TRACING STUDENTS' GROWING UNDERSTANDING OF RATIONAL NUMBERS

by

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To my family, who provided the conditions for continued study, I express my gratitude.

ABSTRACT OF THE DISSERTATION

An Investigation of Fourth Grade Students Growing Understanding of Rational Numbers

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This research, a component of a year long National Science Foundation funded study, traces and documents how rational number ideas are built by students as they move from placing fractions on a line segment (finite concept) to placing fractions on an infinite number line (infinite concept). The evidence is supported by representations used by students to express their ideas, explanations given by students and student justifications about their reasoning. The study was guided by the following research questions:

1. What evidence, if any, is there of the students' understanding of the idea of fraction as number?
2. How do students extend their understanding of fraction ideas to rational numbers?
3. What representations do students use to express their fraction ideas and extend these ideas to rational numbers?

The subjects consisted of a heterogeneous class of twenty-five, fourth grade (nine and early ten year old) students. Digitized videos, transcripts, student work, observation notes, and student overhead transparencies comprised the data from extended classroom sessions, videotaped with three cameras.

The study gives evidence that the students built understanding of fraction ideas such as equivalence and extended these ideas to negative fractions and improper fractions. It also showed that students successfully ordered fractions on line segments, then number lines, after working out distinctions between operator and number ideas. Student ideas revealed in these sessions showed that they were comfortable and successful with basic fraction operations. Lively classroom discussions and arguments worked out obstacles in the placement of fractions on a number line. Engagement in discussions about fraction ideas and negative fractions extended to rational numbers to include improper fractions as students identified equivalent number names for fractions. In the active student-centered environment the students worked together on tasks and shared their personal representations of rational number ideas and density of the rationals. This study provides detailed evidence that students can build understanding of fraction as number and successfully make connections to extend their understanding of number, generating and interest and understanding of fraction ideas that generally are not made accessible to students of this age.

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CHAPTER 1 – INTRODUCTION

1.1 Study Overview

As students progress in schools, it is expected that they gain increasingly greater understanding of the real number system and for those who continue their study of mathematics, an introduction to complex numbers as well. A solid background in understanding how the number system is structured is essential for students' continuing, meaningful study of mathematics. Relatively little work has been done on examining *young* student's understanding of rational numbers. In part this might be explained by students' lack of understanding of the concept of fraction as number. Several recent studies at Rutgers University have investigated how young students build a meaningful understanding of fractions and their operations (Steencken 2001, Reynolds 2005, Bulgar 2000).

All of these studies were components of The Colts Neck Study—a year-long classroom-based teaching experiment designed to investigate how young students build fraction ideas. The Colts Neck Study, a partnership between Rutgers University and the Colts Neck Public School district, was a component of a three year long National Science Foundation supported study conducted by Davis and Maher in three New Jersey school districts including an urban district of New Brunswick, a working class district of Kenilworth and an urban/suburban district in a Colts Neck[1]. The Colts Neck teaching experiment took place during fifty-six one to one and half hour classroom sessions over

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[1] The researcher was directed by Robert B. Davis and Carolyn A. Maher. It was funded in part by grant MDR 9053597 from the National Science Foundation and by grant 93-992022-8001 from the NJ Department of Higher Education, directed by Robert B. Davis and Carolyn A. Maher. Any opinions, findings, conclusions or recommendations expressed in this work are those of the author and do not necessarily reflect the views of the National Science Foundation or the NJ Department of Higher Education.

the school year. For a comprehensive list of the studies and their respective sessions, please see Table 1. For a complete comprehensive list of the Colts Neck 1993-1994 sessions, please see Table 2.

The fourth grade class consisted of twenty five heterogeneously grouped students and their teacher, Mrs. Joan Phillips. Fourth grade was selected since fifth grade at Conover Road School is traditionally the year when students are formally introduced to fractions in their curriculum and introduced to algorithms for computing operations with fractions. Prior to grade 5, students in the district were introduced to ideas related to fraction as operator as part of their primary school education in mathematics.

The earlier data from the Colts Neck Study research showed that students as young as nine and ten years old were able to build the idea of fraction as number and extend their knowledge to equivalent fractions (Steencken 2001), comparing fractions (Reynolds 2005) and division of fractions (Bulgar 2000).

The Colts Neck Study environment was constructed to encourage the students to build personal understandings of fraction ideas through: open discourse; in which individual students, pairs and small group organization could locally discuss ideas as well as whole class discussions where global discussions could connect student ideas across the classroom; class reflections, such as student written solutions that helped students to manage personal ideas as well as provide researchers insights into student thinking; using representations such as manipulatives (rods, rulers, candy bar ideas among others) to make available a tool for students to build new assimilation paradigms as transitions between representations. The tasks and classroom conditions were selected to enhance

Table 1.1: Colts Neck 1993-1994: One year study initiated by Carolyn Maher to explore the hypothesis of using Cuisenaire™ rods to investigate whether working with rods would help student transition in their understanding between fraction as operator and fraction as number.

Dissertation	Session Dates	Page	Activity Description
Steencken, Elena (2001) “.Studying Fourth Graders’ Representations of Fraction Ideas.”	09-20-1993	43	Fractions [Rods]
	09-21-1993	48	Dividing a Blue Rod
	09-24-1993	63	Rods
	09-27-1993	71	1/2 or 1/3
	09-29-1993	76	Is 1/5 = 2/10
	10-01-1993	83	Rods
	10-04-1993	88	Meredith Equivalent
Reynolds, Suzanne (2005) “A Study of Fourth Grade Students’ Exploration into Comparing Fractions.”	10-06-1993	161	Which is larger?
	10-07-1993	171	Which is larger?
	10-08-1993	190	"Trains"
	10-11-1993	199	Rods
	10-29-1993	217	1/4 versus 1/9
	11-01-1993	255	Number Line
Bulgar, Sylvia (2002) “Through a Teacher’s Lens: Children’s Constructions of Division of Fractions.”	12-02-1993	43	Fractions
	12-09-1993	63	Ribbons and Bows
	12-14-1993	126	Ribbons and Bows
	12-15-1993	170	Ribbons and Bows
Yankelewitz, Dina (2009) “The Development of Mathematical Reasoning in Elementary School Students’ Exploration of Fraction Ideas”	09-20-1993	93	Fractions
	09-21-1993	104	Dividing a Blue Rod
	09-24-1993	124	Rods
	09-27-1993	136	1/2 or 1/3
	09-29-1993	153	Is 1/5 = 2/10
	10-01-1993	168	Rods
	10-04-1993	180	Meredith Equivalent
	10-06-1993	192	Which is larger?
	10-07-1993	203	Which is larger?
	10-08-1993	214	"Trains"
	10-11-1993	224	Rods
	10-29-1993	235	Math Sentences w/Rods
	11-01-1993	249	Comparisons
	12-02-1993	255	Division of fractions.
	12-09-1993	267	Holiday Bows
	12-14-1993	281	Holiday Bows
	12-15-1993	293	Holiday Bows
Schmeelk, Suzanna (2010) “Tracing Students’ Growing Understanding of Rational Numbers.”	11-01-1993	32	Infinity & Number Line
	11-03-1993	68	Big Number Line
	11-10-1993	104	Number Lines
	11-12-1993	144	Numbers on Number Line

student exploration so that researchers could observe how the students developed a conceptual understanding of the mathematical ideas and how the students justified their solution.

Key components were identified from the earlier sessions about how the nine/ten year old, fourth grade, students constructed their own personal understandings of fraction ideas. During the fist seven sessions, Steencken (2001) traced the growth of understanding fraction ideas and found that the students showed understanding of several mathematical components of understanding: fraction as operator, fraction as number, identifying the unit for comparing fractions, the use of assimilation paradigms such as the candy bar, equivalence of fractions and fraction comparisons. Steencken examined how the students built their fraction ideas, including a description of the representations they exhibited by words, physical models with rods, drawings, and written work. She traced the flow of fraction ideas throughout the classroom, showing how students built on each other's ideas and challenged the reasonableness of justifications. Steencken found that the students expressed fractional ideas more precisely throughout the sessions within their use of natural language, physical models and notation. As the researchers introduced more precise language, the students' language became more precise, such as expressing the unit of comparison, verbally and in the models and drawings. Steencken also found that a candy bar, used during Sessions 3 and 5, became a means for the assimilation paradigm where the idea of unit was introduced to students. Steencken and Maher (1998) detail the process by which some of the students constructed the idea of equivalent fractions.

Reynolds (2005) studied six later sessions of the same class of students, focusing on their understanding of comparing fractions, with particular attention to the conjectures they made and attempted to validate. The students worked with Cuisenaire Rods to construct models and justify their solutions to the comparison tasks. Reynolds explored three research questions based on student conjectures including: student conjectures that spanned characteristics of a particular number or model with respect to characteristics of oddness and evenness, student conjectures that spanned fractions versus counting numbers, and student conjectures that included the relative model size. She found many students reasoned according to patterns that they identified in building solutions. With respect to conjectures, Reynolds found that each category of conjectures frequently triggered new conjectures or ideas and that students tended to build on each other's ideas as well. She also documented the development of students' use of more precise language as time elapsed through the sessions (also found by Steencken 2001). Ultimately, Reynolds showed that students were able to appropriately compare fractions, building on the idea of equivalence.

Bulgar (2002) examined and documented the Colts Neck students' division of fractions constructions during four sessions in December. She took into account the nature of the researcher interventions, the student ideas that were expressed visually, the representations that were used, and the student reasoning that resulted in their justification of solutions to problems. Bulgar identified twenty ways researchers intervened including: giving information to students, rephrasing student ideas, asking another student to rephrase an idea posed by a student, asking questions, asking for justification and directing students to (construct, discuss, observe) their own or another's idea. Bulgar found that the researcher in the sessions was a pivotal classroom figure but not responsible for students' inward assimilation of knowledge. The research conditions resulted in the development of a sense of community within the classroom, the selection and implementation of appropriate tasks for the students, and appropriately injecting interventions to continue the classroom community's natural progression of exploration and construction. A second finding by Bulgar included how students externally expressed their ideas. The two categories included: (1) if an idea was general or local to a problem, and (2) if an idea expressed was in agreement with others. Bulgar also examined representations used by students and identified a variety of types including words, models, symbols, drawings and gestures used through the sessions. Finally, Bulgar examined and found that student reasoning and justification included the use of real-world representations, metaphors, paradigms and references to earlier representations. Ultimately, Bulgar showed evidence that students had understanding of division of fractions.

Yankelewitz (2009) examined seventeen sessions flagging forms of reasoning as well as factors of both the task and the environment that encouraged the development of reasoning. Yankelewitz found at least four forms of reasoning including the following: generic reasoning, reasoning by cases, recursive reasoning, reasoning using upper and lower bounds. Throughout the sessions, 364 arguments were found where 309 arguments were a claim justification variant and the remaining 55 were counterarguments. Of the arguments, Yankelewitz explained that the majority were direct arguments and minority were indirect arguments. She further explained that the subtle majority of the counterarguments were indirect “elicited” (p. 317) by the claims of others; while direct arguments were used predominately for a claim justification.

1.2 The Study

This study builds on and extends the previous work with the same class of nine/ten year old students in which they investigated strands of problems dealing with fraction ideas later in the year, guided by the following research questions:

(1) What evidence, if any, is there of the students’ understanding of the idea of fraction as number?

(2) How do students extend their understanding of fraction ideas to rational number?

(3) What representations do students use to express their fraction ideas and extend theses ideas to rational number?

In particular, this study examines how students express their understanding of fractions number as they move from rod models to number line representations. Also, it traces their building of the number line model and examines how they use the model to represent understanding of number.

CHAPTER 2 – LITERATURE REVIEW

This chapter begins with a discussion of the theoretical framework that guides this research focusing on the importance of representations in learning. It follows with a review of literature in the following areas: learning and teaching of fractions, and curriculum materials used in schools for the teaching of rational numbers.

2.1 Theoretical Framework

The theoretical framework that guides this work parallels the framework that guided the Kenilworth Longitudinal Project and the Colts Neck Project. The framework provides a lens for studying the developing ideas of students that are captured on video. In particular, the theoretical framework addresses at least three questions into promoting student-learning including: (1) What are the environmental conditions? (2) How do students transition to new ideas? and (4) How do researchers' facilitate students' learning? These questions will be discussed in this section.

What are the environmental conditions that promote student learning? The Davis, Maher and Martino (1992) paper entitled "Using Videotapes to Study the Construction of Mathematical Knowledge of Individual Children Working in Groups" showed that students who had not received formal instruction could develop their own individual solutions given the proper conditions such as working in small groups, discussing ideas/questions and sufficient time to build on and revisit tasks.

How do students transition to new ideas? The notion where students build and restructure their thinking based on previous experiences has been discussed by Davis (1984) through what he calls an *assimilation paradigm*. An assimilation paradigm according to Davis and Maher (1993) is a set of internal mental functions where a learner sees a new experience to be “just like” or as “similar to” earlier experiences.

Davis (1984) describes “the properties” that make an activity an assimilation paradigm. Davis describes four properties as follows: (1) “involves ideas for which virtually all students have powerful representations” (2) “it is a reliably accurate isomorphic image for all [the applicable] operations” (3) “it tells a student how to deal with [the applicable] problem; the story itself guides you to a solution” and (4) it is “simple.”

In Steencken and Maher (2003), an example of an assimilation paradigm was fairly sharing a candy bar. When sharing candy bars fairly, the bars should be the same size. For the idea of keeping fraction units the same when making comparisons, Jessica, “spontaneously introduced the candy bar” (p. 130) to question “the construction of different sized rod models” when they should be the same for comparing fractions (p. 130). Another example of an assimilation paradigm for fraction ideas given by Steencken and Maher (2003) were the rods with variable number names. The students gradually replaced the rods with fractions number names in dealing with fraction problems. “Eventually they referred to the comparisons without physically using the rods” (p.130).

Davis and Maher (1997) discuss how a teacher can provide “a carefully designed experience for a student” (p.99). If the experience is “essentially isomorphic to the relevant mathematics” then the teacher’s method of introducing the idea is known as the *paradigm teaching strategy* (Davis 1984). Davis and Maher explain how the “Pebbles-in-the-Bag” activity can introduce the idea of subtracting integers.

What are the researcher roles? The role of the researcher in the experimental classroom has been described as a facilitator for promoting student explorations. In the research, the facilitator monitors and observes student progress; the facilitator encourages the students to actively building their own understanding of the underlying mathematical notions found in the tasks (Maher 1988; Martino and Maher 1994; Maher and Martino 1996).

The role of the researcher is to promote students' constructing their own "representation structures" (Davis 1984) versus a "formal school approach where pupils are supposed to learn the same thing at the same time" (Sutherland 1992) and representations are imposed on the students. Representation structures mean different things to different people. Davis states:

> Indeed, uncertainties are clearly present, nowhere more evidently than in the nature of large information representation structures whether "chunks" (George Miller), or "frames" (Minsky), or "scripts" (Schank), or "models" (Papert), or "powerful ideas" (again, Papert), or "schemas" (Piaget) or "assimilation paradigms" (Davis). But the true business of mathematics instruction is to help the student to construct, in his or her own mind, a large collection of knowledge representation structures that provide powerful forms of all the key ideas of mathematics [...]. If our goal involves the representations of key ideas in the student's mind, we must be willing to try to talk about such matters (p.356-357).

Certain factors and conditions are required for students to develop a conceptual understanding of mathematics, and, in this case, of rational numbers. This dissertation begins with a discussion on reviewing some historical origins of fractions followed with a discussion on general rational number research studies. It then presents research on how rational numbers are presented to students through curriculum materials and manipulatives. Finally, this dissertation will address cross cultural rational number studies and rational number teaching implications.

2.2 Rational Number Studies

Fractions, historically, according to Gary Davis (2003b) were used by early mathematicians such as Abu Ja'far Muhammad bin Musa al-Khwarizmi. Al-Khwarizmi is accredited for two things: (1) the term ***algorithm*** as defined as "mechanical procedure" and (2) bringing fractions to the Arabic world from India around 800 BC. Davis uses the early mathematicians as examples of times when numbers ***made sense*** to explain the necessity to move from the current school algorithms conveniently used in many schools back to the philosophies and ideas behind the procedures used originally by the great early master mathematicians.

Rational numbers are defined to be any number that can be written as a fraction with an integer numerator and a nonzero integer denominator. Using set-notation the definition is the following: "if ***Q*** is the set of rational numbers, then ***Q*** = {x | x is a number that can be written in the form ***a over b (a/b)***, where ***a*** and ***b*** are integers and ***b*** is not equal to zero}." (Frisk 1993) Other definitions include, "A number is a rational number if it can be expressed in the form of a fraction, x/y, and the denominator is not zero (Ross 1996)." As seen from the literature, there is no single definition for rational number; instead, an abstract idea is conveyed though many different word combinations.

Freudenthal (1983) defines rational numbers and fractions as representing the "same thing." He gives the example, " 2/3 = 4/6 = 6/9 = …" (p. 133) and describes how each fraction in the example is an alias for the ***same*** rational number. He uses whole number alias as a metaphor to describe rational number aliases and says:

> On the left and right of the equality sign, the same object occurs. [In aforementioned example] there is talk again and again of the same thing, only represented in various ways; and this thing is a *rational number*. Well, one can agree to prefer the way 2/3, and in general, for every rational number, the expression by means of a fraction where numerator and denominator have the common divisor 1, the simplified fraction; as one prefers for the number 5 the expression 5 rather than 3+2, 10-5 and so on, though the others are equally well admissible. There is, however, a difference: '5' is not only the *preferred* name of the number 5, it is its *first* name, the name by which it has been introduced to me, and under which I first made acquaintance with it, whereas '3+2' and '10-5' are aliases by which I can also call it up. '2/3,' however, is only the simplest name of a certain rational number, and I would not even be able to say about many rational numbers under which name I first met them. This then is the reason why the various fractional expressions of the same rational live so much more their own lives, and why they are known under a special name: *fraction*. (p. 134)

He also described the correspondence between fractions and rational numbers as the "phenomenological source" (Freudenthal, 1983, p. 134)., indicating that the rational number "object" matters more than the "fraction" (p.134) and writes:

> Fractions – or what corresponds to it in other languages – is the word by which the rational number enters, and in all languages I know it is related to breaking: fracture. Rational number evokes much less violent associations; rational is related to ratio, not in the sense of reason but of proportion, of measure – a learned context, and much more so than fraction. (p. 134)

2.2.1 The Rational Number Project

The Rational Number Project (RNP), 1979 through 2002, is one of the longest federally funded cooperative multi-university research projects in the history of mathematics education to investigate student learning and teacher enhancements for rational numbers. The RNP researchers led by Merlin Behr, Richard Lesh and Tom Post developed three elementary mathematics courses and two curriculum texts (Cramer et al. 1997) through their research. The research was based on primarily elementary school and some middle school classroom observations, student interviews, and written assessments. The RNP spawned over eighty-five research publications on fractions, decimals, ratios, indicated division, measure, operator, among others. The RNP Teacher's Guide to Middle School Mathematics (Level 1) discusses Lesh's *translation model.* The model shows that mathematical ideas can be represented in five ways: written symbols, pictures, real life situations, verbal symbols, and manipulatives. Cramer et. al (1997a) state that, "Children learn by having opportunities to explore ideas in [ways described in Lesh's translation model] and by making connections between the different representations." As shown though the RNP and the Colts Neck Study, when students are given time to personally construct relationships between these mathematical relationships they learn and develop number sense.

RNP research studies, such as Cramer, Post and del Mas (2002) and Cramer and Henry (2002), show that students, when encouraged to construct their own conceptual understanding of rational numbers, relied less on rote procedures when solving mathematical tasks. Cramer, Post and del Mas (2002) examined the achievement of over sixty-six classrooms containing 1600 fourth and fifth grade students over approximately a month. The RNP curricula differed from the Commercial Curricula (CC) by emphasizing physical models and translations between representations. In the study, the students were randomly assigned treatment groups of the traditional commercial curricula contrasting to the RNP fraction curricula. The studies showed that the students who emerged from the RNP curricula had significantly higher mean posttest and retention test scores as well as better quality of thinking and estimation on tasks given during interviews. The test were scaled according to concepts, order, transfer, and estimation and interviews showed the RNP students used an approach where they constructed their own representation structures to solve new problems while CC students relied on algorithms and standard procedures to solve new problems.

Further, Cramer and Henry (2002) examined the role of using manipulative models every day for five instruction weeks to build number sense specifically for the addition of fractions as emphasized in the RNP. Cramer and Henry emphasizes that teachers often transition to symbols from manipulatives too soon. Cramer and Henry show that the RNP students developed an understanding of fraction size through the ability to order fractions, were able to estimate answers to problems and verbalize their thinking. The students who had not received the extensive work with manipulatives (rods, paper folding and chips) through the RNP did not display the same number sense characteristics. Overall, the RNP materials when used correctly has been shown to improve students' number sense characteristics.

2.2.2 Representations fostered by Using Manipulatives

Goldin and Janvier (1998) connect the terms "representation" (1) and "system of representations" with mathematics learning and teaching. According to Goldin and Janvier, representations can be interpreted in the following four contexts: (1) an "external, structured physical situation" that can mathematical be described; (2) a "linguistic embodiment, or a system of language;" (3) a "formal mathematical construct" representing situations through "symbols;" and (4) an "internal, individual cognitive configuration." Using these widely accepted definitions, manipulatives fall into the external representation category.

Gattegno (1961, 1963) wrote about how to use Cuisenaire rods to foster student understanding of fractions. He suggests a four stage introduction to Cuisenaire rods including the following: (1) free play (2) free play accompanied with directed activities with the rods, in which "relationships are observed and discussed without the use of written mathematical notation" (p.1) (3) free play accompanied with directed activities with the rods, in which "mathematical notation is introduced and used without assigning number values to the rods" (p.1) and (4) free play accompanied with directed activities in which "the use of mathematical notation is extended and number values are assigned to the rods" (p.1).

Within the first stage students freely play (work) independently or together "without restriction" (p.2) so that they become "acquainted with the mathematical relationships" (p.2) formally discussed in later stages. The duration of stage one depends on the age of the students allowing younger students more time to become familiar with the rods.

The second stage includes directed activities which aim to bring out "basic mathematical ideas and relationships" (p.2). Gattegno notes that the researcher should "concentrate on experiences rather than on the language used in describing them" (p.2). For example, lengths made from two or more rods, a "train" (p.3), can be compared (example: "larger than," "shorter than," and "equal to") with lengths of a single rod.

The third stage includes free play with a formal introduction to mathematical notation. Gattegno notes that the students will be ready at this stage to "learn to read and write some of the discoveries" the made at the earlier stages. Gattegno encourages using the symbols including the first letter of the rod's color to represent the rod among other signs to represent relationships between rods. For example, students might fill in a missing length in order to complete the writing of a relationship between rods.

The fourth and final stage is the assignment of number values to the rods so that the students "treat the rods as models of numbers" (p.7 1963). For example, in this stage mathematical relationships including order of operations, inverse operations, commutativity properties are meaningfully introduced to students.

Work such as Dienes (2001), Kennedy (2000) and Middleton (1998) show that there are many stages to understanding rational numbers. Dienes explains a theoretical perspective where there are at least six major stages required for students to achieve understanding of rational numbers including: free interaction, looking for rules for ratios, comparing activities, ordering ratios and representations and formalization. Similar Gattegno's methodology, the stages start general (free play) and develop into precise formalism (written notation).

In Dienes, the first stage pertains to freely recognizing real world characteristics such as a very "realistic" picture or movie as the ratios have been preserved. In the second stage, learners develop the sense of proportion where a ratio of small to big is preserved. For example, the ratios between a large house model must precisely match units with a small house model. By comparing activities in stage 3, learners are helped to pinpoint the ratio notion developed in stage 2. Dienes suggests using a table to act as a *dictionary* to map or translate the actions from one activity into the required actions of another. The next stage, stage 4, involves student ordering their representations thus developing meanings. Finally, stage 5, introduces symbols to the representations; and, stage 6 formalizes the symbol system traits which may provide further insights. Dienes' 2001 paper is strictly theoretical as a classroom study was not included in the research.

Kennedy (2000) describes the benefits of using physical models to develop number sense and transitions between representations for secondary school students. Kennedy relates at least three stages beginning with (1) using manipulatives, (2) developing the notions of rational numbers on the number line; and, followed by (3) extending the ideas to percents and equations and discussions. Kennedy explains how the Partnership for Access to Higher Mathematics (PATH) Project students who were "at risk" of not passing mathematics were successfully merged into student centered classrooms where they were given concrete approaches to develop individual reasoning. Within the PATH sessions, transitions were created to scaffold students from familiar representations to new representations. For example, many students who had been introduced to integer addition and subtraction had worked with physical chips and not the continuous number line since many of their instructors had chosen this means to introduce addition and subtraction. Another example was shown through using fraction strips with a fraction mat to connect the number line representation of fractions with their algebraic representations and percents. Students who had participated in the PATH program scored significantly higher on state-mandated mathematics test specifically within the areas of proportional reasoning, equation solving and linear relations. The PATH project developed theses necessary links to scaffold students as they transitioned between representations.

In their paper, "Using bar representations as a model for connecting concepts of rational number," Middleton et al. (1998) suggest that traditional teaching and learning emphasize the differences between the rational number meanings rather than their similarities. Middleton et al. emphasize that students tend to confuse meanings and over generalize properties between the different rational number contexts. Middleton et al. claim that rational number topics are traditionally treated as distinct ideas rather than highlighting the connections between topics. Middleton et al. emphasize the importance for students to learn to use "a variety of equivalent [rational number] forms" and representations. Middleton et al. suggest teaching rational numbers by using "real" representations towards more abstract representations. For example, the research showed vignettes of how fifth-graders accurately used and mastered linear representations of fractions during their first formal introduction to fractions through dividing and fairly sharing three submarine sandwiches among six students. (Other representational suggestions given by Middleton et al. include fruit tape, parking spaces, graduated measuring cups, rulers and routes on a map.) Middleton et al. suggest using the common representation, ***a bar***, as a replacement for the ***real*** objects as a tool "for whole-group communication (p.303)" and as a means to transition middle grade students between rational number concepts including fractions, decimals, percents and ratios. The results of the study suggest these new ways to represent rational numbers will arise naturally in student centered teaching environments and should complement representation uses.

2.2.3 Teaching approaches

Approaches such as Dabell (2003) and Neil Griffiths' "Walter's Windy Washing Line" from Corner To Learn (2007) show that using manipulatives can offer students the opportunity to transition between ideas. John Dabell introduces the work of Alison Borthwick and Constance Tyc who developed the software "Power of the Number Line" for Elementary and Middle school students in the United Kingdom as an enhancement of the mathematics curriculum. The work was initiated by observing classrooms in Hungary where the researchers observed quality teaching of number line concepts. Walter's Windy Washing Line offers a fun and creative way to explore all the math concepts for *stages one* and *two* in the United Kingdom or *kindergarten* and *first grade* in the United States.

2.2.4 Curriculum materials expectations

One approach used in schools for students to build their understanding of number is Houghton-Mifflin Pre-Algebra book (Dolciani, Sorgenfrey & Graham 1985). Rational Numbers are introduced in Chapter 3 which is titled "Rational Numbers." The chapter on rational numbers is broken down (p.iii) into two main categories, "Number Theory and Fractions," and "Operations with Fractions." Surprisingly, rational numbers are never mathematically defined; nor is the term listed in the glossary. Instead, *properties* of rational numbers are sprinkled throughout the ten sections comprising Chapter 3 introducing readers to some "notions" of rational numbers. For example, a property (p.119) within section 10 reads, "Every terminating or repeating decimal represents a rational number."

2.2.5 Cross Cultural Studies

Moseley, Okamoto, and Ishida (2007) presented a cross-cultural study comparing the use of rational number representations between fourth grade classrooms within the United States and Japan. The researchers found two predominate underlying cultural differences between international teaching styles: (1) the reliance on following curriculum textbooks is greater in the United States and (2) United States teachers training is for a "narrower range of grades than their Japanese counterparts," (p. 181) which may limit their understanding of the various rational number perspectives and representations. The paper found through interviewing teachers that cross cultural teaching styles and expectations are very different between the two countries. They reported that district, state and national standard required by United States teachers may limit inquiry and constructivist learning. An exception that was noted was with a classroom of gifted students where inquiry and constructivism are encouraged internationally. The authors suggested a new study examining cultural influences to explain differences between mathematics classrooms in each country.

Subramaniam's (2008) explained the need to develop curriculums that emphasize the equivalence between number notions such as the equivalence between thinking about m/n as a multiplication operation ($m * 1/n$) and as a division operation (taking the n^{th} part of m). Subramaniam referenced the work (Naik and Subramaniam *forthcoming*) from a developing teaching project in India where students are encouraged to develop intuitive number understandings rather than develop the traditional algorithmic understandings; thus, moving away from the goal of "computing an expression" (p. 14) to a deeper symbolic understanding of expressions.

2.2.6 Teaching Implications

Studies such as Burn (1998) and Czyz (2003) show that teaching styles significantly shape student understanding. Bob Burn (1998) describes his personal experience in sixth grade where students were encouraged to accept the standard proof rather than raise important questions. Czyz et al. (2003) suggest showing rational numbers as continuous fractions to emphasize the correspondence between natural numbers, integers, fractions and rational numbers.

Guy Brousseau, Nadine Brousseau and Virginia Warfield (2004; 2007) describe the Brousseau (1987) *Didactique* theory where all students can “create, understand, learn, use and love mathematics under certain conditions.” The Brousseau philosophy of learning and teaching rational numbers is modeled after the *theory of situations* where spontaneous learning is encouraged through which a teacher can use the opportunity to “provide meaning, a context or an objective aim for the knowledge the Situation gives rise to.” Within the model, the teacher is relieved of the “pedagogical stance of teacher as authority,” as well as the student relinquishes the role of “obedient absorber.”

The Brousseau philosophy has three arch-objectives that span fifteen modules. The objectives are (1) fractions non-traditionally through set ordering and rational/decimal topology, (2) teach rational numbers before decimals to emphasize the relationships between natural numbers, rational numbers and decimals and (3) emphasize the relationships between fractions as measurements, fractions as linear mappings and fractions as ratios. The underlying structure of the modules is a game played by groups of students.

> [The Theory of Didactical Situations for the construction of these lessons] are many-faceted adventures that pull together a whole conglomeration of pieces of knowledge that will be provoked, activated, used, modified, invented and verified, around a project of a mathematical nature dealing with an essential mathematical notion. (Guy Brousseau, Nadine Brousseau and Virginia Warfield 2007 p.282)

Brousseau emphasizes the analogy of learning to play Rugby where measuring a learner's progress at fixed intervals is inherently flawed.

The Brousseau bracketing game worked in a multi-stage sequence where each sequence had different objectives. Students were asked to find a bracket for a sum or a set of fractions. Brousseau (2004) described a bracket for a sum of fractions as, "The sum is bigger than __ and smaller than __." For example, at the introductory stage of the game sequence, the teacher wrote at least ten fractions on the board. The teacher would select approximately three fractions from the set and the students were asked to swiftly find an interval (or bracket) in which the numbers could be found. As the game developed, the teacher asked the students to find smaller and smaller intervals. For example, in the fifth-stage of the game the teacher asked the students to find a lower and upper interval bound that was smaller than the whole number one. Throughout the stages, the game was played in different size groups ranging from pair-wise rounds to student-class (student versus class) rounds. Richly, the game promotes both the development of c-knowledge (connaissance knowledge), the concept knowledge, and the s-knowledge (savoir knowledge), facts knowledge (Brousseau et al. 2004).

2.2.7 Rational Numbers Today

In sections 2.2.1-2.2.5 current trends for improving rational number education were presented. Moseley, Okamoto, and Ishida (2007) suggested that schools in the United States are more prone to teaching directly from the textbooks than other countries including Japan. Within the content of teaching rational numbers there follows a discussion on the use of manipulatives and how they can be used to scaffold students to new understandings of the breadth of rational numbers (i.e. fractions, decimals, ratios, etc.) The final sub-section explores conditions perhaps beneficial for fostering student understanding of rational numbers.

Why is having ***number understanding*** important for students? Mathematical thinking is pervasive throughout society and is crucial for building resilient and autonomous citizens. Developing number sense understanding is one fundamental ingredient of all mathematics areas and thinking. Nationally, the National Council of Teachers of Mathematics (NCTM 2002) states that students K-12 should be able to, "naturally decompose number, use particular numbers and referents, solve problems using the relationships among operations and have knowledge about the base-ten system, estimate a reasonable result for a problem, and have a dispositions to make sense of numbers, problems, and results." NCTM also specifies for students' K-12 that they should be able to understand numbers though different representations, relationships, conceptual meanings, conceptual relationships, computations, estimations and systems.

2.2.8 A Way to Characterize Understanding

How is understanding characterized? In their model for studying growth in mathematical understanding, Pirie and Kieren (1994) describe the process used by learners to recursively link previous experiences to current problems as *folding back.*

In their model, Pirie and Kieren explain the *folding back theory* where mathematical understanding can be characterized through a dynamic, iterative and recursive eight-stage model. The stages contain: primitive knowing (initial understanding of everything except the topic being considered), image making (active in developing representations for the topic being considered), image having (developing a mental plan without being tied to visual representation), property naming (connecting properties for topic), formalizing (working with topic without image/activity constraints), observing, structuring, and inventing (ability to take topic and use it appropriately for further thinking). Students move between layers and revisit (*fold back*) to inner layers and they grow in mathematical understanding.

Martin (2008) expands on the Pierie-Kieren Theory by elaborating and characterizing the *folding back* metaphor and introduces the *thickening* notion. *Thickening*, as Martin describes, happens during *folding back* at inner understanding layers as new information is assembled, developed, broadened and restructured to support new outer understanding layers. Martin uses video-data, written work and observational notes of seven small groups of students to explore the theoretical *folding back* phenomenon and explore implications for teaching and learning.

In Martin (2008), the study takes place with students ranging in age from first year secondary school to post-graduate teacher education. Martin finds seventy-nine *folding back* cases that he characterized in three ways: *source* (the cause of *folding back* such as peer, teacher, self or material), *form* (the actions the learner engages in as a result from the *source* such as *collecting* more information at the inner layer, working at the inner layer using existing understanding, or forming a new path for understanding—through causing a discontinuity—at the inner layer), and *outcome* (effectiveness). The research contributed to the Pirie-Kieren Theory by showing that folding back does not necessarily lead to continued mathematical understanding growth and that *folding back* happens for a plurality of causes through a plurality of methods. Martin's work offers insights into predictions into "the kind of actions a learner may need to engage in to facilitate further growth (Martin 2008 p.20)."

CHAPTER 3 – RESEARCH METHODOLOGY AND ANALYSIS

Researching learning and teaching is a complex process. A discussion on the design of the study and research methodology and analysis are found in this section.

3.1 Purpose of the Study

The Colts Neck Project was designed to work with fourth grade students before they were formally introduced to mathematical ideas concerning fractions and their operations through traditional methods in school. The significance of this study is that at this time of the 1995 study a standards base curriculum (NCTM) did not exist.

3.2 Study Design

The study design was purely to explore how students build mathematical ideas, in this case, ideas about fractions. The study was student-centered where the students were neither formally assessed nor consciously influenced from the researchers on to how to think about an idea; however, they were given very specific tasks to explore (as discussed in Chapter 8). The sessions were led by Drs. Carolyn Maher (RT1) and Amy Martino (RT2). Interventions were designed using techniques such as questions seeking explanations or evidence for claims that were made.

3.3 Setting and Population

The Colts Neck Project took place in Colts Neck, New Jersey at Conover Road School. The class consisted of nine and ten year old students—fourteen of whom were girls and eleven of whom were boys. The teacher, Mrs. Joan Phillips, was present throughout the sessions as well as other observers including: graduate students, the Conover Road School principal, Dr. Judith H. Landis, occasional outside researchers (Drs. Alston, Davis and Gjone), graduate student observers, and the camera crew.

3.4 Data Collection

The data were collected from video cameras, observation notes, student overhead transparencies and student work. The data combination (Pirie 1996; Lesh and Lehrer 2000) was designed work towards examining in detail the students' mathematical activities while reducing human and technological biases (Powell, Francisco & Maher 2003).

Table 3.1: A Complete Guide to the Colts Neck Data Library (Date, Title and Views)

Date	Title	#		Date	Title	#
09-20-93	Fractions	3		12-09-93	Ribbons and Bows	3
09-21-93	Dividing a Blue Rod	3		12-14-93	Ribbons and Bows	3
09-24-93	Rods	3		12-15-93	Ribbons and Bows	3
09-27-93	1/2 or 1/3	3		02-01-94	Number Line	2
09-29-93	Is 1/5 = 2/10	3		02-02-94	Number Line	4
10-01-93	Rods	3		02-03-94	Number Line	3
10-04-93	Meredith Equivalent	3		02-07-94	Number Line	2
10-06-93	Which is larger: 3/4 or 1/2?	3		02-08-94	Negative Numbers	3
10-07-93	Which is larger: 3/4 or 2/3?	3		02-15-94	Neg. Numbers	3
10-08-93	"Trains"	1		03-01-94	Neg. Numbers	3
10-11-93	Rods	3		03-02-94	Neg. Numbers	3
10-25-93	Math Sentences w/Rods	3		03-07-94	Neg. Numbers	3
10-27-93	Math Sentences w/Fractions	3		03-08-94	Neg. Numbers	3
10-29-93	Fractions Rods 1/4 versus 1/9	3		03-10-94	Neg. Numbers	3
11-01-93	Number Line	3		03-11-94	Towers of 4	1
11-03-93	Infinity & Number Line	3		03-15-94	-	2
11-10-93	Big Number Line	1		04-11-94	Neg. Numbers	1
11-12-93	Number Lines	3		04-13-94	Dice Games	1
11-18-93	34 Cents Problems	1		04-14-94	Neg. Numbers	2
11-19-93	Fractions, Parent's Visit	3		05-04-94	Neg. Numbers	3
11-22-93	Positive Rational Numbers	3		05-18-94	Neg. Numbers	1
12-02-93	Fractions	3		06-20-94	Assessment of Fractions	1

The Colts Neck Project video data were collected using one to three video camera views of each session. As the Robert B. Davis Institute for Learning's video library grew there arose the need to move to digital conversion of the video collection. Therefore, the analyzed video data has been digitized and is now stored on CD and DVD. A complete guide to the Colts Neck 1993-1994 data (highlighted by sessions explored in this dissertation) can be seen above in Table 3.1.

3.4.1 Video Data

Throughout the Colts Neck Project sessions, as seen in Table 2, the camera views ranged from one to three views with a majority covered from three views. Predominately the views are one of the following: projector, a roving camera or a still camera. A view labeled “projector” was a still camera that strictly captured the projector activity. A view labeled “front” or “side” was a still camera that was set in either the front or side of the class capturing classroom activity. A view labeled “roving” was a moving camera that captured various classroom activities.

This study examines four digitized tapes as follows:

- 11-01-1993 [Projector, Front and Side]
- 11-03-1993 [Projector, Front and Roving]
- 11-10-1993 [Roving]
- 11-12-1993 [Projector, Front and Roving]

This initial work focused on four sessions where the students constructed and ordered rational numbers on the number line. The results from four classroom episodes are presented in chronological order. The sections are organized by events where a new section represents a transition to a new event which changes the classroom dynamics including a new task or a new classroom organization schema.

In the first session, November 1, 1993, the students explored comparisons and ordering of fractions between the interval zero and one. They explored the fractions one half, one third, one fourth, one fifth, one sixth, one seventh, one eight, one ninth and one tenth. During group activities, some students, without formally being requested to do so, explored the placement of one one-hundredth and one one-thousandth on the number line. Finally, RT1 asked the class to explore the placement of the number three-fourths.

The second session, “Infinity and the Number Line,” took place on November 3, 1993 and included three camera views. In this session, the students explore the ruler as a representation for a segment of the number line. The students were asked to consider an infinite number line and are asked questions respectively (i.e. are there biggest and smallest numbers?). The class was asked to place positive and negative whole numbers on the infinite number line and the word “integer” is introduced to the class discussion. The class then explores the notion of infinity between whole numbers (i.e. between zero and one). The students use many external representations to explain their ideas. The end of the session was spent placing numbers (and justification of the placement) on the number line located on the overhead projector including the fractions one half, one and three fourths and two and one half.

The following session, the “Big Number Line,” took place on November 10, 1993. In this session, the class considered Meredith’s number line and her dual labeling for fraction as operator and fraction as number spanning five lines (e.g. the placement of the number three thirds and the region that represents three thirds). RT1 discussed some mathematical labeling conventions to help eliminate confusion. Alan drew a similar representation for the dual notions of thirds on the number line; a class discussion resulted. The end of the session was spent filling in the big (infinite) number line located on the dry erase board at the front of the room. Students place the fractions positive one half, negative one half, positive three fourths, negative one fourth, negative two fourths, negative three fourths and negative four fourths.

The final session, “The Number Line,” occurred on November 12, 1993. In this session, students revisited components of the previous session discussion. They discussed why the fraction four fourths could be another name for the number one and the fraction one half could have another name two fourths.

3.4.2 Documents and Observations

Documents arising from the Colts Neck Project include student notes, researcher notes and overhead transparency work. Session observations include both direct researcher notes as well as post-observations made from the video data.

3.5 Methodology

The data were analyzed based on components of the Powell, Francisco and Maher (2003) method, evolved from the longitudinal research video studies of the Robert B. Davis Institute for Learning (RBDIL). Their method includes several stages for the analysis of video data including: viewing, transcribing, coding, writing analytical commentaries and summaries, and developing a narrative.

The components that were used are elaborated as follows: (1) viewing: each of the four CDs will be viewed multiple times. CDs from earlier and later sessions have also been viewed to develop a more comprehensive picture of the Colts Neck Project. (2) Transcribing and verifying each of the four CDs is the next step. Transcription verification is the process of having another viewer watch the CD to verify that the CD was properly transcribed. The transcription consists of four components including: line number, time, speaker, spoken words and actions. The line numbers are used for reference. The time stamp is usually taken in five minute intervals. Transcriptions then list the speaker’s name followed by both spoken words and actions that are not verbally expressed. The varying camera views are also accounted within the session transcriptions. (3) Coding is the phase where critical events are flagged, traced, compared and categorized. The codes are placed on the transcript in a new column where a viewer can scan the transcript looking for events. (4) During the last phase a summary of the findings and a narrative emerge. The narrative tells the story of what took place during the sessions and references the actual critical events in the transcript by line number.

Each phase is dynamic and the entire process is recursive; descriptions became broader with more views and critical events, "significant conceptual progressions" (Kiczek 2000) or "conceptual leaps from earlier understandings" (Powell, Francisco and Maher 2003) emerged as descriptions grew.

3.6 Coding Scheme

Coding Schemes were developed are used to track and compare critical events. Then, the codes were used to construct a storyline from the sessions.

In this work, the coding scheme included: (1) individual representations including rods, rulers and number line, (2) the idea of number versus the idea of operator, and (3) equivalence ideas. In Table 3, the coding schema is as follows:

Table 3.2: Coding Schema

Focus Students		**Fraction Ideas**	
Al	Alan	a. Representations i. Verbal ii. Writing/Drawing iii.Tools b. Cognitive Obstacle c. Equivalence ideas d. Ordering e. Fraction as operator	f. Fraction as number g. Coordination of operator and number fraction ideas h. Conceptual Change i. Operations with fractions ii. Fraction as number
D	David		
J	Jessica		
M	Meredith		

3.7 Results Organization Schema

The next four chapters present the narrative results from the four sessions with each chapter organized around a single session. Based on the early writings of Plato and Socrates (Plato 1937), the extensive figures are included to portray student affect (Goldin 1998a, 1998b) or dynamics not effectively described by words; words are open to distinct individual interpretations. The final chapter discusses the research findings.

The sessions were led by Carolyn A. Maher (RT1). Amy Martino (RT2) assisted throughout the sessions. Robert B. Davis (RT3) provided a brief introduction of Gunnar Gjone (RT4); both were present at the front of the classroom during at least one session. The classroom teacher, Mrs. Phillips (RT5), occasionally entered the classroom dialogue.

CHAPTER 4 – RESULTS 11-01-1993

4.1 Previous Session Review

On October 27, 1993, the class was divided into three groups where each group had to fairly share one of three candy bars, each containing ten blocks of chocolate. One group consisted of nine people and the other two groups consisted of eight people. During the session the students decided that the group of nine students should each received one and one ninth pieces of the candy bar. The class also determined that the two groups of eight students should each receive one and one fourth pieces of the candy bar.

On October 29, 1993, RT1 asked the students which group of students (eight or nine) got more candy and by how much. Initially some students thought that the difference between the group candy (one and one ninth and one and one fourth) would be one fifth. The entire session was spent working on their solution. Some students determined the answer was five thirty-sixths.

In Session 1, November 1, 1993, the students explored the ordering of fractions—one fourth, one fifth, one ninth—on the number line. They discussed what they had done in the previous sessions discussed above.

4.1.1 Discussion

RT1 (Figure 4.1 left) began the session by welcoming the students back and introducing RT3 and RT4 who were visiting the class (Figure 4.1 right).

RT1 asked the class if they remembered what they did together during the previous session, October 29, 1993. The camera captured the response of five students who responded that they remembered what they had done (see Figure 4.2 left).

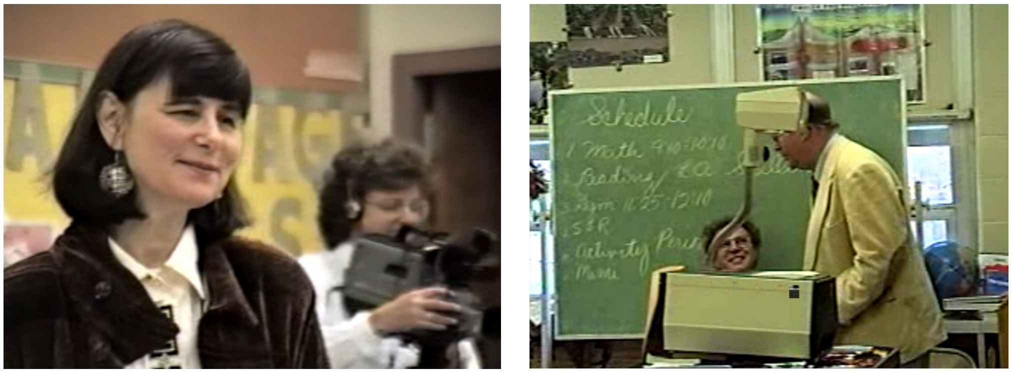

Figure 4.1 (left) RT1 (right) RT3/RT4

Graham, see Figure 4.2 (right), who quickly raised his hand offered, “We had a candy bar on Tuesday. We had to make a problem and use our rods to see who got more and by how much” (10).

Figure 4.2 (left) Students raised their hands to show they remembered the activity (right) Graham described prior session task

RT1 asked how the session ended. She asked “Who got more and by how much?” (11). Mark answered: “The people that got one fourth, got more by five thirty sixths.” (12). RT1 asked the class if they remembered. Figure 4.3 (left) shows a camera view which captured at least five students raising their hands to indicate their understanding.

Figure 4.3 (left) Students raised their hands to show they remembered (right) Students raised their hands to show they agreed

RT1 asked if the students believed what they remembered. Many of the students captured by the camera raised their hands. RT1 commented, “You all seem to believe it, but you do not all quite remember it” (15).

RT1 asked the students how they were able to show that one fourth was larger than one ninth by five thirty sixths. RT1 asked the students, “Can you kind of remember it in your head without the rods how that worked?” (17).

RT1 asked the students if they remembered and many students captured in the camera view raised their hands (see Figure 4.3 right). She commented that prior to the previous session activities every student thought the difference between one fourth and one ninth would be one fifth, RT1 asked Brian to respond. He answered with a question: “What made [the class] think one fifth?” (22). Brian explained that “me and Meredith kind of thought that it was the same as nine minus four equals five” (23).

RT1 responded, “You were thinking whole numbers” (24). Brain said “Yes” (25). RT1 then asked Meredith, “Does it work that way with fractions?” (Meredith’s response is clearer on view 2—Brian’s view.) Meredith responded: “If you put the blue which has a nine under it, and the four plus the five rod then have nine” (27). RT1 then modeled Meredith's response with the rods on the over head projector as seen in Figure 4.4. First, RT1 placed a blue rod on the over head projector. Meredith gave the blue rod the number name nine. RT1 then placed a purple rod on the over head projector.

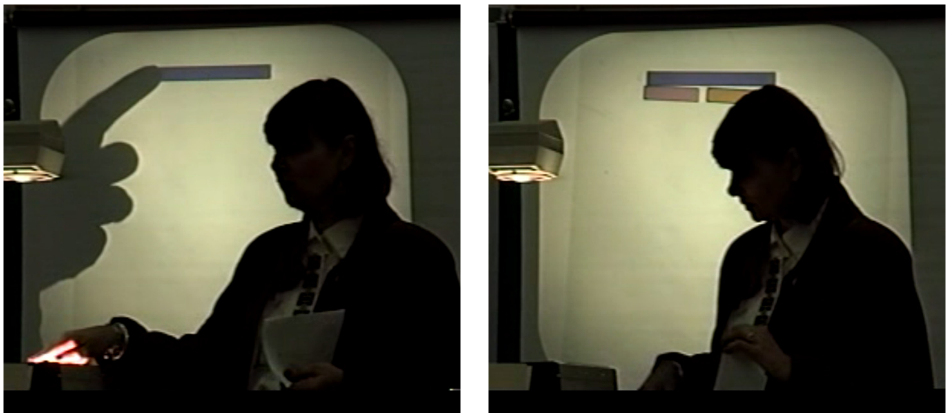

Figure 4.4 RT1 modeled Meredith's response with the rods

Meredith, Figure 4.5 (left), gave the purple rod the number name four. Meredith explained, "The yellow would be the five and it would equal up. That is what I thought at first" (33).

Figure 4.5 (left) Meredith explained her solution
(right) Erik gave a counter argument

At least one student in the camera view shook their head, indicating no. RT1 responded, "What is wrong with that thinking? Five plus four is nine, I can show that." Erik (Figure 4.5 right) interjected:

> I think that it doesn't make sense because how could the blue rod be on ninth of one model and the purple rod be one fourth when the blue rod is larger than the purple rod? I just don't think the way Meredith thought before made a lot of sense. (36)

Meredith responded: "I know. I changed my answer. I just think the five rod equals up to the same of five thirty-sixths" (37). RT1 stated, "You think the five thirty-sixths is somehow related" (38). Meredith responded, "Um-hum" (39). RT1 replied, "That's an interesting idea." She then added an additional rod, with number name one, to the overhead projector. At that point, the one rods totaled nine on the overhead projector as seen in Figure 4.6.

Figure 4.6 RT1 continued to build Meredith's model

RT1 exclaimed that a start with integers can get students confused. The class responded, "yes" (42). RT1 continued, "If you call the blue rod, nine; white rod, one; pink rod, four; yellow rod, five; and you proved five plus four is nine. You actually proved five plus four is nine. It doesn't work that way for fractions, does it?" (42). The class remained quiet.

RT1 asked the class if five thirty-sixths of a candy bar is much of a difference. Jessica (see Figure 4.7 left) replied, "No. I think that there is twenty five people in the class that is an odd number. So, you cannot have all even groups, that is why I think some people got one ninth and some people got one fourth” (46).

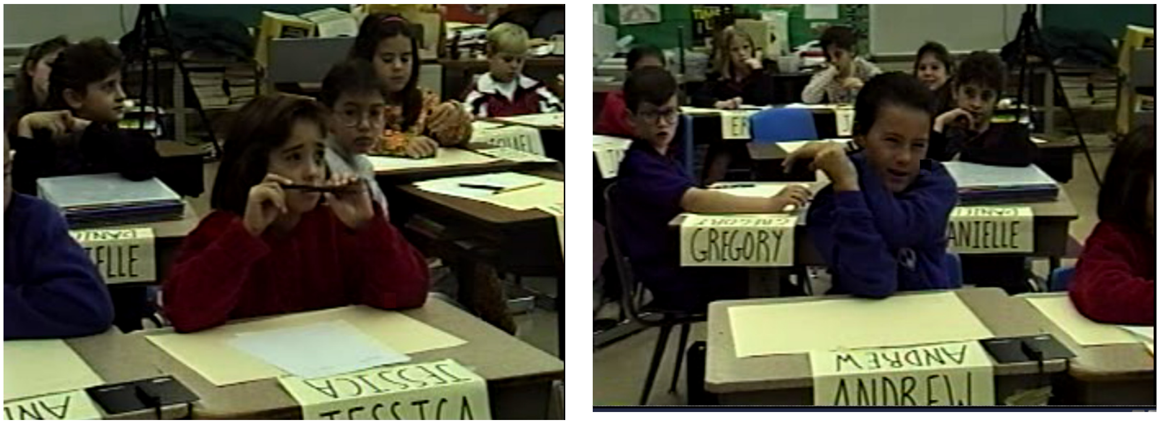

Figure 4.7 (left) Jessica explained why the class got different answers (right) explained a previous candy bar homework problem

RT1 then asked the class to think of a way to fairly share the three candy bars so that everyone would get the same amount. Andrew stated that for homework one day they had to divide it evenly and he “came up with the answer that everyone got—one and one fifth” (48).

RT1 asked: “How did you do that?” (49). Andrew responded: “There were three candy bars and each one had ten rectangles in it. [*The candy bars were scored as a five by two rectangle, making ten pieces for each candy bar*.] I took twenty-five of them and circled it and put a one. Then, the five left, if you divided them up into fives it would be five, ten, fifteen, twenty, twenty-five, so each person would get one and one fifth” (50).

RT1 stated that Andrew had an “interesting conjecture” (51) and asked the class if they followed what Andrew had said. A few students captured in the camera view raised their hands. RT1 asked Andrew to “Draw us a picture or something to show us your way” (53). As seen in Figure 4.7 left, Andrew remained in his seat and described his method in detail while RT1 asked students to imagine what he was saying. Andrew, while gesturing as seen in Figure 4.8, again explained, “I made the three candy bars with the ten pieces in them” (54); and, “Then, I took two candy bars and five pieces of the other one which made twenty-five” (57).

Figure 4.8 Andrew gestured while describing a model for his solution

RT1 responded to Andrew’s explanation and asked, “Okay, so everyone gets one of those thirty pieces and there are how many left over?” The class responded, “five” (59). RT1 asked how many students were following what Andrew was saying and some students captured by the camera raised their hands (61). Andrew then explained how he divided-up the remaining five rectangles.

> Then, those five would be just like one candy bar, but it would be smaller so you divide them into one fifth—five, ten, fifteen, twenty, twenty-five. There are enough people so everyone gets one and one fifth. (63)

RT1 asked the class what they thought and if Andrew’s solution was fair. The camera focused on James (Figure 4.9 left) who appeared to be contemplating.

RT1 asked the class if one and one-fifth was more or less than one and one-fourth. Michael and other students raised their hands as indicated in Figure 4.9 (right).

Figure 4.9 (left) Camera focused on James
(right) Michael raised his hand to answer a question

RT1 asked Danielle what she thought. Danielle responded, “Less” (65). RT1 then asked: “How many think it’s less?” (70). Some students raised their hands as indicated in Figure 4.10 (left). RT1 asked the class why (72) and, as indicated in Figure 4.14, Danielle (seen in Figure 4.10 right) responded: “Because, [five] is a bigger number, so when you have a bigger number, you get less”(73).

Figure 4.10 (left) Students showing they thought 1 & 1/5 less than 1 &1/4
(right) Danielle justified her previous answer

RT1 asked Brian what he thought. Brain (Figure 4.11) responded:

> I agree with her. If you have a bigger number, than you need to take one and one-fifth. If it is one-fifth, then there needs to be five of them in one whole. If there is one fourth, quarters, then you only need four of them to go into one whole. So, five is a bigger number so it needs more to fill up one whole. So, it is less. (77)

Figure 4.11 Brian Agreed with Danielle

4.1.2 Summary

The session began with RT1 introducing the visiting researchers RT3 and RT4 to the class. Then, the students were asked about the previous session and the students reviewed what they remembered. Graham began by describing the task. He stated that they were given a candy bar task where they had to determine who between two group solutions got more and by how much. Mark responded that the people who got one fourth of the candy bar got more by five thirty-sixths. RT1 asked the class how many students remembered the task and many students captured by the camera raised their hands.

RT1 asked the class to remember how the class was able to show that one fourth was larger than one ninth by five thirty-sixths. RT1 commented that before the activity many students thought the difference between one fourth and one ninth would be one fifth. Brian explained that he and Meredith thought before the activity that fractions worked like integers, i.e. the difference between one fourth and one ninth would be the same as nine minus four equals five. RT1 commented that Brian and Meredith were originally thinking whole numbers; Brian agreed. Meredith explained what she originally thought before the activity. Erik disagreed with Meredith's explanation.

RT1 asked the class if a five thirty-sixths difference of a candy bar portions is much of a difference. Jessica explained that the five thirty-sixths was the results of an odd number of people in the class.

RT1 asked the class to think of a way to fairly share three candy bars among twenty-five people. Andrew stated that they worked on a similar problem for homework where they had to divide three candy bars among twenty-five people. Andrew stated that everyone got one and one fifth of the candy bars. He explained that a candy bar had ten squares in it, so everyone got one square. Then, he said, that there were five squares to be shared among twenty-five people, so everyone would get one fifth of a square. RT1 asked the class what they thought.

RT1 next asked the class if one and one fourth is more or less than one and one fifth. Danielle stated that she thought it was less. She explained that five is a bigger number and when you have a bigger number you get less. Brian agreed with Danielle. He added that it takes five one fifths to fill a whole whereas it only takes four one fourths to fill a whole.

4.2 Smaller or Bigger—1/2, 1/3, 1/4 and 1/5?

4.2.1 Discussion

RT1 finished the class introduction by asking,

> If I were to say things like one half, one third, one fourth, one fifth, right? If I were talking about these numbers, then would you know which are bigger and which are smaller? How many think you know which are bigger and smaller? (78)

Many students captured by the camera raised their hands as show in Figure 4.12 (left). RT1 as seen in Figure 4.12 (right) wrote the numbers on the over head projector.

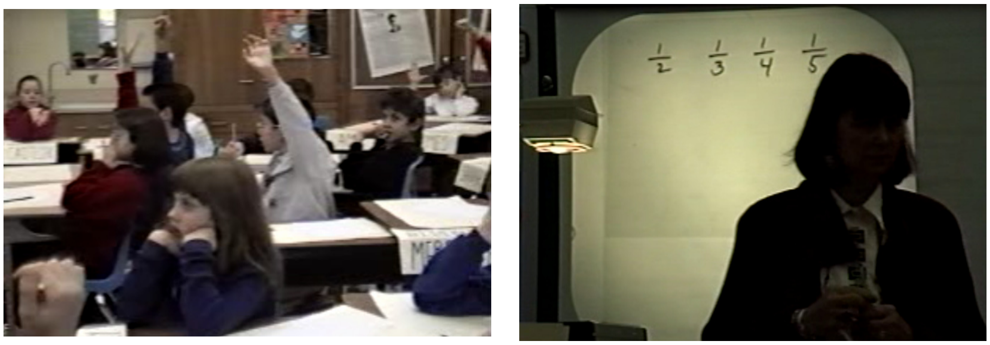

Figure 4.12 (left) Students raised their hands indicating they knew which ones were smaller and which ones were bigger (right) RT1 wrote some fractions

David explained how he would order the numbers by gesturing with his hands to an imaginary segment as seen in Figure 4.13. “If you have one half it cuts right there. If you have one third it cuts right there.” RT1 asked David to draw what he is said on the

Figure 4.13 David gestured while discussing his solution

overhead projector. David walked up to the over head projector at the front of the room. RT1 asked David to draw his “one” (84). David suggested, “maybe the orange” (85). He further explained, “if this is the one here” (87) and drew a rectangle which he labeled “1 whole” (87) as seen in Figure 4.14 (left). Then, David explained “one half would be there” (88) and placed two smaller rectangles below the “1 whole” (88) rectangle. He labeled the two smaller rectangles “1/2” (88) as seen in Figure 4.14 (right).

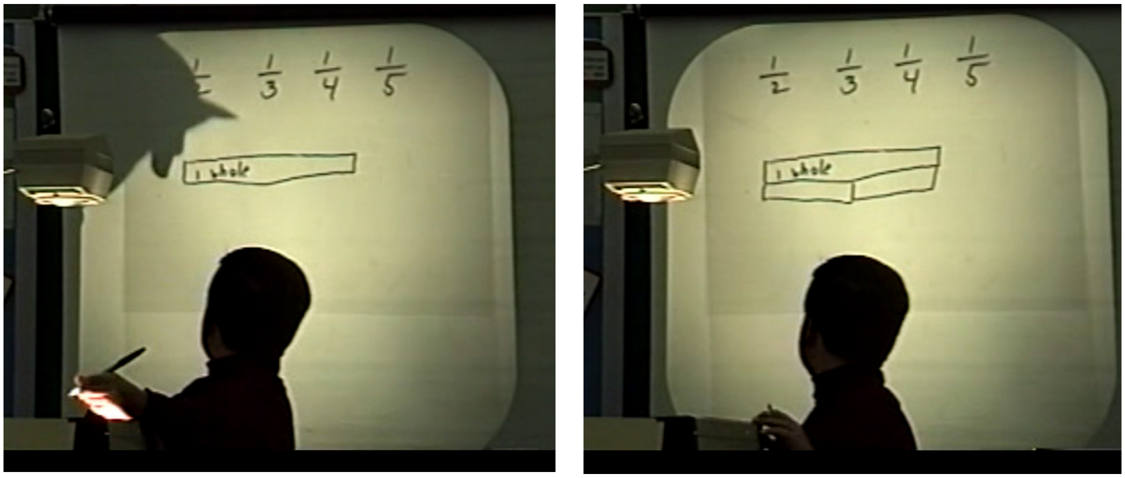

Figure 4.14 David explained his reasoning at the over head projector

David then placed “one third” (92) as three smaller rectangles below the one half rectangles as seen in Figure 4.15 (left). David then placed “one fourth” (93) as four smaller rectangles below the one third rectangles as seen in Figure 4.15 (right)

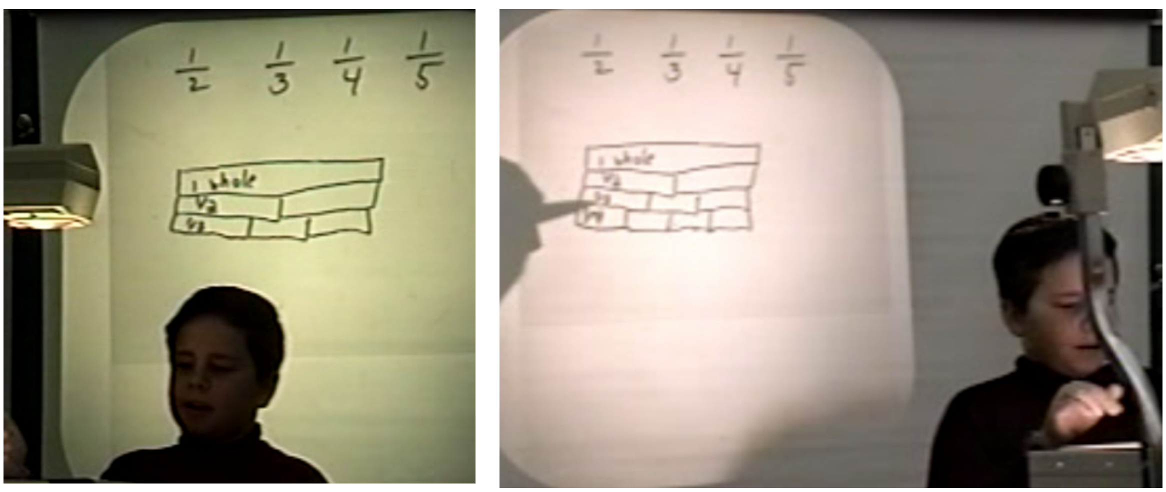

Figure 4.15 David explained his reasoning at the over head projector

RT1 stated “one fifth” as David sat down and asked Meredith where she would place one fifth. Meredith, with her hands on top of her head (Figure 4.16 left), responded that “the whole would be divided into one fifth,” (95) and would be to the “left” (97) of a quarter. RT1 placed corresponding one fifth rectangles below David’s rectangles a seen in Figure 4.16 (right).

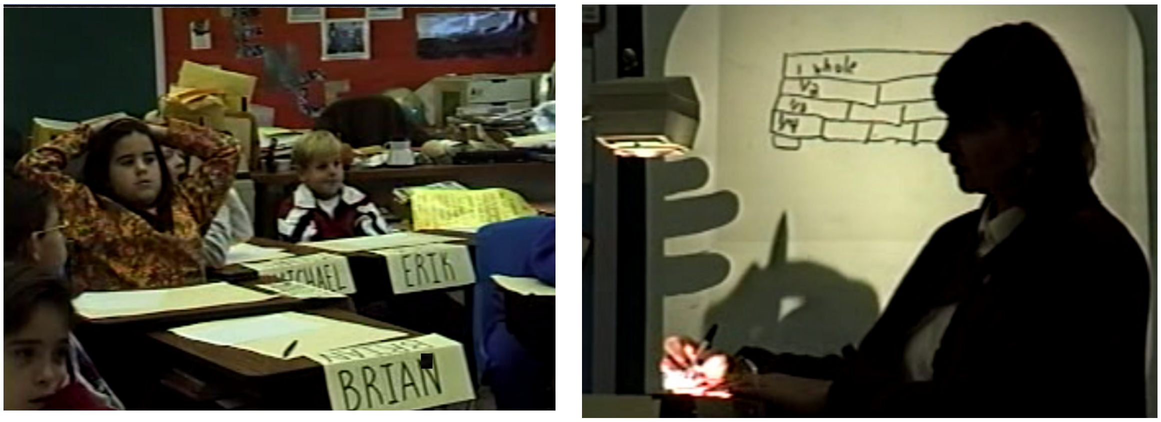

Figure 4.16 (left) Meredith explained where the one fifth would be placed (right) RT1 modeled Meredith's response on the over head projector

4.2.2 Summary

The students explained which fractions they thought were bigger and smaller from the set—one half, one third, one fourth and one fifth. David used rods to model how one whole, one half, one third and one fourth related to each other. Meredith extended David's model using rods to show how one fifth related to the other fractions.

4.3 Rods to Number Line

4.3.1 Discussion

RT1 drew a segment directly below David's rods which she labeled zero on the left-side and she labeled one on the right-side (Figure 4.17 left). RT1 asked the class for a volunteer to mark the placement of one half on the number line, and said: "I'd like someone to come up here and mark where the number one half would be. Michael?" (99). Michael walked up to the over head projector at the front of the room and places a tic mark half-way between zero and one (Figure 4.17 right). RT1 asked the class what they thought the next question would be. Their response was inaudible; however, RT1 then asked the class where they would put one third and one fourth. Erik walked up to the

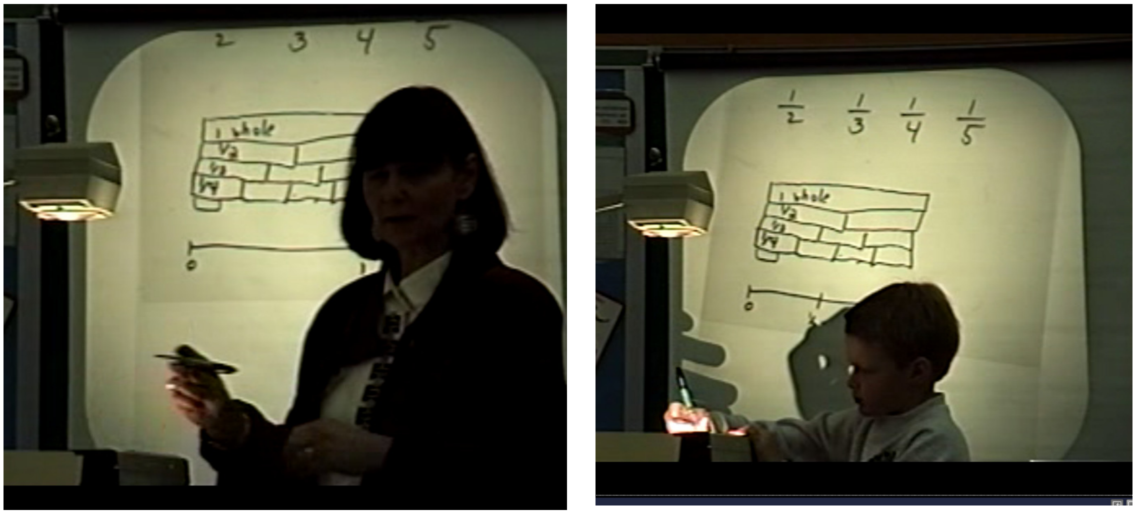

Figure 4.17 (left) RT1 wrote a line segment on over head projector (right) Michael marked one half on the line segment

over head projector at the front of the room and placed one third to the left of one half (see Figure 4.18).

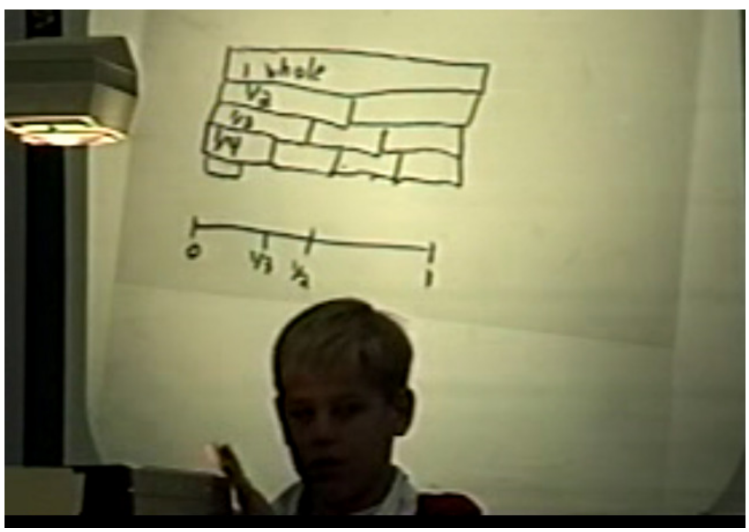

Figure 4.18 Erik marked one third on the line segment

RT1 asked the class: "Everyone okay with this?" (110). Some students captured in the camera view raised their hands. RT1 asked if anyone disagreed (112). Andrew responded: "I do. The one third needs to be a little more over because the one fourth needs to be half of the one half" (115). Andrew gestured with his hand as seen below in Figure 4.19 (left).

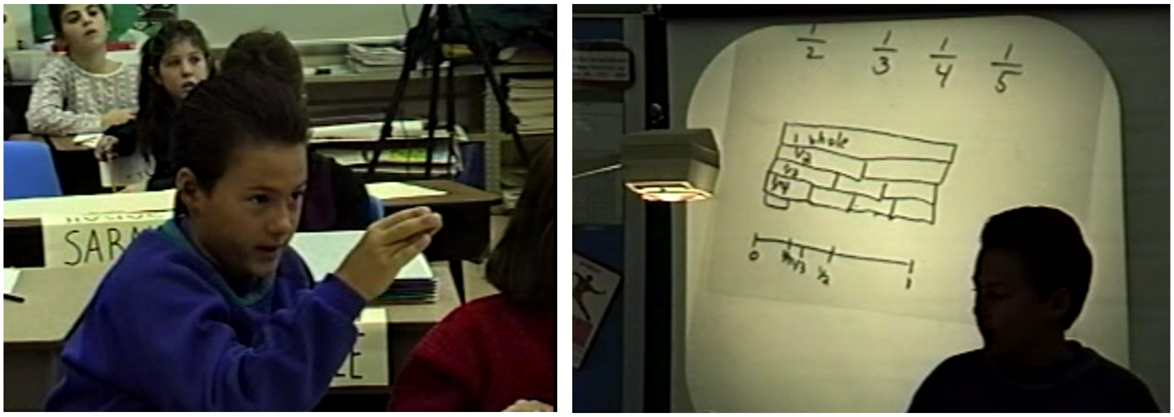

Figure 4.19 (left)Andrew gestured to where one third should be placed (right) Andrew marked one fourth on the line segment

Erik then responded: “It’s just an approximate” (116). RT1 suggested to Erik, “You want to call on Andrew to put in the one fourth?” (117). Andrew chuckled and walked up to the overhead projector at the front of the room and placed the one fourth mid-way between the zero and the one-half (see Figure 4.19 right). Andrew then asked, “Should I call on someone to place one fifth?” (119).

RT1 asked the class: “How many of you agree with what is up there?” (120). The camera captured (see Figure 4.20 left) many students raising their hands. RT1 asked, “Anyone disagree?” (122). As seen in Figure 4.20 (right), no students in the camera view raised their hands.

Figure 4.20 (left) Students raised their hands to agree with diagram (right) Students did not raise their hands to disagree with diagram

RT1 asked about the placement of one fifth. Brain walked up to the over head projector in front of the room and placed one fifth to the left of one fourth as seen in Figure 4.21 (left). RT1 asked the class, “How many agree with that?” (125). A few students in the camera view raised their hands as indicated in Figure 4.21 (right).

Figure 4.21 (left) Brian marked one fifth to the line segment (right) Students raised their hands to agree with placement of one fifth

4.3.2 Summary

RT1 drew a line segment between zero and one under David’s initial rod model. RT1 asked the students where the fractions would go on the line segment. Michael marked one half mid-way between the zero and the one on the line segment. Erik marked one third slightly to the left of one half. Andrew was dissatisfied with the precision of Erik’s one third placement and requested that it moved right slightly to have adequate room to place one fourth. Andrew was called upon to place one fourth on the number line. He placed it midway between the zero and the one half. Brian was called upon to place one fifth on the line segment. He marked it between the one fourth and the zero.

4.4 Where would one tenth go?

RT1 asked the students, “Suppose I asked you to put one tenth up there. Where do you think it would go?” (127).

4.4.1 Mark's Argument

Mark placed one tenth between the zero and the one fifth; he labeled the mark above the number line (see Figure 4.22 left). RT1 asked the class, "What do you think?" (130). Erik disagreed.

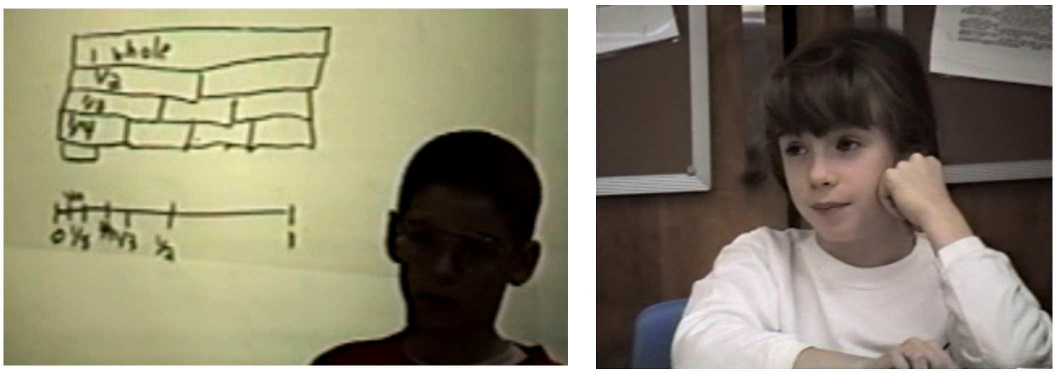

Figure 4.22 (left) Mark marked one tenth on the number line (right) Jakki described where one tenth would go

4.4.2 Jakki's Argument

RT1 asked Jakki why she disagreed. Jakki responded, "Well, if one fifth is next to the end. Then, five plus five is ten, so it would be like in the half" (133). Jakki had her head leaning on her hand as shown in Figure 4.22 (right).

RT1 followed-up Jakki's response with: "Jakki thinks one tenth should go in the middle" (134). Some students mumbled "no" (135). Jakki moved her hands to her forehead as seen in Figure 4.23 (both).

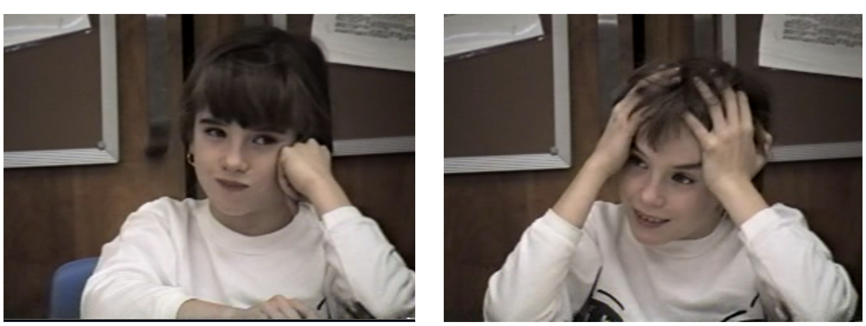

Figure 4.23 Jakki responded to students disagreeing with her idea

4.4.3 James's Argument

RT1 responded, "You disagree. James?" (136). James (seen in Figure 4.24) replied, "I think it should go more towards zero" (137). Some students mumbled, "Yes" (138).

Figure 4.24 James explained his argument

4.4.4 Alan's Argument

Alan stated, "I think that the one tenth should be over just a tiny bit" (140). Alan gestured with his hands (Figure 4.25). Alan continued:

> Up there you have a whole, you are dividing it into tenths and you have a half mark. So as a guideline, you'd have five minus one tenth on one side and five negative one tenth on the other side. Up there, if you to take that little space between the zero and the one fifth, and you cut that five times it wouldn't reach the one half way mark. (143)

Figure 4.25 Alan gestured as he explained his argument

4.4.5 Brian's Argument

Mark mumbles off camera and points to the overhead projector as seen below in Figure 4.26. RT1 asked Brian, "What do you think?" (144). Brain responded from his seat. Brain stated, "I agree. It is a little far back. I think the third should be moved up, then the one fourth should be moved up. I thought the fifth was wrong when I did it because everything was moved back" (145).

4.4.6 Summary

RT1 asked the class to place one tenth on the number line. Mark placed one tenth between the zero and the one fifth on the line segment. RT1 asked the class if they agreed and some students disagreed. As Jakki disagreed, RT1 asked her why. She responded that she thought one tenth should go in the half of the space between zero and

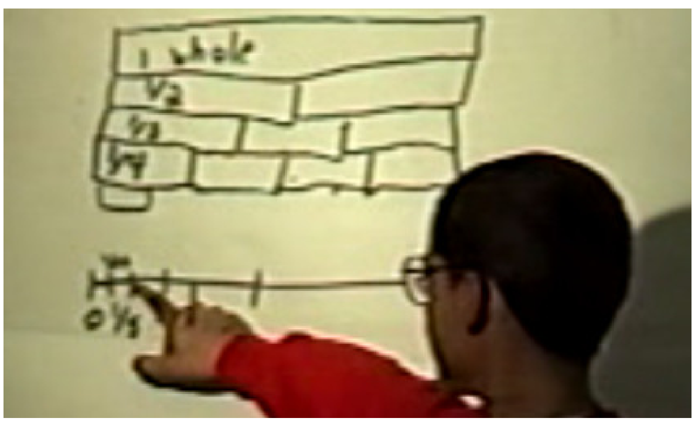

Figure 4.26 Mark pointed to the overhead projector

one fifth because five plus five equaled ten. Some students could be heard on the audio that they disagreed with Jakki's explanation. James said that Mark's placement of one tenth should be moved slightly closer to zero. Alan, too, indicated that one tenth should be moved closer to zero. Alan explained that five one tenths would need to fit on either side of the one half. Brian agreed that the fractions should be moved slightly. He stated that the placement precision was important for him to consider correctness.

4.5 Working in Pairs

Figure 4.27 RT1 asked the class to make their own number lines

RT1 talked to the students (Figure 4.33):

> Know what I would like you to do? Maybe the problem is there isn't a lot of space; when you use the overhead pen it takes a lot of space. I would like you all to make your own number line between zero and one at your seats. [***RT1 gestures with her hands as seen in Figure 4.27***] I would like to see if you could place fractions between zero and one. I'd like you to place all the fractions, one half, one third, one fourth, one fifth, one sixth, one seventh, one eighth, one ninth, one tenth, with your partner. (146)

A student whispered into RT1's ear as the class settled into their small groups.

4.5.1 Pair One

During the next few minutes, the cameras roved capturing the activity of different groups. One camera focused on Jessica and Andrew as seen in Figure 4.28. Andrew was heard counting out ten places on the number line and placed one one-hundredth near the zero. Jessica exclaimed that it needed to be closer to zero. Andrew then placed one one-thousandth on the number line. Jessica stated that one one-thousandth would be closer to zero than one one-hundredth. Jessica further argued that the placement "depends how big the number line"is as to how close to place the given numbers together (162). Andrew countered that size is not important as the numbers would remain in exactly the same spot on every line. Jessica continued to place numbers on the line including one one-hundredth-thousand and one one-millionth. She asked the researcher, “How high are we supposed to go?” (170) Andrew responded that he went as far as one one-hundredth.

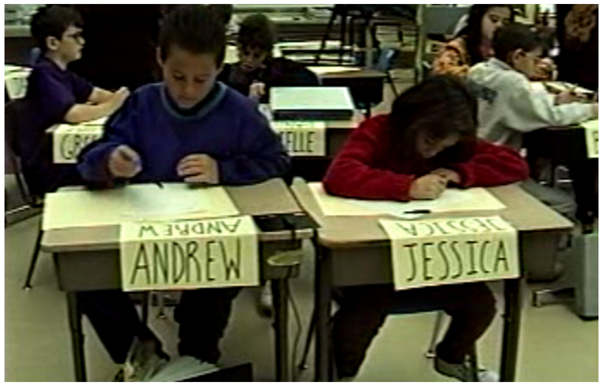

Figure 4.28 Andrew and Jessica working

4.5.2 Pair Two

The camera microphone picked up the conversation of Erik and Michael before it picked up the camera video. Erik stated that one one-thousandth would be “at the window” (174). RT1 repeated Erik’s statement in the form of a question asking Erik if one one-thousandth would be on the line or at the window. Erik replied that the line would need to be made bigger. RT1 again asks if one one-thousandth would be on the line. Erik was not sure that it would be on the line. Later, Michael and Erik bantered about dividing up the line. Erik used a ruler to measure where the numbers would go as seen below in Figure 4.29 (left). Erik’s written work is seen in Figure 4.29 (right).

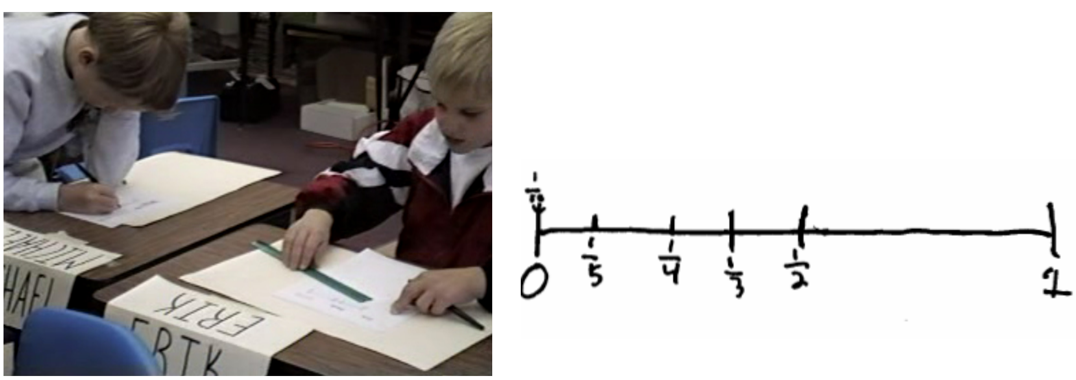

Figure 4.29 (left) Erik and Michael working
(right) Erik’s written work

4.5.3 Pair Three

Meredith’s and Brian’s group was the next group the camera captured. As the camera focused in on Meredith, she was shown counting out five spaces on the line with her fingers. Brian exclaimed that he thought he knew where the one one-hundredths would fit in on the number line. He pointed to where he thought one fifteenth would go—between the zero and the one fifth. He pointed to where he thought the one hundredth would go—between the zero and the one fifteenth. He then pointed with his pen to where he thought the one one-thousandth would go—between the zero and the one hundredth.   Brian exclaimed, “it’s like a pattern” (183).

4.5.4 Summary

RT1 asked the students to make their own “zero to one” number line and place on it the fractions one half, one third, one fourth, one sixth, one seventh, one eight, one ninth and one tenth. The students broke into pairs.

The camera focused on Jessica’s and Andrew’s group. Andrew was heard counting out ten places on the number line. He placed one one-hundredth near the zero. Jessica exclaimed that it should be closer to zero. Jessica continued to place numbers on the line including one one-hundred-thousandth and one one-millionth.

The camera focused Erik and Michael. Erik stated that one one-thousandth would be at the window. RT1 asked Erik if the fraction would be on the line or at the window. Erik was not sure it would be on the line. Later, Erik used a ruler to measure fraction placement.

Next, the camera focused on Brian and Meredith. Meredith counted out five spaces on the line using her fingers. To place one fifteenth, Brian pointed to the space between the zero and the one fifth. For placing one one-hundredth, Brian pointed to the space between the zero and the one fifteenth. He pointed to the space between the zero and the one one-hundredth to place the one one-thousandth. Brian exclaimed that he saw a pattern.

4.6 Where Would Three Fourths Go?

The students continued to work in pairs. RT1 asked the class where they thought three fourths would be placed on the number line.

4.6.1 Pair One

The camera captured the conversation of Jessica and Andrew as seen in Figure 4.30 (left). Off camera, Jessica stated that it would probably be in the middle. Andrew said, “hum” (188). Jessica continued, “it would probably be in the middle of one fourth and one third” (189). RT2 entered the group asking how things were going for them. Andrew motioned with his pen to either side of one half. RT2 asked Andrew why he had one third on both sides of one half. Andrew responded, “Yeah, I did it on both sides” (195). Off camera, Jessica stated that she, too, placed it on both sides. RT2 asked the students to explain why they placed one third in two places. Andrew responded, “it does not matter because I just did it on both sides” (198). Off camera, Jessica exclaimed that she, too, did it on both sides. Andrew motioned that you can “go by that way or you could go by that way” (200) as he pointed to either side of the segment. RT2 exclaimed that she saw a “mirror image” (202). Andrew responded, “yeah” (203). Off camera, Jessica responded, “yeah,” (204), too. RT2 mentioned folding the paper in half. Jessica folded her paper in half.

RT2 commented on Andrew’s placement of one hundredth as seen in Figure 4.36. Andrew responded that he was trying to estimate and count all the way up to one half. RT2 asked Andrew how many times he had to count. Off camera, Jessica exclaimed that the counting had to be all even. Andrew described, “the length, to count all the way up to fifth by the one half, then the other fifty in the other one half” (215). Off camera, Jessica stated that you would have to imagine the length. RT2 asked Andrew how many times he would have to count from zero to one tenth to place one hundredth. Andrew determined that a person would have to count ten times.

Figure 4.30 (left)RT2 worked with Jessica and Andrew
(right) RT2 worked with Caitlin and Brian

4.6.2 Pair Two

RT2 moved over to a new group comprised of Brian and Caitlin as shown in Figure 4.30 (right). The group initial discourse was not caught on microphone and the camera crew immediately placed a microphone next to the group.

RT2 asked Brian how his line worked. RT2 asked Caitlin if she agreed with Brian’s line. Caitlin responded that the one tenth would be placed near the zero. Her work, captured by the video, is seen below in Figure 4.31 (left). Caitlin's next comment was somewhat inaudible; but, it seemed to be about both RT1 and the rods

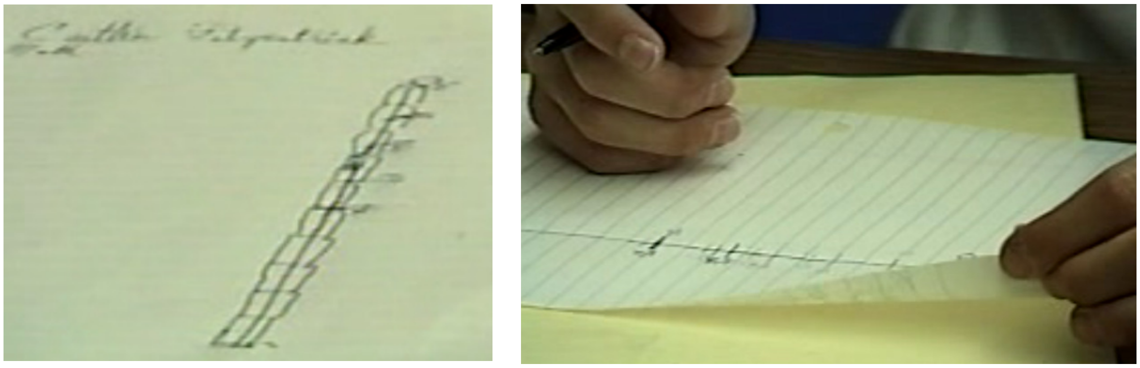

Figure 4.31 (left) Caitlin's written work
(right) Brian's written work

RT2 asked Brian what he was doing. She asked Brian about the placement of equal spaces between the fractions. RT2 asked: "What happens every time I add a new fraction?" (237). While Brian's response was somewhat inaudible, the audio captured his response: "each would be a different sized rod" (238). RT2 asked Brian if he was picturing the rods. Brian's response was inaudible. RT2 commented that one half was in the middle of the line and the one third was between the one half and the zero.

RT2 asked Brian why; but, Brian's response is inaudible. Additionally, he lifted the corner of his paper causing the camera to be obstructed from a clear view of his paper as seen in Figure 4.31 (right). Again, RT2 asked if Brian was imagining something. Again, Brian's response was somewhat inaudible, but the audio captured his response that "each time you add a new fractions it gets smaller each time" (243). Later, Brian erased his work..

4.6.3 Pair Three

The camera roved around a few groups. It finally stopped at the work of Meredith (Figure 4.32 left) and Brian (Figure 4.32 right and Figure 4.33) where someone exclaimed, "it doesn't have to be exact, only approximate" (258).

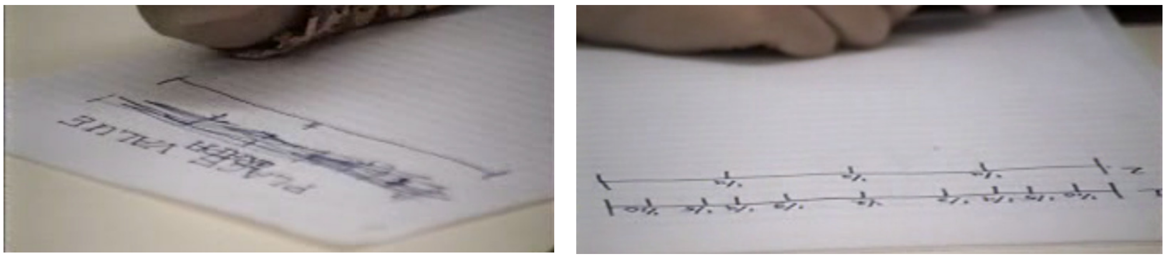

Figure 4.32 (left) Meredith's written work (right) Brian's written work

RT1 spoke to Brian and asked him to mark zero and one on his number line. RT1 stated that she could understand where they placed one third, but could not understand where they placed one fourth. She remarked that the numbers should be getting bigger whereas Brain was arranging them to become smaller. Brian stated that he was getting confused placing the numbers between zero and one. RT1 stated that she knew and asked Brian if he would like to think about it for a while.

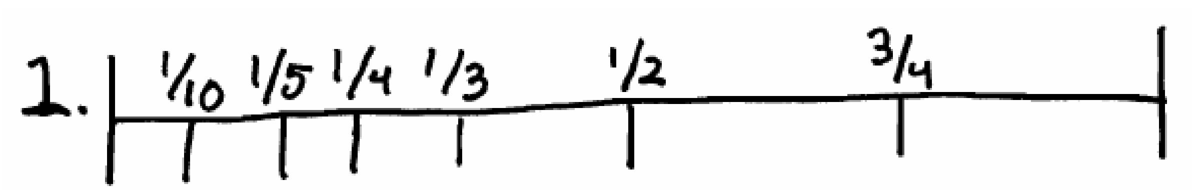

Figure 4.33 Brian's written work

4.6.4 Pair Four

Now, the camera briefly focused on Mark's work as seen in Figure 4.34. RT1 asked mark if he could label zero and one on his line. Mark's partner used a ruler to measure fractions on their line.

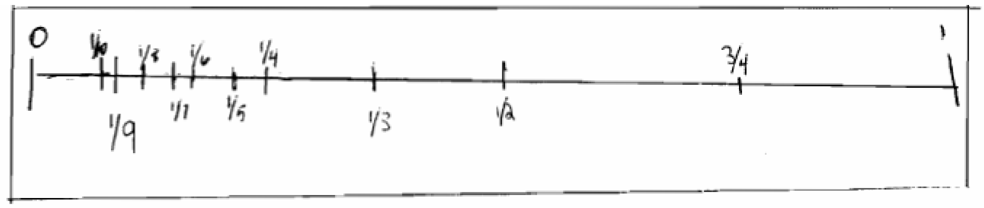

Figure 4.34 Mark's written work

4.6.5 Summary

RT1 asked the students where they thought three fourths would be placed on the number line. The camera focused on four pairs throughout the room—Jessica and Andrew, Caitlin and Brian, Brian and Meredith, Mark and Laura.

Andrew and Jessica discussed where they would place three fourths. Jessica stated that it should be placed in the middle of one fourth and one third. RT2 joined the group and asked Andrew why he placed one third in two places—on both sides of one half. Jessica, off camera, exclaimed that she, too, had done the same thing. Andrew described the line as a mirror image. RT2 discussed folding the paper in half. RT2 commented on Andrew's placement of one one-hundredth. Later, Jessica said that you would have to imagine the length.

RT2 walked over to Caitlin and Brian. Caitlin stated that one tenth would be placed near the zero. RT2 commented on Brian's line consisting of equally spaced fractions and asked him what would happen to the fractions when a new fraction was placed on the line. Brian referred to the rods. Later, Brian erased his work.

Next, the camera focused on Meredith and Brian. Someone exclaimed that the fraction placement did not have to be exact, only approximate. RT1 commented that she could understand why they had placed one third where they did, but that she could not understand why they placed one fourth to the right of one third. RT1 commented that the fractions should be getting larger on the number line, not smaller. Brian stated that he was getting confused between zero and one.

Finally, the camera focused on Mark. RT1 asked him to label his line. Laura used a ruler to help her place fractions on the number line.

4.7 Alan's Line

The class re-organized into a class discussion led by Alan. RT1 asked the students if in the past they had either used the number line or placed numbers on the number line. RT1 drew a new number line from zero to one.

4.7.1 Integers

RT1 asked David where he would put the integer two on the number line. David pointed towards the right side of one. RT1 marked the line accordingly. RT1 asked where to put the integer three on the number line. David again responded, "further over" (274). RT1 rhetorically asked about the integers four and five. RT1 again asked the students if they had placed numbers on a number line. Many students captured in the camera view raised their hands. RT1 asked the students where they would put a thousand on the number line and followed with the question: "Would it be in the building?" (277). The class mumbled no. RT1 asked: "Would it be outside the building?" (279). Many students giggled affirmatively. Alan stated that it would be all the way to "Pittsburg, Pennsylvania" (281).

4.7.2 Fractions

RT1 discussed the difference between placing integers on the number line and placing fractions on the number line. RT1 stated, "We are sort of looking at other pieces of the number line" (284).

Alan went up to the over head and discussed where he would place the one hundredth on the number line (see Figure 4.35). RT1 instructed Alan to talk about three fourths. Alan placed three fourths between the one half and one.

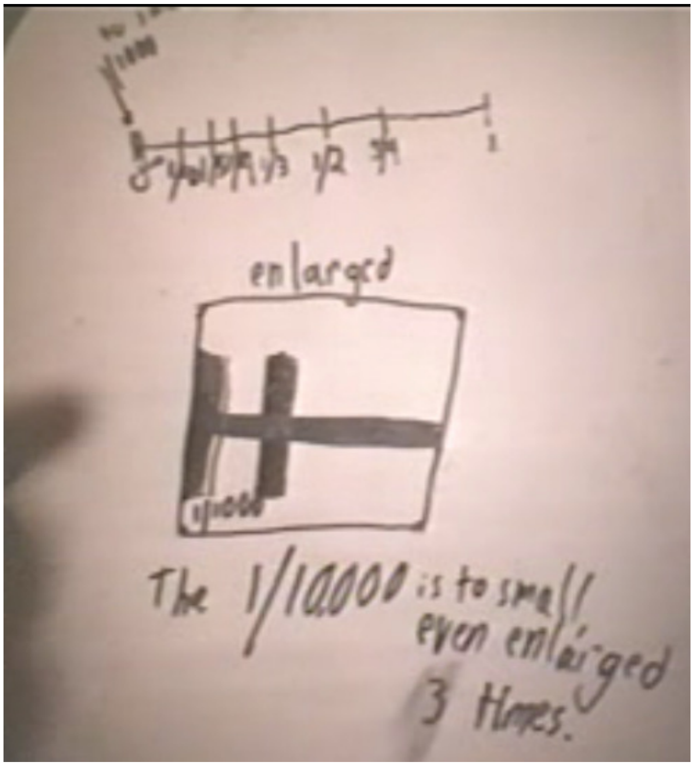

Figure 4.35 Alan's written work projected on the over head projector

He stated: "You would have the one third there and place the one fourth there. It would take three of those [motions to one fourth] to get up to that mark" (287). Alan explained the second number line on the bottom of his transparency. He explained that it was an enlarged portion of the number line because a person could not properly see the top line.

4.7.3 Summary

The class reorganized. Alan was asked to share his work with the class. In response to RT1, David stated that the integer two would be on the right side of the integer one on the number line. David continued to explain that the integer three would be to the right of integer two. RT1 asked the class where they would put the integer one thousand. Alan stated that it would be all the way to Pittsburg, Pennsylvania. RT1 discussed the difference between the placement of integers and fractions on the number line. RT1 stated that the class was looking at other pieces of the number line.

Alan walked up to the over head and shared where he would place the one hundredth on the number line. His drew two lines—a standard number line and an enlarged portion of a piece of the standard line. Alan placed one one-thousandth close to zero. Alan also shared that he placed the three fourths between the one half and the one.

4.8 Placing Thirds

RT1 asked the class how many students had one third to the right of one half. The camera captured some students raising their hands. RT1 encouraged a discussion about the placement since there were varying solutions for the placement.

4.8.1 Alan's Argument

RT1 first asked Alan for his thoughts on the placement of one third to the right of one half. Alan stated that one third could be placed in any one of three places on the line. RT1 asked where the second place for one third would be located. Alan described that it would approximately be near the one half. RT1 asked, "Where would you put two thirds?" (295). Alan pointed to the same spot for a correct placement of one third. Alan explained that if you had thirds you would be dividing the line into three parts each of which could be one third. RT1 stated that she was confused. RT1 asked, "How are you comparing the places where you put the second one third and the two thirds?" (298). Alan replied, "If you use the rods to sort of bracket" (299). Alan pulled out the rods and placed them on the over head projector. RT1 and Alan drew a new line using the rods.

Alan placed a one third one the line. RT1 asked the class if they agreed with Alan's placement of the one third as indicated below in Figure 4.36.

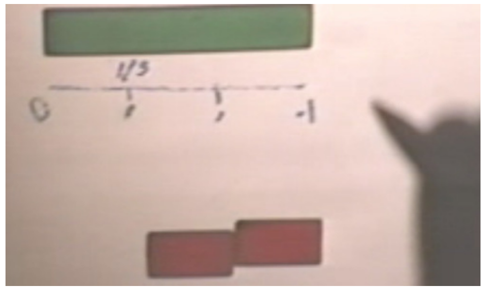

Figure 4.36 Alan placed one third on line segment

The regular classroom teacher, who was seated in the rear of the room and observing the session, asked Alan to move so that he would not be obstructing the overhead image. RT1 elaborated on what Alan did explaining that he was giving the green rod the number name one and that he took the red rods, representing the one third, and "marked off the spot at the end of the red rod" (310). Again, RT1 asked the class if they agreed. Many students were captured by the camera raising their hands.

4.8.2 Mark's Argument

RT1 asked the class how many believed one third should go somewhere else other than where Alan had marked it. A few students captured in the camera view raised their hands. RT1, then, asked the class if it were possible to place one third and two thirds in the same place. RT1 said that she was confused and wanted to know what they were thinking. Mark walked up to the over head projector and placed two thirds over the second tic mark as indicated below in Figure 4.37.

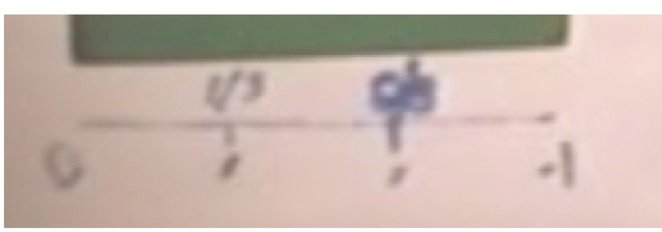

Figure 4.37 Mark placed the number two thirds on line segment

4.8.3 Danielle's Argument

RT1 asked Danielle to come to the over head projector and place three thirds. Danielle walked up to over head projector and placed three thirds above the third tic mark as seen below in Figure 4.38.

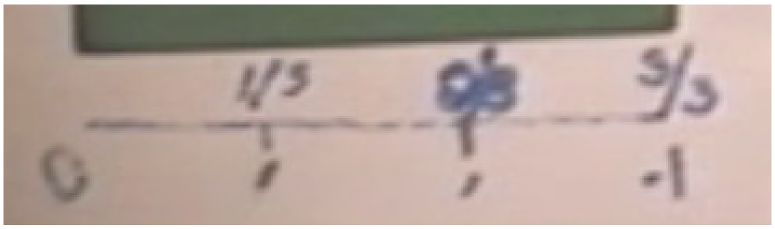

Figure 4.38 Danielle placed the number three thirds on the line segment

4.8.4 Andrew's Argument

RT1 asked Andrew to place zero thirds on the over head projector. Andrew placed it above the first tic mark, zero, as seen in Figure 4.39.

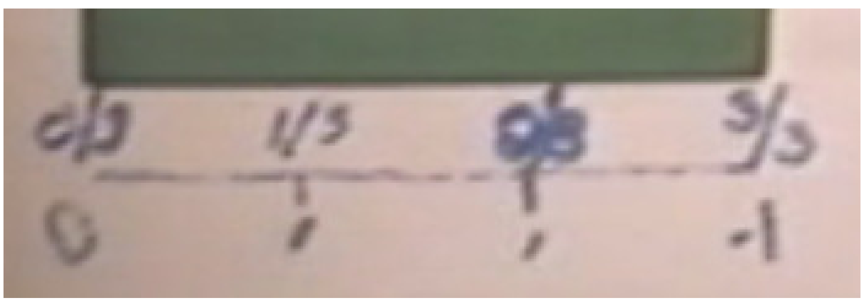

Figure 4.39 Andrew placed the number zero thirds on line segment

4.8.5 Alan's Discussion

RT1 asked the class if it would be okay to put one third on the same spot as two thirds. Alan repeated his earlier argument where each space was one third. He said, "Basically, what comes to mind when you think about fractions is that you cannot always think about the first one" (324). RT1 replied that she believes the space between each was one third. She stated that Alan proved it when he placed the red rods on the line.

Alan repeated his earlier work where he placed the rods on the line and drew on the overhead projector as seen in Figure 4.40.

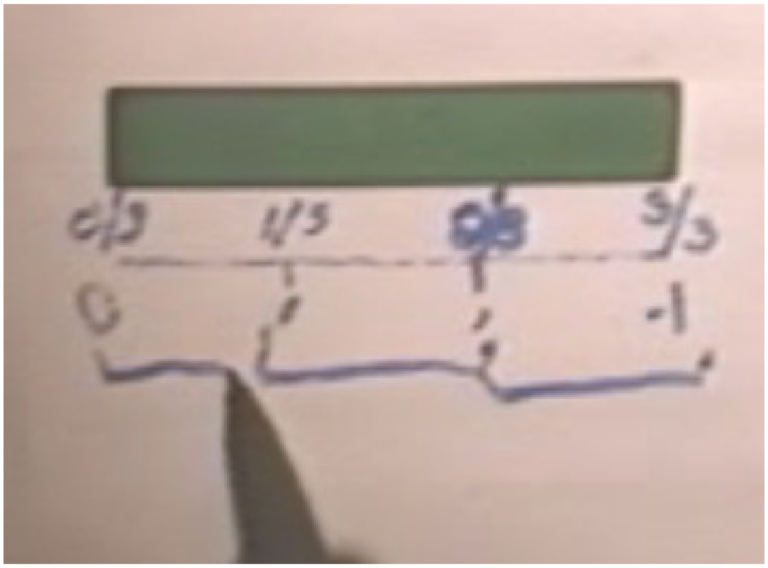

Figure 4.40 Alan showed the regions between thirds

4.8.6 Andrew's Discussion

RT1 asked Andrew what his opinion was of Alan's discussion. Andrew responded that he did not agree with Alan's solution because he did not believe that you could put a red rod in the middle and call it one third without starting at zero.

4.8.7 Class Discussion

Alan responded to Andrew by explaining that every space would be one third. RT1 asked if Alan were describing the length of the rods that happened to be one third. Alan said, "Yeah" (338). Again, RT1 asked Alan if it would be okay to mark one third for every place on the number line. Again, Alan responded that you could not and that you could only write one third in the beginning. RT1 stated that it was like the ruler he drew earlier. Andrew reiterated his discussion that a person must start at zero when giving the line number names.

4.8.8 How Do Integers Work?

RT1 asked the class to think back to inches on a ruler, "Would it be okay on my ruler, once I decided an inch, to make this a one and I mark another one and say this is one again?" (346). RT1 said that it was true they were all one inches in length, but asked the class if it would be okay to call every inch mark one? Sarah exclaimed that you could not mark them all one (see Figure 4.41). Sarah looked at a ruler in her hands and said that the different numbers were there so that a person could "count" (347) them.

Figure 4.41 Sarah described a ruler

RT1 reiterated her interpretation of Alan's argument which was that every tic mark would be one inch, so why not mark them all one. Alan said, "they are the same length" (349). RT1 asked, "How do I mark my ruler?" (350). RT1 said that we are making a ruler for fractions. Alan said, "right," (351) but a ruler gives a person length. Alan described how a person would make a ruler by putting a one at the first inch, a two at the second inch and a three at the third inch.

David quietly remarked: "They may be all the same thing but when you're measuring something then you know that if it is an inch you know how many instead of just counting all of them" (360).

RT1 assigned homework to the class asking them to see what they could do with placing fractions between zero and two on a number line.

4.8.9 Summary

This episode began with RT1 asking the students how many of them had one third to the right of one half. Some students raised their hands affirmatively. Eight students explained their arguments.

Alan stated that one third could be placed anywhere of three places between zero and one. He explained that one third divides up the line into three parts. When asked by RT1 how he was comparing the places, Alan replied that he was using the rods to sort of bracket the places. Alan used one green and three red Cuisenaire rods on the over head projector to bracket the number line according to his argument.

RT1 asked the students to place two thirds, three thirds and zero thirds. Mark walked up to the overhead projector and placed two thirds on the number line between one half and one. Danielle, was then selected, to place three thirds on the overhead projector. Danielle placed three thirds above the one. Andrew was next selected to place zero thirds on the line segment. Andrew placed zero thirds over the zero.

RT1 revisited the question of whether or not it would be okay to put one third in the same spot as two thirds. Alan repeated his earlier argument. Andrew, however, did not agree. He said that he did not believe that a person could put a red rod in the middle on the line and call it one third without starting at zero.

The class continued to debate the question of whether or not one third and two thirds could be placed in the same spot on the number line. RT1 asked the students to think of a ruler and asked them if it would be okay to call every unit on the ruler one. Sarah exclaimed that a person could not mark all the inches as one because a person could not count them. David continued Sarah's explanation and described how a ruler is used. He said that a ruler shows the cumulative inches up to a designated point. David explained that the cumulative measurement would save the ruler user the busy work of adding up all the distinct units. The session ended.

CHAPTER 5 – RESULTS 11-3-1993

5.1 Introduction

During Session 2, November 3, 1993, students explored both number line properties, as well as, the ordering of fractions between zero and two on the number line.

5.1.1 Discussion

RT1 began the session by asking the students how they were doing. As seen in Figure 5.1, RT1 asked the students about their homework. (Recall that in the previous session RT1 asked the students to explore placing fractions between zero and two on a number line for homework..) Brian responded that he was confused on the work they did Monday and indicated that he wrote what he thought (20). RT1 held up some students homework, as seen in Figure 5.1 (right), and suggested that they discuss what they had done.

Figure 5.1RT1 suggested discussing homework

5.1.2 A Ruler

RT1 stated that the class was discussing rulers at the end of the prior session. She suggested that the students perform a private experiment examining the layout of rulers. She stated, “not all rulers are alike” (20). RT1 asked the class, “Do you have a ruler?” (20).   Jessica reached into her desk, got a ruler, handed is to RT1 and exclaimed, “yeah” (21). While pointing to the ruler, RT1 said: “I’m trying to imagine that these are making a rod" (22). RT1 continued by asking the students if they recalled Alan's idea when they were talking about a rod that was twelve inches rod. Many students in the class responded affirmatively. RT1 (Figure 5.2) said, "In my twelve rod, what Alan said very nicely is that all of these lengths [*distance between markings*] are exactly the same" (24). RT1 further stated that it would not matter which inch a person pulled out of a ruler as all the lengths of an inch are the same.

Figure 5.2 RT1 showed the class how a ruler is constructed

RT1 asked the class to recall what David had said during the prior session as to why it would be a good idea not to number inch marking on a ruler as one (24). RT1 continued, "I would have to count up all the ones" (24). RT1 asked David if what she said was correct. David replied, "Yeah" (25). RT1 continued to explain that people may look for a short cut when they get tired of doing the same thing over and over again. She sates, "it is easier when we read five [*for five units of one*] rather than count [*one, five times*]" (26).

RT1 stated, "imagine moving from some of our rods to maybe a ruler; and, then moving from a ruler to what we call a number line" (26). She explained to the students that they would be studying the placement of numbers on the number line for a number of years—through high school. RT1 said that the students should ask themselves to question where the numbers belong on the number line as they go through more advanced classes.

5.1.3 Summary

In this section, RT1 welcomed the students back to a new session, November 03, 1993. RT1 began the session by preparing for a discussion of the homework. First, RT1 had the students think back on the ruler discussion during the prior session. RT1 pointed out that rulers may be constructed differently. She also reminded the students a comment David had made during the previous session where he said that a ruler is a cumulative integer count to save the user the time of adding all the one inches together. RT1 asked the students to imagine moving from rods to a ruler to a number line.

5.2 Number line Ideas

RT1 then asked the class to discuss some number line ideas.

5.2.1 What is the Biggest and Smallest Numbers?

RT1 asked the students, “What is the biggest number?” (26). The students responded, “nothing” (27). RT1 continued by stating that if someone were to call a million the biggest number, then adding one to it would contradict that a million was in fact the biggest number. RT1 asked the class if her statement were true. Many students nodded their heads affirmatively. RT1 furthered the discussion by stating that if the number, a google, was thought to be the biggest number then adding one to a google would show that there was a bigger number. Many students exclaimed yes. RT1 finished the discussion by stating, “we could imagine this line going on and on and on” (32).

RT1 asked the class about the smallest number and said: “some of you talked about numbers that were negative numbers, right?” (34).

5.2.2 Made up of Points

RT1 stated that when the students get to high school, they will learn to think of the number line as being, “made up of lots of points” (34). Michael (Figure 5.3) exclaimed, “Yeah, we studied points in Mrs. Dominica’s class last year we studied” (35). “Segments,” RT5 stated (130). Michael gestures with his two fingers clamping them together and said, “Segments and sections because the number line is just a section of the big line that goes on and on forever and that [segment] is just a little section that is taken out” (37).

Figure 5.3 Michael gestured to show how a number line is constructed

RT1 asked the class how many other students had studied number lines and sections. Approximately six students in the camera view raised their hands. RT1 continued by stating that mathematicians usually use the term, “infinitely” (42) many points. RT1 asked the students, “You have heard of infinity, haven’t you?” (42). Many students captured in the camera view agreed. Brian exclaimed, “a lot” (45). Erik stated, “The definition of infinity is that it keeps going, it never stops” (47). RT1 replies, “Yes. It never stops. It never ends” (48).

RT1, then, asked the students, “How many numbers can we put on this line?” (48). The students responded that it would take too much time (49).

5.2.3 Summary

RT1 then discussed some number line ideas. First, RT1 asked the students if there is a biggest number on the number line. They said nothing was the biggest. RT1 then continued by giving examples of big numbers and finding a bigger number simply by adding one to the big number. RT1 also asked the students if there was a smallest number and mentioned negative numbers. Second, RT1 described that a number line is made up of points. Michael talked about learning segments and sections in their math class the previous year. Michael gestured extensively when describing the studies the previous year. Then, RT1 talked about a term mathematicians use when there are too many to count, infinity. Erik stated that infinity never stops, it keeps going on and on.

5.3 Studying Pieces

RT1 referred to what Michael had said earlier and stated, “The issue then is that we are going to study pieces of this number line” (50). RT1 continued, “The little piece I asked you to think about was the piece between zero and one” (52). She asked, “Isn’t that what I asked you to do between zero and two?” (52). The class responded yes (53). RT1 asked Beth to mark zero to two on the over head projector. RT1 stated that she wanted a discussion from you about where certain points would be (54).

5.3.1 Placing One

Alan stated: “on the number line, I used one as the half mark between zero and two” (55). RT1 said, “if mathematicians want to get really fussy then they will talk about a piece of a line, right, and that is not infinite. We call that finite. They call that a segment and that is what you are talking about" (56). Then, RT1 repeated Alan's argument, "you thought about the number one being half way?" (182). Alan responded affirmatively (57). RT1 asked if anyone else had thought about the number one like Alan had thought about the number one (58).

5.3.2 Finite versus Infinite

RT1 revisited her statement distinguishing between a line segment and a number line. While pointing to the ends of the line segment that Beth had drawn on the over head projector, she said: "I want you to understand that this keeps going. We are kind of restricted to how much room we have on the over head, aren't we?" (58).

RT1 continued, "if I asked you to place three—the whole number three—how many of you think that you know where it would be?" (58). RT1 repeated the question and Jessica along with at least nine other students in the camera view raised their hands.

RT1 called on Jessica to place the number three and said, "Can you sort of kind of point, Jessica?" (62).

Figure 5.4 (left) Jessica pointed to where the number three would be placed (right) Jessica pointed to where the number four would be placed

Jessica took up a ruler to measure the distance between the currently projected points. She, then, used the same distance to point to where the number three would be located as seen in Figure 5.4. RT1, then, asked: "Where would you place four?" (64).

Jessica moved the ruler over one length to the right and pointed as seen in Figure 5.4. RT1 followed with: “Okay, get the idea?” (66). The students responded affirmatively (67). RT1 asked, “Do you know where you would place a million?” (68). Some of the students laughed and one said off camera, “down the hall” (69).

Figure 5.5 (left) RT1 pointed to negative side of line (right) Amy placed negative one

As seen in Figure 5.5, RT1 pointed with a stick to the other side of the line and asked: “notice that I had my arrow going the other way?” (70). RT1 asked the class, “Why?” (70). Amy responded: “because you have negatives” (71). RT1 said okay and asked, “Where do you think I would put negative one?” (72).

Figure 5.6 (left) Amy measured distance between points (middle) Amy commented that the placement of one was a foot (right) Amy pointed to negative one

Amy walked up to the overhead projector and placed a negative one using the ruler to measure the distance (Figures 5.5 and 5.6). RT1 asked the class how many students agreed with the placement of negative one. RT1 directly asked Audra what she thought and Audra remained quite. Amy stated that the placement was not exact. She re-measured the space between zero and one as seen in Figure 5.6. RT1 asked, “How long it would be, Audra, that makes sense, right? But, why would you call that a negative one, right? I think that is what is confusing Audra” (80). RT1 then asked, “Where would you put a negative two?” (80). Amy points to the space left of negative one (see Figure 5.7).

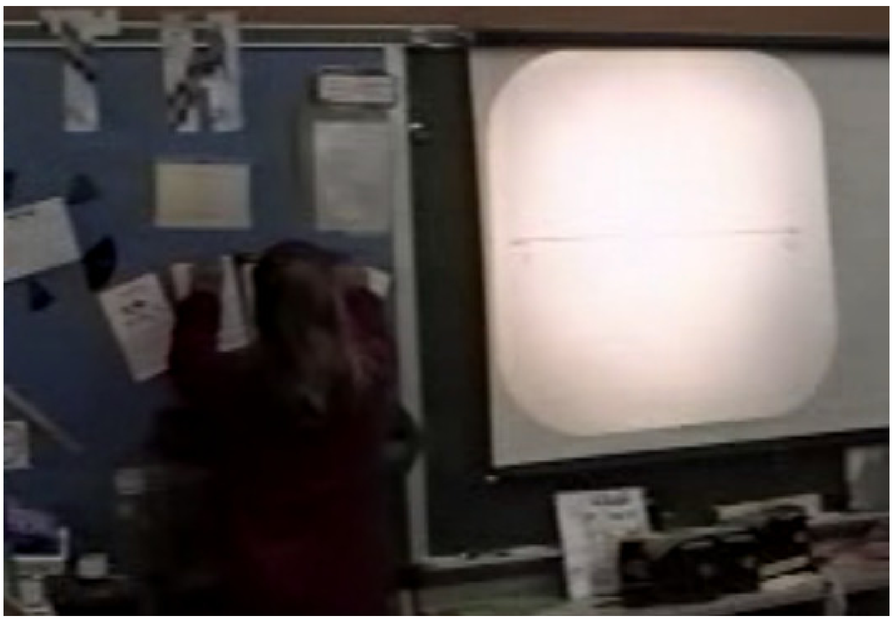

Figure 5.7 Amy placed negative two

RT1 responded: “You are sort of telling me something about where you are putting negative numbers and where you are putting positive numbers” (82). RT1 asked Kimberly what she thought. Kimberly responded: “I think it is negative one back there because it is beyond zero” (83). RT1 followed, “It is beyond zero?” (84). Kimberly said yes. RT1 then asked which way. Kimberly pointed towards the left and said: “that way” (87). RT1 asked: “the positive ones?” (90). Kimberly gestured to the right side as seen in Figure 5.8 and responded, “this way” (93).

Figure 5.8 (left) Kimberly gestured to show placement of negative numbers (right) Kimberly gestured to show placement of positive numbers

RT1 asked: “to the right?” and Kimberly responded yes (95).

RT1, then, asked Alan what he thought. Alan responded, “I think all numbers to the left of zero would be in the negative and all numbers to the right of zero would be in the positive” (97). RT1 asked, “How many of you think that?” (98). At least eight students in the camera view raised their hands.

RT1 followed: “That is what the mathematicians often do, they do exactly that, they put the numbers to the left of zero as the negative and the numbers to the right of zero as the positives. And by the way, since we are already into high school math, we might as well give you a little bit more high school math. Do you know what they call those numbers to the right of zero and to the left of zero. Those whole numbers? Zero, one, two, three, four, five, and so forth without stopping? And, negative one, negative two, negative three, negative four and negative five? You know what they call them?” (100). The class remained quiet. RT1 asked, “Do you want to know?” (102). The students responded affirmatively. RT1 asked, “How many of you want to know?” (104). At least eight students in the camera view raised their hands (105). As seen in Figure 5.9, RT1 wrote the word integers on the over head projector. The students repeated the word, “integers” (107).   RT1 said, “Can you say it?” (108). Again the students said, “integers” (109). Next to the integer word, RT1 wrote “{…,-2,-1,0,1,2,3,…}” as seen in Figure 5.9.

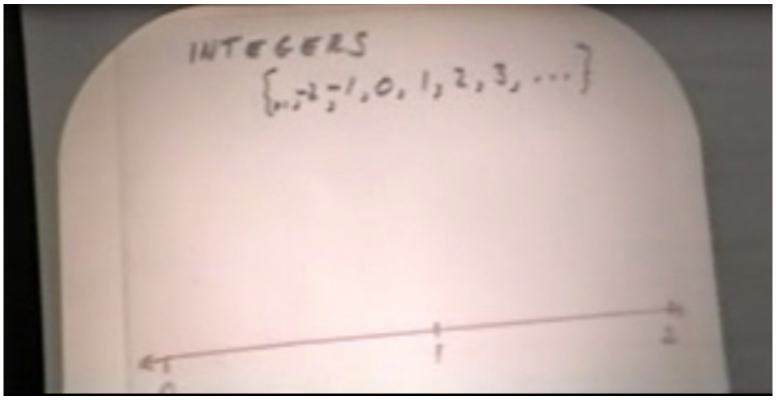

Figure 5.9 RT1 writes integers on over head

RT1 said that it is unnecessary to show all the integers and that the notation she used shows that the numbers keep going to the right and to the left of zero. RT1 said, “they call that a set of numbers” (110). RT1 continued, “so now you know about the set of integers that you usually study when you get to pre-algebra or algebra” (110).

5.3.3 Summary

RT1 stated that the class was to study pieces of the number line. She recalled the homework from the prior session and stated that the class was supposed to place fractions between zero and two. Alan stated that he had placed the number one half way between zero and two. RT1 stated that what she wanted the class to understand was that the number line keeps going on and on and that the over head projector limits the room for showing the number line.

RT1 asked the class where they would place the number three on the number line. Jesssica walked up to the over head projection on the board and measured with a ruler the distance between one and two and used the same length to place two. RT1 followed by asking about the number four. Jessica again used the ruler to copy the length on the projected line to mark four. RT1 asked the class where they would place a million. Someone off camera said "down the hall" (69).

RT1 then asked the class about the arrow going in the opposite direction. Amy responded that it was for negative numbers. Amy came up to the over head and placed the number negative one using a ruler to measure equi-distance between whole number placement.

RT1 asked about the number negative two. Amy followed with placing a negative two. RT1 commented that the class was telling her something about where to place positive and negative numbers. Kimberly said that negatives were beyond zero to the left and that positive numbers would beyond zero to the right on the number line. Alan added that he thought all numbers to the left of zero would be negative and all numbers to the right of zero would be positive.

RT1 asked the class if they knew what to call the whole numbers to the left and right of zero. The class did not know. She asked the class if they wanted to know and they said yes. She introduced the word integer and wrote it on the over head projector. Additionally, she used set notation for integers to include the set of negative and positive whole numbers, indicating that the set is infinite {…, -2, -1, 0, 1, 2, 3,…} next to the word integer.

5.4 Between zero and one are infinitely many numbers

5.4.1 The Positive Size of the Number Line

RT1 referred to the integers written on the over head projector and said, "for now we are going to worry about the ones that are positive now, but we could also worry about the ones that are negative sometime if we want, to" (114). RT1 continued: "what we are concerned about are those numbers between particular integers. What do they kind of look like and where would we place them if we want to put them on that line?" (114). RT1 asked if the students knew where to place integers and many students nodded their head affirmatively. RT1 asked the students if they could make a number line and place the integers. Many students responded affirmatively. RT1 asked them to show her how it would work. Erik said, "I don't want it to go on forever" (119). RT1 followed, "you see what I do, I put arrows" (121). She continued, "mathematicians do not like to do a lot of extra work, they invent notation to make their life easier" (121). RT1 asked if this made sense to the students and many responded yes.

5.4.2 The Notion of Infinitely Many Numbers

RT1 stated, "let's go back to what you assignment was. We were trying to figure out what sort of happens in between [whole numbers]" (123). RT1 explained that similarly to the number line which keeps going and going into infinity, so does the numbers in between zero and one. RT1 stated "between zero and one there are also infinitely many [numbers]" (123). The students followed and repeated, "infinitely many?" (124). RT1 responded affirmatively.

Erik asked, "infinitely many?" (126). RT1 responded, "we have infinitely many" (127). Alan asked, "infinitely many between zero and one?" (128). RT1 responded: "they [mathematicians] also claim that there are infinitely many fractions" (129). An student off camera said, "eternity sort of speak" (132). Erik stated, "I just don't understand how there can be infinitely many numbers between zero to one" (135). He continued, "It doesn't make sense" (138-140).

5.4.3 Zillions, Billions and Googles

Alan stated, “you can divide the line into the smallest of fractions. You could divide it into zillionths” (141). Alan continued, “You could divide it into zillionths and there would still be space in there” (144). Michael interjected, “If you had the longest number line in the world” (145). Erik countered, “Alan. Alan. That doesn’t make sense” (150). Alan responded, “Yes it does” (151). Erik argued, “Even if you were to divide it into zillionths depending on how big your one whole is, if your one whole is ten you cannot divide it into zillionths” (152). Alan countered, ‘form zero to one, you could” (153). Erik continued, “if your one is ten, how could you divide it into zillionths?” (154).

RT1 asked, “What if your one whole is a zillion?” (155). Erik answered, “then you could divide it into a zillionths” (156). RT1 said, “Well, I think, that is what Alan is saying” (157). RT1 continued, “what if your one whole is a billion? What if your one whole is called a google?” (159). Erik laughed, “a google” (160). Alan asked, “what if you called the zero to one a billion?” (161). RT1 said that these were infinity ideas for the students to think about and argue about.

RT1 returned to the idea about the number of fractions between zero and one when she said:

> Just for now to focus on the interval between zero and one. I want to be able to place fractions, as many fractions as you can imagine, and then lets us even talk about some [fractions] that you cannot fit here because it gets hard to squeeze them in; but, you could imagine [that they fit in] (162).

Alan stated, “As I was saying before about the zillionths, you could have a line the size of a dust particle and you could put that on there a zillion times. You would have zillionths” (164). Michael added, “if you had a pin that was smaller than a dust particle”

(164). Erik interjected, "something that is smaller than a dust particle—a dust bug—a hundred dust bugs can fit in to a dust particle" (165). Alan commented: "I was not talking about insects" (166).

5.4.4 Magnifying Glasses, Microscopes & Telescopes

Andrew stated, "if you had a number line and you took a magnifying glass or a microscope, you would see that you have a lot of room left to put the one hundredth and one thousandth" (168). Alan extended Andrew's comment, "if you did put it under a microscope it would look like you had enough room to put another zero to one in there. You could have it enlarged so that the line from the zero would be this big [gestured with his hands as seen in Figure 5.10] and you would still have room there to put more" (171).

Figure 5.10 (left) Alan gestured to describe placing line segment under microscope (right) Brian agreed with Alan's statement about inserting zillionths into segment

RT1 asked, "What happens when scientists discover more and more powerful telescopes" (172). Michael responded, "then the more numbers you could fit onto one number line" (173). RT1 continued, "What do they see in the sky when they look through more and more powerful telescopes? What did they find?" (174). The students responded that scientists would find more stars. RT1 asked, "So, is it that the stars aren't there or is it that we don't have the instruments?" (178). The students respond, "we don't have the instruments" (179). Alan stated, "right" (183). Andrew continued, "you could make more powerful instruments" (183). Brian, as seen in Figure 5.10, interjected, "so, like Alan said, you can put zillionths in." (184).

5.4.5 In Reality, Not Getting More Space

Erik, as seen in Figure 5.11, made a counter statement, "What I don't understand is that if you are using a microscope to get more space, in actual reality you are not getting more space" (188). RT1 followed with, "that is an interesting idea isn't it, Erik" (189). Erik repeated his statement. RT1 asked the class, "what do you have to say about it?" (191).

Andrew responded, "actually you are because the human eye cannot see" (192). Alan continued, "when you enlarge it you can see how much space you have left between the zillionths and the zero" (193). Erik gestured with his hands as seen in Figure 5. and responded, "yeah, but actually, you said before when you use the microscope you get more space in the number line. That is what you were saying before" (194).

Alan responded that what Erik had said was not what he said. Erik countered that his earlier statement was how he "understood" (196) Alan's earlier conjectures. Erik continued, "I though you said if you use a microscope you get more space on the number line. It is not true" (196). Alan continued:

> If you had some really small pen, you could draw a small line in the space you have because you really don't know how much space you have left between the zillionth and the zero. You really don't know that because you can't see it so you look at it under a microscope you could see how much space you have left. (197)

RT1 interjected, "it might be, Erik, when you were thinking more space, you were thinking of extending it" (198). Erik replied, "yeah, the first time the way he said it that's

Figure 5.11 (both) Erik and Alan bantered

why" (199). RT1 gestured with her hand as seen in Figure 5.12 and continued, "both of you had a different picture in your head about the kind of space and (200).

Figure 5.12 RT1 gestured describing the various images in student's head

5.4.6 The Human Eye Cannot See Dust Particles

Alan continued, "like what I'm saying, if you looked at it under a microscope…" (201). Brian interjected, "like the human eye" (202). Alan resumed his thought, "like the human eye" (203). Simultaneously Alan and Brian continued, "you can't see it like a dust particle" (204). Brian stated that the dust particle was like the zillionths and trillionths. Alan agreed and stated that a person would need to magnify the line.

David stated, "I think you really can't see it too well, but if you use a microscope then you are seeing closer and it looks like you are seeing more, but you're really not. You're just looking closer than before" (208).

5.4.7 Summary

In this episode, students discuss the idea of infinite many numbers between zero and one. RT1 points to the integers on the over head projector and says that the will only worry about the positive ones at the moment.

RT1 stated that similarly to the number line going on and on forever, so does the numbers in between zero and one. A student off camera stated that it was like eternity.

The students then explored dividing the line segments into zillionths, billions and googles. Alan commented that a person could divide the smallest of fractions into zillionths. He continued to say that after you divided the smallest of fractions into zillionths that there would still be space in between. Michael added that a person would need the biggest number line in the world. Erik countered that it just did not make sense. Alan replied that it did make sense. RT1 intervened and asked Erik this opinion if the one whole were changed to a zillion. Erik stated if the whole were a zillion, then they could divide it into zillionths. RT1 asked the class to think about a whole billion a billion or a google. Alan asked what would happen if a person called the zero to one a billion. RT1 replied that there infinite ideas for the students to think about and argue about.

Andrew stated that if a person put the number line under a magnifying glass then they would be able to see that there was space left between the one hundredth and the one thousandth. Alan extended the idea and suggested placing the number line under a microscope to enlarge the line and see that there was still more space in there. RT1 asked what do scientists find in the sky when they use more and more powerful telescopes. The students responded that they find more and more stars.

Erik, then, said that he did not understand that when a person uses a microscope to get more space, in actual reality a person is not getting more space. Andrew responded that actually a human eye cannot see the space. Erik and Alan bantered about their statement meanings. Alan followed with a hypothetical situation where a person had a really small pen and drew a small line then the space between the line (the zillionth) and the zero would have unknown space to the human eye, but you could see it under a microscope. RT1 intervened and said that Erik might be interpreting more space to mean extending. Erik replied yes that was what he thought. RT1 commented that they both had different pictures in their heads.

Alan and Brian talked about difficulty for the human eye to see dust particles. Brian commented that a dust particle was like the zillionths and trillionths. Alan agreed. David commented that with a microscope a person is only seeing closer not seeing more.

5.5 Discussion between zero and one

5.5.1 Laura

RT1 wanted to know how other people were thinking about the number line (211). RT1 called on Laura who smiled and shook her head. Erik asked, “are you a little bit lost?” (213). RT1 countered, “I don’t think so, she is listening very carefully to both ideas. What do you think?” (214). Again, Laura shook her head from both sides and mumbled. Again, RT1 asked Laura what she thought between zero and one. Again, Laura shook her head left to right. She opened her mouth as if to speak; however, she remained quiet.

5.5.2 Audra

RT1 called on Audra whom said, "I really do agree with them because"(219). RT1 asked, "with whom?" (220). Audra replied, "with Andrew and Alan because the human eye cannot see it if you are making it that small so if you put it under a microscope you really could see more" (221).

5.5.3 Jessica

Jessica agreed with Alan and Andrew because she said, "you really can't see and if you put it under a microscope you could space" (224).

5.5.4 Small Discussion on Alan's Argument

A student said that Alan was not saying "it is getting bigger, he is just saying that it is not going to stop" (225). Michael stated, "it is sort of like the more you see the more space you have" (226). Alan replied affirmatively. RT1 asked the class what they thought about Michael's statement. The student replied, "it is hard to explain" (229).

5.5.5 Mark

Mark stated that he agreed with Alan and Andrew because, "you can't see the thing but if you put it under a microscope and if it is a really powerful one you would have a huge space there" (233).

5.5.6 David

David stated, "I think you can take the little smallest thing and then put it under a microscope and you will have a lot more space but you don't. It looks like a lot more space but it really isn't. You are just magnifying it" (235). The students agree.

5.5.7 Michael

Michael said, “it looks like you have more space and human take advantage of it and take the really big space and mark these really little lines on it that you really just can’t see on it” (237).

5.5.8 Summary

RT1 asked the class what they thought of the class discussion about microscope, telescopes, zillionths, googles and dust particles. RT1 called on Laura who smiled and shook her head. Erik asked her if she were lost. RT1 countered that she did not think Laura was lost as she was listening very closely. RT1 called on Audra whom said she agreed with Andrew and Alan because a human eye cannot see really small things unless a person puts it under a microscope. Jessica agreed with Alan and Andrew because she said a person really cannot see it unless they put it under a microscope. Mark stated that he agreed with Alan and Andrew because a person cannot see it but if it were put under a really powerful microscope a person would see a huge space between the numbers. David commented that you are just magnifying the number line so that it looks like there is more space, but, in reality there is not more space. Michael repeated the magnification argument.

5.6 Number names on tiny lines

RT1 stated, “Okay, we are going to give number names to all those really really little lines. Won’t that be fun?” (238). RT1 called on Alan.

5.6.1 Alan’s Argument

Alan walked up to the over head projector and marked one hundredth on the line. He then showed the class, as seen in Figure 5.13, that there was, “all that space” between the one hundredth and the zero. He continued, “it looks like it, but you really don’t have that much space. It’s just that if you and it really big that is how much space” (241). He added, “that means you could divide this [space] into halves and thirds and fourths and fifths and all of that” (241). RT1 rephrased Alan’s statement, “you’ve magnified it because you got a very powerful microscope. And, it would be really hard to place one hundredths; but, once you magnify it you will have all this extra space in between. That’s interesting” (244).

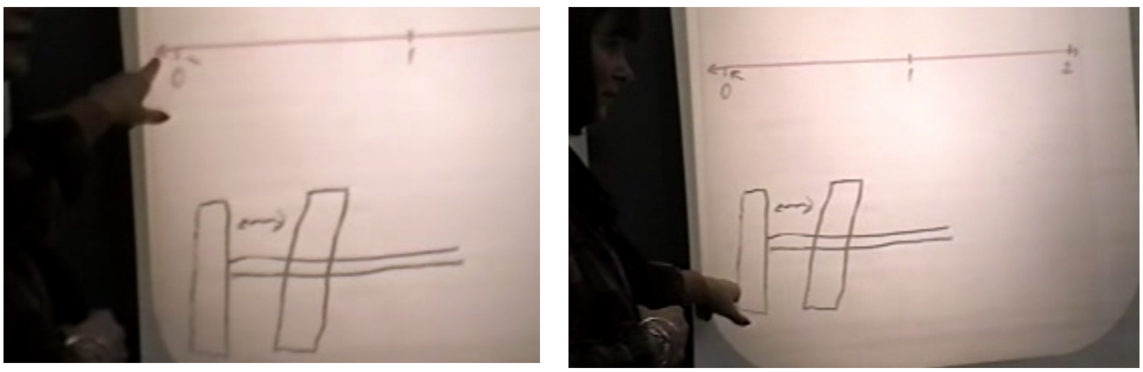

Figure 5.13 Alan presented his written work on the over head projector

Alan replied:

> Yeah, because it looks like you got a lot of space, but you only really have the tincy-wincy little space in between there. I mean you could take like a really small pen and you could divide this up into all those pieces, but if you look at it with your regular eye you couldn’t see that so you would have to make it bigger. (245)

RT1 asked Laura if that helped her. Alan continued to discuss inserting little bars and dividing up the space in between. RT1 thanked Alan and asked the class if they had any questions to ask Alan. A few other students had comments.

5.6.2 David’s Comments

David stated that he did not really have a question. David gestured as he described his written work as seen in Figure 5.14. He said, “on my paper I had a ruler that I put up to it that I was using and I think it was millimeters or something. I had a ten inch number line so I just put it after one millimeter that was one hundredth (251).

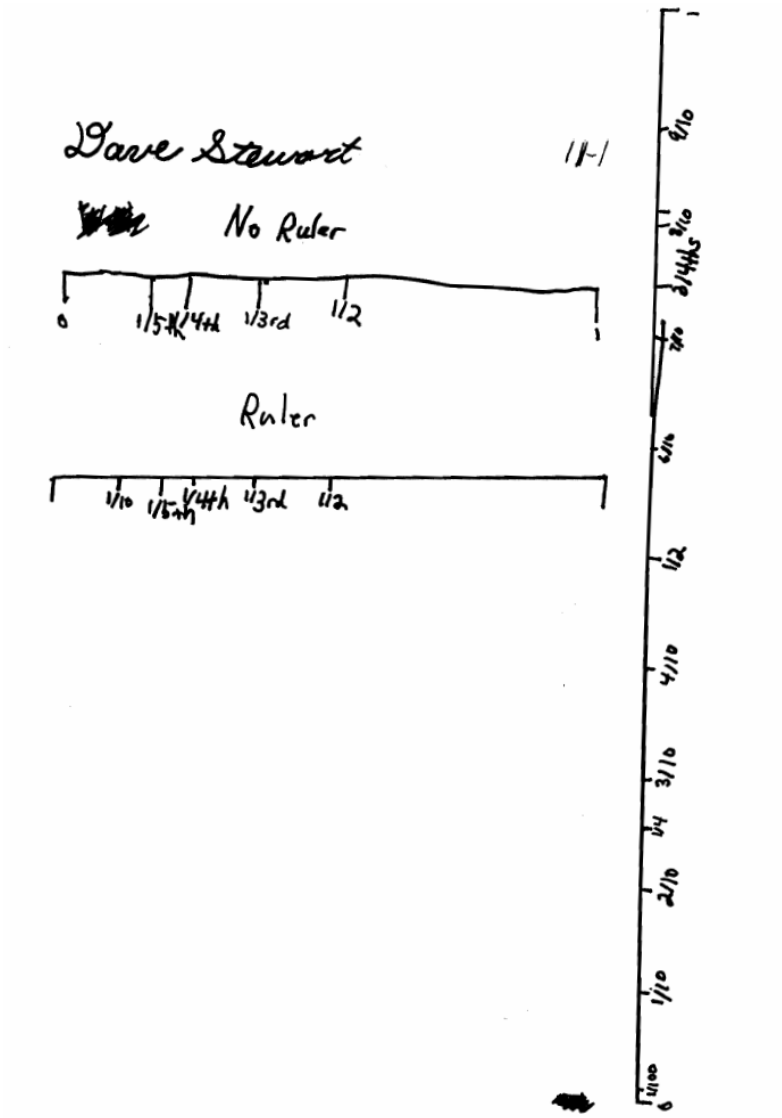

Figure 5.14 David's written work

5.6.3 Brian's Comments

Brian said that he had a comment about what Alan and Andrew had said. He continued, "you see humans don't have powerful enough eyes to see where the zillionths are so there really is a lot of room but you don't see it because the human eye is not as powerful as a microscope" (255).

5.6.4 Instruments Get in the Way

Michael interjected, "Oh, I get it. So, there is a lot of room that you can't see" (256). Alan replied, "say in the future that you come up with this really high powered microscope. You could make that zero bar from the floor to the celing that would maybe let you see it being that big. You could divide it up into such small pieces that when you took off the microscope you wouldn't see anything. It would be so tiny and so small that you couldn't see it, but, there really is space there and if you magnify those really tiny pieces you could divide those up into spaces" (257). David commented, "then you would probably need something with a really small point to write that small" (258). RT1 added, "it sounds like the instruments get in the way, not the numbers" (259).

5.6.5 Additional Comments

RT1 called on James. James stated that he agreed with Alan mostly. He said, "Space in between the zillionths" (260). Alan interjected, "the biggest number you could think of you could make one and so on. You could go on forever with this. I mean you could keep on magnifying it and magnifying it and magnifying it, dividing it, magnifying it dividing it" (261). Brian added, "you could take the number line that has so much little space between it and if you look at it with a very powerful microscope then you would be able to put billionths into it. So, it doesn't matter how bit it is it could be a small as a germ and you could still put germs in it" (262). David commented that he was going to say what Brian had said that the line could be as big as a dust bug (264).

RT1 called on Gregory and asked him what he thought. Gregory said, "no" (266). RT1 called on Meredith. Meredith said that she though what Alan was trying to say was that if you looked through a microscope and saw a lot of space, but if you only used the human eye then there is not as much space (263). RT1 followed with, “that is a nice synthesis” (269).

5.6.6 Summary

RT1 asked the class to give number names to the tiny little lines and called on Alan. Alan walked up to the over head projector. He drew an enlarged portion of the number line and showed the class all the space between the zero and the hundredth on the enlarged portion of the line. He said you could now divide that space into halves, thirds, fourths, fifths and “all of that.” RT1 repeated Alan’s statement. RT1 asked the class if they had any questions. David commented that on his paper he used a ruler which had millimeters or something. He continued that he had a ten inch ruler so he marked one hundredth on the one millimeter.

Brian commented on Alan and Andrew earlier statements and said humans do not have powerful enough eyes and they need powerful microscopes. Michael exclaimed that he “got it” that there is a lot of room that you cannot see. Alan said that in the future you could come up with this really high powered microscope where you could see more accurately. RT1 stated that it sounded like the instruments got in the way not the numbers.

RT1 called on James who agreed with Alan mostly. He said there was space in between the zillionths. Alan exclaimed the biggest number you could think of you could make one and so on. He continued that you could keep on acting on the number via magnifying, magnifying, dividing, magnifying and dividing it. Brian added that if a person took the smallest space and placed it under a telescope they would be able to place a billionth into it. He continued you could still put germs in it. David added that the line could be as big as a dust particle.

RT1 called on Gregory. Gregory said no. Meredith said she thought Alan was trying to say that a microscope helped people see what the human eye could not see. RT1 commented that her statement was a good synthesis.

5.7 Placing fractions

RT1 stated, “I would like everyone to take a turn up here to place some fractions on this number line” (269). RT1 asked Gregory to go first.

5.7.1 Gregory Placed a One Half

RT1 commented that since Gregory did not get a chance to say anything it was his turn to go first. She asked him to place a number between zero and two. Gregory walked to the over head.

RT1 added, “Everyone is going to get a turn so you might think about a number you are going to place between zero and two. Any fraction you want just tell us why you are doing it and you have to get the class to agree that that is a reasonable place” (269).

Gregory wrote the number one half midway between the zero and the one on the over head projector as seen in Figure 5.15 (left). As Gregory was writing, RT1 added:

> You all can be thinking about another number. Someone else may take your number, you know, so you better have a couple of back up numbers. Remember, we have infinitely many to choose from so we are not going to run out of number are we? (271)

The students responded negatively, affirming that they would not run out of numbers.

RT1 asked Jessica her opinion of Gregory’s placement of one half. Jessica pointed with her ruler towards the board and replied, “Are you doing it between zero and two or zero and one?" (273). RT1 stated, "I think you are doing it between zero and one" (275). Jessica stated, "That is not one half" (276). RT1 asked, "How could you change it a little bit? Which way would you move it?" (277). The students responded closer to the negative side. RT1 asked Jessica to offer her ruler to Gregory to help him line up his line. Gregory re-wrote his placement of one half as seen in Figure 5.15 (middle/right).

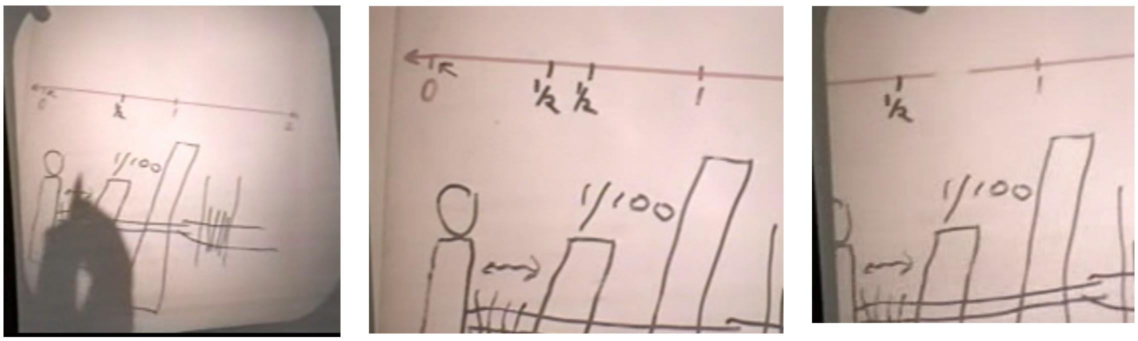

Figure 5.15 (left to right) Gregory placed one half chronological

RT1 asked the class if they were happier with the shuffle of one half slightly to the left. RT1 commented that in the future they should probably use the board as it would be easier. The students responded that the one half should be moved back over slightly to the right. RT1 laughed and asked that the placement not to be exact. The class bantered about placement. RT1 looked to pick another student. Everyone the camera captured raised their hand. RT1 selected Laura.

5.7.2 Laura Placed a One Fourth

Laura placed a one fourth mid way between the zero and the one half. RT1 asked the class what they thought. A student off camera asked if Laura was placing the one fourth between the zero and one or the zero and two. RT1 replied, "Should it matter?" (290).   The students replied no. Brian commented, "You should put it on the one half if it is going to be zero to two" (292). Alan agreed, "Yeah, you could divide all the zero to one into all those fractions or you could divide the zero to two into all those fractions" (293).

RT1 replied, "Laura wanted it right there between zero and one half" (294). She continued, "You want to put it where one is? We have one there. You want to put it where one half is? We have one half there." (294). RT1 asked, "Can one half and one fourth go in the same spot?" (294). Andrew answered, "Well, between the zero and two, the half of it is one and usually the fourths are a half of a half so the half right there would be the fourth" (295). RT1 asked Laura what she thought. There was no audible response. A student off camera stated, "it looks like a third" (297).

RT1 added, "I still want to discuss it. Even if she is going between zero and two, I want to know if it were going between zero and two, should one quarter go someplace else or not" (298). She asked, "My question to you is on this number line can a fraction have more than one place?" (298). The students respond yes.

Michael replied, "Three fourths is on the other side of one half" (300). RT1 stated, "I am very confused" (301). Michael continued, "It could stay there or it could also go somewhere else if you double it or multiply it" (302). RT1 said, "No, I'm talking about the number one quarter" (303). Erik said, "one-quarter should be moved over towards the zero more" (304). Meredith added, "because you need to fit one third on the other side" (305). RT1 asked Laura if she would mind moving one fourth over a tiny bit more.

5.7.3 Can Two Different Points be Named a Quarter

RT1, then, asked the class, "My question is, on that number line where we said there were infinitely many pointes, the point that has the number named a quarter could there be a different point with the number named a quarter" (306). A student responded, "there could be" (307).

Alan commented, "there could be infinite places because if you enlarged the space between two points you could divide that into one fourths" (309). RT1 countered, "that is not what I am saying. I'm not saying divide the line" (310). Alan continued, "well, you could also put the one fourth on the one half and that could be" (311). Erik stated, "No. You can't that an improper fraction. There are fourths for each whole, so if you are dealing with two wholes, then, that would have to be eight fourths. Yes, there has to. You cannot have a one fourth for two whole" (312). Brian interjected, "two wholes can make one whole" (313). Erik countered, "No. They cannot. There are two wholes separate" (314). Alan spared, "No. The zero to the two is what we are thinking about. The one is what we are doing the zero to the one. The one is a half of the zero to the two" (315). Brian agreed again with Alan and stated, "those two wholes put together make one whole so you would put the one fourth on the half between the zero and one" (316).

Erik walked up to the overhead projector and gestured as seen in Figure 5.16. He stated, "Brian, if you say that you are supposed to have two fourths, here, and two

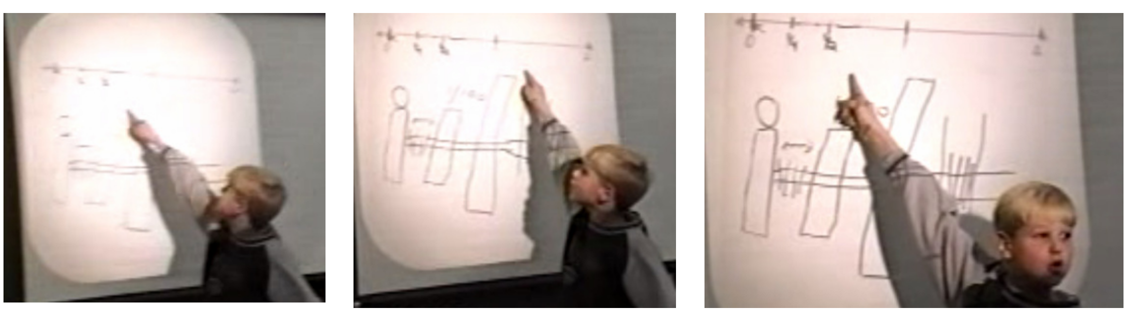

Figure 5.16 (left to right) Erik chronological discussed placement of one fourth

fourths, there, and, then, you divide this into two fourths, then, you are only going to divide this into halves ” (317). Erik, Brian and Alan argued. Alan continued, “You are talking about having them separate, but we are not. We are talking about having them together” (319). Alan stood up. RT1 called a time out. RT1 stated that she was confused and asked if anyone else could explain what is going on. RT1 commented that RT2 and RT5 were confused as well.

RT1 held up a ruler and stated, “there are some things about this ruler that are not like a number line” (320). The students agreed. RT1 continued “There is something about this ruler that is like your Cuisenaire rods and there is something about this ruler that is not like your Cuisenaire rods. Unless we are arguing agreeing about the same thing then we are going to keep arguing past each other” (322). The students agreed.

RT1 continued, “let’s establish a few things to make sure we are in agreement about what we are talking about. There is a number line on the board. Does it end or does it go on and on?” (324). The students repeated the latter—on and on.

RT1 asked how many agreed that the line goes on and on and eight students in the camera view raised their hands. RT1, then, asked, “the ruler in my hands, does it end or does it go on and on?” (328). The students agree with the former—it ends. RT1 stated, “It is a segment. It ends.” (330). RT1 asked, “The Cuisenaire rods that you built, does it end or does it go on and on?”(330). RT1 stated, “It ends. It is a segment. We have some agreement” (330).

Jessica commented, “it goes on and on if you want it to” (331). RT1 agreed, “yeah, but right now, these models, whether they are the Cuisenaire rods or the ruler, are models. Now, the number line that goes on and on without stopping forever, could we build such a model? (332). The students mumbled. RT1 added, “That’s the idea, right?” (334).

RT1 revisited Andrew’s earlier idea, “Once I start taking a piece of the line, … I could say here is a half, right in the middle, right?” (336). The students agreed. RT1 continued, “And what his is doing is that now he is taking the half of the length of this particular ruler, right? But, if you look at the ruler, you don’t see in the middle here one half, do you?” (338). The students responded no. Michael added, “you see six” (340). RT1 agreed, “You see six.” (341). RT1 recalled how Cuisenaire rods worked where a person could call different rods by different names and find half rods. The students agreed.

RT1 stated, “Now what is tricky here is that this line goes on and on without ending, right? So, maybe what helps is to think about pieces of it” (345). The students repeated what RT1 says, “yeah, pieces” (346). RT1 continued, “like the piece between zero and one, but once I call this one you cannot change its name because I’ve already given it the number name one. Do you understand?” (347). The students remained quiet. RT1 added, “We can talk about pieces of it, but try not to get it confused with the ruler and the rods. Try to think that once I put a number name on a particular point that will always be that number name. Okay? The question is where do you fit the other fractions and how do you give them number names?” (349).

5.7.4 Erik Places One and One Half

Erik revisited his earlier argument. Erik argued that two fourths cannot be placed between both zero and one as well as one and two (350). RT1 asked Erik, “If you are talking the distance half way between one and two, that is what you are telling me to do, right?" (351). The students agree. Erik stated, "That is one and one half" (352). RT1 asked Erik, "That would have what number name, Erik?" (358). Erik restated one and one half. Erik stated, "it would not be fourths" (361). Brian repeated his earlier argument. RT1 stated that they were no longer working with rods. She stated that they were now working with the number line.

RT1 asked them to look again between one and two and said, "you know that half way is in the middle, right? Now, if I have to give it a number name, can you give this a number name one half when it is to the right of one?" (365). Erik replied, "No. You have to call it three fourths" (366). Alan countered, "You would have to call it one and one half" (367). RT1 finished, "Some of you think you need to call it three fourths and some of you think you need to call it one and one half. This is a good place to stop. We have so much to think about, don't we?" (368).

David added, "On my paper I put one and one half there [half way between one and two]" (369). RT1 repeated David's statement as a question. David replied yes.

5.7.5 Meredith Places One and Three Fourths

RT1, then, asked, "Where would you put one and three quarters?" (372). Michael stated, "You would probably put it a little to the right of one and one half" (373). RT1 asked, "How much to the right of the one and one half would you put it?" (374). Michael gestured with his fingers as seen in Figure 5.17 and replied, "like that much" (375). RT1 asked if someone could give a number name. Jessica replied, "one and one half. No, I mean [*chuckling*]" (377). RT1 gestured to the projector as seen in Figure 5.17 and asked, "Why did you think the middle, Jessica?" (380). Jessica replied that she is not sure (383). RT1 replies, "something to think about" (384).

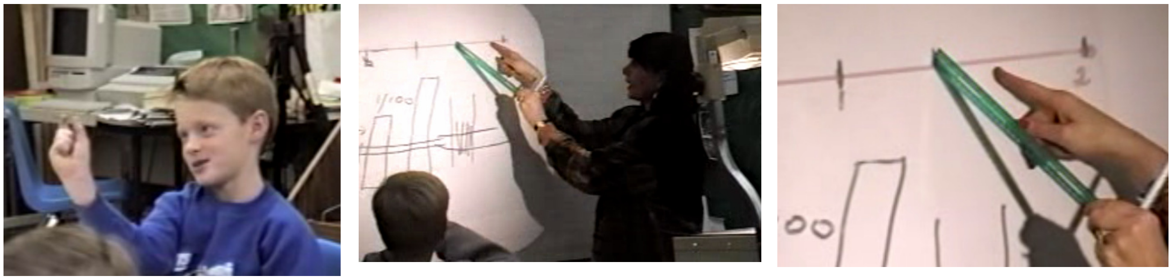

Figure 5.17 (left) Michael gestured his answer
(middle and right) RT1 gestured to the number line

Meredith stated, “If you have two, two fourths, that would be saying one and one fourth … one and one half.” (385). RT1 stated that she is not sure she understands. Meredith walked up to the over head and pointed from one to one and one half and said, “if you have [one and] two fourths it is equal to one and one half. So, then you would have two more fourths [points from the one and one half to the two] would equal another half which would equal a whole” (389). RT1 draws in one and one half. RT1 continues, “You are telling me that if I want another fourth, what would that be?” (392). Meredith stated, “one and three fourths” (393). Michael agreed, “Yeah, that would be right” (396).

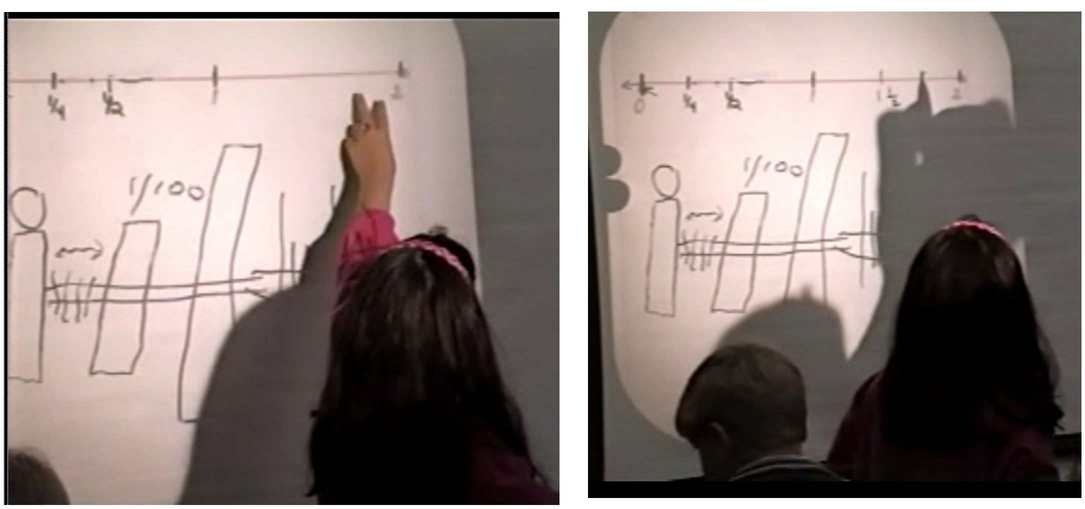

Figure 5.18 (both) Meredith pointed to one and one half

RT1 asked Andrew if he agreed. His response is off camera. RT1 asked Jessica if that was what she was saying. Her response is off camera. Michael added, "I was also saying that I think it would be a fourth because like in a half is two fourths and in that half it should take two fourths. It does if you divide that half in half." (403). Andrew stated, "It's a fourth because anything past the line is one and whatever" (405).

RT1 said that the students were thinking in terms of lengths and asked them not to confuse the lengths with the new number name. RT1 asked the students to think about where two and one half would be placed. Alan stated that it would be behind the two. RT1 asked how many students thought they knew where to place two and one half. At least six students raised their hands. Kelly responded, "a little bit past the two" (415). RT1 asked how much past the two and Kelly replied, "half" (417). RT1 asked, "half way?" (418). Kelly replied yes. RT1 asked, "half of what?" (420). Kelly replied, "half of that ruler" (421). RT1 asked again half of what and Amy replied, "two and three" (423). RT1 asked again, "where would you put two and one half" (424). A student replied half of the ruler (425). RT1 said yes. David interjected, "six inches" (427).

RT1 asked the students to find as many fractions as possible between zero and two. She asked, "And, see if you can find number names for numbers in between if you've changed your mind about anything as a result of today's wonderful discussion" (430). RT1 finished, "I'll see you next week at the end of the week" (430). After the session ends, Erik walks up to talk with RT1. Erik states, "What I'm saying is that two wholes would be an improper fraction which would be eight fourths" (437). RT1 followed, “You are telling me another name for two is eight fourths. I’ll buy that, Erik” (438).

5.7.6 Summary

RT1 asked the students to each come up and take a turn placing some fractions on the number line. She called on Gregory to come up first. He placed a one half on the number line mid way between zero and one. The class had some discussion about the exact placement of one half. RT1 commented that in the future they should probably use the board as the projector exaggerates the placement. Gregory slightly move one half back and forth at the requests of the students.

Laura, then, came up to the projector and place one fourth midway between the zero and one half. Brian commented that she should put it on the one half if they were going from zero to two. Alan agreed. RT1 replied that Laura wanted right where she had placed it. RT1, then, asked if one half and one fourth could go on the same spot. Some students said it depended on what the whole was supposed to be.

RT1 repeated her question asking if a fraction could have more than one place on the number line. She asked if the point named a quarter could be in two places on the number line. A student replied that there could be. Alan said that there could be infinite places because between any two points a person could divide into fourths. RT1 followed by saying that she wasn’t asking about dividing the line. Erik stated that it was impossible; that you would have an improper fraction when a person had two wholes each comprised of fourths would be eight fourths. Brian interjected that two wholes could be renamed a whole. Erik countered that they were two wholes separate.

RT1 asked the class to establish a few things to make sure they were all in agreement. She asked the class if the number line ended or if it went on and on. The students replied that the number line went on and on. RT1 asked the students if a ruler ended or went on and on. The students replied that a ruler ended. RT1 asked the students if the Cuisenaire rods ended or did they go on and on. The students replied that the Cuisenaire rods ended. RT1 stated that the ruler and Cuisenaire rods were segments. Jessica replied that it could go on and on if a person wanted them to go on and on. RT1 stated that right now these models are models whereas the number line goes on and on without stopping forever. RT1 asked if they could build such a model that went on and on forever. She stated that that was the idea. RT1 said that maybe it would help them to think of pieces of the number line and not to get the piece confused with the ruler or the rods. She continued and said that once a number name was given to a particular point then that point will always remain with the same name. She added that the question is where to fit the other fractions and how to give them names.

Erik revisited his earlier argument. RT1 asked Erik what name he would give the midway distance between one and two. Erik stated that it would be one and one half. Erik countered that it would not be fourths. David said he placed one and one half midway between one and two on his paper.

RT1 asked the class where would the place one and three fourths. Michael replied that a person would probably put it a little to the right of one and one half. RT1 asked by how much to the right. Jessica replied that it would go in the middle of one and one half and two. Meredith commented about two fourths. RT1 commented that she did not understand so Meredith came up to the over head. Meredith said that if you had (one and) two fourths that it would be equal to one and one half. She then said if you had two more fourths it would be equal to two. RT1 asked Meredith if she had another fourth what would she have. Meredith said one and three fourths. Michael agreed. Later, he said that it would be a fourth because a half is two fourths and what they were looking at was a half of a half.

RT1 asked that the students not confuse the lengths with the new number names. RT1 asked the students to think about where two and one half would be placed. Alan stated that it would be behind the two. Kelly said that it would be a little bit past the two. RT1 asked how much past the two. Kelly replied half. RT1 asked if Kelly meant half way. Kelly replied that she did mean half way. RT1 asked half of what. Kelly replied half of that ruler. Amy replied two and three. RT1 asked again where to put two and one half. Another student replied half of the ruler. David said six inches.

At the end of the session RT1 asked the students to find as many fractions as possible between zero and two. She said she would see them at the end of next week. Erik walked up to RT1 at the end of the session as other were packing up to leave. He said that if a person had two wholes then there would be an improper fraction which would be eight fourths. RT1 followed with saying that he was giving her another name for two which would be eight fourths. RT1 stated, “I’ll buy that” (438).

CHAPTER 6 – RESULTS 11-10-1993

6.1 Introduction

In Session 3, November 10, 1993, the students explored the placement of various rational numbers between negative two and positive three. Within the interval between negative two and positive three, the students explored fractions, equivalent fractions and improper fractions.

RT1 began the session by welcoming the students and asking them how they were doing. RT1, then, asked the students if they had been working on the number line. RT3 commented that the class had held off. RT1 asked the students again if they had though about the number line anyway.

The camera view captured seven students raising their hands affirmatively, as seen in Figure 6.1. RT1 said to RT5, “see” (11). RT5 reaffirmed, “yeah” (12). RT1 commented that she was “impressed” (13).

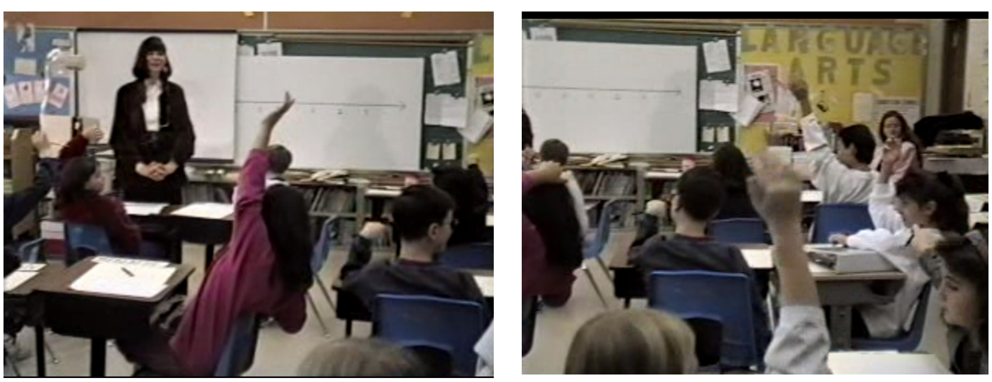

Figure 6.1 (both) Students responded to RT1

6.2 Meredith Line

RT1 stated that she looked at the class number lines and Meredith said that they could share her work. RT1 placed Meredith’s work on the over head projector (Figure 6.2). She, then, asked Meredith to explain her work. RT1 asked the students to discuss with their partners Meredith's work to see if the number line made sense. In pairs, the student discussed Meredith's work. RT1 commented that Mark, Audra, Amy, James, Jacqueline, Graham and Michael all had questions. RT1 said that Michael should go first.

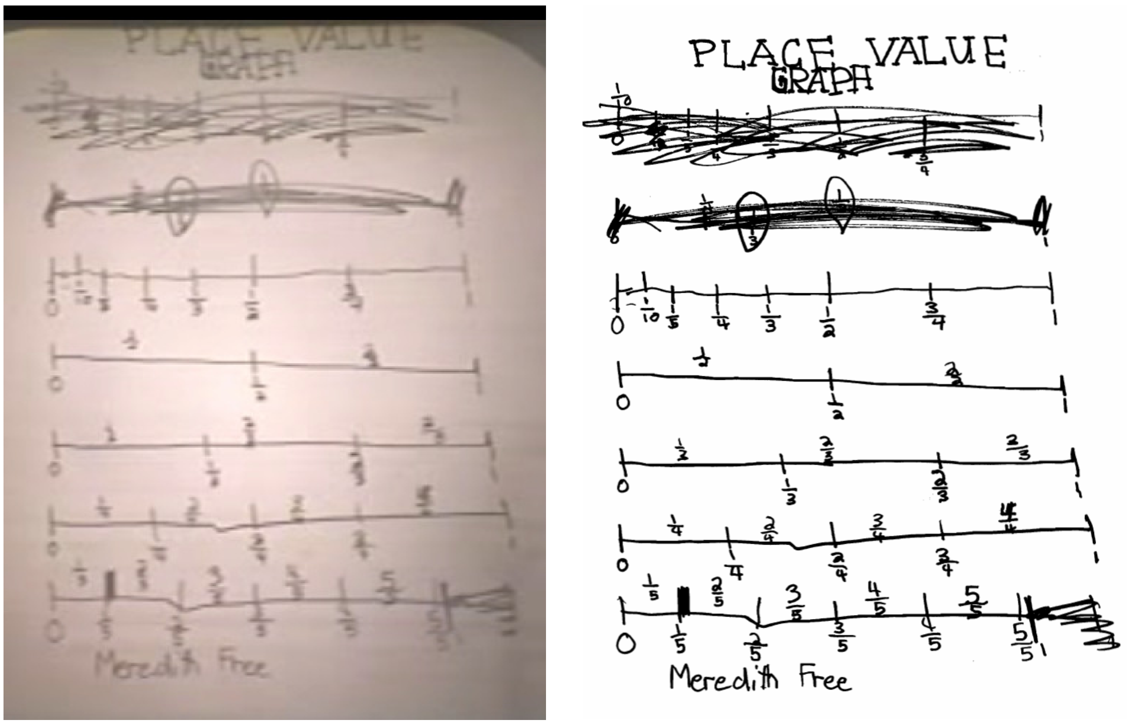

Figure 6.2 Meredith's written work

6.2.1 Michael's Question

Michael asked:

> Why do you have like you have for your half, you have half and half. I am not arguing that. But, in your third number line you put two-thirds as your half … Why are you calling two thirds, a half? It is not half. It is bigger than half, two thirds is bigger than half. (22)

Meredith murmured that she knows and was unsure "where to put it" (23). She walked up to the over head projector and described her lines. Meredith pointed to the third line down from the top (Figure 6.3 left) and said:

> This is what the bottom is. This is the one third [*points to one third*]. This is two thirds [*points to two thirds*]. The area right here is one third [*points to the space between zero and one third*]. The area right here is two thirds [*points to the space between zero and two thirds*]. This area is three thirds. [*points to the space between zero and one*]. (23)

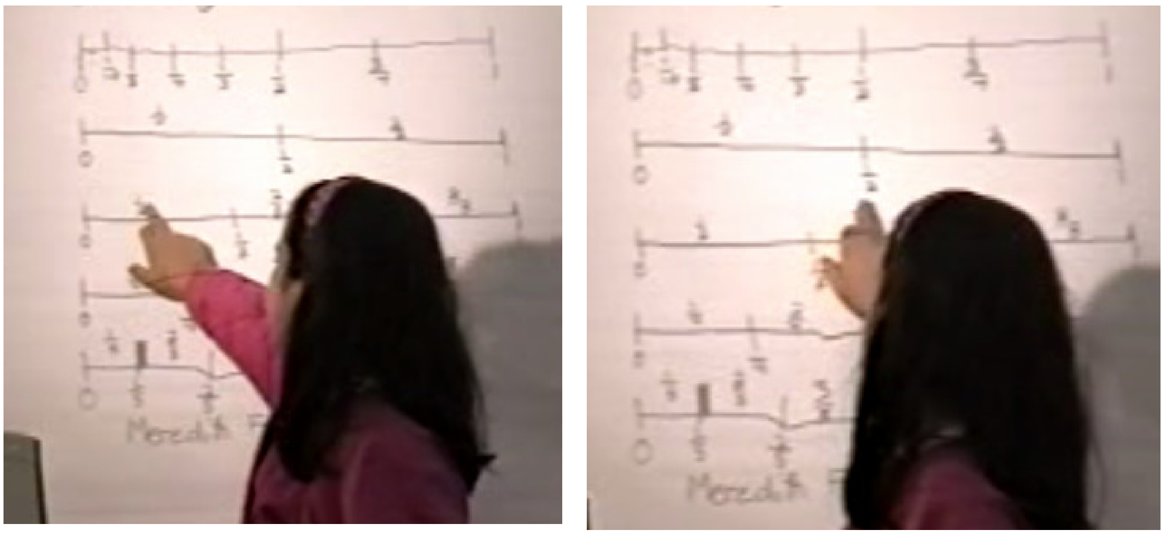

Figure 6.3 Meredith described her work

Michael, Figure 6.4, stated, “You have one third, two thirds, and, then, three thirds” (24). Meredith replied, “one third, two thirds, three thirds. This is the area in the middle” (25).

Figure 6.4 Michael asked Meredith about her lines

6.2.2 Erik's Question

Erik stated, "But, two thirds is not the area in the middle because two thirds is not equal to one half" (26). Meredith said, "This [*pointing to the *?**] is two thirds (27)". Erik countered, "Then, why did you put it there?" (28). Meredith replied that she had. Erik continued, "Why did you put it right under the half?" (29). Meredith pointed and replied, "See, this is the one third area, so I put it there [*on top of the line*]. This is the two thirds area, so I put it there [*on top of the line*]" (31).

6.2.3 A Debate

Erik forcefully responded, "No. Meredith, what you did, where you think is the one-third area and you put it under the line and in the one-third area. It does not make sense. Which one is it?" (32). Meredith replied, "It is the bottom one" (33). Erik continued, "Then why did you put it in the area?" (34). Meredith, Figure 6.5, answered, "Because, I wanted to show the one third, two thirds and three thirds area" (35). Erik argued, "Yeah, but they are in the wrong space. If you wanted to do that you should put it in the exact space or at least approximate" (36). As seen in Figure 6.5, Meredith smiled and shook her head.

Figure 6.5 Meredith reacted to Erik's counter argument

RT1, Figure 6.6, interjected, “Well, I think her spaces are, here, Erik” (38). Meredith replied, “Yeah” (79). Off camera, Erik began to argue, “I know, but”(40). RT1 continued, “What I am hearing her say is, pretend you don’t see these [*covers numbers on top of line*]” (41). Michael exclaims, “Oh, I get it. She is just labeling the spaces like one third of a space, and that is two thirds and three thirds” (42). The students in the class giggle.

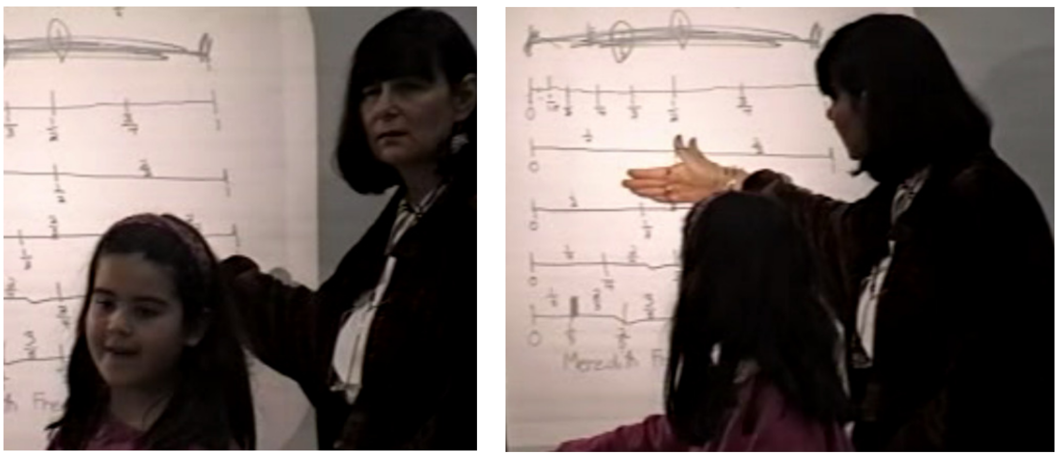

Figure 6.6 RT1 interjected into the debate

6.2.4 Amy’s Question

RT1 asked the students why they were “laughing” (43). She asked Amy what she was thinking. Amy, Figure 6.7, replied, “My question was, why didn’t she just make one big one and not make like five?” (46).

Figure 6.7 Amy’s response

Off camera, either Michael or Erik replied, “It just doesn’t make sense, because you do not need to have them there and a lot of people will think that they are the numbers and it is confusing” (47). As seen in Figure 6.8, RT1 gestured to the work on the overhead projector and commented: “For a moment, let’s up pretend the numbers are not on the top, for a moment” (48). RT1 asked the students if they had trouble pretending and they responded that that did not.

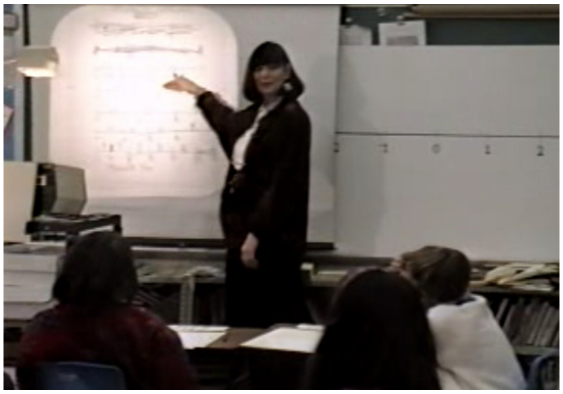

Figure 6.8 RT1 gestured to Meredith’s line

6.2.5 Remember Alan’s Line?

RT1 gestured, as seen in Figure 6.9, and asked, “Remember what Alan did? When Alan magnified a pice of the number line? Do you remember that?” (50). RT1 looked through a folder at the front of the class to find Alan’s over head work. Once found, RT1 placed the work on the over head projector (Figure 6.10 left) and asked, “How many of you remember what Alan did?” (52).   As seen in Figure 6.10 (right), at least four students were captured on camera raising their hands.

Figure 6.9 RT1 gestured as she described Alan's earlier work

RT1 continued:

> Now I'm sort of thinking when I saw this, I was thinking that Meredith maybe was doing some variation of what Alan did; but, what she did rather than magnify a piece of the number line, she showed you particular sections of the number line by making them over again. That is what I think. (52)

RT1 asked Meredith if that was in fact what she did. Meredith replied affirmatively. A student commented that Meredith was "making smaller parts of it" (55). RT1 replied, "Yeah, she is making smaller parts of it" (56).

Figure 6.10 (left) RT1 placed Alan's earlier work on the overhead projector (right) Students raised their hands indicating they remembered Alan's work

6.2.6 RT1 Discusses Meredith's Lines

RT1 asked the class why they thought Meredith made multiple lines. Jessica replied: "It is just easier to see when making the whole number line over. It is just easier to see" (57). RT1 replied, "Yeah, it gets kind of crowded sometimes doesn't it?" (58). RT1 continued that Micheal and Erik had asked different questions than the one she wanted to ask. RT1 stated that the first number line showed her that Meredith knew where to put the numbers. RT1 continued and said that the second number line showed halves. RT1, then, asked, "And the third number line, what were you trying to show?" (62). Meredith replied, "thirds" (63). RT1 asked, "And, the fourth one?" (64). The students answered, "fourths" (65). RT1 continued, "The fifth one?" (66). The class replied, "fifths" (67). RT1 finished, "She was focusing on showing different pieces of it" (68).

RT1 commented that her question still had not been addressed. Erik stated:

> If someone was to look at this for the first time, on the third number line, for the first time, I know when I did I got confused because I thought that in the middle of the section where she put the one third, two thirds and three thirds. I thought that, that is where they would be. So that's why. I think that is what Michael did, too, so that is whey he asked the question. (69)

RT1 asked how many people in the class thought the same thing. Two student on the camera raised their hands. RT1, then, asked how many had not thought the same thing Erik had thought. Four students in the camera view raised their hand. RT1 stated, "so, we all looked at it differently" (74). Jessica commented, "I think what Erik means is that he thought Meredith was making a whole new number line; like, she thought two thirds was the half" (75). Erik replied affirmatively. RT1 continued, "But, that was not what she was doing, was she?" (77).

6.2.7 Summary

The session began by RT1 welcoming the students back and asking them if they were thinking about the number line since they last met. Many students responded that they had been thinking about the number line.

Meredith shared her number line with the class. She had drawn five lines—one main number line and four number lines showing different aspects of the number line as follows: one line showed the placement of halves, one line showed the placement of thirds, one line showed the placement of fourths and one line showed the placement of fifths.

Michael and Erik were very confused by the way Meredith labeled the number line. She had placed area (lengths) above the number line; however, Michael and Erik thought that the fraction areas were the actual fraction numbers. A long debate took place over Meredith's labeling. RT1 intervened into the debate.

Amy asked why Meredith had made five number lines instead of just one. RT1 asked the class to recall Alan's line where he had magnified a piece of the number line. She said that what Meredith had done seemed to be a variation of what Alan had done.

RT1 asked the class why they thought Meredith had made multiple lines. Jessica replied that it was just easier to see. Meredith explained that the first line was a line, the second line showed halves, the third line showed thirds, the fourth line showed fourths and the fifth line showed fifths. RT1 emphasized that Meredith was just focusing on showing different pieces of the number line.

6.3 The Class Discusses Meredith's Line

RT1 asked Alan what he wanted to say. Alan said, "The way I thought, when you divide it up into fractions, then you have a line going to the fractions, but Meredith just put the fractions above the number line without the line, so that meant she was labeling the area" (80). RT1, Figure 6.11, pointed to the two thirds above the third line and said, "In the particular region over here, she has two thirds in this region, see where I'm pointing?" (81). The students murmur affirmatively. RT1 asked, "between the one third and the two thirds, does that region represent two thirds of the line?" (83). Erik said, "it represents one third" (84). RT1 replied, "Okay, it represents one third. How do you know that?" (85). Erik, Figure 6.11,walks to the over head projector.

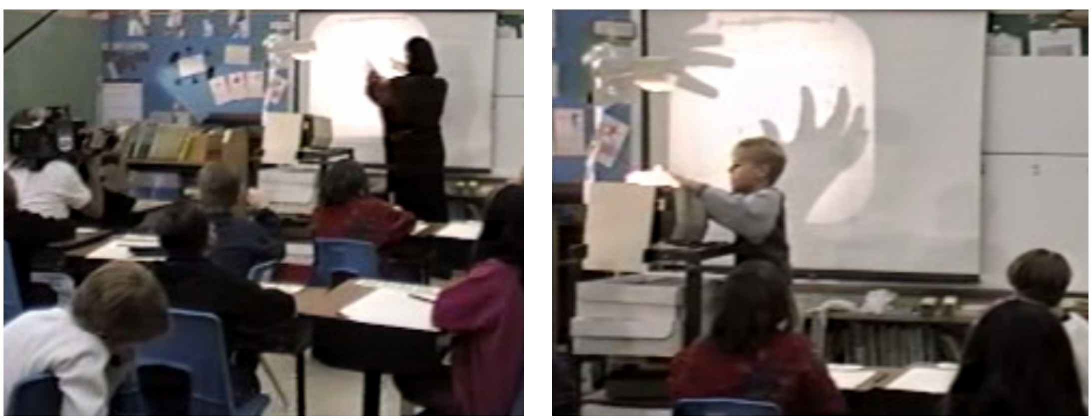

Figure 6.11 (left) RT1 gestured to Meredith's work on the over head projector (right) Erik gestured to Meredith's work on the over head projector

6.3.1 Erik's Discussion

Erik said: "Because the way you said that one segment. Well, if you use both those segments, like the segment here to here [indicating to the region between zero and two thirds], that would be two thirds, but you said the segment here [indicating to region between one third and two thirds], it would be the two thirds segment, but it would be only one segment" (86). RT1 asked Erik to show her what part of the line represented two thirds. Erik pointed, as seen in Figure 6.12, to the region between zero and two thirds and said: “Right there. That would represent two thirds” (88). RT1 asked the class what they thought. The students murmured: “That is true” (90). RT1 asked how many students agreed with Erik. Two students were observed by the camera raising their hand. Alan stood up and walked to the overhead.

Figure 6.12 Erik gestured to region between zero and two thirds

6.1.2 Alan’s Discussion

As seen in Figure 6.13, Alan points to the overhead and interjects:

> Right, but what I think Meredith was trying to do was—you see how it had one third and the one third being here—she was saying that she was labeling this to be the second third of the line and labeling this to be the third, third of the line. (93)

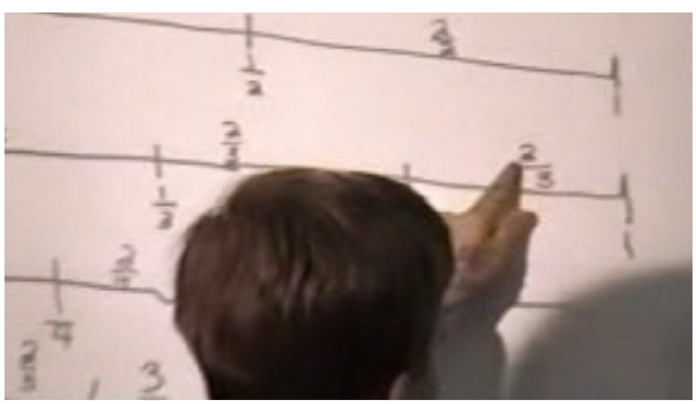

Figure 6.13 Alan described his interpretation of Meredith’s work

6.3.3 Where Would Three Thirds Go?

RT1 asked, “Okay, so where would you put three over three? Where would you write that number if you were to write it on the number line?” (94). Alan responded:

> If you are doing it the way Erik’s talking about it, you would put one third over that area [*points to the space between zero and one third*]. And, you would put two thirds in that area [*points to the space between one third and two thirds*]. And, three thirds in that area [*points to the space between two thirds and one*]. Because, this would be representing one third [*points to one third*] both of those would be two thirds [*points to two thirds*], and three of those would be three thirds [*points to one*]. (95)

RT1 pointed to the overhead and asked, “Okay, so if you were to put the number three thirds on the line? I see over here the numbers zero, one third, two thirds and one. Where would you put three thirds?” (96). Alan replied, “If you put three thirds, you would put it just in that big area because it would be” (97). Erik, Figure 6.14, pointed to one on the over head screen and interjected, “No, you wouldn’t. It would be right there“ (98). Alan replied, “Right, that would be the mark of the three thirds. But, all three of those are the three thirds” (99). Erik, Figure 6.14, pointed to the area above the two

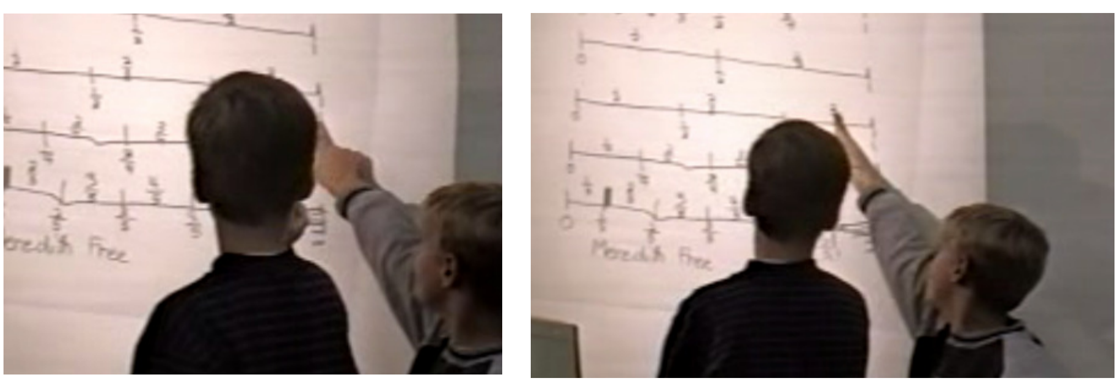

Figure 6.14 Alan and Erik gestured to two thirds region

thirds region and said, “Meredith is saying right here that [the number] three thirds is right here. She says that is one, three thirds in her perspective would be right here” (100).

Meredith interjected, “No” (101). RT1 stated, “No. Let us hear what Meredith has to say” (102). Erik complained, “But” (103). RT1 continued, “Let us hear what Meredith has to say” (104).

Meredith explained, “As I did over here, I am saying this area here [pointing to the three thirds above the line] is three thirds” (105). Erik countered, “I know; but, if someone were to look at this for the one time, they would think that [the three third written above the line] would be one whole and that would be three thirds, because you did not label. You should label one whole and under it put three thirds” (106). RT1 interjected, “But, I did hear Meredith say that this piece is three thirds, did you mean that Meredith?” (107). Meredith (see Figure 6.15) replied, “Yeah, this whole entire piece right here [pointing to the area between two thirds and one]” (108).

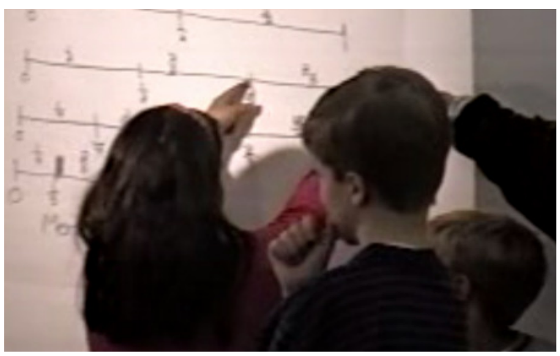

Figure 6.15 Meredith gestured as she explained her line

RT1 again asked, “You mean that?” (109). Meredith replied affirmatively. Erik responded, “I agree that [*points to space between zero and one third*] would be the first piece. That [*points to the interval between one third and two thirds*] is the second piece. But, together they make two-thirds“(113).

RT1 asked, “And, what makes three thirds?” (114). Alan replied, “the entire thing” (115). Erik, Figure 6.16, stated, “three thirds would be the entire thing” (116).

Alan said, "Right, that is just representing the three thirds" (117). Erik said that was what he was talking about. RT1 interjected, "Okay. Okay. Let's hear from what other people are thinking" (119).

Figure 6.16 Erik gestured as he explained his notions

6.3.4 More Student Discussions

Brian stated that he thought he knew what Meredith was trying to say and said:

> She is trying to say—to label—between it. Because, if someone said—if someone looks at it—and they did not know a thing about the number line, they would probably think that the one third would be between the one third and two thirds, that is why she labeled it also in the middle because if someone saw it and did not know what a number line was the would probably think one third is between the one third mark and the two thirds mark, so she just labeling lit in the middle to make it less confusing. (120)

Mark walked up to the overhead and pointed, as seen in Figure 6.17. He said:

> I agree with Brian because if no one ever saw a number line, they would think that this was one third [*points to the one third on top of the number line*] and this was two thirds [*points to the two thirds on top of the number line*]. (124)

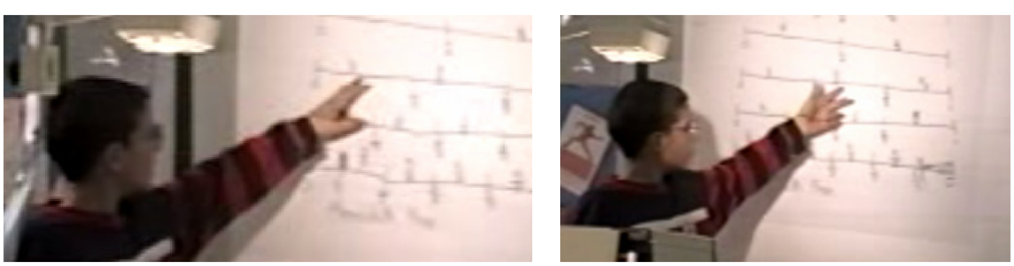

Figure 6.17 Mark gestured to the over head projector

6.3.5 Summary

The class collectively discussed Meredith's line. Again, they began to debate Meredith's labeling schema. Alan stated that the labeling above the line represented the area. Erik stated that the area between each third was one third. Alan added that Meredith was labeling the area above the line collectively, not separately.

RT1 asked where three thirds would go on the line. Erik said that all three thirds would be equal to three thirds. RT1 asked where would they put the number three thirds on the line. Alan said it would be the whole big area. Erik countered and said it would be one. Alan agreed that one would be the mark of three thirds, but added that the whole area would be three thirds. Erik, again, debated Meredith's labeling.

RT1 asked to hear from other people. Brian stated that Meredith was just trying to label the space between. He added that someone might get confused if they had never seen a number line before. Mark agreed with Brian that if no one had ever seen a number line, they might get confused.

6.4 Mathematician Conventions

RT1 interjected:

> Okay. Mathematicians conventionally—what they do so that people are not confused they—kind of agree to a way to makr those numbers on the line, they have a common way to agree. And the common way to agree, you can see here that we have a big number line, we take it apart in a minute, but you see where we put our numbers, we place them where Meredith placed them. That is the way Matehaticians do it, and if you want to know where those numbers go you usually look on the bottom of the line, okay, and that sort of helps us understand the why we have the notation. Alan? (125)

6.4.1 Alan's Lines

Alan said, "We could do that so that we could represent one half, two thirds, and three thirds in pieces" (126). Alan walked up to the overhead and RT1 gives Alan a fresh transparency and marker. Alan began to write and said:

> Here is one way to do it. Now, ,here would be one mark, two marks [*Figure 6.18 left*]. Now, you could take out one of those pieces and say it is one third [*Figure 6.18 left*]. And, then, you could take out another piece this long and you could put two thirds [*Figure 6.18 middle*]. And, then you could use the entire number line and say it three thirds [*Figure 6.18 right*]. (126)

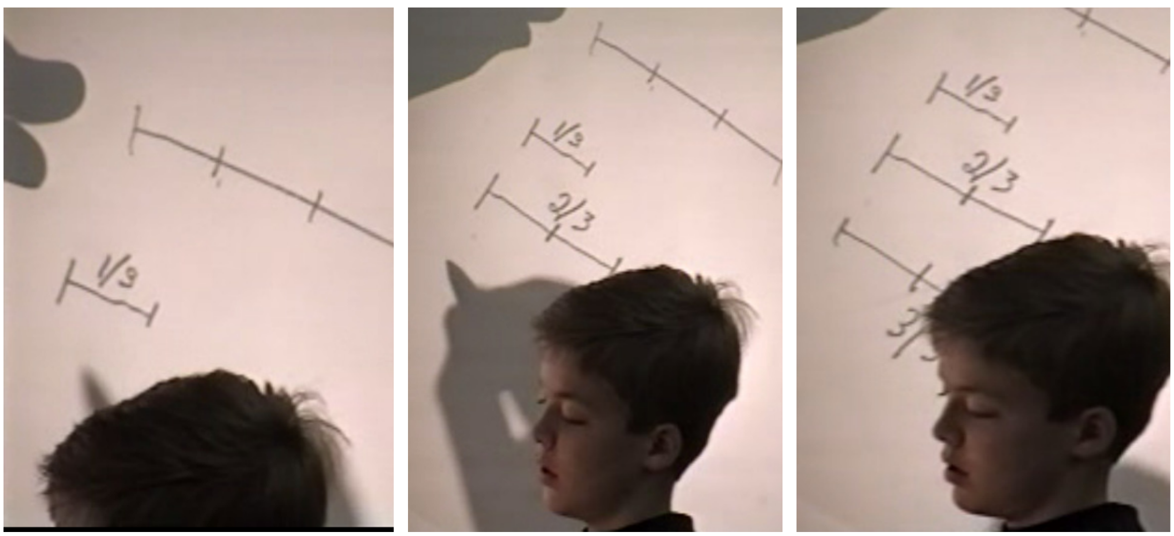

Figure 6.18 (left to right) Alan chronologically marked thirds

RT1 asked the class if they liked what Alan had done. Seven students raised their hands affirmatively. RT1 asked the class if they had any questions. No students in the camera view raised their hands.

6.4.2 Alan Places: 0, 1/3, 2/3 and 3/3

RT1 asked Alan to place the fractions—zero, one third, two thirds and three thirds—on the first number line he had drawn on the overhead. Alan labeled the number

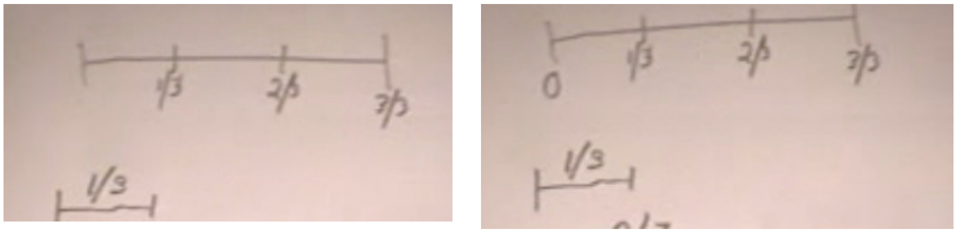

Figure 6.19 (both) Alan labeled the zero to one interval

line as seen in Figure 6. RT1 asked about zero. Alan wrote a zero under the first tic mark (Figure 6.19).

6.4.3 Summary

RT1 discussed that mathematicians develop conventions to help eliminate confusion. She continued by saying that numbers usually go on the bottom of the line. She added that helps understand why there is notation.

Alan got a fresh transparency and drew four parallel lines as follows: one line from zero to one divided into thirds, one short line between zero and one third, one longer line between zero and two thirds, and one line between zero and three thirds matching the top most line. Many students liked what Alan had drawn.

On the top most line of the four lines, Alan labeled the line according to thirds as follows: zero, one third, two thirds, and three thirds.

6.5 Where would 1, 4/4, 7/7 and 1,000,000/1,000,000 go?

RT1 asked for someone to tell here where to place one. RT1 stated that the students could talk with their partner if they so wanted. A student off camera asked which number line and RT1 responded Alan's. The students murmured. RT1 commented that some students wanted the number line longer and other did not want the number line longer. RT1 added that she wanted the students to have reasons for their responses. The students discussed amongst themselves.

RT1 commented that if the students have worked out where they would place one, she also wanted them to think about where they would put four fourths. The students continued to discuss amongst themselves. RT1 walked around the room.

RT1 asked Erik and Michael, where they would put five fifths, seven sevenths and a million millionths. Their response is inaudible; however, RT1 replied, “Okay, so you are telling me that, that is just another name for one?” (153). RT1 reminded Michael and Erik to put their numbers on the bottom as “Erik so forcefully told us that people will be confused” (153).

As seen in Figure 6.20, at least eight students congregated at the over head projector.

Figure 6.20 The students congregate at the over head projector

RT1 asked the students if they were ready to discuss their ideas. The students returned to their seats. Meredith stood in the front and raised her hand. RT1 commented that there were at least two different positions within the room. She stated that she wanted to give Meredith the “first crack at an argument” (155) as it was her number line that raised the question.

6.5.1 Meredith's Argument

Meredith (see Figure 6.21) pointed to the overhead projector and stated:

> You asked me where to place one. I think if you have thirds—one third, two thirds, three thirds—three thirds would be equal to one. See, because one third [*points to the interval between zero and one third*], two thirds [*points to the interval between zero and two thirds*] and three thirds [*points to the interval between zero and one*]—three thirds is the same as saying one. Four fourths is the same as saying one. One hundred one hundredths is the same thing as saying one. (158)

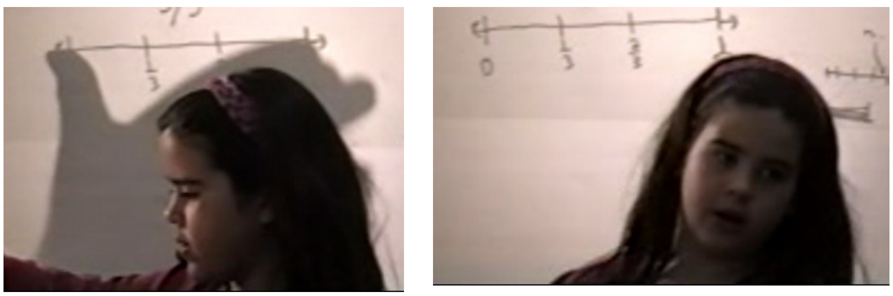

Figure 6.21 Meredith placed the number one on the interval

6.5.2 Michael's Argument

RT1 called on Michael. Michael replied, "I think that if you have a number with the same number on top as in the bottom, then it is always going to be equal to the number named one" (160). RT1 asked the class if they disagreed or had a question about Michael's statement. The room remained silent.

6.5.3 Jessica Agrees

RT1 commented that Jessica was making faces and asked her if she was confused. Jessica replied, "I think, yeah, I agree with her" (168).

6.5.4 Alan's Argument

Alan said:

> What I was saying before when I was talking is that zero to the one third mark is one third. Zero to the two thirds mark

> is two thirds. Zero to the three thirds mark is three thirds. Now, three thirds you cannot have any more thirds or you would have four thirds. Then, you have to make the thirds bigger or not have another. You can only have three thirds. (170)

RT1 suggested, “Let us hold that question about what would happen if we had four thirds” (171).

6.5.5 Meredith’s Improper Fraction Discussion

Meredith commented about four thirds. She said, “You only have four thirds, if you are going to have that you could only have four fourths, not four thirds. You cannot have four thirds” (172). RT1 continued, “You cannot have four thirds in that interval. I wonder if we could have four thirds if we went further” (173). Meredith replied, “Then, we would have to have six thirds” (174). RT1 followed:

> We would have to have six thirds. Okay. Hold on. Meredith just said—I want to make sure you are all able to hear what she just said—I believe she said that you cannot have four thirds in this interval; but, I asked the question, ‘if you extend the interval, could you have four thirds? (175)

Meredith replied, “If you made it two you would have six thirds and then you could place four thirds” (176). RT1 repeated Meredith’s statement.

6.5.6 Brian’s Argument

Brian said:

> Alan thinks that I am placing four thirds and I am not. Because you see, I think that you start on the negative side, like [*walks up to over head projector and points*] For example, you would probably start on the one third and right between the one third and the two thirds would be one third, right. Between the two thirds and the three thirds would be two thirds and, you see, he thinks that when I add that there, he thinks that zero to one third is a fraction, it is not. Because zero is a separate number and it is not a

> fraction, and so I would think that if you start on the one third and anything past that if you keep going up, you would hit two thirds. And, anything between there [pointing to the area between one third and two thirds] would be one third. But, if you start at zero and you go across and you hit the one third, zero is not a fraction, so if you add one more, zero between one third is not a fraction, so it would be four thirds. (178)

RT1 commented: “Some people have written zero as zero thirds” (179).

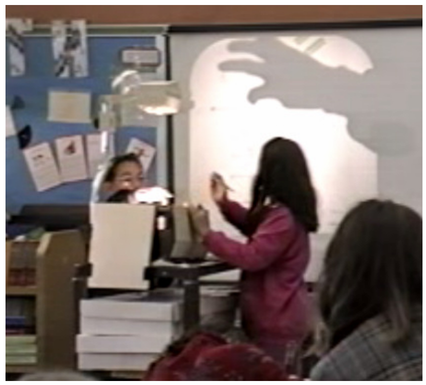

Figure 6.22 Class discussed as Meredith wrote on over head projector

6.5.7 Meredith Places

Alan walked up to the over head. Meanwhile, Meredith began to write on the over head projector. RT1 stated, “Let us see what Meredith is doing here” (181). Meredith extended the zero to one interval to become zero to two and placed all six thirds within the interval. Below the one and two, she placed three thirds and one and three thirds respectively as seen in Figure 6.23.

6.5.8 Alan and Brian Discuss

Alan continued, “What I am saying is that zero to the one third mark—anything from zero to one third—is the one third. Now, anything past the one third mark we would be calling two thirds”(182). As seen in Figure 6.23, Meredith simultaneously drew arrows on either end of the line showing the interval from zero to two. Brian commented, “See one third is higher than zero” (183). Alan replied affirmatively. Brian continued, “So you would not count zero to one third as one third” (185). RT1 interjected, “Maybe let us stop for a moment” (186).

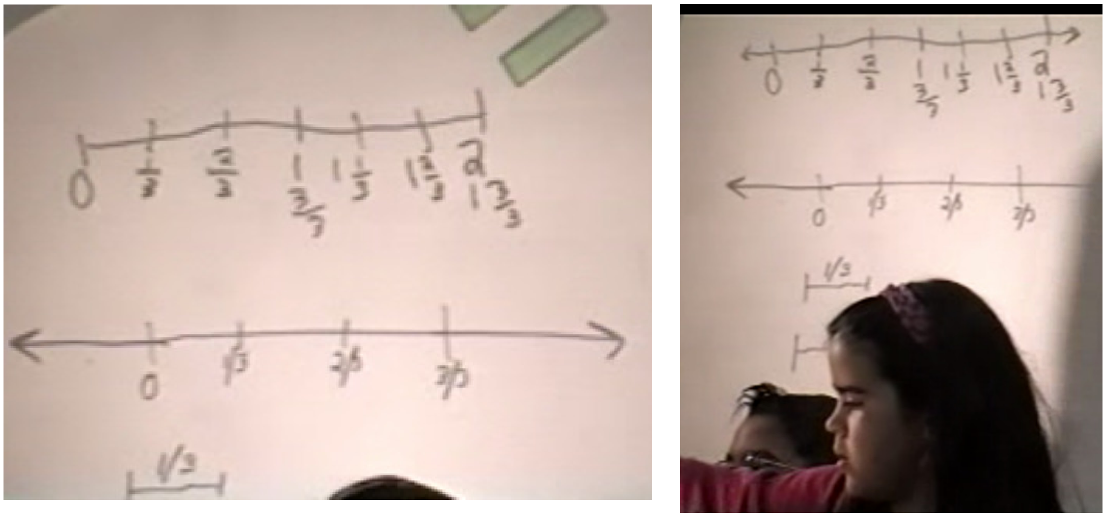

Figure 6.23 Meredith independently placed fractions between one and two

Alan wanted to make an additional comment. He said, “What I am saying is—here is a model of using thirds—suppose that end is zero and that end is one. You are saying from there [zero] to there [one third]—this piece—really has no fraction value in the one third” (187). Brian replied: ”There is no fraction value” (188). Alan replied, “You cannot put that [rod] over there, look, it [past the region from zero to one] has extra room. You only have three spaces—one, two, three—you have one third, from zero to one third; two thirds, from one third to two thirds; and, three thirds area, from two thirds to three thirds” (189).

6.5.9 Meredith Discussion

RT1 stated that they would listen to Meredith and then work on something else. Meredith said, “I put one third, two thirds, and, then, three thirds would equal one. And, then I went from zero to two it would be one and one third, two and two thirds, and, then, one and three thirds“ (191). RT1 corrected, “One and two thirds, right?” (192).

6.5.10 Zero Does Not Have Value

Brian and Alan reiterate their earlier statements. Brian stated, “There is no fraction value between zero and one third, because” (193). Alan pointed as seen in Figure 6.24, and intervened: “If you have no value between zero and one third, then look, you eliminate that [*covers the space between zero and one third*] how many spaces would you have? Two” (194).

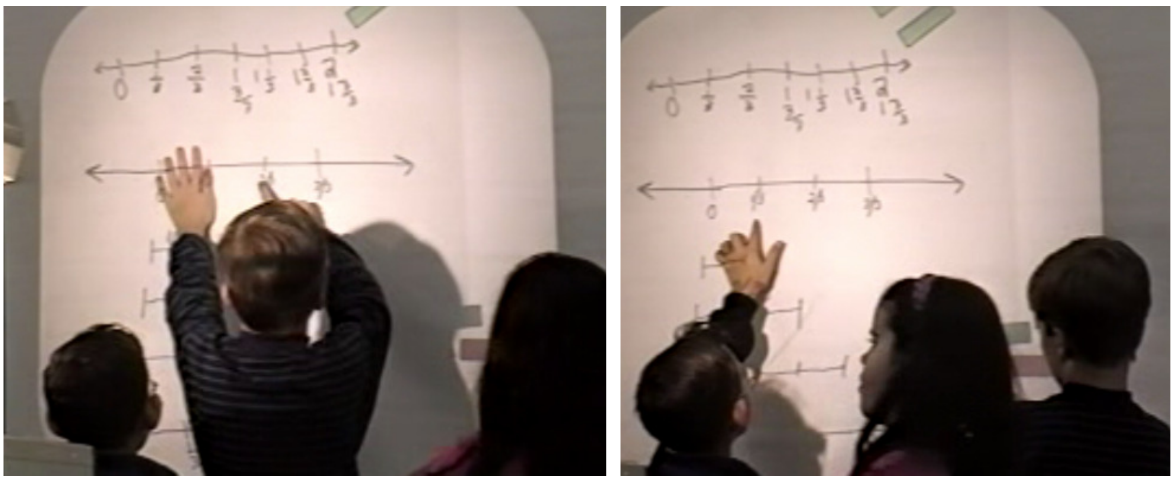

Figure 6.24 Brian described his thinking that zero to one third has no fraction value

Brian replied:

> What I would think would be that if you start on the lower number and then keep going up. And, until you hit that [*points to two thirds*] anything between there would be the one third and the two thirds right there. And, if you keep going up to there you will hit that [*points to three thirds*] that will be the two thirds. And, if you put a bar right there [*points to space between one and one and one thirds*] it would count as one. Two thirds between one would be three thirds. Because, like I was saying, if you started there at zero, then zero does not have value. (195)

6.5.11 Sara’s, Beth’s and Audra’s Argument

RT1 asked the students to return to their seats and stated that Brian was obviously not convinced. RT1 stated that she wanted to give Sarah, Beth and Audra a chance to share their ideas. They walked up to the front of the room and placed their work on the over head projector as seen in Figure 6.25.

Audra began:

> We thought that we did not have to put anything else on the number line, because if we put this from zero to one, and you would mark one third here [*points to second tic mark*] because if you used a ruler here to measure it or something one third would go here [*points to second tic mark*], two thirds would go here [*points to fourth tic mark*], and three thirds would go here [*points to the fifth tic mark*] because the length will be the same as the Cuisenaire rods or something. (197)

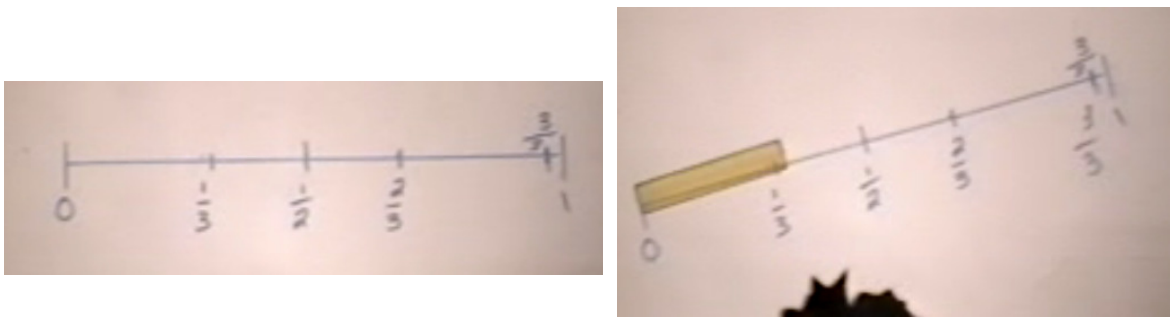

Figure 6.25 Sara, Beth and Audra described their work

RT1 replied that she was not clear about the number written on the top and stated that she thought everyone had agreed to write numbers on the bottom. RT1 followed with: “Where would you place the three thirds?” (198). Audra replies that they would put it “here” (199) which they indicated slightly left of the number one. RT1 asked the students to put the label “underneath the number line and tell us why you would place it there” (200). The group placed three thirds on the number line as seen in Figure 6.25.

RT1 continued: “You are telling me you would place it with one to the right of three thirds?” (200). The group did not verbally respond. RT1 followed, “So this clearly defines where we have differences of opinions, right? Is that true? So, how many of you are agreeing that we should place one to the right of three thirds?   How may of you believe that?" (200). The students responses were not captured on the camera.

RT1 continued, "How may of you believe that it should go to the left of three thirds?" (202). The camera showed Brian raising his hand. RT1 continued: "How many of you believe it should go right on top of the one?" (204). Nine students were shown on camera raising their hands. Audra replied, "That's what we meant. We just could not git it right on top to fit" (206). RT1 responded, "You meant to put it on top?" (207). Michael and some students said to put the number "under it" (208).

RT1 followed, "I see. You just could not fit it in. So, Jessica, how could they do it to put it by the one?" (209). Jessica walked up to the overhead projector and placed the label three thirds under the label one.

RT1 asked, "I am hearing that we have some agreement here then; you all agree that three thirds would go under one. It would go in the same spot. How amny agree with that?" (209). Six students were shown on camera view raising their hands. RT1 continued: "And, I would like to hear again why that would work. Could you tell me Erin?" (209). Erin responded that she would like to think about it a little bit more.

James said, "I think one half and one half makes a whole and four one fourths would make a whole and three thirds would make a whole also. So, it will be right on top of the one"(212). RT1 followed, "are you agreeing with that Jakki?" (213). In which, Jakki replied affirmatively.

6.5.12 Summary

RT1 asked the class where to place one on Alan's number line. (He had three thirds written on his line). RT1 said that if the students had figured that one out, where they would place four fourths, five fifths, seven sevenths and a million millionths. The students worked together briefly.

Meredith stated that three thirds would be equal to one. She said the four fourths would be equal to one and a hundred hundredths would be equal to one. Michael added that if the top and bottom number matched then it would be equal to one. Jessica agreed with Meredith.

Alan stated that there could not be more than three thirds or else there would have to be four thirds. RT1 said to hold the comment about four thirds. Meredith responded that there could not be four thirds. RT1 interjected that there could not be four thirds within the interval zero to one. Meredith continued that in order to have four thirds there would have to be six thirds. RT1 repeated Meredith's statement. Meredith added that if the interval were to be from zero to two then there would be six thirds and could therefore place four thirds.

Brian argued that the space between zero to one third did not count because zero is a separate number and not a fraction. He argued that the first area for one third would be after the one third label up to the two thirds label. RT1 added that some people had written zero as zero thirds.

Meredith placed the thirds between one and two as follows: one and one third, one and two thirds, and one and three thirds.

Alan and Brian continued their discussion as to where the first one third area would begin. Alan stated that zero to one third would be the first one third. Brian stated that one third to two thirds would be the first one third.

Meredith showed how she had labeled the number line between one and two. Brian reiterated that zero to one third had no fraction value. RT1 asked the students to return to their seats. She stated that Brian was obviously not convinced.

Sara, Beth and Audra explained their number line which matched Meredith's and Alan's lines. There was some discussion about how they labeled the three thirds on top of the number line and to the left of one. The three said that they meant three thirds to be equal to one; but, that there was no space to place the label. Michael suggested to put three thirds on top of the label one. Brian thought that three thirds should be to the right of one.

James explained that two halves would make a whole and four fourths would make a whole; so, then, three thirds would make a whole. When asked, Jakki replied affirmatively that she was agreeing with James.

6.6 Filling in the Number Line

RT1 continued: "We only have about fifteen minutes so I would like to try something else" (215). RT1 asked the class what they thought they were going to do now. The students replied: "Fill in that number board" (216). RT1 replied affirmatively and asked the students if they were thinking of their numbers. She commented, "Remember, someone else may take your number so you had better have a few extra numbers ready to go" (217). She continued, "If, indeed, we find that some of you disagree, then you will have to argue why you are placing it. Maybe someone will convince you to change it or maybe you will convince them that you will not change it" (219).

Jessica raised an inaudible question in which RT1 replied, “That is a very good question, Jessica. Let’s have some ground rules, that’s right. In other words, if we were going to put three thirds, we are going to go underneath [the number line]? Right? Where would you put it?” (221). The students replied “on top” (222). RT1 followed, “Well, let’s try to put it underneath. If we run out of room, then, maybe we’ll try to figure out another way. Is that fair?” (223). The students replied affirmatively.

Brian asked, “Should we go from zero to one?” (226). RT1 replied:

> No. We are going zero to—by the way, we have to thank Mrs. Deming for making us this number line. She is running the camera back there—We could keep making it bigger. I do not know if Colt’s Neck will let us break through the walls or we could go on the other side of the walls.(227)

RT1 asked for students to volunteer and asked for someone to lend a ruler to help mark the line.

6.6.1 Audra Places One Half

Audra walks up to the dry erase board bringing a ruler. She places one half below the zero, as seen in Figure 6.26.

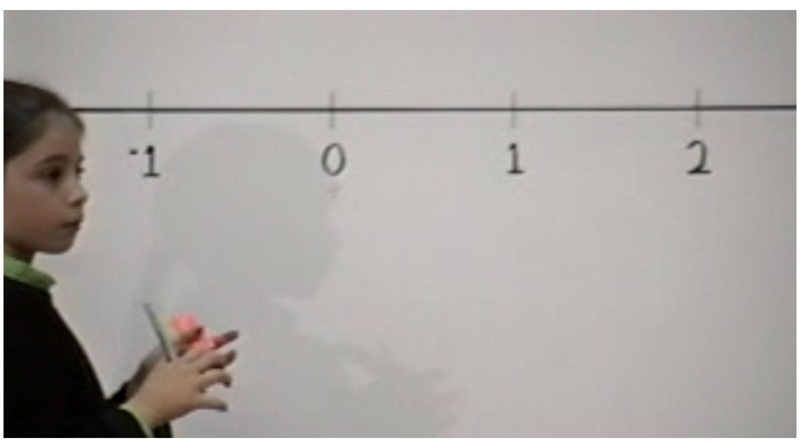

Figure 6.26 Audra placed the number one half under the number labeled zero

6.6.1.1 Michael's response. Michael responded, "No. That's not right. She put one half on the zero. That is half between negative and positive, but that is not a half on the number line" (228). The students briefly discuss what Audra did amongst themselves.

6.6.1.2 Alan's response. Alan said, "She is putting a half there. We have three negative and three positive numbers. She is using negative three as the beginning of the line and positive three as the end" (230)

6.6.1.3 Brian's response. Brian replied, "It should be half between zero and three because on that side is the negative side" (231). RT1 repeated Brian's statement. A student off camera agreed with Brian. Alan stated, "I think she is right" (234). A student off camera replied, "I don't think so because its at the end" (235).

6.6.1.4 Graham's response. Graham replied, "She might have thought the ones before the zero like the three, two, one were not negative" (237). RT1 asked Graham where he would put the one half. Graham responded, "If I did not know it was negative, I would put it under the zero" (239). RT1 repeated Graham's statement as a question. Graham replied, "If I didn't know it was negative" (241). RT1 followed, "Does it matter? Should it really matter? If Mrs. Deming decided to make the line go over here, should it matter? David?" (242).

6.6.1.5 David's response. David said, "I agree with Audra because since it is integers it would go both ways, zero is one half of the whole thing that keeps on going. Because, that is where you start you can keep on going either way, but that is the middle" (243).

6.6.1.6 Erik's response. Erik stated:

> I agree with Audra and David, because there is no way. I heard Michael say that the half would have to be on the positive side, its integers, they keep going, if it is on the positive side its not going to be equal halves, the negatives would be larger than the positives and if you even make the number line bigger, zero is right in the middle, so it is going to halve to be half. (244)

RT1 followed, “Let’s hear from Jessica” (245).

6.6.1.7 Jessica’s response. Jessica walked up to the number line, Figure 6.27, and pointed. She replied:

> On that number line, it [zero] really is half because if you are going to put it on this [positive] side or this [negative] side would be really a quarter. If you put it on the two, it will be a quarter because you will have one, two, three, and, then, four. (246)

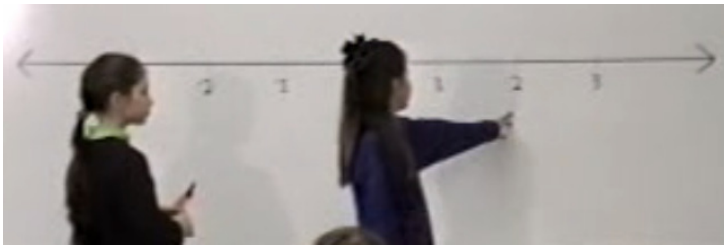

Figure 6.27 Jessica agreed with Audra’s placement of one half

RT1 walks to the overhead, as seen in Figure 6.28, and extends the positive numbers to include four, but does not extend the negative numbers. RT1 stated, “I want to make that my number line” (248). Some students in the class said, “That is not half anymore” (249). RT1 called on Alan.

6.6.1.8 Class discussion. Alan walked up to the front of the room. He said, “Since you added on the four, then, that means you now have four numbers positive and three numbers negative” (251). RT1 interjected, “By the way, do I really have four numbers positive?” (252). Some students replied, “You have four numbers negative. You have five numbers positive—zero, one, two, three and four” (253). RT1 continued, "Is that what we have on the number line? Numbers up to four?" (254). The students murmured.


Figure 6.28 RT1 extended the number line

RT1 asked, "Does the number line end at four?" (257). The students replied no. RT1 asked, "Where does it end?" (261). Some students replied, "no where. It keeps on going. Infinity" (262). RT1 continued, "Where does it end on the other side?" (263). Some students replied, "No where. Infinity" (264).

RT1 stated:

> I should be able to place every single one of my numbers. Just because I have run out of room to write it, that should not get in our way; we have to imagine these numbers going on. I would like to know where I would put negative one half (265)

Some students asked: "Negative one half?" (266).

Alan, as seen in Figure 6.29, wrote on the number line and stated:

> All negative numbers are different than positive. From here [*points to zero*] down you are negative so that means any number here cannot be equivalent to a number over there [*points to positives*]. So, that means if you were dividing this part of up into fractions you would have to put one half mark in the negative, right about there [*points to left side of negative one (Figure 6.29)*]. (268)

RT1 asked, “So you would put a negative one half between negative one and negative two?” (269). Some students replied affirmatively.

RT1 asked Meredith what she thought. Meredith replied, “Because one and one and half and half is three” (272).

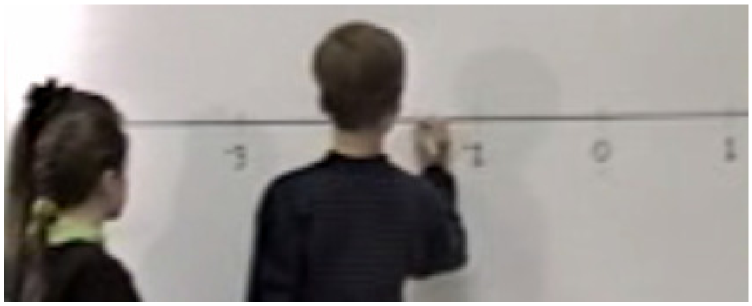

Figure 6.29 Alan placed the number -1 and ½ on the number line

6.6.2 Meredith Places Three Fourths

RT1 asked the class where they would put three quarters. The students asked negative or positive. RT1 replied positive three quarters. Erik stated that it was simple. RT1 continued: “I am confused. Yeah, let use hear what Meredith has to say. Why don’t you all sit down” (277).

Meredith, Figure 6.30, walked to the board and said:

> If you are asking where is three fourths, three fourths would be here [three fourths], here [one and three fourths], and here [two and three fourths]. This [one fourth] would be one quarter, this [two fourths] would be two quarters, this [three fourths] would be three quarters and this [one] would be four quarters. This would be one and one quarter, this would be one and two quarters, this would be one and three quarters, this would be one and four quarters. This would be two; and, so on. (278)

RT1 stated that she was still confused why one half was still on the zero and why two fourths was between zero and one. Meredith replied:

> If you have zero to one, it [half] would not be from zero to four. You would not divide it [the number line] like that. You would divide it [the number line] one fourth, two fourths, three fourths and four fourths. This would be one and one fourth; and, one and two fourths; and, one and three fourths; and, two (280)

RT1 thanked Meredith and asked the class if they agreed with what she did. All the students on camera raised their hands. RT1 commented that everyone agreed.

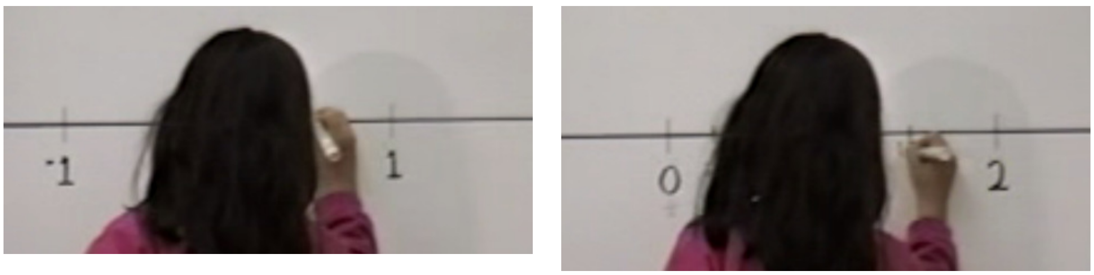

Figure 6.30 Meredith wrote fourths on the number line

6.6.3 Where to Put One Half

RT1 stated that she was still "a little confused with where to put one half" (279). RT1 asked that the students not tell here where to put one half; instead, she asked them if they wanted to keep it under the zero. Meredith replied no. RT1 asked the class if they wanted to keep it under the zero. No students were captured on the camera raising their hands. RT1 asked the class how many of the students wanted to place the one half somewhere else. Ten students were shown on the camera view raising their hands.

Amy stated, "You could keep it there, but you would have to add negative four" (290). A student off camera said, "You have to do something to keep it there" (619).

David stated, "I think Audra is using the whole thing [number line] while Meredith is using from zero to two" (293). Alan commented, "When adding up all those numbers on the negative side would just be like doing nothing" (294).

RT1 interjected: “I’m kind of curious where Meredith would put negative one fourth, negative two fourths, negative three fourths and negative four fourths”(295).

6.6.4 Meredith Places -1/4, -2/4, -3/4, and -4/4

Meredith walked up to the board and placed the four fractions correctly between zero and negative one as seen in Figure 6.31. The students replied, “She is cutting the number line”(297).

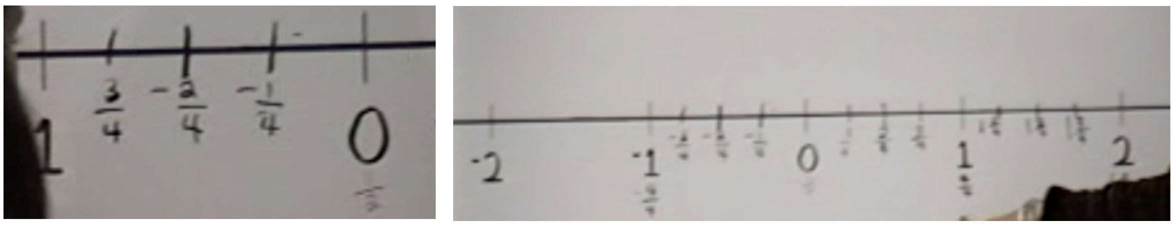

Figure 6.31 Meredith placed fourths on the number line

6.6.5 How Can Zero Have Another Name One Half?

RT1 stated:

> I am still confused about where one half is. I do not know how zero and one half could be the same point. Very confusing to me. If we are given a point with the number name zero, I don’t see how it could have another number name one half. I am so confused. I hope you will straighten me out, because I am so confused. (298)

Meredith replied, “It doesn’t” (299).

6.6.5.1 David’s argument. David continued:

> When you put the one more number, four, you… before it was from negative three to three, when it was from negative three to three so zero was half. But, now that you added the four to the positive side it is not half, both sides of the negative and positive are not equivalent. (300)

RT1 pointed to one and stated:

> But, you told me earlier that I could put four fourths here because that is another name for one. And, I think some of

> you told me that I could put five fifths here because that is another name for one. And, in fact, I thought I even heard Michael say earlier that you could put one thousand one thousandths here because that is another name for one. Did you say that Michael? (301)

Michael responded affirmatively. RT1 continued, "You could put one million one millionths and you could put other names for one here. But, I don't' understand how another name for zero could be one half. I am so confused; this mathematics is confusing to me" (303).

6.6.5.2 James argument. James stood up and gestured as he stated: "I think that one half might go between zero and one. Half of it is negative one, negative two, negative three and the other half is one, two, three and four" (304). RT1 asked James to show the class what he was thinking by writing on the board. As seen in Figure 6.32, he drew one half under the two fourths between zero and one. RT1 continued, "So I am hearing James say that one half is another name for two fourths" (305).

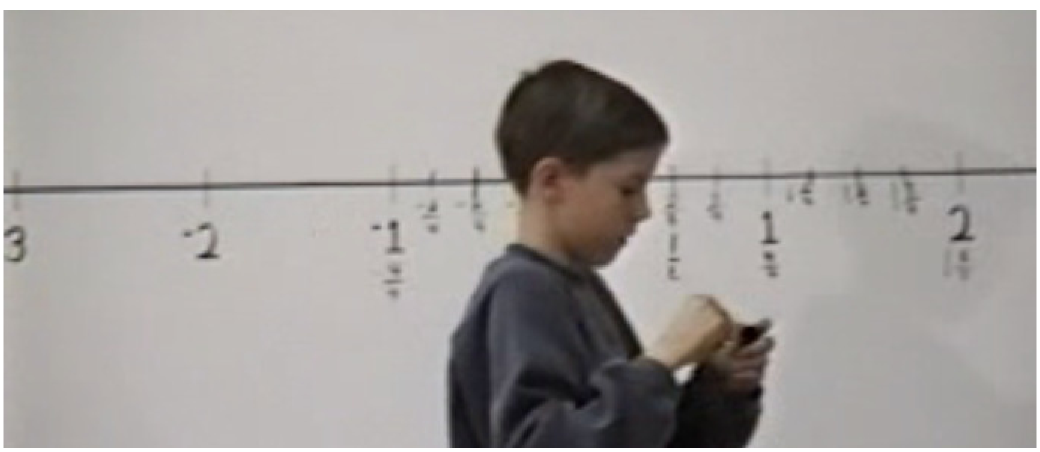

Figure 6.32 James places the number one half on the number line

6.6.6 Audra Changes Her Mind

RT1 called on Audra. She stated, "I agree" (313). Her discussion was inaudible. RT1 asked, "Am I hearing you changed you mind?" (314). Again, Audra response was inaudible. RT1 continued, "Audra is telling me that one half could go here [zero]. Can zero and one half be the same? How do you feel about that? Not sure? Want to think about that?"(316). Audra replied, "Well, one half could be anywhere between these numbers [between every whole number]"(317).

6.6.7 What is the Confusion?

RT1 asked, "Could someone tell me what the confusion is? There is clearly something that we are getting confused about. Andrew, and, then, James" (318).

6.6.7.1 Andrew's response. Andrew replied, "I think what we are getting confused about is the length of the number line. We have five positive numbers and four negative numbers counting the zero. They are not exactly the same so you would have to put one half a little more over to the positive side where two fourths is, now that we have three negative numbers and four positive numbers" (319).

RT1 asked, "How many agree that is what the confusion is?" (320). Some students mumble no.

6.6.7.2 James's response. James responded, "I think the confusion is one half is in the middle of two things. So, it is confusing to see in which place between three to four, two to three, one to two, zero to one, negative one to negative two. I think that is the confusion" (323).

6.6.8 There is one half between all numbers

RT1 pointed to the middle of the number line between two and three and asked, "Would you place the number one half here" (324). Some students responded yeah. RT1 asked, "We are talking about the number one half. I agree that you could find half the length between two and three, I agree, but could you place the number one half here?" (326). Meredith exclaimed, "It has to be two and one half"(327).

RT1 asked, “Is that allowed? Andrew?” (328). Andrew replied, ”It has to be two and one half”(329). RT1 continued, “You could put two and one half here, I could take this and split it in one half but I would not put one half here, I would put two and one half. Agree?” (330).

Brian stated, “You could put a one half between every number. Like [*gets up and runs up to board*]” (331). RT1 stated that they could hear him and motioned to Brian to sit down. Brian continued, “Between the zero and the one would be a one half. Between the one and the two would be one and one half. Between two and three would be two and one half. Between three and four would be three and one half and the same on the negative side” (333).

RT1 asked:

> How many agree with that? In other words, you are saying that the one half you are splitting that and then you know where to place those numbers. You were all telling me to place the number one half on the zero. I got very confused. What are they thinking, Alan? (334)

Alan commented, “They were thinking negative numbers are equal to positive numbers. That is probably what they are thinking. Because, you cannot add negative numbers and have them be positive numbers” (335). Brian added, “Audra was thinking all that is one whole and its not.” (336). Alan followed, “The negative numbers are lower than zero. Zero is said to be the lowest number and they are lower than the lowest” (337).

6.6.9 Meredith Describes a Ruler

RT1 stated that they had time for one more comment. Meredith held up a ruler and pointed to the tic marks and said, “It is like a ruler, here it has the inches—one half, one and one half, two and one half, three and one half, four and one half, five and one half; and, so on." (339).

RT1 asked, "Is it starting to make some sense?" (314). Some students responded affirmatively. RT1 commented that she was so glad Audra had placed one half where she did because it led to such a nice discussion.

RT1 asked the students if they wanted to leave the one half there or take it off. The students exclaimed, "Take it off!" (344). RT1 finished the class by stating that they would continue this on Friday.

6.6.10 Summary

RT1 commented that she wanted to try something else for the final fifteen minutes. When asked what they thought they were going to do, the students said fill in the number line on the board.

Audra was the first to place a number on the board. She placed the number one half under the number zero. Michael said no because zero was half way between the positive and negative numbers but that it was not half on the line. Alan stated that zero was half way between the three positive numbers and three negative numbers. Brian stated that half should be midway between the zero and the three because it was the positive side. A student off camera agreed with Audra. Another student off camera did not agree.

Graham stated that Audra probably thought the numbers before the zero were not negative. When asked by RT1 where Graham would place one half, he said he would put it under the zero if he did not know if it were negative. RT1 asked if the line were to go on forever, would it matter where the one half went.

David agreed with Audra as zero was half way between the integers in both directions. Erik agreed with Audra and David because he said that there is no way Michael's statement that one half would be on the positive side could be correct. Erik continued that zero was equal halves for both sides, right in the middle. Jessica walked up and pointed to the board and said that zero is really half way between both sides.

RT1 walked up and extended the number line so that the positive side included the number four without extending the negative side. She declared this the new number line. Some students exclaimed that zero was no longer half.

Alan walked up and stated that now there were four positive numbers and three negative numbers. RT1 asked if there were only four positive numbers? She asked if the number line ended at four; in which, many students replied no. RT1 asked where the line ended and some students replied no where.

RT1 asked where negative one half would go. Alan replied that negative numbers are different than positive numbers. Alan placed the negative one half mid way between negative one and negative two. Many students agreed.

RT1 asked the class where they would put three quarters. Meredith divided the interval between zero to one in fourths and placed three fourths at the third interval. Many students agreed.

RT1 asked again where one half should be placed, Amy stated that it could be kept at zero as long as a negative four was added to the line. David said that Audra was using the whole thing to determine where to place one half while Meredith was using the interval from zero to two.

RT1 asked Meredith where she would place negative one fourth, negative two fourths, negative three fourths and negative four fourths. Meredith walked up to the board and placed them appropriately between zero and negative one.

RT1 stated that she was still confused how zero could have another name one half. David said that it was possible before four was placed on the line because both sides were equivalent. RT1 asked why earlier four fourths, five fifths and one thousand thousandths could be another name for one; but, how could one half be another name for zero.

James decided one half would go mid way between zero and one. Audra agreed. She said she changed her mind.

RT1 asked the class what the confusion was in the discussion. Andrew stated that they were getting confused about the length of the number line. He continued that since there were more positive numbers than negative numbers, the half should be placed a little more over on the positive side where two fourths is placed.

James replied that one half is in the middle of multiple things. RT1 asked the students if they would place on half between two and three. Some student replied affirmatively. RT1 reiterated that she was talking about the number one half, not the length between the two numbers. Meredith exclaimed that it would have to be two and one half. Andrew agreed. Brian stated that half could go between every number.

Alan said that originally they were thinking negative numbers are equal to positive numbers. Brian added, that they were taking half of the whole line. Alan continued to stated that negative numbers were the lowest numbers.

Meredith held up a ruler and showed how the ruler was divided by inches as one half, one and one half, two and one half, three and one half, and so on.

RT1 added that she was glad Audra placed one half where she placed on half as it led to such a nice discussion. She finished the class by stating that they would continue the discussion on Friday.

CHAPTER 7 – RESULTS 11-12-1993

7.1 Introduction

In session 4, November 12, 1993, the students explored why the fraction with number name two fourths could also have number name one half. The students began by working briefly in groups and then discussing their solutions as a group. During the exploration, the students, in a natural way, successfully added fractions (with the same denominator).

7.1.1 Task Discussion

RT1 opened the session by and reminding the students that their parents would be visiting the following week. RT1 commented that she hoped the students would be teaching the class that day as her teaching assistants. RT1 stated that for the current session she hoped the class, would continue working on the number line and then plan for the visiting day with their parents (12).

7.1.2 Parent Visitation Discussion

RT1 continued to discuss the parents visit. She commented, "Maybe you will be teaching them some things that maybe they haven't had a chance to think about when they were in school because many of them didn't have the materials that you have when they were in the fourth grade" (12).

RT1 commented that there were two ideas that might be considered for the parent visit. One idea was to assign particular parents to a given student; the other, was to have the students teach a parent that was their own. RT1 added that she would like to meet with the parents separately at the beginning of the session. She commented that they could include the parent work on their college resumes and portfolios. She remarked that the students would be on television at some point and that the other schools on television would include one in a one room school house in South Carolina and one school in New Brunswick, New Jersey.

7.1.3 Previous Session Discussion: "We Thought a lot"

RT1 then asked the students if they remembered what they did during the last session. Brian exclaimed, "We thought a lot" (24). RT1 asked again. Many students captured in the camera view raised their hands indicating they agreed. A couple students replied, "We debated" (26). RT1 repeated, "You debated. Okay. You debated a lot. Can you think a minute what it was all about?" (27).

She continued, "Why don't you talk with your partner and see if you can recall what the issues were and how the issues came about for the last time we met" (29). The students then talked amongst themselves.

7.1.4 Summary

In this section, RT1 opened the session stating that she hoped to work a little bit on the number line and discuss the parents visit the following week for the last five or ten minutes. The class then discussed the parents visit briefly. The section ended with a brief discussion about the previous session. When RT1 asked the class what they had done during the previous session, a student replied that they had thought a lot. RT1 suggested that partners discuss briefly together what they had done during the previous session.

7.2 Previous Session Review

When the students had finished their discussion, they began to raise their hands. RT1 stated, "I thought that I would raise some other issues that might push some of you to think about a little bit so some of you on the issues on the debate so that we can temporarily come up with some kind of an agreement to move on"(31).

7.2.1 Is There a Biggest Number?

RT1 asked, "First of all, when we talk about something that goes on and on without end—like if you keep counting whole numbers and I say to you, is there a biggest whole number, what would you tell me?" (31). Many students in the class simultaneously responded, "no" (32). A student exclaimed, "there is no smallest and there is no biggest" (33). RT1 asked if that made sense to the class. Many students responded affirmatively.

RT1 asked, "If I said to you, 'if you gave me a whole number.' And, you say, 'I claim that's the biggest.' I could always say to you that, 'I can find a bigger one'" (36). Beth interjected, "Just add one" (37). RT1 repeated Beth's statement. She then gestured as seen in Figure 7.1 and asked, "And, I could always find one smaller by?" (38). Beth and some other students replied, "By subtracting one" (39). RT1 nodded her head affirmatively.

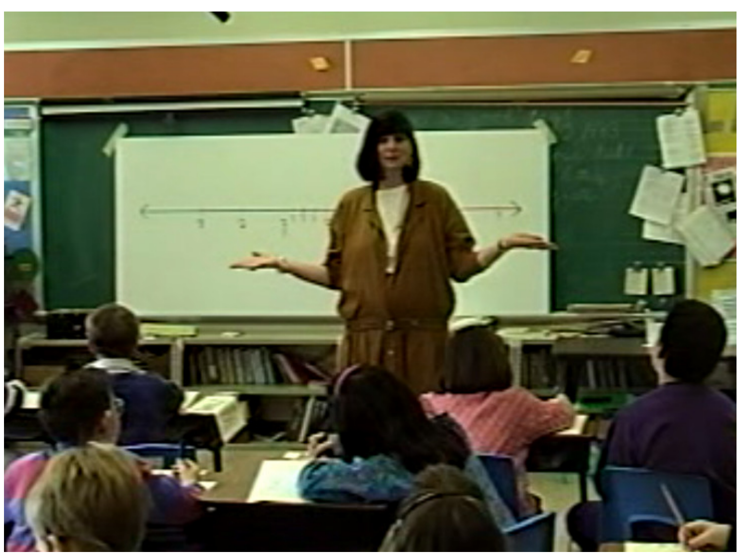

Figure 7.1 How do you find a smaller whole number?

7.2.2 It's Like a Traffic Intersection

RT1 stated:

> The issue of what happens when we talk about what we call sets of number that are infinitely large or infinitely small, the point is they go on and on without end. This notion of without end is a really difficult idea. Mathematicians work hard on ideas that they call infinity—they go on and on without end, OK? Now, if I said to you, imagine a ruler and I said to you find me the half way mark, you can do that. (40)

The students responded affirmatively. RT1 continued, "You can imagine bigger and bigger rulers and finding me that half way mark, right?" (42).

RT1 added, "It gets a little more complicated if my ruler never ever ends" (44). Erik interjected, "I know what the half way point of something that never ends—it would be zero" (45). Other students in the class exclaim "yeah" (46). Erik continued, "If they are equal on both sides it would be zero" (47).

RT1 stated:

> That is an interesting idea and I could understand why you might think that, but it is not quite that simple because mathematicians argue that if it goes on and on without end, no matter what point I take there are infinitely many numbers to the right and infinitely many numbers to the left so you cannot talk about half the same way when you are talking about infinitely many. (48)

Michael replied, "Yes, you can because zero is the starting point, basically. You start at zero and you can go that way [motions to the left side of the number line], or that way [motions to the right side of the number line]" (49).

RT1 rebuttaled, "Well, that is when it ends, Michael, but when it does not end it is much more complicated. You can still go both ways" (50).

Erik interjected:

> You can still go both ways but no matter what, if you start at zero—it's like an intersection—if you start at zero, and you go right you get all the positive numbers and if you go left you get all the negative numbers. So, it doesn't matter if every time, it is equal on both sides like if it is equal to negative five and positive five, then the half way point will be zero. No matter what. (51)

RT1 rebuttaled, "I'm not so sure it is quite that simple, Erik. I'm going to ask you a little bit about, if you are thinking that it ends, I would agree with you. If it doesn't end, it is a little more complicated. Let's hear from Alan and Brian a little bit" (52).

Off camera, Alan stated, "What I think is that when you have negative numbers and positive numbers—to the right are positive numbers and to the left are negative numbers—if you take the numbers zero to negative three or zero to positive three, then zero would not be the half way mark because the numbers to the left are not equal to the positive numbers" (53).

Erik contradicted, "Yes they are" (54). Brian agreed with Alan, "Yeah, Because zero to say positive five" (55). Erik interrupted, "Yes they are" (56). RT1 shushed the class and said "let's hear from Brian" (57).

Brian stated, "they are wholes. Zero to one would be a half mark. One to two would be a half mark. Two to three would be a half mark. Etc. Etc. But, even the same thing on the other side. But, if you are counting negative five to positive five as one whole, then zero would be that half mark. But, as I said before, they are different kinds of—zero to one is a whole, and one to two and so one—Those are all different kinds of wholes so you put a half mark between each one of them. That is what Erik said when we had another argument about this" (58).

RT1 commented, “For now, I don’t think we are going to resolve this now. But, for now, what I would like for us to at least deal with at the moment, is that I would agree with what you are saying if we are talking about something that ends” (59). Michael interrupted, “No. No. No” (60). RT1 stated:

> Let me finish. If it is something that ends I can talk half of it. If it doesn’t end taking half of something is much more complicated. Let it suffice for a moment that I can’t even imagine half of something that doesn’t end. Now these are very difficult ideas and we have to think about these a lot. But, for now if it ends, let’s say that I will agree with you and maybe we’ll have to discuss that at a later time when we bring in some more ideas. But, I want us to imagine this line is never ending. Okay, for now; and, I’m not taking half of that line [*points to the number line on the board as seen in Figure 7.2*]. (61)

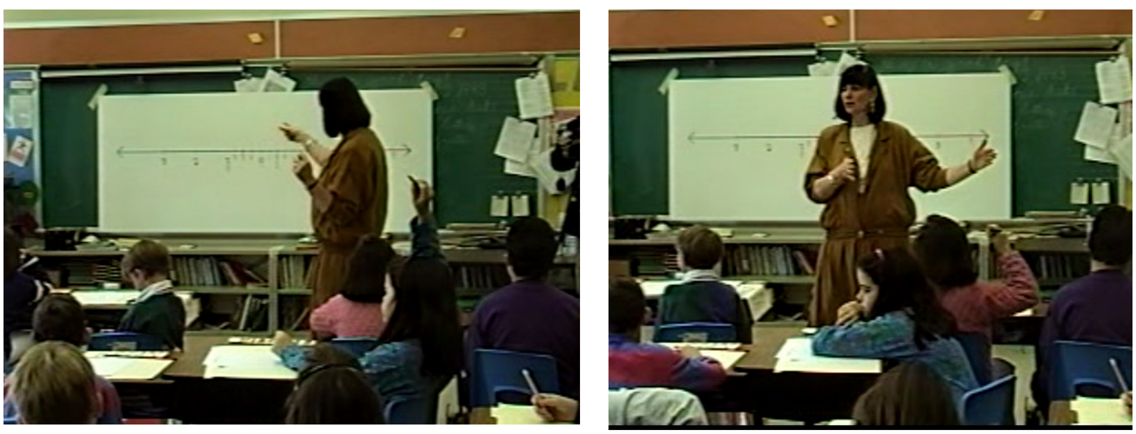

Figure 7.2 (left) Not talking about taking half that line
(right) I can shift the number line all over and start again

Michael stated, “I’m not saying it’s a half, I’m saying it’s a starting point for both sides of the line” (64). Another student off camera stated, “Exactly” (65). Another student exclaimed, “It’s a starting point” (66). RT1 followed, “Sure. I hear you” (67). Michael continued, “That’s what I’m saying” (68). A student off camera interrupted, “be quite” (69).

RT1 motioned from left to right as seen in Figure 7.2 and stated, “I would argue, too, that I can start that anyplace, Michael, I can shift it all over and start it. I can shift it all over and start” (70). Michael interrupted, “I know, but if you want to start at one” (71). Other conversations erupted.

Alan stated, “If you took zero to three on the positive side, then the zero to three on the negative side would be equal to the zero to three on the positive side. Negative numbers are basically lower than zero and zero is said to be the lowest number” (73).

7.2.3 Smaller and Bigger Numbers

RT1 continued:

> Okay. Let’s hold that discussion for now. Okay, that’s a very important discussion and I think you will be thinking about these ideas for maybe the next to twenty years because these ideas about infinity are really rather complicated and they require a lot of mathematics to begin to understand. But, I think those ideas you have are, are a good as a starting point to imagine that we know it goes on and on without end. We agreed upon that right. I agreed with you. I can start with zero and get bigger and bigger and never stop, right? (74)

The students affirmatively replied.

RT1 followed, “I can go to the left of zero and they get smaller and smaller and never stop, right? How many of you agree with that? That’s where we have agreement” (76). A camera captured three students raising their hands. RT1 replied, “How many of you are not sure about that?” (78). The camera view captured no students raising their hands as seen in Figure 7.3.

Figure 7.3 Students did not raise their hands indicating they agreed

RT1 asked, “Kelly, you’re not sure about that?” (80). Kelly replied that she agreed with the first question where RT1 asked how many the students agreed that the numbers to the left of zero get smaller and smaller—never stopping. RT1 repeated Kelly’s statement and asked Meredith if she agreed with the first one.

7.2.4 An Agreement

RT1 continued, “Now, then, I want to establish another thing that I think we al agree on. I can talk about half way of something if it ends. Right?” (82). Students murmured “yes” (83) off camera. RT1 continued, “Now some of you want to think about half of something that doesn’t end, but I’m not prepared to talk about that idea yet. Okay?” (84). Michael interjected, “It’s going to take weeks” (85). RT1 countered:

> No. It’s going to take many years and we will talk about it later. Those are important ideas and we will talk about them, but, not right now. For our purposes, right now, what we are interested, what we are not allowed to do for now is, if it’s okay with you, if we can agree on this. Sort of like a gentleman’s gentlewoman’s agreement, that we’re not ever allowed to give an argument here where we take a piece of this [gestures to number line on board as seen in Figure 7.4] and say that it ends. (86)

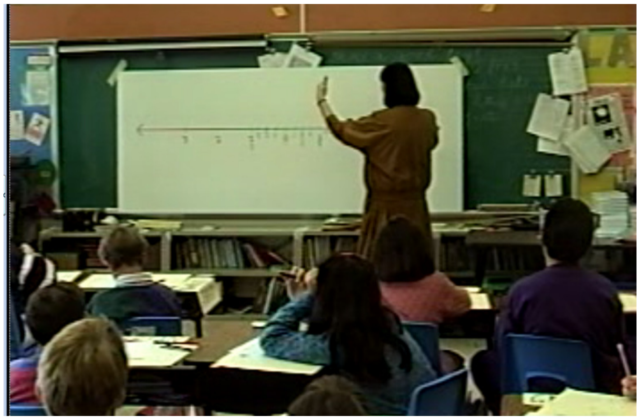

Figure 7.4 The students agreed that the number line never ends

RT1 continued, “for our number line we have to think of this as going on and no without end. Can we agree on that?” (87). Jessica replied affirmatively and said, “So, we’re not allowed to make a half” (88).

RT1 followed, “So, we have to place numbers on pieces of it, if you like but we can’t think of half of a line that never, never ends. Because, then I get into a contradiction and mathematicians don’t like contradictions. I don’t want to have another name for zero to be a half. That is a contradiction” (89).

7.2.5 Four Fourths is Another Name for One

RT1 gestured to the board as seen in Figure 7.5 and continued:

> You see I can only write numbers on my line where it really is another name for that number, you see. Now, some of you told me that four fourths is another name for one. I think it was Meredith. And, Meredith gave me a very good argument. She said, she convinced me, anyway, that if I went, if I thought between zero and one and started to put number names on this interval, I could give number names one fourth, two fourths, three fourths and four fourths. (89)

RT1 continued, “Which means then I would have another name four fourths that is another number name for one” (90).

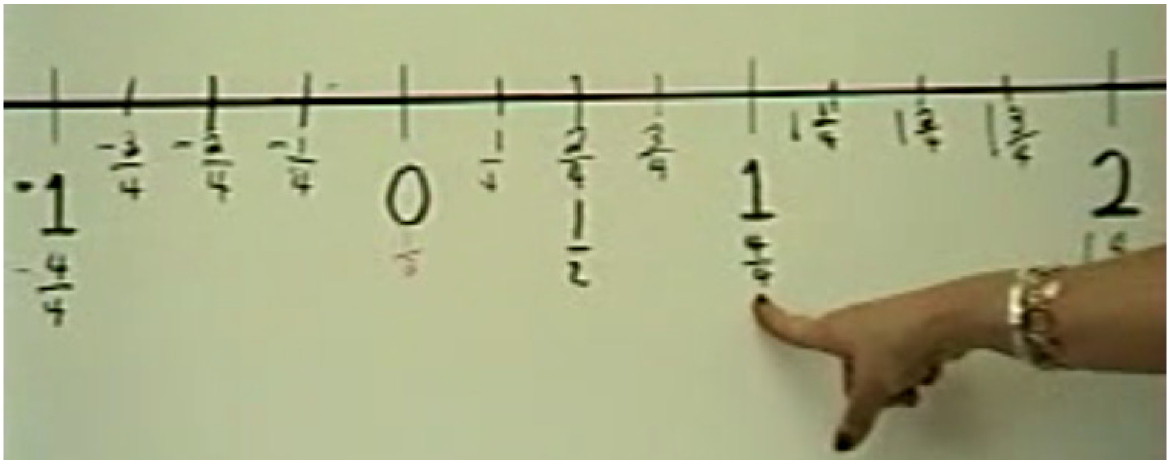

Figure 7.5 RT1 pointed to the interval between zero and one

7.2.6 Infinite Numbers between Zero and One

RT1 continued, “I’ll buy that. That makes a lot of sense to me. Now, Meredith did not have to do four fourths, two fourths, three fourths to get her four fourths or one. She might have found different fractions between zero and one, right?” (91)

Off camera some students exclaimed, “There are infinite numbers between zero and one” (92).

RT1 followed, “There are infinitely many numbers between zero and one, right! So, we shouldn’t run out of numbers, should we. We shouldn’t run out of ideas because what ever ideas we have when we start giving number names is very interesting” (93).

7.2.7 Summary

The students discussed what they had done during the previous session. RT1 stated that she wanted to raise some issues so that the class could come to a temporary agreement and move one.

First, RT1 asked if there is a biggest number. Many students in the class responded that there is not a biggest number on the number line. RT1 asked if there was a smallest number and many students responded that there is not a smallest number on the number line. RT1 asked how, if given a whole number, to find a bigger whole number? The students said to add one. She respectively asked how to find a smaller whole number in which the students replied to subtract one.

RT1 stated that the students could find the half way mark of a ruler. She asked if they could imagine a bigger and bigger ruler and finding a half way mark. RT1 continued by stating that it gets a little more complicated if the ruler never ever ends. Erik interjected that zero would be the half way point of something that never ended. The other students agreed. Erik added that zero would make both sides equal. RT1 said that Erik had an interesting idea and she could understand Erik's point of view; however, she stated that if there were infinitely many numbers then a person could not talk about half the same way as if there were a fixed number numbers on either side of zero.

Michael stated that zero is a starting point. RT1 responded that it is more complicated when the line does not end. Erik stated that zero would be like a traffic intersection where a person could either go left or right. RT1 countered that she did not think it was quite that simple. Alan stated that zero would not be half way where the positives are not equal to negatives. Erik countered that the positives were equal to the negatives.

Brian stated that both positive and negatives numbers are wholes where there would be half marks between each whole. Michael stated that he wasn't trying to say zero was half way, instead he was trying to say that it is a starting point for both sides of the number line. RT1 stated that she would argue that a person could start the line any place and shift it all over.

RT1 asked the class to hold the discussion on finding half way of a number line that never ended. She asked if they agreed that the numbers to the right get bigger and bigger and never stop. The students replied affirmatively. RT1 then asked the class if the left side gets smaller and smaller without ending. The students replied affirmatively.

RT1 asked the class to come to an agreement. She said that the class could talk about half way of something if it ends. She stated that it may take years to understand half way of something that never ends. RT1 asked for a gentleman's gentlewoman's agreement that the class will never give an argument where the class takes a piece of the number line and say that it ends. RT1 continued that for their number line they were to think about the line as going on and on without end; otherwise, they would get into a contradiction—which mathematicians don't like.

RT1 gestured to four fourths and said that some students had said another name for four fourths would be one. She voiced Meredith argument where Meredith had put number names on the interval as one fourth, two fourths, three fourths and four fourths. RT1 said that made a lot of sense to her and asked if Meredith could have used other fractions between zero and one. Some students replied that there are infinite numbers between zero and one. RT1 stated that they should not run out of numbers or ideas.

7.3 Why is it Okay to Call One Half, Two Fourths?

RT1 said that during the prior session Meredith "ended-up with another name for two fourths" (93). As seen in Figure 7.6, RT1 gestured to the tic mark labeled both two fourths and one half on the number line.

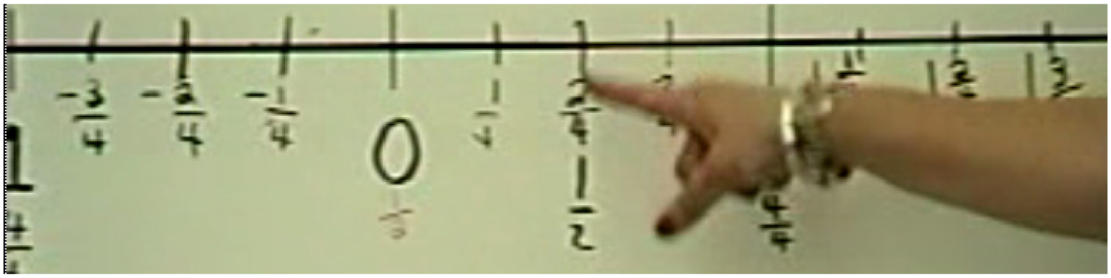

Figure 7.6 RT1 pointed to the interval between zero and one

RT1 asked:

> Can someone help me remember how she ended up with another name for two fourths to be one half? Can you help me remember that? In fact, it would be good to talk and be sure you agree, even though you and your partner have your hands up you may not agree with the reason, so I would like you to talk, because I know that sometime partners have different ideas. So talk to you partner. Why is it okay to give two fourths and one half the same number name. (94)

7.3.1. Students Worked Together

The camera roved as RT1 walked around the room talking with various students.

7.3.1.1 Group one discussion. Jessica stated, “I think because it’s between minus one, one and two” (96). Andrew asked, “What?” (97). Jessica continued, “Just look at the thing because one and two, half of, half of one and two” (98). Andrew replied, “one half is two fourths. Two fourths is half of one” (99). Jessica responded, “I know” (100).

Brian stated,

> Two fourths plus two fourths is equal to, two fourths is half of a whole. One half of something is one half. You can’t move the half mark over a little bit otherwise it won’t be half so like I said. You can’t move the half mark over. You can’t move the half mark over it just makes the half and two fourths. You can’t move the two fourths over a little bit. (101)

RT1 asked Meredith, “Does Brian agree with you? See if Andrew and Jessica agree with you?” (102). Jessica replied that she had not heard what Meredith had said. Meredith followed, “So, one fourth plus one fourth is one half. Two fourths equals one half” (104). Brain interjected, “No wait. Two, two-fourths plus two fourths equals one whole. And, one half plus one half equals one whole. They have to be in the same, same line” (105). Meredith continued, “one half of four is two because one half of four is two” (106). Brian stated, “Because, they’re in the same spot. You have to keep them there. You can’t move everything” (107). Andrew followed, “Just say I agree” (108).

7.3.1.2 Group two discussion. RT2 asked, “What do you two think? I’d like to hear your opinions on this. Laura, what about you? I hardly every get to hear from Laura. Not sure? David, what are you trying?” (111).

David replied, “I’m not really sure what to say” (112).

RT2 continued, “I think what Dr. Maher is asking is, the place on the number line that Meredith called two fourths, can it also have the number name one half? And, if so, why?” (113).

David replied, “Yeah, but in the other discussion I’m confused on what Erik and everything” (114).

RT2 followed:

> Oh, you’re still back on that discussion. Ok, maybe refocus and think about this question. It’s an important question she is asking. Do you kind of understand what she’s asking? Is there a way that would help you think about this? Um, what do you think? I see you’ve drawn a picture . (115)

David replied, “Well, four fourths equals one whole. Two halves equals one whole. And, two fourths equals one half because one half is half of one whole so two fourths is one half” (116).

RT2 replied, “Oh, I see what you’re saying. Lauren do you see what David’s done? You’re thinking about the Cuisenaire rods; am I right?” (117).

David replied, “It’s kind of like I just drew a whole” (118).

RT2 replied, “So, maybe we can think of this one as, is it possible, maybe we can think of this, the segment on the number line between zero and one, is that possible?” (119).

David replied, “Yeah. Ok. Zero and one” (120).

RT2 followed, “Okay. I can see that. Okay. So, we are pretending that this piece right here has a length of one? Going to call that one. Okay. Which of these is one half on David’s drawing?” (121). David marks on his paper off camera.

RT2 continued, “Okay, and what does that have to do with two fourths” (123). David replied, “Well, four fourths equals up to one and two halves equals up to one and so two fourths equals one half of that so a half has to be on the same spot” (124).

RT2 followed, “Okay. So, what you’re saying is that two of these fourths has the same length as one of these halves. Is that sort of what you are saying? I don’t want to put words in your mouth. Is that what you are saying? Do I understand you?” (125).

7.3.1.3 Group three discussion. Erik stated, “It’d be one whole of that part. Four fourths is one whole. Half of four fourths is two fourths. And, half of one whole is one half” (127). RT1 replied, “Okay” (128).

7.3.2 Group Discussion

RT1 said that she thought it was time to discuss what the students were thinking. She stated, “One more thing occurred to me that I think we maybe have to agree with or disagree with. Will there a place to put every number on that number line that is a fraction number or a whole number?” (133). The students replied no. RT1 asked, “Well, I mean theoretically?” (135). The class responded, “Well, theoretically, yes” (136). RT2 followed, “You may have to go underneath it to pull as many names. There should be some place to put it” (137). RT1 gestured with her hands. The students replied affirmatively. RT1 asked if everyone agreed.

7.3.3 Graham’s Argument

RT1 asked Graham if he wanted to tell the class his argument for why one half and two fourths should go in the same place. Off camera, Graham replied, “Well, one half plus two halves equals a whole, and two fourths and two fourths equals a whole” (141). RT1 repeated Grahams argument and asked the class if his argument made sense.

Erik responded, “I think they’re kind of off. It’s true, but they’re kind of off. It’s true that one half plus one half equals one whole, but two fourths plus two fourths equals four fourths which is one whole” (143).

As seen in Figure 7.7, RT1 wrote on the chalk board next to the white board the following: ½ + ½ = 1, 2/4 + 2/4 = 1, 2/4+2/4=4/4/=1. She asked if these equations made sense to the students. The student’s response was off camera. RT1 added, “That is very neat. I hope you notice, [RT5], that they’re adding fractions” (147). RT5 replied that she had noticed.

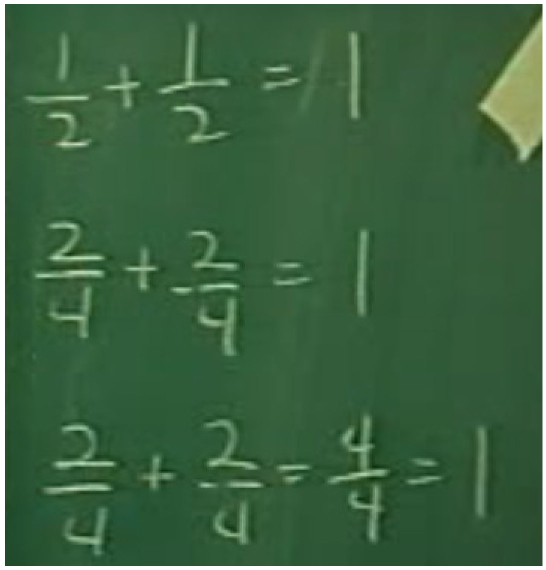

Figure 7.7 RT1 wrote David's statements on the chalk board

7.3.4 David's Argument

RT1 asked David what he had to say. David stated, "I was thinking. That like four fourths equals one half which equals two halves" (150). RT1 asked David to repeat his statement. David replied, "four fourths should be one half" (152). RT1 wrote his statement on the board.

Erik exclaimed, "Four fourths equal to one half? Four fourths? Two fourths" (153). David replied, "two fourths, oh, wait, one whole" (154). Erik replied, "four fourths is equal to one whole" (155). David stated, "Yeah, that is what I mean" (156).

RT replied, "You want me to change this four fourths equal to one whole" (158). David replied affirmatively and RT1 wrote on the board as seen in Figure 7.8.

David continued, "And, two fourths equal to one half" (158). RT1 asked why. David followed, "Two fourths would be equal to right up right next to, is like in the middle of like one whole" (160). RT1 asked, "In the middle between numbers?" (161). David replied, "zero and one" (162). RT1 restated David's answer.

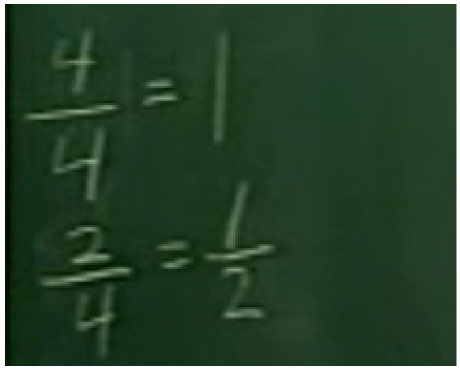

Figure 7.8 RT1 wrote David's statements on the chalk board

David continued, "so, then one half of it would be in the same place" (164). He followed, "because if you put them right next to each other the half would be in the same place as both of them because half would be equal on both sides" (169).

RT2 stated, "What might be helpful is if David drew a picture here which I found very helpful to me in understanding his argument and I think it might be hard for everyone out there to understand what David says" (169). RT1 asked David to draw it on the board. RT2 asked, "Would you please, David? I think that might help. Want to take your picture with you?" (171).

David sketched on the board as seen in Figure 7.9 (left). David said, "I just drew it like that because that's the one whole [*gestures to the top level*]. These are the one fourths [*gestured to the second level*]. And, that's the half [*gestures to the bottom level*]. This would be zero [*gestures to the left most vertical line*] and that would be one [*gestures to the right most vertical line*]" (174).

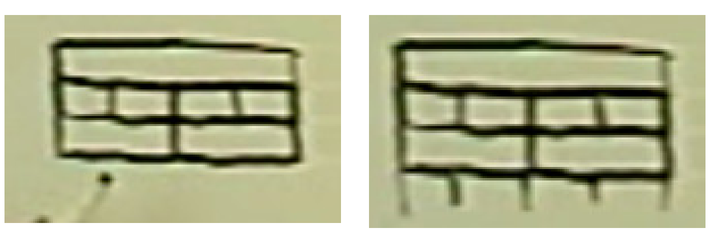

Figure 7.9 (left) David sketched his response on the dry erase board (right) RT1 extended David's model to include tic marks on the bottom

RT1 asked David to place the numbers on the line. RT1 drew in tic marks on the lowest horizontal line according to a number line as seen in Figure 7.9 (right). David asked if RT1 mean half. RT1 replied, “Where zero, go underneath like the number line” (181). David labeled the tic marks zero, one fourth, one half, three fourths, one on the number line as seen in Figure 7.10 (left). He then drew in two fourths as seen in Figure 7.10 (right).

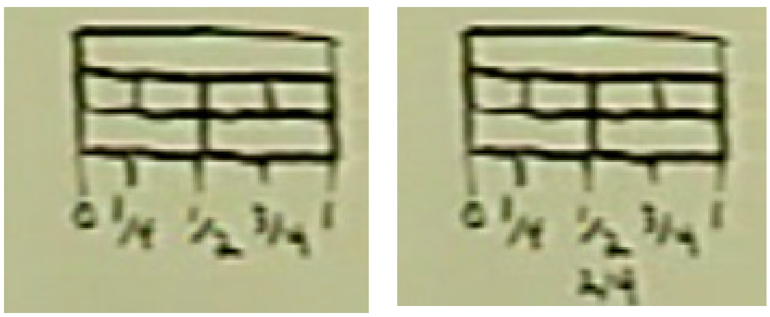

Figure 7.10 David sketched his response on the board

RT1 said: “What I’m imagining when you do that David, I’m imagining the rods and I’m also imagining the number line. That’s very helpful to me. Is that helpful to you what he’s done?” (186). The class murmured affirmatively. RT1 asked the students to raise their hands if they understood what David had done. The camera view captured five students raising their hand. RT1 added that if anyone did not understand that David would be happy to answer questions. No students were captured in the camera view raised their hand to ask David a question. There were no questions.

RT1 followed: “Now what David is suggesting which, I think helps me a lot, I don’t’ know if it helps you, that if you went to place numbers between zero and one, imagine the rods, right, helps you to place those numbers” (188). RT1 gestured to the interval between zero and one on the main number line. The class replied affirmatively.

RT1 gestured to David’s figure and continued:

> Now once you place the numbers and, then, once you imagine the rods it seems to be when the rods would end. Right, where the one-half rod ends, is where you would place the one half where it ended here [gestured to the right most column on David’s figure] he placed a one, right? That’s very nice notation. I like that a lot. What do the rest of you think of that? (190)

A student off camera exclaimed, I like it” (191). RT1 directed a question to Alan and asked, “What do you think?” (192). Alan replied, “I agree with him” (193). RT1 responded, “Isn’t that nice. That’s very nice. How many of you like that?” (194). The camera captured two students raising their hands.

RT1 continued to say that she liked David’s work a lot. She said, “Maybe we can adapt that as an interesting notation. If we were inventing our own notation that would be a very useful one” (196). She called on Jackie who responded: “It’s sort of like the Cuisenaire Rods” (197). RT1 replied, “yes, that helps me a lot doesn’t it?” (198).

Brian said, “It’s supposed to be the purple rod, one fourth would be the white and the half is like the red, or the purple I think” (199). Jessica stated, “I think it is sort of anew way to make a number line” (201). RT1 followed, “It’s a way to build it, isn’t it?” (202). Jessica replied affirmatively. RT1 called on Michael who said:

> When I was working at home trying to make a number line I found out that if you do like one whole divided by two, you would need would get like one half because you would take one plus –like that—and then you could get two fourths if you divided it by two so that would prove that two fourths and one half are the same. (206)

RT1 replied that that was nice and would prove it to her. She asked if what Brian had said would prove it to the others.

7.3.5 James Argument

James exclaimed that he had another way to prove it and that he had drawn a picture. RT1 asked him if he wanted to share his picture with the class. James walked up to the dry erase board and drew his picture as seen in Figure 7.11. James referred to the right circle and said, “this is one half and this is two fourths [*gestured to the left circle*]” (214). James continued, “For two fourths I made a circle and divided it into fourths and colored in two fourths and it shows it equals one half, one side of this [*gestured to the half the circle on the right*] and two sides of this both are one half [*gestured to the half of the circle on the left*]” (216).

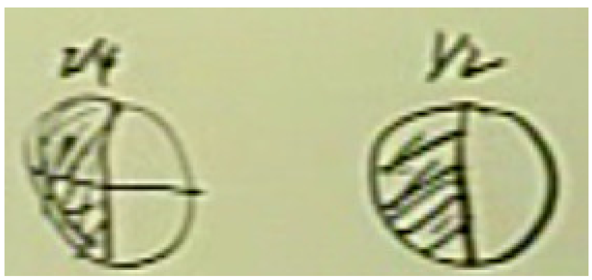

Figure 7.11 James’s pie charts

RT1 asked the class what they thought. The class agreed. RT1 asked how many students agreed with what James had drawn. The camera captured three students raising their hand. Jakki stated, “I agree with it because it is not complicated” (226).

7.3.6 Alan’s Argument

Alan stated, “I used a math problem to figure this out. What I did is I took two fourths and one plus four would equal five and one plus four would equal five. So you add the two together and you get ten. Half of that would be five. I’m saying zero to one is ten. That would be a half” (228). RT1 asked Alan to repeat what he had said; Alan asked if he could draw it. As seen in Figure 7.12, Alan draws his statement.

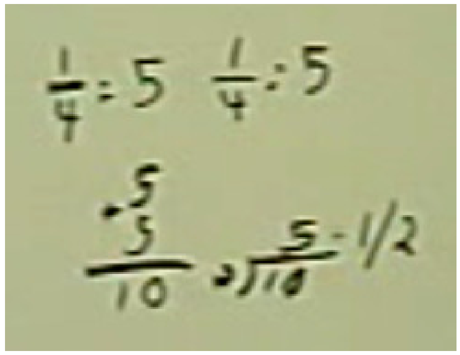

Figure 7.12 Alan's Argument

Michael exclaimed, "I get that" (241). RT1 stated that she did not get it and needed Michael to help her. Another student off camera stated they did not get it either. RT1 said, "You lose me when you say the one fourth equals five" (244).

Michael stated, "What he means is that one plus four equals five" (245). RT1 countered, "But he did not write that. I wish he would write that one fourth equal to five. Would you write what you mean? That confused me. Did that confuse you?" (246). Some students murmured affirmatively in the room. Erik stated that he did understand what Alan had done.

RT1 asked if they could write one plus four equaled five rather than leaving it one fourth equaled five. Michael changed what Alan had written as seen in Figure 7.13.

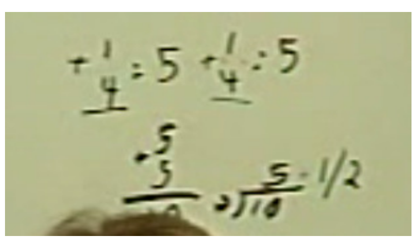

Figure 7.13 Michael Changed Alan's Work

Michael continued, "And, that is just the same as saying that. So, now, he takes two fives because one fourth plus one fourth equals two fourths. And, takes the two fourths and so he calls it five because both are equal to five" (252).

RT1 asked, "Wait two fourths equals five?" (253).

Michael responded, “No. Two fourths equals five because one fourth plus one fourth equals five and five which is ten” (254).

RT1 stated that she was confused.

Alan interjected, “What my problem means is one plus four equals five and one plus four equals five” (256). RT1 stated she understood that one plus for equaled five.

David asked, “Where are you getting one and four from?” (258). Michael stated, “Exactly. One fourth. One fourth!” (259). Meredith commented that “one fourth is a fraction” (261). A student off camera exclaimed “I know” (262). Alan interjected, “Exactly” (263). Several conversations simultaneously erupted.

RT1 asked David what he thought. David replied, “I think that they mean that here, alright one fourth equals five and one fourth equals five. And, take the two fives and put them down here as five plus five equals ten” (272).

Michael interjected, “Now, it’s ten divided by two because you have it takes two halves, fourths, two, two fourths equals a whole equal five with equals one half” (274).

Meredith questioned, “I think the question is where are you getting the one and the four?” (275). A few students make exclamations.

Alan explained, “Okay. Let me explain. What I am trying to say is that. Let me write it over. It will make this easier. What I’m saying is one fourth is equal to five” (280).

RT1 stated that there was where she had a problem. Meredith added, “one fourth does not equal one and four” (283). Andrew commented, “one fourth does not equal five” (284).

Brian commented:

> I know what he's trying to say. But, I know what he is trying to say. He's trying to say five is half of ten and take two fourths. Then two fourths is a half. And, that's why and five is a half. When he says one fourth equals one, five, he means one fourth equals five; but, you see what he's saying one fourth plus one fourth. (286)

Alan stated, "I'm saying that five is basically what I'm calling one fourth" (287).

Meredith added, "You cannot call one fourth equal five. And, five is a half. You cannot call one fourth—one and four" (288). Alan countered, "Well, that's the way I figured it out" (289).

RT1 added:

> Okay. I have an idea. We have to plan for next Friday. I'd like you to write this up for me, of what you think Alan is saying. Michael and what you think. And, Alan will write it up how about that? And, I'll tell you that we, we, it's ten o'clock and I have one last thing. Thank you gentleman. If you write this and we'll pick it up on Monday, okay, and look at this. I think there is some important ideas here I want to try to understand. Brian? (290)

Brian sated, "I have another way" (291).

RT1 replied:

> You have another way? Okay. You are going to write it up for me? What I'd like you all to do. I am going to ask you to do one writing assignment for me between now and next week. I would like you to be able to sketch a number line that goes on and on. So, that the little arrows at the end, what does that tell me? (292)

The students explained, "It goes on and on" (293).

RT1 followed:

> That it goes on and on. That I'm not using a ruler. I'm not cutting it. That means that there are infinitely many large numbers and small numbers. Right? You know? And, I

> want you to label it much like we have here. Maybe zero, one, two, three, as far as you want to go, okay? But, at least to three and negative three. Fair enough? And, I would like , you know, to place some numbers on that number line so I can see how you are thinking about that now because I think we keep changing the way we're thinking about it and I'd like you to put about, maybe, I don't know, put ten numbers on the line at least. That is not too many. If you want to put more that's fine. About ten numbers. But, when you put those ten numbers I would like you to find another number that has, that would be put in the same place as one half and two fourths" (294)

RT1 asked the students not to tell her the numbers they were thinking about.

Erik said, "No. So, what you are saying is if we came up with one number and then we came up with another that has its equal value we'd put it under it and that would count as two numbers" (297).

RT1 responded:

> We would put it under it and that would count as two numbers. But, I want one of the numbers you put to be. I want to see one half there. I want to see two fourths there but I want one another one. Okay. Do you understand what I am saying? Andrew? Okay, now, I would like you to give me a number that's, uh, past one. Place some number past one. And, I'd like you to place some numbers past one. And, I'd like you to place some number. Well, pass one but bigger than one and I would like you to give me a negative number there. I would like you to give me some variety. I don't't' want you to give me, for instance, ten names for one. You can do that in addition if you want. I said at least ten. But, of the ten you give me, I'd like to have a little variety. Some of them could be the same names, okay? I want to get a sense of what you are thinking about on this line. Fair enough? Is that okay? How many of you understand what I am asking you to do? (298)

RT1 added, "We have to decide what to do next Friday. We maybe have ten minutes to decide" (302).

Alan said that he could see a mistake he made up on the board. RT1 suggested that he tell the class quickly. Alan wrote two fourths equals five. Meredith exclaimed, “Then, that just eliminates your theory—two plus four equals six [$2 + 4 = 6$]” (310). Brian interjected, “Yeah but I know what he’s trying to say, two fourths equals one half” (312). Alan replied, “right” (313). Brian continued, “And, five is half of ten” (314).

RT1 asked the class to “hold on” (315). She continued, “Why don’t you write? Time out. If you want to say five is a half of ten is that the way you write it?” (315).

Meredith asked, “Where are you getting the five? Where are you getting the ten? Where are you getting the four?” (316). Many students erupted into discussion over Alan’s work.

RT1 asked, “Why not write five is a half of ten? Brian?” (318). Brian responded, “He just wants to show. I don’t know where he is getting five from but, at least, I just think that he just wants to show what another way one half is of a number. I mean he is just trying to say that five is a half of ten, and” (319). RT1 questioned, “How do we write five is half of ten?” (320). Brian answered, “Well, he wrote it” (321).

RT1 called on David who said, “Well, one half is like one whole” (323).

RT1 continued, “How would we write five is half of ten? We wouldn’t write five as five is half of ten. How do you think we would write five is a half of ten? Let’s here form some people. Graham, what do you think?” (324). Graham replied, “I think you would call it half” (325).

RT1 followed, “Alright. That’s a way. Right. But, he doesn’t want to write. He wants to write five is a half of ten using a five. By the way, is a fraction, we have a numbers on top and a number on the bottom. Isn't that right? Does anyone know the name for the number on top?" (326).

Erik exclaimed, "Oh, I know" (327). RT1 asked, "What?" (328). Erik answered, "The numerator" (329). RT1 repeated Erik statement. Erik continued, "I know what the bottom is" (331). RT1 asked, "Anyone know the name for the bottom?" (332). Multiple students exclaimed that that they knew. RT1 called, "Everybody?" (334). In which, several students exclaimed, "The denominator" (335).

RT1 stated:

> Denominator. Okay, when we write the number five, a whole number—that's also a fraction by the way. All our whole numbers, which mathematicians call fractions, we don't always write the number of the bottom. We just assume we know what it is. Do we know what it is when you don't write it? (336)

Some students in the class responded, "Zero and Ones" (337).

RT1 followed, "It's a one" (338). She continued:

> In other words, when we write a whole number it's five on the top and one on the bottom. The numerator is five. The denominator is a one. Isn't that interesting? So, in some ways, whole numbers are fractions, too, but, they're special fractions. They are fractions and all the denominators are one. And, we don't write it for short cuts because it gets to be tedious. Now, I'm asking you a question. If you're saying five parts out of ten, how would we write the fraction with five in the numerator? Meredith? (339)

Meredith replied, "Oh, wait, well you asked how would you say five is one half of ten? Five plus five equals ten" (340).

RT1 followed, "That's true; but, I want to write a fraction number. How would you write five parts out of ten as a fraction number?" (314).

Michael answered, “Five tenths” (342). As seen in Figure 7.13, Alan wrote on the dry erase board that two fourths equaled five tenths.

RT1 responded, “Yeah, Michael, five tenths. So, if you write five tenths, here, then you’re talking about five parts out of ten” (343). Brian added, “Now it’s getting clear” (344). RT1 called on Brian whom said, “Now it’s clear because at the first time I didn’t know what he was saying by five. I thought he was calling like zero to positive five. I thought that’s what he’s talking about” (346). Alan wrote on dry erase board as seen in Figure 7.14.

Michael added, “What I thought he was doing is—I thought he was forgetting about the fractions from the beginning” (347). RT1 followed, “He had it in his head” (348). Michael continued, “Adding it and, then, he was adding those and, then, he, um, took it and divided it, got a number, and, then, he took a fraction and said it could be called one half” (349).

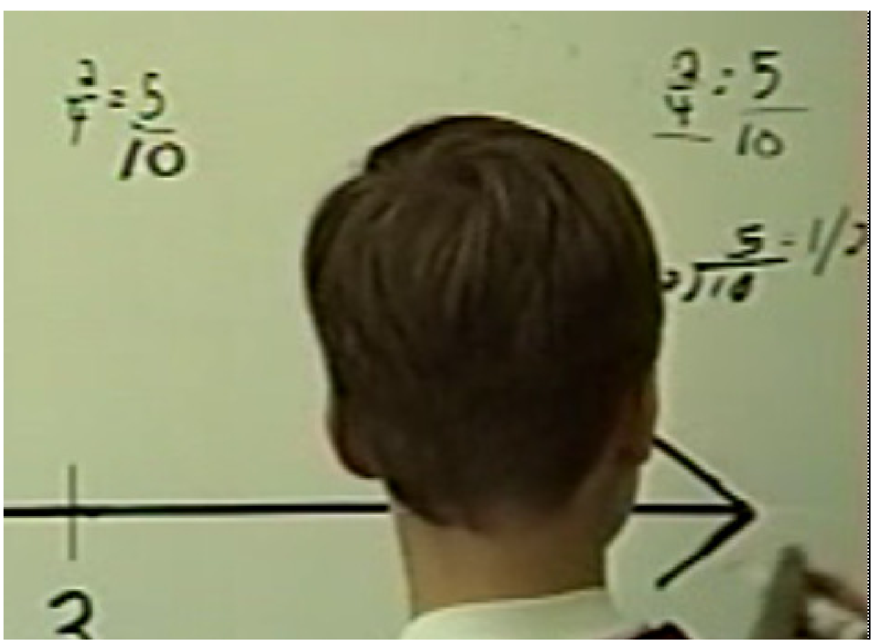

Figure 7.14 Alan wrote on the dry erase board

RT1 stated:

> Sometimes what happens which is really very interesting, sometimes we have ideas in our head, but we don't know how to write them. But, that doesn't mean that the idea is not in our head. And, sometimes we don't even know how to say them even though the ideas are in our heads and that sometimes we say it, it comes out sounding different then what we wanted to say. Does that ever happen to you? And, someone doesn't understand what you are saying. (350)

Michael said, “You can't explain it” (351). Other students exclaimed, “Yeah” (352). RT1 continued, “But, that doesn't mean the idea is not in your head and sometimes we have to learn the language and the notation to say those ideas. Meredith?” (353). Meredith stated, “What I think was confusing everybody was that one and four equaled five” (354).

RT1 followed:

> Yeah. I like that. That was good that he said it, but he had an idea in his head and that's why we have to listen to each other very patiently sometimes to try to figure out the ideas that we really are trying to communicate right even thought it doesn't always come out quite right. That's wonderful. Well, I know he is going to be working on this. [*Alan wrote five tenths plus five tenths equals ten or one wholes*] You don't mean ten to be your whole. Give me a, write this as a fractions. Ten is not a whole. I think we know what he means. What number name is a whole when he wants to have ten in the numerator? Graham?” (355)

Graham replied, “Ten tenths” (356).

RT1 added, “Ten tenths. Do you agree with that Alan?” (357). Alan answered affirmatively. He wrote ten tenths or one whole on the white board as seen in Figure 7.13.

RT1 stated, “Yeah. Very nice. You all see that ten tenths is another name for one isn’t it?” (358). Many students replied affirmatively. RT1 continued:

> Right. Ten is not a number name for one. Be careful. But, he was thinking ten tenths. So, remember we could write lots of names for one. You told me that before. I have an idea. All of you know how to tell me which fractions are bigger or smaller, right? (358)

The students answered affirmatively. RT1 continued, “And, by how much?” (358). The students responded yes. RT1 followed, “We could give you parents some problems like that. If they can’t do it you could help them.” (358).

RT1 finished the session by talking briefly about the parents visit the following week. Alan wrote on dry erase board as seen in Figure 7.15.

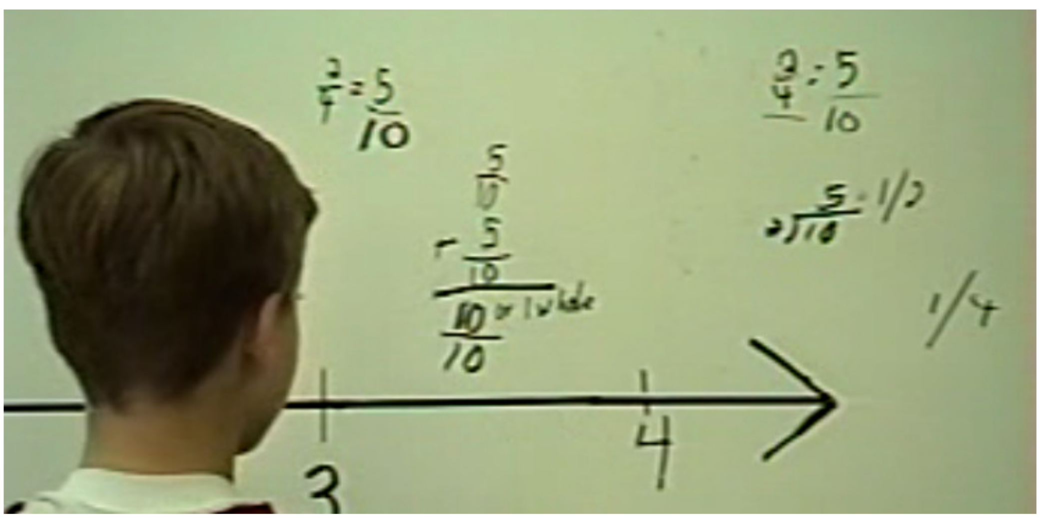

Figure 7.15 Alan’s Writing

7.3.7 Summary

RT1 asked the student why is it okay to call two fourths, one half. She asked the students to talk together in pairs. The camera roved around the room picking up parts of the students discussions.

Brian stated that two fourths plus two fourths is a whole so two fourths was half of a whole. He also stated that one half plus one half is a whole; therefore, they both must be the same.

The camera captured a discussion among David, Laura and RT2. RT2 asked about David's picture and asked him if he were thinking about the Cuisenaire rods. RT2 asked David if they could think of his picture on a segment of the number line between zero and one. David replied affirmatively. David later shared his picture with the class during the group discussion.

The camera captured a discussion between another group where Erik stated that half of four fourths, one whole, would be two fourths. He also stated that half of one whole is one half.

RT1 asked the class to move into a group discussion. She asked the class if there would be a place on the number line for every fraction and whole numbers. The students replied no. RT1 stated that she meant theoretically and the students replied theoretically yes. Additionally, RT1 said that the labels may need to go underneath current labels.

RT1 asked Graham what his argument would be for why two fourths could be labeled one half. He replied that one half plus two halves equals a whole and that two fourths plus two fourths would be a whole. Erik stated that what Graham had said was true; however, he thought the line was off. RT1 wrote Graham's statements on the chalk board. RT1 commented that the students were adding fractions.

RT1 asked David to share his argument. David replied that he was thinking four fourths equals one half which equals two halves. RT1 wrote David's statement on the board. Erik countered the statement by saying four fourths equal to one half. David exclaimed that what Erik had said is what he meant. RT1 fixed the statement on the board accordingly. RT2 suggested that David put his picture on the board; and, David drew a rod model where he gestured the top level to equal a whole, the second level to represent fourths and the third level halves. RT1 extended the model to a number line and asked David to label the tic marks at the lower portion of his drawing. He did so appropriately. RT1 asked the class if that helped them; they replied affirmatively.

James said that he had another way to prove one half was equal to two fourths. He drew two circles on the board: the left circle he divided into fourths and the right circle he broke into halves. He shaded two fourths accordingly on the left circle and shaded one half accordingly on the right circle. He said they were equal. When asked by RT1, the class replied that they agreed with what James had drawn.

Alan shared his transitive argument that two fourths equals five tenths; and, five tenths is half of a whole. Therefore, two fourths equals a half. The class debated the notation Alan used. RT1 asked class what they call the top number; Erik replied numerator. RT1 asked the students what they call the bottom number; many students replied denominator. Alan rewrote his idea using the correct notation. The class understood his argument. RT1 stated that sometimes we have ideas in our heads but we do not know how to write them. She commented that sometimes we do not even know how to say them. She added that sometime our words do not come out the way we mean them so our ideas sound different to listeners. She asked the students if this every happened to them; they replied affirmatively.

The session ended with RT1 discussing the parents visit the following week..

CHAPTER 8 – CONCLUSIONS

This study shows that young students can naturally build an understanding of rational number concepts before traditional instruction. Researchers including Erlwanger (1973), Freudenthal (1983), Maher, Davis and Alston (1993), Middleton et al. (1998), Moss (2005), Steencken (1998), Steencken and Maher (2002) and Streefland (1991) have documented the shortcomings of traditional instruction that focus on rules and algorithms for the learning of fractions and their operations. For example, Benny in (Erlwanger 1973) called mathematics a "wild goose chase;" he learned to write the *correct* answer one half on his paper when the teacher marked the answer two fourths as *incorrect* as the teacher used only the answer key to determine *correctness* of student work..

This dissertation traces and documents student understanding of rational numbers as they naturally move from placing fractions on a line segment (finite concept) to placing fractions on an infinite number line (infinite concept). The study is significant as it documents students naturally learning without having representations imposed on them. The evidence is supported by representations naturally used by students to express their ideas, explanations given by students and student justifications about their reasoning.

Freudenthal (1983), Alston, Davis, Maher, Martino (1994) and Davis and Maher (1993) explain the use of the term "of" to exemplify understanding of fraction as operator. The operator sense of fraction relates parts (or regions) to a concrete object (or sets). With time, fraction as number (Freudenthal 1983, Gattegno 1961/1963; Dienes 2001), an abstract notion (perhaps called *context-free*), develops as students abstract away concrete objects. Freudenthal states, "One badly needs the fraction as *number*, which for that matter may have arisen by applying a fraction *operator* to a unit. This means that in the *fraction operator* one mush distinguish the *operator* from the *fraction*" (1983 p.150).

The process where students build their own understanding of the two number notions is a complex process involving building personal mental representations (Davis 1984) which act as assimilation paradigms (Davis and Maher 1993) to help students grasp the new number notions. Many other factors are required including having adequate conditions for learning, including collaboration, challenging tasks, and opportunities to talk about and share ideas (Maher 1996). These conditions are considered important for students to build and understand fraction ideas and to extend these ideas to rational number.

Representations can take on many forms. Davis (1984) writes:

> Any mathematical concept, or technique, or strategy – or anything else mathematical that involves either information or some means of processing information – if it is to be present in my mind at all, it must be *represented* in some way. An exception may exist for some processing capabilities that are, as computer people say, 'hard wired in'. But such exceptions are surely the exception, and not the rule (p. 203).

External representations, one form of representations, can be found in the physical environment and can be used as mathematical learning tools (Goldin and Janvier 1998; Goldin 1998, 2003). Cramer and Henry (2002) showed that the use of manipulatives (external representations), in the form of rods, paper folding and chips, helped students developed an understanding of fraction size through the ability to order fractions contrasting a groups of students who did not receive the extensive work with manipulatives and did not display the same number sense of fractions.

This research is presented by examining the "student portraits" (Streefland 1991) of four focus students as follows: Alan, David, Jessica and Meredith. The students (with partners) were selected to include boys and girls whose participation in the classroom discourse was capture by the camera views. The communication and sharing of ideas during the four sessions in November 1993 for these students are presented in the setions that follow.

8.1 Alan's Portrait

8.1.1 Representations for Number Ideas

In Session 1, the class discussed extending a line segment projected on the overhead projector. The numbers zero, one and two were written on the line and RT1 added an arrow to the end of the positive side of the line. RT1 asked the class where they would put the whole numbers three, four, five and one thousand. In response to RT1 to the double questions as to where one thousand would be placed and if the placement would occur in the building, Alan verbally replied that it would be all the way to Pittsburg, Pennsylvania (281). In Session 2, Alan commented that he used the number one as the half mark between zero and two (174). When asked what he thought about negative numbers by RT1, Alan commented that "all the numbers to the left of zero would be in the negative" and "all numbers to the right of zero would be in the positive" (259). These episodes exemplify Alan using direction, quantity and geography as an external representation (Goldin 1998, 2003).

8.1.2 Representations for Fraction as Operator Moving to Fraction as Number

In Session 1, Alan described that one third could go in three places "because, if you have thirds you would be dividing [the line] into three parts so you could put it in three different places" (297). To support his description, he suggested to "bracket" (299) the thirds on the number line. Alan used a green rod to represent the whole and three smaller red rods each representing a third. RT1 used the green rod Alan selected and marked the numbers zero and one along the side; as well, she took the red rods and placed tic marks along the same line. Alan then correctly pointed to where the one thirds could be placed. This episode gives evidence that Alan has an understanding of fraction as operator as he divides the segment and states that one third can fit into any of three places.

After placing "the number one third" (307) on the number line, Alan returned to discussion that one third could be placed in any of three "spaces" (322). He used a marker to bracket the three spaces that a third could be placed. Alan stated, "Basically, what comes to mind when you think about fractions is that you cannot always think about the first one" (324). He gestured with a red rod and showed that the red rod could fit into any of the three spaces on the line. He said that the spaces would "still be one third; but, you could put one third, two thirds and three thirds" (328). Once again he moved the red rods along the line and showed that each space would be one third. When asked by RT1 if Alan way saying that the length of all the red rods happen to be one third he replied yes (337-340). This episode gives evidence that Alan has both an understanding of fraction as operator and fraction as number as Alan clearly correctly places "the number one third" (307); as well as, correctly stating that one third can fit into three separate regions within the segment..

In Session 2, in response to Erik's comment that he did not "understand how there can be infinitely many numbers between zero to one" (359) Alan stated, "You can divide that line into the smallest of fractions. You could divide it into zillionths. … You could divide it into zillionths and there would still be space in there" (366-372). This episode gives evidence that Alan is grasping the notion of fraction as operator, "dividing" a number line into the "smallest" of fractions. In addition, this episode shows evidence that Erik is in a state of disequilibrium as he is struggling with building a new notion of fraction as number, "infinitely many numbers" (G. Davis 1991?).

The students select their own meaningful external representations to express their mathematical ideas. Alan used a dust particle to represent his idea, "As I was saying before about the zillionths, you could have a line the size of a dust particle and you could put that on there a zillion times. You would have zillionths" (418).

Andrew introduced a microscope as an external representation that helped develop a standardized discourse where many students took part and continued to use the representation throughout the class. Alan commented, "If you did put it under a microscope it would look like you had enough room to put another zero to one in there. It would look like that. You could have it enlarged so that the line from the zero would be this [gestures with hands to a space approximately a foot] big and you still have room there to put more" (431).

Alan continued, "When you enlarge it you can see how much space you have left between the zillionth and the zero" (470). He, then, introduced another representation, "What I mean is, look if you had some really small pen you could draw a small line in the space you have because you really don't know how much space you have left between the zillionth and the zero. You don't really know that because you can't see it so you look at it under a microscope you could see how much space you have left" (479).

Alan consistently expresses his ideas on fraction as operator. He talked about how many thirds can be found in a whole, “Zero to the one third mark is one third. Zero to the two thirds mark is two thirds. Zero to the three thirds mark is three thirds. Now, three thirds. You cannot have any more thirds or you would have four thirds. Then you have to make the thirds bigger or not have another. You can only have three thirds” (341).

In Session 4, when discussing positive and negative numbers Alan stated, “If you took zero to three on the positive side, than the zero to three on the negative side would be equal to the zero to three on the positive side. Negative numbers are basically lower than zero and zero is said to be the lowest number” (70). In this statement, Alan then expanded his discussion from placing fractions on a small positive segment to a larger infinite number line.

8.1.3 Representations for Fraction as Number

Alan made many statements (following) that demonstrated his ability to negotiate between both fractions as number and fractions as operator. During these negotiations he is folding back to his fraction ideas knowledge base (Pirie and Kieren 1994; Martin 2008).

In Session 1, Alan presented a two number line representation where the first number line represented the segment from zero to one and the second number line represented an enlarged version of first number line. He placed one one-thousandth on the second (enlarged) number line because he said that one one-thousandth could not be seen on the first line (285). He wrote on the over head that even the enlarged portion of the number line was not large enough to draw in one ten thousandths on the line.

RT1 asked Alan where he would place the number one third on the number line (306). On the number line scored by thirds created earlier in the session, Alan said he would place "the number one third" at the first label (307).

In response to Andrew's comment that it did not make sense for the one third to be placed independently in the middle of the segment and still call it one third, Alan stated "Right, It's true. You can put one third in anyone of these places but basically what comes to mind once you think of fractions is that you always think of the first one. It could go in anyone of these" (336). This episode exemplifies Alan using another student's idea to monitor his own thinking.

RT1 asked if it would be okay to label each tic mark on the line one third, Alan replied, "No. You can put that in the beginning on the number line" (342). Alan stated that the length of each of three red rods was the same. Thus, their "fraction value" (343) was the same, one third.

RT1 asked how does a ruler get marked (350). Alan responded that a ruler "shows you how long something is" (351). He gestured to the red rods on the overhead projector. He said "The red [rod] is one inch. And, if you add another one inch on there then that would be two inches. And, If you add another inch on there it would be three inches" (351-354).

RT1 asked what "would I mark where the one inch ended?" (354). Alan replied that the number one would be marked at the end of the first rod, the number two would be marked at the end of the second rod and the number three would be marked at the end of the third rod (355). He stated that he placed the marking as he did because one rod would be one inch, two rods would be two inches and three rods would be three inches (358).

In Session 2, Alan walked up to the overhead projector to draw his ideas. A number line scored from zero to two was already on the projector. Alan drew a second, magnified, line below and placed one one-hundredths on the line. He then commented on the space between the zero and the one one-hundredth on the enlarged number line: “You would have all space in there. It looks like it; but, you really don’t have the much space. It’s just that if you had it really big [enlarged] that is how much space you would think you could see” (566). He continued to talk about the space between the zero and the one one-hundredth on the enlarged number line “you could divide this into halves and thirds and fourths and fifths and all of that” (575).

Alan continued to use the now standardized external representation, the microscope: “Say in the future that you come up with this really high powered microscope you could make that zero bar from the floor to the ceiling. That would maybe let you see it being that big. You could divide it up into such small pieces that when you took off the microscope you wouldn’t see anything. It would be so tiny and so small that you couldn’t see it; but, there really is space there. And, if you magnify those really tiny pieces you could divide those up into spaces” (619).

In Session 3, Alan used Meredith’s work to assess his own thinking. He responded to RT1’s question about which numbers represented the thirds region on Meredith’s (thirds) number line (as seen in Figure 6.2). Alan walked up to the overhead projector and pointed to the numbers above the line and says, “What I think Meredith was trying to do was… You see how it had one third? And, the one third being here [top of the line]. She was saying that she was labeling this [top of the line between one third and two thirds] to be the second third of the line and labeling this [top of the line between two thirds and three thirds] to be the third, third of the line" (180).

This following episodes show Alan reflecting on the ideas of Erik and Meredith to situate (assess) his own thinking supporting the notion that students build off other student ideas (Reynolds 2005; Steencken 2001). RT1 asked where would a person place three over three on the number line. Alan replied that if they did the way Erik is "talking about" (180) then all three pieces would be three thirds. (Erik earlier explained that the two thirds area would be written over the entire region between the zero to the two thirds (168). Erik's labeling differed from Meredith's as Meredith labeled the two thirds region between the one third and the two thirds.) RT1 asked again where the "number" three thirds would go on the line (192). Alan stated that "you would put it just in that big area" (195) referring to the region between the zero and the one. Erik interrupted, "No, you wouldn't. It would be right there [points to one]" (197). Alan continued, "Right; that would be the mark of the three thirds, but all three of those [points to the three regions between the number zero and the number one] are the three thirds" (198).

To clarify the two different ways of thinking about thirds on Meredith's line, Alan drew four new lines: a line scored by thirds, a line representing one third, a line representing two thirds and a line representing three thirds. He said, "Here is one way to do it. Now, here would be the one mark, two marks. Now, you could take out one of those pieces and say it is one third. And, then you could take out another piece this long and you could put two thirds. And, then you could use the entire number line and say it is three thirds" (258). RT1 asked Alan to place the fractions zero, one third, two thirds and three thirds on his first line. He labels it correctly (257).

In response to Audra's placement of one half on the class number line—under the zero, Alan stated, "All negative numbers are different than positive [numbers]. From here [zero] down you are negative so that means any number here cannot be equivalent to a number over there [positive side]. So, that means if you were dividing this part up into fractions you would have to put one half mark in the negative, right about there [points to area between negative one and negative two]" (578). RT1 commented, "So, you would put a negative one half between negative one and negative two?" (582).

8.2 David's Portrait

8.2.1 Representations for Number Ideas

In Session 1, David used direction as an external representation to standardize discourse with the researcher. He told RT1 that the number two would be placed on the number line to the right of the number one (272). RT1 asked where the number three would be placed (273); to which, David replied, "further over" (274). David discussed how a ruler worked—the numbers represent a cumulative integer count (360).

In Session 2, David referred to the external representation first used by Andrew. He described what a number line would look like under a microscope, "I think that you really can't see it too well. But, if you use a microscope then you are seeing closer. And, it looks like you are seeing more; but, you're really not. You're just looking closer than before" (504).

David revisited the microscope representation, "I think that you can take the little smallest thing; and, then put it under a microscope and you will have a lot more space;

but you don't. It looks like a lot more space, but it really isn't. You are just magnifying it" (553).

8.2.2 Representations for Fraction as Operator Moving to Fraction as Number

In Session 1, David responded to a question about which of four fractions (one half, one third, one fourth and one fifth) was bigger by gesturing to an imaginary number line with his hand (81). He motioned to midway on his number line and stated "if you have one half it cuts right there" (81). He then said that one third would "cut" (81) closer to zero than one half and that there would be "three pieces" (81). This episode exemplifies David's awareness that a number both "cuts" (fraction as number) the line and comprises "pieces" (fraction as operator) of the line. He appears to be negotiating between both fraction as operator and fraction as number.

David was asked to show his ideas on the overhead projector. David drew five levels of rods where the first level represented the one rod, the second level he labeled halves (two rods), the third level he labeled thirds (three rods), the fourth level he labeled fourths (four rods) and the fifth level he just drew one short rod which he labeled one fifth (84-98). In this episode, David used a familiar representation, rods, to assimilate notions of fraction of operator with the final "placement" of fraction as number.

8.2.3 Representations for Fraction as Number

In Session 1, David used a ruler representation to assimilate the placement of fractions on the number line (fraction as number). During the group work, David drew two separate number lines on his paper. On each he placed the fractions one half, one third, one fourth and one fifth between the interval zero and one. One line he labeled "no ruler" and the other line he labeled "ruler" (259).

In Session 2, David commented that he had placed one one-hundredth using a ruler. He stated that he used a “millimeter or something” (608). He continued “I had a ten inch number line; so, I just put it after one millimeter that was one one-hundredth” (608).

In response to a question posed by RT1 about where they put one and one half on the number line, David replied that he had put it midway between one and two (912) implying that he was folding back to fraction as operator to divide the line in half thus appropriately placing the fraction

In Session 3, David responded to where Audra placed the number one half on the number line—under the zero. David said that he agrees with Audra’s placement “because since it is integers it would go both ways. Zero is one half of the whole thing that keeps on going, because that is where you start. You can keep on going either way, but that is the middle” (534). David’s agreement with Audra’s placement of the number one half on the number labeled zero shows David’s understanding of fraction as number was not yet fully developed. As David says zero is “half of the whole thing” emphasizing fraction as operator.

He continued to talk about Audra’s placement of one half under the zero, “When you put one more number, four. Before it was from negative three to three so zero was one half. But, now, that you added the four to the positive side it is not one half. Both sides of the negative and positive are note equivalent” (626). This episode shows evidence that David is using Audra’s idea to situate (assess) his own thinking.

In Session 4, when discussing how two fourths can be another name for one half, David says to his partner, “Four fourths equals one whole. Two halves equal one whole.

Two fourths equals one half because one half is half of one whole so two fourths is one half" (111). He continued, "Two fourths equals one half of [a whole] so a half has to be on the same spot" (119). During the class discussion, David said restated his earlier discussion with his partner. RT1 wrote on the board with mathematical notation what David verbally stated. David clearly had developed the equivalence notion that one half was equal to two fourths. He was able to correctly add unit fractions as he stated that two halves were equal to one whole.

RT1 asked David to draw his picture on the dry erase board. David sketched rods three levels deep where the top level consisted of one rod, the second level consisted of four rods and the third level consisted of two rods (170). He then showed the class that two of the four rods equaled one of the half rods (172). RT1 asked David to place his numbers on the line. She drew in tic marks on the bottom of the rods, turning the rod model into a number line. David wrote in the following numbers left to right respectively: zero, one fourth, one half, three fourths and one. Under the one half David wrote two fourths. Jessica commented that David's model was "sort of a new way to make a number line" (196). This episode gives evidence that David is using Cuisenaire rods as an assimilation paradigm (Davis 1984) to assimilate the use of fraction as number with the use of fraction as operator. He uses rods (fraction as operator) to show the division of the whole into multiple regions, fourths and halves. David, then, used the regions to label the fraction number on the line correctly. He further showed that two fourths is equivalent to one half.

8.3 Jessica's Portrait

8.3.1 Representations for Fraction Ideas

In Session 2, Jessica is asked to point to where the number three would be placed on the number line at the overhead projector. Jessica used a ruler to measure the length between the integers one and two. She then used the same length to point to where the integer three would be placed (200). RT1 asked where the integer four would be placed (202). Again, Jessica uses the ruler to determine the length and points (205). As Jessica consistently measures the distance for the placement of whole numbers, Jessica demonstrating evidence that she has an understanding of unit iteration.

After Alan and Andrew introduced the microscope representation to the class discussion for examining small fractions on the number line, Jessica stated she agreed, "because you really can't see. And, if you put it under a microscope you could see spaces" (532). This episode gives evidence that Jessica is using another student's ideas for the placement of very small number on the number line to assess her own thinking. Additionally, she is using the classroom standardized external representation, the microscope to express her ideas.

8.3.2 Representations for Fraction as Operator Moving to Fraction as Number

In Session 1, When RT2 asked Jessica and Andrew why they had written one third on both sides of the line, Andrew described that you could look at the line from both directions. Jessica folded her paper in half (207) to show the "mirror image" (202). This episode gives evidence that Jessica (and Andrew) are still using the operator notion of fraction as they are labeling the regions rather than the numbers.

8.3.3 Representations for Fraction as Number

In Session 1, Jessica and Andrew worked together during the small group activities. Jessica stated that one tenth would "be on top of the zero more" (155). She then stated, "One hundredth would be close to zero. One thousandth would be right on the zero" (157). Jessica stated that for one ten-thousandth the line would have to be "bigger" (159); otherwise, everything would have to be "squish[ed]" all in (164). Although Jessica and Andrew had "mirror images" on their line, they correctly placed the very small fraction on the number line suggesting that there is evidence the are developing a sense for fraction as number.

In Session 3, Jessica commented on Meredith's work where Meredith had drawn five number lines representing different aspects of the top most number line, i.e. halves, thirds, fourths, and fifths. Jessica stated, "It is just easier to see when making the whole number line over. It is just easier to see" (111). Jessica is using Meredith's notions to assess her own thinking. Clearly she agrees with Meredith's multi-lined number line and acknowledges its usefulness to clarify for her the placement of fractions.

8.4 Meredith's Portrait

8.4.1 Representations for Fraction Ideas

In Session 1, Meredith verbally discussed an early solution to the problem of how much bigger one fourth was than one ninth. She used rods to support her solution (27). She stated, "if you put the blue [rod] which has nine ones in it; and, then, the four [purple rod] plus the five [yellow] rod then you have nine" (27). Meredith stated that that is "what I thought at first" (33). This episode show that Meredith had changed her mind from exploring a candy bar a task in October 1993.

8.4.2 Representations for Fraction as Operator Moving to Fraction as Number

In Session 1, when asked about fifths, Meredith stated that the "whole would be divided into fifths" (95). In this episode Meredith is dividing a "whole" to achieve fifths showing evidence of using fraction as operator.

In Session 2, Meredith talked about using the microscope as a tool to look at the number line. She stated, "If you look at it through the microscope then there is a lot of space; but, if you just look at it through the human eye then there isn't very much space in between [numbers]" (651). Meredith gives evidence in this episode that she, too, is using the standard classroom external representation, the microscope to build her own ideas.

8.4.3 Representations for Fraction as Number

In Session 2, in response to a question posed by RT1 as to where the number one and three-fourths would be placed, Meredith pointed to the space between the number one and the number one and one half and says, "it is equal to two fourths" (938). She then pointed to the space between the number one and one half and the number two and says, "you would have two more fourths" (940). RT1 placed on the number line the numbers given by Meredith. RT1 asked Meredith what number she would place on "another fourth" after one and one half (945). Meredith stated, "one and three fourths" (947). This episode exemplifies Meredith's understanding of equivalent numbers. She correctly places the number one and three fourths based on equivalent fractions.

In Session 3, Meredith portrayed strong and elegant evidence of her understanding of both fraction as operator and fraction as number. She presented five number lines to the class with the first number line showing the correct placement of fractions (fraction as number) and the remaining four number lines showing the correct regional divisions for the respective fraction (fraction as operator).

She presented her number line(s) to the class (as seen in Figure 6.2). She had five stacked number lines. From bottom to top she has placed the following numbers on the number line: fifths, fourths, thirds, halves, and all. Additionally, on each line, she has labeled the area between the numbers. Erik expressed confusion both as to why there were five separate lines, as well, as the meaning of the labels (attached to the regions) above the lines. For example, Erik referred to the numbers above the line and contrasted them with the numbers below the lines and said,:

> I see the half, but is not exactly the half way. Then, I think the three fifths are either too small or too big. The two fourths are fine. The halves are fine. It is the two thirds and three fifths. Why are you calling two thirds, one half? [referring to the two thirds above the line] Because, it is not one half. It is bigger than one half. Two thirds is bigger than one half. We did that once. (50)

Meredith explained:

> I know it. I just didn't know where to put it. [She walked up to the overhead.] Well, this is what the bottom is [pointed to the labels on the bottom]. This is one third. This is two thirds [pointed to the two thirds on the third number line]. [Pointed to the labels above the line] The area right here [pointed to the space between zero and one third on the third number line] is one third. The area right here [pointed to the space between one third and two thirds] is two thirds. The area right here [pointed to the space between two thirds and one] is three thirds. (54)

This episode displays evidence that Meredith is assessing and supporting her ideas as Erik asks questions.

RT1 asked Meredith a set of questions as to what she was trying to represent on the five different lines. RT1 states, “I kind of thought that was neat to show me all the different pieces of the line. Also, on the first number line Meredith also kind of showed me she knew where to put those numbers [one half, one third, one fourth and one fifth], right? The first number line tells me Meredith knew where to put them” (118). RT1 asks, “I got the feeling that the second [line] you were showing halves and the third number line what were you trying to show?” (127). Meredith replied, “thirds” (129). RT1 asked, “The fourth one?” (130). Meredith and the other students replied, “fourths” (131). RT1 asked, “the fifth one?” (132). Meredith and the other students replied, “fifths” (133). Meredith’s multiple number lines shows that she is aware of the distinction between fraction as number and fraction as operator, while she is aware that she may not always know how to represent her knowledge (e.g. “I know, I just didn’t know where to put it” (54))

On Alan’s line of thirds on the overhead projector, RT1 asked the students to place the number one. Meredith responded that the number one would go on the number three thirds (written earlier by Alan). Meredith said, “Three thirds is the same as saying one. Four fourths is the same as saying one. A hundred hundredths is the same thing as saying one” (326). Meredith’s statements shows strong evidence of Meredith’s understating of equivalence

Meredith described what would happen if you had more than three thirds indicating her knowledge of improper fractions, “You only have four thirds, if you are going to have that you could only have four fourths not four thirds. You cannot have four thirds” (348). RT1 commented that “you can have four thirds, just not in that interval”

(350). Meredith adds that then you would have to have “six thirds” (352). In this episode Meredith showed evidence that she was folding back to fraction as operator ideas to support the new placement of number on the number line (fraction as number) as she expressed that the improper fraction four thirds cannot be found within the current interval so the interval would need to be expanded to six thirds. In this segment, Meredith introduced a new fraction idea to the class.

As the class discussed the placement of four thirds, Meredith extended the thirds from one to two—one and one third, one and two thirds, and one and three thirds. She wrote one and three third under the number labeled two. In this episode Meredith shows strong evidence of her understanding of equivalence as she correctly shows that one and three thirds is equivalent to two. In this episode, Meredith extended the idea of mixed number to the class.

The class discussed the representation of a larger number line written on the dry erase board. Each student was asked to place a number on the number line. Audra placed the number one half under the zero. A class debate resulted. In response to where the number three fourths would be on the number line, Meredith correctly placed fourths between the zero to one (594). Meredith wrote the number four fourths under the number labeled one. This episode shows more evidence on Meredith’s growing understanding of equivalence.

After a brief discussion with the class on her placement of fourths, Meredith correctly placed negative fourths on the number line (622). The students exclaimed, “she is like cutting the number line” (623). Later James wrote the number one half under the number Meredith labeled two fourths (646).

In response to the students discussion that between any two integers is a half, RT1 asked what number name would they give the midway point between one and two (678). Meredith exclaimed that it needs to be “one and one half” (683). At the end of the session Meredith held up a ruler and said, “It’s like a ruler. Here, it has the inches one half, one and one half, two and one half, three and one half, four and one half, five and one half” (707). At the end of the third session, Meredith revisited the ruler representation which now was a standard external representation used by the class.

In Session 4, RT1 asked the students how during the previous session a student had given the number two fourths another name, one half. Meredith talked with her partner and said, “Two fourths plus two fourths equals one whole. Two fourths equals one half” (99). She continued, “One half of four is two” (101). This episode shows evidence of Meredith’s growing understanding of equivalence ideas; as well, giving evidence for the ability to add fractions.

8.5 Summary

The finding of this study suggest that at least the four focus students were all at various levels of fraction understanding by the end of the four sessions. Alan early grasped the difference between the notion of fraction as operator and fraction as number. Alan showed he understood fraction as operator in Session 1, as he described how one third could go in any of three places. In Session 1, he also showed he understood fraction as number as he placed “the number one third” (297) on the number line.

In Session 1, David folded back to Cuisenaire Rods to back up his placement of fractions on the number line. In Session 3, when Audra placed the number one half on the number zero, David agreed using the argument that “zero is one half of the whole thing that keeps on going" (534).David's remark in Session 3, is open for interpretation as to whether or not he is backing Audra's idea or if he, too, is using fraction as operator. In Session 4, when explaining how the number named two fourths could also be named one half, David used Cuisenaire Rods to justify his solution.

Jessica, in Session 1, exhibited an understanding of fraction as operator. She stated that she had placed one third on both sides of one half. She used the term "mirror image" to describe her line. In Session 3, when working in small groups she correctly placed the number one tenth, one one-hundredth and one one-thousandth explaining that the line should be "bigger" (159) or else everything would have to be "squish[ed]" together (164).

Meredith exhibited multiple notions of fraction ideas. In Session 1, when asked about fifths, Meredith stated that a "whole would be divided into fifths" (95). In Session 2, Meredith accurately placed the number one and three fourths between after explaining that there were four fourths in a whole. In Session 3, Meredith presented her number line to the class. She explained to the class that she had used different lines to present different fraction aspects (i.e. halves, thirds, fourths and fifths). On the top of her number lines she labeled the area (fraction as operator) and on the bottom of the line she labeled the fraction number (fraction as number). Many students in the class were confused by her notation. In Session 3, Meredith correctly placed negative fourths between zero and negative one on the number line. In Session 4, Meredith explains equivalent fraction ideas by explaining why one half can also be named two fourths. She stated that two fourths plus two fourths would equal a whole; thus, two fourths would equal a half (99).

At least two of the four focus students, Alan and Meredith, appeared to fluctuate (fold back) between the understanding of fraction as number and fraction as operator. These two students appeared to distinguish between the two fraction notions.

The students naturally used several representations for explaining their fraction ideas. Verbally they used representations consisting of geographic locations, small objects (including dust particles, dust bugs, and small pens), they used the cosmos (including the universe and stars) and they used tools (including microscopes, telescopes and magnifying glasses).

The students also used a variety of natural written representations to express their fraction ideas. They drew representations including the number line, Cuisenaire rods, pie charts, rulers and personal notations.

Tools were another form of representations used by students to express their fraction ideas. Students used models, number lines, Cuisenaire Rods, rulers and candy bars.

The year long Colts Neck research project gives evidence that the students thoughtfully engaged in discussions about fraction ideas. This study reports how some of the students built the ideas related to rational numbers. What is impressive in the work of the students is that this occurred without traditional instruction where students were forced to learn through teacher-driven representations. In a student-centered environment where students are given appropriate tasks, freedom to select personal representations, given time to explore and play with ideas and when communication is encouraged and respected, engaged students can build powerful mathematical ideas. This study provides detailed evidence about how connections were made between fraction ideas and traces how those ideas were extended to negative numbers.

8.6 Study Significance

This research is significant to document both for the fact that there is evidence of student's growing understanding of rational numbers without using traditional instructional methods and for examining representations that naturally resonate with students (rather than examining representations that are *imposed* on students) as they thoughtfully engage in rational number discussions. As described at the beginning of Chapter 8, everything mathematical can be thought of in terms of representations. As human understanding cannot be thought of as "snapshots" (Davis 1992), eluding to human mathematical understanding takes an indirect approach by examining thinking patterns and representations used by an individual. Thus, through studying representations, this dissertation gives evidence of students growing understanding of rational number ideas.

8.7 Limitations and Implications for Future Work

This study is limited in that it looked at four student portraits in detail. A future study may examine the work and reasoning of other students throughout the sessions. Additionally, only three cameras captured student discourse; thereby, limiting the audio and visual data when students worked with partners or in small groups.

Future work might examine the nature of researcher interventions, forms of reasoning, classroom dynamics and assessment mechanisms with respect to the student-centered approach to learning fraction ideas during these four sessions of the Colt's Neck study could be studied. Additionally, similar studies might include students within different age ranges, perhaps middle school, high school or even adult, college-aged students or teachers for whom fraction ideas are not well understood.

It might be useful to explore the nature of the researcher interventions that elicited among students some disequilibrium and conceptual conflict, as well as researcher moves that resulted in engagement of students. These might include, but are not limited to, an analysis of teacher questions, tasks and the moves that trigger student extension/generalization of ideas, as well as moves that encourage student justifications and connections. In earlier sessions Bulgar (2002) and Reynolds (2005) examined researcher moves along these lines. The videos could also be examined for research interventions that tended to result in specific student learning.

In earlier work, researchers examined the forms of reasoning used by students to justify their ideas. Mueller (2007), Mueller and Maher (2008) and Yankelewitz (2009) examine ways students build their personal understanding of number ideas. One might examine forms of reasoning identified in the earlier studies to include direct reasoning, reasoning by cases, reasoning using upper and lower bounds, and reasoning by contradiction. Future work might consider an examination of students' forms of reasoning in their placement of rational numbers on the number line.

The classroom dynamics that were observed in these sessions include student communication, group collaboration and mathematical-play. The centrality of classroom dynamics has been cited by Freudenthal (1983), Brousseau, Brousseau and Warfield (2004), Dienes (1963), Davis (1997), Maher (1996) Steencken (2001) and Francisco and Maher (2005). Conditions including giving students adequate time for students to adequately examine tasks and ideas, presenting opportunities for students to make connections and giving students time for mathematical play, are essential for student-centered learning.

In Streefland's book (1991), a teacher at the Fatima Jozef school asks his sixth grade class to solve a fraction problem (p. 306). After giving them the task he says:

> You won't get any more information. Think up one solution and then another way of approaching it. It's up to you whether you work with fractions, ratio or a drawing. The idea is: Next year, if the students don't understand my approach, then maybe I can use yours. That's why I need so many different solutions. (p.306).

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APPENDICES