these questions as Romina would approach the Towers task. In her 2009 interview, when faced with the same Towers problem from fourth grade in 1992, Romina said she would need to think of the problem "from scratch." We must consider the scope of seventeen years of videodata – from 1992 to 2009. What pictures and patterns emerge? Let us construct a representation for Romina's ideas and the culture which surrounded them from scratch, building what towers of meaning we can from the blocks of videodata that exist. What comes into view? A girl, randomly selected to be part of a longitudinal study in fourth grade, was given complex tasks to explore over extended periods of time with other students in collaborative settings. Romina became someone who *talked, argued,* and *justified* mathematics. Not only did she build models of her ideas and make connections among mathematical concepts, but she also constructed "comfortable" relationships with her peers and teachers. Working through times of what she described as frustration and confusion, Romina became a learner who persevered, asked more questions, managed, collaborated, and succeeded.

9.2 Behaviors

Three research questions guided this study. This section seeks to address how the data analyses provide insight into the first two research questions:

1. Within the context of problem-solving situations, how do Romina's representations and justifications for her ideas develop over time?
2. To what extent, if at all, does Romina collaborate and incorporate the ideas of others into her own ideas?

The two research questions above address the nature of Romina's problem-solving behavior, specifically in terms of representation, justification, and collaboration. It is appropriate then to review and summarize what themes emerged from the videodata of Romina's problem solving in fourth, sixth, tenth, and twelfth grades.

Recall Persi Diaconis' characterization of mathematicians as quoted in *Mathematical People: Profiles and Interviews* (1985) by Donald Albers and Gerald Alexanderson: "Mathematical people enjoy talking to each other… collaboration for me means enjoying talking and explaining, false starts, and the interactions of personalities" (p. 74 - 75). Under the definition of Persi Diaconis, Romina's behavior in the problem-solving sessions seems to be particularly that of a "mathematical" person. No matter which year of videodata chosen, one hears Romina talking--*talking* and *explaining* mathematics to others over and over again. What words could characterize Romina's behavior then as evidenced in the videodata? In re-reading the data analysis of the problem-solving tasks, notice the participles used again and again in the narrative descriptions for Romina's actions: talking, explaining, arguing, naming, relating, convincing, justifying, asking, and suggesting. In the sections that follow, let us parse out more specifically how to define Romina's problem-solving behavior.

9.2.1 Representation and Justification of Mathematical Ideas

When we consider the development of Romina's representation and justification in problem solving situations over time, we recall that the terms representation and justification cover a vast terrain of both dynamic processes and static products. The videodata samples from fourth, sixth, tenth, and twelfth grades as well as the later interviews allow consideration of Romina's utterances, writings, figures, and manipulative models. Recall the literature review in chapter two of the extensive documentation that already exists for students' use of representations to build, interpret, justify, and communicate their mathematical ideas as well as the teacher/researcher's role in supporting and probing the learners' representations in longitudinal studies (Bellisio, 1999; Davis & Maher, 1997; Davis, Maher, & Martino, 1992; Francisco & Maher, 2005; Kiczek, 2000; Kiczek, Maher & Speiser, 2001; Maher & Martino 1996a, 1996b; Maher & Speiser, 1997; Uptegrove, 2005; Uptegrove & Maher, 2004a, 2004b). Much of this literature suggests that "abstract ideas" emerged only after students built, rebuilt, revisited, and discussed their personal representations for an extended length of time. The development of Romina's representations seems to support that suggestion. For example, after years of building, rebuilding, and discussing her representations for the Towers Problem, Romina's more "abstract" ideas emerge: an exponential rule for the total number of towers, a proof by cases for Ankur's Challenge, or an inductive argument for Pascal's Identity.

9.2.1.1 Constructing Ideas by Building Relationships among Concepts

The data analysis of previous chapters suggests that Romina builds mathematical ideas through *association* and *relationship*, both literally and figuratively. As she builds models for her mathematical ideas, she constructs associations among concepts. The more time that passed with a problem, the more conceptual knowledge could accumulate to inform her problem-solving approach. Similarly, over the years, she built strong relationships with her peers and the teacher-researchers. In her May 1999 interview, Romina recalled, "in fourth grade, I didn't know anything – I didn't know who you were. Now, we're comfortable with you" (16). A reciprocal nature seems to exist between

Romina's personal and conceptual relationships. The more "comfortable" she felt herself to be with her peers and the teacher-researchers, the more likely she became to ask questions and share ideas. Knowing others became linked to knowing ideas. Romina progressed from a fourth grade "I" with a narrow conceptual framework to an adult "we" with an expansive mental toolbox of assimilation paradigms from which to work on problem solving.

Romina's belief about how she learns seems to embody the theoretical framework of Davis and Maher whereby to "do mathematics" is to construct a set of individual mental representations that can be applied, revisited, and modified as new experiences are encountered (Davis, 1984; Davis & Maher, 1990, 1997). In her 2006 interview, Romina recalls that her group from the longitudinal study "built that whole concept" of Pascal's Identity before being formally introduced to any combinatorics formulas:

> See and for me, like for me, like we built that whole concept and then we were introduced with this formula, so like that formula I, when I look at that because I remember I had to do it my first year in college and I remember looking at that and it's not like I can memorize a formula but I would look at that formula and I was like okay, so this means that I have my options for this could be a tower 5 tall and I have 3 blues and 2 whites, and that's how I remembered it and where the numbers went. So for me I really took probability and combinations a lot very conceptual. (Reflections IV – May 12, 2006, line II-311)

Though Romina claims that she cannot "memorize a formula," she can look at a formula like the combinatorial ${}_nC_r$ and she will know what it "means" in terms of a personal representation she built with different color Unifix cubes. For instance, ${}_5C_3$ becomes the set of towers 5-high from a choice of two-colors with three of one color present. Following the "assimilation paradigm" theory (Piaget, 1967), meaning for Romina seems to involve a dynamic, conceptual relationship between her previous knowledge of mathematical models and the new information or problem with which she is presented.

Consider over the years the frequency and consistency with which Romina employs types of *human relationships* in her labels for *conceptual relationships*. Consider the first recorded task session of Romina when she was in fourth grade. For the second half of the Towers 5-high problem-solving session in 1992, Romina and her partner, Brian, employed a particular heuristic that assumed for any tower, one could find another tower in which each position's color was reversed. For instance, for the tower of five blue cubes, one could find a tower of five white cubes. In this session, Romina introduced and applied several different anthropomorphic names to this conceptual relationship: "opposites," "matches," "husband and wife," and pairs that take "strolls in the park." In 2000 when Romina explores Taxicab Geometry as a twelfth grader, she directs the others to "see a pattern" by recounting the number of paths to each intersection of smaller sub-grids. When considering a 3x2 sub-grid with Jeff who talks in terms of "opposites," she introduces the language of "couples" to describe the relationship between certain paths. For example, a path going down two units and then across three units would create a "couple" with a path going across three and then down two units. Very soon after she attempts to find "couples" among the taxicab routes, she again asks the others to relate the Taxicab problem to Towers. In an illustration of relating a new problem back to her previous knowledge of mathematical models, Romina asks the others if they could do something "like in towers" where the two different directions: across ("lines over") and down ("lines down") would be associated with the two different color choices in the Towers problem. She wonders aloud if Pascal's Triangle might be involved as well.

Compare Romina's use of conceptual relationships from 1992 when she first encountered the Towers 5-high task to when she revisited the very same task in 2009. Initially asking and wondering whether "n to the x" applies, Romina acknowledges she cannot remember whether the total number of towers 4-high with two colors would be "four squared" or "two to the fourth." In order to remember, Romina says she will have to build "from scratch." She begins to record sequences of the letters "Y" and "B" for yellow and blue cubes on her paper. Romina also writes rows of Pascal's Triangle to help her "figure how many combinations there'd be." Although she mentions a variety of abstract ideas within the first few minutes - possible algebraic rules, Pascal's triangle, and combinations - Romina re-builds the Towers by recording sequences in order to make sense of the rule and the use of Pascal's Triangle. While discussing and rebuilding the Towers problem, Romina refers to each lettered sequence as a "person" who can be in a relationship she identifies with synonymous names: "inverse," "opposite," and a "couple." When asked to elaborate on what she means by her terminology of "little inverse guys" and "the couples," Romina defines her "ideal couple" as the "person" BBYY with the "person" YYBB. Though they may not be taking strolls in the park as in fourth grade, the towers still live, literally and figuratively, in very personal and conceptual relationships for Romina seventeen years later.

9.2.1.2 Justification through Iterations of Personal Representation

The Reasoning and Proof Standard (NCTM 2000) identifies four objectives for instructional programs from prekindergarten through grade 12. Specifically, programs should enable all students to:

- "Recognize reasoning and proof as fundamental aspects of mathematics;

- ➢ Make and investigate conjectures;
- ➢ Develop and evaluate mathematical arguments and proofs; and
- ➢ Select and use various types of reasoning and methods of proof" (p. 56).

Combined, these four goals suggest what mathematical sense-making should look like in students. Recall that Harel and Sowder (2007) propose that any description of the process of proof-making must be student-centered and include two important, interrelated facets: a student (or community) *ascertains* for him or herself the truth of an assertion and a student (or community) *persuades* others of an assertion's truth. Throughout the years of the longitudinal study, Romina's behavior seems to meet the NCTM objectives for Reasoning and Proof as well as the various definitions for mathematical sense-making. Consider examples of her specific proof-related behaviors like making and investigating conjectures and developing and evaluating mathematical arguments as suggested by the data analyses in the previous chapters:

- ***2/6/1992 - 4th Grade – Towers 5-High***: Much of the beginning portion of the task is spent making sense of the towers and developing spontaneous heuristics ("how about…" suggestions) and local organizations ("opposites" and "husband and wife" pairs). After questioning by T/R2, Romina engages in proof-related behaviors when she conjectures that a tower will "always" have an "opposite," Brian expresses doubt about her assertion, and then the two students investigate the conjecture by testing each of their towers to see if their exists a tower "without an opposite." Later, Romina and Brian also conjecture that the total number of towers 5-high with a choice of two colors must be an even number.

- ***10/1/1993 – 6th Grade – Guess My Rule***: When investigating the various tables of values, Romina develops several conjectures of what it means to write a function rule and distinguish between linear and non-linear functions. She asks and explores whether the "same equation" with the "same plus number" (y-intercept) must be used for the same table of values. She conjectures that the "plus number" (y-intercept) of each function rule will be the output value for the input value 0 (for instance "plus five" if there is a zero in the box column and five in the triangle column). She and Brian also make sense of the "between" numbers (finite differences in the table) as they relate to slope. For linear functions, they develop and test a strategy of identifying the y-intercept from the first triangle entry and the slope from the "between" finite differences. They develop and evaluate arguments of the meaning of the "between" numbers with quadratic function tables. Here, they question and test "the code" other groups of students might have determined.

- ***1/9/1998 – 10th Grade – Ankur's Challenge***: As a tenth-grader, Romina engages in the proof-related behaviors of developing conjectures, evaluating arguments, and presenting more abstract justifications of her assertions. Using notation of "ones, zeroes, and Xs" for the three colors of Ankur's Challenge, Romina develops an argument for why there would be 36 towers that meet his criteria. Over the course of forty minutes, Romina tries to convince the others of this. She moves from vague language like "you put them somewhere and switch them around" to more specific supporting statements for where to place the two 1s given four possible positions and how to fill the remaining positions with Xs and 0s. The students also challenge each other. At one point Romina asserts that there are 84 total towers four-high with a choice of three colors. Rather than accepting her conjecture, Michael insists on a "proof the other way around." The students then develop arguments for what both the set of towers meeting Ankur's criteria and that set's complement would look like.

- ***5/5/2000 – 12th Grade – Taxicab Geometry***: At the beginning of the session, Romina and her group develop and investigate conjectures like the "theory" that you take the number of line segments to a terminal point and "divide it by two" to get the total number of paths or that perhaps the number of paths will always be a "prime number." Dismissing many of these conjectures after testing, Romina suggests using "towers" to solve the problem. Developing an argument for this connection takes quite some time. The students employ heuristics like trying

> "simpler" cases and using local organization for the types of paths as "couples." The students investigate "why" a relationship between Taxicab, Towers, and Pascal's Triangle exists and develop an inductive argument.

Coupled with Romina's verbal arguments, it is helpful to juxtapose her written representations and physical models from the various years to illustrate the development of Romina's mathematical ideas. In each problem-solving session, Romina's representations undergo continued revisiting, revising, and reiterating. Through these many iterations, Romina's personal representations undergo an <u>evolution</u> in the original sense of the word. Darwin actually used the word evolution only once in his closing paragraph of *The Origin of Species* (1859); he preferred the phrase "descent with modification." The word comes from the Latin *evolvere* meaning "to unroll, open, or unfold" and thus literally means the "unrolling of a book." Appropriately then, we can see the unfolding and opening of Romina's mathematical ideas through the physical representations she offers. So what does justification look like as we review the data analysis? We can consider Romina's representations during the justification process as pages from the whole book of her meaning – indeed, for any student, we can hope at most to only see glimpses, mere pages or parts of pages from the narratives of their thoughts.

Consider the evolution of Romina's written representations both within the span of a particular problem-solving session and then in relation to each other over the course of years. First, recall Romina's work with Brian in fourth grade with the towers. Notice in Figure 9-1 there are no written representations, but rather only physical models that Romina and Brian offer as support for the various conjectures they make and investigate.

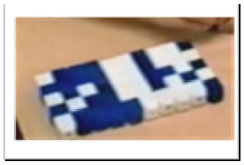
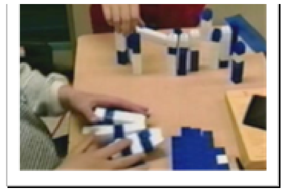
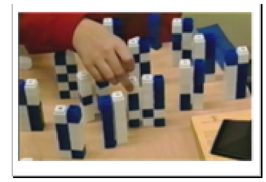
Romina and Brian's models for ***Towers 5-High*** *from a choice of two colors* - **4th Grade, 1992** – (lines from Appendix A)					
Romina's Written Representation					
Time and transcript line number	**00:09:40** (line 87)	**00:14:59** (line 168)	**00:23:16** (line 288)	**00:25:57** (line 321)	**00:31:39** (line 390)
Description	Romina and Brian build a ***total of ten towers*** by creating pairs towers with colors replaced. For instance, notice the towers on either end of the set: Romina built white-blue-white-blue-white and then Brian built blue-white-blue-white-blue.	Employing a variety of spontaneous heuristics, Romina and Brian build eleven more towers a ***total of twenty-one***. Whenever one suggests a color pattern, they reverse colors to build a second tower. Notice that duplicate towers exist in their set.	Under questioning by T/R2, the students define "opposites" as a pair of towers with the colors switched in corresponding positions and conjecture that all towers have an opposite. Brian and Romina begin ***checking to see if each of their towers can pair with an "opposite."***	Having removed the duplicate, Romina moves ***"husband and wife" pairs*** together – towers with a "match" of reversed colors in each corresponding position. Brian and Romina have a ***total of twenty towers***.	Using a strategy Romina calls ***"strolling in the park,"*** Brian suggests new possible tower sequences and Romina checks them by moving the tower through the existing "husband and wife" pairs. By the end of the session, Romina and Brian have built a ***total of twenty-six***.

Figure 9-1. **Romina and Brian's models for Towers 5-High – 4th Grade, 1992**

Notice that over a time span of about twenty minutes, Romina and Brian move from a representation of only ten possible towers as their solution (which was Romina's initial guess at the answer) to a total of twenty-six towers five-high from a choice of two colors. Though they never determine the final solution, they develop new strategies for checking their work and describing the relationships that they see among the towers.

Now flash forward from 1992 to 1998 when Romina and her group consider the number of towers 5-high that have only two of a particular color. Notice that now instead of physical models, Romina employs lettered sequences to represent each of the towers (see Figure 9-2). Over the span of about five minutes, she records a list using the letters "Y" and "R," then conjectures that there might be a function relationship between the number of yellow cubes and the total number of towers, and finally adopts Michael's binary code to represent and organize the number of towers with only two red cubes.

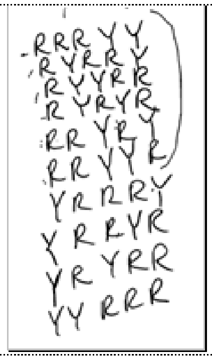
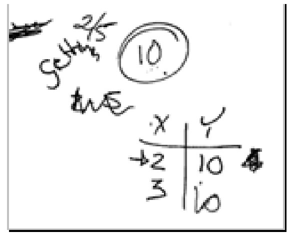
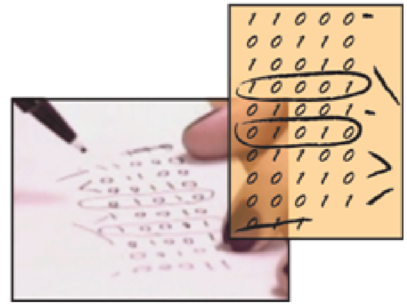
Representations for ***Towers 5-high*** *from a choice of two colors with exactly two red cubes* – **10th Grade, 1998** – (lines from Appendix C)			
Romina's Written Representation			
Time and transcript line number	**00:04:24** (line Ad32)	**00:05:34** (line Ad41)	**00:10:30** (line Ad75)
Description	***Using the letters "R" and "Y"*** to represent the two colors red and yellow, Romina records a list of towers five-high based on sequences Brian and Jeff call out and she thinks of on her own. She wonders if this set of ten is complete.	Perhaps investigating a ***possible function relationship*** between the number of yellow cubes and the number of towers, Romina creates an input-output table. She suggests the answer is "2/5" and then asks "why is it ten?"	Adopting Michael's ***binary representation***, Romina records a list of ten towers five-high with "0" representing yellow and "1" representing red cubes. With dashes, Romina indicates which towers that are "opposites." She circles the pair they consider to be "palindromes."

Figure 9-2. **Romina's representations for Towers 5-high – 10th Grade, 1998**

Within five minutes, Romina written representations evolve from lettered sequences without a clear organization to a binary list annotated by pairings she defines as "opposites" and "palindromes."

Transitioning from the towers task of 5-high with exactly two red cubes to Ankur's Challenge of towers 4-high with a choice of three colors where each color is represented, Romina's initial written representations undergo a similar process of iteration by revision later during the same problem-solving session in 1998 (see Figure 9-3). Moving from an initial list that incorporates a variable "blank" to be filled by the three colors she has represented as 1, 0, or X, then to a locally grouped but incomplete list, and finally to a complete list of thirty-six towers that meet Ankur's criteria.

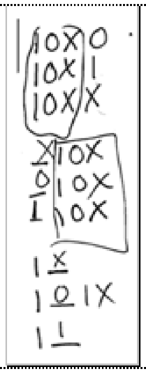
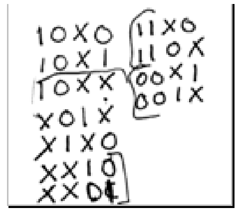
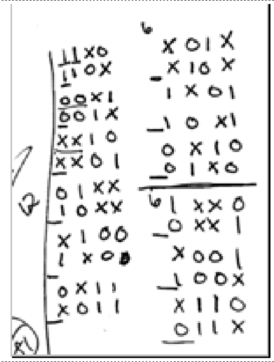
Romina's first three written representations for ***Ankur's Challenge* – 10th Grade, 1998** – (lines from Appendix C)			
Romina's Written Representation			
Time and transcript line number	**00:30:43** (line Ad270)	**00:34:51** (line Ad318)	**00:38:00** (line Ad353)
Description	Romina records an ***initial list incorporating a variable "blank"*** that can be filled by the three "colors": 1, 0 or X. Romina experiments with the position of the blank space among the four positions of the tower.	Romina organizes a new list based on ***grouping "three or two of them put together."*** For the four pairs, she keeps the first two tower positions constant and switches the color in the third and forth positions.	Romina records an expanded list by ***employing a pairing strategy***. For instance, 11X0 and 110X form a pair where the first two positions are held constant and the third and fourth positions vary among X and 0). She now has the ***correct total*** of 12 + 6 + 6 = 24 tower sequences.

***Figure 9-3*. Romina's first three written representations for Ankur's Challenge – 10th Grade, 1998**

Here, within the span of eight minutes, Romina develops and investigates her conjecture that there will be thirty-six towers that meet the criteria of Ankur's Challenge.

After ascertaining the truth of her assertion with written representations (see Figure 9-3) that there are thirty-six towers, Romina next attempts to persuade the others of the truth of her assertion. Over the course of forty minutes, the four iterations of her mathematical arguments can be seen in a sequence of written representations evolving in terms of accuracy, detail, and abstraction (see Figure 9-4). First, she puts forth her reasoning of "working with sixes" to T/R1 and Brian. Then, she tries to convince Jeff and Brian. Next, Romina develops a rectangular grid to persuade Ankur. Finally, she summarizes her justification at the chalkboard for Michael.

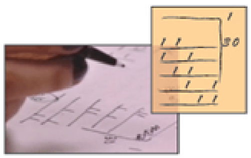
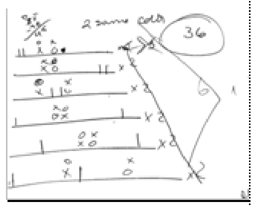
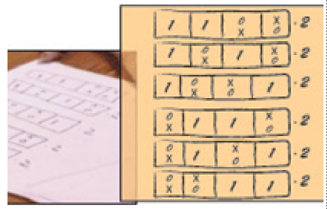
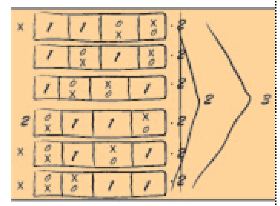
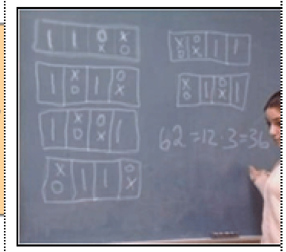
Sequence of Romina's representations used for presenting her solution to ***Ankur's Challenge*** **- 10th Grade, 1998 –** (lines from Appendix C)					
Romina's Written Representation					
Time and transcript line number	**00:45:55** (line Ad361)	**00:48:00** (line Ad367)	**00:54:05** (line 729)	**00:54:35** (line 741)	**01:20:29** (line 1177)
Description	***Solution version 1***: Romina presents to Brian and T/R1 in order to convince them that there are thirty-six total towers that meet Ankur's Challenge requirements. Romina uses five horizontal lines and the digits "1" and "1" for the color held constant.	***Solution version 2***: Romina presents to Jeff and Brian. She tries to re-explain why there are thirty-six total towers. Romina now uses six horizontal lines, the symbols 1, X, and 0 as well as the annotations about color and doubles.	***Solution version 3***: Romina presents to Ankur and Michael by using six rows of enclosed rectangles instead of lines. She explains that are six combinations for two 1's given four possible positions: "next to each other," "separated by one," "on the ends," "in the middle," in the "two and fourth spot," or in the "third and fourth spot"	***Revision to version 3***: Romina crosses out the six separate "*2" on her paper. She explains that the previous calculation was wrong. Her new algorithm is 6 arrangements of the repeated color times 2 for the "X" and "0" colors then times 3 for the three possible colors (6 *2 * 3 = 36)	***Solution version 4***: Romina presents her solution at the chalkboard to Ankur and Michael. Note that Romina does not place the 1's digits in exactly the same way for any of her solution representations. Here, her fifth tower has the 1's in the third and fourth positions.

Figure 9-4. **Romina's final sequence of representations for Ankur's Challenge – 10th Grade, 1998**

Similar to the many iterations Romina's representations undergo during the Ankur's Challenge task, the sequence of Romina's written representations during the Taxicab session in 2000 evolves in accuracy, detail, and abstraction (see Figure 9-5). Working with Jeff on "easier" cases, Romina records the number of paths to each intersection point and initially considers it to be "like a multiplication table." Then she and Jeff investigate possible local organization of the types of paths with "couples" and the symmetry in the numerical sequence that emerges. After testing more specific cases within the large Taxicab grid, Romina observes the numerical sequence she has on her transparency paper is Pascal's Triangle – after which she develops an inductive argument to predict the number of paths with an augmented Pascal's triangle she records.

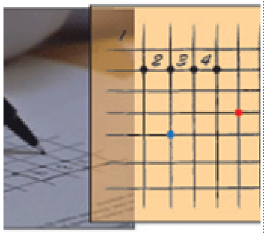
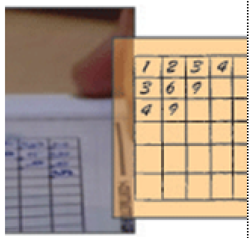
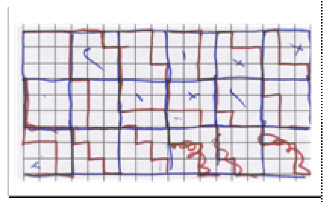
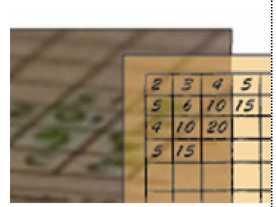
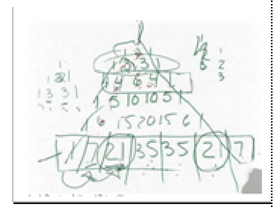
Romina's written representations for ***Taxicab Geometry*** - **12th Grade, 2000** – (lines from Appendix D)					
Romina's Written Representation					
Time and transcript line number	**00:15:49** (line 314)	**00:27:58** (line 506)	**00:32:14** (line 560)	**00:47:40** (line 778)	**01:24:26** (line 1320)
Description	After Jeff suggests that they try "easier" cases, Romina begins to record the number of paths on the grid itself. She labels the first row of intersection points as "one, two three, four…" She wonders if this is "going to be dumb and stuff" observing that with the numerical labels, "***it looks like a multiplication table***."	While Jeff counts paths on his grid, Romina records them. When Jeff mentions finding "opposite" paths, Romina asks whether they should be ***counting the paths "in couples."*** They conjecture that certain paths do not have a "couple" since they are getting some odd totals included among the even. Jeff also suggests there is symmetry in the pattern.	Under Jeff's direction, Romina ***partitions a new piece of grid paper into fifteen 3x3 sub-grids*** with blue marker and then records a separate path (in red marker) within each like "down one over two down two over one." Both Romina and Jeff make suggestions for new paths. They find fifteen total and wonder if that is correct.	Romina asks Brian and Michael what numbers they got for different sub-grids. Brian shares his paths on the chalkboard for the 3x3 sub-grid. Romina observes that their previous label of 9 should actually be 10. Jeff notices another mistake and includes the 20. Romina corrects and rewrites her transparency. She observes that "***it's Pascal's Triangle.***"	Romina creates an augmented representation of Pascal's Triangle during her explanation ***relating Pascal's triangle to the number of paths on the Taxicab grid***. Under Mike's and Brian's questioning, Romina marks each entry in the seventh row and describes how it corresponds to Taxicab – like "21" is "five across and two down"

Figure 9-5. **Romina's sequence of written representations for Taxicab Geometry – 12th Grade, 2000**

In addition to displaying proof-like behavior in her problem-solving tasks, Romina reflects on herself as someone for whom personal understanding necessitates sense-making activity. In each interview, Romina discusses reasoning and justification as fundamental aspects of her mathematical learning. Listen to Romina talk about and reflect upon her own reasoning in problem-solving:

- ***5/18/1999:*** "I just know everything in my own way – ***everything has Romina's definition to it***" (Reflections I, line 58)

- *7/21/1999*: "***When we do come up with something it's so much better*** because we came up by ourselves without someone holding our hand and walking us through it" (Reflections II, line 83); "We kind of had to like invent anything – we had to choose what path we were going to take" (Reflections II, line 69)

- *3/11/2002*: "***Everything has to make sense in my terms***… someone else might have done it already in a book, but I just don't understand it unless I try it myself and put it in my own terms" (Reflections III, line 428)

- *5/12/2006*: "***We learned so much from just getting up in front and explaining*** what we thought of concepts versus someone just telling me what it was" (Reflections IV, line A-291)

- *7/15/2009*: "I think you obtain expertise through just a lot of hours and understanding the fundamental aspect – like understanding every point of the way versus certain aspects" (Reflections V, line 155); "But it's very similar to what I do in consulting… it's a lot of just socializing, communicating with a lot of different groups… so you have to kind of ***work with everyone to convince them***" (Reflections V, line 96).

Romina emphasizes the importance of developing conjectures and evaluating mathematical arguments for herself and her group. She describes the "getting up in front and explaining what we thought of concepts" as a necessary part of their problem-solving – they "learned so much" through that process. Further, this process is not only an integral part of her school experience but also of her workplace – in consulting, she says that the majority of what she does it "communicating with a lot of different groups" and working to "convince" those groups of different arguments. Romina indicates the fundamental aspect of her sense-making – for her, "everything has to make sense in my terms" and have "Romina's definition to it."

9.2.2 Collaboration

A little over a week after the final interview with Romina in the summer, an article appeared in the *New York Times* on July 28, 2009 entitled, "Netflix Competitors Learn the Power of Teamwork." In the article, Steve Lohr described how the million dollar contest set up by Netflix to improve its movie recommendation model had finally ended in a dead heat between two teams. The movie rental company had started the contest in 2006 with the prize of $1 million dollars to be awarded to whoever could improve its movie recommendations by at least ten percent. In the end, the key to a million dollars was not a particularly brilliant formula, but rather the element of human collaboration in problem solving:

> The biggest lesson learned, according to members of the two top teams, was the power of collaboration. It was not a single insight, algorithm or concept that allowed both teams to surpass the goal Netflix, the movie rental company, set nearly three years ago: to improve the movie recommendations made by its internal software by at least 10 percent, as measured by predicted versus actual one-through-five-star ratings by customers. Instead, they say, the formula for success was to bring together people with complementary skills and combine different methods of problem-solving. (Lohr, 2009, p. B1)

In the beginning of the contest, contestants had worked alone for the most part. Then, as the deadline for submissions approached, teams began to merge. As the members of the top teams reported, they learned the "power of collaboration" as they brought together people with "complementary skills" that would combine various methods of problem-solving. One of the finalists extolled the "collaborative approach," saying that by putting all of their component algorithms together, they created a product that "exceeded our expectations."

The message of "the power of collaboration" in the *New York Times* article complements the themes that Romina brought forth in her interview during the same month of 2009. Romina discussed problem solving as something were "we always came at it from different perspectives and different ways… we disagreed a lot and then came to a conclusion together, which is better" (41). When asked about how she would characterize herself in the longitudinal study, Romina rejected defining herself in the first person. Instead, she reflected, "I don't think it was an individual thing… we would always try to be on the same page" (61). She recalled each member of the group having "different perspectives," but they would come to a conclusion "together" as a cohesive whole. She asserted that "it's not one person [who] knows the right or wrong answer" - instead, "it's kind of all coming to an agreement and just eventually it's the group saying this is right" (163). There is a short answer then to the second research question of this study as to what extent, if at all, did Romina collaborate and incorporate the ideas of others into her own ideas. As she would later assert in interview after interview and the previous videodata would support, Romina collaborated to the fullest extent she could with her peers in the longitudinal study and found incorporating her ideas with others a necessary component to solving problems.

What was the nature of Romina's collaboration? How did she converse with others? Certain constructivists distinguish between "didactic talk" where the speaker's intention is report his own ideas to "really talking" where the speaker's objective is to share ideas:

> 'Really talking' requires careful listening; it implies a mutually shared agreement that together you are creating the optimum setting so that half-baked are emergent ideas can grow. 'Real talk' reaches deep into the experience of each participant; it also draws on the analytical abilities of each. Conversation, as constructivists describe it, includes discourse and exploration, talking and listening, questions, argument, speculation, and sharing. (Belenky et al, 1986, p. 144)

A conversation of "really talking" necessitates many elements: exploring, talking, listening, questioning, arguing, speculating, and sharing. This type of conversation draws on both the "experiences" and "analytical abilities" of each participant. Such conversations facilitate collaboration. Given in previous chapters, videodata analysis of Romina's problem-solving conversations suggest that she engages in "real talk." In the following sections we will summarize the major themes of those analyses as they contribute to better defining the nature of Romina's collaboration.

9.2.2.1 Questioning Others' Ideas: "Arguing" and Asking Why

"Arguing" is one of the words Romina uses most frequently when recalling her interaction with her peers in the longitudinal study. Argument takes on the role of catalyst for problem solving. In fact, the sense in which Romina uses the word "argue" might be closer to the original meaning of the word: argue comes from the Old French *arguere* "to make clear, demonstrate" as any word with the base *arg*- itself originates with the Latin and Greek meaning the shine of silver. For Romina, arguing brings clarity and perhaps the shining light of insight. She remembers the arguments as not only helpful, but also necessary to solving problems:

> Like we could never do any of the things, well I don't think I could ever do any of the things we do alone. Like they just help you bring out things you didn't know were there. And we have a relationship where we argue a lot, so, like through arguing is where we come up with most of our answers. (Reflections I, line 20)

Romina explains that neither "we" nor "I" could "ever do any of the things we do alone" – problem solving necessitates group work for her. Crediting collaboration for their success, she explains that the others in the group "help you bring out things you didn't know were there." If one were to solve the problem alone then, one might conclude that knowledge would stay hidden and unexpressed "there." She characterizes the "relationship we have" in the group as one in which "we argue a lot." This collaboration takes the form of argumentation. Romina summarizes that "through arguing is where we come up with most of our answers."

Evidence of this "arguing" and "asking why" abounds throughout the videodata. Consider examples of her specific collaborative behaviors as suggested by the data analyses in previous chapters:

- ***2/6/1992 - 4th Grade – Towers 5-High***: Throughout the session, there exist sections where Romina asks Brian a high frequency of questions. For instance, early in the session (00:09:40 – 00:15:49), Romina asks thirteen questions within the span of six minutes. About half of the questions are informative in nature – asking Brian to provide a piece of information like, “Which one did we just make?” The other half of the questions allows Romina the opportunity to offer suggestions like, “How about we put one there?” or seek verification of an assertion like, “Wait a minutes, didn’t we just do that?” Interestingly, only the teacher-researchers ever ask a question with the word “why” during this session, such as when Dr. A asks the students, “Why can’t you have twenty-five” towers, after Romina asserts that “you can’t have twenty-five.” The students then argue and explore the use of even numbers in their solution and whether they have all the possible pairings.

- ***10/1/1993 – 6th Grade – Guess My Rule***: As Romina and Brian begin problem number two of the task, Romina questions Brian, Bobby, and Amy-Lynn a total of fourteen times within the span of ten minutes. Her collaboration here involves making suggestions, asking for information, seeking verification, expanding on others’ ideas, or reiterating others’ statements. For instance, her questions range from seeking information like, “Did you only change the square and the triangle or did you change the whole entire equation?” to questions that make suggestions, “Don’t we have to use the zero and the five?” and inquiries that seek clarification or explanation like, “Can you run that by me?” or “Would it work with other problems?” Later, when working on the first quadratic function of the task, problem number six, Romina again asks many questions – eight questions in under four minutes. She makes suggestions about the y-intercept like, “One’s the first number – then wouldn’t it be plus one?” Also, she asks “why” questions like, “why do you think” the finite differences are behaving differently for this table as opposed to the other previous ones. Romina asks Brian questions that probe “what,” “where,” and “why” he is applying a particular algebraic rule.

- ***1/9/1998 – 10th Grade – Ankur's Challenge***: Throughout the problem-solving session, Romina asks questions that involve seeking information, making suggestions, asking for explanation, or reiterating others' ideas. For instance, when they work on the towers 5-high problem, Romina questions "why" there are ten towers total and seeks information about how the boys are defining opposites, "what would be the opposite to this?" She again asks frequent questions when they encounter the Ankur's Challenge task. Before Brian, Jeff, and Romina generate their first representation for Ankur's Challenge, Romina asks five questions within the space of about a minute and a half. Her questions alternately seek information about the problem itself like, "We put it in every space, right?" or seek explanation like, "Can you explain what you did here?" Later, Romina asks for information about the sequences the others are generating like, "What is that one on the end?" and questions her own strategy, "What happens when I change the ones around?" Meanwhile, the other students ask Romina questions as well like when Jeff asks, "Do you understand?" about towers 5-high or "So how do we justify this even more?" after he hears Romina's justification for 36 towers for Ankur's Challenge. At one point in the video an argument erupts about how many total towers exist that are 4-high with a choice of three colors. Brian observes that they are having a "brawl" when Ankur asks, "Can I tell you now why it's not eighty-four?" and Romina challenges, "Hold on - can I tell you why it could be eighty-four?"

- ***5/5/2000 – 12th Grade – Taxicab Geometry***: Throughout the almost two-hour long problem-solving session, Romina asks questions. In the beginning, Romina asks questions that make suggestions like, "Can't we do towers on this?" or "I mean there are ten blocks - like ten lines to that thing, right?" She asks for information, "How do we do that?" Romina's interaction with Michael at the beginning of the session reveals several examples of her asking questions, proposing strategies that incorporate Towers, and then questioning Michael's suggestions for a strategy that counts the number of interior line segments. For instance, she asks Michael, "How many are there in here?" and "How are you keeping track?" After Jeff proposes that they try "easier" cases with sub-grids, Romina continues questions like, "You want to do them in couples?";" You got that?" and "Shouldn't we draw them just to make sure though?" As Romina continues to propose that they incorporate Towers and Pascal's Triangle into the problem, Jeff and Michael question her. For instance, Michael asks her, "How do you know?" and Jeff questions, "Why is this – why does the Pascal's Triangle work for this?."

In later interviews, Romina describes all of the questioning back and forth that occurred during the problem-solving sessions as "arguing" and "disagreeing," but emphasizes that it was a necessary component for her learning. As reviewed in an earlier chapter, recall that a growing body of research into collective mathematical learning and reasoning assumes that "individual learning can be seen as an inherently social process" (Bowers & Nickerson, 2001, p. 2). Previously discussed research also suggested the importance of collaborative experiences for individual learning during the longitudinal study (Alston & Maher, 1993; Francisco & Maher, 2005; Maher & Martino, 1998, 2000; Maher, 2005). One might liken Romina's term "arguing" to the "negotiatory interlocution" discussed by Powell (2006) and the development of "co-constructed ideas" analyzed by Mueller (2007).

In addition to witnessing various forms of collective mathematical learning through "arguing" in the problem-solving sessions, Romina identifies "arguing," "disagreeing," and "asking why" as the critical components of her collaboration during problem solving. Listen to Romina elaborate on her meaning:

- ***5/18/1999:*** "But if I disagree with someone, they'll have to explain it to me, ***and if you're explaining it, they're either going to find something right or they're going to find something more***. So, if I don't agree with it, they're going to explain it to me, but if they find something wrong, maybe I can help, and then someone else may disagree with me. And that's how we get through everything. We just disagree." (Reflections I, line 22)

- ***7/21/1999***: "We did a lot of thinking – like we just sat and thought for hours a day and we came up with a lot of interesting things and we were able to go in front of a large audience and just ***talk about our ideas and then argue our points and prove our point***s." (Reflections II, line 81)

- ***3/11/2002***: "***I think a big way we learn is we tend to argue a lot***. So that's how we get to places because we argue. And then we have to take their argument into consideration. When it's just me, I don't have much to argue about with myself because I think I'm right. Jeff doesn't have the same ideas as me." (Reflections III, line 275)

- ***5/12/2006***: "…but I know that we used to drive our teachers crazy because ***we'd always be like, why***? And they would have to go to the next level – like in our chemistry class, we'd be like, but we don't understand exactly why that happened…" (Reflections IV, line B-348)

- ***7/15/2009***: "But ***to promote the group dynamic and get a better output, you should disagree and just ask enticing questions***. I think it's a probe – I think that's what we did. I don't think we ever thought someone was completely wrong. It's just that not everyone may have understood it. So, I just keep asking them questions so that they can dissect their whole thought process." (Reflections V, line 45 - 47)

To arrive at understanding, Romina emphasizes the importance of alternately asking others for explanation and trying to persuade others of one's own ideas. What she terms as "arguing," "disagreeing," or "asking why" encapsulates the process through which effective discourse occurs for her during collaboration. She observes that by explaining, you either "find something right" or "find something more." Thus, for Romina, there is *more than just a "right" answer* and collaborative discourse leads to that "something more." She strings together a series of actions: we "talk about our ideas" and then "argue our points and prove our points." Talking mathematics leads to arguing about ideas which in turn allows for proof-building of assertions – in other words, this is a process through which "we get to places." Romina states that asking "why" in order to prompt explanation that would lead to justification and deeper understanding is something that pervades all aspects of life – whether in school during chemistry class, after school in a problem-solving session, or at work in a consulting job. Romina's "arguing" serves a valuable service – she describes her questions as necessary for the group's success. Romina views herself as someone who needs to "keep asking them questions" so that she can "promote the group dynamic" and help others "dissect their whole thought process."

9.2.2.2 Interacting with the Teacher-Researcher

Each problem-solving session included sections coded as "critical events" in which Romina's interactions with a teacher/researcher resulted in a significant change in her understanding. For instance, about twenty minutes into the Towers 5-High task in fourth grade, a critical event occurs when a teacher-researcher asks the students to "tell me what you're thinking about" (256). Romina and Brian describe how they have built twenty-one towers by finding "opposites." They clarify their definition of "opposite." The teacher-researcher then asks, "Do they always have an opposite?" When Brian expresses doubt that all towers will have opposites, Romina argues that in fact they will. They then decide to go back and check over all of their towers to see if there exists a tower "without an opposite." Thus, the brief interaction with the teacher-researcher serves as a pivotal moment in which the students clarify their mathematical ideas and begin to think about justifying a more general and abstract conjecture about towers "always" having opposites. They will soon make another conjecture that the total number of towers must be an even number. In sixth grade, another critical event occurs when Romina calls the teacher-researcher RBD over to see what they have gotten for number six. She expresses that "we're not totally sure." When Ankur and Michelle interject that they have "figured out how to write it," RBD encourages the four of them to share their findings with the camera. This leads to the four students sharing their work with each other as well. In so doing, Brian explains how he and Romina used a rule to generate first order difference and Ankur hints that there is "a different way to do it." Brian and Romina continue working but soon collaborate more closely with Ankur and Michelle, who have indeed found the quadratic rule.

In the high school problem-solving sessions, the data analysis also suggests examples of critical teacher-researcher interventions. For instance, in tenth grade, after they have been working on Ankur's Challenge for some time (it is about 45 minutes into the session), T/R1 asks the entire table of students what "ideas" they are pursing and if they would like to "share with each other" (589). As a result of T/R1's question, Jeff shares the list of towers he is generating and turns to work with Michael and Ankur. Romina says to T/R1 and Brian that she things it "might be thirty-six" because she is "working with sixes now." She proceeds to explain her initial representation of the 36 towers. T/R1's question thus serves to offer Romina an opportunity to give her first justification of for what will later become her much more abstract proof of Ankur's Challenge. By twelfth grade, the students seem to anticipate that the teacher-researchers will ask them to justify their reasoning. After Romina has suggested and explained a possible connection between the Taxicab problem and Pascal's Triangle, Michael comments that "they're going to ask us" and Jeff finishes, "the next question is why" (909-910). When T/R1 does join the group and Romina and Jeff offer their narrative about how they solved the problem, T/R1 asks the students for more justification. T/R1 asks, "Why do those numbers seem to work? How could you explain those numbers?"

(1166). After T/R1's inquiries, the students return to the problem. Claiming that she is "having trouble seeing Pascal's Triangle," Romina draws a new augmented Pascal's Triangle on a separate sheet of paper. Romina develops a more rigorous justification that uses the letters "A" and "B" to represent the horizontal and vertical directions. She builds an inductive argument to relate corresponding elements between the Taxicab and Pascal numerical arrays. T/R1's series of questions about "why" the number works thus seem to serve as a catalyst for a much more intense examination of the problem and justification of Romina's assertions.

9.2.2.3 Working through Frustration

In an autobiographical essay included in *Mathematical People: Profiles and Interviews* (1985) by Donald Albers and Gerald Alexanderson, the mathematician Olga Taussky-Todd asserts that talent is not enough to make someone a mathematician. She comments that there are many talented people in mathematics who never become actual mathematicians. Why not? For her, the deciding factor for who makes it as a mathematician and who gets "lost" is frustration. Mathematicians are those people "who work their frustrations out" (p. 314). Taussky-Todd's observation that mathematics necessitates working through frustration parallels this research's consideration of Romina's variable affect as another dimension of her behavior. Recall that DeBellis and Goldin (2006) define "affect" to be a representational structure that includes both "local affect" involving the variable states of emotion during problem solving and "global affect" encompassing the longer-term constructs established for local affect (p. 133). They describe "affective pathways" as the sequences of local states of emotion as they interact with cognitive configurations. When faced with a problem, for example, one might first feel bewilderment, fall into frustration, and abandon the problem. Or perhaps a student might experience feelings of pleasure and satisfaction after finding a pattern and continue to work. How would we characterize Romina's affective pathways as evidenced in the various problem-solving sessions? Perseverance seems to characterize Romina's affective pathway during problem-solving: she is someone who encountered frustration in each problem and yet she *worked her frustrations out* and found a way to arrive collaboratively at solutions.

Reflect upon examples of how Romina persisted as she encountered frustrations in the problem-solving sessions included in this research. For instance, in sixth grade, within a five-minute span of working on problem number three, Romina acknowledges that she "messed up," expresses "I didn't care," describes that the first time she "copied off you guys," sticks her tongue out at students across the table and taunts, "I got the answer," clenches her fists to her head when Brian says her rule does not work, laughs and tells Brian that "it works," and then hums a tune "dum, dum, dum, dum, dum – you're so slow." Later in the same session when she begins work on problem nine, she displays another example of strong and variable affect. She first asks for time "to think," exclaims "Ahhh!," expresses frustration that "this is hard," claims to have "messed up," rewrites the finite differences, and then observes that "I was right."

Compare Romina's affect in sixth grade to her affect in tenth grade. When the students revisit the towers 5-high task, Brian comments that he does not have a "breakthrough" nor will he ever "in his life" have a breakthrough. As Jeff observes that he is "getting a little frustrated," Romina puts her head on the table and responds that, "I have no clue." Then she lifts her head off the table and asks a couple of new questions, probing, “What’s the total?” and “What’s it doing?” When Jeff and Brian do not respond, she returns to a comment of negative affect saying, “I don’t know what I’m doing.” Later, while working on Ankur’s Challenge, Romina criticizes herself that she “has such trouble with simple stuff” and “I’m an idiot.” Then as she looks at her initial representations for Ankur’s Challenge and observes that “this is getting really confusing.” As she attempts to write out possible sequences with her notation of 0s, 1s, and Xs, Romina comments that “I just don’t want to do this,” “it’s confusing me,” and she has “written the same thing ten times now.” However, two minutes after these comments expressing frustration and confusion, Romina suggests that the answer “might be thirty-six” and she spends the rest of the session attempting to convince the others that her assertion is correct. Notice that for about the first thirty-five minutes of the session, Romina must work through feelings of frustration and persist in problem solving as a solution is not apparent.

Consider the amount of time that Romina spent on each of the problems included in this study: 40 minutes with her first exposure to Towers 5-High as a fourth grader, 45 minutes with the nine Guess My Rule algebraic tables in sixth grade, 93 minutes during Ankur’s Challenge in tenth grade, and 112 minutes with the Taxicab problem as a twelfth grader. Remember also that these are only a sample of all the problem-solving sessions in which Romina participated over the years with the longitudinal study. In her 2009 interview, Romina recalled that problem solving during the longitudinal study would take “a few hours at a time” and they would still revisit the problems weeks, months, or even years later. Now, in her consulting job, problem solving stretches out over similarly long periods – she estimates that "our standard project is probably six to eight weeks" and projects usually require extensions.

In *Outliers*, Gladwell (2008) describes the work of Alan Schoenfeld as it relates to success in the field of mathematics and he includes a vignette about the importance of time working problems. Although Schoenfeld has videotaped "countless students" problem solving over the course of his career, he professes that a videotape of a student named Renee working for twenty-two minutes to solve a problem of slope as "one of his favorites" because Renee persists where so many other students in his experience would give up after a few minutes:

> We sometimes think of being good at mathematics as an innate ability. You either have "it" or you don't. But to Schoenfeld, it's not so much ability as *attitude*. You master the mathematics if you are willing to try. That's what Schoenfeld attempts to teach his students. Success is a function of persistence and doggedness and the willingness to work hard for twenty-two minutes to make sense of something that most people would give up on after thirty seconds. (Gladwell, 2008, p. 246)

Gladwell concludes that a country made up of classrooms of Renees would be a country "good at math." Now imagine classrooms with students like Romina for whom not just twenty-two minutes, but hours, months, or even years of problem-solving could be the societal norm.

9.2.2.4 Romina's "Role": Secretary, Manager, or Something Else?

In the July 2009 Reflections V Interview, when asked to characterize herself as a member of the longitudinal study, Romina comments that she was the "most compassionate considering they were all men and it was just me" (68). Romina seems to be referring to high school sessions when the group would consist of her, Jeff, Michael, Brian, and sometimes Bobby. She defines the roles of her high school group in the longitudinal study:

> Like Bobby or Mike coming up with some binary code; we would expect that from them. I think, like I asked a lot of questions, so prying that way. I think Brian and Jeff were like our presenters to the outside world and they were very good at communicating our ideas to everybody. (Reflections V, line 127)

She describes Bobby and Mike as the thinkers who would develop "binary code." Romina calls herself the one who "asked a lot of questions" and would be "prying." She identifies Brian and Jeff as "our presenters to the outside world" because they were "very good at communicating." Romina's comments lead one to wonder how accurate her team role labels really are. What were the roles of each student in the longitudinal study? Did Romina fall into a set category? Later in the same 2009 interview Romina defines herself as not only the questioner, but also the "secretary" of the group. She recalls that throughout high school she kept bags of Unifix cubes in her locker so that the students could work on problems outside of school. Laughing, she labels herself the "secretary slash holder of stuff" for the other students during the longitudinal study (223). Asked if she thought her gender every played a role in her collaboration, she responds that she thinks it did:

> Yes, I was always the secretary. I was always the one - to this day, I'm still the one who has to get Brian, Ankur. No, they worked and they went to Rutgers. So I think now. But I used to have get dragged (sic) them into every after school program like I was their personal secretary. In the thing, I was always the one writing. That came up. I don't know if you guys caught that on camera or it was after camera—we had a discussion one day. (Reflections V, line 70)

How accurate is Romina's assessment of herself and others? Was she "always the secretary" and "always the one writing"? Consider examples from the problem solving sessions from elementary through high school. What does her role seem to be? As we have seen in the previous section, Romina is a group member who will often question the others – of course, questioning and "disagreeing" seems to be the hallmark of much of their interaction. Examples from the earlier section summarizing the data analysis of Romina's collaborative behavior do support her assertion that she was the one who "asked a lot of questions." What else do the data indicate about her role? In addition to the many examples of Romina questioning Brian throughout the Towers 5-High task in fourth grade, there also seems to be evidence that Romina sometimes takes on an authoritative role. For instance, by the end of the session, Romina directs Brian in how to reorganize the tower groupings on the desk. She tells Brian to "get the matches together" and then shows him specifically how she wants them to be paired (as opposed to the long row that Brian wanted). By the end of the excerpt, Brian is not even placing the towers himself, but rather handing them to Romina to place in visually distinct "matches" on the desk. Romina says that she wants to arrange the pairs so they are "going for strolls in the park" – Brian likens her direction to doll-playing by commenting that "it's like playing with Barbie dolls." Disregarding the possible negative connotation of this remark, Romina continues to direct the arrangement of the towers on their desk.

During the high school tasks, Romina's role within the group seems to be dynamic as her function fluidly moves among many possible positions: a *facilitator* who questions and encourages, a *manager* who directs the others' work, a *communicator* who presents justifications of her solution to the others, and finally a *secretary* who records the others' dictated representations. For example, as the students revisit the towers task in tenth grade, Romina asks them many questions about "why" the answer is ten and then encourages them – "we know this – it's Friday – don't panic." Her frequent questioning throughout the session seems to facilitate deeper exploration. Similarly in twelfth grade, she facilitates the sharing of ideas when she asks the others at the beginning of the Taxicab problem about “how many” paths they have – at the time the boys are all working quietly and individually as the count paths. She continues to question how the others are counting paths and reasoning. In both tenth and twelfth grade she also acts as a manager. For example, in tenth grade, she tells Brian and Jeff to “hold on” as she takes Jeff’s paper and directs them to clarify the meaning of certain sequences they have recorded. With their input, she re-writes their list to generate the ten towers with exactly two red cubes. In twelfth grade, she gets the boys’ attention with “all right, guys – this is what we’re trying to do” as she observes that “we’re getting confused.” She directs that they systematically record their results for smaller sub-grids as she and Jeff have been doing so that they will “see a pattern.” Later, as she directs Brian and Michael to check entries on her numerical array (to Brian “Do that cool number thing” and “Mike, do three over and two down”). As a result of getting the others’ input and checks, she realizes her numerical array is Pascal’s Triangle. We have seen in earlier sections that in both tenth and twelfth grades, Romina spent a great deal of time communicating her results to both the other students her group and to the teacher-researchers.

Finally, there are indeed instances in both high school sessions where Romina acts as the secretary, if we define “secretary” as one who keeps records, takes notes, or handles clerical work for others. For instance, after Romina has presented her “proof” of the 36 towers for Ankur’s Challenge, Ankur requests that she write “exactly without changing a thing” of what he dictates to her of his and Mike’s solution. Romina stands at the board and records what Ankur reads off his paper. During the Taxicab task, Romina begins to record various paths on sub-grids under Jeff's dictation. He gives her directions like, "we're just doing all the ones that are going like one across – just don't blow it." Later, all of the students use Romina's written representation of Pascal's triangle and the taxicab grid numbers to support their justifications. Notice, however, that Romina's role is not static – in both high school sessions she alternates among many different roles in relationship to the others: facilitator, manager, communicator, and secretary. In both sessions, Romina also contributes an important piece of mathematical insight – her grouping by "sixes" of the X, 0, 1 combinations for Ankur's Challenge and the isomorphism between Towers, Pascal's Triangle and Taxicab paths. Thus, although her memory in 2009 is that only Michael and Bobby had the discerning ideas like "binary code," the video data suggests she too contributed necessary mathematical connections. Similarly, she was also a communicator - one of the "presenters to the outside world" as she terms Brian and Jeff.

9.3 Beliefs

This section seeks to address how the data analyses provide insight into the final research question of the study that was primarily epistemological in nature. Specifically, how do Romina's later adult views about learning relate to evidence of her earlier mathematical behavior in terms of her descriptions of knowledge, the conditions for learning environments, and the learning process?

Schoenfeld (1992) asserts that students' beliefs have "extraordinarily powerful" consequences for their behavior and called for mathematics education to more closely examine the development of students' epistemological beliefs in relationship to their problem-solving behaviors. Analysis of Schoenfeld's year-long case study of a 10th

grade geometry class indicated that the students developed a series of four “unhealthy” beliefs (1988). Specifically, the students in the study believed the processes of formal mathematics had “little or nothing to do with discovery or invention;” students who understand math should be able to solve problems “in five minutes or less;” “only geniuses are capable of discovering, creating, or really understanding mathematics;” and students succeed in school by completing tasks as assigned by a teacher (Schoenfeld, 1988, p. 151). Romina’s professed beliefs as she describes them over the course of interviews from 1999 to 2009 stand out in sharp contrast to the “unhealthy” beliefs of those high school students. Indeed, Romina asserts the following beliefs in each of the five interviews conducted over the course of the ten year span:

1. Knowledge is an active construct that is personal and conceptual, rather than passive or procedural.
2. A successful learning environment should foster “comfortable” relationships with teacher-researchers and peers, encourage collaboration, and include tasks that “interest us” and allow for a sufficiently long time of exploration.
3. Learning mathematics involves making personal and real world connections and engaging in “group thinking” where ideas are shared, questioned, and argued.

The goal of the structural descriptive narratives included within the previous data analysis chapters was to clarify and organize the relationships put forth by Romina’s statements in order to gain insight into the phenomenon being studied: namely, her beliefs about mathematical learning. Contextual analysis suggested three broad themes into which Romina’s “significant statements” fell during her interviews: ontological and epistemological descriptions of mathematical learning, descriptions of the conditions of learning environments, and descriptions about the process of activities that contributed to her mathematical learning. It is from each of these three broad thematic categories that her three main beliefs about knowledge, learning environments, and processes of learning emerged. The following three sections summarize the data analysis that suggested these findings about Romina's beliefs.

9.3.1 Knowledge and Knowing

This section attempts to demonstrate and elucidate Romina's first belief – namely, ***knowledge is an active construct that is personal and conceptual, rather than passive or procedural***. Recall Romina's March 2002 interview in which she stated that "everything has to make sense in my terms" and that "I just don't understand unless I do try it myself and put it in my own terms" (Reflections III, line 428). Or, when, in July 2009 she was asked what it means to her to know something really well, she responded that knowing means "just to understand where it comes from" (Reflections V, line 131). Notice that Romina often uses the word "understanding" rather than "knowledge" in her epistemological definitions – even when asked specifically about her definition of knowledge. What could be the significance of Romina's repeated use of the word "understanding" in her descriptions of knowledge? The authors of *Women's Ways of Knowing* make a distinction between understanding and knowledge:

> By *understanding* we mean something akin to the German word *kennen*, the French *connaitre*, the Spanish *conocer*, or the Greek *gnosis*, implying personal acquaintance with an object (usually but not always a person). Understanding involves intimacy and equality between self and object, while knowledge (*wissen*, *savior*, *saber*) implies separation form the object and mastery over it. (Belenky et al., 1997, pp. 100 – 101)

If "understanding" connotes "intimacy and equality" with a subject whereas "knowledge" indicates a separate "mastery" over topics, then perhaps Romina's use of the word understanding in her epistemological descriptions is associated with her emphasis (as discussed in an earlier section) on developing both conceptual and human *relationships*. Indeed, Romina's descriptions of knowledge and knowing stress the value of personal and conceptual relationships cultivated through active construction. Consider other examples from each of the five interviews over the ten-year span from 1999 to 2009 in which Romina discusses knowledge and knowing:

- ***5/18/1999:*** "You can't live your whole life being told what to do. You're going to eventually have to do it yourself and they're going to have more knowledge about everything… You have to go deeper, you have to, ***if you understand something from the beginning, you always understand it***" (Reflections I, line 52); "… But I know it in my way, not their way. And everything I explain is in my words, not in anyone else's words. It's not from some mathematician from thousands of years ago, because I don't know that. Like I didn't know what the pyramid – Pascal's – was called. ***I just know everything in my own way. Everything has Romina's definition to it.***" (Reflections I, line 58)

- *7/21/1999*: "***I think there's (sic) two big areas of math: one of them is like the thinking involved and one of them is just like spitting out numbers***." (Reflections II, line 85); "…You do like a real life version of it you can see what the cat's doing – you can understand… when we did it we could see it was accelerating, it came to a peak and then it was slowing down like it just makes sense of all the math." (Reflections II, line 65); "Maybe they don't think we are as smart as we can like as we are but if you give real problems like problems that actually matter not just spitting back numbers and then memorization, we can apply – we can come up with pretty interesting things" (Reflections II, line 101)

- *3/11/2002*: "***We each needed to know from the absolute, like, beginnings***, because if we didn't, you would ask…" (Reflections III, line 88); "I took calculus all last year… I didn't know a lot of, like, the simple notation, and I would work with a friend, and ***she could spit out all the formulas, and she didn't understand it, and I only knew the background behind every formula***." (Reflections III, line 121); "…someone else might have done it already in a book, but ***I just don't understand it unless I do try it myself and put it in my own terms***" (Reflections III, line 432)

- *5/12/2006*: "***When you're able to explain it to someone else and they ask you every question under the sun and you can still answer it – I think then you know it***." (Reflections IV, line A-160); "We built that whole concept… it's not like I can memorize a formula… I'm completely confusing it because it doesn't align with my conceptual knowledge." (Reflections IV, line B-311)

- *7/15/2009*: "It's a little quantitative thing to me, ***but it's more – understanding how slope works versus actually figuring out the slope***. It's much more higher level – I have tools which help me do like the basic, the number crunching – I have Excel, I don't need – it's much more understanding and setting up a problem in more of a quantitative in an easy to see, easy to calculate way." (Reflections V, line 135)

Romina describes a dichotomy in her view of mathematical knowledge – there is "thinking" and then there is "spitting out numbers." Romina's "thinking" knowledge is active, personal, and conceptual. She refers to building understanding "from the beginning" and knowing the "beginnings" or "background" of more abstract formulas. Constructing physical and mental representations for concepts and then explaining them to others allows a person to "always understand." She recalls how, in the longitudinal study, "we built that whole concept" of combinations with the Towers problem. Similarly, during the summer experience of the Catwalk problem, building personal "real life version" representations "makes sense of all the math" - it was only "when we did it" by actively constructing a physical model that "you can understand." She claims that she and her peers can "come up with interesting things" when you provide "problems that actually matter." Romina rejects the passive and procedural knowledge inherent within "spitting out numbers." She recalls her friend in college who could "spit out all the formulas" but "didn't understand" the mathematics. She, on the other hand, did understand the "background behind every formula" although she struggled sometimes with the actual computation. In her most recent 2009 interview, she offered the example of slope. She explained that, for her, knowledge implied "understanding how slope works" as opposed to "actually figuring out the slope" – concept trumps formula. Indeed, she asserts that she can "build a whole concept."

Romina often provides an object study of what does *not* support her knowledge. In 1999, she referred to a negative experience of high school geometry (a year in which she was not able to participate in the longitudinal study sessions) – she recalls how she taught herself and the "rest of my class did really bad, because they weren't used to that – they looked to the book for answers" (Reflections I, line 56). In 2002, she recounts "horrible" college calculus and the "evil things" they did. She describes "ten-page exams" where "they only want an answer" as opposed to an explanation or derivation (Reflections III, line 132). In 2006, she discusses the "forced frameworks" of business school student learning. She suggests that what all of these negative learning experiences had in common was a focus on procedural pedagogy that positioned the learner as a passive recipient of formulas.

When we consider such statements as "everything has Romina's definition to it" and "I just don't understand it unless I do try it myself and put it in my own terms," we hear someone who lives a life of constructed knowledge. Romina's belief about her own knowledge seems to integrate what she has learned from others with personal representations through continuous self-reflection. It is fitting that in 1999, she describes her mathematical understanding as a process in which "we kind of had to like invent anything" and "choose what path we were going to take" (Reflections II, line 69). Her statement recalls Piaget, who famously wrote in *To Understand is to Invent* (1973) that "to understand is to discover, or reconstruct by rediscovery, and such conditions must be complied with if in the future individuals are to be formed who are capable of production and creativity and not simply repetition" (p. 20). Romina's dichotomy of "thinking" versus "spitting out numbers" seems to echo Piaget's description of future individuals who need to be capable of "production and creativity" as opposed to simple repetition. Again, as Romina reminds us, "you can't live your whole life being told what to do" – if you learn *on your own terms,* then you will have "more knowledge about everything."

9.3.2 Conditions of the Learning Environment

This section turns to the second main belief that emerged from Romina's interviews – specifically, ***a successful learning environment should foster "comfortable" relationships with teacher-researchers and peers, encourage collaboration, and include tasks that "interest us" and allow for a sufficiently long time of exploration***. When considering the conditions that Romina asserts during the interviews as necessary for her own learning, one should also recall the conditions purposefully included in the longitudinal study and extensively described in the literature (Alston & Maher, 1993; Davis, 1984; Davis & Maher, 1990, 1997; Francisco & Maher, 2005; Maher & Martino, 1998, 2000; Maher, 2005). Note that the Rutgers researchers sought to design an environment that would support a "culture of sense-making" by including four specific components: *complex and coherent tasks* – inviting students to explore mathematically rich problems; *sufficient time* – providing extended time for investigation and reinvention, *stimulating teacher/researcher interactions* – the educator carefully listening to and questioning student reasoning to stimulate reexamination, justification, and generalization; and *collaboration* – promoting the exchange of ideas in groups where students share their representations and make convincing arguments.

All of the components the researchers put in place seem to be related to the conditions that, during her interviews, Romina professes to value in her learning environments. In the previous chapters, data analysis of the interviews suggests that

Romina identifies four conditions for a successful learning environment: ***"comfortable" relationships*** with teacher-researchers and peers, ***collaboration***, complex ***tasks that "we're interested in,"*** and sufficiently ***long time spans*** during which to explore problems. Recall some of the facets for what Romina discussed as necessary for a supportive learning environment:

- ***5/18/1999:*** "In fourth grade, I didn't know anything. I didn't know who you were***. Now, we're comfortable with you***." (Reflections I, line 16); "But if you come into my math class, we're all in a big circle, and our teacher is in the middle sometimes and sometimes he just kind of sits down and let's us do our own thing. He gives us problems that we want to know the answer to – that we're interested in, and then he doesn't give us an equation. ***We all just kind of talk about it, and then come to a point***. And you're kind of socializing while you do your math… I'm a friend to my math teacher, and I can talk to him, like not only about math. And he's got a comfortable relationship with me." (Reflections I, lines 68 - 70)

- ***7/21/1999***: "…***We will talk about it forever – we will argue about it forever***. We will do anything that's required. We'll come up with anything, like we will come up with weird things too. We will keep going as far as we can with the problem if we are interested in it." (Reflections II, line 18); "It was typical Rutgers. They give us something – they give us like very little information about something and see what we take it to." (Reflections II, line 49)

- ***3/11/2002***: "…I don't learn well. Like if you give me a book. I didn't even really use textbooks in high school 'cause I mean for math I never really had a textbook ever, and I don't learn well like that and that's - ***I'm having a lot of trouble in college now with that because I don't even know who my teacher is***. Like if I saw, if they saw me on the street, they wouldn't recognize me and most of 'em it's like you have to read a book and then you're tested from what's in the book and I never learned like that so I'm just – I, ***I'm better at learning if like thinking about things, discussions, group work***, and I've always been, and now when I, now I'm not, I'm not doing as well as I think I could be doing in college because we're just not taught like that anymore…" (Reflections III, line 226)

- ***5/12/2006***: "We'd have tables, no desks, tables. I don't know, ***we'd all sit in groups of 4 or 5*** and we'd rotate periodically so we could work with different people all the time so we'd have to re-learn how to work with people." "We had a math teacher right who was getting, still getting his Ph.D., is still getting his Ph.D., and he'll get it you know, but ***we had someone who really dedicated a lot of time into his own education and learning about how people think and learning about how people learn*** and like he just spent all these years learning and applied them all on us and tested then out. And I know he, you couldn't, you wouldn't know, but I know he spent hours thinking up our lessons, and then we went to other classes"; "…***we were so comfortable with each other***, so I think I was fine not knowing something and being like, I don't know this, you guys, we have to go back and explain something to me for the tenth time because I don't understand this…" (Reflections IV, lines A-277, B-37, and B-136)

- *7/15/2009*: "I mean, I had a lot of experience. I did that a lot; through this program, growing up with the same twelve people all the time. And, at my job, that's what we did; ***we worked in small rooms with each other all the time***…"; "…It was a long time; ***we built a lot of relationships with them*** [the teacher-researchers]. I think Mr. Pantozzi kind of indirectly came out of that; we always saw him as a kind of Rutgers person, so I don't know if they thought of him like that. He was pretty significant to all of us. We had him for three years in math and he was very just invested in our learning…"; "…But, even with us, I think our sessions were like a few hours at a time—maybe 3 or 4 hours—we'd come up with an answer. But, we'd always go back and refine it. So I think that was what—that's why we'd get to right answers eventually, because ***we weren't scared, even after 4 hours, to say, 'You know what? We need to go back to this*** **-** we need to go back a few steps and start this from step 5.' Not all the way to the beginning because we had some basis, but we started over a lot…" (Reflections V, line 27, 78, 175)

The value of collaborative, small group working environments with thought-provoking teacher-researchers emerges as a theme of Romina's descriptions. In the longitudinal study she recalls working in "small rooms with each other" where, given a problem task, they would "talk about it forever." Romina claims that "we will keep going as far as we can" with a problem "if we are interested in it." Sessions could last "maybe 3 or 4 hours," but even after that amount of sustained exploration, Romina recalls that "we'd always go back and refine it." Saying that they would work on problems "forever" may not be entirely hyperbolic – indeed, Romina continued to revisit problems throughout the seventeen years. Romina's depiction of her high school math class with Mr. Pantozzi, one of the Rutgers researchers involved with the longitudinal study, similarly reveal a group of students "in a big circle" where they get "problems we want to know the answer to" and "we all just kind of talk about it." Quite frequently in her later interviews, Romina describes the "comfortable" relationships she developed while learning mathematics. She contrasts how, in fourth grade she "didn't know anything" and "didn't know who you were" to now being "comfortable with you." She recalls how "we were so comfortable with each other" and gives that as the reason she and here group were comfortable with "not knowing something" and asking for explanation. She identifies Mr. Pantozzi specifically as a teacher-researcher who was "significant to all of us" because he was "invested in our learning." Just as she "built" these "comfortable" relationships with the teacher-researchers and fellow longitudinal study participants, Romina built lasting relationships among mathematical concepts.

9.3.3 Learning Process

Now we explore the third major belief that came forward in Romina's interviews: ***learning mathematics involves making personal and real world connections and engaging in "group thinking" where ideas are shared, questioned, and argued.*** In her last July 2009 Reflections V interview, Romina observes that, "I think I'm pretty good at this point just getting a lot of information and being able to – organizing it to see what the problem is… it's more like that process I'm good at" (137). Later, she details more specifically the "process" through which she solves a problem and emphasizes the importance of "solving it together" with others after going a period of her own personal organization and "digesting time." Romina assertions about her learning process recall Vygotsky (1978) who argued that all higher-order cognition results from a personal internal reconstruction of interaction with others. Let us summon up a small sample of the comments Romina made over the ten-year period about this learning process:

- ***5/18/1999:*** "***If people learned the way I did with, like, group talking, I think people would learn more and be able to do more*** because if someone that was taught with just a teacher teaching them, if you're given something in, like, the real world, you're not going to know how to handle it. Whereas I would probably question it, and like, throw different ideas in the air. Other people, they get intimidated, and they don't know how to do that" (Reflections I, line 52)

- ***7/21/1999***: "… ***we were able to put like real life things into it*** and like what can it affect it like not math like real things"; "… first of all, I wouldn't be like finding the solution for a big problem by myself. I would – a lot of other peoples they'd be like – we would have to have some sort of arguing like to bring up points that maybe I don't see that could help the solution. ***And people arguing will help and people would just keep talking about it and we have to find as many solutions as possible and go from there to see which one's the best solution***… Like I am a more verbal person. I can speak well and I can communicate my ideas where other people might like my same age level can't because they never had to – they don't know. They're intimidated where I was kind of put on the spot and had to and it just develops your idea and maybe when we are like running the world, we can come up with better solutions cause we know more and we can like we've practiced and ***we have been able to have like group thinking and solutions***. (Reflections II, lines 61 and 97 - 98)

- ***3/11/2002***: "***I think we learned more of a thought process*** and how we deal when we were first given questions, which is how I always deal with how I'm given questions now. And that's how we do it; we talked it out, like, between my friend and I and then we came up with the how are we going to do this."; "We used to call each other and we'd just discuss ideas and what happened and details and things. And ***that's how I learn – and it's a group setting. It's - I learn in groups***."; (Reflections III, lines 181 and 259)

- ***5/12/2006***: "And it's just being able to pick things up quickly and ask the right questions to get an answer, it's going to take me a long time to get to my solution or my answer, but just in the fact that I know what questions to ask and I put the effort in and ***I know how to learn and how to absorb information***, the right information, and weed through it. I mean, that's all we have to learn, to know how to do"; "I hate learning things that don't, like I feel like if you learn one concept that doesn't connect to other concepts, like you're learning something almost useless." (Reflections IV, lines A-262 and B-242)

- ***7/15/2009***: "I think I'm very—I need a little bit of quiet time, digesting time, at the beginning. I need to really understand something. Have some alone time to really think through my own thoughts - Then, it's like, ***I only get to a certain point by myself by kind of organizing the problem. I like to talk about it with other people***. Kind of be like, "Is this what you think? The issues? How are we going to tackle this?" Then, work on it together. And then, come up with some kind of plan. And I think it works out well, because then we get it a little bit

> further. I don't like going down and doing a problem all by myself, because the chances of me getting to the right answer that everyone else gets to, is gonna be—it's probably not going to happen. ***So, it's just bringing people along and then solving it together***" (Reflections V, line 151)

Data analysis in earlier chapters suggests that both Romina's beliefs and behaviors regarding her learning process involve a period of "alone time" followed by a period of "group thinking" where ideas are shared, questioned, and argued. Indeed, Romina comments that "I only get to a certain point by myself" and that "I like to talk about it with other people." Over the years, she maintains that she has learned "how to learn and how to absorb information" by making personal, "real life" connections. Through "talking" with others, "we have to find as many solutions as possible" and then "see which one's the best solution." She asserts that this process "works out well" because, collectively, everyone moves "further" ahead. She predicts that "it's probably not going to happen" that she can get an answer "all by myself," but that success follows "bringing people along and solving it together." Romina describes this process alternately as "group talking," "people arguing," and "group thinking" and claims that if others experience this during problem solving, "people would learn more and be able to do more." Going even further into the future, she envisages that "maybe when we are like running the world," it will be through this process of group thinking that "we can come up with better solutions" and develop more powerful ideas for whatever problems await.

9.4 Limitations

Inherent within any research are limitations. As a Rutgers University graduate student, math teacher by profession, and interviewer for one of the video data sets included as part of this work, I am a participant in the phenomenon being studied by this research. Thus, themes that emerged during coding the beliefs and behaviors of a member of the Rutgers longitudinal study must be acknowledged as emic in nature. Further, not all the video data sets that exist for Romina in the Rutgers videodata archive were included in this research. Why were certain sessions chosen? In an effort to select representative data for behavior that showed Romina working in full class settings during school time as well as in the voluntary after-school small group settings, video data sets were chosen from fourth, sixth, tenth, and twelfth grades. However, Romina participated in many other problem-solving sessions in those school years (1992 – 2000) as well beyond high school, as in 2003 when she revisited the Fundamental Theorem of Calculus with two female students, Magda and Angela. Findings are limited to only the sessions chosen here; this researcher would welcome future exploration of those other sessions. For instance, one might observe that all four sessions included here show Romina with Brian and all but the first featured Romina with Brian, Jeff, and Michael. Maybe if sessions were studied in which Romina worked with other students in the longitudinal study, different behavioral themes would emerge. In addition, no interview sessions concerning Romina's beliefs existed in the elementary grades and, therefore, belief findings are limited to the interviews during and after high school (1999 – 2009). Finally, why choose Romina in the first place as the subject for a case study? One must acknowledge that Romina's tendency to verbalize and the fact that she had been a consistent longitudinal study participant since 1992 made her a more amenable candidate for videodata analysis then a student who worked quietly or who had joined or left the study in a later year.

A case study in and of itself presents limitations considering that it focuses on a bounded system. Stake (1995) cautions that a case study will not establish a generalization or a modification of one:

> The real business of case study is particularization, not generalization. We take a particular case and come to know it well, not primarily as to how it is different from others but what it is, what it does. There is emphasis on uniqueness, and that implies knowledge of others that the case is different from, but the first emphasis is on understanding the case itself. (Stake, 1995, p. 8)

Here, then, we can only highlight the "particularization" of Romina's case and not attempt to generalize for all mathematics students or even all students within the longitudinal study. In considering the "uniqueness" of Romina's experience, one may infer some knowledge of other cases, but the emphasis must be on "understanding the case itself." However much one may want to expound upon the experience of others, any results are specifically Romina's and only as suggested by the specific number of data sets provided in the scope of the study. Stake (1995) warns against drawing hasty interpretations from a small database and thus this researcher sought to include a variety of data over a seventeen-year span. He defines "good case study" as "patient, reflective, willing to see another view" and he offers this advice, "an ethic of caution is not contradictory to an ethic of interpretation" (p. 12). He acknowledges that "ultimately, the interpretations of the researcher are likely to be emphasized more than the interpretations of those people studied, but the qualitative case researcher tries to preserve the multiple realities, the different and even contradictory views of what is happening" (p. 8). In order to preserve the "multiple realities" and possible contradictory views of what is happening, Stake suggests extensive verification of a case study by using "triangulation of information" whereby the researcher attempts to confirm the study's assertions.

Certain researchers involved in the longitudinal study did read and provide feedback on the data analysis of this research. Though Romina herself never read the rough drafts to examine the accuracy of assertions concerning her words and actions, Romina was asked to comment on much of what she had said in earlier interviews during the last Reflections V interview held in 2009. Further, sessions were videotaped in which Romina watches and comments upon herself solving problems (as she does during the 2002 Reflections III Interview).

Given that other interpretations could exist than those of this researcher, care was taken to include full transcriptions independently verified by at least one other person for each of the nine sessions included here. All transcripts are provided in the appendices for review. A total of 674 minutes (11 hours and 14 minutes) for 16 disks of video data were analyzed. A balance was attempted between the amount of time from behavioral data (4 hours, 50 minutes from 7 disks) and interview data (6 hours, 24 minutes from 9 disks) – it should be noted that the July 2009 interview incorporated an hour of problem-solving. One can only hope that the findings preserved as much as possible the "multiple realities" of what happened and feature the "particularization" of Romina's experience as grounded and well-supported by the variety of informational sources considered over a seventeen-year time span – interviews, observations, documents, and audio-visual materials.

9.5 Implications

What do the results of this study imply? During her July 2009 Interview, Romina comments on "our little group" and muses, "I was really impressed by us sometimes – how did we do that?" (Reflections V, line 117). Later, she professes that she feels that her participation in the longitudinal study is "an accomplishment of mine" (474). How did Romina "do" mathematics over the years and why does she feel this was such an "accomplishment"? The results of this case study propose answers to both questions. Very little knowledge or understanding has existed in math education research about the relationships among students' beliefs, behaviors, and learning environment over an extended time. The literature has been also relatively silent as to how to stimulate positive beliefs about math learning. Romina's case study spanning seventeen years begins to address the need for a *systematic* and *simultaneous* examination of a student's mathematical views and problem-solving behaviors. The results of this study have implications for specific instructional interventions that support the development of mathematical ideas and reasoning from elementary grades through college and into the workplace. In addition to calling for further study into the relationship between mathematical beliefs and behavior over time, the outcomes of this research contribute to a larger body of work considering how social interaction, teacher questioning, and task design affect students' cognitive growth in terms of mathematical justification, proof, and generalization (Alston & Maher, 1993; Maher, 2002, 2005; Maher & Martino, 1996a, 1996b, 1999, 2000; Maher & Speiser, 1997; Martino, 1992; Muter, 1999; Powell, 2003; Uptegrove, 2005; Uptegrove & Maher, 2004a, 2004b). Romina's story encourages deeper reflection on our current mathematics curriculum and school structure. Here we watch a student who seems to have engaged in range of collaborative behaviors and produced continuously evolving personal representations that promoted mathematical understanding. Simultaneously, the data suggest she developed three very "healthy" mathematical beliefs involving the active construction of conceptual knowledge, learning environments built around "comfortable" collaborative relationships and sustained engagement in complex tasks, and, finally, a learning process of “group thinking” where personally relevant problems are shared, questioned, and argued.

Robert B. Davis would probably be most able to translate the results of Romina’s case study to mathematics curricula today. From his time with the Madison Project through his research at Rutgers University, Davis made a career of thinking about and acting on math education ideas. As a result of his wealth of experience and insight, much of Davis’s writing reflects upon mathematics curriculum in schools. In 1972 he outlined a “Piaget-based curriculum” on which future mathematics learning could be based. He identified the “major task of schools” in teaching mathematics to be “not to tell the adult version, but to work with a child on describing, elucidating, and improving his ideas” (Davis, 1972, p. 8). He advocated a “developmental” approach that allowed students to develop “schemata” as they encountered carefully designed “assimilation paradigm”-building experiences. Lamenting that most curricula he witnessed in classrooms were dangerously “severed” from the real world and unfortunately consisted of “meaningless bits and pieces,” Davis (1992) surveyed new approaches being taken across the country. He highlighted the “particularly important series of studies” that Carolyn Maher had undertaken with the inception of the Kenilworth longitudinal study at Rutgers (p. 731). He looked hopefully toward how the small-group work with mathematically rich tasks given plenty of time and manipulative materials could be incorporated into schools. Indeed, the mode through which Rutgers used such investigations as the tasks which Romina experienced suggests a very powerful vision for curriculum implementation today. Imagine schools where Romina had her way and “group thinking” and talking “forever” about meaningful problems were the norm!

Open any newspaper or do a quick search of the internet - one cannot help but see a myriad of articles addressing education. Standardized test scores, "failing schools," vouchers, teacher unions, No Child Left Behind, Race to the Top… education is a topic of constant national, state, and local debate. NCTM (2009) opens its most recent book on *Focus in High School Mathematics: Reason and Sense Making* with a list of challenges facing our students in the future as the demands for mathematical literacy increase. The chapter concludes with a call for "restructuring" mathematics programs around reasoning and sense making to support "students' development of both the content and process knowledge they need to be successful in their continuing study of mathematics and in their lives" (NCTM, 2009, p. 7). So where do we go from here? What can Romina's story tell us? Perhaps the results of Romina's case study might respond to the criticisms typified by an Op-Ed piece (February 2, 2010) in the *New York Times* in which Susan Engel considers recent educational reforms and calls for a "theoretical classroom" with a "curriculum designed to raise children, rather than test scores." One wonders what Romina would say to Engel's provocative critique of the national pedagogical climate:

> Our current educational approach – and the testing that is driving it – is completely at odds with what scientists understand about how children develop during elementary school years and has led to a curriculum that is strangling children and teachers alike. (Engel, 2010, Op-Ed)

I imagine Romina would agree with Engel that certain current approaches are "at odds" with how she saw herself develop as a learner and, likewise, would support Engel's call for a classroom that would "provide lots of time for children to learn to collaborate with each other" so that they can "construct knowledge" (Engel, 2010). In 2006, during the Reflections IV Interview, Romina speaks specifically about how, if she had her way, she would "get rid of the formalized classroom" with multiple choice tests and "rows of chairs" because "it's not realistic" and does not produce "thinkers" or "leaders." Later, Romina states that, "I think our education fails at inspiring our thinking" (line 288). She talks about how she would replace the "formalized classroom" with a structure like that which she experienced in the longitudinal study: small, collaborative groups "talking" about complex mathematical tasks over extended periods of time and "explaining what we thought of concepts."

The nineteenth century Russian mathematician Sofia Kovalevskaya worried that too many people confuse mathematics with arithmetic – she famously said that, "It is impossible to be a mathematician without being a poet in soul." Kovalevskaya would no doubt have agreed with Romina's blunt assessment of the dichotomy that exists in current mathematics programs where "spitting back numbers" lives alongside "thinking." Romina's case study offers the reader perspective through a magnifying glass seventeen years long and one student wide on how to see reasoning and sense-making in action. To bring an end to this story about mathematical learning, let us reflect on a poem. A haiku from Issa sums up how large meaning can be found in the careful consideration of small and seemingly inconsequential things:

The distant mountains
are reflected in the eye
of the dragonfly

Issa's poem gives us hope that, through the reflections in the eye of a single student, one can see great, "distant mountains" of meaning. Perhaps through the particular lens of this case study, Romina's experiences can offer us new insight on the development of mathematical understanding over time and suggest a vision for instructional interventions for the future.

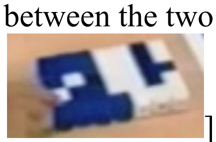
APPENDIX A: TRANSCRIPT – TOWERS 5-High: February 6, 1992 (4^{th} Grade)

2 Camera Views: "People View" - Romina and Brian for audio
and "Work View" for screen shots
Date of filming: 1992-02-06
Harding public school, Kenilworth NJ, Towers 5-high with choice of two colors
Transcribed by: Maria Steffero
Date of transcription: March 2009
Verified by: Margaret Steffero
Date of verification: April 2009
* Where possible, screen shots from the "Work View" camera view are provided.
Where a screen shot is not clear, □ or "W" will represent a white cube and ■ or "B"
will represent a blue cube.

Line	Time	Speaker	Transcript
1	00:00:41:25	Romina	What do you mean? Who asked you?
2		Brian	Miss Ansonia yesterday morning. They gave quizzes.
3		Romina	To who?
4		Brian	The kids in our class.
5	00:00:52:28	Romina	That's gonna be cool. I wanna do that. Cause I wanna be a teacher when I grow up.
6		Brian	Mrs. Barnes. [Shows container to teacher standing off screen].
7		Mrs. Barnes	Who else had [inaudible]?
8		Brian	Jeff.
9		Mrs. Barnes	Where's Jeff?
10		Brian	He's in English.
11		Mrs. Barnes	Oh, that's right. When he comes back.
12		Brian	I did it five times.
13		Mrs. Barnes	Take some paper.
14	00:01:17:00	Brian	Look at these cameras. They've got like them TV cameras.
15		Romina	[Leans in to Brian] That's cause we're one of the best.
16		Brian	Yeah, we're gonna be on TV. [Looks around] Might as well put this on the ground. [Puts container that he showed earlier on floor]
17		Romina	[Gestures to container] Was that easy?
18		Brian	Yeah, well sorta hard.
19		Brian	Candace is working alone. Look at that one. [Points and looks up]. They're huge.
20		Mrs.	T.V., huh?

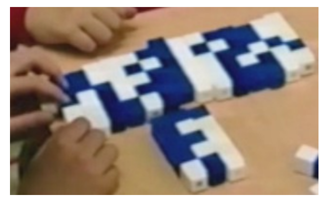
Line	Time	Speaker	Transcript
		Barnes	
21		Brian	Yeah.
22		Mrs. Barnes	Don't worry about it.
23	00:02:05:02 – 00:02:48:00		[Brian and Romina sit in silence].
24		Brian	What is that?
25		Romina	What?
26		Brian	This.
27		Romina	A microphone thingie.
28		Mrs. Barnes?	Put your name tags on the front of your desk. We'll move them back tomorrow. [Romina and Brian move their name tags]
29		Mrs. Barnes?	[Voices in front of room] Oh, you want them on top?
30		Mrs. Barnes?	You wanna take your books off your desk so you have a lot of room.
31		Brian	Look at the TV. [Brian points behind him]
32		Romina	Oh god. [Brian and Romina laugh and cover their faces with their hands].
33		Brian	You're on Candid Camera!
34	00:03:54:04	T/R1	Okay, you remember – um, you know who I am?
35		Student	No.
36		T/R1	How many of you have seen me before? Okay, this is a new group. Okay, I'm Dr. Maher. I'm from Rutgers and I've been in some of your classes and I'm very happy to be with you today because your teacher tells me that you like to solve problems. Is that true? [Brian and Romina begin nodding their heads] And that you're very good at it. [Brian continues nodding his head] Is that correct? [Brian and Romina both nod their heads]. Have you seen these before? [Indicates Unifix Cubes]
37		Brian & Romina	Yeah.
38		T/R1	How have you used these before? Eric?
39		Eric	[Voice inaudible]
40		T/R1	Okay, did anyone use them in any other way? [Another student speaks off camera – voice inaudible]
41		T/R1	Yes? [Romina is raising her hand]
42		Romina	We used them to make patterns.
43		T/R1	To make patterns? How?
44		Romina	Well, our teacher gave us a lot of different colors and we had to put them in patterns by – sometimes she said put them in patterns by three colors and everything.
45	00:05:14:03	T/R1	Ah huh. Does anyone else here remember using them to make patterns? Some other people too. Okay, well we're gonna use them a little

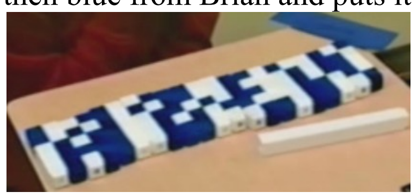
Line	Time	Speaker	Transcript
			differently maybe to do problems we have here on the board. Can you all read this? It's called building a tower. We'll talk about what a tower is.
46		Brian	What one?
47		Romina	The whole thing.
48	00:05:37:00	T/R1	Building a tower. Well, what we're going to ask you to do is have a partner. Some people have partners, but if you don't, you might have to find a partner. And with your partner, you will be using the blue and white cubes. You know they come apart, right? Because you've used them before. Right?
49		Brian	Yeah.
50		T/R1	And using the blue and white cubes, you're going to work together to build as many different towers that you can that either use white, blue, or blue and white together. But it has to be five cubes tall. Okay? Is this a tower? Five cubes tall -
51		Brian	Yeah.
52	00:06:22:19	T/R1	This isn't a tower? Is this okay? Our towers will have sort of like a bump on the top – a chimney, if you like. We're not gonna count it upside down as a tower. That will be part of our rule. So again, the rule is, you have to either use white cubes, blue cubes, or blue and white. Can somebody just reach in your box and make me a tower just five cubes tall? Using white, blue, or blue and white together.
53		Romina	[To Brian] What are you doing?
54			[Brian puts down two towers of all white cubes. Romina places two towers of all blue cubes in front of her] .
55		T/R1	You think you have one? Okay – we have a tower here that Alex made. Is that what we mean?
56		Brian	Yeah.
57		T/R1	Okay, this is an interesting tower and he has it standing up so it has a chimney on top. Now we want to find all possible towers that are five cubes high. Do you think there are more?
58	00:07:23:12	Brian	Yes.
59		T/R1	Does anybody have any idea how many more? Let's take some guesses. Eric? Eric thinks there are two more.
60		Brian	Oh! [Shakes his head]
61		T/R1	What do you think?
62		Brian	I think there are like six more.
63		T/R1	You think there are six more. Does anyone else have any guesses here?
64		Student	Eight.
65		T/R1	Eight.
66		Romina	[Quietly to Brian] Ten.

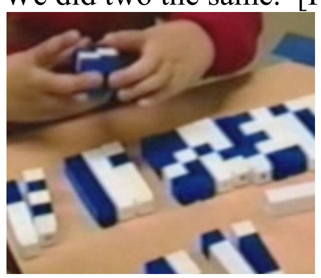
Line	Time	Speaker	Transcript
67		Student	Nine.
68		T/R1	Nine. Alex?
69		Alex	Twelve.
70		T/R1	Twelve. You have some interesting guesses here.
71		Romina	I got ten.
72		T/R1	Well the problem for you is to find all possible ones and try to be able to convince us and each other and Mrs. Barnes that you have found all possible towers and that you haven't missed any. Okay? They could be made of blue cubes, white cubes, or you could use blue and white together. Okay? Go to it!
73	00:08:19:19	Romina	Yo, Brian, there has to be ten.
74		T/R1	After you've built them, we're going to ask you to somehow keep a record. But right now let's just worry about building them, okay?
75		Romina	There has to be ten. First we have one way – one white and one blue.
76		Brian	What is that – six?
77		Romina	No. [Takes off one cube from her tower. Then she moves the two towers they have already built closer]. Can't let anybody see these.
78	00:08:52:12	Brian	Same thing as Alex did. Make Alex's. No, I need a white.
79		Romina	There's gonna be ten, Brian.
80		[There are now four towers now in front of them ].	
81		Brian	Okay. Wait, I got one. Do we have this one? Do we have like this? No. [Brian adds a tower at the end of four blues and one white. At the same time, Romina adds a tower at the other end of four whites and one blue. ]
82		Brian	[Brian adds another tower of four blues and one white. He points between the two towers he has made with four blues and one white. ] Oh! Move it up.
83		[Romina adds a tower after Brian's. ]	
84		Brian	Oh, right! Do the same thing with blues. [Romina drops some cubes on the ground].
85		Romina	I'm doing this one. [Romina points to the last tower she built of alternating color cubes].
86	00:09:40:29	Brian	I'm doing it, I'm doing it like that.
87		Romina	How does that look?
88		[Romina adds a tower of four whites and one blue at the beginning of their row	

Line	Time	Speaker	Transcript
			next to the other tower of four whites and one blue she had built earlier:  . She pushes all of them together and claps her hands.]
89		Romina	How ‘bout one with one on the bottom? One on the bottom.
90		Brian	Here. [Puts down another tower]. Two, four, six, eight, ten.
91		Romina	[Points and smiles] See, I told you! 
92		Brian	Yeah.
93		Romina	How about one on the bottom?
94		Brian	Yeah, that’s right. Wait, check – four [runs his finger over the other towers] Yeah.
95		Romina	[Puts down a tower with a blue on the bottom of four white cubes] We need more – wait – one, two, three, four [Romina counts out four blue cubes to put on the top of one white cube]
96		Brian	We have one like that. No, we don’t.
97	00:10:14:25	Romina	No we don’t. [Rolls her eyes] We need more cubes.
98		Brian	They’re back there.
99		Romina	Can you get any? [Looks at camera]
100		Brian	Yeah, look over there. [Points off camera] Told ya.
101		Romina	Okay, I’ll get these white cubes. [Reaches back and gets a stack of white cubes]
102		Romina	Umm. [drums her fingers against the desk]
103		Brian	Do we have a white one right here? [A teacher/researcher puts a handful of blue cube stacks and white cube stacks on the students’ desk]. A white one right there? [Brian runs his finger over the towers they have] A blue, white?
104		Romina	[Begins to make a new tower] A blue, white.
105		Brian	Blue, blue. [Puts more cubes on the tower Romina is making] These things stick. [Brian puts down their new tower with the existing group]. 
106		Romina	Do we have them like this, but only with white on the top?
107		Brian	Four blues and a white?
108		Romina	Mmhmm.
109		Brian	I got it. I got the four blues.

Line	Time	Speaker	Transcript
110		Romina	And I've got the one white. [Romina hands Brian a white cube] Oh god, this is a lot.
111		Brian	[Assembles the new tower and holds it over their existing group]. Yeah, right there. [The tower they just made is a duplicate]
112		Romina	Oh duh.
113		Brian	Four blues and a white there. [Points to another tower]
114		Romina	Which ones are those that we did… this one and this one. [Romina pulls two different pairs of towers down from their large group]. How about we put one there? [Romina points to a location on the existing towers] No, there!
115		Brian	No, take the kinds that are like this. [Brian pulls a different pairing from their group].
116		Romina	Brian, look, I have one. [Romina makes a new tower] Well, I thought of it first! 
117		Brian	We have that.
118		Romina	Which one did we just make?
119		Brian	That one.
120		Romina	That one? Well, if we go -
121		Brian	Wait. White, white, white. White blue, white-white-white. We don't have that. No, yes, we do right here. [Pushes all the towers in two rows]
122		Romina	[she counts the number of existing towers they have on the desk] One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen. Whoa. 
123		Romina	Umm. [drums her fingers against her head]
124		Brian	Ahh. Any others?
125		Romina	Yeah, think.
126		Brian	Yo, white-white-blue?
127		Romina	White-white-blue? Two whites. [Hands Brian two white cubes]
128		Brian	White-white-blue [Puts two white cubes on top of one blue cube] Another two whites. [Romina hands him another two white cubes] Do we have that?
129		Romina	Yeah, we have that!

Line	Time	Speaker	Transcript
130		Brian	Where? [Romina points] No. white Ahhh!
131	00:12:48:15	Romina	No, I thought you meant like this. [Romina builds a different tower]
132		Brian	Two blues and three whites?
133		Romina	One, two, three, four, five.
134		Brian	No, we don't have that. [Romina puts down a new tower] How about we do that with the blues?
135		Romina	Yeah, that's what I thought.
136		Brian	A three and two.
137		Romina	Two blues. [Romina builds a new tower]
138		Brian	Take the blue off. Yeah. And now we can do it like this. Three and two with whites like this. [Brian now builds a tower].
139		Romina	Wait a minute, didn't we just do that?
140		Brian	No.
141		Romina	Oh, we did it upside-down.
142		Brian	[Puts down his tower and points] Three. 
143		Romina	How about -
144	00:13:32:29	Brian	Three whites and two blues.
145		Romina	Three whites and two blues.
146		Brian	Going up this way.
147		Romina	You don't have that one.
148		Brian	[Builds again] We're doing good. [Romina pushes the towers they've built down closer to her on the table in a long row]. It's going off the desk! 
149		Romina	How about
150		Brian	We have all these blues. [Brings the long stacks of blue cubes closer].
151		Romina	And all these whites. [She points to the stacks of white cubes]. Ummm. [She drums her fingers against the table]
152		Brian	A blue on the bottom and four whites?
153	00:14:06:05	Romina	Yeah, we did that. I think that was one of first [drags finger across the row of their existing towers].
154		Brian	Yeah, right there. [Brian points to the tower] a blue On the top and four whites? [Romina points to that tower] Oh.
155		Romina	Umm. [Pulls at her necklace]. Ew. [Frowns]
156		Brian	[Laughs] Four whites and one blue?
157		Romina	Four whites, one blue [she points to the tower]
158		Brian	Oh, that was the one I just said.
159		Romina	And we have that one too, so… [picks up the tower with four whites and one blue] We may as well put this one here. [Rearranges the row so the

Line	Time	Speaker	Transcript
			tower with four whites and one blue is next to the tower with four blues and one white] 
160		Brian	[Counts the towers] Nineteen.
161		Romina	Whoa. [Drums her fingers against the desk] If we only get one more, we'll get twenty.
162		Brian	Wait. Do we have a white and then three blues and then a white?
163		Romina	[Drags her finger down the row] A white -
164		Brian	We don't have it -we don't have it! [Brian and Romina begin to build] A white, three blues, and then a white.
165	00:14:59:09	Romina	Here. [Hands Brian a cube] Sure we don't have it?
166		Brian	It's going off the desk. [Puts the new tower down at the end of the row]
167		Brian	Now blue then white, white. No. blue White, white, white. We need another white. [Romina hands him a cube] How many do we have now? Like twenty now?
168		Romina	Twenty-one. I counted. [Takes the tower of blue, three whites, and then blue from Brian and puts it down] 
169		Brian	Don't it look like it has a word? In the little squares? [Brian points to the row of the towers they've built]
170		Romina	I know. [Straightens the row]. Hmm. Did we… Do you think we have all of them?
171	00:15:49:13	Brian	No, there has to be [A teacher/researcher approaches. Brian answers a question from off camera] Twenty-one.
172		T/R2	Twenty-one?
173		Romina	One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, twenty, twenty-one. Yeah, twenty-one.
174		T/R2	Do you think you have them all?
175		Brian	Yeah, that's what she just said, but we're still working it. Sort of looks like it makes words like that.
176		T/R2	Words? Okay, what words do you see?
177		Brian	It looks like it. Like that's an "H" and that's an "E"
178	00:16:24:12	T/R2	That's interesting. That does look like an "H." Did you begin to see any kind of pattern with them?
179		Brian	Um.
180		Romina	We can always find. Well, this and [picks up one of the towers]. Wait, where's the other one? The one - white, blue, blue.
181		Brian	Right there.

Line	Time	Speaker	Transcript
182	00:16:42:12	Romina	[Holds up two towers] We can always do the opposite.
183		T/R2	Okay. You can do the opposite. Do you have any pairs that are opposites of each other?
184		Brian	I had one. I just had one.
185		Romina	This is opposite. [Puts down the white with three blues and a white next two the blue with three whites and then a blue] 
186		T/R2	Okay, that's opposite.
187		Brian	What do you have – three, two.
188		Romina	[puts another two towers together] This one's opposite.
189		Brian	Ah! [Romina holds up BBBWB] Wait, wait, wait, wait.
190		Romina	We did two the same. [Holds up another BBBWB] 
191		T/R2	Okay, so that doesn't count.
192		Brian	Ah! Three whites, two blues!
193	00:17:10:16	T/R2	Do you see any other pairs that are opposites?
194		Brian	Wait, these. We have 3 blues
195		Romina	No, wait.
196		Brian	Hey, we got one.
197		Romina	[Puts another pair down] We got another one.
198		Brian	[Holds up a tower] Where's the one for this?
199		Romina	Is it this one?
200	00:17:27:29	T/R2	Okay, so one thing you found out is that you could have opposites, but is there another kind of pattern you could have besides having opposites?
201		Brian	Putting them in different orders? Opposites?
202		T/R2	No, besides opposites. Is there something else you could find that's a different kind of pattern besides them just being opposite of each other?
203		Brian	Ah.
204		Romina	[Shifts in her seat] Ah.
205		T/R2	Why don't you work on seeing if you can find some more and then you maybe you can see if there's a pattern.
206	00:17:59:03	Romina	Let's see if we have some of the same.
207		Brian	Let's put the sames in an order.

Line	Time	Speaker	Transcript
208		Romina	[Takes a tower and passes it along the top of the existing row. She makes a driving sound.] Vrooooom.
209		Brian	No, wait. Put the pairs with the opposites.
210		Romina	Some of them could be the same.
211		Brian	[Brian takes the tower Romina was holding] Oh, good idea. [Brian repeats the action Romina was doing by passing the tower along the top of the row and making a driving sound]
212		Romina	Neeeeerr. [Passes a tower along top of the row] Nothing matches with this.
213		Brian	Wait, where is this?
214		Romina	Wait. One almost matched.
215	00:18:37:08	T/R2	[Returns to their table and leans over]. Look at that. What do you think that's making now?
216		Brian	[Looks at the new row they've created] "M" – yeah.
217		Romina	How bout after this we can try making letters?
218		Brian	Yeah!
219		T/R2	Making what?
220		Brian	Letters!
221		T/R2	Okay, but first try to find as many combinations as you can. I think you can come up with more combinations.
222		Romina	More than twenty-one?
223		T/R2	Exactly.
224		Brian	Let's see. More than twenty-one?
225	00:19:04:06	T/R2	I think so. Why don't you try it.
226		Brian	Do we have white-blue-white-blue-white?
227		T/R2	Do you need more – oh, you have enough.
228		Romina	White blue?
229		Brian	We have it.
230		T/R2	Brian, why don't you stand them up? You could see them better.
231	00:19:17:16	Brian	Yeah. [Brian stands the towers up on the desk instead of laying them flat as they have been up to this point]
232		Brian	[Romina begins standing up the towers as well]. Don't stand them up in front of each other because then you can't see the back.
233		Romina	[She moves the towers along the side] Happy? 
234		Brian	Yes.
235		Romina	Now I don't see the back part.
236		Brian	Then we can add um up like
237	00:19:33:12	Romina	See the front part. See if two that are exactly the same.
238		Brian	Wait. I've got some more. Almost. Wait. This one better fit in there.
239		Romina	[Taps her finger against her chin] Well, give me some of these white

Line	Time	Speaker	Transcript
			and blues in case I find more [Pulls the stacks of white and blue cubes closer to her] How about three and two? Do we have that? Three and two?
240		Brian	Three and two? Three and two! [Brian watches as Romina drags the new tower down along the row of previously made towers]
241		Romina	Three-two? Three-two?
242		Brian	Yes, right there.
243		Romina	Nuts. But, wait. Yeah, it's the same.
244	00:20:10:29	Brian	How about three blues and two whites or four blues and one white? [Romina pulls up an example tower of each] We have that? Four blues and one white?
245		Romina	[Holds up a tower] Four blues and one white. We're almost – I think we have all of them. We have all the whites. We have all blues.
246		Brian	How about one blue on the bottom and four whites up?
247		Romina	Um. [Drags her finger across the row]
248		Brian	We don't have that! We don't have that!
249		Romina	Um. [Pulls up the tower of one blue and four whites. She laughs and Brian puts his hands to his temples]
250		Brian	One white and four blues up?
251		Romina	One white and four blues.
252	00:21:03:02	Dr. A.	[Leans down and looks at the row of towers with the students]. Oh, how many do you think you have?
253		Brian	I don't know but we had twenty-one.
254		Dr. A.	Oh, you have a lot.
255		Romina	One, two, three, four, … [Romina quietly counts the towers]
256		Dr. A.	Can you tell me what you're thinking about?
257	00:21:12:26	Brian	Well, once when we find one we just do the opposite.
258		Dr. A.	What do you mean 'the opposite'?
259		Brian	Like, when we found this one out [holds up WBBBW]
260		Dr. A.	Yeah?
261		Brian	We just put two blues on the top and three whites in the middle. [Brian holds up the BWWWB]
262		Dr. A.	Oh. Do they always have an opposite?
263		Romina	Yeah.
264		Brian	Yes. Well, not. Yeah. [Romina nods her head] Well not like ones that have two in the middle.
265		Dr. A.	Hmm. So it works sometimes?
266		Brian	Like if you have two here. You can't do that. Switch it around.
267		Romina	What? You can switch this around. You can put two whites and three blues.
268		Dr. A.	What would be the opposite to that one?
269		Romina	Which one? That one?
270		Brian	Do we have it? Yes.
271		Romina	This one.
272		Dr. A.	Oh, so that's what you mean by 'opposite.'

Line	Time	Speaker	Transcript
273		Romina	Yeah.
274		Dr. A.	So that's the way you've been working?
275		Brian	Yeah. Let's see if we have any one without an opposite.
276		Romina	Yeah, that's a good idea. [Dr. A. moves away from their desk]
277		Brian	I found one already!
278	00:22:31:02	Romina	Which one?
279		Brian	Three blues. No, three blues, one white and one blue. [Romina pulls two towers – BWBBB and BBBWB - out of the row] Ah!
280		Romina	Look for one, I can't find - two blues, one white and two blues.
281		Brian	Two blues, one white, and two blues?
282		Romina	Oh nuts. [Holds up the BBWBB]
283		Brian	Two whites, one blue, two blues. Two whites, one blues
284		Romina	They're opposites. Take the ones that are opposites and put them over here.
285		Brian	[Separates a group of towers] All of them are opposites.
286		Romina	Maybe you can find one that's not. Well these are [Holds up WWWWW and BBBBB]
287		Brian	Wait, I found one, I found one! This one. Oh no, I found it.
288	00:23:16:29	Romina	[Hums and moves the tower] Da, da, da, da. 
289		Brian	With two whites on top? Right there.
290		Romina	Oh, I guess I have bad eyesight. [Hands towers to Brian who puts them in a separate pile laying flat against the desk]
291		Brian	We don't have it, we don't have it, we don't have it! Ah! This one we don't have.
292		Romina	What do you mean, this one we don't have?
293		Brian	We don't have one blue and four whites.
294		Romina	[Holds up a tower] One blue and four whites. [Brian grabs two more towers out of her hand] Whoa, whoa, whoa, whoa.
295		Brian	They're the same, look. Yeah, no opposites.
296		Romina	All opposites you mean.
297		Brian	Yeah, no sames.
298		Romina	Oh god. Umm. [Puts her chin in her palm and looks down at the desk].
299		Brian	Ah! You shouldn't have done that, because now we can't see which one we did. [Brian starts to pick up the towers that are now all lying flat and stand them up on the desk]
300	00:24:24:18	Romina	Well, get the matches together. Which one – is this right? [Romina holds up WWWWB and BWWWW together]
301		Brian	Yeah.
302		Romina	No it's not.

Line	Time	Speaker	Transcript
303		Brian	Wait.
304		Romina	No, we had to find four blues and one white.
305		Brian	Four blues and one white? Did you find it?
306		Romina	No
307		Brian	Oh! We might not have it.
308		Romina	But we do. Four blues – [Romina leans over the pile of towers]
309		Brian	We don't have it. We don't have it. We don't have it. [Romina picks up a tower BBBBW and shows it to Brian]. Oh, we have it! [Holds his hands up to his face] Oh! So close.
310	00:25:03:22	Romina	We have this one. We have this one. [Romina stands pairs of towers up on the desk in front of her: WBBBB and BWWWW, BBBBW and WWWWB]
311		Brian	Two whites, three blues. [Brian stands up WWBBB and BBWWW next to the two pairs Romina put up]
312		Romina	Don't put them together. [She separates the six towers into pairs].
313		Brian	No, we already know they're whatcha-macall.
314		Romina	They're going for strolls in the park.
315		Brian	[Laughs and leans back] It's like playing with two Barbie dolls. Here - match. [Brian hands Romina more towers to stand up"BWBBB and WBWWW]
316		Brian	[Looks at the pair WWBWW and WWWBW that Romina just stood up] That ain't no match. 
317		Romina	I know that. Hmm. [Looks over at the other pile of towers]
318		Brian	[Brian pairs WWBWW and BBWBB together and then picks up two more towers] Got three whites. [Romina picks up the pair Brian puts down and then puts it down again. Romina and Brian stand up more pairs on the table until there are a total of twenty total towers standing or 10 pairs]
319		Romina	[Sighs. Leans over and picks up the only tower that is not standing up now: BBBWB] Do we have this one?
320	00:25:57:29	Brian	What? Is there any without a pair? Any without a pair? [Romina holds the tower BBBWB against a duplicate already standing up] Yeah, same thing. 

Line	Time	Speaker	Transcript
321		Romina	[Romina moves two of the opposite pair towers closer to each other]. Husband and wife. [Then she knocks down several of the standing towers]
322		Brian	Strike.
323		Romina	Whoops.
324		Romina	[She stands up the fallen towers again] Um. I can't see any more. I can't. What about two blues -
325	00:26:48:25	Brian	Two blues, two whites – Two blues, two whites, and a blue!
326		Romina	[Romina assembles the tower BWWBB and nods] Correct.
327		Brian	Please no, please no. We don't have it!
328		Romina	[Places the new tower BWWBB against each of the existing towers standing up] Oh.
329		Brian	We don't have it, we don't, we don't, we don't. We don't have it [pumps arms in the air]. Oh yeah!
330	00:27:10:00	Romina	Hallelujah!
331		Brian	[laughs] Two whites
332		Romina	Two blues
333		Brian	Two blues and a white. [Builds the tower WBBWW]
334		Romina	Here's your white.
335		Brian	I got it already.
336		Romina	Well I got it first.
337		Brian	I did all that work for nothing. [Puts the WBBWW next to the BWWBB]
338		Romina	Um [Puts chin in her hands] 
339		Brian	Maybe three blues, a white, and a blue?
340		Romina	Three blues, a white, and a blue?
341		Brian	We have that.
342		Romina	[Points] We have that already. Three whites, a blue -
343		Brian	One blue, two whites, two blues.
344		Romina	One blue?
345		Brian	One blue, two whites, and two blues.
346		Romina	[Begins to assemble a new tower] Two whites and two blues?
347	00:27:58:23	Brian	Yes. We don't have that, I don't think. We don't have that! Ow. Wait. Match them up. You gotta match it up. [Romina holds the tower up against the first pair]

Line	Time	Speaker	Transcript
			
348		Romina	Go strolling again. [Holds the tower BBWWB against each of the other existing pairs towers]. Wait a minute, isn't this? No. Opposite?
349		Brian	Yes, we don't have it! We don't [Pumps hands in the air].
350		Romina	[Brian knocks down some towers and then picks them back up] Whoops.
351		Brian	Strike. Another strike. Going for a stair. Where's the one we just did?
352		Romina	Um, we need
353	00:28:33:29	Brian	Oh yes! We have twenty-five.
354		Dr. A	You have twenty-five?
355		Brian	Yeah, two, four, six, eight, ten, twelve, fourteen, sixteen, eighteen, twenty, … twenty-four.
356		Romina	You can't have twenty-five. Twenty-four. *<audible on "Work View" only>
357		Dr. A	Why can't you have twenty-five? *<audible on "Work View" only>
358		Brian	Cause there's even numbers. *<audible on "Work View" only>
359		Romina	Yeah. **<audible on "Work View" only>
360		Dr. A	Oh.
361		Brian	Whoa how about two blues, three whites -
362		Romina	Two blues, three whites?
363		Brian	We have it. Right there. [The bell rings] It's time to go already.
364		Dr. A	I think you have another fifteen minutes.
365		Brian	Yeah!
366		Dr. A.	Cause you don't think you have them all yet?
367		Brian	Oh no, twelve forty-six.
368	00:29:19:15	Dr. A	I'll come back in a minute and see if you found another. Okay?
369		Romina	Let's experiment. One blue. [Holds up one blue cube and then reaches for the white cubes]
370		Brian	How bout three blues, one white and one blue?
371		Romina	Three blues, one white.
372		Brian	You have that? No, we have that, right here. [He points].
373	00:29:52:06	Romina	Let's try something with three.
374		Brian	This is never gonna – two blues, a white, a blue, and a white. Yeah! Two blues, a white -
375		Romina	How about three and two? [Holds up BBWWW]
376		Brian	No.
377		Romina	Yeah, we found it. What did you say? Brian – what did you say last time?
378		Brian	Three blues, a white and a white. Do we have that?
379		Romina	That's what I just said. A white, and a white

Line	Time	Speaker	Transcript
380		Brian	No, we had that. Ah! No, we don't. [Romina holds the new tower up against an existing one]
381		Romina	No, unless you want two girls and two boys. That would be odd.
382		Brian	Do we have that? Oh, yes we do.
383	00:30:48:02	Romina	What?
384		Brian	Do we have a white, two blues, and two whites?
385		Romina	A white?
386		Brian	Oh, yes we do. We have it right there.
387		Brian	Do we have a blue, a white, two blues, and a white?
388		Romina	A blue? A blue, white
389		Brian	A blue, a white, two blues, and a white. [Builds tower] Do we have it, no! We have another lift-off.
390		Romina	Will you wait on. [Picks up tower and holds it against the existing towers on the desk]. Let's check.
391		Brian	[Leans down and watches Romina]. We have lift-off. We have ignition. [A couple towers fall over]
392		Romina	We got a strike.
393		Brian	[Laughs] No, where are the pairs? [Picks up the towers that fell] Hey, we're missing a pair, dude.
394	00:31:39:09	Romina	We're not missing a pair.
395		Brian	Okay, I got the one white.
396		Romina	Whoa, whoa. Dun. Dun. [Puts the new tower against each existing pair]

Line	Time	Speaker	Transcript
397		Brian	White. Where's that
398		Romina	Hey, I don't think we have this. What do you need a white?
399		Brian	A white, a blue, two whites, - two whites. I got the whites over here.
400		Romina	White.
401		Brian	Two whites and a blue. I need a blue, I need a blue! I have it. Gimme it. Twenty-six. How many groups do we have? Twenty-six.
402		Romina	One, two, three, four
403		Brian	One, two, wait. Three, six -
404		Romina	Five, six, seven, eight, nine, ten, eleven, twelve
405		Brian	Twenty-six.
406	00:32:19:28	Romina	Let's put em in rows, that way - Yeah, we do. [Romina moves some of the pairs of towers and some fall down].
407		Brian	Would you quit getting strikes?

Line	Time	Speaker	Transcript
408		Romina	[Sighs] 
409		Brian	Do we have a white, a blue, a white, and two blues?
410		Romina	A white, a blue – what did you say?
411		Brian	Oh yeah, right there.
412		T/R1	How many think there are more than twenty?
413		Students	Me. I know.
414	00:32:48:15	T/R1	Raise your hand if you think there are more than twenty. [Romina and Brian raise their hands]
415		T/R1	How many of you think there are more than thirty? [Romina and Brian put down their hands]. Anybody find more than thirty? How many did you find? [T/R1 speaks to students off camera] You found forty? You have forty there? And they're all different? No two are alike? Absolutely sure of that?
416		Brian	[Smiles] Oh!
417		T/R1	How many did you find?
418		Romina	You have to be kidding.
419		T/R1	Forty? Show us those. You have thirty-nine and they're all different?
420		Romina	I'm going to start thinking.
421	00:33:24:07	T/R1	Anybody think there are more than -
422		Brian	Fifty.
423		T/R1	Fifty-five?
424		Romina	We have that.
425		Brian	No we don't. Yes we do, right there.
426		T/R1	Well I'm interested in knowing if you know how many exactly
427		Romina	[Hands Brian a tower] Check if we have this.
428		Brian	Two blues,
429		T/R1	How many exactly you should find or is that something you don't know. How many different towers are there?
430		Brian	We don't! [Romina takes tower out of his hand and checks] We have it. Ha, ha, ha. [Romina puts it down next to a duplicate tower and leans back]. It ain't it – that ain't the same. [Romina holds the two duplicates up and waves them] Oh.
431		T/R1	And Alex estimates forty-nine. Are there any other estimates?
432		Brian	White, blue, white, white, white.
433		Romina	[Makes the tower and hands it to Brian] We have this.
434		Brian	Wait – we don't have it! White, blue, white
435	00:34:54:07	T/R1	Okay, I'm seeing some duplicates
436		Brian	Oh we better not have this.
437		Romina	[Takes the tower from Brian: WWWBW] Wait, let me see.
438		Brian	White, blue, white, white, white. We have it right over here.

Line	Time	Speaker	Transcript
439		Brian	How about a blue, three whites, and a blue?
440		Romina	What? A blue
441		Brian	We don't have it!
442		Romina	I need another white.
443		Brian	A blue, three whites, and a blue.
444		Romina	We have this. I can't -
445	00:35:39:26	Brian	Three whites in the middle, three whites in the middle! [Romina begins to pass the new tower through the pairs of existing towers] Ah ha – we have it!
446		T/R1	Okay, I think you might want some more time to check what you found. And you might want some more time to find some more. So we're going to save them and would you like to finish this tomorrow?
447		Brian	Yeah.
448		T/R1	And then maybe share with each other what you found and maybe think about how many there are?
449		Brian	[Nods] There's gotta be one more.
450		Romina	There's gotta be one more. No, if we find one, there's got to be two.
451	00:36:20:27	Brian	Blue, white, white, blue, white?
452		Grad student	Do you want to arrange them - the cubes? Set them down in a strip and I'm gonna put some tape across then I want you to put your name on a sheet of paper.
453		Brian	Get them in a row.
454		Grad student	That's it. It doesn't have to be one row. You can have two rows. Whatever is good for you.
455		Romina	How bout we put them in two rows?
456		Grad student	Whatever you'd like.
457		Romina	How much do we have?
458		Brian	We have twenty-eight.
459		Romina	… fifteen, sixteen, seventeen, eighteen, nineteen, twenty, twenty-one, twenty-two, twenty-three, twenty-four, twenty-five, twenty-six.
460		Brian	Twenty-six then.
461	00:37:05:05	Romina	How can we separate them?
462		Brian	What do you mean?
463		Romina	How can we put them in two equal groups?
464		Gradstudent	Write your name on here and the number you have. [Hands Brian a paper and pen]
465		Brian	What number?
466		Romina	We have twenty- One, two, three, four, five, six, seven, eight, nine, ten, eleven twelve… twenty-two, twenty-three, twenty-four, twenty-five, Yeah, we do have twenty-six.
467		Brian	[Hands Romina a paper and pen] Write your name and how many we have.

Line	Time	Speaker	Transcript
			
468		Romina	Do we write the last name?
469		Brian	Brian's the first. Hey
470		Romina	How do we separate twenty-six into two equal
471		Brian	Thirteen.
472		Romina	Thirteen!
473		Brian	Two, four, six, eight, ten, twelve, One more.
474		Romina	One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen.
475	00:38:13:29	Brian	The last one in this row is the first one in this row. There.
476		Romina	Put them like this.
477		T/R1	Okay, tomorrow you can finish.
478		Grad student	Do you want to put this right on top? [Takes paper from Romina and puts it over the two rows of towers. She tapes them down] Excuse me, Brian. Grab that one, would you? Whoa. I'm no good at this. [The graduate student drops the tower row the students made]
479		Brian	You're on Candid Camera! Smile, smile!
480		Romina	Put your head down. Put your head here.
481		Brian	A blackout. Smile, smile – you're on Candid Camera!
482	00:40:28:10	Romina	You watch that? Ow, my feet are hurting. I don't think they got forty.

APPENDIX B: TRANSCRIPT – GUESS MY RULE

October 1, 1993 (6th Grade)

Camera View: Class RC
Date of filming: 10/1/93
Harding public school, Kenilworth NJ, Guess my Rule (GMR) problem
Transcribed by: Poroshat Shakoor
Date of transcription: 7/25/2006
Verified by: Dina Honigwachs
Date of verification: 7/2007
Format revised by: Patricia Giordano
Date of revision: 8/2007
Verification revised by: Maria Steffero
Date of revision: 11/2008

Time	Speaker	Transcription
06:14	RBD	Okay I guess we're ready to start. Hi.
	Student	Hello.
	RBD	Um, I wanted to talk a little bit some more about that question of secrets because I think that's an interesting part of what we do. Um, what sort of things do scientists do?
	Jeff	Discover, invent.
	Romina	Explore
	RBD	Yeah, that's right. Um, are there any problems you hope scientists will solve in the next few years?
	Jeff	Cure certain diseases.
	AmyLynn	AIDS.
	RBD	Yeah, that would be on the top of my list. Yeah I think that's right.
	Brian	Make solar powered cars.
	Student 2	Yeah, like solar powered cars.
	RBD	Now, now there's a sense the sort of a two sided ah thing about ah secrets. When people, obviously since you don't know, people don't know how to deal with certain kinds of cancer and things like that, there are secrets, it's not because somebody is keeping the secret, it is because nobody knows, right, and people are trying to find out what it is. Um, now, I think it's clear that the people who make some of these discoveries are very proud of it and they like their name attached to it. Do you know the names of any scientists?
	Jeff	Do I think the people now or famous people?
	RBD	Famous people.
	Student	Or people dead.
	Jeff	Thomas Edison

Student	Alex Graham Bell.
RBD	Alexander Graham Bell. What did he do?
Student	He invented the phone.
RBD	Yeah.
Student	Thomas Edison.
Student	Einstein, I guess
RBD	Einstein would probably be the first thing that would occur to me
Student	He's a scientist
RBD	Okay…okay. So you have some idea of that and people really are very proud the theory of relativity, which is the thing Einstein, thought about and worked out. Yeah, I'm sure he's very proud that it's attached to his name. And so there's a sense in which people um… clearly there are secrets because nobody knows what to do until somebody figures it out or finds it. Uh, and then when they do they like to get credit for it. So, I mean that's the sense in which okay there are secrets… but do scientists also share this information?
Jeff	Sometimes.
RBD	Yeah, I think that Jeff got it probably exactly right, sometimes. They have to, they have to actually because in the long-run…uh in the long-run no single person could do it all by themselves, so they have to share. Matt.
Matt	I was wondering what does the mean anyway, e equals m c squared?
RBD	Well that's interesting…
Matt	How do you use that?
Student	Energy equals…
RBD	Yeah that's right e stands for energy do you know what the m stands for?
Student	m c squared.
RBD	What does the m stand for? E equals m c squared.
Bobby	Mechanical.
Student	We could look it up.
RBD	Well, we could talk about that, but probably not today. Why don't you see if you can find out and we can talk about that, but I think maybe I propose not to do that today?
Student	Maybe a dictionary.
RBD	So the main thing is uh we need we need to do both of these things. I think that you do want to try to find secrets occasionally because that's, that's sort of fun and that's what you do in science and, and in mathematics, but we also want to share them too, and so we have to work out which we're doing. Maybe the first time you find a secret you keep it a secret for little bit so other people can think about it too and see if they can find it. And then at some point, probably, we want to share it. Um, okay, now there was really a sort of neat thing that happened last time… oh what were we

		working on? Remember, we were doing equations that were box times box minus something times box plus something equals zero. What were we taught to do? Do you remember what you were trying to find some numbers what did those numbers do?
	Michelle I	The numbers replaced like the empty boxes or triangles.
	RBD	And they made a true statement didn't they… when you did it… said it was equal to zero and that was true. Okay, and we did quite a few of those and you got to be quite good at that I think. And various people found the secret and I guess by now everybody knows what it and we didn't quite agree whether it's was one or two secrets, most people say it's two, but I think some people here like you persuaded us it's one. Um, what's the secret to that?
	Milin	It's one big secret.
	RBD	It's one big secret? Matt.
	Matt	That the… the two multiple… the two num the numbers have to like when you add them up it has to equal… it has to equal the number to the, to the left and be multiples of the number to the right.
	RBD	Well you might not really mean multiples, when you multiply them…
	Matt	Yeah, be able to multiply them…
	RBD	Yeah, yeah right when you multiply them they give you the number on the right and that's certainly right. Okay and I think that everybody was good at that. And then we started working on, well maybe before I leave that… uh those two equations on the bottom came up because uh, uh, Milin actually proposed one of them and then somebody proposed the other one. Jeff, what was special about them.
	Jeff	Cause, there were two prime numbers in it so it was like impossible…
	Student	No.
	Jeff	…or you had to go into decimals or whatever.
	RBD	OK, we left that hanging a little bit and I think I'm going to leave it hanging again today, but it's a very interesting problem and it certainly looks like it might be impossible doesn't it? And we might have to use some other kinds of numbers or something. OK... um now then we started working on the sort of thing that's on the top up there. Um, we started with that equation box times two plus one equals triangle. Right, and what did we do then, Stephanie what did we do?
	Stephanie	Well, we had to put a number in the box and a number in the triangle so that the equation was true.
	RBD	Exactly what we were doing, and when we did that if we put zero in that box what number did we put in the triangle?
+5:00	Stephanie	One.

RBD	One. And we made that table there, right. Okay, and now then we, in fact actually um Michelle where…yeah um I'm sorry … Michelle R. Uh, you remember what you wrote on your paper.
Michelle R	No.
RBD	You want to take it and maybe write it here so that everybody can see it. Here, just stand there. Well a couple of them anyhow. [Michele goes to write on the board]
RBD	Well, you suppose you can get it if you wrote small do you suppose you could get it up by the table the way you did it on your paper?
Michelle R	Up here?
RBD	Yeah, cause that was sort of neat the way you did that.
RBD	[Michelle R writes on board ($\square \times 2 + 1 = \Delta$] And you left out one parenthesis; do you see where you left it out?
Michelle R	Oh. [Michelle R closes the parenthesis $(\square \times 2) + 1 = \Delta$ and places a zero in the box and one in the triangle.] Should I do more?
RBD	Well that's probably enough, but she went down and did that, and you agree that that's what we were doing?
Student	Yeah.
RBD	Now, what did we do then? We, then we turned the problem around and did something different. Michael what'd we do then? Michelle?
Michelle I.	We tried to find a secret to it with a pattern like how the numbers…
RBD	Okay, and some of you did find a very interesting secret and it might be an appropriate one to share, um no, Ankur says that we shouldn't do that.
Jeff	Yes we should.
RBD	Well, okay, well we won't we won't do it just now we will sooner or later. We will sooner or later okay, uh, but we started, we started turning the problem around didn't we and for the other problems I gave you the table. Here, here I gave you the equation and we made the table, right, but now in the other problems, I gave you the table and what are you supposed to do?
Romina	Find the equation.
RBD	Yeah, find the equation. Uh, and now for the second problem, let me pass this back to you. This is Stephanie's, uh the other ones, uh Michelle's, that's Ankur's, that's Amy-Lynn's, that's yours, that's yours, and who's is this?
Student	That's mine.
RBD	That's oh yeah, okay good. Ah, some people didn't get one.
Milin	Whose is that?
Brian	I got a really neat one here.
RBD	That's his. That's his. Well okay, let me, uh, let's see what we can do here. [Children talking.]

	RBD	Okay, who didn't get one back? Anybody didn't get one back? Okay, that's that. I'm sorry…who's is that?
	Brian	I have no idea, this ain't the one we're doing right now
	RBD	Yeah, but it's the one we're about to do isn't it?
	Brian	No, that's the one we did before.
	RBD	Oh I see okay, okay.
	Milin	Whose is this?
	Jeff	I need the one you have there.
	Brian	This thing?
	Michael	I need the one that you have in your folder.
	RBD	Well we can pass out some new ones. Who doesn't have one now? Jeff you don't have one.
	Milin	Whose is this?
	RBD	And, okay, Michael you don't have one. Okay, why don't you put your names on them right away so you make sure they get back to the right people? Ah, why don't you talk to your neighbors and see what you can do with problem two. We know about problem one. So, problem two, you've got the table and you're trying to find the equation, just what Romina told us.
	Ankur	We're finished
10:00	RBD	You're finished. All of them or just number two?
	Ankur	Just number two
	RBD	Number two. OK, come show me would you?
	Ankur	[Ankur's paper 2.

□	△
0	5
1	7
2	9
3	11
4	13

$(0 * 2) + 5 = \Delta$]

	Brian	Zero times two plus one.
	Romina	What?
	Brian	We have to get the equation.
	Romina	Zero times two plus one?
	Brian	Yeah, like her. Look at that.
	Romina	Yeah but she used next number, and the next number is five.
	Brian	I know but the next number shouldn't have been five. She used zero and one
	Romina	But zero and one so zero times two plus one
	Romina	Don't we have to use the zero and the five? Aren't we supposed to go down to number two?
	Brian	Oh yeah.
	Romina	Yeah duh.

Brian	Der. Zero times one plus four.
Romina	Whatever we are getting zero times something.
Brian	Zero times one plus four equals five. Oh yeah.
Romina	Can you run that past me?
Brian	We have to use that in the square, that in the five and that in that little thing.
Romina	Triangle
Brian	OK. Zero. We got the "x" in the zero, the square in the zero.
RBD	If anybody needs more paper here is some paper.
Brian	Zero times one
Romina	That's zero
Brian	Oh wait, wait, wait, wait, OK, yeah ok. Zero times four plus
Romina	plus five, equals five
Brian	Equals five
Romina	Would it work with the other problems? One times seven; one times one, one plus
Brian	seven
Romina	seven would equal eight.
Brian	Oh wait wait, der, one times one plus six.
Romina	One times one times six?
Brian	One times one plus six
Romina	Doesn't it have to be all the same equation?
Ankur	Ankur's paper: 3. $(0 \times 3) + 1 = \Delta$
Brian	Yeah! Let's figure. These go up by two every time and these just go up by one.
Romina	Yeah
Brian	I can do this.
Brian	Ohh!
Romina	What?
Brian	Did you get it? [To Bobby]
Romina	Yeah he got it.
Brian	Bobby's got the big, giant, enormous, egghead brain.
Romina	[laughs] No you, duh, you just don't use it.
Brian	Do we have to have the same plus there too? The same plus number?
Romina	[to Bobby] Is it all the same equation?
Bobby	What do you mean by that?
Romina	What do you mean what do I mean by that?
Brian	See this? This has got to be the same for every one except you change the numbers in there?
Romina	[Turns and holds up her paper to Bobby and Amy-Lynn] Okay, look, you guys, did you only change the square and the triangle or did you change the whole entire equation?
Bobby	No

	Romina	What did you do?
	Bobby	We changed numbers.
	Romina	The whole equation or just the ones in the square and triangle?
	Bobby	Just the ones in the square and triangle.
	Romina	And you got them? [Romina looks at Bobby's paper]
	AmyLynn	The one in the square.
	Bobby	Yeah, the one in the square. See we changed it. [Bobby points to his paper and AmyLynn holds up her paper to show Romina]
	Brian	Get over here. Are they giving you the answer?
	Romina	Here's what they got. [Looks over again at Bobby's paper]
	Bobby	Hey!
	Brian	It's not going to work.
	Romina	Times two plus five.
	Brian	I had the plus right! [Romina laughs]
	RBD	How many people have got problem two done?
	Bobby	We got three.
	RBD	You got three?
	Brian	Bobby's head's so small but it's all filled with brains and it's coming out of his ears and stuff.
	RBD	Okay, I need somebody to come and do two. Amy Lynn would you do three? You've got three would you do three. I need somebody to do two. Ok Michelle would you do two? Show everybody how you did two?
21:00	Romina	Oh, duh. Zero times two is in between so this one have three in it.
	Brian	Oh duh.
	Romina	(together with Brian) duh.
	Romina	Three, yeah. Oh now I figure it out.
	RBD	Ok we need to talk about that. Is it okay to give away secrets or is it too early to do that?
	Romina	Too early.
	Brian	Ankur, Ankur. She got it. I can't believe it.
	Romina	[Laughs] Oh, thanks.
	Brian	Let's see if it works for every one.
	Romina	It does. Bobby has it.
	Brian	Zero times two plus five equals.
	Romina	Hold on we have to write this down.
	Brian	Does it? It doesn't. Oh it does. One times two plus seven, it does. Oh my God it does. It does.
	Romina	No, duh, that's why I'm writing it down.
	Brian	Zero times two.
	Romina	Don't say it out loud.
	Brian	Do we have to write the numbers in there too?
	Romina	I'm not writing them.
	Brian	[looks up across the table] We got some. Oh my, I can't believe we got that! [silently writes on paper]

Brian	[addresses another student] No, do you know the secret? It's easy. [gestures to Romina] Romina got it.
Romina	Oh, that makes me feel real good. Oh come on.
Brian	There's stuff on my nose.
Bobby	[points to Romina's paper] You can't change this. It has to stay the same.
Romina	[turns to Bobby] What?
Bobby	[points to Romina's work on problem 2] This has to stay the same.
Romina	Oh, der, I messed up. [Romina and Bobby laugh]
Brian	What'd you do? [Looks over at Romina's paper as she begins to erase her previous work] Der. [Brian laughs]
Romina	Hey, I was just making up numbers. I didn't care.
Brian	We got it. I can't believe that. [Romina laughs] And Jeff's just gonna squeeze his brains out of his head – ahh!!
Romina	[Answers question from across the table] Yeah, cause you have Michelle in your group.
Jeff	She didn't say anything yet.
Michelle	Oh, thanks a lot. That makes me feel good.
Brian	[looks up] Why's it gonna make you feel bad?
Stephanie	Because she goes the only reason you got anything is because Michelle's in your group.
Romina	Well, it's true. [laughs] Mm, I got the answer. [Sticks out her tongue and makes a face].
Bobby	Cause you copied off of us.
AmyLynn	Yeah, you copied off of us.
Romina	Yeah, but first time I copied off of you guys I didn't even get it.
Brian	Ok now we gotta do number three. Let's go. Oh, okay.
RBD	Could I get some idea how we're coming along here?
Brian	Very good
Romina	[not looking up] Very
RBD	What's the - How many problems have you people done?
Brian	We can just fly through this. We just have to write it down.
Romina	We just have to write it down.
RBD	And Bobby and Amy Lynn you are what number? Four?
AmyLynn	four
RBD	Number four. Ankur and Michelle you are on what number?
Ankur	Six. We found the secret.
RBD	You found the secret? You want to be careful. There might be more than one secret.
Romina	Hold on what did you write? It's not two this time, it's one this time isn't it?
Brian	It doesn't work.
Romina	Oh, no. [Clenches her fists and puts them against her head]. Zero times three, zero how come it doesn't work?

Brian	I know. [Romina takes Brian's pen, erases something on his table, and begins writing on his paper]. It still doesn't work - it's still zero plus
Romina	Zero times three is zero, plus one is one
Brian	Oh, okay one, okay.
Romina	I don't know where you got this one… It works!
Brian	Okay, okay. [Both Romina and Brian begin writing again. Romina finishes her table for number three and turns the page.]
Brian	[Still writing] Yo, wait up. What is that four? Okay, two times three is
Romina	[Leans on her elbow and hums] Dum, dum, dum, dum. You're so slow.
Brian	Well, I'm sorry, I work like… times three plus one equals seven.
Brian	Okay. [Brian turns his page to the next problem]
Romina	This one's ten.
Brian	We're on four. Ten?
Bobby	It's the first number, it's the plus number.
Romina	[Turns toward Bobby and AmyLynn] Der, you didn't know that?
Bobby	No, we heard you guys.
AmyLynn	No, I got it myself.
Romina	No, I got that one by myself.
Brian	Oh, I know how, I know what the multiple is [turns to the previous page]
Romina	How? [Romina leans over and points to his paper] Der, all you have to do, Brian, is take the first number and add it
Brian	Okay, okay
Romina	That's what I told you in the beginning, but no
Brian	This is zero. I'm just writing this out. This is blank times four right? [Looks over at Romina's paper] No, times seven.
Romina	[Laughs and points at Brian's table for number four] Whatever is between seven and seventeen.
Brian	Ten - Oh, okay
Romina	Yeah, yeah, yeah, whatever.
Brian	Now this is plus
Romina	Hold on, plus what?
Brian	Seven
Romina	Oh yeah
Brian	I'm just writing this out. Like this just putting numbers.
Michelle?	Ankur, we have two secrets.
Ankur	We have totally different secrets. We've got a secret.
Romina	[Looks up writing and looks around] We've had a secret.
Jeff	Be quiet, Matt. I want to hear theirs.
Brian	[Looks up from writing. Pumps his hands out]. So do we! [Looks back down at paper and writes again. Then looks up and addresses a student off camera] You've got the secret? Oh my god! What is it?

	Ankur	We got the secret. We finally got the secret!
	Brian	Der, we got it. Look. [Turns to previous page]
	Ankur	There's more than one secret.
	Romina	[Looks up] We've had the secret. Look at Steph and the little microphone.
	Ankur	How many secrets are there?
	Brian	[Mimics talking into a little microphone] Hello! [Romina laughs] You wrote 'em all out?
	Romina	No I'm still working on it. I'm on number two. Oops. [Erases]
	RBD	I wanted some people to have a chance to talk to the camera so they can have a chance to tell the camera…
	Brian	Talking into this little cheap microphone.
	Jeff	She's all ready to give an answer. [Romina looks in the direction of Jeff and then looks down again]
	Brian	This is easy
	Romina	I know.
	Bobby	I got it. Another secret.
	Brian	[without looking up] Good for you, Rob.
	Bobby	It's the first number in the answer.
	Brian	Okay, der, this is way easier.
	Romina	I know, we could have done this in the beginning and we would have been done already.
	Brian	Twenty-seven, thirty-seven, forty-seven.
	Romina	Hey, wait, we're doing the [looks over at Brian's paper] This time you beat me.
	Brian	[Brian puts down pen and sticks out his tongue at Romina] Okay, this goes up by…
	Romina	Is that a minus two? [Indicating the first entry in the triangle column for PROBLEM #5]
	Jeff	No, it's a plus two
	Brian	Yeah, It's minus two so it goes up by ten.
	Romina	Ten again. Okay, negative two. Brian?
	Bobby?	Did you get this one?
	Romina	Hey! You guys can't look at ours.
27:00	Brian	Yeah.
	Stephanie	[Stephanie's paper:

2.

□	△	
0	5	$\times 2 + 5 = 5$
1	7	$\times 2 + 5 = 7$
2	9	$\times 2 + 5 = 9$
3	11	$\times 2 + 5 = 11$
4	13	$\times 2 + 5 = 13$

3. $\times 3 + 1 = 1$

	$\times 3 + 1 = 4$ $\times 3 + 1 = 7$]
Romina	Let's see you get it. Yeah, unfortunately. Brian, Brian? Zero times ten minus two - it does not equal, no it does equal ok.
Bobby	Oh, I get it now. This is a lot easier way.
Romina	Yeah, I know. When you know the answer it's like…
Brian	Boom, boom, boom, boom – you get done with the answer.
Bobby	Oh, I've got this now. You multiply that and minus two instead of plusing.
Romina	Oh, get out of here.
Brian	What are you talking about?
Romina	Don't, Bobby is in his own little world right now.
Brian	What am I doing?
Romina	Everything is negative two? Oh yeah.
Bobby	Everything is minus two
Romina	Minus two – oh, it's the same thing!
Brian	We got it too!
??	She's telling it to the camera
Michelle I	[Michelle I's paper: 2. $(0 \times 2) + 5 = 5$ 3. $(0 \times 3) + 1 = 1$]
Brian	Our one goes with everything. Duh.
Romina	Der. This is easy once you get the hang of it. Oh, you're so slow.
Brian	You're done?
Romina	Yeah, I'm done.
Brian	Okay – what does this go by? [Begin PROBLEM#6]
29:00Romina	I get it. You multiply. One times one, two, two plus [Makes crying sound] Oh! Why do they do this to us?
Amy Lynn	[AmyLynn's paper

	□	△	
2.	0	5	$(\square \times 2) + 5 = 5$
	1	7	$(\square \times 2) + 5 = 5$
	2	9	$(\square \times 2) + 5 = 9$
	3	11	$(\square \times 2) + 5 = 11$
	4	13	$(\square \times 2) + 5 = 13$

]

	□	△	
3.	0	1	$(0 \times 3) + 1 = 1$
	1	4	$(1 \times 3) + 1 = 4$
	2	7	$(2 \times 3) + 1 = 7$

3	10

$(3 \times 3) + 1 = 10$

	Brian	Wait. One times two
	Romina	Wait. One times one equals one.
	Brian	Oh, oh, oh, oh! See, five times two is ten. Ten, seven. Wait.
	Romina	Wait – what's between each one?
	Brian	Wait, one,
	Romina	Three
	Brian	Five
	Romina	Five, seven, nine, it goes up by two. No.
	Romina	Yeah, what's between it, it goes up by two.
	Romina	This is not fair.
	Brian	Zero. Oh! Oh, oh, oh.
	Romina	Oh, oh, oh, what?
	Brian	Where do you put the plus - You don't have the same number there. Oh, you can't do that. [Makes crying sound]. Oh!
	Romina	[Laughs] It's not fair.
	Brian	Let me try something out.
	Romina	Guys (calling another group) - did you get number six?
	Brian	Did you get number six?
	??	We got number six.
	Romina	Oh, I hate them.
	RBD	Do you have a secret you want to say to the camera?
	Romina	We know it, Bri.
	Brian	Ahhh!
	Romina	We know the secret but we're stuck on six.
	RBD	A lot of people are saying they know the secret but they're stuck on a certain problem that's giving them difficulties. Has anybody found six?
	Brian	We know what it does, we just can't put it into the thing.
	Romina	Yeah, we know-
	Brian	It keeps on going up by two.
	Romina	No, what's in between goes
	Brian	Yeah, one, three, five,
	Romina	No he means like this doesn't go up by two, but what's in between this goes up by two.
31:00	RBD	Can you come up? Let's erase this and come and show it to me.
	Romina	Come on Brian.
	Brian	Yeah!
	Romina	Will you come with me?
	Brian	Okay. (Romina gets up to go to RBD, Brian follows.) You can definitely say it.
	Romina	[Romina's work:

6.

□	△
0	1

>1 ______________

1	2	>3
2	5	>5
3	10	>7
4	17	>9
5	26	

RBD	Did anybody figure out the equation for 6?
Ankur	Brian, did you guys?
Brian	No.
Ankur	We got it, we got it.
Romina	I hate them.
Brian	I'm gonna eavesdrop.
Romina	[She laughs] That wouldn't surprise me, Brian.
[Ankur and Michelle's work for number 6	
Michelle	(0 * 0) + 1 = 1 (1 * 1) + 1 = 2 (2 * 2) + 1 = 5]
Romina	Brian, maybe if we – maybe if we…
Brian	It's got to do something with that one, two, five, seven, nine thing.
Romina	Well, when'd you figure that out?
Brian	Maybe we just put plus two at the back.
Romina	Ok, here we got the problem.
Brian	Plus one?
Romina	One's the first number. Then wouldn't it be plus one? We did that with all the other ones.
Brian	One - oh, yeah.
Romina	Der.
Brian	What does that multiply?
Romina	Why are you 'oh yeah'?
Brian	It's gotta be a zero
Romina	Okay, the answer has to be one so.
Brian	It could be any multiple there, so far.
Romina	So far, yeah.
34:30 RBD	Okay, I guess I would like to do one on the board here so everybody gets the see one.
Jeff	Did you hear us?
Romina	He's trying to eavesdrop.
RBD	Okay, could we get everybody to think about one problem here, the same problem for a minute and let's do one of the first five I think that's what people felt the happiest about.
Brian	[Whispering] It's one.

RBD Who is going to come and explain one? Who has not had the chance to talk to the camera? Mike, why don't you come and explain one. It's your choice, one, two or three or four or five.
Romina That's what
Brian Look - Two times zero plus one is one. It's these numbers. See look. Two times one plus one is three. Two times two four plus one is five. Three times two is six plus one is seven. Yeah seven.
Romina [Points to her paper] But, what - But these are the answers you're supposed to be getting right now.
Brian But it goes, look at it.
Romina But did you get five for this one?
Brian Four times two plus one is what's that – nine. Nine.
36:32 Romina Yeah, but those aren't the answers, Brian. Those aren't the ones we're supposed to be getting right now.
Brian But they are are impossible. They are the ones we can use.
Romina Why do you think –
Brian It goes.
Romina Yeah, but - but, one, three, five, seven, and nine are not the ones we're supposed to get in the triangle. What is that?
Brian Look, this would be eleven now.
Romina So?
Brian This goes here. That goes there. That goes there. That goes there. It works.
Romina But these are the numbers which are supposed to be in the triangle.
RBD Did everybody hear what Mike was saying?
Brian Look. Told you.
RBD He says we start with box times two and I take it that everybody agrees with that isn't that right? You saw where he got the two.
Brian Boo-yah.
Romina Each one's gonna have to be a different number.
RBD So now you want to say where you get the five from? He is doing a sort of complicated thing here but I'm wondering if there is something easier.
Brian Look, this is exactly what it is, two times five plus one is eleven.
Romina Okay, and where is eleven?
Brian Down there, eleven by the twenty-six and it's going to keep on going up by two.
Romina Yeah, but aren't we supposed to get these numbers in the triangle, not these?
Brian It doesn't say it.
Romina [laughs] No, but
RBD Is everybody happy with that? Do you all know that? You know that's a very important set of ideas in mathematics.
38:00 Romina [To RBD] I think we've got something for six, but we're not totally sure
RBD You think you've got something for six?

Ankur	[Michelle and Ankur wave their hands] We found out how to write it!
Michelle	We figured out how to write it!
RBD	Ok, show me how to write it. Come and show people. (Ankur and Michelle I go up to the camera) Well wait, maybe we don't want to do that.
Michelle	Can we just show you?
RBD	Just show the camera.
Brian	Just show everybody.
Brian	Ankur, Ankur can I show you mine and see if it's right?
Romina	No, first let's show…
Bobby?	Do you guys know how to do number six?
Brian	We're done with number six! She doesn't believe me though! The way I have it works though. I'm putting it down. I don't care.
Romina	Did you hear what [] said? She's trying to answer everything but she's making Ankur do it because she really doesn't know it. She wants to do that because yesterday she didn't know it. [Brian and Romina laugh]
Ankur Michelle I.	[Ankur and Michelle wrote "The numbers in the bracket are always the same. The number after the bracket is always one"]
Brian	Yo, Matt, Matt, Matt. Did you get six?
Romina	We got it.
Brian	Don't tell them. Wait, I just have to see if it works.
Romina	Come on. This time you're talking.
Brian	No, this time you are.
Romina	You are.

[Brian's work

	□	△		
6.	0	1	>1	(0*2) +
	1	2	>3	(1*2) +
	2	5	>5	(2*2) +
	3	10		(3*2) +
	4	17	>7	(4*2)
	5	26	>9	(5*2)

RBD	May I get everybody's attention to what Michael did I'm not sure if everybody caught it but it's a very good idea. He has written this formula using the box and triangle. Some of you have done very clever things but you're writing them in words. The question is can you go and do it with the box and triangle.
student	[A student's work (□ * 4) + 1 = Δ
Michelle I	(3 * 3) + 1 = 10
or	(0 * __) + __ = 1

	Ankur ?	(2 * 2) + 1 = 5]
	Romina	Doesn't the equation have to stay the same? Only the box and triangle change? Told ya! Ask him. [Points to Dr. Davis]
	Brian	No.
	Romina	I forget his name.
	RBD	So I guess the hard problem that people are working on is number six
	Bobby	We finished six. We got six.
	RBD	You got six? Did you show it to the camera yet?
	Bobby	No
	RBD	Why don't you come do that?
	Ankur	Did you guys get it?
	Brian	Yeah, look.
		[End of CD 1 of 2]
		[CD 2 of 2]
0:00	Amy Lynn	[Amy Lynn's paper

6.

□	△	
0	1	(0 ×1) + 1 = 1
1	2	(1 ×1) + 1 = 2
2	5	(2 ×2) + 1 = 5
3	10	(3 ×3) + 1 = 10
4	17	(4 ×4) + 1 = 17
5	26	(5 ×5) + 1 = 26

]

Romina	Yeah, switch them all, Brian.
Brian	Guys, guys, you know how to solve it? Just change all these numbers, to one three five seven…
Romina	He switched all the numbers in the triangle section.
Michelle and Ankur	[(□ ×□) + 1 = Δ]
Romina	Doesn't the equation have to stay the same though?
RBD	How many people have got number six?
Bobby	We got the secret for number six.
Romina	But they changed the equation, other than the triangle. The changed almost the whole entire equation.
Romina	[asks RBD] So the equation can change?
Bobby	As long as it develops a pattern
Brian	Can you change the numbers in the triangle? Please say yes.
Romina	Yeah the triangle yeah you can change the numbers.
Brian	We did that and it works the way I have it.
RBD	I think Mike is making a point I'd like to pursue. I think maybe it would be ok try sharing a little bit of the secret without telling everybody everything. [Michelle I raises her hand]

RBD	Michelle, without saying what you and Ankur have done, can you tell people what you said in the original? Do you remember what you said?
Jeff	Just explain the basics.
Romina	Ankur, Ankur.
RBD	They're discussing how much of the secret they're prepared to publish
RBD	OK, it would be worth listening because this is a very interesting idea and they are trying to be careful. They are still giving you a chance to invent it but they are going to tell you something to help you.
Michelle I	[Michelle I and Ankur go to board.] The number you add after the bracket is always one.
RBD	I think everybody is pretty well agreed on that.
Brian and Romina	Yeah, we got that.
Michelle I	And the number that is here is always the number that's in the box cause if you put zero here, zero times zero is zero, plus one equals one.
Brian and Romina	What code?
RBD	Can we get it quiet please because I want to hear what Michael is saying?
Michael	For number three the difference between one and four is three, the difference between [inaudible] and seven is two and the difference between seven and ten is three so that is going to be the first number but on number 6, the difference between one and two is one and [inaudible] is three so that goes in the box.
RBD	Yeah, why don't you come and write that on the white board so everybody can see what you're doing. What you just said.
Brian	We got it.
Romina	Yeah but we have to make a code or something.
Brian	The code you mean we have to write something out?
RBD	Well with boxes and triangles.
Brian	Hey tell me is that right?
Romina	It's right but you have to get the code for it
Brian	The code? Let's write a code on the back.
RBD	Let me keep track of where we stand because I'm getting a little confused.
Brian	We have to write a code? How we got it or something?
RBD	Do just the same thing you've been doing.
Romina	Once we have it. You guys come here could this be right? You just add. You just add a triangle or something.
Michelle I	No. I'll give you one hint. Give me your pen. This, the box here and then you have the line here. The only thing you have to change is the line what do you change it to?

Brian	Ok, the box. That box right there?
Stephanie	Look you guys what do you have to do with this number always you like multiply it or something?
Romina	I don't know Steph.
Michelle I	Then you have plus one equals
Stephanie	Chell, Chell, does this number always multiply itself or something?
Brian	We have that see?
Romina	Yeah we have that much Chell, but
Brian	We just have to change it?
Stephanie	Oh sure help them Chell.
Brian	Oh just put zero, one, two, three, four, five?
Jeff	[Jeff's paper

6.

□	△	
0	1	$(0 \times 0) + 1 = 1$
1	2	$(1 \times 1) + 1 = 2$
2	5	$(2 \times 2) + 1 = 5$
3	10	$(3 \times 3) + 1 = 10$
4	17	$(4 \times 4) + 1 = 17$
5	26	

]

Michelle I	Remember I said what ever number that's in the square is going to be this number [pointing to the square column]
Brian	Yes, we did that, see?
Michelle I	So what shape is going to be here? A square. Because if it's going to be the same number as
Brian	AHHH!
10:00 Romina	Ah that's cheap.
Brian	O.K.
Romina	Tell him - quietly. And this time you're explaining it Brian 'cause I'm not saying anything
Brian	[Brian M.'s paper

6.

□	△	
0	1	([0] ×0) +1 = 1 (inside a triangle)
1	2	([1] ×1) + 1 = 2(inside a triangle)
2	5	([2]×2) + 1 = 5(inside a triangle)
3	10	([3] ×3) + 1 = 10(inside a triangle)
4	17	([4] ×4) + 1 = 17(inside a triangle)
5	26	([5] ×5) + 1 = 26(inside a triangle)

□ ×____) + 1 = Δ]

		[Brian and Romina write on a paper $(\square \times \square) + 1 = \Delta$]
	Romina	Finally, right? Now we have to get to the rest of the problem. Not again.
	Brian	Five, what the … maybe even numbers are the same. One, three, five, seven. It's the same thing as it was last time. It's this.
	Romina	Yeah, I know.
	RBD	Ok. Let me say the people with the secret would like to publish it. You know when scientist really think they discovered something they do what they call publishing.
	Romina	Yeah, why not.
	Brian	Five, five, seven, eleven.
	RBD	Are you ready for them to publish this? Is that all right?
	Brian	Here, just write it out.
	RBD	Ok. Could we get it quiet please? So they say they're going to tell you what they've discovered.
	Michelle I	We have the box and then it's always one equals the triangle.
	Romina	Chell, it's times.
	Michelle I	Oh, oh well
	RBD	Why don't you erase it? Ok. Ankur says tell him a number and he'll show you how it works.
	Bobby	Eighty six
	Brian	Three
	Student	Oh I thought you said you could solve for anything.
	RBD	Oh, you could but it'll take a while
	Jeff	Just show us how you do it.
	Student	Come on, who really cares? Just tell us the answer
	Jeff	They are going to do eighty-six just to make us wait, aren't they? They are going to get it wrong.
	Michelle I	Eighty-six times eighty-six. You said eighty-six now figure that out.
	Romina	Ok. I'll do it. Eighty-six, eighty-six, six, three,
	Michelle I	If you have the number here, the number here is going to be the same number so you have to take what that would be for the code.
	RBD	This is really the key point so I would pay to listen very carefully. Now they're really going into the secret.
	Michelle	If the number here is going to be the same as the number here, then what shape would that be?
	Students	Square
	Michelle	Yes, that's it. That's the code
	Jeff	That's the code? Square times square plus one…
	Michelle	That's the code you were asked. That's the code.
	Jeff	That's the code. Square times square, plus one equals triangle.
15:00	Romina	Yeah,
	Michelle	Yeah, that's it.

RBD	Yeah, that's what you had to do
Amy Lynn	That is corny
Jeff	That isn't very corny though. If we knew what it was we just had to put it down how it was supposed to be.
RBD	Ok. Does every body understand that ok?
Jeff	Yeah it works.
RBD	Ok. Let's see if any body can do number seven.
Bobby	[Bobby raises his hand] We know seven. We have that done. We have seven.
RBD	What?
Bobby	We have seven
RBD	You've got seven already? Come say it to the camera
Bobby and Amy Lynn	[Bobby and Amy Lynn get up]
A student	It's the same code.
Bobby and Amy Lynn	[Bobby and Amy Lynn's work

7.

□	△	
0	5	(0 ×0) + 5 = 5 (inside a triangle)
1	6	(1 ×1) + 5 = 6 (inside a triangle)
2	9	(2 ×2) + 5 = 9 (inside a triangle)
3	14	(3×3) + 5 = 14 (inside a triangle)
4	21	(4 ×4) + 5 = 21 (inside a triangle)

]

Brian	We're done. Let's go to number eight.
Romina	Why is eight all the way down there?
Romina	Zero, two, four
Brian	Six, eight...
Romina	Two!
Brian	So we do the same thing we've been doing.
Romina	Yeah, it's the same thing. This is really ticking me off. Zero times
Brian	Zero
Romina	Plus zero equals zero.
Brian	One times zero plus zero, what the ___ no…
Amy Lynn and Bobby	[Amy Lynn and Bobby's paper (□ ×□) + 5 = Δ]
Brian	This is messed up. I know what it is. Yes I do, yes I do. Look, this is exactly like that. This ain't two times two
Romina	Brian, you are confusing me so much.
Brian	Look, see, two times one is two and then three times two and four times two…
Romina	Brian,
Brian	What?
Romina	Aren't we supposed to be using a code?

Brian — I hate number eight. Number eight stinks. We don't have to use a code for the whole entire thing.

Brian — OHHH! Look at this. How many times does two go into two?

Romina — Once

Brian — How many times does three go into six? Two. How many times does four go into twelve? Three. It's the same thing.

Romina — The code.

Jeff — [Jeff wrote

$0 \times 0 + 5 = 5$
$1 \times 1 + 5 = 6$
$2 \times 2 + 5 = 9$
$3 \times 3 + 5 = 14$
$4 \times 4 + 5 = 21$

$(\square \times \square) + 5 = \Delta$]

Brian — Hey look at this one, how many times does two go into two? Once. How many times does three go into six? Twice. How many times does four go into twelve? Three times. How many times does five go into twenty? Four times. How many times…

20:00 Romina — Don't talk so loud.

Brian — These stay like this but these go up.

Romina — One, and then two, and then three and four….

Brian — We got eight. See, Look. See. Look. Two goes into two once. Three goes into six twice. Four goes into twelve three times. Five goes into twenty four times. Six goes into thirty five times. And since the zero is there, it's a plus.

Michelle R — [Michelle R's paper

8.

□	△	
0	0	
1	0	$1 \times 1 - 1 = 0$
2	2	$2 \times 2 - 2 = 2$
3	6	
4	12	
5	20	
6	30	

]

A student — No, divide the number in the parenthesis by the number in the square.

Bobby — I know. Mr.…., what's his name? Oh, We got nine.

RBD — You have nine?

	Brian	We have another code.
	Romina	We have another code. And this time all four of us wrote the same.
	Ankur	Divide the triangle by the square. We've got nine.
	Romina	Ankur… they were listening the whole time.
	Brian	Divide the number in the square by the number in the triangle.
	Romina	In the triangle to the number in the square.
	Brian	In the triangle to the number in the square.
	Ankur	Now we got to work on number nine and after we've got number nine we can put our secrets into that camera.
	Romina	All of us? Who is going to talk?
	Romina	We'll take turns.
	Brian	I put square
	Romina	Oh no…
	Brian	I know it - I know what it does
	A student	What does it do?
	Brian	Doubles
	Amy Lynn	[Amy Lynn's work

9.

	$\square$	$\triangle$	
9.	0	0	(0 ×0) + 0 = 0 (inside a triangle)
	1	½	(1×0)+1/2 = 1/2 (inside a triangle)
	2	2	(2 ×0) +2 = 2 (inside a triangle)
	3	4 ½	(3 × 0)+41/2=41/2(inside a triangle)
	4	8	(4 × 0) + 8 = 8 (inside a triangle)
	5	12	(5×0) = 121/2(inside a triangle)
	6	18	(6 ×0) = 18(inside a triangle)

]

	Romina	I got to think. AHHH! This is hard. Brian, is this right?
	Brian	Ok. That goes in that four. That goes in that five. Six times eight, ahhhhh!
	Student	Look, look. This is one. That's one and a half.
	Romina	That's what I just did.
	Student	Let me see, let me see, let me see.
	Romina	I erased it now.
	Brian	Half, two and a half, five and a half?
	Romina	I messed up, I know.
	Brian	Ok, that goes one and a half, two and a half, three and a half, four and a half… you were right, you were right.
	Romina	Thank you.
	Brian	It's the code you had before this one, this one.
	Michelle I	There has to be another code for nine. As long as you change a code around it works all the time. You just got to change it.
26:00	Brian	What's number ten? It's nothing. Number ten is blank.

Ankur	[RBD is pointing to Ankur's paper but the conversation is inaudible. Ankur writes (Δ ÷□) +]
Ankur	[Ankur's paper

8.

□	△
0	0
1	0
2	2
3	6
4	12
5	20
6	30

Divide the two numbers Δ ÷□]

Michelle	Oh I got the thing for this.
Romina	What?
Michelle	You add zero, you add two, you add four, you add six, you add eight, you add ten.
Michelle	We have a pattern for number eight. It's two, four, six, eight, ten.
Brian	We're done. We just have to write it down.
Romina	Yeah, we know nine.

Michelle R [Michelle R.'s paper

	□	△	
9.	0	0	
	1	½	1 ×1 - 1/2
	2	2	2 ×2 - 2 = 2
	3	4 ½	3 ×3 - 4 ½ = 4 ½
	4	8	
	5	12	
	6	18	

]

Romina Isn't this five and a half?
Brian It keeps on going by one whole.
Michelle How did you get one and a half for that?
Brian The difference between here. One and a half between here, two and a half between here, three and a half between here, four and a half between here, five and a half between here
RBD I noticed that there are different kinds of secrets different people are making up but this kind of thing which is known as a formula - what mathematicians call a formula, lets you, if I tell you the number in the box that lets you to find what the number in the triangle is. Ok? Now some of you have some very interesting secrets. I'm not saying don't use it. But some of you use something that depends on knowing what the number in the triangle is, but you see, what we've got here doesn't Ok? It only depends on knowing the number in the box. If I tell you the number in the box then you can find the number in the triangle. So what we particularly looking for are formulas like this where you don't need to know the number in the triangle. All you need is to put in the number in the box and it will tell you what the number in the triangle is. Ok? I think we are really close being out of time here. Anybody anything you want to say about this?
Bobby Yeah, we have a secret about the whole thing.
RBD You have a secret about the whole thing?
Student That works in every problem?
Bobby Yeah. Basically
Brian We have the secret for the whole thing too.

30:00

Bobby [Bobby's paper

8.

□	△
0	0
1	0
2	2
3	6
4	12
5	20
6	30

]

Brian We'll figure it out. The first, the only code for the whole thing is so easy. Plus a zero

Brian Wait, wait, wait. Look at every single one.

Romina Guys why don't we use scrap paper.

Brian Five, five, one, one, seven, seven, minus two, minus two.

Romina I'll do the writing. Ok, so we're pretty sure that it is plus zero.

Brian It is definitely plus zero.

RBD They have an interesting thing that maybe you all thought of.

Romina We knew that.

Michelle We knew that.

Brian Three no that's four, six, eight, what about twelve? It would go twelve cause we have the…for thirteen, or six to three. We just put the three.

Romina We knew that

Jeff Box times box plus one equals triangle. Box times box minus one equals a triangle.

Brian Minus? Never mind this double zero. You never had to divide, you never had to subtract an eight. You had to divide to get the answer.

RBD How many people got this? Some people have the formula. Who else had the formula for this?

Michelle We have the formula for it. Why don't we go explain what we got?

Romina We already explained that

Michelle Not to everybody.

Brian We only explained it to the camera

35:00 Michelle I don't really know what to say, so

Brian Ok, every body goes up. [Michelle, Romina and Brian leave their seats]

RBD I'm not sure if everybody could follow that. It's a nice idea, but it's not quite this formula because they depend on using the number in the triangle where you really want something where you don't know the number in the triangle. This is a nifty idea. You could use it but it does something different.

	Brian	That's the number that will always come up. Six times three is two. Multiply three times two you get six. That's the answer. Duh!
	Students	[Students start copying another problem from the board]
	Michelle I	I can't see
	Brian	Is there a "z". Yeah there is a "z".
	RBD	Now we're certainly are not going to solve that today. But I bet you; you are going to solve it in the next few weeks.
	Brian	I know what to do. Wait, wait, wait. What was I saying?
	Michelle	You weren't saying anything. You were thinking Brian.
	Brian	Ok, look, the difference between there is four,
	Romina	Yeah
	Brian	Four times four is eight and that's the difference between there. Eight times eight is sixteen…
	Romina	Four times four is eight?
	Brian	Where?
	Romina	You said four times four is eight.
	Michelle I	Four plus four is eight not four times four.
	Brian	Yeah four plus there we go
	Romina	Yeah there we go
	Brian	Four plus four is eight and eight goes there. Eight plus eight is sixteen, sixteen plus fifteen is thirty one, sixteen plus sixteen goes over there… I know how to do it. [Brian talks to Ankur] you use the eight here, plus it and you get sixteen
	Ankur	Where did you get the eight from?
	Brian	Four plus four. And then you do eight plus eight is sixteen and then you add it to thirty one.
40:00	Michelle I	Could you explain that to us?
	Romina	Remember us over here?
	RBD	Brian would you like to explain that for the camera?
	Brian	We got to make sure it works, it does, it works that's sixty three
	Michelle	What about us?
	Romina	Yeah, you know Oh boy
	Brian	[To Ankur] See, there is one difference there. One plus one is two. Two plus two is four, that's the difference between here. Four plus four is eight, that's the difference between here. Eight plus eight is sixteen, that's the difference between here. Sixteen plus sixteen is thirty two, that's the difference between here. Should we say? Camera, OK.

	Brian	[Brian's paper 10. <table><tr><th>□</th><th>△</th></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>3</td></tr><tr><td>3</td><td>7</td></tr><tr><td>4</td><td>15</td></tr><tr><td>5</td><td>31</td></tr><tr><td>6</td><td>63</td></tr><tr><td>7</td><td>127</td></tr><tr><td>8</td><td>255</td></tr></table>]
	RBD	OK. We are really running out of time. Anybody has anything they want to say? Can we take one minute to talk about this question about keeping secrets and so forth?
	Bobby	We shouldn't keep secrets
	RBD	We shouldn't keep secrets?
	Romina	I'm sure if you guys had it you would.
	RBD	Seems to me there were two sides to it. What's the bad thing about keeping secrets?
	Michelle I	As long as you eventually tell them I think it's all right. This isn't something that's serious though.
	Romina	when you guys had secrets you didn't tell anybody
	RBD	Let me turn it around and argue the case another way a little bit. I think we do need to keep thinking about it. We want to find a way to do this that every body is comfortable with. But there is a case also to be made for keeping secrets because really what I've said sometimes to people is suppose say Michael and I went to the gym. Michael did a lot of weight lifting, and I watched him, who gets stronger?
	Michelle	The other person has to try to figure it out like if you would want to get stronger, you would have to weight lift. If they want to find it out, they've got to figure it out. They've got to at least try hard.
	RBD	If you want to get good at figuring out you better practice figuring out.
	Student	Yeah, but maybe we want to watch to make sure what it is before we go ahead and do it.
45:00	RBD	We really are out of time. Thank you very much.

APPENDIX C: TRANSCRIPT – "ANKUR'S CHALLENGE" January 9, 1998 (10^{th} Grade)

1 Camera View: "Ankur's Challenge" (Two disks)
Date of filming: 1998-Jan-09
David Brearly High School, Kenilworth NJ, "Ankur's Challenge"
Transcribed by: Anna Brophy (Disks 1 & 2), John Zengerle (Disk 1 – Addendum)
Date of transcription: 2008 (Disks 1 & 2), 2009 (Disk 1 – Addendum)
Verified by: John Zengerle (Disk 1), Melissa Lieberman (Disk 2), Maria Steffero (Disks 1 & 2)
Date of verification: June 2009

Notes:

- Ankur's Challenge – How many towers can you build four high, selecting from cubes available in three different colors, so that the resulting towers contain at least one of each color?
- This episode took place on January 9, 1998. The student participants in this episode are Ankur, Michael, Brian, Jeff, and Romina. At the time, they are in the tenth grade and are from the David Brearly High School in Kenilworth, New Jersey. This session is after school and is part of an after school enrichment program sponsored by Rutgers University that met on a regular basis on Friday afternoons.
- Prior to Ankur's Challenge, the students were working on a problem that they did in fourth grade. The question was: how many towers can you build five tall selecting from red or yellow that have exactly two red cubes. This is where the tape begins.

*<u>Disc One</u> of Two

Line	Time	Speaker	Transcript
1	00:00:01	T/R1	So, so the question is, now, do you understand the question? Okay, first, maybe there aren't any. Maybe there are twelve. There have to be ten because they have found ten, right?
2	00:00:15	Romina	Do you have ten there?
3	00:00:16	Ankur	Yeah.
4	00:00:17	T/R1	So now you have to convince me that you found them. That there couldn't be any others. So why don't you think about that for a minute. That wasn't really what I was going to ask you to do today. But certainly a good way to start given where maybe we should be going. You can talk to each other or...
			The camera focuses on the group composed of Ankur and Michael. T/R1 joins them. In the background, we can slightly hear the other group Romina, Jeff, and Brian working
5	00:00:39	Ankur	Don't you mean like if this is yellow and that's red. But she said...
6	00:00:42	Mike	Twenty.
7	00:00:42	Ankur	No, she said with only two red and three yellow. She didn't say three red and two

Line	Time	Speaker	Transcript
			yellow. Know what I mean?
8	00:00:49	Michael	Dr. More...
9	00:00:51	Ankur	Trust me, trust me.
10	00:00:52	Michael	Dr. More?
11	00:00:54	Ankur	Maher.
12	00:00:55	Michael	Maher. Uh, if we had like this, let's say like, the ones are like red and those are like yellow.
13	00:01:03	T/R1	Okay.
14	00:01:04	Michael	Could we like, could this be...
15	00:01:05	Ankur	He wants to know if we could make this....
16	00:01:06	Michael	Two yellows and three reds be a different one?
17	00:01:10	T/R1	What do you think?
18	00:01:10	Michael	You said, two... well, I don't know if you wanted two...
19	00:01:12	Ankur	You want two red and three yellow, how many can you make with that? And it's ten.
20	00:01:15	Michael	And could it be two yellow and three red... you just said...
21	00:01:17	Ankur	No, she said two red and three yellow.
22	00:01:18	Michael	No, no, she said...
23	00:01:20	Ankur	She didn't say two of one color and three of....
24	00:01:21	T/R1	I did say both. So you can answer either question. I will accept your answer to either question.
25	00:01:25	Michael	Ten and twenty.
26	00:01:26	Ankur	Ten and twenty.
27	00:01:27	Michael	Or twenty, which ever...
28	00:01:27	T/R1	Okay, so how can you convince me that you found them?
29	00:01:29	Michael	Cause like, I just like...
30	00:01:30	Ankur	Cause like it goes with the first number, it's a one there. And then it's a one here and then the rest are zeros.
31	00:01:37	Michael	See this the the first tower was a color...
32	00:01:41	Ankur	Red.
33	00:01:41	Michael	You would have this one and could also have this one.
34	00:01:43	Ankur	And then that one...
35	00:01:45	Michael	Alright then you would go for the second tower, I mean the second space from the top,
36	00:01:50	Ankur	Is red...
37	00:01:50	Michael	Always, that color. You have...you couldn't have one up here because you would have one here. I'm having that problem with the lines. And then...
38	00:02:03	Ankur	And then the third one..
39	00:02:04	Michael	And then that one...I probably could make little lines probably with, you know with the one, two, three, or something like that.
40	00:02:11	T/R1	Uh-hum. That's another way you could probably do it. But this works, you say, that works. It's the same thing.
41	00:02:15	Michael	Yeah.
42	00:02:15	T/R1	Okay.
43	00:02:16	Ankur	Are you convinced? [The following is written on the paper:]

Line	Time	Speaker	Transcript
			1 0 0 0 1 1 1 0 0 0
			1 1 0 0 0 0 0 1 0 1
			0 1 1 0 1 0 0 0 1 0
			0 0 1 1 0 1 0 1 0 0
			0 0 0 1 0 0 1 0 1 1
			[And underneath the above set of zero's and ones, they have:]
			1 1 1 1 0 0 0 0 0 0
			1 0 0 0 1 1 1 0 0 0
			0 1 0 0 1 0 0 1 1 0
			0 0 1 0 0 1 0 1 0 1
			0 0 0 1 0 0 1 0 1 1
			[Underneath this they have a one, two, and three written with two lines coming from the one and two and one line coming from the three.]
44	00:02:17	T/R1	Yeah, I am convinced. Um, so, you might be curious to know what you did when you were in the fourth grade. Probably the same thing.
45	00:02:25	Ankur	Did we have the same answer?
46	00:02:26	T/R1	Do you think you dealt with ones and zeros? You got the same answer.
47	00:02:28	Michael	Well, I thought that...
48	00:02:30	T/R1	Do you think you used the same argument?
49	00:02:31	Ankur	We probably had, like yeah.
50	00:02:33	Michael	Wait a minute.
51	00:02:34	Ankur	Did we have towers out?
52	00:02:35	T/R1	Yes.
53	00:02:36	Ankur	Then we probably built them.
54	00:02:37	T/R1	You did. You did built them. But after you built them.
55	00:02:41	Michael	What a waste of time, building towers.
56	00:02:42	T/R1	You built them and you pulled out ten of them but how did I know you had them all. You had to organize them and you needed to find a way to convince me that you had them.
57	00:02:49	Ankur	We organized them like this.
58	00:02:51	T/R1	Very much so but you did them with the towers. Okay. Patiently watching them.
59	00:03:02	Michael	Are they doing the same thing?
60	00:03:06	Romina	You guys proved it already?
61	00:03:07	Ankur	Yeah.
62	00:03:09	Brian	Don't ask....
63	00:03:11	T/R1	No, no, we are going wait to hear what you do after you hear what they did. Take your time.
64	00:03:15	Brian	Why don't you tell us now?
65	00:03:21	T/R1	What do you think?
66	00:03:22	Michael	[inaudible] The ones and the zeros..
67	00:03:27	T/R1	So you have a very powerful strategy here to use.
68	00:03:32	Michael	I could apply that to anything you give me even if, like.
69	00:03:37	Ankur	We used it to apply to everything you did give us, practically.

Line	Time	Speaker	Transcript
70	00:03:39	T/R1	Let me ask you another question. How would you account, suppose you were building the towers four tall and you can select from three colors?
71	00:04:00	Michael	Three colors?
72	00:04:03	T/R1	Right? I want to know how many, how many there...You raise the question. What would be a reasonable question? That's somewhat different then this question.
73	00:04:18	Ankur	How many with at least one of each color? What were you going to ask?
74	00:04:26	T/R1	That's a good question. Your question is as good as mine.
75	00:04:29	Ankur	Four Tall?
76	00:04:31	T/R1	Four tall, now you can select from three colors. First of all, how many towers can you build? And then how many...[inaudible]
77	00:04:41	Ankur	That's easy, look two colors [inaudible] Three colors...
		The camera focuses on the group containing Jeff, Brian, and Romina. However, the microphone picks up the voices of Ankur and Mike at the same time. Ankur and Mike are working on Ankur's challenge while Romina's group is working on the original problem.	
78	00:04:51	Michael	Sometimes they have towers that are missing one color. That have three of one color.
79	00:04:59	Ankur	That would be with one of each color, probably.
80	00:05:02	Michael	No, because we have to count like red and then four yellows. You could also do that.
81	00:05:14	Ankur	No, she said with total numbers and then one of each. One of each would be one, two, and zero, one, two.
82	00:05:19	Mike	[inaudible] ..whatever
83	00:05:22	Ankur	Okay, then I'll do three and then one, one at the bottom.
84	00:05:28	Micheal	One, two, three, this could be anything. It could be any color in there, of those, any of those.
85	00:05:34	Ankur	No, it would be... it could three of these, three of these. Put one, two, three here.
86	00:05:39	Michael	You could put one, two, and three, and any color here, it doesn't matter.
87	00:05:47	Ankur	No, wait, start over. Do it like this.
88	00:05:48	Michael	[inaudible] I'll squeeze it in.
89	00:05:52	Ankur	And then do it like this, I guess like.
90	00:05:56	Romina	You guys,why is it ten? Are they on a different problem already?
91	00:05:59	Ankur	[Ankur and Michael do not acknowledge Romina's question. The camera now focuses back on Ankur and Michael] One, two, three. So that..
92	00:06:04	Michael	And then do one, 'o'
93	00:06:05	Ankur	One, three, two...
94	00:06:09	Michael	Then do one, 'o', ...'o'.
95	00:06:13	Ankur	Make the 'o' right here, now, because those are the.... Let's do all of the ones where the 'o' is last, know what I mean?
96	00:06:19	Michael	You can't do anything else? I want to do all of the ones with the one first.
97	00:06:24	Ankur	Okay. [The camera focuses back on Romina's group]
98	00:06:32	Ankur	One, two, 'o', two. Put one at the top.
99	00:06:37	Michael	Red?
100	00:06:38	Ankur	Yes. Three of these, three of these, three of these, three of these, three of these, three of these.

Line	Time	Speaker	Transcript
101	00:06:44	Michael	This time three.
102	00:06:46	Ankur	Yeah, this time three, because there are three of these, two...
103	00:06:55	Michael	It's going to be the same exact thing.
104	00:06:57	Ankur	No, wherever there is a two, put a one. Put zero at the end so like..Three, three, three...The zero last.
105	00:07:24	Michael	Um, what is it?
106	00:07:27	Ankur	One, two, two, one. One, two, two one, One, two, two, one.
107	00:07:32	Michael	See if you can find anything else.
108	00:07:35	Ankur	Six, twelve, eighteen times three, 54.
109	00:07:38	Michael	Where do you get times three?
110	00:07:39	Ankur	Because you have three of each. You can put one, two, and three here. Do you know what I mean? In these empty spaces put..
111	00:07:47	Michael	How about this?
112	00:07:51	Ankur	No, one, two, three, there's probably more than that.
113	00:08:09	Ankur	Do it down here, put a one here. Another one, three, two. Two, two, three, one. Three, one, two, three...so it's twelve, eighteen, twenty-four, thirty....
114	00:08:30	Micheal	Why times three?
115	00:08:31	Ankur	Because look you can put a one, two, or three here, do you know what I mean? Look there's three colors with four towers. So, three, three, three, so all of these are three. So, twenty-four times three is seventy-two.
116	00:08:49	Michael	Sixty.
117	00:08:49	Ankur	Seventy. Wait. You'll have to do three to the fourth power, what's that? Three times three times...
118	00:08:57	Michael	I don't know...three times three... nine...
119	00:09:02	Ankur	Twenty-seven.
120	00:09:09	Michael	Twenty-seven. Why don't we just do plus one? [They laugh]
121	00:09:15	Ankur	Let's just erase that.
122	00:09:17	Michael	What's the answer?
123	00:09:21	Ankur	Nothing.
124	00:09:31	Michael	This is just a matter of [inaudible]
125	00:10:12	Ankur	Yeah. Because zero represents...
126	00:10:14	Michael	Any of the three...
127	00:10:15	Ankur	One, two or three can go there. Because it doesn't matter.
128	00:10:18	Michael	Cause you can have anyone of those three in there.
129	00:10:20	T/R1	Okay, I'm thinking towers.
130	00:10:22	Ankur	Okay, this is red, blue, yellow, the zero can be...
131	00:10:28	Michael	Can be any of those three. It could be one, two, three, anyone. The zero is like an x, a variable, it can be any of those three. So you would have...
132	00:10:38	Ankur	You would have one on top. Six, but since there's.... you could have three of each, do you understand that? Okay, you could have eighteen with the first color of red.
133	00:10:51	Micheal	Okay, listen, listen. I'll tell you how we got six first. How do we know it's not seven.
134	00:10:56	Ankur	Okay. Cause we did it..
135	00:10:57	Micheal	Okay, we had one..
136	00:10:59	Ankur	And then you could have two, three, or three, two.

Line	Time	Speaker	Transcript
137	00:11:01	Michael	We have one with the variable on the bottom, that would be a two and a three, or a three and a two. Do you think it could be anything else? So then we put the variable on the top. Two, three, three, two. Two, three, three, two. Nothing else.
138	00:11:25	Ankur	Red as the top color.
139	00:11:27	Michael	That's six. Same thing, go for the two, except it would be one three, three one.
140	00:11:33	Ankur	Except everywhere there's a two, there would be a one. It's like switching the ones and the twos. Do you understand that? And over here it is switching the ones and the threes.
141	00:11:43	Michael	And then we had the variable on top. But then, that would be...well, we had to do times three because...
142	00:11:51	Ankur	There is three of each kind for the variable. You can put red.
143	00:11:54	Michael	Cause you could have..you are going to have to make, 1, 1, 2, 3; 2, 1, 2, 3; 3, 1, 2, 3. That...[inaudible] Times three for every single one of these.
144	00:12:13	Ankur	Twenty-four total. So that would make it seventy-two.
145	00:12:16	T/R1	Okay, so out of the total which you said is?
146	00:12:21	Michael	We didn't do that yet.
147	00:12:22	Ankur	Total? We didn't do that yet.
148	00:12:22	T/R1	Yes, you did, I heard you. Three colors, four-tall.
149	00:12:32	Ankur	Three to the fourth?
150	00:12:35	Michael	Which was... the pizza one. It would be three different colors up here and...
151	00:12:43	Ankur	Three to the fourth.
152	00:12:44	Michael	Four to the third.
153	00:12:44	Michael	With the pizza one, you could either have only a topping or not, so it's a two up there.
154	00:12:55	Ankur	[The camera focuses back on Ankur, Mike, and T/R1] Remember with two colors you had like, because there is two colors there was two and they were five high, it was two to the fifth, right?
155	00:13:04	Michael	Okay. I think so, I don't know. I think it was two to the n, right? Or was it x to the two?
156	00:13:11	T/R1	Think about it.
157	00:13:12	Michael	I just have a bad memory.
158	00:13:13	T/R1	I know, so think about it again.
159	00:13:16	Ankur	Because we had five tall so it was two to the fifth, so if you had four tall, it would be two to the fourth. N up there represents the height of the thing.
160	00:13:23	Michael	Or the amount of toppings on the pizzas.
161	00:13:27	Ankur	Yeah.
162	00:13:28	Michael	So, it's two the the fourth. Nine, twenty-seven, we have....
163	00:13:36	Ankur	...eighty-one.
164	00:13:38	Michael	Seventy-one.
165	00:13:39	Ankur	Eighty-one. [laughing]
166	00:13:40	Michael	Eighty-one. [laughing]
167	00:13:42	T/R1	So you are telling me that there are only nine of them.
168	00:13:46	Michael	That don't have....
169	00:13:48	T/R1	Can you find them, there's only nine?
170	00:13:51	Michael	Uh, there is probably more.

Line	Time	Speaker	Transcript
171	00:13:55	T/R1	That's what you are telling me, right? So you should be able to find them pretty easily.
172	00:13:58	Ankur	There is more than that - you could have a red, red, red, yellow. Red, red, red, blue.
173	00:14:06	Michael	There's more.....uh....
174	00:14:26	Ankur	We'll do it the same way....do three...[The camera focuses on Romina's group again]
175	00:14:28	Michael	Yeah, we'll do two, two, it is nine cause the only way, we would have three of...
176	00:14:38	Ankur	Why can't you have two of one number...why can't you have one, one, two, two.
177	00:14:42	Michael	Well, if you have two of one number, you have three of the other number...
178	00:14:46	Ankur	What?
179	00:14:48	Michael	If you have two of one number...
180	00:14:50	Ankur	Yeah, why can't it be one, one, two, two?
181	00:14:54	Michael	No.
182	00:15:16	Michael	That's six or eight
183	00:15:27	Ankur	There is a lot more than nine.
184	00:15:28	T/R1	Do you think? You've changed your mind?
185	00:15:32	Michael	I'm not changing my mind yet.
186	00:15:35	Ankur	You don't think there is more than nine?
187	00:15:37	Michael	There is, I just don't wanna...
188	00:15:39	T/R1	He's not ready to...
189	00:15:40	Ankur	Admit it.
190	00:15:45	Michael	Fifteen.
191	00:15:48	T/R1	So what's wrong with [inaudible] something somewhere.
192	00:15:53	Michael	What is four to the third? Maybe that's...
193	00:15:55	Ankur	Sixty-four and that's less than seventy-two.
194	00:16:10	Michael	It's right. Do you think we screwed up in here?
195	00:16:16	Ankur	No, we did this right.
196	00:16:18	Michael	We did this right, too.
197	00:16:20	Ankur	Cause, look.
198	00:16:21	Michael	We did this right. I'll show you exactly what we did.
199	00:16:23	Ankur	Watch.
200	00:16:29	Michael	No, look, I'll show you right here.
201	00:16:30	Ankur	Hold on.
202	00:16:31	Michael	No, watch. With this one...
203	00:16:35	Ankur	Yeah.
204	00:16:36	Michael	This is if you have three of one color. You could only have, you have three different possibilities for that one.
205	00:16:39	Ankur	Yeah. Three for that one.
206	00:16:41	Michael	Three times three is nine. If you have two over here, you only have two possibilities in there. You either have two or one. I mean, you either have....
207	00:16:52	Ankur	No, no, no. This is what you got to do. Alright, this is nine, right? And you have two of one color. One, one, and this is open. This could be one also. Know what I mean? There is nine of those. Two, two, two, two...
208	00:17:13	Michael	Alright, then, there's a lot.

Line	Time	Speaker	Transcript
209	00:17:20	Ankur	Nine of these. Erase this garbage.
210	00:17:26	Michael	That's not garbage.
211	00:17:27	Ankur	Yeah, it is.
212	00:17:28	Michael	No, it's not.
213	00:17:43	Ankur	Put zero at the top. Do that one. That's thirty-six right there.
214	00:17:58	Ankur	It's thirty-six right there.
215	00:18:00	Michael	[inaudible]
216	00:18:06	Ankur	Huh...
217	00:18:13	Ankur	You can have like one, one, two, zero, know what I mean? Cause this one's not up there.
218	00:18:40	Ankur	I got it. I got it.
219	00:18:42	Michael	This this, one, one, one, zero.
220	00:18:50	Ankur	No there aren't. Like if you turn it upside down but that's not [inaudible].
221	00:18:57	Michael	Here, do whatever you want.
222	00:19:01	Ankur	Find all the possibilities with three colors, like three towers high and then you multiply that by four because say the numbers two, three, four, right?
223	00:19:14	Michael	Yeah, but [inaudible]
224	00:19:26	Ankur	[inaudible] So you multiply that by four. Do you know why I multiply by four?
225	00:19:30	Michael	Wait. Wait. Wait.
226	00:19:31	Ankur	Listen. If the number is two, two, four, right? It could be zero, two, two, four. [inaudible] Two, three, zero, four. So if you just multiply that by four, this could be...
227	00:19:49	Michael	Wait. To find all of the possibilities you would have... I remember I did this last time. You could have four different. You could have four colors here, four colors here...
228	00:19:59	Ankur	Four colors here...
229	00:20:00	Michael	So it's four to the fourth. Isn't it? So, that eighty-one is totally bull...
230	00:20:09	Ankur	Yeah, so it's four times four, sixty-four.
231	00:20:26	Michael	Sixty-four times four.
232	00:20:31	Ankur	We got it now.
233	00:20:49	Michael	No, it's not.
234	00:20:50	Ankur	Why not?
235	00:20:51	Michael	Sixty-four times four – that's a big number.
236	00:20:53	Ankur	That is a big number. Three colors, three colors. Three colors, you would have to use all three colors [inaudible] we got that...
237	00:21:11	Michael	This would be seventy two. But, see, you know that eighty-one...
238	00:21:17	Ankur	Four possibilities for the first number, four possibilities for the second number, that would be four to the fourth.
239	00:21:22	Michael	Three to the fourth. We would have four possibilities for the first, four possibilities for the second.
240	00:21:29	T/R1	Tell me what you mean. Where are the possibilities coming from?
241	00:21:32	Ankur	You could have red..
242	00:21:33	Michael	What was that? Three?
243	00:21:35	Ankur	Three possibilities.
244	00:21:37	Michael	Ah, okay, screw it, scratch that.

Line	Time	Speaker	Transcript
245	00:21:40	Ankur	What was it, nine?
246	00:21:44	Michael	We're the biggest idiots.
247	00:21:51	Ankur	Why isn't it working now?
248	00:21:52	Michael	I don't know, it's not working...
249	00:21:54	Ankur	It should be nine, nine times three is twenty-seven.
250	00:21:59	Michael	Because watch. A lot of these are repeats.
251	00:22:02	Ankur	[inaudible]
252	00:22:05	Michael	I think a lot of these are.. but it's not...[inaudible]
253	00:22:07	Ankur	[inaudible] it is not one of each color.
254	00:22:13	Michael	Why did we start doing fours now? We had no fourth color.
255	00:22:22	Ankur	[laughing]
256	00:22:33	Ankur	It's not there.
257	00:22:55	Michael	Are we sure that three to the fourth is...
258	00:23:18	T/R1	There agonizing over there. [Ankur and Michael laugh] Why don't we put that on hold for a little bit?
259	00:23:33	Michael	No, no.
260	00:23:34	T/R1	No, [inaudible] I guess we are not going to put that on hold.
261	00:23:36	Ankur	I got it.
262	00:23:37	Michael	You got it?
263	00:23:38	Ankur	I think so, cause look...You can just like, guess that this is a one, like one, two three one. No what I mean?
264	00:23:41	T/R1	Do you know the problem that they are working on?
265	00:23:42	Romina/Jeff/ Brian	No.
266	00:23:44	Jeff	Ankur started to explain it but...
267	00:23:47	T/R1	Ankur, do you want to tell them the problem that you are working on again?
268	00:23:50	Ankur	I told Jeff.
269	00:23:52	T/R1	Say it.. everyone is listening now.
270	00:23:52	Ankur	You heard me.
271	00:23:54	T/R1	He didn't really hear it. You said the words but he didn't hear it.
272	00:23:55	Jeff	What is it you need to use, it is four high and you need to use three?
273	00:23:59	Ankur	You have four high and three colors and you have to use at least one of each color in each tower
274	00:24:05	Jeff	And... what's the answer?
275	00:24:07	Michael	We have that.
276	00:24:09	Ankur	We have that.
277	00:24:09	Michael	But it's not like working.
278	00:24:10	T/R1	They think it's now... They have a conjecture but they can't prove it.
279	00:24:10	Ankur	But that answer is right. That answer is right.
280	00:24:13	Jeff	What did you... what did you come up with?
281	00:24:17	Ankur	Um, seventy-two.
282	00:24:18	Jeff	That's a lot. That's a lot
283	00:24:19	Romina	Seventy-two? With four high?
284	00:24:21	Ankur	Do you want to try it? Four high, you have to use one of each color.

Line	Time	Speaker	Transcript
285	00:24:25	Romina	And you have seventy-two?
286	00:24:26	Ankur	Yes. Trust us.
287	00:24:30	Brian	Even if we agree with you, it's going to make no difference.
288	00:24:32	Ankur	She believes us.
289	00:24:34	Jeff	Yeah, I am not doubting that you guys are wrong, I was just trying to figure...
290	00:24:36	Ankur	It's hard to believe
291	00:24:38	Romina	That is kind of weird. I wasn't expecting that.
292	00:24:40	Ankur	Just, if you start to do it, then you'll will realize.
293	00:24:43	Ankur	I got it.
294	00:24:44	Michael	No, I got it. We should have never put this...
295	00:24:48	Ankur	No.
296	00:24:48	Michael	No, no, we should have never put this in. I would [inaudible]
297	00:24:53	Michael	No, no, no... you see....zero, one, two..[inaudible] No, no, no, listen to me [laughing]. Look, look, see this one right here, look, look, look, look...
298	00:25:07	Ankur	Stop, stop, stop, stop, stop, just stop one second, okay, go. See this, these are the ones we missed. Shut up. Don't even speak. Look these are everything with two colors. Cause look, this is like red, blue, yellow, red, you know what I mean?
299	00:25:33	Michael	I know.
300	00:25:34	Ankur	So we got everything with two colors now we just got to do everything with three colors.
301	00:25:37	Michael	But listen...
302	00:25:38	Ankur	That's all we have to do.. that's all we have to do...
303	00:25:39	Michael	Can I explain... No, no, no, can I explain something to this.... What about four colors?
304	00:25:45	Ankur	Yeah, the four...
305	00:25:45	Michael	Listen, no, look... See this one right here, these are x's. All of these are the doubles. I'll find you the doubles. [inaudible] or six. Why not? Watch. Zero, one, three, two. Is the same as two, one, three, zero. You could have a two here or you could have a two here. [inaudible]
306	00:26:18	Ankur	Alright, so take out seventy-two. Take out [inaudible]
307	00:26:24	Michael	[inaudible] Times three.
308	00:26:27	Ankur	Oh, this works out perfectly. No, stop, stop, stop, stop, I have to do one more thing. There's nine of these, right? And then zero here, there's nine more. Right? Twenty-seven, right? So there's nine, nine, nine. Know what I mean? With three colors?
309	00:26:57	Michael	What about, um...
310	00:26:58	Ankur	Four colors?
311	00:26:59	Michael	What about that? Look at this, she is going to ask us about it.
312	00:27:03	Ankur	No, this is part of this.
313	00:27:05	Brian	Seventy-two?
314	00:27:05	Jeff	Wait. Ankur, Ankur, I have a question.
315	00:27:08	Ankur	It's not seventy-two.
316	00:27:08	Jeff	Are you saying that, alright. Say we are using three numbers,right or.. say these are our things...
317	00:27:11	Brian	What did you get?

Line	Time	Speaker	Transcript
318	00:27:13	Ankur	Fifty-four.
319	00:27:14	Jeff	Oh, I thought it was seventy-two. I have a question now. This...Say x, o, o, right?
320	00:27:25	Ankur	I have no idea.
321	00:27:27	Jeff	That's different then this, right? [shows them something on his paper.] These are different?
322	00:27:33	Ankur	That's what we just found out. Look. Look. It's not different.
323	00:27:37	Jeff	It's not.
324	00:27:37	Ankur	It's not.
325	00:27:38	Jeff	We are saying it's the same thing.
326	00:27:41	Ankur	You know when you put an x. You can put in...
		Romina and Brian are having a conversation at the same time but the camera is focused on Ankur and Jeff. Brian says "In order to get to that seventy. You'd have to add eighteen more...."	
327	00:27:43	Jeff	Here, look, look....this is what we are saying that is, right there [he shows Ankur a set of towers.] We're going o, o, one x. We're going x, one, those are not the same...
328	00:28:02	Ankur	Yeah, those are different.
329	00:28:04	Jeff	Yeah, and that is exactly what that says.
330	00:28:10	Ankur	Those are different.
331	00:28:11	Jeff	And that, that's the same thing there.
332	00:28:12	Ankur	Yeah.
333	00:28:13	Jeff	And you are saying x, one, zero zero. And you're saying zero, zero, one x, so they are different.
334	00:28:18	Ankur	Yeah.
335	00:28:19	Michael	Jeff, give me a piece of paper.
		Jeff joins his group of Romina and Brian. The camera is focused on this group but we can still hear Ankur and Michael.	
336	00:28:31	Ankur	You found out.
337	00:28:33	Michael	I still found...
338	00:28:36	Ankur	There isn't a [inaudible] Definitely.
339	00:28:43	Michael	You could have like a one here and a one there...
340	00:28:50	Ankur	Okay, so minus one – don't write on top of this because then it will get confusing. Cross it out.
341	00:29:03	Michael	If you put two here, two here.
342	00:29:05	Ankur	These two are the same thing.
343	00:29:08	Michael	So that's two. If you have a...
344	00:29:13	Ankur	They are the same. Minus two. And then in this one..
345	00:29:18	Michael	So all of them are just.... Instead of times-ing them by three, times them by [inaudible]
346	00:29:26	Ankur	Brian, it is not fifty-four, it is less then that.
347	00:29:30	Brian.	Alright. We'll prove our [inaudible]
348	00:29:32	Michael	This one... A one and one
349	00:29:37	Ankur	So that's minus three.
350	00:29:40	Michael	I'm just putting all of the possibilities, you know? The last one is possible. So instead of times-ing everything by three, so it's really, it is really going to be

Line	Time	Speaker	Transcript
			what? Fifteen,eighteen
351	00:29:52	Ankur	All you did was take away four Mike.
352	00:29:54	Micheal	I took away six.
353	00:29:58	Ankur	No you didn't.
354	00:29:58	Michael	Yes, I did. There's only two possibilities for each of those.
355	00:30:03	Ankur	For this 'o' and which 'o'?
356	00:30:05	Michael	If you have a one over here.
357	00:30:06	Ankur	Yeah.
358	00:30:07	Michael	And then you put a one over there.
359	00:30:09	Ankur	A three over there. A one over here, okay.
360	00:30:15	Michael	Two, one, can be one. Two, one, three, one.
361	00:30:19	Ankur	Okay. But if you put a one over here, right? This is two one. And you put a three. Oh, okay, You just took away one. Cause one of them...
362	00:30:30	Michael	Which one, which one.
363	00:30:34	Ankur	No, cause look, if you took away one from these two because like one of them still counts. Both of them don't count one of them still counts. Know what I mean?
364	00:30:50	Ankur	So it's minus three here. Fifty-four minus nine.
365	00:30:55	Michael	Times it by three. Minus three.
366	00:30:58	Ankur	No, minus three for each. Cause you can do the same thing here, can't you?
367	00:31:02	Michael	Yeah, this is for one. Times it by three. So what is it? Six times three eighteen, right? Minus three, fifteen.
368	00:31:13	Ankur	Forty-five. Do eighty-one minus forty-five.
369	00:31:28	Michael	Thirty-six. And watch. Now here is my other reasoning. Watch. Seventy-two to thirty-six. That's half.
370	00:31:37	Brian	Thirty-six? I can beat that.
371	00:31:40	Michael	What do you have?
372	00:31:44	Ankur	We could prove you wrong then.
373	00:31:45	Brian	We went from seventy-two to your fifty-four.
374	00:31:48	Ankur	We went to seventy-two, to fifty-four, to
375	00:31:51	Michael	to thirty-six....
376	00:31:51	Ankur	to forty-five.
377	00:31:53	Michael	Forty-five, I mean.
378	00:31:57	Jeff	You are just going down by multiples of five, each time.
379	00:32:00	Ankur	[laughs] Seems like it.
380	00:32:03	Michael	I don't think we are stopping at thirty-six, I mean forty-five.
381	00:32:05	Ankur	What do you have?
382	00:32:06	Brian	Forty-eight.
383	00:32:08	Jeff	No we don't. Brian just picked a number out of his head. [They all laugh]
384	00:32:11	Ankur	You're a bum, you just wanted to beat us.
385	00:32:12	Brian	No, I had an idea in my head. I probably have been right so many times in my life but I just didn't want to say anything.
386	00:32:21	Romina	You just didn't want to prove it?
387	00:32:24	Michael	You want to have something like one, two, one, one.
388	00:32:29	Ankur	I know. Listen.

Line	Time	Speaker	Transcript
389	00:32:32	Michael	We have to...
390	00:32:34	Ankur	I know. Just write them all out.
391	00:32:41	Michael	And then we have...
392	00:32:42	Ankur	We definetly have it now.
393	00:32:44	Michael	Don't be too sure.
394	00:32:45	Ankur	No, I am sure. Write one, two, three again. And zero, zero, zero. And one two three.
395	00:32:55	Michael	[inaudible] maybe.
396	00:32:57	Ankur	Yeah, we got to have that.
397	00:33:02	Michael	If we had a one here and a one here.
398	00:33:09	Ankur	Alright, so look, if you have [inaudible]
399	00:33:11	Michael	This could be, this can't be a one, a minus one right here. Could this be a two. This can't be two. Could this be a two? Yes, it could be a two. The same thing minus one.
400	00:33:26	Ankur	Why? Why over here?
401	00:33:28	Michael	Because one.
402	00:33:30	Ankur	Okay.
403	00:33:32	Michael	It would equal the same as one of these.
404	00:33:34	Ankur	Okay.
405	00:33:40	Ankur	No, no, listen, no, leave this, forget all this. Just multiply by...
406	00:33:45	Michael	No, I want to see if it works for all of em...
407	00:33:46	Ankur	No, listen, you can only have two other colors here...so multiply all of these by two, instead of three.
408	00:33:50	Michael	Hold on...[inaudible]
409	00:33:56	Ankur	Do you know what I mean? You can multiply by two other colors.
410	00:34:07	Michael:	[he is writing] Nice, very nice.
411	00:34:09	Ankur	I know you are pumped.
412	00:34:20	Ankur	One, two, three [Mike is writing]
413	00:34:33	Romina	Did we cancel out fifty-four as a possibility?
414	00:34:36	Ankur	It's not, it's less then fifty-four.
415	00:34:38	Romina	Definitely?
416	00:34:39	Ankur	Yeah.
417	00:34:40	Brian	What do you guys have a formula?
418	00:34:41	Ankur	Trying to get it.
419	00:34:41	Michael	We don't have a formula.
420	00:34:43	Brian	So, then how can you going to proof it?
421	00:34:45	Michael	Don't worry about it.
422	00:34:49	Ankur	Listen, shut up, in this place you can put either a two or a three, you can't put a one. Listen, wait, wait, wait.
423	00:34:58	Michael	No, we are leaving this and we can put three here.
425	00:35:01	Ankur	No, no, no, listen, cause we are going to do this. We are going to do: one, one, one, one. Two, two, two, two. And then plus three [inaudible], do you know what I mean?
426	00:35:10	Michael	But the one, one, one, one, we're just going leave it right there? We can have three over here. And this one?

Line	Time	Speaker	Transcript
427	00:35:18	Ankur	Put two. [inaudible]
428	00:35:21	Michael	You could also...[inaudible]
429	00:35:24	Ankur	You could only have two. Two, two and two. And then two, two and two. Two, two, and two.
430	00:35:37	Michael	So, what does this equal? Six, twelve,...
431	00:35:41	Ankur	Why don't you just multiply all these by two and then just add three? [Michael has written on the paper the following:]

1	2	3	1	2	3	1	2	3	0	0	0
1	2	3	1	2	3	0	0	0	1	2	3
1	2	3	0	0	0	1	2	3	1	2	3
0	0	0	1	2	3	1	2	3	1	2	3
3	3	3	2	2	2	2	2	2	2	2	2

Line	Time	Speaker	Transcript
432	00:35:45	Michael	eighteen plus nine...
433	00:35:49	Ankur	eighteen plus nine is twenty-seven. Alright, and then we gotta to find the nine more.
434	00:36:09	Ankur	Do you know what it is? I know what it is. It's like this, one....
435	00:36:17	Michael	You have like one, one, ...
436	00:36:18	Ankur	Two, two, one, one, three, three. Write a one.
437	00:36:22	Michael	Or you could have two, two, one, one. There's got to be more than that, no? That's only with two colors. Two colors, four places. These guys, that's like uh... That would go on with this wouldn't it?
438		They have on their paper the following:	

1	2
2	1
2	2
1	1

Line	Time	Speaker	Transcript
439	00:36:41	Ankur	No, it won't. Cause look, that is not one of each color, you know what I mean? Just write this...
440	00:36:48	Michael	How about the one we did before? No, no.
441	00:36:51	Ankur	Write this. Now, write this 1, 3, 3, 1. No, next to it. And then write 2, 1, 1, 2. Three, no, 2, 3, 3, 2. Cross that out now. One, one. It's two, four, six. And then you got to do those.
442	Mike is writing this on his paper:		

3	3	1	1	2	2	1	1	2	2	3	3
2	1	2	3	1	3	2	3	1	3	1	2
2	1	2	3	1	3	1	1	2	2	3	3
3	3	1	1	2	2	2	3	1	3	1	2

(A column that says 2, 1, 2, 1 is crossed out – it was between the 2, 3, 3, 2 and the 1, 2, 1, 2 columns.)

Line	Time	Speaker	Transcript
443	00:37:28	Ankur	There's some doubles that we missed, probably. We missed this...
444	00:37:41	Michael	[He is counting the columns] One, two, three, four, ...[quietly counting], twelve
445	00:37:45	Ankur	We missed three somewhere.
446	00:37:50	Michael	Ah, there's too much now.
447	00:37:54	Ankur	I know. There's probably a double. It is probably here. (As they look at the other

Line	Time	Speaker	Transcript
			paper)
448	00:38:04	Michael	No, it's not. Because these are with three different colors. These are just two different colors.
449	00:38:08	Ankur	No, we are probably missing an extra double over here somewhere. Without this. Like remember when we only minused three? It might be minus four. [Ankur sneezes]
450	00:38:24	Michael	Bless you.
451	00:38:25	Ankur	Thank you.
452	00:38:34	Ankur	What is this?
453	00:38:40	Jeff	Are you still guys thinking it's the same thing?
454	00:38:42	Ankur	We think it's still forty-five.
455	00:38:44	Jeff	You think it's forty-five?
456	00:38:44	Romina	Forty-five?
457	00:38:45	Jeff	There's no way its forty-five.
458	00:38:47	Ankur	You think it's no way it's forty-five?
459	00:38:48	Jeff	Cause, I got...
460	00:38:48	Brian	What happened to thirty-six?
461	00:38:51	Romina	I know
462	00:38:51	Ankur	No, it was never thirty-six.
463	00:38:54	Romina	Yes, it was.
464	00:38:54	Ankur	He said the wrong number.
465	00:38:54	Michael	That was thirty-six, that was the extra...
466	00:38:58	Jeff	I think it is thirty-six, though.
467	00:38:58	Ankur	There's eighty-one total.
468	00:39:00	Jeff	Of these?
469	00:39:02	Ankur	No, of like everything.
470	00:39:02	Michael	Combinations that you could have.
471	00:39:04	Jeff	Oh, oh, well you're saying...
472	00:39:04	Michael	Forty-five plus thirty-six.
473	00:39:07	Ankur	We found the other ones and now we are trying to find the other thirty-six.
474	00:39:10	Romina	Oh, then there's eight...
475	00:39:12	Jeff	Eighty-one total. Cause we are saying that you have to use one of each. But they are saying if you could make any combination, that that's what it would be. But I think this is the thirty-six. I don't think there's forty-five of these.
476	00:39:22	Romina	Okay, Brian, go, give me another set.
477	00:39:24	Jeff	There is no way it is forty-five, there's too many.
478	00:39:27	Michael	That's what we are trying to figure out.
479	00:39:28	Jeff	Cause now, I have thirty-seven right now. I have the same thing somewhere but I don't know where it is...
480	00:39:33	Ankur	If it's not forty-five, then it's probably forty-two. But, either one of those two.
481	00:39:38	Brian	How could you have an odd though?
482	00:39:40	Ankur	We found thirty-nine other ones.
483	00:39:42	Romina	But don't they have like pairs?
484	00:39:42	Jeff	You found thirty-nine of these?
485	00:39:44	Brian	Doesn't each one have a pair?

Line	Time	Speaker	Transcript
486	00:39:45	Ankur	No, like [pause].
487	00:39:48	Romina	Yeah, doesn't, don't they have pairs?
488	00:39:49	Jeff	Don't they have a pair?
489	00:39:49	Ankur	Don't they have a pair?
490	00:39:51	Michael	Yeah. Watch – [there is laughter and they are talking all at once]
491	00:39:54	Romina	'Cause they're all odd.
492	00:39:55	Ankur	'Cause like when you have like all red, and all blues and all yellows. [pause]
493	00:39:59	Brian	You can do that?
494	00:40:00	Romina	You can't do that...
495	00:40:01	Ankur	No, when we find...
496	00:40:03	Michael	Don't worry about it.
497	00:40:03	Ankur	Were you listening to a word me and Jeff were talking to, about?
498	00:40:05	Jeff	There's eighty-one total of these. You weren't listening.
499	00:40:08	Ankur	And we found...
500	00:40:09	Romina	You weren't talking to me.
501	00:40:11	Ankur	You butted in our conversation and then...
502	00:40:14	Romina	You have a conversation between yourselves.
503	00:40:18	Ankur	I think they are calling you, Jeff.
504	00:40:22	Romina	Hold on. What, okay. Could you run the conversation by me one more time then?
505	00:40:25	Jeff	There's eight-one total things you could have.
506	00:40:28	Romina	How did you get eighty-one?
507	00:40:29	Ankur	Do it and you'll figure it out.
508	00:40:30	Jeff	The x times the y deal.
509	00:40:31	Romina	No, Ankur
510	00:40:32	Jeff	Alright. X times the y. What is it?
511	00:40:33	Ankur	The x times the y deal. Remember when we?
512	00:40:36	Jeff	Wait, wait. X is three? X was three.
513	00:40:39	Ankur	It's three to the fourth.
514	00:40:39	Jeff	To the fourth.
515	00:40:41	Ankur	'Cause look.
516	00:40:43	Jeff	Three times three is nine times three is twenty-seven.
517	00:40:45	Jeff/Ankur	Times three is eighty-one.
518	00:40:47	Ankur:	You want to know why we multiplied it like that?
519	00:40:47	Jeff	Yeah, do you understand that? I am not being..
520	00:40:52	Romina	You're being. Sorry you guys.
521	00:40:52	Ankur	'Cause, look. You have four spaces. In the first space you have three. In the second space you could have three [pause]
522	00:40:58	Jeff	You understand how it's eighty-one total?
523	00:40:59	Romina	I understand. Yes.
524	00:41:01	Jeff	That's how it is eighty-one total
525	00:41:02	Romina	Okay.
526	00:41:02	Jeff	Now you know.
527	00:41:05	Romina	And there's no doubles then?

Line	Time	Speaker	Transcript
528	00:41:08	Jeff	This is the method we came up last week with, proving this.
529	00:41:09	Romina	I understand but that was for a different thing.
530	00:41:14	Jeff	How did get thirty-two for the first problem we did that took us three hours to do? We just took it for granted that it was x to the y, right? That it was two to the five?
531	00:41:22	Romina	Yeah.
532	00:41:23	Jeff	So we are not going to take this for granted that it's three to the fourth?
533	00:41:24	Romina	Okay. Calm down a little bit. Okay? You guys are all too hyper.
534	00:41:32	Brian	Isn't that the same thing we had for the problem a few weeks ago?
535	00:41:35	Ankur	Yeah, that's what we got. [inaudible]
536	00:41:37	Jeff	[inaudible] tower problems like that no more. We have different ones because I know how to do those.
537	00:41:41	Ankur	So instead of finding all the ones that we can use one of each color we found the other ones.
538	00:41:46	Jeff	Because if she just said that just find all of the ones you can do four towers, three high, we would have been done three hours ago.
539	00:41:52	Romina	Okay, so that's not the problem. So what is the problem?
540	00:41:54	Jeff	The problem is how many using one in each slot, using, well you have to use all three colors.
541	00:41:56	Ankur	We just used eighty-one to try to help us to find the other one, the other side. Do you know what I mean?
542	00:42:01	Jeff	[inaudible]
543	00:42:02	Ankur	Instead of one of each color, we just found, we were just trying to figure out like not ones with each other.
544	00:42:12	Romina	See that is where I missed understood you. I thought you meant eighty-one of these things.
545	00:41:20	Brian	You know that wouldn't work for like...
546	00:41:21	Michael	I can't think.
547	00:41:21	Brian	You know that wouldn't work with like two colors, five high because if it's two to the fifth.
548	00:41:27	Ankur	Bri, whoever asked you to talk?
549	00:42:30	Brian	If you want these roll of cubes coming at your neck.
550	00:42:35	Ankur	If you want these roll of cubes coming back at you.
551	00:42:40	Romina	We have such short attention spans.
552	00:42:42	Jeff	It's not that, it's just you sit there and try to do a problem for three hours...
553	00:42:44	Romina	Okay, I have a short attention span.
554	00:42:45	Ankur	Hey, Mike did you figure it out?
555	00:42:47	Michael	No, I'm trying..
556	00:42:47	Ankur	While we are arguing, he is still trying to figure it out.
557	00:42:49	Romina	Mike is the only one that doesn't have a short attention span.
558	00:42:50	Brian	Mike, do you remember? I got to talk to Mike about something real quick.
559	00:42:54	Romina	Okay, [inaudible]
560	00:42:55	Brian	I just remembered something.
561	00:42:56	Ankur	Why don't you say it in front of the camera like you do with everything else?
562	00:42:58	Brian	Because I can't. I can't.

Line	Time	Speaker	Transcript
563	00:43:00	Ankur	There's other things you couldn't say in front of the camera. [inaudible]
564	00:43:03	Brian	Like what? Oh.
565	00:43:07	Brian	Do you remember those little wooden cubes? Those Cuisinare cubes?
566	00:43:11	Michael	You put them in your nose, right? Erin picked one up.
567	00:43:15	Romina	What?
568	00:43:16	Michael	Nothing. When we were in eighth grade. He use to to little kids [inaudible]. And you picked one up. [inaudible] and you picked it up. [They are all laughing] And me and Brian started cracking up. No one knew what we were talking about except for you, me, and Sara, too. And you were like [inaudible]. Nothing, nothing.
569	00:43:39	Ankur	Put the camera on Brian, he is going to turn bright red. [They are all laughing]
570	00:43:46	Romina	[inaudible] That's what made me laugh.
571	00:43:50	T/R1	Poor Brian. Why are you picking on poor Brian?
572	00:43:52	Ankur	You got a little color there.
573	00:43:54	T/R1	So what did you find so far?
574	00:43:56	Romina	Nothing.
575	00:43:57	T/R1	Nothing?
576	00:43:58	Romina	No, we are getting there. We're going.
577	00:44:01	T/R1	Is there anything you are sure of?
578	00:44:03	Jeff	That there is eighty-one total ones.
579	00:44:05	Ankur	We are sure of that.
580	00:44:06	T/R1	You are sure that there are eighty-one total.
581	00:44:09	Romina	Yeah, for the tower problem. Don't ask them any questions, they might freak out.
582	00:44:14	T/R1	Okay.
583	00:44:15	Michael	Are you sure? That's there eighty-one total?
584	00:44:19	Jeff	[laughing, inaudible] up your nose..
585	00:44:28	T/R1	So what else are you sure of?
586	00:44:31	Michael	Nothing else yet.
587	00:44:32	T/R1	Nothing else yet.
588	00:44:36	Ankur	We have ideas.
589	00:44:39	T/R1	Okay, what are ideas that you are pursing? Any of them that you want to share with each other?
590	00:44:45	Jeff	Um, I'm trying to figure out cause I'm close to thirty-two, thirty-six. I'm at thirty-seven and I think I have the same ones somewhere. And if that's the case, it will be thirty-six is the next number that I kind of try to work with. I will try to figure something out. I got to at least look at combinations.
591	00:45:01	Romina	You know it might be thirty-six cause I'm working with six's now.
		Jeff and Ankur talk while Romina talks with T/R1. The following transcript is the conversation between Ankur and Jeff.	
592	00:45:04	Ankur	Where you have x, what does that mean?
593	00:45:06	Jeff	Alright, here is the deal. This is like 'x' is red and one is yellow and zero is white, that's all it means.
594	00:45:17	Ankur	Okay.
595	00:45:18	Jeff	And um these, do you understand how these go? These are all three. That's...

Line	Time	Speaker	Transcript
596	00:45:23	Ankur	Yeah, you can have three of these.
597	00:45:23	Jeff	That's 'x', one, 'o', 'x'. 'X', one, 'o', one.
598	00:45:25	Ankur	Yeah. That is the same thing. We did that with everything and then we subtracted out three out of each and we found out.
599	00:45:36	Jeff	Alright. But just look in here and see if you see two of the same ones somewhere. Because they're somewhere I just don't know where it is.
600	00:45:47	Ankur	I think now it's forty-two.
601	00:45:49	Jeff	Do you? Or do you think that?
602	00:45:50	Ankur	Because we did this, we got forty-five but...
603	00:45:54	Michael	No, we had.... First we had seventy-two and then we forgot that...
604	00:45:57	Ankur	It's definitely not seventy-two.
605	00:45:58	Michael	Then we had like um, what was it?
606	00:46:01	Ankur	Then we found the ones that don't have one of each color and we got thirty-nine.
607	00:46:04	Michael	And then we found a lot of repeats in here so, I'm still thinking ...
608	00:46:08	Jeff	They could be thirty-six.
609	00:46:13	Michael	We know how much ....
610	00:46:14	Jeff	But Mike I got thirty-seven and I can't think of no more.
611	00:46:18	Ankur	We found all of the ones without one of each color and we got....
612	00:46:22	Jeff	Thirty-nine?
613	00:46:22	Romina	You guys for thirty-six.
614	00:46:24	Brian	[inaudible] right there...
615	00:46:27	Romina	Oh, that's not it then. Hold on.
616	00:46:29	T/R1	What did you find?
617	00:46:31	Ankur	You told us to find all the ones with one of each color. We found the ones that…..
618	00:46:35	T/R1	What did I ask you to find, Ankur?
619	00:46:37	Ankur	All the ones with one of each color in each tower so… We knew there were 81. So we figured out all the ones without at least one of each tower.
620	00:46:48	T/R1	And how many did you find?
621	00:46:49	Ankur	Thirty-nine.
622	00:46:50	T/R1	Explain to me how you did that.
623	00:46:58	Ankur	Give us a minute?
624	00:46:59	T/R1	Sure.
625	00:47:03	Ankur	[inaudible] Alright this is what we have to do. We have to explain this and this.
626	00:47:14	Michael	How much do we have here? Twelve?
627	00:47:17	Ankur	One, two, three, four, five, six, one, two, three, twelve. This is definitely right, there's no doubt.
628	00:47:25	Michael	And this. Definitely right.
629	00:47:29	Ankur	Alright. Dr. Maher [T/R1 joins their group] Alright, we did the same thing as we did before.
630	00:47:38	T/R1	Explain to me where… You're going to show me the ones that…
631	00:47:42	Ankur	Don't have at least one of each color.
632	00:47:44	T/R1	Right.
634	00:47:45	Ankur	So we have, we have the x-variable which can be either one of those… [Ankur is pointing to the zero at the bottom of the first column]

Line	Time	Speaker	Transcript
635	The camera focuses on their paper and this is what is written on it:		

1	2	3	1	2	3	1	2	3	0	0	0
1	2	3	1	2	3	0	0	0	1	2	3
1	2	3	0	0	0	1	2	3	1	2	3
0	0	0	1	2	3	1	2	3	1	2	3

And below each of these columns, they have these numbers:

3	3	3	2	2	2	2	2	2	2	2	2

Line	Time	Speaker	Transcript
636	00:47:51	T/R1	I have trouble with that x-variable. Can you just tell me….
637	00:47:55	Ankur	The x-variable can either…. on this side the x variable can be either 1, 2, or 3. Do you understand that?
638	00:48:05	T/R1	Well, not really.
639	00:48:06	Ankur	This is red, red, red, this could be red. It could be red, red, red, blue. Red, red, red, yellow. [Ankur is saying this as he points to the column that says 1,1,1,0]
640	00:48:14	T/R1	But then it will have one of each other. I want the ones that do not have one of each color.
641	00:48:17	Ankur	That's not one of each color. Because the tower is red, red, red, yellow.
642	00:48:21	T/R1	Oh, okay. Okay, I heard, I see. I didn't hear that right.
643	00:48:24	Ankur	And then you're going to have blue, blue, blue and then the next set. But over here it cannot be red, red, red, red, because we already have that over here. [He is referring to the fourth column]. So in this case, since this is red, red, and red, x can only be blue or yellow. Do you understand that? And then we changed the x-variable up here in the same situation. [He is pointing to the seventh column]. It can' be, x can't be red again in this column because then it will overlap over here. And we did all this and then we just did it with the one on top. [He points to the last three columns]. And we got 3, 6, 9, 6, 12, 18 plus nine which equals 27. Do you understand that half?
644	00:49:09	T/R1	I'm not really sure. Um...
645	00:49:13	Michael	What are you unsure about?
646	00:49:13	T/R1	What I'm unsure...How have you accounted for duplicates? Pick one color.
647	00:49:20	Ankur	Pick one color.
648	00:49:21	T/R1	How many do you have that don't have red? Can you tell me that?
649	00:49:25	Ankur	That don't have red?
650	00:49:26	T/R1	Uh-hum.
651	00:49:28	Ankur	I probably could. But if you just look at it the way I just explained it, it would be easier.
652	00:49:35	T/R1	Can you show me the ones that don't have red here?
653	00:49:37	Ankur	Probably.
654	00:49:38	T/R1	Okay, which are they?
655	00:49:41	Ankur	That don't have red at all?
656	00:49:42	T/R1	Uh-hum.
657	00:49:43	Ankur	Well there is two here that don't have red. [He points to the first column of 1, 1, 1, 0] Do you understand that? Oh, wait. There's two here that don't... [He is pointing to the third column that contains 3, 3, 3, 0] Yes, two here that don't have red because it's blue, blue, blue. It could either be blue or yellow over here. [He

Line	Time	Speaker	Transcript
			is pointing to the zero at the bottom of the column] And over here there are two that don't have red, so that's four. Are you with me?

658 The paper that he is pointing to has the following written.

1	2	3	1	2	3	1	2	3	0	0	0
1	2	3	1	2	3	0	0	0	1	2	3
1	2	3	0	0	0	1	2	3	1	2	3
0	0	0	1	2	3	1	2	3	1	2	3

And below each of these columns, they have these numbers:

3	3	3	2	2	2	2	2	2	2	2	2

Line	Time	Speaker	Transcript
659	00:50:05	T/R1	I'm listening.
660	00:50:07	Ankur	But are you with me?
661	00:50:08	T/R1	It depends on where you're going.
662	00:50:11	Ankur	This one already has red in it, so that's discarded [pointing to the fourth column that says 1, 1, 0, 1]. And this one can't be blue and it can't be red, as we said, so it's only one [pointing to the fifth column that says 2, 2, 0, 2]. It can only be yellow over here [pointing to the zero in the column of 2, 2, 0, 2]. So that's two, four, five. And this one can only be, since three is yellow, can't be red, so it can only be blue, so it's one [pointing to the sixth column that says 3, 3, 0, 3]. So that's six. And then this one is already discarded [pointing to the seventh column that says 1, 0, 1, 1]. And this can't be blue or red, so it can only be yellow [pointing to the eight column that says 2, 0, 2, 2] and this can only be yellow so it can only be blue. [pointing to the ninth column that says 3, 0, 3, 3]. Are you still with me?
663	00:50:44	T/R1	Uh-hum.
664	00:50:45	Ankur	So that's two, four, five, six, seven, eight. And this can either be yellow, that's it [pointing to the zero at the top of the eleventh column that says 0, 2, 2, 2]. And this can only be blue [pointing to the zero at the top of the twelve column that says 0, 3, 3, 3]. Because ten without red. Not total yet. I didn't show you up here yet. There's ten down her without red.
665	00:51:06	T/R1	Is Mike following you? Do you agree?
666	00:51:17	Michael	I found some more doubles.
667	00:51:20	Ankur	How many did you find, three?
678	00:51:21	Michael	I don't know.
679	00:51:24	Ankur	So I did that for absolutely no reason.
680	00:51:26	Michael	What is this? What does she want me to explain?
681	00:51:29	Ankur	She wanted me to total the towers without red in it.
682	00:51:31	Michael	Without red in it?
683	00:51:33	Ankur	Look, this one you can't do because it already has red in it. One's red, right?
684	00:51:37	Michael	It doesn't matter what one is.
685	00:51:38	Ankur	Just say one is red.
686	00:51:39	Michael	Yeah.
687	00:51:40	Ankur	And then over here you can have two, two, two. You could have yellow or blue. That's two, right?
688	00:51:47	Michael	Yeah.

Line	Time	Speaker	Transcript
689	00:51:48	Ankur	And then there's two here because you can't have red here [pointing to the third column]. So that's four. But over here [pointing to the fourth column].
690	00:51:53	Michael	Did she give us another problem?
691	00:51:54	Ankur	Just listen. All the towers without red. Over here, this already has red [pointing to the fourth column]. And over here, it's two, two, you can't have two again [pointing to the fifth column] because that be the same as one of these, right? You can only have one and all of the rest will be one because, because the same reason as that.

692 The paper that he is pointing to has the following written.

1	2	3	1	2	3	1	2	3	0	0	0
1	2	3	1	2	3	0	0	0	1	2	3
1	2	3	0	0	0	1	2	3	1	2	3
0	0	0	1	2	3	1	2	3	1	2	3

And below each of these columns, they have these numbers:

3	3	3	2	2	2	2	2	2	2	2	2

693	00:52:12	Michael	How does that..[inaudible] to what I was thinking.
694	00:52:15	Ankur	She wanted to know all of the towers without red.
695	00:52:16	Michael	Okay.
696	00:52:19	Ankur	Do you agree with me?
697	00:52:20	Michael	Yeah. Now, I found some more doubles which... which is making me thinking that there's like three. Listen, listen, you have one, two, three. All the doubles that I found, remember? I found the one was one, two, three...
698	00:52:43	Jeff	Here, pay attention.
699	00:52:44	Ankur	Hold on.
700	00:52:45	Jeff	Just listen.
701	00:52:45	Ankur	He wants to just tell me something real fast.
702	00:52:47	T/R1	Let them finish.
703	00:52:48	Michael	Now if this was a two in there.
704	00:52:50	Ankur	Yeah.
705	00:52:51	Michael	We also have one, 'o', three, two. This and this are a double.
706	00:52:57	Ankur	Okay, you can't put, so it's minus four.
707	00:53:01	Michael	I don't know. For each one there's like two doubles. Hold on, let me... you do whatever you want, I'm just gonna...
708	00:53:08	Jeff	No. Just pay attention, though. 'Cause you're gonna think it's thirty-six when we're done.
709	00:53:11	Michael	How do you know?
710	00:53:12	Jeff	What?
711	00:53:13	Michael	Hold on.
712	00:53:13	Jeff	Just pay attention. Put your paper...
713	00:53:14	Michael	Go. Go. Go.
714	00:53:14	Jeff	Give me your paper for a second. Let me see your paper.
715	00:53:15	Michael	Go.
716	00:53:17	Jeff	You see... 'Cause you're still gonna pay attention over there. You're not gonna listen to us.

Line	Time	Speaker	Transcript
717	00:53:19	Michael	It's under the table. I'm not looking at it.
718	00:53:21	Romina	Okay. Look. We had, we had to have them, the two, we had to have two of the same color, right? In one of them if we're gonna have all three same, all three colors, right? Do we agree?
719	00:53:35	Ankur	I have no idea what you just said.
720	00:53:35	Jeff	All right.
721	00:53:38	Romina	Okay [She and Jeff begin talking at the same time and it is not possible to understand what they are individually saying.] Need to have two of them. We need to have two of the colors.
722	00:53:40	Ankur	Okay, Okay, Okay.
723	00:53:41	Romina	So you have to organize them so they... so that you don't have any doubles. So either you can have them next to each other. You can have them separated by one. You have them on the ends, in the middle, two and fourth spot, and third and fourth spot. Right?
724	00:53:53	Ankur	Yes.
725	00:53:53	Romina	So that's six.
726	00:53:55	Ankur	Yes.
727	00:53:56	Romina	Okay. Now you, in the other spots you can have an o and an x. Those are colors. Like these are three different colors – an o and an x and an x and an o. Right?
728	00:54:04	Ankur	Mhm.
729	00:54:05	Romina	So you have to multiply each of these six by two.
730	00:54:06	Jeff	And you couldn't have like x x because that wouldn't fit the requirement.
731	00:54:09	Ankur	That would be against...
732	00:54:10	Romina	And then
733	00:54:11	Jeff	So you multiply that – each one by two. So that would give you twelve. Correct? 'Cause that means you could have this. You could have either the bottom or the top.
734	00:54:16	Brian	You could have the x's in the first spot, the o's in the first spot.
735	00:54:19	Ankur	Okay. Hold up. I just want to think about it for a second.
736	00:54:23	Romina	Six times two, twelve; six times two, twelve; six times two, twelve; six times two, twelve; six times two, twelve; six times two, twelve.
737	00:54:29	Ankur	Yeah, now when you add them...
738	00:54:20	Jeff	Why do you keep saying six times two?
739	00:54:32	Michael	You get thirty-six for the ones without...
740	00:54:33	Jeff	Why do you keep crossing that out?
741	00:54:35	Romina	'Cause it's wrong! No, you multiply all this by two. Right? And then you multiply all that by three, because of the three different colors.
742	00:54:44	Jeff	Yeah, yeah, no.
743	00:54:46	Romina	So that is what we were trying to say but we wrote it bad.
744	00:54:46	Jeff	We were saying that but she wrote it funny.
745	00:54:50	Ankur	Okay.
746	00:54:51	Romina	Okay, so you can multiply these all by two, right? Because you have one color or the other.
747	00:54:55	Ankur	An o or an x or an x and an o.
748	00:54:57	Romina	Right? Then you have to multiply all by three because the ones can be any

Line	Time	Speaker	Transcript
			colors.
749	00:54:59	Jeff	And then you could switch the numbers around, x's and then you could bring...
750	00:55:02	Romina	It could be the three colors.
751	00:55:03	Jeff	There's like... there's twelve this way. And there would be twelve if you took the x's put them here. And took the one's and put them there, that's twelve more. And there's twelves more if you took the zeros and put them here and put the x's back over there with the ones.
752	00:55:16	Romina	So it's thirty-six.
753	00:55:16	Jeff	Thirty-six.
754	00:55:19	Brian	Ankur's drawing blanks.
755	00:55:21	Ankur	No, I followed.
756	00:55:25	Brian	What about you Mike? Mr. Binary?
757	00:55:25	Jeff	Do you think it is thirty-six?
758	00:55:26	Romina	Do you agree?
759	00:55:27	Michael	Yeah sure
760	00:55:28	Ankur	Probably.
761	00:55:30	Romina	But do you agree?
762	00:55:31	Ankur	Probably.
763	00:55:33	Brian	Probably?
764	00:55:33	Ankur	Yeah.
765	00:55:37	Jeff	What do you think?
766	00:55:39	T/R1	So there's thirty-six, you're saying that have exactly....
767	00:55:41	Jeff	One of each color.
768	00:55:41	T/R1	One of each color. And so how many would there be that do not have?
769	00:55:47	Jeff	There would be forty-five. And it would normally be an even number but since there are three colors.
770	00:55:53	Romina	Because there is three colors. All x's, all zero's and all one's.
771	00:55:56	Jeff	Yeah. So that would make the extra set of all of one color would make it odd. And that's why the whole number is odd. Eighty-one total.
772	00:56:03	T/R1	I don't know. Are you convinced Ankur? That's not what you found is it? No. Why don't you explain what you did.
773	00:56:12	Ankur	I didn't do anything. There's makes good sense. The way like she drew that out. The way we drew it out, we just had a mistake.
774	00:56:30	Michael	Many mistakes.
775	00:56:32	Ankur	A couple mistakes.
776	00:56:32	Jeff	Ninety-four, thirty-six, seventy-two...
777	00:56:34	Michael	We just thought we could do it like real fast, like...
778	00:56:37	Ankur	Yeah, we didn't stop to think about it.
779	00:56:42	Michael	So what is thirty-six. The number you could have with all..
780	00:56:44	Romina	With all three in it.
781	00:56:44	Jeff	With three. And then forty-five would be the number of the rest of them because the eighty-one is the total number.
782	00:56:49	Romina	The x and the y.
783	00:56:51	Michael	How many, can you prove that there is forty-five?
784	00:56:53	Jeff	Of the other ones?

Line	Time	Speaker	Transcript
785	00:56:54	Michael/ Ankur	Yeah.
786	00:56:55	Ankur	You proved that there is thirty-six of these ones.
787	00:56:55	Romina	That was, that was, we just eighty-one minus thirty-six.
788	00:56:58	Jeff	Yeah, we're saying that cause, wait, eighty-one...
789	00:57:01	Ankur	Listen, can't you just do the same thing and...?
790	00:57:04	Jeff	Yeah, but it would be harder because then this, because then, because then it would make no sense. Because then you could have one, one, one, one.
791	00:57:09	Brian	You could have three of one color.
792	00:57:11	Jeff	And you could have one, one, x, one.
793	00:57:12	Romina	Without a color completely.
794	00:57:14	Jeff	Like, so that would take some more time to figure out but..
795	00:57:17	Romina	Hold on, wouldn't we, couldn't we just do like..
796	00:57:21	Jeff	That's what I started to do that. This is what I started to do here.
797	00:57:24	Michael	I want, [inaudible] I want proof that...
798	00:57:27	Jeff	You can prove forty-five...
799	00:57:28	Michael	No, I want proof the other way around. For that there's, 'cause that's what we did.
800	00:57:32	Jeff	There's forty-five.
801	00:57:33	Ankur	We proved the other side.
802	00:57:33	Michael	We came up with seventy-two. Okay, then we just, if you were right, then eight-one minus seventy-two that is only nine.
803	00:57:39	Jeff	Yeah, that's what I'm saying. So we could be wrong. That's what I was starting to do there.
804	00:57:41	Michael	I want you to prove
805	00:57:43	Jeff	The other one.
806	00:57:43	Michael	the other one.
807	00:57:44	Ankur	The only way you could prove that you were right is to prove the other side.
808	00:57:45	Michael	We proved the other one. But I don't. That's not enough for me. I want to prove the other.
809	00:57:50	Jeff	Yeah. Cause if, cause if. You can't just say 'alright, we take eighty-one for granted' and we said thirty-two to ourselves
810	00:57:53	Ankur	Look we proved seventy-two to ourselves but then we tried to prove the other side and it was wrong. So we figured out that seventy-two was wrong.
811	00:57:58	Michael	We proved all of the ones that don't have all three.
812	00:58:01	Jeff	Yeah, which is thirty-six, no.
813	00:58:02	Michael	We have thirty-nine.
814	00:58:03	Ankur	Thirty-nine.
815	00:58:04	Jeff	As the ones that have three?
816	00:58:05	Michael	That don't have all three.
817	00:58:06	Ankur	Don't have all three.
818	00:58:08	Ankur	According to you, it's forty-five.
819	00:58:10	Michael	That. It should be forty-three or something like that.
820	00:58:13	Jeff	Forty-five.
821	00:58:16	Ankur	Do you want us to prove thirty-nine to you?
822	00:58:18	Jeff	Yeah.

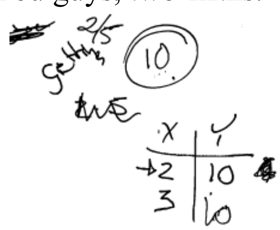
Line	Time	Speaker	Transcript
823	00:58:20	Ankur	Do you want to prove thirty-nine to him?
824	00:58:22	Michael	You do it.
825	00:58:23	Ankur	You do it.
826	00:58:23	Michael	You do it because I am stuck.
827	00:58:25	Ankur	You are the one that wrote it all.
828	00:58:26	Michael	I found a couple doubles. You prove it to him because um.. it's right here, right?. And maybe he'll find some faults.
829	00:58:37	Ankur	Alright, this is what we did.
	00:58:39		TAPE STOPS

ADDENDUM DISK 1 of 2

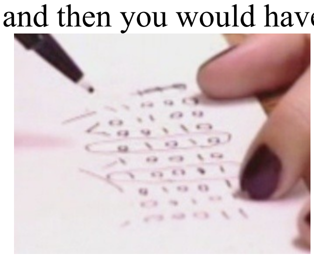
Line	Time	Speaker	Transcript
Ad-1	00:00:54	Jeff	It would be a fraction.
Ad-2	00:00:55	Romina	I'm saying that's what it would be.
Ad-3	00:01:10	Romina	Maybe (inaudible)
Ad-4	00:01:30	Romina	See this
Ad-5	00:01:31	Jeff	I do see that
Ad-6	00:01:41	Jeff	How come you're saying there's three now?
Ad-7	00:01:45	Romina	I just know, okay.
Ad-8	00:01:50	Jeff	Not two high, I mean two…
Ad-9	00:01:51	Romina	What if there were three, would there be three yellow?
Ad-10	00:02:03	Jeff	Red is like having two yellows?
Ad-11	00:02:05	Brian	Same goes for that's two yellows
Ad-12	00:02:06	Jeff	So it won't be the same thing opposite.
Ad-13			(inaudible)
Ad-14	00:02:41	Romina	Alright, first, second, now this
Ad-15	00:03:10	Romina	(To Ankur and Michael at the other end of the table) You guys proved it already?
Ad-16	00:03:18	Brian	Just tell us now.
Ad-17	00:03:20	Romina	They don't want to tell us now.
Ad-18	00:03:21	Jeff	So you're saying that's for … This is
Ad-19	00:03:22	Romina	Three reds
Ad-20	00:03:32	Brian	That's all you're gonna have. Y, R, Y, R, Y, R
Ad-21	00:04:00	Romina	Y, R, Y, R, Y, R, Y, R, Y, R
Ad-22	00:04:02	Brian	Yeah
Ad-23	00:04:04	Brian	You could have the opposite
Ad-24	00:04:05	Romina	It'd be …
Ad-25	00:04:06	Brian	Y, Y, R, R, R
Ad-26	00:04:08	Brian	How many?
Ad-27	00:04:13	Jeff	Yeah, but, look. You could use three. Sliding these two over here. [Puts a tower of two white and three yellow on top in front of them]
Ad-28	00:04:18	Romina	[Romina holds two towers together: WWYYY and YYYWW] How do we get those two over there?

Ad-29	00:04:19	Brian	You mean all the same color?
Ad-30	00:04:20	Jeff	Different colors
Ad-31		Brian	I don't know. Bottom up.
Ad-32	00:04:24	Romina	So is that it? I don't know.

Ad-33		Brian	Y, Y, R, R, R? Der. Do you have Y, R, R. No, that ain't working.
Ad-34		Brian	You have Y, R, R, Y, R?
Ad-35		Brian	You have R, R, Y, Y, R?
Ad-36	00:04:50	Jeff	What did you say?
Ad-37		Romina	That would be four reds.
Ad-38		Brian	You got them. R, R, Y, Y, R. (Holds out a finger of one of his hands as he repeats each letter until all five of his fingers are extended)
Ad-39		Brian	How many is that? Ten. Boom.
Ad-40		Jeff	Twenty. It's twenty total.
Ad-41	00:05: 34	Romina	Come on. That doesn't make an equation. 0, 1, 1 .. You have the same output for two inputs. Why is it you guys? We know this. It's Friday… Don't panic. You guys, two-fifths.



Ad-42		Brian	It's not two fifths.
Ad-43	00:05:41	Romina	I know it's not. I'm just saying. Why is it ten? (To Ankur and Michael) You guys, why is it ten? Are they on a different problem already?
Ad-44	00:06:08	Brian	Can we just say we got it, and then go to another one?
Ad-45	00:06:11	Jeff	How are we supposed to get the next one, if we can't get the first one?
Ad-46	00:06:14	T/R1	You have to convince me that you have them. You can't just say "I have 10." You have to be able to prove to me that there can't be more. Sixteen, eighteen?
Ad-47	00:06:33	Romina	This is so frickin… We don't even have, hold on, we don't even have it cause they did it …
Ad-48	00:06:37	T/R1	You didn't even have to do this in the fourth grade.
Ad-49	00:06:39	Brian	I don't remember.
Ad-50	00:06:42	Romina	We need to go on. Let's do the five of the two cause they already did that. We didn't do that. We just came out right with what we had. Does that really matter?

Ad-51	00:06:51	Brian	Look it's going to be the same thing. The Y's can be the R's in that the R's can be the Y's in that. So it still comes out.
Ad-52	00:07:00	Romina	That wouldn't be it. No, wrong. No, that's wrong (Looking at paper and crossing things out.)
Ad-53	00:07:07	Brian	Having two R's and three Y's is the same thing as having two Y's and three R's.
Ad-54	00:07:11	Romina	What?
Ad-55	00:07:13	Brian	You still gonna have the same amount cause they're still three and two
Ad-56	00:07:20	Jeff	What's the problem?
Ad-57	00:07:30	T/R1	Towers five tall. You have two colors, say red and yellow, and you are selecting (inaudible) some of these towers will have exactly two reds which means they'll have three yellows. How many of them will have two reds and three yellows? And how do you know you've found them? Now you could tell me how many have two yellows and three reds if you want to solve that problem. I don't care. But I asked for (inaudible). That was the problem that I saw on the tape when you were in fourth grade.
Ad-58	00:08:14	Romina	I know, fourth grade it was easy. I hope we can do it now.
Ad-59	00:08:19	Brian	I don't remember stuff that happened last week.
Ad-60			(Inaudible)
Ad-61	00:08:46	Brian	Jeff, you're the bomb.
Ad-62	00:08:50	Jeff	[writing – Romina and Brian watch]. (Inaudible)
Ad-63	00:08:52	Brian	What have you been doing?
Ad-64	00:09:03	Romina	You guys, stop. Brian, Stop. (Three of them looking at Jeff's Paper)
Ad-65	00:09:33	Jeff	Did you get ten with your reds and yellows trick?
Ad-66	00:09:35	Romina	No, I didn't do it because (inaudible)
Ad-67	00:09:39	Jeff	Yeah, but why are you trying to figure this out?
Ad-68	00:09:43	Brian	Would 0, 0, 0, 1, 1 be the same thing as 1, 1, 0, 0, 0? (Jeff and Romina point to paper) I didn't see that. Never mind.
Ad-69	00:10:02	Jeff	Oh, I know. No, I don't know. 1, 1, 0, 0
Ad-70	00:10:03	Romina	[Takes the paper from Jeff and points with the pen]. Hold on, here we go then. Where is the opposite to this? [Taking the paper and pen from Jeff] This. This. What would be the opposite to this?
Ad-71	00:10:14	Brian	0, 1, 1, 1, 0
Ad-72	00:10:17	Jeff	The difference of that would be because these two are opposites.
Ad-73	00:10:20	Brian	It's like read them backwards
Ad-74	00:10:22	Jeff	This one doesn't have a backwards 'cause it's the same thing.
Ad-75	00:10:30	Romina	They proved it already. They could be like that, and then this, and this and this, and then you would have one. This one. This one you would have. 
Ad-76	00:10:43	Jeff	Which one?
Ad-77	00:10:44	Romina	So,

Ad-78	00:10:45	Jeff	So, it would be 0, 0, 1, 0, 1?
Ad-79	00:10:48	Romina	0
Ad-80	00:10:49	Jeff	0, 1, 0, 1 [Jeff takes paper and pen back]
Ad-81			(inaudible)
Ad-82	00:11:08	Romina	Yup. No. Five?
Ad-83	00:11:38	Brian	Listen, if you there's three reds, it's still five high. You're still going to have 10.
Ad-84	00:11:39	Romina	Mm, hm
Ad-85	00:11:44	Jeff	Yeah, but you're saying if you had two yellows it would be ten.
Ad-86		Brian	As long as it's five high, it's still three of one color and two of another.
Ad-87		Romina	Why? Cause that's what she wants to know.
Ad-88	00:11:49	Brian	You're gonna have ten. You could have this and this (holding up the blocks WWWYY and YYYWW). You would have ten, and then if you just got.
Ad-89	00:12:00	Romina	Oh, that, yeah.
Ad-90	00:12:07	Brian	It don't matter if you have three. The height is going to make the amount itself.
Ad-91	00:12:09	Romina	Oh, I know that. I'm just saying I thought you had like…
Ad-92	00:12:12	Brian	I don't have no breakthrough. I don't have breakthroughs in my life.
Ad-93	00:12:28	Jeff	I'm getting a little frustrated.
Ad-94	00:12:30	Romina	Oh, okay then.
Ad-95	00:12:35	Brian	School's just gonna go on.
Ad-96	00:12:39	Romina	[Puts her head down on the desk. Jeff has the pen and paper.] I have no clue.
Ad-97	00:12:46	Romina	[Picks up her head and looks on Jeff's paper]. What's the total? What's it doing? All right?
Ad-98			(inaudible)
Ad-99	00:13:14	Romina	I don't know what I'm doing.
Ad-100	00:13:24	Brian	(inaudible) like a regular response then.
Ad-101	00:13:28	Romina	Maybe we should just do the binary system. This could have been easier.
Ad-102	00:13:32	Jeff	But Mike's the mad scientist.
Ad-103	00:13:37	Jeff	This before was just like (inaudible). That's one, right? And then this (inaudible) That means that there's four right here. That's four I got it.
Ad-104	00:14:00	Romina	That would only make four, right?
Ad-105	00:14:04	Brian	Look, one, two …
Ad-106	00:14:05	Romina	No, there would be …
Ad-107	00:14:06	Jeff	One, there's five.
Ad-108	00:14:08	Romina	Say we had four red and one yellow. Then that would be five.
Ad-109	00:14:14	Brian	I'm thinking of something right now.
Ad-110	00:14:32	Romina	I don't know what the binary system is. I don't know how to do this. I think I was also absent when you did this.
Ad-111	00:14:43	Brian	Probably shouldn't have said my name (inaudible)
Ad-112	00:15:03	Jeff	I'm missing two. Oh, the four ones. Alright. [writing]
Ad-113	00:15:07	Romina	What are you doing? [To Jeff who is writing]
Ad-114	00:15:13	Jeff	Can I think for one second? I think we're all set.
Ad-115	00:15:14	Romina	You're getting mad.
Ad-116	00:15:15	Jeff	How many times are you going to ask me that? How many times did you tell me to write something down?
Ad-117	00:15:17	Romina	Okay, go!

Ad-118	00:15:19	Jeff	You know how you were complaining about the old guy in English cause every time you write something he (inaudible).
Ad-119	00:15:21	Romina	Okay, go.
Ad-120	00:15:32	Jeff	These are all the combinations that we have. This and the other, the opposite. So, that would be twenty. This. For a total of thirty-two.
Ad-121	00:15:44	Romina	Mm-hm
Ad-122	00:15:45	Jeff	[Shows Romina from his paper] Ten of these. Five of these. Two of these. Oh, no. One of these. And that's sixteen. And then everyone has the opposite colors and so forth. Five, five, one color (inaudible) I thought it was one deal. [Brian is building towers]. Five high and zero of the other color. Five high and two of the other colors. And just half. Do you understand? Five high…
Ad-123	00:16:33	Romina	This is ten…
Ad-124	00:16:34	Jeff	Yeah, five high with no other colors in there.
Ad-125	00:16:39	Romina	So how is it ten?
Ad-126	00:16:40	Jeff	Zero.
Ad-127	00:16:41	Romina	Okay.
Ad-128	00:16:49	Jeff	Wait.
Ad-129	00:16:50	Romina	Yeah, that equals zero. I understand.
Ad-130	00:16:51	Jeff	Wait, look what I'm (inaudible)
Ad-131	00:16:54	Jeff	Five high
Ad-132	00:16:55	Romina	Five times two equals ten
Ad-133	00:16:58	Jeff	Five high one other color equals that. Five high two other colors equals that.
Ad-134	00:17:02	Romina	So with one color there would only be five different. Okay.
Ad-135	00:17:03	Jeff	Yeah.
Ad-136	00:17:04	Romina	Okay.
Ad-137	00:17:04	Jeff	So. It's that plus the opposite. Does that make sense?
Ad-138	00:17:09	Romina	Yes. At least, it makes sense on my end.
Ad-139	00:17:10	Jeff	How do we explain about the zero would get here if there's zero when there's one?
Ad-140	00:17:19	Romina	We have a zero color cause we have just one color. We can do that. Okay.
Ad-141	00:17:32	Romina	I don't know. They probably know.
Ad-142	00:17:38	Brian	So what? What do you have?
Ad-143	00:17:44	Brian	Listen, listen, listen, listen. There two that don't have an opposite. (Holding up towers that he built) This doesn't have an opposite. I didn't make this one, but there's another one.
Ad-144	00:17:54	Jeff	10001
Ad-145	00:17:56	Romina	Yeah, don't.
Ad-146	00:17:58	Jeff	Yes it does. The opposite colors right here.
Ad-147	00:18:01	Romina	No, this one don't have an opposite. These three don't have an opposite.
Ad-148	00:18:06	Brian	Yeah, then forget that.
Ad-149			(inaudible)
Ad-150	00:18:15	Brian	Listen, listen, listen, listen, listen. Ones can be put into four different positions than I don't know how. Below, down, down, I don't know. How can you … (refers to block towers)
Ad-151	00:18:20	Jeff	How come you have two whites on top of that one over there? Oh, yeah, two whites on top, alright. How about …

Ad-152	00:18:32	Brian	This one doesn't have an opposite either (holding up a tower)
Ad-153	00:18:33	Jeff	How about this one? (holding up another tower) Opposite.
Ad-154	00:18:39	Brian	That's part of the thing I'm trying to do. Ones are placed in four original positions, times two because of having an opposite.
Ad-155	00:18:47	Romina	Plus two.
Ad-156	00:18:48	Brian	Plus the other two.
Ad-157	00:18:50	Jeff	So at five high
Ad-158	00:18:51	Brian	So at five high
Ad-159	00:18:52	Romina	So, that like just figuring it out.
Ad-160	00:18:54	Brian	So it's just like an equation. If it's five high, it's what it's going to be.
Ad-161	00:18:58	Romina	We have to figure it out.
Ad-162	00:18:59	Brian	Five high.
Ad-163	00:19:00	Romina	That's not an equation. I think they want an equation. You just figured it out.
Ad-164	00:19:04	Jeff	So what are you saying. You have five high, and what are you doing with that?
Ad-165	00:19:07	Brian	What do you mean what am I doing with it?
Ad-166	00:19:08	Romina	Now, look at what he did. Can I just show you?
Ad-167	00:19:11	Jeff	(inaudible)
Ad-168	00:19:12	Romina	Okay
Ad-169	00:19:13	Jeff	Explain it again.
Ad-170	00:19:19	Brian	Four places and opposite.
Ad-171	00:19:20	Jeff	So that's eight. So that would be four times two.
Ad-172	00:19:23	Romina	Plus the two that don't have …
Ad-173	00:19:24	Brian	Plus the two that are impossible to get opposites cause you change the number of what it should have. So this, or whatever.
Ad-174	00:19:35	Jeff	(inaudible)
Ad-175	00:19:37	Romina	What?
Ad-176	00:19:38	Brian	Other than that, you end up having three whites, and you just get screwed over.
Ad-177	00:19:40	Romina	(Talking to Jeff) Isn't that exactly what you did?
Ad-178	00:19:46	Jeff	(inaudible)
Ad-179	00:19:46	Romina	Not until they do. Let me see what you did.
Ad-180	00:19:56	Jeff	Brian, really just like that (referring to his paper) with one you have up top. A tower of five, five with one block of a color it's like one orange would be five times one cause it could go in one of five spots and then you've got that and then times its opposite so it's times two.
Ad-181	00:20:14	Brian	I'm fine.
Ad-182	00:20:18	Jeff	And with this you have two cause there's two. They can go in two spots at a time. Five spots, and then double that would be thirty. And then this just is ones and zeros. We did nothing else. We have the call her over.
Ad-183	00:20:38	Brian	That's a nice little thing you've got there. Cause that'll work.
Ad-184	00:20:39	Jeff	Dr. Maher?
Ad-185	00:20:39	Romina	But then, this one wouldn't be times zero cause five times zero is zero. Right?
Ad-186	00:20:42	Jeff	Then that's not gonna work. We'll cross this out. Thanks.
Ad-187	00:20:49	Brian	Nobody saw that. Really it is zero.
Ad-188	00:20:55	Romina	Yeah, it is zero, but you have to work it out to make it equal one. And then you do all the opposites.
Ad-189	00:21:03	Brian	Now is there only five for that one? Yeah, alright.

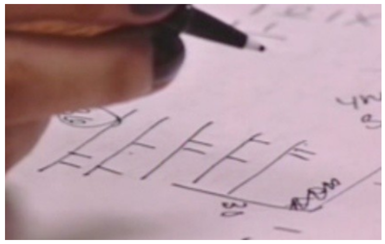
Ad-190	00:21:05	Romina	Yeah, cause look (pointing to paper).
Ad-191	00:21:06	Brian	Yeah, yeah, yeah, yeah.
Ad-192	00:21:10	Romina	This one would be five, this would be five, and this would be ten. And this would be one. Sixteen times two.
Ad-193	00:21:14	Brian	Alright.
Ad-194	00:21:23	Jeff	(inaudible)
Ad-195	00:21:24	Romina	I know. Now we just sit and wait.
Ad-196	00:21:27	Jeff	It's just, it's like not that hard if you do one grouping by itself. (inaudible)
Ad-197	00:21:37	Romina	Did this happen in the class now?
Ad-198	00:21:40	Jeff	With us?
Ad-199	00:21:41	Romina	And it's not their list.
Ad-200	00:21:54	Jeff	(Talking to T/R1) We really just said that you have a thing of five high and if you have one color in it. One color times the five different spots it can go in would be five, and then …
Ad-201	00:22:08	T/R1	This isn't it here, is it?
Ad-202	00:22:09	Jeff/Romina	That's not it. That's what he said.
Ad-203	00:22:10	Jeff	And then, you have the opposite colors so you can go say one's red and zero is yellow, then you can go yellow, red, red, red, so there would be ten for that, right? And if there's ten, we did there's five high times the two of one color in it, and that gave us ten. Flipped over the other way would give you twenty. Twenty plus ten is thirty. Excuse me. And then there's the zeroes or like all reds or all yellows which makes thirty-two which is the total that you can get. And that's how they divide up into and that's the number of ones that have…
Ad-204	00:22:45	Romina	Just simple multiplication. It's just simple multiplication. (pointing to paper) This one with one would be five times one, and this would be five times two, like how many with two colors. Five times two.
Ad-205	00:22:58	T/R1	Okay. So, you answered even more than I asked, right? You didn't just tell me how many with just two reds.
Ad-206	00:23:03	Jeff	Yeah, but, in order for me to get that, I had to like… I just tried to find some kind of connection between
Ad-207	00:23:07	Brian	The other stuff and this.
Ad-208	00:23:08	T/R1	Okay, very nice. Okay then, that's very interesting then. It sort of connects to some of the stuff that I wanted us to talk about today.
Ad-209			
Ad-210	00:24:46	Romina	Okay, I'm going to use ones, zeroes, and x's.
Ad-211	00:24:47	Jeff	Ones, zeroes and x's? I want to use hearts, squares, and o's.
Ad-212	00:24:54	Romina	[Puts down pen and pushes paper away from herself]. Fine. You do it.
Ad-213	00:24:57	Jeff	It was a joke - that's a great idea.
Ad-214	00:24:58	Romina	[Picks up the pen again]. Shut up.
Ad-215	00:24:59	Jeff	How 'bout, we just use three letters though? Or, three numbers?
Ad-216	00:25:02	Romina	I don't want to…
Ad-217	00:25:09	Jeff	I said let's just use one thing rather than …
Ad-218	00:25:10	Romina	I'm not going to stop you. (Working on her paper) 1, 0, 0, X (inaudible)
Ad-219	00:25:22	Romina	Actually, okay …
Ad-220	00:25:24	Jeff	Alright, well then, figure with this one, it could have four right? Right? No.

Ad-221	00:25:31	Romina	Three.
Ad-222	00:25:33	Jeff	X, O, 1 Right? And then you go O, 1, X That could have three. (inaudible) Nine. Come on, Romina, I know you can (inaudible). Then you can do X, O, 1. Nine.
Ad-223	00:25:52	Romina	Soon as you have like three (inaudible)
Ad-224	00:25:54	Jeff	Nine times three is seventy-two. Remember, three minus one is two plus the seven is nine?
Ad-225	00:26:01	Romina	Nine times three?
Ad-226	00:26:04	Jeff	Nine times three is seventy…
Ad-227	00:26:05	Romina	Twenty-seven.
Ad-228	00:26:06	Jeff	I got it backwards. Nine times eight is seventy-two. (Romina laughs)
Ad-229	00:26:14	Brian	(inaudible) It's twenty-nine and it's like twenty-six.
Ad-230	00:26:19	Romina	I was like oh my god. [laughing] I have such trouble with simple stuff.
Ad-231	00:26:21	Jeff	My god, I just got them totally like mixed up. Okay.
Ad-232	00:26:23	Brian	I'm happy. I'm the only one who hasn't screwed up yet. First time in my whole life.
Ad-233	00:26:25	Romina	Okay, come on. Two times three; I'm an idiot. Twenty-four.
Ad-234	00:26:34	Jeff	Two and one …
Ad-235	00:26:35	Romina	Right here is twenty-nine.
Ad-236	00:26:37	Brian	What's up with the divide by?
Ad-237	00:26:38	Romina	(Showing Brian her paper) Well, because look. Like keeping here.
Ad-238	00:26:40	Brian	Are we only using like two colors?
Ad-239	00:26:43	Romina	Yeah.
Ad-240	00:26:44	Brian	Alright.
Ad-241	00:26:45	Jeff	(Working by himself) One, Zero …
Ad-242	00:26:46	Brian	And that's four high?
Ad-243	00:26:47	Romina	Yeah.
Ad-244	00:26:51	Jeff	That's three, six, nine, twelve, fourteen, fifteen.
Ad-245	00:26:58	Romina	All right, we put it in every space, right? And that would give us like four times three. That would give us twelve.
Ad-246	00:27:05	Jeff	Wait, Ankur, Ankur. I have a question. (He joins Ankur and Michael. The conversation between Romina and Brian continues in the background.)
Ad-247	00:27:19	Brian	Fifteen times three.
Ad-248	00:27:20	Romina	For what?
Ad-249	00:27:21	Brian	Look what Jeff just did. Then you multiply by three because that's how many colors there are, and that makes by four.
Ad-250	00:27:30	Romina	But I thought that (inaudible)
Ad-251	00:27:51	Brian	But in order to get to that seventy two, you would have to add eighteen more, which I don't know where you gonna get that.
Ad-252	00:27:54	Romina	It is seventy-two?
Ad-253	00:27:55	Brian	No.
Ad-254	00:27:58	Romina	It's fifty-four.
Ad-255	00:27:59	Brian	'Cause I don't really want to tell you what Jeff just did right there.
Ad-256	00:28:05	Romina	Five, six. So he got? I don't know what he did.
Ad-257	00:28:08	Brian	He's going three, five…three, six, nine, twelve, fifteen, eighteen.
Ad-258	00:28:22	Romina	He… (inaudible) [To Jeff] Can you explain what you did there?

Ad-259	00:28:25	Brian	You changed the first space on each one.
Ad-260	00:28:26	Jeff	No, I changed the last space. I just did. All right. X you could only have. You could have, X, O, 1 and you could get three for each of them. X, O, 1, X; X, O, 1, 1; X, O, 1, O.
Ad-261	00:28:41	Brian	And you multiply that by three.
Ad-262	00:28:42	Jeff	Yeah, so this would be like one, two, three, four, five, six. Now, you could say take out the middle one, so we have X, blank, O, X.
Ad-263	00:28:57	Romina	One.
Ad-264	00:28:58	Jeff	Blank, O, 1; O, 1, O Right. So, let's go So we could also go…
Ad-265	00:29:07	Romina	It would be six.
Ad-266	00:29:09	Jeff	Wait. Now. And this one, this would have to be X if you wanted… I mean this would have to be …This could be 1, 3. It could be any of the three. And this could be any of the three. And then you would go O, 1, X; O, X, 1;
Ad-267	00:29:31	Brian	Alright. We'll prove ourselves wrong again.
Ad-268	00:29:34	Romina	This would be a 1.
Ad-269	00:29:35	Jeff	Look, look, look, look. If you put a 1 there, that's the same as O, 1, X, 1. It's the same as the one we already counted that's right there. So, it's not good.
Ad-270	00:29:54	Brian	So, why don't we just put like X, something, blank, whatever?
Ad-271	00:29:58	Romina	You have the right idea.
Ad-272	00:30:03	Jeff	Because X, O, blank, 1. (inaudible) Blank, O, X. Can't do that. Then you go X, O, O, 1. We didn't write these up here, because these are the only ones you can get three out of. You can't get three out of these because if you put X, O, O, you can't put another X. It's not going to work, because you can't put X, O, O, O, so there's only one of these. X, O, O, O.
Ad-273	00:30:43	Romina	It was easy. Give him a number and he multiplies. Why don't I. We'll do the first …
Ad-274	00:30:58	Brian	(inaudible)
Ad-275	00:31:00	Romina	Okay, four times three (inaudible). I'm not sure where you get the four.
Ad-276	00:31:08	Brian	'Cause you could move that blank into four different spots. Like you could come up with a certain amount for this one…
Ad-277	00:31:11	Romina	Make it double.
Ad-278	00:31:12	Brian	A certain amount for that.
Ad-279	00:31:15	Jeff	So, X…
Ad-280	00:31:18	Brian	So, come up with a formula. Which will be… It can't be that, doubles.
Ad-281	00:31:25	Romina	That's what we're trying to do. All right. (Jeff continues to figure quietly).
Ad-282	00:31:37	Brian	(Overhears Ankur and Michael) Thirty-six? I can beat that.
Ad-283	00:31:40	Michael	What do you have?
Ad-284	00:31:44	Ankur	We could prove you wrong then.
Ad-285	00:31:45	Brian	We went from seventy-two to your fifty-four.
Ad-286	00:31:48	Ankur	We went to seventy-two, to fifty-four, to
Ad-287	00:31:51	Michael	to thirty-six....
Ad-288	00:31:51	Ankur	to forty-five.
Ad-289	00:31:53	Michael	Forty-five, I mean.
Ad-290	00:31:57	Jeff	You are just going down by multiples of five, each time.
Ad-291	00:32:00	Ankur	[laughs] Seems like it.
Ad-292	00:32:03	Michael	I don't think we are stopping at thirty-six, I mean forty-five.

Ad-293	00:32:05	Ankur	What do you have?
Ad-294	00:32:06	Brian	Forty-eight.
Ad-295	00:32:08	Jeff	No we don't. Brian just picked a number out of his head. [They all laugh]
Ad-296	00:32:11	Ankur	You're a bum, you just wanted to beat us.
Ad-297	00:32:12	Brian	No, I had an idea in my head. I probably have been right so many times in my life but I just didn't want to say anything.
Ad-298	00:32:21	Romina	You just didn't want to prove it?
Ad-299	00:32:23	Brian	Yeah, 'cause I, I flip tables when you try to prove me wrong. No smirking around in the background.
Ad-300	00:32:47	Romina	Okay, this is getting really confusing.
Ad-301	00:32:50	Brian	Just all three, or can you just have like…
Ad-302	00:32:52	Romina	You have to have all three
Ad-303	00:32:54	Brian	You can't just use like …
Ad-304	00:32:55	Romina	And
Ad-305	00:32:57	Brian	All 1's, all O's, all X's?
Ad-306	00:33:01	Romina	I keep on writing the same thing over and over again.
Ad-307	00:33:06	Romina and Jeff	X, O, …
Ad-308	00:33:07	Brian	Why did you write 1, O, X, O?
Ad-309	00:33:13	Romina	'Cause you have to have three. It's got to be four high…
Ad-310	00:33:15	Brian	Oh, right. Alright.
Ad-311	00:33:22	Brian	(looks for the time) I thought they put in a new clock system in this place.
Ad-312	00:33:30		(the three of them continue to write 1's and 0's and X's on their papers)
Ad-313	00:34:33	Romina	Did we cancel out fifty-four as a possibility?
Ad-314	00:34:36	Ankur	It's not, it's less than fifty-four.
Ad-315	00:34:38	Romina	Definitely?
Ad-316	00:34:39	Ankur	Yeah.
Ad-317	00:34:40	Brian	What do you guys have a formula?
Ad-318	00:34:41	Ankur	Trying to get it.
Ad-319	00:34:41	Michael	We don't have a formula.
Ad-320	00:34:43	Brian	So, then how can you going to prove that?
Ad-321	00:34:45	Michael	Don't worry about it. Shut up, shut up.
Ad-322	00:34:50	Brian	I'm on thirty right now. (The three of them work together again as the camera continues to focus on Michael and Ankur)
Ad-323	00:34:51	Romina	I don't know how I got it. You have three or two of them put together. (inaudible) and then you mostly (inaudible)
Ad-324	00:35:07	Jeff	Two next to each other and there are six of them.
Ad-325	00:35:10	Romina	Yeah. And the other one is (inaudible)
Ad-326	00:35:17	Jeff	There's, there's three times six of these that's eighteen. One's with two next to each other in the middle is six, and then that's…
Ad-327	00:35:31	Romina	(inaudible)
Ad-328	00:35:32	Jeff	No it's not.
Ad-329	00:35:33	Brian	That's thirty so far.
Ad-330	00:35:34	Jeff	Yeah, thirty.
Ad-331	00:35:35	Brian	And the other two. Thirty-two.
Ad-332	00:35:36	Romina	And what is that one on the end? Double at the end.

Ad-333	00:35:42	Jeff	I don’t get it now.
Ad-334	00:35:44	Romina	Are you boys done? I just don’t want to do this.
Ad-335	00:35:49	Jeff	Um, X, X,
Ad-336	00:35:54	Brian	(inaudible)
Ad-337	00:35:54	Romina	It’s it’s. I don’t know why, it’s just that I’m looking at it and it’s confusing me.
Ad-338	00:36:05	Jeff	0, 0 …
Ad-339	00:36:07	Brian	Aren’t we doing four high?
Ad-340	00:36:09	Romina	Yes.
Ad-341	00:36:15	Jeff	Wait, wait. I can’t believe (inaudible).
Ad-342	00:36:21	Romina	Dude, I’ve written the same thing ten times now.
Ad-343	00:36:23	Jeff	O, 1, O. That’s … Bring the X’s down. (inaudible) Two, Four, Six. Two, Four, Six.
Ad-344	00:36:52	Brian	Does all of them have six?
Ad-345	00:37:03	Jeff	Eight, Ten, Twelve, Fourteen, Sixteen, Eighteen, Twenty.
Ad-346	00:37:10	Brian	So what’s that, thirty-eight?
Ad-347	00:37:11	Jeff	If what they’re saying is right, then I messed up.
Ad-348	00:37:13	Brian	They didn’t have thirteen.
Ad-349	00:37:18	Jeff	They didn’t have thirty-one.
Ad-350	00:37:19	Romina	Okay, what happens when I change the 1’s around?
Ad-351	00:37:22	Jeff	I have the same thing somewhere. (inaudible) There’s one.
Ad-352	00:37:59	Jeff	I probably missed something.
Ad-353	00:38:00	Romina	Alright. I came up with all the combinations.
Ad-354	00:38:06	Brian	Put that one under there. This over that.
Ad-355	00:38:10	Romina	(inaudible) At the end.
Ad-356	00:38:14	Jeff	Minus one.
Ad-357	00:38:16	Romina	(inaudible) Get this in there.
Ad-358		Jeff	…thirty-six will be the next number I work with [Turns to Ankur and Michael]
Ad-359	00:45:00	Romina	[To Brian and T/R1] It might be thirty-six, because I’m working with sixes now. I mean. Okay. You put them. You pair them up. ‘Cause you’re only going to have. Okay. [Holds up her hands] Let me organize my thoughts a little. You can have ‘em together, together, like here these are together, these are together, these are together. Like two of the same color like in a pattern and then you put them somewhere and you like switch them around. So, I’m up to twenty-four now and I’m going to put them the same way here and here. (Referring to her paper). So that’s thirty. I’m going to put the same one here, here, um.
Ad-360	00:45:43	Brian	Can you do that maybe right there? (pointing to paper)
Ad-361	00:45:44	Romina	Here, here, that’s one. Here, here. I didn’t put them yet. And there’s your thirty-six ‘cause one, two, thirty-six. Right? (See figure I-6) That’s four and that’s six, and then (makes more marks on the paper and counts lines) How did I just get thirty-six again? Okay, we know that this is… Six, six, six. Okay, that’s thirty. Oh, no, you guys for thirty-six?



Ad-362	00:46:24	Brian	You had the way right there...
Ad-363	00:46:27	Romina	Oh, that's not it then. [Crosses off her 36 answer on the paper] Hold on.
Ad-364	00:46:47	Jeff	First and third.
Ad-365	00:46:48	Romina	[Gets a new sheet of paper. Explains to Jeff and Brian] First and third. Okay. Thirty-six. Ready? The way we did it. You’ve got two of the same color, right? Two of the same color which stands for where I’m putting these and these. You’re going to have ‘em in. And the rest you fill up, right? And you going to have ‘em in. And there’s only two other ones that you could have. So you have this one which you’re going to multiply by two. Hold on. One, two, three, four, five, six.
Ad-366	00:47:16	Jeff	So that’s only twelve.
Ad-367	00:47:23	Romina	Okay. No, you multiply this by two, and this by two. Multiply this by two and this by two. By two, and by two. Then, how much is that? One, two, three, four, five, six.
Ad-368	00:47:34	Jeff	Is this the way?
Ad-369	00:47:35	Romina	No. I’m just (inaudible). You know what?
Ad-370	00:47:39	Brian	You multiply by two ‘cause this is what you can get with one color.
Ad-371	00:47:40	Jeff	But here’s what I’m saying. Hold on. Hold on. Is here the three? (inaudible)
Ad-372	00:47:48	Jeff	It’s what I’m saying. You crossed everything out in the middle of your explanation.
Ad-373	00:47:53	Romina	Okay, guys. One, two, three, four, five, six, right? (inaudible) For each one here, you have six other combinations. You have two for this one,
Ad-374	00:48:00	Jeff	That’s why we multiply by two.
Ad-375	00:48:01	Romina	You multiply by two, and then you multiply this by two. (inaudible)
Ad-376	00:48:10	Brian	Multiply by two. Multiply all of them by two not the whole thing by three.
Ad-377	00:48:12	Jeff	The whole thing by three by three you’re saying?
Ad-378	00:48:13	Romina	Yeah.
Ad-379	00:48:14	Brian	‘Cause each one has three.
Ad-380	00:48:18	Romina	You want to make a neater one?
Ad-381	00:48:21	Jeff	Then we can put (inaudible) X, O, O, X.
Ad-382	00:48:27	Romina	It’s only four high.
Ad-383	00:48:34	Jeff	X, O, O, X
Ad-384	00:48:39	Romina	Everywhere there’s a space, you just do that. So that’s why you multiply by two.
Ad-385	00:48:43	Brian	The amount of colors, you would multiply that by four ‘cause if there was four different…
Ad-386	00:48:51	Jeff	X, O, O, X. And then, yeah. X, O, O, X. You can find it. Oh, wait.
Ad-387	00:49:02	Romina	I’m making a neater copy for her.
Ad-388	00:49:06	Jeff	Then you multiply by three ‘cause there are three different colors. So that would be two, four, six, eight, …

Ad-389	00:49:13	Brian	Thirty-six. Twelve times three.
Ad-390	00:49:15	Jeff	Times three.
Ad-391	00:49:18	Brian	I know it's going to be wrong 'cause
Ad-392	00:49:19	Jeff	No, we're not even (inaudible)
Ad-393	00:49:28	Romina	(inaudible) Okay, one, two, three, four, five, six.
Ad-394	00:49:31	Brian	The society of the summer session. Did you hear me, or are you not listening?
Ad-395	00:49:35	Romina	I kept up all my notes. What do you got?
Ad-396	00:49:38	Brian	Did you even read the book? How did I know, like, everything about the story?
Ad-397	00:49:51	Romina	Uh, oh.
Ad-398	00:49:51	Jeff	What?
Ad-399	00:49:59	Romina	(inaudible) Times two, times two, times two…
Ad-400	00:50:00	Jeff	(inaudible) to justify what the O (inaudible). Right, the O, X and the X, O above that (inaudible). Beautiful.
Ad-401	00:50:18	Brian	Nice and simple. (inaudible)
Ad-402	00:50:19	Jeff	(inaudible)
Ad-403	00:50:23	Jeff	So how do we justify this even more? Um, this, we have thirty-six of these, right? That means that there's fifty…
Ad-404	00:50:33	Brian	Forty-six
Ad-405	00:50:36	Jeff	Forty-eight. No, forty-one. It's forty-one.
Ad-406	00:50:38	Brian	Four colors?
Ad-407	00:50:39	Romina	I don't know if it's going to be (inaudible)
Ad-408	00:50:40	Jeff	It is eighty-one.
Ad-409	00:50:41	Romina	It has to be eighty-one, but …
Ad-410	00:50:43	Jeff	But that means that forty…
Ad-411	00:50:46	Romina	What are you doing? Eighty-one minus thirty-six?
Ad-412	00:50:55	Jeff	That means that there's forty-five other possibilities.
Ad-413	00:50:57	Brian	Of what?
Ad-414	00:51:00	Jeff	Of not with, with no requirements.
Ad-415	00:51:02	Romina	(inaudible)
Ad-416	00:51:11	Romina	[To T/R1 – who has been with Ankur and Michael] You want to see how we got thirty-six?
Ad-417	00:51:13	Jeff	We got it. We got it. It's thirty-six.
Ad-418	00:51:17	Romina	You want to see how?
Ad-419	00:51:35	Jeff	(inaudible)
Ad-420	00:51:36	Brian	(inaudible)
Ad-421	00:51:39	Jeff	Here's how you know that (inaudible)
Ad-422	00:51:42	Romina	We have like the basic. You know why it might be forty-one though, because …
Ad-423	00:51:48	Brian	Forty-one? Forty-five?
Ad-424	00:51:49	Romina	Forty-five. You know how we have an odd number in here. There's three colors (inaudible)
Ad-425	00:51:57	Jeff	That takes it from, yeah. And then you could have all X's.
Ad-426	00:52:02	Romina	We have to leave that out (inaudible)
Ad-427	00:52:05	Brian	That proves (inaudible)
Ad-428	00:52:05	Jeff	I'm just looking at something. All 0's, all 1's (inaudible). Then you could do (inaudible).

Ad-429	00:52:25	Jeff	That should be (inaudible)
Ad-430	00:52:35	Romina	We have an explanation for you. Okay, for. We figured thirty-six because …
Ad-431	00:52:43	Jeff	Here, pay attention.
Ad-432	00:52:44	Ankur	Hold on.
Ad-433	00:52:45	Jeff	Just listen.
Ad-434	00:52:45	Ankur	He wants to just tell me something real fast.
Ad-435	00:52:47	T/R1	Let them finish.
Ad-436	00:52:58	Jeff	(inaudible)
Ad-437	00:53:00	Romina	(inaudible)

***Disc TWO** of Two

Line	Time	Speaker	Transcript
829	00:00:01	Ankur	I'll prove it to you.
830	00:00:04	Jeff	Wait, I'll be back in a second.
831	00:00:04	Romina	What is eight plus six? Eight, nine, ten, eleven, twelve, thirteen, fourteen. Thank you.
832	00:00:07	Brian	Oh my God. I'm telling you.
833	00:00:09	Romina	You guys I got eighty-four.
834	00:00:11	Michael	Eight-four what, total?
835	00:00:12	Romina	Total.
836	00:00:13	Jeff	And wait, what was your number?
837	00:00:13	Romina	Hold on. But I got, you guys, it makes sense.
838	00:00:15	Jeff	What was your number?
839	00:00:16	Ankur	We had eighty-one total.
840	00:00:17	Jeff	I mean, your other number? Thirty-nine?
841	00:00:18	Ankur	Thirty-nine.
842	00:00:19	Jeff	And we got thirty-six?
843	00:00:21	Romina	You guys, cause the other... did we include the all x's, the all one's, and the all o's and the other one?
844	00:00:25	Jeff	No, we have to add the three because we had to do that for the pizza. Like the plain...
845	00:00:26	Romina	Because look at what I did, look...
846	00:00:28	Ankur	No, but we added to the pizza but we didn't add it to the tower problem.
847	00:00:30	Romina	You guys, you guys you know how we have our x to the y system? Oh, I'm just talking to myself.
848	00:00:35	Jeff	No, we were...
849	00:00:35	Ankur	Can I tell you right now why it's not eight-four?
850	00:00:37	Romina	Hold on can I tell you why it could be eight-four?
851	00:00:39	Ankur	Can I tell you first?
852	00:00:40	Romina	No, I don't want to, no, okay, go ahead.
853	00:00:42	Ankur	Cause look...
854	00:00:43	Brian	We don't have to have a brawl like we do in history.
855	00:00:43	Ankur	there's four spots, right? So for the first one, there's three colors...
856	00:00:47	Romina	But Ankur, do you agree, hold on, do you agree that you other thing works?
857	00:00:49	Ankur	Just cut me off.

Line	Time	Speaker	Transcript
858	00:00:51	Romina	Ankur, I'm just doing your other thing.
859	00:00:51	Brian	Where do you think you've been for the last sixteen years of our life?
860	00:00:55	Romina	I know.
861	00:00:56	Ankur	She's like, 'okay, I'll let you explain'. I start to explain.
862	00:00:58	Romina	Okay, go, go, go.
863	00:01:00	Jeff	Go.
864	00:01:00	Ankur	There could be three colors for the first one, three colors for the second one, three colors for the third one, three colors for the fourth one. Right?
865	00:01:05	Romina	Yes.
866	00:01:05	Ankur	Multiply them and you get eighty-one. Now there's no way there can be eighty-four now.
867	00:01:11	Romina	But there could be because you have to add the three.
868	00:01:13	Ankur	But..
869	00:01:15	Jeff	He's including them. Three colors, three colors, three colors, three colors.
870	00:01:18	Romina	But it works. It just works, I don't know why.
871	00:01:21	Ankur	What do you mean it just works?
872	00:01:22	Romina	Hold on. Look at. I am not saying that I am right. I'm not saying that I'm right.
873	00:01:23	Jeff	You prove what you thought, prove what you think.
874	00:01:26	Romina	Okay, you have the thirty-six, right?
875	00:01:28	T/R1	Here. Here. [T/R1 hands her paper.]
876	00:01:29	Romina	Okay, you have the thirty-six.
877	00:01:31	Ankur	Thirty-six what? What you have?
878	00:01:33	Jeff	Uh-hum.
879	00:01:33	Romina	And then you're going... yeah, and then you are going with the x, y deal, right? And say you can't work 'em in all at the same time so you figure one of them might be dropped. Cause that's what we did we worked them in all at the same time and one of them has got to be dropped the other ways we do it. So then it would be two to the fourth because there's two colors, right? And for each one you have to multiply that...
880	00:01:49	Ankur	What's the fourth one?
881	00:01:50	Romina	That's how high it is. That is like your x to the y system. And that equals sixteen. And then there's colors.
882	00:01:56	Ankur	Two to the fourth equals sixteen?
883	00:01:58	Jeff	Yeah.
884	00:01:58	Romina	Two times two.
885	00:01:59	Jeff	Is four times two is twelve, times two is sixteen. Two to the three is sixteen. No, two to the four is.
886	00:02:07	Ankur	Yeah, eight, yeah, is sixteen. Okay.
887	00:02:09	Jeff	Okay.
888	00:02:08	Romina	Okay and then you multiply that by three. I was getting, okay.
889	00:02:11	Ankur	Why by three?
890	00:02:12	Romina	Because three different colors. Right? So one of them is going to be dropped out one time, and then the other one and then the other one. So that's three. So what's that? I didn't do this...
891	00:02:20	Ankur	Sixty-four.
892	00:02:21	Romina	Sixty-four.

Line	Time	Speaker	Transcript
893	00:02:22	Ankur	No.
894	00:02:23	Jeff	Six times three is eighteen..
895	00:02:24	Ankur	Forty-eight.
896	00:02:26	Jeff	Carry the one [inaudible].
897	00:02:27	Romina	And add that to the thirty-six, eighty-four.
898	00:02:32	Ankur	But we've use this method all the time.
899	00:02:34	Romina	Well, I'm just saying that could be..
900	00:02:36	Brian	So, things are subject to change over a lifetime.
901	00:02:38	Romina	But I could, I could
902	00:02:39	Jeff	So we are saying that we have to go back and reprove all of the other problems that we did because we did this wrong?
903	00:02:42	Romina	I could be completely wrong, you guys. Chill out, I could be completely wrong. I'm just saying, couldn't work like that?
904	00:02:46	Ankur	There's probably a mistake in there, somewhere.
905	00:02:47	Jeff	Couldn't you say.. wait, wait, whoa...
906	00:02:50	Romina	There's three doubles, in there.
907	00:02:52	Jeff	Yeah, and those are the three of each one. That double, that double, and that double.
908	00:02:57	Romina	Out of this? So there's three doubles in there. So then there's eighty-one, there.
909	00:03:02	Jeff	What are you saying?
910	00:03:02	Romina	And then there's these doubles, because those go over there?
911	00:03:04	Jeff	No, we are counting these as the three doubles that you just subtracted? Not just any [inaudible]
912	00:03:10	Ankur	What number does that leave us with [laughing]?
913	00:03:11	Romina	I'm just saying...
914	00:03:11	Jeff	Eighty-one.
915	00:03:14	Romina	Why did...?
916	00:03:15	Ankur	I'm kidding. I just...
917	00:03:16	Brian	We do not need any fighting like there is in history every time we do [inaudible] project. No cat fights.
918	00:03:22	Jeff	[inaudible]
919	00:03:26	Romina	What?
920	00:03:32	Brian	Dead silence. Nine thirty.
921	00:03:34	Romina	So alright could this fit into this as our doubles?
922	00:03:39	Ankur	Are you sitting on the bench?
923	00:03:40	Brian	Yeah, I am.
924	00:03:41	Ankur	Are you really?
925	00:03:42	Romina	Ankur, in doing our x and y could this fit into this as our doubles, could we have, could we have two of these?
926	00:03:46	Brian	I have to get up at 8:30 tomorrow morning.
927	00:03:49	Ankur	I have no idea what you said.
928	00:03:51	Jeff	She is saying where are these three doubles in this part?
929	00:03:54	Ankur	Somewhere.
930	00:03:54	Jeff	Where?
931	00:03:55	Romina	Could like, could they fit in there?
932	00:04:01	Brian	What's Mike doing?

Line	Time	Speaker	Transcript
933	00:04:03	Michael	Don't worry about it. [They all laugh] Don't worry about it.
934	00:04:06	Ankur	He's gonna [inaudible]. He's going to prove us all wrong.
935	00:04:10	Jeff	He's going to be walking around with his tweeds and his Birkenstocks.
936	00:04:15	Michael	Don't worry about it.
937	00:04:16	T/R1	While Mike is busy doing that. Let's give you something to think about. Let me ask you a question. I left you with something to think about the last time do any of you remember what that was?
938	00:04:25	Romina	No, we were talking about that yesterday in class...
939	00:04:27	Ankur	I think I know where your mistake is.
940	00:04:30	T/R1	Go head.
941	00:04:31	Brian	Dante.
942	00:04:32	Jeff	Dante?
943	00:04:33	Ankur	Alright, we need the two to the fourth. Just explain that two to the fourth part over.
944	00:04:36	Romina	Okay. At one point we did the one where we have all three of them included. At one point you drop them because you know you have x, x, o, o, you know, or x, x, x, o. So you do like two colors, you do two cause they're four high. Like that. And then you multiply by three because you could do that with three different colors like this could be the ones and this could be the x and this could be.. you know? Do you understand why I multiplied that by three?
945	00:04:56	Ankur	Alright.
946	00:04:57	Romina	Ankur, I don't know, I'm just saying like, I'm working with numbers that's what.... And then you do.
947	00:05:01	Ankur	Yeah, I understand. Alright. When you did, x, x, x, o, right?When you multiply it by three, right now this could be x, x, x, one.
948	00:05:12	Romina	One.
949	00:05:13	Ankur	Or x, x, x...
950	00:05:16	Jeff	x
951	00:05:16	Ankur	x
952	00:05:16	Romina	See that's probably where the doubles would be.
953	00:05:17	Jeff	That's where the x's are.
954	00:05:18	Ankur	When you multiply it by three. Again.
955	00:05:21	Jeff	You multiply those three.
956	00:05:22	Ankur	It could be x, x, x, x, again.
957	00:05:25	Romina	So that's why I subtracted three for this.
958	00:05:27	Ankur	Yeah, and you get eighty-one which is the same as this. That's where your mistake was.
959	00:05:32	Romina	That's what I just said ten minutes ago.
960	00:05:33	Ankur	Yeah.
961	00:05:33	Jeff	I know but before you were yelling and screaming it was eighty-four.
962	00:05:36	Ankur	Yeah.
963	00:05:36	Romina	I asked you I never said I was right. I told you that I could of have been wrong in the beginning.
964	00:05:39	Jeff	We understood that and we agreed and we listened to it and we came up with a logical reason why your answer [inaudible].
965	00:05:44	Romina	I said that ten minutes ago.

Line	Time	Speaker	Transcript
966	00:05:47	Jeff	Mike, what you get?
967	00:05:47	Ankur	Mike.
968	00:05:51	T/R1	So I guess I heard Ankur said that in order to be convinced, right, he would want to know..
969	00:05:58	Jeff	The other ones.
970	00:05:58	T/R1	The other side of the argument. So that will give you something to think about, right? Um, actually this isn't what I had planned for you to do. This is Ankur's problem.
971	00:06:07	Ankur	My problem?
972	00:06:08	Michael	Ankur made it up [inaudible]
973	00:06:08	T/R1	Remember I asked you to make up a problem because you were finished with the other one.
974	00:06:12	Ankur	I made this up?
975	00:06:13	Michael	Yeah. Like you...
976	00:06:14	Ankur	Did I?
977	00:06:15	Michael	when she asked you [inaudible]
978	00:06:16	Ankur	Oh [They laugh].
979	00:06:19	T/R1	Ankur's problem is a lot harder then this problem.
980	00:06:22	Jeff	Yeah, thanks a lot Ankur.
981	00:06:25	Romina	[inaudible]
982	00:06:25	T/R1	But it's a good problem. Don't you think it is a good problem?
983	00:06:27	Brian	Yes, I do.
984	00:06:27	T/R1	I like the problem. So Ankur, you know, why don't you think about the other side and think about your arguments. I know you don't like to write things up.
985	00:06:36	Romina	No..
986	00:06:36	Ankur	How did you [inaudible]
987	00:06:37	Romina	But we were going to yesterday. But we were sitting there, I'm like okay, let's write something up.
988	00:06:41	T/R1	What you were suppose to write up?
989	00:06:42	Romina	Yeah, I was like what was the problem?
990	00:06:43	T/R1	Okay, let me help, remind you. And then I will, I was going to give you something else to think about. Okay? Remember you were, um, Brian was working with these and you were looking at something like a plus b quantity squared. And a plus b quantity cubed.
991	00:06:59	Jeff	Yeah, we were working on the..
992	00:07:00	T/R1	Remember that? What do you remember Jeff?
993	00:07:04	Jeff	That we were looking on, yeah, the cube part. Like making it a cube.
994	00:07:08	Ankur	The different parts of the cube?
995	00:07:10	T/R1	Right. So that was what you were suppose to do...
996	00:07:10	Jeff	That was hard.
997	00:07:11	T/R1	What I am interested in, I don't think...Mike, you might not be listening anyway. Mike and Ankur heard your argument for finding how many of exactly two reds and then you even solved more than that. So I'd really appreciate what this group worked with earlier if you could show that. Because I want to show you something after you do that.
998	00:07:31	Jeff	Alright. Do you want me to go up or?

Line	Time	Speaker	Transcript
999	00:07:33	T/R1	Why don't you do that Jeff since you are going to have to leave. And then Romina can probably take over.
1000	00:07:37	Romina	What, just rewrite this?
1001	00:07:38	T/R1	No, no, the first problem.
1002	00:07:38	Jeff	No, we are talking about the first problem.
1003	00:07:40	T/R1	Okay? Um.
1004	00:07:45	Romina	This one?
1005	00:07:47	Jeff	This one.
1006	00:07:49	Romina	Okay.
1007	00:07:55	Jeff	We were stuck on this one and we felt really stupid because they took them like three seconds. And we were having...
1008	00:07:58	Michael	It's thirty-six.
1009	00:08:01	Ankur	You found the doubles?
1010	00:08:01	Jeff	And we were having problems. Thirty-six Micheal?
1011	00:08:05	Michael	It's thirty-six.
1012	00:08:05	T/R1	Well, we will hear from Mike in a minute, yes.
1013	00:08:07	Michael	It is.
1014	00:08:08	Jeff	Well just that we were trying to figure out something and none of us could get anything. So then I said well, alright, well, let's try to find a math kind of connection to it. So we looked at what we had and we had ten, if we got.. I'll be done in a second.

1015 Jeff is at the board but we can hear Michael and Ankur's conversation. While Michael and Ankur are quietly talking to each other, Jeff is writing the following on the board:

10001	10000
11000	01000
10100	00100
10010	00010
01001	00001
01100	
00110	
00101	
01010	

Jeff, Romina, T/R1, and Brian are quiet. The conversation between Michael and Ankur is as follows.

Line	Time	Speaker	Transcript
1016	00:08:28	Ankur	How about what we originally made. We had a double for each one.
1017	00:08:32	Michael	Now we have to find 45 that aren’t.
1018	00:08:35	Ankur	We found 39. Where’s the other six?
1019	00:08:40	Michael	I don't know.
1020	00:08:47	Michael	So both of us don't know. I want to know how I got that six.
1021	00:08:54	Ankur	I believe. [inaudible]
1022	00:08:58	Michael	No, just for one section.
1023	00:09:00	Ankur	Oh.
1024	00:09:01	Michael	I categorized them as all that end in three, all that end in one, all that end in two. I checked the doubles off. There's six in each.
1025	00:09:10	Ankur	Yeah, okay.
1026	00:09:11	Michael	So it's actually [inaudible]. So it's didn't we just do five [inaudible]

Line	Time	Speaker	Transcript
1027	00:09:13	T/R1	I'm going to need you to write what you did, a write up. [Talking to Romina]
1028	00:09:17	Romina	On this problem, or the other one?
1029	00:09:19	T/R1	The other problem.
1030	00:09:19	Romina	The other one.
1031	00:09:20	Jeff	Alright, on this one.
1032	00:09:22	T/R1	Okay, let's see what Jeff did.
1033	00:09:23	Jeff	Alright, what we did on this one was that we looked at this. And we had, we were just writing out each thing, and we got ten for if there was two in each thing saying a one was red and two was say, blue. And then we knew that this also had a flipped side where that it could be zero, one, one, zero. We knew that, we took that for granted.
1034	00:09:45	T/R1	One was red and two was blue?
1035	00:09:48	Romina	Zero is blue.
1036	00:09:49	Ankur	Jeff, there's no two.
1037	00:09:51	Jeff	What?
1038	00:09:51	Ankur	Make the zero blue.
1039	00:09:53	Jeff	Oh, excuse me, my bad. We took it for granted that there is a flip for each one. And so then, we're looking at this, and there's ten here and so there was.. first...
1040	00:10:04	Ankur	Certain ones you can't flip, right?
1041	00:10:06	Jeff	Well first we were looking at this here.. No you could flip all of them just different colors like...
1042	00:10:11	Romina	[inaudible]
1043	00:10:11	Jeff	Red, red, blue, red, blue and then red, blue, red, blue, blue. It's just a different colors. We looked at this too and there's five and these all flipped too, so ten. So we were looking at this and we say well there's five colors, it's five long, five, and there's one color in each, one color going in to it, one color is taking one space times one equals five, and then plus the flip so that would equal ten. Times two again. So five times one, five spaces times one block is five times two, the flip side. Then we took this and we said alright, well there's two in here. So five blocks times two. These different color taking up two spots would equal ten and plus the flip side would equal twenty. So that gives you thirty and then there's these and these. So that's it. That's your shortcut.
1044	00:11:14	T/R1	Do you have any questions?
1045	00:11:18	T/R1	Now, would that work...
1046	00:11:19	Ankur	He just proved thirty-two.
1047	00:11:21	T/R1	Would that work if you were doing it with blocks four tall and I wanted to know where two reds?
1048	00:11:26	Jeff	And then we would have said. If we were doing this with four tall we would have said well we have four blocks times one.
1049	00:11:27	Ankur	Two of a colors, five tall.
1050	00:11:30	Michael	So wait, it's not this problem?
1051	00:11:32	Ankur	No. It's the first problem we did. [Michael and Ankur laugh]
1052	00:11:37	Jeff	We would do four blocks times one block taking up each one would give us eight. And then we flip it which would be sixteen. And then we would say four blocks, one block, four times four would be four times two would give us eight. Which would be plus eight. And then plus each one filled up would be plus two. Wait. That would

Line	Time	Speaker	Transcript
			be...And how do we usually figure these out?
1053	00:12:04	Romina	What is it, how many different colors?
1054	00:12:07	Jeff	Two, four high so...
1055	00:12:09	Ankur	Two to the fourth. Sixteen.
1056	00:12:13	Jeff	Times two, sixteen? No, two times two..
1057	00:12:17	Ankur	Is four, times two is eight, times two is sixteen.
1058	00:12:17	Romina	Times two is eight, times two is sixteen.
1059	00:12:18	Jeff	Sixteen. So then I messed up somewhere. So that wouldn't work for that. Wait, wait, let me just think for a second.
1060	00:12:26	Romina	Wouldn't be eight?
1061	00:12:26	Ankur	Why do we need a new way to do it?
1062	00:12:28	Jeff	Oh, oh, yeah, that's where I messed up. Four times... For the first one, four times one is eight, would equal eight.
1063	00:12:34	Ankur	Yeah.
1064	00:12:36	Jeff	And then, then.
1065	00:12:37	Romina	Four times...
1066	00:12:38	Ankur	Four times two is.
1067	00:12:39	Jeff	Yeah, but wait but you could flip them though, couldn't you? I can right?
1068	00:12:43	Ankur	Yeah. So then there's sixteen.
1069	00:12:44	Jeff	Yeah, no, we flipped already. Four times one is four times two is eight. Like four times one doesn't equal eight. Four times one times two equals eight.
1070	00:12:55	Ankur	Oh, okay.
1071	00:12:55	Jeff	And then we said four times two actually equals eight times two is sixteen. And then we got thirty-two, though. That would be thirty-two. Cause, sixteen...no, it would be twenty-four, sixteen plus eight. I'm still messed up somewhere.
1072	00:13:10	Romina	Yeah, that, we....
1073	00:13:10	Ankur	Now why do we need a new way to count up the total [inaudible]?
1074	00:13:12	T/R1	Well.
1075	00:13:13	Jeff	Well we were just looking back at. She asked would it work for four. And it should work with four if that works and it's not working with four.
1076	00:13:21	T/R1	Okay, let me try to answer Ankur. I'm trying to understand why that rule works. So what I like to ask you to think about, you have this rule and you sort of believe in it. So this group did some kind of analysis and accounted for all they could find and did get thirty-two for a variety of ways of organizing it. When you try to do it with four spots, it sort of didn't work.
1077	00:13:55	Romina	Yeah.
1078	00:13:56	T/R1	I want you to think about that I want you to think about it for fives, fours, threes, sixes...
1079	00:13:57	Michael	[inaudible] all of these and then I'm going to eliminate those. [He is talking to Ankur as T/R1 is talking]
1080	00:14:01	T/R1	And see if you can make sense of the rule by counting for those pieces. Does that make sense? Do you understand the question? Should it make sense? Should it work out? Just like what you're working out with the, what's left over. It should work out both ways, right?
1081	00:14:23	Romina	It should.

Line	Time	Speaker	Transcript
1082	00:14:24	T/R1	So I'm asking you to do the same thing here. Do you understand, Ankur? Cause you said to Jeff that you weren't going to be convinced unless you worked it out the other way. Well, I'm saying I'm not going to be convinced unless I see it both ways.
1083		The two groups split up. Ankur and Michael look over their papers as Jeff, Romina, and Brian look over their papers. The camera focuses on the Jeff, Romina, and Brian. They are talking about the problem. The following transcript is the conversational between Ankur and Michael.	
1084	00:14:42	Michael	Do you know what I want you to prove? When I'm doing this, I want you to prove..
1085	00:14:49	Ankur	There's probably more over here.
1086	00:14:51	Michael	No, I just want you to prove this is it.
1087	00:14:53	Ankur	I probably can't.
1088	00:14:54	Michael	Just do it.
1089	00:14:55	Ankur	Let's see.
1090	00:14:58	Michael	Now I'm doing [inaudible]
1091	00:15:29	Ankur	[inaudible] Six more.
1092	00:15:37	Michael	Just times.... Just look to see if you find anything else, anything wrong, any doubles, anything
1093	00:16:14	Ankur	What about these... 1, 1, 2, 2. I found the six…1,1,3….
1094	00:16:24	Michael	Exactly, so...
1095	00:16:26	Ankur	There's 45
1096	00:16:30	Michael	No, show me, write it down.
1097	00:16:34	Ankur	One, one, two, two. It's looking right at us.
1098	00:16:38	Michael	I believe that. [inaudible] I'm just making, I'm being a 100% sure. Double check.
1099	00:16:46	Ankur	Three, four, five, six. Thirteen, fourteen, fifteen, eighteen plus how many do you have?
1100	00:16:55	Michael	I have..down here?
1101	00:16:56	Ankur	Twenty-seven.
1102	00:16:57	Michael	Yeah, twenty-seven or something like that.
1103	00:17:01	Ankur	Forty-five.
1104	00:17:06	Michael	Yeah, but listen. Ankur, [inaudible]
1105	00:17:17	Ankur	Can you explain this?
1106	00:17:19	Michael	Yeah. [inaudible] Cause I found those doubles over there.
1107	00:17:24	Ankur	No, I can explain the forty-five.
1108	00:17:27	Michael	Some of these are two doubles. And I don't want to explain this.
1109	00:17:32	Ankur	No, they explained the thirty-six and we will explain the forty-five. I can explain the forty-five with just this paper.
1110	00:17:36	Michael	Right. [inaudible]
1111	00:17:41	Ankur	I know, I can explain it, I can explain it.
1112	00:17:44	Michael	I'll double check [inaudible]
1113	00:17:56	Ankur	You guys are right with thirty-six.
1114	00:17:59	Brian	What?
1115	00:17:59	Ankur	You guys are right with thirty-six.
1116	00:18:00	Michael	We can explain the forty-five.
1117	00:18:02	T/R1	We're going to have them explain the forty-five.

Line	Time	Speaker	Transcript
1118	00:18:03	Romina	With this one?
1119	00:18:04	Ankur	You know how you got thirty-six?
1120	00:18:06	Romina	With this?
1121	00:18:06	T/R1	Do you want to hear the explanation Jeff? Do you have to run?
1122	00:18:07	Jeff	I really have to go.
1123	00:18:09	T/R1	Okay, so you can explain it to Jeff tomorrow.
1124	00:18:09	Ankur	Yeah. You guys have [inaudible] explain the forty-five.
1125	00:18:10	T/R1	They'll write I up for you.
1126	00:18:12	Jeff	Oh, they're going to write it up for me?
1127	00:18:13	Ankur	Are we?
1128	00:18:14	T/R1	Sure.
1129	00:18:15	Jeff	Alright. Thank you very much. See you all later.
1130	00:18:17	T/R1	Bye Jeff. [Jeff leaves]
1131	00:18:18	Romina	We could, we could explain the forty-five [inaudible]
1132	00:18:20	Michael	Exactly. So you should do this the [inaudible] part.
1133	00:18:23	Ankur	Alright, no, since I...
1134	00:18:23	T/R1	Okay, that would be good if you explain this...
1135	00:18:27	Romina	I mean, I don't know.
1136	00:18:28	Brian	With the eighty-four?
1137	00:18:28	Michael	Are you going to explain the thirty-six? Cause.. I don't know.
1138	00:18:29	Romina	Well we got the eighty-four and then we subtract the three.
1139	00:18:33	Ankur	Alright.
1140	00:18:35	T/R1	Why don't I give you all a couple of more minutes to think about how you did the formula.
1141	00:18:36	Michael	How about this... explain the thirty-six one more time because I was not paying attention. I was..
1142	00:18:39	T/R1	Do you want to go to the board and do that?
1143	00:18:40	Brian	That's why [inaudible]
1144	00:18:42	Michael	Now I want to know. I want you to explain it. Was it a good explanation?
1145	00:18:46	Ankur	It was a good..
1146	00:18:47	Michael	Not on the board cause...
1147	00:18:49	Ankur	It would be easier with the paper.
1148	00:18:50	R2	Yeah, erase what's there.
1149	00:18:52	Ankur	Romina, it would be easier with the paper because you already have everything written down.
1150	00:18:54	Michael	I wasn't paying attention. And I could explain the forty-five now.
1151	00:18:56	Romina	I knew you weren't paying attention.
1152	00:18:57	Michael	Forty-five is easy.
1153	00:18:59	Romina	Okay. You have, we have all three colors, right? So then when we add one.
1154	00:19:06	Michael	What's the one? What's o and what's x?
1155	00:19:07	Romina	They're three different colors. Like it could be...
1156	00:19:10	Michael	Don't say, don't say anything more. I understand.
1157	00:19:11	Romina	We have three different colors and then we, you know that they have to be paired up. Like the fourth color had to be, has to be the same as one of them that's already there, right?

Line	Time	Speaker	Transcript
1158	00:19:20	Michael	The fourth color....
1159	00:19:22	Romina	Okay, see
1160	00:19:22	Michael	... has to be the same, yes.
1161	00:19:23	Romina	Yeah cause you have.
1162	00:19:24	Michael	Yeah.
1163	00:19:26	Romina	Okay. So what we did, we, well, let's say there's are your different ones. [Someone sneezes] And we came up with six different like possibilities for like the, the match it could be. It would be here and here, the same. Here and here. Here and here. Come on. Which one am I missing?
1164	00:20:02	Ankur	The second.
1165	00:20:02	Romina	Okay.
1166	00:20:03	Ankur	And the last.
1167	00:20:04	Romina	Yeah, the second one and the last. Okay. Do you agree with me? And then each one, this is either going to be an o or an x.
1168	00:20:12	Brian	Or an x.
1169	00:20:13	Ankur	Or an x or an o. So each one, there's two of each one. You can't have x and x.
1170	00:20:18	Michael	Yeah. I get that.
1171	00:20:18	Romina	You get that?
1172	00:20:19	Michael	Yeah.
1173	00:20:19	Romina	So should I...?
1174	00:20:26	Michael	What are you doing?
1175	00:20:28	Romina	I'm writing.
1176	00:20:28	Michael	No. I was talking to Brian.
1177	00:20:29	Romina	Oh. Okay. So so far we have six. And then we have to multiply the six by the two for all of these so you get twelve. Right? And multiply the twelve times the three to get thirty-six. You multiply it because it's three different colors.
1178	00:20:44	Michael	Yeah. The one's can be any color.
1179	00:20:45	Romina	So each one here can be three.
1180	00:20:46	Brian	Yeah.
1181	00:20:46	Romina	Yeah. So you multiply that to get thirty-six.
1182	00:20:51	Michael	Okay.
1183	00:20:52	T/R1	Show us the forty-five.
1184	00:20:54	Ankur	It's not as simple as that. [They laugh]
1185	00:20:56	T/R1	It's not as simple.
1186	00:20:57	Ankur	You do it.
1187	00:20:58	Michael	I'm just...
1188	00:20:58	T/R1	Well, do it together. Do it together. You can help each other.
1189	00:21:01	Michael	Well talk amongst yourselves because I got to, I got to finish something.
1190	00:21:04	T/R1	We'll wait for you.
1191	00:21:05	Michael	Well then they're going to be staring at me and I don't like that.
1192	00:21:07	T/R1	No, we won't. Why don't you get your presentation ready and ...
1193	00:21:10	Ankur	We got to do it up there?
1194	00:21:11	T/R1	Yeah.
1195	00:21:12	Ankur	I don't like writing.
1196	00:21:14	Brian	Tough.

Line	Time	Speaker	Transcript
1197	00:21:14	T/R1	It's easier for me to understand. Maybe someone will write it for you.
1198	00:21:18	Romina	I'll write for you. So you don't have to write.
1199	00:21:21	T/R1	Romina will [inaudible]. What do you think? Will you accept?
1200	00:21:27	Ankur	I decline.
1201	00:21:28	Romina	But you're going to be aggravated, you're just going to want to do it yourself, anyway. So why don't you just do it yourself.
1202	00:21:34	Brian	You lazy bum.
1203	00:21:35	Ankur	That's right.
1204	00:21:37	T/R1	Come on.
1205	00:21:39	Ankur	I gotta wait for him.
1206	00:21:41	Michael	You don't have to wait for me.
1207	00:21:42	Ankur	You have to do this part.
1208	00:21:43	Michael	Write this exactly. See how it looks on the paper? Write it up there but neater. [They laugh] Write it up there. Do that.
1209	00:21:52	Ankur	Did you, did you write them out?
1210	00:21:54	Michael	Don't worry about it. Look at how [inaudible]
1211	00:21:56	Ankur	That's what you got to write up there.
1212	00:21:57	Michael	I'm not going to write that up there. I'm....
1213	00:22:00	Ankur	You came up with this.
1214	00:22:01	Michael	I'm just doing it for myself because I want to be 100% sure that is, what is that eighteen? What did we say for that one, eighteen, right?
1215	00:22:11	Ankur	Three, six, nine... twenty-seven cause you got [inaudible] up here.
1216	00:22:16	Michael	Twenty-seven, yeah. I just want to make sure, exactly, that it's twenty-seven. Just do what you have to do up there. Just draw it.
1217	00:22:22	Ankur	Alright, Romina?
1218	00:22:23	Romina	Yes.
1219	00:22:23	Ankur	Could you write something for me please?
1220	00:22:25	Romina	Yes. That exactly without changing a thing? [She goes to the board]
1221	00:22:28	Ankur	Write..
1222	00:22:28	Michael	Well draw a box around it. She's going to write the stuff at the top.
1223	00:22:30	Ankur	Write this stuff.
1224	00:22:30	Brian	Are you telling me that you can't?
1225	00:22:32	Michael	He's lazy.
1226	00:22:32	Romina	They're going to get so aggravated with me. So...
1227	00:22:36	Ankur	Alright, wait, I'll just read it to you.
1228	00:22:37	Brian	I'll offer you something... [He erases the board]
1229	00:22:40	Romina	Do I draw a big box or…?
1230	00:22:41	Ankur	No, just write whatever I read, write going down.
1231	00:22:45	Romina	Going down?
1232	00:22:47	Ankur	One, one, one, zero…like that's [inaudible] and then...
1233	00:22:53	Romina	Yeah, [inaudible]
1234	00:22:55	Ankur	Two, two, two, zero. Three, three, three, zero. Then like skip just a little like space and then, not that much….Like one, one, zero, one. Try to keep it even – it it will be easier to see [laughing]. I was just trying to get you mad. Two, two, zero, two. You see where this is going.?

Line	**Time**	**Speaker**	**Transcript**
1235	00:23:30	Romina	Yeah, I can.
1236	00:23:32	Ankur	And then you do the same thing but the zero's are like in the second.
1237	00:23:38	Brian	You couldn't do that?
1238	00:23:40	Ankur	Not really.
1239	00:23:43	Romina	And do I have all the zero's?
1240	00:23:44	Ankur	Yeah.
1241		Brian and Ankur are having a quiet conversation as she writes. Their conversation is inaudible.	
1242	00:23:58	Romina	Okay.
1243	00:24:01	Ankur	Alright. Now make, make an entire line going across the bottom, under them. Okay. Say one equals red, two equals blue, you don't have to write it, and three....
1244	00:24:13	T/R1	Yeah, write it down.
1245	00:24:23	Romina	Equals...
1246	00:24:24	Ankur	yellow and zero equals any one of the three.
1247	Romina has written the following on the board:		

1	2	3	1	2	3	1	2	3	0	0	0
1	2	3	1	2	3	0	0	0	1	2	3
1	2	3	0	0	0	1	2	3	1	2	3
0	0	0	1	2	3	1	2	3	1	2	3

1= red
2=blue
3=yellow
0=any one of 3

1248	00:24:38	Ankur	Okay, now, I'm just trying to think this out, I don't know how to say it. Alright now, there's not going to one of each color in this cause you can just see that, right? Like there's not going to be….
1249	00:24:58	T/R1	Brian, do you agree with that?
1250	00:25:00	Romina	What do you mean by that?
1251	00:25:00	Brian	I see what he's talking about.
1252	00:25:02	T/R1	Romina?
1253	00:25:02	Ankur	There's not going to be like a red….Like for the thirty-six there was a red, a blue, and a yellow. There's not going to be a red, a blue, and a yellow in any one of these. Could you see that? Can you see why?
1254	00:25:15	Romina	What do you mean [inaudible]?
1255	00:25:16	Ankur	Like, remember in thirty six? There had to be a red, a blue, and a yellow in each one?
1256	00:25:21	Brian	Yeah.
1257	00:25:22	Romina	Isn't that what we are suppose to do?
1258	00:25:22	Ankur	Yeah, but in this one I am proving the forty five. There can't be a red...
1259	00:25:25	Romina	Oh, you're proving the forty five. Okay, now you can see. Yeah.
1260	00:25:27	Ankur	Yeah, so.
1261	00:25:28	Romina	Yeah.
1262	00:25:29	Ankur	Can you see why there can't be a red, a yellow and a blue?
1263	00:25:32	Romina	Yeah.
1264	00:25:34	Ankur	Ok, I just wanted to, you don't see why?
1265	00:28:37	R2	No, I want you to explain it.

Line	Time	Speaker	Transcript
1266	00:25:39	Ankur	Explain why there can’t be a red, a yellow?
1267	00:25:41	R2	No, I understand there can’t be three in each one because of the reason from before.
1268	00:25:48	Brian	We already figured that out.
1269	00:25:49	Ankur	Ok…right now, this isn’t the entire part of the forty-five.
1270	00:25:52	R2	Oh, okay.
1271	00:25:53	Ankur	This is just a section of it.
1272	00:25:54	R2	A piece of it, okay.
1273	00:25:56	Ankur	This is the ones with three of one color and, and then...
1274	00:26:01	R2	Something else….
1275	00:26:01	Ankur	Something else.
1276	00:26:03	Romina	Okay. Hold on. Three of one color……
1277	00:26:05	Ankur	Three of one color... and another color.
1278	00:26:06	Brian	Anything besides, all three.
1279	00:26:09	Romina	Yes, I can see that.
1280	00:26:10	Ankur	But there’s just one problem in this. When, like... [Ankur goes up to the board.]
1281	00:26:15	Romina	Told yah. [Romina tries to hand him the chalk]
1282	00:26:16	Ankur	I don’t need chalk When you look at this half, it could be red, red, red, and red. But you go over here and you do red, red, red and red, then those two are the same. So instead of having... zero really represents not any one of the three but the other two that are not present. Do you understand that?
1283	00:26:44	T/R1	Do you want to change that? You want to change what zero is?
1284	00:26:47	Ankur	Can you write that? [Ankur asks Romina to re-write the definition of zero]
1285	00:26:51	Romina	Any one of the three except the one that is present.
1286	Romina rewrite the definition of zero. The board now says:		

1	2	3	1	2	3	1	2	3	0	0	0
1	2	3	1	2	3	0	0	0	1	2	3
1	2	3	0	0	0	1	2	3	1	2	3
0	0	0	1	2	3	1	2	3	1	2	3

1= red
2=blue
3=yellow
0=any one of 3 except the ones that are present.

Line	Time	Speaker	Transcript
1287	00:27:01	Ankur	So it could be red, red, red, blue; red, red, red, yellow. For every single one.
1288	00:27:07	T/R1	So you’re not allowed to ever have red, red, red, red?
1289	00:27:09	Ankur	No, you’re allowed to have that.
1290	00:27:11	T/R1	No here.
1291	00:27:12	Ankur	Not in this situation, cause we’re doing three of one color and…
1292	00:27:15	T/R1	What do you think about that Brian?
1293	00:27:18	Brian	I totally understand what he’s trying to talk about. It's just he keeps saying the same thing
1294	00:27:22	Ankur	So, alright. So there’s eighteen cause there’s three, six, [he counts silently]. No, I mean twenty four. Twenty four. And then when you, when you…
1295	00:27:38	T/R1	Why twenty four?
1296	00:27:41	Romina	Cause you have the, hold on, one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve…times two, right? Am I wrong?

Line	Time	Speaker	Transcript
1297	00:27:50	Ankur	You’re right.
1298	00:27:51	Romina	Okay.
1299	00:27:52	Ankur	So there’s twenty four and you can add red, red, red, red; yellow, yellow, yellow, yellow and blue, blue, blue, blue. So that’s twenty seven. And now you got to erase that. [He turns to Romina. Romina goes up and erases the whole board.]
1300	00:28:13	Ankur	Are you guys still with me or ?
1301	00:27:15	Romina	Okay, we have around twenty seven, right?
1302	00:27:17	Ankur	Now, I still got to do…
1303	00:28:20	Romina	Do I have to write anything?
1304	00:28:21	Ankur	Yeah.
1305	00:28:28	Ankur	Write one, one, two, two; one, one, three, three; two, two, one, one. [She continues and writes the next three columns, see board work below]. Okay now you got to write one, two, two, one; one, three, three, one; three, I mean two, one, one, two; you all see where this is leading? [She continues and writes the next three columns on the board].
1306	00:29:24	Ankur	Now go on the bottom and write one, two, one, two; one, three, one, three; two, one, two, one; two, three, two, three; three, one, three, one. And three, two, three.
1307	00:29:49	Ankur	Now this is the other, two, four, six, twelve, eighteen. And twenty-seven plus eighteen equals forty five. Now these are the ones, these still don’t have three of each color.

1308 The board now says:

1	1	2	2	3	3	1	1	2	2	3	3
1	1	2	2	3	3	2	3	1	3	1	2
2	3	1	3	1	2	2	3	1	3	1	2
2	3	1	3	1	2	1	1	2	2	3	3

1	1	2	2	3	3
2	3	1	3	1	2
1	1	2	2	3	3
2	3	1	3	1	2

Line	Time	Speaker	Transcript
1309	00:30:06	T/R1	So which ones are these?
1310	00:30:08	Ankur	Theses are two of one color and two of another color.
1311	00:30:12	T/R1	And you have all possibilities?
1312	00:30:13	Ankur	Yes.
1313	00:30:17	T/R1	Are you convinced of that? [Asking Brian and Romina]
1314	00:30:19	Romina	It’s kind of like an extended version of what we did.
1315	00:30:22	T/R1	In what ways is it an extended version of what you did?
1316	00:30:24	Romina	Because we didn’t actually go through, we just went on the math. Which, that's why I said we could have been wrong because we didn’t actually go through them. This one showed us every single possibility. Very good. So we’re sure.
1317	00:30:40	T/R1	So you’re convinced? You believe, Ankur?
1318	00:30:42	Romina	Yeah.
1319	00:30:43	T/R1	Ankur believes you?
1320	00:30:44	Ankur	Yes.
1321	00:30:46	T/R1	What about, what do you think Mike? Any idea what he did?

Line	Time	Speaker	Transcript
1322	00:30:49	Michael	Yeah, I did it, I believe them. Something.
1323	00:30:54	T/R1	So, I'm curious, I'm kind of curious when you started out…um…where do you think…um…your notion of variable got you in trouble? You found duplicates later, Mike. Where did they come in?
1324	00:31:08	Ankur	We made a mistake when we did the first thing we did. When we showed you this.
1325	00:31:12	T/R1	Right.
1326	00:31:13	Ankur	When we put the zero's on top.
1327	00:31:15	T/R1	Right.
1328	00:31:16	Ankur	It turned out that…even if this was a one, like, could we do…. Explain how you did it [Ankur says this to Michael who gets up and goes to the board].
1329	00:31:24	Michael	Do you guys, you don't need that, do you? [He points to the work on the board.]
1330	00:31:27	T/R1	We have it already.
1331	00:31:42	Michael	We had like the dumbest way. In the beginning we did this, to find the ones that have all three. And this "o" could be like a variable, any three of the numbers, any three.
1332	00:31:58	Ankur	Are you going to write all that on the board?
1333	00:32:00	Michael	No, I am not going to write one section of it – just six.
1334	00:32:02	Ankur	Okay…one more.
1335	00:32:08	Michael	You could have the variable "one" up here or up here.
1336	00:32:12	Romina	Yeah…it's….
1337	00:32:14	Michael	Just wait though. We have doubles in here.
1338	00:32:18	Ankur	Which we didn't realize until later.
1339	00:32:20	Michael	Too far ahead. Okay, do you see that? Now there's like six…no, not six.
1340	Michael has written the following on the board:		

1	1	1	1	1	1
2	3	0	0	2	3
3	2	2	3	0	0
0	0	3	2	3	2

Line	Time	Speaker	Transcript
1341	00:32:31	Ankur	Yeah, its six…cause there's six…
1342	00:32:35	Michael	Okay, watch. There's six doubles in here. Because the way we thought it, here are like three ways to do it, you would do times three, times..
1343	00:32:41	Romina	There's only six up there?
1344	00:32:42	Michael	I know, watch.
1345	00:32:43	Ankur	No, but you would times three because the zero could be either one of the three.
1346	00:32:43	Michael	You would do times three because you could have three different ones in there.
1347	00:32:47	Romina	Okay.
1348	00:32:48	Micheal	So six times three, eighteen. But there's three doubles in there.
1349	00:32:51	Ankur	But then you could have like two off two….
1350	00:32:53	Michael	Wait, let me explain. You could have, let's take the first one. One, two, three, and this is "o." Now, what was the other one that I was looking at? We took this one – one, 'o', three, two. If you have a two as this variable and a two as this variable, the variable, it's the same thing. And you would also have another one. One, two, three…no it would be….
1351	00:33:27	Ankur	One, zero….
1352	00:33:29	Michael	I think it's this one.

Line	Time	Speaker	Transcript
1353	00:33:30	Ankur	Yeah.
1354	00:33:31	Michael	One, two, “o” and a three. If this is three and this is three, it's the same thing. [He is referring to the two zero's that are contained in the last two columns – see board work below.]
1355	Michael now has the following on the board:		

1	1	1	1	1	1	1	1
2	3	0	0	2	3	2	2
3	2	2	3	0	0	3	0
0	0	3	2	3	2	0	3

Line	Time	Speaker	Transcript
1356	00:33:37	Michael	And there’s a total of six in there. And this, this would be only for the ones with the one, the certain color. The second color, we did like instead of a one there, we had a two there. And wherever the two’s were, we’d put a one. Two, two, two. I’ll erase the top row.
1357	Michael now has the following on the board:		

2	2	2	2	2	2
2	3	0	0	2	3
3	2	2	3	0	0
0	0	3	2	3	2

Line	Time	Speaker	Transcript
1358	00:34:05	Michael	And we ran out of time total because we found all those doubles in there. And I didn’t notice that until late.
1359	00:34:09	Ankur	And then we did the same thing, put the three’s across.
1360	00:34:10	Michael	It came out to be three, thirty-six. And to explain it, you guys wouldn't understand me, so we used your explanation. That was best.
1361	00:34:18	Ankur	That's a lot easier.
1362	00:34:19	Michael	And since we had some sort of explanation for the forty-five which is still confusing.
1363	00:34:24	Romina	Yeah.
1364	00:34:26	Michael	We could still prove it that it's thirty-six….forty-five? What did we get [inaudible]?
1365	00:34:33	Ankur	Thirty-six.
1366	00:34:34	Michael	Thirty-six and forty-five.
1367	00:34:35	Ankur	Thirty-six and forty-five.
1368	00:34:37	T/R1	So suppose you were doing towers five tall, would that same reasoning work?
1369	00:34:44	Ankur	I really don't care. No one is going to ask me that on the street.

APPENDIX D: TRANSCRIPT – TAXICAB GEOMETRY

May 5, 2000 (12th Grade)

Powell, A. B. (2003), Appendix C of *"So Let's Prove It!": Emergent and elaborated mathematical ideas and reasoning in the discourse and inscriptions of learners engaged in a combinatorial task.* Unpublished doctoral dissertation, Rutgers University, NJ.

1. T/R1: Thank you all, thank you all for coming.
2. BRIAN: These tables aren't [inaudible]
3. MICHAEL: Yeah, I know, I was looking at-
4. T/R1: Yeah, well, you know you can move back in. I have a problem for you.
5. JEFF: All right.
6. BRIAN: Yes! [Punching the air with is right fist.]
7. T/R1: You're all set.
8. BRIAN: Let's do this.
9. T/R1: Okay.
10. JEFF: We're going to do taxicab geometry?
11. T/R1: Do you know about it?
12. JEFF: I have no clue.
13. T/R1: Did you ever hear of it? I understand that you all love geometry. I was listening to your interviews.
14. JEFF: Awe. [Wiping the left side of his face with his left hand].
15. T/R1: I though we would end with a smash of a problem in taxicab geometry. Okay. Why don't I just give you the problem, okay? Um, I'll give you a chance to look at it and see whether you understand the problem. [Leaving the table.]
16. T/R1: Why don't I just give you the problem, okay? I'll give you a chance to look at it and see whether you understand the problem
17. JEFF: You have to stay on the lines, right? Those would be streets?
18. T/R1: Exactly.
19. JEFF: I agree.
20. ROMINA: Isn't it like anyway you go-
21. BRIAN: Pretty much, because look-
22. ROMINA: As long as you don't go like past it. [Facing Brian's direction.]
23. BRIAN: The first one- No, 'cause.
24. MICHAEL: Well what if you go to the last one-
25. BRIAN: You can go all the way down and go over and go down three and go over two. [Tracing the routes above the problem sheet with a black marker in his right hand.]
26. ROMINA: Isn't it- Don't they all come out to be the same amount of blocks? [Jeff beginning to draw.]
27. BRIAN: Five.
28. JEFF: Five?
29. ROMINA: Five? I got seven.
30. JEFF: Uh, which one- Yeah, we were both looking at the red one.

31. BRIAN: I'm looking at blue. [Michael tapping his pen on the grid along intersection points.]
32. JEFF: Yeah.
33. ROMINA: Oh, okay.
34. JEFF: All right. I mean pretty much.
35. ROMINA: As long as you don't go like past it you're fine. So it's the same thing.
36. BRIAN: So, let's prove it.
37. T/R1: Okay, does somebody want to tell me what you think you understand the problem to be asking?
38. JEFF: Um, what's the shortest route from there to here staying on the streets, right?
39. T/R1: Okay, is there more than one shortest route?
40. BRIAN: Yes.
41. ROMINA: Yeah.
42. T/R1: In other words, if there is, how many?
43. ROMINA: Ah-
44. BRIAN: Let's do the blue.
45. T/R1: Okay?
46. JEFF: All right, how many different shortest routes are there?
47. T/R1: Yes.
48. JEFF: Is what you're asking right now? //All right.
49. T/R1: //Mm hm.
50. BRIAN: Blue's got five.
51. T/R1: Okay. And how do you know? You're going to have to convince us. Okay.
52. BRIAN: All right.
53. T/R1: If you need us call me or Gina. [Inaudible].
54. ROMINA: I have five.
55. JEFF: Can we have like a- You have colored like markers? Word! [Responding to T/R2's statement that she will give them some markers.]
56. BRAIN: For what?
57. JEFF: Because then we can just do each route a different color. To like- [Waving his hand.]
58. ROMINA: Yeah, but they all kind of go on top of each other.
59. JEFF: Well, I mean, I don't know. I mean, let's see what it looks like. If it get too ugly then- Which one are you doing?
60. ROMINA: Which one do you want to do?
61. JEFF: I'll go to red.
62. ROMINA: I've got blue.
63. BRIAN: I did blue.
64. JEFF: Brian already-
65. ROMINA: One-
66. BRIAN: It's just going to look like you're filling //in the boxes.
67. ROMINA: //Two. Yeah, it is.
68. JEFF: That's what it's going to end up looking like, right?
69. ROMINA: Yeah so screw it. There's- Okay, so we know five-

70. JEFF: Well,- [Romina writing "Blue 5" on her paper to the right of the grid and tracing routes with her pen on the grid.]
71. BRIAN: Just count them and then make sure you know how you got them. You know? [Jeff and Romina counting by tapping their pen or marker on the grid. Each of them counts on their own grid.]
72. JEFF: Yeah.
73. ROMINA: One, two-
74. JEFF: So why- why is it the same every time?
75. MICHAEL: You're going left and right.
76. ROMINA: Ours is a four by one, right?
77. MICHAEL: Yeah, it's a four by one, unless you go backwards a couple of times.
78. ROMINA: You can't go, well-
79. MICHAEL: I know that would be dumb.//
80. BRIAN: //[inaudible] the shortest route only if you go forward.
81. MICHAEL But the only- You can't go diagonal so you have to go up and down. So if the thing is down this many and//
82. JEFF: //Over that many, //it's the same
83. MICHAEL: //It's the same-
84. ROMINA: //It's the same area
85. MICHAEL: No matter how you do it, no matter how you do it it's you have to- you can't //get around doing that. [Pointing and gesturing around his grid]
86. ROMINA: //All right.
87. MICHAEL: //You can't get around going four down and right one 'cause -.
88. JEFF: All right, yeah. All right.
89. MICHAEL: You can't go over there. You can't get around doing that.
90. JEFF: Yeah.
91. ROMINA: What if I were to go like to the red when I go one, two, three, four- [Pointing at her problem sheet.]
92. MICHAEL: But they're not asking for like a //[Inaudible].
93. ROMINA: //Five, //six, seven.
94. JEFF: //Five, six, seven. //It's the same thing.
95. ROMINA: //Like //how- how am I going to- like //how would I-
96. JEFF: //It's the same thing.
97. MICHAEL: //It's the same.
98. ROMINA: -devise an area for that? Like this- this area up here? [Motioning with her pen on her grid, indicating the area of the rectangular space whose vertices are taxi stand and the red pick-up point.]
99. BRIAN: Like plus and [Inaudible].
100. JEFF: Well, it's not area.
101. MICHAEL: It's not area. It's //just a-
102. JEFF: //It's the perimeter. It's like //each one being one.
103. MICHAEL: //One, two, three, four, five, six, seven. [Pointing at Romina's paper and counting the length of a route to the red pick-up point.] [Jeff scratching his head.]
104. ROMINA: All right.
105. MICHAEL: There's no way you can get around going- [Gesturing with his hands]

106. JEFF: //Going seven blocks.
107. ROMINA: //No, yeah, I understand.
108. MICHAEL: Across that many and down that many because you can't go diagonally. Can't- [Gesturing with his hands over his problem sheet across to the left and then down]
109. JEFF: Yeah.
110. MICHAEL: Can't get around it, so- [gesturing with his hands]
111. JEFF: I mean, that's the most sensible way I think to say that. Right? And they want to know how many though.
112. BRIAN: Are there seven possibilities, though? You know how like blue was five? There's five possibilities but-
113. JEFF: Ah, so-
114. BRIAN: You know how it's only like five spaces. Like one, two, three, four, five. [Pointing at the grid on his problem sheet.]
115. ROMINA: Yeah, so if it goes more.
116. BRIAN: Is there seven for blue, I mean red?
117. JEFF: Well, check it out.
118. BRIAN You've got one- [Pointing at the grid on his problem sheet]
119. ROMINA: Here, I'll- //Me and Michael do
120. MICHAEL: //Is that the shortest routes?
121. ROMINA: Me and Michael do greens. The green one.
122. BRIAN: All right.
123. MICHAEL: //Oh, like that's the biggest one. [Pointing at paper]
124. ROMINA: //And they'll do red.
125. BRIAN: Green is nine I think. [Then he begins to check thisidea.]
126. ROMINA: Well //count how many ways. [They use their pens or markers to count on the grid.]
127. JEFF: //All right, we'll look for it.
128. MICHAEL: One, two- [counting and pointing at paper]
129. BRIAN: Ten. My bad. [Correcting himself on the length of a shortest path to green.]
130. MICHAEL: There's a lot.
131. ROMINA: Yes I know. I'm trying to devise a- like a-
132. JEFF: The- the way to do it?
133. ROMINA: Yeah.
134. JEFF: This is hard. [Romina draws routes on her grid with her pen.] (00:06:02)
135. ROMINA: Two-
136. JEFF: How many was there? For, um, for the blue dot. How many different ways.
137. BRIAN: Five.
138. ROMINA: Ha…I already lost count. [of the number of shortest routes to the green pick-up point.]
139. JEFF: How many //you got for red so far? [Talking to Brian]
140. ROMINA: //Well, I'm saying like if you go //all the way over. [Leaning over and pointing with her finger at the grid on Michael's problem sheet.]
141. BRIAN: //Two, three- [pointing at paper]
142. ROMINA: And then //you go all the way// over and leave only one space. [Romina points to Michael's grid and motions with her finger.]

143. MICHAEL: //Yeah. One, two, three- Yeah, one, two, three, four, five, six. Six going like that. [Outlining routes on his problem sheet.]
144. BRIAN: One, two, three, //four.
145. JEFF: //You only got five?
146. BRIAN No I'm just.
147. JEFF: Oh, I can't. //I can't keep //track of what I'm doing. [While Romina watches, Michael traces routes with his marker on the grid, without writing.]
148. MICHAEL: //Six this way. //Then you got-
149. JEFF: You know what I'm //saying?
150. MICHAEL: //possibility of doing this. //One, two-
151. ROMINA: //Yeah. How do we get that.
152. MICHAEL: -three, four. Oh, got one. But then you got // Ah,
this is a lot
153. ROMINA: //Yeah, you could do this. [Michael counting by tracing with his pen.]
154. MICHAEL: You guys want to do the green? We'll do the blue.
155. JEFF: No that's all right. //We already did the blue.
156. BRIAN: //We already did the blue.
157. ROMINA: Yeah, the blue is fine.
158. BRIAN: We're doing red.
159. ROMINA: Okay, we can't count. Like we need a- can't we- can't we do towers on this? (00:07:07)
160. JEFF: That's what I'm saying. Look, all right, you go to here
161. ROMINA: And they're like blocks.
162. JEFF: All right, you go to here and you got a choice of going there or there. Right? [Indicating a choice of across or down at an intersection point of the grid on his problem sheet.] So then you pick one of those and then you got a choice of there or there. When you get to you know what I'm saying? Maybe we can add all those up or something and get like a whole- [Explaining routes on grid paper.]
163. ROMINA: All right.
164. MICHAEL: There's a lot.
165. ROMINA: Okay, for ours there's ten //
166. MICHAEL: There's more than ten.
167. ROMINA: No. I mean there's ten blocks. Like ten lines to get to that thing, right?
168. MICHAEL: Yeah, six by five.
169. ROMINA: So if there's ten, ten could be like the number of blocks we have in the tower. (00:07:52)
170. MICHAEL: This is one-
171. ROMINA: How do we do that? Two to the n? [Moving her pen cap on and off of her pen.]
172. MICHAEL: How- how many? This was five they said? [Pointing to the blue pick-up point on his problem sheet.]
173. ROMINA: Yeah. [Looking back to her problem sheet.]
174. MICHAEL: How much you guys get for the red? Still doing that one?
175. ROMINA: How could-
176. MICHAEL: It's got to be some kind of pattern.
177. ROMINA: Okay, there's ten lines- ten lines-

178. MICHAEL: Ten ways of getting there. So you can do. Like you got to
179. ROMINA: There's ten different lines to get there.
180. MICHAEL: Think of the possibilities of doing this and then doing that. [Pointing at an intersection on his problem sheet grid and gesturing downward and then rightward.]
181. ROMINA: Well how many- okay, there's ten. How many lines //end up in the thing?
182. BRIAN: //What are you doing man?
183. JEFF: I'm just- I'm not, uh, trying to- [Drawing routes on grid paper.]
184. ROMINA: Two, //three, four, five, six, seven, eight.
185. MICHAEL: //Three, four, five.
186. JEFF: -get easier.
187. MICHAEL: There's thirty plus- I have thirty. About sixty I think.
[Pointing with the pen on the grid.]
188. MICHAEL: You might want to-
189. ROMINA: So- It couldn't be like a block ten high in six different colors, type deal? That would be- [Counting on the grid with her pen,]
190. MICHAEL: There's like- there's ten line- there's ten like lines in here and the answer was five. So I'm waiting for them That's like a half or something.
191. ROMINA: So maybe it's thirty? [Counting the number of rows in the array and then draws a symbol. Jeff adds an "L" to the third row, left hand corner box and then adds an "L" to the fourth row left corner box.]
192. MICHAEL: It'd be nice if it was.
193. ROMINA: How many are there in here. One, two, three, four, twelve, twenty-. You guys got at least twenty-four yet?
194. JEFF: Uh, which, wait a sec-
195. BRIAN: I'm at eight. What to do you think? What are you guys thinking?
196. ROMINA: To get to this one, there could also be five times two but there's ten lines-
197. BRIAN: I've counted it.
198. ROMINA: And there's five ways to go.
199. JEFF: Wait, five?
200. ROMINA: For the blue one.
201. JEFF: There's ten lines?
202. ROMINA: //[Inaudible].
203. MICHAEL: //You got eight for red. I only have nine ways.
204. JEFF: //No but I'm like-
205. MICHAEL: //You have eight?
206. BRIAN: I'm drawing them. I'm not stumped; I'm just like not speeding through it. You know. Did you count the middle lines?
207. MICHAEL: No, I just got eight from the- you know, just- just
//[Inaudible].
208. ROMINA: //I didn't- I didn't do it.
209. BRIAN: All right.
210. MICHAEL: I was thinking about that-
211. ROMINA: So ten-
212. MICHAEL: Let's- let's try doing the red one. Try doing the red one.

213. ROMINA: Yeah but, how you going to know when we- how are you keeping track though? [Romina places her hand on her head.]
214. MICHAEL: I don't know. I'm just- see like if I can just not forget. Are you going to like //just write them down?
215. ROMINA: //We can do what Brian's doing. Like we'll just draw a big thing on the board.
216. JEFF: And just go over each way to do it? There's got to there's got to be
217. ROMINA: //There's got to be something.
218. JEFF: //some kind of math- You know what I'm saying?
[Placing his hand on his head.]
219. ROMINA: All right.
220. BRIAN: How many do you think for red? Twenty-four?
221. MICHAEL: I was guessing.
222. BRIAN: See that.
223. ROMINA: [Inaudible.]
224. BRIAN: How'd you count that?
225. ROMINA: Or- hold on. There's-
226. MICHAEL: Uh, //that's not really. It's- it's just a guess.
227. ROMINA: //No, there's twenty- No it'd be twelve. Wouldn't it be twelve?
228. MICHAEL: I don't know. How- how much is this?
229. ROMINA: There's ten lines and there's five ways. So if there's //twenty-four lines there would be twelve ways. [Pointing to Michael's problem sheet.]
230. MICHAEL: //but there's one, two, three, four- It's twelve, yeah. We're, we're guessing twelve but that's probably not it. I doubt it. [Counting routes on grid of problem sheet. Romina leaning over her problem sheet and outlining routes.]
231. JEFF: All right, you- you're here. [Speaking to Brian, he points with his black marker to an intersection point on his problem sheet.]
232. BRIAN: Uh hmm.
233. JEFF: You get to go over or you can go up. [From † $(5,1)$,moving his pen to the left one unit and back and then up one unit [SK_1, 0:11:09- 0:11:12].]
234. BRIAN: Mm hmm.
235. JEFF: So like here you can go over or up. [On the right side of the problem sheet, drawing a point and from it two lines, producing a binary tree.]
236. ROMINA: What are you doing?
237. JEFF: I don't know. I'm not doing anything. I'm just trying to think. [Returning to Brian.] And then you get to here. You can either go over or up again. And the same thing. But I don't know what that has to do with anything. My brain is like- just looking at this right now and going like- [Inaudible.] It's just not working. [Jeff waves his hand.]
238. ROMINA: But you know, I am- //I understand what you're doing-
239. BRIAN: //Just look at the lines and see where you're getting five.
240. ROMINA: -but like for this one you know what sucks with this one, is because if you're there you have either one of two choices.
241. JEFF: Mm hmm.
242. ROMINA: When you get here you have one or two choices, you know, this just doesn't-

243. JEFF: Well, yeah, you're here, you can either go there or there. You get here- [Tracing routes on the grid of his problem sheet.]
244. ROMINA: Yeah.
245. JEFF: -you can go there or there. But if you're here, you're only going to go down. [Pointing at an intersection point on the grid of his problem sheet.]
246. ROMINA: Yeah. //That- that- exactly.
247. JEFF: //Because you're going out of your way.
248. ROMINA: That's exactly what I was doing.
249. BRIAN: See this is- //this is what I was thinking of.
250. JEFF: //Then you're here and you're only going down or over. Again, this is just down and you can just follow all the routes to the end point- I don't know. [Pointing to the binary tree that he drew on the right side of his problem sheet.]
251. BRIAN: I don't know. That doesn't sound right. That's one. That's two, three, four, five. That's what I was doing with all of them. That's how I got twenty-four for this one. [Referring to the red pick-up point, pointing at Jeff's paper with his pen.]
252. JEFF: And that's what you thought it was Mike?
253. MICHAEL: //What'd you do?
254. ROMINA: //Yeah.
255. JEFF: Wait, what'd you do? How'd you do it? //That's one-
256. ROMINA: //No, not twenty-four.
257. BRIAN: //That's two. [Brian points to the grid with pen.]
258. JEFF: //That's two, three four //and five. [Brian pointing at Jeff's paper with his pen.]
259. ROMINA: //Twelve. Twelve would work.
260. MICHAEL: But that was not like-
261. JEFF: So then that's one, //that's two- [Pointing to his paper.]
262. MICHAEL: //Good guess.
263. JEFF: And you counted those up for twenty-four?
264. BRIAN: Three, four. [Pointing at paper with pen]
265. JEFF: See, that's what I'm saying.
266. BRIAN: Wait-
267. JEFF: And then the side streets.
268. ROMINA: But then there's more. [Brian counting with his pen on the grid.]
269. JEFF: There's more than fourteen?
270. ROMINA: No, I don't know how many there are.
271. BRIAN: Are you sure you got-
272. ROMINA: No, I was just saying like if- that wouldn't work with our theory.
273. JEFF: What theory is that?
274. MICHAEL: Divide //it by two.
275. ROMINA: //Divide it by two. It's like a highly- it was like a-
276. JEFF: Was it- like what divided by two? All the- add them all up //[Inaudible]. [Pointing at paper]
277. ROMINA: //Because there's ten lines- ten lines like that are all within this rectangle. [Pointing at paper with pen]
278. JEFF: All right.

279. ROMINA: There's five ways to get to it. So if there are twenty-four lines there would be twelve different lines to get to it. But, it's hard to prove. [Pointing to her grid with a pen.]
280. BRIAN: Actually, this whole thing, if you count the middle lines there's thirteen. [Referring rectangular region between the to the blue pick-up point and the taxi stand.]
281. JEFF: There is. That's why I- //as soon as I got to thirteen I
stopped working because there's none- it's prime.
282. MICHAEL: //[Inaudible], right?
283. ROMINA: One, two, three, //four, five-
284. BRIAN: //It's four on the sides, eight, nine, ten, eleven, twelve, thirteen. [Brian uses his two hands to show routes in the air.]
285. ROMINA: //-six, seven, eight, nine, ten, eleven, twelve- There is thirteen.
286. MICHAEL: Thirteen what?
287. JEFF: Lines //over here.
288. ROMINA: //Lines.
289. JEFF: That's why I- I threw that out. I wrote- Oh, that's a thirteen but I was like, oh man, prime numbers. [Jeff puts his head in his arms.] No good.
290. ROMINA: //thirteen.
291. JEFF: There's like no way it could work with a prime number- like you can't even like make something up.
292. BRIAN: All right.
293. ROMINA: I think we're going to have to break it apart and draw as many as possible.
294. BRIAN: Yeah, //that's what I'm going to do.
295. JEFF: //And then have that lead us to something? What if we do- why don't we do easier ones? You know what I'm saying? What if the- the thing- Do you have another one of these papers? [Speaking to T/R2.] (00:15:00)
296. ROMINA: Here, to make it simple, just draw on here.
297. JEFF: All right. Well, yeah. We're just going to make a grid.
298. T/R2: Oh, we got grid paper.
299. JEFF: Oh yeah? We could get some grid paper? Or those.
300. BRIAN: Whatever.
301. T/R2: To tell you the truth..[Inaudible].
302. JEFF: We're not there yet. We're not-
303. T/R2: No, I mean like so that you can cover it
304. JEFF: Whatever. We're flexible.
305. T/R2: Okay, here's some more copies if that helps. Okay.
And I'll get you
306. JEFF: All right. So-
307. ROMINA: Pick a dot.
308. JEFF: Right there.
309. ROMINA: One, two.
310. JEFF: Two. All right. Here.
311. T/R2: We also have more to choose from.
312. JEFF: Jesus.
313. T/R2 There's graph paper there. Okay

314. ROMINA: Okay. So one, two, three- Oh, is this going to be dumb and stuff? One, two, three, four- It looks like a multiplication table. (00:15:49)
315. JEFF: All right. Uh, one-, two [Inaudible]. [Brian draws his eighth symbol on the right side of the grid and writes “1, 4, 2.” On the top of “1, 4, 2” he writes “DRD.” He also goes back to 7 and writes “D3, R1”. He has written a number with each of the first 6 symbols on Brian’s paper, too.]
316. ROMINA: All right.
317. JEFF: Why don’t you just- here, use blue. It doesn’t matter.
318. ROMINA: Yeah. One-
319. JEFF: One- //two.
320. ROMINA: //Two. Three. //Four.
321. JEFF: //Four.
322. ROMINA: Five?
323. JEFF: Where are you? Wait was that one, two over? The fourth spot? One, two- three- four- five. I don’t- I can’t remember what I- [Jeff draws routes on a 2 by 2 rectangle.]
324. ROMINA: I think it’s five. I think it’s five. [Brian draws his ninth symbol for a specific route, with the numbers “2, 4, 1” next to each line on the symbol.]
325. JEFF: I think it is five.
326. ROMINA: All right. Do the next one. Don’t- don’t count out loud and we’ll see if we get the same thing.
327. ROMINA: [After working silently.] What’d you get?
328. JEFF: Nothing. I’ve got to start all over again. And what is that? Six?
329. MICHAEL: What ‘s that?
330. ROMINA: That’s…how many I can-
331. JEFF: For each of those points?
332. ROMINA: Yeah. Like the point diagonally down.
333. MICHAEL: Yeah.
334. ROMINA: I’m not sure if I’m right though. I’m not sure if I’m counting right.
335. JEFF: // One, two, three-
336. ROMINA: I mean this one- this one looks to be like prime numbers- I know this one going up- [Romina points to an intersection on the rectangle with her pen.]
337. JEFF: How many did you get?
338. ROMINA: Hol’- I think- Seven.
339. JEFF: All right. Well, the only thing I’m seeing right now with this right, is those together with that and those together with that.
340. ROMINA: //Well I-
341. JEFF: //So hopefully-
342. ROMINA: I’m going two, three, four, five, six. Two, three, four, five, six. Five- three, five, //seven, nine.
343. JEFF: //Seven, nine.
344. ROMINA: Eleven and then we’re going to go up again?
345. JEFF: Well go for it. Yeah.
346. ROMINA: Here go- go and we’ll have to [Inaudible]. [Brian writes tally marks on the top of his grid. Brian crosses out two of the tallies on his paper.]
347. JEFF: Wait. Why don’t we give one of these like to-

348. MICHAEL: Brian, how many did you get, get so far? [Romina and Jeff wrote a number in each of the squares in a three by two rectangle that they drew on their grid. The top row contains the numbers 2, 3 and 4 and the bottom row 3, 5 and 7.]
349. JEFF: To like here. [Pointing to the intersection point † $(9, 3)$ on the grid of the problem sheet in front of Romina.] No//and then-
350. ROMINA: ///Because it's going to be too much. Well, //go down and see like when we go down and we do all these and all of these that go out one more and see how much you get. [Pointing to intersection points †$(5, 4)$,†$(6, 4)$,†$(4,1)$,†$(5, 2)$,†$(6, 3)$, and†$(7, 4)$.]
351. JEFF: //For the red one, sorry.
352. JEFF: All right.
353. ROMINA: One- [Romina starts tracing routes to†$(5, 4)$ with her pen on the grid.]
354. BRIAN: I'm not good at this kind of stuff.
355. ROMINA: //one, two, three, four, five.
356. JEFF: Where- where you going to?
357. ROMINA: Here, this is- this is five. [Writing a 5 in the†$(5, 4)$ square.] And, go to this one now because- [Pointing at intersection point †$(6, 4)$.]//I mean that one I'm pretty sure. [Referring to the result obtain for the point †$(5, 4)$.]
358. JEFF: //Was it four by four?
359. ROMINA: Uh, four by two.
360. JEFF: That's what I meant. I was drawing the right thing.
[Jeff draws a four-by-two sub-grid on a sheet of 1-centimeter grid paper and draws routes within the subgrid.]
361. ROMINA: Yeah, it's working. [Romina writes a 9 in the †$(6, 4)$ square.]
362. JEFF: Wait, you only got nine for that!?
363. ROMINA: Uh hmm. [Romina writes a 4 in the square in the third row, under the 3 in the second row, after counting routes with a pen on the grid.]
364. JEFF: All right, wait a second. Check it out. Um, all right.
You go one-
365. ROMINA: All right.
366. JEFF: Wait- just wait a second.
367. ROMINA: No, I know. I'm just- One-
368. JEFF: One. Then two. [Drawing routes on grid paper.]
369. ROMINA: Uh hmm.
370. JEFF: And then- Uh, three, four, five. [Drawing routes on grid paper.]
371. ROMINA: Uh hmm.
372. JEFF: Six, seven, eight, nine, ten, eleven- you know what I'm saying? We're missing- [drawing routes on the grid]
373. ROMINA: Okay, what am I missing?
374. JEFF: You're- we're like
375. ROMINA: Did we do that for seven?
376. JEFF: Well you're- I don't know. You're not going like over two down one. //Over two over one. [Jeff motions with his pen on the grid.]

377. ROMINA: //I'm not doing [inaudible]. I'm not doing that.
378. JEFF: So-
379. ROMINA: Okay.
380. JEFF: You want to go back from the //beginning.
381. ROMINA: //Go back to- //go back to seven.
382. JEFF: //You got to go- Well, how do you know- we did five right?
383. ROMINA: We had to have done five because there was like-
384. JEFF: As long as it's right I don't- I don't care. Just as long as it's right. All right, so, which one's the seven one? Two by three?
385. ROMINA: I got eight for that, right? [Jeff draws routes on the grid.]
386. JEFF: Seven, eight- I got more than that. All right, wait. We got to go through this, and you got to watch.
387. BRIAN: I got at least twenty-two for red. I assure you of that.
388. JEFF: Assure you?
389. BRIAN: It's not raining no more. I'm sweating.
390. ROMINA: Yeah.
391. JEFF: Yeah, it's like the hot seat. All right, check it out. One- [On 1-centimeter grid paper, drawing a route in a two-by-three sub-grid.]
392. ROMINA: Mm hmm.
393. JEFF: All right. There's only one you can go by going two down. I'm trying to like figure out ways to like cross them out. You know what I'm saying? And then going one down, you can go one, two, three- There's no other ways to go. [Drawing more routes on his 1-centimeter grid paper.]
394. ROMINA: Mm hmm.
395. JEFF: What about like that? Four?
396. ROMINA: Mm hmm. Mm hmm.
397. JEFF: And then, five, six, seven, eight- [Counting the routes as he draws them.]
398. ROMINA: You already did that one.
399. BRAIN: // I don't remember if I did that.
400. JEFF: Which one?
401. BRIAN: //There's definitely twenty-three.
402. ROMINA: All right guys. This is what we're trying to do. Why don't we try to do this- [Taking a blank piece of 1-centimeter grid paper.]
403. JEFF: All right, what's-
404. ROMINA: We're getting all confused. You see how we're like going to like we're drawing like we're going to here. How many it takes to get to that point and then we're going to here and it's like a- this is just going up like one, two, three- two, three, four, five and then we go down to here and there's the same thing and then like how much we'll get to this point and how much we'll get to that point. [Pointing to intersection points on a blank 1-centimeter grid paper.] Why don't we all try to do that because we're getting confused and we're- (00:22:00)
405. JEFF: Yeah.
406. ROMINA: We're doing the same mistakes.
407. JEFF: And it's like real hard. My brain-
408. ROMINA: If we do that and we see a pattern I'm sure we'll be able to- uh

409. JEFF: Hey, you know what we could even do, we could, uh where are those transparencies? We could exploit the fact that we have those. You know what I'm saying? Like- [Michael silently writes.]
410. ROMINA: Well I was going to go over like to see how far we've gone. That's good. Oh, that's not the same side.
[Romina takes a transparency with a grid on it.]
411. JEFF: No, but even- I mean you could say, all right, um, on on the- you could do, um, a hundred six squares here. You could do- [Pointing at paper.]
412. ROMINA: Yeah.
413. JEFF: You know what I'm saying? And then just-
414. ROMINA: So- we definitely know this is two, right?
415. JEFF: Here, knock yourself out.
416. ROMINA: We definitely know that's two, right? Now give me a blank. [Romina writes 2, 3 and 4 in the first row of the grid and a 3 directly below the 2.]
417. JEFF: Well go to- which one are we having the most trouble with right now?
418. ROMINA: Well, I'm just going to write numbers. Two-
419. JEFF: Yeah, but I'm saying like before we get involved in all this, let's find out like how many there are and-
[Michael and Brian silently write.]
420. ROMINA: Okay, let's make sure that's- Let's make sure how much that is. I'm going to go with that's- [Jeff draws rows of two by twos while Romina rewrites her numbers in the squares, only writing the top 3 numbers and the number in the second row, first position on the left.]
421. ROMINA: I think that's five.
422. JEFF: That is five? [On centimeter grid paper, drawing three horizontal lines across the page, creating two sets of parallel lines 2 centimeters away from each other.]
423. ROMINA: Mm hmm. Oh well, you do it too.
424. JEFF: Oh.
425. ROMINA: I mean-
426. JEFF: Which- by what by what?
427. ROMINA: Two by two.
428. JEFF: Two by two?
429. ROMINA: Let's get that done.
430. JEFF: One. All right. One, two, three, four, five- [Using a transparency of a centimeter grid paper, traces in the air shortest routes for a 2 by 2 square.]
431. ROMINA: You're counting one twice.
432. JEFF: Six- All right, wait. That's why- here watch.
433. ROMINA: Maybe, yeah.
434. JEFF: You just go make two by twos. [Drawing three vertical lines on his centimeter paper to create two-by-two subgrids.]
435. ROMINA: Mm hmm.
436. JEFF: You could go- [Drawing two-by-two sub-grids.]
437. ROMINA: Yeah, make at least six at the moment.
438. JEFF: All right. You can go this way. [Drawing 1 two-down route.]
439. ROMINA: Yeah that's one.

440. JEFF: That's one. Now that's all the ways you can go-
[Drawing a route.]
441. ROMINA: Yeah.
442. JEFF: -by two down. So then you can go like this…
443. ROMINA: Two.
444. JEFF: You can go like this. [Drawing 2 one-down routes.]
445. ROMINA: Three.
446. JEFF: Is there any other ways to go by going down? No.
447. ROMINA: Okay.
448. JEFF: All right. So then you could- you could go like that?
[Drawing 2 one-over routes.]
449. ROMINA: Mm hmm.
450. JEFF: You could go like that. [Drawing 1 two-over route.]
451. ROMINA: Mm hmm. Or you go all the way top to bottom.
452. JEFF: There's nothing else to do? Right? Now that would be the opposite of that one. That would be the opposite of that one and that would be the opposite of that one. //They're all covered. [Pointing to pairs of routes on the grid with a pen.]
453. ROMINA: //So we got six. Good. Good thing we did that over again. (00:24:57)
454. JEFF: All right, well. Yeah, good because at least we're- you know, //we're- we're making progress. [Romina writing 6 on her transparency of centimeter paper in the square that represents a two-by-two grid.]
455. ROMINA: //Yeah, all right. And go-
456. JEFF: All right. //The three-
457. ROMINA: //Three and two. [Jeff draws three vertical lines, creating four three-by-two rectangles.]
458. JEFF: The greatest MC in the world. [Singing.] Look at that. Beautiful. [Drawing three-by-two rectangles on the grid and crossing out the others]
459. ROMINA: Tell me you know how to count those. All right. [Jeff crossing out the 6 different 2 by 2s he just drew shortest routes on.]
460. JEFF: All right. We can go like this, and that's the only way-
[Drawing the one 2-down, 3-across route.]
461. ROMINA: Right.
462. JEFF: -to do that.
463. ROMINA: You want-, you want to do them in couples? (00:25:30)
464. JEFF: Now the opposite of that is that right there. So that's that covers those two. [Underneath the previous route, drawing the one three-over route.] Now, the other way- now we've got to go one down like that. [Using a red marker to draw a route one-down, three-over.] And the couple of that would be- //I'm not-
465. ROMINA: //[Inaudible].
466. JEFF: -not exactly sure so wait.
467. ROMINA: We can't go in couples I mean.
468. JEFF: Yeah well-
469. ROMINA: All right, I'm going to open the windows.
470. JEFF: Ah yeah? [Draws 2 more one-down routes in his 3-by-2 grids.]
471. Cameraperson: What's making noise here? [Inaudible]?
472. ROMINA: [Inaudible].

473. Cameraperson: I understand.
474. JEFF: All right.
475. ROMINA: What'd you get?
476. JEFF: I don't know. I'm waiting for you, man.
477. ROMINA: All right. One, two-
478. JEFF: All right. That's that- [With his pen, pointing at the different three-by-two routes on the grid in which the first move is one down.]
479. ROMINA: Mm hmm.
480. JEFF: And that's that. And then, you know, that's going one over. It's going two over. It's going //three over.
481. ROMINA: //Three over.
482. JEFF: That covers all going through the middle.
483. ROMINA: Mm hmm.
484. JEFF: Correct?
485. ROMINA: Yes.
486. JEFF: All right. So now we've got to start going to the top. You can go one over, down, over. You can go one over or two over, down. You could also go one over, down two and over. You could also go- [Drawing the route.]
487. ROMINA: We've got eight so far, right?
488. JEFF: Could also go, um, two over, two down and over.
[Draws the route.]
489. ROMINA: Mm hmm.
490. JEFF: Anything else? That's one, two, three, four, five, six, seven, eight, nine. Oh, it's nine because that one doesn't have a couple.
491. ROMINA: Yeah, //okay.
492. JEFF: //Those are couples, uh, this one and that one are couples, uh- [Pointing to routes and matching them with marker.]
493. ROMINA: //The one going-
494. JEFF: //These two are couples. [Pointing with marker.]
495. ROMINA: The one going all the way across in the middle is never
going to have a //couple.
496. JEFF: //Never going to have a couple.
497. ROMINA: Because-
498. JEFF: That's- //so that will always be odd.
499. ROMINA: //All right, so you can't [Inaudible].
500. JEFF: So every other one will be odd because there will be one going fully across the middle. Right? That's why that's nine.
501. ROMINA: Well that can't be odd because it's-
502. JEFF: Hey- 'cause that- that won't-
503. ROMINA: Maybe any one with an odd length or width.
504. JEFF: Which would be every other one.
505. ROMINA: Yeah.
506. JEFF: All right. So now where are we at? [Romina writing a 5 next to the 4 and I notice she has a 6 and 9 next to the 3 in the second row. She also writes a 4 under the 3 in the second row.] (00:27:58)

507. ROMINA: This is five. Okay, do you want to go down this- has to be four [Inaudible]. This should be nine too. [Pointing to the square below the 6.]
508. JEFF: Right. Because that is the- that is the same as that. [Jeff rotates the grid that Romina is writing numbers on.] Exactly. So that should be nine too. And- all right, do you want to go three by three? [Pointing to the routes and matching them.]
509. ROMINA: Yeah.
510. JEFF: You write it.
511. ROMINA: Um,-
512. JEFF: Cut a long one.
513. ROMINA: [Inaudible].
514. JEFF: It's you?
515. ROMINA: No. I thought it was you.
516. JEFF: Are you serious?
517. ROMINA: No, I didn't know.
518. JEFF: Oh man. It's the greatest MC in the world. [Singing.]
[Brian working silently. He has a symbol for 10 now.]
519. JEFF: You're going to draw the lines on this one because it's-
[Jeff draws rows of 3 by 3 rectangles.]
520. ROMINA: So I can mess it up?
521. JEFF: Yup. Because I can't handle it no more.
522. BRIAN: Oh man.
523. ROMINA: He's getting a little like kidish.
524. BRIAN: What are you starting?
525. JEFF: None of that.
526. ROMINA: One-
527. BRIAN: Isn't your head like
528. ROMINA: Two, right? [Drawing a route in the first and second three by three on the grid.]
529. JEFF: Opposites. Uh, wait, wait, wait. Stick with a pattern.
Do all the ones like going down first. You know what I'm saying? [Pointing to paper in front of Romina.]
530. ROMINA: I was going to do all the ones going across first.
[Pointing to paper]
531. JEFF: All right. Then do that. But, [Inaudible] we're just doing all the ones that are going like one across.
532. ROMINA: Like I'm going to do two to one. Instead of one to-
533. JEFF: All right. Just don't blow it.
534. ROMINA: One.
535. JEFF: Where you going with that?
536. ROMINA: Uh, well, you know what //I meant.
537. JEFF: //Go for the whole deal now.
538. ROMINA: Should I- on the next one should I go all the way down?
[Drawing routes]
539. JEFF: Yeah. That's another two over piece.
540. ROMINA: Okay. I'm going to go one- one over, one two over, one three over. All those? [Drawing routes]

541. JEFF: Yeah. And those are all the ways that you can go from the top over?
542. ROMINA: Yeah. Now going down.
543. JEFF: Now- now before you even do that, can't you just move these all the other way?
544. ROMINA: Yeah I know but- shouldn't we draw them just to make sure though?
545. JEFF: Yeah. Well let's do the opposites then like the same way we did the other thing.
546. ROMINA: Okay, so now we're going to go //two down. [Pointing to paper]
547. JEFF: //So, two down over-
548. ROMINA: Over- //All the way?
549. JEFF: //All the way.
550. ROMINA: Down.
551. JEFF: Right? Where's that there? That's- that's that right there. One over- [Jeff marks off two of the 3 by 3s to show couples.]
552. ROMINA: All right.
553. JEFF: So you could either just like, uh- [Drawing routes.]
554. ROMINA: Down- Here? [Drawing routes.]
555. JEFF: You can do whatever you want and we'll just- all right- [Marking off two more 3 by 3s with routes drawn.]
556. ROMINA: Down over-
557. JEFF: Uh, where do you see- All right. Um, where's that one the other way? [He marks off 2 more three by threes with routes drawn.]
558. ROMINA: Hold on not yet- I'm All messed up now. Okay, so now I'm going to go down one- [drawing routes]
559. JEFF: Mm hmm.
560. ROMINA: Over down across. [Drawing a horizontal line of their grid paper] Down one- down one over two. Okay.
561. JEFF: All right, wait, I 've got a question. Where's the other one of this? That's the other one of that? But, that's already the other one of that. So there's more ways that you can have two boxes open- Yeah, it's getting- it's getting heavy. All right. Well, just continue. [pointing at paper]
562. ROMINA: Down one. Down one over two. I already went down two. I could do- I could do one of these little babies. (00:32:14) [drawing a "staircase" route]
563. JEFF: What? You don't know? What about the opposite of that. Of that one. Got that?
564. ROMINA: I already did the ones that are in bold. [Drawing another "staircase" route]
565. JEFF: How many are there?
566. ROMINA: One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen - It doesn't work.
567. JEFF: That's all of them?
568. ROMINA: Well, no it's working. It's going. It's going.
569. JEFF: [Inaudible]. [Counting routes in the air with his marker silently.]
570. JEFF: Hmm.
571. ROMINA: Hmm. [Jeff points to the numbers on Romina's grid.]

572. JEFF: Uh, but are we sure there's only fifteen? Is my question. Coming out [singing, inaudible] What about like, um, down one- Where's all the down ones? All starting here?
573. ROMINA: Second row.
574. JEFF: Uh, down one over two- Where's that? You got that?
575. ROMINA: Down one- down one's are over here.
576. JEFF: Down one, over two. Hmm- [Drawing a route then crosses it out.]
577. ROMINA: You just draw, I'll find it. [Inaudible] yeah-
578. JEFF: That one's already there. Huh? Come here. Maybe there is only fifteen. Or- I wish we knew like-
579. ROMINA: Three by three? Is that what we're doing?
580. JEFF: I guess. Put it in for now. [Romina writing a 15 in the square next to the 9 so the 3rd row reads 4, 9, 15.]
581. ROMINA: What are you working on? //The red?
582. MICHAEL: //Uh, yeah. Did you do the red ones?
583. JEFF: We're working on, uh-
584. ROMINA: Oh, we're getting there.
585. JEFF: We're //just working on stuff.
586. ROMINA: //Where's red? Three by four? That's our next one.
587. JEFF: Nah, [inaudible]. So, if we could get to there it would be, uh, big time you know what I'm saying?
588. ROMINA: It's this one.
589. JEFF: This one right here?
590. ROMINA: Mm hmm.
591. JEFF: Three by-
592. ROMINA: Yeah.
593. JEFF: All right, well we're on- what are we on? Two by four? [Brian writing rows of numbers 0, 1, 2 or 3 silently. Michael has routes drawn all over his paper. He continues working.]
594. ROMINA: Mm hmm. That's not going to be enough.
595. JEFF: That's all right. We can make more.
596. ROMINA: Now I'm really starting to hate doing this.
597. JEFF: Oh yeah?
598. ROMINA: One.
599. JEFF: Yep.
600. ROMINA: Two. All right.
601. JEFF: Go all the way across.
602. ROMINA: Two down? [Romina draws routes.]
603. JEFF: We already did all the two downs. [Pointing to paper]
604. ROMINA: What if we go two down and across. //Did that.
605. JEFF: //We already did that. Um, what about- All right that's all of //them.
606. ROMINA: //Over one. [pointing to paper]
607. JEFF: All right.
608. ROMINA: Over one. [Romina draws routes for each four by two rectangle.]
609. JEFF: Uh um. Wait, stay with- go over ones. Do all the over ones.
610. ROMINA: Oh I was going- I was doing this //one. [Pointing to paper] [She continues drawing routes.]

611. JEFF: //All right then. Well, whatever you want to do. You're out of control.
612. ROMINA: Now, I want to go over one.
613. MICHAEL: What'd you get so far for the red one? What are you up to // now?
614. ROMINA: // Can I go anymore?
615. BRIAN: Before I got at least like thirty.
616. MICHAEL: How many? Thirty?
617. BRIAN: Right now I'm actually writing them out, I got like seventeen.
618. JEFF: Over.
619. ROMINA: No that's this one.
620. BRIAN: What do you have?
621. MICHAEL: I have to count them up.
622. JEFF: Oh man. Wait, there should be one more.
623. ROMINA: Yeah, I'm drawing-
624. JEFF: I'm saying - I think there should be one more.
625. MICHAEL: I got thirty-four.
626. BRIAN: So far?
627. MICHAEL: That might be it. [Counting routes on Brian's grid. Romina counting with a pen over grid in air.]
628. JEFF: Um- No, over to there, down there, like that. [Drawing routes]
629. ROMINA: [Inaudible]. [Brian working with the rows of numbers 0,1,2 or 3 and adding more rows.]
630. JEFF: You got that?
631. BRIAN: Man, I'm giving up on it.
632. ROMINA: Hum, that's kind of weird.
633. BRIAN: Mike here's the list. So far I've got some of the things [Inaudible].
634. MICHAEL: How many have you got there, thirty?
635. BRIAN: No that's only like //[Inaudible].
636. JEFF: //I think- I think we should like [inaudible] on like the next one we do, I think we should just like //do all ones over one. [Motioning across with his pen.]
637. BRIAN: //D's is like down- //down one.
638. JEFF: //All those, you know what I'm saying?
639. BRIAN: Like the order.
640. JEFF: What else? Is there anything else?
641. ROMINA: No.
642. JEFF: That should be it.
643. ROMINA: That looks nice too, what they're doing.
644. JEFF: What?
645. ROMINA: Brian, see that looks like a much- when you do like the-
646. BRIAN: Opposites and all that?
647. ROMINA: All right, this is- start counting- see, uh,- Which one's this?
648. JEFF: Well it's- well it should //be twelve.
649. BRIAN: //Oh man.
650. ROMINA: Twelve-
651. JEFF: You got the [Inaudible] right now.
652. BRIAN: All right //now this-

653. ROMINA: //That makes no sense. Oh well they're all – oh yeah they're all factors of three.
654. JEFF: Yeah, now this is the one that's- that's making it tough.
655. ROMINA: But this one has to be nine. One, two, three- Here
Brian, do a box-
656. JEFF: Well- well you know what //that is?
657. ROMINA: //Two by three.
658. JEFF: That's plus five, plus six, maybe it's plus seven there? That's plus one, plus one, plus one. That's all plus threes. Or that's- I don't know what that is. I don't know. But that's plus- You know what I'm saying? Plus five, plus six? Plus fifteen plus seven um- twenty-two? [Placing his finger over each number on Romina's grid. Romina wrote a "12" next to the "9" in the second row of her grid.]
659. ROMINA: Mm hm.
660. JEFF: Is that an option for what that is?
661. ROMINA: How many did you guys get by the- the-
662. JEFF: To the red.
663. ROMINA: Three by- or four by three? To the red?
664. MICHAEL: I got thirty-four.
665. BRIAN: [Inaudible]. So what am I doing with the box two by three?
666. ROMINA: How many- how many to the one at this one. [Pointing to Brian's grid.]
667. BRIAN: From here to here?
668. ROMINA: Yeah. Do your down- do that cool number thing.
669. JEFF: Cool number thing. [Brian beginning writing rows of 2 numbers, then 3 numbers in a row.]
670. ROMINA: We have to have some of these wrong.
671. JEFF: Well just- I don't know, is this just twelve? I mean I'm saying we found this one like in a second. That's it? We quit after we found that one? [Pointing to paper]
672. ROMINA: Only because it's not like long enough to be going like zigzagging through. [Romina zigzagging with a pen in the air.]
673. MICHAEL: Which one, that point?
674. ROMINA: Mm hmm.
675. MICHAEL: What's that's two by-//.
676. ROMINA: He's doing two by three, now you do four by two.
677. MICHAEL: Let me finish the two by four.
678. JEFF: Here Mike, you got all this man.
679. BRIAN: Ohaa.
680. JEFF: Mike, because then you could- you see how we're doing it? Like you could just do, you know, on all different-
681. ROMINA: Because we're going to be working on the one that you just did now. What is this? Four by two?
682. JEFF: [Inaudible].
683. BRIAN: Do you have like a formula that you're wanting to see if it works with this one?
684. ROMINA: No, we're just guessing.
685. JEFF: What are you doing?
686. ROMINA: Nothing, I was just going to put these under here so-

687. JEFF: All right, just- [inaudible]
688. ROMINA: Sure I'll do it. Are you doing four by three?
689. JEFF: It's big money.
690. JEFF: Now you're cooking with oil. Good. She's really good.
691. ROMINA: [Inaudible]. [Brian continuing by writing rows of 4 numbers 0, 1, 2 or 3. Romina makes rows of 4 by 3s on the grid.]
692. JEFF: Oh yeah?
693. ROMINA: Here, and we'll show- we'll even show you our patterns.
694. JEFF: Well wait, let him do his first.
695. ROMINA: Yeah, then-
696. JEFF: Because you're going to- it'll- gets in your [Inaudible] brain.
697. ROMINA: All right. Here why- if you have an organized way why don't you do it?
698. JEFF: All right- All the ones that you can go three down and get- Right, come over here. There's that and that's all the ones you can go three down and get. Right?
699. ROMINA: Mm hmm.
700. JEFF: All right. So going two down, if you go two down then you could either go, um,-
701. ROMINA: Over one down? [Michael counts with the pen on grid silently.]
702. JEFF: Over one down and over. If you go two down, you can go over two, down and over. You go two down, you can go over three down and over. You go two down, you can go over four and down. Um, is there any other place you can go if you go two down? No. What about- yes you can. You can go two down, over two, uh- No you can't. I was going to say and then down and over but we //already got that. [drawing routes on the 4 by 3 rectangles]
703. ROMINA: //You just messed up the box. [Romina cross out the box, 4x3 sub-grid, in which Jeff drew an incorrect route.]
704. JEFF: You're out of control.
705. ROMINA: All right. Now go one down?
706. JEFF: You go one down. You could go all the way over. You go one down you can go almost all the way over. You go one down, you can go two over. You go one down, you can go one over. Now you can go one down- [Drawing routes on the 4 by 3 rectangles.]
707. BRIAN: //How many did you think was going to be for this one?
708. ROMINA: Nine.
709. JEFF: What'd you get?
710. BRIAN: Ten.
711. JEFF: Do you know what they are?
712. BRIAN: Yeah.
713. JEFF: Can you do them- can you do it //on something like this?
714. ROMINA: //Here- Where's- where's ours?
715. JEFF: Which one is-
716. MICHAEL: I got twelve for the one you've just got-
717. JEFF: For the one we got twelve for? //All right.
718. ROMINA: //Here. Those are the ones we have for that one.
719. MICHAEL: They probably the same thing
720. ROMINA: Yes.

721. JEFF: What are you looking for [inaudible]
722. ROMINA: Did- we did that one.
723. JEFF: All right.
724. BRIAN: What did- do you know what your twelve are?
725. ROMINA: Nine.
726. JEFF: One, two-
727. MICHAEL: Me?
728. BRIAN: //Him.
729. ROMINA: //Oh.
730. MICHAEL: I haven't- don't have them written down but I know-
731. ROMINA: These are our twelve. [Handing Brian a sheet containing her and Jeff's work counting routes on a 3x2 sub-grid.]
732. BRIAN: All right, let me do it on the board for you.
733. ROMINA: Mike, do three over and two down.
734. MICHAEL: Huh?
735. ROMINA: three over and two down.
736. BRIAN: Writing up on the board. [Jeff drawing a four by 3, draws in routes, then crosses it out with his pen in the air. Brian draws his symbols or taxonomy of routes on the board with a number next to each edge.]
737. JEFF: One, two- Uh, that's [Inaudible].
738. ROMINA: Couldn't we just do something like in towers where like lines over are like the color and the lines down are the, um, number of blocks?
739. JEFF: All right. And that would?
740. ROMINA: Because, okay, lines over- because what is it- the number of blocks to the number of colors?
741. JEFF: I don't know what you're- what- what's that?
742. ROMINA: Two to the *n*. Two is the amount of blocks or the colors? (00:44:39)
743. MICHAEL: For what? Like towers on them?
744. ROMINA: Yeah.
745. JEFF: Colors. *n* is the number of blocks. I think. I don't know. I'm not sure.
746. MICHAEL: Well you figure a block has this- you got two- two ten over like this. Or two colors actually. I think it's, uh, the colors and *n* is the blocks.
747. ROMINA: Color two- //right. [Writing the words "color" and "blocks" on a piece of paper.]
748. JEFF: //Same thing.
749. ROMINA: All right, here we have one color- nah; it doesn't work for the first one.
750. ROMINA: Scratch that idea. [Crossing out the words on her paper.]
751. JEFF: Well- why- you know, what happened to- to what we were doing?
752. ROMINA: No, I know. Just keep on going. [Jeff, Brian and Michael working silently.]
753. JEFF: All right.
754. ROMINA: You're right [inaudible] three by two.
755. JEFF: Can you help me out?
756. ROMINA: What- what [Inaudible] //by two of this sheet?
757. BRIAN: //That's what I got so far.
758. ROMINA: //You need one [Inaudible]?

759. BRIAN: //That's how far right there. It's on the board. //The board.
760. ROMINA: //I know, I'm looking for- [Jeff continuing to draw routes.]
761. BRIAN: Mike do you see anything that I'm not getting?
762. ROMINA: //Three by three.
763. MICHAEL: //Which one you doing?
764. BRIAN: Two by three.
765. ROMINA: Three by two. All right, here. This is what we got.
766. JEFF: It's really hot in here.
767. ROMINA: All right, we got down two over three. Over three, down two. [Brian drawing routes on the chalkboard while Romina reads off her possibilities.]
768. BRIAN: //Okay.
769. ROMINA: //That's one of those? The first one.
770. BRIAN: [Inaudible].
771. ROMINA: All right we got those. [Brian continues writing on the chalkboard.] Got down one over three.
772. ROMINA: Except they don't have one, one, one, one, one, that one.
773. JEFF: That's one we don't have?
774. ROMINA: We don't have his last one over there. Check. I think that was the only one. So that nine does equal ten. [Brian writing, "start over" on the chalkboard and the word "Moves" up top.] (00:47:18)
775. JEFF: I don't see uh- Um-
776. ROMINA: Because we don't have that one?
777. JEFF: No, we don't have that one. [Inaudible]. [Romina erases the 9s and writes in 10s. She also writes a 5 under the 4.]
778. ROMINA: All right. It's, um, - it's Pascal's triangle. [Looking at the numerical array of the 1-centimeter-grid transparency.] (00:47:40)
779. MICHAEL: What is that? Two by three? [Looking and pointing to Brian's inscription on the classroom chalkboard.]
780. JEFF: It is?
781. ROMINA: Yeah.
782. JEFF: Let me see.
783. ROMINA: All right. Yeah it is.
784. MICHAEL: What?
785. ROMINA: It's Pascal's triangle.
786. MICHAEL: Two, three-
787. ROMINA: No, it's not. It doesn't work out.
788. JEFF: See look at- Here, Mike-
789. ROMINA: Because twelve that doesn't-
790. JEFF: Mike look- just look at it in this thing. You got the 6 and the 4 and the 6 are the 10. That should be a 15- //that's should be a 20- [Pointing to the 1-centimeter transparency grid that is in front of Romina.]
791. ROMINA: //But that's not a 15. That is a twelve because he even got the 12.
792. JEFF: Well- that should- that should be a 20 right there. [Pointing to the square † (6,3) on the transparency that contains the datum 15.]
793. ROMINA: [Inaudible].
794. MICHAEL: Up to here is been a one, one, one, one and-

795. JEFF: Huh.
796. BRIAN: So what's wrong?
797. MICHAEL: It should be six- fifteen.
798. ROMINA: Do- do a four by two.
799. MICHAEL: Yeah.
800. JEFF: You do the four by two, and it should put us, uh, in business.
801. BRIAN: All right.
802. ROMINA: And then- because we'll compare it to all-
803. JEFF: If this comes through it just-
804. ROMINA: If it's Pascal's triangle it'll just give us problems.
805. JEFF: No but it- it's just nice how- you start- like when you start from nothing. You know what I'm saying? Like we have no clue what we're doing. [Putting his hand on his forehead and then waves his hand in the air by his forehead.]
806. ROMINA: But he even got twelve when he did it.
807. MICHAEL: I might be missing two.
808. JEFF: It could be- it's not hard to miss three, right? [Jeff waves his hand in the air.]
809. MICHAEL: Two.
810. JEFF: Three.
811. ROMINA: So for the next one Jeff we missed five?
812. JEFF: It's very easy. I mean, //there's a lot of things going on.
813. MICHAEL: //That's kind of a lot.
814. JEFF: We like blew like a lot of these. You know what I'm saying? [Waving his hand in the air and puts it back on his forehead.]
815. ROMINA: Yeah. I think we, uh, got a few wrong. So what.
816. JEFF: That's what I'm saying. So why- like it wouldn't be totally out of control. [Removing his hand from of his forehead and waving his hand toward the grid.]
817. BRIAN: Oh.
818. ROMINA: Do- do it the other way. Just turn it around. That'll make our life- that- because that's we did. It's the same thing but- [Brian writing rows of numbers silently, this time adding a 4 too.]
819. BRIAN: Is that the air that just turned on?
820. JEFF: Yeah. But, it don't work though.
821. ROMINA: I'll be right back. [Leaving the room.]
822. JEFF: So how do you do your deal? I don't know how to do your deal.
823. BRIAN: It's nothing, the ones with two moves, the ones with three moves so I just go like three moves over- starting over first. over three down two boom, boom boom, boom boom. Then, then I go to over down, over down. This row gets eliminated pretty much. [Jeff nods his head at Brian.]
824. JEFF: All right. But you're not going to get there. I hear you.
[Jeff shakes his head "no" and then him and Brian work silently.]
825. JEFF: It'd be so much easier if some of us were lefties. [Brian already wrote his rows of 2 numbers, 3 numbers and now is writing rows of 4 numbers choosing from 0, 1, 2, 3 and 4]
826. BRIAN: Why?

827. JEFF: You'd just block like, uh, you try to see what someone does and it's just like- I mean like what is Mike looking at when I'm writing right now. //You know what I'm saying?
828. MICHAEL: //Oh yeah. [Michael is drawing routes in the air on top of the 4 by 2 rectangle drawn on his grid.]
829. JEFF: It's like, what- Which one are you doing, man?
830. MICHAEL: We're looking for fifteen for this one, right? [Brian works silently.]
831. JEFF: Didn't you get that?
832. MICHAEL: Hmmm. Don't know, yet. [Jeff works silently.]
833. BRIAN: What number are we looking for on this one?
834. JEFF: Fifteen. How many you got?
835. BRIAN: Eight. [Jeff is drawing routes on his 4 by 2 rectangles.]
836. JEFF: When it rained he went home and as soon as it stopped he came back out. Annoying bastard. I can't take it no more. [Inaudible singing]. [He is referring to the driver of an ice cream truck the noise that the truck makes.]
837. MICHAEL: What is that?
838. BRIAN: Zero.
839. MICHAEL: I got twelve so far.
840. JEFF: Yeah, twelve's like the number that we got stuck on last time.
841. MICHAEL: I think I got it.
842. BRIAN: [inaudible] it's fifteen.
843. JEFF: All right. All right, what if we even went- let me know when you're done. All right. Because there's an easier way to [Inaudible]. Listen to me for one sec.
844. MICHAEL: Go ahead.
845. JEFF: All right. If- all right. Say in a situation where it's like, uh a two by four. [Drawing a two-by-four sub-grid on 1-centimeter paper.]
846. MICHAEL: Uh hum.
847. JEFF: All right. If we know that in a four-by-four [really meaning a two-by-two] it's six [shortest routes] if you figure out all the ways to get to the beginning parts of this, this would all just be six different ways to get from here to here. So you figure out all the ways to get there and you could just add six- you know. [Subdividing the two-by-four sub-grid into two two-by-two sub-grids.]
848. MICHAEL: If you have the two, you could find out how many ways it's to get to here and add that where every two is. [Leaning over to Jeff's paper and pointing.]
849. JEFF: You know what I'm saying? So like from- from-
850. BRIAN: I got fifteen.
851. JEFF: You did?
852. BRIAN: Yeah.
853. JEFF: All right. 'Cause from there to there you have six different ways. And then from there, there's one way. To there there's one way and from there- //
854. BRIAN: //Haaa. Tell me when you're done.
855. JEFF: Sure. one- two- there's three ways. Um-
856. MICHAEL: I got fifteen also.
857. BRIAN: Yeah Mike. [Inaudible]. [Leaving the room.]
858. MICHAEL: So what does that mean?
859. JEFF: It means that it is the triangle. Right here? [Pointing to paper]

860. MICHAEL: Mm hm.
861. JEFF: You have fifteen there?
862. MICHAEL: I got fifteen.
863. JEFF: That's good. Yeah, because then- Yeah. This- then in a three by three it should be twenty. That'll be, uh,- [Pointing to paper and writing a 6 in the lower right hand square.]
864. MICHAEL: Is nine blocks for that one? [Pointing to intersection point (6,3) on the transparency]
865. JEFF: In the nine block it should be twenty. [Jeff writes the numbers 1, 3, and 6 in squares vertically with two 3s to the left of the other 3.] [inaudible]
866. MICHAEL: Where'd they go?
867. JEFF: What?
868. MICHAEL: Where'd they go?
869. [Brian returns.]
870. JEFF: Well, Brian's just standing there. I don't know. I need to go to the bathroom though. I'm like going to pass out. [Jeff leaves.]
871. BRIAN: So what are we doing now.
872. MICHAEL: No idea.
873. BRIAN: Did you figure out the five by five?
874. MICHAEL: Five by five? I'm doing three by three right now.
875. BRIAN: It's the green. If we already know what it is then we have to figure out-
876. MICHAEL: I just want to make sure that's twenty first, so-
[Counting routes with his pen on his grid.]
877. MICHAEL: I'm missing two. That's probably right though.
878. BRIAN: Did you get the, uh, staircase one?
879. MICHAEL: Which one? For the three by three?
880. BRIAN: Yeah. [Romina returning.] All right, I got twelve, fifteen, right.
881. ROMINA: Oh, you guys went and wrote on this didn't you?
882. MICHAEL: I didn't do it.
883. ROMINA: All right.
884. BRIAN: Did Jeff tell you?
885. ROMINA: What?
886. BRIAN: I got fifteen for this one-
887. ROMINA: For which one?
888. MICHAEL: //For-
889. BRIAN: //Four by two.
890. ROMINA: So you did get fifteen? So now it's working? And then the two by four has to be fifteen too. Now if we do three by three and that's twenty, then we're done.
891. BRIAN: That's what he's doing?
892. ROMINA: What?
893. BRIAN: He said he was off by two. [Romina erasing the numbers on the grid transparency then takes a new transparency with a grid on it.] You can just get another one.
894. ROMINA: I'll just turn this around. [Referring to the transparency of a centimeter grid. She then writes 2 and 3 in the squares of the first row of the transparency.]
895. BRIAN: It's only a couple of numbers.

896. ROMINA: Did it again. You got twelve for this one? Fifteen, I mean? [Rewriting the numbers on the grid and adds a 15 to the right of the 10 and under the 10.]
897. BRIAN: Yep. Now, which one are you expecting to be twenty?
Three by three?
898. BRIAN: I guess I'll do it. Check it out.
899. ROMINA: I don't think- it's here- he has- He was just doing three by three wasn't he? [Looking through her papers.]
900. BRIAN: Yeah. It's no big deal.
901. ROMINA: I'm already stuck. [Brian drawing a three-by-three subgrid on his paper. Romina draws in shortest routes for the "imaginary" three-by-threes on her grid. Romina's pen stops when drawing a route.]
902. JEFF: You shouldn't be. Where you going?
903. ROMINA: Three by three. [Showing the paper to Jeff.]
904. JEFF: You said F making the- the boxes.
905. MICHAEL: Yeah, I got twenty for that one.
906. JEFF: For three by three?
907. MICHAEL: Yeah.
908. JEFF: All right well then- I mean can't we explain why we think- well- all right. [Waving his hand.]
909. MICHAEL: //They're going to ask us-
910. JEFF: //All right then the next question is why- //why
911. ROMINA: //Now-
912. MICHAEL: //How do you know-
913. ROMINA: //Just relate this back to the //blocks. [Pointing to the 1-centimer grid on the transparency with his marker.]
914. JEFF: //Wait- Why is this- why does the Pascal's triangle work for this is the question.
915. ROMINA: //Exactly. Relate it to the blocks.
916. MICHAEL: //Just think first how do you know it's twenty? You know, how do you know it's not nothing else?
917. JEFF: Well F that. If we could explain-
918. ROMINA: Stop saying that.
919. JEFF: Why- why this is the Pascal's triangle up to here [Pointing to the numerical array on the transparency 1-centimeter grid.], we don't need to explain how we're positive this is twenty. //You know what I'm saying?
920. ROMINA: How does it go- this is-
921. JEFF: One-
922. MICHAEL: It should be ones on all the sides. [Jeff writing ones on the outside lines of their numeral array on the transparency 1-centimer grid.]
923. ROMINA: Yeah right. So- [Writing out Pascal's triangle.]
924. JEFF: That's six-
925. ROMINA: This is just one, two, three. //So-
926. JEFF: //What's that?
927. ROMINA: With one- //there's only one possibility.
928. MICHAEL: //All right, how- //how is he getting them?
929. ROMINA: //Two-

930. MICHAEL: How are you getting yours? Maybe the way you're doing will give us
931. JEFF: Has some kind of- Yeah, we can work something out.
932. BRIAN: [Inaudible].
933. MICHAEL: Do you just like- are you guessing or do you have some kind of pattern?
934. BRIAN: I'm just- doing it man. I'm just- you know-
935. MICHAEL: Ah- [Romina pointing to the numbers on her transparency with a marker.]
936. BRIAN: I know there's a way to make two and get there in two moves. I know there's a way to make it in three moves. Four moves.
937. MICHAEL: So you're going by the moves, right?
938. BRIAN: Yeah.
939. JEFF: Don't use that one.
940. ROMINA: Hold on. For the Pascal's triangle-
941. MICHAEL: Yeah.
942. JEFF: You're making thumbprints again.
943. ROMINA: The one, //two, one-
944. JEFF: //Bringing it back to eighty-six.
945. ROMINA: -that's with what? With?
946. MICHAEL: Um-
947. ROMINA: Two colors- It's, it's two to the *x* like that? [Pointing to the second diagonal "row" of the array of numbers on the 1-centimeter-square transparency, containing the numerals 1, 2, 1.]
948. MICHAEL: Yeah it's two.
949. ROMINA: So it's two colors-
950. MICHAEL: Think of it as zero, one, two- you only have two colors of choices - zero, one, two. Three
951. ROMINA: Huh
952. MICHAEL: Three toppings on a pizza.
953. ROMINA: Yeah, like- so then how could this- this is two what? Two? Two different ways- like- [Pointing to the top numbers on the transparency with her marker.]
954. MICHAEL: Two- Uh- it's the total. One, two, three- That's, that's the total length that you can get, have to get there- to get there. [Pointing at numbers on transparency with marker.]
955. ROMINA: Yeah, okay.
956. MICHAEL: You know?
957. ROMINA: So for this one, the total length is three.
958. MICHAEL: But then this one is one, two, three, four, five and you get ten. You know? [Pointing at the 6 on the transparency grid]
959. ROMINA: But you're in the second row. [Pointing at triangle]
960. MICHAEL: Yeah. [Romina taps her marker on the table.] Right. This is one, two, three- four, five, six and you get twenty. [Pointing at the 20 on the transparency grid.]
961. ROMINA: All right.
962. BRIAN: All right.

963. ROMINA: I'm going to write it this way because I'm having a- I don't know about you people but- How does this go? It's not like in the blocks, is it?
964. JEFF: What? For the thing?
965. ROMINA: Yeah.
966. JEFF: Yeah, it'll fit. Why- why don't you start like in the middle like here.
967. ROMINA: Yeah.
968. JEFF: Or why don't you use a different transparency?
969. ROMINA: Well I just want it like- I'm just doing it so I can see it.
970. MICHAEL: Why don't you do it like //that that way we can see it.
[Pointing to the transparency grid with his marker.]
971. JEFF: //Why do you keep- //you're starting all the way over on the side every time.
972. BRIAN: //All right. There's twenty
973. MICHAEL: Start like this. It's easier to figure out like a two by two box. Over here [Inaudible].
974. ROMINA: No, I know. It's just- it's just so I can see it so that's one block, two block, three block, ok.
975. JEFF: All right.
976. MICHAEL: That would be seven- twenty-one- thirty-five-
977. JEFF: All right. You want to, um- You want to try and explain this and then wherever we get like confused along the way, you know maybe that's how we'll be able to- as we talk through it we could even- Oh sorry. I tried to stop it from hitting your leg. I don't even see it.
978. MICHAEL: It's a wet erase marker it will come off with water.
979. BRIAN: Oh man.
980. JEFF: At least you don't have grease all over your pants.
981. ROMINA: Yeah but this is like my favorite-
982. JEFF: Oh I hate these pens.
983. ROMINA: Would you lick it because my fingers have blue on them? Lick your fingers Jeff.
984. JEFF: It'll be all right. It's going to be there for a little while, you'll have to deal with it. All right, can we try to explain this?
985. MICHAEL: To who?
986. JEFF: Anyone who wants to hear it.
987. JEFF: Here's some water for it.
988. BRIAN: Jeff-
989. JEFF: All right. Uh-
990. MICHAEL: Put the caps on so they don't roll.
991. BRIAN: We've got like five minutes.
992. BRIAN: You're just going to spread it all over the place.
993. JEFF: Well don't get mad at me. Relax. I'm trying to work this out here. What do you think? Should //I continue.
994. BRIAN: //You ain't working nothing that way.
995. JEFF: What about a tissue that we could dab- we could-
996. BRIAN: What about Romina, when you go home-
997. JEFF: Yeah, you put a little stain stick on it-
998. BRIAN: A little Shout. //It'll Shout it out.

999. JEFF: Shout it out.
1000. ROMINA: I don't have those things.
1001. JEFF: You could go to the //store and pick them up.
1002. BRIAN: //Go to the store. You could buy little Shout wipes.
1003. JEFF: Yeah, they're real cheap. You could clean it up and you'll never have to worry about it.
1004. ROMINA: I'm very upset right now.
1005. BRIAN: How many they got?
1006. JEFF: Do you see this?
1007. UNKNOWN: Romina do you want a baby wipe?
1008. ROMINA: Yeah please. We'll try this.
1009. JEFF: Can I get one of those just for my hands?
1010. UNKNOWN: Yes.
1011. ROMINA: Yeah, my hands are-
1012. UNKNOWN: Anybody else? Baby wipe Brian?
1013. BRIAN: Nah, I'm clean man.
1014. UNKNOWN: Mike?
1015. MICHAEL: No I'm good.
1016. JEFF: Let me see your hands.
1017. BRIAN: I ain't working with the markers.
1018. JEFF: Oh that's-
1019. ROMINA: So when I asked someone to lick my shirt you were obviously not going to.
1020. BRIAN: Well Romina, I'm going to come over and lick your shirt. That's what I'm going to do.
1021. ROMINA: Lick your fingers. I didn't mean lick my shirt.
1022. BRIAN: And you see what it did? It spread it all over your shirt.
1023. JEFF: Why don't you go for that on your shirt?
1024. ROMINA: Because I'm going to try to-
1025. BRIAN: I love the smell of baby wipes dude.
1026. JEFF: They do smell good.
1027. ROMINA: Oh. It's just getting worse.
1028. BRIAN: Now it's going to be a wet stain.
1029. JEFF: Ah-
1030. BRIAN: Romina has it- Romina if it didn't-
1031. MICHAEL: You just better leave it.
1032. JEFF: If it- Just stop.
1033. BRIAN: You're making it worse.
1034. ROMINA: It's already bad.
1035. BRIAN: You're going to ruin it beyond repair.
1036. BRIAN: Is she busy? She can't come and visit us?
1037. UNKNOWN: She's just all the way down the hall.
1038. BRIAN: All right.
1039. JEFF: How- //how are those kids doing?
1040. ROMINA: //Yeah you know they're probably done with the assignment.
1041. JEFF: Are they- //what are they doing?

1042. ROMINA: //They're on problem five.
1043. MICHAEL: This takes a long time to figure this out.
1044. BRIAN: You know probably- if we just think about it what do we work on every single //time.
1045. JEFF: //Yeah I know, but we go to-
1046. MICHAEL: We got to explain it.
1047. JEFF: You got to figure it out. You know what I'm saying? You got to go through it.
1048. BRIAN: She's going to look at these, and she's like I have no idea what you're doing.
1049. JEFF: I'm out- I've got to leave //in five minutes.
1050. ROMINA: //No she's going to go like this. You're still on this?
1051. JEFF: She won't say that.
1052. MICHAEL: No they got- they got a different problem than us.
1053. JEFF: They have the same- do they have the same problem down- down there?
1054. ROMINA: Are they done?
1055. T/R1: Um, they're working on a different problem.
1056. JEFF: They have a different problem?
1057. BRIAN: All right.
1058. ROMINA: Like they didn't get //this one to work on?
1059. MICHAEL: //We- we can't justify our answer but we're- we're, uh-
1060. JEFF: We're going to talk through it.
1061. ROMINA: //[Inaudible].
1062. JEFF: //And we want to see where that takes us. I'm going to have to leave in five minutes though.
1063. T/R1: Oh so, you've got to talk fast.
1064. JEFF: So we're going to talk fast. All right.
1065. T/R1: Okay, the problem I really wanted to give you was for all the points on the grid.
1066. ROMINA: Oh good. //We did that.
1067. JEFF: //Oh yeah. That's what we did.
1068. ROMINA: Why don't we do points-
1069. BRIAN: We got it. We got it.
1070. JEFF: You know-
1071. ROMINA: Points up to-
1072. T/R1: All right. So tell me, tell me.
1073. BRIAN: Pens are flying now.
1074. T/R1: Yeah. Did you like the problem?
1075. ROMINA: No. Nah, it was okay.
1076. JEFF: It's just-, doing all this kind of stuff really hurts your brain, but other than that //it was all right.
1077. ROMINA: //It- your eyes. All right. What we did is we- we took it
1078. JEFF: We broke it down.
1079. ROMINA: Yeah, we just went from point to point on the thing.
1080. JEFF: Yeah. Like we even- we'll just say we started making the box like that. How many different ways can you get from this point to this point? You know, make an easier

problem. Like the basic math- deal. [Romina draws in points with her marker and points to the numbers on the transparency grid.]
1081. ROMINA: So we did like up to this point there's two. Up to this point there's three, four, six, three- So that- those are our numbers. Those are up to the points like down and diagonal. And what we got is Pascal's triangle. [Jeff points to the numbers on the transparency grid with his marker.]
1082. JEFF: Yeah. We started, you know, and then as we started, you know, like it takes two to get to there. Three to you know, to get there as Romina just went through and did. And then as we started filling it out we noticed that if you tilt it like that [Rotating the transparency.] and throw ones on the outside and a one on top, I mean you're looking at Pascal's triangle. And so we stopped at this point [Jeff points to a point on the transparency grid with his marker.] because I mean making, you know, like thirty plus different things like this it gets- it just gets confusing you know. [Drawing a curve on his paper.]
1083. T/R1: Hm.
1084. JEFF: And so Brian had a- Brian was get- like doing, you know, we were- some of us were drawing out all the ways. [Jeff begins to draw on his paper.] Brian had another method of finding out the ways to do it. //You know. //And we just- [Jeff waves his hand to Brian.]
1085. ROMINA: //And then we just compared //them. And like //whatever he didn't have-
1086. JEFF: //-brought it all together and then that's kind of what we're looking at right now.
1087. T/R1: So you found those numbers, all of them, by counting?
[Referring to the numbers on the taxi grid]
1088. ROMINA: Yeah. //The ones we have written. Yeah.
1089. JEFF: //Well up- up to here. [Jeff points to a number on the transparency grid.] Right. What is written we counted through them. [Making a circular motion with his hand.]
1090. T/R1: Okay. So is there anyway you can justify if I were to say to pick- you said these are like rows, like so this one, two, one would be what row? These points here //of the triangle? [Pointing to 3 vertices on the grid.]
1091. JEFF: //What? Um, I'm not-
1092. T/R1: You put ones on the side I noticed.
1093. JEFF: Yeah.
1094. T/R1: So if you were to look at //one, two, one-
1095. MICHAEL: //Do you mean like this row? [Pointing to triangle]
1096. T/R1: Well pick any row.
1097. JEFF: All right. All right. We'll say one, two, one because that's an easy place to start from.
1098. T/R1: Right.
1099. JEFF: What's the question though?
1100. T/R1: Right. So-
1101. BRIAN: //One, two, three, one-
1102. T/R1: //-that's the second row.
1103. JEFF: //Yeah.

1104. MICHAEL: //I mean I guess we're saying-
1105. JEFF: That's the second- yeah. [Pointing to triangle]
1106. MICHAEL: Things with, uh, one- one block. [Pointing to the transparency grid with pen.]
1107. T/R1: Okay.
1108. MICHAEL: Two blocks, three blocks, four blocks.
1109. JEFF: And then this would be five blocks then- [Pointing to triangle]
1110. MICHAEL: Not four. That doesn't make sense.
1111. T/R1: How would that be?
1112. MICHAEL: Six, you could, hum, things that- I don't know.
1113. JEFF: See I'm still not exactly sure what you're asking.
1114. MICHAEL: //Yeah, I don't know.
1115. ROMINA: //Yeah.
1116. T/R1: I didn't ask anything yet.
1117. JEFF: I'm all-
1118. T/R1: I was- I was saying that you-
1119. JEFF: What are you trying to-
1120. T/R1: you're showing me that's Pascal's triangle but I don't see it. Help me see it.
1121. JEFF: You don't see it?
1122. T/R1: Right. Can you //show it to me?
1123. JEFF: //All right, well here. The one, one two one, one three three one, one four six four one- [Pointing to transparency grid with marker]
1124. T/R1: Okay let- show me again where's the four?
1125. JEFF: All right. We're on- all right. The- that one right there- [Pointing to grid]
1126. T/R1: Mm hm.
1127. JEFF: -that we added in and this two is the three. The two in that one there is a three and there's two ones on the outside- [Placing his finger on each number as he speaks.]
1128. T/R1: Mm hum.
1129. JEFF: So you get one three three one. And then one- the one and the three for the four. Three and the three for the six. The three and the one for the other four and then the other one on the end and then continuing through the four and the one together is the five. The four and the six is the ten. Six and the four is the other ten. Four and the one is the five. //Do you see it? [Pointing with his finger and marker to the numbers on the transparency grid.]
1130. T/R1: //Okay. So can you explain to me, for instance, why that works? Why- where this ten comes from when you know- you're just saying well there's a pattern here because you found them, but is there a way where you
haven't' found them that makes sense to predict the number of paths of one you haven't found?
1131. JEFF: Well like to here, I mean we //would say-
1132. T/R1: //You understand my question?
1133. JEFF: Well like to here we would say it was thirty-five. [Pointing to a square on the transparency grid with marker]
1134. T/R1: Right. How would you- how- where //would the thirty-five come from?

1135. MICHAEL: //You can't justify it because- You can justify these because you can say you counted. You can't justify that because you can't say you counted.
1136. JEFF: Yeah because we didn't count it. We're saying we're following the pattern- [Waving his hand.]
1137. T/R1: Right.
1138. JEFF: That's- that is our justification as of now.
1139. ROMINA: [Inaudible].
1140. T/R1: Right.
1141. JEFF: That we're just following //the pattern.
1142. T/R1: Do you understand my next question Jeff? What I'm sking?
1143. JEFF: //Yeah.
1144. ROMINA: //What if three- what if Pascal-
1145. T/R1: Because so you notice this pattern and the pattern fits Pascal's triangle.
1146. BRIAN: So does that mean there's //thirty-five for the red one?
[Romina and Mike are counting. Mike writes something.]
1147. MICHAEL: //Only these are zeros. This is like one topping- you know on the pizza? [With Jeff looking at the transparency grid. Jeff pointing to a number on the grid.]
1148. ROMINA: Yeah, one topping, two toppings.
1149. BRAIN: Remember how- Mike you had thirty-four for the red one, right?
1150. MICHAEL: Um- Yeah I think that was the problem.
1151. BRIAN: It's thirty-five.
1152. JEFF: Yeah, it's thirty-five.
1153. MICHAEL: Oh, I probably missed one.
1154. JEFF: Good, uh, deduction.
1155. T/R1: So you counted thirty-four by brute force //and you're saying that by this pattern, um, you would feel more comfortable with the pattern in saying thirty-five.
1156. JEFF: //Yeah.
1157. BRIAN: But-
1158. T/R1: Right?
1159. ROMINA: Did you actually get thirty-five?
1160. MICHAEL: I got //thirty-four.
1161. BRIAN: //He got thirty-four but you know he's been off by like one cause you know. Yeah, it could- it could of //been one.
1162. ROMINA: //Natural tendencies? Um,-
1163. MICHAEL: Stop that.
1164. T/R1: Okay. So why is- why is that-
1165. ROMINA: All right. [With Jeff studying the transparency grid.]
1166. T/R1: Why do you think that- Why do those numbers seem to work? How could you explain those numbers? That's that's really- isn't that interesting?
1167. JEFF: Yeah. It- it hurts though. It really does.
1168. ROMINA: Yeah, I'm having trouble seeing Pascal's triangle.
[Rewriting the triangle the way she is used to seeing it.]
1169. T/R1: It's hard to see the other way, isn't it?
1170. ROMINA: All right. So for this one the two comes from when there's- [Pointing to numbers in the triangle with her marker]
1171. JEFF: One block.

1172. ROMINA: One-
1173. JEFF: Block.
1174. ROMINA: Is that-
1175. JEFF: One block.
1176. ROMINA: Isn't that two blocks?
1177. JEFF: One, two.
1178. ROMINA: No. Um, let's go back to towers. The two comes from this is one block. This is two blocks with two colors. [Continuing to point to numbers in the triangle with her marker.]
1179. JEFF: I have to leave. I'm kind of out.
1180. ROMINA: Hold on. How's this go? Just tell me where this comes from.
1181. MICHAEL: What happened?
1182. ROMINA: Okay. This is with- with just one block?
1183. MICHAEL: This is nothing.
1184. ROMINA: This is nothing? This is one block?
1185. MICHAEL: This is like- yeah, //one. All right.
1186. ROMINA: //One block, two- this one tells how many blocks.
1187. MICHAEL: One block. Two blocks. [Pointing to the 1 and 2 in the triangle Romina redrew.] Not two blocks but like- [He points to the numbers on the transparency grid.] [Inaudible.]
1188. ROMINA: One block, two blocks, three blocks- Oh no, this is zero block, one block, two block?
1189. MICHAEL: For one block you get two. Right? Or two blocks-
1190. ROMINA: All right.
1191. MICHAEL: Three- three- three blocks. One- So you can't really say it because there's three for three and then you get four here. You can't really- I don't think you can use that. That- that row thing. [Pointing to the numbers on the transparency grid.]
1192. ROMINA: All right. Yeah. I know. I'm just trying to- because for like-
1193. MICHAEL: There's got to be some type of, you know, way. If I could see-
1194. ROMINA: Can't you just go one, two, three, four?
1195. MICHAEL: Uh hum [nodding his head in agreement]
1196. ROMINA: That's what it goes one, two, three, four? Because then okay for this one for the three. If we name all the ones going horizontal- *A*s and ones going down same with *B*. And this would be with two *A*s and one *B* there's three and then there's two *B*s with one *A*, three. [Pointing with a green marker at the intersections Points † $(3,2)$ and † $(3,1)$ on the transparency grid.] And for this one remember like two *A*s two *B*s- //six. [Now pointing to the intersections point † $(4,2)$ and on the transparency grid.]
1197. MICHAEL: //You could say, um-
1198. ROMINA: Do you understand what I'm saying?
1199. MICHAEL: Like yeah, these are like this row is everything with perimeter two. I mean I half the perimeter, like. [Pointing with his marker to numbers on the transparency grid.]
1200. ROMINA: //Well no I'm saying so to get that-

1201. MICHAEL: //In order to get to that point you have to go over one and down, uh, one or down one and over one. [Pointing to the intersection point † $(2,1)$.] Just like that row. Everything in this row, over two and down two and over one.
1202. ROMINA: Yeah but like I'm just saying like if she were to pick anything like right there we could say it's like eight *A*s and like six *B*s. [Tracing a rectangle on the transparency grid.] You know like- and then we could tell you where you it is in this one. [Pointing to the redrawn Pascal's triangle on the piece of paper.]
1203. MICHAEL: Well you could say all- everything in this row, the shortest route is two. Everything in this row shortest route is three. This one shortest route is four. [Pointing to a diagonal of numbers—1 4 6 4 1—in the transparency grid.]
1204. ROMINA: Yeah.
1205. MICHAEL: The shortest route is five, six and so on. So that's how you could, you know, name them. This is row six because it has everything in the row has shortest route of six. [Pointing with a marker to diagonals of numbers on the transparency grid.]
1206. ROMINA: No, I understand. I'm just saying like-
1207. MICHAEL: There's a, you know-
1208. ROMINA: To get it-
1209. MICHAEL: //To- to say it like, oh I'll pick this block-
1210. ROMINA: //Because isn't that how- isn't that how we get like these? Like doesn't the two- there's- that I mean, that's one- that means it's one of *A* color, one of *B* color. [Pointing to the 2 on the redrawn triangle on paper.] Here's one- it's either one- either way you go. It's one of across and one down. [Pointing to a number on the transparency grid and motions with her pen to go across and down.] And for three that means there's two *A* color and one *B* color [pointing to the 3 on the redrawn triangle], so here it's two across, one down or the other way [tracing across and down on the transparency grid] you can get three is //two down- [Pointing to the grid.]
1211. MICHAEL: //You mean like one *A* color and two-
1212. ROMINA: Yeah.
1213. MICHAEL: This is one-
1214. ROMINA: Like two blues, one red. Two across, one down or this is two reds, one blue, two down, one across. And that's how we would get the Pascal's triangle. [Pointing to numbers on the redrawn grid and transparency grid.]
1215. MICHAEL: But there's like- you know, there's got to be a way that we could just say, all right this one's three. //So five down this has to be this because of some kind of-
1216. ROMINA: //I know, I'm just saying-
1217. ROMINA: So if it were-
1218. MICHAEL: Pattern- I mean like, you know, reasoning. You can't just say I counted them.
1219. ROMINA: I know. I'm just saying so like- and then that could relate back to this but that is this, so it's believable- and for-
1220. MICHAEL: So what- what are you looking for right now?
1221. ROMINA: Yeah like-
1222. T/R1: I think Romina knows what I'm looking for. I think she's said it very articulately. That if I were to pick any point right on-
1223. MICHAEL: Mm hmm.

1224. T/R1: If I were to make a larger grid- right Brian? I think he //knows what I'm looking for.
1225. BRIAN: //Yeah.
1226. T/R1: She's looking for a way to come up with a particular pattern that she's identifying that. I think I'm hearing you say- you're trying to look at blocks-
1227. ROMINA: Mm hmm.
1228. T/R1: Colors?
1229. ROMINA: Yeah.
1230. T/R1: And then you're doing *A*s and *B*s.
1231. ROMINA: Mm hmm.
1232. T/R1: That's what I'm hearing you say? And you were trying to say maybe that could get you to some general point. Why don't you try saying that again? I- I thought I followed you, but I'm not so sure that Brian and Michael followed what you said.
1233. ROMINA: Like why-
1234. T/R1: So say it again. What you were-
1235. ROMINA: Like why this and this are related?
1236. T/R1: Yeah.
1237. ROMINA: Well-
1238. T/R1: Throw out your idea //again for them so we can hear it.
1239. ROMINA: //When we look- whenever we do this we always- we always talk about towers and how this is like a tower of two high with two different colors and there's one- one tower you can make that makes one color and one and one and then all the other color. And- and then for this one it's three high and this is all one color. There's two of one color and one of the other, whatever. And for this it's basically the same thing because this is- let's see. This is- this is two but usually you go one across and one down so there's two different ways to get to that one. And for this one there's going to be two across and one down. Or to go down here it's two down and one across which is basically the same thing and it just goes on. Do you understand? Understand? Was that good? Or, do you want more? [Connecting the data from the grid and the triangle drawings by pointing to the numbers on each back and forth.]
1240. BRIAN: Yeah.
1241. ROMINA: Or do you want more?
1242. T/R1: I don't know. I don't know if Michael-
1243. BRIAN: Mike do you understand?
1244. MICHAEL: Yeah, I understand what you're talking about.
1245. ROMINA: Yeah. Yeah.
1246. MICHAEL: This would be, um, one- we'll think of it as pizza because that's the thing I like but-
1247. ROMINA: Think of towers.
1248. MICHAEL: Or towers. I mean this will be a tower of three-
1249. T/R1: Think of it as pizzas.
1250. MICHAEL: A pizza. A pizza with, um, three possible choices for toppings and- I like the tower.
1251. ROMINA: Yeah, the tower is easier.
1252. MICHAEL: You have, you have a tower of three and you have, you know, two colors. So one- it's either you know- Color x and two of color y. Well this is direction x

and two, two directions of *y*, you know-// [Pointing with a marker to the redrawn Pascal's triangle.]
1253. ROMINA: //Yeah.
1254. MICHAEL: //of *y*. So that makes- that makes sense.
1255. ROMINA: So for like the three, it would be two *x*, one *y* or two *y*, one *x*//
[Referring to the taxi grid.]
1256. MICHAEL: //Yeah, I got that.
1257. ROMINA: And this would be
1258. T/R1: Okay. Well- where I'm still having a little trouble is, um, - Okay, so you're talking about these blocks, right?
1259. ROMINA: Mm hmm.
1260. T/R1: So what are you labeling them? These blocks?
[Referring to blocks on the taxi grid.] Which is the *A* and which is the *B* and why is it //okay to call them *A*s and *B*s?
1261. ROMINA: //We'll do it- how about x and y?
1262. T/R1: Sure.
1263. ROMINA: *x* will be the ones that go horizontal. [Motioning across with her marker on transparency grid.]
1264. T/R1: Okay.
1265. ROMINA: And y will be the ones that go over there, basic graphing skills.
[Moving her marker down.]
1266. T/R1: Does that make any sense Brian?
1267. BRIAN: Yeah.
1268. T/R1: Brian, do you think so?
1269. BRIAN:
1270. I think so. Yeah. I'm- I'm hanging out. I'm doing good now. You know what I'm saying. Oh, I was like what is that? A research paper.
1271. T/R1: T/R3, T/R2, do you have any questions?
1272. T/R3: Well I mean I have a very simple question. That is, it's still not clear to me how- how they know that the- to get to any particular corner corresponds to one of the numbers in Pascal's triangle.
1273. ROMINA: You see I haven't done that either yet.
1274. T/R1: Okay, why don't you think about that for a couple of minutes?
1275. ROMINA: All right let's say- [Drawing on Michael's representation of Pascal's triangle]
1276. T/R1: Let me just leave you be while you think.
1277. ROMINA: What would that be anyway?
1278. BRIAN: We'll say thirty-five there. [Romina writing 35 on the transparency grid.]
1279. ROMINA: You know, why don't we do this one?
1280. BRIAN: Thirty there.
1281. ROMINA: This is thirty?
1282. MICHAEL: No, no that's uh-//
1283. ROMINA: No.
1284. BRIAN: No //twenty-one.
1285. ROMINA: //This is twenty?
1286. BRIAN: Twenty-one.

1287. ROMINA: No, you know, why don't we do it this way.
1288. MICHAEL: That should be twenty-one.
1289. BRIAN: That one right there should be twenty-one.
1290. ROMINA: One, six- [Drawing on triangle.]
1291. BRIAN: And that should be a six. Fifteen plus six, twenty-one. And twenty- [Pointing to a number on triangle]
1292. ROMINA: Like that? Is that one of them? One //six-
1293. MICHAEL: //No. The next one. The next one.
1294. ROMINA: All right so that's one, seven- [Writing more rows of the triangle on paper.]
1295. MICHAEL: Twenty-one.
1296. ROMINA: Okay, I' not just- I, I'm doing
1297. BRIAN: Thirty-five.
1298. ROMINA: And one- //seven.
1299. BRIAN: //Seven.
1300. MICHAEL: Like we know it is that.
1301. ROMINA: Okay- //So this-
1302. MICHAEL: Without- //without just saying //oh it follows the pattern.
1303. BRIAN: //[Inaudible] why.
1304. MICHAEL: He wants to know why. Yeah.
1305. ROMINA: So this one is- is that thirty-five again? Or no, this one's thirty-five. [Writing the numbers in on the transparency grid.]
1306. BRIAN: This one's thirty-five.
1307. ROMINA: This one's thirty-five so then this one is?
1308. BRIAN: Twenty-one.
1309. ROMINA: Twenty-one. So let's see. One, two, three, four, five, one, two- I don't know. I see how it would go. [She draws lines in between the numbers in the 7th row.]
1310. MICHAEL: I- I know- we know it follows a pattern but he wants to know-
1311. ROMINA: Okay. Five-
1312. MICHAEL: Without saying oh it just follows a pattern. //Why is it-
1313. ROMINA: //Okay, five and two- five and two, just add that. That's how many blocks there are. That's seven. So you got to go one, two- no. One, two, three, four, five, six, seven. Gets you down to seven. And five of one thing and two of another thing, so you just- you don't count- we won't count the one because that doesn't involve that. [Pointing between the transparency grid and the redrawn, augmented version of Romina's Pascal's triangle.]
1314. MICHAEL: What do you mean five and seven?
1315. ROMINA: What?
1316. MICHAEL: What are you talking about five and seven?
1317. BRIAN: Five across //and two down.
1318. ROMINA: //Five across and two down. Like you just count in. It goes- that's with one of one color and that's with two of two- of another color. That's with three, that's with four, that's with five. So it's either the two or the five. Both of them are the same thing. Yeah, we can explain this. Right? If anyone you pick like this one, you know it's one, two, three, four, five, six, seven. You know it's seven and it's going to be one, two,

three- six of one color so it's going to be seven. [Pointing to both the redrawn, augmented triangle and numbers on the transparency grid.]
1319. MICHAEL: Are you saying five across- one, two, three, four, five, one, two. [Working with a figure of the first six rows of Pascal's triangle.]
1320. ROMINA: So- either way- no, but it's seven blocks. It's five plus two. That's how many blocks you had. For seven blocks you go down. Go one, two, three, four, five, six to the seventh row. And now you know it's five by two so it means there's five of one color, two of another color so if I go to the second one this has to- this is all one color. This is one with one color this is two. So it's either twenty-one or there's three of one color, there's four of one color, and this is five of one color or twenty-one again. [Circling the two 21s on the redrawn, augmented version of Romina's Pascal's triangle.] (01:24:26)
1321. MICHAEL: But suppose you were to say not colors but like //ups and downs, you know-
1322. ROMINA: //Or like that- this is //with two- two-
1323. MICHAEL: //But why- you know, why is it thirty-five? If you go- Or why is it- let's go- go a little easier. Why is it, you know, four if- of, um
1324. ROMINA: All right. Four, right? Four is- all right, why don't we do six? Six is a little harder. Six is one two- the one with six. There's one, two, three, four- you know there's four- [Pointing to triangle.]
1325. MICHAEL: It's two and two. All right. Two, four-
1326. ROMINA: This one.
1327. MICHAEL: One, two, three, four.
1328. ROMINA: It's because it's four blocks. No matter how you go there you had to take four spaces. And any direction you take has to be four spaces, right? So that means it's four- it's four blocks high. So you go to the fourth one. So you know it's in here. [Circling the 4th row of the triangle.] And it's- to get here it's two across and two down. So whatever, like you know- Do you understand? Whatever route you take you'll end up two across two down. So it means there's-
1329. MICHAEL: Two across and two down that would be this one because this would be one across and two down and this is two across and two down and this is- Wait, two down- two down and one across. One across and two down and this is two across and one down. [Pointing to redrawn triangle.]
1330. ROMINA: No, this is three across one down.
1331. MICHAEL: Oh whatever. Three.
1332. ROMINA: And this is //three down-
1333. MICHAEL: //No it's imposs-. //It doesn't make sense.
1334. ROMINA: //Three across.
1335. MICHAEL: Three across would be at- you'd be in- you'd be somewhere else.
1336. ROMINA: No you won't. Three across, one down is still in that row.
1337. MICHAEL: Yeah but you- you're doing this- this square right here, right? Two and two.
1338. ROMINA: I'm doing the six, right? You want me to do the six?
1339. MICHAEL: Yeah. That square right there. [Pointing at taxi grid transparency]
1340. ROMINA: That's still a four.
1341. MICHAEL: Mm hum.

1342. ROMINA: That's two across two down. That's four so you're in the four blocks. And then it's this- to get to here the only way to get to here is somewhere you got to go two across and two down. So there's two of one color and two of another. This is all one color. This is one and three. Two and two. Three and one.// [Pointing to grid and redrawn triangle]
1343. MICHAEL: //All right. Yeah - That makes sense//
1344. ROMINA: //All one color. And the- the four is still three and one but then it's three across and one down so it means it's three of one color and one of the other color. [Pointing to triangle]
1345. MICHAEL: That- that's a pretty good explanation.
1346. BRIAN: It's cool.
1347. ROMINA: Who's calling them in? (01:26:24)
1348. BRIAN: Don't call them in yet. Let's hang out. I'm going to go home //I'm going to weigh a hundred and ten pounds.
1349. ROMINA: //Does it look better?
1350. MICHAEL: Yeah.
1351. BRIAN: You didn't have to get them.
1352. T/R1: Oh.
1353. ROMINA: We're ready for his question.
1354. BRIAN: Romina's got something good.
1355. T/R1: Okay, ready for your question.
1356. ROMINA: Come on down.
1357. T/R1: RESEARCHER 3.
1358. BRIAN: RESEARCHER 3.
1359. ROMINA: He's our summer buddy.
1360. MICHAEL: All right. Ask- ask your question again so we know what we're-
1361. ROMINA: Exactly what you're saying.
1362. RESEARCHER 3: Uh, my question was you said that you found Pascal's triangle here and um, it wasn't clear to me that if you go, let's take-
1363. MICHAEL: You want a like reason why- how it relates?
1364. RESEARCHER 3: Yeah.
1365. ROMINA: Okay.
1366. MICHAEL: Not because it looks like it? You want to know why.
1367. ROMINA: Now we just picked any point. Let's say we picked this point. No matter how you get to this point-
1368. MICHAEL: Do the six one. The six one-
1369. ROMINA: Well we'll do the six and the four.
1370. MICHAEL: All right.
1371. ROMINA: Okay, to this point you know you need to take at least you have to take four moves. That's the shortest amount of moves because just like a simple one, two, three, four. So that means it's- let's say you we're relating back to this four moves equals four blocks. So I'd have to go down to the four block area. So that's one, two, three, four. [Pointing to the fourth row of her Pascal's triangle.] And now here you're going three across and one down. Or- so- [Illustrating the moves on the taxi grid and pointing to the numbers on the grid and redrawn triangle.]
1372. MICHAEL: There's no possible way you could-

1373. ROMINA: //Do anything else.
1374. MICHAEL: //You have to- no matter how or which way you go you have to go three and then one.
1375. ROMINA: Right. In any move you're going one down and three across no matter- in any direction you take. So the three across and one down, that relates to three colors and then-
1376. MICHAEL: Of one-
1377. ROMINA: Three of one color and one of another. So you go and you look in here. Say- Okay, here's with all one color. This is with one of one color-
1378. MICHAEL: That's- that's nothing.
1379. ROMINA: No that's all one color but we're not using that because you can't all go all in the same direction. That's all one color. That's with one of one color and three of the other. So that's four and that's what we have and if you go down to here this is two and two and this is three and one which is the same thing. So there's your other four. And if you go to the sixth, the only way you can get there again is by four moves. It goes one- one, two, three, four. So you're in the four block again but this time you have to take, no matter what you do, you go two across and two down anyway you do it. So that would be two and two which is your six but you're still in like the four block area. [Relating the taxi grid to Pascal's triangle.]
1380. MICHAEL: Like you know what the uhm- let me write this down.
Like when you write the Pascal's triangle, this is really like- like- all right, let's say-
1381. ROMINA: [Inaudible].
1382. MICHAEL: Let's say it's like, uh- I don't know how to say it- like, um, like a pizza or something. All right, you have choice of four toppings.
1383. RESEARCHER 3: Okay.
1384. MICHAEL: All right. This one is the pizza with nothing. So you'll only- there's only one possibility without any toppings on the pizza. [Pointing at the triangle.]
1385. RESEARCHER 3: Uh hum.
1386. MICHAEL: Now if you have one choice of topping you get four. II see it but I don't know how to like say it. [Waving both hands.]
1387. RESEARCHER 3: Maybe you can help me see how you're relating the number of toppings and the number of //blocks.
1388. MICHAEL: //To this?
1389. RESEARCHER 3: Yeah. To the- get- getting to any- to a particular corner.
1390. MICHAEL: I like see something and I- if I say it'll- it'll make it a lot clearer but I just don't- don't know how to say it.
1391. RESEARCHER 3: Why don't you just try saying it?
1392. MICHAEL: All right. Well- I'm trying to think of like a- a way//
1393. ROMINA: //Mike, if we were to use pizza could you explain this 'cause I don't know how to do this, okay, that means you have four toppings- [Pointing with Michael to the 4th row of the triangle.]
1394. MICHAEL: This is, um,- Yeah, four toppings.
1395. ROMINA: //Plain. [Pointing to the 1st number in the 4th row.]
1396. MICHAEL: //You have one topping, you're going to make //four different kinds of pizzas.

1397. ROMINA: //One topping. //Two toppings. [Pointing to the 2nd # in the 4th row]
1398. MICHAEL: //Two toppings. [Pointing to the 3rd #]
1399. ROMINA: //Three toppings. [Pointing to the 4th number]
1400. MICHAEL: //You can make six.
1401. ROMINA: Four toppings. [Pointing to the 5th number]
1402. MICHAEL: Yeah.
1403. ROMINA: All right. So, you can do that. Just do-
1404. MICHAEL: Don't know where to go from there though. Like how to relate toppings to that.
1405. ROMINA: Just the same way I just did with the blocks. It's the same thing.
1406. MICHAEL: All right, think of a topping as like, um, being able to go across so if you're only able to go across one time then you could do it four different ways and-
1407. ROMINA: That's one topping.
1408. MICHAEL: Here. You could do this- This- this one right here. Go across this time and go down this time and go down this time and that time. The rest is all going down. The rest of your moves are all going down. [Tracing moves on grid]
1409. RESEARCHER 3: So you're say one topping-
1410. MICHAEL: Yeah. Yeah, one topping would be like you're only able to walk across or go across or drive across actually it's a taxi, one time- one block.
1411. RESEARCHER 3: Okay.
1412. MICHAEL: Now the six would mean you're able to drive two blocks across and two down. Um, four would be you're able to drive three across and the last- and this one right here is you're able to drive- wait four, um, you're able to drive four across which- I mean, drive four down- no, nothing across. I'm trying- I'm trying to say- I can't really say-
1413. BRIAN: Good job.
1414. MICHAEL: Yeah, this would mean you would drive nothing across. It wouldn't even get you to that- wouldn't even get you there. So, that's why, you know, the ones don't really count in our- in our like model. Like- [motioning with fingers in air and pointing to redrawn triangle and grid triangle]
1415. ROMINA: The ones- the ones //would be if you just could-
1416. MICHAEL: //The only thing-
1417. ROMINA: -if you're going just to this point because it's only you're only going in one direction. Like you can't get to any of the inside points because you have to use two directions.
1418. MICHAEL: Yeah. So on the odd do you see like four-
1419. RESEARCHER 3: What I understood you say- you're saying is that the number of toppings related to-
1420. MICHAEL: To the number of times you go across.
1421. RESEARCHER 3: Okay. So that one that you have at the corner there-
1422. MICHAEL: This one right there? [Pointing to a number in the redrawn triangle]
1423. RESEARCHER 3: Uh hum. How many toppings is that one?
1424. MICHAEL: That's all the toppings. But you really- you can't get there by going all- you know- um-
1425. ROMINA: Those would be like the across- toppings.

1426. MICHAEL: Yeah. This one actually- this would be, uh, all toppings, which would really mean all down.
1427. T/R1: So are you telling me that some of those are across and some of those are down?
1428. MICHAEL: Yeah, like how I was saying it.
1429. ROMINA: This one would be two across- [Pointing at 4 in the triangle]
1430. MICHAEL: No, no. This would be one across and-[pointing at 4 in triangle]
1431. ROMINA: One across, yeah.
1432. MICHAEL: -and three down. All right? That's- [Pointing at one by three in grid]
1433. ROMINA: No-
1434. MICHAEL: No, one across and three down. [Pointing at grid]
1435. ROMINA: Yeah, that one
1436. MICHAEL: All right, this one you go two across and two down and three across and one- and one down. [Pointing at grid]
1437. T/R1: So how does that work with the A's and the B's and the toppings? So I see what you mean by across and down but now if I'm thinking of *A*s and *B*s or *x'*s and *y'*s, right. Would you say that just one more time? I know that you've said it.
1438. MICHAEL: I- I said it?
1439. T/R1: No. Somehow it came out of the conversation.
Somebody said it.
1440. BRIAN: Romina was bringing it up.
1441. ROMINA: Um, I'm sorry. What am I trying?
1442. BRIAN: *x*'s and *y*'s like-
1443. T/R1: I think it was Romina who did it, yes. She used *x*'s and
y's for across and downs.
1444. ROMINA: Okay, so if we're doing the same one with, um, with no- no *x*'s then you'd have to go four down- four *y*'s down and that would be this one. [Motioning across and down on grid] But you're not going to get there. Whatever. But if you're trying to get there it's one *x* and then you go three *y*'s. So that's your four. If you're trying to get to this one over here it's two *x*'s, two *y*'s then three *x*'s, one *y* and they all- they all equal four but they all have different amounts of *x*'s and *y*'s and that's how we get this. Yes? No? [Referring to the drawing of Pascal's triangle.]
1445. RESEARCHER 3: And the *x*'s and *y*'s- What does *x* correspond to again?
1446. ROMINA: //*x* is across.
1447. BRIAN: //Going across. And *y* is //down.
1448. ROMINA: //Or a topping or a color. All the same thing. And all our *y*'s are down, toppings, color.
1449. T/R1: Could you use zeros and ones?
1450. ROMINA: Sure.
1451. T/R1: How does that work
1452. ROMINA: That's his area.
1453. MICHAEL: I don't believe it.
1454. BRIAN: Come on Mike.
1455. T/R1 Is that Michael's area?
1456. ROMINA: Come on Mike. Zero, one.
1457. BRIAN: //Break out the binary.

1458. T/R1: //Does that work with zeros and ones?
1459. MICHAEL: Uh man, I haven't seen that in a while. Uh, I really don't remember.
1460. ROMINA: Well just- the same thing-
1461. MICHAEL: Oh like-
1462. ROMINA: One would be every time across-
1463. MICHAEL: Yeah, one-
1464. ROMINA: Zero would be every time down.
1465. MICHAEL: Just- All right, this- right there. This group is, you know, everything that has one, one and two zeros. [Writing binary codes: 100, 010, and 001.]
1466. T/R1: Uh hum.
1467. MICHAEL: That's that. The next one would be- [Writing binary codes: 110,011 and 101] two ones and one zero. That's this. And I guess the one you could call going across and two down. Across and two down. Twice and down. You know you go two ones- [Pointing to a two by one on the grid.]
1468. T/R1: //Mm hm.
1469. MICHAEL: //or two across' and one down there's a zero. That's a is that good?
1470. T/R1: I don't know. Is that another way?
1471. MICHAEL: Do you- like do you see how you can relate the zeros //across and down.
1472. BRIAN: //The same thing.
1473. T/R1: Brian- //Brian thinks-
1474. MICHAEL: The one moving across and the zero would mean down.
1475. T/R1: Romina?
1476. ROMINA: Yeah, see I can't work like that. I work in, um, towers.
1477. T/R1: You're working in towers.
1478. ROMINA: He works in pizzas and binary.
1479. T/R1: Brian are you- work both ways Brian?
1480. BRIAN: No. No I'm totally not a binary kid. I don't-
1481. ROMINA: We- see me and Brian were absent when we did binaries in like sixth grade.
1482. BRIAN: I missed a week.
1483. ROMINA: We obviously weren't there.
1484. BRIAN: What class was that?
1485. MICHAEL: Seventh grade.
1486. ROMINA: Seventh grade. We weren't there.
1487. BRIAN: I wasn't in that class all year man.
1488. ROMINA: I was in surgery.
1489. BRIAN: I was playing basketball all year in that class.
1490. T/R1: Wow. That's really neat. Do you have anything else to add?
1491. BRIAN: Um, no. I mean I'm
1492. MICHAEL: I mean- I mean did that convince you?
1493. RESEARCHER 3: Well sort of.
1494. T/R1: Well I see- I see how you get the numbers. I see how you get those numbers.
1495. MICHAEL: How you figure-
1496. T/R1: I guess my- my question still is suppose once we get just a general number there, um-

1497. ROMINA: Okay, that-
1498. T/R1: How would you talk about some general numbers?
1499. ROMINA: All right. We'll just pick this one. [Drawing the intersection point † $(10, 5)$.]
1500. T/R1: Um hum.
1501. MICHAEL: We've proved to you that you understand why it relates to the Pascal's triangle.
1502. ROMINA: Yeah.
1503. T/R1: Oh yeah.
1504. MICHAEL: So you give us a general number, we look at the triangle.
1505. ROMINA: You pick a general number=
1506. MICHAEL: That's basically-
1507. ROMINA: To get the simplest way you're going to go all your overs and all your downs at one time so that'll tell you this is going to be one, two, three, four, five- five across so one and //then one, two, three, four, five and five down. [Counting with marker on grid.]
1508. MICHAEL: //And five down.
1509. ROMINA: So you know there's going to be a total of ten blocks.
1510. RESEARCHER 3: Mm hm.
1511. ROMINA: And then- so you've got your ten block row and then you're going to know it's five of one color and five of the other color.
1512. MICHAEL: There's going to be one right //in the middle.
1513. ROMINA: //There's going to be a number- yeah, it's going to be the one right in the middle. It's going to- Well I don't know. I don't know what it's going to be but-
1514. MICHAEL: The one that like-
1515. ROMINA: The one right in the middle of everything.
1516. MICHAEL: I don't- That's- that's-
1517. RESEARCHER 3: Which- which row?
1518. MICHAEL: -that's way up there. That's-
1519. ROMINA: It's going to be the tenth row because you took ten moves to get there. So you're going to go down to the tenth row.
1520. MICHAEL: Yeah, it's going to be the tenth row because you have-
1521. ROMINA: And the tenth row that has five of one color and five of the other color, that's your number and that's how many ways you can get to that point.
1522. MICHAEL: Which that one will be in the middle.
1523. ROMINA: Uh hum.
1524. MICHAEL: Because just the way it's set up. That one will end up in the middle.
1525. ROMINA: Plus it's like an even //the square.
1526. MICHAEL: //One- yeah, it's- no it's an odd number. That's why it's in the middle.
1527. ROMINA: Yeah it's a square.
1528. MICHAEL: Even numbers- there is no-
1529. RESEARCHER 3: How do you know it's the tenth row?
1530. T/R1: Yeah.
1531. ROMINA: Because it took us five moves to get- uh, ten moves to get there.
1532. MICHAEL: Because you have ten spots. Ten toppings and-
1533. ROMINA: Because you know //you can always-

1534. MICHAEL: //Ten different places to put these numbers.
1535. ROMINA: Yeah.
1536. MICHAEL: Which is ten.
1537. ROMINA: And you know- and this ten there's- there's only ten moves you can take because this is like the simplest way. You go all the way across and all the way down.
1538. RESEARCHER 3: Mm hmm.
1539. ROMINA: And that's going to be like the simplest way and that's going to mean that's the shortest way to get there. Like-
1540. RESEARCHER 3: Maybe help me understand that by running us through-
1541. ROMINA: Okay //like-
1542. RESEARCHER 3: //-each story //from the first row-
1543. ROMINA: //-this one?
1544. T/R1: Yeah, what's the first row.
1545. RESEARCHER 3: -of Pascal's triangle.
1546. ROMINA: This one, there's only two moves you can get to this one. You go over one down one. //Two moves.
1547. MICHAEL: //You mean like the first row that would be-
1548. ROMINA: To the second row because there's two high in block terms. And for this one it's two across and one down-
1549. MICHAEL: I mean, like I said before, the rows correspond to the //shortest distance.
1550. ROMINA: //Yeah.
1551. MICHAEL: I mean the //shortest route.
1552. ROMINA: //Yeah. So this is //three moves.
1553. MICHAEL: //Everything in this row, two. [Pointing to triangle]
1554. ROMINA: Third row.
1555. MICHAEL: And this one three. So that's how-
1556. RESEARCHER 3: Say it again please.
1557. ROMINA: Okay, this one. There's three moves.
1558. BRIAN: One, two, three.
1559. ROMINA: And this is the third row. [Pointing at row 3]
1560. RESEARCHER 3: So the-
1561. ROMINA: This one's four moves, fourth row. [Pointing at row 4]
1562. MICHAEL: If the shortest route is ten, then it- then it's //in the tenth row.
1563. ROMINA: //Tenth row.
1564. RESEARCHER 3: I'm still a little confused.
1565. MICHAEL: All right. If you pick any point on- [Pointing to the grid.]
1566. RESEARCHER 3: Start- start from the very first row please.
1567. MICHAEL: The first- the first one.
1568. ROMINA: The first-
1569. MICHAEL: All right.
1570. ROMINA: No moves. There's only- you're stationary there. That's one. Just one. [Pointing at first row of her Pascal's triangle]
1571. RESEARCHER 3: So it's the top row of-
1572. ROMINA: Yeah, that's just //your Pascal's.

1573. RESEARCHER 3: //-Pascal's triangle?
1574. ROMINA: Yeah. You go down to here. There- You're going to go over one, down one. There's only //two moves.
[Pointing to the grid]
1575. RESEARCHER 3: //Two.
1576. ROMINA: That's the simplest way you can go.
1577. RESEARCHER 3: Uh hum.
1578. ROMINA: So that's Pascal's like second row, two blocks, two toppings, whatever you want to say. [Pointing to the redrawn triangle.]
1579. RESEARCHER 3: Uh hum.
1580. ROMINA: And this one, you go over two and down one so that's a total of three moves. The simplest moves so that's the
third row and you can go-
1581. RESEARCHER 3: So it's the second going over two blocks-
1582. ROMINA: Yeah.
1583. RESEARCHER 3: -and it's which row of Pascal's triangle?
1584. MICHAEL: //That's in the third row.
1585. ROMINA: //The third row. [Pointing to the third row in the triangle.]
1586. MICHAEL: //Because it takes three to get there.
1587. ROMINA: //Because you have two and one. And you're going over two over one. You're doing three complete moves. And that move just happens to be two and one. [Inaudible]. [Gesturing across and down on the grid.]
1588. RESEARCHER 3: Uh hum.
1589. ROMINA: And- and this one here you're making- you're going over three and down one so that's a total of four moves. That's the fourth row.
1590. T/R1: So, what about the r^{th} row?
1591. MICHAEL: Would be-
1592. ROMINA: The r^{th} row would be- r moves
1593. MICHAEL: Yeah, *r* moves *r* shortest distance. Whatever-
1594. ROMINA: Yeah.
1595. RESEARCHER 3: Uh hum.
1596. MICHAEL: *r* half the perimeter whichever, you know-
1597. RESEARCHER 3: Okay.
1598. T/R1: Are you convinced?
1599. RESEARCHER 3: Yeah.
1600. T/R1: It's really very interesting. Interesting problem. Did you ever do anything like this before?
1601. MICHAEL: No, no I've never seen it before in my life.
1602. ROMINA: We just discovered Pascal's triangle.
1603. BRIAN: Didn't we have to- didn't we have to do something in Pantozzi's class with the subway?
1604. T/R1: What's that?
1605. ROMINA: Yeah but we didn't do it though.
1606. BRIAN: Uh, no. Something- I don't know, somewhere like-
1607. ROMINA: We can-

1608. BRIAN: If a person is let off at like this subway station and they want to go to this building what's the shortest way to go or something?
1609. MICHAEL: No it was like- no it was a bunch of subway stops
1610. ROMINA: Yeah.
1611. MICHAEL: And there's some subway stop is three blocks away from this building- something- //which stop should he get off at?
1612. BRIAN: //Something like this.
1613. MICHAEL: In order to get there. And //then-
1614. BRIAN: //It wasn't exact. It wasn't exact so we're not going to get into it.
1615. T/R1: So some of the same kind of-
1616. MICHAEL: Yeah.
1617. T/R1: -reasoning you used.
1618. MICHAEL: Yeah. That was last year though.
1619. T/R1: You- you are wonderful for staying and working this hard. I just have one general question to ask you. You're going to be the last ones here [Inaudible] for coming and staying so long.

APPENDIX E: TRANSCRIPT – REFLECTIONS I

May 18, 1999 (11^{th} Grade)

1 Camera View: PUP Math Interview
Date of filming: 1999-05-18
Kenilworth, NJ, Reflection for the Kenilworth Longitudinal Study
Transcribed by: John Francisco
Date of transcription: 2004
Verified by: Maria Steffero
Date of verification: June 2009

Line	Time	Speaker	Transcript
1	00:27	T/R1	Romina, tell us some of your favorite activities. What do you do after school? Volunteer work?
2		Romina	I have a full day of school and then from here, I go to softball practice, and usually I have a part-time job at an ice cream parlor, and I'm there almost all the time, because I work every weekend. I'm class president; I run the Interact Club, which is a voluntary club, like here with the Rotary Club. I'm on the Student Council. I… I teach CCD, which is like a Catholic religious thing for kindergarteners. They're real cute. Let's see, what else do I do? I'm part of the FBLA, the Future Business Leaders of America. And I'm the President of the Honors Society, and I think that's it.
3	01:21	T/R1	Wow. What do you enjoy most?
4		Romina	You mean like spare time, most? Or out of all those things I just mentioned?
5		T/R1	Spare time.
6	01:33	Romina	I just like to sleep. With all that stuff, I don't get time to do much. I'm always running around. So, I just like to rest at home. That's it.
7		T/R1	So, your plan is to rest this summer?
8		Romina	Oh, I'm not resting this summer. I have a program with Rutgers for two weeks. This summer, I have a program with Rutgers for two weeks. I'm planning to take a lot of vacations with my friends, because this is our junior year and we're going into senior year. And then I'm probably going to be working a lot, because the ice cream parlor gets busy in the summer.
9	02:10	T/R1	We've known each other for a long time. Do you have first memories about when Rutgers came into the school that you could tell us about?
10		Romina	My first memories of Rutgers were, I got pulled out of class one day, and I didn't know why, and I got put into a special class, which is kind of scary, because you don't want to be different back then. Kind of young. And then I remember you came in and we all sat there and we had these cameras and you just gave us things to play with, and it wasn't that bad.

			[Laughter] And everyone kept asking why, and we never understood that until about this year. It's hard to believe. We were scared, but then you made it fine. It wasn't that bad.
11		T/R1	Say more about that.
12	02:57	Romina	Like, we were never told why anyone was there, and why we were getting filmed, like, until this year, because this year is when we were first able to speak of our past experiences. And when you explained to us that this was like research for you, and you want to see how you could help, like, people understand math, and how to teach math. And so now like, we're more willing to do it than before. We were scared. Like the why thing. Everyone wanted to know why we did everything. We didn't know why we did everything. And we didn't know why things worked.
13		T/R1	What grade? Do you remember when it was?
14		Romina	I think it was fourth grade, when we began. Well I did, anyway.
15		T/R1	So, was it different than it is now, in terms of when we come in and work with you?
16	03:43	Romina	In fourth grade, I didn't know anything. I didn't know who you were. Now, we're comfortable with you. Like you've been like our teachers for ten years. That's what you've been to us. And so, now it's easier, and we know what's expected of us, and what we have to do. And before we would wait for you to give us a little start or a little push and point us in the direction. Whereas now you hand us a problem and you just kind of leave, and we just do it ourselves. And we just start experimenting and see what we can give you.
17		T/R2	What did you think was going on back in fourth grade, when we came in?
18	04:19	Romina	We called ourselves your guinea pigs. Because we were never sure. We thought you thought that we were smart, and we didn't think we were all that smart, and we were kind of scared. And when you came up, we thought you just wanted to experiment to see what our capabilities were. And that was very scary when you're young, because none of us had any confidence. And most of us still don't but…
19	04:47	T/R1	So, is the way you worked together then different than it is now?
20	04:59	Romina	The way we interact? I think we were very unique. All of us, the way we associate with each other, because I think we have like this… that's why we were picked, like the group of kids, we were picked because we just worked really well together. Like it's almost like we bring out the things we never thought we could bring out. Like we could never do any of the things, well I don't think I could ever do any of the things we do alone. Like they just help you bring out things you didn't know were there. And we have a relationship where we argue a lot, so, like through arguing is where we come up with most of our answers.
21		T/R2	How does arguing help you? I'm not sure I understand you.
22	05:38	Romina	Because if you're like passive, and I'm like, "This is what I think it is," and everyone is, "Okay, that's what it is," we all sit back and we all take

			that and we never go any further. But if I disagree with someone, they'll have to explain it to me, and if you're explaining it, they're either going to find something right, or they're going to find something more. So, if I don't agree with it, they're going to explain it to me, but if they find something wrong, maybe I can help, and then someone else may disagree with me. And that's how we get through everything. We just disagree.
23	06:05	T/R1	Do you remember any particular events or problems where that happened?
24		Romina	I don't know it was disagreeing, or if we just didn't understand, but like the binary system, where we threw that one into the combinations, and our tower problems. That one was hard for us to accept. And now I think I understand what the binary system is, and so does everyone else. But we didn't know what that was when we started. Only Michael did, and I was arguing with him, because I thought it was wrong the whole time. And we just argued with him and he explained it to me, and he explained it to Jeff and Brian. And now we understand it. And we use that all the time and that helps us a lot in our combination problems.
25	06:50	T/R1	Do you all know what she's talking about?
26		T/R2	No, I don't know what she's talking about. [Laughter]
27		T/R1	Help them a little.
28		Romina	If you want me to back through class today a little. What we do, well when we were very young, we started off in fourth grade with towers. And what they did is they wanted to know how many different combinations we had with, like two different colors, three high. And we built them. We actually, with the unifix cubes, we built them. And we discovered that what that was A plus B to a certain exponent, and when we like today in class, that's what we did, and we discovered that we can figure that out with the pyramid, what's that called?
29		T/R1	Pascal's Triangle
30		Romina	Pascal's Triangle. I should know that, huh? And then we just associated the cubes with Pascal's Triangle. And then the binary system was just a way of expressing with two colors the different combinations like well, one would be like a red, and the zero would be the blue. And you can go through it, and that's just a more organized way of just having them written. And you usually don't have doubles with that.
31	07:58	T/R1	So, you keep track?
32		Romina	Yeah, that's how we keep track.
33		T/R1	And so what does that have to do with combinations?
34		Romina	That's just how we kept track, and then by adding them, by adding certain ones, we get like totals, like without having to write them all out. That's how we do it.
35		T/R2	I have a question, going back to this thing with the fourth grade, was it strange for you in the fourth grade, to find yourself suddenly arguing about math.

36	08:28	Romina	Yeah, we thought we were real weird. Like, fourth graders interested in math and arguing with their own friends about it? And we still think this today, like, why do we sit here and argue about math? It's math. It's not going to, it's weird for us, and like, in fourth grade, we – I don't know what happened there that we started arguing. But it just like got us so far. And what happened is we amazed ourselves with the things we got, and where it led us to now, that I guess we're not that weird anymore.
37		T/R2	Do you do this in other classes?
38	09:08	Romina	Well math is where the most arguing is. Like, you can't do this in other classes. It's not like, in English, you read. You don't argue; it's there. It's written. And in history, you don't do the same. In math, it's like, well especially the way I've been taught, because I have never actually had a math teacher that's said, "This is the equation, put in the numbers and do it." I've always had to argue to get somewhere, because they never actually told me where we were heading with anything. So, through arguing, that's how I discovered… that's the only reason I know math. Because I did it myself, all these years.
39		T/R1	But now you went to a different school, your first year. Was that similar?
40	09:44	Romina	Ask me one question about geometry. Because I won't know it.
41		T/R1	Why is that?
42	09:49	Romina	Because my geometry teacher wasn't aware of this, and she, it was a completely different town. And when we went there, she was amazed at how much math knowledge we knew. And that was just through what we thought. Because we weren't afraid to come out with our ideas, because that's how we were taught. If we were sitting there in math, I would argue with her, I would ask her a question, and she was so surprised, she didn't know what to do with us. So she tried to keep us quiet in the corner, she handed us a book the first day of school, and they told us to get a notebook, and I gave my notebook to Mr. Pantozzi, "I'd really like you to look at." [Laughter]. What we did was, every night, we'd read something in the book, we'd answer 20 questions on it. We'd go in the next day, she'd give us the right answer, and then we'd just do the next one. And that whole class was graded on how neatly you did your notebook.
43		T/R1	Now that's a contrast. Did you do well?
44	10:38	Romina	Oh yeah. I think I can learn out of the book, and I can learn this way too. Like, we're experimenting, so I don't have a problem with that.
45		T/R2	How did it make you feel, though, when you were kind of like pushed in the corner and not allowed to ask questions?
46	10:50	Romina	I was not interested in geometry. When I went to school the first day, we were just talking, it was a regular day, and she brought up something about a line. And I was telling her about Y equals MX plus B. And every kid in the class turned around and looked at me going, "What are you talking about? I have no clue what you're talking about" So, I took it

			upon myself, just like we do in Rutgers, I started explaining it to them, and I got up, I was like, “This is what you do, and this is what M equals, and how it equals that,” and the teacher was kind of upset with me, because she didn’t want me teaching them. She wanted her to be teaching them. And whenever there was a question to be asked, I’d raise my hand. Or when I had a question, I’d raise my hand, and she wouldn’t answer. Like, she wouldn’t answer any of my questions, or she wouldn’t call on me. So, it was weird, and it turned my off from math completely. And then the next year, I came back, and Mr. Pantozzi had me, and I didn’t know him, and I didn’t want to do anything. I just sat there. Until he pushed me to start again. It wasn’t a good year.
47		T/R1	And what about the next year?
48	11:54	Romina	The next year was different too, because we were sophomores and we were in a class with juniors. And they had never been taught the way I’d been taught. So, in class when Mr. Pantozzi started us, it was Ankur, Jeff, and Michael, and we kind of just jumped into it, and we started throwing ideas around, and the rest of the class didn’t know what we were doing. They were used to just getting an equation, and then just figuring it out. And everyone else was lost, and Mr. Pantozzi had to kind of single us out, and like put us in the corner by ourselves to work with each other, while he tried to get the rest of the class to do what we were doing, until we could finally all work together. It was hard.
49		T/R2	What do you think would happen if all the classes were sort of done the way that you were used to?
50	12:39	Romina	I think kids would be able to do more.
51		T/R1	Repeat the question.
52		Romina	If people learned the way I did with, like, group talking, I think people would learn more and be able to do more because if someone that was taught with just a teacher teaching them, if you’re given something in, like, the real world, you’re not going to know how to handle it. Whereas I would probably question it, and like, throw different ideas in the air. Other people, they get intimidated, and they don’t know how to do that. And, like, if they’re not specifically told, and you can’t live your whole life being told what to do. You’re going to eventually have to do it yourself. And they’re going to have more knowledge about everything. Because everything I do I understand, because it’s more than just numbers to me. It’s like you have to go deeper, you have to, if you understand something from the beginning, you’re going to always understand it. You can’t forget something like that. And like an equation, I don’t really know any equations. It’s like things, I don’t know any solid equations, but I could explain to you something and work from there. And you’re likely to forget an equation.
53		T/R1	You mean a rule?
54	13:43	Romina	A rule, or just like anything. Yeah, that’s what I meant.

55		T/R2	When did you realize - in all the years you were doing this, when did you realize that you were starting to get something different from this?
56		Romina	Probably my freshman year, when I went to a different school, and I saw how everyone else was taught, and what everyone else knew about math. I got through most of my tests, because I went back to sixth, seventh, and eighth grade, and what I learned then, and what I could put together. I taught myself, basically, that year, from what I knew. Whereas other kids did really badly in the class. The rest of my class did really bad, because they weren't used to that. They looked to the book for answers. And they didn't understand the book, and the teacher wouldn't help them. They were lost. And they couldn't do anything for themselves.
57		T/R3	But why do you say you're not confident in your abilities?
58	14:41	Romina	I'm not confident because, I know I can do a lot, and I can do it. But when I try to explain to a person what I know, I can't explain to you what I know. They might throw out, "Oh, do you know this rule, and this guy and all this stuff?" and I'm like, "No, but if you sit me down, maybe I know do know it." But I know it in my way, not in their way. And everything I explain is in my words, not in anyone else's words. It's not from some mathematician from thousands of years ago, because I don't know that. Like I didn't know what the pyramid – Pascal's - was called. I just know everything in my own way. Everything has Romina's definition to it.
59		T/R2	Some people feel it's very important to know the work of others. Have you ever had to argue with people who have a different perspective?
60	15:33	Romina	Yeah, a lot of times. Because a lot of my friends, they know about my programs and things, and they come up to me because they think I'm a math genius, which I am not. Bu they come up to me and they're like, "But you didn't use the equation I used," and I'm like, "Well did I get the right answer?" I'm like, "Do you know the equation? Can you figure it out using your equation?" And they can't. But once I explain it to them, and I say, "Maybe you could do it a different way, some way you understand it." Then, we've had arguments but I've helped them, so it's been okay.
61		T/R1	What do you like most about high school?
62	16:10	Romina	What I like most about high school - probably not the actual school aspect. I like the socializing and, I don't know, I like being involved in things, not, I don't come to school to do homework and work and all that. I'm a normal person.
63		T/R1	So when you have to come, what are things you don't like?
64		Romina	I don't like having to go from class to class and do homework, and take tests and quizzes like anyone else.
65		T/R1	So, if someone asked you, how would you make a difference?
66	16:48	Romina	Come and watch Mr. Pantozzi's class, if you want different.
67		T/R1	So that's a math class. So you're saying that, what are you saying?

68	16:55	Romina	I'm saying, in order for someone to enjoy school, you have to give them things that they'll like, a situation that they'll like. In my math class, I don't have problems about going to the market and buying apples. I have problems that have to do with me. Like with having enough money to buy clothes, and things like that. And then having, like, a socialized class. If you all sit there in neat rows and have to look at the teacher, and listen to them, you're going to be bored, you're not going to pay attention. You're going to hate school. But if you come into my math class, we're all in a big circle, and our teacher is in the middle sometimes, and sometimes he just kind of sits down and let's us do our own thing. He gives us problems that we want to know the answer to, that we're interested in, and then he doesn't have to give us an equation. We all just kind of talk about it, and then come to a point. And you're kind of socializing while you do your math. And you get an answer and you weren't that bored.
69		T/R2	Did you ever have parents or friends of yours or somebody wonder if the teacher is just sort of standing in the middle of the room and not giving you anything? What's the point of school? What's the teacher doing?
70	18:04	Romina	That's a very interesting question. A lot of people I know, not only my parents, but a lot of my older cousins take a very big interest in my schooling, think it's too weird to be good. They don't like the idea that I'm a friend with my math teacher, and I can talk to him, like not only about math. And he's got a comfortable relationship with me. And they think that's very odd. And they think that my teacher should give me homework every night, in the book, and I should bring a nice big thick math book, with a whole bunch of numbers in it, and a notebook. And my parents got to meet Mr. Pantozzi and they've come to a lot of the things, they're just now understanding, because I've worked with Rutgers for so many years, and they can see where I've brought this to. But like, my cousin always asks about this, and she thinks it's the weirdest thing in the world that my math teacher sits there in the middle of the room and everything is so disorganized. No one appreciates it or understands it or thinks it works.
71		T/R2	What do you think?
72	19:02	Romina	I think the way I've been taught math worked for me. And I think it's just amazing, like, I know, I go to Mr. Pantozzi a lot after school, because I have an individual project from a different class with him, and everyday I walk in and I see someone that I didn't think was that smart, who they didn't think they were that smart, and they didn't know anything about math. And I remember going there in September and it was the easiest thing that came to me. Just like that, it was like two over four equals what over eight? And I just knew what it was, and that person didn't know what it was, and now you go there, and they're been working by himself the whole year, and with Mr. Pantozzi, experimenting with things, he can do that stuff now. By himself too. Like he doesn't need anyone to, like, tell him what to do.

73		T/R1	So you’re an advocate?
74	19:47	Romina	Yeah, I am.
75		T/R4	One thing we should say, from a technical point of view, if you could just tell us, start off by saying, a little bit about ninth grade, what happened to you, how you changed schools.
76		Romina	I think it was my seventh grade year. They closed the high school down in my town. And after that, we were all sent to our neighboring town, to that high school, and in ninth grade I started there.
77		T/R2	Is there anything you might want to tell us that we didn’t ask you about?
78	20:47	Romina	Let me think. I would like to, like, support this idea. I know you’ve been working very hard with this, and I know you’ve been going out to the whole world, and trying to tell them. And I’ve been doing it for ten years, and I think it really pays off. I’ve had amazing teachers who have gotten so involved in what we’re doing. And they’re not the regular book teachers. And it shows that, maybe, if you’re a little different, it does work. You just have to take the risk to do it.
79	21:21	T/R1	Thank you.

APPENDIX F: TRANSCRIPT – REFLECTIONS II

July 21, 1999 (12th Grade)

1 Camera View: PUP Math Interview – Romina (1 Disk)
Date of filming: 1999-July-21
Kenilworth, NJ, Reflection for the Kenilworth Longitudinal Study
Transcribed by: Melissa Lieberman
Date of transcription: June 2009
Verified by: Maria Steffero
Date of verification: June 2009

Line	Time	Speaker	Transcript
1	00:30	R1	I have two questions for you, um, well the first one, I am just curious when they brought those boxes out from third grade and what was that like
2	0:48	Romina	Um, Too many memories, I don't know….we used to, I knew it had something to do with probability cause we used we used to problems like that all the time it was just, when we were small it was just tormenting cause we did not understand anything , you don't know probability when you're young and now when they brought it out this time, we were glad, you know, we knew it was going to be easy for us this time ….but it just brought back all those memories from years ago when we couldn't do it
3	1:10	R1	Did you ever see those tapes…Carolyn was saying she was going to show them
4	1:16	Romina	I've never, I think we've seen one but we weren't actually doing math in it people were coloring, they showed us they thought it was funny how someone was so not amused with the problem they would start coloring their thumbs and making thumb prints and that's the only I've ever seen, I've never seen a tape
5	1:28	R1	Would you like to see it
6	1:29	Romina	Yes I would, it should be funny, um to see how much we have progressed too
7	1:35	R1	Your tables seemed to be very quiet compared to the other tables when the boxes came out, I didn't I mean after you got over the big surprise, did you remember the problem pretty well?
8	1:50	Romina	Yea I knew exactly what the problem was, it was just just it was like as soon as you take out the boxes you knew you had to guess how many of something were in there and our table and I think our table was more used to it, I think we're more, more experienced group and the other tables had taken a long break or hadn't done it so we just kinda knew what we were going to do so it wasn't even a problem for us. It didn't even bother us
9	2:13	R1	What was it like hearing some of the ideas from the other kids?
10	2:16	Romina	It was, you know they were some of the same ideas , we were all like, it's that's why we're strange, we all had like we've all worked in different

			schools and everything but we all have the same ideas, we all approach things very similarly so it's just funny
11	2:31	R1	I was interested in your reaction to this whole thing because if if I remember right usually do like maybe an evening or an afternoon and this was like a two weeks something like that.. what was it like doing that this whole time compared to the way you normally do
12	2:52	Romina	It wasn't as bad as I thought it would be to be honest, um usually it's, I think its more stressful not this way the two week way period but the other way we do it because you have to come up with something by the end of that time period and usually they want like an explanation, they want an answer and everything. this we had more time to sit think about it we could of approach the problem in like several different ways we had like different graphs and we represented the problem in different ways and we like all talked about it for a long time and then we came up with the answers by ourselves and then we would like be able to share it so we got other groups ideas too. It was a lot. It was paced better and I think it was more productive too
13	3:37	R1	Was there anything you didn't like about it?
14	3:41	Romina	Nah it was okay I mean the four hours got a little long and our attention span is very short if we are not occupied with something that is like difficult for us we are gonna lose track very easily but I think it was fine. It was pretty good. It was organized well.
15	3:57	R1	You said you know um something about um if it's not difficult. Could you tell me more about that?
16	4:07	Romina	We, We all have very short attention spans. If it is something that seems too easy or something we can get done in a matter of two minutes then we will do it real quick. We will like we will throw an answer on a sheet and we will stop I mean the problem really has to interest us and it has to cause it usually works best when we disagree with each other cause that way we will be like it it helps so much we are the type of people that if we disagree with each other just cause we are disagreeing we will work on any problem you'll give us we will go on and on until we figure which one of us wins kind of and if it's not intriguing to us we won't do it we'll just sit around and just look at each other that what it takes
17	4:48	R1	It does not seem like you have a short attention span when you working on something that you are into
18	4:54	Romina	Yeah, when we're into something we are into something. We'll uh we'll give you any we will talk about it forever - we will argue about it forever. We will do anything that's required. We'll come up with anything, like we will come up with weird things too. We will keep going as far as we can with the problem if we are interested in it.
19	5:11	R1	This must have been a bit of a reunion for you right?
20	5:14	Romina	Yea yea um. I remember working a lot with those kids and uh Matt, Milin in like uh sixth grade . just seeing them again it was just weird. Yeah, we all got along now like I know Matt's coming home with me, we've got an hour ride so it was nice we all got along well. We all remembered each

			other and our tendencies
21	5:36	R1	Were you surprised by how they looked and people changed a lot
22	5:40	Romina	Actually no they didn't they look a little like they have they look like they don't have they're they're not as like the baby fat anymore but they're the same people they - they just grew up a little
23	5:52	R1	Did have you ever mean in the situation where you have not seen someone in a long like that and saw them
24	6:00	Romina	Yea
25	6:00	R1	What was unusual
26	6:01	Romina	It is unusual. Most people your if you're from Kenilworth you're from Kenilworth forever you know you don't move from that you it is such a small town we don't a lot of people don't move and come back if you're gone you're gone so this this was very unusual. I hadn't seen them in years
27	6:16	R1	I was just curious what you though people were you were you expecting the kids you seen you knew them when you were little kids so were you expecting them to look really different or
28	6:26	Romina	I was I didn't know I was I didn't think I was gonna recognize either of them but I did they came and I was like Milin and Matt it wasn't even a surprise I knew what they were going to look like I I I thought I knew what they were gonna look like. They didn't change much
29	6:42	R1	And then how bout um there was some new people to the group too that wasn't there before what was it like
30	6:48	Romina	I felt bad for them but cause we all had I mean Matt and Milin like I felt bad for them too cause we were all like we've been we've grown up together we've been together for years and they were kinda like the outsiders and I'm I'm not sure if the kids from New Brunswick had done this before they said they hadn't so I it must like must have been so awkward for them they did very well but they by end of this week they were into everything they were doing they were just with us. We all we got along well
31	7:16	R1	Why do you think it might have been hard for them
32	7:17	Romina	Cause they were especially the first day they were all thrown they were all separated they were all in different group the bunch of kids we all knew we we socialized a little you know and they just kinda sat there they didn't know what was expected of them they didn't know we were all like ready to we knew what Rutgers wanted you know so we were able to give them the answers and the discussions and they didn't they didn't know they wanted to talk about the problem and things like that so, until they caught onto that part they were a little lost
33	7:47	R1	Did any of them come up to you and and talk to you about was going on
34	7:53	Romina	Yea, Victor did a little I think Victor did the best getting into the whole group he uh he was he was I kept on asking him yea I like pushed him along a lot throughout the thing cause when he'd go up there and

			presenting I would ask him questions and he hated that so much but by the end he was like it's alright I expect questions from me he like helped me out a lot up there he was ok we got along
35	8:21	R1	Um I think to some point somebody from the outside who has never seen this set up would seem like a really strange thing. Could you do me a favor and just describe what it is like to just walk in the first day and you know see whatever they had set up for you?
36	8:36	Romina	The first day was so intimidating it was I had expected like the usual Dr. Maher and Gina they're like always there and they're like we're we're like acquaintances. We are like we get along well and all that but the first you walked in their was like three tables and you're all separated and there was not cameras than we had expected and there was like a special camera for each table and and so many people there were so many adults there that we hadn't expected we thought it was only going to be like three maybe but then there was like twenty and they were so they were hanging on every word we said we had two note takers each like to each table that was so scary because like you had to do something you couldn't sit there for four hours and not do a thing but everyone's waiting so so it was scary
37	9:21	R1	Usually when you do this there are also people sitting around watching how was this different
38	9:29	Romina	Well the thing with when we do it after school sessions we are in a room and we have a camera man but usually the adults leave and they're not they're not there and they leave they used to not do it but recently bringing random people in one by one but we never we were never hit by all of them at the same time
39	9:49	R1	Does that affect how you work I mean
40	9:52	Romina	It's pressure but we all work we all work very well under pressure we just we're more productive under pressure any way so I guess it was it was a good idea just make us do something
41	10:15	Romina	You know what else was different about this just to continue when when we were people were just brought in new people we were able to talk about things we had done that we knew well and we were able to just explain to them you know we thought we knew what we were talking about and knew the answers already we were worked on the problem like prior to them being there so that was easy just explain to them but this time they were new problems at us and actually watching us work which is very different
42	10:44	R1	Could you describe the cat problem for us if somebody hadn't seen it? What's that about?
43	10:50	Romina	Okay, it's about how a cat someone took pictures of a cat in .031 seconds like intervals we just had to see how far the cat moved using just a grid and 24 pictures that's it
44	11:10	R1	That doesn't sound like a problem
45	11:14	Romina	There wasn't much to work with. They just wanted to know they gave us twenty four pictures and they were like here figure out how far the cat

			went and how fast it went that was hard we didn't know where to start and then each line is five centimeters and we took it from there see like we didn't that's what they didn't give us any information that the first hour I say we just sat going, 'what what do they want?' but after that it was easy we just take a ruler and do some multiplying and its fine. It wasn't that bad
46	11:45	R1	Well what were some of the questions you had sort of had to settle with in that first hour before you could move along
47	11:50	Romina	What the problem was. How we're gonna like they gave us little tiny pictures on one just one sheet and we were suppose to see how far the cat went and in most pictures it didn't even look like the cat was even moving and like we could measure anything cause like our ruler was too big for the pictures and we just didn't know how to do it we didn't realize it was in like centimeters like we are used to miles per hour not centimeters per second we had to like establish like what our variables were and work from there.
48	12:20	R1	Do you think it was a badly framed question cause you didn't know what to do?
49	12:25	Romina	I think they did it on purpose cause it wasn't they just didn't give us any information, they didn't give us like it wasn't structured - they didn't give us all, 'This is what it is' and 'This is what we want you to figure out.' It was just typical Rutgers. They give us something - they give us like very little information about something and see what we take it to and it think we did very good we got as far as I think we could of with our knowledge.
50	12:52	R1	Where did you end up taking it to?
51	12:55	Romina	We we were able to figure out how far the cat moved and that time span, what velocity he was going out between each frame we we went in depth like how far the cat moved in like what was it doing when it was walking running we were able to make graphs to see how fast it was going at certain times and we related that to when it was walking and running.
52	13:17	R1	Yea I wasn't there for much of this but I I I I heard that they had laid out some tape on floor could you tell me about that?
53	13:27	Romina	Yea we uh we had numbers for each time frame and we had um we uh we knew how far the cat moved and how fast it did it and well it was .031 seconds and we knew how fast it moved in every interval, so we did we kind of multiplied that by 50 like a life like version of it and put it out in the hallway and then just to see because we were having a problem we didn't know where the cat was accelerating if it was accelerating throughout so what we did was ran the course and saw that we had to get one place to another in 2 seconds so we had to see what we went from walking taking little steps to like running and then we were just sprinting as fast as we could reach them we were able to tell how the cat was moving.
54	14:12	R1	Why did you multiply by 50?
55	14:13	Romina	To make it so we can follow it cause the actual size of the paper layout was 130 centimeters so we couldn't really feel it in there then we made it

			a little bit bigger in the library but it wasn’t you couldn’t feel the full effect because at first like in the span of two feet was the first ten intervals so you couldn’t really do it so we figured we’d multiply it by 50 so that way we would actually have to be walking from the piece we marked with tape from tape to tape we would have to be walking and then running
56	14:49	R1	What did you find out by doing this yourself in the hall?
57	14:52	Romina	Well we were they were asked cause they were really stuck on this frame ten and we didn’t know what was happening in frame ten cause they knew it was it was we ranged at cause we started at between 9 and cause we went uh we measured the time between the interval like between that time we measured the velocities we couldn’t do it for the exact frame we did it in between but we could between from 9 to 10 I could remember getting somewhere between 8 cm/sec and the 10^{th} to 11^{th} going 120 so something big happened there but we were missing it so what we did when we laid it out we saw like that you would have to come to like a stop and we go from a walk and coming to a pause and then you started running so then we figured out that was what the cat was doing so we were able so that what we got out of it
58	15:36	R1	If you were to describe with your hand what the cat was doing just like …
59	15:42	Romina	The cat the cat was walking like this you can’t even see it in pictures the cat walking and then it brought I think it was scared or something startled it cause someone wanted it to run for the picture sake and its two hind legs came together and it started like a gallop so it was like a big change there from just walking from step to step and then a gallop.
60	16:09	R1	This is a very unusual kind of activity I’ve never seen anything like it. What do you think you got out of it?
61	16:19	Romina	I think we see things almost at a higher level we don’t see it as just numbers and a grid we saw it like like we put like variables that no one thinks of like what the cat just doesn’t just gradually walk and then running no it like something had to have happened there to make it run and then we we were able to put like real life things into it and like what can it affect it like not math like real things
62	16:56	R1	You mentioned something about real life could you go into that more cause I really could not follow it
63	16:59	Romina	So like when you do math you don’t like this was our problem like we have a real big problem going from one thing to another when you do math we do math two plus two equals four - there is nothing involved and when we do like word problems we never take anything else into consideration like you take when in real life like little things like a person like just running that just doesn’t happen all of a sudden you have to be kind of gradual to it or just things like air resistance and things just friction there’s like real life situations that math doesn’t account for that we have to.
64	17:37	R1	And did bringing those things in sort of change how you did the math?
65	17:42	Romina	It made sense of it. Like we had the numbers there but we they did not

			make sense to us like we didn't understand how things could change so fast what was going on why it would change speeds so fast you do do like a real life version of it you can see what the cat's doing you can understand and like our graphs would go up all of a sudden and fly down and but when we did we could see it was accelerating, it came to a peak and then it was slowing down like it just makes sense of all the math
66	18:20	R1	Did any of this experience in the last couple weeks you know it was pretty intense couple of weeks did any of it make you change how you're thinking about math?
67	18:30	Romina	They have been working up to this. It hasn't been like all of a sudden it made me I was I don't know I was very surprised by what we did this week cause they gave us very random things or just like our two problems were just pictures and that was all they were from the picture we developed so many things like we had all these numbers and then we were able to get graphs for all these different things so it was just like pretty interesting how we came up with so much many different like point of views and areas and methods and like we had hour conversations about our math which I didn't think was possible not a lot of people think you can talk about math but we it was just surprising what you can do and what like how controversial it could get like how many different opinions and ideas and like we all had we all knew we were working with the same things but we had so many different ideas like how to go about it so.
68	19:21	R1	So that's something you've been doing before with probability problems um was there anything about these problems you found especially surprising or unusual having to spend two weeks working on them
69	19:40	Romina	The probability problems they gave us specific numbers and they gave us ratios and they there wasn't there really weren't other factors in it fractions and you kept adding and multiplying fractions until you came up with something with this it was just so like random it was just like a seashell there is not much you can do with a seashell and then like we kind of had to like invent anything we kinda had to like what we had to choose what path we were going to take and what we were going to do. With other things we had we had things given to us, with this we kind of had to make up on our own we all had to agree with the other group kind of had to come up with common let's do something standard so we all get the same thing so it was a lot more compromising too
70	20:29	R1	Could you um so we can get to see it could you describe that third grade probability problem what was the problem
71	20:41	Romina	They put ten marbles into a box and it's either you get a choice of two colors and they cut a hole out of the box and then when you shake the box only one - we could only see one marble and from shaking the box you have to guess how many first of all there are ten marbles in there you have to guess how many are one color and how many are the other color and you just do that by you just trial that's all you do
72	21:03	R1	So you shake the box, then a marble falls out and do you get to keep shaking it until you empty it all out

73	21:11	Romina	No, it's only one at a time you shake it and the hole is not big enough for the marble to fall it it is only big enough to see part of 1 marble to see what color it is and then you shake it and only 1 marble comes out and you shake it again and they all kind of the 10 get mixed together again and then falls out well it doesn't fall out it come to corner
74	21:31	R1	Did you do it by trial and error or did you solve the problem by trial and error or is there a way of thinking about it?
75	21:41	Romina	Well, if you do it enough times you just have to keep on doing it and recording it is just a lot of data and note taking you just keep writing down what you get until actually your going to do it 100 times and yellow came up 70 times there is probably 7 yellows in there and the 3 of the other color and you do it's some logic but like if there is more of one color its more likely to come up and if you keep on doing it enough like if you came up if you were to the millionth you would eventually get a reasonable probably right number of how many are in there.
76	22:14	R1	Is it possible that you could do it a million times and get all yellows?
77	22:19	Romina	Its possible but see when you do it more times the chance of getting say there is 1 yellow and 9 blacks for the first 10 times it's a possibility to could get yellow 10 times but when you go up to when you do it more and more times its less likely your chances are smaller so.
78	22:40	R1	Some people might say you know I mean if if like you know if the yellow shows up 3 times in a row it's more likely the yellow is going to keep showing up
79	22:53	Romina	No I don't… It's It's like flipping a coin The first time you can flip a heads or a tail but if you do it 2 times getting tail like twice in a row is harder than just getting it once it it always like it the probability becomes like the probability of getting something in a row like a lot of times becomes less likely.
80	22:42	R1	Maybe you should tell me is there anything you would like to you say that we have not been asking you about about this experience?
81	23:52	Romina	I don't I think it was a good experience over all I don't know it just for us we don't know why we all have very low self esteem about everything and we didn't think we were capable we were very scared coming to this two weeks cause we thought a lot was expected from us and we were not going to be able to perform under all the pressure but I don't know we came out we I think we did I don't know what do you guys think while we came and we did a lot of problem solving we did a lot of thinking like we just sat and thought for hours a day and we came up with a lot of interesting things and we were able to go in front of a large audience and just talk about our ideas and then argue our points and prove our points so I think it was a very good experience
82	24:37	R1	Would you feel more comfortable if people reassured you more about whether you were on the right track, not on the right track or do you feel
83	24:46	Romina	That's um almost I don't know I don't like being reassured in like the problem like I look for reassurance but if they gave it to me it's almost like they're like they're treating me like a little child like you're good

			keep going with that keep on going like that this is like when we do come up with something it's so much better because we came up by ourselves without someone holding our hand and walking us through it like they if they walk us through it but yea we are going to get to the right answer but if they do this they don't know what's going to happen what direction we are going to take so it makes it all the better.
84	25:18	R1	So this problem that you brought up about self esteem is really a big problem in math that especially like a lot of people feel you know they're not good in math and whether they are good or not they should sometimes feel they're not good I mean I mean if you were to advise a teacher on how you would build up someone's self-esteem in math do you have any thoughts on that?
85	25:41	Romina	It's hard though because math is just so different I think there's two big different areas of math: one of them is like the thinking involved and one of them is just like spitting out numbers. I know I was never good at the spitting out the number thing and everything but I was decent at the thinking about it. If they can incorporate both of those into one and so I think most kids either go on one track or the other and if they can they if the teacher has both of them in a class the kids are bound to do one or the other, like you think he can do math for the time being
86	26:12	R1	If you were to make um if you were talking to someone who just wasn't familiar with what math is maybe there is some older generation someone who has not had the opportunities you had you know what what how you define math I mean what is mathematics?
87	26:34	Romina	I don't know. It's just so - I think it's problem solving. It's taking up a lot of things into consideration coming up with a reasonable answer to something or solution. It's nothing - math is just so vague and in so many areas of something it's everywhere unfortunately like it's like everywhere you can't get away from it - it's everywhere you can find and every situation you could possible think of you always think of probability, cause and effect things and that's math
88	27:05	R1	I think I think a lot of people think of math as arithmetic the number adding part and all that
89	27:11	Romina	I think that's what scares most people away from math the fact that they think it's just like long division or something and its not that it's just you have to apply it to different things and it's so much so…I think that if you just open and you keep opening math up like this for everybody eventually we are going to have a good math background and enough to do what we can do.
90	27:43	R2	Um Dr. Speiser talked about how much it meant to him how he was here for the two weeks and what he got out of it um and how it benefitted him I want to hear how you think it benefitted you guys you?
91	27:59	Romina	I think it helped us out a lot it almost like we were testing ourselves like we had to come up at first it didn't tell us half of the people in there were from Harvard and um they were professors at like universities like and like going in there and finding that out it was almost like a test to see if

			we can actually do it I know a lot of have been especially this year I don't know what happened with half of us in there but we were already turned off by math and we already thought we couldn't do it and that was it our math career was over when a lot of us had hoped to pursue math in the future but this changes it around a little because if we were able to go in there like professors we were to have in years to come were like kinda impressed by what we were doing and how we were thinking, maybe we can do it so maybe it changed us a lot. Changed our opinion.
92	28:54	R1	I guess one thing that I'm wondering is, you know I I am very impressed by what you do and I'm I'm you know I hadn't really seen people do the kinds of things you do but I'm sure there are people who do you know but given that you're in high school and that these are new ideas for you I'm just wondering why you're concerned or you think maybe people may think its you're not doing a fabulous job and
93	29:24	Romina	Cause it doesn't it doesn't seem a lot for us. Like if you gave us like this big long test with all these problems that seems like a lot for us cause it's either right or wrong but like when we come in here we are just sharing our ideas and like working in groups to come out with an answer like this this it's not easier for us it's completely different but we usually we don't think people just expect that so when we do that like we don't understand why that's such a big deal for some people like we never understand like until recently we never really understood the research that is behind this whole project but we didn't we didn't see how us sitting in a room talking about our ideas can be interesting to someone or benefit anyone
94	30:11	R1	Well you know well if you had a job in math do you think doing let's say somebody gave you some problem to solve they wouldn't give you a cat probably you know but it might be something else that's a real world problem, do you think you would know what to do with it?
95	30:31	Romina	I think if you me if we all went in there and they gave us the problem and we work at it until we get somewhere until we start off in the right direction it might take us a long time but pretty much anything like things that were given to us were like this was all calculus and we pretty much have no experience in because we haven't taken calculus yet but we were able to break it apart from what we knew we have like the basic math background but whenever we have to take it to the higher levels and like make that like use it in situations and apply it and I think we could do if we really we would have to work at it but from what we know I think we can handle it the most part.
96	31:11	R1	Well I guess I'm I'm just wondering do you think if you had a job that involved math the kinds of things you'd be doing in that job would be more like this or more like the kinds of things that maybe taught in some of the other classes like in textbooks?
97	31:30	Romina	No it would definitely be like this because first of all I wouldn't be like like finding the solution for a big problem by myself I would a lot of other peoples they'd be like we would have to have some sort of arguing like to bring up points that maybe I don't see that could help the solution cause it

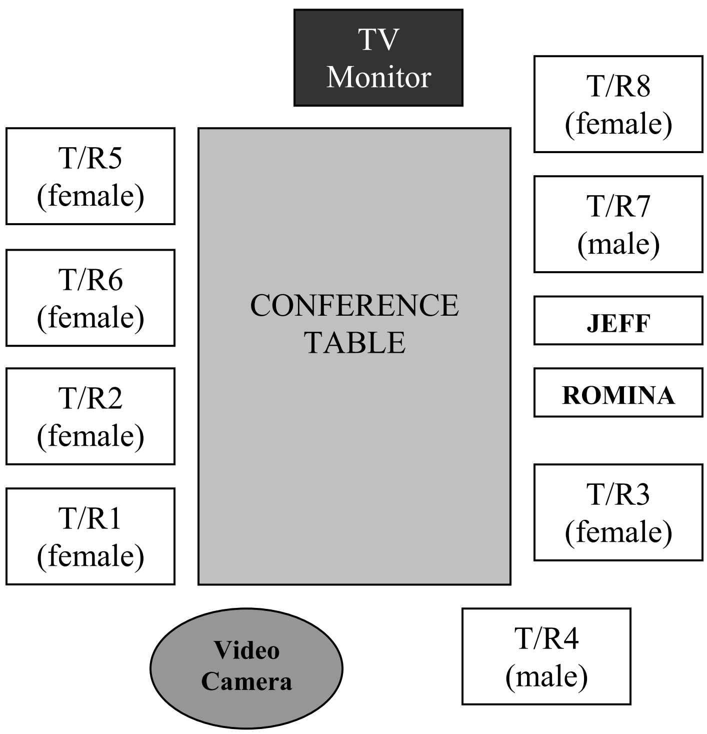
			it's gonna affect people mine 2 + 2 = 4 is not going to affect anyone on a test but when I go out in the real world and have to find like like real life situations like different variables that would affect it and people arguing will help and people would just keep talking about it and we have to find as many solutions as possible and go from there to see which ones the best solution it will definitely help this way
98	32:15	R1	<inaudible question>
99	32:25	Romina	I still don't understand how beneficial this research is they tell us but I think we don't think we are doing enough to help you but I think we're showing you that maybe like the textbook way isn't always the best way it may be it might be good to learn like we used it the first couple of years like 1st to 3rd grade you know just giving us the basics but after that if you just - people underestimate what we can do and if you can just give us problems and keep working at it it like builds us up. Makes us more. Like I am a more verbal person I can speak well and I can communicate my ideas where other people might like my same age level can't because they never had to they don't know their intimidated where I was kind of put on the spot and had to and it just develops your idea and maybe when we are like running the world we can come up with better solutions cause we know more and we can like we've practiced and we have been able to have like group thinking and solutions.
100	33:24	R1	You've said you had a lot people underestimate what you can do can you tell me more about it?
101	33:28	Romina	Like because we don't always get to a right answer like the do fractions and logarithms and things like that like maybe they don't think we are as smart as we can like as we are but if you give us like real problems like problems that actually matter not just spitting back numbers and then memorization we can apply its almost like higher level thinking like like real life solutions and we can sometimes like we can we are pretty rational people when we can come with interesting things like point of views and ideas that no one else really looks at especially for our age level they just think that we can memorize things that's it and if we can't memorize things we are not that intelligent.
102	34:35	R1	By underestimating you in that way
103	34:36	Romina	We don't we are not able to achieve if not everyone if no one else thinks we can do it why would we think we can do it um like if you're telling the only way I can be smart is if I can memorize all the answers to the like science that wouldn't be very I could never do that I could but it would take a lot of studying where I can be using my time for more like thinking like thinking up my own ideas connecting it where as the sine of 30 will never mean anything to me whereas but if I thought about it and could see a graph in my head and see where a point fell on a graph and what it was that would mean a lot more cause then I could I could use like that graph in other situations where the sine 30 I couldn't I could just that is all it would be worth.

104	35:27	R2	I was just wondering. You said that you really did not know what this research had to do with anything and um you might be able to articulate that but you've learned a lot through being in that class and being in this project and I'm wondering if you were if a new teacher someone who hadn't taught before came to you and said listen Romina know more about learning math than I do how would you what should my math class look like what is the most important thing I should do for my students? What would you say?
105	36:00	Romina	I think they would have to incorporate the two maths I was talking about before I think we need a little bit of like the memorization and like this times this is this just like quick math to get by little things the SATs to get by that but then I think they have to have like have to have us able to talk in class talk about our ideas and give us like a problem about speeds and cars and let us go on for about a week or two just talking about it and experimenting with it and then once we are done with that they can show us like the easy math went in that and if we can do that we would be able to do both. We'll be able to do our own thinking and we will be able to do the quick solving at the same time.
106	36:44	R1	I know of work you have been doing have had lot of use of calculators, what's the point of that?
107	36:53	Romina	Our calculators are to do like the math that would take us just a little longer to do by hand and like calculator now is not just like you can add divide multiply and subtract it's it does so much more for us like with a calculator we were, its almost like we were making something visual for us to see something so if we input all our numbers in the calculator and it came out with this nice graph for us and then we can see like the cat was like accelerating and like decelerating we can see like like almost like how to show it was almost like an exponential graph it was going up by so much that we could see that when we graphed whereas when we were just looking at the numbers if a pattern doesn't jump out at me I wont be able to tell but you put in on a calculator you can see it.
108	37:47	R1	Anything else? Do you have anything to add?
109	37:49	Romina	No, I'm good.
110	37:51	R1	Well thank you very much.

APPENDIX G: TRANSCRIPT - REFLECTIONS III

March 11, 2002 (College Sophomore)

1 Camera View: Romina & Jeff Reflections (*3 Disks*)
Date of filming: 2002-03-11
Kenilworth, NJ, Reflection for the Kenilworth Longitudinal Study
Transcribed by:
John Zengerle (Disk 1), Margaret Steffero (Disk 2), Melissa Lieberman (Disk 3)
Date of transcription: June 2009
Verified by: Maria Steffero
Date of verification: June 2009



DISK 1 of 3

Line	Time	Speaker	Transcript
1	00:00:20	T/R1	When you talk about it. So I said, "Why don't you make up a problem?" And Ankur came up with a problem that he and Michael started to work on it looked like you and Jeff partnered to work on it. Whether or not Brian was there or not, I don't remember myself, but I remember that you two were working on it, and it was called… We called it Ankur's problem. Didn't we?
2	00:00:39	T/R3	Ankur's Challenge.

3	00:00:41	T/R1	Ankur's Challenge. Okay. You want to tell us what the problem is?
4	00:00:46	T/R3	Yes. I have copies.
5	00:00:47	T/R1	Yes. She has copies.
6	00:00:48	T/R3	Some people got them by email. (reading from paper) "Find as many towers as possible that are four cubes tall, if you can select from three colors. There must be at least one of each color in each tower. How do you know that you have found all the possibilities? Build the solutions selecting from three colors of Unifix cubes. Convince your peers that you have found all the possibilities, no more, no fewer."
7	00:01:14	T/R1	Okay. So the group in here, is there anyone here who has never seen this problem before? (several people raise their hands) So Jenna hasn't seen it, and Marik hasn't seen it, and the rest of you have all seen this problem?
8	00:01:30	T/R3	(inaudible)
9	00:01:33	T/R1	Okay. So, I don't know. Do you want a minute to think about it? Or, do you want to watch the way they worked on it? (laughter) Jenny, do you have any preference? (She shakes her head, no) Okay, well, you can still think about it. Why don't we watch the video? Okay, since most people have seen the problem. Now again, before you put that on, what I, what I'm asking, um, Jeff and Romina to do as they look at this, um. They're now looking at it from the outside in. Now, I don't know how much you'll remember. As I look at old tapes from myself, I sometimes say "I'm saying that? I'm doing that?" It comes as a total surprise to me that I did what I did when I did it. So I'm not surprised when you say you don't remember, you know? And some things I do remember saying, but some things I don't. But you do have an opportunity to look from another lens, and so you could take the stance, I think, all of us can, is "What are they doing? How are they working on this?" And, what are we noting from the way they're working? Obviously, a solution comes out of this, right? A rather elegant solution comes out of this, but this process produced this, right? So, how do we describe that? What's your window or lens for describing that work? Does that make sense? As for everybody else here, I kinda want to know your thoughts as well as Jeff's and Romina's. Okay. (The tape starts)
10	00:03:11	T/R3	You might need to up the volume on the TV set. (They watch the video – Ankur's Challenge 01-09-1998 – when Romina argues "So you have to organize them" – disk 1 00:53:00 as excerpted in PUPMath – ends with narrator's voiceover)
11	00:08:22	T/R1	Do you want to stop that? (Romina laughs. The video stops) Okay. There's the later version of Romina's that you can pass out now, where, uh, some of you don't might not like six times two equals twelve times three equals thirty-six. She doesn't have that in the second time. (Romina laughs)
12	00:08:51	T/R1	The reason I point this out, the first time I showed this tape, I had a conference with Aski was there and he saw that so I pointed out Romina's revision. So this is the product and you saw the process. Do you need to

			see it again?
13	00:09:21	T/R3	Not me. Anybody else?
14	00:09:22	T/R1	Nobody here? So this is one of the first problems that we did. One of the first new problems given the tower problems that Ankur came up with. Let me start with a, with maybe more a specific question, um, to Romina and to Jeff. The way you worked here, was this sort of typical, do you think?
15	00:09:54	Jeff	Yes, it was. I mean that's, that's how we kinda worked every time. I mean there would be a group of us. In that case, was there five of us? And, uh, I mean we didn't really know what we were doing, and, I mean, that's, that's how we worked throughout time. I mean, we'd all kinda yell at each other for a little while, and it gets a little loud and, uh, hard to listen to, but then, you know, someone has an idea and by all of us talking together we could see, we could - one idea could come out as the prevailing idea, the one that seems to be the right answer. You know, it comes out, and then we could all manipulate it from there and make it the right … what we feel is the right answer in the end.
16	00:10:36	Romina	I think this is really typical of how Michael and Ankur paired off, and Jeffrey and I did, and Brian just, uh… (Laughs) wherever the wind blew, I don't know, uh. I did that, and then we'd both come up with a solution, and then, whoever, we'd argue it out and then probably take ideas from each other and then worked from there and come back.
17	00:10:56	T/R3	Yeah, that's something they didn't show on the tape actually, but before we saw yours, Ankur and Michael had a different solution. Do you remember that?
18	00:11:03	Romina	I'm assuming they did. I don't remember, but they usually do. Yeah.
19	00:11:12	T/R1	Does anybody else have a question? Or comment for what they saw in this tape? I know you laughed in a few places. What struck you as being funny?
20	00:11:25	Jeff	Besides the way we looked? (Laughter) Um.
21	00:11:28	Romina	Us arguing.
22	00:11:32	Jeff	Yeah, I mean, it's just that's, that's a piece of our life right there. You know, that's how we did. We did this for a long time, and that's how, that's how we did it, and it's just funny to see that again. 'Cause I, you know it…
23	00:11:44	T/R1	Well, you sort of laughed when I, and I think many of you did here, but if you did, you know, you certainly tell me that wasn't what you were laughing at. When Michael said "I wasn't paying attention. I wasn't paying attention. I want to hear what you have to say now."
24	00:11:58	T/R3	Yeah. "Now I'm ready to listen."
25	00:11:59	T/R1	"Now I'm ready to listen."
26	00:12:00	Romina	Well, Michael. I think Michael's on a different level than we are at many times. So he blocked us out until he's done thinking about his problem and coming up with his solution. He doesn't do it on purpose, and then he

			comes back and he says "Okay, now I'll listen to what you have to say." After he…
27	00:12:13	Jeff	He started to recognize that, that what she had to offer might have been the right answer, and maybe that was the right way to go about doing it, and, you know, when you, when you think you have your own way of doing it, you're not, it's hard to listen to what other people have to say a lot of the times and you know it takes a point in time when you kind of realize that maybe I'm doing it the wrong way and maybe you have, you know, you have something that I could listen to that could maybe help me out before you kind of sit down and relax and start to listen and see what other people have to say.
28	00:12:37	Romina	He also prepares his argument very well. Like he'll watch you and then you, when he thinks you messed up, or when he thinks his way would work better, then he bring it up, but, yes, he thinks it through all the time.
29	00:12:49	T/R1	Now, this "I don't want to think about it, yet. Now I'm ready." Do you think that was just Michael or were there times, over the years, that you don't want to talk to somebody, you want to be ready to talk to somebody? Is that so unique to Michael?
30	00:13:05	Romina	If I didn't understand, if I didn't understand a problem, or if I didn't work enough through it by myself to understand where like… I guess Michael didn't know where I was heading with what I was doing, and if I didn't understand where the other person was heading, I liked to work on it before I form a couple options and see which one he takes.
31	00:13:24	T/R1	Right, so what Michael did, you sort of sometimes did also?
32	00:13:27	Romina	Mh, hm. (Nods)
33	00:13:29	T/R1	Jeff
34	00:13:30	Jeff	Well, yeah. I think everyone wanted to have their own, their own way of doing it at the beginning. You know, everyone wanted to come up with something by themselves to have something to say "you know, this is what I think about it." And then, you know, until you're ready to share that with other people, you're not really interested. You know, you don't want to hear what other people are doing. You know you want to do your thing, and you want to get it kinda together, try to set it up a little bit, and then when, when there's some time, and you know everyone's kinda done yelling with each other or doing whatever they're doing, you know, you could kinda say "alright, this is what I think," and you can kinda discuss it a little bit, and get somewhere.
35	00:14:01	Romina	We're also a little competitive. We didn't want to be proved wrong.
36	00:14:04	Jeff	Yeah, certainly, none of us ever wanted to be wrong.
37	00:14:07	T/R1	Okay. So that was a reason for being careful.
38	00:14:08	Romina	Mm, hm.
39	00:14:09	T/R1	So, with each other. You had your own, sort of culture. 'Cause there was no. You know I wasn't there. There were no adults, researcher, teachers there for this whole session here except to ask, you know, what you did in

			this particular one. I wanted to show you a tape, um, clip where, um, I'm there more. We call it the Night Session. I want to tell you more about that before you put it in to help you remember. It actually took place, um, right before your junior prom. The day before. Do you remember that time when you stayed really late? Do you remember that, Romina?
40	00:14:54	Romina	I was the only girl. I had things to do.
41	00:14:56	T/R1	Yes. (Laughter) I bet you had things to do, right? And it was really late. I don't think you… It was about 10:00 when we ended. And I remember Brian came in late. They had the wrong-sized tux for him.
42	00:15:09	Jeff	(inaudible) good excuses.
43	00:15:10	T/R1	Do you remember that? Excuses. Do think that was a story, Jeff?
44	00:15:14	Jeff	He was late a lot. That's all I'm saying. More of us than others were late more often than other times.
45	00:15:22	T/R1	But, in that session, um… I don't want you to start it yet, though. Okay? Um, we just started very informally. We had some visitors from the Harvard group there. If you recall, it was a different camera crew was there that night. (Romina and Jeff nod) And, we started by my asking you what you did that day in class. Is that right, Liz?
46	00:15:44	T/R3	Yeah.
47	00:15:45	T/R1	You want to say a little bit more about how it started?
48	00:15:48	T/R3	Well, yeah, you say you needed an explanation. You didn't understand what they were doing in class that day and there was, um, Jeff started telling you about they were doing the, um, the rules for the binomial expansion. The coefficients. How to figure out the coefficients of the numbers in the binomial expansion like a plus b to the two or to the three or to the four, and that got in to… started talking about towers. Um, talking about combinatorics, you know arranging people on a line. How you got three places on the line and two people or something like that. How are you going to arrange them? And then it got in to, well, like you said, it went on for hours. (Laughter)
49	00:16:28	T/R1	Okay, it went on and there was a point in it, um, where, uh, somehow the triangle, Pascal's triangle came up, and, um, there was a, uh, reference made to that triangle. I remember asking the question, "How did the triangle grow?" and there was an explanation provided, uh, for that, and that lead to some other stuff that became, um, more general. So we were talking about particular rows of the triangle. We were talking about how those particular rows grew, and your explanation evolved from an argument Michael had presented earlier, but I don't think it was being communicated by Michael. It might have been communicated by you, Jeff, but it was, um, had to do with pizzas and more toppings on pizzas. Either you give a pizza a topping or you don't. Do you remember that?
50	00:17:32	Jeff	Uh, yeah. Well, I got. I got the tape, um, and…
51	00:17:36	T/R1	So you looked at that, right.

52	00:17:37	Jeff	Yeah, we looked.
53	00:17:38	T/R1	You know more about it than I do.
54	00:17:39	Jeff	Yeah, we looked at it a little bit. There was some things that I was, um… It was kinda funny 'cause there were some things that I was a little sketchy about. I didn't really remember what we were doing, and, um, asking her what's going on there? And it was really silly 'cause now she is showing me. "Why is it two?" It was really embarrassing; it was like straight out of one of our discussions, and it was really…
55	00:17:58	T/R1	We didn't tape it. (Laughter)
56	00:17:59	Jeff	I know. It wasn't there, but, I mean, we were, 'cause, you know, you forget how you do things, uh, if you don't do them in a long time, and it was, it was really kinda wild.
57	00:18:07	T/R1	You were able to reconstruct some of it?
58	00:18:08	Jeff	Yeah, we started to get a little bit of it back. Uh, but, yeah, it was really kinda funny because it was, like, straight out of one of our scenes.
59	00:18:14	T/R1	You saw it too, Romina?
60	00:18:15	Jeff	She saw a very little bit.
61	00:18:16	Romina	I saw the beginning and then we got in a fight, so we stopped.
62	00:18:17	Jeff	Yeah, that was… we started arguing a little bit.
63	00:18:19	T/R1	About what you were seeing or… ?
64	00:18:20	Jeff	Yeah, whatever.
65	00:18:21	Romina	Yeah. (Laughter)
66	00:18:26	Jeff	So, um, but yeah. So we watched a little bit of the tape.
67	00:18:29	T/R1	So, let me tell you what the issue is here. Um, Liz has spent a lot of looking at and studying this tape, um, and I have also spent a lot of time on this tape with another colleague who is from Austria, uh, Professor Willie Doerfler, who was visiting with us last year. Um, and what occurred when Professor Doerfler first looked at this tape, he sort of came up with the notion that this was more like a teaching experiment because I'm in it a lot, and, um,
68	00:19:05	Romina	That was the first time. I remember that because it was the first time you ever sat with us and taught us.
69	00:19:10	T/R1	Let's get that to the camera. (Laughter)
70	00:19:14	Romina	That's why I remember it, 'cause then when you taught us the, uh, how to write it actually what we were doing for years, and we never knew that, and that's what we were arguing about because he didn't understand it.
71	00:19:22	T/R1	Okay, so Liz, um, also thought that this is how we worked and so, you see that you can take a piece like this and you could sort of make an inference about the teacher-researcher works that maybe isn't representative. So this was different, and I believe it was different. You're saying it's different, and I think, you know, one could do a very formal study and

			look at all of them and see whether it is different or not, but for now let's accept that there's clearly a difference in my role here and my role in the one we just saw in which I wasn't even there, and what I'd like us all to do here, together, is, um, try to, um, get some sense of what do you think I was trying to do? I mean, I'll tell you what I think I was trying to do, but I'm wondering what you think I was trying to do. Is that fair? And why I was in so much here, where, maybe not in other places. Is there a way I shouldn't have been so much here, or was it appropriate or not appropriate? Sort of, let's have some conversation about that because all of us get into it different ways different times, and we never really know if we're getting into it too much or not enough or whether we should so we're trying to understand it, we're trying to study it. Fair enough? And Liz really needs this. (Laughter) Okay. But I just want to make sure, Jeff, you don't think this is representative either of my…
72	00:20:43	Jeff	No, no. Certainly not. This was a different kind of tape. (They watch the tape of "The Night Session" from PUPMath with the narrator voice-over. Periodically, Jeff and Romina whisper to each other)
73	00:26:18	Jeff	(reacting to the tape) That's what I was trying to do a half hour ago. (Romina laughs)
74	00:26:51	T/R1	Do you want to stop it? (The tape ends) So, does anybody want to point out the interventions of the researcher that they saw in this tape? The times that I intervened? You want to start?
75	00:26:05	Jeff	I thought a lot of it was when we were, we were asking you if what we did was enough, and then, when you suggested maybe we should write it in the factorial notation which, I mean, changes, that changes the whole equation. That brought it to a different place. Um, other than that, I didn't see too much interaction with you in that tape.
76	00:27:26	Romina	I think you tied it in for us 'cause, I mean, that equation, I've seen that now. I see that in my calc classes, and we, I mean we worked on this what, since we were in first grade, and we worked on a lot of the same problems and we never, we never formalized like we never had 'cause we didn't have this every day so we never had a set equation or we just, we had a way of thinking about this and we always pretty much tried this same way, but we needed to end it almost, and that's how we ended it. We came up with that formula, and then we actually use that formula now.
77	00:27:57	T/R1	Okay. So anybody else have a …?
78	00:27:58	T/R2	I thought it was interesting that you asked them to do quite specific things sometimes like, "Can you write this in this notation?" and they didn't always do what it was that you said. When you first asked for "Can you write this in this new notation?" Michael didn't just write it in the new notation, he was explaining what something on the blackboard about when you add these guys here and these guys here, and he wrote some completely different things before that happened so he didn't respond in a way that immediately… He tried to use it. (shrugs)
79	00:28:30	Romina	I think it's 'cause we've never actually… I don't think we've ever

			actually done that. I don't think we've ever actually written it in a notation form so the only way, whenever you asked us to do anything we had to explain it, we had to go through that whole…
80	00:28:40	T/R2	So that's why you think that's why you did it?
81	00:28:41	Romina	Yeah.
82	00:28:43	T/R1	Which is what Jeff said the last time.
83	00:28:46	Jeff	Yeah, yeah. There's a lot of things that re… decline to say 'cause I've said them a couple times, and you can only say the same thing so many times.
84	00:28:53	T/R1	But I thought that was, that was really, um, interesting. I don't want to, want to try to, uh, say what I thought I heard Jeff said, but the notion is that the expectation would be that you would have to explain it from, from really the basic details so the way of talking about what you did always made that assumption and went back to basic details.
85	00:29:17	Jeff	We needed to do that for ourselves, though, too, to know what we're talking about. I mean 'cause, if there was, if we tried to just present a final thing, and really didn't know it from the beginning; we couldn't explain it in a way that you would accept from us. So in order to explain it in the way that you would accept, we'd really have to start from, from bare bones, from the beginning.
86	00:29:39	T/R1	I mean, I noticed when, when you worked together, you tend to demand that same thing of each other.
87	00:29:46	Jeff	Well, I mean, if she knows something, I want to know it. (Laughter) You know, and if we're gonna use it together for to do a problem, I mean if I can't understand part of it and have her understand everything and expect me to be any help, like helping out trying to do anything because you need, I need to know the whole deal.
88	00:30:05	Romina	We each needed to know from the absolute, like, beginnings, because if we didn't, you would ask, you would know that and single me out. (Laughter) It's true.
89	00:30:12	Jeff	Yeah, yeah. You could see. Yeah, you could see who didn't know.
90	00:30:17	Romina	You would ask me, and I would be like "I really don't know," and then I'd try to ask Michael.
91	00:30:19	Jeff	And then you would ask one of us to explain to them.
92	00:30:21	Romina	And then you'd leave the room.
93	00:30:23	T/R1	Well, you give me more credit than I think I deserve (laughter), but Jeff just said something that I, um, think does make sense. I would ask another person to explain it. Now, you notice I did that?
94	00:30:34	Jeff	Well, yeah, yeah.
95	00:30:36	T/R1	As, um, so I would… If I heard Michael explain it, and I've heard him already, now I might ask someone else like Jeff or Romina. That tended to be typical, or to explain to someone else even if not to me. I'd say…
96	00:30:52	Romina	We had to be prepared.

97	00:30:53	T/R1	So you had to be prepared. So there was a sense that all of you had to own it in some way. That’s interesting, I hadn’t really thought about that as driving some of the motivation, but I find that very interesting. And it sort of makes sense, doesn’t it? (Laughter)
98	00:31:09	T/R1	Sean, come sit down. Eden, come sit down to. (inaudible) Okay. So, anyone else have a comment or observation or question?
99	00:31:19	T/R2	Does Liz have anything about that?
100	00:31:20	T/R1	Yeah.
101	00:31:20	T/R3	Well, I had something I thought about that you didn’t actually see on this portion of the tape. You know the tape I gave you and the transcript was the whole session, whereas this was a piece of it for this TV show. Um, and something you did in the other tape was you actually went up to the board at one point and helped them write some of the… I think you helped them write Pascal’s triangle to show what you wanted to choose notation, and I found that interesting, and I wondered if you remembered that, if you had any comments about that?
102	00:31:50	Jeff	I think, well, I mean, just from watching the tape briefly earlier this afternoon, we didn’t, we didn’t know anything about this choose notation stuff. Like, we didn’t even know how to write it as any kind of notation. You know, we didn’t know even how to put it on the board. I was putting, uh, you know it looked like two over zero. Like I was writing stuff like that when we were trying to write it, and …
103	00:32:10	Romina	It looked like division.
104	00:32:11	Jeff	Yeah, and I mean we really, this was like really fresh stuff for us, and we needed a little help. We needed some direction to get somewhere and I think that’s why you were more intensively involved in this because we really didn’t know what we were doing.
105	00:32:24	Romina	And I think, earlier in class, Mr. Pantozzi had written that and we all, he’s like “you should know this.” And we all looked at him like “I don’t know what you’re talking about.” And he was like this, and he tried to relate it back for us, and we just didn’t see how we reached from what we, from the work we had done to that formula. So we had to start at the very bottom and then she showed us. She showed us that extra step that we were missing.
106	00:32:48	T/R3	Well, that’s something, I mean, …
107	00:32:49	T/R1	That’s interesting.
108	00:32:50	T/R3	That made it even more remarkable in the sense that you never even saw the notation before, and yet, you still came out with this equation.
109	00:32:55	Romina	Yeah, we’re supposed to know it inside and outside and we didn’t.
110	00:32:58	T/R7	Could be right, I’m sure. ‘Cause Michael writes on the board “N choose A” and the person holds N choose R all the way through the tape.
111	00:33:06	T/R1	Well, individual had a little bit of it. Not everybody.
112	00:33:10	Jeff	Well, I mean, we started. If we, if we…

113	00:33:12	Romina	Michael had some of it.
114	00:33:13	Jeff	Were doing that, it was right, it was like that day in Math class. That was the day that we started to talk about it.
115	00:33:16	Romina	That we learned in class.
116	00:33:18	T/R4	Right.
117	00:33:24	T/R3	Yeah, there was some discussion at the beginning of the tape about the buttons on the calculator. You push these buttons to get the answer. (Laughter)
118	00:33:30	Jeff	Yeah, yeah.
119	00:33:31	Romina	That's what we're learning, but we're not quite sure what it means.
120	00:33:33	T/R1	Well, the, um. This is a standard notation that's a convention. There's no way you should have known it, unless someone showed it to you. You had the idea behind it. Now it was just simple introducing you to the standard notation which ended up being very trivial for you once you saw that. Um, and I think that's when I think I used to try to step in, when you were at the point when you had built something, mathematically, that fit a particular structure, or was an idea that was universal, I would then try to show a notation, and Bob Davis too. Um, Ella has been looking at the tapes from the Towers of Hanoi in great detail, and you dealt with ideas of exponents, but you didn't know how to represent an exponential function. You had to be, sort of, shown that. You were solving quadratic equations when you were in fifth grade, but you didn't know that you were solving quadratic equations, but, somehow, then that structure was shown to you, "by the way, this is what you're doing" and then you just moved on. It was, sort of, you were solving linear and exponential, which is, I guess, not fifth-grade curriculum or sixth-grade curriculum, but, so when that came, it was, sort of, a label, and, uh, so there is a time when it would be unfair to you not to say "by the way, this is what you've been doing." But what I find is interesting is, and I'm curious about, you've taken more Math since then in college, both of you, and, I would guess that most of the Math is taught at the formal, symbolic level.
121	00:35:14	Romina	It's funny, um, I took Calculus all last year, Calc I and Calc II, and it was very difficult for me, and a major part of that was, um, I didn't know a lot of, like, the simple notation, and I would work with a friend, and she could spit out all the formulas, and she didn't understand it, and I only knew the background behind every formula. So,… (Laughter) No, it was, and…
122	00:35:34	T/R1	What a pair!
123	00:35:35	Romina	Yeah, no, it was, and I had to, and I, and she would, we started off the semester with probability. So, it's like "this is easy." So I brought out towers and I was like, "Say you have towers four high, and you have two colors" 'cause we had four choose two or something. I'm like, and then, relating it to… It was horrible. (Laughs) 'Cause I knew how to do it, like I understood, like, "say this one is four choose zero, so you have none of

			this color, and now you have one of this color, four choose one," and I went through this whole explanation. She's just looking at me. She's like "you claim you can't even…" I'm like, "no, 'cause I don't know the formulas." I don't know that that means that, but this is if we were to think of it like that, this is the reasoning. And that's how we worked, and we worked the whole semester like that, and she, I broke… I had to explain to her all this, and she thought it was so funny 'cause I knew so much detail that everything you could possibly think of, even with, um, like even when we got to the more difficult subjects. Mr. Pantozzi showed us like that too. I was like "Imagine the graph, if *u p* it adds this exponential here, like it moves like this and…." (Gesturing) She was just like, "I don't understand how you know that and you can't just come up with the formula," and I never could, and that's why I had so much trouble in Calculus. I couldn't apply the formulas.
124	00:36:40	T/R1	You must have a question to ask, Lara, at this point.
125	00:36:43	T/R2	Um, no. I just want to hear more about it.
126	00:36:44	T/R1	Yeah.
127	00:36:47	T/R3	You mean you were expected… You weren't expected to explain answers, you were just expected…
128	00:36:51	Romina	No, my calculus classes, both of them, were very… They gave you homework, if you did the homework, and then, on exams, it was a scantron. Like, you had had your exam. It was ten-page exam, a problem on each page.
129	00:37:03	Jeff	They gave you scantron exams?
130	00:37:02	Romina	Yeah. No. Scantron.
131	00:37:04	Jeff	That's insanity. That's crazy.
132	00:37:07	Romina	That's why I, I did so horribly and they… And it was a ten-page exam so you hand in your exam, and they had the question on top of the page, gave you all the room to work on it, but at the end, you take the exam home with you. They only want the answer.
133	00:37:19	Jeff	That's really crazy.
134	00:37:20	Romina	And they did evil things, like… It was like 2.5e to the -3 and .25e to the… It was horrible, like they made 'em all really close so if you were off even one little… like you didn't get any credit for it. It was all or none.
135	00:37:35	T/R3	So, it was basically give me the answer and don't tell me how you got there? (Romina shakes her head in agreement)
136	00:37:40	T/R2	Can you, sorry, for my benefit, scantron.
137	00:37:42	Romina	Oh, what is a scantron?
138	00:37:43	T/R2	Yes.
139	00:37:44	Romina	It's like those… They just have ovals and they give you…
140	00:37:47	T/R2	Oh, and you fill in the little…

141	00:37:47	Romina	Yeah. And they give you A through E.
142	00:37:49	T/R3	It's a multiple choice test.
143	00:37:50	Romina and Jeff	Yeah.
144	00:37:53	T/R3	Scored by computer.
145	00:37:54	T/R7	Six times as effective.
146	00:37:55	T/R2	Absolutely. (Laughter)
147	00:37:58	Romina	That's all my Math exams were.
148	00:38:01	T/R3	Was that a typical Math department thing? If you had a different Math teacher, where they different, or was it all the same?
149	00:38:05	Romina	Well, the way we did it, we were taught, each of us were taught by, um, there were two hundred lecture, like two-hundred people in a lecture, and then our exams were at night, and then everyone, everyone in class, one-fifty say it was, took the same exact, the same exam at the same time.
150	00:38:21	T/R3	And it was all scantron?
151	00:38:22	Romina	The same, yeah.
152	00:38:23	T/R3	Final exam too?
153	00:38:24	Romina	Yes.
154	00:38:25	T/R3	Wow. Homework? Did they, did they give homework?
155	00:38:29	Romina	They gave homework, and they graded homework, but they didn't grade homework, um, more or less, 'cause I remembered knowing that the problems are wrong, and still getting good grades on the homework, they just checked to make sure that you did it. Like make sure every problem is there.
156	00:38:42	T/R5	That must have been really discouraging.
157	00:38:44	Romina	Yeah. I did horribly.
158	00:38:47	T/R5	Did other students feel as you did? That this isn't fair, that they don't look at our work? (inaudible)
159	00:38:50	Romina	No, because I think a lot of the other kids in my school, and I don't know, I have a very… They say it's a very professional-oriented school so they don't deal with a lot. They just give you an answer, and it's like that in all my classes. They don't… They never have to explain anything; they just… that's how they were taught. So it was easier for them, but I struggled through Calculus and they didn't 'cause they just knew the formula, they just put the numbers in and they got an answer.
160	00:39:15	T/R2	Did… Did they do better than you?
161	00:39:16	Romina	Yes.
162	00:39:17	T/R2	By a large? Right.
163	00:39:18	Romina	Like, I didn't know basic things: how to manipulate log. Like you know how… I don't know kinda like if you multiply two different logs and you

			get… I didn't know how to do that. I had to learn that to take my exams, or the things with *e*. I didn't know how to… I didn't even know how to add, like, exponents 'cause I just never thought of it like that. Just looking at it.
164	00:39:38	T/R2	When you did sit down to learn it, did you find you could, or was that difficult as well?
165	00:39:40	Romina	I mean, it was a lot. I just had to invest a lot more time in it than most other people did, and I didn't invest it until after the first exam.
166	00:39:51	T/R7	Can I ask something of, uh, of you both? I just picked up this book on the weekend. It's a book of, about my favorite topic about gory operations usually on the brain. I love it. (Laughter) I sort of wake up 3:00 in the morning can't get back to sleep, so I pick these books up and read these stories, and the most recent book I got is a guy who tells you how he becomes a brain surgeon. He didn't really want to be a brain surgeon, and the last Math class he took was a math of physics class and the instructor said at the beginning "The exam is just going to have a question on it. You have to do a calculation. You get it right in three significant figures in which case you get an A+ or you could fail." And the class said "That's not fair. How 'bout our reasoning?" The guy said "This is real life." He said "When you build a bridge, and it falls down, it's your fault. When you operate on someone, and you cause an aneurysm and they die, it's your fault. You get it right or you get out." So, the question is, what do you feel about that now? Do you think there's something to that point of view, and how does that relate to your sort of experiences?
167	00:40:52	Romina	I think it's not gonna… For me, I think this math helped me more for what I planned to do in the long run. Whereas, I understand that engineer can't… They don't have room for reasoning. They do have room for reasoning for the basic, but they have to get it right answer.
168	00:41:05	Jeff	But wouldn't you rather have somebody who knew what he was doing, and knew how to do…? You know, I was saying yes, you have to get the right answer, but I'm saying the key to the whole thing is being able to reason and get the right answer, and that's the kinda guy that I want building my bridges or …
169	00:41:20	T/R7	Operating on your brain?
170	00:41:21	Jeff	Exactly. (Laughter) I mean I don't know if. I mean I think the reasoning's better for certain things, but I think that in order to be able to do both successfully, I mean I think those are the people that could really do stuff. That's all.
171	00:41:34	T/R1	There is nothing to add to that.
172	00:41:36	T/R7	I don't think there is. (Laughter) That's the answer (inaudible)
173	00:41:39	T/R1	Marybeth?
174	00:41:40	T/R6	Romina, how did you study for those things that you didn't like, *e* and things like that?
175	00:41:46	Romina	My friend made me a worksheet. (Laughter) And it was, um, it was all the basic, uh, the basic things I needed to know, and then I just took old

			exams, and we just did the problems over and over again, but every time it was the same thing; I'd get stuck at the part where I'd actually have to solve it and get the answer, and then she'd carry me on through there.
176	00:42:05	T/R6	Right.
177	00:42:06	Romina	And then we'd just…
178	00:42:07	T/R6	Was there anything from the things that you used before, you know when you met with Rutgers that apply?
179	00:42:13	Romina	Well, yeah. I mean that's how I arrived at most of my answers. I thought of them like that, and I came up with this is how I would do it, now what formulas would I use to get the answers if I were to do it like this?
180	00:42:23	T/R6	So, even when you learned new things, this is how you kind of worked?
181	00:42:25	Romina	Yeah, this is how I… This is my thought process. I don't… I think we learned more of a thought process and how we deal when we were first given questions, which is how I always deal with how I'm given questions now. And that's how we do it; we talked it out, like, between my friend and I and then we came up with the how are we going to do this.
182	00:42:43	T/R7	Do you feel, Romina, that with these tests that you're given in Math, I think you said something about calculus where you couldn't find the formulas, do you honestly feel that if you had the time to work through the problem, like you had a day or something, that you would get it?
183	00:42:58	Romina	Yeah, uh hm.
184	00:42:59	T/R7	And so, I think what I'm hearing you say, from what you said before, is that you haven't really forgot the basics?
185	00:43:02	Romina	No.
186	00:43:03	T/R7	They're there. Maybe under certain exam conditions, you're going to get stressed out.
187	00:43:06	Romina	Well, I, I only have, it was about, it averaged out to about three to four minutes per question.
188	00:43:13	T/R3	Yeah, I noticed that. I can jump in as a math student too. I took a math course. (Laughter) And she was my teacher. (pointing to T/R2) (Laughter)
189	00:43:24	T/R7	That's what it was like (inaudible)
190	00:43:28	T/R3	You gave us great exams and they required thoughtfulness and so on, but I had a lot of trouble getting them done in the time frame because I had to sit and think about them, and if I didn't have everything right at my fingertips, I couldn't get them done in time.
191	00:43:43	T/R2	Can I just point out that Liz came out with well over 90 percent, which was considerably higher than the average scores on this exam, before everyone starts to think that I was some kind of evil (inaudible)
192	00:43:51	T/R7	I understand that.
193	00:43:52	T/R2	(inaudible)
194	00:43:53	T/R1	She makes a good point. I have to tell you a personal experience. Um, I took a summer course at Rutgers. That's where I met my husband. It was

			a small course. It was a geometry course; it was projective geometry. And it was maybe six or seven students in it, and we were having our first exam. I didn't really know him, and so he was chatting with me before, and he said to me "Can you prove Desargues's Theorem?" I said, "Yeah, if I thought about it." He says "No, no, can you prove it for the test?" I said "You've got to be kidding." Guess what was on the test? "Prove Desargues's Theorem." I decided I better talk to him a little bit more about what he thought was going to be on the test. (Laughter) We got to know each other.
195	00:44:35	Jeff	He really knew something.
196	00:44:37	T/R1	Um, and it's this kind of culture shock that it wasn't really a question of what you understood, it's a question of what you can produce during this time. You had to have it so much on your fingers. It was almost like, I think the dancer or the skater, there's no room for figuring it out on the mark; it's got to be that perfect performance on the spot. It seems to me, however, that the fallacy in that is that folks could get through doing well, and do it in a very superficial way, and not have what Jeff alluded to earlier as that deep understanding. Um, and I would worry about their bridges ultimately if they were engineers or whatever, and bridges do collapse, we know that, and structures do collapse when they're not supposed to, buildings and so forth, and you wonder about that.
197	00:45:30	T/R7	And surgeons kill people.
198	00:45:31	T/R1	And so my, and surgeons kill people. And so, in some ways, we can't say that, we can't say that we advocate that. That, in itself, is not a good goal, is it? Um, and yet, you can't just sort of have an approximate either so there's got to be something in between, and, of course, professors all say "But I don't have time to get to know what everyone's thinking is. Look how many students." Like the size of your lectures. Right? There must have been hundreds of students taking the course. There's no way to give grades, grade papers. So the system itself, something else is going on. So then I think to myself, okay suppose I were to replicate this twelve-year study, and anticipated, perhaps what might be obstacles for you guys going along. What might I have done differently to prepare you for it? I ask myself this question. Um, and I think that there are some things, I think that if there were more time really at some point to, to help that transition, you know, to the more formal, to the more symbolic, but, but you know that where's the time? All the time you gave was volunteer time after school. You certainly weren't wasting your time in your math classes dealing with what you were doing. That time was certainly looking deeply at certain problems too. So the question is how does that get done? Something has to change, right? Something has to change in the way Math is delivered and organized, you know, and how people work together. We're not even close to that. We're really not. I think about. I think I have a better sense of what needs to be done. I think we have to have more things like these night session things, right? Where things get pulled together and there's more chance to generalize and

whatever, and I was pushing for that, and I'll tell you why. I knew you all could do it. I believed you all could do it. I wanted the world to see you all do it, and I sort of ran with where you all were going because that's what the world values, unfortunately. They're going to say that this, this is the real math, you know. You can do it in symbols, you can do it abstractly, you can do it in the standard notation of the generalizations. That's the real stuff. Well, I don't think that's the real stuff. I think that's part of it, you know, and I think you have to get there to even go to the next step, or to go on in your study of Mathematics and even to a higher level, you have to be able to think in the more general language of mathematicians and the more symbolic, but I think you have to be able to think and have the meaning behind it. I mean everyone falls apart one way… Students who don't have the meaning behind it can't go on after some point even if it appears that they can work symbolically.

199	00:48:31	T/R2	There's often a crash, yeah.
200	00:48:32	T/R1	There's often a crash, and so, you need both. It's not one or the other.
201	00:48:37	T/R2	Yeah, I would disagree with you a bit with what you are saying.
202	00:48:38	T/R1	Go ahead.
203	00:48:39	T/R2	I think that mathematicians at least, do enormously value the meaning behind things and would definitely claim that they wanted this to be there.
204	00:48:48	T/R1	Did I say that they didn't?
205	00:48:49	T/R2	You said that the world didn't which is not quite the same. The visible part of it is the stuff that you know that you can write down in order to be assessed in these things. (inaudible)
206	00:48:58	T/R1	Are they not mathematicians who make these tests?
207	00:49:01	T/R2	Yeah but, if you are going to test, if you are going to test people and you're going to test a great number of people, you do that by having them all sit in one place at one time and write some stuff down for you, and the written stuff, therefore, becomes the thing that everyone thinks is valued even though that might not be what the person teaching it would tell you that they value. See what I mean?
208	00:49:24	T/R1	You can value it, but you don't live your values. Then I don't call those values.
209	00:49:27	T/R2	Yeah, but… yeah.
210	00:49:28	T/R1	I mean if you value something, you live your life by your values.
211	00:49:32	T/R2	Now we're having this big debate and we've gotten completely away from you.
212	00:49:33	T/R1	No, I think this is important. I think we should all have this debate and think about this because even those who come and study here and go back to world of teaching and classes, and they say well all this is very nice, but in my world this doesn't work. This is what I have to do, and we need to think about what is it? Where are the values? What do you have to do? And if there really are your values, you need to work towards them even

			if you don't achieve them you have to work towards getting closer to your values, and to, to send one message if this is your values and then to act another way, I have trouble with that. Maybe it's me. I'd like to hear other people.
213	00:50:09	T/R8	This discussion reminds me of the argument that's going on in the *New York Times* about AP exams. How one of the Ivy League presidents said he didn't even want to accept the five on the AP exam because he found those students were being just taught for the test, how to answer those particular problems that they knew were going to be on the test, but they fall flat on their face when they get to a college-level course because they couldn't reason. They just knew those answers which were gonna be on the test. And so there is some discussion now on doing away with the AP. I mean it's probably a long way down the road, but it's exactly what you're talking about.
214	00:50:40	Romina	It's not that far; my school's having a big problem with that because I know for some kids they didn't accept their AP credit and I know my, one of my very good friends took Calculus with me, and he, uh, he did receive a five on the exam, but he got kicked out of school, his high school a little early, so our college wouldn't take 'em, and he ended up getting, he barely passed. He got a C in 150 and I think he got a D in 151, and he got fives on the exam.
215	00:51:11	T/R3	He got. And the fives were intended for him to skip those classes?
216	00:51:12	Romina	Yeah.
217	00:51:12	T/R3	And he didn't skip 'em; he took 'em, and barely passed.
218	00:51:14	Romina	And they're fine 'cause they skip and they don't have to take 'em so they don't have to prove that they, you know, that they did well, but they really can't do what we did even though they passed the exams.
219	00:51:23	T/R1	Well maybe if he had a deeper understanding and the test itself didn't let that understanding come out. That's certainly a possibility.
220	00:51:29	T/R7	I don't want to wander too far away from the discussion, but just on Romina's point. The son of a colleague of mine, uh, was a Math Olympiad student, not a Math Olympiad, a Physics Olympiad student back in Australia. That's pretty good by world standards. He and another Olympiad student started first-year physics at the University of Melbourne, and this guy barely passed, and his fellow student failed. So two of the top students in the world in this department and they fail one of them and they barely pass the other. The one who barely passed then went on to do a PhD in Mathematics at Stanford and he's now absolutely running red hot. Now, both those kids were very good at handling symbolic cognition and thinking, but they're both good. But somehow there comes a time in a person's life especially when you're young when you think "I just don't want to do this anymore." So that can be an issue, I think. In terms of understanding, we just think "It just doesn't mean anything." You don't buy into it.
221	00:52:32	T/R6	You also bring another issue. I'm coming from a different world so it's, it

			might be…
222	00:52:38	T/R1	Tell a little bit about your background so…
223	00:52:39	T/R6	Oh, I, I ventured down from third floor. I'm a doing a PhD in educational psychology so I'm not doing Math or anything like that, but I figured I'll be brave enough to go down and see what other people in the world are doing. But, um.
224	00:52:54	T/R2	I bring right up a tower.
225	00:52:54	T/R6	It's great, you get free food, and …(inaudible) but, um, you brought up the issue of symbolic, and symbols are a different way. And what I heard was that they are a different way of representation, and that may be also an issue. People may understand things in a different way and they represent their knowledge. Their knowledge is being represented in a different way. Um, symbols, you know, whatever way that, um. I mean you doing the towers and the taxi problem is also a way of using symbols or visual, taking something visual and visually representing it to getting an answer. And you, you can do that on your own time, and you are not really expected, or you don't have time to do that when their issue of standardized testing where you expect to deliver an answer in three minutes or less. So that's just something that. I thought I would mention that.
226	00:53:51	Romina	I think that. I think about that a lot because I think it's very hard to mold education around… Like I know. I don't know. I think we pretty much learn the same way (looking at Jeff). I don't learn well. Like if you give me a book. I didn't even really use textbooks in high school 'cause I mean for math I never really had a textbook ever, and I don't learn well like that and that's… I'm having a lot of trouble in college now with that because I don't even know who my teacher is. Like if I saw, if they saw me on the street, they wouldn't recognize me, and most of 'em it's like you have to read a book and then you're tested from what's in the book and I never learned like that so I'm just. I, I'm better at learning if like thinking about things, discussions, group work, and I've always been, and now when I, now I'm not, I'm not doing as well as I think I could be doing in college because we're just not taught like that anymore, and I, maybe I should have during high school and during my earlier years, I should have practiced more learning the other way. That way I would have been prepared for college. And…
227	00:54:48	T/R4	I have a question. Sorry to interrupt.
228	00:54:49	T/R7	I'll be quiet and keep thinking. (Laughter) Sorry, John.
229	00:54:55	T/R4	I know you've been asked again and again and again. Off your experience with Rutgers, and now where you are, what things did you feel, that you took from your Rutgers experience, have been a plus in your work? I just want to situate your Rutgers experience in the context of where you are now. What strengths you felt that you have then, where you felt you had worked differently. This question may have been asked

			here, but I'm still curious to …
230	00:55:28	T/R7	John, when you say "by their Rutgers experience" you mean their experience with the Rutgers people, right?
231	00:55:32	T/R4	Yes, I want you to concentrate on your Rutgers experience. All you learned and, and how this reflected in where you are now. Positively, or it could be strengths or weak… Whatever you want to say.
232	00:55:47	Romina	I think I, I deal with groups. I work very well with groups. Um, I do some of my best work with other people so that's helped me because I'm assuming that in the long run, I'm going to have to be, I hope to be in some sort of leadership role where I'm going to have deal with people and delegate, and I do that very well. I think. And another, I just noticed this, and I was… I had an interview last, in January. It was my first real interview, and, um, and I was not qualified for the job. I was not in the age group; it's not in my major; it's not, but it's something I really wanted and, um, I have no trouble walking into a room full of people that are supposedly, like I mean, like, I've dealt with professors and I have no problem walking into a room and sitting down and just discussing things, and I don't get, I get nervous, because it was odd, someone new, but I didn't, I performed well, performed well under pressure especially when, like, when people are older.
233	00:56:40	Jeff	They take out cameras and they put them on you. (Laughter)
234	00:56:45	Romina	I mean it's pretty. Yeah. I don't, I don't get nervous and I ended up getting the job because I, in the, he even told me on the phone. He's like "Your interviewing skills," he's like "the way you presented yourself to me, I was very impressed with that," and he's like "and that's why you have the job 'cause you're not qualified for it." (Laughter)
235	00:57:01	Jeff	Nice.
236	00:57:03	T/R1	Can you tell us what that job is, Romina?
237	00:57:04	Romina	It's just an internship position at Saks Fifth Avenue, so.
238	00:57:09	Jeff	Folding cloths. A lot of folding, and…
239	00:57:10	Romina	Yeah, I know I'm not qualified.
240	00:57:11	T/R1	Big discount.
241	00:57:12	Jeff	That's what I'm saying. Big discount.
242	00:57:17	T/R4	How about with respect to the learning of mathematics that you, you've been taking at college. What things that…
243	00:57:21	T/R2	She's said that.
244	00:57:24	Romina	What have I, how have I …
245	00:57:25	T/R4	Applied your Rutgers experience in a…
246	00:57:27	Romina	I don't, I don't really have a chance to apply it much in college. Um, I mean, I'm right now I wish I did because I wish I could, this, like, I just have. In my college, I'm learning in a completely different way and I have to. I'm reteaching myself to learn from different, in different ways

			and hopefully now that I'm getting into higher-level classes it's not going to be as much like that, but I haven't taken, I've taken mostly lectures of hundreds of people so…
247	00:57:53	T/R1	So as you move up next year you'll be a junior. Your classes may be smaller. We may have to talk to you next year again.
248	00:57:59	Romina	Yeah. But I mean I've had, I have a couple of smaller classes this year and it's not… (shaking head) It's different. I think they have different goals then we have. I know my one professor wrote our book so you'd think if he's teaching it, he wrote the book, I'd understand what's going on. I took an exam last week, and I don't have any. They just have different… I mean he doesn't even show up to our exams and… He just has different goals for… He already, he's done with what he has to do and he just tries to relate information. If we get it we get it; if we don't, it's our fault.
249	00:58:37	T/R2	This is going back a bit, but you said something a while ago about, uh, your thinking process and the way that you learn things and, you know, how you're starting to learn the stuff that you have to learn now in a different way. Can you describe what that process is? I mean does it have stages that you can say "I do this and then I do this," or is it more vague than that.
250	00:58:53	Romina	(to Jeff) Am I the only one talking now?
251	00:58:54	Jeff	No, this is more directed towards you. Unfortunately, I'm not. It's alright.
252	00:58:58	T/R2	I was actually, but I'd like to hear what Jeff thinks about that as well if you have a similar thing.
253	00:58:59	Romina	I know you are…
254	00:59:00	Jeff	No, this is… We are talking about you right now.
255	00:59:02	Romina	No, I'm. Wait, I'm sorry. It was how I teach or something?
256	00:59:06	T/R2	No, you said, you said something about, um, something not corresponding, well not necessarily corresponding to your thought processes and the way you think and learn things, and I just wondered if you could describe what that is. You were talking about when you were working with your friend, I think, and that this is the way you went through things and then you would work with her back and forwards and it sounded like you had something quite specific in mind when you talked about the way you would go about learning something. I wondered if you do.
257	00:59:32	Romina	I, how I, I have a. Like, I'm not math. Let's say, uh, history. Like I always… I know we used to this together (referring to Jeff) we used to… some of us didn't take notes in class.
258	00:59:46		**End of Disk One**

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Line	Time	Speaker	Transcript
259	00:00:00	Romina	We used to call each other and we'd just discuss ideas and what happened and details and things. And that's how I learn – and it's a group setting. It's - I learn in groups. I don't know, it's just – I think this has a big part to do with it.
260	00:00:10	T/R2	Right.
261	00:00:12	Romina	'Cause that's how we always did it. And, um, now in college, I try to do that. Now that I'm getting – and I don't know anyone in my classes, so it's harder and I have to sit there and instead I have to memorize everything myself. And I don't talk to anyone. I just keep reading over and I just don't retain the information. As soon as I – if I do sit up all night, and try to memorize my sixty pages of notes and I write down what I know on the exam. As soon as I walk out of there it's done. I don't remember anything.
262	00:00:40	T/R2	Yeah.
263	00:00:42	Romina	It's over. Like I remember just for that and I don't retain any information.
264	00:00:47	T/R1	It sounds like me.
265	00:00:49	T/R2	Yeah, no one ever learns
266	00:00:52	Jeff	Yeah, but is that learning then if you don't remember anything when you walk out?
267	00:01:00	T/R2	Rote learning. Just memorizing.
268	00:01:02	T/R6	I have a crazy idea. I'm sorry - if you just
269	00:01:05	T/R1	Do we take Laraof's crazy idea next or do we take Ella?
270	00:01:13	T/R6	Just crazy – do you go into a closed room and just talk to yourself and try to explain yourself the problem?
271	00:01:19	Jeff	It's hard. But talking to yourself ain't as much fun as it is to talk to other people. You know?
272	00:01:24	T/R6	True, true.
273	00:01:25	Jeff	It's like, what am I doing?
274	00:01:26	T/R1	What kind of feedback do you get from yourself? You say, 'Oh yeah, yeah – no, I don't agree with you. Why are you thinking that? You know, I have another idea.' [Laughter]
275	00:01:34	Romina	I think a big way we learn is we tend to argue a lot. So that's how we get places because we argue. And then we have to take their argument into consideration. When it's just me, I don't have much to argue about with myself because I think I'm right. Jeff doesn't have the same ideas as me. [Laughter]
276	00:01:52	T/R8	Can't you say to yourself, 'Now what would Jeff say about this?'
277	00:01:55	Jeff	Yeah, that's all you need.

278	00:01:56	T/R8	And you probably know him well enough.
279	00:01:58	Romina	Yeah, he should have come to school with me but he chose not to.
280	00:01:59	Jeff	They didn't want me down there.
281	00:02:00	T/R4	Quick question, I wanted to share -
282	00:02:01	T/R1	No, Ella has a question.
283	00:02:05	T/R5	Just going back to what Dr. Maher said about taking this twelve-year study and thinking what would you change. I would not change anything at all. I think it very unique and very interesting and that's how it's supposed to be. Because we had different probably goals then the school but how I feel it – that Romina and I heard Jeff last time had exactly the same problems with math courses and the rest of the courses. Maybe the high school teachers have to look at they approach – it's very nice to teach students one way and it's a nice way. But maybe we're preparing students for the college and not to give them a hard time there. Maybe we should take a different approach a little bit or maybe to balance some different skills so they would not have should not have this hard time in the college. They're very bright and very smart. It should not be like that.
284	00:03:06	T/R3	Maybe it's the colleges that should change.
285	00:03:09	Jeff	That's what I say.
286	00:03:10	T/R1	Yes, especially if the professors really do value what Romina says they do value and I think they do value it in their own work and the way they think and reflect upon their productive contributions. But, somehow, the system moves a particular way and it doesn't—some things don't work well. In the system as it is, as students move on to more advanced level graduate work, I think the faculty are the first to say the students aren't prepared for that graduate work as they should be. And yet, they're the ones who have been preparing the students as undergraduates. The model is one of a factory model. It really is. It's questions of economics and cost—it's not always about learning, is it?
287	00:04:10	T/R2	There's also the question of whether people don't (inaudible) feel they do have control of other things and they do—it's a frustrating situation— Yes, from the other side, I would actually try to avoid teaching like that if I possibly could. But, I think often people find, the only experience they have is the way they were taught. So it cycles round and round. So what happens is even if you have someone who really cares and really tries—wants to do well—but often what they think that means is giving what to them is a really clear explanation and the result is that, you know, to some of our students isn't, of course. So then, you set a test and lots of people do very badly. And this is a very--if you have put a lot of effort in, you know, to your mind you have put a lot of effort in, it's a really nasty experience to feel that you have completely failed a lot of these students. So, people end up feeling really quite negative about this whole thing and then don't try to change things in different ways. People don't see the

			options.
288	00:05:11	Romina	I think it works the same way the other way around. In the beginning of my first year, I put a lot of effort into some of my classes, and you get your grades back and you're graded on a curve. So about this many people (puts fingers together) get a good grade and the rest of us all get B minuses. So it really doesn't matter. It's a huge range —I give up. That's horrible to say, but, after a while, I just gave up. It just wasn't worth my time.
289	00:05:31	T/R2	Did your grades get worse when that happened?
290	00:05:33	Romina	No, they were still a B minus. The range was so big that it really doesn't matter.
291	00:05:41	T/R3	Do a lot of places do that? This may be off the subject, but I can't imagine grading on a curve. Do you mean that it has to be the same number of A's and failures and so on. And most of them are C's.
292	00:05:52	T/R1	Especially when you are from such a selective institution…
293	00:05:57	T/R3	To me, you have goals and if the student learns what you want them to learn,
294	00:06:00	Romina	We argued that, and they did that for us so you don't have grade inflation. Because, according to the people that are going to my school, if we weren't graded on a curve, then we would all probably do in the A, B plus range.
295	00:06:16	T/R3	Well, if you mastered what they wanted you to master…
296	00:06:19	Romina	They don't want that. They don't want grade inflation.
297	00:06:20	T/R7	The argument for that, Liz, You're talking about criterion-based learning, which, I think, is the only learning there is. But, other people say we're talking about normative, so we just gotta judge you compared to other people. Right, so I think it's criminal. But, what can I say? John was going to ask a question.
298	00:06:35	T/R3	As a teacher that doesn't make any sense…
299	00:06:44	T/R4	Just a related experience - I work with students now who work with physics. This is just to share an experience…to find what you think. And sometimes we meet with teachers; these are TA's. (You must be from England).And one of the things…They said once; I was sitting there. They were frustrated. And one of the TA's said, he could not understand… How can I put this? The complaints that students don't know the algebra. My understanding was, that for this TA, there was physics and there was algebra. And the students weren't doing well in physics because they don't know algebra. They felt it was not their job to teach algebra to these students. The closest situation I can see between what we did here - I may be wrong - is that algebra, symbol manipulation or close to that and, on the other hand, you play with towers. (Romina smiles and raises eyebrows).You did things; you did some concept building there. So, do you also see these tools this way? We have algebra

			and doing towers; or we have algebra and physics…have any of you taken physics?
300	00:08:20	Romina	No way. (shakes head to indicate “No”) Others say No.
301	00:08:24	others	I will not take physics. Tried it in high school.
302	00:08:25	Romina	Our high school teacher had to teach us the algebra and I remember he taught us derivatives and how to manipulate. He had to teach us that before he taught us (inaudible). Because we didn’t know how to do that.
303	00:08:37	T/R1	That came before the calculus.
304	00:08:41	Jeff	Yeah, we were there; we just got there before they got to that part…
305	00:08:41	T/R1	Very difficult…I learned it in physics before I learned it in calculus. I had physics here.
306	00:08:47	Jeff	Physics teachers get nasty; they really don’t like it that you don’t know algebra. It’s not really algebra; it’s more advanced algebra.
307	00:08:58	Romina	It was derivatives. And he didn’t teach us what derivatives was; he said, “Listen, you take this number and you bring it down here (motions with hands) and you subtract one, and that’s it. He sent a couple of notes to our math teacher; he wasn’t happy.
308	00:09:05	Jeff	Yeah
309	00:09:11	T/R3	That’s another issue you see in schools. You just reminded me. What John was saying reminded me. The college teachers blame the high school teachers for not preparing the students; who blame the middle school teachers for not preparing who blame the elementary school teachers for not preparing or the parents. They all say it’s the people before me who prepared the students badly. All the way up through the whole system.
310	00:09:22	T/R1	Who blame the parents
311	00:09:32	T/R2	…John’s question and it was an interesting one. I think he trying to say, and correct me if I’m wrong, do you see a difference between, or do you see links between, the kind of stuff you did with the Rutgers people—all this reasoning stuff—and the written version? Which is kind of what Carolyn was trying to get you towards at the end.
312	00:09:51	Jeff	I just think that we didn’t get to that point to very—to the very end of us being together. So a lot of it to us seemed like just reasoning for a long time. And I think we didn’t start to make the connection between what we were doing and what it really was until very, very late in what we did.
313	00:10:10	Romina	The time we did that you have to think that it was the spring of our junior year so we had the summer and the senior year and that was it. And then we were off to college.
314	00:10:17	T/R1	This may be a good time to show another clip.
315	00:10:23	T/R7	May I just ask a very little question? I’d like to ask it of Jeff because (inaudible). This is actually about Romina (laughter). I tried to write down something here that Romina said; it was something about rote memorizing stuff for exams that you forgot the night after. You just memorized it. I’m

			trying to reconcile that with your feelings about getting back to basics and really understanding things. Because, it seems to me, that's part of what you believe in about learning. You need to get to the bottom of things, understand them properly; they'll stick with you forever. So why was it that you just rote memorized this stuff?
316	00:10:58	Romina	Cause I don't have time. I have 4 other classes other than that one that I have to…and it's not like that's all I have to do. And they give you so much work for that one class. So you just have to get through as much as you can as fast as you can.
317	00:11:14	T/R7	Was it less intrinsically interesting?
318	00:11:19	Romina	Yeah, it was horrible. I can say I don't really enjoy any of my classes.
319	00:11:23	Jeff	That's not good. That's a shame.
320	00:11:25	T/R7	I'm just thinking that most of us don't like to spend a lot of time trying to understand things we don't like. We try to avoid that.
321	00:11:32	Jeff	And, you know, a lot of the stuff that you don't—that you do understand, that's the stuff you like though, because you do understand it. There's some stuff that naturally clicks for people, you know. You just get things. That's the stuff you try to stick with because, hey, I could do things really easily like this and it doesn't take that much effort and I get it. Some things you don't get and it takes a lot of effort and why even waste my time doing all that?
322	00:11:57	T/R7	<inaudible>
323	00:11:58	Jeff	Exactly.
324	00:11:58	T/R8	Before you decided to go to the university, did you visit classes? Oh, you didn't. you didn't know what you were getting into.
325	00:12:09	Jeff	I didn't visit any of the schools I applied to.
326	00:12:10	Romina	Let's just say that I decided I visited…I wasn't very excited about college so I wasn't putting a fair amount of time in. So I visited about 1 week before I had to have my final decision in. I went in and sat down in a summer session and I had an eye infection…I had pink eye. I wasn't feeling well. I didn't even go on a tour. And then I left.
327	00:12:38	Jeff	I think at the end of the year when you're in high school and it's at the end of your high school career, you don't really care where you're going next. You're just like, I'm trying to get out of here. I'll go somewhere. And this place sounds good. And people have nice things to say about it. But, you know, I don't—I didn't visit anywhere. I just applied to a couple of places that sounded alright. I don't know.
328	00:12:57	Romina	I think my decision was made for me.
329	00:12:58	Jeff	How could you go somewhere else? You're going to go Ivy League? What are you going to do?
330	00:13:00	Romina	Yeah, yeah.
331	00:13:09	T/R3	Even if you do do all the things you're supposed to do, it's hard to know before you get there what it's going to be like.

332	00:13:11	T/R1	And it might not be very different from a lot of these places.
333	00:13:16	T/R8	No
334	00:13:29	T/R1	A matter of more or less. I want to speak to Jeff's last comment about getting to the maybe more symbolic representations—some of the more general forms of things which is helpful if one gets there, there's no question, especially if you build it up. But I thought we would take a look, not every one in this room has seen it. Some of you have seen this clip several times. But I always enjoy seeing it. Jeff is in it, not Michael, or Ankur,or Brian. It's one we call, "The Gang of Four." It was…I'll tell you a little bit of the background of the clip. The reason we chose to put these four students together…were fourth grade students at the time…is because they all had very different ideas before they went into the session about the way they were thinking about the particular…their solutions to the problem. And so we were, we wondered what would happen…in fact, Milin is one of them and Milin is trying to interview Alice and me for us to tell him how Stephanie was thinking about it at that time because you saw us interview her and so forth. So, it was this kind of curiosity how our…Alice said to Milin at one point, "I'm not going to tell you; why don't you ask her?" So we came up with this idea of having what we called a group assessment; we had never really done that before. We had never really pulled, and we had just gotten funded for this grant from the National Science Foundation. We had just worked; you had all been working on building towers five tall. And I think you, Romina, and Brian were not even in the same class. You were in a different section and you and Brian were partners in the building tower task. And I'm sure you must have seen clips on this which we have.
335	00:15:20	Romina	Yeah (inaudible comments and laughter). But I'm only in 4^{th} grade.
336	00:15:30	T/R1	You're only in fourth grade; we're not going to show you. We're going to show the Gang of 4. And I'll tell you what I want you all to look at with me for this, and it's the fact that you guys—all of you—were working with these blocks and a lot of your arguments that you were giving came from the structures you were building. You would put them down flat. We're going to show a piece of Jeff working with Michelle later if we have a couple of minutes. But, you would put them down and Jeff talked the last time about the importance of finding patterns—you were doing that too—you were all finding patterns. But, in this little clip, which was a little sort of interesting communication particularly between Stephanie and Jeff, where Jeff is questioning Stephanie and Stephanie is saying, "Well, that's how I do it." "The way you do it, is the way I do it." But what's more important, is that there are no blocks. What's being done here is all being done with symbols. And you're actually—and notations—and no one ever taught you about a notation and no one ever told you not to use blocks or to use them or to argue this way. If you even watch, and you may not have seen this in the earlier ones where you look at the way Milin is thinking. I just want you to look at this—I want you to

			think that there was no teacher here—no intervener to say, This is how you should represent it; this is how you should think about it. And I also want all of you to pay attention to the researchers here and what we do—what is the nature of intervention. Because, I think, this one is a little bit like The Night Session in that there are certain things the researcher was wanting to see come on the table. But I'd like you to look at this and this is a long one and it's one I think you'll enjoy especially since you're much older. I don't know, Jeff, if you'll enjoy it still.
337	00:17:26	Jeff	Yeah, I know; it's rough. (laughter)
338	00:17:34	T/R1	Jeff, I just showed a piece of you working with Michelle, and who is it that said, "That's Jeff?"
339	00:17:40	Romina	It was me; sorry.
340	00:17:42	Jeff	Those were rough years. (laughter) They were rough.
341	00:17:43	T/R1	But Jeff never looked the same to me.
342	00:17:45	T/R2	I've noticed this.
343	00:17:49	T/R1	Your physical appearance is different.
344	00:17:52	Jeff	I stayed big for awhile. (Insert Romina: I stayed the same.)
345	00:17:55	T/R1	Romina looks like Romina all the way through. But Jeff doesn't. You have a different look at different years.
346	00:18:02	Jeff	I tried to grow a little bit.
347	00:18:03	T/R1	You know what, Jeff? You could just deny it. Nobody would know.
348			(Inaudible)
349	00:18:09	T/R1	The Gang of Four
350			Film (PUPMath film clip)
351	00:32:02	T/R1	OK, you can stop this. Do you know the music for this series follows a particular pattern? Did you know that?
352	00:32:12	Jeff	No, I didn't know that.
353	00:32:13	T/R1	Did you all know that?
354	00:32:19	T/R1	Fibonacci—there's little acknowledgment in the back. They made the music to follow that pattern. The Harvard group.
355	00:32:28	Jeff	Must be the Harvard people that would do something like that.
356	00:32:32	T/R3	Carolyn, would it be okay to take a little break? He needs to get a new tape.
357	00:32:33	T/R1	That's a good reason. We might want to get some food. And, I guess what I'm…I want you to look back, you're the adults looking at these younger folks. And look at the reasoning…the symbolic, the power, the way they question each other. (inaudible) hear about that. Sounds like your college classes, right? From all of you. So, okay, let's take a break.
358	00:33:03	T/R3	Five minutes, ten?

359	00:33:04	T/R1	I'll say five; it'll be ten.
360			**END OF DISK TWO**

DISK __3__ of 3

Line	**Time**	**Speaker**	**Transcript**
361	00:15	T/R1	So I guess my question for this episode tape is what is going on here in terms of students expressing their ideas, their thoughts and what was the role of teacher-researcher, namely me? Did you look at that at all?
362	00:45	Jeff	Well I think you definitely needed to be there for us at that age we were just we were really just yelling out at each other then, I mean everyone wanted their point to to be it and the loudest one person kind of wins at that age.
363	00:57	T/R1	Ok, so uh keeping keeping, giving everyone a chance to talk?
364	1:00	Jeff	Yea Yea and the - I don't know. What do you think?
365	1:07	Romina	I agree
366	1:10	T/R1	Let's for the record that Romina agrees. Ok but what about what was coming out of the discussion um in terms of the reasoning about the building five- tall towers. That was not a trivial task, right?
367	1:25	Jeff	No
368	1:26	Romina	I think with any group like that it's hard to, um, it's hard to have like organizational skills and reasoning skills and I think you were just helping us along with that.
369	1:33	Jeff	I mean I could even see the point of the pattern there I mean that's that's essentially what we used to prove everything we ever proved to you. I mean I couldn't of even comprehend even why we want to use them why we would waste our time trying to find a pattern.
370	1:45	T/R1	So you really didn't understand why we wanted to find a pattern?
371	1:49	Jeff	I'm assuming but videotape.
372	1:51	T/R1	You're guessing now looking at it -
373	1:53	Jeff	Yea yea and I mean and then that's was really the basis of a lot of the stuff that we did and then starting with and we started with figuring out how to do two and adding just knowing that there was just the two on top I mean that's
374	2:05	T/R1	That that was Milin.
375	2:08	Jeff	Yeah but that was that was what we are doing now I mean that's how we do everything that is the beginning of us learning how to do what we do now.
376	2:18	T/R1	I was very impressed at that session with the forms of reasoning that came out from nine year olds, maybe nine becoming ten year olds, it's pretty

			young right and all you had done was played with the blocks, long class session, a little bit the year before, and uh in grade three, four-tall towers and here you had one three-tall and really it's three forms of proof emerged from that conversation. It was pretty impressive. Don't you think?
377	2:59	Romina	I I was telling outside I I'm working with kids that age now and younger and a little bit older actually um they range from about 6 to 13 and I I never thought that what Jeff and them was doing was that impressive. I thought it was normal because we were all doing it and looking back on it now I think it is because I don't think the kids that I work with would be able to understand that at all. I tried to like grasp I just tried to show them like we tried to sit there and talk problems out and they're not capable of doing it, they can't, they don't understand me. In math it is one of the areas that they have the most trouble with, I and I don't know how to get through to them.
378	3:38	T/R1	What kind of math?
379	3:39	Romina	You know, fractions, it's absolutely impossible with them
380	3:44	T/R1	Aren't you all shocked?
381	3:47	Romina	I mean and I draw pictures and I separate oranges and they just they can't can't get it.
382	3:53	Jeff	Why Why is that hard to Why can't some people get that? Do we know? Does anybody know? [Romina's cell phone rings and she steps away from the table]
383	4:00	T/R1	Actually there are a couple of dissertations, one of them was done by um Elena that was done last year and another was done by Sylvia. There is a lot of research done on learning fractions and there are a lot of theories, um, and that's not an easy question to answer in a brief amount of time but to give you the short abbreviated answer in my opinion which some people may not agree with [Romina steps back to the table] is that you have to build up the ideas before you get the formulas and notation so that the idea is there and notation can be applied to the idea. And, and this is not an easy transition for young kids who probably know about fraction names and parts. They could learn what's one half. They can see lots of different representations of a half or third or quarter and they could think of half of objects, a third of objects a quarter but that transition to think of one half in general and think of that number that goes across many representations is not a trivial move in the development of kids. So that takes work. That movement we call fraction as an operator of something, one half of to number, the abstraction ½ and I don't think in teaching that's thought about, the difficulty of that movement is thought about or the opportunity of for students to engage in right the right types of investigations and activities so they can own these ideas themselves. I did a year long teaching experiment with fourth graders in another school, they would be the same age, they'd be a year younger than you I guess they would have been third graders when you were fourth graders so you

			were fifth graders when they were fourth graders. And the reason why I did it, by own personal motivation for doing this was because I was not happy with your knowledge of fractions because you were you know you were told a lot you were given a lot I didn't think you built it so it was my little experiment where we had 50 sessions. This is still being analyzed these sessions and I think it is possible but not with the way curriculum is currently organized. They threw it into fourth grade curriculum as problem solving because the kids were not expected to learn fractions until 5th grade so we got away with it and so they were able to build these ideas and build the notion of equivalence and really extend theses ideas of fractions, generalize these ideas and did some beautiful beautiful mathematics as fourth graders. So I think the kids are very willing to do this but under what conditions how much time you know what I am saying um what's accepted you know they have to like your task it seems to me Romina. They have to achieve a certain day and know it by then it's artificial already but if they are working on the problem until the meaning is theirs they own it they um I think they are more willing to engage, their more willing to think, there is not a threat you know what I am saying there is not that anxiety I may not be able to this, it is not making sense to me you know what I am saying it's complicated.
384	7:34	Jeff	But then there is a line there before where if there is isn't an exam though. I mean we might be able to sit down and do things but there is a point when yourself come out of and produce stuff.
385	7:45	T/R1	Exactly
386	7:46	Jeff	And I think
387	7:46	T/R1	That's right
388	7:47	Jeff	That's a very fine line
389	7:48	T/R1	You need both you need both but it really means the whole change in curriculum and the time that is allotted for things. I won't argue that you don't need both. Um unfortunately it seems to be one or the other you know. You get some movements that it is a lot of this free exploration and play but it does not get to a form that becomes more more of the what is accepted in the community that is the form to build on and in others its all the formalism without the meaning behind it or not enough of one or the other you know and you have to sort of think about about it how much is so individual. You are making great demands on schools and teachers for all students to build. You have to be willing to give some of it away. That is why the small groups are nice. In other words, you can't engage, you just can't engage with all the students in an equal way but students can engage with each other and you have to use that tool and learn how to use that you you in the study learned to use that pretty well. The engagement was sort of there too after a while because you just knew what to do. I don't think, I don't think that should be taken for granted though. You told me some stories when you were in sophomore year and you were a year ahead of some of the students in Mr. Pantozzi's class. Right, they

			were a year ahead of you. You’re behind them.
390	09:36	Jeff	Well there was some seniors and juniors. It was me, you, and Mike.
391	09:38	Romina	No
392	09:41	T/R1	It was just a few of you and there was a lot of students
393	09:43	Romina	They were older than us. We were ahead of them
394	09:46	T/R1	Exactly. But the point is that. I remember your telling me that Why don’t they talk to each other? Why don’t they go up and there was sort of a habit set for them where the belief that they couldn’t. I don’t know or a mind set. I don’t know how you would interpret it. I remember in those days you would sort of comment that you would try to go up but they didn’t join that. So it wasn’t a part of their culture.
395	10:12	Romina	We got moved outside. We did. We had three desks outside the door, didn’t we? In the hallway. That is what we had, what we taught ourselves outside in the hallway.
396	10:22	T/R1	I didn’t know that but that’s very interesting
397	10:23	Romina	We were very loud
398	10:25	T/R1	So its almost like These are good questions. There are really hard questions. They’re complicated. How do you begin to change you know that way of working? You know I think its possible. One of our students Barbara Glass, did a wonderful study of college students where she spent probably a third of her semester, that’s not a long time, but trying to change the culture of their working where they would really buy into it and then she started to introduce some of these problems and her goal to bring it to much more formal object ways and to get some other problems that were rich and extensions. And she did this over a few semesters and had gotten very very good results from these college students and she went to do the study by the way as to show it was impossible so late to do what you guys did.
399	11:28	T/R2	And she was wrong.
400	11:28	T/R1	She was pleasantly surprised, not to 100% but she had gotten really amazingly good results and far better than she had expected. She learned a lot. How many of you were at Barbara’s defense? Would you say that was a fair way what I am saying.
401	11:44	T/R3	It was very impressive.
402	11:45	T/R1	It was very impressive.
403	11:45	T/R3	And surprised.
404	11:45	T/R1	Yes . And And I wasn’t surprised. I believe and this is my comments to you and I am not really down on mathematicians. I don’t think they try hard enough if they believe how you get them to I think they are capable of doing lots and lots more in their classroom. Its not. No one is ever suggesting that a research situation can become a classroom condition. But you learn from research to see what you can do in your classroom

			situation to improve it and and the things you value that believe are important and I do believe they value them, their worth
405	12:12	T/R2	Yea, What I was going to say about that which more not directed to you guys sorry but the people in the room, the enormous amount of excitement in this group in how wonderful it is see these kids so young, you and the others, you know doing this kind of reasoning that you simply wouldn't expect even a college student necessarily to be able to engage in to be used to engaging and this is clearly so much better than having people memorized things without understanding you everyone feels this quite strongly but you have to remember is that there are official form of these things for a reason and the reason is that so everyone across the world can communicate with each other in these mathematical terms so that you are not just having you know to be able to invent and come to a fantastic thing or eventually you can bring about what kind of notation we are going to use and all those things cause we can't contribute and the whole subject can't grow because you need a basis everyone can agree about on this is what we mean on these symbols for the other people around the world to be able to do this and I can see why the focus can comes learning these certain words and symbols because the edges are disappearing faster and faster more and more work is being done in order to get to a stage where you're going to be able to contribute if you go into any of that kind of stuff, you need to know a lot of this stuff and I can see why people say you know need know this is basics in order to be able to do these things so I can see I I don't think its right to swing so far in that direction but I can see why it does because it is so important to the subject as a whole to have this solid so a lot of people can all know. So
406	14:00	T/R4	I'm not sure it's a lot that goes on in classroom that attends to communicate in that sort of universal way
407	14:0	T/R2	No no not in an overt way but sort of that is in the background. That is sort of what drives it at some level. I mean I think that's what drives from higher in the curriculum and that kind of filters down into what everyone believes is important some how
408	14:23	T/R4	And I think there is a difference between what we want to know and how we come to know it
409	14:30	T/R2	Yeah but this is obviously glossed over
410	14:36	T/R4	Um so on the other hand when I think about it is that a lot of what we do and what they do it's also giving meaning to what counts as meaning in some sense. I think if you become a mathematics professor. The university is calling an idea of what meaning is. If you tell him that you are doing these things without meaning he is very likely going to be shocked.
411	15:06	T/R2	No No this has meaning
412	15:08	T/R4	What I'm talking about. This becomes an issue of contention.
413	15:11	T/R2	Meaning for whom. How you come to construct the meaning just becomes an area of contention I think. My experience has been these

			people their belief on what they are doing
414	15:21	T/R4	Absolutely. They believe they are conveying that meaning
415	15:25	T/R2	They have a different idea of what counts as meaning and how you come to know these meanings and this is what it is. It's a It says if a whole different mathematics they're doing.
416	15:34	T/R1	I don't I don't know I don't think they thought much about how you come to know these meanings. One reason might be they seem them pretty easily and quickly themselves because they are so brilliant and they are not representative of the world and so.
417	15:51	T/R2	I think we probably all agree on that one.
418	15:52	T/R1	So So they don't really have to they just think it's there you know it's they see it and um and I think it's hard to expect them to think that way but I think as as educators and those responsible for the whole learning of the vast majority of the people you know all these few outliers are not wholly responsible for working with. We are doing a pretty bad job and um and we keep going to them for guidance because that is not their job to think about it. Their job is to invent more mathematics, to be creative, and contribute to that pool and it's our job to to help our citizens get as deep an understanding as they can so they can whatever they do in the world solve problems that haven't been solved before which they don't do in isolation. They do in groups. They do in collaboration and they have never known a company to say I'm going to hire you Liz to solve the problems you studied in your books and you're not allowed to talk to anybody and you got to do it in 5 minutes.
419	16:57	T/R3	And your not allowed to use references.
420	16:58	T/R1	Not allowed to used references. I don't think there is any job like that at all and what they look for is to get good thinkers and good reasoners is what they pay the money for. Talking about out of academics, I'm talking about in companies to bring the right teams together, to solve the new problems that have never been solved before and at least to come to some acceptable solution for now with the tools and the best brain power we had until we come up you know advanced more so in a sense I think what the expectation has been doesn't match what the needs are for some out into the world in the sense we do them big disservice by putting them through those courses. It is almost like um um it's almost like an obstacle course to get through that doesn't make necessarily success in those other ways and and I'm not. I applaud the interviewer for Romina for seeing that. That probably is a person who hires good people, can see beyond the person who could do exactly what they are told to do and do it you know the way it is expected to be done so you I think those are bigger issues. We all grapple with them. We all still have to go through the rite of passage until once you get your degree. It is not going to matter if they are being As and B minues you know especially if I'm going to an ivy league school and I wouldn't worry much about it if you think you are making good choices. You didn't ask me opinion but that's fine.

421	18:32	T/R4	I would say about the tape, I would think the problem and the way I, um just going back to what we just saw. I think it is interesting that you the question was asking how many towers you can build. If you in a context of the mathematicians they are more likely to ask how many towers are there.
422	18:52	T/R2	Yes
423	18:54	T/R4	When you ask how many towers you build you make a epistemological problem in some sense or to whom. Romina can build eight given a certain way of thinking about it but if you say how many towers are there, you you there's a tendency to say we say what is the answer it is. I mean for me I see those questions different questions.
424	19:19	T/R2	Oh, I agree. The phrasing is very different
425	19:20	T/R4	I don't know what you think. What do you think Romina, Jeff? Is there a difference in saying how many towers can you build and how many towers are there of two choosing from three?
426	19:33	Jeff	I think the first questions makes it more of you know you get more involved than you start you know it becomes a personal thing rather than someone already built towers and knows the answers and you know it is kind of you. Yea, I think there is a lot. A big difference there. I don't know how much I can really say about that but I think that that it makes it much more personal when you ask like how many can you build rather than how many there are.
427	19:54	T/R4	And I remember reading Romina's interviews, uh, you may not remember this. At some point you said "Everything has to have Romina's definition of it." Do you remember saying that though? I have a transcript but I'm not you were explaining it and talking at some point "everything has to had to have Romina's definition of it." I mean in your mathematics that you're doing.
428	20:14	Romina	I don't remember saying that. I mean I understand it cause everything has to make sense in my terms other than I can't like I someone else might have done it already in a book but I just don't understand it unless I do try it myself and put it in my own terms.
429	10:32	T/R2	Can I ask is there anything. I am trying to phrase this right. Sometimes I say things that don't come out right.
430	20:40	T/R1	You can go a second time.
431	20:42	T/R2	And I talk funny anyways as Carolyn is fond of pointing out. So things that you are trying to learn which are now in a textbook. This is a new way for you to have to learn this stuff . Can you think of? Has there been an example of anything that you have successfully got to grips with in that way. And can you describe that too? Maybe not the detail but do you remember that kind of experience? I just wonder if there is anything that is more accessible that way where some things are less accessible in that way?
432	21:15	Romina	Like that I have learned successfully through a textbook? I can't think of anything right now.

433	21:23	T/R2	Ok.
434	21:23	T/R1	How about you, Lara?
435	21:24	T/R2	Me I had very similar experiences to Romina I should say. I had extremely unusual A level experience. Oh god now I am going to have to explain.
436	21:35	T/R1	You need to explain that.
437	21:35	T/R2	Sixteen to eighteen you do in England you, um, well when I did it most people did 3 subjects so we specialize as much as we need to early. I did four which is not uncommon for people to go on to top universities but I had a math teacher who took interest in me in school and used to ask me sort of interesting questions on the sides, give me interesting stuff to do and I wanted to math and theory of math there are two separate ones and this did not normally happen in my school. It was a small old school by the time it got to that stage you know it was not generally that academic, not many people did it and he said that he would teach me this course if I wanted to do it during lunch times and stuff and I would go and see him so I basically invented A-level maths. I don't remember one single time. No, I do. That's a total lie. Wow, I do remember exactly one single where he stopped me and said this is how you do this. The rest of the time he asked me questions and I sent me all exam questions. I would go see him in the staff room when I would get stuck and I'd say 'Sir, I'm stuck.' And he would say 'go away and think about it some more Alcock.' And a lot of the time I would then be able to do it. So this was not working in a group with other kids cause there was no one else doing that in my school but it was working very much with him. I would ask questions and he would ask me questions and then I arrived at university and every single class I took first year was a math class and there was 200 of us in a big blank lecture theater being talked at it was the kind of stuff you do in advanced calc and the proof courses because everyone specializes early. I started and this was a complete shock to me and I thought I was going to fail for quite a long time because I just I had no idea how to learn in this environment. You know I invented all this math previously basically and I have no idea what to do when someone just told me all this stuff and it was a huge shock.
438	23:22	T/R1	So clearly you survived and did very well.
439	23:23	T/R2	I did
440	23:23	T/R1	well set
441	25:25	T/R2	I happen to get a couple of good teachers in the smaller groups that we did work with
442	23:30	T/R1	The smaller groups
443	23:31	T/R2	Well we would have a group of 4 of us and we happen to get
444	23:33	T/R1	Ah great

445	23:33	T/R2	good, well, no no no to work with once a week in that group for an hour and go over things we were stuck with from the lectures and things and I happen to half way through our first year to get someone who was very good in math actually. I think she turned it round for me.
446	23:47	T/R1	That's great.
447	23:49	T/R2	But it was a lot of work. So I know what you mean about that experience and the extremely frustrating conditions I went through feeling like this was mine. I could do this right to feeling like I had no clue what was going on but yea then I got a first class.
448	24:08	T/R1	Interesting.
449	24:08	T/R2	But I remember a similar experience. But then that you see is very much lecture just notes and we really didn't have textbooks the same way people do in this country. It is not like everyone in the course has to buy this textbook and the course really runs according to this textbook. It is much more a model of just a lecture giving notes and asking questions based on notes and situations.
450	24:28	T/R1	By the way, Romina if you really are working with these kids and you want to have some interesting investigations with fractions, you should come back.
451	24:39	Romina	I can't my kids to get fractions. They won't do it.
452	24:42	T/R1	They would know they were doing fraction.
453	24:44	Romina	Ok
454	24:44	T/R1	But when their all done. The whole idea is not to let them know. I mean did we tell you you were doing proofs in fourth grade and doing reasoning by induction? Did we tell you that you would be doing algebra in fifth grade? We didn't tell you that until after you were doing. The trick is to get them to do without knowing their doing it and later on saying by the way. You can't forget the by the way. It should be more by the way.
455	25:09	T/R2	(to T/R1) But us um, as Alice said, that is not what you were doing. You weren't trying to be the best teacher you can be, you were trying to do research on what happened in this very other extreme situation.
456	25:18	T/R1	But there were implications for this. There are things you could do and you learned a lot so I do believe they can. I really do. We we have the same situation even with adults who say they can't learn algebra. We have a course called 0
457	25:33	T/R2	Half the country, half the world thinks that can't do algebra.
458	25:35	T/R1	But I'm talking about students at Rutgers who can graduate except they can't get through algebra. We have created a course called Mission Algebra which is um Bob Davis's creation and other people have taught it over the years including Elena, several other people but there's pieces of this course which is very much exactly like the things you did and there's pieces of it which are very traditional so it's kind of a mixture. It's sort of helping people get more confidence. I imagine I could guess. I could be way off but I imagine these kids don't have a lot of confidence and

			probably do not have a lot of good affect when they think about things in math. I bet you when you say things like fractions it conjures up a lot of failure and those are not good emotions you know it doesn't make you feel good about it. It's complicated you know I don't. I still think it's possible to do a lot more than we think we can you need some different conditions. You need time. You need sometimes to not to let them think they are doing math. Of the teachers at Kenilworth told us when we used to come in that you never as a group, when we were in the early years, ever said you were doing math. You were doing Rutgers.
459	26:59	Romina	I think we still say that.
460	27:00	T/R1	You made a distinction that only started to look later on you know more like some of the things that were going on though so maybe that was a piece of it. If you start from the beginning it doesn't have to be a piece of it you know in math it's too bad that is the case but so much of the elements that you described in college were very characteristic of I think lot of kids schooling all the way from the time they enter. Little kids hating math early, having to memorize a lot and having to give it back fast and quick without error. You know stories of parents saying how kids are literally ill going to school and you know being stressed, feeling that their not up to it. None of that's good. I mean, there are certain conditions that need to be in place and it all I believe has to be fun. You mean you have to enjoy it you know if that's being with people you like to be, fooling around a little bit and teasing each other. That's part of it. I've never seen any adults work in anything together when they don't do that and they and this talk about always being on task. I've never seen adults always being on task at every level, have you? I mean
461	28:16	T/R3	We are always on task right here. You know what I mean.
462	28:22	T/R1	So it's complicated. Does anybody else have any other questions for Romina and Jeff?
463	28:27	T/R2	I was wondering whether, did we get to your, do we feel that we answered your original question I've now forgotten what it was, Carolyn.
464	28:34	T/R1	I don't know. I guess I can email them if I have any more. Can I email you two?
465	28:37	Romina	Yes
466	28:38	T/R1	Can I email you any more questions?
467	28:39	Jeff	Certainly
468	28:40	T/R1	Um, I might want to do that. I'm going to have to pull away from a little bit and reflect. I think. I think I've heard. Do you have anything to ask me? Anybody? Oh that's so good.
469	28:57	T/R2	Ok I got a question then. We've asked Jeff this a lot. Um that was a little disturbing.
470	29:04	Jeff	I'm sorry
471	29:05	T/R2	I'm losing my train now clearly. Um, ok, now this is a strange question but um this relates to another one. Why are you here today Romina?

472	29:17	Romina	Yea I don’t know. No I I
473	29:21	Jeff	A little bit of pressure.
474	29:21	Romina	I think today was a I don’t know. It just I worked a lot on this for so many years and know I enjoy it because now I don’t have the pressure of doing math so now I mean I enjoy coming here talking with you about it. And I feel that this is now an accomplishment of mine. I never viewed it like that before until I went away to college and I got shot down in every other area. This is the one thing that makes me feel sometimes alright about myself. No so I enjoy talking about it with people.
475	29:50	T/R2	Good. And is that similar to your thoughts Jeff about the fact that you guys came out of school all those times did this kind of thing. Did you ever resent it or did you want to go?
476	30:02	Jeff	I don’t know if we ever really wanted to go. If you think about it after school I can think of a lot of things you would rather be doing than math.
477	30:10	Romina	And it was kind of stressful sometimes so of course we
478	30:13	Jeff	Yea
479	30:13	Romina	And we knew it was going to get like that once we got there.
480	30:15	Jeff	But you know we started out and we started doing it we did it in school and to get out of school to do something kind of neat was always a great thing. I mean to get out of school to go anything even if it more school it makes you think like you’re not in school, you know.
481	30:28	T/R2	I see what you’re saying
482	30:30	Jeff	And then you know and then week after week you can’t just quit you know now that we now that we have to give some of our time doesn’t mean we can be like no, we’re sorry we don’t want to be involved anymore and it’s interesting I mean apparently we must have been doing something good if they kept coming back you know so we felt like alright you know we’re kind of you know special or something
483	30:49	Romina	We owed it to them.
484	30:49	Jeff	Yea and you guys spent a lot of time with us you know it was the least we could do
485	30:52	Romina	We were the unique, the 12 year research, we couldn’t after ten go
486	30:57	Jeff & Romina	Ehhh
487	30:59	Jeff	Call it the 10-year study
488	31:05	T/R1	14 now. That’s you were unique. You were special. All of that is true.
489	31:11	T/R3	Still are
490	31:12	T/R1	And we still enjoyed it very much and of all the things we all had to do when we were with you always it was a lift. You know how you talk about getting out of class to do, that’s how we felt.
491	31:24	Jeff	Yea Yea

492	31:25	T/R1	Coming to work with you, getting out of
493	31:28	Romina	We entertained you a little bit.
494	31:30	T/R2	When I was doing my interviews for my Ph.D research, I used to come out that I just had this nice interesting chat with these two people who just arrived at the university you know and I sat there and I listened and I made them do some math that may not liked too much but you know it was cool and I came out of this and thought this is work. You know I was sitting here with these nice chats. It was great! So I know Caroyln thinks that. Yea I had a similar. Did I tell you I had similar thing with at least one of my pairs of people. You know I would make them do some advanced calc stuff per se and give them questions they may not get the answer but they would work together and maybe or maybe not you know really get somewhere and earlier. And one pair would really thank me when they walked out the door and I always thought this was hilarious. I was like oh thank you. And it was just that we would have these nice chats I think.
495	32:17	T/R1	I don't really think that people really get listened to enough
496	32:22	T/R2	I think they recognized on some level that it did help them with the math as well to sit and talk about it without someone saying you know let's do this. At the time I was like no, thank you. You just given up your time.
497	32:33	Romina	It just seems like so many times they were impressed by what we would do and we would just sit there and be like we are doing anything of any value.
498	32:42	T/R1	Can't you know what impressed us?
499	32:43	Romina	Well, when I was sitting there I was like I had no idea.
500	32:44	T/R1	Right but now that you look. Is it impressive?
501	32:47	Romina	I think it is.
502	32:50	T/R1	Jeff, really?
503	32:51	Jeff	Yeah, I mean we really came a long way
504	32:54	T/R1	You really did come a long way. I mean I remember when we shown these tapes pretty much many parts of the world and all the world and I have folks say that there is no way you can orchestrate this. There is no way you could -
505	33:06	T/R2	Fake it
506	33:07	T/R1	Fake it right. You can get people to this. It was such a natural way that you just spoke and
507	33:14	Jeff	Yea but you just you don't think that if you had a group of different kids they wouldn't have done the same thing we did?
508	33:18	T/R1	Yea I think they would have done different things but they would have done. I believe truly I do that all kids are capable of doing this and I think it is the conditions that have a lot to do with it. And I think the conditions have to be in place. I think in college mathematics the students there are

			capable of doing the most incredible stuff when the conditions are not in place. Fortunately, the brilliant genius professors don't get a chance to see it because they don't set the conditions, they don't know what to look for. That's sort of maybe a big challenge isn't it? I mean you're going to be the next generation and you've gotta think of what education is going to be like you know for the next generation of kids. What would you like it to be like, you know? Give you something to think about.
509	34:20	T/R3	That's the question I have. We talk about college professors being disappointed in their students. Do you think it makes a big difference that people tend to think math is something that you got it or you don't got it.
510	34:29	Jeff	I think it's a lot easier for people to write off math and say I don't have it and then you just don't you know. I can say I am not a math student and math's not my thing and then it's just easy to just shy away from math and I think that breeds being bad at math in the end cause you use that as an excuse your whole time about how it just wasn't your thing and you know you just don't really learn it.
511	34:51	T/R3	That personally. I think that's what a lot of kids learn in school. They don't learn math; they learn that they're not good in math.
512	34:57	Jeff	Yea I think that is very legitimate.
513	35:00	T/R3	And I think that teachers tend to think that's it's a, it sort of goes with what you said. The teacher's trying really hard and if the kid can't learn it is because the kid is not good at math.
514	35:11	T/R2	It's much easier to believe that than to believe you need to do something very different. Even if they start having some kind of imagination of what the thing might be. None of your experience was different and you always sat there and you always learned. I mean people generalize from their own experience, right? So you naturally you tend to assume what worked for you would work for other people and you know you have some good examples so maybe you have this teacher who did this but who did it really well so you kind of try to emulate that rather than just change it. Of course this is just replicated cause those are the people who do get to the top and their able to teach it.
515	35:45	Romina	You see with us we in our grade, we were separated as the we were good in math supposedly but we never. We didn't think so but that's what everyone classified and everyone was bad in math and it just. I don't know if this was a direct correlation but we were the ones who did better in school in general than like. It just like
516	36:02	T/R5	It's called the Horton effect.
517	36:03	Romina	Like our group of kids. We were in the top in the class. Out of everyone involved in this program we were all pretty much 1 through 10. That was us and then everybody else and if anyone out of our group. We were like the ones who did more math-oriented college-y things whereas I know my friend she was not in this program; she went to college. She found out the first day that she could not use a calculator in her chemistry class and she dropped pre-med. And she didn't even you know so scared

518	36:30	T/R3	She couldn't do it without a calculator
519	36:30	Romina	Yea. I know she could. She was so intimidated. She was like that something you could do but it's not true. I never used a calculator. I rarely used it during all this.
520	36:40	T/R1	So there's confidence building.
521	36:41	Romina	Yea
522	36:42	T/R1	Really important and
523	36:44	Jeff	Didn't use a calculator until the summer program.
524	36:47	Romina	Yea we never
525	36:47	T/R1	That was the first time you ever used it. That's interesting.
526	36:47	Jeff	Yea it was the first time we used it. That was the first time we were allowed.
527	36:50	T/R3	What you got some calculators in the night group but I don't know if you used them much.
528	36:53	Jeff	Yea
529	36:55	T/R3	Yea if only you used your nCr root button
530	36:57	Romina	Yea but even if we had this fancy TI-89 I don't know how to use it. Supposedly that can solve all these amazing things for me but I don't use it. I just solve it by hand. I don't
531	37:08	T/R5	They were all very confident. Remember the time we did World Series and they found a mistake in the teacher's way of doing the problem. It was confidence because it was teacher's opinion against 10^{th}/11^{th} graders opinion and they were very confident and convinced everybody that it was our mistake versus theirs.
532	37:34	Romina	Was it your mistake?
533	37:39	T/R1	Well you didn't see, What you didn't see were the tapes that were going on in our seminar where our graduate students did the problem wrong and you did the same problem and you did it correctly. Not all of them but there were people with strong mathematics, you know masters in math is pretty strong wouldn't you say?
534	38:00	Jeff	I would agree with that.
535	38:01	T/R1	Would you think so, Lara?
536	38:02	T/R2	Yea I would think so. Yea. I have to. I have one.
537	38:06	T/R1	So they used all the formulas. They didn't build anything from the bottom up as you did. They used all the formulas and I asked one of them to look at your tape and they reported after looking at your tape where you had solved it correctly that you did not get it right. So remember that? So I showed you their solution and do you remember that?
538	38:32	Romina	I remember you said they
539	38:33	T/R1	At first you were saying they must be right cause

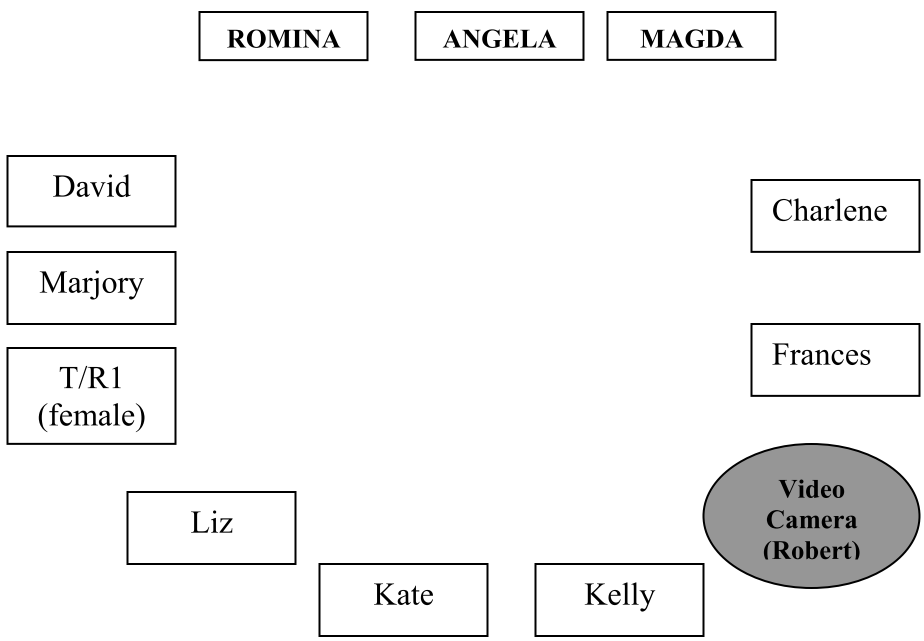
540	38:36	Jeff	College kids
541	38:36	T/R1	But then I think it was Jeff or someone who said wait a minute we spent a long thinking about this and we had convinced us so you know went back and you trusted your solution and you gave even a stronger argument and it was wonderful learning experience for our graduate students
542	38:56	Romina	I wouldn't like this too much if I were your graduate students.
543	38:48	T/R1	Well I protect their privacy. I don't show the tapes. I haven't written about that but I thought you might want to know that. Right Ella, we do protect them.
544	39:11	Jeff	It comes across as interesting watching these tapes you know. What kind of preconceived notions about what kind of people we are?
545	39:17	T/R1	I think the point is that is look you said it earlier yourself. You gotta trust your own deep understanding. You really got to and they know that too in some ways. You know they fell into a little trap of trusting rules that they hadn't really thought through deeply enough and the mistake was understandable that they made doing that which I think happens. Right? It is not unusual to happen but it always and I think we have a wonderful very famous mathematician here at Rutgers his name is Gelfand and he has must be almost 90
546	39:57	T/R2	I think Jerry told me he was over 90
547	39:58	T/R1	Possibly and every great mathematician under the Soviet Union worked under him and he holds seminars here regularly for very famous mathematicians all over the world and they'd walk into his seminars trembling because he always asked them questions that made them build up from the bottom and will not accept anything that is in any way just generalized knowledge or rules and whatever. Yet they keep going to him and they say that he wants to have high school students at some of the seminars. Maybe next year you might want to go to a couple seminars. It might be fun. Lara will take.
548	40:39	T/R2	I have not been to one. I'll try to go yes.
549	40:41	T/R1	So um I mean the man is brilliant. He is a genius. There is no question. He's interested in this work on combinatorics. He's seen some of the tapes and likes them very much from way back. Um, but don't lose what you believe in I mean don't lose it just there is certain rights of passage for right now. Do you know what I am saying? Don't compromise. I mean you got to survive. Do what you have to do. Use good judgment and get your degree. I'm not telling you not to do that but don't compromise that. That's not good.
550	41:19	T/R2	And don't worry about what people think about the tapes. Because in my experience and I just got here a little while go but anyone who has seen just any one of them has developed tremendous affection for every single person in them that. It is almost impossible for people come across
551	41:34	Romina	It is a lot of them
552	41:35	T/R1	I think you see more than they do.

553	41:38	Romina	Yeah I know I see more than you do but some of it you can’t help but pick up and I’m just like aww. They really don’t need to know that -
554	41:48	T/R1	Ok you know we are passed our time here and really thank you for sharing and we want to get Michael back here and even if Romina can’t be here next time maybe we can get Jeff and Michael. Maybe another time. We are going to get Jeff an independent study.
555	42:09	Jeff	We are working on it. That’s all. That’s what we got to do.
556	42:12	T/R1	We need him. Right? But thank you all.
557			**END OF DISK THREE**

APPENDIX H: TRANSCRIPT - REFLECTIONS IV

May 12, 2006

1 Camera View: Romina, Magda, & Angela (*2 Disks*)
Date of filming: 2006-05-12
Location: Holmdel, NJ, Reflection for the Kenilworth Longitudinal Study
Transcribed by: Kristen Lunny
Date of transcription: June 2007
Verified by: Maria Steffero
Date of verification: June 2008



Participants: T/R1, Romina, Magda, Angela, Marjory, Liz, Charlene, Frances, David, Kate, Kelly, Robert

DISK __1_ of 2

Line	**Time**	**Speaker**	**Transcript**
1	00:31	T/R1	We don’t want to have Robert edit anything out, Jim. (laughter) So, that’s my husband, Jim, if you haven’t met him.
2		T/R1	Right, so we have this lovely space and we’ve been meeting in a new conference room we have there and that’s where we had our little interview with Amy and Brian and Robert. And that was great fun.
3	01:00	T/R1	So the seminar group which consists of – this is a subset of them – David
4		David	Yes

5		T/R1	Who really wasn't in the seminar, but he was in the seminar. He was in another course but did the seminar as the practicum.
6	01:12	David	Yeah.
7		T/R1	A master's student taking his last course and now a doctoral student.
8		David	Yup.
9		T/R1	And, um, Marjory. And we also have Charlene and Frances and some others who aren't here. About five others. The semester's pretty much over, so they came after the semester.
10	01:34	T/R1	And you've already, you've already met Kate and you've met Kelly. But why don't you say something about yourself to everybody here, so they get to know a little bit about – you've told some people. Where you came from and where you're from and how you might know Arthur. She's also been in South Africa working a long time. A very interesting background Kelly has, I think.
11	01:53	Kelly	Um, so my Ph. D. is in math education and, um, when I started my degree it was with a math educator who is South African in the early nineties. And early into my work with him, he decided he wanted to go home. So um
12	02:09	T/R1	Say who he is.
13	02:12	Kelly	Oh, his name is John Volmink. And his area is critical math education. So that's an area that sort-of I got into that's related to ethnomath, which is one of Arthur's specialties. One of many.
14	02:23	Kelly	And so there was this sort-of group of people internationally in the early nineties who knew each other. And so I actually met Arthur in the early nineties at University of London. See you're bringing up all these old, old – and then again in South Africa and then again, I think, at Rutgers Newark at one point.
15	02:41	Kelly	So, so, so my degree took me to a math department for a few years and, um, that wasn't right for me. It just wasn't right. And so then I was in Oregon for a few years and that was fine
16	02:54	T/R1	Portland?
17	02:55	Kelly	Yeah, Portland, Oregon but as we were talking about earlier, Oregon is not an education state and so teachers were being laid off and having their years cut – their days of school cut – in the middle of a school year. No money, fifteen days taken off the school year, in the middle of the school year.
18	03:10	T/R1	Really?
19	03:11	Kelly	Yeah, it was the subject of many Doonesbury cartoons. So my program was cut. Right when I was granted tenure. It was lovely – I got tenure and my program was cut.
20	03:27	Kelly	So I came back to New York and it was good. And so I've been in New York for three years at Bard College – no, two years Bard College. And um, all of my students learn Carolyn's work. So probably for four years now all of the students – people who are becoming teachers – it's one of

			the requirements to learn about this body of research and what the findings are.
21	03:57	Kelly	And so Kate is one of those former students who is now a teaching at Fannie Lou Hamer High School in the South Bronx.
22	04:02	T/R1	Say the name of the school again?
23		Kelly	Fannie Lou Hamer Freedom High School. And um so we invited Carolyn to come speak at our college and that's when we met.
24	04:18	Kelly	Kate is at a very interesting high school. Um it's in a cluster of schools called "The Autonomy Zone" in New York City. I don't know if you know about them, but they're – they have all sorts of variances from the usual mandates. So they're not required to use any particular curriculum materials. Her classes are two hours long. She stays – do you stay with a group of students?
25	04:42	Kate	Um, we stay with a group of students for two years. We have Regents waivers from the traditional seven Regents that you have to, um, pass to graduate. We have portfolios instead of tests typically.
26	04:59	Marjory	Is it like a charter school or is it like part of the regular public school system?
27	05:05	Kate	It's public but
28	05:05	Marjory	More like a specialized magnet school?
29	05:07	Kate	We don't have any screening. Um, there is a lot of small schools popping up in the Bronx and all of New York City and parents will have filled out a form to say that we don't want to just be P.S. or C.S. whatever
30	05:23	Marjory	Mmhmm.
31	05:24	Kate	And some of the schools in it will accept by application but ours, most of our kids are from the neighborhood and they don't have any say -
32	05:32	T/R1	So tell us, tell the folks where you came from before you went to Bard.
33	05:35	Kate	Well, I was an undergraduate at Bard College from two thousand to two thousand and four as a math major. And the year I was graduating, was looking at teaching schools, and it was the first year that the M.A.T. program was opening and it was just the right time and the right place. It was perfect.
34	05:59	T/R1	Fantastic. I don't know if you all know Liz? You all know Liz? Liz is now teaching at Felician College. Do you all know that?
35		(several)	Congratulations.
36	06:06	Liz	Thank you.
37	06:06	T/R1	She's a professor. Doctor, professor, Liz and been working with us on lots of stuff since then. And continues to be part of our group and it's very exciting. And that's everyone here. Good, so did we introduce everybody here?
38	06:27	T/R1	So we want to hear about you guys. We have some questions we want to ask that we really want your opinion about. We're not going to ask you about yourself, we want your opinions about things. But before that, why don't you just tell folks sort of a little bit about, not everybody knows

			where you're working, so everyone knows a little bit about what you do, what's the great fun about it. Anything just to start. Who wants to go first?
39	06:48	Romina	Me? I work at Deloitte consulting. I am in business analyst right now. I'm going to be promoted this year.
40		T/R1	To?
41		Romina	Consultant. Yeah, I know, I work in a consulting firm and I'm not actually a consultant yet. I'm in - I work in the strategy and operations field, so it's mostly management and consulting. But what I do, I can't really define.
42	07:06	T/R1	Sounds like my son. This worries me always. I never know what he's doing.
43	07:11	Romina	No, I have no defined skill set yet. [Laughter]. But I mean, you guys would love this. Cause I basically go in a room, we sit there and whiteboard and brainstorm and come up with an answer that we just, you know, we don't know anything when we walk in and we come up with an answer and share it with a client.
44		T/R1	Why do you think we would like that?
45	07:27	Romina	Because that's what you did to me for four years – for twelve years of my life. [Laughter] I said four years, but that was only high school.
46		T/R1	So that's what you're doing now. Do you like doing that?
47	07:36	Romina	Yeah I do actually, it's something I find I'm pretty good at so.
48		Unknown	Can you give us an example of something? Like what you go in to talk about?
49	07:44	Romina	Well no, right now I'm actually working on, I've been, I haven't had the luxury of changing too many clients, I actually get stuck in one place, which is both good and bad. Right now I'm working on an internal project, and we're trying to become a tier one strategy consulting firm and compete with the McKinseys and Bates and BCG's. So I'm looking at our HR model and it was very interesting cause we…
50		T/R1	HR?
51	08:10	Romina	Human Resources model, sorry. I apologize - all I know are acronyms now, so just let me know.
52		T/R1	Sounds like my husband, he talks in acronyms.
53	08:17	Romina	So we're researching our recruiting processes, like our deployment compensation all these different. We just walked in and I didn't know anything about HR. We walked in and they gave us all this information and now we're off - we've been off designing our new a completely new HR system, and we've been sharing it with our CEO and our COO's. It's very - It's a great experience for me because I get so much exposure to senior level clients, or executives really, they're not clients.
54		David	What is management consulting? I always hear that term.
55	08:48	Romina	Yeah, it's very big. It's basically just advising CXO's, so that, yeah.

56		David	Okay.
57		Unknown	C what?
58		David	C X O, C star O.
59		Romina	CEO’s, COO’s, CFO’s Chief Operating Officers.
60		T/R1	Advising COO’s. Oh, right, right, right. I’ve heard something about that. He talks about CFO’s, CEO’s. It’s like a new math language.
61		Romina	Chief Operating Officers
62		Unknown	It is a new language. Completely.
63		Romina	Yeah, I had to learn it too. But now I’m so absorbed in it.
64		T/R1	You talk that way.
65	09:20	Romina	I know. It’s so bad. I hate it cause I hated it when I joined consulting.
66	09:23	Marjory	It’s okay, I understand you because before I came to Rutgers I worked at Andersen Consulting which I guess is now called Accenture unless it’s gone out of business.
67		Romina	No, it’s Accenture.
68		Marjory	Or eaten up by one of the other firms. And I worked it’s internal clients – I worked for human resources.
69	09:39	Romina	Oh really?
70		Marjory	Yes. When you worked for external clients, were they in a particular field? Or a particular industry?
71	09:45	Romina	Yeah, I worked for a pharmaceutical industry, BMS actually so it’s down in Princeton. I worked in the New York office. I worked with their Sarbanes-Oxley. We created a whole… you know that?
72		Unknown	I did that too!
73		Romina	SOX.
74		T/R1	What? What is it?
75	10:03	Romina	It’s a new regulation that came out. When everything happened with Enron, they came out with Sarbanes, Mr. Sarbanes and Mr. Oxley, with a new regulation that requires us, corporations to have all these procedures and processes and people signing off
76		Unknown	Documenting for years and years and storing
77	10:20	Romina	Yeah, documenting. So we went in and designed. There’s no organization, there’s no infrastructure for who’s gonna to do all this, so we went in and we organized the companies so they had a whole division dedicated to this, and we trained them and we took the first stab at actually documenting all their controls.
78		T/R1	Okay, very interesting. Angela, tell us what you’re doing. You’re doing two things.

79		Angela	Kind of.
80		T/R1	We know a little bit about the teaching piece, but tell us about both.
81	10:50	Angela	I'm working as a marketing assistant in an IT company. We do a lot of stuff with stocks. We provide hardware for all the servers and storage and we sell hardware for operational systems. But we also make identity management. Excuse me, my allergies have been bothering me. Identity management stuff for all that stuff they [looks at Romina] have to keep track of, so we make software stuff too. And mostly what I do is some PR work. I do a lot of writing for the company's collateral sheets and the company's, I do a lot of their internal communications. I throw parties! Fun! And I'm doing a lot of learning at that job so far, because I mean, I was an English major, so I don't really have "marketable skills" [she puts those in quotes] except for the communication one. So, I've also have taught one class.
82	11:46	Romina	She's really good. We went and sat in.
83		Angela	Oh yeah! They observed.
84		T/R1	You should let us come in with our camera.
85	11:51	Romina	No, I know she's not going to say it, but we were so impressed, because you know, she's our friend.
86		T/R1	Well tell us about it. What were you impressed about?
87		Romina	She did, the way she just commanded the whole audience. They loved her.
88		Angela	I'm an actress.
89	12:04	Romina	But she, she was asking all these questions. I mean, I know she's intelligent, but I've seen her always in a different setting. I don't know, they loved her, and they were all learning, and they were all asking these really good questions, and she was answering them and pushing them to think harder.
90		Angela	They're such great students.
91		David	What type of class was this?
92	12:20	Angela	Just an English comp class, but the way that Kean's writing, I'm teaching at Kean.
93		David	Oh, okay
94	12:26	Angela	The way that their writing program works, well, is supposed to work, I mean, if you have all the competent professors and stuff, is that you really introduce them in that first course to all the stuff that I was learning in 2000 level courses in college, the divisions of criticism, all that stuff. You talk about cloning, and you actually have a debate going on while you taught them how to write, because the first key to being able to write well is to have something to talk about. So if you don't have that, if they don't

			have a reason to write, they're not going to write good papers, they're not going to be interested in the class. So you know, we'd go every week because it was a night course, and we'd sort of have that unit where one week would be feminism, the next week would be science, the next week would be technology, you know, so on and so forth, religion, all those sorts of things.
95	13:19	Angela	And, they were just, you know, great students, great students. They were so interested, they always did their work, but it was because, I think they felt comfortable with me, one, being younger, you know, they kind of related to me, and two, I didn't underestimate them, at all. I told them from the get go, I told them, I know this is a 3000 level, is not a 3000 level course, it's your first English class of college, but I'm not going to underestimate you because I know you can do what I can do. So, I think at first they hated it because I was really harsh on them, but I mean, from the first papers they gave me to the last papers, it was such an improvement, and I was so proud of all of them!
96		T/R1	How many did you have in class?
97		Angela	I had twenty-two students.
98		T/R1	Wow, it's a big class.
99	13:59	Angela	It was freshmen. And some of them were older then me, you know. But they, they came to realize… I mean, the first day I walked in, they were like, you're our teacher, cause you know, I got my master's in one year, and then got a teaching job the next year, an adjunct position the next semester. And um, but the other ones, the ones older than me, they couldn't believe that I was younger than them. So you know, it's a matter of confidence when you're up there. If you know what you're talking about, if you have something to say, then you can say it. With confidence.
100		T/R1	So, you wouldn't want to do that full time?
101		Angela	Not now.
102		T/R1	Not yet. Okay, so tell us about your other job.
103	14:37	Angela	Well what about it? I don't know, I really like it. I like learning, I love the PR, that's what I really love. I love the writing stuff. I'm using a lot of what I learned, I'm in this theater group that we just started. Some of my friends from St. John's, we all just graduated. We didn't really, I don't know, we wanted to start a theater group where we did charity too. So all of our proceeds go to charity. Every dime. And it goes to, you know, production costs of course. It goes to the Children's Aid Society in Staten Island. What they do is they always have a camp for the kids from the Bronx and from Brooklyn and stuff whose parents can't, have no place for them for the summer because they're not in school, and they can't afford it. They're kids of all ages, they're kids of everything. And um they put them in, they do theater, they do sports. They really encourage them to get involved with these things. And last summer we

			did a whole program with them during the summer where they did theater. And they loved it, you know. And it's, I don't know, I feel like that's really important. Charity's important. You have to use your talents to help other people.
104		T/R1	So you go to Staten Island to do this?
105		Angela	Yes
106		T/R1	It's on weekends?
107		Angela	Um, I don't know. This week it was pretty much every day during the week.
108		T/R1	You're doing it during the week also?
109	15:56	Angela	Yeah, next week is tech week, so I'm going to be there until 12 o'clock in the morning if not later every night, if not later, and go to work. But I've been doing PR for that. So I'm using what I'm learning at work to promote the show and get more people to go. Um, we're in three papers already. We've got a photographer coming to rehearsal.
110		T/R1	You need to let us know about it.
111	16:16	Angela	Our show is May 19. May 19 and 20th. So. That's what I'm doing.
112		T/R1	Very nice. Magda, what about you?
113		Romina	Maybe we should switch seats, we have the boring job.
114		Magda	Well, you should have gone last.
115	16:32	Angela	I ship things, I ship things too. That's part of my job.
116		T/R1	What do you do?
117		Angela	I do shipping. I order games, and toys. That's not so much interesting.
118		Magda	No, you made it sound really good though.
119		Marjory	Haha, that's the PR strength.
120	16:45	Romina	Auditing is so fun, come on.
121		Magda	Yeah, no, so I'm an auditor. I work at Deloitte. Basically my client has been a pharmaceutical company. Um, I do a lot of Sarbanes-Oxley stuff too, it's not fun.
122		Angela	Should use my company, we'll solve the problems for you.
123	17:08	Magda	No, nothing will solve the problems. Basically, that's it. I've been with the same client. Actually I'm trying to get onto BMS now. Where like, cause she did the consulting side of it, now we took them over on the auditing side. I'm trying to get onto something different.
124		T/R1	So you'd like a change maybe?
125		Magda	Yeah, no definitely.
126	17:27	T/R1	Is it hard to make those changes?
127		Magda	Um, people they don't want to let you go.

128		T/R1	You're doing a lot of good stuff.
129	17:35	Magda	Yeah, it's turnover and just you know the client and you know the people. Just the whole cycle of them reteaching you and going through everything, and clients don't like change. They don't like to be you know.
130		Marjory	That's the nature of management consulting is that well you know, these big firms, first of all they recruit a ton of people, but a lot of the people that are recruited aren't really I don't know well-suited for the company. And they you know either they go, or whatever. But once you get people that are on the job, the clients like to have the same people.
131		Magda	They like to have recurring people.
132		Marjory	The same people coming back and it's kind of like if it ain't broke don't fix it kind of thing. So like if you've got personnel assigned to projects and it's working out, it's hard to get assigned to something else, even if the worker doesn't like the project they're assigned to.
133	18:28	Magda	Yeah it's like my manager doesn't want to let me go basically and it's you have to go through your counselor and do all this stuff and it's just, but. I need to do something
134		T/R1	How long are you there with the same…
135		Magda	Well I've been working for two years –
136		T/R1	That's what I thought.
137		Magda	And practically I've been on the same client.
138		T/R1	In industry, that's a long time.
139	18:49	Liz	What firm is this? What pharmaceutical company is this?
140		Magda	It's Aventis Health. They do outsourcing for all these pharmaceutical companies. They do like marketing for them, sales outsourcing, like basically just stuff like that, so I don't know. It's interesting in a sense, but I mean, if you do it long enough, and I'm not on the operations side of it either. I'm the auditor and I go in and check what they're doing, see the client. So you kind of want to see different things, different clients.
141	19:27	T/R1	Well maybe next year you'll tell us a lot of stories. We have some questions that we would like you as a group to answer for us if you can, if you want to. If not we'll go to some extra, we'll create some new ones.
142		Angela	They're not math problems, are they?
143		T/R1	Well that calculus one, did you [turns to Liz], did you get that one?
144	19:46	Romina	You'd be proud, I've been doing math every weekend for the last month.
145		T/R1	You're tutoring somebody?
146		Romina	No, I'm saying for the GMAT's.
147		T/R1	Oh, you're studying.
148		Romina	I have all the books. I'm re-learning Geometry, all the good stuff.

149	19:58	T/R1	Good for you, good for you. Okay, so. This is a general question. These were made up by the seminar students. The question is what does it mean to know something really well and deeply? What does it mean to know something really well and deeply?
150		Romina	You start off.
151		Angela	What?
152		Romina	You start off, you had this one, you answer it.
153		Angela	[Laughs]
154		T/R1	Well to any of you, what does it mean to you -
155	20:20	Angela	To be confident with it, to understand it, to want to learn more, you know. Even when you feel like you know everything to know there is to know about it. If you could stand in front of a class to teach it, I feel like you really know it.
156	20:32	T/R1	We have a lot of people standing in front of a class and teaching, but the students don't always think they really know it.
157		Angela	Well to teach it well and to generate discussion, and…
158	20:42	T/R1	But for yourself as a learner, when do you feel you know something really well?
159	20:45	Magda	I guess when you can explain something to someone else.
160	20:52	Romina	Kind of. When you're able to explain it to someone else and they ask you every question under the sun and you can still answer it. I think then you know it
161	20:58	T/R1	It sounds like you've been in that position.
162		Romina	A couple of times.
163	21:04	Angela	Basically the same. When you can teach it… But you know what, when you start teaching it, you learn things at the same time, so you know. I don't think you really ever know anything a hundred percent. You're just always learning.
164		T/R1	That's interesting. Anybody else? Robert, did you have something?
165		Robert	No, just scratching. [Laughter]
166		Angela	You don't want to take a stab at that Robert?
167	21:24	Robert	No, I already answered that.
168		T/R1	I know you did. Do you want to know what Robert said?
169		Romina	Yes.
170		Angela	Sure, what did he say?
171		T/R1	What did you say Robert?
172		Robert	I don't remember. No, I forget really.
173		T/R1	You remember.

174	21:36	Robert	No, I really don't. [Laughter]
175		T/R1	I don't even remember if he answered it. Does anyone remember if Robert answered this one?
176		Liz	We have to check the video.
177		Angela	You guys have to take better notes.
178	21:48	T/R1	Alright, you now are all working out in industry in the world. And now you may be at the other end of. Let's suppose you're going to be interviewing people to come in to work in your companies and maybe even with you, right? They may even be helping you. So let's pretend that you're looking for someone. What do you think are important skills for young adults just entering the job market to have?
179	22:21	Romina	To be organized. Well for my firm, I'm not actually allowed to interview yet. But I'm very involved in the recruiting, and something we, we actually test them for the logic and structure and the ability to just set up a problem, not even solve it. But in our interviews we give them case studies, and we sit there and we watch how they think through it. And they're usually too hard for them to actually know the business aspect of it. But we watch them, and we give them paper and pen, and they explain to us how they would figure it out, and it's really interesting to see how different people think through problems and just them talking to you about it and depending on how well, how much they convince you even though they have no idea what they're doing –
180	23:00	T/R1	Problem-solving?
181	23:01	Romina	Yeah, problem solving it and just basic personality tests.
182		T/R1	What, why is that important. What do you mean personality?
183		Angela	People skills.
184	23:12	Magda	Yeah, I was just gonna say that. We look for, I do a lot of recruiting stuff so we look for social skills. You need to be, know how to talk to people and stuff like that, and teamwork.
185		T/R1	Say more about that.
186	23:24	Magda	Um, well basically within my company – my company, it's not my company.
187		Romina	That's okay, we know…
188	23:35	Magda	Um, like we work in teams, so you need to get along with people, you need to bounce ideas off each other, and that's kinda how you come up with things. That's basically what we look for.
189		Frances	How can you tell that in one interview, whether somebody would be good in doing something like that?
190	23:52	Magda	Well, it's not only one interview, we do a lot of recruiting programs too, so we have. Like I'm involved in this Mentor program. So basically it's five meetings during the summer, and you do interactions. One day is like leadership, another day is teamwork events, another day something else and you see them interacting with other people, and that's basically

			how like the whole process works. So it's not only one interview. It's like you get one interview to get into the program, and then you do all these other things, and then you know, it's you evaluate them, see how they interact, if they have more leadership skills or if they you know, working in the team kinda thing.
191	24:40	Romina	One thing we do, we have these case competitions. We put all these college kids in a room, we break them up into teams of four and five. And we give them a problem, and they have all day to just sit and break this problem out and we're observing the whole time. So it's groups offive and, kinda like we used to do, they just work on it all day. At the end they have to present their findings so a board of partners. And you know, just one day is probably all you need you can tell how people work in groups and who contributes.
192		Charlene	Did you say that you sit in the room and watch them?
193	25:12	Romina	Well yeah, we watch them, cause that, I think a lot of people are more concerned with how they're going to come off during the presentation, but they have to understand, I've sat in the team room with three people for seventeen hours. So, I mean during the presentation is a lot different than those seventeen hours that we sat in that room together. So you observe them to make sure everyone's working together, make sure everyone's contributing ideas and pushing each other to you know, pushing their thought so
194		Charlene	Did you have to do that? So, to get the job there?
195		Romina	No, I didn't. They didn't come to my school, so I didn't. But I did get case interviews. Yeah, so I had to sit there. It was more one on one and explained to them. They gave me a case like that and I'd come up with my thought structure. But it was more one on one. They didn't actually see me interact with other people.
196		Liz	And do you do the one-on-one things also? Or just the team things?
197		Romina	No, we're not allowed to interview. You have to be a senior manager level to interview.
198		Liz	So you only see the –
199	26:11	Romina	I do more the personality check, behind the scenes. Because we're so close in age, I can tell. And I know how it is to prepare for interviews and what lies you have to tell. So I tell them not, you know, they look too well-prepared.
200	26:27	Liz	Yeah, I was going to ask, what kinds of different, what's the range you see of how they solve problems from the best to the worst?
201	26:35	Romina	Um, I find that, and this is a personal thing, because I went to a liberal arts school, and I was a liberal arts major, so I didn't have the technical business. And it's so funny, because you have those kids that went to business school, and like they've been doing it since day one. And you give them a problem and they're starting to talk about all these frameworks and all these Porters Five Forces and SWOT analysis and they're doing it. And I am working for two years have still not done this,

			so they're to impress you, and you're like you don't actually use that every day. Like it's a framework, you don't actually use it to solve a problem. And then you have those kids that are so intimidated because they've never done business before. But they get in there and they just think about a problem, and they come up with a better solution, because they're not just trying to force all this knowledge that they already have onto this problem and they're just looking at it as if they've, they've never seen anything like that before. So it's interesting to hear what they would come up with, because sometimes they're really original ideas that you wouldn't have thought of because you're so constrained by this business mentality.
202		Angela	Are a lot of the other people that you're talking about, are they mostly liberal arts majors?
203		Romina	Yeah
204		Angela	Yeah, you can tell. It's just a different -
205		Romina	Because they're not, they wouldn't know those business concepts.
206	27:44	Angela	Yeah, a lot of my friends went to the business college of St. John's and it was more like, they would try to, in everything, every day stuff, they would just try to take something there and solve it with that. They would just fit it in the frame. But I'm, I don't know, I'm like the exact opposite
207		T/R1	Let's find a formula and use that formula? Gotta use that formula.
208	28:03	Romina	And that's what would drive me crazy! In my calc class, because I took my math courses in my preliminary economics courses with all my Wharton counterparts, and we would get to Calculus, and we would all be studying because these exams were just impossible, and they could not tell you, and I loved the probability stuff, because I knew the probability stuff inside and outside, but I didn't know any of the formulas. And I couldn't - they were like computation, permutation formulas, which one do we apply? And I was like I don't know! You just think about it, and they couldn't understand the concept behind it. They never thought about it. And I'm like, well say you had I don't know, 4 colors, and you had to make –
209		Romina	[Laughter] Towers, and I'm like, god, and I just assumed, because I've probably told you this before. I walked in, it was just funny to see how much of a different learning experience we had, because I walked them through the concept, but I don't know how to solve it through the formula, I'm like I don't know. And we'd get that far, and I'd be like, I don't know what that means, but I could solve it using just
210		Angela	It's better to understand it.
211	29:03	Romina	Yeah, but my grades weren't that good though. They don't really care if you understand it, the professor doesn't but…
212	29:11	Liz	I have another question about the interviews and the people you were watching. Did you ever find anybody who just, who just couldn't do it? Who was so intimidated they just couldn't solve the problem?

213	29:22	Romina	No, that happened to me though. That happened to me when I was preparing for case studies. And it's not like you can't do it, it's just so intimidating and a setting that you're not comfortable with that you just don't think it through and most of the kids that interviewed for our positions are very, very prepared. Even if they have liberal arts backgrounds they are very good to go. They've read all their books, they've had all their consulting knowledge all shoved into them before the interview, so.
214	30:00	Frances	I have a question for Magda. You mentioned that you see them four or five times. Are their personalities consistent through these four or five times? Do they contribute in the same way?
215		Magda	Yeah, pretty much. You kinda, maybe the first meeting, maybe because everybody's shy and nervous and whatever, but cause what you do is you evaluate them after every meeting and you pretty much at least I find that you pretty much write the same thing every time. Not the same exact thing, but pretty much, it's like a recurring thing. It will be like, oh this person, you know, volunteered all the time. This person when you have to go up to the board or something like that, it will be like oh this person is volunteering all the time, so this person has is taking the leadership role. It would be like recurring stuff I would say. Yeah, basically, I mean, at least that's what I noticed.
216	31:07	T/R1	Okay, any other comments? So, you said some things about the skills for young adults. You talked about it from the perspective of your company. You want to leave your company for a moment and talk about your own? Now you're allowed to interview, now you're allowed to make these decisions. You know what I'm saying? You have a say in the final voice. What would you be looking for?
217		Romina	I would –
218		T/R1	I just made this one up by the way.
219	31:35	Romina	Through my first experiences I'm very biased against business because I didn't have that structured – my, our learning environment wasn't that structured, we didn't have this homework and problems sets and all that. So I'm, if I ever had to change anything, and I'm just, it's very personal thing.
220		T/R1	No go ahead, it's on my list of questions later, you can answer it now.
221	31:59	Romina	I'm just so frustrated that we are, and I feel like in high school I was much more confident about my abilities than I am now. Because they always they're testing me with things that aren't relevant to how successful I'm going to be in the business world. Like, for example, to get my job I had to go through this big case interview. What they'd like to see, it depends on who, like for example, when I look at people's thinking, I don't know the business models because I still haven't had a chance to work with them, so that doesn't interest me, that doesn't impress me, because I don't understand them anyway. It's just genuine thinking impresses me but when I was interviewed, one of the people interviewing me was very

			upset that I didn't have all this business knowledge and they would give me all these things, throw numbers at me to see how fast I could spit out spit back numbers, and that is NOT how I grew up. I never had to do that, ever. So I'm not used to that, and they would get really upset, and that by no means indicates how I've done. Because I perform very well at work and I get like top ratings all the time, and because I can't shoot out answers within two seconds that you throw at me, they think I'm not, like my intelligence is underrated.
222		Magda	But when do you really have to have an answer like this? [Snaps fingers]
223		Romina	You don't! But the people who test you, that's how they test you, the SAT's –
224	33:13	Angela	That's what I'm saying, the SAT, it's silly when you have to take an SAT to get into college, but the statistics don't match up on those sorts of things either.
225	33:19	Romina	The GMAT's, it's like I have to do every single problem in under a minute. Why? It's never going to be, it's not going to test how well I do –
226	33:30	T/R1	We had a retreat today with our whole faculty, and that came up. I think that's a good thing by the way, that that came up. And we have these tests, why? Bob Davis used to say that we have tests to eliminate people.
227		Romina	And it's true, it's how well you prepare and how dedicated you are.
228	33:46	T/R1	But it's just to eliminate, you can't possibly interview everybody, you can't possibly allow everyone in. So your first cut is just to eliminate, that's what Bob would say.
229		Romina	But what about people who slip through the cracks?
230		T/R1	I know, that's a shame.
231		Romina	I'm going to be one of those people.
232	34:00	Angela	But that's when they show up someplace else. I don't know. When I applied for my job, a lot of it is editing, like I edit the company, the corporate newsletter, and um, she said that she got I think three hundred applications through email and she cut out 120 of them just because they had a typo. [laughter]
233		Angela	Well you know for an editing job you have to –
234		T/R1	Let this all be a lesson!
235		Angela	You have to find some line to get rid of a portion, but it's not like all those people are never going to find a job, they'll find a job someplace else.
236	34:38	Romina	See I cut people on that, because you have like four hundred resumes and I'm like, they spelled something wrong [motions with hands]
237	34:44	Angela	But that's the thing, think about how many people apply to college, you know, there has to be some place I guess, I don't know, I think it's silly to place a lot of emphasis on an SAT to say okay, this person's going to perform well in college. I mean, I didn't do so phenomenal on my SATs, but in college I graduated 3.91, I mean so I did well in college. So I don't

			know, I think I understand why they're there, I just don't think that it's a great gauge to say what a person's mind is worth.
238	35:14	Romina	We probably don't do as well on them, that's probably why we personally hate them. They're holding us back!
239		Angela	No, I mean, a friend of mine –
240		T/R1	It's hard to notice where it's holding any of you back
241		Angela	She got fifteen-something and then flunked out of college. I mean it doesn't mean anything – I mean it means something, but it doesn't mean enough to base everything –
242		David	It's part of a grand, it's part of a lot of something.
243		Angela	Right.
244		Romina	Well you can be prepped for them, which isn't fair.
245	35:34	T/R1	Okay, but what would you look for though? Because now you're in charge.
246		Angela	Oh.
247		T/R1	We know what doesn't impress you, so what would you look for?
248	35:43	Angela	Someone who's competent and eager. Like you don't have to know everything from the get go, but you have to be willing to learn it. You have to want, you know, have an interest in what you're about to do. You have to like what you're about to do.
249		T/R1	Like, want, have an interest, willing to learn –
250		Romina	Strong work ethic.
251		T/R1	Say what you mean by that. Strong work ethic.
252		Romina	Umm,
253		T/R1	Now like Magda, you work from eight in the morning til nine at night.
254		Romina	I know, poor Magda
255		T/R1	She really has a strong work ethic.
256		Romina	No just,
257		T/R1	Marjory
258	36:11	Romina	Just a desire to learn, like we, the most unsuccessful people that come into my class are those people that just want to get by on very little and not invest in the time, invest in the time upfront to learn, invest in the time to produce quality deliverables. Investing the time to just, they want to be in and out in three hours, and it doesn't work like that. So if you have a strong work ethic, you can teach those people, I mean, most of the people that couldn't do it are weeded out during an interview process, or weeded out in the first year. Now we're all in our second or third year. I mean, we can all do it, it's just a matter of how much time we're going to put into it.
259	36:48	Magda	Well I think like when you're, I mean I don't know, for me at least, I

			learned a lot on the job. So, I don't think it's really the technical skills that you're looking for. It's more, is this person willing to learn, is this person sociable, is this person, you know, I mean, cause you basically learn on the job. You get taught whatever you need to do. It's not like you have to come in and be like, okay, I'm a superstar, I know everything. You're not, you don't know anything coming in.
260		T/R1	So I'm hearing you say that – if this is not what you're suggesting then tell me – but I'm hearing you say that learning, knowing how to learn to learn, something about learning to learn. You should have learned how to learn, something new, something different. If you haven't learned how to learn, you're in trouble in the work.
261	37:36	Angela	Yes. You're in trouble in everything! If you don't learn how to learn, then
262	37:42	Romina	I feel, like that's completely applicable to what I do, because even at this stage, we don't specialize in anything, and we're in industry, and if you go in thinking that you know what you're going to do every time I've changed a client, or every time I've gotten a new project or a new task on my project, it's completely new. And it's just being able to pick things up quickly and ask the right questions to get an answer, it's going to take me a long time to get to my solution or my answer, but just in the fact that I know what questions to ask and I put the effort in and I know how to learn and how to absorb information, the right information, and weed through it. I mean, that's all we have to learn, to know how to do. I mean, I don't know about you guys with your jobs
263	38:24	Angela	Absolutely. I mean, I never knew a thing before about data compliance in my entire life, and now I can answer anything about it.
264	38:31	T/R1	So now you're, now you've been appointed to this presidential commission, and we all know they need help, and you're supposed to advise this commission about what young people need to become professionally successful adults. So what would you tell this, you're members of this commission - what would be your recommendations? You're talking about kids who are in school, high school, elementary school, you know. But to have these kinds of skills that you've talked about, they have to learn some things. Schools would have to teach them some things. What can school teach them? Or how does school teach them or whatever to be professionally successful adults?
265	39:09	Angela	I would say first and foremost, an emphasis on understanding concepts versus memorizing things. Because memorizing everything you got it there, but what good is it going to do you, what good is knowing all those frameworks if you're not really going to use them? It's a matter of those, oh, you learned it for the test, great, you got a good grade. But I don't, I don't think memorizing things proves anything about what type of thinker you are. So an emphasis on understanding concepts. So of course you have to teach them the important things, like you know, that you teach them, when you teach them to memorize. But if you understand how something works, you can always answer a question. Or if you don't, you

			can always find out where the answer is.
266		T/R1	So learning how to find out is another thing you say. Do you want to add anything to that Romina or Magda?
267	39:59	Romina	I'd get rid of standardized tests, that'd be my first thing. [Laughter]
268		T/R1	Hey, you're on that commission, you're on the team. What do you say?
269		Romina	Um, no I'd also get rid of the formalized classroom.
270		T/R1	Okay, say what you'd have instead.
271		Romina	I would have –
272		T/R1	Or why you would get rid of the formalized…
273	40:17	Romina	I don't think it produces thinkers. I don't think it produces leaders, it's not realistic, I mean, you're never - No, it is, but very rarely are you going to sit in a room and just work by yourself. And I think a lot of the skills that I learned growing up are very, they're people skills, being able to interact with a group and kind of assessing someone's strengths and capitalizing them and then delegating work well. And that's what we have to do all the time in the workplace, so I'd get rid of all those classrooms that are set up like with the rows of chairs, and I'd, yeah, I mean, that's we loved –
274		Angela	Those differences, I mean those small things make a difference.
275	40:53	Romina	Yeah, I think that's what we loved, and that's why college was so hard for me. I went from always leaning on Magda to explain something I wasn't going to get or when I couldn't write something very well I'd turn to Angela and be like, could you rewrite this for me. I mean we all had our strengths and we taught each other and we learned from each other and we got to college and it was completely different. No one worked in groups, and we all just sat there and we listened with three hundred other people and I think I lost a lot of the leadership capabilities I had and the speaking, I mean, I'm the most not confident, I hate speaking in front of crowds now, and I used to do it all the time. Like I did every day in school, I could do it. And it's just so different, and it's not like that in the workplace. So why even do that?
276		T/R1	So tell me a little bit about what these classrooms would look like?
277	41:41	Romina	We'd have tables, no desks, tables. I don't know, we'd all sit in groups of 4 or 5 and we'd rotate periodically so we could work with different people all the time so we'd have to re-learn how to work with people.
278		Angela	I had a hippie major, so everything was like that, like everything was a discussion versus –
279		Marjory	A hippie major?
280		Angela	English [throws up hands]
281		Marjory	[Laughter] Sorry
282	42:01	Angela	That's what all the hippies do. So, I'm one of them. No, but most of my, grad courses of course, but most of my courses for English were just like

			tables, like any time a teacher could get out of the classroom, and we'd be in the English teacher's conference room all the time.
283		T/R1	Was that typical with all the classes or just English?
284		Angela	I, I don't-
285		T/R1	You don't know.
286		Angela	Well I mean like my other courses, you know the philosophy courses, they were regular setups -
287		T/R1	Did you have discussions?
288	42:34	Angela	They were, oh no, absolutely. I think just about all the classes were discussion classes except for my one math class. I only had to take one math class, and it was like I was in second grade all over again. She was like checking our homework every day. I was like this is college, I'm pretty sure.
289	42:53	Romina	Yeah, and the thing with, one of the big things that I think our education fails at is inspiring thinking, and when we have, when you sit everyone in a classroom like that and you lecture and you have problem sets you just, you're letting them be mediocre and not contribute because they don't have to. And I think we were challenged the most when we were in those classes and we had to be prepared the most for the classes we were going to speak, because we don't want to look , you know, it's not that we were competitive, but I didn't want to let my classmates down by not being prepared, not knowing what was going on, because they would have to pick up the slack and drag me along because we were a team.
290		Angela	I had to speak in class everyday. Every single class every day it was like you had to contribute something so
291	43:34	Romina	You're always prepared for those classes where you have to contribute something and discuss things you're actually taught. I remember, we learned so much from just getting up, and this is back in high school, not in college. We learned so much from just getting up in front and explaining what we thought of concepts versus someone just telling me what it was, and I'm always going to remember what an integral is – it's the area of a
292	43:58	Magda	Funny that you say that, my sister was taking a class, a calc class at Rutgers, and she asked me for help, and I'm like she was doing integrals or something, I don't remember, and I'm like it's the area between – and she's like, no just tell me the formula, I need to apply it, and that's all she wanted to know, and I was like well I don't really remember
293		Angela	She just wanted to know it for the test
294		Magda	Yeah, she wanted to, yeah
295	44:17	T/R1	So, when you talk about changing things, you're on this commission and we're trying to prepare people to enter the job market, so you can take this even through college. So you're experience was really different, and so what Amy Lynn said in her interview on, I'm not going to have it exactly, but she now has her classes, and she's a teaching assistant, and she says

			that they actually gave her a name, I won't say what is, what her students call her, what her mentor told her that her students call her, and she actually insists they go in groups
296		David	Yeah, I remember
297		T/R1	And you were paying attention, so that the students, no one else, none of the other TAs, the other faculty get their students in groups. And she insists that they work in groups, that they problem solve and she's at them all the time, and she's apparently quite successful, her students do really well. So she's actually in a situation where she's actually doing this now in her own teaching responsibility now at the graduate level at the university, which I guess is unusual in this particular
298		Marjory	She'd be teaching undergrads at this point?
299	45:36	T/R1	Probably teaching undergrads. Yeah, I'd guess that's true. So um, but we're talking about public schools or universities where this isn't done, and you think this is a good idea that you had this display? Would you like to have had some lectures?
300	45:53	Angela	Well you know I mean like even though I teach my classes kind of modeled after the way I was taught in college and kind of particularly my grad courses, which were kind of like, you know, Rutgers math stuff. But um, I would lecture for maybe fifteen, twenty minutes tops, give them the information that they don't have, that they need. And then, they would have prepared with the readings, and I would go over the readings. And it's more like I would act as a guide as opposed to this is what this means, this is symbolism, this is, you know the author saying abortion is bad because of this, and this argument is different from, I didn't, I tried to stay away, steer from far as that as possible because I never learned that way, you know. So getting the students involved with discussion, you guys saw, they weren't just sitting there, they were talking, they were thinking about it, they were asking me questions. But it's more of acting as a guide for the students to learn.
301		T/R1	So, so you're agreeing with Romina then in terms of the approach in –
302	47:00	Angela	Yeah, and I mean, I also did like some group work too cause it was a writing course, so half of the class was, like, the first half of class would be that short little lecture and discussion, and the second half of class would be a writing exercise. Either I'd have to lecture on you know what a thesis is, what a, you know how you pick your examples, blah blah blah, that kind of stuff, or I'd have them actually do a writing exercise, but they did a lot of peer reviews, I mean, the whole thing was a process, it wasn't like okay a paper is due next week, turn it in, that's your grade. It was, okay, first week, submit your outline, submit your prewriting. Next week, I want to see, and switch with somebody, and I'm going to grade you on your comments to those papers too. Because part of the whole learning how to write is learning how to critique it too, in my opinion, because if you can pinpoint something that's not flowing in someone else's writing, like a gap in the logic, you can pinpoint that in your own writing, you can

			say, oh well you know, so and so's paper didn't explain why that quote was important maybe I'm not doing that too. So I mean, they always worked in groups, smaller groups, of course, but I think that's really important, I mean, somebody else is going to fill in the blanks for you for awhile.
303		T/R1	So we have Magda who's studied mathematics at a different university, same as Robert, the two of you studied. And so –
304		Angela	I think that's my phone, I'm sorry [Leaves room]
305	48:30	T/R1	So, Magda, what would you recommend? Would you keep everything the same? Would you change things?
306		Magda	Um, I definitely not. I mean basically I minored in math when I was at Rutgers and basically every single class I took was okay, this is a formula, this is how you do it, plug it in, stuff like that. But I think the biggest thing was that the classes were so big, you didn't really have the opportunity to get together and bounce ideas off of each other. And like stuff like that. So I think that was the biggest thing. And but I definitely agree with Angela and Romina about the whole idea of sitting in groups and you know, I mean, cause when you really, I mean you can memorize every single formula and just plug it in. But in the long run, you're not really going to remember it. It's like when I was helping my sister it's not like I remembered the stuff that I did in college, it's more from like Pantozzi's class, where we actually, you know
307		T/R1	Thought about the idea.
308		Magda	Yeah, so, you know.
309		T/R1	Well we'll be sure to let him know that we talked about his class on tape, but we're not going to tell him what it's on. He didn't come.
310		Angela	He was this horrible teacher, we couldn't stand him. [Laughter] Jumped up on tables, it was very…
311	49:52	T/R1	Is anyone else, are you satisfied with these answers to that question? Should we move to the next one? The rest of you?

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Line	Time	Speaker	Transcript
1	00:11	Robert	No, I'm just saying that when we're filming in schools, and they're trying to have them create perimeter from nothing just from looking at stuff, and a lot of kids are having problems, so I don't know if it's better to start from scratch with certain things or have an established answer and work backwards. I'm just curious.
2	00:23	Magda	Well, sometimes you can give a formula and stuff like that and kind-of show them how it works and stuff like that, but I don't feel like if you show them a formula and be like okay, these are the numbers, plug them

			in and you get the answer, I don't think that works.
3	00:36	Angela	Well I think it depends on the student too. I mean it might work both ways. If you give them the answer and have them work backwards, they're still going to understand it, it's just not building up in steps that way, it's going backwards. Like, this is corny, but like when I'm in plays, I memorize the lines first. I'm not good at delivering them until I understand them and have the motivation behind it, but I'm working backwards like that. So I don't think either is a wrong or right way of doing it, it's just a different way.
4	01:05	T/R1	Or you could say here's a formula, why does it work?
5		Angela	Right
6	01:10	Romina	I mean, also it depends how much time you have on it.
7		Angela	I'm pretty sure that's happened to us before.
8	01:13	Romina	Right, like I mean sometimes I think if you just give them that. Like I was telling someone this story the other day about how we thought we'd discovered Pascal's Triangle. Was that the one we were talking about? Remember that?
9		Angela	Yes! [Laughs]
10		Romina	We thought we were the cool-. We were so disappointed when we found out that someone else
11		Angela	We did come up with it.
12		T/R1	She thought so too.
13		Magda	We thought we had discovered Pascal's Pyramid.
14		Romina	That's what it was called.
15		Angela	Yeah, that's what it was. The one, two
16		Romina	The one, two (gestures with hands)
17		T/R1	This was the 3-dimensional version.
18	01:39	Romina	Oh, we never got there.
19	01:44	Angela	Yeah, we thought we'd discovered it. But we did, we did discover it. Someone else just did it before us. There's nothing wrong with that. But that happens with everything.
20	01:50	Romina	But it depends on how much time you have. We had weeks and hours to spend on just trying to figure this out. But if you don't, you can just give it to them and. It's like the shortcut almost. You know, we all understand it very well and have applied, we thought we created it, but
21	02:05	Angela	Yeah, but how much more did we also learn in spending a lot of time on it. You know like, maybe you pick up all those other things but –
22	02:13	Romina	I'm not sure we, I'm sure we went in circles, I remember desks being thrown. I'm sure there was frustration.
23		Angela	I think that was in 6th grade, right?
24	02:21	Marjory	Well, don't you think that any time there is teaching and learning going on there have to be some explanations or definitions of things before you can do, because it sounded like when you were talking about your literature teaching that well, you have to say well, look this is what a thesis is or this is what a denouement is, or you don't tell someone what

			it is, then they're not going to be able to talk about it.
25	02:46	Angela	Exactly. But then they end up learning in practice, or they when they write their paper I'm going to say that this isn't a good thesis because it doesn't answer the initial question you're asking. So you know, they, you have to give them sometimes that expository part of it and then let them work from that. I mean, cause, that's how, most of my students learned from lectures all their lives.
26	03:08	Romina	I don't think it's, I mean, I don't think it's necessary all the time. I mean, it definitely helps, but I don't think you have to. You said you have to. I don't think you have to. I mean, we've learned so many, like we had nothing and we still managed to learn-
27	03:23	Magda	I think it's the whole time constraint though. Like, if you have a whole –
28		Romina	Yeah, it is.
29	03:38	Magda	Like if you have a whole semester to do one little thing yes, you can be like. You know what I'm saying? Yeah, okay. But, if you have only a class or two to teach the idea, I don't think it's like you can't just be like do it yourself, figure it out, and just kind of guide them through. I don't think it will work.
30	03:47	Angela	Well maybe you can, it would just require a little more guidance.
31		Magda	Yeah, but that's the whole thing, you're defining what a thesis is.
32	03:56	Angela	Yeah, but that's what I'm saying, that's, it's a lot, but well, also asking more specific questions that they can get to the answers. Like if you ask the question the right way. You're still guiding them, but you're guiding them more specifically.
33		Romina	[audible]
34	04:10	T/R1	You said something Magda, that there may be certain conditions. Like you may only have a few classes or this much time to do something. We hear this all the time from folks who are teachers at every level. Really, every single level. So, what, well, this is also interesting, let's take something we know that adults have trouble learning, and there's been a lot of time spent in school with time allotted to learning it but never any large chunks of time to do it the way we might want to develop it, to be, maybe so many weeks here, and then you revisit it the next year, and you spend so many weeks and find out the students didn't understand it, and then you go, and find out the students didn't understand it the last two years. It's like fractions, that's a good example. So fractions, we should think adults should be able to learn, but that would be an example of where we haven't really succeeded well in school. We don't teach fractions to children well. So, my question to you is that you've given one example of it, there may not be time, there's an explanation of it, maybe it's how you go about it, maybe you're not dealing with ideas, maybe you're just giving rules and formulas that are not understood. I mean you can come up with a lot of reasons why people aren't learning fractions, right? Depending on who the people who are talking are, they might differ in fine detail. But, recall something, maybe that's not a good example, maybe you can think of others, calculus, the

			Fundamental Theorem of Calculus, whatever. If you wanted to have some ideal learning environment, what might be the features of the ideal learning environment? You're on this commission, you can restructure schools. I mean, this is heavy duty stuff. By the way, as you're thinking about it driving home, don't get in a wreck or anything, but if you have thoughts later, you know, we want to hear about this, because you're on this Presidential Commission, and we, we're finding that Americans can't read, you hear about this in writing and literacy, we're finding that children can't read, we're finding lots of difficulties in our schools. But what would you do differently in an ideal learning environment where you really could have your ideal. What would the ideal look like? Romina, you started before saying you'd change some things.
35	06:20	Romina	Well, I think a huge component, I think this is what we're missing in a lot of cases, is our, this is bad, but teachers who are genuinely invested in –
36		Angela	And qualified too.
37	06:30	Romina	Yeah, I think it's, I mean, we had a math teacher right who was getting, still getting his Ph.D, is still getting his Ph.D, and he'll get it you know, but we had someone who really dedicated a lot of time into his own education and learning about how people think and learning about how people learn and like he just spent all these years learning and applied them all on us and tested then out. And I know he, you couldn't, you wouldn't know, but I know he spent hours thinking up our lessons, and then we went to other classes –
38	07:00	Angela	It was his life, videotaping himself, for a math problem… [Laughter]
39	07:10	Romina	And he'd think, he'd customize every lesson because he'd be like okay, Bobby's going to say this and no one's going to understand him, so then Angela's going to ask, and Romina, he's going to have to explain it to Romina, and then Mike is going to get it, you know? And he went through all these different scenarios about how people learn and then you go to another class where your teacher gives you the same thing that she's been using or he's been using for the last ten years. And it's like this same paper, years and years before. So I think it's a lot about how invested your teachers are going to be too, that's probably one of the first things you have to change before your –
40	07:40	Angela	We had it pretty good even when we had it pretty bad with teachers who used the same stuff every day.
41		Romina	Well, yeah
42	07:46	Angela	The same stuff everyday. Like my sister just finished student teaching in Elizabeth, and to get photocopies, she, I mean it was impossible for her, you know? Some of the administration didn't really care, they were just there because they wanted a paycheck. And that's not of course not all teachers, you know, you just have to make sure you're not just filling positions, you know, you have to make sure that the people are there because they want to be there. And like our friend Renee, she loves it, you know. She's really invested in teaching. She doesn't love where

			she is, but she cares about the students.
43	08:13	T/R1	Okay, so ideal learning environment is caring teacher who knows the students.
44		Romina	Inspiring. You have to inspire too.
45		Unknown	Inspire? Say more.
46	08:25	Romina	Well because you have to, if we're having this different learning style right now, it's so easy, it's so easy to slack because you do have your team members to fall back on and you don't, you're not being tested. So do I really have to learn this concept? No, because I'm not being tested on this concept ever. So there's no like penalty for not learning as much as the next person learns, for not investing enough time in it. But, I know for us, like, we found him to be very inspiring, and we had like this unspoken commitment that we wanted to perform our best, I don't know.
47	08:55	Angela	But we were tested too. I mean –
48	09:00	Romina	No, but I mean, like we could do our tests over and explain to him things –
49		Angela	I know, but I mean –
50		Romina	No, but –
51		Angela	But the teacher had the emphasis on the value of learning. It wasn't you're going to get this grade and that's it. He always encouraged us to improve, that's the important thing.
52	09:10	Romina	He had to give us tests because I think our school like confined him to this
53	09:15	T/R1	Yeah, there are these bureaucratic laws
54		Romina	But even if he had never given us tests, we went in there and we spent hours giving him our all, you know? Even without theses tests.
55	09:26	Magda	Yeah, that was us, but there's some people that like were going to be like, whatever-
56		Romina	That's what I'm saying –
57		Angela	A couple of my friends had him too Pantozzi the next year, and they hated it, because they weren't used to it. And they didn't feel like they were learning anything, and blah, blah blah.
58	09:35	Romina	They hated it because they weren't used to it.
59		Angela	I was going to say, they weren't receptive to it. So maybe in the beginning years you have to teach kids to memorize a little bit, but maybe introduce this sort of flavor of learning yourselves, of teaching yourselves with guidance early on too, I mean, you need a little bit of everything I think, because everybody learns differently, everybody has a different way of thinking, and I don't know –
60		Romina	Which is why good teachers are able to adapt to that.
61		Angela	Absolutely, absolutely.
62	10:05	T/R1	So anything else in that ideal learning environment? You probably said some things earlier that might fit.
63		Marjory	Yeah, like not having desks in a row.

64	10:19	Angela	I think extracurriculars are very important. They teach you social skills as well and leadership abilities.
65		Romina	Yeah, but so does teamwork.
66		T/R1	So teamwork in the classroom as a way to engage students, afterschool to build on that.
67		Angela	Teaches you to think right away too, with that sort of logic that you develop.
68		T/R1	Can you think of anything else?
69	10:43	Romina	I think we should all have to present for teachers.
70	10:47	Angela	Have enough money for books.
71		Romina	What?
72		Angela	I always get so upset when I think about my sister and where she was.
73		T/R1	Well I think that your sister is seeing really, it's very interesting just to see the differences too.
74		Romina	Well see, I was actually talking about us the other day
75		Angela	Did you?
76	11:06	Romina	To someone at work, because he was, they all, I don't know, we were talking about the GMATs of course, because that's what my life revolves around right now, but he like I'd been pretty stressed about them, and he's like, I don't understand why you're so stressed, it's such easy stuff. I'm like, okay, so I have to be like okay, so, I didn't learn math like this. And he's like, what are you talking about? So I went and I explained like our whole program to him and how I learned math and how I used to learn math a certain way, but then someone that graduated from this program became our teacher for all of high school, so then we went through, like this is how I was taught, this was my everyday learning, and he was like, he was actually was kinda like, well that's the dumbest thing I've ever heard, why would they teach you like that? And I'm like, well, I mean, it's taught me different skills, so I went into all the benefits of the program. And he came back to me, and he's like an MIT grad, like very quantitative, very numbers crunching person, and he's like, well, he's like this is why I think, our country's so into doing, I guess Americans, he was like we're into doing such conceptual things we're not teaching people basic skills. And that's what I wanted to ask you today, what you guys about that. Because I have, I mean I only know from my own –
77		T/R1	He thinks we're teaching them concepts?
78		Romina	Like, it's just,
79	12:22	T/R1	He thinks we're teaching them concepts? Because that's the part that's interesting to me.
80		Romina	Well no, just like -
81		Angela	I disagree one hundred percent.
82		David	What does he mean by basic skills?
83	12:32	Romina	Well, it's like, so conceptual
84		T/R1	He thinks it's conceptual, what we're doing?

85	12:34	Romina	He thinks it's too conceptual, and too fluffy. He's like you can't talk about math, you just do math. And I'm like, well we used to talk about math.
86	12:45	Angela	But math is theory, like you have to talk about it.
87		Romina	I didn't say I agree with him, I just wanted everybody's perspective.
88		David	You do both, you talk about it and do it.
89	12:51	Romina	Right, but he's saying like people are much more, like, I think he was talking about in Chinese schools they're, they don't sit around and talk about anything. They do, they do, they do, they do, and they're much, they're doing better, supposedly on testing, and they're doing better on –
90		Angela	Yeah, but it's teaching them on testing.
91	13:09	Romina	Don't argue with me! I want to get their opinion. I already fought this fight.
92		David	You're saying in China –
93		T/R1	Are you asking if we have an opinion?
94	13:18	Kelly	What's coming to mind is what you just said before, though, when you were describing I think the Wharton people in your job,
95		Romina	Yes, but not –
96		Kelly	I'm not done. Interviewing, and you're interviewing with these peers who were trained in this very skills, memorize the facts, time is of the essence way, and your experience as a thinker, let's say, of that, in your workplace world, um, that's what came to mind to me right away, was that that's a mini version of this portrait that your colleague's painting, no?
97		Romina	I
98		Kelly	No, you don't think so?
99	14:03	Romina	No, I haven't had enough, I'm not sure, because I look at other people that I'm working with now and that I'm in classes with now, and they didn't learn like I learned at all and they definitely have a very you know like just doing concept and they're doing well, they have like that intelligence, like they have that raw intelligence and we're all at the place, the same place, and they did really well and they can do, I'm sure if they had to they can think strategically.
100		Angela	If they had to?
101	14:36	Romina	Well I mean, technically you have to for our job, but I mean, I don't think they're any worse off, and in some areas they do better than I do on testing, and they do better than I do at certain like quick thinking things. I mean, they're not in a bad spot.
102	14:49	Liz	But you are seeing only the successes. If you go back and look at kids in first, second, and third grade – I'm trying to teach now people who will eventually be first, second, and third grade teachers. And, you don't see the ones who don't make it to college. And they're the ones who as Robert was saying, or was something he was alluding to, they never get those basic skills because there are kids who never really, never ever understand decimals, and place value, and fractions and things like that.

			And, you know, the kids who get to college having had bad teachers can still manage and they somehow still understand what they have to do. But there are other kids who just fail because they never get that the conceptual understanding at the younger ages. So personally I think yeah, conceptual is very important, although you do have to know how to do it. I mean, you have to know how to perform the operations.
103	15:39	T/R1	I just want to just make an observation to build on what Liz is saying is that, I don't know about people in this room, but I wasn't taught conceptual. So some people, I don't know about you Kelly, I don't know about you Kate, but some people are able to figure it out, you know, despite, and they're very few. And that builds on the argument here. Your colleague, or those colleagues that are successful, are probably among those very few too. And the question is to make this and apply it to everyone is really faulty, because we have evidence, and evidence, and mounds and mounds of evidence that it doesn't work. And how do we know? Well schools work this way, and it doesn't work. Schools who try to change it, there is evidence that that is working better, and people, maybe not enough evidence because we're only starting to change it, because remember to change it, there have to be people that understand what the other part is to be in those situations to change it, and there are very few of those. You allude to the teachers, and how do you get the teachers with that kind of knowledge and understanding and depth and development. You know, how does that happen? There are always exceptions that somehow it happens, you know, and we all know that. But how do you now educate your –
104		Kelly	The masses.
105	17:08	T/R1	Well the masses significant enough to make that difference? That's, that's not such a trivial problem, that's part of the hard problem here.
106	17:16	Romina	I should have you come in and talk to him. I was like, I was like –
107		T/R1	What makes you think that he would believe me? Bob Davis used to say, and I would always say –
108	17:23	Romina	You have much more credentials than I do.
109	17:24	T/R1	That has nothing to do with it. See, people, you can give people lots and lots of evidence, but if they believe something, they're going to find a flaw in it. You know, they're not necessarily going to change what they believe. Beliefs are really very strong, and the argument well it worked for me, so it can work for other people. That argument doesn't hold either. And I guess, I often believe wait until they become parents and they try it with their kids, and they'll find out soon enough. Do you know what I'm saying? But it doesn't quite work that way. But they wouldn't believe it, they really wouldn't believe it, unless they had to experience, unless they had to work directly with kids or directly with teachers who work with kids, or in that classroom with Kate and her students and really understood that it doesn't work and what they call it and figured out and built on maps into nothing of the cognition of the learner. It sort of makes no sense, you might as well have been speaking

			a totally foreign language. And it's what they hear, even. I remember someone once saying you know, you let the students go, and they had an answer that was wrong and you didn't tell them? I often think of the fact that, how many, how many times kids leave classrooms where teachers tell them the right answer and they don't remember it. What makes you think they're going to remember a wrong answer that they can't reconstruct with any meaning. Do you know what I'm saying? So it's, I don't think there's, I like your explanation the best that it's far more complicated. But David has been trying to say something.
110	18:58	David	Well I think that, umm, -
111		T/R1	Oh I'm sorry, I interrupted Liz, let Liz finish first.
112		David	Okay, sure.
113	19:05	Liz	Well just, the thing about China. There is some evidence that Chinese elementary school math teachers are actually much better at teaching conceptual knowledge than American teachers. So when he says they teach the basics he they're really teaching conceptual understanding.
114	19:19	Romina	I'm going to go back with that argument.
115	19:27	T/R1	It varies, we notice, we're known in the United States for teaching a mile wide and an inch deep curriculum, which is something one of you alluded to, you know the time factor, we teach so many things, do we really need to teach so many things? You said this earlier, why not teach fewer and well and deep? So that you learn how to figure things out. Um, I visited several Japanese classrooms and these were not probably representative because it was at a university that was relatively new and they were very proud of what they were doing in their schools, and I went in, and this was ranging from middle to high school in Skuba, Japan, and I went in, and I was absolutely amazed with the teachers could have been working in the same way we were working with you guys in every single class with differences in personalities and styles and you know whatever. But what happens? Students were given a problem, and they worked together on the problem, and they had lots of time, and they went and they explained, they argued, they discussed, and it was really, I was, I don't know how representative it was, but that's what I saw. And I saw it in every single classroom I visited. Now, if they wanted to set us up to impress us as visitors, we're having a conference there, I still am impressed, because to get that many teachers to be able to do this, was impressive enough. It's not so easy. So I don't know if that answers your questions.
116	20:49	Kelly	You should give him *The Teaching Gap*, about the TIMSS study.
117		Romina	What is that?
118		Kelly	It's a book called *The Teaching Gap*.
119		Romina	I'm going to write that down.
120		Kelly	But yeah, give him the TIMSS study.
121		Romina	He also probably is one of those people who also
122		Kelly	There's lots of research on it.

123		Romina	Yeah
124	21:03	David	I think that thinking is what you do with the basic skills. At some point, you have to start somewhere, you know, and in one place I tutor right now, sometimes I'm working with first grade, second grade students, kids who have to subtract two numbers. Well, if you give them some sort of representation and let them kind-of figure it out, but, there's stuff, you have to have something to build on. So I think, you know, you take the basic skills, you build a conceptual understanding with those, and you use that to build more what we call basic skills until they build on top of each other. As far as the Chinese schools, or the Japanese schools, or wherever, again, I'm not familiar with Chinese or Japanese schools, I don't know how they work over there. But I think just you know, it's not a copout, but I think in their culture also, there's a learning in all subjects, not just in math but in all their subjects, but in math in particular compared to the United States is valuable. So I mean, I don't want to, I said this is not a copout because I don't want to say they value it more therefore it's okay to do all the drill and kill, but I think they value it more.
125	22:15	Angela	My friend, well my coworker, she's actually Korean, and like the whole, they way it works in high school there, is that high school is more like college, cause like you pick whatever your major is, and that's basically what you do all through high school. And she was very interested in art, and she draws very well, all that sort of thing, and her parents told her there was no way, no way on earth that she was ever doing that, and now she's an accountant. You know, like, that's what, and she said her brother too, loves art, and her parents won't let them, you know, and she said that's very typical, you know, I mean, not that, I can speak for a whole country or anything or she does, but I think that what people value, what they put emphasis on definitely has something to do with it too.
126	23:02	T/R1	Just a response also to your question is there's a difference in that those who become teachers - classes are bigger, but teachers teach less. So you might have a class of 45 students, but they teach fewer hours a week, not the way our teachers teach very long days with many different subjects. Different subjects, large classes. But in fact they're trained in the discipline, and have a very strong understanding in the discipline, however they learn it, I don't know, but they're not a, their approach is not through liberal arts. It's through specialization of a field. And learning how to teach is on, is on the spot. They spend a lot of time, you talk about Japanese lessons, they spend a lot of time developing these lessons that work and they have these books that you study all your life, you keep getting more, you spend a lot of time making this lesson, watching children do it, seeing what the problems are, the issues, the way they think, you know, and they don't do a lesson until they know it works. So, there is something to be said about some of these things. But there is a much more homogeneous population too. They don't

			have the issues with diversity we have, the issues with language we have, so much variability we have, there's a lot of I'm giving you the short answer to I think an interesting question you could have lots of lots of conversations with your colleagues. You have a question.
127	24:31	Romina	No, I have a comment. About the question that you asked us before about what we would change. I would make all the class sizes over 30 significantly smaller.
128		T/R1	What would you think would be about –
129		Romina	Uh, twelve. Twelve, fifteen.
130		T/R1	We would go, we'd like that.
131		Romina	Twenty, I could go twenty max.
132	24:53	David	Well if they're too small, you don't get as much discussion too.
133		Romina	Twenty, and then if it's too big, you can't cater to everyone's – twenty, I'll settle on twenty.
134	25:02	Angela	You know I had a grad course that had thirty-five students in it, it was huge, I couldn't believe it, because most of our grad courses had fifteen, twenty tops. And I thought it was awesome, like it was a shame that we didn't get to speak as much in class, but the comments that came out of people were just so different, everyone had such different life experiences, especially in grad courses when there were like, you know older people, there were people my age, one girl was younger than me, they were from all New York, all over NJ, other states, that kind of stuff. So I mean, I don't know, I guess when you're younger though, you all come from the same place
135		T/R1	So you like the diversity, and what that brings.
136	25:41	Romina	I don't think they should be, see I think they should start of small, because they have to kind of teach – we were so comfortable with each other, so I think I was fine not knowing something and being like, I don't know this, you guys, we have to go back and explain something to me for the tenth time because I don't understand this. And you don't have that comfort in bigger classes. Until you get older, and you gain that confidence, and that ability to accept that you just aren't going to know everything, which I know is hard for us. Like, then you can get the class sizes bigger. But I'm going to argue that they should be smaller when you're younger to kind of instill these habits and this, I think a lot of us have trouble learning because we won't ever fess up when we don't understand something or we don't feel comfortable actually voicing our opinions.
137	26:23	T/R1	It's trust
138		Romina	Yeah
139		T/R1	The risks you take.
140		Magda	Well you don't want to be embarrassed.
141	26:29	Romina	Well yeah, but we weren't. The things that would come out, we were never embarrassed with each other.
142		Angela	I don't really remember, I don't remember how I was dynamically in a

			group when I was younger.
143		T/R1	Wanna look at your tapes? You're probably not in too many, but we're very happy to have you look at your tapes, Angela. What do you say, Robert?
144		Robert	Okay.
145	26:49	Romina	Well I'm not saying young young, I'm pretty sure I'm saying high school.
146		Angela	I'm pretty sure I'm talking about gum all over them, or boys, or something completely unrelated.
147		T/R1	You were not, you were not in the program until later.
148		Angela	Right, sixth grade.
149	27:00	Romina	Well I didn't even mean to say that, I mean even in high school. I'm older now, and now voicing my opinion in a room of people I don't know that's like forty or fifty people is really hard for me, and the things I would say in front of you,
150		Angela	[Inaudible]
151		Romina	I think it's different though. But in high school, we could do like, anything. I wasn't afraid to like ask you guys anything or
152		Angela	No no, I know what you're saying
153		Romina	Yeah, because we'd probably been doing it for years.
154	27:21	T/R1	So the issue is like right now, how comfortable are you in new learning situations? Right now.
155		Romina	In what type of learning situations?
156	27:31	Angela	I don't know, I'd say I'm pretty good with being uncomfortable, you know.
157		Romina	She's a nutcase.
158		Angela	I know, I said I'm pretty comfortable with being uncomfortable. Like, I'm okay with –
159		T/R1	You don't mind being in a new learning situation.
160	27:45	Angela	Yeah, I like learning new things. At first you're like, eh, I don't know if I can do that, I don't know if I want to know that, it's just another thing for me to remember, but
161		T/R1	You jump in
162		Angela	I'm usually glad I do.
163	27:56	Romina	I've had very different experiences, because on the one hand, like I'm comfortable going into something I don't know, like when I get on a new project, at first I was uncomfortable because I'm too nervous about everything, but now I'm like so comfortable with being uncomfortable, not knowing what I'm going to do because I know I'll figure it out. I'm not that concerned, but it's funny because I just started a GMAT class where he sits up there and puts a problem on the board and he's like everyone what's the answer and you have to like scream it out at him, and it is the most uncomfortable, public
164		Angela	That's awful
165		Romina	Or the best is when he makes you hold up fingers for A, B, C, D, or E,

			and then whoever gets it wrong, he's like Romina, why'd you pick that, and I'm like ugh, it was wrong. And then I mean that is, so I've had two different learning experiences in the last year. One I'm completely fine with, the one where I know I'm not going to be put on the spot and told I'm wrong. But now this new one where I thought I was going to be fine and I went in the first day being like oh this is going to be fine, and –
166		T/R1	How long are you doing this?
167		Romina	Two months.
168	28:54	Angela	But being put on the spot and being wrong, kinda makes you comfortable with it. It's how they train law students, I mean, like they don't know your name, they point at you and you have to answer the question.
169	29:03	Romina	I'm not really comfortable with it. But I mean, I'm learning, but it's not like, I'm trying to learn basic math things, like he's like you didn't add right, and I'm like, public humiliation-
170		Angela	I'm not saying it's okay
171		Romina	But in front of 25 people there I don't know
172		Angela	But what I'm saying is like I guess you get used to it, maybe it's not right, but you get used to it
173		Romina	Maybe I'll get used to it
174		T/R1	Well maybe you won't. Magda, what about you?
175	29:23	Magda	I don't know, um, I'm generally a shy person so I won't like step up to the plate. When I find, like, when I find is usually I know the answer, but I won't step up, so
176		T/R1	Mmm hmm, mmm hmm. But if someone came to you and asked you, you –
177		Magda	Yeah, I think I'm better at one on one situations
178		T/R1	Or small –
179		Magda	Yeah
180		T/R1	So that's the kind of learning situation you like.
181		Magda	Yeah, oh definitely.
182		T/R1	And we, oh I'm sorry go ahead
183		Charlene	When we thought of this question, I don't think any of us were thinking of a learning situation where like the spotlight was on you, and you had to enter and you were humiliated, so think about a question where the environment is safe, or somewhat safe.
184	30:00	Romina	No, I'm very comfortable, and one thing I'm comfortable with is I'm one of the youngest people at all, almost all the time on my project teams, and like, I feel very comfortable asking questions. I mean, I think we've always worked, we've always worked kinda facing older peers, so I mean, I feel very comfortable asking questions or taking the lead or questioning people when I don't think their logic is right. And, I mean, maybe I shouldn't feel as comfortable questioning my superiors, but I am. But they seem to like it. And I also work with people that

			think and work like I, like we've grown up with
185		Magda	Yeah, but it's usually like smaller groups that you work with.
186		Romina	Yeah, it's smaller groups, and they like it. They love it, they're like thank you because you're constantly pushing them, and they push me back and it's great. So it's a good learning environment, because you just learn more.
187	30:54	T/R1	So do you like to work with a group, rather than by yourself? With a small group, with a large group?
188		Angela	I like to work by myself.
189		T/R1	I asked this question to the other group. You like to work by yourself?
190		Angela	Yeah, I don't know.
191		T/R1	You want to know how Robert answered this question?
192		Romina	Robert, how did you answer?
193		Robert	I don't remember. Something about group work, about people with similar skill level.
194		T/R1	We do, we do – yeah, but if he had his druthers he'd rather work with a computer.
195		Robert	Yes.
196	31:23	Frances	His very study partner is Google.
197		Robert	Yes. It is.
198		T/R1	But tell them why, because I think why is really important.
199		Robert	Oh, well because –
200		T/R1	I found that to be really interesting as a response.
201	31:35	Robert	Well I mean as you probably know, if you put a word problem, like quotes everything, you'll get answers to math problems, because a lot of people use the same textbooks and they put answers online, but I think the good thing is then you get other teachers how they presented the material is differently than how you might have been presented to it. So you get different points of view. So I find that really helpful.
202		T/R1	These are his partners in his group.
203		Robert	Yeah, so basically –
204		Romina	It's like a nonspeaking group.
205	32:02	Robert	Yeah so I guess the professor from the other school is actually your partner indirectly.
206		T/R1	What do you think of that Kelly?
207		Kelly	It's a new way of thinking.
208		Robert	So it's like taking a class.
209	32:16	Angela	I like getting to my own answer to myself first though.
210		T/R1	But doesn't everybody? You have to have something to talk about first though. You have to try to do something. I mean you can't try to talk to somebody about something unless you have an idea. When you guys used to work on stuff that I observed you were quiet in the beginning. And then when you, either if you were stuck you would start talking to someone or if you had an idea you would throw it out. Isn't that how

			people –
211		Magda	I was going to say that now, I'm studying for a CPA exam.
212	32:44	T/R1	Oh you're another one studying for an exam. You're doing it by yourself?
213		Magda	Well, the thing is I have a friend that's taking it at the same time, and he's like oh, study with me, study with me. And I'm like no, I need to go through the material first so I get comfortable with the stuff and then once I have read it once or twice we can do problems and stuff like that, so, I think initially, maybe to just get yourself comfortable with it, yes, do it by yourself. But ultimately I think like working in a group is good.
214		T/R1	And what do you gain by now, by working with the other person? Why don't you just do it yourself?
215	33:14	Magda	Well you can do it by yourself, but I find that you can always, like if you're stuck you can ask the person, you know how did you get to that answer, or can you help me out, or something like that. I mean, I think that helps a lot.
216		T/R1	Did you like to do that at all, Angela?
217		Angela	Yeah, no absolutely, I mean in my classes, like, I like to, I really like the whole discussion aspect.
218		T/R1	But now you as a learner, I'm interested in you as a learner.
219		Angela	Yeah, that's what I'm talking about. Like in my classes I loved sitting and –
220		T/R1	Talked to other people
221	33:41	Angela	Absolutely, because everyone's got their own frame of reference, everybody has something different to bring to the table, that sort of thing. But I always, even you know with everything, with every class I've ever taken, I always like to get to the answer first, try to get to the answer first myself. Because you get a little sense of accomplishment I guess, I don't know. Or without distraction.
222		T/R1	It's your own idea, you want your right
223		Romina	Define the answer though, what answer are you getting to?
224		Angela	Well no specific answer, I mean, I don't know, with anything. Whatever the challenge goes to, if it's read this book and tell me what you think about it, you know,
225		T/R1	You want to have something.
226		Angela	Exactly, I want to have my notes, my page of notes on how it's representative of money, or whatever. I want to have that before I come to class and start a discussion or participate in a discussion. Which I mean, I said I love the whole discussion aspect of it, I think it brings so much more to learning.
227		Magda	Yeah, but it's, you're preparing, that's the whole thing.
228	34:41	Angela	Exactly, that's why I said I like to get to, I hated groupwork in high school that was like, alright, here's your assignment, and I'm going to break you up into groups, and one person takes notes, and one person write this section, and one person –

229		Romina	But then you're working individually
230		Angela	I hate it because then you have the slackers
231		Romina	It's like individual work that you have to combine together.
232		Angela	Yeah, but even when it's not just individual work, it's always like, it's such a chore. It's like you have to get with those people after school, you like have to find a time where your schedules meet. And then one person doesn't do a thing, and they still get the same grade as you.
233		Romina	Angela was the person that was on the team that did all the work.
234		Angela	I'm a control freak. [Throws up hands] I'm okay with that. But no, I usually teamed up with Magda, so we all did our share of work.
235	35:27	T/R1	Okay, I just have really one question and then sort of want to segue I know get to an idea that's particularly, that Charlene has a more specific question. This has to do with math. So if you can put yourself into the math mode again, it's a general question. And the question is, is it important to make connections about mathematical ideas? Do you think it's important, yes or no, and if yes or no, why?
236		Romina	What do you mean,
237		Angela	Like in terms of learning? Like this why it is?
238	36:00	T/R1	If you have one idea in mathematics, can you imagine a mathematical idea where it would connect to another mathematical idea?
239	36:10	Romina	Oh, oh okay.
240	36:12	T/R1	Is that important to have connections of an idea to another idea?
241	36:18	Angela	I don't think it can hurt to have those connections, I mean, you get an idea of the bigger picture.
242	36:22	Romina	I hate learning things that don't, like I feel like if you learn one concept that doesn't connect to other concepts, like you're learning something almost useless. Because it's never ever going to be presented to you –
243		T/R1	That's a yes answer.
244	36:32	Romina	Yes, well it's never going to be presented to you. Nothing is ever going to be that simple. Nothing's going to be presented to you as just one little issue that if you figure that out it's all done. It's all interconnected as it is in the real world.
245	36:45	Angela	It's like maps, you know. If you know one little section of say, Staten Island, like if you know the bottom, you know the top, it helps if you know how to get from the bottom to the top. So, they're all, it's one big thing that makes sense.
246	36:57	Romina	It also allows you to like grow conceptually, because if you have like the first part and you go to something a little bit harder it just, without knowing it, your mind is growing with the concept.
247	37:10	Magda	Well, it's the whole idea I need to know how to add to do multiplication.
248		Romina	Yeah
249		Angela	Yeah
250		Magda	So, I'd start it like you were saying, you need to have the good foundation in first, second, and third grade to be able to succeed and do calculus problems in like high school or college or something.

251	37:28	T/R1	So okay, do you think that's true outside of math as well?
252		Magda	Yes.
253		Angela	Yes. Everything. There's no harm in seeing the bigger picture.
254	37:41	T/R1	Okay, Charlene is working on some video tapes that involve all of you, you were all here in the 8th grade, right? And um,
255		Angela	My sweatpant days. I said I had a sweatshirt with me -
256	38:00	T/R1	I'm just curious, I don't know if Charlene's going to ask you, if she's going to bring out the problem at all or anything, but I'm just curious even if you remember in the 8th grade.
257		Charlene	June of your 8th grade year.
258		Robert	I just watched the video, so I know that.
259		Charlene	It was a long time ago.
260	38:20	T/R1	So in the 8th grade you worked on a problem, right, I was there for a little while, and it had to do with surface area and volume. Do you remember that?
261		Charlene	You were using the rods, do you remember?
262		Angela	I remember the rods.
263		Romina	What room were we in?
264		Angela	What room were we in?
265		Romina	I don't know, that always helps me.
266		Angela	Yeah.
267		T/R1	You were in your 8th grade classroom.
268		Angela	Mr. Poe
269		Robert	Mr. Poe
270		Romina	Oh, Mr. Poe's class, okay.
271		T/ R1	No I think Mary, Ms. Toy was your 8th grade teacher.
272		Charlene	Well the tape said Mr. Poe.
273		Robert	It was in Mr. Poe's classroom.
274		T/R1	It was in his classroom, but he wasn't the teacher.
275		Robert	Ms. Toy was the teacher.
276		T/R1	Go ahead and tell them Robert. She needs to know that.
277		T/R1	It was in a larger room, and do you remember the problem, with surface area?
278	39:10	Romina	What was the problem?
279	39:12	Charlene	You take let's say three red rods, and put them on top of each other, and T/R1 asked you to figure out the surface area using the small white rod, as a one unit stamp. So you would stamp it –
280		Angela	It's kind of a little familiar
281	39:32	Romina	Yeah, it is [rubs head] No towers?
282	39:33	Angela	That was like a really long time ago
283	39:33	T/R1	Do you remember towers?
284	39:34	Romina	I will never forget towers.
285		T/R1	So, so the notion is that um, you were, you configured these rods and

			also considered different sets of colored rods a variety of ways, and you were actually dealt with these as three-dimensional figures, and so you could think of the stamp as surface area and you could also think of the rod that was one unit as one unit, we called it one square unit, as a measure of volume, and you were asked to find the volume and the surface area of these, we would create them sort of as stairs. First we stacked them like a suitcase, and then we –
286	40:22	Romina	I'm vaguely remembering…
287		T/R1	Is it coming back?
288		Romina	No yeah, I do remember that.
289		Magda	I don't remember.
290	40:32	T/R1	Okay, that's very interesting. Well she's working on looking at how you worked on that, and one of the interesting questions is that you were asked to solve some specific problems, specific surface area and volume problems, and you ended up solving general problems. You weren't asked to do that, really, but you did it.
291	40:50	Romina	I remember that, like coming up with a formula? Yeah, I remember that.
292	40:55	Angela	Well, that's what we used to do all the time, right?
293	41:00	Charlene	So actually I guess, Romina, it will be like some kinda question mostly for you, but it will be for all of you. You talk a lot now about having some kind of, I'm hearing it as sort of a resistance to the formality of mathematics as opposed to what you really are looking for is understanding the concepts. When you were in high school, and they really started it in this class in the 8th grade when you were moving towards the formula, if you're able to, I don't know how to ask this so it's not very leading, but, in thinking, especially looking at this problem where the conceptual came first, and then you created the formula, do you find that the formula stays with you longer when you're thinking about the conceptual first?
294	41:47	Romina	I, yes, and this has actually happened to me a couple of times, because one of the formulas, what's the computation, combination formula? There's a C, and then there's a letter here and a letter here [gestures with hands]. Come on –
295		T/R1	Robert is allowed to help you
296		Romina	Yeah, what is that? And then you do like this one factorial over this one factorial
297	42:09	Robert	Isn't it like n factorial over n minus r factorial? Or r factorial?
298		Romina	Is that…? Yeah, so
299		Robert	Where you're choosing r items from n…
300		Romina	Honestly, I haven't seen this in probably like six years
301		Angela	I took one math class in college, and it was like, I don't know, horrible
302	42:28	Romina	Because we, I don't know if you, but we studied probably probability and combinations for years, and years and years and years, and we never had that formula til like high school our junior year you probably told us. Umm, and –

303		T/R1	The night session
304		Romina	Yeah, cause we
305		T/R1	Well, might have told you before
306		Liz	They had it before then, or somebody had it before
307		T/R1	But it was that year
308	42:47	Liz	Michael got it from Robert he said, I think
309		T/R1	I think Robert discovered it though, I don't think I told you before Robert discovered it and told Michael
310		Robert	Yeah, I remember in Mr. Pantozzi's class –
311	42:59	Romina	See and for me, like for me, like we built that whole concept and then we were introduced with this formula, so like that formula I, when I look at that because I remember I had to do it my first year in college and I remember looking at that and it's not like I can memorize a formula but I would look at that formula and I was like okay, so this means that I have my options for this could be a tower 5 tall and I have 3 blues and 2 whites, and that's how I remembered it and where the numbers went [gestures]. So for me I really took probability and combinations a lot very conceptual. And even now I'm trying to relearn it, and the way, I haven't done this in years, but the way they're teaching us now, it's similar and I think they taught us a whole new concept that I'm trying to learn now in my class, and I am completely confusing it because it doesn't align with my conceptual knowledge of like, okay, how many spaces are there, which is how high it is [more gesturing with hands]. How many different colors do I have, and it's really mixing me up because they're trying to teach me in a different, with a whole formula that's different, I can't associate conceptually so I'm having so much trouble just memorizing this one formula, I can't do it. Like it's very simple, it's like yeses and nos, and I can't do it cause it doesn't, cause I can't associate it conceptually. Does that answer your question?
312		T/R1	You need to spend some time with Robert
313		Romina	Yeah, Robert, you want to be my tutor?
314		T/R1	He could be a very good tutor.
315		Robert	Okay, sure
316		T/R1	He really could.
317	44:17	Charlene	So, it's really important to you to be able to make a connection from that formula to where that formula comes from.
318	44:22	Romina	Well and especially because I find it like in a lot of problems what I would have trouble with is I don't automatically associate a formula when I read a problem. I think about a problem. So if I'm thinking about a problem and I kind of understand what it's asking me first and I have to draw some sort of picture, and like, I still do this a lot, I still draw some sort of picture. Then, the formula will hit me much later after I thought about the problem and thought about the picture. I'm like, oh, so this is that formula where we used this. But it doesn't, it's not an automatic association if I have just the formula. I have to have a

			concept in mind, and it's not good because it takes me a long time to do stuff and I don't learn it right away.
319	45:04	T/R1	Well, I've asked all the questions we had plus a bunch of others. Is anybody, you want to ask anybody here a question or does anybody want to ask a question? We can open it up for questions or you can say you're tired, you don't want to talk anymore and get something to eat and drink, it's your call.
320		Robert	I have a general question.
321		T/R1	Go ahead Robert.
322	45:22	Robert	So I don't know much about like this, I know problem solving's good, but why is problem solving like construction of an idea, emphasized more than deconstruction? You know what I mean? Like, when I do problems, I always deconstruct and cause when I took Advanced Calc I had a horrible time, cause I didn't understand how to write proofs, I'd never done it before. But like, the way I actually learned it was I deconstructed everything. I actually kinda constructed a spider web, and I saw that every idea was connected to everything else, but like I, in all the interviews, everyone says problem solving, problem solving construction. But I don't understand why deconstruction is –
323	46:10	T/R1	I'm going to take a stab at this one and hope Kelly can join me and other people. But, there are certain kinds of, within certain problems, with certain content areas, certain ways of doing things. Certain forms of proofs that are acceptable to certain communities, and certain content domains, if that makes sense. And in a sense that there's sort of the rules of that ritual, of that discipline. And that's not something you're going to discover, you have to be enculturated into it. And the enculturation is, people don't know how to enculturate people into it, and if they knew how to do it, students would be successful at doing this. And what you've just suggested to me, I don't want to read into something you didn't say, but, here you are here you're faced with this. You haven't been enculturated into how to do this, and, and I think that's a problem. I think this is something that I think people who teach the courses ought to be worried about, and there are more and more people who are worried about it. You know I think people who teach at this level are starting to deal with these issues and meet and have conferences about. So you found a way to do it, and in a sense you're looking for well what is the acceptable model in this content domain of what is going to be acceptable as, for, it's not just a reasoning or end state of reasoning, I can give you an argument, I'm convinced by it. What's acceptable in another domain or another level, you've got to figure out what it is as you move through the hierarchy within that discipline. And, and, what you're, you have a very interesting idea here. I mean, you're making a very good suggestion of one way to do it, and um, we should probably have more conversations about it, but I don't think, I think that you're not going to discover this, you're really not. I mean, you could discover the idea behind a proof, you could have the

			insight and conceptual idea, but it's still not going to be acceptable because someone may not know that you have it and not accept your case unless you follow what is acceptable. I don't know, Kelly? You may add or subtract or disagree.
324	48:10	Kelly	No, I don't disagree. But, but, it's, just tell me if I'm misunderstanding you. It sounds like you're talking about reasoning within trying to prove something that kind of problem solving.
325		Robert	Yeah, like I don't know, I guess I was like I don't understand problem –
326		T/R1	Proving, he's talking about proving.
327	48:29	Kelly	You're talking about proving? So, as [T/R1] said, there's a whole body of convention, first of all, and ways of thinking about things. There's also what a proof attempts to do, anybody's proof, a seven year old's proof, um, is convince someone of something. And so it attempts to be a reasonable way of making sense or a reasoned way of making sense out of something. And often, least with mathematics, and this is where it gets interesting, is this now convention or historical enculturation only. Back to Euclid, we've always started from as few definitions and as few axioms. That's the game, it's like how much little can we take for granted and then build up what we're trying to prove. So, or assert with certainty. And so part of what you're being brought into by being taught that way is that aspect of the way the game is played. Mathematician back there [points behind her]. Umm, and that's reasonable, but what you're saying is, that's interesting and wise is ultimately your goal is to understand the notions and the concepts, and you did that by taking it apart. And so that's
328	49:51	Kate	Is what you're saying that you're not really doing the math by taking it away?
329		T/R1	Sure you are
330		Kelly	Oh, is that what you're saying?
331		Kate	No, or –
332	50:03	Robert	I guess I was saying is how come they emphasize building from nothing instead of taking apart. And then like why is one method better than the other.
333		T/R1	I think, do they just expect you to build from the bottom? Do they ever do a take apart?
334		Robert	I think like all they, well I don't know with combinatorics, I don't know, I think that basically –
335		T/R1	It sort of lends itself. Certain things lend itself.
336		Robert	Yeah
337		T/R1	Certain things lends itself. Like why do we work in that domain instead of another? Well it lends itself to it, and you could build up. Certain domains are not so easy to build up that way.
338		Robert	Yeah, that's what I was saying.
339	50:30	T/R1	And this is the argument that before so, you know, well don't worry about that right now just believe it and then move on. Take this as okay without having to give evidence because you don't really have the skills

			to do that now. But you do have the skills to do the reasoning you know to get to the question you're really being asked. That's called, I call that hand-waving. And a lot of that goes on, and the people who that really bothers – I really can't go on, because I really don't know what that is, and you're telling me to just trust you. When does that trust stop? When am I supposed to know to just trust you and that's troublesome, isn't it? That's really tricky. That just shows you how much work there is to be done. I mean, that really is a challenge in teaching and learning for all of us to find these ways to help people to really, to have that deep understanding, to want to build on that deep understanding. Some people don't want to have that, you know, and I think in every domain that's true. Like, you know, you must all know this – please just fix my computer, I don't really care what you're doing to it, I just need this to work right now, and I really don't care I'll figure out why this didn't work maybe later, but right now I just want it to work. You know what I'm saying? And so, that's a domain, or another domain, or the car you know? Some people want to know why that car works and some want to be able to do these things and some people want to say you get a competent person to fix this vehicle. And so I think that there are some analogies like that too, get me through, this is not my real thing.
340	52:03	Romina	Well, how do you think though, I mean, I know at our school, our high school when we tried to switch over to more of like the Pantozzi method if teaching there –
341		T/R1	There was a rebellion.
342		Romina	Our high school went up in – I have never seen, I don't think our town cared about anything
343		T/R1	Yeah, but you have to understand something Romina too that is part of the politics of it all. You're, you had very traditional teachers from a very traditional background
344		Romina	But even the students fought against it
345		T/R1	But don't you understand? They're part of the system.
346		Angela	Because that's what they learned from
347	52:39	T/R1	But the part of how they - yeah, listen, I remember when my husband and I were teaching in Georgia, Augusta, Georgia, this was a riot. He was in the military and we were teaching you know, in the schools. He was teaching at the college in the evening which was part time and I was teaching in the high school and at the time, they were making a big move, this was called modern math, and they had adopted statewide, it was on the recommendation of the University of Georgia. Was the Dolciani math books. Some of you know those books, some of you don't, some of you who have been around. But the Dolciani math books was supposed to promote understanding, and those were books that I used as a young math teacher, I was your age at the time, and so they were doing this massive change in the high school and this very, very traditional teacher stands up, teacher gives test, teacher fails 60% of the class all the time, all the rows, and this was extreme way of teaching.

			And so the teacher says, ha, we have new books look at what they put you through, look at all the steps you have to go through, I'm going to make it easy. All you have to do is move this, and divide this, and get an answer. Kids loved it, and they'd test them on that, and look how, look what I'm saving you from. The teacher is coming there, really overtly making statements like this, not probably understanding those steps that are there either. No teacher development there, no opportunity for the teachers to learn it, and that's part of, they were, the stories can go on and on. But you see it's very complicated, because it's very, very hard to change. There's a lot of politics that went on in that community. If you think of the politics of your community even before the resistance to change –
348	54:15	Romina	But I think of, because we were all in the same classes so maybe it was that we were conditioned to it, but I know that we used to drive our teachers crazy because we'd always be like why? And they would have to go to the next level, like in our chemistry class we'd be like but we don't understand exactly why that happened. She's like you just do this and we're like no, why? And that used to drive them insane
349	54:39	T/R1	We actually have an interview of a teacher who is, one of the teachers you have, I'm not going to give the name on the camera, but the teacher really resisted change and blamed all of the problems the students were having on the teachers of those students. But those teachers were trying to change, you know, trying to introduce more thoughtful and conceptual understanding, especially if they could, and this teacher was still writing on the board, writing on the board, and wouldn't change because this teacher knew the math, and um, it was everybody else who had to change. Until that teacher had you guys, and she, the teacher had to change, because the teacher said they wouldn't. She actually had to get new furniture, had to get new tables, and that teacher did change.
350		Romina	In high school?
351		T/R1	No, this was not in high school.
352		Robert	I think I know who this is.
353		T/R1	Turn it off
354		Romina	But people, don't want to, they were resisting, actively resisting learning. The fact that he asked them why, I could not understand that, and they blamed all their failures and –
355		Angela	Mob mentality, something very strong.
356		Liz	I get that, I get that, with my students who are going to be teachers, because they already know the rules, and a lot of them don't want to know why.
357		T/R1	Let me show them how to do it, get them through it.
358		Magda	I mean, my sister had Mr. Lombino, and then she had Mr. Pantozzi, and she hated Pantozzi.
359		Angela	Same with my friend Jaime
360		Romina	I never had the other side of it, but
361	56:00	Magda	Yeah, because Mr. Lombino was like these are the formulas this is how

			you do it [motions with hands], and that's how, she, whatever. But now I can tell you that she has trouble with Algebra. That's like exactly what I was teaching her when she was asking for help. It wasn't like she wasn't getting the whole concept of integrals or derivatives or – the Algebra was missing. That's what it was.
362		Angela	I think part of it though was
363		T/R1	She was rule bound you're saying, just rules, rules rules
364	56:30	Angela	Why people felt so, why people hated it, why your sister hated it, why Jaime hated it, was because it made you feel so uncomfortable. You have to learn. You can't just sit there in a classroom and be like okay, I'll pay attention to this later in my book, you have to participate in class, you have to, there's a whole lot more responsibility on the student that way because you really have to understand it
365		Romina	What I don't understand though, and I wouldn't def – There was a certain success rate that was involved with our group that was associated with our group. Fair?
366		Angela?	Fair
367	57:07	Romina	Fair. I would say in our high school, right? And we, not only were we, okay, so we were taught math this way, and we were also very successful. And this could happen by chance this way, I don't know, but we were also very successful in a lot of areas of academic studies as well as extracurriculars, as well as, you know like overall, well, we were –
368		Angela	Overachievers.
369	57:24	Romina	No, I mean, I don't know if we were overachievers though. I wouldn't say that we were all overachievers. We, and people say that and still fought against it which is what I don't understand –
370		Marjory	People don't like change.
371		Kelly	There was some, a colleague of ours did a study maybe three or four I don't know, let's say three or four years ago in Iowa in large school districts in Iowa, and they were trying to implement a reform-based curriculum. Right, a curriculum in high school math
372		T/R1	Which is thoughtful math.
373		Kelly	Using problems like –
374		T/R1	Problem-based.
375	57:58	Kelly	But extended problem based, like you work on a problem for a few weeks and then you did some practice but you were still really involved in this concept. And her finding, and she went to the board meeting, she did a whole bunch of stuff. And her findings basically, it's a bit of an overgeneralization, but, is that the people who opted, it became an option to keep the old traditional way do you want your child to opt for this, it was a parental decision. The finding overall was that the professional families, college educated families, opted for the reform, and the non professional families, and her work is about social class which is a very, very complicated thing and so I won't even go there. But so upper social class versus not upper social class chose traditional, and part of what was studying to her was that she thought that the

			working people would sort of want what the professional classes saw was right for their children. But it's not the case, at least not in Iowa right now.
376	58:48	T/R1	Well it's not only in Iowa, but I know of a very affluent privileged district, very wealthy and one of my former students was a principal there, no longer but was, and they had some really wonderful teachers doing wonderful things and I said what do you do with the teachers who refuse to change and so forth, and she said oh don't worry, there are parents who sign up early and they want their, there's a great demand for the parents to put the children in those classes. She says, it's amazing, she said it's surprising to her, and she doesn't understand it, but there's a sort of sense well this worked for me, so this is what I want, this teaches you discipline, but they don't look, this is why we're asking you these questions. I mean, all of you are examples in opinion, or were a long time ago, but talk about existence proof, but all of you have achieved, really, amazing success. So young. I mean, there's not a question about that, but really the story's still being written, but look at now, you're young and you're doing these wonderful things. Okay, so maybe you could argue we have an unusual sample. We have a sample of all gifted people and gifted in every one of these aspects you know, and it's possible. The probabilities may be very, very small, but it's certainly possible. You said Romina, it's certainly within the realm. I don't think I believe that, but I believe there are certain things you've learned, that you're together and caring, because you're telling me these things, and I had not imagined any of this. I mean, I was there because I didn't like the way the math was taught and I wanted you to meaning in your math and build it. But there were other by products of this, there's more here. And it raises lots of interesting questions, so this story's still being written. So just to say that, but you should also know that people are interested in this story. There's some that, a significant and growing number of people who are interested in this story. Who may not been interested even five years ago. It's really growing, recently even more so. In fact it will grow more, cause you're all on, what's this national digital library, NSDR library, and I'll send you the URL for it. You can go find it yourself. And it happens to be when in the summer institute they had the catwalk, remember that one? And it's just really lovely? Have you seen that?
377		Kelly	Oh yeah, yeah, yeah yeah
378	01:01: 02	T/R1	I know you've seen the Catwalk, have you seen the NSDR library?
379		Kelly	No, not yet.
380		T/R1	This is a Harvard Smithsonian Astrophysics, they have a
381		Marjory	They're the people that helped us make the private universe project
382	01:01: 35	T/ R1	Yeah, but what they did is they had another project. The project was for science, they do science a lot, and it's to show inquiry science and to have this, I was on the advisory board for a grant, that's how I know

			about it. But I didn't know about the project until our last visit with Robert and Marjory and some others, and in it, they showed us some of this, what they did was they took a few pieces of ours, of our tapes and they put it on the library. Now, there's another tape. Do you remember Amy Martino?
383		Unknown	Yes
384	01:02:09	T/R1	Um, she teaches at a school and she does wonderful teaching, she does fourth grades. And there's video of her teaching her class that never got in PUPmath. And that's they put that piece on the inquiry science library. There's the other piece that's on the World Series problem, which I can't get the video to work Robert, you should check it out and see if you can get it to work or tell them?
385		Robert	Okay
386		T/R1	And then the other one, this is the summer institute, this is the 1999 summer institute, remember with the Catwalk? [Lots of talking]
387		T/R1	No, the World Series you did in high school. You did in 9^{th} or 10^{th} grade, yeah. You were there for that? You remember the World Series?
388		Romina	We did it in class?
389		T/R1	No
390		David	That was Kiczek's dissertation, the one I was looking –
391		Romina	No I think we, Pantozzi brought it into the classroom too
392		Robert	No, we were in a different class, we were in Mr. Shuster's class.
393		Romina	Oh, who was in our class?
394	01:03:01	Robert	Because I remember I did it over the summer with Mike because I wasn't exposed to it. We didn't have the same class.
395		T/R1	Oh, okay, so we must have gone into do it.
396		Robert	Yeah, me, Angela, Magda, and a couple other people had a different Algebra class.
397		T/R1	That's right, you were not in Gina's dissertation. You were later on in Lynn Tarlow's.
398		Robert	Actually I was in Gina's because we did it over the summer with um, me and Mike did that and Pascal's Points…
399		T/R1	Right, right, right. That's because of Michael.
400	01:03:37	Magda	We were with Nicole, Warner, and a whole bunch of people who hated us…

APPENDIX I: TRANSCRIPT – REFLECTIONS V

July 15, 2009

1 Camera View: Romina Reflections (*2 Disks*)
Date of filming: 2009-07-15
Room 236, Graduate School of Education
New Brunswick, NJ, Reflection for the Kenilworth Longitudinal Study
Transcribed by: Margaret Steffero
Date of transcription: July 2009
Verified by: Maria Steffero
Date of verification: July 2009

DISK __1_ of 2

Line	Time	Speaker	Transcript
1	03:24	Romina	I just did empirical method in strategy. So it was challenging. So I figured I'd challenge myself in my last quarter of school.
2	03:33	T/R	What was challenging? I'm interested to know.
3	03:39	Romina	It's a very undefined class so you had to use a lot of statistical methods to kind of come up with a theory and present your theory. It's kind of a little consulting case every couple weeks. So he doesn't…he just kind of gives you hordes of data…thousands and thousands and thousands of excel rows of just numbers and you had to come up with some sort of hypothesis and test it and make it statistically significant. So he doesn't really tell you how to do that. So you just have to figure it out. And we have to work in groups. So that was really difficult.
4	04:07	T/R	And you found that difficult too…
5	04:10	Romina	No, I'm luckily used to it because of this program probably. So it's kind of funny because business school was pretty much this every day in every single class. So we always work in groups but it's a little difficult to get five people looking at the same numbers thinking the same thing
6	04:29	T/R	Oh, boy. Okay.
7	04:30	Romina	Dr. Maher would love to hear that I do this every day now. (laughter) In every class. So even writing a paper in groups is challenging, but we do it.
8	04:41	T/R	Wow, ok, I definitely want to return to that later then. So…
9	04:50	Romina	I also just taught someone how to do a derivative and what it means based on my little graph and my…I don't know if you ever saw it…the graph in the shaded area that Mr. Pantozzi taught us…he taught us that..I just taught it to someone which is sad because at the age of thirty we're business school students and we should know how to do that.

10	05:04	T/R	Is it something you're very comfortable with?
11	05:06	Romina	Yes, because we did it so much and I don't think other people did…They just kind of memorize a formula and when you forget the formula, it was kind of hard to figure out how to do a derivative. So, the professor says, "You just take the derivative," and everyone is kind of, "Ugh"…
12	05:17	T/R	So that was something then that was familiar to you.
13	05:22	Romina	Yeah, I'm not going to…I'm not like Bobby…but I can remember a couple of things.
14	05:31	T/R	So when you were talking about derivative—so what is a derivative for you?
15	05:37	Romina	The area underneath a line, which I know is very basic…the rate of change, I guess, which is measured by the area. I know, that's probably wrong at this point, but that's how I'll always remember it and explain it.
16	05:45	T/R	You gotta forgive me too, I don't know as much about the business - what is
17	05:57	Romina	Elasticity of demand. So the elasticity of demand is essentially just change. It's just a formula, just a derivative and that's all it is.
18	06:04	T/R	And that's something you would be looking at in your daily job or something…
19	06:06	Romina	I hope not; it depends. I have had projects where you look at that. But, that's very kind of academic. And I don't think…we don't really use that in the real business world. That's a little too high level; but in some cases you will. But it's people who are much more advanced in that kind of stuff than I am…like actual econometrics
20	06:29	T/R	And that was something…
21	06:30	Romina	In class, we'd do that a lot.
22	06:32	T/R	Interesting…so you're teaching me something about…well, maybe at the end, I'll want to hear more about the visit. So, if you just had to summarize, what was something you liked best about business school?
23	06:45	Romina	I mean, I think it was the people I would say. There are 600 people who are kind of like me, we're kind of a little anal, a little overachieving, You put us all in a room. It's really funny, I've learned a lot more about group dynamics and stuff, because we had to work in groups for everything and Kellogg is just---I mean, every school is different-- but that's what their schtick is—teamwork. That's why I wanted to go there. You learn a lot about people, and people's priorities and how to manage that. We had to hand everything in, in groups. It's a group project for…even papers, I've had to write twenty page papers with people, so…
24	07:24	T/R	With the other people sitting…
25	07:25	Romina	We had to figure out how to do it, the most effectively…
26	07:28	T/R	So, what's something you learned new about group dynamics that you didn't know before?
27	07:38	Romina	I mean, I had a lot of experience. I did that a lot; through this program, growing up with the same twelve people all the time. And, at my job,

			that's what we did; we worked in small rooms with each other all the time. I just think it's always surprising what motivates people and figuring out what motivates people is kind of like a new thing every single time. Even when you assume someone…you know, you're at a good business school everyone should have this as a top priority and it's not. And figuring out how to divide and conquer sometimes; or you have to sit in the room for three hours and just work together on coming up with theories…
28	08:13	T/R	That's interesting. And here I promise you we'll go backwards in time. So, you have participated in a longitudinal study for a long time. What are your first memories? So…
29	08:32	Romina	…the towers, right there? (laughter) Yeah, those towers, two colors, four high. I don't know if that is my actual first memory, but that's the first thing I remember.
30	08:44	T/R	So, when, so that's literally the first…you remember making the towers…Did you start in first grade? When did you actually start? Do you remember?
31	08:54	Romina	I don't…Bobby, do you remember? They came, they started coming in first grade, but I became a regular, kind of member in the fourth grade, I believe. In Mrs. Barnes' class…I'm looking at Bobby because he might remember. I don't know. I might have been one of the first grade originals; I don't know.
32	09:16	T/R	I remember there was the one with you and Brian in the fourth grade…
33	09:19	Romina	I was so mean to him, I know. He doesn't let me live that down. (laughter)
34	09:23	T/R	Why do you say you were mean?
35	09:24	Romina	I don't know. I just remember being like, "Can't you see this?" And he was like, "No." I pretty much told him, "I told you so." I still remember that video.
36	09:38	T/R	Would you say that was your usual dynamic with Brian?
37	09:41	Romina	I don't know. I think I got nicer when I got older. I think I was a little too arrogant as a fourth grader, but, Brian? I think I fought more with Jeff than I did with Brian. I think that was rare for me and Brian. But you've probably seen more of the tapes than I have. (laughter) Is that how I usually treated Brian?
38	10:03	T/R	No, it's just interesting to get your point of view.
39	10:04	Romina	I think we all fought a little bit. I think that's why we worked well together. Because we were more like siblings than anything else. I think we all fought a little at the beginning until, then, ok, someone would come up with a good point and we'd work towards it. But we'd always started off a little bit rocky. I don't know if that was your opinion. Some of the time, we'd try to hide it, but I don't know how well we did that.
40	10:30	T/R	Well, you talked about in a couple of interviews, you talked about disagreeing. Is that something you thought was throughout your time in the longitudinal study?

41	10:38	Romina	I think so. I don't think we ever, we never walked in and said, "Oh, this was it." We always came it from very different perspectives and different ways. We always had a very different way of thinking; we disagreed a lot and then came to a conclusion together. Which is better; it was never, "Oh, ok, that's how you do it." We always questioned each other a lot.
42	10:58	T/R	Is it something you find yourself still doing?
43	11:01	Romina	Yeah, now it's probably a little more critical for me to do. Because a lot of time this influences…people are paying us a lot of money to come up with these opinions. And, we just, we—even in school I have to do it. Because the professor asks that one question we're not ready for. That impacts my grade and it's embarrassing in front of the class. So, we do it a lot now too.
44	11:23	T/R	Ok, interesting… to disagree.
45	11:25	Romina	Sometimes, it's easier..”Oh, yeah, sure." It is easier. But to promote the group dynamic and getting a better output, you should disagree or just ask enticing questions..
46	11:37	T/R	What do you mean by that?
47		Romina	I think it's a probe—I think that's what we did. I don't think we ever thought someone was completely wrong. It's just that not everyone may have understood it. So, I just keep asking them questions so that they can dissect their whole thought process. And, if I agree with them, fine. If not,…
48	11:52	T/R	So this would be the people you're working with?
49	11:54	Romina	In the groups, at work my project team.
50	11:58	T/R	So, it's something you find yourself doing whether you're in business, in school…
51	12:01	Romina	Now, I have to. Before, it was kind of like testing my peers. Now, it's kind of like we - it has more of an impact now.
52	12:12	T/R	So, it's more of a necessity now. Interesting. So, cause I was going to ask you how you feel working with groups.
53	12:20	Romina	I like it; I just, sometimes, I think I revert back to I have very little patience. I should, at this age, have more patience. I never had it, so. I don't know if you saw that in a couple of the tapes. But, I really like it. I really like working like that; I always have. It might have been from this program; I'm not sure. But it's how I'm comfortable working.
54	12:45	T/R	Like, when you say patience too, what is the amount of time—do you feel you're giving yourself enough time?
55	12:53	Romina	I think it's more of when people aren't on my page as fast as I am on my page. I get a little bit - this is what people at work tell me on my reviews (laughter). No, it's not a bad thing; I'm not mean or anything. I think I just, when I have to explain something too many times, I think "why don't you just get this?" So, in that sense, I lose my patience. Not that quickly, but probably quicker than I should.

56	13:17	T/R	Now, when you say, “being on your page,” is there something that you think is uniquely you when you’re going in to a problem?
57	13:23	Romina	I think I’m really quick to jump to something, and then can explain it really quick. I’ve always talked really fast and I’m really animated. I’m very visual, too. So I draw these charts. So, I’m like, “You put this on the y axis and you put this on the x axis. You get it, right” And, they’re, “No.” So, I don’t think people always get my kind of visual interpretations.
58	13:52	T/R	Ok, is there somebody, have you found yourself working with certain people who have that same, who work in the same way that you do?
59	13:55	Romina	I have. And that’s like probably project teams at work who were probably a little bit easier for me to work with. And at school, too. A lot of people…At school it’s a lot easier to find people who think in y and x all the time or in some sort of a chart or a picture. I know it’s surprising, but a lot of people at school think like that. So, I didn’t have that problem at school necessarily.
60	14:22	T/R	Interesting, thinking in x and y. Do you remember - I can leave this kind of open, too - helping anyone, specifically in the longitudinal study? Like how would you, would you characterize yourself as someone who helped other in the longitudinal study?
61	14:43	Romina	Would you characterize me as someone who helps…(laughter) I don’t know; I guess. Not even with the study. I’m just thinking of the people that were in it. I pretty much carried Jeff through most of grammar school and high school, so…I’m assuming I had to have helped him if he needed it. But, I think we were all pretty…I don’t think it was an individual thing, so especially Ankur, Brian, Mike, Bobby, Jeff and I worked a lot together . We had after school. We definitely; I think we did. At least, we tried to. I don’t think we were ever, “Oh you don’t get this; that’s it.” We would always try to be on the same page. I know they would leave the room and then we’d have an hour before they got back. By that hour, we were very good at being on the same page and understanding what Mike or Bobby understood and trying to get there.
62	15:30	T/R	So, do you remember…So you’re saying that helping each other…
63	15:37	Romina	I think so. We may not have always done it in the nicest way.
64	15:40	T/R	What do you mean by that?
65	15:41	Romina	I’m sure we weren’t always…we were together for twelve years and it was just us. We’d fight a lot. I don’t know if that was always visible on camera. We could get a little snippy. But I think we always had good intentions. And I think we still, to this day, always try to help each other out. I mean if they need anything, like…I still talk to Brian and Jeff pretty regularly.
66		Romina	Why, did you think we helped each other?
67	16:16	T/R	I was just interested to get a sense how you would characterize yourself. Sometimes, when I’m doing something versus what I think I’m doing.

68	16:24	Romina	I think I would be the most compassionate, considering they were all men and it was just me. Now, if you're throwing in Angela and Magda, maybe not.
69	16:35	T/R	So, did you think gender ever played a role, then?
70	16:36	Romina	Yes, I was always the secretary. I was always the one - to this day, I'm still the one who has to get Brian, Ankur. No, they worked and they went to Rutgers. So I think now. But I used to have get dragged them into every after school program like I was their personal secretary. In the thing, I was always the one writing. That came up. I don't know if you guys caught that on camera or it was after camera—we had a discussion one day.
71	17:01	T/R	What did they feel about that?
72	17:04	Romina	No, they thought I was probably crazy. No, sometimes, I think they—they tended to be more talkers than I was and take the spotlight when people came into the room. Anything that is kind of a little bit gender, a little bit how we always interacted. So…
73	17:23	T/R	Is it something at work? You said "secretary"; do you find yourself…
74	17:29	Romina	I think so. I work with mostly men. (inaudible) I was the one always ordering dinner every night and doing all our grunt work. But I don't know if that was low level or it's a little bit of both.
75	17:40	T/R	In your groups when you were in business school…
76	17:44	Romina	It happens too. I mean, it's kind of sad; we just talked about it. There's a lot less girls at school, but we always get together and talk about it. We kind of made a conscious effort, you know. We're not going to be the ones who - we have to set up meetings on Outlook; it's a very kind of business atmosphere. We always try to - I think it happened the first year—you're trying not to stir the water; everyone get along. And, talk to some of the girls and, "did you ever notice that you're the one always taking notes and setting up the meetings?" Yep, so we tried to be more conscious of not doing it. A lot of guys don't think like that. I was really surprised, especially at business school when we're supposed to be on the same level, they're like, "Can't you just do it?" They're like, "I don't work my own schedule." We're in business school; you're gonna work your own schedule. I was really surprised to find it there, but yeah.
77	18:36	T/R	Wow, when you look back then, in terms of relationships, what would be relationships that would be important for you that you found, I don't know…if I just said important relationships, what relationships were important for you in the longitudinal study?
78	18:59	Romina	I guess the relationship with everyone who came in all the time. With Dr. Maher, we had it pretty and with a lot of other people. She was probably the most regular the whole time. But I think a lot of other people came for five years at a time. It was a long time; we built a lot of relationships with them. I think Mr. Pantozzi kind of indirectly came out of that; we always saw him as a kind of Rutgers person, so I don't know if they thought of him like that. He was pretty significant to all of us. We had him for three

			years in math and he was very just invested in our learning. He's the reason I went to Penn. He said, "No, you're going to do this." And he wrote all my recommendations to college too. I still talk to him too. I heard he just got his too, finally.(laughter) He was like, "I just did it." I said, "It's been awhile."
79		T/R	So you're both -
80		Romina	Even if, I don't know if it was the study, or these are the guys I went to school with for twelve years and we only graduated with, at least, 67 kids, so I don't know if it was because of the study, but I became pretty close with them and with Angela and Magda. Still. I was just with them last night.
81	20:21	T/R	How was everybody? Are they all doing well?
82	20:26	Romina	Everyone's like an adult. They're buying houses and stuff. I think Jeff just got a promotion. I haven't talked to him yet since I got back. But him and I catch up all the time. No, everyone's doing really well.
83	20:37	T/R	I hope we can do something in August. Now, I was wondering…
84	20:45	Romina	They're coming to our ten year reunion. Which you're coming to (directed to Bobby)
85	20:49	T/R	Are you going to be in charge of…
86	20:52	Romina	Yeah, I have to…Jeff and I were senior class officers; so we're in charge. We just started emailing about it. Apparently, it's next year. I didn't know.
87		T/R	So you're going to be coordinating that?
88		Romina	Well, Angela was the one who emailed and said, "Did you know it was next year? Don't you have to plan…" "I guess so; I'll get right on that."
89	21:10	T/R	Dr. Maher will definitely be interested in that. She'll probably bring some towers. Now, one of my colleagues is working with pre-service teachers and she's noticed that the adults don't always feel comfortable working in front of the camera, like talking or convincing others. What do you think about that? Now here are pre-service teachers having some trouble with that.
90	21:33	Romina	It's funny, because I think as…it never really bothered me. I don't really tell a lot of people that I was in this math study. It's not the coolest thing to talk about. When I talk about it, it's funny because now, I have a lot of trouble, kind of, in front of classrooms. At business school, I had a lot of trouble with it. Even at my internship last year. I had this kind of a nervous panic attack. I lost my voice, and it's funny to me because I've been doing this since first grade. Videotaped…I just think we don't realize when we're younger and when you're just kind of…Then, when you're in high school, I've known these people for ten years, so it's not that…it's a level of comfort, but. I'm actually not surprised, because I'm actually very uncomfortable now talking in front of big crowds. I gave my graduation speech in high school and now if I were to do that, I think I'd pass out. So, I don't really know what happens; I just really think when

			we're kids, we don't think about it. And I don't think we ever thought about it until…I think in high school it hit us that these are very well educated people coming in and we're telling them about math. It's really, we shouldn't…this is not our place. So, as we get older, we start to realize it's probably kind of odd.
91	22:50	T/R	You said your internship. What were you doing in your internship?
92	22:52	Romina	I worked at Target last summer in their corporate headquarters. I've always tried to be a buyer, but I've never really done it. I've always interned as a buyer in places. It's actually a very—I like it a lot. I just never made the full jump away from consulting.
93	23:08	T/R	And it's something…What would you have to do within that?
94	23:10	Romina	It's just—I would get a department—well, you know Target—I worked in women's plus sizes. And you just manage that whole area from buying the clothes to kind of managing it through the sales. It's pretty quantitative. It's actually surprisingly very analytical. So I kind of liked it—a lot of strategy, too. It's the best of both worlds.
95	23:30	T/R	So what types of…If you could give me an example like -
96	23:35	Romina	What I did last summer. I wasn't a buyer obviously; I was an intern. So their business had not been doing well—this is the women's plus business—for about five years it was declining. And, so, they were kind of, "Can you fix that? Can you come up with some ideas?" So I was like, Ok. But it's very similar to what I do in consulting. I kind of…I did a competitor kind of research and analysis and came up with a…I did launch a juniors type of line. Which I did – it's out in the stores. It's really exciting. And it's a lot of just socializing…communicating with a lot of different groups. You're kind of the center of this type of wheel, they call it, The buyer—you're the person who defines the strategy but you don't necessarily do anything. You don't make the clothes; you don't create the marketing or the advertising. So you have to kind of work with everyone to convince them.
97	24:25	T/R	Now you can walk into a store and see what…
98	24:28	Romina	Yeah, I just did. They don't have them out where I live. But they have them around here. It's really exciting. But—I liked the internship a lot but I just wasn't ready to move to Minneapolis, I don't think. Real cold.
99	24:48	T/R	That's for sure. That would be enough reason for me. And you had to do public speaking within that too?
100	24:50	Romina	Well, I mean…Well, for most MBA's, this internship thing is a little bit extreme, I think. Ten very long weeks. But, at the end of it, what they have you do is invite your whole division, vp's. Most people that graduate have you come and present what you've been working on. And, so, I had to come and present my findings. It was a powerpoint and a little fashion show 'cause I had a line. So, it's a little intimidating; you had to stand up there. It's a big room and you're in a suit and there's like a projector behind you…

101	25:23	T/R	How many people…?
102	25:24	Romina	Thirty-ish.
103	25:28	T/R	Now, when—would you do…I know within in the longitudinal study, you do a lot of convincing a group…
104	25:35	Romina	That I don't have a problem with…That's what I do in school. Most days I sit in a small room with six other people and you just argue your point. And that's what I'm fine with. It's the more formal presentations. I don't think I liked it that much in the study either. Usually I let Jeff take that or Brian. When they made us stand…I really tried to stay away from the board unless they made me.
105	26:06	T/R	That's funny. So actually – cause I was interested—sometimes you talk about in some of the interviews "comfort" and "being comfortable." You've said, "We were so comfortable with each other," and you felt comfortable asking questions. I was just wondering if you could talk more about that—comfort.
106	26:26	Romina	I think we were in class; it's not that I ever felt that, "Oh, if I ask that question, they'll think I'm dumb." Because they're not. They know my abilities at that point. They know me pretty well. It was just a lot easier to ask questions. You're like, "Should I ask this?" Whatever. What are they going to say? I think all of us were like that; we had no problem…sometimes, when I really didn't get it, I didn't mind, being like, "I don't get that. You'll have to explain that again and again and again." So, I think that comes with comfort and new environments you don't necessarily do that as much, because you're everyone else seems to get it, so I'm going to get it, too. So, I think that's why we worked so well together. Once they brought me along and I got it, I could probably add something later on. So, that's how we came to a better end product, I would say.
107	27:11	T/R	Is it something—so you'd still characterize yourself as comfortable then within groups?
108	27:17	Romina	Yeah, within groups. But, within that specific group I was, like, the most comfortable I'll ever be. But, now I'm…I think I am comfortable…I don't mind asking questions, especially after you establish yourself at work or at school or anything. But, I don't think it'll ever be at that level of comfort where I will just keep asking over... At one point, you're just, "ok, move on without me. If I don't get it, just go. I'm stalling the group." But I still am pretty comfortable within groups.
109	27:48	T/R	There was - when you were being interviewed for PUPMath that was back in…
110	27:55	Romina	What math?
111	27:59	T/R2	Private Universe Project
112	28:00	T/R	The Cat Walk and the Placenticeras… That was 1999.
113	28:10	Romina	We came back from our shore house for that.
114	28:15	T/R	You're always so good coming back. You said in an interview, "We all

			have very low self-esteem about everything. We didn't think we were capable and we were scared." Do you remember feeling low self-esteem?
115	28:25	Romina	Yeah, I think…I don't know if I should have said, "We all." I think that is something I've kind of suffered with a lot. Even now, still. It's probably why I said that. I don't know why I'm talking about these things in public. But, I just think we were always, not all of us—I can't speak for all of us—I think it was a little daunting when we were—I don't know how random it was—if we were randomly chosen. But, to some extent, we thought we were randomly chosen and we kept doing this. These people would come in and say this was really important—we'd have no idea what we were doing. And, half the time we'd spend five hours in a room where four of those hours, we were just sitting there beating our heads against the wall going,"I don't know how to do this; I have no idea." Somehow, it would work out and we'd figure something out by the time we had to present or present our findings. But, to all of us, we're not capable of this high-level —once the Rutgers group left, we'd kind of talk amongst ourselves and our teachers would talk to us: "Oh, you guys are doing such high level math and you don't even know and we're like, " Yeah, right, we had no idea. So, I don't think we ever felt that confident to walk into one of these sessions and say, "We're about to amaze people right now." It was kind of, "Ugh, how do we do this?" Every single time, I felt like that happened.
116	29:53	T/R	Did you feel proud once you had?
117	29:54	Romina	I think so. I think my group, our little group, I was really impressed by us sometimes. How did we do that? Because, I mean, I don't think everyone, obviously the tapes were rolling and the microphones were rolling. But, to us, it was like people would walk out—everyone was silent. No one would talk to us for hours; so it would just be us sitting there. There were points when you were like, "We're not ever going to get this. It's never going to happen for us." And then, we'd always manage to get something. Yeah, I think the way we built on ideas—I think it was more interesting as we got older and we were able to figure out, go from towers to kind of an equation to kind of like a standard theorem, you know. That kind of stuff was a little bit—when we started connecting that…
118	30:42	T/R	So, would you say then, there's another entry where you talked about—you said, "People underestimate us sometimes." Is that something you still feel like…
119	30:53	Romina	If I still feel what?
120	30:56	T/R	Do you still feel, 'cause you mentioned that sometimes "people underestimate us." Would you still agree with yourself?

121	31:08	Romina	I said that? I'm surprised I said—we may have underestimated ourselves, but I really think they kept coming back because they wanted us—they were really behind us coming up with something. But I think so. I think we're—if you were around when we would first get these problems…I'm sure people have watched tapes of us for the first hour or two when we're just sitting there going….I think everyone at first does. I think some of our teachers we got growing up—Because we weren't necessarily--not all of us were always on the ball in every class. So the fact that we'd get pulled out of classes to go do this very exciting math stuff; I don't know if they all bought it. They had no idea what we were doing, but…
122	31:53	T/R	They didn't know what you were doing…
123	31:54	Romina	I wouldn't say that. To this day, if you put the group of us in a room, I think we'd still come up with something pretty good. I think we're all very good working with each other; we know each other so well. We all have very different strengths, I think.
124	32:13	T/R	When you did the summer institute, that's when you were with a group you were not as familiar with, right? Was, did you feel the same way—that you were able to…
125	32:22	Romina	Well, I think we came up with a lot; I don't think it was the same. It introduced people—I didn't even know these people. So, it was a little bit different. Because we all started off at the same point, and we always remembered towers in the fourth grade, you know. "Remember that; it's kind of just like the towers but with four colors?" So we always had that basis—we also, coming into it I knew who was good at what. That's really important when you're working in groups. Like I knew to expect certain things from certain people, so that made it go a little bit faster. Versus, I think, when it's a whole different group of people—we still did an ok job, but, if it was just the six of us, it would have been a lot more comfortable with each other.
126	33:12	T/R	So, what would be things…you said you knew what to expect from certain people. What were things that you would expect?
127	33:17	Romina	Like Bobby or Mike coming up with some binary code; we would expect that from them. I think, like I asked a lot of questions, so prying that way. I think Brian and Jeff were like the our presenters to the outside world and they were very good at communicating once our ideas to everybody. Bobby and Mike got the real intense math, like they thought in a different…A lot of times we'd ask them questions, they'd start with an idea. And based on that idea, we could really take it far. Things like that.
128	33:59	T/R	Maybe I'll ask you a more abstract question then. I know you've talked about this in other interviews, but I'm eager to hear what you would say today.
129	34:05	Romina	Do you have what I said written down? (Laughter)
130	34:14	T/R	What it means to know something really well. What does it mean to you to know something really well?
131	34:19	Romina	Just to understand where it comes from…to be able to not have thought

			about it or even talked about it for five years, then still recall something about it. I mean, I think that's what we did with a lot of these—the way we learned. Getting a little bit off track. With that, I mean I'm not really good at instant recall, crunching numbers type of—the normal thing. But, to this day, I've still—I've talked about this before—in college, when everyone was failing calculus, I could talk to all of them and explain. I don't know if I could do it now. It's been nine years. But, I could probably explain to them the fundamental theorem of calculus and kind of explain to them how all these things happened and worked. Visually, how everything was represented. But then when I went to go do it, that was a whole different thing. I could actually do it, and this complex working of the equation I couldn't necessarily do, but I understood it. So, being able to explain that to people—for them to be able to understand it—to explain the mechanics behind, just moving numbers around…I think that's very—back then I was very frustrated that I couldn't do the mechanical part of it. But, now, as I'm getting older, I don't have to do that. No one really does all that, really. Like logging things. We don't do that. So I understand kind of the basic idea behind it. I'm going to get through life just fine with that.
132	35:49	T/R	So how would you define math, now? What would be…
133	35:55	Romina	Wow. How would I define just math? I don't…
134	36:04	T/R	You said that there's the number crunching, the number part. But then you described this other thing you were doing.
135	36:11	Romina	I guess I don't know how to describe that. It's a little quantitative thing to me, but it's more—understanding how slope works versus actually figuring out the slope. It's much more higher level—I have tools which help me do like the basic, the number crunching—I have Excel, I don't need…It's much more understanding and setting up a problem in more of a quantitative in an easy to see, easy to calculate type of way. That's for me…
136	36:51	T/R	What would be something you would say you know really well?
137	36:56	Romina	From a math perspective? I don't know. I don't think it's a specific thing. I think I'm pretty good at this point just getting a lot of information and being able to - organizing it to see what the problem is and then working to find the solution. It's more of like that process that I'm good at, not necessarily all the little details that go along with it
138	37:31	T/R	That's interesting. Because there was, you mentioned—this is another one from 1999—this is from July, this is literally ten years ago—You said, "I think there's two different areas of math. One of them is the thinking involved, and one of them is just spitting out numbers. I know I was never good at spitting out numbers thing, but I was decent at the thinking about it."
139	38:00	Romina	Yeah, I'd still stand by that.
140	38:01	T/R	You'd agree.

141	38:04	Romina	I notice that more now. This last class I took, I think the professor got really upset with me because I couldn't figure out how to do the algebraic logging part of the equation, because I don't remember what log is. And stuff like that. I was in his office and he, "It's really annoying because you get all the other stuff." Like, yeah - I had the entire problem figured out; I knew how to analyze it. I knew what to do, but then, when it came to actually doing it, I'm a little confused with these little parts. But, I think that's more important, because you can always find someone to help you with—how do I log both sides of this equation versus thinking about this whole problem. So, I still stand by that. I still think I'm not that great with the numbers part.
142	38:52	T/R	So, what happened with the professor, then? Did you guys…
143	38:54	Romina	It's fine, I got a good grade, but he really humiliated me in class a few times because I didn't know how to do the algebra part of it.
144	39:04	T/R	Would they call on you specifically?
145	39:07	Romina	Yeah, business school is very—we have little name tags we have to put out in every class—Miss D'Andrea--They'd talk to me about certain things. It's all about public humiliation.
146	39:19	T/R	So there would be a problem, and they would call on you specifically?
147	39:22	Romina	Yeah, we would do a lot of, we would work on—it's case- based, so you'd go and read a case and you'd kind of work on it within your group. Then, you'd get to class and they'd ask you very specific questions based on the case. Some financial statements—in this case, it was a statistical approach.
148	39:41	T/R	So, sometimes you'd see those movies about law school…
149	39:42	Romina	It's kind of like that, but not those type of greetings but it's like that, yeah.
150	39:52	T/R	So, how would you describe yourself as a problem solver? So, would you say, just as a problem solver—whether you talk about it now or in the longitudinal study, how would you describe yourself as a problem solver?
151	40:01	Romina	The process? How I would approach? I think I'm very—I need a little bit of quiet time, digesting time, at the beginning. I need to really understand something. Have some alone time to really think through my own thoughts. I actually—I don't know if I was like this before. This is how I am now; I don't know if this was the way I was during the study. Then, it's like, I only get to a certain point by myself by kind of organizing the problem. I like to talk about it with other people. Kind of be like, "Is this what you think? The issues? How are we going to tackle this?" Then, work on it together. And then, come up with some kind of plan. And I think it works out well, because then we get it a little bit further. I don't like going down and doing a problem all by myself, because the chances of me getting to the right answer that everyone else gets to, is gonna be—it's probably not going to happen. So, it's just bringing people along and then solving it together.
152	40:58	T/R	Is that how you think someone else would describe you too?

153	41:02	Romina	I definitely think a lot of people would say I like my alone time, my quiet time at the beginning. I think so.
154	41:13	T/R	How would you know if someone is an expert at something? What's an expert to you?
155	41:18	Romina	Probably it's someone who worked with something for a very long time. I think you obtain expertise through just a lot of hours. And understanding the fundamental aspect— like understanding every point of the way versus certain aspects.
156	41:38	T/R	Would you consider yourself an expert then?
157	41:43	Romina	At nothing. Not yet. What was the rest of the question, sorry.
158	41:50	T/R	Is there something you're trying to become an expert at?
159	41:54	Romina	I don't know yet. I don't think I'm an expert at anything yet. And that's with always meeting new people and finding where they're at with things. No, I still have a little way to go with everything. I haven't really chosen what I want to become an expert.
160	42:15 `	T/R	So with this idea of an expert or with a problem—how do you know an answer to a problem is right or that it is true? How do you know; how do you judge that?
161	42:25	Romina	I don't know if like-- from a math problem?
162	42:32	T/R	Within your job today: how do you know if someone is right even if they're called an expert? How do you know they're right?
163	42:37	Romina	I just figured out that no one is—you can be right in many different ways. Especially at work, I mean. Even at school, we'd come up with so many different answers to the problem that we're all right—no one is wrong. It's kind of all coming to an agreement and just eventually it's the group saying this is right. It's not one person knows the right or wrong answer. I think everything is up for debate. Maybe not in a math equation. Most other things.
164	43:06	T/R	What would be an example where you're saying, "Everyone in the group could be right." What would be an example?
165	43:13	Romina	You see - do you want me to go back to the longitudinal…I'm thinking a lot of the problems I saw about work or school, you don't know. I mean, what should a company do? You really don't know if that was the right decision until way after. In hindsight, yeah, this was the right decision; but you don't know that at the time. So it's kind of just assessing everything around you and just being able to kind of take everything into consideration…this is the best decision we can make. From the study perspective, I think what we – I mean, we just kept testing it. We'd come up with…that was a little bit different because there usually was right answer. Like if we made an equation, it pretty much had to work in several different circumstances. I think Dr. Maher and the rest of the gang were very good at coming up with different circumstances to kind of see if our theory still held up.
166	44:13	T/R	So, would you—cause I remember when there was one time an issue between what your group thought vs. like a graduate class thought was the

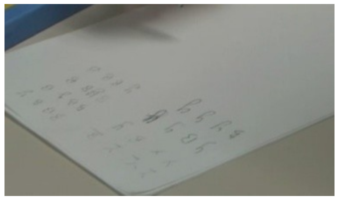
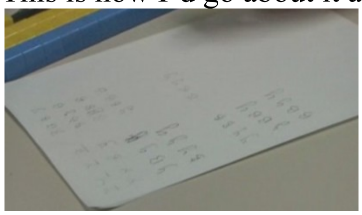
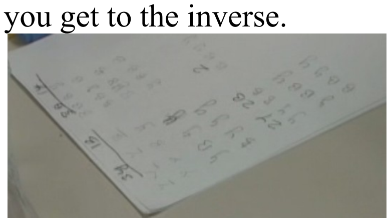
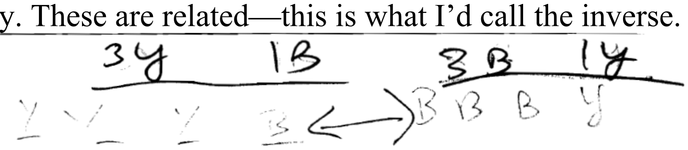
			right answer.
167	44:22	Romina	Which problem was that?
168	44:24	T/R	I think that was the World Series problem.
169	44:26	Romina	I was going to say that—the World Series problem—I don't remember exactly what—do we know who was right?
170	44:34	T/R	You guys were the ones who were right.
171	44:35	Romina	Really?
172	44:39	T/R	Sometimes a graduate class could be considered the more expert class. But you guys were definitely the ones…who were right.
173	44:49	Romina	I don't remember that – I remember there being an issue.
174	44:52	T/R	Now, in general, how long would you say it takes typically a problem to solve?
175	45:00	Romina	I don't know. I mean, hours for us. But, even with us, I think our sessions were like a few hours at a time—maybe 3 or 4 hours—we'd come up with an answer. But, we'd always go back and refine it. So I think that was what—that's why we'd get to right answers eventually, because we weren't scared, even after 4 hours, to say, "You know what? We need to go back to this - we need to go back a few steps and start this from step 5. Not all the way to the beginning because we had some basis, but we started over a lot. Once you get older, I don't think you do that as much. I think that was unique to our group that we were like, "ok, we've done four hours of work on this, but it's not heading in the direction we want. So let's just bag the last two and start over again." And we would even—in new sessions—go back to old ideas that maybe weren't working in that session and refine them.
176	45:57	T/R	Is that something that now, too…
177	45:58	Romina	I think people don't do that now, and that's probably why the graduate students came up with an answer. "You know, I've done so much work on this for us to go back now." But, I think it's a much more exaggerated time scale. You work on something for a month and it's not perfect, you think, "Well, we're going to go with not perfect right now."
178	46:14	T/R	At work, how long will it take you to work on a problem?
179	46:17	Romina	We're a little bit different. There are much larger, company-wide issues. Our standard project is probably six to eight weeks. And a lot of other stuff comes up and usually we get extended to another six to eight weeks for everything.
180	46:39	T/R	Do you see yourself as a learner different now than when you were first in the longitudinal study? You, learner, now versus then.
181	46:50	Romina	No, I don't know which one influenced us, so it might have been the study which influenced the type of learner I am. It was just very conducive to the way I learn; I'm very visual. I have a lot—I can't just hear something which made college real hard in lectures. So, I'm still a very visual, hands on , and I'm also—I think I can learn—I can read something and go ok. I

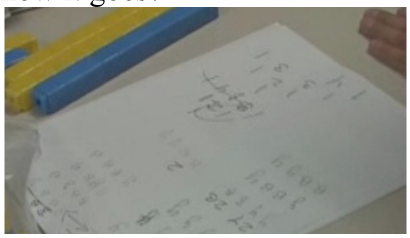
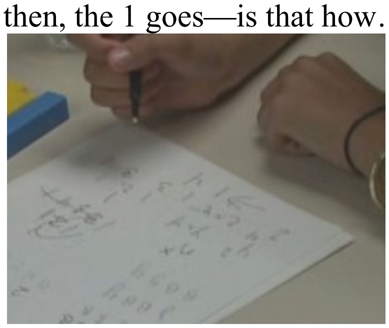
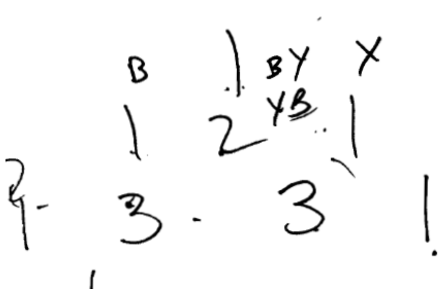
			can see someone else doing it but I'm all about doing it myself. That's the only way I can really learn something is once I do it myself.
182	47:26	T/R	And that's something even through business school?
183	47:31	Romina	Especially now —even at work, too.
184	47:37	T/R	So is that the way you learn best then?
185	47:38	Romina	Yes. I think so.
186	47:42	T/R	So, the learning process for you then…your ideal situation, if you had to leran something brand new right now. Because you said, I remember, in some of your interviews, geometry was something that you had to… So if there was this new topic in geometry you had to learn what would be the ideal situation for you to learn it?
187	48:02	Romina	I think it would have to be---probably someone talking to me about it. From being very visual, I can't just learn again listening to something. So, I'm definitely very visual and then, just walking me through how to do something. After that, I can try it on my own and be fine. That's all it really takes.
188	48:26	T/R	So you think it would be enough if I show you how to do a problem…
189	48:29	Romina	Are you going to do that? (laughter) Yeah, and I'd have to walk through it a few times myself. It's my own time type of thing. Then, I'm usually ok.
190	48:45	T/R	What do you think will stay with you from your experiences in the longitudinal study? What is something lasting that's just going to stay?
191	48:50	Romina	I think that whole problem-solving aspect of it and that whole kind of being comfortable which I think is really very important. Being comfortable being put in a situation where I have no idea how to do this. Like, "I don't even know what you're talking about." Being able to break it into smaller parts and organize yourself and get the information you need for each part. Then, work in a group is the other big thing. Work in a group to kind of figure it out—something that is a daunting task, but if you work in a group, you figure it out together. Those two things.
192	49:25	T/R	So, is there something with those things that you're—you would say that you're doing now in your job?
193	49:33	Romina	That is what I do, at school, at my job. Yeah. So, I kind of like it apparently.
194	49:45	T/R	What would you see yourself doing five years from now? Do you have a plan?
195	49:49	Romina	No. [Shakes head]. Sorry—graduating, everyone keeps asking me this. No, I don't know. I honestly don't know. I really like that whole—consulting is literally that every three months—I get pointed to a completely new situation where I have no idea how I'm going to do this. You just kind of spend a week get your bearings and figure it out. I really like that; it keeps it interesting for me. So I don't know and I keep thinking I want to do – settle into just one job, but every time I do that, this is going to get boring. I get this itch every three months to do something completely new. Which I can do with consulting. So, I don't

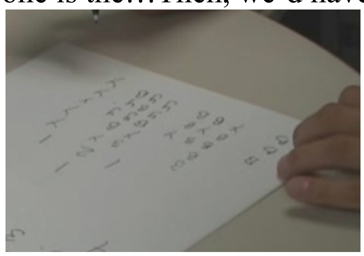
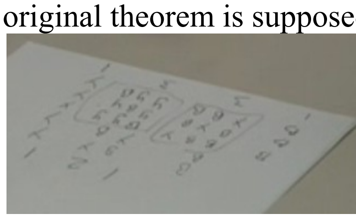
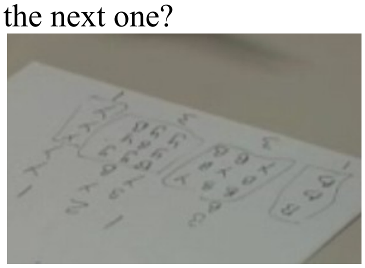
			know. Either that or I hope to start my own business. I just need a really great business idea.
196	50:32	T/R	What type of business would it be?
197	50:33	Romina	Something with a product….marketing, that's what I did in school. And, I'd still like to be (inaudible) and I can do that. And work in Excel for a little while a day.
198	50:49	T/R	So, it would be like Romina's…
199	50:50	Romina	I don't know; I don't know. I'm still thinking about it. That's what I started to get interested in the last couple of years. I've practiced writing a few business plans, but…I don't think I'm really going to start yet.
200	51:02	T/R	So, why did you get interested in business…what led you that this was the thing for you?
201	51:13	Romina	I don't know. In college, I liked econ. Econ was like the perfect mix of the analytical part—but kind of like the theory too—like the higher level impact. I liked that whole part. I liked doing—I don't ever see myself as an academic. I don't know what else is there but business out there. I like the whole—it's like the right level of—because I am an analytical, quantitative kind of person. So, it's the right level for me, because I'm not going to be—I'll never be a statistician. I'm never going to be sitting behind a computer the whole day, all day, doing this—getting into the details. But, I like it enough where you can do some high level analysis and come up with a recommendation and impact an entire company.
202	52:06	T/R	Now, I know we have the towers, but do you have any questions for me or for us?
203	52:08	Romina	No, I'd like to hear what you're doing. If that's ok…
204	52:14	T/R	I can tell you, too, what we're doing…and with timing, too. I don't know how your schedule is too. But, from the dissertation point of view that would be in November; it looks like that would be the time. Of course, you'd be invited. You want to hear about yourself.
205	52:41	Romina	Are you just doing it on me? Really?
206	52:43	T/R	Several different graduate students are following different people from the longitudinal study.
207	52:50	Romina	That is really funny.
208	52:51	TR2	Maybe a better way of trying to contextualize this is… For a long time, Dr. Maher did not really want any of the graduate students to focus on a particular student. Mainly because at various points in time, everyone had sort of their shining moment of when, you know, either they had some insight or maybe they were the one to stand up and share the idea the group was doing, you know. But now that there have been a lot of different studies – people look at how students use like representation. You talked about being visual – how they draw, build, whatever – contributes towards developing math ideas or developing ideas about reasoning and making convincing arguments in math. Now they're – because there have been these other studies done, there are some graduate

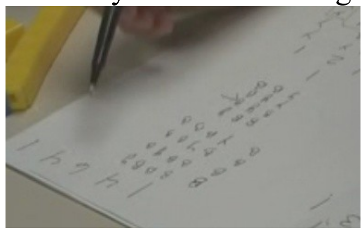
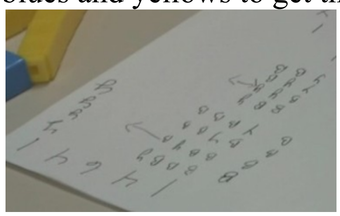
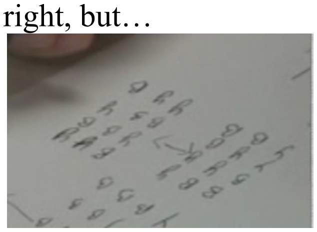
			students interested to look at the development of an individual
209	53:45	Romina	Oh, I'd love to hear what you think!
210		TR2	Over time. And so, um, and so there's a handful of students who are doing basically a case study to look to see how Romina's ideas grow and change over time or how Jeff's. Actually, no one's picked Jeff yet. Or Milin. I don't know if you remember. Someone's working on Robert too.
211	54:08	Romina	I would have picked Robert.
212	54:10	TR2	And he's around for interviews and he's had to sit through a few. So yeah, these are some of the other studies. My research is looking at how people learn in groups and the process of problem solving in groups.
213	54:30	Romina	Are you still a group advocate?
214	54:35	TR2	Am I a group advocate? I mean, like, sure. So kind of like – to talk about my big dreams or whatever – I'd like to show. I'm around all these math people, but I'm really more interested in language. How people talk about these ideas. How discussing, exchanging ideas contributes to problem solving ability. Because any field that you go in…
215	55:16	Romina	That was good that you picked me because this is what I do every day. So you could say there have been long term effects of it. I wondered—because now working in business school, we've had a lot of issues with groups. It's just how I'm used to working, so I'm always comfortable in that setting. But, it's been funny, because some people felt that they can't learn. Because the group moves—it's been hard because you're getting very accomplished people in a room who like to talk. It's really—I wonder how…I think it's different because I've done it from such an early age like I didn't have an ego problem in our—or a free rider problem—Because we were all pretty accountable, or if we weren't, we'd walk out and, "Dude, you didn't do anything today." So I wonder the difference if you started it young or you started in high school. Because we were doing it since we were little kids.
216	56:23	TR2	Yeah, that raises some interesting points. I think people generally underestimate what's involved with having people work in a group and have the group be effective, productive. I think there's some naiveté – just because you put two people down at a table or put six people in a group and say, Hey you're going to solve problems. I think there's a lot in the shadows – it's overshadowed by what gets produced by the group and there's not enough people who have looked at how people are learning in groups.
217			**DISK 2 of 2**
218	00:03	Romina	The draw from your results? It happens sometimes. Like, when it's like this is due in two hours—I don't have time to learn it.
219	00:23	T/R	I have the towers--Could I ask one question about the towers? (Pulls out bag of towers).
220	00:33	Romina	You know, I had these in my high school locker for four years? I had bags of towers in my locker.

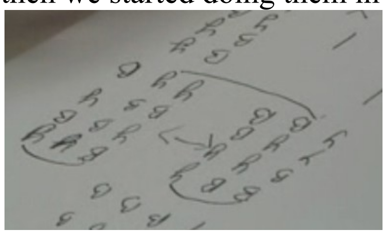
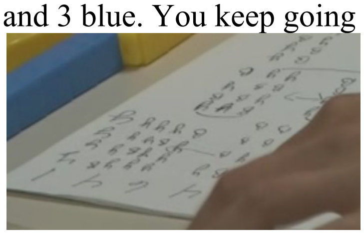
221	00:36	T/R	Why did you have them in your locker?
222	00:42	Romina	Because—I don't know why we ended up coming back to them in high school. They gave them to me, and we worked on a problem outside of the Rutgers thing.
223	00:48	T/R	Did everybody in your – did Jeff?
224	00:49	Romina	I was the secretary slash holder of stuff. (laughter)
225	00:56	T/R	So they were safe in your locker. So what do you remember about towers?
226	01:04	Romina	Isn't it n to the x? The whole - I hope it is. I guess I just remembered that this was how they showed us combinations and permutations and all that. Right? This was the basis we learned in fourth grade and then we carried it on over and over until we figured out if you have, like. We started out: There's tower 4 high and there are two colors, so how many different towers can you make? That eventually led us into (writes with her finger on the desk) four squared or two to the fourth. I don't remember. It's one of those. I can't remember which one n is: the number of blocks? You don't have to tell me.
227	02:05	T/R	I bet you could rebuild it.
228	02:11	Romina	(T/R pushes the bag of blocks toward Romina) Help me. I can't do this alone! No, I'm kidding.
229	02:18	T/R	Here, if you want…we brought two bags…
230	02:19	Romina	Do you want me to recreate the original towers problem? Wasn't that 4 high, 2 different colors?
231	02:25	T/R	Do you want to do that problem?
232	02:26	Romina	Sorry, do you have a different problem? That's the only one I remember.
233	02:28	T/R	In fourth grade, you guys were doing one with 5 high. You were looking at 5 high.
234	02:37	Romina	That's one of the variables that changes.
235	02:39	T/R	So you can look at either one. You want to do 4 high or 5 high…
236	02:43	Romina	Do you actually want me to build all the towers?
237	02:46	T/R	What would be the way you would think of this?
238	02:54	Romina	The way I would think of it from scratch? I would probably build the towers—we got pretty quick at it—we just got pretty quick (uses a pen to write charts) Didn't we do that? Are we seriously doing this? You told me we didn't have to do math. Did you find that I was the anal one about patterns?
239	03:27	T/R	So, what do you mean by patterns?
240	03:28	Romina	(keeps writing) B follows through. I think we all got like this to make sure we didn't forget any one. Do you want me to work through the whole problem?

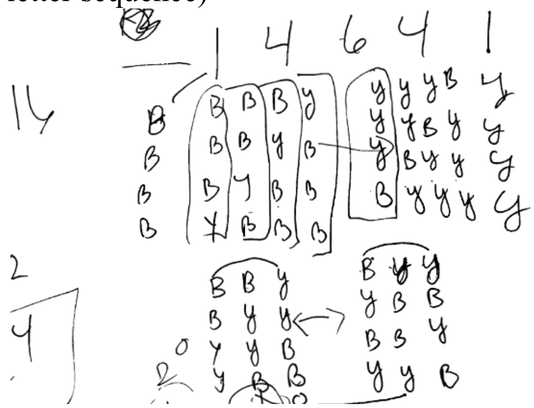
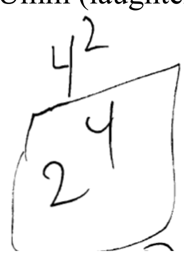
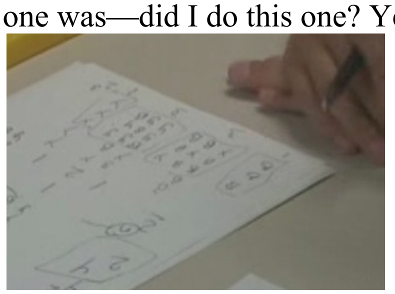
			
241	03:42	T/R	Sure…I'm interested.
242	03:45	Romina	This is how I'd go about it and add like (writes more) Then, do that again. 
243	04:06	T/R	What are these groups?(points at paper)
244	04:07	Romina	I think this is how I used to do it—I'm not sure but. I take…this is 4 high, 2 colors –yellow and blue, obviously (points at paper). I work in little couples, I guess you could say. I start with 3 yellow—3y,1b group This is the 3b, 1y group.(T/R gives Romina a new marker).3y, 1b group. Then, 3b, 1 y group. This is going to be, eventually, the same thing. So, I think what we used to do-- we double checked ourselves on everything. We used to kind of write them all out and even though this we knew… We'd just cross out the ones that are visually like the same thing. I would just keep going…Then, you'd have the 4b. We'd have 4 blue and 4 yellow and then we'd work on them like that. …introduce one new color until you get to the inverse. 
245	05:16	T/R	The inverse—so what do you mean by the inverse?
246	05:18	Romina	The inverse—wherever this was yellow, you'd turn to blues—to get to the opposite of that little tower.
247	05:27	T/R	So the opposite—so inverse and opposite, is that …
248	05:34	Romina	Um, I don't know if it's mathematically is the same thing, but, like the opposing couple to this person (indicating with her marker) would be this guy. These are related—this is what I'd call the inverse. 
249	05:48	T/R	So, when you said "a couple" before, is this a couple?
250	05:49	Romina	Yes, this is a couple. This is a group. (indicating on paper with her marker) So, that's the way I would think about it. Let me see, so this would be (writing on paper)—is that how we do it?—this high(crosses out

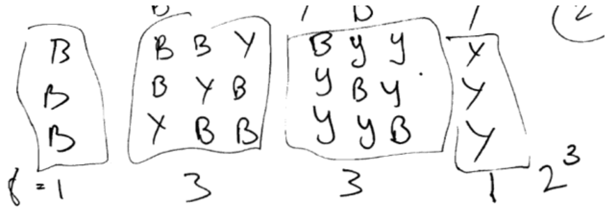
			on paper); sorry. (Begins writing again) So we would be—is this how it goes?—I don't remember how this one. Bobby, can you tell me if this is how it goes? 
251	06:21	TR2	You're gesturing up with your pen as you write; are you doing something?
252	06:36	T/R	So, what is it you're building?
253	06:37	Romina	I'm building Pascal's Triangle, to figure out the fourth row of it. To figure how many combinations there'd be; then, I'd go back to figure out which was this little…But, see my problem is—this is where the technical stuff comes in—so, 4 squared is 4 times 4, right? 2 to the fourth - 2 times 2. So, that would be the same thing anyway. So that's the way I—Sometimes I have to build it up from scratch. But, I'm building—I love Pascal's Triangle. We thought we discovered it. 1 goes on the outside; you add these 2 to get this (she is pointing and writing with the marker). And, then, the 1 goes—is that how…I feel like… 
254	07:24	T/R	So where did the (points with finger) So is the 1; So where did the 2 come from?
255	07:27	Romina	I just know it goes there. Is that…This is just one color; so this would be 2; so this would be all blue; this would be all yellow. There would be two: One's blue/yellow; one's yellow/blue. That's how I remember that line. Is this line wrong; could you tell me that? 
256	07:44	T/R	Just explain to me where…
257	07:48	Romina	One color is one high, one color…

258	07:53	T/R	So what would it be? (pushes rod toward her)
259	07:54	Romina	I think this (she puts a yellow cube on paper); right?
260	07:59	T/R	That's the one?
261	08:02	Romina	(Takes new paper) So, I think this one…the whole thing is for two colors actually. Is there is just one?
262	08:17	T/R	When you did this, what was this referring to? (points on paper)
263	08:19	Romina	The way I remember this, and I could be thinking about this wrong, is: this one would be my blue/blue; this one would be my yellow/yellow; this would be blue and yellow. So, that's why I was thinking it would be…the one is the…Then, we'd have three of (keeps writing) 
264	09:09	T/R	So, can you explain…you said you'd have three…
265	09:12	Romina	This is the 1,3,1
266	09:16	T/R	3…So, what do these have in common?
267	09:19	Romina	1 yellow—is that 2 yellows? (Draws boxes around letter sequences on paper) My—I may be grouping them incorrectly though from what the original theorem is supposed to be. 
268	09:28	T/R	Then, when you wrote 1 under this?
269	09:29	Romina	This is all yellow. (Draws a box around a YYY sequence)
270	09:31	T/R	So, what would this group then…?
271	09:34	Romina	All blue. (Romina draws a box around the BBB sequence). So, it's two colors. (T/R repeats,"ok") The next would be…So, do you want me to do the next one? 
272	09:46	T/R	Yes. I'm interested to see where that would go then.

273	09:53	Romina	Me, too. I'm surprised they always had me be secretary, because I don't have very neat handwriting. (keeps writing) This would be the… 
274	10:37	T/R	So is that part of this group?
275	10:40	Romina	This would be part of the 6. Then you do the little inverse guys…get the other 3 of these to get the 6…Then, you do the inverse of these; switch the blues and yellows to get the other 4…these would be all my yellows. 
276	10:52	T/R	I'm not sure I understand. Can you just explain…(laughter) Write it out. Write it out for me?
277	11:04	Romina	I don't know - This would be…let me see if this one doesn't work exactly like... This isn't working; maybe that's the 4. I didn't do these couples right, but… 
278	11:39	T/R	So, can you explain what you mean by the couples here?
279	11:40	Romina	It's essentially like - These are the combinations–they call it the combinations--you can make. They're not, because…
280	11:50	T/R	What is the couple? (points on paper)
281	11:53	Romina	I know; I know; I messed up the couples….My ideal couple to this person would be this person (writes) B Y B Y Y B Y B
282	12:03	T/R	How come? Why is this an ideal couple?
283	12:06	Romina	This is like the exact opposite—the inverse type of relationship.
284	12:12	T/R	When you said "the exact opposite," what did you mean?
285	12:14	Romina	Wherever there's a blue here, there's a yellow here. So they switch off. But, I didn't do it right. I was dragging the yellows through. So that would

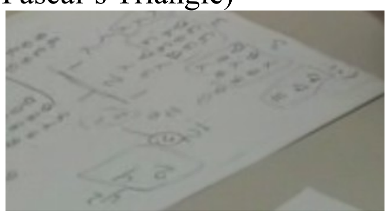
			be the…that's what I tried to do…This one and this one are little couples, and this one and this one are little couples. (Draws a line from one sequence to another) And, these two are couples. That would make up my six, because the whole idea that you have, like, repeats—that's why you have 6 and not 8 Um - You can talk to me about how poorly I communicate my ideas throughout the twelve years. The reason that was 6 and not 8 and instead of adding the 4, because as you move the couples, they become the same thing eventually. The way I used to do it very systematically with carrying the Y's through; if you do that, and do that the opposite with the B's, eventually you get to the issue where you carried it through and then it's the same pattern. It's the same tower. So, then we started doing them in couples. I think that's how I – 
286	13:32	T/R	You said, "carrying through"—what's the carrying through?
287	13:37	Romina	It's just a pattern; and, then, I lift the two y's up to make sure—just to be systematic about the different—So you can just see it and build it up systematically. So you don't have repeats instead of just building it all out. Whatever comes to your mind. You start with one yellow; and then you bring the yellow through and make all the combinations: one yellow and 3 blue. You keep going like that. So this would be (writes); then, 
288	14:22	T/R	You did that pretty quickly. Where did you come up with that…
289	14:25	Romina	I just took the opposite of these—So, wherever this was a B, I made it a yellow.
290	14:37	T/R	Ok, so..are there any more, then?
291	14:40	Romina	I hope not, because that would mean I didn't remember it right. There's 12; no there should be 16 there.
292	14:52	T/R	So, how many towers did you..?
293	14:53	T/R2	Where did the 12 come from?
294	14:55	Romina	Don't you add them up? Is that how you do that thing? Ah, that wouldn't be 12. That's 16. Good. That makes….whew. See, it's the actual math I'm not so good…
295	15:08	T/R2	I thought you got the 12 from counting some of the things you built.
296	15:13	Romina	No, I was adding up these numbers. (points to numbers on paper)

297	15:18	T/R	Are there 16 here, too?
298	15:19	Romina	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16…yeah, I think so. Each one of these is one tower. So there are 16 groupings (draws boxes around each letter sequence) 
299	15:36	T/R	So, when you said, “n to the x” before…
300	15:38	Romina	So, this is my issue: 4 to 2, 2 to 4,--it’s the same…
301	15:42	T/R	They both go to 16.
302	15:43	Romina	So, that’s why I count -
303	15:47	T/R	So, how would you resolve that then?
304	15:48	Romina	Umm (laughter)—I think it’s this one (boxes the 2 to the fourth power)  —I think it’s that one because it would be changing 2 to the n; n would be the row that you’re on—and 2, because there are always 2 colors. This one was—did I do this one? Yeah, because this one would be 2 to the 3. 
305	16:28	T/R	(points at paper) So how come?
306	16:30	Romina	So this equals…There are 8 combinations here on the 3rd row. 3 times 3 would be 9…I’m hoping it’s 2, yeah.



307 16:41 T/R So, does that always work then?

308 16:48 Romina So, this would be 2 to the 2 (writes 2^2 next to the 1 2 1 line and 2^1 next to the 1) That doesn't work, does it? What is 2 to the - There's a strong possibility… I might be missing something here (draws line in her Pascal's Triangle)



309 17:04 T/R Hold on…these are…when you wrote these out, these are ones that are…

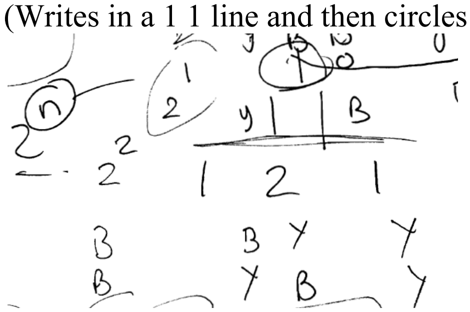
310 17:12 Romina So, this is this row (points on paper)

311 17:14 T/R That row: those are the 2 high.

312 17:17 Romina This is the 3 high.

313 17:20 T/R What would come before that, then?

314 17:23 Romina Maybe, two 1's, can I do that? (laughter) This would be a yellow, then the blue. I don't know what this one would be—the beginning of all towers? (Writes in a 1 1 line and then circles the single 1 at the top)



315 17:43 T/R2 But what if you - you've got things moving up, right?

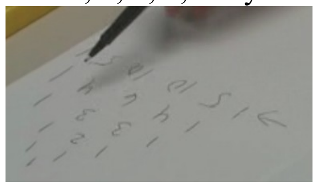
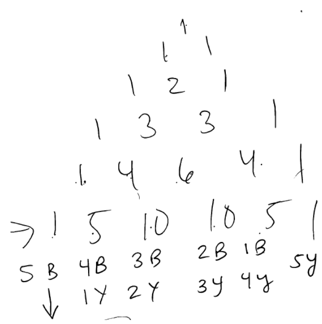
316 17:47 Romina So, this would be zero. This would be 2 to the one. This would be 2 to the zero, which is one, isn't it?

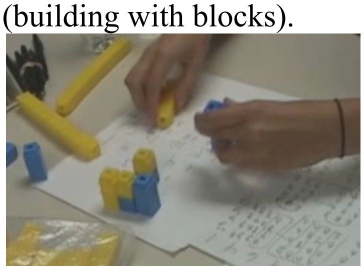
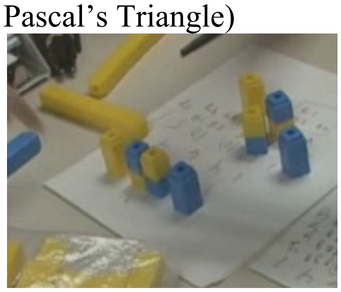
317 17:56 T/R2 Isn't that one of those rules of math? That's one of those few things, I think, that you just have to accept and nobody has ever explained that to me. No one's ever explained that to me, do you know what I mean?

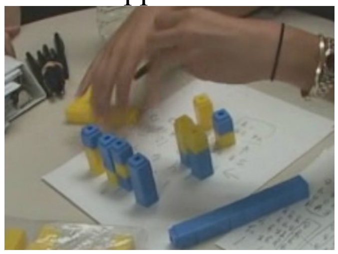
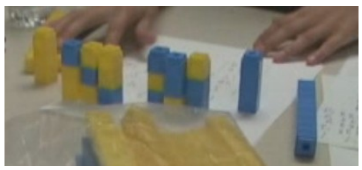
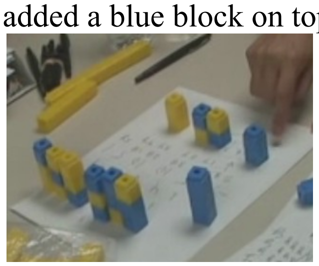
318 18:17 Romina I wasn't consistent..we just never worked with this part of the triangle. (Points to the 1 on the top of the triangle). They never asked us…but I think when we built it out, we built the whole triangle.

319 18:32 T/R So, if you had to solve the 5 high, then, what would you do based on this, then?

320 18:37 Romina I'd just---(writes 2^5) eventually, you'd get to that. I would draw out the

			whole (begins to write on a new sheet of paper) So, this would be my 5 row. 0, 1, 2, 3, 4—yeah. 
321	19:00	T/R	So what…
322	19:04	Romina	Are you asking me what am I doing? Sorry. So, I take—you carry down the 1's the whole way because that's going to be your tower that's all one color anyway…so you're going to have one tower that's all one color.
323	19:20	T/R	So, how do you know if that's the yellow or the blue, then?
324	19:21	Romina	I don't think it—I don't think it matters—whatever this is, this is the opposite. So if I started with blue, this would be my yellow. So, I would work with…this would be my all blue…this would be 4 blue, one yellow…these would be 3 blue, 2 yellow…Then, we'd switch here to 3 yellow, 2 blue…this would be my – oh, I should write this out.
325	19:46	T/R	Let's go through it one more time to be sure I understand.
326	19:48	Romina	This would be my all blue towers. (writes below the fifth row of Pascal's Triangle) This would be—this is 5 high, right? So this is 5 blues. This would be 4 blues, 1 yellow. This would be 3 blues, 2 yellows. This would be 2 blues, 3 yellow. This would be 1 blue, 4 yellow. And this would be 5 yellow. So, I just kind of gradually (motions across the top of the row). 
327	20:17	T/R	Huh…that's interesting, ok.
328	20:19	Romina	So, the way I get this is the 1; then 1 plus 4 equals 5. 4 plus 6 equals 10.
329	20:30	T/R	So, you're saying these two (points to the 6 and 4 entries)
330	20:32	Romina	Add up to that one (points to the 10 entry).
331	20:35	T/R	So, why did those (indicates on paper)?
332	20:36	Romina	Because you're just adding. You're adding an extra block, so that adds that many more combinations to that set. I also just know that this is how you do it. (laughter) I'm sure I had a good reason at one point, but now…I didn't have any thought behind why I'm going…I just know that is how to do it and that is what it means. I think when we did it a long time ago, it was…this up here would be 4…so what you do is you add—you're adding on—your 4 all-blue here, when you add an extra block, this

			becomes…you're adding extra block and then the 3 blue, one yellow to each one of these to become…you assume it's one blue box. These two become the same grouping. (Writes on her paper) 4B 3B 1y
333	21:32	T/R	Hmm…that's interesting
334	21:33	Romina	I don't remember how we exactly…
335	21:40	T/R2	I think I followed what you said: If this is a tower with 4 blue; from that, when you go to build it—towers one cube taller, you could either put a blue on it or a yellow on it?
336	21:55	Romina	Yeah, so it changes, so that these…we combine these (circling on paper)—this is like one extra yellow, one extra blue. What happens to your towers, then?
337	22:08	T/R	That's interesting, so if you had to, like…
338	22:11	Romina	Yeah, so, each time, if you're just building this, it would be like…We'll start with (picks up blocks) the two high, right? Then, you're just essentially, you have that and like what happens is this duplicates (building with blocks). 
339	22:36	T/R	So, this is the two…(indicates blocks)
340	22:38	Romina	No, now, what happens, then, is you get…you have all of this. It's been awhile. So, you're gonna kind of have these guys again (illustrates with blocks). And, then, you're just adding.
341	23:02	T/R	So, what is the…that's the (points to blocks)
342	23:03	Romina	This is your-- this guy. (places four towers over the row 1 2 1 on the Pascal's Triangle) 
343	23:06	T/R	Oh, that row? I see.
344	23:10	Romina	Then, you're gonna have to add…It's just as if you were systematically…
345	23:20	T/R	Now you're adding a blue block to …

346	23:22	Romina	…get to the next. A blue block. You can also add a yellow block. What should happen is…this should cancel out, I hope. Let me see if I have this. 
347	23:42	T/R	You're saying this will make the third row?
348	23:43	Romina	Yeah, I hope…hope this works out. (rearranges the towers) Did I make that third row? 
349	24:05	T/R	So, the way you grouped these, then…can you just explain…
350	24:08	Romina	So, this is the all blue…this is two blue, one yellow…two yellows, one blue...all yellow…
351	24:17	T/R	Huh, look at that. So, you went from the top row to…the second row to the third row. Explain the process you did again. So you…
352	24:29	Romina	So, I…hold on…(moves blocks) Let's just rebuild the …other row.
353	24:35	T/R	That's really interesting.
354	24:42	Romina	So, this is the row we started with. And, then, initially, I made a duplicating row that looked exactly like this. And, then, to this one I added a blue block on top. To this one, I added a yellow block on top. 
355	24:55	T/R	Oh, yes, I was wondering why you duplicated the row?
356	24:58	Romina	Cause I needed…you can either add a blue or yellow to each one of these So, I did it so I could add…just to make it easier.
357	25:04	T/R	Huh, that's interesting. So, that's the rationale for going…for adding the two entries above to get the next entry?
358	25:15	Romina	Any time in here, you can add a blue or yellow to jump down to the next number of high towers.
359	25:23	T/R	That's interesting…- parking meter - that's really interesting – okay - Here, I can stop us here. That's such neat stuff.
360	25:43	T/R2	Can I ask you a couple of questions?

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