# TRACING MILIN'S DEVELOPMENT OF INDUCTIVE REASONING: A CASE STUDY 

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ABSTRACT OF THE DISSERTATION<br>Tracing Milin's Development of Inductive Reasoning: A Case Study<br>By<br>MANJIT KAUR SRAN<br>Dissertation Director: Carolyn A. Maher

This study examined how Milin, a nine-year old student, justified his solutions to towers of a variety of heights over a 13 month period. Specifically, it sought to identify heuristics, strategies, and forms of reasoning and argumentation used by Milin in building and supporting his reasoning by partial cases, cases, and then an inductive argument. The research also traced how Milin's ideas traveled to other students. Videotape recordings of Milin's work on towers task and its extensions were analyzed along with his written work, written assessments and the researcher's field notes. The video data consisted of two problem-solving sessions, three individual task-based interviews, a small group assessment, and a whole class discussion.

As Milin searched for and sought to justify a global solution for the towers problem, he constructed mathematical ideas by continuously evolving the heuristics and strategies employed. He started by making random towers using a "guess and check" method, where he would randomly create a tower and then compared it with existing towers to identify duplicates. He then proceeded to use local organization strategies to create pairs of towers. This included opposites by color, opposites by inverting and a hybrid strategy. Later, Milin moved towards more refined local organizations such as staircase patterns. When these schemes also proved inadequate to justify a complete
solution, Milin developed a family strategy, based on a doubling pattern he had uncovered. This strategy gave him a global organization method. The progression to the global solution was an iterative process in which Milin revisited earlier strategies.

Milin also used various forms of reasoning to account for all towers. These included amount of time elapsed between building towers, the concept of "partner" towers, justification by contradiction, cases, doubling rule, and the family strategy.

Milin shared his inductive argument with three other students during a small group assessment session. Almost one year later, he re-explained his inductive argument to his partner, Michelle, while working on another task. In turn she shared this argument with other class mates, culminating with one student presenting it to the entire class. The students appeared to understand and retain Milin's strategy better when involved in solving the problem themselves.

This case study contributes the body of research in several ways. It documents strategies used by young students to build models of reasoning and argumentation. It also provides support for Davis and Maher's idea that building understanding is not a linear process in that new ideas are built from previous ideas. Finally, this study contributes to the broader collection of case studies from the longitudinal study at Rutgers University.

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## 1 INTRODUCTION

### 1.1 Statement of the Problem

The National Council for Teachers of Mathematics (NCTM) in its document, Principles and Standards for School Mathematics (2000), identifies five process standards, with one of them focusing on reasoning and proof. This document states:

Being able to reason is essential to understanding mathematics. By developing ideas, exploring phenomenon, justifying results, and using mathematical conjectures in all content areas and - with different expectations of sophistication - at all grade levels of rigor, students should see and expect that mathematics makes sense. (p. 56)

The NCTM Standards also state that mathematics curriculum should include ideas and experiences that provide students with numerous opportunities for development of mathematical reasoning and proof making. The Standards indicate: "Reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts" (NCTM, 2000, p. 56).

In the last twenty years, some theories about how mathematical understanding is acquired (Davis, 1984; Davis, 1992; Davis \& Maher, 1995) and grows (Kieren, 1990; Pirie \&Kieren, 1992; Pirie \& Kieren, 1994; Maher \& Davis, 1995) have shifted attention from rule-based learning to meaningful learning. The present research will focus on these two central perspectives about the growth of understanding.

Davis (1992) makes a distinction between the new views of mathematics education and the older rule-based views. In talking about the differences, he indicates that based on the older views, mathematics consisted of "symbols written on paper" and memorized facts, whereas the new way of thinking of mathematics involves "mental
representations." Davis further suggests that these mental representations are not words but rather images based on a verbal description. He states,

The mental representation of a 7-foot-tall man is not primarily about the numeral " 7 ", not about the three - letter word "man", but rather about a very tall male human being (maybe a basketball player?). (p. 227)

Davis refers to a learner's personal representational structures as assimilation paradigms. Davis and Maher (1992) suggest that when a learner is faced with new information it is viewed as "similar to" some existing experience and the learner uses this to accommodate the new knowledge.

How might meaningful learning be acquired? Davis (1997) indicates that there is a need for change from a teacher-centered learning environment to a more student-centered environment. According to Davis, in this new type of environment, students would be able to build up mathematical ideas themselves. Davis (1992) notes, "one gets a feeling of understanding when a new idea can be fitted into larger framework of previouslyassembled ideas" (p.228).

Pirie and Kieren (1992), in their model for understanding, identify eight potential levels: primitive knowing, image making, image having, property noticing, formalizing, observing, structuring, and finally inventising (p. 245). They also indicate that, "any level of understanding has embedded in it all other more inner levels of understanding and is itself embedded in all outer layers" (p. 248). They suggest that growth in understanding is represented by the back-and-forth movement among the levels. They call this movement "folding back". They state:

When faced with a question or circumstance at any level of understanding activity, which is not immediately resolvable, we argue that one can fold back to any inner level of understanding activity, in order to extend one's current, inadequate understanding. ...The new inner level behavior is
informed and shaped by both the outer level intervention which prompted the folding back and by the existing outer level understanding itself ( p . 248)

In addition, they discuss three types of interventions that can be used to trigger folding back, namely provocative, invocative, and validating. Validating interventions are used to get the student to explain their current thinking as opposed to provocative interventions that are designed to extend student's understanding, and invocative interventions that are designed to point out an "obstacle" and encourages folding back to inner layers (p. 248).

Maher and Martino (1999) assert that if students are provided with a supportive environment, they are more likely to see the value in discussing their ideas with other students. Yackel and Cobb (1996) suggest the notion of sociomathematical norms as "normative aspects of mathematics discussions specific to students’ mathematical activity" (p. 459). They further suggest that these norms are shared by a community of learners in that the learners understand what is expected of them during a mathematical discussion.

In their paper on teacher questioning, Maher and Martino (1999) also discuss the role of interventions of the teacher/researcher in promoting student activity in learning. They examine how timely questioning by a teacher helped students in third and fourth grade to build justifications to their solutions. They suggest teacher questioning can also encourage students to generalize their solutions, recognize the isomorphic structure of problems, and understand strategies offered by other students.

Francisco and Maher (2005), looking at conditions that promote reasoning in problem solving, examine the mathematical experience of a group of students in a longitudinal study. They stress the importance of emphasizing justification, rather than
formal proofs, in school mathematics. They found that by encouraging young students to offer justifications to convince others of their solutions, the students would engage in proof-like activities without having to struggle unnecessarily. Francisco and Maher (2005) indicate that even young students can engage in proof making. They further suggest that the students' engagement in proof making is "enhanced when the focus is on building convincing justifications for their mathematical claims" and not on writing formal proofs. They argue that when this approach is taken as opposed to trying to fit ideas into established way of writing proof, then "proving becomes an integral, not separate, part of the problem-solving process and promotes the building of personally meaningful arguments and ways of articulating them" (p. 368).

Ball and Bass (2003) suggest that mathematical reasoning is a basic mathematical skill that is essential for mathematical understanding. Maher and Davis (1995) indicate that young children can "propose conjectures, reflect on them, and try to make sense of their ideas while also trying to convince others of the reasonableness of their arguments" (p. 87). Maher and Davis' research shows that elementary school children are able to construct elegant arguments that have proof like structures.

The rule-based approach to teaching requires students to memorize rules and procedures. However, many students find these rules meaningless (Davis, 1994). Erlwanger (1973) gives an example of Benny, a twelve-year-old sixth grader who is using an early computer-based instruction system called Individually Prescribed Instruction (IPI). In Benny's eyes, mathematics is not a rational or logical subject and thus does not require one to reason, analyze, or generalize. He does not think there is a
need to verify answers because mathematics is like a game where you find rules to solve problems.

### 1.2 Background of Longitudinal Study

The present study is grounded in an extensive body of research at Rutgers University that has been conducted as part of a longitudinal study, now in its $22^{\text {nd }}$ year, of the development of mathematical ideas in learners. The purpose of the study was to explore, first, the development of students' mathematical ideas and, later, reasoning. The study took place in the public schools in Kenilworth, New Jersey, a working class community.

The first eight years of the research was conducted in classrooms at the Harding Elementary school. The students coming from the school were having difficulty with mathematics at the high school level. Students were being taught mathematics in half hour sessions using mostly rote methods. This resulted in a partnership with Rutgers University that began in 1984, first, for professional development and then to study the effect of intervention on student learning. For the first three years, this collaboration took the form of a teacher development project. Dr. Carolyn A. Maher, professor of mathematics education at Rutgers University and director of the longitudinal study, and her team of graduate students worked with teachers to help them build an understanding of the mathematics they were teaching their students.

In 1987, a formal longitudinal study was initiated by Dr. Maher. In 1992, when the students in the study were in fourth grade, the Rutgers research team received funding from the National Science Foundation. This research has been partially funded by NSF grant MDR 9053597, directed by Robert B. Davis and Carolyn A. Maher as Principal

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This research followed a cohort of students from first grade through twelfth grade and into adulthood. During the students' early school years, researchers visited classrooms, and during the later school years, after-school sessions were held to engage the students in rich, open-ended problem-solving tasks in various mathematical strands including but not limited to counting and combinatorics, algebra, and conditional probability. Care was taken by the researchers to design investigations that were not, at that time, a part of the school curriculum. The task design and the sequence of tasks introduced to the students were developed by the research team to provide meaningful opportunities for students to build mathematical ideas in ways that encouraged sensemaking and collaboration.

Students worked together in pairs or small groups so that individual thinking could be made public. For every task, researchers required that students offer convincing arguments to justify their solutions. Explanations of how a task was solved were communicated to researchers, usually in verbal form as well as in written work. Students were invited to share and discuss their solutions with other students using various methods including discussions within the group, discussions with other groups, and discussions involving the whole class.

Certain tasks, such as building towers of a specific height selecting from Unifix cubes of either two or three different colors, were revisited on multiple occasions, often with variations or extensions to the task that was originally given to students. In addition, tasks that appeared different on the surface but that had the same mathematical structure
were given to students so that they would have opportunities to discover the isomorphism and make connections among mathematical ideas. The requirement to convince others of the validity of solutions fostered the development of mathematical reasoning and proof making (Francisco \& Maher, 2005; Maher, 2002; 2005).

Throughout the study, researchers conducted interviews with one or more students at a time to enable deeper exploration of students' ideas and to challenge students to articulate their ideas more clearly through verbal, written, and gestural communication. The problem-solving sessions and interviews were videotaped. Multiple cameras were used in almost all cases to capture actions of different groups of students or complementary viewing angles of the same group with one camera capturing the students and the other capturing their work.

### 1.3 Purpose of the Study

This study traces the origin of Milin's inductive reasoning as well as how his idea traveled through the community of learners. Milin was part of the original cohort of students from Kenilworth that participated in the longitudinal study since the first grade. His family moved away after he completed the fifth grade. He returned to participate in an NSF sponsored summer institute in 1999. This research focuses on Milin's problem solving as a participant in the study during grades 4 and 5 while working on counting tasks. Specifically, it examines the development of Milin's understanding of inductive reasoning and it traces how Milin's ideas traveled to other learners. In particular, the study examines the heuristics, strategies, and justifications offered by Milin during his work with a partner, in individual interviews, and in small group work.

### 1.4 Research Questions

The following research questions guide the investigations:

1. What heuristics, strategies, and forms of reasoning and argumentation are used by Milin in his building of an inductive argument?
2. How do Milin's ideas spread to other students?

## 2 LITERATURE REVIEW/THEORETICAL FRAMEWORK

### 2.1 Introduction

This research is based on the view that when learners are presented with welldesigned and challenging problem-solving tasks in an environment that is supportive, they begin by building personal representations of components of the problem task (Davis \& Maher, 1990). Maher (1998b) indicates that as students build connections between and among the representational systems, their reasoning also develops. She also suggests that when students revisit, modify, and extend their earlier representations and are asked to justify their ideas, the cycling process helps in their development of arguments that are proof like. This literature review will first discuss important ideas related to views on understanding, representations, and reasoning and justification, and then discuss the research that has been carried out related to these areas in the context of elementary school mathematics.

### 2.2 Theoretical Framework

### 2.2.1 Mathematical Understanding

The NCTM (2000) in its Learning Principle states, "Learning with understanding is essential to enable students to solve the new kinds of problems they will inevitably face in the future." Davis has played a key role in emphasizing the importance in students' gaining mathematical understanding. Davis (1984), in his seminal book, Learning Mathematics: A Cognitive Science Approach, argues that "understanding" involves building schemes and fitting new ideas into existing schemes or building new schemes to
accommodate the new ideas. These schemes are personal representational systems that are built by the learner using various tools such as "spoken and written language, physical models, drawings and diagrams, and mathematical notation" (p. 88). According to Davis, understanding is achieved when a learner can fit new ideas into already formulated concepts. He also uses a metaphor of a "jigsaw" puzzle and states, "Each new candidate piece, like each new idea, can be used only if it fits into the aggregate of pieces that have previously been assembled" (p. 228). In order to work on a mathematical scenario, a person may go through a sequence of iterations. The learner first builds a personal representation of the data based on a search of the previous knowledge then the learner makes decides to accept, reject or modify the existing representations.

If, in the course of solving a mathematical problem, a person goes through these iterations and builds a satisfactory mapping needed to solve the problem at hand, the existing schemes are adequate. However, some problems require solutions for which the schema are not yet built or fully developed. In summary, as students modify and form new schema, in their efforts to make sense of new ideas, there is opportunity for their understanding to grow. When students are afforded chances to share and justify their solutions during classroom discussions, teachers are helped in assessing the ideas put forth by the students. The problem process involves the use of personally meaningfully representations that are internal and cannot be observed and external that is observed through what the learner says. These internal representations can only be inferred by analyzing the external representations and the actions performed by learners upon them. These representations can range from mental representations, to paper and pencil
representations, to concrete representations. According to Maher (1998), these "building blocks" come from an individual's experiences.

Davis (1980) conveys that students build knowledge and understanding by "building upon previous experience" indicating that earlier ideas serve as the foundation for later experiences, and that the earlier ideas can come from acting on concrete objects. Davis $(1984,1992)$ uses assimilation paradigms as metaphors for describing the process by which learners can build ideas through meaningful actions. An assimilation paradigm provides a way, according to Davis for students to create "frames" that make possible the connection of old knowledge already in existing schemas to new knowledge by appropriate modification of these schema. Davis (1992) states,

Students are determined to understand, and they create their own ways of understanding. What they learn thereafter is built upon this foundation of previously-built-up understanding (and future learning is therefore limited by the form of this previous understanding). (p. 226)

An activity called "Pebbles in a Bag" is an example offered by Davis and Maher as an assimilation paradigm for students learning how to operate with positive and negative integers (Davis \& Maher, 1997). Davis and Maher suggest the use of a "paradigm teaching strategy" that requires a teacher to provide carefully designed experiences that are similar in structure to the relevant mathematics. These experiences also referred to as "assimilation paradigms," serve as a conceptual framework for viewing the relevant mathematical topic. As indicated by Davis and Maher these experiences can then be used to build representations that are more abstract. Giordano (2008) gives another example of an assimilation paradigm used by Davis; it was the activity called "Guess My Rule". Giordano describes that during the activity, the students were expected to build their own understanding of the concept of function. This activity provided an experience that
helped the students to move on to explore multiple representations of functions including tables, equations, and graphs.

In the process of sharing and justifying solutions to problems during small group work, individual interviews, group interviews, or classroom discussions, the students' ideas are made public. This process helps the teacher/researchers to monitor the representations and strategies of their students. The teacher/researchers can explore further the developing ideas of the students by posing questions that seek explanation and justification of the developing ideas and ways of reasoning. This can help students build deeper and enduring mathematical understanding (Maher \& Martino, 1992a). The teacher/researcher's understanding of students' reasoning about a mathematical task is crucial, and this helps the teacher/researcher to guide students' developing mathematical understanding (Maher \& Davis, 1990).

Skemp's views on mathematical knowledge are especially applicable to this study. Skemp (1976) identifies two distinct types of understanding: instrumental and relational. He contends that developing an instrumental understanding is widespread; it serves as the basis for memorizing different rules and procedures for different types of problems. Relational understanding, on the other hand, involves student's acquisition of conceptual knowledge, which can be adapted, modified, and applied to new problems. This type of understanding helps students to apply generalized rules to specific cases. Skemp also suggests that relational understanding is more enduring in contrast to the short-term nature of instrumental understanding. Finally, according to Skemp, relational understanding serves as a motivational tool for students to explore new topics and extend their previous knowledge.

Skemp (1979) also identifies logical (or formal) understanding, which is another type of understanding based on justification. This type of understanding occurs when students use relational understanding and explain their reasoning to others in a learning community. Relational understanding helps learners to convince themselves, whereas logical understanding helps others to be able to use the mathematical ideas put before them. Skemp asserts that even very young children are capable of using logical understanding. Therefore, when students are encouraged to communicate their ideas to others in an encouraging and supportive environment they are able to use this logical understanding to convey their ideas.

Hiebert and Leferve (1986) also indicate the differences between conceptual and procedural knowledge. They convey that conceptual knowledge has to be linked to other information and is of use when acquired with personal meaning. They describe conceptual knowledge as "rich in relationships" (p. 3), and assert that it cannot be accumulated through rote methods, just as Davis (1984) conveys that understanding is built when students can fit new information into already existing information. In contrast to conceptual knowledge, Hiebert and Leferve describe procedural knowledge as structured knowledge that does not have be linked to other information much like instrumental understanding as described by Skemp (1976).

### 2.2.2 Reasoning and Justification

Justification and reasoning are important aspects of mathematics education. Mathematical reasoning is based on a learner's ability to construct solid and convincing arguments. The NCTM (2000) in its document, identifies reasoning and proof as a process standard. NCTM also state that mathematics curriculum should include ideas
and experiences that provide students with numerous chances for development of mathematical reasoning and proof making by the students. Further, the NCTM Standards indicate: "Reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts" (p. 56). According to Thompson (1996), mathematical reasoning can be described as "purposeful inference, deduction, induction, and association in the areas of quantity and structure" (p. 267).

Given the fact that understanding is crucial for reasoning in mathematics, it is important to look at different ways of thinking that demonstrate understanding. According to Rips (1994), reasoning is a "mental process that creates new ideas from old ones" (p. 10). Ball and Bass (2003) maintain that mathematical understanding is not possible without reasoning, because it would only be procedural knowledge. Ball and Bass also distinguish between sense making and reasoning in that they view sense making as an individual process and reasoning as a set of norms shared by the community in a given discipline. They suggest that reasoning is carried out when various ideas are examined in order to reach a common conclusion in a given community. They relate "reasoning of inquiry" to discovery learning, and "reasoning of justification" to situations that call for justifying and proving. Similarly, Polya (1954) distinguishes between "plausible reasoning" used for discovery and "demonstrative reasoning" for formal proofs.

Francisco and Maher (2005) emphasize the centrality of establishing a culture where students are invited to work on well-defined, open-ended mathematical investigations that call for justifying solutions to problems whose solutions are built by careful reasoning. These activities, they suggest, provide the foundation for building
durable knowledge. They also make a distinction between justification and proof, and state, "Justification refers to how students explain their mathematical actions and decisions. Proof is the formal and rigorous argument, which helps mathematicians explain their ideas" (p. 371). Francisco and Maher further assert that emphasizing justification, rather than "rigorous proof," helps promote reasoning in students without the possibility of their getting lost in having to write a formal proof. The authors of the study further suggest that "explanatory proof," rather than formal proofs should be emphasized to promote mathematical understanding.

Recently, there has been a call for curricular and teaching practice changes with respect to formal argumentation or proof. The position of proof has been significantly elevated in the recent national standards document (NCTM, 2000) as compared to the previous document (NCTM, 1989). In the 1989 document, proof was defined as "careful sequence of steps with each step following logically from an assumed or previously proved statements and from previous steps" (p. 144). In contrast the new document written eleven years later states,

By the end of secondary school, students should be able to understand and produce mathematical proofs-arguments consisting of logically rigorous deductions of conclusion from hypotheses- and should appreciate the value of arguments. (p. 56)

The nature of proof has undergone a great deal of rethinking in recent years. Balacheff (1991) also comments on the role of social aspect of proof and distinguishes between justification verses constructing rigorous mathematical proof. He indicates that proof is an argument that is accepted by the mathematical community (p. 178). Balacheff (1988), Hanna (1989), and DeVilliers (1990) describe proof as a way to communicate, explain, verify, and discover.

Yackel and Hanna (2003) also recognize the social aspect of reasoning, and describe reasoning as a communal activity where learners interact with others to solve mathematical problems (p.228). The authors also stress that even elementary school students can participate in inductive and deductive reasoning when provided with a supportive environment. They also mention that creating this type of environment, where justification and argumentation are integral part of the learning experience, takes a great deal of effort and time. Ball and Bass (2003) characterize the use of justification or proof as one of many "mathematical practices" that should be incorporated in the teaching of mathematics across grade levels. In summary, the perspective for this research is based on the view that the evidence for an individual's mathematical reasoning ability is based on his/her ability to build convincing arguments or justifications.

### 2.2.3 Representations

The NCTM (2000) asserts, "the term representation refers both to process and to product - in other words, to the act of capturing a mathematical concept or relationship in some form and the form itself" (p. 67). According to Janvier (1987), "A representation can be considered as a combination of three components: symbols (written), real objects, and mental images" (p. 68). Davis (1992), writing about mental representations indicates:
"Mathematics" is a way of thinking that involves mental representations of problem situations and of relevant knowledge, that involves dealing with these mental representations, and that involves heuristics. It may make use of written symbols (or even physical representations with manipulateable materials), but the real essence is something that takes place within the student's mind. (p. 226)

Representations are not meaningful by themselves, because they can take on various meanings based on the context. Goldin and Shteingold (2001) state:

A mathematical representation cannot be understood in isolation. A specific formula or equation, a concrete arrangement of base-ten blocks, or a particular graph in Cartesian coordinates makes sense only as a part of a wider system within which meanings and conventions have been established. (p.1)

Students' representations play an important role in their justifications of tasks that are mathematical in nature. Davis (1984) states, "Representations are fundamental to mathematical thought" (p. 78). He suggests that a person's ability to solve a problem depends upon the representations of the current problem and the representations of the past knowledge. Therefore examining representations built by students can provide a window of insight into the students' understanding through justification.

According to Davis and Maher (1996), there is a great need for a change in mathematical instruction from memorizing to building representations and models that
enhance students' thinking and helps to make sense of mathematics. Goldin (1998) proposes a "representational system" (p. 143) that includes internal and external representations. The internal representations correspond to mental images and the external representations correspond to written symbols. According to Davis (1984), when a learner looks at a problem, he/she has to first, build a representation of the problem situation, and then recall or construct a representation of the relevant knowledge needed to help solve the current problem. This step helps create a mapping between the two representations; this mapping is then checked for accuracy and can be modified as needed by cycling through the above steps.

Davis and Maher (1997) indicate that the representations built by students for a mathematical idea can take many different forms. These representations become increasingly sophisticated as students build new schemas. When students work with new problems, they construct more elaborate and sophisticated representations helping students to expand existing knowledge (Maher, Martino \& Alston, 1993).

### 2.3 Related Research

### 2.3.1 Mathematical Understanding

Both Davis (1984) and Skemp (1976) emphasize the importance of learning mathematics with conceptual understanding and distinguishing it from procedural learning. Alston and Maher (1989) and Davis and Maher (1990), in a case study about a student named Ling Chen, illustrate a distinction in the two types of understanding. Ling Chen, a student in an urban district, finished fifth grade and was interviewed during the summer when she was participating in a program for gifted students. This interview was
videotaped as she worked on the following problem: Karen had a whole candy bar. She gives $\frac{1}{2}$ to Kathy. She also gives $\frac{1}{3}$ to Paul. How much does she have left?


Figure 2-1. Ling Chen's Drawing.


Figure 2-2. Ling Chen's pattern block construction.

Ling Chen used pattern blocks (a new tool for her) (Figure 2-2) to build a representation of the problem (Figure 2-1). When asked by the interviewer whether she could do the problem with numbers, Ling Chen incorrectly elected to use the "invert and multiply" rule and she came up with an answer that differed from her representation. She was finally able to match her numeric answer to her representation by writing one-third divided by on-half equals one-third multiplied by one-half, "producing" her answer of one-sixth (Figure 2-3).

$$
\begin{aligned}
& \frac{1}{3} \div \frac{1}{2}=\frac{1}{3} \times \frac{2}{1}=\frac{2}{3} \\
& \frac{1}{2} \div \frac{1}{3}=\frac{1}{2} \times \frac{3}{1}=\frac{3}{2} \\
& \frac{1}{3} \div \frac{1}{2}=\frac{1}{3} \times \frac{1}{2}=\frac{1}{6}
\end{aligned}
$$

Figure 2-3. Ling Chen's three attempts to solve the problem.

The researchers concluded that while working on mathematical problems, learners should first build representations, which will help them to understand the appropriate algorithms for the given problem. Even though Ling Chen had memorized the algorithm for the division of fractions, she did not know how it related to the task. However, she was confident in her own justification based on her representation that she chose the correct representation of the solution.

### 2.3.2 Reasoning and Justification

Yackel and Cobb (1994) analyzed the nature of second grade children's arguments. They found that children used arguments in various capacities such as to specify their understanding of the problem, to explain their solution methods, restating answers from other students, to convince others of their solution, extending an existing argument, convincing others of an error, and finally to generalize (p. 8).

Yackel and Cobb, give an example of the students' use of argumentation for a problem in which an array of sixteen dots arranged in rows of four is flashed before the students using an overhead projector. Students, in their explanations, demonstrated that they had interpreted the problem correctly. Anita, a student from the class, offered the answer of sixteen and explained as she pointed to the rows that "16. I counted four and four right here and four right here and four right here (pointing to the rows)" and then gave the following explanation "I saw four 4's and all that added up was 16 cause the two fours and another two fours was 8 plus 8 . And 8 plus 8 is 16 ." This part of the argument is used to inform her classmates of her thinking and to clarify. The authors give other examples that illustrate different functions of argumentation and argue that in talking about the examples from the study the authors emphasize the need for
mathematical reasoning and the need for the students to share their ideas with others and make sense of the ideas offered by others (p. 20).

Lampert (1990), in analyzing a discussion in an elementary classroom, reported a session in which students investigated the pattern in the last digits of the squares of natural numbers (such as $1^{2}, 2^{2}, 3^{2}, 4^{2}$, and so on). After the students investigated the pattern: $1,4,9,6,5,6,9,4,1$ etc., the teacher asked the students to find the last digit in the pattern: $5^{4}, 6^{4}$, and $7^{4}$. Initially, the students gave a specific answer; later a student offered a generalization by arguing that any power of five would have five as the last digit. Next, the students thought about the pattern of the last digit of number 7 raised to increasing powers. By the end of the session, fourteen out of the eighteen students had contributed to the class discussion with their respective mathematical ideas and arguments. Lampert concluded that the classroom discussion modeled the doing of mathematics that Polya (1954) advocates, and closely followed the pattern of discourse of the discipline.

To illustrate the meaning of the reasoning and proof standard, NCTM refers to a longitudinal research case study by Martino and Maher (1996a) that traces the development of justification for an elementary school student. This example consists of a fifth grade student's (Stephanie) "proof by cases" for the solution of building all possible three-tall towers when selecting from two colors of Unifix cubes. In this case, the data was then organized into eleven critical events for the progression of Stephanie's justifications from grade 1 through 5. The heuristics used by Stephanie ranged from trial and error, pairing towers by "opposites" or "cousins," "upside-down and opposite," to "proof by contradiction," and finally using symbols in a grid to present her "proof by
cases" by controlling for variables. NCTM uses this example to show that when students are given supportive environments within a mathematical community of learners, as described by Maher and Martino, students are able to see that mathematics makes sense.

Francisco and Maher (2005), in their qualitative longitudinal/cross-sectional study, worked with students who were videotaped over a period of three years to eighteen years. These students were from three New Jersey school districts. Critical events were identified as "the student's different forms of mathematical reasoning and the research conditions associated with them" (p.363). The results for the study were based on a subset of four high school students, as they worked on, and then gave justification for, a probability task known as the "World Series Problem." The authors also talked about "the role of basic ideas, complex tasks, strands of problems, students' ownership of their mathematical activity, justification of ideas, and student collaborative work" (p. 371), as some of the key factors in promoting mathematical reasoning.

Mueller (2007) analyzed the various forms of reasoning used by two groups of sixth grade students from an urban district during an informal after school program. During five sessions, students were videotaped while working with Cuisenaire Rods. These sessions were transcribed and verified, and then coded according to the forms of reasoning used by the students. There were four types of reasoning that were flagged: direct reasoning, reasoning by cases, reasoning by contradiction, and finally reasoning by upper and lower bounds. The analysis showed that students were able to co-construct their arguments by questioning and extending each other's ideas.

The Rutgers longitudinal study has documented the development of proof-like arguments by young children. A brief account of the research conducted from this study is discussed below.

On February 6, 1992, the fourth grade students at Kenilworth site were asked to find all possible five cube tall towers when selecting from cubes of two colors. A case study of one of the students named Stephanie was conducted as she explored this and other related problems during second, third, and fourth grades. Initially Stephanie and her partner Dana began by using trial and error strategy to find different towers. Finally, Stephanie and her partner were able to find 32 towers using "upside down" and "opposite" patterns. During the class discussion that followed, the idea of "staircase pattern" was introduced. Following this, Stephanie was able to able to provide indirect reasoning as to why she had found all towers with one cube of one color and four cubes of the second color. She was also able to give reasoning for the number of towers where two cubes of one color were moving in a staircase pattern. Towards the end of the activity, she was beginning to think about exhaustive methods of finding towers five cubes tall when choosing from two colors (Martino, 1992).

Stephanie's work and her development of mathematical understanding and of mathematical proof during fourth grade and beyond have been documented in numerous studies. In Maher and Martino (1996b), Stephanie, in process of sharing her findings with the class, introduced a method of holding a color in one position constant while she changed the colors in the other positions to find various towers.

In addition to Stephanie, Martino (1992) also documented the reasoning of another student, Milin, during his work on the same problem. Milin and his partner

Michael also started with a trial and error method and eventually found all thirty-two towers by using "opposites." When the researcher asked them if they had found all towers, they replied that since they had not found any new towers in the last ten minutes, there could not be any more towers. On the next day during an individual interview, Milin too was able to conclude that there were only two possible one-color towers. He also used "staircase pattern" and by using contradiction, he was able to show that there could be only five towers with one red cube and four towers with two red cubes next to each other. In addition to this, Milin also concluded that the same would be true for towers with one yellow cube in a "staircase pattern" and two yellow cubes in a "staircase pattern."

Milin's problem solving is described elsewhere in Maher \& Martino (1996a) and Alston \& Maher (1993). During subsequent interviews, Milin proposed an inductive argument to generate all possible towers of a given height. Both Milin and Stephanie participated in a small group assessment session with two other students, known as the "Gang of Four." During this group assessment, Stephanie presented a proof by cases as justification for the problems; Milin presented an inductive argument (Maher, Sran, and Yankelewitz, in press).

### 2.3.3 Representations

Davis, Maher, and Martino (1992) analyze the variety of representations used by six students working on a combinatorial task called "Shirts and Pants":

Stephen has a white shirt, a blue shirt, and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make? (p.178)

A group of three students, Dana, Stephanie, and Michael, had a chance to work on the above problem when it was assigned in second grade. These students used a variety of representational strategies. Michael suggested that there were only two possible outfits because for him blue jeans go with blue shirt and white jeans only go with white shirt. Both Dana and Stephanie informed Michael that they had to find all possible outfits. Stephanie recorded her answers by using groups of two letters to represent different colors; the first letter stood for the color of the shirt and the second letter for the pants. Dana on the other hand drew pictures of three shirts and two pants each labeled with a letter to represent the color; and she drew connecting lines to find outfits. Dana did not include a line connecting a yellow shirt with white jeans because "... yellow can't go with white" (Davis, Maher, \& Martino, 1992, p. 181). Dana's representation of the problem was based on her personal experience.


Figure 2-4. Dana's, Stephanie's, and Michael's second grade work.

Even though all three students worked in a group, they all had slightly different approaches to the problem and thus different representations. None of the students came up with all six possibilities but they seemed to be comfortable with their own solutions (Figure 2-4). The students had an opportunity to revisit this problem five months later when the students were in third grade. This time they found all six possible outfits and
both Stephanie and Michael had incorporated the strategy of connecting lines used by Dana in second grade (Figure 2-5).


Figure 2-5. Dana's, Stephanie's, and Michael's third grade work.

The researchers talk about the growth of meaning and representations used by the students in third grade as compared to second grade. The researchers note that understanding students' "premathematical building blocks" is a crucial first step for an educator while tracing mathematical ideas built by students. In order to build "abstract ideas" students had to be able to build, rebuild, revisit, and talk about their representations with their peers over an extended period of time (p. 188).

The case study of Ling Chen discussed earlier in this review is a striking example of the need for creating representations of the problem situation (Alston \& Maher, 1989). Maher and Martino (1998) describe a case study in which they trace the mental representations of a fourth grade student named Brandon over a period of two years while working on the Tower Problem and the Pizza Problem from the combinatorics strand. Martino and Maher show how Brandon was able to discover the isomorphic nature of the two problems that on the surface appeared quite different. The students worked with partners and for each problem task, the students worked for about ninety minutes over two days. Brandon worked on the tower problem with one partner first, then four months
later he worked on the pizza problem with a new partner. During their work on the tower problem Brandon and his partner started with trial and error and later on they moved to strategies like "partner" and "opposite" and finally they used more sophisticated strategies of "upside-down" pairs to find all possible towers that were four cubes tall when selecting from two colors (Martino \& Maher, 1998 p. 77). Brandon and his partner were able to develop increasingly sophisticated strategies to find all sixteen possible towers. During his work on the Pizza problem Brandon created a table using " 1 " to show the presence of a topping and " 0 " to show the absence of a given topping. He used guess and check strategy to generate different pizzas. Brandon's partner for this problem also used similar method but he used check marks instead of numerical notation like Brandon. Later on in the session Brandon was able to group various pizzas into cases: pizzas with zero toppings, pizzas with one topping, and pizzas with two toppings. Brandon made a chart and used abbreviations to indicate the type of topping for each column. He again used a " 1 " to indicate the presence of a topping and " 0 " to indicate the absence of a topping (Figure 2-6).


Figure 2-6. Brandon's work.

Brandon systematically moved the " 1 's" to find all the possibilities (Figure 2-6). He then gave a justification for his solution to his partner with a proof by cases.

Finally, during his clinical interview on April 5, 1993, Brandon revisited the pizza problem and again used numerical sequences to represent solutions in groups by caseszero toppings, one topping, two toppings, three toppings, and finally four toppings. When he was asked if this problem reminded him of any other problem, Brandon brought up the tower problem. He used red and yellow cubes to rebuild all of the sixteen towers by using his organization by opposites. Later on, he regrouped his towers into three groups. When the interviewer asked Brandon to focus on a single color, he changed his organization into five groups using the yellow cube to make groups with zero yellow cubes, one yellow cube, two yellow cubes, three yellow cubes, and four yellow cubes. In response to the interviewer call for clarification, Brandon mapped each tower to a corresponding pizza in his chart. He explained that a yellow cube is like a one and the red cube is like a zero so that " 1111 " in the pizza chart is the same as an all yellow tower or a pizza with all four toppings. Martino and Maher (1998) contend that Brandon was able to build the isomorphism because of the environment he was working in and included building representations and modifying them, having an extended time to work on problems, and collaborating with others. The authors also note that this process of revisiting problems and teacher questioning was crucial to building justifications that resemble proofs and helped Brandon see the isomorphism in two seemingly unrelated problems.

The use of representations by students to interpret, build, justify and share their mathematical ideas have been documented in numerous reports from the longitudinal study In summary, students' representations provide an important way to evaluate their
mathematical understanding. Representations also help students to communicate their reasoning to other students and their teachers.

## 3 METHODOLOGY

### 3.1 Setting

This study is situated in the ongoing longitudinal study on children's mathematical thinking in the Harding Elementary School in Kenilworth New Jersey. It is a part of the larger study conducted at the above location while the students were in fourth and fifth grade during 1992 and 1993. This study analyzes video data from six days ranging from February 6, 1992 through February 26, 1993. In addition to the video data, two written assessments, one given near the end of the fourth grade and the second given in the beginning of the fifth grade, are analyzed.

In the longitudinal study, the students were given well-designed open-ended tasks to work on and they had an opportunity to revisit problems later; they were asked to justify their solutions. They were not told if their answers were correct. Rather, they were asked to convince themselves and each other of the correctness of their solutions. The students worked in pairs or small groups; they were encouraged to share their ideas with others in small and whole group discussions. The requirement to justify the solutions in the classroom helped students in the development of mathematical reasoning and building proofs (Francisco \& Maher, 2005; Maher, 2002; 2005).

The research design for the longitudinal study called for minimal researcher intervention, never "telling" or "showing" but sometimes asking questions in order to get clarification, to get students to elaborate on ideas and to get students to extend their current thinking particularly when a student felt like he/she was "stuck." Martino and Maher (1999) convey that questioning helps a teacher/researcher to monitor the present
thinking of a student. These questions can encourage a student to explain his/her thinking while working on the problem.

### 3.2 Data Sources

The data for the present study comes from three main sources. The first is the database of video recordings of every session and interview. One to three cameras were used to capture the data. Sometimes a video captured the students, their expressions, and another captured their work. The video recordings serve as the primary source of data for this study. Video data provides an opportunity for in-depth study of the student activity because it lends itself to multiple viewings during analysis. In addition, video data was used to inform the present study through screen-shots to capture students' actions and representations. This step allowed for in-depth study of the representations used by the students during the problem solving tasks. This allowed the researcher to use these in conjunction with the transcripts to analyze students' work effectively.

The second data source is the written work produced by the students during the problem solving sessions, individual interviews, and small group interview. Homework, if assigned during an interview, was also collected for analysis. These documents are directly related to the researcher interventions and play an important role in the analysis. In this study, this data is be used to supplement the video data in order to construct a thorough analysis. This data source is also invaluable because it contains recordings of students' representations, models, and justifications during in-class work and outside work, on researcher posed problems related to the intervention. This added another dimension to the data being analyzed and helped the researcher to construct complete
storyline documenting the reasoning and mathematical understanding used by the students.

A third data source consists of the field notes recorded by the researchers present at the problem solving sessions. These notes were made throughout the progression of the session. The field notes act as a substitute for being present at the session during taping.

Multiple data sources allow for triangulation and helps ensure the validity of data collection. This study analyzes the data from the following sessions and written assessments (Table 3-1).

| "Tower" Activities Timeline |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Activity | Task | Grade | Type of work | Date | Length |
| Classroom Lesson Episode 1 | Build five cubes tall towers and justify your solution | Grade 4 | Partnered With Michael work | 2/6/92 | 40 min . |
| Classroom <br> Discussion <br> Episode 2 | Build five cubes tall towers and justify your solution | Grade 4 | Whole class | 2/6/92 | 50 min |
| First Interview Episode 3 | Reconstruction of solution | Grade 4 | Individual Interview | 2/7/92 | 54 min . |
| Second Interview Episode 4 | Extension problem with three colors | Grade 4 | Individual Interview | 2/21/92 | 33 min . |
| Third Interview Episode 5 | "Families" of towers from one tall to five tall | Grade 4 | Individual Interview | 3/6/92 | 30 min . |
| Group interview Episode 6 | Justifications for towers three tall with Milin, Stephanie, Michelle and Jeff | Grade 4 | Small Group Interview | 3/10/92 | 45 min . |
| Written assessment | Towers |  | Written work | $\begin{aligned} & \text { May } \\ & 1992 \end{aligned}$ |  |
| Written assessment | Towers |  | Written work | October 1992 |  |
| Guess My Towers Episode 7 | Probability space | Grade 5 | Partnered with Michelle | 2/26/93 | 120 min . |

Table 3-1
Time line for Milin's work on tower activities

The data for the present study consists of approximately six and a half hours of video data. This data consists of a group work session, followed by a whole group discussion, three follow-up individual interviews, each lasting between $30-45$ minutes, a
small group interview, two written assessments, and another group work session followed by presentations.

During this study, Milin has been videotaped during classroom problem solving activity sessions and task based individual and group interviews since first grade. On February 6, 1992, during grade four Milin and his classmates were assigned the following extension of the tower problem.

Your group has two colors of Unifix cubes. Work together and make as many different towers five cubes tall as is possible when selecting from two colors. See if you can plan a good way to find all the towers five cubes tall.

The students worked in pairs to find all the towers that were five cubes tall when selecting from cubes of two colors. The group work session was followed by a whole class sharing session and by three individual interviews that were videotaped. There was also an interview done in a small group comprised of Milin and three fellow students. In addition to the above problem, Milin also wanted to explore towers of various heights when selecting from three colors. Finally, in grade five, Milin's class was given the following "Guess My Tower" activity that introduced probability ideas to the students.

You have been invited to participate in a Quiz Show and have the opportunity to win a vacation to Disney World. The game is played by choosing one of the four possibilities for winning and then picking a tower out of a covered box. If the tower you pick matches your choice, you win. You are told that the box contains all possible towers that are three tall that can be build when you select cubes of two colors, red and yellow.

You are given the following possibilities for a winning tower:

1. All cubes are exactly the same color;
2. There is only one red cube;
3. Exactly two cubes are red;
4. At least two cubes are yellow.

Q1. Which choice would you make and why would this choice be better than any of the others?

Assuming you won, you can play again for the Grand Prize which means you can take a friend to Disney World. But now your box has all possible towers that are four tall (built by selecting from the two colors yellow and red). You are to select from the same four possibilities for a winning tower. Which choice would you make this time and why would this choice be better than any of the others?

All the episodes from the tower activities used for the present study were transcribed, verified, and analyzed to find key moments in Milin's mathematical reasoning.

### 3.3 Method of Analysis

Davis, Maher, and Martino (1992) indicate, "Videotaping small groups of students in a regular classroom environment makes it possible to study individual student in a social setting". The authors further stress their point by using an analogy of a microscope for biology. They point out that videotaping lends itself to make to study "mathematical ideas" in detail, over an extended period. The present study uses components of the analytical model reported by Powell, Francisco, and Maher (2003) to analyze video data. The model is described as consisting of seven non-linear phases of analysis that includes, viewing video, describing segments, identifying critical events, transcribing of video data, coding the data, constructing a storyline, and finally composing a narrative (p. 413).

Below is the description of each one of the phases involved in the video analysis for this research study.

### 3.3.1 Viewing

Researcher viewed and listened to the video data multiple times in order to become familiar with the contents of the data. During this phase, the researcher just views the data without any lens in mind (Powell, Francisco, \& Maher, 2003). This phase helps the researcher to identify critical portions that might require further investigating.

### 3.3.2 Transcribing and Verifying

All the videos were transcribed in their entirety by the researcher to enable a more detailed and thorough analysis of the data. The transcription allowed for accurate coding of critical events within the larger context of the complete session. This sheds insight into the chronology of the occurrence of the critical events and allows the researcher to construct a storyline more accurately.

All transcripts were verified by at least one independent researcher. During this phase of the verification process, the transcript was checked for accuracy by viewing the videotapes. The researcher, as well as another graduate student, independently verified the transcripts. To ensure accuracy the researcher verified these sessions a second time.

### 3.3.3 Identifying Critical Events

The video data was used to look at students' thinking over a given period. All the video data was transcribed and critical events were flagged based on the analysis being
done. The events that were coded as being critical depended on the researcher questions the study is answering. According to Maher (2002):

The analysis begins with the identification of critical events. The mathematical content of each critical event is identified and described, taking into account the context in which the event appears, the identifiable student strategies and/or heuristics employed earlier evidence for the origin of the idea, and subsequent mathematical developments that follow its emergence. (p.35)

Maher also states, "Each critical event defines a timeline, consisting of a past, a present and a future" and illustrates with the following figure (2002, p. 35).


Figure 3-1. Maher's notion of a critical event.

Given that the critical event represents the present, it is important to study the events prior to the critical event and the events after the critical event to see the influence on the future events. Kiczek (2000) and Steencken (2001) define critical events that are related and lead to the growth of understanding as a pivotal strand or a pivotal mathematical strand.

### 3.3.4 Coding

During the coding stage, themes were identified that aided the researcher in interpreting the data. This phase also helped in answering the research questions. The codes used for this study are discussed below.

### 3.3.4.1 Codes

Table 3-2 gives the codes that are used for the analysis of the data. The codes have been broken into four groups including codes for heuristics, for strategies and monitoring answers, for building methods used, and finally for justifications used by Milin. The researcher coded the data in an attempt to identify events in the data that can be used to answer the research questions. All the transcripts used during the present study are included in the appendices section.

| Codes for heuristics used |  |  |
| :---: | :---: | :---: |
| 1. | Used guess and check or trial and error | H1 |
| 2. | Used pattern recognition | H2 |
| 3. | Recalled a similar problem | H3 |
| 4. | Used a picture or diagram | H4 |
| Codes for strategies for finding and monitoring answers |  |  |
| 5. | Used estimation to get an idea of the answer | Sl |
| 6. | Used estimation to check the reasonableness of answers | S2 |
| 7. | Monitored work by checking | S3 |
| 8. | Checked whether the answers made sense | S4 |
| 9. | Used paper and pencil to keep track | S5 |
| Codes for building methods used |  |  |
| 10. | Used the "Opposite pair" strategy to build towers | B1 |
| 11. | Used the "Mobile blocks within a tower" strategy to build towers | B2 |
| 12. | Used the "Family Tree" strategy to build a tower | B3 |
| Codes for justification |  |  |
| 13. | Gives a justification of work | J1 |
| 14. | Justifies the solution in a different way | J2 |

Table 3-2
Coding scheme

### 3.3.5 Constructing a Storyline

The next phase in the analytical model described by Powell, Francisco, and Maher is that of constructing a storyline. In this phase, the results from the coding phase are put together as the researcher tries to make sense of the results and begins to identify an "emerging narrative about the data" (p. 430). Other data sources, such as field notes and students' work, were also used during this phase to develop a comprehensive perspective
of the video events. During this process, the researcher identified traces, which are a collection of events that lend insight into a student's growth in mathematical understanding. After the data was coded, the researcher was able to identify the heuristics, strategies, representations, reasoning and justifications used by Milin during his work on the problem tasks.

### 3.3.6 Composing a Narrative

Powell, Francisco, and Maher (2003) indicate that this step, even though listed last, actually occurs throughout the research. They describe the entire process of describing the data, identifying critical events, coding events, and constructing a storyline as nonlinear, and that it can all occur in any order. My final narrative composition involved constructing an interpretation of the events described in the storyline, from the perspective of mathematical understanding. Finally, the findings that were derived from the coded data and other non-video sources are described in a narrative that provides a clear overview of the results of the study.

### 3.4 Verification of Validity

To ensure validity of results, the following three steps were taken throughout the process of data collection and analysis:

1. Triangulation of data with the use of researcher field notes, student work, and video recordings validates the accuracy of the storyline that is constructed.
2. In addition to the researcher, an independent researcher verified transcription and coding, and differences were discussed until they were resolved.
3. The researcher tried to express the data in a manner that should enables readers to arrive at similar conclusions, independently, from the description provided.

## 4 RESULTS: FIVE-TALL TOWERS

| Date: | February 6, 1992 |
| :--- | :--- |
| Grade: | Grade 4 |
| Task: | Towers (Five-tall) |
| Participants: | Milin and Michael (Group work and sharing) |
| Researchers: | Alice S. Alston (R1), Carolyn A. Maher (R2) <br> and Amy M. Martino (R3) |

### 4.1 Overview

On February 6, 1992, Milin's class had the opportunity to work on the following task.

Your group has two colors of Unifix cubes. Work together and make as many different towers five cubes tall as is possible when selecting from two colors. See if you and your partner can plan a good way to find all the towers five cubes tall.

When Milin was in fourth grade, he had the opportunity to work on a tower task with Michael as his partner. The students in the class worked in pairs for about 40 minutes and the entire class came together for a sharing session, which lasted for about 50 minutes, where they discussed how different people had worked on the problem. Before the students began their work on the problem, one of the researchers (R2) gave students the instructions, and the class as a whole came to a consensus as to what was allowed when making different towers and what was not. Figure 4-1 shows Milin and Michael listening to the instructions, prior to their work on the problem.


Figure 4-1. Milin and Michael listening to the instructions.

During the group work, Milin and Michael generated towers using guess and check. Both students used local organization by first making a tower and then generating its partner tower by switching the colors of the corresponding cubes or by flipping the first tower upside down. Throughout their work on this task, Milin and Michael monitored their work by visually checking each new tower against the already made towers to eliminate any duplicates. Using the guess and check strategy and local organization of pairs the towers, both students were able to find all thirty-two towers using red and yellow cubes.

During the sharing session, some of the groups had used certain patterns based on certain characteristics of these towers. Some of the patterns shared during this session included solid towers, towers with one red cube and four yellow cubes where the red cube moved through the different floors of each tower to make a staircase, and another set of towers with two red cubes next to each other moving in a staircase pattern. Other patterns included towers with two red cubes separated by one, two, or three yellow cubes.

Table 4-1 gives an overview of the heuristics and strategies with the types of reasoning and argumentation that emerged during the group work and the classroom discussion.

|  | OVERVIEW (FEBRUARY 6, 1992) |  |  |
| :---: | :---: | :---: | :---: |
|  | Participants | Heuristics/Strategies | Reasoning/ Argumentation |
| Group Work | Milin and Michael | - Guess and check <br> - Monitor by comparing <br> - Opposite by color <br> - Opposite by inverting | - It is taking too long <br> - Keep finding duplicates |
| Sharing Session | Whole Class | - Opposites by colors <br> - Patterns | - Has to be an even number of towers <br> - Each tower has a partner <br> - By cases based on number of cubes of one color <br> - By contradiction |

Table 4-1
Overview of heuristics, strategies, types of reasoning and argumentation

### 4.2 Result Details

In the following sections, this researcher discusses in detail the various heuristics, strategies, representations, and justifications and reasoning Milin utilized during his group work and during the sharing session.

### 4.2.1 Heuristics and Strategies Observed During Group Work

The strategies used by Milin and Michael on February 6, 1992 included:

1. Guess and check
2. Building an opposite tower

### 4.2.1.1 Guess and Check

During this session, both Milin and Michael generated each new tower by using guess and check and then making another tower as a "partner" to complete each pair of towers. Both Michael and Milin monitored their work by checking each new tower against the previously built towers by moving the new tower over the other towers and visually comparing the towers (Figure 4-2).


Figure 4-2. Milin monitoring his work to check for duplicates.

When R3 asked them about how they knew that there are not any duplicate towers, Milin explained how they checked their work to make sure by demonstrating their method.

### 4.2.1.2 "Opposite" Pair Strategy

When Milin and Michael worked on the task during the classroom session, they started out by making a random tower and then made an "opposite" tower. Sometimes, Milin and Michael referred to the second tower as "perfect match." During the group work, Milin made partner towers by using "opposite" by either switching the colors of the corresponding cubes or by inverting the original tower (Figure 4-3).


Figure 4-3. Milin's two ways of making "partner."

Both Michael and Milin made their towers in pairs. The following section gives all the tower pairs generated by the two students. The tower pairs can be divided into three categories based on who was responsible for generating the towers in a given pair.

### 4.2.1.2.1 Both Towers Made by Michael

Throughout the group work session, Michael consistently made the partner tower using the "opposite by color" strategy. He constructed the second tower in the pair by switching the color of each corresponding cube. Figure 4-4 shows the four complete pairs of towers that were constructed by Michael.


Figure 4-4. Michael's tower pairs.

### 4.2.1.2.2 Both Towers Made by Milin

Milin, on the other hand used two different "opposite" strategies for finding partner towers. He sometimes used the method of changing the color of each
corresponding cube to make the second tower in a pair. For the sake of convenience, in this analysis, this strategy will be referred to as "opposites by color." Milin made six complete pairs of towers using this strategy (Figure 4-5).


Figure 4-5. Tower pairs generated by Milin by switching colors.

At other times, Milin used the inverted tower strategy to make the partner tower. For the sake of convenience, in this analysis, this strategy will be referred to as "opposites by inverting." Milin made two complete pairs of towers using this method (Figure 4-6).


Figure 4-6. Tower pairs generated by Milin by inverting the original tower.

### 4.2.1.2.3 Tower Pairs Where Milin Made One of the Towers

As opposed to the towers in Figure 4-4, Figure 4-5, and Figure 4-6, the remaining tower pairs were built through collaboration with each other. In some cases, Milin made the first tower, and in others, Michael made the first tower. The following sections give details of these towers.

### 4.2.1.2.3.1 Pairs Where Milin Made the First Tower

Figure 4-7 shows all the tower pairs where Milin generated the first tower using guess and check, and Michael generated the partner tower of the pair by switching the colors.


Figure 4-7. Tower pairs where Milin made the first tower.

### 4.2.1.2.3.2 Pairs Where Milin Made the Second Tower

Figure 4-8 shows all the tower pairs where Michael generated the first tower. Milin continued to use the two strategies of changing colors and inverting the original tower to generate its partner. As shown in Figure 4-8, the first and the third pair were generated by using opposites by color strategy and the second pair was made by using the opposites by inverting strategy. Altogether, Milin and Michael generated 18 pairs of towers, out of which two of the duplicate pairs were later discarded, resulting in 16 valid pairs for a total of 32 unique five-tall towers when selecting from red and yellow cubes.


Figure 4-8. Michael made the first tower of these pairs.

### 4.2.2 Other Justifications for the Solution

Milin expressed his certainty that there can only be two solid towers when selecting from two colors when he reasoned, "we already know that we made five of these (yellow) and five of the reds so we are not gonna try that again" (221).

Michael and Milin had made 28 towers when R1 came to see what the two students were working on. Milin explained how they had made the towers in pairs by changing the colors of the cubes. He referred to the tower pairs as "doubles". When the researcher asked them if they thought there were more towers, Milin replied that there were. When the researcher questioned them as to how they would know when they were done, Milin replied, "when we lose all these - use up the cubes" (181) as he pointed to the cubes on the table in front of him.

After Milin and Michael had made 34 towers, R1 came back to talk to the two students about their progress on the problem task. Milin again explained how they had monitored their work while building the towers to make sure they were not making duplicates of the towers they had already made. During their conversation and explanation, R1 noticed a duplicate tower in the arrangement of towers on the table. Michael and Milin removed the duplicate tower and its partner from their collection. When R1 asked them, why they took out two towers rather than just one, Milin explained that each tower had a partner. At the end, they had used the same strategy as they removed the duplicate towers in pairs.

After the removal of the duplicate pairs, R1 asked if there was any way to tell whether they had found all the towers or not. Milin first reasoned that, "... if we keep on doing [building new towers] and we keep on getting duplicates" (328). Milin also
indicated that time is another way of knowing whether they were done or not and since they had not found a new tower for a long time. He said, "so that's more than ten minutes and we still didn't find one" (354). Milin's comments were supported by an analysis of the time elapsed between the building of towers. Figure 4-9 shows that for the first 28 towers, the elapsed time between the building of towers did not exceed 100 seconds. However, after that point the elapsed time varied greatly with the high being 783 seconds. For the first 28 towers, the mean was 28.86 seconds and the median was 20.5 seconds. For the last eight towers, the mean was 120.45 seconds and the median was 58.5 seconds.


Figure 4-9. Elapsed time between building of towers.

The time of the completion of the 36 towers is shown in Table 4-2. The students started their work on the towers after the directions and Michael built the first tower at 5:30.

| Approximate Time Codes for Each Tower During Milin's and Michael's Partner Work (From Work-View Video) |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| TOWER\# | BUILT BY | TIMIE CODE | TOWER \# | BUILT BY | TIME CODE |
| 1 | Michael | 05:30 | 19 | Milin | 12:50 |
| 2 | Milin | 05:45 | 20 | Milin | 13:19 |
| 3 | Michael | 05:52 | 21 | Michael | 13:37 |
| 4 | Milin | 06:13 | 22 | Milin | 14:01 |
| 5 | Michael | 06:24 | 23 | Milin | 15:41 |
| 6 | Michael | 06:55 | 24 | Milin | 16:10 |
| 7 | Michael | 06:55 | 25 | Milin | 17:50 |
| 8 | Milin | 07:00 | 26 | Milin | 18:09 |
| 9 | Milin | 07:14 | 27 | Michael | 18:34 |
| 10 | Michael | 07:33 | 28 | Michael | 18:58 |
| 11 | Milin | 08:02 | 29 | Michael | 24:40 |
| 12 | Milin | 08:22 | 30 | Michael | 26:13 |
| 13 | Milin | 09:03 | 31 | Milin | 27:39 |
| 14 | Milin | 09:20 | 32 | Michael | 28:10 |
| 15 | Milin | 09:32 | 33 | Milin | 28:30 |
| 16 | Milin | 09:52 | 34 | Michael | 28:56 |
| 17 | Michael | 11:17 | 35 | Milin | 41:59 |
| 18 | Michael | 11:40 | 36 | Milin | 42:24 |

Table 4-2
Approximate time codes for each tower from work-view video

Figure $4-10$ shows the 36 towers made by Milin and Michael. The numbers next to the towers indicate the order in which the two students built the towers. Out of this collection of towers, Milin and Michael removed tower numbers 15, 16, 27, and 28 because they were duplicates.


Figure 4-10. All 36 towers made by Milin and Michael.

After removing the duplicate towers, Milin and Michael rearranged the remaining 32 towers (Figure 4-11). The pairings of towers 4 and 3, and towers 9 and 8 were
inconsistent with other pairings. For these two pairs, Milin and Michael used the opposites by inverting strategy, whereas the remaining pairs were identified based on the opposites by color strategy.


Figure 4-11. Final thirty-two towers after removing the duplicate towers.

### 4.3 Sharing Session (2/6/1992)

At the conclusion of the group work the entire class came together to share how different groups had found their solution for the tower task (Figure 4-12). R2 went to different groups and asked the students to present their work and share the strategies they had used to find the answer.


Figure 4-12. Sharing session from 2/6/92.

### 4.3.1 Group Results

At the beginning of the sharing session one of the groups (Robert and Sebastian) claimed that, they had found 35 towers. When R2 posed a question as to whether it was possible to have an odd number of towers. Milin responded, "Um, we got 36 before, but then we found duplicates...but now we um got 32 and we keep on duplicating it by changing the color. So, you can't get an odd number unless you don't duplicate it and get all of them" (28). Milin was able to reason that since every tower had a "partner," there had to be an even number of towers. At the researcher's invitation, the students helped Robert and Sebastian to find the duplicates. Upon further examination, it turned out that Robert and Sebastian actually had 34 towers instead of 35 . After taking two duplicate towers away, Robert and Sebastian's group also had 32 towers.

A few of the other groups in the class also had their towers arranged in pairs (Figure 4-13) to check for duplicates. Even though when Milin and Michael constructed their towers they referred to the pairs as "partners," "opposites," "perfect matches," "duplicates," and "groups" interchangeably, Milin was able to relate to the vocabulary used by the other students in the class. Relating to the use of switching of colors by another group, he stated, "I know what they mean. See this yellow turns into red on this one and all of these reds turn into yellow in this one" (135).


Figure 4-13. Other groups that had their towers arranged in pairs.

Next, another group, Michael and Paul, presented their solution. Michael explained how they had used the opposites to make matches. As the students looked at the collection of towers made by Michael and his partner, they found duplicate towers. Another student, Ankur, shared his group's strategies for finding all towers. He explained his pattern of towers to the researcher while his classmates listened to his explanations (Figure 4-14). Ankur also explained that in addition to using patterns for some towers, they used guess and check for the others.


Figure 4-14. Ankur shares his patterns.

Michael and Milin had inadvertently picked up two of the discarded towers and now had 34 on their desk. However, they quickly found the duplicates. When the researcher asked them about how many towers they had, Milin quickly replied, " 32 ". When R2 questioned Milin about how he knew that there are only 32 towers in the solution, he explained that any additional towers they developed seemed to be duplicates of the existing towers. He also went on to explain that they had changed their pairings so that all of their pairs were now based on switching the colors (Figure 4-15).


Figure 4-15. Milin and Michael's final pairs during Sharing.

### 4.3.2 Role of Researcher Interventions

After the researcher went around to all the groups in the classroom, she brought all of them together for a whole class discussion. It is important to notice that researcher's questioning played an important role in the students' mathematical development without giving them the answers. The researcher during the sharing session was able to ask meaningful questions that helped students to think about the local organizations of the towers based on certain parameters.

R2 asked Ankur if it would be okay to share his strategy for organizing the towers with the class. Ankur and his partner then showed the entire class how they had kept track of some of their towers using patterns. Ankur explained how the towers were arranged based on the position and number of cubes of each color (308). The researcher then shared Ankur and Joey's organization and introduced the first pattern as "exactly one yellow" and "exactly one red" and how the red cubes were on the second, third or fourth "floors" of the towers (Figure 4-16). R2 also pointed out that they used a solid red tower followed by a tower with four red cubes on the top with one yellow cube on the bottom. The next tower in the pattern had three red cubes on the top followed by two
yellow cubes. This pattern of decreasing the red cubes by one and increasing the yellow cubes in each successive tower in this pattern ends with a tower that has is a solid yellow tower (Figure 4-16).


Figure 4-16. Researcher (R2) presenting Ankur's organization to the class.

### 4.3.2.1 Discussion Questions that Promoted Reasoning

Researcher (R2) asked students to work with their partners to see what they could figure out about the patterns in towers with "exactly two reds". She also encouraged the students to find and study all the towers with "exactly two reds". One of the students volunteered that there were ten five-tall towers that had exactly two red cubes. R2 then prompted the students to think of ways they could convince her that there were exactly ten towers. R2 then shared a tower with two red cubes on the bottom floors followed by three yellow cubes on the top and another tower with a yellow cube on the bottom and with two red cubes in the second and third floor followed by two yellow cubes. With the help of the students, the researcher uncovered four towers where two adjacent red cubes were positioned at all possible locations in the tower.

When the researcher asked students if there was a way to have two reds separated by four floors some of the students replied "yes." Rather than telling the students that they were wrong, R2 invited them to build a five-tall tower for her that had two red cubes
separated by four floors. The researcher (R2) used these inquiries to elicit reasoning from students and to get the students to think about the problem in a different way. As R2 was asking students these questions the students were actively engaged in the process and provided R2 with towers that fit the description. Table 4-3 lists the inquiries used by the researcher (R2) during the sharing session.

| RESEARCHER INTERVENTIONS USED DURING SHARING SESSION |  |  |
| :---: | :--- | :--- |
| Number | Inquiry by the Researcher (R2) | Students' response |
| 1. | Did anybody look for patterns with exactly two reds or exactly two <br> yellows? <br> Do the two reds always have to be together? | yes |
| 2. | Can they be separated by a floor? | No |
| 3. | Can they be separated by two floors? | Yes |
| 4. | Can they be separated by three floors? | Yes |
| 5. | Can they be separated by four floors? | Yes |
| 7. | Is there only one way to have two red separated by one? | No! but some reply |
| 8. | What about two reds separated by two yellows? | yes |
| 9. | Is there another way to have two reds, two reds separated by <br> three? | No |

Table 4-3
Researcher interventions used during the sharing session to elicit reasoning

As R2 challenged the students to make these towers for her, another pair of students handed her another tower with two reds separated by one floor. Once she had three distinct towers with two reds separated by a floor (

Figure 4-17), she asked students for another tower. The students argued that it was not possible because the towers could only be five cubes tall.


Figure 4-17. R2 shares towers with two reds separated by a floor.
When she inquired about towers with two reds separated by two yellows, students started offering her two towers that fit that description. When she asked for another tower with two reds separated by two yellows (Figure 4-18), Michael handed her a tower with two reds separated by three yellows, which prompted Milin to explain that there were not anymore because "on ones there is only three. On twos, there is only two. And on threes there is only one" (387).


Figure 4-18. R2 shares towers with two reds separated by two floors.

R3 then displayed all the towers with two red cubes that were handed to R 2 by the students, on the board (Figure 4-19).


Figure 4-19. Ten towers with exactly two red cubes during sharing.

## 5 RESULTS: MILIN'S INDIVIDUAL TASK-BASED INTERVIEWS

### 5.1 Milin's First Individual Task-based Interview

| Date: | February 7,1992 |
| :--- | :--- |
| Grade: | Grade 4 |
| Tasks: | Towers |
| Participant: | Milin (Individual Interview) |
| Researchers: | Alice S. Alston (R1) and Mrs. O'Brien (TR) |

### 5.2 Overview

This interview with Milin was conducted in order to determine how he found the total number of possible five-tall towers when selecting from red and yellow cubes. The interview also sought to determine Milin's depth of understanding of towers that were four, three, two, or one-tall when selecting from two colors of Unifix cubes. The interview was conducted by R1 and one of the teachers, Mrs. O (Figure 5-1).


Figure 5-1. Milin during his first interview on February 7, 1992.

During this individual interview, Milin reconstructed all possible five-tall towers when selecting from two colors of cubes. He used various building methods to find a solution to this problem. In addition to the earlier building methods used by Milin during his group work the previous day, Milin also used patterns and partial cases to find some
of the towers. Milin continued to use his opposites by color strategy to build more towers and for local organization of his towers. Milin was also able to modify his opposites by color strategy to justify "duplicate groups" of towers rather than just pairs of towers. During this hour-long interview, the methods for building towers, for monitoring work, and types of reasoning and justification used by Milin are summarized below (Table 5-1).


Table 5-1

During this individual task-based interview, Milin used local organizational strategies in addition to random towers to find all five-tall towers. He was able to account for ten towers using the staircase patterns with one red cube and with one yellow cube. He also used the staircase pattern with two red cubes in consecutive positions and two yellow cubes in consecutive positions to account for another eight towers. Milin was also able to justify that there would be only two solid towers, one with all red cubes and the other with all yellow cubes.

For the remainder of the towers, Milin initially used guess and check to find two towers and then utilized his earlier strategy of opposites by color to complete each pair of towers just as he had done during his initial group work the day before. There is evidence in his work that he used opposite by inverting strategy to generate new towers and then found their partners by using opposite by color. Milin continued to monitor his work by visually comparing his towers. Milin now also used local cases to monitor his work in bigger chunks rather than by individual towers exclusively as he had done during his group work the previous day.

Unlike during his initial group work, Milin no longer used time to justify that he had found all his five-tall towers but used justification by contradiction and patterns to justify the number of towers in a given group of towers.

### 5.3 Result Details

In the following sections, this researcher discusses in detail the various heuristics, strategies, representations, and justifications and reasoning Milin utilized during this interview.

### 5.3.1 "Opposite" Pair Strategy

When R1 asked Milin how he solved the problem, Milin explained that he and his partner Michael used the concept of making a tower by guess and check and then made its partner. He explained this by stating, "...when Michael and I kept on ... building [towers] and putting another [tower] exactly like that but different colors" (20). R1 asked Milin to show her what he meant. As he built a couple of towers, Milin explained how he and Michael had changed the colors of each of the corresponding cubes to make each "group" of towers. Milin also explained that a "group" was a pair of towers with the colors changed (32).

### 5.3.2 Monitoring

In addition to explaining how he and Michael had built new towers, Milin also demonstrated how they monitored their work to check for duplicates by moving each new tower over all the previously made towers. He then added that they would then proceed to make the partner tower by changing the colors and they then repeated the process in order to make a new "group" of towers.

### 5.3.3 Staircase Pattern with One Cube and Proof by Contradiction

After the class discussion and sharing of ideas from the previous day, Milin started thinking about the towers in terms of the staircase pattern with one color and four cubes of the other color. When R2 asked him to show her what he was talking about, Milin reconstructed the five towers using one red cube for in each tower (Figure 5-2). The five
towers started with a red cube in the bottom position and moved up one level in each successive tower.


Figure 5-2. Milin's staircase pattern with exactly one red cube.

After Milin made the group of five towers, he used his opposite by color strategy to justify that the same would also be true for the second group of towers with exactly one yellow cube moving in a staircase pattern (91-103). Milin, by stating, "they all have partners" (107) showed his use of prior knowledge of every tower having an opposite by color to justify that these two groups of towers would yield ten total towers. Figure 5-3 shows Milin's written record of the number of towers in these two sets. He also offered a proof by contradiction when he explained that you could only have five towers in this pattern because the towers were not allowed to be more than five cubes tall.


Figure 5-3. Milin's written record.

### 5.3.4 Staircase Pattern with Two Cubes Next to Each Other

Milin continued his explanation with the next group of towers with a staircase pattern where two cubes of one color moved together. When R1 asked Milin about the number of towers with two red cubes in a staircase pattern Milin quickly replied, "um four ... four of each, eight" (113). Milin again demonstrated his understanding of the fact that each tower has a partner tower that can be made by changing the colors of the corresponding cubes (111-135). Figure 5-4 shows Milin's group of towers with two red cubes moving in a staircase pattern.


Figure 5-4. Milin's finished staircase pattern with two red cubes

After Milin was done with this staircase, he stated that the only way to have two red cubes and three yellow cubes in a tower was to have a staircase pattern where the two reds were moving together (135-137). About twelve minutes into the interview, Milin had built two groups of towers with the staircase pattern: one group with exactly one red cube and four yellow cubes and the second group with two consecutive red cubes. R1 then asked Milin about how he was going to get from those nine towers to thirty-two towers. Milin replied that the first group which had "doubles" with a yellow cube would give him ten towers and the second group would double to eight resulting in eighteen towers so far (144-151).

At R1's request Milin built the second group of towers with two yellow cubes in a staircase pattern. Milin utilized his opposite by color strategy to build the second group of towers. Milin made all the towers with three red cubes and two yellow cubes moving together and placed them under the corresponding tower in the staircase with two red cubes moving in the staircase pattern (Figure 5-5).


Figure 5-5. Milin's staircase pattern with two yellow cubes.

### 5.3.5 Milin's Struggle with the Next Group of Towers

With the completion of groups of towers using the staircase pattern with one cube and two cubes, he had accounted for 18 towers. In talking about the next set of towers he concluded, "and then on the [the towers with three cubes of one color together] there would be probably three on each" (151). When R1 asked him to show her what he meant, Milin responded by building a tower with three yellow cubes in the top three positions on the top followed by two red cubes. However this ended up duplicating the first tower in the staircase pattern with two red cubes (Figure 5-6).


Figure 5-6. Milin built a tower with three yellows together and found the duplicate.

Even though the tower he had built had three yellow cubes together, Milin did not realize that it also had two red cubes together which he had already made. When he tried to build another tower with three yellow cubes and two red cubes, he built a duplicate of the second tower in the staircase pattern. (Figure 5-7). When Milin was unable to find new towers with three yellow cubes together, he seemed unsure of his original answer of 32 towers from the previous day (216-218). R1 again asked him if using the staircase pattern was a must. Milin replied that he thought that staircase pattern was the only way to arrange the cubes.


Figure 5-7. Milin tries another tower and finds a duplicate.

### 5.3.6 Towers with Three Red and Two Yellow Cubes Separated

When R1 asked Milin if he remembered any other patterns other than staircases, Milin responded by showing her the solid red tower and the solid yellow (Figure 5-8).


Figure 5-8. Milin shows two solid towers that do not have the staircase pattern.

Next, R1 asked Milin a question that would help him to construct a more complete argument. She asked him to pay attention to a particular component of the towers, and not to give up. R1 picked up a duplicate tower with two yellow cubes in the second and third position from the top and red cubes elsewhere and asked him if there was any way to rearrange that tower to create a tower that he did not already have. Milin quickly rearranged the cubes and made a new tower by separating the two yellows cubes and moving them to the ends of the tower. R1 constructed a duplicate of this new tower and asked Milin if he could come up with yet another tower that looked different (233-290). During his work on this set of towers, Milin pointed out that the new towers differed from his staircase pattern towers and reasoned, "because all of these [two cubes of one color] are together" (242) as he pointed to the towers in the staircase patterns where two cubes of red/yellow were in consecutive positions.

As Milin was working on these new towers, he also utilized his previous strategy of opposite by color to construct the group of towers with three yellow cubes with two red cubes separated. During his work on these groups of towers, Milin also built a tower with two red cubes on the top followed by a yellow cube and then two red cubes on the bottom and pointed out that this tower was the duplicate of the third tower in his staircase pattern with one red cube. Milin discarded this tower because he realized that this tower was already counted as a part of the group he had not built (staircase pattern with one yellow cube) but had already counted in his total.

Table 5-2 summarizes the 12 towers built by Milin with three cubes one color and two separated cubes of the other color. Throughout his work on these groups of towers,

Milin monitored his work by comparing each new tower to the previously built towers just as he had done during his initial work on this problem with Michael.

| Tower Given by R1 to rearrange | Tower built by Milin | Group by number of cubes of each color | "Partner" tower built by Milin | Group by number of cubes of each color |
| :---: | :---: | :---: | :---: | :---: |
|  |  | Two yellows and three reds | $\theta$ | Two reds and three yellows |
|  |  | Two yellows and three reds |  | Two reds and three yellows |
| None |  | Two yellows and three reds |  | Two reds and three yellows |
| None |  | Two yellows and three reds |  | Two reds and three yellows |
| None | $1$ | Two yellows and three reds |  | Two reds and three yellows |
| None |  | Two yellows and three reds |  | Two reds and three yellows |

Table 5-2
Milin's work on towers with two yellow/red cubes separated

Milin then constructed a new tower by using his opposite by color strategy to make a partner. He consistently used this strategy during his interview. It appears that the third and fourth towers and the fifth and sixth towers are opposites by inverting.

Opposites by inverting was initially used by Milin to make partners during his initial work. Here it appears that Milin is utilizing this strategy to make new tower and then was making the partner tower. The last pair of towers built by Milin completed the 32 five-tall towers. When the researcher asked Milin if he thought he was done Milin replied "yes" (389-414). Table 5-3 summarizes the information for all the groups of towers. The table also shows which groups of towers were actually constructed during the interview by Milin and the number of towers in each tower group.

| Tower Group | Made <br> or not | Number of <br> towers in the <br> group | Total towers |
| :---: | :---: | :---: | :---: |
| Staircase with one <br> red cube | yes | 5 | 5 |
| Staircase with one <br> yellow cube | no | 5 | 10 |
| Staircase pattern <br> with two red cubes <br> together | yes | 4 | 14 |
| Staircase pattern <br> with yellow cubes <br> together | yes | 4 | 18 |
| Solid red and solid <br> yellow | yes | 2 | 20 |
| Tower with three <br> reds and two yellow <br> cubes separated | yes | 6 | 26 |
| Tower with three <br> yellows and two red <br> cubes separated | yes | 6 | 32 |

Table 5-3
Milin's five-tall towers by groups and the number of towers in a given group

Figure 5-9 shows the 20 towers with three cubes of one color and two cubes of the other color.


Figure 5-9. Milin's new sets of towers with 2 red/yellow cubes separated.

In an effort to justify that he had found all possible towers, in Figure 5-10, Milin proceeded to rearrange these new towers to make patterns (389-414). Milin took the 12 towers from his two new groups of towers, which he built using the guess and check, opposite by color, and opposite by inverting strategies, and arranged them into a new pattern. He pointed out how the towers had the yellow cube moving diagonally up and then down with the last five towers with the red cube moving diagonally (Figure 5-10).


Figure 5-10. Milin's collection of rearranged towers.

### 5.3.7 Estimation Skills

After Milin was finished with the five-tall towers, the researcher asked Milin to predict how many four-tall towers there might be. Milin used paper and pencil to write a few numbers down and estimated that there would be 20 possible four-tall towers initially. R2 asked questions to prompt Milin to explain his rationale for the guess and

Milin stated, "eight plus six plus two plus eight" (480). Milin estimated that there would be eight towers with exactly one red/yellow cube, six towers with exactly two red/yellow cubes in consecutive positions. He at first estimated that the group of towers where the two cubes of one color are separated would yield four towers and then changed the four to an eight. Finally, the two represented the two solid towers that would still be the same number. Milin's initial and final estimates are summarized in Table 5-4 below:


Table 5-4

## Milin's work on estimating four-tall towers using five-tall towers

Figure 5-11 shows all the towers built by Milin for the five-tall towers task. Milin actually built all the groups except Group D. He used his opposite by color strategy to reason that these towers were also possible. For his final estimate for four-tall towers,

Milin explained how he used the number of towers from each group to get each number in his final estimate. The number eight in his final estimate came from the groups C and D. Milin reasoned that since there would be fewer four-tall towers there would be only eight towers for four-tall. Next, the number six came from eight towers from groups E and F by reducing that number by two to account for the lower height. The next eight came from groups G and H by reducing the number of towers in each group by two towers.


Figure 5-11. Milin's five-tall towers arranged by groups.

Finally, the number two came from groups A and B that consist of towers of one color. Milin explained that this number stayed the same regardless of the height because there is only one possible tower with all yellow cubes and one possible tower with all red cubes.

Milin further explained that the number of four-tall towers had to be less than the number of five-tall towers.

### 5.3.8 Towers One, Two, Three, and Four-tall

When R1 asked Milin for his estimate for three-tall towers, Milin replied, "it's probably going to be less than towers of four" (498). When R1 asked him to make a guess Milin quickly replied, "18" (502). When R1 asked Milin about the two-tall towers, he chose to build the four towers in less than 30 seconds (Figure 5-12) and concluded that, "on one's there would only be two" (520). As Milin was working on his towers of various heights, he recorded his results on a piece of paper.


Figure 5-12. Milin's four two-tall towers.

When R1 asked Milin to build the three-tall towers, he started with two solid towers and then added two towers; one with two yellow cubes on the bottom and one red cube on the top and the other its opposite by color. Then he made a tower with two yellows on the top with one red on the bottom and its opposite by color. It is important to notice that his third pair of towers was also the inverted opposite of the second pair of towers he had constructed. Finally, Milin made a tower with a yellow cube in the middle of two red cubes and its opposite by color (Figure 5-13).


Figure 5-13. Milin's four pairs of 3-tall towers.

Milin originally thought that there would be more three-tall towers but quickly decided that there are only eight towers because when he tried to make more he ended up with duplicates.

### 5.3.9 Written Representation

After Milin was done making the smaller towers, he used paper and pencil to keep a record of the number of towers of various heights on the paper with his earlier estimate for four-tall towers. He recorded the number of towers for heights two, one and three. Milin also added the number of five-tall towers to his written work from the interview up to this point (Figure 5-14).


Figure 5-14. Milin's written work for towers one, two, three, and five-tall and his final estimate for towers four-tall.

So far, Milin had found towers that are five-tall, three-tall, two-tall, and one-tall. The only towers he was missing were four-tall. Milin again showed his understanding
that the four-tall towers would have to be more than the number of towers that are threetall. After writing the number of towers of other heights on his paper, Milin changed his estimate for four-tall towers from 24 to 16 and started building random four-tall towers again using his "opposite" pair strategy. He built six four-tall towers by first building a tower and then its "opposite" by changing the colors of the corresponding cubes. When R1 asked him about his new estimate, Milin replied that he guessed that it would be four times four because for towers three-tall, three times three is nine, and the number of towers for three-tall was eight which is very close (lines 621-624). At the conclusion of the interview, Milin was asked to finish his work on four-tall tower problem at home.

### 5.4 Milin's Second Individual Task-based Interview

| Date: | February 21, 1992 |
| :--- | :--- |
| Grade: | Grade 4 |
| Tasks: | Towers |
| Participant: | Milin (Individual Interview) |
| Researcher: | Alice S. Alston (R1) |

### 5.5 Overview

The second individual interview was held two weeks after the first interview. At the conclusion of the first interview, Milin had been asked to work on towers four-tall when selecting from two colors. At the onset of this interview, Milin declared that there were sixteen four-tall towers. When the researcher asked Milin if he was sure of his answer, he replied yes. His written work from the previous interview now included the number of towers that were four-tall (Figure 5-15).

$$
\begin{aligned}
& 1 \text { yellow wind } 4 \text { rede are } 5 \\
& 1 \text { red a } 4 \text { re } 110 \text { w } s \text { are } 5 \\
& 1 \text { towers of } 2 \text { are } 4 \\
& \text { towers of } 1 \text { tare } 2 \\
& \text { towers of } 3 \text { are } 8 \\
& \text { towers of } 5 \text { are } 32 \\
& \text { tow ens } 0 \text { in } 4 \text { are } 16
\end{aligned}
$$

Figure 5-15. Milin's written work with towers four-tall results from his work at home.

He came back with the towers he had built at home and explained how he had built the towers that had exactly one red or yellow cube using a staircase pattern. He also demonstrated that the towers with two red and two yellow cubes could also be arranged in a staircase pattern, either by concentrating on the red cubes or the yellow cubes. He
also stated that there are two solid towers and two towers with alternating colors. Table
5-5 gives the overview of methods for generating towers, for monitoring work and the types of reasoning and justifications used.


## Methods for Monitoring Work

1. Visually by comparing
2. Locally with cases

## Reasoning and Justifications Used

1. Proof by contradiction
2. Exhaustive case with solid towers
3. Even number of towers because they come in pairs when choosing from two colors
4. Odd number of towers "pairs of three" when choosing from three colors

Table 5-5
Overview of Milin's work from the second interview

During his work on the three-tall towers, Milin also stated how he used the opposite by color strategy to complete a pair and opposite by inverting strategy to generate a brand new tower. There was an evidence of this modified strategy during his first interview during his work on five-tall and later during the three-tall towers. During the second interview, Milin actually admitted to always using this opposite by inverting strategy to create new towers.

In addition to working with towers choosing from two colors, Milin also worked on towers one-tall and two-tall choosing from three colors. Milin was also able to conclude that when choosing from three colors the towers always came in "pairs of three". He was also able to reason that choosing from two colors resulted in an even number of towers whereas choosing from three colors, resulted in an odd number of towers.

### 5.6 Result Details

In the following sections, this researcher discusses in detail the various heuristics, strategies, representations, and justifications and reasoning Milin utilized during this interview.

### 5.6.1 Milin's Work at Home on Four-tall Towers

Milin was asked to work on the possible four-tall towers when selecting from cubes of two colors. When R1 asked Milin to show her what he had done at home, he started out by taking the towers he had built at home and checking his towers by pairing each tower with its opposite by color as a way of monitoring if he had all the towers. Milin's work on this problem is described in the following sections.

### 5.6.1.1 Opposite Pairs Using Initial Local Organization

Milin started the interview by organizing his towers in pairs as a way to make sure he was not missing anything. This again demonstrated his understanding of the fact that each tower had a partner that was made by switching the colors of the corresponding cubes. When R1 asked him to explain, what he was working on, Milin explained how he checked his work by pairing the towers by opposites (17-20). He used a pair of towers to illustrate and explained how the reds in one tower corresponded to yellow cubes in the same positions in the other tower.

### 5.6.1.2 Staircase Pattern Local Organization with Reorganization

When the researcher asked Milin how he was going to be sure that he had all the towers, Milin proceeded to explain how the towers with one red cube could be arranged in a staircase pattern. He then started picking up all the towers with exactly one red cube and arranged them with the red cube moving up from left to right in a staircase pattern. Milin then proceeded to make another staircase this time with one yellow cube and three red cubes, with the yellow cube moving down in a diagonal pattern from left to right (Figure 5-16). Next Milin added his two solid towers to his staircase patterns, giving him ten towers (60-80).


Figure 5-16. Milin's staircase pattern with one red/yellow and three yellow/red cubes.

### 5.6.1.3 Staircase Pattern with Two Red and Two Yellow Cubes

When R1 asked Milin about the remaining six towers on the table, he started to arrange some of the towers with two red cubes next to each other in another staircase pattern with two red cubes moving together. Milin explained why the staircase with two cubes of one color next to each other could be made with arranging the two red cubes going in a staircase pattern or the two yellow cubes arranged in a staircase pattern (100115). Milin demonstrated his reasoning by first using his third, first, and fourth tower to make a staircase with two red cubes moving up in a staircase pattern from left to right and then used fourth, second, and third tower to make a staircase with two yellow cubes moving up from left to right (Figure 5-17). Milin explained that you had to pick whether to make the pattern with the red cubes or the yellow cubes.


Figure 5-17. Milin's possible staircases based on two red cubes or two yellow cubes.

### 5.6.1.4 Towers with Alternating Color Cubes

Next, Milin picked up his two towers where the red and yellow cubes alternate (Figure 5-18) and explained that there were only two ways to make the towers where red
and yellow cubes alternate and that they were opposites of each other (116-124). Just like the reasoning he used for his solid towers Milin again stated that there were only two possible towers where the colors alternated.


Figure 5-18. Alternating color tower pair.

### 5.6.2 Work on Smaller Towers

During this interview, Milin also made the four two-tall towers and the two onetall towers. When R1 inquired about the missing towers, Milin quickly replied that there were eight three-tall towers and proceeded to make these towers. He started by making the two solid towers, one with all three yellow cubes and one with all three red cubes. He used his opposite by color strategy from before to make these towers in pairs. He finished by making all eight three-tall towers.

### 5.6.2.1 Milin's Explanation for Building New Towers

Milin then picked up two of the four pairs of three-tall towers and explained to R1 how these two pairs are the same when he inverted one pair of towers (Figure 5-19). He also explained, "that's what I always work on" as a strategy to find new towers (170178).


Figure 5-19. Milin explains how the two pairs of towers are the same when one pair is turned upside down.

### 5.6.2.2 Solid and Alternating Color Towers

Finally, Milin picked up his alternating color-cube towers and the two solid towers and showed how the solid color towers are the same as four-tall solid towers except smaller in height (Figure 5-20) as he indicated, "these fall into the same hands as these" (182).


Figure 5-20. Milin's solid and alternating color towers.

### 5.6.3 Estimation for Six-tall Towers

When the researcher asked Milin to estimate how many different towers there would be for six-tall towers, Milin indicated that each time the height of the towers increases there are more towers possible because you can change more things on a taller tower as he explained "it has more and you could change more stuff on it" (204). He also mentioned that "you could build a bigger staircase" (206).

### 5.6.4 Recalling Choosing from Three Colors

When R2 asked Milin if building towers reminded him of any other problem, he recalled working on a tower problem earlier which involved choosing from three colors. When R1 asked Milin if having three colors available to build towers would change anything, Milin right away pointed out that with three colors it would be possible to make three solid towers. Milin also responded that for one-tall towers, there would be one more than one-tall towers when choosing from two colors. When the researcher asked him about the two-tall towers when choosing from three colors, Milin decided to build these towers. He built the solid white tower and placed it next to his two solid towers of red and yellow cubes he already had on the table. Next, Milin built the two towers with white cube on the bottom and then the two towers with the white cube on the top (Figure 5-21). Milin then built the rest of the towers and then paired the remaining six towers but he failed to consider the three solid towers when he was pairing these towers and was unable to come up with "pairs of three."


Figure 5-21. Milin's nine two-tall towers choosing from three colors.

When the researcher asked him if he was sure that he had found all of them, he replied that he could keep on building and checking. As he scanned the towers in front of him, he replied that "there'll be $2,4,6,8,9$ for this, and I'm sure of that!" (258)

### 5.6.5 Milin's Reasoning about Doubling and Tripling Rule

In response to R1's inquiry about being sure of the number of towers, Milin pointed out that the colors available to choose from make a difference for towers built. He indicated that when working on towers of two you multiplied by two to get four towers when choosing from two colors (262), and three times three to get nine when choosing from three colors (266). Milin wrote his observation. (276) (Figure 5-22).


Figure 5-22. Milin's written representation for doubling and tripling rule.

### 5.6.5.1 Odd and Even Number of Towers Based on Color Choices

After Milin noticed this pattern about two-tall towers, he conjectured about the number of three-tall towers and indicated, "I'd guess around 17" (299) when selecting from three colors. He also reasoned that "it would be an odd number because of this" (304), as he used his two-tall towers grouped in threes to demonstrate. When the researcher asked him how he got that number, he explained that he took the three-tall towers when choosing from two colors and doubled it then added one (312). He also stated that he could have subtracted one to get fifteen towers. Milin recorded his
conjecture for three-tall towers when selecting from three colors on his sheet of paper (Figure 5-23).


Figure 5-23. Milin's written representation for his estimate for three-tall towers choosing from three colors.

When R1 asked Milin if there was a way to figure out the answers for sure, Milin suggested time as a factor by saying that it would take you " 10 hours of your time" (326). R1 then asked Milin to guess the number of six-tall towers when selecting from two colors. Even though Milan originally guessed 45 towers for six-tall, he later suggested that the answer had to be an even number because each tower has a partner and he indicated that, "two is an even number and, ... it's got to be even because you can make pairs of them" (360). As the interview was concluding, the researcher wrote down the two problems that Milin had to work on at home (Figure 5-24).


Figure 5-24. Milin's two assignments at the conclusion of the interview.

The researcher made sure that her copy and Milin's copy of his written work had the same information as she asked Milin to add the new information to her sheet as well.

### 5.7 Milin's Third Individual Task-based Interview

| Date: | March 6, 1992 |
| :--- | :--- |
| Grade: | Grade 4 |
| Tasks: | Towers |
| Participant: | Milin (Individual Interview) |
| Researcher: | Alice S. Alston (R1) |

### 5.8 Overview

During this third individual interview, Milin discovered his "family" strategy. This helped him with the global organization of his towers of various heights and assisted him in making personal meaning for the doubling rule for the number of tower built when choosing from two colors. The same was true for the pattern for the towers when selecting from three colors. Since Milin still appeared to be unsure about his doubling and tripling rule working for towers of all heights, R1 assigned these two problems to Milin to work on again.

Milin started his third interview by recording his results from his earlier work. He recorded his towers of different heights on a piece of paper (Figure 5-25).


Figure 5-25. Milin's record of his towers.

Milin then built four two-tall towers when choosing from blue and black cubes and was able to explain his understanding of the doubling pattern by reasoning that each tower
could be turned into two new towers in the next height by adding a blue cube on top for the first tower and then adding a black cube on top for the second tower. He extended this by explaining that any two-tall tower could be made into a taller tower using this approach. He referred to all towers originating from a one-tall black tower as one "family" and those originating from a one-tall blue tower as another "family". This family strategy provided a global organization scheme for Milin. Milin was able to conclude that this strategy would also work for towers that were four-tall and five-tall. Milin was unsure about this strategy working for six-tall towers. During his work at home on six-tall towers, Milin was only able to construct 50 towers, not 64 towers as suggested by his new strategy. During this interview, Milin made several conjectures as to why the "family" strategy would not work for six-tall towers.

As Milin continued to build six-tall towers from his five-tall towers utilizing his "family" strategy, he seemed to be doubting his earlier conclusion that the pattern of doubling would suddenly breakdown for six-tall towers. This interview concluded with Milin's summarization of his results from his three-tall towers choosing from three colors.

During this interview, Milin was able to refine his justifications and reasoning even further and was able to provide a convincing argument for the doubling pattern which he had noticed as he was recording all his results in the beginning of the interview. He admitted that this written work had prompted him to recognize the doubling pattern which resulted in the development of his "family" strategy. Even though Milin understood how each tower turned into two new towers, he continued to use his earlier strategy of opposite by color to monitor his work throughout this interview.

The methods for building towers, monitoring towers and the types of reasoning and justifications used by Milin during his third interview are summarized in Table 5-6.

## OVERVIEW OF MILIN'S THIRD INDIVIDUAL INTERVIEW FROM MARCH 6, 1992

Methods for Generating Towers

1. Doubling pattern by adding a blue or a black cube on top: Example
2. Family tree: Example


3. "opposites" by color: Example
4. Tripling by adding a red, yellow or white cube: Example


Methods for Monitoring Work

1. Doubling by adding two color cubes
2. Tripling by adding three color cubes
3. Opposites by color

## Reasoning and Justifications Used

1. Doubling
2. Tripling
3. Family Strategy

Table 5-6
Overview of Milin's work from the third interview

### 5.9 Result Details

In the following sections, this researcher discusses various heuristics, strategies, representations, and justifications and reasoning Milin utilized during this interview in detail in order to better understand Milin's development of an argument by induction.

### 5.9.1 Taller Towers from Smaller Towers

### 5.9.1.1 Two-tall Towers from One-tall Towers

When R1 asked about what the towers of one and two-tall would look like, Milin built all one and two-tall towers as indicated in Figure 5-26. When R1 asked "why did that happen" (47), Milin responded that he had to put another cube on top of one-tall towers to make the two-tall towers as he explained, "if you had blue you could put another blue or a black on it " (52-54).


Figure 5-26. Milin's one and two-tall towers when choosing from two colors.

As R1 asked Milin to elaborate further, Milin explained that in order to get two-tall towers he could only add either a blue or a black cube on top of a blue cube since he only had two colors available. To demonstrate, Milin proceeded by putting his one-tall black tower in front of two-tall towers with black bottoms (Figure 5-27).


Figure 5-27. Milin shows how each one tall tower turns into two two-tall towers.

### 5.9.1.2 Three-tall Towers from Two-tall Towers

Next, R1 asked Milin about three-tall towers choosing from two colors and asked him to explain why there were only eight towers. Milin started by taking a blue cube and reasoned that, "see, just put another one on top of this, this, this" (86) as he moved the blue cube along the row of two-tall towers in front of them. R1 asked him to show her what he meant. R1 then gave Milin a duplicate two-tall tower with two blue cubes and Milin added a blue cube on top. Milin then added a black cube to the two-tall tower with two black cubes. During his work on these eight towers it is interesting to note that, for the first set of four three-tall towers, Milin matched the color of the new cube he added to the color of the top cube from two-tall tower (Figure 5-28). Also, even though Milin understood that a blue or a black cube could be added to each tower to get all towers in the next height, he still made the towers using his previous strategy of opposites by color. For example, the second tower he built was the opposite of his first tower and the fourth tower was the opposite of the third tower he had built.


Figure 5-28. Milin's set of first four three-tall towers.

Milin then made the second set of towers by adding the second-color cube on top. He first made the black/blue/black (fifth) tower as he reasoned, "this one, and the blue is still in the middle" (118), because he had already added the blue cube when he had made
the third tower. When R1 asked Milin if the black/blue tower of two could be turned into anything else, Milin moved his head side to side to imply no. Milin then proceeded to make the remaining three towers to complete his eight three-tall towers (Figure 5-29). He continued to used his opposite by color strategy in these towers to monitor his work.


Figure 5-29. Milin's eight three-tall towers built by adding another blue or black cube.

When Milin was finished with all eight towers, he rearranged the three-tall towers next to each two-tall tower as shown (Figure 5-30) by grouping them so that the bottom two cubes in the two-tall and the three-tall towers were the same. He also introduced the word "family" to refer to the group of towers originating from one-tall towers (138).


Figure 5-30. Milin's eight three-tall towers from four two-tall towers.

### 5.9.1.2.1 Opposite Pairs

During his explanation for building three-tall towers, Milin also pointed out how the blue/black/blue tower was similar to the black/blue/black tower he had already made and reasoned, "this is a duplicate of this" (126) as he compared the two towers to show that the two are the "opposites" of each other by color. It is interesting to note that even though Milin demonstrated an understanding of how each tower can result in two towers by adding a blue or a black cube, he still built them as opposites as shown in Figure 5-31.


Figure 5-31. Milin's opposites.

### 5.9.1.3 Four-tall Towers by Adding Two Different Color Cubes on Top

Next, R1 asked Milin if his method of adding two different color cubes on top would work for four-tall towers. Milin replied that it would (148). R1 then asked Milin to show her how the first three-tall tower in his collection would be transformed using his method (Figure 5-32).


Figure 5-32. R1 asks Milin to show what would happen to the first tower.

Milin took the two duplicates of the first three-tall tower with one black cube on the bottom and two blues cubes on the top, and then he added a black cube on top of one of the duplicates of the first tower and then added a blue cube on top of the second duplicate tower (Figure 5-33).


Figure 5-33. Milin builds the two four-tall towers by adding a blue and a black cube.

Milin further reasoned that the same would be true for all the other three-tall towers, and each tower would become two four-tall towers by adding a blue cube on top and a black cube on top. He also reasoned that there would be sixteen four-tall towers. He pointed to each three-tall tower as he argued, "two for this, two for this" (168). Milin was able to convey his understanding of the doubling pattern when choosing from cubes of two colors.

### 5.9.1.4 Extending to Five-tall Towers

Once Milin was able to make personal meaning for the doubling rule, Milin was able to extend this pattern to five-tall towers resulting in 32 five-tall towers and he stated, "And once you get to16 (Milin points to the column with five and thirty-two on his paper) you get all of them and you get 32 " (170).

### 5.9.2 Trouble with Six-tall Towers

Since Milin's answer for the six-tall tower problem that he had from his work did not match this doubling pattern, he announced that the pattern did not work for six-tall towers because it was "different". When R1 asked him to explain why it was different, Milin proceeded to show her his towers of six that he had worked on at home. He then told R1 that he had gotten 50 towers and he pulled out his paper with his written record on it with the answers for the two questions he was assigned from the previous interview (Figure 5-34).


Figure 5-34. Milin's written answers for the two problems added to his work record.

### 5.9.2.1 Milin's Explanation of His Work on Six-tall Towers

During his explanation for his work done at home, Milin stated that after he made a staircase he knew that there would be another "duplicate" staircase just like it with opposite colors. He started his explanation with one yellow cube moving in a staircase pattern. He also told R1 that he had other staircase patterns but was unable to find all the six-tall towers he had made at home. In his effort to reconcile this difference in his
answer from the work he had done at home and the doubling pattern he had just explained to R1, he suggested that "once you get to five it's an odd number or something, but it doesn't work" (200).

### 5.9.3 Introduction of "Families" (Global Organization)

At this point, R1 brought Milin's attention back to his towers he had built earlier. The researcher started pulling the towers back with Milin's help. During his explanation, about 12 minutes into the interview Milin introduced the word "family" (220) to the towers that are built up from the black one-tall tower as he reasoned, "it has to have a black on the bottom" (222) (Figure 5-35).


Figure 5-35. Milin talks about the "family" with black on the bottom.

Milin helped R1 to reconstruct the "family" up to and including the eight three-tall towers. Milin took each three-tall tower and arranged it in a family tree using his "family" strategy (Figure 5-36).


Figure 5-36. "Family tree" up to three tall towers.

As Milin worked on this stage of the family tree, he again declared that it would not work for the towers six-tall (250). He then added, "You could go to fives" (254) as he referred to his family originating from the black cube.

### 5.9.3.1 Extending the Family Tree

R1 suggested that they work on four-tall towers first. She also suggested that they should concentrate on one little family, and she separated the family tree collection into two smaller families. Then R1 added the two four-tall towers which Milin had built earlier during the interview to the first family (Figure 5-37).


Figure 5-37. First three-tall tower turned into two four-tall towers.

When R1 asked him about the five-tall towers for the first four-tall tower from the smaller family they were looking at, Milin told her that one black cube on the bottom all the rest would be blue, and the researcher built that tower for Milin and then they added the second tower to the family (Figure 5-38).


Figure 5-38. Two five-tall towers added to the family.

Milin reasoned that the row of towers with five-tall towers would have 32 towers. When the researcher asked him if he was sure about that, he replied that he was sure because he had done this problem in class and had gotten 32 towers. He further reasoned, that "if you follow the pattern up to this" (296), as he pointed to his paper, "it'll do the same thing [and] keep on doubling" (298). In other words he demonstrated his understanding of the doubling pattern.

### 5.9.3.1.1 Work on Six-tall Towers with Family Strategy

As R1 pointed to the first five-tall tower with four blue cubes on the top followed by a black cube on the bottom, she asked Milin about how this tower could be made into a six-tall tower. Milin replied that she could put either another black or a blue on top. He talked R1 through to make the tower for him. Milin then explained how the bottom part of the six-tall tower was the same as the five-tall tower as he argued that "you just put another, either a black or a blue on" (300) (Figure 5-39).


Figure 5-39. Two six-tall towers from the first five-tall tower.

### 5.9.3.1.1.1 Milin's Doubt

About 19 minutes into the interview, Milin again suggested that the pattern would break down for six-tall towers. When the researcher asked him for a reason why that
might be happening he replied that, "some families can't afford them." Nonetheless, Milin thought he might be "wrong" (332) as he had created only 50 six-tall towers, not 64 as suggested by the doubling pattern.

Milin continued to look for reasons why he only got 50 six-tall towers, using other strategies that had worked earlier for smaller towers. As Milin was trying to justify the number of towers he created at home he conjectured that "after five" (336) the pattern broke down. His reasoned that this might have happened because "ten's an even number and you can divide by five or something like that" (336). The researcher again drew Milin's attention to the second five-tall tower and asked him how it could be turned into a six-tall tower. As Milin appeared to be getting restless, R1 started to make the six-tall towers for him while Milin guided her (Figure 5-40).


Figure 5-40. Second five-tall tower turned into two six-tall towers.

Nonetheless, Milin still thought that the doubling pattern "just doesn't work on this one" (350) and that it would break down somewhere.

### 5.9.3.1.2 More Four-tall towers in the Family Tree

Since Milin was still doubtful that his family tree pattern would work for six-tall towers, R1 suggested that they finish the family tree. She offered to make the towers for Milin if he would tell her what to build. After they made the next two four-tall towers, R1 again asked Milin about what would happen to these towers in the next stage. Milin
replied that the four four-tall towers would turn into eight five-tall towers and those would turn into sixteen six-tall towers. When R1 asked Milin if he could think of any that would not work as he moved to six tall towers, he admitted, " not yet" (376) (Figure 5-41).


Figure 5-41. Doubling pattern up to six-tall.

At this point in the interview, Milin asked if anyone else was working on these problems. He asked about the number of towers Stephanie had gotten for this problem (401-407). It was at this point that R2 joined the group and suggested that it might not be a bad idea to bring a few students together to share their ideas. Even as the interview was concluding, Milin again suggested that the doubling pattern might work for six-tall towers and that he might be wrong that there are only 50 six-tall towers.

### 5.9.4 Three-tall Towers Choosing from Three Colors

Next, when R1 asked Milin about one-tall towers selecting from three colors, he replied that there would be three towers. She encouraged Milin to write down his answer on the paper he had used earlier to write down the towers when selecting from two colors. Milin drew a line to separate his answers with three colors from those from two colors. When R2 asked what Milin's guess was for three-tall towers when choosing from
three colors he replied 17 or 15 and he added that the real answer was 25 towers, referring to his answer for this problem from his homework assignment (Figure 5-42).


Figure 5-42. Milin's estimate and answers for the two questions.

When R1 asked Milin about the two-tall towers, Milin replied "six" (488). R1 then asked Milin to explain his answer. Milin started to build the towers, and after about 30 seconds he declared that there would only be nine two-tall towers (487-497). As Milin was constructing these towers, the first set of three towers had the top cube yellow and the bottom cube yellow/white/red. The second set had the white cube on the top with the bottom cubes white/red/yellow. He continued to make the last set in which the tops were all red and the bottoms were red/yellow/white (screen shot 1 in Figure 5-43) and added his results on his paper (screen shot 2 in Figure 5-43)


Figure 5-43. Milin's last set of two-tall towers and his written record.

When R1 asked him why the number of two-tall towers could not be six when selecting from three colors, Milin reasoned that because "there's 3 colors" (513), it could not be six.

Milin quickly added to his written work that there were 25 three-tall towers when selecting from three colors. When R1 asked him to explain, Milin responded by picking up the tower with red top and yellow bottom and reasoned, "this one would have another three" $(531)$. He proceeded to reason that the other two towers would also yield three towers each as he stated, "another three...another three...that would be nine times three...26" (533). When R1 asked him what nine times three was, he replied 27. Milin again suggested that the pattern would not work out as he looked at his paper. He did not think that selecting from three colors could follow the same kind of pattern as the towers selecting from two colors were following (537). He suggested that the two are different because the first pattern with the two colors was times two and this new one with three colors has to be different. R1 indicated to Milin that the pattern with tripling worked from one-tall to two-tall towers and Milin agreed with her. Then she asked him if it would work for the first two-tall tower with a red cube on the top followed by a yellow cube on the bottom and Milin, after a few seconds, replied that it might. The researcher then built the three three-tall towers from the first two-tall tower, as Milin guided her by telling her what to do (Figure 5-44).


Figure 5-44. Milin's three-tall towers from the first two-tall tower.

Still, Milin did not appear to be convinced. He again suggested that this pattern "breaks up somewhere"(559), as did the previous pattern after 32 towers.

### 5.9.5 Milin's Other Strategies

As the interview was concluding, R1 told Milin that the family strategy he had just utilized was impressive. When R1 asked him about what other strategies he had worked with, Milin explained that he had also used staircases, and then he suddenly started talking about how the family strategy just came to him during the interview when he was looking at the number of towers recorded on his sheet. He had "found this out ... today when [he] was reading this paper, so [he] just went along with it" (573). As Milin was discussing his strategies with R1, he added that building staircases was the wrong approach (581). When R2 asked him why the staircases helped him in other cases but not for this one, Milin stated that, "staircases don't have a couple of things," such as nice patterns (584) .

Towards the end of the interview, R1 asked Milin to think about these problems he had worked on and write about them. Even after the formal interview was over, he seemed to be still trying to consolidate all his results. He finally declared that he thought he did something wrong from 32 five-tall towers to six-tall towers and admitted that, "I don't think that pattern would break down" (598).

At this point both R1 and R2 told Milin that he was ready to talk to Stephanie, who has also been working on these problems. R1 suggested that Milin share his threecolor tower problem with Stephanie since she had not worked on that problem. She
asked Milin to again work on both problems involving six-tall towers when selecting from two colors and three-tall towers when selecting from three colors.

## 6 RESULTS: SMALL GROUP ASSESSMENT AND WRITTEN ASSESSMENTS

6.1 "Gang of Four" Group Assessment from March 10, 1992

| Date: | March 10,1992 |
| :--- | :--- |
| Grade: | Grade 4 |
| Tasks: | Three-tall Towers (Group Assessment) |
| Participants: | Milin, Michelle, Jeff, and Stephanie |
| Researchers: | Carolyn A. Maher (R2), Alice S. Alston (R1), and <br> Mrs. O'Brien (TR) |

### 6.2 Overview

This small group assessment was held with four students, Milin, Michelle, Jeff, and Stephanie, on March 10, 1992, at the request of Milin, who had expressed an interest in knowing the strategies used by his classmates. These students had been working on different tower problems, and they were brought together to share some of their ideas with each other. At the conclusion of his third individual interview, Milin had been asked to think about the six-tall towers when choosing from two colors along with the three-tall towers choosing from three colors.

During this group assessment, Milin, Michelle, Stephanie, and Jeff were able to present their ideas to each other and to the researcher. Milin was able to convince R2 and the other three students that each tower in a given height resulted in two distinct towers in the next height since there were only two colors to choose from. Thus he was able to demonstrate his understanding of the doubling pattern. Milin used letters to represent the colors of the cubes and was able to offer his inductive argument on multiple occasions throughout the group assessment. In each instance Milin was able to explain how a
shorter tower could be used to generate two distinct taller towers by first adding the cube of the first color to the base to generate the first taller tower and then adding the cube of the second color to that base to generate the second taller tower. Whereas he had doubt from his third interview, Milin was now certain and he concluded that there was no reason why this pattern of adding two cubes would not work for any height of towers.

Michelle, on the other hand, was able to identify the number pattern of doubling but initially she was unable to justify why the towers were doubling each time. She also made her towers randomly but she was able to find all eight three-tall towers based on her understanding of the pattern. Later, she was able to demonstrate that she understood why the towers doubled using Milin's inductive reasoning in conjunction with Stephanie's case-based reasoning.

Stephanie was also able to convince the other three students and the researcher of her reasoning using an argument by cases. She was able to arrange the eight towers into cases (no blue, exactly one blue, exactly two blue cubes "stuck-together", three blues and finally "two blues separated"). Even though Stephanie recognized the existence of the doubling pattern with numbers, she was unable to offer any reasons for why this pattern worked.

Jeff's approach during this assessment was not systematic. This is perhaps due to the fact that Jeff had been absent from class. He questioned the approaches used by the other three students. These questions elicited more refined explanations and reasoning from the other students. In the end, Jeff seemed to accept the approaches put forth by others.

### 6.3 Result Details

The small group assessment lasted for approximately 30 minutes. Details of this assessment are presented here.

At the beginning of the assessment, when R2 asked the students how many towers of six would there be, Milin replied, "probably 64" (8). When R2 asked him why? Milin replied that there was a doubling pattern and he stated, "you just times them [number of five-tall towers] by two" (12). Michelle also tried to explain how the next set of towers could be obtained by multiplying the number of towers at previous height by two. R2 then asked the group to explain their reasoning. Milin explained that for each one of the one-tall towers, one could add either a blue cube or a red cube on top to make two twotall towers from each one-tall tower, thus resulting in four two-tall towers (lines 43-51).

R2 then asked how many three-tall towers were possible. Michelle and Milin answered that there were eight such towers. Michelle explained that from the two towers that were one-tall you had to multiply them by two to get four two-tall towers and you would have to multiply the four by two to get the eight three-tall towers. Jeff stated that if the numbers were a pattern they should go 2, 4, 6 not 8; both Milin and Stephanie disagreed with Jeff. Stephanie explained that the pattern they observed was 2, 4, and then 8 . When R2 asked them to explain why, Milin explained that for each tower, "you could put another blue or another red" (86); i.e. each tower from the previous stage resulted in two distinct towers in the next stage (Figure 6-1). After Milin gave his explanation, R1 asked Jeff if he understood what Milin was explaining and Jeff replied, "yeah" (96).


Figure 6-1. Milin's explanation for eight three-tall towers.

Then, as R2 pointed to the tower drawn on the paper with red on the bottom and blue on the top, Milin again indicated that you could add a blue or another red. Milin explained further that this would be true for each one of the two-tall towers as he indicated that you could get two three-tall towers by adding a blue or a red cube on top. He also added that this could be done with all four two-high towers which would result in eight three-tall towers. When R1 asked all the members in the group if they understood what Milin was saying, they all agreed. At this point, Jeff appeared to be confused by what Michelle was drawing as Milin was giving his explanation. Jeff suggested that Michelle was alternating the red and blue in the top row of her grid and the bottom row of her grid. As Jeff started noticing duplicates, Milin pointed out that Michelle was not yet finished because the grid was supposed to be for three-high towers. When R2 asked the four students to explain how to get the eight towers, Milin reiterated that the only possibilities would be to add two colors to the top of each tower because there were only two color choices available. When the researcher asked what would happen next, Milin confidently replied that there would be " 16 " (134) towers that are four-tall. Milin was able to extend his doubling scheme to towers of different heights.

As Milin was explaining his scheme, Jeff was still not convinced about the eight towers and he wanted Michelle to show him the three-tall towers she was working on. Michelle then finished all eight three-tall towers using a grid with letters to represent the colors (Figure 6-2).


Figure 6-2. Michelle's representation for three-tall towers.

Milin too started drawing his representation. Milin initially drew a grid and used letters to represent the colors of different cubes in each tower just like Michelle's representation (Figure 6-3).


Figure 6-3. Milin's representation using a grid.

As the students were working on their representations, R2 called Milin's attention to Jeff's paper where he was representing towers by trying to add blocks to the bottom of his towers from the previous stage rather than to the top as Milin had done. Milin pointed out that cubes had to be added to the top. When Milin replied that the cube had to be
added to the top for his method to work, Jeff argued that it should not make a difference whether the cube was added to the top of a tower or to the bottom of a tower. This disagreement about whether to add a cube on the top or on the bottom led R2 to draw everybody's attention to the four two-high towers that Michelle had drawn earlier (Figure 6-4). R2 then asked everyone to work from those four towers in order to explore the idea introduced by Milin.


Figure 6-4. R2 asks students to use Milin's idea with the four two-tall towers.

After R2 asked all the students to work from the four towers exactly as they were drawn, Milin crossed out his earlier work using representation of a grid (Figure 6-5) and started anew by making separate towers resembling the towers drawn by Michelle (lines 172-187).


Figure 6-5. Milin crosses out his work.

Michelle started making another grid using letters to represent the color of each cube (Figure 6-6). After looking at Michelle's representation, R2 pointed out that she only wanted to see the four towers because they were trying to work on Milin's strategy.


Figure 6-6. Michelle's representation.

In response to R2's comments that Michelle's representation looked different from Milin's individual towers, he explained that even though Michelle drew a chart, the towers are just "put together" (186) rather than being separated and should not make a difference. Milin was able to demonstrate his ability to understand multiple representations of the same problem.

Milin now had four separate towers on his paper with letters representing the color of each cube in the tower. R2 asked the group to imagine the first tower in their minds and then use Milin's strategy. At this point Milin again indicated that you could just put another blue or another red on top. R2 asked him to show what the towers would look like to the whole group. In response, Milin drew two towers using his strategy by starting off with the first two-tall tower with a red cube on the top and a blue cube on the bottom and for the first three-tall tower he added a red on top and for the second tower he added a blue on top (Figure 6-7) (lines 189-217).


Figure 6-7. Milin's drawings the two three-tall towers.

R1 asked Milin to show everybody what he had done on his paper. Milin explained by pointing to his first two-tall tower (Figure 6-8) and adding that he took that first tower and added a blue cube on the top to get first three-tall tower. Milin went on to explain that he then took the same two-tall tower and this time added the red cube on top to get the second three-tall tower.


Figure 6-8. Milin explains how his first two-tall tower turned into two three-tall towers.

R2 reiterated what Milin had done by drawing her own representation (Figure 6-9) as she started with the two-tall tower and drew a line from this tower to the first three-tall tower and a second line to the second tower.


Figure 6-9. Researcher's representation of Milin's explanation.

When the researcher asked Michelle what she had done, Michelle indicated that she had not done what Milin had done, and that she just made the towers randomly and did not find any duplicates (218-220).

Stephanie also indicated that she just drew the lines and made table and she did not do it like Milin. At this point Milin asked Stephanie if she could convince the
researcher that there are only eight and that she has found them all. Stephanie then went on to present a justification by cases for the eight three-tall towers by explaining how she had drawn her towers with no blues, one blue, two blues "stuck together", three blues, and finally two blues separated (Figure 6-10) to account for all possibilities.


Figure 6-10. Stephanie's cases.

While all of them were listening to Stephanie's explanation for her eight towers and her pattern Jeff asked the group if it had to be a pattern. All the students explained that having a pattern helped you be sure that you had found all the towers. Milin replied that it is easier. During this discussion, Michelle took Stephanie's paper and wrote " 2 " on top of every three-tall tower (Figure 6-11) to show that each one of those eight towers would give two four-tall towers, thus incorporating Milin's strategy into her understanding of how the number of towers doubled each time.


Figure 6-11. Michelle adds 2's to Stephanie's chart.

When R2 asked about the five-tall towers and if that those towers also worked the same way the students all replied "yes." Stephanie responded that making the pattern
was the hardest part. She also indicated that now they knew how to find towers of different heights saying that she already knew what the ten-tall towers would be (400412). Stephanie explained how she had found the towers of ten to be 1,024 . This prompted Milin to quickly do the calculations to confirm Stephanie's assertion (Figure 6-12) and he replied that Stephanie was right.


Figure 6-12. Milin's calculations to find ten-tall towers.

The interview concluded with R2 challenging the students to explore why Stephanie's idea of multiplying 64 by 8 in a single shot did not work when she was trying to find the number of ten-tall towers (485-501).

### 6.4 Milin's Written Assessment: June 15, 1992

Towards the conclusion of the fourth grade, on June15, 1992 Milin was given a written assessment (Figure 6-13).

> Chris and Alex have been arguing about how many different towers can be built from Unifix cubes if there are two colors available andif each tower must be three cubes tall.
> Will you please settle the argument in a way that shows every possible tower and will convince Chris and Alex that you have not left any out and that there are no duplicates.
> Use whatever materials you like to work out the problem. Let us wam you, though, that we can send on to Chris and Alex only the pages on which you have recorded what you have done. We cannot send actual plastic cubes.
> Please be careful to write enough so that Chris and Alex will be convinced.

Figure 6-13. Written assessment from June15, 1992.

The students worked on this assessment in pairs. Milin was partnered with Stephanie for this assessment, but both students produced written justifications for the solution individually. Figure $6-14$ shows Milin's written response to this assessment. Milin started out by making a grid like the one he had used during the "Gang of Four" group assessment.


Figure 6-14. Milin's written response to the assessment from June 15, 1992.

In this grid, he used letters to represent the two colors, " B " for one color and "G" for the other color (Figure 6-15). Milin's first two towers were the two solid towers. Towers number three, four and five had exactly two blue cubes and one green cube.

Towers number six, seven and eight were the partners of tower number three, four and five using opposite by color strategy. Milin was able to record all eight towers using letters placed in a grid.


Figure 6-15. Milin's representation of three-tall towers.

A close examination of Milin's towers showed that the towers were also arranged by cases, as was done by Stephanie during the small group assessment a little over three months before this written assessment. The first tower in Milin's grid has all green cubes and the next tower has no green cubes. Tower numbers three, four and five all have exactly one green cube whereas tower numbers six, seven, and eight all have exactly two green cubes. This representation bears a striking resemblance to Stephanie's first representation during the small group assessment.

Since the students were asked to justify their solutions, Milin took each tower from his grid and paired it with its opposite by color or its opposite by inverting, as he had done during his initial work on the five-tall towers on February 6, 1992 (Figure 6-16).


Figure 6-16. Milin puts the towers in pairs.

The first two pairs of towers were opposites by color and the second two pairs were opposites by inverting, also referred to as "cousins" by some students in the class.

Milin also demonstrated his "doubling rule" on page three of this response. He showed his calculations for the tower heights ranging from one to ten-tall in the left hand and the right hand columns of his work and wrote out the results in the middle column of his work (Figure 6-17).


Figure 6-17. Milin's work for "doubling rule" and record of number of towers for heights one-tall to ten-tall.

### 6.5 Milin's Written Assessment: October 25, 1992

In the beginning of fifth grade, the students were given an individual written assessment (Figure 6-18). Figure 6-19 shows two pages of Milin's written response. The durability of Milin's argument can be seen from his written response. Milin represented each tower individually and used letters to represent colors of each cube of the each tower just as he had done during his group assessment from March 10, 1992 seven months earlier.

Please send a letter to a student who is ill and unable to come to school. Describe all of the towers that you have built that are three cubes tall, when you had two colors available to work with. Why were you sure that you had made every possible tower and had not left any out?

Figure 6-18. Written assessment from October 25, 1992.

Milin was able to generalize the doubling rule by listing the number of towers for various heights. He started his letter by indicating that there are two one-tall towers, four two-tall towers, eight three-tall towers, 16 four-tall towers and finally 32 five-tall towers (page 1 in Figure 6-19). Milin also included his representations of the individual towers with letters to represent the two colors of the cubes in each tower. He used R for red cubes and W for white cubes. On the second page of his response, Milin explained his reasoning for the "doubling pattern" by using the tower representations for one and twotall towers (page 2 in Figure 6-19). Milin also included the third iteration of the towers by drawing the representations for the three-tall towers on the bottom of page 1 of his response. In addition to including the three iterations of the tower heights, Milin also included the doubling number pattern in his justification.


Figure 6-19. Milin's written response to the assessment from October 25, 1992.
This response bears resemblance to Milin's "family" strategy that he developed during his third individual task based interview. In his explanation for the doubling pattern Milin used two one-tall and four two-tall towers to justify his three-tall towers. He appeared to have built these towers using his method for building taller towers from his third interview, where he first added a cube of one color of the four two-tall towers to make the first four three-tall towers and then added the opposite color cube to generate the remaining four three-tall towers. This demonstrated the durability of Milin's understanding of the doubling pattern in the heights of the towers and his inductive reasoning he had demonstrated during the small group assessment.

## 7 RESULTS: GUESS MY TOWER

| Date: | February 26, 1993 |
| :--- | :--- |
| Grade: | Grade 5 |
| Task: | Towers (Extension) |
| Participants: | Milin and Michelle (Group Work) |
| Researchers: | Alice S. Alston (R1), Carolyn A. Maher (R2), and Dr. Kelly (Visitor) |

### 7.1 Overview

On February 26, 1993, Milin now in grade 5, and his classmates had an opportunity to work on an extension of the towers task involving conditional probability.

You have been invited to participate in a TV Quiz Show and the opportunity to win a vacation to Disney World. The game is played by choosing one of four possibilities for winning and then picking a tower out of a covered box. If the tower you pick matches your choice, you win. You are told that the box contains all possible towers that are three tall that can be built when you select from cubes of two colors, red and yellow. You are given the following possibilities for a winning tower:

- All cubes are exactly the same color.
- There is only one red cube.
- Exactly two cubes are red.
- At least two cubes are yellow.

Which choice would you make and why would this choice be better than any of the others?

Assuming you won, you can play again for the Grand Prize which means you can take a friend to Disney World. But now your box has all possible towers that are four tall (built by selecting from the two colors yellow and red). You are to select from the same four possibilities for a winning tower. Which choice would you make this time and why would this choice be better than any of the others?

This new problem given above called for students to know the sample space for three-tall and four-tall towers when choosing from two colors of cubes. During this problemsolving activity almost eleven months after Milin's development of the "family" strategy, Milin was quickly able to answer the first question unlike his partner Michelle.

Milin once again explained his reasoning using an inductive argument in order to help Michelle. Even though Michelle had already had the opportunity to listen to Milin's inductive argument during the small group assessment, she did not make personal sense of this strategy until she was faced with the problem of finding all possible four-tall towers when choosing from two colors. She was then able to explain this reasoning to Stephanie and Matt, who had both been unsuccessful in finding all possible four-tall towers, despite Stephanie's conviction of the doubling rule.

Michelle was able to explain Milin's reasoning to Stephanie and Matt. Matt then shared this reasoning with another group of students as Stephanie listened. Finally, at the conclusion of the class session, Stephanie was able to share Milin's reasoning with the entire class. Even though both Michelle and Stephanie had listened to Milin during the small group assessment as he explained the doubling pattern using an inductive argument, they did not fully comprehend until they were faced with a problem. Since both students were unable to find all 16 four-tall towers using their recollection of the doubling pattern, they needed to find a new way to arrive at the solution to the problem at hand.

During this session, Milin and Michelle were also able to extend their answers to five-tall and six-tall towers.

### 7.2 Result Details

In the following sections, this researcher discusses various heuristics, strategies, representations, and justifications and reasoning Milin utilized during this session in chronological detail in order to better understand how Milin's ideas traveled to others in the classroom community.

### 7.2.1 Milin's and Michelle's Group Work

After the statement for the task was shared with the entire class, the whole class decided what was being asked in the problem and the students started working on the problem with a partner. During the discussion of the problem, Milin shared with the class his understanding of the problem and stated, "when it says that, it means ... which [choice] would probably have... the most possible ways that you could do it" (8).

The students were then asked to come up with their choice and their arguments for that choice. Milin and Michelle started working on the problem and when Michelle suggested that choice two or four would result in the same number of towers, Milin disagreed with her by indicating that in using choice four, three yellow cubes were also allowed. Milin quickly concluded that the fourth possibility would be the best. Michelle still appeared confused by the last choice because she was not sure what "at least" implied. Milin pointed out to Michelle that "exactly and at least are two different things" (50). Milin again insisted that choice four would be a lot better than choice two. Michelle still appeared to be confused by Milin's assertion. Milin quickly started to write down numbers next to his possibilities on his paper (Figure 7-1) and stated, "that'd be easy, this has only two ways, this has three ways nah this- yeah this has three ways..." as he wrote down his answers next to each possibility on his paper (54). Michelle replied that she did not understand what Milin was saying. Despite her confusion, she proceeded to tell Milin to pick number four and started writing the reason for picking that answer. They both wrote, "Because you could win if its two yellow or 3 yellow" on their respective sheets with the problem statement.


Figure 7-1. Milin's possibilities written next to each question.

Milin and Michelle then decided to represent their towers by drawing them using two different colored markers. They both drew two towers each but because the markers they chose were too similar in color, they decided to change them (62-68). They quickly started working on the second question with four-tall towers. Michelle asked Milin if they should choose the same possibility and Milin suggested that they build the towers (70-74). Michelle then stated that just like the last problem, with the four-tall towers, for choice number four you could have towers with two yellow cubes, three yellow cubes or four yellow cubes. Milin agreed with Michelle on her conclusion (74-75), and they proceeded to write their answer on the paper (75-76).

When R1 came to their table, Michelle asked her about the "at least" possibility even though she had gone along with Milin's explanation earlier and both students had picked the same answer. In response to Michelle's inquiry, R1 asked both of them, "What do you think?" (88) Milin appeared very confident and answered "you have to have two yellows or more" (93). In response to Milin's explanation, the researcher asked him how this information related to the problem and Milin argued that it is "the most ways you could have it" (97). When R1 told them that they had to be able to convince
others of their answers, Milin and Michelle decided to draw pictures of their towers to represent their solution. Although Milin indicated that he would rather build the towers than draw them, he still continued to draw.

### 7.2.2 Three-tall Towers For Possibility Number One

He started by drawing the solid towers (Figure 7-2) and grouping them by drawing a box around the two towers. Unlike his earlier representations of towers using letters to represent colors Milin used two different colored markers to represent his towers.


Figure 7-2. Milin's three-tall towers for the first possibility.

Michelle also drew her two solid towers (Figure 7-3). She stated that now she understood what the numbers on Milin's paper meant from earlier on. She was able to relate the two towers she had drawn to the two Milin had described next to his possibilities for the first question.


Figure 7-3. Michelle's representation for the first possibility.

### 7.2.3 Three-tall Towers for the Second Possibility

For the representation of his second possibility, Milin started by drawing three red cubes (screen shot 1 in Figure 7-4) starting from the bottom left and ending in the top right, making the staircase pattern with single red cubes (screen shots 2 and 3 in Figure 7-4). This was the same way he had made the models of his towers using the actual Unifix cubes during his earlier task based interviews.


Figure 7-4. Milin's representation for the second choice.

Michelle also drew three towers for the second possibility. As she was working on her representation, she asked Milin to see if she was right. During their work on this part of the problem Michelle asserted that " there is only two possible [towers] for this cause if it is on the top or the bottom it is the same thing all you have to do is turn it around" (133). She suggested that a tower with two blues on the top and a tower with two blues on the bottom were the same and therefore for this possibility there would only be two possible towers. Milin quickly suggested that if Michelle were to build these towers using Unifix cubes, all three towers would be different. Michelle seemed to have ignored Milin's explanation, and she still appeared to be confused about the number of towers for the second possibility trying to convince Milin of her reasoning. In the middle of their discussion, R1 came back to their table and Michelle asked R1 about the two towers in
question. Michelle demonstrated her reasoning by building the tower using the Unifix cubes and then flipped it upside down. Milin then actually built the two towers using the Unifix cubes and again mentioned that in first tower, the cube was on the top and in the second, the cube is on the bottom. He also stated that he "was right" (182), and that he could build three distinct towers for that possibility. Once Michelle indicated that she understood, the two students decided to build the towers using the unifix cubes rather than drawing them. After Milin and Michelle built the actual towers, Michelle separated them into groups by possibility (Figure 7-5).


Figure 7-5. Milin and Michelle's towers in groups for first three possibilities.

### 7.2.4 Explanation for Three-tall Towers Using Unifix Cubes

Milin and Michelle were able to come up with the answer to the first three questions quickly. For "all cubes are exactly the same color" they had two towers: one all red and the other all yellow (Figure 7-6).


Figure 7-6. Milin and Michelle's possibilities for first possibility.

For "there is only one red cube," they had three towers with the red cube in the top, middle, and bottom positions of the tower (Figure 7-7).


Figure 7-7. Milin and Michelle's possibilities for second possibility.

For "exactly two cubes are red", Milin and Michelle had three towers with the yellow cube in the top, middle, and bottom positions. Michelle started by making the first tower with two red cubes in the bottom positions. When Milin put all the towers in one group, Michelle told him to arrange the towers by possibilities (screen shot 1 in Figure 7-8). Michelle then built the tower with two red cubes on the top. Milin watched Michelle as she built the towers for their explanation. Michelle then built the last tower with the red cubes in the top and bottom positions (screen shots 2 and 3 in Figure 7-8).


Figure 7-8. Milin and Michelle's possibilities for third possibility.

After R1's discussion with Milin and Michelle, the researcher addressed the whole class to reach mutual understanding about what is allowed to be in the box. The class decided as a whole that in order for the game to be fair, there should be only one tower of each
kind. The researcher added that when a tower looked similar to another tower when turned upside down, it would not count as a duplicate.

As Milin and Michelle were set to find the towers for the fourth possibility, Michelle started building the towers and Milin noticed that the towers for the fourth possibility were already done as a part of the previous three possibilities. Milin started by first picking up the three towers that were already made for the second possibility. Then, he combined these with the solid yellow tower that was made for the first possibility to represent all the towers that were possible for the fourth possibility. He exclaimed that they could "just use this, this, this and this" (226), referring to the towers in screen shots 1-3 in Figure 7-9. Either Michelle was not paying attention to what Milin was saying or she chose to ignore him and instead insisted on making an "explanation" for the fourth possibility (171-174).


Figure 7-9. Milin's recognition of the towers for the fourth possibility from the already made towers.

### 7.2.5 Explanation for the Fourth Possibility

Milin and Michelle then built the four towers for the fourth possibility, giving the two students a total of 12 towers because the four towers for the fourth possibility were the duplicates of towers that were made as part of the other possibilities, as Milin had
pointed out earlier. Milin picked up this explanation for the fourth possibility and stated, "I guess this one" (230), referring to their answer for the question posed (Figure 7-10). After they were done building, Michelle explained how the eight towers were the only ones that would be in the box because they consisted of all the towers that represented all four possibilities and took out the four towers that were duplicates. This explanation was already given by Milin earlier (171), but Michelle either did not hear it or chose to ignore it.


Figure 7-10. Milin holding the explanation for the fourth possibility.

After Michelle took out the duplicate towers, Milin appeared to be a little unsure about the number of towers that were going to be in the "box," or the sample space. Before R1 came back to their table, Michelle explained to Milin how the eight towers can be rearranged to explain all four possibilities. She put the eight towers together and stated that if they had drawn pictures, those eight towers would be in the box (243). When R1 asked them to explain which towers would be in the box, Milin answered by pointing to the eight towers and commented that "these are the only towers that could be in the box" (272). When R1 asked them about the twelve towers, both Milin and Michelle explained that the twelve towers represented the explanations for the four possibilities. When R1
stated, "this is an explanation, it's like a picture, is it true?" (327), Milin agreed with her. Finally, Milin proposed that "if it has two or more yellows, then we win" (356).

### 7.2.5.1 Fifty-Fifty Chance

Referring to the four towers for the fourth possibility, Milin held them up and stated that there was "a fifty-fifty chance" (362) of achieving that result. After R1 asked him for his reasoning and just as Milin started to explain, Michelle cut him off and said that she thought it was not fifty-fifty (367).


Figure 7-11. Milin explaining his fifty-fifty chance.

R1 again asked Milin why he believed it was a fifty-fifty chance and Milin explained that "it is like half of it would be the fifty we won't have a chance and this half would be the fifty we would have a chance" (371) (Figure 7-11).

Michelle suggestion that if they pick number three as the answer, as she picked up the three towers for possibility number three, lead Milin to conclude "you won't have fifty-fifty chance" (373). When R1 asked Milin what chance this choice represented he replied that he did not know. R1 then asked whether it would be more or less than fiftyfifty, and Milin quickly replied, "less" (377).

### 7.2.5.1.1 Role of Researcher Intervention

Researcher intervention played an important role in aiding Milin to extend his concept of fifty-fifty chance to possibility number three. When R1 asked Milin what would happen if number three was picked, Milin reasoned that the chance of winning with this possibility would be less than fifty-fifty (377). Milin explained further that "the only way that we won't win is if we pick one of these (screen shots 1 in Figure 7-12) but the only way that we won't win for number one is if we pick any of these (screen shots 2 in Figure 7-12)" (406).


Figure 7-12. Milin explains when they will not win by picking up the towers.

Before R1 left, she asked both students to draw their explanations so that she could remember their discussion. Milin's representation for the three-tall towers was grouped by possibility (Figure 7-13).


Figure 7-13. Milin's representations for the four possibilities for three-tall towers

Since Milin switched the two markers he was using to draw his representation, for choice number three Milin drew two blue cubes rather than two red cubes but wrote letter "Y" next to the red cubes and letter "R" next to the blue cubes (Figure 7-14). Milin used the same method for his fourth possibility. Milin also included a representation for the "box" for three-tall towers representing the sample space for the problem. He used his earlier strategy of staircase pattern to complete this representation. Figure 7-14. Milin's representation of the "box" for three-tall towers is given in Figure 7-14.


Figure 7-14. Milin's representation of the "box" for three-tall towers

### 7.2.6 Four-tall Towers

As Milin and Michelle were working on their representations, Milin declared "now I know how to do it" (419). Since they had eight three-tall towers and Milin remembered the doubling pattern he concluded that they were "going to have to make $16 "(419)$. He continued working on his representation. As Milin and Michelle got ready to work on the possibilities for four-tall towers, R1 came back to their table and Milin declared that they would pick the "same choice" (443). While Michelle started by drawing the explanations for different possibilities, Milin drew a grid (screen shot 1 in Figure 7-15) and started making his box (using red and orange) by first utilizing the staircase pattern with one red cube (screen shot 2 and 3 in Figure 7-15).


Figure 7-15. Milin's partial work on his box up to tower number 8.

It was interesting to note that even before Milin started coloring the grid, he made exactly sixteen columns to represent the sixteen four-tall towers.


Figure 7-16. Milin's representation with cases before he crossed \#5 and \#12 towers.

Milin started out by making the towers that had one red cube moving in a staircase pattern followed by three towers that had the two red cubes moving in a staircase pattern. The next tower he added was a solid red tower followed by an orange tower. Then he continued the pattern by making tower number ten, the opposite of tower number seven, and he made the staircase patterns with two and one orange cubes as shown in Figure 7-16. The tower representations made by Milin were perfectly symmetrical with the second set of eight towers constructed as a mirror image of the first eight towers by employing opposite by color strategy. When he showed his "box" to Michelle she pointed out that there were duplicate towers. after Michelle's comment, out of the sixteen towers he made, Milin only kept fourteen towers. A closer look at Milin's
representation showed that Milin had two towers crossed out with black ink (Figure 7-17). At this point, the two students resorted to making the towers with cubes rather than drawing pictures.


Figure 7-17. Milin's final representation with alternating color towers missing.

### 7.2.6.1 Explanation for Four-tall Towers Using Unifix Cubes

Milin quickly built the four towers with one yellow cube moving in a "staircase" pattern, as he had done during his individual interviews. Meanwhile, Michelle put all the yellow cubes and the red cubes in groups of ten by disassembling the old towers. Milin continued to make more towers and made a tower with two red cubes on the top followed by two yellow cubes. At this point during their work, R2 came to their table and Milin and Michelle made the two solid towers next. When R2 asked them how many towers they could make, Milin replied "16" (482). R2 then asked Milin to explain his reasoning behind that answer. Milin explained "that for blocks of two, you have to multiply two from the first tower; that's two times two so second one is four, then two times four is eight times one more would be times two would be 16"(484). As Milin gave his
explanation, Michelle did not appear too sure. Milin suggested that he could convince her by building the towers using blocks (488).

### 7.2.6.1.1 Milin's Explanation for Towers One, Two, and Three-tall

Milin started by explaining that there were two one-tall towers. Milin then added a red cube on top of the red tower and a yellow cube on top of the yellow tower. Next Milin added a yellow cube on top of a red cube and a red cube on top of cube. R2 then placed two one-tall towers on the table, and Milin put the towers with the yellow bottoms in front of the yellow cube and the towers with the red bottoms in front of the red cube (Figure 7-18). Milin went on to add that they had already found the eight towers for three-tall towers (508).


Figure 7-18. Milin's one and two-tall towers.

As R2 asked him questions, Milin started to group his towers by putting them next to the tower from the previous stage. Michelle helped him with the explanation and she moved two three-tall towers closer to the solid red two-tall tower. During Milin's work, when R2 pointed out that one two-tall tower had two three-tall towers and the other one had
one, Milin made the second tower, giving him the first four three-tall towers (Figure 7-19).


Figure 7-19. Milin builds first four three-tall towers from two two-tall towers.

To explain his reasoning further, Milin took off the tops of the two three-tall towers (screen shot 1 in Figure 7-20), and he pointed out that the smaller towers looked just like the two-tall tower. Milin then added a red cube to one tower and a yellow cube to the other (screen shot 2 in Figure 7-20), as he already had before this explanation.


Figure 7-20. Milin's Explanation to explain how a red and yellow two-tall tower became two three-tall towers.

Michelle indicated that her understanding of the doubling pattern "because there is two different colors" (553). Milin also added to the explanation by reasoning that "every single time you have two off of each one" (556). Researcher intervention played a key role in getting Milin to develop a valid proof for his "doubling pattern."

### 7.2.6.1.2 Michelle's Explanation

Once Michelle demonstrated an understanding of Milin's reasoning, she was able to explain to R2 how the towers were being generated. She rearranged the towers as she explained how each lower height tower became two taller towers in the next stage. Michelle then commented that it was a lot simpler explanation than the last time. She thought this "because last time $\ldots$ we didn't do it like this. It is easier to explain it when you have it like this even though we didn't ... I think the answer is 16 because eight times two is 16 from every one $\ldots$ of these you add on two" (578). Milin, talking about the doubling pattern, pointed out that Stephanie was also familiar with this method. When Stephanie and Matt joined Milin and Michelle at their table, R2 asked Michelle to explain how the towers were growing as she commented, "Michelle, ... why don't you do it because Milin just explained it to Michelle. Let's see if Michelle knows it, Okay? Milin just explained it to me too" (597). As Michelle explained her reasoning to the group, Matt referred to the group of towers as a "family tree" and demonstrated his understanding of what Michelle was explaining as he commented, "see, you add a yellow or red on top of that" (602) as he pointed to a tower. R2 complimented Milin and stated, "that's wonderful! Milin, see you are helping people remember" (607), and suggested that maybe Matt could explain this to the class as she added, "cause Milin explained it to Michelle and Michelle explained to me" (616).

### 7.2.6.1.2.1 Milin and Michelle's Family Tree for Towers Including Four-tall

Milin and Michelle set out to make their family tree by clearing out a space on their table in order to arrange the towers one, two, three, and four-tall in a "family tree"
pattern (Figure 7-21). As they started working on this explanation, R1 joined their table and watched as the two students worked on their representation.


Figure 7-21. Milin and Michelle work on their "family tree" up to four-tall towers.

As Michelle added the sixteenth tower, R1 pointed to one tower just added by Michelle and stated that she was confused by that tower. Milin immediately agreed he did not "understand this one either" (732), and removed this tower from the tree. Michelle took the old tower and made the correct tower to add to the "family tree" (Figure 7-22).


Figure 7-22. Michelle takes the old tower and rearranges to make the new tower.

Milin and Michelle's final "family tree" is shown in Figure 7-23. Milin started with one-tall towers and then he demonstrated that each one-tall tower resulted in two two-tall towers by adding a red and then a yellow cube on top of each one-tall tower, resulting in four two-tall towers. Michelle helped Milin to finish the final stage of fourtall towers.


Figure 7-23. Milin and Michelle's final "family tree" with towers one, two, three, and four-tall.

Then R1 asked the two students, "What's the point?" (736), referring to their "family tree." Michelle appeared unsure of how to answer, but Milin quickly reasoned, "this is the box" (738), as he pointed to the row of four-tall towers in front of them. R1 asked the two students for the number of towers for each possibility and the two students picked up different towers from the sixteen four-tall towers and showed them to her. Figure 7-24 shows the tower choices for each possibility as identified by the two students from their "family tree."


Figure 7-24. Milin and Michelle's answers for each possibility.

Both Milin and Michelle worked in a very methodical way as they identified the towers for each possibility. For the third possibility they started on the right side of their family tree and identified all the towers as they moved to the left side of the tree. For the fourth possibility the two students started identifying the towers starting from the left side of the family tree as they moved to the right side of the tree.

When R1 asked them about their answer for the second question, both Michelle and Milin picked possibility number four as their answer. When R1 asked them about the two games, the three-tall towers and the four-tall towers game, they both replied that they would rather play the four-tall game, and Milin indicated that "you have a better chance" (806) with the four-tall game.

### 7.2.6.1.3 Milin and Michelle's Extension to Five-tall Towers

When R1 asked the two students about the number of five-tall towers, Milin quickly replied " 36 " rather than 32 , making an arithmetic mistake. Instead of correcting him R1 asked them to show her by counting from their four-tall towers. Both students counted the towers and realized that they had made a mistake in doubling. Michelle checked the answer with paper and pencil while Milin recounted the towers to make sure. Milin went on to reason that the eleven towers that were winners for the four-tall towers would still be winners for the five-tall towers by stating that "this one is going to be a winner also" (884) (Figure 7-25), as he pointed to a tower in their family tree.


Figure 7-25. Milin points out another four-tall tower that would become a winner in fivetall towers game.

Milin proceeded to give a justification for his reasoning and explained that it is "because you are going to put a red and aellow on top of that" (889). R1 then helped the students by making a five-tall tower from that tower, as Milin guided her by adding a yellow cube on top. R1 questioned Milin about the tower with a red cube on the bottom followed by two yellow cubes and a red cube on the top. and Milin quickly responded, "two" (895). Milin and Michelle then went through all the four-tall towers and decided how many winners would result from each four-tall tower (Figure 7-26).


Figure 7-26. Milin and Michelle count off how many five-tall winner towers will result from each four-tall tower.

As Milin was counting the towers from their "family tree", Michelle kept a written record of the number of winning towers. Milin then assisted Michelle in figuring out the total number of winning five-tall towers. He started off by counting the number of two's in their written record and entering this information in his watch calculator (Figure 7-27) and declared that " 11 times two, $22,23,24,25,26$ " (938), while he added the four ones. R1 checked the number again with the help of Milin and they confirmed 26.


Figure 7-27. Milin uses his calculator to find total of five-tall winning towers.

R1 then posed the question, "How many losers?" (941) and Milin quickly responded, "six" (945). R1 and Milin quickly counted these towers from the four-tall towers (Figure 7-28). At this point, both Michelle and Milin concluded that the fourth possibility was the best choice.


Figure 7-28. Milin counted the number of losing towers coming from four-tall towers.

### 7.2.6.2 Generalizing from Three, Four, and Five-tall to $\boldsymbol{N}$-tall

R1 asked Milin and Michelle focusing questions that helped the two students summarize all their results for the three different heights of towers. R1 asked Milin and Michelle questions that helped them to generalize their answer for the problem for any height of towers. As R1 asked them questions about the number of winners and losers for the three heights of towers they had figured out, Michelle kept a written record at the request of the researcher (Figure 7-29).


Figure 7-29. Michelle kept a written record of winners and losers.

When R1 asked them about the number of towers of height six, Milin predicted " 64 " (990). R1 then asked them to predict the number of losers in the six-tall towers, and Michelle quickly predicted seven and explained that the number of losers was increasing by one each time as she pointed to the paper in front of her. When R1 asked the number of winning towers for six-tall, Milin replied "57" (1018). In talking about the new patterns, Milin added that, "for every height, you add one more loser" (1025).

Table 7-1 shows the summary of the results obtained by Milin and Michelle for towers three, four, and five-tall. In addition, it also shows their prediction for the towers of height six-tall.

|  |  |  |  |
| :---: | :---: | :---: | :---: |
| Height | Total Towers | Number of Winning <br> Towers | Number of Losing |
| Three-tall | 8 | 4 | Towers |
| Four-tall | 16 | 11 | 4 |
| Five-tall | 32 | 26 | 5 |
| Six-tall | 64 | 57 | 7 |

Table 7-1
Milin and Michelle's predictions for number of winning and losing towers

### 7.2.6.2.1 Milin and Michelle Share with R2 and Dr. Kelly

R2 asked Milin why they had picked possibility number four for the three-tall tower problem, and he reasoned that "there's a fifty-fifty chance of that and everything else was less than fifty-fifty chance" (1037). R2 then asked Milin to make a new table for her based on the number of yellow cubes in a tower. They used their "family tree" to find all the categories of the towers. This provided the students with another way of convincing others of their answers for towers three and four-tall and for picking the fourth possibility. Milin was able to show that the number of towers for three-tall towers was four because the answer included all the towers that had two or more yellow cubes (Table 7-2).


Table 7-2
Summary of three-tall and four-tall towers by cases

As the class was concluding, Dr. Kelly noticed the rule Milin and Michelle had stated about the number of losers increasing by one as the height of the towers increased. Dr. Kelly asked the students to explain how that rule worked and Milin volunteered to explain. Milin showed them the written work and explained that for three-tall towers they had four losers, and for four-tall towers they had five losers. At this point, R2 drew their attention to the table she had asked them to make with the number of towers based on the number of yellow cubes (Table 7-2). She asked them to find the number of towers for the fourth possibility, and they got 11 towers, which gave them five losers as Milin had stated earlier. When R2 questioned them about their prediction for six-tall towers, Milin was able to state that there would be seven losers and 57 winners. With this, R2 summarized what various groups of students were working on and suggested that they share ideas for the few minutes they had left in the session.

### 7.2.6.2.2 Stephanie's Presentation

Figure 7-30 shows how Milin's idea of mathematical induction traveled through his classmates and resulted in Stephanie making a presentation to the entire class using Milin's family tree strategy.


Figure 7-30. Dissemination of Milin's idea of Mathematical Induction.

Even though Stephanie had the opportunity to listen to Milin's inductive pattern during the group assessment and then during her visit to Milin and Michelle's table during the
present (Guess My Tower) task, she did not make personal meaning of the strategy till later, after she heard Matt explain the strategy to Robert and Michelle R.

## 8 CONCLUSIONS

Milin had been involved in the Rutgers Longitudinal Study since his first grade. During Milin's school years, researchers visited classrooms to engage students in rich, open-ended problem-solving tasks in various mathematical strands. The task design and the sequence of tasks introduced to the students were carefully developed by the research team to provide opportunities for students to build mathematical ideas in ways that encouraged sense-making and collaboration, and required students to provide justifications for the validity of their solutions. In the early years (third, fourth, and fifth grade) the counting tasks that were designed by the researchers called for students to classify, organize and reorganize information.

As a participant in the longitudinal study, Milin had been videotaped solving problems with other children and in follow-up task-based individual interviews since his first grade. In third grade, Milin worked with a partner on a counting problem inviting them to build all possible four-tall towers Unifix cubes of two colors. Questions that were not resolved by the students were left for future exploration.

In general, the tower problem requires students to determine the number of $n$-tall towers that can be built when selecting from cubes of two colors. Since the choice for the color of each cube in the tower is independent of the color choice of any other cube in the tower, there are exactly two possible color choices for each cube in the tower.

The present study explored the process by which nine-year old Milin developed solutions for the towers task during fourth and fifth grades. The first step in this process was a problem-solving session where Milin and his partner, Michael, developed five-tall towers choosing from two colors. This session was followed by a class discussion, three
individual task-based interviews, and a small group assessment with four students. In addition, Milin also had an opportunity to work on two written assessments to determine the durability of his arguments and reasoning. The first written assessment was given at the end of the fourth grade and the second at the beginning of the fifth grade. Finally, during fifth grade, Milin had an opportunity to work on an extension of the towers task involving conditional probability.

Through problem solving, Milin developed a justification that took the form of inductive reasoning to convince the researcher and his classmates of the validity of his solution. One might ask how Milin developed this form of argument. An important contribution of this work is that it documents and traces the details of his building the structure of his inductive argument. The following sections discuss the heuristics, strategies, and types of justifications and reasoning used by Milin.

### 8.1 Heuristics and Strategies Used by Milin

According to Polya (1945), heuristics are, "mental operations typically useful for the solution of problems". Schoenfeld (1985), in discussing the attention given to teaching of heuristic strategies stated, "...the attention devoted to them has not, in general, been adequately repaid with success; attempts to teach students to use heuristic strategies have consistently produced less than was hoped for". Researchers such as Polya, and Schoenfeld recognize heuristics as strategies that should be taught in order to enhance their problem solving ability. On the other hand, Maher and Martino (1996b, 1996c), Maher and Speiser (2001), and Maher, Martino and Alston( 1993) argue that learners, in a supportive culture of sense-making, can develop their own heuristics and strategies over time.

In the course of his exploration of n-tall towers, Milin developed and implemented several heuristics and strategies. These included guess and check, opposite by color, opposite by inverting, staircases and other patterns, estimation, considering simpler problems, generating taller towers using smaller towers and family tree. These heuristics and strategies are discussed below.

### 8.1.1 Random Tower Strategy: Guess and Check

Milin used the guess and check strategy to find new towers during his work on the five-tall tower task. Michael and Milin started out by building random towers. They then checked to make sure that they had not made that tower already by comparing it to the previously made towers. Milin accomplished this by moving a newly made tower over the previously made towers to identify duplicates. He continued to use this strategy during his first and second interview. Milin used this strategy for local organization of his towers until he was able to come up with a global organization for keeping track of his towers being built during the extension of the original towers task.

### 8.1.2 Local Organization Strategies: Opposites and Patterns

Local organization strategies allow a learner to find a group of mutually exclusive solutions. However, the solution set is not exhaustive. Furthermore, the solutions generated using different strategies could overlap.

Milin discovered and implemented several local organization strategies. Some let him generate pairs of tower, while others let him generate as many as five five-tall towers. These strategies included: opposite by color, opposite by inverting, and pattern based strategies. He further combined strategies to develop hybrid strategies.

### 8.1.2.1 Opposite by Color Strategy

Early on, Milin and Michael realized that each tower had a partner - i.e. a second tower with colors of the corresponding cubes changed. During their initial group work session on February 6, 1992, each time Milin and his partner found a new tower using guess and check, they both utilized the opposite by color strategy to find a second tower of the pair. Milin continued to use this strategy during his individual interviews as well. Later during his work, Milin used this strategy not only to generate pairs of towers but also to create and justify opposite groups of towers.

Opposite by color is a local organization heuristic. It was used by Milin and Michael to view towers in groups of two, thus providing them with an effective method for managing their collection of towers. For example, if they uncovered a duplicate tower, they knew that they had to delete not just one tower, but rather a pair of towers.

### 8.1.2.2 Opposite by Inverting Strategy

Milin also used the opposite by inverting strategy to make some of partner towers during his group work on February 6, 1992, and to verify that he had a comprehensive solution for the three-tall tower problem in a written assessment June 15, 1992. With this strategy, Milin used the up-side down orientation of a newly found tower to complete a pair of towers. This was also another local organization heuristic used by Milin.

### 8.1.2.3 Opposite Hybrid Strategy

During his second individual interview from February 21, 1992, Milin demonstrated that he had developed a new strategy to identify new towers. Unlike guess and check, this strategy was not random. Rather, he combined the systematic opposite by color and
opposite by inverting strategies. He would use opposite by color to generate a partner to complete a pair. He would then apply opposite by inverting to the original or the partner tower to create totally different tower. To complete the second pair, he would use opposite by color. In effect, after finding one unique tower, he could systematically find three more towers (Figure 8-1).


Figure 8-1. Milin's Opposite Hybrid Strategy.

Even though Milin shared this strategy with the researcher during the second interview, there was evidence that Milin had used this strategy during his First interview also. First during his work on five-tall towers and later during his work on three-tall towers.

### 8.1.2.4 Pattern-based Strategies

During the sharing session, Milin had the opportunity to see some of the patterns used by other students based. Most of these were based on what the students called the "staircase" pattern. Not only did these patterns help him to construct towers but they also helped him to monitor for duplicates by looking at groups of towers.

### 8.1.2.4.1 Staircase Pattern with One Red/Yellow Cube

During the first interview, Milin utilized this strategy to build a group of towers. He made five five-tall towers by moving a cube of one color from the first floor of the first tower, to the second floor of the second tower, progressing to the fifth floor of the fifth tower. He continued to use this pattern to find four-tall towers and during his work on "Guess My Tower" problem. Milin first used this pattern in his written representation of towers and later during his model building.

### 8.1.2.4.2 Staircase Pattern with Two Red/Yellow Cubes

In his first interview, Milin demonstrated that staircase pattern could also be used by moving two cubes of one color. Using this approach, he created eight unique five-tall towers, four with two red cubes and four with two yellow cubes. As demonstrated in Figure 5-17, using this strategy will result in six towers, with two of them being
duplicates. Because of this, Milin argued that we have to decide whether to build the staircase pattern based on red cubes or yellow cubes.

### 8.1.2.4.3 Solid and Alternating Color Patterns

Milin was able to argue that there were only two possible solid towers since there were only two colors to choose from. He also demonstrated that there are two alternating color towers regardless of the height of the tower. This approach resulted in uncovering more unique towers.

### 8.1.2.4.4 Other Color Patterns

In addition to the staircase patterns, Milin also made other patterns with towers that had two cubes of one color and three cubes of the other color. During the sharing session for five-tall towers, Milin was able to reason that there would be three ways to have two reds separated by one yellow, two ways to have two reds separated by two yellows, and one way to have two reds separated by three yellows. During his first individual interview, Milin found the six towers where two yellows were separated by red cubes using his earlier strategy of guess and check and constructed the partner towers using his opposite by color strategy.

### 8.1.2.4.5 Doubling Pattern

On February 21, 1992, during his third interview, in his efforts to find a way to justify that he had found all possible towers of a given height, Milin started looking at how taller towers were being generated from smaller towers. During this interview Milin recorded the results of all his work up to this point and in so doing realized the doubling
pattern - there are two one-tall towers, four two-tall towers, eight three-tall towers, etc.
Figure 5.28 shows how he used this knowledge to create eight three-tall towers.

### 8.1.3 Global Organization Strategy: "Family Tree"

As mentioned earlier, a local organization strategy generates a limited number of solutions and results of several such strategies could result in duplicate solutions. If a learner were to apply multiple strategies to a tower problem he/she would have to eliminate all the duplicate towers and find a way to justify that he/she had found all possible towers. On the other hand, a global strategy would generate all mutually exclusive and collectively exhaustive solutions. It was Milin's understanding of the doubling pattern, that seemed to have led him to invent a "family tree" strategy.

Milin first demonstrated an inductive argument during his third interview when he started arranging the towers as families. Milin started this interview by recording the number of towers of various heights when choosing from two colors. He built the one-tall and two-tall towers using the blue and black cubes and explained that the only way to turn a one-tall tower into two-tall towers was by adding either a blue cube on top or a black cube on top. When Milin built the three-tall towers, he started with four two-tall towers and employed the same strategy. He also monitored his work by building towers using his earlier strategy of opposite by color. He then arranged the towers into families, where all towers in each family originated from the same tower of the previous height. Milin was able to extend his families to six-tall towers by adding a blue or a black cube on top of each tower from the previous height.

Milin again used this argument based on his inductive reasoning during the group assessment on March 6, 1992. During this assessment, Milin used pictures of towers,
with letters representing the color of each cube, to demonstrate how the same two-tall tower could be made into two different three-tall towers by adding the cube of the first color on top and by adding the cube of the second color on top. Milin used his inductive argument multiple times during this assessment. Milin was able to explain to the researcher why the doubling pattern worked.

### 8.2 Types of Justification Used by Milin

Milin used various methods to justify that he had found comprehensive solutions to the towers problems. Eventually, he generalized the form of the inductive argument after going through several iterations. During the beginning work on the problem task, Milin used guess and check, opposite by color, and opposite by inverting to create towers and their partners. During this time, the only justification he was able to provide for the completeness of his solution was the amount of time it was taking to find new towers and that he was unable to find any more towers using his random methods of building towers because he kept finding duplicates.

During his first and second individual interviews, Milin moved more towards other local organizations using strategies such as the staircase pattern. Milin was able to make a group of towers and reasoned that the group will have an "opposite" group by color that could be created. He was also able to provide justification by contradiction for staircase pattern.

During his work on six-tall towers Milin pointed out that these patterns become harder to keep track of as the towers get taller because there are more places to change things. During this phase, Milin was only able to provide partial justification for his solution. The only full justification he could provide was in one instance of an
exhaustive case dealing with the solid towers. Milin reasoned that when selecting from two colors, there are only two "solid" towers possible regardless of the height of the towers.

The need to be able to convince others that he had found all possible towers and that there were no more towers, led Milin to search for a global organization. Milin came up with proof by partial cases based on number of cubes of one color in a given tower and finally, Milin was able to provide an argument by mathematical induction that from any tower, one can make two new towers by adding the first color or the second color cube on top. Milin was able to demonstrate how the doubling rule worked.

Heuristics, strategies, and types of reasoning and argumentation used by Milin during his work on tower tasks, from February 1992 through February 1993 are summarized in Table 8-1.

| Type of Activity | Other <br> Participants | Task | Heuristics and Strategies used by Milin | Types of Reasoning and Argumentation used by Milin |
| :---: | :---: | :---: | :---: | :---: |
| Group Work <br> 2/6/92 | $\begin{aligned} & \text { Michael, R1, } \\ & \text { R2, R3 } \end{aligned}$ | To build towers five-tall selecting from two colors <br> To convince others that you had found all possible towers | Unifix cubes Trial and Error <br> "Opposite" by changing color <br> "Opposite" by changing orientation Building in pairs Monitoring work to check for duplicates | Taking too much time Only two towers that are solid Keep finding duplicates |
| Sharing Session 2/6/92 | Whole Class, R1 | To build towers five-tall selecting from two colors <br> To convince others that you had found all possible towers | Unifix cubes Guess and check "Opposites" by changing color and orientation to build pairs <br> Building by cases based on number of cubes of given color Patterns | Has to be an even number of towers <br> Keep finding duplicates Cases with two cubes of a color separated by one, two, and three cubes of the second color <br> Staircase pattern <br> By contradiction Opposites |


| Type of Activity | Other Participants | Task | Heuristics and Strategies used by Milin | Types of Reasoning and Argumentation used by Milin |
| :---: | :---: | :---: | :---: | :---: |
| First task <br> based <br> Interview <br> 2/7/92 | R1, Mrs. O'Brien | Towers five, one, two, and three-tall | Partial cases with staircase pattern "Opposite" by changing color <br> Guess and check Estimation skills Paper and pencil | Proof by contradiction Reasoning by cases Opposites Patterns |
| Second task <br> based <br> Interview <br> 2/21/92 | R1 | Towers fourtall when choosing from two colors Towers one, two, and threetall when Choosing from three colors | "Opposite" pairs <br> Local organization <br> Patterns <br> Estimation skills <br> Doubling and tripling rule <br> Paper and pencil | Proof by contradiction Towers of higher height will have more towers Even number of towers when choosing from two colors Odd number of towers when choosing from three colors |
| Third task based Interview 3/6/92 | R1, R2 | Towers one, two, three, four, five, and six tall choosing from two colors | Opposites <br> Patterns <br> Doubling with families Paper and pencil | Staircase patterns <br> Harder to keep track of taller towers using staircases Doubling rule with family strategy |
| Small <br> Group <br> assessment <br> 3/10/92 | R2, Stephanie, Jeff, and Michelle | Towers threetall choosing from two colors | Doubling rule Inductive <br> Opposites Grid with letters to represent towers Pictures of individual towers Paper and pencil | Each tower gives two new towers by either adding a blue or a red cube on top Patterns help with keeping track of towers "Times two" doubling rule |
| Written Assessment 6/15/92 | Stephanie | Towers threetall Milin's Individual Response | Cases <br> Opposites by color Opposites by inverting Doubling rule | Patterns help with keeping track of towers; "Times two" doubling rule; Cases |
| Written Assessment 10/25/92 | Milin | Towers threetall | Doubling rule Towers of smaller heights to show doubling | Used pictures of towers to justify the doubling rule Generalizes for different heights |
|  | Michelle, R1, R2 | "Guess my tower" | Cases <br> Family tree <br> Written representations | Cases <br> "Doubling rule" with family tree Inductive reasoning |

Table 8-1
Summary of Milin's work from 2/6/92 through 2/26/93

### 8.3 Other Findings

As discussed in chapter 2, according to Davis and Maher (1990), even young students are capable of working on challenging problems when provided with the right environment and conditions. In the present study, the researchers required that students offer convincing arguments to justify their solutions. Explanations of how a task was solved were communicated to researchers, usually in verbal form as well as in written work. In the early years, certain tasks, such as building towers of a specific height selecting from Unifix cubes of either two or three different colors, were revisited on multiple occasions, often with variations or extensions to the task that was originally given to students. The requirement to convince others of the validity of solutions fostered the development of mathematical reasoning and proof making (Francisco \& Maher, 2005; Maher, 2002; 2005). Davis (1997) also conveyed that in a student-centered environment students are able to build mathematical ideas and understanding for themselves. The evidence from chapters 4 through 7 suggests that Milin was successful in developing meaningful mathematical reasoning skills. As Milin had opportunities to work on towers of various heights ranging from one-tall to six-tall, and revisit his strategies, he observed the doubling pattern for consecutive heights of towers. This observation led Milin to generalize using his argument by mathematical induction. This development was not achieved in isolation. As Milin had the opportunities to listen to the explanations of other students and present his own ideas to others, he was able to modify and build on his earlier representations of the solution. On March 10, 1992, Milin had an opportunity to listen to Stephanie's proof by cases during the small group assessment. In his written
assessment from June 15, 1992, Milin drew his representation in which he had his towers grouped by cases. During Stephanie's justification, Milin and Michelle had wanted her to include all the towers with two cubes of one color and the third cube of the second color in a single case. Milin's written assessment was arranged so that all the towers with same number of cubes of a color are grouped together. Milin started out by making the two solid towers, three towers with two blue cubes and one green cube, and the three towers with one blue cube and two green cubes. Milin's representation bears a striking resemblance to Stephanie's Initial drawing from the small group assessment (Figure 8-2). During this assessment Milin also conveyed his understanding of the doubling pattern by including the heights of towers ranging from one-tall to ten-tall.


Figure 8-2. Influence of Stephanie's reasoning on Milin's solution

Milin used a variety of representations during his work on the towers tasks, including, models with Unifix cubes, grids, pictures of towers with letters to represent
different colors, and written patterns and calculations. Milin's personal representations went through various iterations during his work on the original tower problem and its extensions in order for him to develop a stable and convincing argument. The extensions of the tower problem provided Milin with opportunities to build upon his earlier understanding of the doubling pattern of the tower problem.

As indicated by Davis (1992), students acquire understanding when they are able to fit new ideas into their already existing schemes. During Milin's work on the tower tasks considered in this study, he was able to demonstrate this. Milin approached the tower problem by using guess and check to build random towers and then monitored his work for duplicates. To simplify the task and to keep track of the towers Milin developed local organizational schemes like partners and opposites. His individual task based interviews gave Milin a chance to look at his own solutions and extend his own mathematical understanding and ideas. When Milin was unable to justify whether or not he had all of the towers using his previous local organizations he focused reorganizing his towers based on cases using certain patterns. Using these patterns helped Milin to provide a proof by contradiction for the staircase pattern towers. Later, when these patterns proved inadequate for taller towers Milin was forced to modify his earlier schemes and think about a more global organization. Once Milin built this global organization using his "family" strategy to justify the doubling pattern in the heights of the towers, he was able to use this understanding to justify his solution for three-tall towers during his written assessment almost seven later. A close examination of the letter written by Milin shows similarities in how he had built the three-tall towers during his third individual task-based interview on $3 / 6 / 92$ and in his written representations drawn as a part of his work on the
written assessment from 10/25/95. During both instances Milin built the first four towers by adding a cube on one of the available colors and then added the second color to make the second set of four towers (Figure 8-3). One difference that is evident is that during his initial work, Milin also use the opposite by color strategy to monitor his work. The towers were consistently built in order, as opposites by color during the third individual task-based interview.


Figure 8-3. Durability of Milin's "family" strategy

The way Milin made personal meaning is also consistent with the model of understanding indicated by Pirie and Kieren (1992, 1994). As discussed in chapter 2, according to Pirie and Kieren (1992), in their model for understanding, growth in understanding is seen as back-and-forth movement (also referred to as folding back) among the various levels. Milin was able to modify and use his intuitive ideas to "formalize" his understanding of the doubling pattern, returning with deeper understanding of the problems. This personal meaning later helped him to convince others of the validity of his argument.

### 8.4 Dissemination of Milin's Ideas

Milin started thinking about the "family" strategy of constructing taller towers from previous towers of smaller heights during his third individual task based interview on March 6, 1992 in an effort to understand the doubling pattern in towers of various height when choosing from two colors. Milin then had a chance to share this "family tree" reasoning with Michelle, Jeff, and Stephanie during the "Gang of Four" small group assessment on March 10, 1992. Milin demonstrated this by using a representation of towers with letters to signify the colors of the cubes used in each tower. Since all four students agreed that, there were only four two-tall towers possible, Milin used the first two-tall tower with a red cube on the top and a blue cube on the bottom to explain his doubling rule. He reasoned that by adding a red cube or a blue cube on top of this twotall tower it was possible to get two new three tall towers. He demonstrated by drawing two new tower representations with the two-tall tower as the base, and one tower with a blue cube on top and the second tower with a red cube on top as discussed earlier in chapter 6. As Milin and Michelle explained to Jeff that the doubling pattern they had noticed in the problem was a result of the two colors of the available cubes, Stephanie was silently listening to this explanation. When the researcher suggested that the students build three tall towers using Milin's doubling method, Stephanie still chose to build her towers using cases. It appears that since Stephanie's method made sense to her she did not feel the need to use Milin's method.

Finally, during the "Guess My Tower" problem task on February 26, 1993, Milin again used this family tree strategy to convince Michelle that from a tower of any height, there were always two possible towers of the next height by adding the cubes of the two
available colors. Even though Michelle had explained the doubling pattern to Jeff using Stephanie's cases and Milin's reasoning about eleven months earlier during the small group assessment, she did not remember it either, just like Stephanie.

Milin explained his strategy again to Michelle using a simpler problem using actual cubes rather than drawings he had used eleven months earlier during the small group assessment. This time he took his one-tall towers and demonstrated by actually building the towers that he could either add a red cube on top or a yellow cube on top of each onetall tower resulting in four two-tall towers. He built his family tree up to three-tall towers and then Michelle helped him to complete their family up to four-tall towers. It can be concluded that when Michelle was unable to use her method of building random towers and then finding duplicates to solve the four-tall tower problem as she had done with the three-tall towers during the small group assessment eleven months earlier. This need for a solution might have forced Michelle to really listen to Milin's explanation the second time around.

During his explanation of his strategy, Milin suggested that Stephanie should also know this strategy because they had shared this during the small group assessment held on March 10, 1992. This prompted R2 to invite Matt and Stephanie to Milin and Michelle's table, and Michelle was able to demonstrate her understanding of Milin's method to Matt and Stephanie. Later during the same session Stephanie tried to explain Milin's "family" strategy to Bobby and Michelle but was unable to convey how the doubling pattern was working. It appears that during this explanation Stephanie was just looking for confirmation for her doubling number pattern and when she recognized the
towers she and her partner, Matt were missing she did not really look at Milin's method as Michelle was explaining.

Stephanie's failure to convey why the doubling pattern was working prompted Matt, her partner, to quickly tell Stephanie to "move over" and then explain Milin's inductive method to Bobby and Michelle. It was during Matt's explanation that Stephanie seemed to grasp this reasoning as she was listening intently. Later, Stephanie was fully able to share the "family tree" strategy with the entire class at the conclusion of the group work on February 26, 1993. Figure 8-4 gives the path that Milin's ideas followed before Stephanie was able to present her understanding of Milin's inductive argument.


Figure 8-4. Flow of Milin's inductive reasoning.

### 8.5 Limitations

The study provided evidence for how mathematical ideas were built by a nine-yearold Milin when certain conditions were in place. The present case study documents how a single student built his inductive argument and, while important and informative, cannot be generalized to other students. Also, the particular conditions of the study were fixed. By design, there was minimal researcher intervention. The extent to which these conditions, where students were invited to do mathematics, affected the development of mathematical ideas needs further study.

### 8.6 Implications for Further Study

The strategies, heuristics and justifications presented in this study were developed by a student during his fourth and fifth grade years. This study examined how his ideas traveled to other students. Similar studies involving students at the same grade level would help to validate the findings of this study and similar studies at higher grade levels would be helpful in developing more generalized models of how students develop and disseminate models of mathematical reasoning. Some of these studies are currently underway as a part of the broader collection of case studies from the longitudinal study at Rutgers University.

Another area warranting further investigation is the impact of researcher intervention. Although researcher intervention was a part of the environment in this study, its impact was not specifically analyzed. It was considered a condition of the environment. Further studies looking at the role the researcher interventions may have
played in the development of students' models of mathematical reasoning would also be helpful.

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## 10 APPENDICES

### 10.1 COMBINATORICS PROBLEMS

The following is a list of the combinatorics problems encountered by the students in the longitudinal study as of February 26, 1993.

### 10.1.1 Shirts and Jeans (May, 1990; Grade 2 \& Oct., 1990; Grade 3)

Stephen has a white shirt, a blue shirt, and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make?

### 10.1.2 Towers 4-tall (Oct., 1990; Grade 3 \& Dec., 1992; Grade 5)

Your group has two colors of Unifix cubes. Work together and make as many different towers four cubes tall as is possible when selecting from two colors. See if you and your partner can plan a good way to find all the towers four cubes tall.

### 10.1.3 Cups, Bowls, and Plates (April, 1991; Grade 3)

Pretend that there is a birthday party in your class today. It's your job to set the places with cups, bowls, and plates. The cups and bowls are blue or yellow. The plates are either blue, yellow, or orange. Is it possible for 10 children at the party each to have a different combination of cup, bowl, and plate? Show how you figured out the answer to this question.

### 10.1.4 Relay Race (October, 1991; Grade 4)

This Saturday there will be a 500-meter relay race at the high school. Each team that participates in the race must have a different uniform (a uniform consists of a solid colored shirt and a solid colored pair of shorts). The colors available for shirts are yellow, orange, blue, or red. The colors for shorts are brown, green, purple, or white. How many different relay teams can participate in the race?

### 10.1.5 Towers 5-Tall (February, 1992; Grade 4)

Your group has two colors of Unifix cubes. Work together and make as many different towers five cubes tall as is possible when selecting from two colors. See if you and your partner can plan a good way to find all the towers five cubes tall.

### 10.1.6 Towers 4-Tall with Three Colors (February, 1992; Grade 4)

Your group has three colors of Unifix cubes. Work together and make as many different towers four cubes tall as is possible when selecting from three colors. See if you and your partner can plan a good way to find all the towers four cubes tall.

### 10.1.7 Guess My Tower (February, 1993; Grade 5)

You have been invited to participate in a TV Quiz Show and the opportunity to win a vacation to Disney World. The game is played by choosing one of four possibilities for winning and then picking a tower out of a covered box. If the tower you pick matches your choice, you win. You are told that the box contains all possible towers that are three tall that can be built when you select from cubes of two colors, red and yellow. You are
given the following possibilities for a winning tower:

- All cubes are exactly the same color.
- There is only one red cube.
- Exactly two cubes are red.
- At least two cubes are yellow.

Which choice would you make and why would this choice be better than any of the others?

Assuming you won, you can play again for the Grand Prize which means you can take a friend to Disney World. But now your box has all possible towers that are four tall (built by selecting from the two colors yellow and red). You are to select from the same four possibilities for a winning tower. Which choice would you make this time and why would this choice be better than any of the others?

### 10.2 RESEARCHERS

R1: Dr. Alston<br>R2: Dr. Maher<br>R3: Amy Martino<br>TR: Mrs. O’brien

### 10.3 TRANSCRIPT - GROUP WORK ON 2/6/92

| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 1. | 00:00:30 <br> People 4:17 work | R2 | It has like a chimney on top. Do you see that? Okay, How many do you think you can build, under those conditions?... <br> The researcher is giving directions to the entire class. The following is the transcript for the group work for Milin and Michael. |
| 2. | $\begin{aligned} & 00: 00.37 \\ & \text { People } \end{aligned}$ | Michael: | What'd you do, what'd you do with the tower? $1,2,3$, $4,5,6,7,8,9,10$. (counts towers while the researcher is still asking questions) |
| 3. | $\begin{aligned} & \text { 00:01:19 } \\ & \text { 00:05:06 } \\ & \text { work } \end{aligned}$ | Milin: | We did this last year. (the researcher is still giving directions) |
| 4. |  | Michael: | Really? (the researcher is still giving directions) <br> They start working on the problem while the researcher is still talking |
| 5. | $00: 01: 34$ <br> people | Michael: | Uh huh. Alright you gotta go like this. You gotta put them together like... you gotta build them like ... |
| 6. |  | Milin: | I know, I know |
| 7. |  | Michael: | With no, with no under...(Michael builds a tower with 4reds on the bottom and lyellow on the top) you can't, they can't be, they have to be, they can't be different I mean, they gotta be the same. |
| 8. | $\begin{aligned} & 00: 01: 58 \\ & 0: 05: 44 \end{aligned}$ | Milin: | Here (Milin built one tower with three yellow cubes on the bottom and a red cube on the top ) |
| 9. |  | Michael: | Put one more (Milin adds another yellow on the bottom) |
| 10. |  | Milin: | Okay, Mike, lets do it like this |
| 11. | 00:02:05 | Michael: | Yeah |
| 12. |  | Milin: | You do this and that |
| 13. |  | Michael: | Make the opposite of that. Let me see if I can make another one. |
| 14. |  | Milin: | Na , no, no, gimme these two |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 15. |  | Michael: | Ou |
| 16. | 00:02:16 | Milin: | These ... these two go like this. Look, look, look, look. |
| 17. |  | Michael: | Here you made the opposite of it. Oops |
| 18. | 00:02:35 | Milin: | Oh yeah I forgot. |
| 19. | 00:06:24 <br> work | Michael: | Got this one (Michael makes and adds a tower starting from bottom with yellow/red/yellow/red/yellow) |
| 20. |  | Milin: | You'll have to put this here, right? Mike? (Milin points to the tower with two yellows on the top with three reds on the bottom) |
| 21. |  | Michael: | Yeah ... Wait a minute. What'd you do? This and ... (Michael picks up the tower with three reds on the top and two red on the bottom) |
| 22. | $\begin{aligned} & \text { 00:02:48 } \\ & \text { 00:06:34 } \\ & \text { work } \end{aligned}$ | Milin: | See (Milin picks up the 3 reds on bottom and 2 yellows on the top) |
| 23. |  | Michael: | Which ones of these are opposites, this? (Milin hands Michael the 3 reds on bottom and 2 yellows on the top tower) first this and this. First this ... no |
| 24. |  | Milin: | This |
| 25. | 00:03:02 | Michael: | Yeah. Uh ... I just got an idea |
| 26. |  | Milin: | There is going to be these two (he continues to build towers) |
| 27. |  | Michael: | There's this and this. Make the one like this and start it with red. <br> Start one with red. What'd you do? (Groans) where'd this goes? |
| 28. | $\begin{aligned} & \text { 00:07:08 } \\ & \text { work } \end{aligned}$ | Milin: | This goes with this |
| 29. |  | Michael: | Alright |
| 30. |  | Milin: | We only have this many. |
| 31. |  | Michael: | It's easy |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 32. |  | Milin: | Yeah yeah but Mike what if we could make more and we can't use these blocks |
| 33. |  | Michael: | It can't be the same so ... |
| 34. |  | Milin: | Uh, where's, where's this $\ldots$ or something like that. |
| 35. | 00:03:48 | Michael: | Where's the one like that? Right there. Alright. Did we try, we didn't try this one |
| 36. |  | Milin: | Did we do this, Mike? |
| 37. |  | Michael: | No. make the opposite of that |
| 38. | 00:04:15 | Milin: | Make the opposite with this. See two yellows on this side and two yellows on this side and one red. I'll do it. |
| 39. |  | Michael: | (Michael flips the towers around) Lot easier |
| 40. |  | Milin: | This? |
| 41. | 00:04:43 | Michael: | Oh, we got that already.(Michael snatches the tower from Milin) I did that already. |
| 42. |  | Milin: | I know, but I'm going to ... |
| 43. | 00:04:48 | Michael: | This is wrong. We've got twenty already...Counted by ones. (Michael looks at what Milin is making) We did it. ...that? ... Wait a minute. Where's that one you put down... |
| 44. |  | Milin: | (Milin Points to the tower he had done) did I do this? |
| 45. |  | Michael: | No |
| 46. | 00:05:16 | Milin: | Two reds and ... |
| 47. |  | Michael: | Here, I'll do this. I'll do that one. Take another one |
| 48. |  | Milin | I already got this. No, Mike don't do that 'cause I already got it. No, I already got ... |
| 49. |  | Michael: | Ha ha (Michael makes the tower before Milin) Gotcha. I beat ya. |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 50. | 00:05:38 | Milin: | Mike we can't do it like this can we? |
| 51. |  | Michael: | If we didn't do it, yeah, no, turn it. Go to the next one. No. |
| 52. |  | Milin: | Yeah we could |
| 53. | 00:05:57 | Michael: | Yeah. I got this one. I got it Mil. How many do we have (counts) $6,8,10,12,13 \ldots$ oh, wait a minute. 2,4 , $6,8 \ldots 16$. This one goes over here. So altogether, how many do we have? |
| 54. |  | Milin: | (shrugs his shoulders) sixteen |
| 55. |  | Michael: | What's sixteen times two? What's sixteen times two? |
| 56. |  | Milin: | Huh? |
| 57. |  | Michael: | What's sixteen times two? |
| 58. |  | Milin: | No see $2,4,6,8 \ldots 16$. |
| 59. | 00:06:41 | Michael: | Yeah what's sixteen plus sixteen? |
| 60. |  | Milin: | You can't do that 'cause I counted by 2's |
| 61. |  | Michael: | 1, 2, 3, $4 \ldots$ |
| 62. | $00: 10: 40$ <br> work | Milin: | Sixteen. Right! |
| 63. |  | Michael: | Never mind. |
| 64. |  | Milin: | See? 2, 4, 6, 8, 10, 12, 14, 16. |
| 65. |  | Michael: | Never mind |
| 66. | 00:10:52 | Milin: | I just counted by two's |
| 67. |  | Michael: | Yeah, but I still get that. Did we do this one yet? (Michael is holding a tower with four yellows on the top with one red on the bottom) |
| 68. | $00: 11: 00$ <br> work | Milin: | I just had this one (Milin adds Michael's tower to their collection) |
| 69. |  | Michael: | No you didn't do that. |


| 70. | 00:07:19 | Milin: | Yeah |
| :---: | :---: | :---: | :---: |
| 71. |  | Michael: | You did |
| 72. |  | Milin: | I did. The other way. See right here. |
| 73. |  | Michael: | But you didn't have an opposite to it so . Yes, you do. Something ain't right... |
| 74. |  | Milin: | See, this goes like this, so this stays on the bottom, see? This stays on the bottom and this stays on the bottom. |
| 75. |  | Michael: | we got another pair |
| 76. |  | Milin: | So that one would be the same as this |
| 77. |  | Michael: | No it wouldn't |
| 78. |  | Milin: | And this one would go like this (Milin flips that tower) |
| 79. |  | Michael: | Yeah |
| 80. |  | Milin: | No, different like this then it would be the same as this. (Milin flips the tower back the way it was) |
| 81. |  | Michael: | Yeah, well it's a different match. So put it this way. It's a different match. (Michael makes the new tower and put it in front of the previous tower) Gotcha. Eighteen. I did it Mil. |
| 82. |  | Milin: | I know, I know, you did? What about for this. (points to tower) |
| 83. |  | Michael: | I did that |
| 84. |  | Milin: | How? Okay. This goes with what? |
| 85. |  | Michael: | This, Mil. Look, these ... |
| 86. |  | Milin: | Oh yeah. 8:27 |
| 87. |  | Michael: | Go together. These go together. |
| 88. |  | Milin: | Okay, okay. |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 89. |  | Michael: | These go together. |
| 90. |  | Milin: | No, no, no, no, these two could be the same. Let me see. |
| 91. |  | Michael: | These go together look. These go together. These go together. These go together |
| 92. |  | Milin: | Yeah, you're right |
| 93. |  | Michael: | These go together. These go together. Two I wonder if we did this one. No. maybe not |
| 94. |  | Milin: | Did we do this? |
| 95. |  | Michael: | Let's see |
| 96. |  | Milin: | No |
| 97. |  | Michael: | I don't think so |
| 98. |  | Milin: | We couldn't have. I got this one. I got this one okay? (Michael tries a tower to see if it is the opposite and Milin Finishes the opposite tower) |
| 99. |  | Michael: | Yeah |
| 100. |  | Milin: | All of these reds turned yellow. |
| 101. |  | Michael: | Yeah, yeah, yeah, yeah. Did we do this? 9.51 |
| 102. |  | Milin: | I think ... could ... here $\ldots$. uh oh. |
| 103. |  | Michael: | Ah! |
| 104. |  | Milin: | Switch them to yellows. (Milin starts building the opposite tower) I got that 10.06 |
| 105. |  | Michael: | (Michael Moves the new pair) |
| 106. |  | Milin: | Did I do this one? |
| 107. |  | Michael: | Yeah. Its fine ... 20 altogether. |
| 108. |  | Milin: | So far twenty. So anybody that said fifteen is wrong, right? |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 109. |  | Michael: | Yeah. Twenty? What? |
| 110. |  | Milin: | So anybody that said fifteen must be wrong or something. |
| 111. |  | Michael: | I don't know |
| 112. |  | Milin: | But didn't you say if you go like um this (turned a tower upside down and knocked the other tower off the table) |
| 113. |  | Michael: | But you can't do this.(Milin switches the tower back and Michael puts the fallen tower next to it) They didn't say you could do that. so ... |
| 114. |  | Milin: | But we might have. |
| 115. |  | Michael: | No, we didn't. (Michael starts rearranging towers)I checked it personally. Here It'll give us more room. In case we have more to do as ... hey you got something... |
| 116. |  | Milin: | This |
| 117. |  | Michael: | Yeah alright |
| 118. |  | Milin: | Do we have three on the bottom and one on the top? |
| 119. | $00: 15: 15$ <br> work | Michael: | That's only four. |
| 120. |  | Milin: | No, but do we have the opposite of this? Uh huh. Do we? |
| 121. |  | Michael: | Here put it ...11:44 |
| 122. |  | Milin: | Nah |
| 123. |  | Michael: | Put another red on top. |
| 124. |  | Milin: | Do we have This. No. (as Mike compares new tower to all towers) |
| 125. |  | Michael: | Something ain't right. We didn't make a match for something |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 126. |  | Milin: | Three on the top ... |
| 127. |  | Michael: | That's a match. Put that together before. Don't do it. We didn't make a match for something. |
| 128. |  | Milin: | Why? |
| 129. |  | Michael: | 'cause it doesn't make any sense. (Milin adds the inverted pair to the collection) |
| 130. |  | Milin: | $2,4,6,8,12 \ldots 12$ |
| 131. |  | Michael: | Oh I guess so |
| 132. |  | Milin: | We haven't missed a match |
| 133. |  | Michael: | Are we sure everything matched |
| 134. |  | Milin: | That we're, we're losing ... do we do three... |
| 135. |  | Michael: | I think I did that. Yeah we did that. |
| 136. |  | Milin: | Where? |
| 137. |  | Michael: | Oh, two on the bottom and two on the top? |
| 138. |  | Milin: | This? |
| 139. |  | Michael: | Yeah, we did it. |
| 140. |  | Milin: | No, we didn't. |
| 141. |  | Michael: | Right here (Michaels checks the tower Milin had built) |
| 142. |  | Milin: | How? (Milin puts the tower back) |
| 143. |  | Michael: | No, we didn't make a match for it. See. I know we didn't make a match for something. This goes together. This goes together. This goes together. |
| 144. |  | Milin: | These go together |
| 145. |  | Michael: | This goes together. See we did it. No we didn't. |
| 146. |  | Milin: | We didn't |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 147. |  | Michael: | Alright |
| 148. |  | Milin: | But this has to, no, yeah but then these two go together too, 'cause these have to ... ah! |
| 149. |  | Michael: | Oh yeah we did it. |
| 150. |  | Milin: | Where? |
| 151. |  | Michael: | Right here. |
| 152. |  | Milin: | No |
| 153. |  | Michael: | Yeah |
| 154. |  | Milin: | No, there (takes tower apart) did we do this? Did we, uh, in the middle |
| 155. |  | Michael: | Um that me |
| 156. |  | Milin: | I don't think so |
| 157. |  | Michael: | Uh huh |
| 158. |  | Milin: | Not three yellows in the middle not three reds in the middle. I need this one. These two go together. 14.10 |
| 159. |  | Michael: | We could make a lot more if we had one more color |
| 160. |  | Milin: | Blue. Yellow. If we had green I could put a green here. |
| 161. |  | Michael: | Wait, I think I just got one. I think I just got one. I think I just got one. Did I make two |
| 162. |  | Milin: | One on the bottom? (Michael compares to check) |
| 163. |  | Michael: | Yeah. I think I got one |
| 164. |  | Milin: | One on the bottom? |
| 165. |  | Michael: | We didn't do it. We didn't do it Mil. |
| 166. |  | Milin: | I think twenty-eight uh huh (R1 arrives to talk to them) |
| 167. |  | R1 | How are you working at it? (Mike counts) |


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| :---: | :---: | :---: | :---: |
| 168. |  | Michael: | We got twenty-eight |
| 169. |  | Milin: | Because, see every time we make it like this, right? 15.25 |
| 170. |  | R1 | Yeah |
| 171. |  | Milin: | Then we change the color, like this. |
| 172. |  | R1 | Oh, so they come like that. 15.33 |
| 173. |  | Milin: | So we get doubles of this and this and all this. |
| 174. |  | R1 | Uh huh. So that's what you have done all along. |
| 175. |  | Milin: | Uh huh. |
| 176. |  | R1 | Do you think there are more? |
| 177. |  | Michael: | May be |
| 178. | $\begin{aligned} & \text { 00:19:29 } \\ & \text { work } \end{aligned}$ | Milin: | Yeah |
| 179. |  | R1 | How are you gonna be able to tell... |
| 180. |  | Michael: | I don't know. Put them together and see if they ... |
| 181. |  | Milin: | When we lose all these use up the cubes |
| 182. |  | R1 | Uh, we have plenty more. But you have 28 now. |
| 183. |  | Michael: | Yeah |
| 184. | 00:16:02 <br> People | R1 | Well, I'll come back and see if you have any more. |
| 185. |  | Milin: | Do we have ... |
| 186. |  | Michael: | Hey. We did ...let me see |
| 187. |  | Milin: | This |
| 188. |  | Michael: | Yellow, red yellow, yellow |
| 189. |  | Milin: | Yes. Double. Lets see if this has a double. |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 190. |  | Michael: | We just did this one |
| 191. |  | Milin: | No, this doesn't, until now. This ... ok two yellows |
| 192. |  | Michael: | We did it |
| 193. |  | Milin: | Phew |
| 194. |  | R3: | Hi there. |
| 195. | $\begin{aligned} & \text { 00:21:17 } \\ & \text { work } \end{aligned}$ | Milin: | So far we only got twenty-eight |
| 196. |  | R3: | Twenty-eight |
| 197. |  | Michael: | That's pretty ... |
| 198. |  | R3: | What did you say Michael? I'm sorry I didn't hear that. |
| 199. |  | Michael: | No I said that's ... he said we only had twenty-eight. |
| 200. |  | R3: | Okay, do you think there are any more? 17.41 |
| 201. |  | Milin: | Maybe. |
| 202. |  | Michael: | Ooo ... |
| 203. |  | Milin: | Yeah |
| 204. |  | R3: | Michael. |
| 205. |  | Michael: | Maybe |
| 206. |  | R3: | Yes and maybe. Okay. You probably want to work on this a little bit more. Are you sure that they're all different towers? |
| 207. |  | Michael: | Yeah. |
| 208. |  | Milin: | Yes. |
| 209. |  | R3: | How do you know that? |
| 210. | $00: 21: 41$ <br> work | Milin: | Because everything we get, we make it like this, right? |
| 211. |  | R3: | Uh hum. |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 212. |  | Milin: | Right now I am going to check, see its not ... |
| 213. |  | Michael: | We made that one. |
| 214. |  | Milin: | Where? (Michael points to the tower) |
| 215. |  | R3: | Ah, so you check by moving it along here? |
| 216. |  | Milin: | Yeah. |
| 217. |  | Michael: | Yeah. |
| 218. |  | R3: | Okay. |
| 219. | $00: 18: 13$ <br> people | Milin: | Did we make it this way? |
| 220. |  | R3: | Also is there anything else that helps you to make sets or make towers? |
| 221. |  | Milin: | Um, we just keep on checking to see if ... there's any $\ldots$ and when we try to do it every way like we get that um ... we already know that we made five of these and five of the reds so we are not gonna try that again. (Milin talks about his solid towers) |
| 222. |  | R3: | Okay. I have a question. There is something here that interests me and I saw Michael doing this. Michael maybe you could tell me about this. I noticed that your towers, there seems to be something interesting about them. What about these two? |
| 223. |  | Michael: | We make ... |
| 224. |  | Milin: | See this goes like this and this turns to yellows from reds and see this turns from ... |
| 225. |  | Michael: | They are like opposites. |
| 226. |  | Milin: | Yellow to red ... |
| 227. |  | R3: | Like opposites what do you mean by opposites? |
| 228. |  | Milin: | Like um when have this we change the color to the other color. |


| Line | Time | Name | Transcript |
| :--- | :--- | :--- | :--- |
| 229. |  | R3: | You agree with that Michael? Sran |
| 230. |  | Michael: | Yeah like here. |
| 231. |  | R3: | Show me. |
| 232. |  | Michael: | We changed, we changed from, we made this one and <br> then we changed yellow to red, these from red to <br> yellow. From red to yellow. From red to yellow. |
| 233. | 00:23:13 | R3: | Okay and that's interesting. Did you do that a lot here? |
| work |  | Milin: | Yes |
| 234. |  | Michael: | Yeah. That's how we got all these. |
| 235. |  | R3: | That's how you got all of them. Okay. I am gonna let <br> you continue to work these. Okay. Call me when you |
| think you've found them all. |  |  |  |


| 249. |  | Michael: |
| :--- | :--- | :--- |
| 250. |  | We do? |
| Milin: | No, I made this opposite make this its opposite. We <br> don't got this but we, we have this an, an opposite <br> instead. (Milin points with the tower he is holding) |  |
| 252. |  | Michael: |
| No, I you turn it upside down. Go ahead. Turn it |  |  |
| upside down. |  |  |, | 253. |
| :--- |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 267. |  | Michael: | We do. I made it remember. |
| 268. |  | Milin: | Where? Yellow/red/yellow/red/red |
| 269. |  | Michael: | Maybe not |
| 270. |  | Milin: | Yeah thirty-two. Thirty-two, I think we're done. |
| 271. |  | Michael: | Hmm. |
| 272. |  | Milin: | Just see. I think I got ... I'm on to one, yeah, this. None of those none of these. |
| 273. |  | Michael: | Okay, I got it already, Mil maybe we ought to put it the other way. |
| 274. |  | Milin: | No, there's |
| 275. |  | Michael: | Thirty things in the way. No, don't move that its taped. (Milin and Michael are rearranging the set of towers on the table) |
| 276. |  | Milin: | How many do we have altogether? |
| 277. |  | Michael: | Thirty-two |
| 278. |  | Milin: | I just want to make sure because I think some ... |
| 279. |  | Michael: | Oh, no thirty-four. We just made one. |
| 280. |  | Milin: | Oh, yeah, yeah thirty-four. I think I am convinced. |
| 281. | $\begin{aligned} & 00: 25: 49 \\ & 00: 29: 34 \end{aligned}$ | Michael: | No, there's got to be one more. There's got to be a match like |
| 282. |  | Milin: | See if you do this right? Then this, right? Then we could do this the opposite of that. see? This on the other way. They don't know if... |
| 283. |  | Michael: | Hey, I think you have got that. |
| 284. |  | Milin: | So that means we have to come up with sixty some what? |
| 285. |  | Michael: | No, I don't think so. |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 286. |  | R1 | About done? |
| 287. |  | Milin: | We have about thirty-four now. |
| 288. |  | R1 | About? Exactly or about? |
| 289. |  | Milin: | Exactly, exactly. |
| 290. |  | R1 | Show me. |
| 291. |  | Michael: | I don't think so |
| 292. |  | Milin: | All that plus 2 more. |
| 293. |  | Michael: | Thirty-six |
| 294. |  | R1 | You have thirty-six? |
| 295. |  | Milin: | Thirty-six |
| 296. |  | R1 | Now how are we going to decide if you have any that are duplicates? |
| 297. |  | Milin: | Because see we still keep on going like this and see this? |
| 298. |  | R1 | Yeah. |
| 299. |  | Milin: | It's a duplicate of this so we can't use this. |
| 300. |  | R1 | Oh, I understand that but what about the ones that you have? How can you be sure that there aren't any duplicates there? |
| 301. |  | Milin: | Because we always keep on going like this and, and then ... if we find any duplicates, in our way... Mike, this can't be used. We need five |
| 302. |  | R1 | Yeah because that's a copy of one |
| 303. |  | Michael: | Yeah, I know. I just want to check something out. |
| 304. |  | R1 | You already have? |
| 305. |  | Michael: | Think there's one more left? |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 306. |  | Milin: | If there is there's... |
| 307. |  | R1 | You know I am just busy looking and I think I see one that looks like this over towards the bottom down there. See if you don't see if you want to check that out. |
| 308. |  | Milin: | No |
| 309. |  | R1 | Another one besides ... |
| 310. |  | Milin: | This? This. Yeah, this is a dupli... |
| 311. |  | R1 | What do you think? |
| 312. |  | Milin: | It's a duplicate |
| 313. |  | R1 | Why'd you take two? |
| 314. |  | Michael: | 'cause they're, we made them like we made them opposite. |
| 315. |  | Milin: | These two, one of this and this is out ... these two ... this is out. |
| 316. |  | Michael: | Took the bottom? |
| 317. |  | R1 | How many do you have now? |
| 318. |  | Milin: | Thirty-four, thirty-four, we had thirty-six and then she came (Mike counts) just trying to figure out if we have any more duplicates. These two are alright. |
| 319. |  | R1 | Count them for me, I can't do it. |
| 320. |  | Michael: | $2,4,6,8,10 \ldots 32$ |
| 321. |  | R1 | Thirty-two |
| 322. |  | Milin: | How did we lose four? |
| 323. |  | R1 | Maybe there's some that you took out? |
| 324. |  | Milin: | Let's see if this one is a duplicate. |
| 325. |  | Michael: | Remember the one we were about to make, but we didn't make when she came? I counted that in. that's |


|  |  | how we got the four left. The four out, I just told you <br> what I did. |
| :--- | :--- | :--- |
| 326. |  | Milin: |
| Nah, this, this doesn't have any exact duplicates. This |  |  |
| might not either. |  |  |, | 327. |
| :--- |


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| 343. |  | Milin: | But that's only if you are really not done. |
| 344. |  | R1 | Only if you are really not done. So you're saying there really is a there really is a done... you really think there really is a special number, there's not any more. |
| 345. |  | Michael: | Yeah |
| 346. |  | Milin: | Because see ... |
| 347. |  | R1 | Why? |
| 348. |  | Milin: | Um. We could make a duplicate of this see, we make a duplicate of each one but we, we change the colors so then that would be different and then we keep on doing it |
| 349. |  | R1 | You mean different colors. |
| 350. |  | Michael: | Yeah |
| 351. |  | R1 | But we can't do that 'cause we just counted red and yellows. |
| 352. |  | Milin: | See if we have this, right? We change all the threes to yellow and these two to reds ... |
| 353. |  | R1 | Hmm, but you've already done that, haven't we? Isn't that the way you did it? I wonder I wish I could come up with a way of thinking whether |
| 354. | 00:31:56 | Milin: | Maybe there's an answer sheet? I think we're done. I think we're done. So does Michael. So that's more than ten minutes and we still didn't find one. |
| 355. |  | R1 | Has it really been a long time? |
| 356. |  | Milin: | Cannot have one |
| 357. |  | R1 | I keep wondering if there's some way you can tell when you've finished. |
| 358. |  | Milin: | Uh, there is one way. If you take hundred hours and still haven't found any. |


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| 359. |  | R1 | Its just a matter of ... do you know what you are supposed to do when you think you are done? Let me go find out. |
| 360. |  | Milin: | We have gotta be done. |
| 361. |  | Michael: | mm . |
| 362. |  | Milin: | Kids think they are done, right? Only up to 15 or something. 20? |
| 363. |  | Michael: | And we got thirty-four, thirty-four |
| 364. |  | Milin: | If she finds one more for us we got a lot more to do. I think we have to do it by today. |
| 365. |  | Michael: | We still got tomorrow I think. |
| 366. |  | Milin: | Yeah, but tomorrow, this is gonna be all wrecked. Don't ya think? |
| 367. |  | Michael: | No, not really |
| 368. |  | Milin: | Could be. |
| 369. |  | Michael: | Not if she puts them in a bag. |
| 370. |  | Milin: | With our names on it. I'll go get and ... okay? Okay? We have thirty-four |
| 371. |  | R3: | Okay you have thirty-four |
| 372. |  | Michael: | Thirty-two wasn't it? |
| 373. |  | R3: | I don't know |
| 374. |  | Milin: | No, thirty-two |
| 375. |  | R3: | I am trusting in the two of you. So can you check for me? |
| 376. |  | Michael: | Mike counts |
| 377. |  | Milin: | How much do you get so far? |
| 378. |  | Michael: | Thirty-two |


| 379. | R3: | Thirty-two. Alright. How did you find all these? |
| :---: | :---: | :---: |
| 380. | Milin: | You made duplicates. But you have to change the colors around. |
| 381. | R3: | Okay, explain to me what duplicates are? |
| 382. | Milin: | Like, see |
| 383. | R3: | Where you change the color. |
| 384. | Milin: | See, you have this right? |
| 385. | R3: | Hmm. |
| 386. | Milin: | These two look the same but see this changes to yellow, this changes to red, this changes to yellow, and this changes to red. |
| 387. | R3: | Okay, Michael you show me a pair of duplicates that are switched around. |
| 388. | Michael: | This, this yellow compares to red, red compares to yellow, red compares to yellow and these yellows compare to the these reds. |
| 389. | Milin: | And all these compare to each other |
| 390. | R3: | Alright. Why do you think you found them all? |
| 391. | Milin: | Because it took us one minute to find another one and now its like 10 minutes left? |
| 392. | R3: | Its taking a long time to find it. Have you found any in the past ten minutes? |
| 393. | Milin: | Nah uh |
| 394. | R3: | No? |
| 395. | Milin: | We just lost some |
| 396. | R3: | What do you mean you just lost some? |
| 397. | Milin: | Because we made a duplicate of two |


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| :---: | :---: | :---: | :---: |
| 398. |  | Michael: | We made like these two the same and these two the same. |
| 399. |  | R3: | I see, so you, you had a pair there that was the same as another pair? |
| 400. |  | Milin: | Yeah |
| 401. |  | Michael: | Yeah |
| 402. |  | R3: | Okay. And you feel pretty convinced about this? |
| 403. |  | Michael: | Uh hum |
| 404. |  | Milin: | Yeah |
| 405. |  | R3: | Alright okay, just sit tight, then. I think we're gonna talk about these in a minute. Alright. |
| 406. |  | Milin: | I thought we were supposed to leave five minutes ago? |
| 407. |  | R3: | No, this is, we're having an extended math class today, so you'll all be here for a little while, okay you'll be here for a double period, today. |
| 408. |  | Milin: | Oh! |
| 409. |  | Michael: | Uh huh |
| 410. |  | R3: | You see, now make sure that you've thought about this and there aren't any others. <br> Okay? |
| 411. |  | Milin: | Could this be? |
| 412. |  | Michael: | Oh, great they're out of here. |
| 413. |  | Milin: | How come? |
| 414. |  | Michael: | Check it out dude |
| 415. |  | Milin: | Let me see |
| 416. |  | Michael: | That's what I was actually checking for. Well we don't have thirty two anymore we only have thirty |


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| 417. |  | Milin: | There's gotta be more. (Someone says thirty four) thirty <br> four they are gonna lose some right now I'm telling ya. <br> I wish you could make it out of 10. Really then I <br> would get like fifty. There's gotta be thirty four. At <br> least thirty-two. |
| 418. |  | Michael: | Well we had thirty-six and then we lost four actually <br> six. |
| 419. |  | Milin: | Check if we have this. Yellow, yellow, yellow |
| 420. |  | R3: | We're ------to now talk about ... |
| 421. |  | Michael | This looks familiar |
| 422. |  | Milin | Sighs |
| 423. |  | Milin | We have got to get one more |
| 424. |  | Michael | There's gotta be- |
| 425. |  | Milin | I think this was there |
| 426. |  | Milin | Check if we have any of these two |
| 427. |  | Milin | Guess not. Now how many do we have? |

### 10.4 TRANSCRIPT - SHARING SESSION ON 2/6/92

| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 1. |  | Jeff: | We Have... |
| 2. | 00:00:00 | R2: | Okay. But hold on a minute Jeff:, I am gonna want you to explain. How many did you find? |
| 3. |  | Jeff: | We are not sure |
| 4. |  | Michelle: | Well so far we found thirty two. |
| 5. |  | R2: | Okay Michelle says thirty two, but the way you're gonna explain it- you made some extras you are telling me to explain it |
| 6. |  | Michelle: | Yeah we found some extras. |
| 7. |  | Jeff: | We need some more colors though we don't have any more |
| 8. | 00:00:24 | R2: | Oh! We can get you some more colors if you need it. Okay, this table here how many did you make? |
| 9. |  | Stephanie: | We made thirty four but we are still checking so there are probably one or two duplicates |
| 10. |  | R2: | Oh! Did anybody else here get thirty four? Okay, so you, this group has thirty four. What about this group over here? |
| 11. |  | Michael: | Thirty two. |
| 12. |  | Milin: | Thirty two. |
| 13. |  | R2: | You have thirty two. Did any other group get thirty two? There are lots of groups here in the front that got thirty two but what about your group Sebastian? |
| 14. |  | Sebastian | Thirty five. |
| 15. |  | R2: | You have thirty five. Okay, do you think it is possible to have an odd number? |
| 16. |  | Students: | No. |
| 17. |  | R2: | They have an odd number thirty five. |
| 18. |  | Michael 2: | You can't because when you have a number you could |

\(\left.$$
\begin{array}{|l|l|l|l|}\hline & & & \begin{array}{l}\text { have the opposite if you have one of this then you have } \\
\text { another one of this because it is the opposite...If you } \\
\text { have ten of these and you have another one that's } \\
\text { opposite so it makes twenty. }\end{array} \\
\hline \text { 19. } & & \text { R2: } & \begin{array}{l}\text { So what you are telling us, Sebastian and his group that } \\
\text { they got to have thirty four or thirty six? Do you } \\
\text { believe that? Do you understand what he is saying? } \\
\text { Does that make any sense? }\end{array} \\
\hline \text { 20. } & & \text { Students: } & \begin{array}{l}\text { Uh huh! }\end{array} \\
\hline \text { 21. } & & \text { R2: } & \begin{array}{l}\text { What do the rest of you think? Do you think that } \\
\text { makes sense? What do you think Jennifer? }\end{array} \\
\hline \text { 22. } & \text { 00:01:28 } & \text { Jennifer: } & \begin{array}{l}\text { It makes sense, but he could have what ever number he } \\
\text { wants. It just depends if he put it opposite or not } \\
\text { opposite. }\end{array} \\
\hline \text { 23. } & & \text { R2: } & \begin{array}{l}\text { So he might not have used an opposite way of doing it } \\
\text { you are suggesting? What do you think about that? }\end{array} \\
\hline \text { 24. } & & & \text { Rtudent: }\end{array}
$$ \begin{array}{l}I think originally he has thirty six but if you make <br>
doubles they cut it down to not doubles like umm...but <br>
different colors like one goes red, yellow, red, red, red <br>
or it could go yellow, red, yellow, yellow, yellow and <br>

that could make it go to thirty four could be the same\end{array}\right\}\)| R2: |
| :--- |


|  |  |  | than thirty uh who think there are only thirty two is to take a look at some of the patterns of the people who found more to see if they found any duplicates. What do you think about that? |
| :---: | :---: | :---: | :---: |
| 30. | 00:03:00 | Dina: | When Robert said that he had thirty five, he could split, when you get ten of 'em of these in a same color like this like this (holds up a long blue tower then takes a ten tall tower of all brown) when you get one of these you could split them into five like two fives and that makes uh that makes an odd number so maybe he could have he could have five in each group and so we could split this into fives. |
| 31. |  | R2: | Why don't we take a look to see what Sebastian and his group has. He claims he has thirty five. Some of you here, if you could turn around and take a look and see what was built by Robert and Sebastian. What do you think? Do you see any duplicates there? Maybe they have thirty five. Maybe there are thirty five gee, I don't know. Did any of you find duplicates? Some of you think you have really good methods of finding duplicates. You want to come and peek? Anyone here find a duplicate? |
| 32. |  | Milin: | There's thirty four! (there are really thirty four) |
| 33. |  | R2: | There are thirty four? |
| 34. |  | Student: | Yeah. (Student counts the towers made by Sebastian and Robert) thirty four thirty five. (miscounted) |
| 35. |  | R2: | You have thirty five. Those of you who found thirty two are telling me that they have three that ought to be duplicates. Now if that's true you should be able to find them. Want to study this for a few minutes? Okay, I will give you a few minutes to study it. |
| 36. |  | Student: | I found two |
| 37. |  | Ankur: | We found a duplicate! |
| 38. |  | R2: | Oh! You found a duplicate. Show me the duplicate. You are not gonna throw both of those away. You are gonna keep one and throw the other away. Very Good! Do you agree? Do you agree Robert and Sebastian |


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| :---: | :---: | :---: | :---: |
|  |  |  | that's a duplicate? Okay. |
| 39. |  | Michael: | I think there is another one. |
| 40. |  | Ankur: | Found one. Here is another duplicate |
| 41. |  | R2: | Okay! You boys agree there's another duplicate? |
| 42. | 00:05:13 | Milin: | Yes! |
| 43. |  | R2: | Okay. Now how many do you have? |
| 44. |  | Student: | Thirty two |
| 45. |  | Ankur: | Thirty two oh yeah thirty three |
| 46. |  | Milin: | There's got to be thirty two |
| 47. |  | Student: | There is no more duplicates. |
| 48. |  | Student: | Has to be, there has to be. |
| 49. |  | Michael: | Let me see that one. |
| 50. |  | R2: | Okay. Ones that are out why we don't put them aside so that they don't get mixed up. Is that another duplicate? |
| 51. |  | Michael: | Yeah. |
| 52. |  | R2: | Okay, we found another. Now let's see what we have left. Will you count then? What did you think Alex? |
| 53. |  | Michael 2: | One two three...thirty. I missed some. |
| 54. | 00:06:10 | R2: | What about the ones that Mrs. Barnes has? |
| 55. |  | Ankur: | Those are duplicates. |
| 56. |  | Milin: | Those are duplicates. |
| 57. |  | Michael 2: | Two, four, six, eight, ten...eighteen was it these two? Eighteen...thirty, thirty one. |
| 58. |  | R2: | So. Is it possible to have thirty one? |


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| 59. |  | Students: | No |
| 60. |  | R2: | Okay. Got to find that other one then or else take one away. |
| 61. |  | Student: | There is two of the same one. |
| 62. |  | Teacher: | Okay, hand me one. |
| 63. |  | Ankur: | Okay, so they have thirty two. |
| 64. |  | R2: | How many? |
| 65. |  | Ankur: | Thirty two. |
| 66. | 00:06:50 | R2: | Okay, you absolutely convinced? What does Sebastian and Robert think? Are they convinced? |
| 67. |  | Students: | $14,16,18,20,22 \ldots 30,31$ (counting the towers) |
| 68. |  | Student: | There is one more duplicate |
| 69. |  | Teacher: | Got to figure out which one it is. |
| 70. |  | Milin: | It's either that or we took more out than there was supposed to be. |
| 71. |  | Teacher: | Well maybe somebody is not counting right. |
| 72. |  | Students: | Sebastian said he found a duplicate |
| 73. |  | Teacher: | There is one on the floor there. |
| 74. |  | Student : | This is confusing. |
| 75. |  | Teacher: | Somebody is not counting them right. |
| 76. |  | Student: | Wait wait wait |
| 77. |  | Teacher: | Let's have one person count them instead of everybody counting them at once. Let Michael count them. Michael, stand them all up and count them. |
| 78. |  | Michael 2: | Which ones? These? (he points to a group of towers) |
| 79. |  | Teacher: | Count them all. Bobby says these are the ones we took |


|  |  |  | out |
| :---: | :---: | :---: | :---: |
| 80. |  | Ankur: | Those are the duplicates. |
| 81. |  | Teacher: | Put them up. One person should count them |
| 82. |  | Student: | There should be thirty two |
| 83. |  | Teacher: | Yes. There is three out and you started with thirty five |
| 84. |  | Michael 2: | Two, four, six, eight, ten, twelve, fourteen, sixteen, eighteen, twenty, twenty two, twenty four, twenty six, twenty eight, thirty, thirty one. |
| 85. | 00:08:26 | R2: | Now let me ask you a question here, how could you, how could you figure out if you think there is an extra one or you think there is one missing ? |
| 86. |  | Ankur: | One missing |
| 87. |  | R2: | How could you... You have a strategy for figuring out the one that is missing. How could you do that? |
| 88. |  | Milin: | You could take one to thirty six. |
| 89. |  | Student: | - and then which ever one doesn't have a opposite |
| 90. |  | R2: | Is that a good strategy? Why don't you work on that strategy? What is the prize for finding that one if there is one? |
| 91. | 00:08:58 | Milin: | Its either that or you could check with our own (walks away to the back of the room) |
| 92. |  | Michael 2: | I found a match a match Yes I did this is a match. |
| 93. |  | Teacher: | Leave them here till you find matches (she put the towers on the table) |
| 94. |  | Students: | This is a match. These are matches. |
| 95. |  | Milin: | This has got to get a match |
| 96. |  | Student: | Wait. |
| 97. |  | Student: | Found another match |


| 98. |  | Student: | These are two the same take one. |
| :---: | :---: | :---: | :---: |
| 99. |  | Teacher: | Okay, choose one person alright here. |
| 100. |  | Student: | Now we have thirty |
| 101. |  | Teacher: | Okay, Bobby |
| 102. |  | Student: | Now we have thirty |
| 103. |  | Teacher: | No, we put the others back in. |
| 104. |  | Student: | We should have all duplicates. |
| 105. |  | Ankur: | Did you put the duplicates back in? |
| 106. |  | Teacher: | Because we didn't know which ones are the duplicates. |
| 107. |  | Ankur: | There's more duplicates |
| 108. |  | Michael 2: | Found another one |
| 109. |  | Teacher: | Michael what do you mean by duplicates? |
| 110. |  | Milin: | We say matches ( he is holding up a tower with three reds and two yellows) because this changes these two could be red and these |
| 111. |  | Michael: | I just called it a match. |
| 112. |  | Teacher: | What do you mean by duplicates? |
| 113. |  | Michael 2: | I didn't say duplicate he did (point to the Ankur next to him) |
| 114. |  | Teacher: | What are you saying? Find a match? What do you mean by a match? |
| 115. |  | Michael 2: | They go together see? |
| 116. | 00:10:34 | Milin: | See, right here all these turn yellow (he is holding a tower with three red and two yellows) and these two turn red |
| 117. |  | Michael 2: | Its' a match |


| 118. | Student: | Hey! Here's a match |
| :---: | :---: | :---: |
| 119. | Michael 2: | No, that's not a match. |
| 120. | Milin: | This could be a match. |
| 121. | Matt: | No its not (he is holding the inverted pair given by Milin) |
| 122. | Milin: | If you put it the other way yeah. |
| 123. | Student: | I found a match. |
| 124. | Student: | I found a match. |
| 125. | Ankur: | Let me see those two. This is a match. Mike look here's a match. |
| 126. | Student: | Aren't these the same? |
| 127. | R2: | Where are you putting your matches? Are you keeping them next to each other? Are you putting them along side each other? Okay. You keeping a record of them here? |
| 128. | Teacher: | Let's let Sebastian and Bobby find it. Let's go over to Michael. |
| 129. | R2: | You boys need to move back |
| 130. | Teacher: | Bobby and, Joey you need to move around. Okay, before, I asked the question what did you mean by the opposite or the match. Michael, want to show us? Because you have yours in pairs you would you explain to us about your pairs. Joe you want to come around here so you can see better. Joe come here there is plenty of room. Okay, Michael I would like you to explain |
| 131. | Michael 2: | See when it is the opposite see we call it a match because they go together. |
| 132. | Teacher: | Okay, can you explain a little bit more about how they go together? |


| 133. | 00:12:43 | Michael 2: | You do it. (Michael asks his partner to explain) |
| :---: | :---: | :---: | :---: |
| 134. |  | Student: | Well, These are a pair because these two are red and these two are yellow and this is yellow and this is red, red yellow, yellow red ( he compares the colors in the left tower to the colors in the right tower) |
| 135. | 00:12:58 | Milin: | I know what they mean. See this yellow turns into red on this one and all of these reds turn into yellow in this one |
| 136. |  | R2: | Oh good! Okay, I see what you mean. I see what you mean. That helps me. |
| 137. |  | Ankur: | You switch them |
| 138. |  | R2: | What are you switching? |
| 139. |  | Milin: | This color changes into this ( he points to the towers he had used earlier) |
| 140. |  | Students: | You switch the colors around |
| 141. |  | R2: | You are switching the colors. Oh I see you are switching the red to the yellow and yellow to the red that's what you mean by opposites? Okay. |
| 142. |  | Teacher: | Michael what, how did you use that as a strategy for finding them? |
| 143. |  | Michael 2: | See, what I- |
| 144. |  | Teacher: | What made to come up with that? |
| 145. |  | Michael 2: | See when I found it I found another one I made one that looked like this and I said to Paul that We should make al these the ones that are like if we have ten we get twelve.. |
| 146. | 00:13:48 | Milin: | They got doubles |
| 147. |  | Teacher: | May be they have them mixed up a little. |
| 148. |  | Paul: | We still have doubles |
| 149. |  | Teacher: | Don't have a double. |


| 150. | Michael 2: | Let me see. |
| :---: | :---: | :---: |
| 151. | R2: | Ah huh! |
| 152. | Milin: | Doubles. Doubles (Michael is looking at four towers) no no Mike, Mike this and this are doubles you can't take those two again. Milin points to second and the fourth towers. |
| 153. | R2: | Which is the top and which is the bottom? How do you know which is the top of the tower? Can you show me what's the top of the tower? |
| 154. | Student: | Shows the chimney side of a tower |
| 155. | R2: | On the ones you think are doubles can you stand them up for me so I can see them? So they look the same to me I see. Okay, why don't you work on that? Maybe we should hear from somebody else. |
| 156. | Michael 2: | Yeah we had some of these left. I thought they were like, we did it and we forgot to break them apart. |
| 157. | Teacher: | Let's go over to Ankur. Hey Ankur I want you to tell us how you went about going and getting yours. |
| 158. | Ankur: | We ... |
| 159. | R2: | Hold on a minute Ankur, hold on a minute |
| 160. | Teacher: | I want you to step back a little so we can see. Ankur I want you to explain how you did yours. |
| 161. | Ankur: | I made one all red (he points to an all red tower) then I took a red away and put one yellow and then I took another yellow away red away and I made yellow and left the bottom the same. I did that all the way down |
| 162. | R2: | Oh! That's very neat. Okay, I see how you made all of these that's really neat. What about the others? |
| 163. | Ankur: | We just made anyone that way. |
| 164. | R2: | Oh! |

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| 165. |  | Teacher: | What made you stop following it? |
| :---: | :---: | :---: | :---: |
| 166. |  | Ankur: | We couldn't find any other ones. |
| 167. |  | Teacher: | Then after you stopped here and you couldn't find any more what made you decide to go this way. |
| 168. |  | Ankur: | We did that from the bottom up |
| 169. |  | R2: | Let me ask you a question. In these I see here you have one red, two reds, three reds, four reds right? ( she starts from the right side and counts) |
| 170. |  | Ankur: | Uh huh! |
| 171. |  | R2: | I guess may be, five reds and here you have one yellow, two yellows three yellows, four yellows, five yellows. Alright, so that's a pattern I could see very easily. You want to put this over here to have five. ( a student picks up the five red tower and moves it next to the others) What about here though I have trouble seeing that can you help me see that? (points to the remaining towers) |
| 172. |  | Ankur: | We didn't do anything here. |
| 173. |  | R2: | You want to think about that how you can better explain that to me. That is very neat. Okay, think about how you can explain. This is very nice. Can we hear from somebody else? |
| 174. |  | Student: | Not yet. Not yet. |
| 175. |  | R2: | Not yet? |
| 176. |  | Stephen: | Over here. Over here |
| 177. | 00:17:04 | R2: | Okay, give you a minute to get set up. |
| 178. |  | Teacher: | Joe I want you to sit over there cause you can see just as well from there. Are you ready yet Stephen? |
| 179. | 00:17:37 | Stephen: | Almost. |
| 180. |  | Teacher: | Okay, Stephen how did you go about solving that problem? |

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| 181. |  | Stephen: | Uh! |
| :---: | :---: | :---: | :---: |
| 182. |  | Focundo: | It just came out. |
| 183. |  | T eacher: | What do you mean it just came out? |
| 184. |  | Focundo: | I just thought of them. |
| 185. |  | Teacher: | Okay, what did you start with? Do you remember the one you started with? |
| 186. |  | Stephen: | Yeah, we started with ( his partner picked up a yellow and four red tower) |
| 187. |  | Teacher: | And after you decided to do it that way what was your next? |
| 188. | 00:18:36 | Stephen: | This one (picks up four yellows and one red) |
| 189. |  | Teacher: | What made you do it that way? |
| 190. |  | Stephen: | Because it is the opposite see red yellow, red yellow ( he shows an opposite pair) |
| 191. |  | Teacher: | And where did you go from there? |
| 192. |  | Stephen: | Then we did, |
| 193. |  | Focundo: | These two. (gives the two solid towers to Stephen) |
| 194. |  | Stephen: | These two. |
| 195. |  | Teacher: | Okay, why those two? Look at them. Put them down and then you can tell me. |
| 196. |  | Stephen: | They are all plain. All you had to do was put five yellow and five red. You put them together and you see you got a match. |
| 197. |  | Teacher: | Where did you go next? Did you have anything in mind when you chose the next one or were you just going to guess? |
| 198. |  | Stephen: | (he finds the inverted pairs of the first towers) They are not the same 20:07 |

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| 199. |  | Teacher: | What is different about those two? |
| :---: | :---: | :---: | :---: |
| 200. |  | Stephen: | Because this one is down here is yellow and up here is yellow. Goes red-yellow, red-red, red-red, red-red, and yellow-red |
| 201. |  | Teacher: | Now when I saw you working at the beginning um, you were working and getting groups of them together and Focundo was getting his group together. And then I noticed that you were each getting your own set and I told you that your set had to be the same. |
| 202. |  | Focundo: | Same. |
| 203. |  | Teacher: | Not the same but you had to come up with one set as a pair. How did you straighten it out hen because Stephen was working by himself and you were working by yourself so I said it had to be a group effort |
| 204. |  | Stephen: | Well, Focundo made them and we had them and then I just checked them with the others to see if there if there was a double. |
| 205. |  | Teacher: | Did you have any doubles? |
| 206. |  | Stephen: | Yeah, like five |
| 207. |  | Teacher: | Did you fight over who was going to keep whose double? |
| 208. |  | Stephen: | No. |
| 209. |  | R2: | Where is he? |
| 210. |  | Matt | I am not sure. |
| 211. |  | R2: | Okay, We are going to move on to someone else and come back. Okay Jeff:. |
| 212. | 00:21:37 | Jeff: | We got a whole bunch. Twelve, fourteen.... Forty seven |
| 213. |  | Michelle: | There is a duplicate of this ( she removes a duplicate tower) |
| 214. |  | Jeff: | Okay. |

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| 215. | R2: | So you think there are other duplicates here |
| :---: | :---: | :---: |
| 216. | Jeff: | I don't. I will put them all in a straight line |
| 217. | Michelle: | I am a mover. (she removes a duplicate)I think there are thirty two |
| 218. | R2: | You think there are still thirty two. I think you found more. Well maybe you need to work on this some more. Now, Jeff: I see you are organizing them differently. Tell me a little bit about how you are organizing them. |
| 219. | Jeff: | First we made them all in patterns and now we are checking them for duplicates. |
| 220. | R2: | Can you show me? Can you tell me about your patterns? |
| 221. | Jeff: | You see we just went up like that like that we did all different ones. |
| 222. | R2: | Okay, here you went up with these two reds. What about here? |
| 223. | Jeff: | I am not sure because they all got mixed up 23.07 |
| 224. | R2: | You want to work on that a little more? |
| 225. | Jeff: | Yeah. |
| 226. | R2: | Give you a little bit more time. Well you check that we are going to see what Stephanie and Dana are did |
| 227. | Dana: | Twenty eight |
| 228. | Stephanie: | Twenty eight |
| 229. | R2: | Oh! You lost some |
| 230. | Stephanie: | We are checking. We have a lot left. |
| 231. | R2: | Now you are back to twenty eight |
| 232. | Stephanie: | Yeah |

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| 233. |  | R2: | Are you finished or are you still working on it? What about you? What are you convinced of now? |
| :---: | :---: | :---: | :---: |
| 234. |  | Stephanie: | We are convinced that you always have to think there are more |
| 235. |  | R2: | Well that's interesting. |
| 236. | 00:23:38 | Stephanie: | Well you do. You don't know the answer. There is no way, you could not go into your head and say I can figure this out in my head, you couldn't. you always have to think this isn't like the problem you gave us like there were five shirts and four pairs of pants where you could go in your head and figure it out. For this one you have to go and say like you have to keep going and say I have twenty eight but there might be thirty two as a total. |
| 237. |  | R2: | You don't think there is any way you ever know that you have them all and there can't possibly be any more. |
| 238. | 00:24:15 | Stephanie: | No, because you could buy like, the biggest, you could have reds and yellows all over this room and people could still get ideas. You would not know that one person could have forty four and other person could have, be having, would be having fifty eight and still going for more because they you don't know until you are finished until you are absolutely positively sure. |
| 239. |  | R2: | How do you become absolutely positive? |
| 240. |  | Stephanie: | That's |
| 241. |  | R2: | You haven't gotten there yet. You are absolutely positive. Okay. Can you tell me a little bit how you have them arranged? |
| 242. |  | Stephanie: | In groups. |
| 243. |  | R2: | So you have them in groups |
| 244. |  | Stephanie: | This is a group, This is another group. |
| 245. |  | R2: | How did you choose your groups? Dana tell me about |


|  |  |  |
| :--- | :--- | :--- |
| 246. |  | Dana: |
| hat. |  | Well when we looked, we made one we just took the <br> other colors and did like, say I Made this one (holding <br> a tower with red on top yellow, red yellow, yellow) <br> Stephanie would take the yellow first then the red then <br> the yellow how I have the red in the middle and then <br> the two reds how have the two yellows. |
| 00:25:16 | R2: | Okay I see. Okay so keep working and see what you <br> can come up with I like to know when you think you <br> know. Okay. Now let's see, did we hear back here yet <br> did we hear from Mike and Milin? |
| 248. |  | Milin: | | We just got to fix this stuff real quick |
| :--- |
| 249. |

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| 263. |  | Milin: | Sure! See |
| :---: | :---: | :---: | :---: |
| 264. |  | R2: | Now hold on Mike and Milin how many do you think there are? |
| 265. |  | Milin: | Thirty two now. |
| 266. |  | R2: | You think there are thirty two? You want to put these others aside? Okay. How do you think you know that? |
| 267. | 26.56 | Milin: | Because right now we can't find any doubles any more. |
| 268. |  | R2: | Okay. |
| 269. |  | Milin: | Now. See we had a little problem because we put one, this, in the wrong place. I mean yeah this in the wrong place I put this here so (points to two pairs of towers) These two got stuck together and these two got stuck together and now I just switched them back cause I found that out. So now we still think there is thirty two. (Milin points to these two corrected pairs). |
| 270. |  | R2: | You still think there is thirty two. Okay. That's interesting. Okay. Did we hear from everybody? Did everyone have a chance to explain? There is one more group who didn't have a chance to explain and I like to talk to- Save what you have. I want to talk to all of you in a moment don't mess up what you have. Okay, can we have our last group here? Let's hear from Matt and Jenifer then we go to Ankur and see what he has done. Okay. What did you find? How many? |
| 271. |  | Matt | We found thirty two. |
| 272. |  | R2: | That what you believe there are? How did you do it? |
| 273. |  | Matt: | Well every time we found a pattern we would put the opposite color say we have yellow, yellow, yellow, red, yellow then red, red, red yellow, red. |
| 274. |  | R2: | Okay. How did you work with Jennifer? How did you- |
| 275. |  | Jennifer: | I made them. |
| 276. |  | Matt: | Every time I found a pattern I would tell her to |


|  |  |  | duplicate and I keep on finding a pattern and then she found a pattern I would duplicate. |
| :---: | :---: | :---: | :---: |
| 277. |  | R2: | Uh huh! So you took turns doing that. Is there any way you can be sure you have found them all? |
| 278. |  | Matt: | Well we kept on trying to get more but every time we try- |
| 279. |  | R2: | You couldn't find any more. I see. That's interesting. Let's hear what Ankur has to tell us about his way of trying to find all of them. Now this is interesting. |
| 280. |  | Ankur: | Over here we made three, we had a pattern like this going down |
| 281. |  | R2: | Okay now, let's look at this. Let's look at these patterns. What's special about these patterns? |
| 282. |  | Ankur: | Well - you go (lets Joey explain) |
| 283. |  | Joey: | We kind of mixed them up like, over here we have three yellows then red and yellow over here we have three reds and a yellow and red here we got two reds/one yellow/ two reds. Two yellows/one red and two yellows, and one red/one yellow/three reds.......... |
| 284. |  | Ankur: | We put them in the same place with different colors |
| 285. |  | Joey: | One yellow/one red/ and three yellows. |
| 286. | 00:29:52 | R2: | This set I set you only have one red and here you have one yellow in all of these and they are in the middle position I see that. Now here I also notice you have one yellow but you put it here can you explain to me why you put it with this group? You didn't put it with this group you put it with this group. Can you tell me why? That's interesting. |
| 287. |  | Ankur: | Because in here it is one, this is two, then three, and then four |
| 288. |  | R2: | And which one would this be? How many yellows in here? |
| 289. |  | Ankur: | None |


| 290. |  | R2: | So you have none, one, two, three and four oh and then <br> you have five if you want to. Then you have the <br> opposites here. No red, one red that's very interesting! <br> I wonder if we should share what you have done here <br> with the class. Is there a way that we could share this? <br> Do you have some tape? |
| :--- | :--- | :--- | :--- |
| 291. |  | Amy: | Yes |
| 292. | 00:30:47 | R2: | Cause' what you have done here I haven't seen anyone <br> else do and I'd like people to think about it. Okay. <br> Amy is going to help us tape this together. |
| 293. |  | Amy: | You want all this in one piece? |


| 306. |  | Amy: | Powerful tape! |
| :---: | :---: | :---: | :---: |
| 307. |  | R2: | Why don't you come and stand up here and explain to the class. Now you all kind of can see what they did here and tell me if you think that you might have done something like this if it were interesting. Why don't you hold it up and explain to the class what you did. |
| 308. | 00:32:03 | Ankur: | Over here we used one all red. We used one yellow, then two yellow, then three yellows and then four yellows all five yellows and then in this one we used one red two reds, three reds, and then four reds. And over- |
| 309. | 00:32:29 | R2: | Let me see, here. Can you all see this? Can you all see this here? I just want be sure you...I guess Ankur is having us look at this piece here. I am covering the other part with my hand. See where they are all reds. He told me before how many yellows there were there. How many yellows were there on this one? This first one (she points to the tower with her thumb) |
| 310. |  | Ankur: | Zero |
| 311. |  | R2: | There were zero yellows. And the next one had how many yellows? |
| 312. |  | Ankur: | One |
| 313. |  | R2: | And the next one had two, three, four and five. So he saw a pattern that way. You see that? The same thing similar thing... did anybody else do something like that? Anybody else? Did you do something like that there? (looks at the towers made by a group in the front) |
| 314. | 00:33: | Student: | We did like a yellow then red, yellow, yellow then red...we made like stairs |
| 315. |  | R2: | So you made stairs like that. What about this end in here? (she points to the left side of the taped set of towers the towers with one yellow or one red in the middle positions) This is interesting here. This is another way of looking for...I will hold and you talk |
| 316. |  | Joey: | For these right here and these we kind of did the same |


|  |  |  | thing as them like we put three red then yellow and then another red, and here we got two and yellow and two, one a yellow then three and then a yellow red and we did the same thing here we put a yellow red three yellows, two yellows a red and two yellows, three one and one and then we got a whole row. |
| :---: | :---: | :---: | :---: |
| 317. |  | R2: | Okay now. I saw other people do something like this where they had exactly one red and here they put it in the middle but here but I was really curious to understand they did something different. When you see here they have a red. Let's say the second floor. Can we say this is the second floor of the tower red on the third floor, only one red in this tower, a red on the fourth floor? I say gee! Where is your tower where you only have one red on the bottom floor and only one red on the top floor? Cause' that's what most of you did. Isn't it? When you showed the one red you showed them on every floor. How many of you did that? Show one red on every floor. Raise your hand if you did that. I saw a few of you use that as a strategy. Or you used one yellow on every floor. But they didn't do that. They didn't do that for this pattern here. And I asked them why uh they did that. But notice what they did because they used that one red on the top floor for very different pattern this other pattern. I noticed Jeff: saw that too. And when Jeff: saw that what he ended up doing I think you ended up using it twice then didn't you Jeff:? You were using it for one pattern and then you picked it up and used it for the other pattern isn't that right? |
| 318. | 00:35:35 | Jeff: | Yeah well |
| 319. |  | R2: | And you ending up with more than thirty two? |
| 320. |  | Jeff: | Yeah we got more than thirty two but we rounded back down to thirty |
| 321. |  | R2: | Ahh! So you are down to thirty. But you were looking different ways to help me follow your pattern you see that because there are different ways of organizing these. Now this ids exactly one red or exactly one yellow. Um, I am kind of interested in what you did with may be exactly two reds or exactly two yellows did anybody look for patterns with exactly two reds or |


|  |  |  | exactly two yellows? I am kind of curious about that. You want to talk to your partner and think and take a look at what you built. What do you see there? Do you see any thing in that? |
| :---: | :---: | :---: | :---: |
| 322. |  | Milin: | Holds up a tower |
| 323. |  | R2: | Now that's exactly two. Let's worry about exactly two. Let's worry about exactly two reds. Tell me what you can tell me about patterns that have exactly two reds. Can you help me with that? If I want to be sure you didn't miss any and you didn't count any twice. Can you find me patterns that have exactly two red cubes in your towers? That has four in it. That has three (students were holding up towers) |
| 324. | 00:36:53 | Student: | I have one. |
| 325. |  | R2: | But I want to see all of them with exactly two reds. |
| 326. |  | Student: | Here! Here! |
| 327. |  | R2: | Is that the only one you have? Why don't you study those for a minute your towers with exactly two reds. |
| 328. |  | Students: | Here! |
| 329. |  | R2: | You only have four towers with exactly two reds? (Students were holding up four towers) here is another one. |
| 330. |  | Student: | We have all these. |
| 331. |  | R2: | Why don't you study those for a minute? You have exactly two reds together? Show me all your. Convince me that you have to have all of them and there are no more. Just work on these for a minute. But you have to convince me by looking at a pattern that you have not missed any. You can take that apart we will put it together another way (talking about Ankur's and Joey's Taped towers and helps them take the tape off). Okay you all listening? Now somebody at this table told me that when I look at all the towers with exactly two reds, there were how many of them? (she is walking around and kids are holding up towers to show her) |


| 332. |  | Student: | Ten |
| :--- | :--- | :--- | :--- |
| 333. |  | R2: | How many got ten towers with exactly two reds? How <br> many? Okay. I want you to think tomorrow is how <br> you can convince me that what you found are ten that <br> there can't be eleven or twelve or eight or nine or six. <br> How can you- you study those towers and you find a <br> way of convincing me that you have all of them. Now <br> let me give you something to think about that might be <br> interesting. Alex showed me this. Now these two reds <br> are both on the bottom floors, right? Is that right? So I <br> can keep track of this in my head easily these two reds <br> are in the bottom floor. And he showed me that when <br> we look at these two reds next to them they are on the <br> second and third floor right? You see that? The first <br> two reds are on the first and second floor these two are <br> on the second and third floor. What is another <br> possibility when I have these two reds together? Any <br> ideas? Ankur? |
| 334. |  |  | Ankur: | | Students: |
| :--- |
| 335. |

Line Time
Name
Transcript
Manjit K. Sran

| 341. |  | R2: | No. Now here they are always together. Can they be <br> separated by a floor? |
| :--- | :--- | :--- | :--- |
| 342. |  | Students: | Yes. |
| 343. |  | R2: | Can they be separated by two floors? |
| 344. | $00: 40: 52$ | Students: | Yes. |
| 345. |  | R2: | Can they be separated by three floors? |
| 346. |  | Students: | Yes. |
| 347. |  | R2: | Can they be separated by four floors? |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 358. |  | Student: | Here, I have one |
| 359. |  | R2: | That's not two reds that is four reds. That is three reds. We could only have two reds. |
| 360. |  | Student: | Here, here, here |
| 361. |  | R2: | That's two reds separated by two. I want... |
| 362. |  | Students: | I got one, I got one |
| 363. |  | R2: | This is the same I have this one |
| 364. |  | Steven: | I have one |
| 365. |  | R2: | That's the same as this. Ah here we go. We have two reds separated by one. I want another one. Can you make me another one with two reds separated by exactly one. Suppose to think real hard. Stephanie are you thinking real hard? We have that one, look. |
| 366. |  | Alex: | You can't |
| 367. |  | R2: | Why cant you Alex? You mean I can't have two reds where I can have my red in the fourth floor and my yellow in the fifth floor and my red in the sixth floor? I Can't do that? |
| 368. | 00:42:58 | Student: | No, you can't do any. |
| 369. |  | R2: | Why? |
| 370. |  | Student: | You only have five blocks so you can't make it up to the seven. |
| 371. |  | R2: | Ohh! So these are all that's possible to have two reds separated by one yellow? |
| 372. |  | Students: | Yes. |
| 373. | 00:43:09 | R2: | So what about two reds separated by- what else- two yellows? Alright, find me all the possibilities. |
| 374. |  | Michael 2: | I got one here |
| 375. |  | Stephen: | Here |


| 376. |  | R2: | Alright when you get one raise your hand. Hold on. Oh here is the first floor Where is the second floor? Where is the third floor? I want the third floor. Where is the red in the third floor? Think. Think. I am asking you to find me exactly two reds separated by two. |
| :---: | :---: | :---: | :---: |
| 377. |  | Students: | Right here. Right here. |
| 378. |  | Milin: | Got one. Uhg. |
| 379. |  | Ankur: | There are only two |
| 380. |  | R2: | Why are there only two? He claims there are only two. How many of you, if you think you know the answer raise your hand if you think you know. Don't speak out. Why do you think there are no more? Just raise your hand if you think you know. Again, exactly two reds separated by two why do you think there are no more? You thinking hard? See a lot of hands up here. Yes. |
| 381. | 00:44:26 | Michael 2: | Because if you needed one more you would need more than five because you need another one |
| 382. |  | R2: | Wonderful? You would need another block? Let's put this here. Is there another way to have two reds separated by three? |
| 383. |  | Student: | I have it right here. |
| 384. |  | R2: | Ah hah! I want to two reds separated by three another way. |
| 385. |  | Milin: | There isn't any |
| 386. |  | R2: | Why not? |
| 387. |  | Milin: | On ones there is only three. On two's there is only two. And on threes there is only one. |
| 388. |  | R2: | You believe that? How many of you agree? Ahh! Look now I have exactly two reds how many ways? Can you tell me? Exactly two reds how many different |


|  |  |  | ways do I have them? If you know raise your hand. Don't speak out. How many ways are there to have exactly two reds? |
| :---: | :---: | :---: | :---: |
| 389. |  | Michael 2: | I know |
| 390. |  | R2: | How many ways are there to have exactly two reds? Jaime. |
| 391. |  | Jaime: | Ten |
| 392. |  | R2: | Could there be any more Jaime? |
| 393. |  | Jaime: | No. |
| 394. | 00:45:26 | R2: | Why Not? How many of you are convinced there can be no more? You really think that if you had to go to another teacher or if we invited a new teacher in that you can convince the new teacher you found all of the two reds and there are no others. What do you think Stephanie? |
| 395. | 00:45:45 | Stephanie: | I think so. Because with the four you can only make, with the four you can make, with the that the first one with two together you can make four with one in between you can make three, ,with two in between you can make two, with one in between you can make one I mean with three in between you can make one. But you can't make with four in between five in between or any thing else you don't have enough you can't you can only use five blocks. |
| 396. |  | R2: | That's really great. You know I am going to ask you the sixty four thousand dollar question. Did you ever hear about the quiz show where you get sixty four thousand dollars if you answer it right? |
| 397. |  | Ankur: | Yeah. Yeah. |
| 398. | 00:46:27 | R2: | Okay. Now that you have all possible ways for building your towers there are ten with exactly two reds, what do you automatically know the answer to? Look at the hands going up. You know some more towers without doing any building you see them in your mind don't you? The minute you see them in your mind you |


|  |  |  | didn't even have to make them. What do you see in your mind? |
| :---: | :---: | :---: | :---: |
| 399. | 00:46:53 | Ankur: | You could just make these yellows red and this red yellow and switch the colors around. |
| 400. |  | R2: | How many thought of that? About just switching the colors around. So, how many more towers do we know with exactly two of one color? How many more do we have? |
| 401. |  | Student: | Um, how many we... |
| 402. |  | R2: | We have ten here with exactly two red, how many more do we have with exactly two yellow? |
| 403. |  | Michael 2: | I know |
| 404. |  | Student: | Ten. |
| 405. | 00:47:20 | R2: | Ten so, all together with exactly two, red or yellow, how many towers do we have already? |
| 406. |  | Student: | Altogether? |
| 407. |  | R2: | Yes. |
| 408. |  | Student: | Twenty. |
| 409. |  | R2: | We have twenty towers with exactly two, and you all convinced you can take a stranger and tell them there are no more, right? Now remember when I walked around and you were convinced cause you worked so hard doing this problem, I mean you really worked hard I don't ever remember seeing a fourth grade class work so long on a math problem ever, as you worked without stopping you were incredible! But you now can see that this is how many you can find with exactly two. I want to ask you a question. Could you use the same kind of reasoning when you are here tomorrow to find out how many there are with exactly three? You already did exactly one. Everybody in this class showed me how many towers you can build with exactly one, remember that, you did that. How many were there by the way I don't remember? With exactly one red for instance. With exactly one red how many |


|  |  | towers did you have? Can you see that in your mind don't look at the towers imagine it? If you can imagine it raise your hand. Oh! Look how many people can imagine it. Great! Wonderful! Okay, anybody else imagining it? Okay, Jeff: back there, Jeff: tell me how many with exactly one red in your mind. |
| :---: | :---: | :---: |
| 410. | Jeff: | Um, um, |
| 411. | R2: | You can consult with your partner. |
| 412. | Jeff: | Okay. One. |
| 413. | Milin: | No. |
| 414. | R2: | Oh so we don't agree. Stephanie? |
| 415. | Stephanie: | With exactly one red five towers high? Uh you can build five. |
| 416. | R2: | How many agree with Stephanie? Five. Show me. |
| 417. | Stephanie: | Alright, well, you have one red |
| 418. | R2: | Well you have them build just take them from your table and show me. |
| 419. | Stephanie: | Alright! |
| 420. | R2: | Get your friend to help you - get you the five with one red |
| 421. | Stephanie: | Here's one. This is the second one, this is the first one, this is the third one |
| 422. | R2: | Okay, you get the picture? See you can see the five already coming up. You all see that? Jeff: you agree? |
| 423. | Stephanie: | Here we go! |
| 424. | R2: | Okay, how many with exactly one yellow? |
| 425. | Stephanie: | Five! |
| 426. | R2: | So we now built how many more towers? |


| Line | Nime |  | Transcript |
| :--- | :--- | :--- | :--- |
| 427. |  | Stephanie: | Ten |
| 428. |  | R2: | We have twenty . Sran |
| 429. |  | Stephanie: | Uh huh. |
| 430. |  | R2: | That's interesting. So we have thirty? I don't have built another ten, right? <br> thirty two. What did I miss? |
| 431. |  | Student: | You don't have all red |
| 432. |  | R2: | Oh no, with five red. Okay, how many with five red? |
| 433. |  | Ankur: | Two |
| 434. |  | R2: | With five red? |
| 435. |  | Milin: | Only one! |
| 436. |  | R2: | Okay, so that gives me thirty one. |
| 437. |  | Ankur: | And you have all yellow. |
| 438. |  | R2: | Oh that gives me thirty two. That makes me feel <br> better. I am going to go home and sleep better tonight <br> cause I really believe there are thirty two. I really <br> believe that. I think you are great! Let's give you a <br> round of applause. |

## INTERVIEW ON 2/7/92



Towers from Group Gand Group H rearranged to form

| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 1. | 00:00:14 | R1: | What did you think about the activity yesterday? Did you enjoy it? |
| 2. |  | Milin: | Yes. |
| 3. |  | R1: | Why? |
| 4. |  | Milin: | Um... before we had something like this, in second grade |
| 5. |  | R1: | Second grade? |
| 6. |  | Milin: | I don't know. Second or third. I don't know. |
| 7. |  | R1: | Yeah. What do you mean, "something like this"? |
| 8. |  | Milin: | 'Cause we had to urn build them and ... I don't know which grade but ... we had to build them so it was much easier this time. |
| 9. |  | R1: | Okay, so this time was easier because you'd done it before? When you did it before was it exactly the same, or do you remember? |
| 10. |  | Milin: | I don't remember. (He chuckles) |
| 11. |  | R1: | You really don't remember, but you felt like you were sort of doing it... again. |
| 12. | 00:00:52 | Milin: | Yeah. |
| 13. |  | Mrs. O: | Built towers... |
| 14. |  | Milin: | Yeah. |
| 15. |  | R1: | And how did that make it easier yesterday? |
| 16. | 00:05:57 | Milin: | Because um when ... because last time I didn't um go for about the amount, but this time I thought it would be around 30 or 40 , so ... |
| 17. | 00:01:12 | R1: | Oh, because you remembered from |
| 18. | 00:01:15 | Milin: | Yeah |
| 19. |  | R1: | Last time that there were more than you had thought or something, and so you kept trying. Oh, what, how did |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
|  |  |  | you solve this problem? |
| 20. |  | Milin: | Um, when Michael and I kept on um building them and putting another one exactly like that but different colors ... |
| 21. | 00:01:33 | R1: | What do you mean by that? |
| 22. |  | Milin: | Cause like Michael and um Paul did ... we um, we looked at the colors and all the yellows turned to reds, and all the reds turned to yellows ... |
| 23. |  | R1: | Show me. Maybe we could use some Unifix Cubes and show me what you mean as an example for what you did. |
| 24. | 00:01:55 | Milin: | Like if you take this for example. (YYYRR) |
| 25. |  | R1: | Uh huh |
| 26. |  | Milin: | We did this (Milin refers to towers YYYRR and RRRYY) and now we have this. All of these yellows turned to these reds, and all these two reds turned to these two yellows. |
| 27. |  | R1: | Oh, I see. Is that always a way you worked? I mean, did that always work? |
| 28. |  | Milin: | Yeah, uh huh. Because see if ... if we had something like ... this (He builds YRRYY) we would always check to see if it would be like this (he compares YRRYY to YYYRR and $R R R Y Y) \ldots$ or if this was a red or something. |
| 29. |  | R1: | Uh huh. |
| 30. |  | Milin: | And then we, (he builds tower RYYRR) now, it would be like this. And they'll be the same okay ... and these two, so we put it like this (he groups all four towers together) and every time we see if one goes to that group and then either just put it away if it does, and um, if it doesn't then we got a new group. |
| 31. | 00:03:13 | R1: | New group? What do you mean by new group? |
| 32. | 00:03:15 | Milin: | Because see, these two are a group because this turns into this ...( He compares tower YRRYY and RYYRR and points to opposite red and yellow positions) and this turns into this. |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 33. | 00:03:19 | R1: | Okay, so they came in two's always. |
| 34. |  | Milin: | Yeah. |
| 35. |  | R1: | And any one you found always would have a partner |
| 36. |  | Milin: | Yeah. |
| 37. |  | R1: | ... or whatever? |
| 38. |  | Milin: | Yes, Uh huh. |
| 39. |  | R1: | And that's always going to be true? |
| 40. |  | Milin: | Uh huh. |
| 41. |  | R1: | That's really interesting. Um, how many towers did you find? |
| 42. | 00:03:37 | Milin: | Thirty-two. (Confidently) |
| 43. |  | R1: | You think you found them all? |
| 44. |  | Milin: | Yeah. |
| 45. |  | R1: | Why? |
| 46. |  | Milin: | Because as you guys explained yesterday. Also, because um we did all that towers and going up one ... |
| 47. |  | R1: | What, what do you mean? Oh, I don't know. That was when Dr. Maher was up there and you all were talking together. |
| 48. | 00:03:56 | Milin: | Yeah. |
| 49. | 00:03:57 | R1: | Can you help me remember what that was all about? |
| 50. |  | Milin: | That was about seeing if we had all of them. |
| 51. |  | R1: | Oh, and how did that help us see that we had all of them? |
| 52. |  | Milin: | Because most people thought that we had 32 but then some people thought that we had 35 and 34 but then they found extras and then she went up and we found it like |


|  |  |  | this. All the ... starting from the reds (He points to the bottom two reds on tower $Y Y Y R R$ ) one red then another red on the next floor (he points to imaginary red cube on the second floor of tower RRRYY) so they'll be four, five reds. |
| :---: | :---: | :---: | :---: |
| 53. |  | R1: | Can you show me what you're talking about there? Let's keep these. (R1 puts YRRYY, RYYRR, YYYRR and RRRYY aside, Milin builds YYYYR) Maybe because these were your pairs that you were showing me for examples and I want to know what you're talking about. |
| 54. | 00:04:40 | Milin: | Sighs Like this and then three... (He builds YYYRY) See this goes in the staircase ( he compares YYYRY and YYYYR and points out the beginning of the staircase pattern) and keeps on going |
| 55. |  | R1: | Oh. |
| 56. |  | Milin: | To the third, and forth and fifth. (He points to imaginary red cube in the third, fourth, and fifth position.) |
| 57. |  | R1: | How many would there be? |
| 58. |  | Milin: | Five! |
| 59. |  | R1: | Why? |
| 60. |  | Milin: | Because when you keep on going up that would be five, five of them. |
| 61. |  | R1: | Why can't there be more? |
| 62. | 00:05:08 | Milin: | Because there's only five of these (he points to YYYRY with five cubes in $i t$ ), so one on each block. |
| 63. | 00:05:13 | R1: | Oh, Okay. And so that means then with one red block (R1 points to the bottom red cube on YYYYR) |
| 64. |  | Milin: | Yeah. |
| 65. |  | R1: | There's how many? |
| 66. |  | Milin: | On one red block um there's four yellows. On another red block on the second floor there's three yellows above it. (He refers to YYYRY) |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 67. |  | R1: | How many yellows all together though, in that tower? |
| 68. |  | Milin: | Four. |
| 69. |  | R1: | Oh. |
| 70. |  | Milin: | Always four yellows if you're talking about one, but if you're talking about two they'll be three yellows. |
| 71. | 00:05:38 | R1: | Oh, like |
| 72. |  | Milin: | Like on this, I mean |
| 73. |  | R1: | Which one over here has ... (She refers to the first four she set aside) |
| 74. |  | Milin: | This ... (Milin points to YYYRR) |
| 75. |  | R1: | This one (she points YYYRR) and that one (she points to YRRYY). Yeah, yeah ... I see. So this one over here has (she points to $Y Y Y Y R$ )... I mean with two reds and three yellows (she points to $Y R R Y Y \& Y Y Y R R$ ) and over here is one red and ... (she points to YYYRY and YYYYR) |
| 76. |  | R1: | Four yellows. And how many were there that had one red and four yellows? |
| 77. | 00:05:59 | Milin: | Five. |
| 78. | 00:06:01 | R1: | Yeah. Okay, how many were there that had one yellow and four reds? |
| 79. | 00:06:07 | Milin: | One yellow and four reds? |
| 80. |  | R1: | Uh huh. |
| 81. |  | Milin: | Um ... five. |
| 82. |  | R1: | Why? |
| 83. |  | Milin: | Because if you had this (he points to YYYYR) then you could do the same thing but they'll be reds on this and yellows on this. Kind of like what I was showing you on these two (he points to YYYRR \& RRRYY) and all that. |
| 84. |  | R1: | Oh. Because of those ... |


| 85. |  | Milin: | Yeah. |
| :--- | :--- | :--- | :--- |
| 86. |  | R1: | Those pair things you were talking about. Oh, I see. And <br> so this was part of how we figured out how many there <br> were all together? |
| $\mathbf{8 7 .}$ | $00: 06: 32$ | Milin: | Uh huh. |
| $\mathbf{8 8 .}$ |  | R1: | Do you remember and, so how many were there that had <br> one red and four yellows? (She points to YYYYR) |
| $\mathbf{8 9 .}$ |  | Milin: | Five. |
| 90. |  | R1: | Maybe we should write these things down. I'm having a <br> hard time remembering all these numbers. |
| $\mathbf{9 1 .}$ |  | R1: | Maybe we could show them too and then that'll help. |
| 92. |  | Milin: | Yeah. <br> Okay then they'll ... (He tries to take YRRYY from the |
| first pile set aside) |  |  |  |


|  |  |  | the red cubes in the downward pattern of Group C) ... for the other... |
| :---: | :---: | :---: | :---: |
| 103. | 00:07:47 | Milin: | Yeah. Same thing because see ... you know ... same thing as this (he refers to the staircase pattern of Group C) see this has a partner like one yellow and four reds (he points to RYYYY) right here going down. This has the same thing (he points to YRYYY) but it's going the other way. |
| 104. |  | R1: | (She nods and says, "Uh huh" throughout his explanation) Uh huh. |
| 105. |  | Milin: | This is in the second floor. (He points to YYRYY) |
| 106. |  | R1: | Uh huh. |
| 107. |  | Milin: | They all have partners (he refers to Group C), so this would probably be ten. (Planning ahead - he refers to the un-built Group D) |
| 108. |  | R1: | Yeah. That's what I'd really would like for us to keep a record of. Maybe you could write down how many there were that had one red and four yellows (she points to RYYYY), and one yellow and four reds. Here's a pen you can use. |
| 109. | 00:08:21 | Milin: | Um ...( he writes " 1 yellow and 4 reds are 5 ") See, because these five (refers to Group B) and four reds go like that so they'll all have something ... a partner so then that'll be one red (he writes below the first line "1 red and 4 yellows are $5^{\prime \prime}$ )and ... see yellows are five and reds are five because you can't go any more. (He points to Group C) |
| 110. | 00:09:35 | R1: | (R1 nods and says, "Uh huh" throughout his explanation) Yeah, okay. And then after you did one yellow and four reds, and one red and four yellows, then what other possibilities were there? |
| 111. |  | Milin: | Um, there were ones that if you could use two of them. |
| 112. |  | R1: | How many of those were there? |
| 113. |  | Milin: | Um four ... four of each. Eight. |
| 114. |  | R1: | Let's see. |

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| 115. |  | Mrs. O | Did you just remember four just now, or what were you thinking about? |
| :---: | :---: | :---: | :---: |
| 116. |  | Milin: | No. I was thinking about because if this had this (he takes two reds put together and shows how you only have four patterns with two reds "stuck" together by traveling up RYYYY) then there would probably be two like this, two and then another two here and another two here and another two here. |
| 117. | 00:10:06 | R1: | Can you show me that? Can you show me that kind or show Mrs. O'Brien? |
| 118. |  | Milin: | It's like this. (He picks up YRYYY) |
| 119. |  | Mrs. O | Oh. |
| 120. |  | R1: | Oh, okay. You mean this is where you have two's moving. Okay. |
| 121. |  | Milin: | (He compares YRRYY \& YYYRR ) See, but there's one missing right there. |
| 122. |  | R1: | Can you build it? |
| 123. | 00:10:22 | Milin: | Can I use these two? (He shows R1 YRRYY \& YYYRR) |
| 124. | 00:10:24 | R1: | Oh sure, use those two and then, but let's don't tear up these (R1moves Group C away - Milin: builds YYRRY). Oh I see what you're talking about. |
| 125. |  | Milin: | See, this goes from here, here, here and then there's going to be another two. One here and one here. (He points to imaginary RRYYY) |
| 126. |  | R1: | Can you make another two? (He builds YRYYY) |
| 127. |  | Milin: | Whoops. |
| 128. |  | R1: | It's gotta have... |
| 129. |  | Milin: | Two reds (He builds another YRYYY - Thinking aloud) Two reds ... |
| 130. |  | R1: | But you got that one. |


| 131. |  | Milin: | Ah. (He builds RRYYY and adds it to the pile - Sighs) |
| :---: | :---: | :---: | :---: |
| 132. | 00:11:16 | R1: | Okay. And that's what you guessed, wasn't it? That there're four? (She points to Group E) |
| 133. |  | Milin: | Uh huh. |
| 134. |  | R1: | Okay, and so this has two reds and three yellows for a staircase. |
| 135. |  | Milin: | Two reds and three yellows on all of them. (He refers to Group E) |
| 136. |  | R1: | On all of them? Is there any other way to have a tower that has two reds and three yellows except in a staircase? (R1 is guiding him to look for alternatives) |
| 137. | 00:11:39 | Milin: | No, (he shakes his head) because ... see ...there's not gonna be any because see if you put ... (he refers to Group $E$ - he points out the downward pattern of 2 reds in the staircase) it'll only be one if you have three because see these two could go in there, these two, these two and these two. That's it. |
| 138. | 00:11:57 | R1: | Uh huh. To make a staircase okay. And so there's these four ... (she refers to Group E) |
| 139. | 00:12:02 | Milin: | Uh huh. |
| 140. |  | R1: | you had five here ... (She refers to Group C) |
| 141. |  | Milin: | And these four ... (He refers to Group E) |
| 142. |  | R1: | Yeah, and these are staircases |
| 143. |  | Milin: | Uh huh. |
| 144. |  | R1: | Okay, but I don't see how you're going to get from here |
| 145. |  | Milin: | See... |
| 146. |  | R1: | to thirty-two. |
| 147. |  | Milin: | See, this is double because of the yellows. (He points to Group C) |


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| 148. |  | R1: | Uh huhit K. Sran |


| 165. | 00:13:12 | R1: | Show me |
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| 166. |  | Mrs. 0 | I don't... understand. |
| 167. |  | R1: | I don't understand why this is the double for this one? (She points to RYYRR and YYRRY) |
| 168. |  | Milin: | No, no ... This is the double for this. Because, nah, oh I didn't see that. (He figures out his mistake and puts RYYRR and RRRYY aside) These don't go over there. |
| 169. |  | R1: | Okay. Just make me the doubles, that'll help. |
| 170. |  | Milin: | This is the double for this. (He builds another RRRYY and puts it next to YYYRR) |
| 171. |  | R1: | Hum, yeah, that is the same as this one. (She points out the original RRRYY) |
| 172. |  | Mrs. 0 | That's okay. |
| 173. |  | Milin: | I'll just put it underneath. (Meta-analysis - RRRYY underneath YYYRR) |
| 174. | 00:13:44 | R1: | Oh, maybe that would be easier. |
| 175. | 00:13:54 | Milin: | (Thinking aloud) I need two yellows (he builds another RYYRR) and the one on the top and that'll match ... (he sigh)s oh yeah, this one ... |
| 176. |  | R1: | Uh huh |
| 177. |  | Milin: | This will match with this (he puts it underneath YRRYY) |
| 178. |  | R1: | Uh huh. |
| 179. |  | Milin: | and (he sighs and builds RRYYR), this will match with this ... (he puts it underneath YYRRY) |
| 180. |  | R1: | Uh huh. |
| 181. |  | Milin: | And (he builds YYRRR), this will match with this. (He puts it underneath RRYYY) |
| 182. |  | R1: | Oh, oh, yeah. You told me there was going to be eight. That's right. |


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| 183. |  | Milin: | That matches. See all of these (he points to Group F) look exactly like all of these. (He points to Group E) |
| 184. |  | R1: | Yeah they're the doubles. Yeah, I understand that. Okay, and so you have ten of these (she points to Group C) and you have eight of these. (She points to Group $E \& F$ ) |
| 185. |  | Milin: | And on the three's they'll probably be um ... |
| 186. | 00:14:53 | R1: | (Interrupting) Show me what a three would look like. |
| 187. |  | Milin: | On the yellows, I'll do it right now (he builds another YYYRR) ... like this. |
| 188. |  | R1: | But don't you already have that? |
| 189. |  | Milin: | (He thinks )Yeah ... see right here. (He puts the new YYYRR on top of the old YYYRR) |
| 190. |  | Mrs. 0 | So are you gonna count that? |
| 191. | 00:15:14 | Milin: | No. (He pulls YYYRR apart) |
| 192. | 00:15:15 | R1: | So can you count it again? Okay, are there any other towers? We've got... you said this was all the towers that had one red and four yellows |
| 193. |  | Milin: | Uh huh. |
| 194. |  | R1: | Okay, is this all the towers that have two reds and three yellows? (She refers to Group E) |
| 195. |  | Milin: | Uh huh. (He rearranges the duplicate YYYRR he built earlier) |
| 196. |  | R1: | And so we just have eighteen? |
| 197. | 00:15:40 | Milin: | But then what about the three's? |
| 198. |  | Mrs. 0 | Show us some three's. |
| 199. |  | R1: | Show us some three's. |
| 200. |  | Mrs. 0 | You showed us one and this was ... |


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| 201. |  | Milin: | Oh yeah. |
| 202. |  | Mrs. 0 | You said was the same as this one. Are there any other, maybe any other three's you could make? |
| 203. |  | Milin: | Probably not. (He chuckles) |
| 204. |  | Mrs. O | Do you want to try something? |
| 205. |  | R1: | Try something out. |
| 206. |  | Milin: | (He builds another YYRRY) There's this (he realizes it's a duplicate and points it out)... yeah, because see all these yellows have three so you can't make any others with three. |
| 207. |  | R1: | Does it have to be a staircase? |
| 208. | 00:16:17 | Milin: | Um, yeah. Because then you would have to have something like this (he puts the duplicated YYRRY on top of the old YYYRY) and then you have to have another one. |
| 209. | 00:16:29 | R1: | Uh huh. Yesterday you had thirty-two |
| 210. |  | Milin: | Yeah...Maybe we had doubles of something? (he pulls duplicate YYYRY apart. He is getting fidgety, nervous and seems unsure of himself) |
| 211. |  | R1: | Oh I don't know! |
| 212. |  | Milin: | We got two's. we've got four of these (points to towers with 2 reds and three yellows) |
| 213. |  | R1: | Uh huh |
| 214. |  | Milin: | Ones all of them like...(has a tower of all reds in his hand) |
| 215. |  | R1: | You did your ones (points to the group of towers with four yellows and one red) |
| 216. |  | Milin: | This...uh you have to go to fours (he smiles and puts \#1 aside.) That's where I messed up. Yeah. See if you have um four yellows it will look like this so... (he puts four yellows together and waves it over group E. he adds one more yellow to make YYYYY) |


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| 217. |  | R1: | It's right here. But you already have one (she points to the set of towers with four yellows and one red) |
| 218. |  | Milin: | Guess there probably would be 22 |
| 219. |  | R1: | Oh no! Uh what I am really wondering is... were there any other patterns you had yesterday? Except for staircases. |
| 220. |  | Milin: | Umm... |
| 221. |  | R1: | In terms of your towers can you remember any other patterns that you might have had? The way you put your blocks together to make towers that were... |
| 222. |  | Milin: | This (holds up a tower of all yellows) |
| 223. |  | R1: | There's some. How many of them were there? |
| 224. |  | Milin: | Two. |
| 225. | 00:17:48 | R1: | Yeah... okay. |
| 226. |  | Mrs. 0 | Put those here if you want. (He puts them next to Group C) |
| 227. |  | R1: | Uh huh. |
| 228. |  | Milin: | Four's. There should ... probably be like ... three. |
| 229. |  | Mrs. 0 | What do you mean? |
| 230. |  | Milin: | On four's, there should be three maybe... (He guessing) ... I don't know, but that would go with this. (He points to Group C) |
| 231. |  | R1: | Uh huh. |
| 232. |  | Milin: | So maybe we made doubles on the staircase? |
| 233. |  | R1: | I wonder ... I wonder if you could take this tower right here? (She give him YRRYY and then pulls it back) Actually, I'm going to give you this one (she stands up $R Y Y R R$ ) because we have a double of it. Is there a way you could rearrange those blocks in some way so that it |


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| 234. |  | Milin: |
| 235. |  | R1: |
| (He builds YRRRY) This? Yeah, this |  |  |
| 236. |  | Milin: | | (Pause) Yeah, so that's one other one. |
| :--- |
| So guess there might be thirty-two still. |
| 237. |


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| 252. |  | Milin: | Yeah. |
| 253. |  | R1: | Of two yellows and three reds. |
| 254. |  | Milin: | This is another one that goes with this. (He builds YRYRY and puts it next to RYRYR) |
| 255. |  | R1: | You mean that's its partner? |
| 256. |  | Milin: | Yeah. |
| 257. |  | R1: | Okay, does it have two yellows and three reds? |
| 258. |  | Milin: | No, but it has ... see the yellows turn to reds and the reds turn to yellows. (He points to YRYRY and RYRYR) |
| 259. |  | R1: | Oh, oh, okay. So if ... alright. And what about this one? (She points to YRRRY) |
| 260. |  | Milin: | (He builds RYYYR) Has this as a partner. |
| 261. |  | R1: | Uh huh. |
| 262. |  | Milin: | So there's so far on this, there will be eighteen and twenty, twenty-two and twenty-four (He sighs as he counts towers of staircases) |
| 263. |  | R1: | Uh huh. |
| 264. |  | Milin: | And now we have to get er ... eight... um (He sighs and tries out new towers of five) |
| 265. |  | R1: | I'm thinking ... if this has got two yellows and three reds and this ones got two yellows and three reds ... (She starts Groups G and H) |
| 266. |  | Milin: | This could work. (He builds YRRYR) |
| 267. |  | R1: | Let me see ... has it got two yellows and three reds also? |
| 268. |  | Milin: | No ... yeah. So now I'll get two reds and three yellows. (Meta-analysis) |
| 269. |  | R1: | Oh, to go over here? (She refers to Group G) |


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| 270. |  | Milin: | (He sighs and builds RYYRY )This is because one of them is separated, so you can't make it like this. (He points to Group F) |
| 271. |  | R1: | Oh, explain what you mean, what you mean by that? |
| 272. | 00:22:06 | Milin: | So ... if one separated ... (he points to RYYRY) See on this the red separated ... um ... but ... I mean the yellow's separated by the red ... so this and these two |
| 273. |  | R1: | Oh, I see. |
| 274. |  | Milin: | And this, they're just putting it like three. (He points to Group F) |
| 275. |  | R1: | Oh, I see. So that's the difference. |
| 276. | 00:22:27 | Milin: | Together ... uh huh. |
| 277. |  | R1: | Yeah, and so for these are the ones that are separated. (She points to the yellows on RYYRY) |
| 278. |  | Milin: | Uh huh. |
| 279. |  | R1: | But now this one (she points to RYYRY) is a partner to one of these, isn't it? (She points to YRRYR and RYRYR) |
| 280. | 00:22:38 | Milin: | It's a partner to ... |
| 281. | 00:22:40 | R1: | This is the opposite. |
| 282. |  | Milin: | This. (He holds up YRRYR) |
| 283. |  | R1: | To that one, yeah. Okay, so if these ... these |
| 284. |  | Milin: | And these two are partners to these two. (He groups together partners RYYYR and YRRRY, YRYRY and RYRYR basically reorganizing R1's Groups $G$ and H into his own type group of partners) |
| 285. |  | R1: | Yeah, yeah, yeah, okay. Which ones are the ones with two yellows and three red because that's what we were working on? |
| 286. |  | Milin: | Um, the two yellows and three reds are this one, this ... |


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| 288. |  | R1: | This one, okay. (She picks up YRYRY) Sran |
| 289. |  | Milin: | No, two yellows and three reds is this one. (He picks up <br> RYRYR) |
| 290. |  | Milin: | Yeah, okay. So we'll put it here, and this partner right <br> over here ... (She rearranges his patterns into Groups G <br> and H) |
| The group) |  |  |  |


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| 304. |  | Milin: | these two are down here ... (He points to the yellow cubes on YRYYR) |
| 305. |  | R1: | I see ... that's kind of a double staircase. Does it have a partner? |
| 306. |  | Milin: | Yeah ... (he builds RYRRY) this ... see |
| 307. |  | R1: | Uh huh. Yeah. Which group does it go with? Two yellows and three reds? Or yeah ... |
| 308. |  | Milin: | See ... this. (He puts RYRRY with Group H) |
| 309. |  | R1: | Okay. |
| 310. |  | Milin: | That would be eighteen, twenty ... (Counting to himself) |
| 311. | 00:24:47 | R1: | Okay. Can.. is there any others that you could see, that would follow those kinds of |
| 312. |  | Milin: | Um, there's probably another pair. |
| 313. |  | R1: | Uh huh. |
| 314. |  | Milin: | I'll try this, this, and this ... (he builds a duplicate RYYRR) Do we have this yet? |
| 315. |  | R1: | Beats me. Which is that? Two reds and three yellows or two ... which ... which ... |
| 316. |  | Milin: | Three reds and two yellows ... |
| 317. |  | R1: | So which group does it go in? |
| 318. |  | Milin: | This ... (He refers to Group H) |
| 319. |  | R1: | This down here. Oh I see. What did you do that time? |
| 320. | 00:25:24 | Milin: | This time I separated by two yellows, then over here. |
| 321. |  | Mrs. O | Did we have two yellows anywhere else? |
| 322. |  | Milin: | No. Not together. Up here. (He points to the top of duplicate RYYRR) |


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| 323. |  | R1: | Uh huh. |
| 324. |  | Milin: | And see if you had a red here (he points to the bottom yellow cube of RYYRY) then it would be the same as this. (He points to RYYRR) |
| 325. |  | R1: | Uh huh. Looks good to me. |
| 326. | 00:25:44 | Mrs. 0 | We don't... so you're just now checking in here cause you know that you've already used all those? |
| 327. |  | R1: | You don't need to check over here? (She points to Group F) |
| 328. |  | Milin: | Because these (he points to Groups C, $E$ and $F$ ) are all together, so right now I'm splitting these apart so they... |
| 329. |  | Mrs. 0 | What are you splitting apart here? (She points to duplicate RYYRR) |
| 330. |  | Milin: | Here I'm splitting apart these so ... all of these reds ... so they won't go ... |
| 331. |  | R1: | Oh. And you don't think there's anything over here (she points to Group F) that's the same way? |
| 332. |  | Milin: | There can't be because on these (he points to Group F) they're all together. |
| 333. |  | Mrs. O | What's together over here? (She points to Group F) |
| 334. |  | Milin: | On this, see they're all three together and these two are together. (He points to Group F) |
| 335. |  | Mrs. 0 | But not here (she points to RYYRR) and not here. (She points to RRYYR) |
| 336. | 00:26:22 | Milin: | Oh yeah, you're right. (He realizes he has a duplicate RYYRR) This goes with this ... um ...(he compares duplicate RYYRR to original RYYRR -sighs - builds YRRYR and compares it to Group H) we have this ... (he builds RRYRR) this ... yeah I think ...(He is trying to create a new tower ) |
| 337. |  | R1: | Yeah, that's what ... |


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| 338. |  | Milin: | That's (he points to rryrr the double to this. (He points to yyryy) |
| 339. |  | R1: | Yeah, yeah ... we said ... you said you really thought there were only ten of these. (She points to Group C) |
| 340. |  | Milin: | Uh huh. There has to be because there's doubles. |
| 341. |  | R1: | Yeah, yeah. So we have to be playing with two of one color and three of another and you have these. (She points to Groups $A$ and B) |
| 342. |  | Milin: | (He builds YRYRR and compares it to all the other towers) This ... |
| 343. |  | R1: | What, you have two yellows there? |
| 344. |  | Milin: | Here ... |
| 345. |  | R1: | Yeah. |
| 346. |  | Milin: | (He refers to YRYRR) This works so probably ... (He sighs -he's getting nervous - he builds RYRYY - he sighs again) This would work, see ... because it's the double of this. (He points to YRYRR) |
| 347. |  | R1: | Uh huh. And which group does it go with? |
| 348. | 00:28:26 | Milin: | Goes with this. (He puts YRYRR with Group G) |
| 349. |  | R1: | Uhhuh. Yeah. |
| 350. |  | Milin: | (He counts all the towers that are completed) That'll be ten, twenty, thirty, I think |
| 351. |  | R1: | Uhhuh. |
| 352. |  | Milin: | Yeah thirty, so I need another pair. |
| 353. |  | R1: | You need one more pair. Hey, we're getting close, aren't we? How are you keeping track? How did you know you had thirty? |
| 354. |  | Milin: | Um, because this (he points to Group C) is um twenty and these two ... (He points to Groups $G$ and $H$ ) |


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| 355. |  | R1: | What do you mean "this is twenty"? |
| 356. |  | Milin: | Twenty, I mean ten. |
| 357. |  | R1: | Oh, okay. |
| 358. |  | Milin: | Because of this and ... this and this are twenty altogether with this and then these two have another ten. That'll be thirty. (He points to Groups C, E, F, G and H) |
| 359. | 00:29:11 | R1: | We're getting close. Mrs. O'Brien, I'm getting excited. (They both laugh ) And you think it has to have two yellows and three reds? (Milin: is concentrating on building the last pair) |
| 360. |  | Milin: | Could have four ... um ... yellows and one red, I mean no it can't be. (He laughs) |
| 361. |  | R1: | Really, no. |
| 362. |  | Mrs. 0 | (She laughs) He saw that. I think I saw his eye go up. |
| 363. |  | R1: | Yeah and he said "uh-oh", so it has to, doesn't it. Gosh (she whispers) I bet we can find it. |
| 364. |  | Milin: | Um ... (He builds RRYRY) This has to be with one of these (he points to Group H) or it's not in, because see, on these (he points to Groups C, E and F) you didn't split them. |
| 365. |  | R1: | Okay, say that one again. How are you checking? Is that ... are you explaining how you got to check it? |
| 366. |  | Milin: | Yeah, see if I don't see them in these ... (He compares to Groups $G$ and $H$ ) I don't ... there can't be any of these in here. (He points to Group E) |
| 367. |  | R1: | Uh huh. |
| 368. |  | Milin: | Because there's three of these ... (He points to the yellows on Group E ) |
| 369. |  | R1: | Uh huh |
| 370. |  | Milin: | and only ... see ... it's split |


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| 371. |  | Mrs. 0 | It's split |
| 372. |  | Milin: | and see ... |
| 373. |  | R1: | You mean ... what's split? |
| 374. |  | Milin: | So it's split. |
| 375. | 00:30:43 | R1: | The yellows. |
| 376. |  | Milin: | So we don't have this and these two. |
| 377. |  | R1: | Yeah ... over here they're not split. |
| 378. |  | Milin: | Yeah. |
| 379. |  | R1: | Okay. |
| 380. |  | Milin: | This would probably work. (He points to RRYRY) |
| 381. |  | R1: | Probably. |
| 382. |  | Milin: | Cause there's not two reds on top of anywhere. |
| 383. | 00:30:57 | R1: | Oh I see. Here's the ... show me the two reds on the bottom. |
| 384. |  | Milin: | This. (He points to the two reds on YRYRR) |
| 385. |  | R1: | Okay. |
| 386. |  | Milin: | And there'll be ... (he builds YYRYR) ... this. (He puts YYRYR with Group G) |
| 387. |  | R1: | So now we've ... have we found 32 ? |
| 388. |  | Milin: | Uh huh. |
| 389. |  | R1: | Do you think you're done? |
| 390. |  | Milin: | Yeah. |
| 391. |  | R1: | Why? |
| 392. |  | Milin: | Because ... see if you could still make a staircase out of this probably ... see ... this ... this could go here. (He |


|  |  |  | arranges Group G into a staircase pattern) |
| :---: | :---: | :---: | :---: |
| 393. |  | R1: | Is that one of these? |
| 394. |  | Milin: | This could go here ... goes to this, this ... whew really. I don't know, but, see we can't ... if you're going on yellows ... then two ... this. (Meta-analysis) |
| 395. |  | R1: | Uh huh. ah |
| 396. | 00:32:13 | Milin: | Here. |
| 397. |  | R1: | Uh huh, that's one. |
| 398. |  | Milin: | This one cause see it goes all the way up here and then it's going down. |
| 399. |  | R1: | That's a pretty pattern ... yeah. |
| 400. |  | Mrs. 0 | Where's the pattern? I'm not sure I see it Milin:. |
| 401. |  | Milin: | See ... goes from here to here ... here ... (He points to the yellow in the staircase pattern of Group $G$ ) |
| 402. |  | Mrs. 0 | Oh, I see |
| 403. |  | Milin: | Here ... here and then it's going ... then it's going down. |
| 404. |  | Mrs. 0 | Oh. |
| 405. |  | Milin: | (Thinking aloud ) Um ... two ... two ... two ... (he takes towers from Group $H$ and continues them in a staircase pattern with Group $G$ ) you could put that here ... let's get four ... and two. (He sighs) But I can't find a three but I could go from here to ... from this |
| 406. |  | R1: | Yeah. |
| 407. |  | Mrs. 0 | Oh |
| 408. |  | Milin: | And this ... |
| 409. |  | R1: | That's really nice. Yeah, there's lots of ways to fit those together. But you really think you have them all? |
| 410. | 00:33:18 | Milin: | Uh huh. |

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| 411. |  | R1: | You know you have all of these. (She points to Groups E and $F$ ) |
| :---: | :---: | :---: | :---: |
| 412. |  | Milin: | Uh huh. |
| 413. |  | R1: | You know you have all of these. (She points to Group C) |
| 414. |  | Milin: | But when we came to the separating spot we kept on checking them |
| 415. |  | R1: | Uh huh. |
| 416. |  | Milin: | And we took about 10 minutes and still didn't find any. |
| 417. | 00:33:39 | R1: | Yeah, yeah, yeah, yeah. |
| 418. |  | Milin: | Then we got a couple of duplicates. |
| 419. |  | R1: | Yeah, yeah.. |
| 420. |  | Milin: | And we checked them. |
| 421. |  | R1: | And so, how many of these (she points to Groups $G$ and $H)$ with two and three did we find? |
| 422. | 00:33:52 | Milin: | Um, I think twelve. |
| 423. |  | R1: | How many do you have in front of you? Twelve? Yeah I think twelve too. |
| 424. |  | Milin: | Twelve. |
| 425. |  | R1: | Yeah. Okay, those had ... |
| 426. | 00:34:03 | Milin: | Um. |
| 427. |  | R1: | Those had two and three. (She points to Groups $G$ and H) |
| 428. |  | Milin: | Yeah |
| 429. |  | R1: | Did these have two and three also? (She points to groups $E$ and F) |
| 430. | 00:34:08 | Milin: | Yeah. |

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| 431. |  | R1: | So how many with two and three altogether? |
| :---: | :---: | :---: | :---: |
| 432. |  | Milin: | Um (he counts to himself) two ... twenty. |
| 433. |  | R1: | Yeah, and then how many with four and one? |
| 434. |  | Milin: | Um, ten. |
| 435. |  | R1: | And how many ... What are these things? (She points to $R R R R R$ and YYYYY) |
| 436. |  | Milin: | These are all (he picks up RRRRR and $Y Y Y Y Y$ ) ... all of these are red and all of these are yellow. |
| 437. |  | R1: | And we think we've got them. |
| 438. |  | Milin: | Uh huh. |
| 439. |  | R1: | Okay, can you ... let's just to keep our notes complete. Ah (she reads what he has written so far) ... we said two yellows and four reds are five. Two reds and four yellows are five. I mean one red and four yellows. Then you said two reds and four yellows. |
| 440. |  | Milin: | See I didn't get up to these. (He refers to all the groups which have 2 and 3 of each color) |
| 441. |  | R1: | Well, oh sure. Well we can wait. We can come back and do that. I'd like to ask you another question. |
| 442. |  | Milin: | What? |
| 443. |  | R1: | Um, you said you'd done activities like this before. |
| 444. |  | Milin: | Uh huh. |
| 445. |  | R1: | Last year ... |
| 446. | 00:35:08 | Milin: | Yeah. |
| 447. |  | R1: | Or sometime ... |
| 448. |  | Milin: | Yeah. |
| 449. |  | R1: | Yeah. If I was going to say, "Gosh, we really worked so |


|  |  |  | hard to figure out towers of five", and Mrs. O'Brien and I sort of left and said, "Golly". Could you have made towers of four instead of five? |
| :---: | :---: | :---: | :---: |
| 450. |  | Milin: | Yes. |
| 451. |  | R1: | If you made towers of four, how many do you think there would have been? |
| 452. |  | Milin: | Um (thinks) ... Can I do this on a math problem? |
| 453. | 00:35:41 | R1: | You can do it any way you want to. |
| 454. |  | Milin: | He writes a column of " $8+6+4+2$ " (and then adds them together and writes " 20 " )About twenty. |
| 455. | 00:36:03 | R1: | How did you figure that out? |
| 456. |  | Milin: | Um, because it's like this. On this (he points to Group C) ... twenty, about twenty because of these. (He points to Groups C, E and F) |
| 457. |  | R1: | Okay, what do you mean by these? What is this? (She refers to Group C) |
| 458. |  | Milin: | This would be about ten (he refers to Group C, specifically RYYYY) but if you take one away because of the four's ... (He uses a planning ahead strategy to describe the reduction of towers of five to towers of four) |
| 459. |  | R1: | Uh huh. |
| 460. |  | Milin: | then that will be four and that will be eight ... |
| 461. |  | R1: | Okay. |
| 462. |  | Milin: | and on the three's take this (he refers to Group E, specifically YYYYRR and RRRYY )and this away and that'll be six. |
| 463. |  | R1: | Uh huh. |
| 464. |  | Milin: | But then when it comes to the two's then that. (He refers to Group I ) But then I don't know how many of these so, it's gonna be around twenty somewhat. ( He approximates) |


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| 465. |  | R1: | Okay. So you know, you know there's eight of these and you know there's six of these and you know there's two of these. (She refers to all the groups) |
| 466. |  | Milin: | But to be exact then there's twelve of these (he points to Group I)and then they'll probably be ... makes this ... twenty-eight. (He crosses off the total " 20 " on his paper and writes " 28 ") |
| 467. |  | R1: | I'm confused. |
| 468. |  | Milin: | See, cause on this, right... (He refers to Group I) |
| 469. |  | R1: | Yeah. |
| 470. |  | Milin: | It'll take about four off of this because ... |
| 471. |  | R1: | So you take maybe. How many of these were there, twelve? (She refers to Group I) |
| 472. |  | Milin: | Uh huh. So take four off of that and that'll be eight. |
| 473. |  | R1: | Okay, and so then there were six of these (she points to Group E, means six of four cubes high), and eight of these (she points to Group C, means eight of four cubes high), and two of these (she points to YYYYY and $R R R R R$ ), and eight of these. (She separates a random eight of twelve towers from Group I) |
| 474. |  | Milin: | Uh huh. |
| 475. |  | R1: | How many of that was that all together? |
| 476. |  | Milin: | I think twenty eight and um ... eight and um ... |
| 477. |  | R1: | Eight of these and six of these and two of these ... |
| 478. |  | Milin: | and ... |
| 479. |  | R1: | (R1restates the new amounts there would be in towers of four high. He rechecks the numbers in the column on his paper) ... and eight of these. |
| 480. |  | Milin: | Okay, eight plus six plus two plus eight |


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| 481. |  | R1: | Yeah. How did you decide about the eight of these? (She points to Group I) |
| 482. |  | Milin: | Um... I was thinking that there would be eight because ... see, these four (he refers to the four from Group I that R1 put aside) would probably get out because um there's only four, so take these things off so ... |
| 483. |  | R1: | Okay, can I divide ... can, you know what would help me a lot Milin: (he doesn't listen, he rechecks his \#'s on the paper and writes " 24 "), is if we split these. (She points to RYRRY in Group I) |
| 484. |  | Milin: | I got twenty-four. |
| 485. |  | R1: | Can we split these back into ... (she points to Group I) these are two yellows and three reds (she puts YRRRY next to RYRRY), two yellow and three reds. (Milin is not listening) |
| 486. |  | Milin: | It has to be less than five's because um five is a higher number and ... |
| 487. |  | R1: | Yeah. |
| 488. |  | Milin: | It'll probably be more ... |
| 489. |  | R1: | It would be less than ... If there were thirty-two for the other, it would be less. |
| 490. |  | Milin: | That's about it. |
| 491. |  | R1: | Yeah. What about if you only had towers of three? |
| 492. |  | Milin: | Then take another one off of these ... (He points to , YYRRY and RRYYR) |
| 493. |  | R1: | Yeah. |
| 494. |  | Milin: | That'll be ... |
| 495. |  | R1: | Show me what a tower of three would look like. |
| 496. |  | Milin: | This. (He has a tower of two red and one yellow in his hand from a tower of five he pulled apart). |


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| 497. |  | R1: | Yeah, hah hah, yeah. How many do you think there'd be for towers of three? |
| 498. |  | Milin: | It's probably going to be less than towers of four. |
| 499. |  | R1: | If you had to guess, what do you think? |
| 500. | 00:39:34 | Milin: | Um... |
| 501. |  | R1: | Can you imagine? |
| 502. |  | Milin: | Eighteen. |
| 503. |  | R1: | You think there'd be eighteen of those? |
| 504. |  | Milin: | Uh huh. |
| 505. |  | R1: | Yeah, yeah, yeah. What about towers of two? |
| 506. | 00:39:50 | Milin: | Um ... |
| 507. |  | R1: | Can you tell me what in your mind? Can you imagine what they would look like and how many there would be? Can you build them? |
| 508. |  | Milin: | It would be like this. (He shows her 2 red cubes together ( $R R$ ) which he has in his hand.) |
| 509. |  | R1: | Yup. Okay. What would they look like then? Tell me. |
| 510. |  | Milin: | Um... |
| 511. |  | R1: | And how many would there be? |
| 512. |  | Milin: | I could do this right now. |
| 513. |  | R1: | Well sure. |
| 514. |  | Milin: | (Thinking aloud) One of these, switch that around. (He builds RY) |
| 515. |  | R1: | Uh huh. |
| 516. |  | Milin: | (He builds YR) Like this. |
| 517. |  | R1: | Uh huh. |


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| 518. |  | Milin: | (He builds YY) About four. (He laughs) |
| 519. |  | R1: | You think? |
| 520. |  | Milin: | On one's there would only be two. |
| 521. |  | R1: | That's right. Could you write that down so we can remember? So towers of one ... Here's one. (She puts one red cube aside) What's the other one? |
| 522. |  | Milin: | One yellow. (He chuckles) |
| 523. |  | R1: | (She chuckles) Okay, and towers of two? (Milin writes " 1 towers of 2 are 4") |
| 524. |  | Milin: | Um, four |
| 525. | 00:40:54 | R1: | What about towers of three? |
| 526. |  | Milin: | Towers of three? (He writes "Towers" and puts down the pen.) Okay |
| 527. |  | R1: | Can you think about it and imagine them now that you've built the towers of two, or would you have to start all over? What do you think? |
| 528. |  | Milin: | You could put another one of these on this. (He adds a red to $R R$ to make $R R R$ ) There's gonna be about seventeen or some, I mean sixteen (approximates) or something like that. (He adds a yellow to YY to make YYY) |
| 529. |  | R1: | I don't know. Okay. |
| 530. |  | Milin: | Towers of one would be ... (next to "Towers" he writes "of 1 are 2") |
| 531. |  | R1: | You said there would be two ... uh, huh. |
| 532. |  | Milin: | Cause then only one tower ... |
| 533. |  | R1: | (Interrupting )And towers of two would be four and towers of three ... (she stands up RRR and YYY) I see two of them. |


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| 534. | 00:41:46 | Milin: | Probably be six. |
| 535. |  | R1: | I wonder. Why, because of ... Why did you say six? |
| 536. |  | Milin: | Because um ... it could be because if we had towers of two. (He stands up RY and $Y R$ ) One would be two, four I mean. Two would be four. |
| 537. |  | R1: | Uh huh. |
| 538. |  | Milin: | Three could be six. |
| 539. |  | R1: | Could be? But awhile ago you said it might be eighteen. Ah, I'm curious; do you think it will take a long time to do towers of three? |
| 540. |  | Milin: | Ah yeah. |
| 541. |  | R1: | Let's think a minute. You got these. (She refers to $R R R$ and $Y Y Y$ ) What would be the next ones you'd put in there? |
| 542. |  | Milin: | (He builds RYY) This. |
| 543. |  | R1: | Uh huh. |
| 544. |  | Milin: | (He builds YRR) This. See that would be another pair of three |
| 545. |  | R1: | Uh huh, okay. |
| 546. | 00:42:43 | Milin: | There's two yellows and one red on the bottom. (He builds YYR) |
| 547. |  | R1: | Okay. |
| 548. |  | Milin: | (Thinking aloud) And two ... two reds and one on the bottom (he builds RRY) ..so that's already six, but I know that you can make more than that. |
| 549. | 00:43:01 | R1: | What do you think? What else could you make? You have um with the one on the top ... you have um ... |
| 550. |  | Milin: | (He builds RYR) See, another. See if you have ... um ... |
| 551. |  | R1: | Okay, is there anymore? |


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| 552. |  | Milin: | Uh huh. But see if you have three right, |
| 553. |  | R1: | Yир. |
| 554. |  | Milin: | I would think about eighteen (R1 puts her arms out and has a puzzled look on her face) because there could be about six pairs. |
| 555. |  | R1: | Well let's see. Okay, does that one have |
| 556. | 00:43:33 | Milin: | Uh huh. |
| 557. |  | R1: | A pair? (She refers to RYR ) |
| 558. |  | Milin: | See, (he builds YRY) like this. |
| 559. |  | R1: | Okay, now here is this. Is ... what else could you do? |
| 560. |  | Milin: | (There's so far eight). He plays around with the cubes, looking for missing towers |
| 561. |  | R1: | Here were the ones that were just one color ... (she refers t rrr and yyy) isn't that right? |
| 562. |  | Milin: | I think about... (he realizes he can't make any more) only eight |
| 563. | 00:44:16 | R1: | Boy, that's a big change. You told me eighteen just a minu ago. You think about only eight. Why? |
| 564. |  | Milin: | Because now I will get another duplicate of this on that. (He points to the whole group of eight) See if you had about four it would be about twelve. (He's referring to towers of four) |
| 565. |  | R1: | You're really changing your mind. Okay, but you're saying there's only eight ... (She brings him back to towers of three) |
| 566. | 00:44:41 | Milin: | Yeah, because see if you did this ... (He refers to RYR and YRY) |
| 567. | 00:44:43 | R1: | Uh huh. |
| 568. |  | Milin: | and these four could probably go together (he groups RYY with RRY, and YYR with YRR) because urn this ... see. (He |


|  |  |  | points out a pattern - Meta-analysis) |
| :---: | :---: | :---: | :---: |
| 569. |  | R1: | The same as over here? (She points to the towers of five) Is that what you're thinking? |
| 570. |  | Milin: | Yeah. |
| 571. |  | R1: | Yeah. |
| 572. |  | Milin: | This would be a pair of four. (He refers to this last group o four) This is only a pair of two. (He refers to RYR and YRY |
| 573. | 00:45:02 | R1: | Uh huh. |
| 574. |  | Milin: | With this. (He puts RYR and YRY with RRR and YYY) |
| 575. |  | R1: | And there's no more? |
| 576. |  | Milin: | Yeah. (He sighs) |
| 577. |  | R1: | My goodness! So you really changed your mind. Okay, so towers of three have how many |
| 578. | 00:45:18 | Milin: | Um... eight. |
| 579. |  | R1: | Can you write that down? (He puts one hand on his head while he writes "Towers of 3 are 8 ") What was the first kind of towers we did, five? |
| 580. |  | Milin: | No, yellows and reds are five. |
| 581. |  | R1: | Yeah, no, yeah. But these? |
| 582. |  | Milin: | One yellow and four red. |
| 583. |  | R1: | No, no, no. But the ... the problem you did yesterday were towers of five. (Milin nods in agreement) And how many did you say there were for that? |
| 584. | 00:45:48 | Milin: | Thirty-two. Should I write it down? (He anticipates her request) |
| 585. |  | R1: | If you don't mind. |
| 586. |  | Milin: | (He writes "Towers of 5 are 32")Okay. |


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| 587. |  | R1: | Okay, and so how many do we have (she refers to what he wrote down) ... towers of one? |
| 588. |  | Milin: | One, two, three and five. |
| 589. |  | R1: | Okay, the only one we missed writing down is towers of four. |
| 590. |  | Milin: | That's gonna be more than three, I know that. |
| 591. |  | R1: | It's going to be more than three ... |
| 592. | 00:46:23 | Milin: | Because see ... |
| 593. |  | R1: | Do you want to change your ... |
| 594. |  | Milin: | Towers of four could be like this (he adds a red to RYR to make RRYR )or there could be another staircase ... (he takes two yellows off of YYYYY by mistake) |
| 595. |  | R1: | Um, Your guess awhile ago was twenty-four. (He puts YYYYY back together) Do you want to change your guess? |
| 596. | 00:46:44 | Milin: | Um ... |
| 597. |  | R1: | Or do you want to stay with that? What do you think? |
| 598. |  | Milin: | About ... (he pulls YYYYY apart again) sixteen. |
| 599. |  | R1: | Why in the world would you say sixteen? |
| 600. |  | Milin: | Hah, hah. It's just a guess. |
| 601. |  | R1: | What made you guess sixteen? |
| 602. |  | Milin: | Nothing. (He adds a yellow to YRY to make YYRY )I just guessed that. |
| 603. |  | R1: | Yeah? |
| 604. |  | Milin: | Guessed numbers ... |
| 605. |  | R1: | You guessed twenty-four first and then you guessed 16 second. |


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| 606. |  | Milin: | (Thinking aloud) Two ...(he builds YYRR and RRYY) another. (He sighs) Then there's also this, (he .builds $R Y R Y$ ) the most popular. |
| 607. |  | R1: | Uh huh |
| 608. |  | Milin: | Exactly like this (he compares RYRY to RYRYR) but only some |
| 609. |  | R1: | Oh ...but shorter yeah. |
| 610. | 00:47:49 | Milin: | Starts like this. |
| 611. |  | R1: | Yeah. What kind of start ... (Milin builds YRYR) What are you thinking when you're going at these? You just build it? |
| 612. |  | Milin: | No, I try something out and then I make the um duplicate but um a different way. ( Meta-analysis) |
| 613. |  | R1: | Yeah. Okay, all of these are towers of four that have ... |
| 614. |  | Milin: | So far, I got um six. (He puts all six towers of four together) |
| 615. |  | R1: | Yeah and you think ... if you had to guess, how many do you think there're gonna be? |
| 616. |  | Milin: | (Quickly replies) sixteen |
| 617. |  | R1: | You think there are gonna be sixteen. Yeah, yeah. You know why? |
| 618. |  | Milin: | What? |
| 619. |  | R1: | Um ... um |
| 620. |  | Milin: | (He's getting restless and looking around ) Because um I took a lousy guess, because um four times four ... |
| 621. |  | R1: | Is sixteen. That's one way to think about it. |
| 622. | 00:48:51 | Milin: | Um, like I guessed um three times three, it would be eight. That would be very near. |
| 623. |  | R1: | Three times three? |


| 624. |  | Milin: | Yeah, that would be nine and um eight is near so ... |
| :---: | :---: | :---: | :---: |
| 625. |  | R1: | Yeah, oh, yeah. |
| 626. |  | Milin: | So now I think there will be about ... |
| 627. |  | R1: | But five times five? |
| 628. |  | Milin: | Huh? |
| 629. |  | R1: | So that was a guess. So anyway, but your real guess is sixteen. Is that right? Yeah. |
| 630. |  | Milin: | Because see,(he refers to his written sheet tower) of three are eight, so and towers of five ... I only had twenty-five so see ... |
| 631. | 00:49:33 | R1: | What were towers of one? |
| 632. |  | Milin: | Towers of one? |
| 633. |  | R1: | Uh huh. |
| 634. |  | Milin: | Two |
| 635. |  | R1: | Uh huh |
| 636. |  | Milin: | Because you can't make any more. |
| 637. |  | R1: | (Laughs) Yeah, that's pretty smart. And you said towers of two? |
| 638. |  | Milin: | Um four. |
| 639. |  | R1: | Uh huh. |
| 640. |  | Milin: | Towers of three ... eight, hah, hah. And towers of four ... I don't know cause I didn't get through that yet. |
| 641. | 00:49:57 | R1: | Yeah. But you guessed that was probably sixteen. |
| 642. |  | Milin: | Yeah. Around sixteen. |
| 643. |  | R1: | Yeah. Something around sixteen |


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| 644. |  | Milin: | Eighteen I guess or ... |
| 645. |  | R1: | Yeah. As I remember, when you were thinking about it before (she pulls his paper closer to him) you came up with another way of thinking about it. You said that there were gonna be- |
| 646. |  | Milin: | Take one out of this ... (he refers to group $C$ - he stands up with a burst of energy) |
| 647. |  | R1: | This Group here? (She refers to Group C also) What do you mean? |
| 648. |  | Milin: | And that ... |
| 649. |  | R1: | Take one out? You take... |
| 650. |  | Milin: | No, from the fives so that'll be four and then there'll be ... this will be eight all together. This would be six. |
| 651. |  | R1: | (Interrupting ) yeah. How ... if you were going to make that into a tower of ... she stands up group B that's still a tower of five. How could you make it into a tower of four? |
| 652. | 00:50:34 | Milin: | Take these off. (He refers to the tops of Group C) |
| 653. |  | R1: | Do it. (He takes the tops off of Group B) oh, I see. |
| 654. |  | Milin: | See. |
| 655. |  | R1: | So is that what you were thinking about? |
| 656. |  | Milin: | Yeah, see ... (he sits down) |
| 657. |  | R1: | Uh huh |
| 658. |  | Milin: | Um and they'll do the same thing ... |
| 659. |  | R1: | With the other color. |
| 660. |  | Milin: | But it's gonna use up only less. |
| 661. |  | R1: | Yeah. |
| 662. |  | Milin: | So on this (he points to RRYYR and YYYRR ) if you take |


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| :---: | :---: | :---: | :---: |
|  |  |  | the top off it'll do the same thing and ... everything, it'll do the same thing. |
| 663. |  | R1: | I see... yeah, I see. So that was when you took the top off |
| 664. | 00:51:10 | Milin: | Yeah. |
| 665. |  | R1: | Or actually you could take the bottom off, couldn't you? Where was the other one that went with this? (She refers to Group E Here it is. Okay, and you said that was gonna shrink into being ... |
| 666. |  | Milin: | (He stands up) Yeah. See take ...)this one (he points to RRYYY you take the bottom off I guess ... hah ... and all nah.. |
| 667. |  | R1: | But anyway, you were saying there would just be three (she refers to Group E) that had the staircase going up. |
| 668. |  | Milin: | Yes, so take this one ... (he picks up YYYRR) |
| 669. |  | R1: | Away yeah. |
| 670. |  | Milin: | No, one of them is away. |
| 671. |  | R1: | Yeah. |
| 672. |  | Milin: | Take this one away ... (he puts RRYYY aside) |
| 673. |  | R1: | Uh huh, yeah. |
| 674. |  | Milin: | And take these off ... (he takes the yellow tops off of YRRYY, YYRRY and YYYRR) |
| 675. |  | R1: | Uh huh, yeah. |
| 676. |  | Milin: | And then do the duplicates of that but put the other away. So that would be six plus eight. Right here, six plus eight. This I guess would be about um eight, (he refers to what is left of Group I) that's my guess because you take all of these off. And this our two. (He picks up two solid towers and takes the tops off) |
| 677. |  | R1: | (R1 nods and says "uh huh" throughout his explanation) That's easy. Yeah. |


| 678. |  | Milin: | And I got twenty-four. (He refers to his column of numbers on his paper) eight plus two... |
| :---: | :---: | :---: | :---: |
| 679. | 00:52:20 | R1: | Yeah. I think we're gonna ...I want you to really keep on thinking about that and maybe if we come back again you could have proved whether ... cause you've got two guesses. You had a guess it was sixteen (she points to his paper) and a guess it was twenty-four and so how would you prove it? |
| 680. | 00:52:42 | Milin: | By keep on building them. |
| 681. |  | R1: | By keeping on building them. Okay let me ask you one more thing cause it's almost time for you to go, you got to eat lunch. Uh, do you remember any other activities that you've done, other than the towers one that we've done together? |
| 682. |  | Milin: | Um ... the dishes and cups. I forget what else, um ... |
| 683. |  | R1: | Yeah. |
| 684. |  | Milin: | Was on that one but ... |
| 685. | 00:53:06 | R1: | Does this remind you of any of those others? (He sits down) |
| 686. |  | Milin: | The ones that we did with the Unifix Cubes but then we had that um thing with candy hearts ... |
| 687. | 00:53:16 | R1: | Yeah, yeah. |
| 688. |  | Milin: | A couple of times, um ... |
| 689. |  | R1: | Yeah, yeah. But this partic... these Unifix Cube ones? Are they like any other or are they different? |
| 690. |  | Milin: | Just one. |
| 691. |  | R1: | Just on other Unifix Cube one. Yeah I agree with you on that. |
| 692. |  | Milin: | That's it. |
| 693. |  | R1: | Yeah, that's it. Well it's time for you to go to have |

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|  |  |  | lunch. Will you promise me that you'll try to figure out <br> $\ldots$ |
| :--- | :--- | :--- | :--- |
| $\mathbf{6 9 4 .}$ |  | Milin: | Okay. |
| $\mathbf{6 9 5 .}$ |  | R1: | Which the towers of four are so that we can solve that <br> mystery. Okay? |
| $\mathbf{6 9 6 .}$ |  | Milin: | Uh huh. |
| $\mathbf{6 9 7 .}$ |  | R1: | Thank you so much Milin. |

10.6 TRANSCRIPT - MILIN'S SECOND INDIVIDUAL TASK-BASED INTERVIEW ON 2/21/92

| 1. |  | R1: | Are we ready? Um-Had you thought anything about what we had talked about? Tell me what you think. |
| :---: | :---: | :---: | :---: |
| 2. |  | Milin: | (taking blocks in his hands) The um.....the towers of four are 16 . |
| 3. |  | R1: | Oh real-That is what the question that -we hadn't answered yet? Wasn't it - are you sure? |
| 4. |  | Milin: | Ah huh. |
| 5. |  | R1: | How can we be sure? |
| 6. | 00:00:47 | Milin: | Um. ...I tried it. |
| 7. |  | R1: | Oh...Can we, can we get them out? Can you figure how they're supposed to be so you can show me what you did? |
| 8. |  | Milin: | ( Milin is taking out cubes from the large bag and making towers of 4, looking at what he has to check he has the blocks he needs) |
| 9. |  | R1: | How are we gonna know if whether we have the right ones or not? You have more than 16 already ... (pointing to the blocks Milin is taking out of the package and counting them) |
| 10. |  | Milin: | Huh? |
| 11. |  | R1: | How many do we have out here now......Oh....(pointing to the group of cube towers on the table that are out of the bag) |
| 12. |  | Milin: | $2,4,6,8,10$ (counts softly) $15 \ldots .$. and the $16^{\text {th }}$ one. |
| 13. |  | R1: | Okay-now how am I gonna know that, that we have every one and that they're all different? |
| 14. | 00:01:51 | Milin: | Ah-one thing is, right now I want to check if there's any..... |
| 15. | 00:01:56 | R1: | Yeah, would you check for me? And sorta let's organize them in any way that is good for you-I'm gonna move these (the large bag of cubes) over here a little bit so we can see.... |
| 16. |  | Milin: | (positioning the cube towers of 4 he has made on the table in some sort of order) |


| 17. | 00:02:39 | R1: | What are you-urn-what strategy are you doing right now? |
| :---: | :---: | :---: | :---: |
| 18. |  | Milin: | Um-how to find everything by-urn-see-all the yellows turn reds and the reds turn to yellows (pointing to 2 towers- 3 yellow/ 1 red; 3 red/ 1 yellow) One thing is that these 2 are together because all of them are different (holding up 2 towers of every other color) but which every way they are the same. (raising up 2 towers of all red and all yellow) |
| 19. |  | R1: | How do you mean? |
| 20. |  | Milin: | By see-this and this go together (comparing 2 towers of opposites) this and this go together ( 2 more oppositesdifferent pattern), this and this go together ( 2 more Opposites-another pattern), and these 2 go together (another different pattern). |
| 21. |  | R1: | Yeah-they're opposites of each other is that what you mean? |
| 22. |  | Milin: | Uh huh. |
| 23. |  | R1: | And is that the way you have decided each of these two up? So that means you have |
| 24. |  | Milin: | (counting his paired towers softly) six... eight... 16 |
| 25. |  | R1: | If you have eight pairs of them... |
| 26. | 00:03:36 | Milin: | Uh huh... |
| 27. |  | R1: | Uh-how are, how are we gonna be for sure that we don't have any that are the same and that we have them all? |
| 28. |  | Milin: | Um....one thing is there's only a couple with ones...( pointing to 2 towers- 3 yellow/1 red; 3 red/ 1 yellow) |
| 29. |  | R1: | How do you mean with "ones"? |
| 30. | 00:03:52 | Milin: | Like you could build a staircase up to four like that..... |
| 31. |  | R1: | Show me what you mean.... |
| 32. |  | Milin: | (building 4 towers of 1 red moving up 1 position in each |


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| :---: | :---: | :---: | :---: |
|  |  |  | tower of 4 from bottom to top-left to right) |
| 33. | 00:04:10 | R1: | Oh...okay, so these are the ones with-Oh, I remember that's a part of what this stuff over here was -wasn't it? Ah... and so there's 4... |
| 34. |  | Milin: | And then remember when I changed my Mind to $16 ?$ |
| 35. |  | R1: | ....from 24... |
| 36. |  | Milin: | Yeah... |
| 37. |  | R1: | ....from 24-I sort of remember that-urn-why did you change your mind? |
| 38. |  | Milin: | Because-um-see you know when they were on the 5's? There would be about 5 or 6 about of 'em that on the reds that would be made up as a staircase so I took that and I did that with that |
| 39. | 00:04:52 | R1: | Uh huh.... |
| 40. |  | Milin: | So.... |
| 41. |  | R1: | You did that with five of them? |
| 42. |  | Milin: | Yeah...uh huh...and I got, I got 16. |
| 43. | 00:05:01 | R1: | For the fives or the fours? |
| 44. |  | Milin: | fours. |
| 45. |  | R1: | For the fours? But before you had said 24.... |
| 46. | 00:05:10 | Milin: | Yeah... |
| 47. |  | R1: | And you changed your mind? |
| 48. |  | Milin: | Uh huh... |
| 49. |  | R1: | Because of these staircases? |
| 50. |  | Milin: | Yeah. |
| 51. |  | R1: | Okay-so here was 4 and then-but somehow you said there was eight-why? |


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| :---: | :---: | :---: | :---: |
| 52. |  | Milin: | Because you can make this way a staircase ( 3 yellows/1 red towers group) and you can make this way a staircase (3 red/ 1 yellow towers group) with yellows and reds. |
| 53. | 00:05:32 | R1: | Do you have that too? Can you show me? |
| 54. |  | Milin: | I'll do it backwards here... |
| 55. |  | R1: | Okay. |
| 56. |  | Milin: | (Making staircase towers with 1 yellow/3 red, yellow cube changing position. He is placing these towers next to those with 1 red $/ 3$ yellow) |
| 57. |  | R1: | (Pointing to Milin's towers...) and you skipped one...(skipped a stair level in the group of towers) |
| 58. | 00:05:48 | Milin: | Oh yeah-this goes on the bottom-right here..... (moving towers) |
| 59. |  | R1: | Oh-I see-so this is going to be a go down staircase. |
| 60. |  | Milin: | And after that there are all of them the same...(adding 2 towers; 1 all yellow, 1 all red at the end of his staircase arrangement) |
| 61. |  | R1: | And how many of them were these-there were two of those- |
| 62. |  | Milin: | ...of these (holding the 2 solid towers) |
| 63. |  | R1: | Okay, and so that was what this 2 was...( pointing to Milin's worksheet paper laying on the table) |
| 64. |  | Milin: | Yeah... |
| 65. |  | R1: | Okay-and so that was the eight and the two and thenwhat else could you have? This is with one color, one of one color and three of the other and this was all of one color... (reviewing the staircase towers arrangement on the table) |
| 66. | 00:06:32 | Milin: | ...and three of the other and then I put these two on the sides...(referring to his all red tower and all yellow tower) |


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| :---: | :---: | :---: | :---: |
| 67. |  | R1: | Sure... |
| 68. |  | Milin: | Because-see-this, after this (pointing to second staircase of predominant red) there's gonna be another staircase but since there's not... |
| 69. |  | R1: | Uh huh.... |
| 70. |  | Milin: | These should really go in the Middle...(moving the 2 solid towers in between the 2 different staircases) |
| 71. |  | R1: | Well-yeah-whichever...it depends on the sort of-what you are thinking your pattern is-don't you think? |
| 72. | 00:06:50 | Milin: | Uh huh. |
| 73. |  | R1: | However, How many of these are there all together? (pointing to all the towers) |
| 74. |  | Milin: | Um... two, four...ten. |
| 75. | 00:07:00 | R1: | ten. So its ten of those... and two of them were.... |
| 76. |  | Milin: | ....were almost the same. |
| 77. |  | R1: | ...and two of them are |
| 78. |  | Milin: | ...are almost the same |
| 79. |  | R1: | ...are all of the one color. |
| 80. |  | Milin: | You see all of these are the same as these (arranging the 2 sets of staircase towers\} but, but they are switched by the colors. |
| 81. |  | R1: | Uh huh-well-what about this-the group over herewhat do they have in common? (pointing to 4 towers: 2 yellow/ 2 red; 2 red $/ 2$ yellow; 1 red/ 2 yellow/l red; 1 yellow/2 red/1 yellow) |
| 82. |  | Milin: | These have in common is that they have some of, most of them have two yellows |
| 83. | 00:07:33 | R1: | And |
| 84. |  | Milin: | ...and two reds. |


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| :---: | :---: | :---: | :---: |
| 85. |  | R1: | Most of them? |
| 86. |  | Milin: | Um...all of them, actually. |
| 87. |  | R1: | Do they? |
| 88. |  | Milin: | Uh huh.... |
| 89. |  | R1: | Are you sure? |
| 90. |  | Milin: | Uh...huh. (Grouping in order of opposites) |
| 91. |  | R1: | Okay, so all of those have two yellows and two reds. What about all these? All of these have... |
| 92. |  | Milin: | They have three yellows and onered on this side (right side) |
| 93. |  | R1: | Uh huh... |
| 94. |  | Milin: | -and three reds and one yellow on this side. (left side) |
| 95. |  | R1: | Uh huh...and then in the Middle? |
| 96. |  | Milin: | In the Middle are the two ones that-um-are all one color. |
| 97. |  | R1: | Okay, and so you have three of one color and one of the other and then you have four of one color ... and you have two and two...is there anything else you can add? |
| 98. | 00:08:20 | Milin: | Ah huh...because if you want four of one color it would be these two be coming into these two-if you want three of this color it would come into these two columns (pointing to the specific applicable staircase towers) but then on the twos you could only make six. |
| 99. |  | R1: | Uh huh. Why can't you add three of one color and two of another? |
| 100. |  | Milin: | See, if you go together right? (Arranging the 3 towers of 2 red/ 2 yellow combinations) it will go like this... and this.. ( 2 red/ 2 yellow; 1 yellow/ 2 red/ 1 yellow; 2 reds $/ 2$ yellows; lyellow/2 reds/ 1 yellow; 2 yellow $/ 2$ red) |
| 101. |  | R1: | Oh, I see...so that..(pointing to the 2 reds together towers) |


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| 102. |  | Milin: | So that it'll be three. .. (pointing to the 3 possible 2 red towers) |
| 103. |  | R1: | Uh, that means if the reds are together-okay |
| 104. |  | Milin: | ...but-um-for yellows together it won't work. |
| 105. | 00:09:00 | R1: | It won't work? You can't do that? |
| 106. |  | Milin: | Because you see-this goes with this and goes up-See? (showing that the 4th tower of 2 yellow won't allow them to touch each other - a yellow on top and yellow on bottom; placing it at the right of his 2 yellow/2 red) |
| 107. |  | R1: | Oh, it doesn't do quite the same as this group over here? (pointing to the 3 red staircase group) |
| 108. |  | Milin: | ...and this here-(moving the $2 / 2$ combination towers to the right of the yellow/2 red/yellow tower) See? So... |
| 109. |  | R1: | Uh huh... |
| 110. |  | Milin: | This has to go like this (from top to bottom-2 yellow/2 red; lyellow/2 red/I yellow; 2 red/ 2 yellow; 1 red/' 2 yellow/ 1 red; from left to right) |
| 111. | 00:09:14 | R1: | So you have to choose either yellow or red for each staircase... |
| 112. |  | Milin: | Yes. |
| 113. |  | R1: | And then what about those over there ... So how many of these are there that had the staircase thing? |
| 114. |  | Milin: | Only three of them that can make a staircase but then if you want to start all over-you need one more but that'll be this (still handling the 2 each color combination towers) |
| 115. |  | R1: | Yeah ... you could keep going- goodness! Yeah- |
| 116. | 00:09:37 | Milin: | And-um-these are here because when you do this (red/ yellow/red/yellow tower) it'll be a coming into these two (placing it next to the all red tower) but see (yellow /red/ yellow /red tower) you want 2 of each-but you are separating them. |
| 117. |  | R1: | Uh huh... |


| 118. |  | Milin: | See-you could put these here (red separated) if you want <br> and it'll be the same thing as this...( yellow/ red/ yellow/red <br> tower) |
| :--- | :--- | :--- | :--- |
| 119. |  | R1: | Uh, so there's only two-so is that what you are saying? <br> And so altogether? |
| 120. | $00: 10: 01$ | Milin: | They became 16. |
| 121. |  | R1: | And there are not any others? |$|$| M22. |
| :--- |
| 123. |


| 134. | $00: 10: 56$ | Milin: | This is supposed to be like this...(arranges 4 towers of 2) |
| :--- | :--- | :--- | :--- |
| 135. |  | R1: | Uh huh.... |
| 136. |  | Milin: | See? (holding the towers of 2) |
| 137. | $00: 11: 00$ | R1: | Oh, I see-so these are the towers of four and there's not <br> any more? |
| 138. |  | Milin: | Uh-huh. Because, I mean, these are the towers of two. |
| 139. |  | R1: | And there are four? |$|$| Milin: |
| :--- |
| 140. |
| Because... see, you can't make any more because see, these |
| two...(points to the paired YR and RY) |


|  |  |  | different ways ( towers arranged) but I'm going to put them right here -for right now...what was the other...did we...what was in between? |
| :---: | :---: | :---: | :---: |
| 152. |  | Milin: | Um...the two um all reds and all yellows. |
| 153. | 00:12:24 | R1: | Yeah...wel1...yeah, I know that. But here's towers of two, towers of one, towers of two, and towers of four...(pointing to his constructed towers respectively) |
| 154. | 00:12:32 | Milin: | Towers of three are eight. |
| 155. |  | R1: | Really? |
| 156. |  | Milin: | Uh huh. |
| 157. |  | R1: | Did that really work? |
| 158. |  | Milin: | Ah huh...we tried it out in here. |
| 159. |  | R1: | And there were eight? |
| 160. | 00:12:41 | Milin: | Ah huh...(deep sigh) (Milin reaches -for the bag and begins to pull more blocks deliberately to create the towers) |
| 161. | 00:12:53 | R1: | I see you're using the same ideas...mm...mmm. |
| 162. |  | Milin: | Milin has already placed 2 towers of 3 with all red and yellows in each) |
| 163. |  | R1: | Well, there's four of them. |
| 164. |  | Milin: | (Referring to 2 all red and all yellow, red-yellow-red and yellow-red-yellow) |
| 165. |  | R1: | (Milin continues to builds 2 red-1 yellow, 2 yellow-1 red) |
| 166. | 00:13:24 | Milin: | Then there's... (Continues building)...oops. |
| 167. | 00:13:33 | R1: | Oops. (a couple blocks shift) |
| 168. | 00:13:36 | Milin: | Bottom. . (places 1 red on the bottom of 2 yellow)...that goes on the bottom (whispering to himself)...Here are all eight (displaying his towers) See, these two -fall into the same hands... because (stands up the towers of 2 red- 1 yellow, 2 yellow- 1 red, 1 red- 2 yellow, 1 yellow- 2 red) |

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| 169. | 00:13:54 | R1: | Tell me what you mean. |
| :---: | :---: | :---: | :---: |
| 170. |  | Milin: | These two groups.. .because.. .see... if you go like this...(turns one paired group over to match up the colors) |
| 171. |  | R1: | (chuckles) Oh, they're Just standing on the heads, yeah. |
| 172. | 00:14:04 | Milin: | They'll be the same...these two would be the same...(points to the top of one yellow-two red and two red-one yellow) |
| 173. |  | R1: | So, they remind you of each other. |
| 174. |  | Milin: | Yeah. |
| 175. |  | R1: | Yeah. |
| 176. |  | Milin: | But, if you flip it over, they'll both be different. So... |
| 177. |  | R1: | Yeah... Yeah. |
| 178. | 00:14:12 | Milin: | ...that's what I always work on (mumbling) |
| 179. |  | R1: | Yeah...yeah, in that kind of way...oh really, did...that was no? |
| 180. |  | Milin: | Uh huh, but then there's also these two. (Shows yellow-redyellow and red-yellow-red) |
| 181. |  | R1: | Uh huh. |
| 182. |  | Milin: | ... but they don't fall into hands but these and these two. (all yellow and all red) These fall into the same hands as these, (picks up 2 towers of all red and all yellow to compare with the all red and all yellow towers of 4) |
| 183. |  | R1: | Yeah...those are in every group, aren't they? |
| 184. |  | Milin: | Uh huh. |
| 185. |  | R1: | Yeah |
| 186. | 00:14:34 | Milin: | Except for zeros! |
| 187. |  | R1: | Except for zeros. What could we do with zeros? Okay, and you when in your classroom, when you did towers of five, you said there were 32. (Pointing to his paperwork) |


| 188. | 00:14:44 | Milin: | Uh |
| :---: | :---: | :---: | :---: |
| 189. |  | R1: | Suppose I ask you to guess about towers of six. |
| 190. |  | Milin: | (Thinks -for about 8 seconds) Around forty -something. |
| 191. | 00:14:55 | R1: | You think forty- something. |
| 192. |  | Milin: | Uh hum. |
| 193. |  | R1: | Yeah, yeah...Let's... |
| 194. |  | Milin: | (Reaches for his already built towers) My guess would be thirty or forty something because... |
| 195. | 00:15:04 | R1: | You got 32 -for five (she is lining up his towers of 3) ...um have you ...that's interesting, how many of them were here? (points to towers of 1 with 2 colors) |
| 196. |  | Milin: | two...and then here are four (points to towers of 2 with 2 colors)...and here are eight (has his hand over towers of 3) |
| 197. |  | R1: | Uh huh. |
| 198. |  | Milin: | (Reaches over to the towers of 4 with 2 colors) And here is a big group of $16 \ldots$...and then there's even a bigger group of fives. (points to his paper work) |
| 199. |  | R1: | Which is how much? |
| 200. | 00:15:30 | Milin: | 32...and six, whew, that's even a bigger group... because...see...urn...everyone is smaller because...see if I had a seven one (takes a large stack of all reds from the side and counts to make sure it has 7)...a seven one like this ... |
| 201. |  | R1: | Uh huh. |
| 202. |  | Milin: | It would be, um, more than sixes... |
| 203. | 00:15:57 | R1: | Uh hum. |
| 204. |  | Milin: | ...in all of these because it has more and you could change more stuff on it. (reaches for a yellow to demonstrate) |


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| :---: | :---: | :---: | :---: |
| 205. |  | R1: | Mm hum... ah hum . . ,uh . . |
| 206. |  | Milin: | You could build, you could build a bigger staircase |
| 207. |  | R1: | Sure, you could build bigger staircases and all kinds of things. Um...that's really interesting .. .um, if you were going to go from towers of, well from, towers of one and make them into towers of two? (points to towers of 1) How would you do that? |
| 208. | 00:16:33 | Milin: | Simple. |
| 209. |  | R1: | Yeah, what? |
| 210. |  | Milin: | You put one tower on top of the other. (Smiling) |
| 211. |  | R1: | Is that the only way you could've gone from a tower of one to a tower of two? |
| 212. |  | Milin: | Yup! |
| 213. |  | R1: | Okay, which one are you going to choose for the bottom |
| 214. |  | Milin: | (Chooses the yellow as he builds a tower of 2) Here it is! (Places it next to the same patterned tower of 2 in the already built group) |
| 215. | 00:16:50 | R1: | Oh yeah! Well, that's Just one though! That's, that's, that's wrong though! (picking up his tower) That means that if you went from a tower of one to a tower of two, you'd only have one. You told me there were four...I wonder about that...see? Does this remind you of any of the other problems that you've ever done? |
| 216. |  | Milin: | Urn...maybe the one where in second grade you were using Unifix cubes. |
| 217. |  | R1: | Yeah...yeah...second grade or third grade...yeah, mmm...or something like yeah...when you were using Unifix cubes and you were building the same kinds of towers. If you would going to... |
| 218. |  | Milin: | Yeah, but we had three colors, but... |
| 219. |  | R1: | Oh really. So you could do it with three colors couldn't you? |


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| :---: | :---: | :---: | :---: |
| 220. |  | Milin: | Uh huh. |
| 221. |  | R1: | Instead of with two. Would that make it different? |
| 222. |  | Milin: | Uh huh... if there's three colors then you could make, um three of these (holds up a tower of one color), three of these... .there would probably be eight more for the fours. |
| 223. | 00:17:43 | R1: | You think? |
| 224. |  | Milin: | Uh huh. |
| 225. |  | R1: | There Might be eight more, why? |
| 226. |  | Milin: | Hmm...just a lucky guess. |
| 227. |  | R1: | How many more would there be for...the ones? |
| 228. | 00:17:54 | Milin: | One more. (chuckles) |
| 229. |  | R1: | Why? |
| 230. |  | Milin: | Give me one more color? |
| 231. |  | R1: | (She hands him white unifix cubes. He immediately pulls one cube and places it next to the towers of 1 yellow and 1 red) |
| 232. | 00:18:02 | Milin: | ...one more tower. (chuckles) |
| 233. |  | R1: | Uh hmm. What about for the twos? |
| 234. |  | Milin: | Twos... (he continues to pull more white Unifix cubes to build a tower of 2 of all whites) There would be this falling into the same hands. (places his white tower of 2 next to the towers of 2 of all yellow and all red) There's going to be a lot more twos, I'll tell you that. |
| 235. | 00:18:20 | R1: | You think? |
| 236. |  | Milin: | (He continues to build) (Speaking to himself)...put a red on top of that...(being a white. He places it next to a tower of 2 with white on the bottom and yellow on the top, as well as a tower of yellow-red)...by this there'll be pairs of three. (chuckles) |

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| 237. | 00:18:41 | R1: | What do you mean? |
| :---: | :---: | :---: | :---: |
| 238. |  | Milin: | See. . .because .. .see ... (pulls his towers of 2 with whiteyellow and white-red) ... on this there'd have to be a pair of three somehow, (he pulls over a tower of 2 with yellow-red and places it next to the white-red) |
| 239. |  | R1: | Hmmm. |
| 240. | 00:18:50 | Milin: | This would -fall into the same kind of hands, at least, pairs of three... |
| 241. |  | R1: | Hmm...hmm. |
| 242. |  | Milin: | Then this (a tower of red-yellow), there's going to be more...like this (a yellow-white), and ...(continues building a red-yellow) |
| 243. | 00:19:07 | R1: | How many do you think they're gonna be? |
| 244. |  | Milin: | Phew...hmm...around ten. |
| 245. |  | R1: | You think...yeah. |
| 246. | 00:19:16 | Milin: | That's a lot more than four |
| 247. |  | R1: | Uh huh. |
| 248. |  | Milin: | (While pointing to his towers of 2, whispers his count) two, four, six, eight, t...nine. Well, this would be an even or odd number. These two would -fall into the same exact hands. (pulls the pair of yellow-red and red-yellow away from the group) |
| 249. |  | R1: | Uh hmm. Yeah, you had them already. Yeah. |
| 250. | 00:19:39 | Milin: | These two...(he continues to group towers together according to their patterns. Looks at the white-yellow and red-yellow, then leaves them)...oh... |
| 251. |  | R1: | Do you have them all or not? |
| 252. |  | Milin: | These two...(pulls the yellow-white and white-yellow), these (pulls the white-red and red-white)... |


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| :---: | :---: | :---: | :---: |
| 253. |  | R1: | How are you going to tell whether you have them all or not? |
| 254. | 00:19:50 | Milin: | S...um, one thing is... (goes -for more cubes as he looks back at his already built towers)...I could keep on building. |
| 255. |  | R1: | Yeah, and make 'em work, yeah. |
| 256. |  | Milin: | (He continues to build) |
| 257. |  | R1: | (reaching for Milin's towers) Is there were three that were in towers of one... |
| 258. | 00:20:02 | Milin: | (looks at his towers of 1 with 3 colors) If they had this... (gestures towards his group of towers of 2 with 3 colors) there'll be two, four, six eight,...nine for this, and I'm sure of that! |
| 259. |  | R1: | You're sure of that. You're sure you have them all? |
| 260. |  | Milin: | Uh hmm... .because |
| 261. | 00:20:14 | R1: | Why? |
| 262. |  | Milin: | See, you know when we had two? we times it...if you times... it by two, you'd have four just like on this. (refers to the towers of 2 near him) |
| 263. |  | R1: | Uh huh. |
| 264. |  | Milin: | But if you had three you could times it by two and you have...(counts his towers of 2 with 3 colors)..two, four, six...(stops) |
| 265. |  | R1: | You times it by two? |
| 266. |  | Milin: | But, I mean by three, because see, there's three of these (holds up his towers of 1 of 3 colors)... two, four, six eight,...nine. (counts his towers of 2 of 3 colors again) |
| 267. | 00:20:37 | R1: | Oh! That's really interesting. So what's you're saying is that for the twos...(takes a paper and pen to write) |
| 268. |  | Milin: | Uh huh. |
| 269. |  | R1: | You times it by two...can you write that down so I can |


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| :---: | :---: | :---: | :---: |
|  |  |  | remember it? |
| 270. |  | Milin: | (Milin takes the pen and begins writing down his explanation) |
| 271. | 00:20:54 | R1: | Actually, why don't you use this one so we could see it better on the camera? (Milin switches to a felt pen) |
| 272. |  | Milin: | I'll write right over it? |
| 273. |  | R1: | Yeah. With so many towers it hard to find room to write . |
| 274. | 00:21:06 | Milin: | Here (hands her the paper) |
| 275. |  | R1: | Okay, and then what? |
| 276. |  | Milin: | (pulls back the paper) And then there's three times three... nine, because... |
| 277. |  | R1: | Okay, okay, now these were for the towers that were how high, two high? |
| 278. |  | Milin: | Uh huh. And so are these, but that's with this third. (holds up the towers of 1 with 3 colors) |
| 279. |  | R1: | Oh, with the third one. Okay. So it was two times two was four. (points to his paper work) |
| 280. | 00:21:28 | Milin: | Uh huh. See... you know on sixes I'm going to do what I did with the fours and more...um...I'll try to do six times six...and that'll be... |
| 281. |  | R1: | And that would be how much? |
| 282. | 00:21:42 | Milin: | Three-six...but it's going to be more than that. |
| 283. |  | R1: | Yeah because you already had 32. |
| 284. |  | Milin: | Yeah, see. I just keep on doing that and (they speak simultaneously) make sure they'll be more than that. |
| 285. |  | R1: | Okay, yeah . . . I think so...Now for the towers of three...with two colors...was how many? (She pulls over sample towers) |
| 286. | 00:22:06 | Milin: | ...um...I think about eight. |


| 287. |  | R1: | (has the towers grouped) I think exactly eight. We've seem to have gotten an extra. Which one's extra? |
| :---: | :---: | :---: | :---: |
| 288. |  | Milin: | Let me see. (Removes the tower of 3 with all red) |
| 289. |  | R1: | Oh, that red one...yeah...okay. |
| 290. |  | Milin: | Oh yeah, this was that I was using for all of 'em. |
| 291. |  | R1: | Oh...using it to build them. Okay, and so -for towers of two it was four. |
| 292. |  | Milin: | Uh huh. |
| 293. |  | R1: | And for towers of two that were three colors it was nine. (she refers to his paper and pen work as she speaks) |
| 294. |  | Milin: | Uh huh. |
| 295. |  | R1: | And for towers of three with two colors... |
| 296. | 00:22:38 | Milin: | ...um, it was eight. It would've been near nine so... |
| 297. |  | R1: | ...had to be near nine. |
| 298. |  | Milin: | ...but, it was eight. |
| 299. |  | R1: | Yeah, it was eight. If you had to guess if you were gonna make towers of three that were three color? |
| 300. | 00:23:07 | Milin: | Uh huh...I'd guess around...(thinks for 9 seconds). .seventeen. |
| 301. |  | R1: | You think seven...Where'd you get that? |
| 302. |  | Milin: | Out of the air! |
| 303. |  | R1: | Oh, I know it, but if you had to, well...I bet you didn't get it out of the air. I bet you were thinking somehow. Where did that seventeen come from? |
| 304. | 00:23:20 | Milin: | Well, it would be an odd number because of this. (shows his tower of 2 with 3 colors) I thought it would be an odd number. |
| 305. |  | R1: | It's gonna, it's gonna be an odd number because of what? |


| 306. |  | Milin: | Because of this and this. (groups his towers of 2 more <br> closely) |
| :--- | :--- | :--- | :--- |
| 307. |  | R1: | Okay, because this was three and then nine. |
| 308. |  | Milin: | If this was a city, it's all made up of twos, but there's three <br> colors. |
| 309. |  | R1: | Uh huh. <br> 310. |
| 311. | $00: 23: 47$ | R1: | Und you were asking me of three colors of three...see. <br> (refers to the towers) |
| 312. |  | Milin: | But then I got this. (towers of 2 with 3 colors) Instead I <br> took this (towers of 3 with 2 colors) and I times it by two <br> and added one more because of this. (picks up a few white) |
| I actually just pulled it out of the air. |  |  |  |$|$| Milin: |
| :--- |
| 313. |


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| :---: | :---: | :---: | :---: |
| 322. |  | Milin: | Yeah. |
| 323. |  | R1: | Yeah. |
| 324. | 00:25:00 | Milin: | eight...put a little box around it and a bigger one in case you need it. (diagrams on paper) |
| 325. |  | R1: | Yeah...um...This is sort of....don't you think these are sort of interesting to try to figure out? Do you think...do you think there's a way to figure it out? |
| 326. |  | Milin: | Uh-huh if you take ten hours of your time of course. (estimates time and chuckles) |
| 327. |  | R1: | If you take ten hours of your time.......Ah-okay, okayand so towers of two.... |
| 328. | 00:25:29 | Milin: | I found the towers of four in about....three Minutes. |
| 329. |  | R1: | That was quick wasn't it? And it was 16..... |
| 330. |  | Milin: | My guess....... |
| 331. | 00:25:39 | R1: | It was 16 , yeah. And so the towers of five were 32 . Which would be the interesting problem to work on next in terms of what you're thinking about? Would you rather try to find towers of three in three colors? |
| 332. |  | Milin: | Uh uh....towers of three in ten colors. |
| 333. |  | R1: | Oh!" (gasping) |
| 334. |  | Milin: | No...towers of ten in three colors. |
| 335. |  | R1: | Towers of ten in three colors.... |
| 336. |  | Milin: | Well, that would be about the same. |
| 337. | 00:26:08 | R1: | As towers of three in ten...... |
| 338. |  | Milin: | Umm...... |
| 339. |  | R1: | But you're saying you'd think there'd be 17 towers of three in- |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 340. |  | Milin: | three colors |
| 341. |  | R1: | three colors and you think there would be how many towers of six in two colors? |
| 342. | 00:26:28 | Milin: | six in two colors? Umm |
| 343. |  | R1: | How many were there for five in two colors? |
| 344. |  | Milin: | 32. ...(thinks for awhile).... 45 (referring to the question of 6 of 2 colors) |
| 345. | 00:26:46 | R1: | You think 45, were did you get that one? |
| 346. |  | Milin: | Out of the air. |
| 347. |  | R1: | You don't get things out of the air. Where did you get the 45 ? |
| 348. |  | Milin: | One thing is..... |
| 349. |  | R1: | (writing on a paper while Milin watches on) one, two, three, four |
| 350. |  | Milin: | Fives. |
| 351. |  | R1: | Is this going to be an odd number or an even number? |
| 352. |  | Milin: | Odd. (Whispers) 45. |
| 353. |  | R1: | For towers of, towers of two colors? |
| 354. |  | Milin: | No, not um, I mean, um, 46. (Holding his hand up towards his head-shaking his head back and forth) |
| 355. |  | R1: | Why did you change your Mind? Why did it have...do you think is has to be even? |
| 356. |  | Milin: | (Shaking his head) I mean 44. |
| 357. |  | R1: | Do you think it has to be even? |
| 358. | 00:27:24 | Milin: | Yeah! |
| 359. |  | R1: | Why? |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 360. |  | Milin: | Because two is an even number and, um, it's got to be even because you can make pairs of them-but, it you had three you can't make pairs of them because of this. (Holds up a 3 cubes of different colors) If you make pairs of them there'll be in twos maybe. But these three would make a difference! (points to cubes) |
| 361. | 00:27:44 | R1: | Uh, the three originals will always make a difference. Yeah, okay, do you know what? I think we're going to come back again. Can I leave you with two problems? |
| 362. |  | Milin: | Okay. |
| 363. |  | R1: | Okay...and it'll probably be awhile. Don't you think a couple of weeks or so before we come back (talking to Amy in background) What are the two problems that I want to know? What do you think? |
| 364. | 00:28:05 | Milin: | How many of threes with three colors and |
| 365. |  | R1: | Okay, can I write that down for you. |
| 366. |  | Milin: | and six with two colors. |
| 367. |  | R1: | Um..um... |
| 368. | 00:28:16 | Milin: | That's the thing that we didn't figure out. So...... |
| 369. |  | R1: | Yeah, those are the next ones (writing). How many towers of three high.... isn't that right? using..... |
| 370. |  | Milin: | um....three colors. |
| 371. |  | R1: | three colors, and you guessed? |
| 372. |  | Milin: | (yawns) Um, I guess that.... |
| 373. | 00:28:42 | R1: | Either 15 or 17. |
| 374. |  | Milin: | 15 or 17. |
| 375. |  | R1: | If you had to go with one of those? |
| 376. |  | Milin: | 15 (chuckling) |
| 377. |  | R1: | Okay ...and I wanna say or 17. Okay, now that's question |


|  |  |  | number one. What is question number two? |
| :---: | :---: | :---: | :---: |
| 378. |  | Milin: | Um...about two-two colors for ix...but if it was one color of six that would be easy, just two. |
| 379. | 00:29:12 | R1: | You're right... is that always going to be true? |
| 380. |  | Milin: | Well, if there's one color for anything it'll just be one. |
| 381. |  | R1: | Yeah... how many towers of 6 high using.... |
| 382. | 00:29:30 | Milin: | two colors. |
| 383. |  | R1: | two colors and what was your last guess? |
| 384. |  | Milin: | That was.... 45. |
| 385. | 00:29:38 | R1: | Really? Okay, you guessed 45. That was towers with two colors that was six high. Okay, now I'm going to let you take (she looks over the papers as Amy Martino lets her know what cubes he can take) Okay, um, maybe I'll give him-that one. Then this is my picture of what you did last time, remember? |
| 386. |  | Milin: | Ah huh. |
| 387. | 00:30:15 | R1: | Okay, it says Milin 2/7/92. What did you add to it before you came back in? |
| 388. |  | Milin: | Um....after that? |
| 389. |  | R1: | Yeah, what's the one thing that you put on there-.what's the difference between my paper and your paper? |
| 390. |  | Milin: | Um......I put towers of four, are 16 |
| 391. |  | R1: | Okay, can you add that to Mine so I can keep record of what we've done? (Milin adds to the paper) Okay, so that ones for me and I'm going to let you keep this one and this one. Now, what did, what did we learn? What did we do today? We said that... what was this four and nine stuff up here? (She holds up his work paper to question him as he begins to explain. He is handling a piece of folded paper as she talks, leaning on the table? |
| 392. | 00:31:12 | Milin: | That I put three times three for three colors of two, and that |


|  |  |  | would be nine, and two times tow for the one before that; and it would still come up with four. |
| :---: | :---: | :---: | :---: |
| 393. |  | R1: | Okay, that was the towers of two with two colors and towers of two with three colors (gesturing towards his built towers); and this was your guess (pointing towards his written work). Can I keep this? Ok, that's mine |
| 394. |  | Milin: | Yeah. |
| 395. | 00:31:39 | R1: | Now I'm going to just write down on the bottom, uh, this one, so I can remember our two questions of two with three colors, is that right?... and how many towers.......Do you ever write? |
| 396. |  | Milin: | Huh? |
| 397. |  | R1: | How many towers of |
| 398. |  | Milin: | six |
| 399. |  | R1: | What six? six with two colors-Do you ever keep records when you do these? |
| 400. |  | Milin: | Uh huh..... |
| 401. |  | R1: | Sometimes use a pencil and do with it......What are you going to need? |
| 402. | 00:32:21 | Milin: | Well, I usually put it on the computer I have. |
| 403. |  | R1: | Do you really? I did not know you could do that. What are you going to need to take some of this stuff home? |
| 404. |  | Milin: | I am going to need another color. |
| 405. |  | R1: | You're going to need another color. |
| 406. | 00:32:31 | Milin: | And I'm going to need a lot more cubes |
| 407. |  | R1: | You think? Okay, well what I'm going to do because I didn't give you very many of.....okay, you can use the red and the yellow to do this one (the towers of 6 with 2) |
| 408. |  | Milin: | Yeah. |
| 409. |  | R1: | Okay. |


| 410. |  | Milin: | But, then I am going to need the red and yellow to do this one (pointing to the towers of white she is holding) |
| :---: | :---: | :---: | :---: |
| 411. | 00:32:58 | R1: | Um..hm... But you may not have to keep them all. You can maybe....you can draw pictures of something.....You can use them over. |
| 412. |  | Milin: | Or I...actually, can I keep these with me? (holding up his towers of 3 with 2 cubes, showing more enthusiasm and energy) |
| 413. |  | R1: | Sure. |
| 414. | 00:33:09 | Milin: | So I'd have to do these. |
| 415. |  | R1: | Those are the towers of 3, so you're not going to have to start over again...okay, why don't you put them back in? |
| 416. |  | Amy: | Do you need another color here? Sure, if you don't mind let me get you some more |
| 417. |  | R1: | Is that okay Amy? |
| 418. |  | Amy: | Oh sure. |
| 419. |  | R1: | If he takes some....okay, what he's decided he needs to keep is his towers of 3 in 2 colors, so he doesn't have to make them over again. Would you have the same ones? |
| 420. |  | Milin: | Um...yes because, and then I would, I, uh...put it- these with the reds and then these with the yellows (pointing to the towers of all white she holds in her hand) So we'll need a little more than that. |
| 421. |  | R1: | But you have some, yeah. |
| 422. |  | Milin: | And then....(begins to play with his folded paper) |
| 423. |  | R1: | And so your strategy would be to start with these and then do some of |
| 424. | 00:33:57 | Milin: | .....make towers |
| 425. |  | R1: | And make towers and then make some with these (the white)... and then make some with yellows...okay....alright, |


|  |  |  | so when we come back... |
| :--- | :--- | :--- | :--- |
| 426. |  | Milin: | And these (places his work papers inside) |
| 427. | $00: 34: 10$ | R1: | If you run out I'll bet Mrs. Barns could always loan you a <br> few more.... don't you think? Or could help you get some <br> more. I think you have enough. |
| 428. |  | Milin: | Okay. |
| 429. | $00: 34: 24$ | R1: | Okay. Thank you very much Milin, I'll see you in a couple <br> of weeks. |

### 10.7 TRANSCRIPT - MILIN'S THIRD INDIVIDUAL TASK-BASED INTERVIEW ON 3/6/92

| Line | Time | Transcript |  |
| :--- | :--- | :--- | :--- |
| 1. |  | R1: | Hanjit K. Sran many different... what have we done... before |
| 2. |  | Milin: | Um we....we did |
| 3. |  | R1: | And maybe we can do it in some kind of order so that I can <br> remember. I have some notes from what you've done too. |
| 4. |  | Milin: | First of all we did towers of five and then we also did |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 20. |  | R3: | Oh he's got white? |
| 21. |  | R1: | Uh uh Blue and- |
| 22. |  | Milin: | Blue and black |
| 23. |  | R1: | Blue and black would be great. |
| 24. |  | R3: | You'd like blue and back? |
| 25. |  | R1: | It doesn't make any difference. Just something we don't have. Blue and black would be great. Light blue. |
| 26. |  | R3: | Light blue I think |
| 27. |  | R1: | Uh huh |
| 28. |  | Milin: | Dark blue |
| 29. |  | R1: | Well dark blue and black are so much alike, maybe if we put them on the camera we can't see them. Okay, I am concerned at how you knew when you had everything. Okay, but let's keep going with that with that thing. Towers of three you said there were eight (R1points to eight on Milin's paper). You did towers of two sometime didn't you? (R1 points to Milin's original paper) |
| 30. |  | Milin: | Two must be around four. |
| 31. |  | R1: | You think? Let's see (R1 picks up Milin's original paper and points to it) |
| 32. |  | Milin: | Yeah |
| 33. |  | R1: | Hey, you guessed it (Milin makes a column with 2 and 4 on paper) |
| 34. |  | Milin: | Towers of one, one. |
| 35. |  | R1: | Really? |
| 36. |  | Milin: | No two, actually two. |
| 37. |  | R1: | What would they be? |
| 38. |  | Milin: | One block of red (Milin holds up his thumb on one hand) |


| 39. | 00:01:31 | R1: | Uh huh |
| :--- | :--- | :--- | :--- |
| 40. |  | Milin: | And one block of yellow (Milin writes a one on his paper <br> with a line on his paper) |
| 41. |  | R1: | Uh huh, except that if we were doing blue and black...(Milin <br> puts a block of blue and a block of black on the table) okay, <br> so these are the towers of one. |
| 42. |  | Milin: | Uh huh <br> 43. |
| R1: | Okay if I had towers of one, what would the towers of two <br> look like?...(Milin starts to build towers of two using towers <br> of one as the bottom block) Is that All? (Milin shakes his <br> head) Here, wait just a minute (R1 puts back the tower of <br> one) Okay. (Milin builds the remaining two towers of two <br> using the towers of one as the bottom block) Let's now keep <br> no, those are, those are my towers of one. You keep taking <br> them away. Okay....okay here are the towers of one? (R1 <br> puts back the tower of one) |  |  |
| 44. |  |  | Milin: | | Uh huh |
| :--- |
| 45. |
| 00:02:18 |
| R1: | | Okay and here's the towers of two (Rlmoves towers of two |
| :--- |
| into a line) |


| 52. | 00:02:49 | Milin: | Uh, if you had blue (Milin picks up blue tower of one) you could put another blue |
| :---: | :---: | :---: | :---: |
| 53. |  | R1: | Uh huh |
| 54. |  | Milin: | Or a black on it (Milin moves the blue tower of one by the blue black tower of two) |
| 55. |  | R1: | Show me those |
| 56. |  | Milin: | See (Milin points to the blue/black tower and moves the blue/blue tower over it) |
| 57. |  | R1: | You could put another blue |
| 58. |  | Milin: | This and this and black you could put a blue or a black on (Milin moves black tower of one and two towers of two with black bottom together) |
| 59. | 00:03:05 | R1: | Oh, and so when you had towers of one and you went to towers of two (R1 points to the set of towers) |
| 60. |  | Milin: | Uh huh |
| 61. |  | R1: | Could you have done anything else? |
| 62. | 00:03:13 | Milin: | Uh huh because there's not enough unifix cubes for this (Milin picks up the set of towers with a black bottom) |
| 63. |  | R1: | Well even if there were more unifix ... what do you mean? |
| 64. | 00:03:21 | Milin: | Like if there were three, then three and for two they'll be around six or eight or something like that. |
| 65. |  | R1: | What do you think for three? |
| 66. | 00:03:31 | Milin: | Let's see, you could have these (Milin puts his hand over the set of towers with blue bottom) |
| 67. |  | R1: | Okay, here you've your towers of one. (R1 moves the towers of one away from the towers of two) |
| 68. |  | Milin: | Okay you could have these four |
| 69. |  | R1: | Um |


| 70. |  | Milin: | Another four (Milin encloses the towers of two between his two hands) |
| :---: | :---: | :---: | :---: |
| 71. |  | R1: | I don't understand what you're saying |
| 72. | 00:03:42 | Milin: | Because you could have... if it was using white (Milin picks up towers of two with blue bottom) you could have white on this or |
| 73. | 00:03:46 | R1: | No, no we're not going to another color. We've still just got two colors. And we've got towers. Here's the towers of (R1 points to the towers of two) here's the towers of one (R1 starts to move the towers of one) and you said that went to this and this went to this. (R1 moves the towers of one closer to the sets of towers of two). Is that right? |
| 74. | 00:04:00 | Milin: | Because this is the black here (Milin moves the black tower of one to the other side of the towers of two) |
| 75. |  | R1: | Uh huh |
| 76. |  | Milin: | For both of these and this is the blue here (Milin moves the blue tower of one to the other side of the towers of two) for both of these. |
| 77. | 00:04:07 | R1: | Uh huh, okay, let's keep it on that side (Rlmoves towers of one back where they were) because then I can think about it. Now uh, that was towers of two. (Milin picks up the pen). There were four (R1 points to the column with 2 and 4 on Milin's paper) now what did you say about, oh sure. What about towers of three? (R1 points to column with 3 and 8 on Milin's paper). |
| 78. | 00:04:18 | Milin: | Eight |
| 79. |  | R1: | Why? |
| 80. |  | Milin: | Eight is because you can't get any more. |
| 81. | 00:04:24 | R1: | Why? |
| 82. |  | Milin: | Cause, uh |
| 83. |  | R1: | If you had these for towers of two (R1 points to towers of $t w o$ ) then you went to the towers of three. |


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| :---: | :---: | :---: | :---: |
| 84. |  | Milin: | You could always put another one on top of that. (Milin points to the towers of two) |
| 85. | 00:04:40 | R1: | Explain |
| 86. |  | Milin: | See, (Milin picks up a blue cube) just put another one on top of this, this, this (Milin moves the blue block over towers of two) |
| 87. | 00:04:45 | R1: | Okay, let's show me how you could do it. But, don't let's tear up those. Okay, here's this one (R1 takes a new tower of two blues and puts it on the table and then points to the original tower with two blues) |
| 88. |  | Milin: | And one more (Milin puts another blue cube on top of the two blues) |
| 89. | 00:04:53 | R1: | Okay |
| 90. |  | Milin: | Two blacks (Milin takes a tower of two blacks) and one more (Milin puts one more black cube on top) |
| 91. |  | R1: | (R1 moves towers of three blue blocks towards the other towers) I'm confused. You said you had uh the two blues here. (R1 points to the tower of two blues) |
| 92. |  | Milin: | Uh huh |
| 93. |  | R1: | And you put a |
| 94. |  | Milin: | Now three blues (both point to towers of three blues) |
| 95. |  | R1: | And three blues |
| 96. |  | Milin: | For this one (he points to his paper) |
| 97. |  | R1: | Okay |
| 98. |  | Milin: | See? (Milin picks up tower of three blacks and points to the tower of two blacks) |
| 99. |  | R1: | Yeah |
| 100. |  | Milin: | This (he picks up black tower of one) |

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| 101. | 00:05:11 | R1: | Now don't take away that tower of one. |
| :---: | :---: | :---: | :---: |
| 102. |  | Milin: | Right (Milin shakes his head and puts back the tower of one) |
| 103. |  | R1: | I don't want to forget |
| 104. | 00:05:16 | Milin: | Then this (Milin makes a black/blue tower and holds it up) see |
| 105. |  | R1: | Oh |
| 106. |  | Milin: | For that (Milin points to the original black/blue tower) |
| 107. |  | R1: | Uh huh |
| 108. |  | Milin: | Put another one on top of that. (Milin puts another blue cube on top of his new tower and places it on the table next to three blacks) |
| 109. | 00:05:27 | R1: | Okay, so this one grew to be this one. This one grew to be this one. This one grew to be this one. (R1 moves the towers of two and three together and spreads them out) Wait, we need another one for that. (R1 points to the tower of two that is not paired with a tower of three) |
| 110. |  | Milin: | (Milin builds another tower with blue/black/black) see the duplicate of this is that (Milin holds the tower he just built next to black/blue/blue tower) so it goes here. (Milin puts the blue/black/black tower into the empty space). |
| 111. | 00:05:50 | R1: | Uh, okay (Milin knocks over the black/black/black tower of three then picks it up along with the black/black tower of two and reverses position with blue/black and blue/ blue/ black towers). Okay, what we're doing (unintelligible). Okay, now wait. Uh, so this one turned into this one. (RI points to the black/blue tower and then the black/blue/blue tower). Could it have turned into anything else? (R1 points to black/blue turning into black/ blue/ blue) |
| 112. |  | Milin: | Uh huh (Milin nods his head) |
| 113. |  | R1: | What |
| 114. | 00:06:00 | Milin: | It could have turned into black/blue and another black. (Milin builds a tower with black/blue/black) |


| 115. |  | R1: | Uh huh |
| :---: | :---: | :---: | :---: |
| 116. | 00:06:12 | Milin: | See, cause that black is still down there (Milin points to the black tower of one) and the blue is still in the middle. |
| 117. |  | R1: | Oh, what do you mean that black is still down there? |
| 118. | 00:06:18 | Milin: | This one and the blue is still in the middle. (Milin points to black/blue tower of two and then puts black/blue/black tower down next to black/blue/blue tower knocking over some of the other towers in the process) |
| 119. |  | R1: | Uh huh... uh, oh, okay, (R1 touches the two towers of three with the black bottoms) and so could it have turned into anything else? |
| 120. |  | Milin: | Um |
| 121. | 00:06:28 | R1: | Could this tower (R1 points to black/blue tower of two and the two corresponding towers of three) have become anything different? |
| 122. |  | Milin: | Uh, uh (Milin shakes his head) |
| 123. |  | R1: | With just two colors? |
| 124. | 00:06:34 | Milin: | But, this one (Milin points to blue/black/black tower of three) would be something like it, because |
| 125. |  | R1: | Um, how do you mean? |
| 126. | 00:06:39 | Milin: | This - this (Milin builds a blue/black/blue tower) see is a duplicate of this (Milin holds the tower he just built next to the black/blue/black tower) |
| 127. |  | R1: | Uh huh |
| 128. | 00:06:45 | Milin: | Has to go on this one. (Milin puts the tower he just built next to the blue/black/black tower) |
| 129. |  | R1: | Uh huh, uh huh, okay, so these are the towers of three. |
| 130. | 00:06:55 | Milin: | Yeah, and still on that |
| 131. |  | R1: | Uh huh |


| 132. |  | Milin: | That's it, two, four, six (Milin points to the towers of three as he counts) and eight would be (Milin starts to pull more cubes apart) |
| :---: | :---: | :---: | :---: |
| 133. | 00:07:04 | R1: | What about this one? (R1 points to towers with all blacks. One of Milin's cubes flies over and knocks down one of the towers) oops (R1 picks up the tower that was knocked down) this one turned into this one. (R1 points to black/black tower of two and then to black/black/black tower of three) |
| 134. | 00:07:15 | Milin: | Uh huh, but, if you wanted eight. (Milin starts to build another tower) |
| 135. |  | R1: | Now we're going to get eight. |
| 136. | 00:07:21 | Milin: | Oops (Milin picks up a black cube) we want blue (Milin puts down black cube and picks up a blue cube to finish building black/black/blue tower) |
| 137. |  | R1: | Yeah |
| 138. |  | Milin: | See... that would go into this family (Milin points to black/black/black tower) because of this. (Milin points to black/black tower of two) |
| 139. |  | R1: | Yeah uh huh |
| 140. | 00:07:29 | Milin: | And two blues (Milin builds blue/blue/black tower) and if you wanted two of something on top, it will go with either this or this. (Milin points to the two sets of towers on the right). So, here (Milin puts the blue/blue/black tower in place) |
| 141. |  | R1: | Okay and so then it would turn into |
| 142. |  | Milin: | Two, four, six, eight (Milin points to the four groups of towers as he counts) |
| 143. | 00:07:45 | R1: | Oh I see. (R1 turns two of the towers of three so that they are next to each other perpendicular to the corresponding towers of two) So now we have these eight that come from (Milin turns the other two groups of towers of three). Those four. (Milin waves his hand over the towers of two) |
| 144. |  | Milin: | Uh huh |


| 145. | 00:07:56 | R1: | That's really interesting. That was this one. (R1 points to column on Milin's paper with 3 and 8) |
| :---: | :---: | :---: | :---: |
| 146. |  | Milin: | Uh huh |
| 147. | 00:08:04 | R1: | What about this one? (R1 points to column on Milin's paper with 4 and 16) |
| 148. | 00:08:05 | Milin: | Sixteen. You could put one more on top of that, but (Milin points to towers on the table) but (Milin hits his head with his hand) |
| 149. |  | R1: | Would, well, would that work? |
| 150. |  | Milin: | Uh huh (Milin nods his head) we tried it out here before |
| 151. |  | R1: | And so you put one more on top of this. (Milin starts to play with his mike) (R1 points to black/black/blue tower of three) what would you put on top of ... How would, how would that one work? |
| 152. | 00:08:20 | Milin: | Uh, you put anything on top of that. |
| 153. |  | R1: | Okay, you're saying then (RI builds a black/ blue /blue tower and gives it to Milin) |
| 154. |  | Milin: | You put either a black or a blue on it. (R1 builds another black/blue/blue tower) |
| 155. | 00:08:34 | R1: | Uh huh, like show me. Just for that one. (Milin takes a black cube from the bottom of the tower that R1 had just built and puts it on top of the other black/blue/blue tower) okay? |
| 156. |  | Milin: | See |
| 157. |  | R1: | Yup |
| 158. |  | Milin: | Just put something on top of it |
| 159. | 00:08:46 | R1: | I've got it. And then you do another one that had a blue on top of, I don't, I don't understand. What would the other one look like? (R1 points to the towers) |
| 160. | 00:08:52 | Milin: | It would be a, a black. |

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| 161. |  | R1: | You need this, this underneath it? (R1 hands Milin a black cube) |
| :---: | :---: | :---: | :---: |
| 162. |  | Milin: | A blue, another blue, cause it's four. (Milin builds a black/blue/blue tower) |
| 163. | 00:09:05 | R1: | Uh huh, okay and so that means. Then from here to here (R1 picks up the two towers of four that Milin just built and turns them so that they are next to each other). |
| 164. |  | Milin: | And that would work with all these too. (Milin points to row of towers of three) |
| 165. | 00:09:11 | R1: | Okay, so how many are there gonna be? (R1 straightens up towers of three) |
| 166. |  | Milin: | Sixteen |
| 167. | 00:09:15 | R1: | Okay, so there'd be sixteen of those (R1 points to towers of four) |
| 168. |  | Milin: | Yeah. Two for this, two for this, two for this, two for this , two for this, two for this, two for this. (Milin points to each of the towers of three) |
| 169. |  | R1: | Yeah |
| 170. |  | Milin: | And once you get to sixteen (Milin points to the column with 5 and 32 on his paper) you get all of them and you get thirty two. |
| 171. | 00:09:30 | R1: | Oh really |
| 172. |  | Milin: | But, it doesn't work on six, towers of six. (Milin pushes himself out of his chair) |
| 173. |  | R1: | Why? |
| 174. | 00:09:36 | Milin: | Cause it's different |
| 175. |  | R1: | Why is it different for towers of six? |
| 176. | 00:09:39 | Milin: | Uh (Milin shrugs) |
| 177. |  | R1: | Can you show me your towers of (Milin starts to reach for |


|  |  |  | something. R1 pushes towers away a little bit) now I'm gonna keep these here). You're saying that this pattern breaks up after five. (R1 points to Milin's paper, then draws a vertical line on Milin's paper to the right of the column with 5 and 32). |
| :---: | :---: | :---: | :---: |
| 178. |  | Milin: | Uh (Milin nods his head) |
| 179. |  | R1: | Is that right? |
| 180. | 00:10:00 | Milin: | See (Milin takes his original paper). Cause I got fifty. I made staircases and I made all of that |
| 181. |  | R1: | Okay can we look at those? I'm gonna... can I, can I keep these over here, so that when we come back to them we can do it? (R1 stands up and pushes the blue and black towers to the other side of the table) you have your bag full. Don't you? Do you have all your towers of six in that bag down there on the floor? |
| 182. |  | Milin: | Uh huh |
| 183. |  | R1: | Uh, can we look at them? Oh, here they are over here. (RI finds the bag on the other end of the table) Okay (Milin pushes away some of the blue and black towers) |
| 184. |  | Milin: | Not on the floor but |
| 185. |  | R1: | Okay, okay these are the towers of six. These are also some of your things with three colors aren't they? (R1 starts to take towers out of the bag) |
| 186. | 00:10:25 | Milin: | Uh huh (Milin stands up) |
| 187. |  | R1: | Okay, what you said you-you decided to do a different kind of strategy with staircases and things like that for towers of six. (R1 hands Milin some of the towers from the bag) |
| 188. | 00:10:38 | Milin: | No, I put the same strategy into it, but |
| 189. |  | R1: | Uh huh (R1 continues to hand Milin towers) |
| 190. |  | Milin: | But I just got up to fifty that time. |
| 191. | 00:10:47 | R1: | You have them all in here? I don't think you do. (R1 dumps everything out of the bag) I don't think you have fifty in |


|  |  |  | here. Do you? |
| :---: | :---: | :---: | :---: |
| 192. |  | Milin: | Yeah, most of them at least. |
| 193. |  | R1: | You have tons of them. Okay, talk to me about what you did. |
| 194. |  | Milin: | See, once I made a staircase of it I knew there'd be another duplicate of all those, so, because you just change this to red and put this to yellow. (Milin points to yellow cube then a red cube in the towers which are in front of him on the table) |
| 195. | 00:11:09 | R1: | What do you mean by a staircase? |
| 196. |  | Milin: | Like all one yellow and all reds or some. (Milin picks up towers that are on the table as he looks through them). There's got to be something in there. (Milin looks through the pile of the towers) I made something like that one yellow and all the rest were reds. (Milin finds a tower with one yellow and all the rest reds. R1 picks it up) and then see (Milin holds up a red/yellow/red/red/yellow tower) this one of them with another staircase. |
| 197. | 00:11:30 | R1: | Another kind of pattern or another kind of staircase pattern? |
| 198. |  | Milin: | But, (Milin shrugs) I can't really find all of mine in here. |
| 199. |  | R1: | Oh you can't, yeah, because they're all sort of in here. Okay, I don't understand why what you were doing there (R1 points to blue and black towers) wouldn't keep working |
| 200. | 00:11:47 | Milin: | Because once you get to five it's an odd number or something, but it doesn't work. (Milin has two of his red and yellow towers in his hand that he starts to play with) |
| 201. |  | R1: | You sure? |
| 202. |  | Milin: | Uh |
| 203. |  | R1: | It works. Maybe, I'm so confused, Milin that we have all of these. I want, can we pull all these things back here. (RI moves the blue and black towers back) can you build it for me again so I can think about this. These were the ones (R1 finds towers of ones) and the ones were one of each. Is that right? |

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| 204. | 00:12:24 | Milin: | Uh huh (Milin is playing with his red and yellow towers) |
| :---: | :---: | :---: | :---: |
| 205. |  | R1: | And then there were the twos and how did the twos work? |
| 206. | 00:12:29 | Milin: | Put another one on top of them. (R1 pulls towers of two from the pile) |
| 207. |  | R1: | Yeah there were two of them. |
| 208. |  | Milin: | For each |
| 209. |  | R1: | And where did this one go? (R1 picks up the blue/blue tower of two) which one, which family was it with? |
| 210. |  | Milin: | Uh |
| 211. |  | R1: | Did it go here or did it go here? (R1 points to two towers of one) |
| 212. | 00:12:40 | Milin: | Here (Milin points to where it goes) |
| 213. |  | R1: | Okay |
| 214. |  | Milin: | Cause it has to go here because (Milin points to towers) |
| 215. |  | R1: | Yeah and then what about this one. (R1 picks up black/blue tower of two) Where did it go? (Milin points to where it goes) |
| 216. | 00:12:47 | Milin: | Cause the black is on the bottom. |
| 217. |  | R1: | Yeah |
| 218. |  | Milin: | See (Milin picks up tower of one and points to towers of two with black on the bottom) |
| 219. |  | R1: | Yeah |
| 220. |  | Milin: | Look at this family. |
| 221. |  | R1: | Yeah |
| 222. |  | Milin: | It has to have a black on the bottom |
| 223. |  | R1: | That family has to have a black on the bottom? |


| 224. | 00:12:55 | Milin: | And this family has to get a blue on the bottom. (Milin <br> picks up blue tower of one and points to towers of two with <br> blue on the bottom) |
| :--- | :--- | :--- | :--- |
| 225. |  | R1: | Had to have a blue on the bottom. Okay, and so it works for <br> that. (R1 straightens up towers) Okay, now |
| 226. |  | Milin: | Up to five (Milin nods his head and picks up one of his red <br> and yellow towers of six) |
| 227. |  | R1: | Yeah and now, now for the three. Okay for going from two <br> to three, (R1 pushes red and yellow tower away) how many <br> families are we gonna have going from two up to three? |
| 228. | $00: 13: 13$ | Milin: | Um, eight, because, see, there's four, and two for each one. <br> (Milin points to towers of two) |
| 229. |  | R1: | Yeah, okay, where does this go? Yeah, where doe s this one <br> go? That one goes to that family? Where does this one go? <br> (R1 picks up towers of three ad Milin points to where each |
| should go) |  |  |  |

Line Time Name Transcript

| 239. | 00:14:01 | R1: | Okay so that's the (R1 reaches across the table. Milin pushes her hands away and pushes the groups of two together) |
| :---: | :---: | :---: | :---: |
| 240. |  | Milin: | Twos |
| 241. |  | R1: | Yeah |
| 242. |  | Milin: | Twos |
| 243. |  | R1: | Yeah |
| 244. |  | Milin: | Twos |
| 245. |  | R1: | Yeah |
| 246. |  | Milin: | And twos |
| 247. |  | R1: | Yeah, yeah, okay, so if we line them this way (R1 turns the groups of towers of three so that they are next to each other) now how many do we have? |
| 248. | 00:14:12 | Milin: | Eight |
| 249. |  | R1: | Okay |
| 250. |  | Milin: | It won't work on six because |
| 251. | 00:14:18 | R1: | I don't understand, we go from three to four. (R1 puts down a tower of four). Maybe we could take one little family. Here's a one (R1 moves black tower of one) and what did it go to? Two here? (R1 moves towers of two with black bottoms) is that right? And then went here. (R1 moves towers of three with black bottoms). Okay now here's the other family that had the blue bottom. Is that right? (RI moves towers with blue bottoms). |
| 252. |  | Milin: | Yeah |
| 253. |  | R1: | Okay |
| 254. |  | Milin: | Uh, you could go to fives. (Milin puts his hand over towers with black bottoms) |
| 255. |  | R1: | Well, let's go to fours first. |


| 256. |  | Milin: | Fours first |
| :---: | :---: | :---: | :---: |
| 257. |  | R1: | Okay here's this one and here's the other one that goes with that. (Rl puts two towers of four into pattern) |
| 258. | 00:14:51 | Milin: | Now these guys have to separate a little. (Milin pulls the towers of three further apart) |
| 259. |  | R1: | Yeah they have to separate and so |
| 260. | 00:15:01 | Milin: | So they have their own family (Milin knocks over one of the towers of three, looks for where it went and puts it down on the side) |
| 261. |  | R1: | Yeah, so, they're, he, he, he belongs in here (R1 moves the tower of three back to the right place). Okay, alright and so you have these two. (R1 points to the two towers of four) |
| 262. |  | Milin: | Uh huh and then the other two |
| 263. |  | R1: | And then you went to fives. What would the fives look like for that? (R1 points to one of the towers of four) |
| 264. | 00:15:14 | Milin: | Tsk, one would have a black bottom |
| 265. |  | R1: | Uh huh |
| 266. |  | Milin: | And all the rest blues for this (Milin points to the black/blue/blue/blue tower of four) |
| 267. | 00:15:20 | R1: | Oh like (R1 builds a tower with black and four blues) |
| 268. |  | Milin: | And one |
| 269. |  | R1: | Like this? That would be one that came from there? (R1 puts the tower she just built by the tower of four) |
| 270. | 00:15:28 | Milin: | Uh huh |
| 271. |  | R1: | Yeah |
| 272. |  | Milin: | And then you could always have another black on top of that. |
| 273. | 00:15:33 | R1: | What do you mean another (Milin coughs) another one that |


|  |  |  | goes |
| :---: | :---: | :---: | :---: |
| 274. |  | Milin: | Like this, but instead of that blue put a black on there. (Milin points to the blue cube on top of the tower of five) |
| 275. |  | R1: | (R1 starts to build tower) How many? It, it, it, would be just like this (R1 hands Milin a tower with black/blue/blue/blue) |
| 276. |  | Milin: | It would be exactly like this, but just put an ordinary black instead of an ordinary blue (Milin adds a black cube to the top of the tower) |
| 277. | 00:15:49 | R1: | Okay |
| 278. |  | Milin: | It'll go here. (Milin puts the new tower next to the other tower of five) |
| 279. |  | R1: | And so that means it would go with that? |
| 280. |  | Milin: | Uh huh |
| 281. |  | R1: | Okay, and so there were four of these (R1 points to towers of four) and how many in the next line? |
| 282. | 00:16:00 | Milin: | So far two. |
| 283. |  | R1: | No, well |
| 284. |  | Milin: | But there's going to be thirty-two |
| 285. |  | R1: | No, uh, okay this was eight (R1 points across the line of towers of three) and the next line was sixteen (R1 points across the line of towers of four) |
| 286. |  | Milin: | Uh huh |
| 287. |  | R1: | And this line, this one had two. Is this gonna have two? (R1 points to the next tower of four then the next tower of three) |
| 288. |  | Milin: | Uh huh |
| 289. |  | R1: | And then this one's gonna have two (R1 points to the next tower of four) |
| 290. |  | Milin: | Uh huh |

Line Time

| 291. | 00:16:15 | R1: | How many did you say were gonna be in this line going across? (R1 Points across the line of towers of five) |
| :---: | :---: | :---: | :---: |
| 292. |  | Milin: | Thirty-two. |
| 293. |  | R1: | You sure, why? |
| 294. | 00:16:21 | Milin: | We did it in class. That's one thing. (Milin is playing with his mike and someone comes over to fix the mike) |
| 295. |  | R1: | Oh, and you remember that, okay |
| 296. |  | Milin: | Um, another thing is, um, because if you follow the pattern up to this (Milin points to his paper) |
| 297. | 00:16:33 | R1: | Uh huh |
| 298. |  | Milin: | It'll do the same thing, keep on doubling |
| 299. |  | R1: | Okay now these are the fives. What about him. How could you make him into a tower of six? (R1 points to a tower of five) |
| 300. | 00:16:44 | Milin: | Him, you just put another, either a black or a blue on. (Milin puts his hand over the tower of five and then in the air) |
| 301. |  | R1: | Okay, here would be, wait you have one black. (R1 points to tower of five and then starts to build another tower) |
| 302. |  | Milin: | Yeah and all the rest blues. |
| 303. |  | R1: | Four blues and one more. Okay, what else could it have been? |
| 304. |  | Milin: | Um all of these, but instead of this blue put a black on. (Milin points to the blue on top of the tower of six. R1 starts to build the tower) |
| 305. |  | R1: | Yeah (R1 points to the tower of six and counts) one, two, three, how many blues does it need? |
| 306. |  | Milin: | Four, one more. (Milin points to the tower of six) |
| 307. | 00:17:20 | R1: | One more? (R1 puts on another blue cube) |


| 308. |  | Milin: | Then a black |
| :---: | :---: | :---: | :---: |
| 309. |  | R1: | And then a black one? (R1 puts on a black cube) |
| 310. | 00:17:25 | Milin: | See it has to have this (Milin picks up the tower of five) |
| 311. |  | R1: | Uh huh |
| 312. |  | Milin: | On the bottom like this, see? (Milin points to the bottom of the two towers of six) |
| 313. |  | R1: | Uh huh |
| 314. | 00:17:30 | Milin: | And then you put either a black or a blue on. (Milin points to the top of tower of six) |
| 315. |  | R1: | Yeah, okay, but you're saying you couldn't do that for all of them going from five to six. (R1 straightens towers) |
| 316. |  | Milin: | Uh huh, because there's going to be less. |
| 317. |  | R1: | Why? |
| 318. | 00:17:43 | Milin: | Because some of the families can't actually afford them. (both laugh) |
| 319. |  | R1: | Oh, I see and so you are saying there's gonna be less because you're out of unifix cubes? |
| 320. |  | Milin: | No (Milin shakes his head) |
| 321. |  | R1: | What if you had plenty of unifix cubes? You mean this one couldn't do it? (R1 points to one of the families of five) |
| 322. | 00:17:59 | Milin: | This one see (Milin points to the same family of five) if you could do that, right |
| 323. |  | R1: | Uh huh |
| 324. |  | Milin: | You could only put a black or a blue on, but somewhere in there (Milin plays with his mike Again) there's going to be this place where this one can't afford it. (Milin laughs) |
| 325. |  | R1: | Yeah |


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| :---: | :---: | :---: | :---: |
| 326. | 00:18:14 | Milin: | But |
| 327. |  | R1: | So you're saying |
| 328. |  | Milin: | And you can't get one more on top of it, so (Milin points to towers) |
| 329. |  | R1: | Why? |
| 330. |  | Milin: | Cause (Milin shrugs his shoulders) for certain reasons |
| 331. |  | R1: | Yeah, you just don't think you could. |
| 332. |  | Milin: | (Milin shrugs his shoulders) unless I'm wrong. |
| 333. | 00:18:29 | R1: | I wonder. When you, when you were, when you were finding them and you were using other strategies you say you only found fifty. |
| 334. |  | Milin: | Um (Milin nods his head) |
| 335. |  | R1: | Does this strategy then just not work after a while? (RI points to the towers) |
| 336. | 00:18:43 | Milin: | Uh huh, after five, maybe cause (Milin pulls on his paper which is on the table) ten's an even number and you can divide by five or something like that. |
| 337. |  | R1: | Yeah, yeah, okay so, but this one which is our tower of five (R1 picks up tower of five) turned into these two (R1 points to the towers of six). This was a tower of five (R1 picks up the other tower of five) |
| 338. |  | Milin: | Turned into these, this |
| 339. |  | R1: | What would it turn into? |
| 340. |  | Milin: | Um, black, three blues, a black and a blue, or two blacks instead of one more blue. (Milin points at the tower of five) |
| 341. | 00:19:10 | R1: | Okay, a black and a blue? Is that right? (RI builds a tower of six) |
| 342. |  | Milin: | Or another black |
| 343. |  | R1: | And a black and three blues |


| 344. |  | Milin: | Another black |
| :---: | :---: | :---: | :---: |
| 345. |  | R1: | And a black and what? (R1 is building a tower) |
| 346. |  | Milin: | And uh, let's see (Milin picks up the tower) |
| 347. | 00:19:27 | R1: | That's coming from this one (R1 points to the tower of five) |
| 348. |  | Milin: | And another black |
| 349. |  | R1: | And another black, okay. (R1 puts another black on and places it in row) |
| 350. |  | Milin: | But just doesn't work on this one |
| 351. | 00:19:36 | R1: | Oh it just doesn't, you're just convinced that it's not gonna work on this one... but what if I keep not. I just don't understand. This was such a good way to get up to the fives. What would we have, what would you have to convince me? |
| 352. |  | Milin: | Uh, make them. |
| 353. |  | R1: | What would we have to build? You know what a pain. (Milin sighs) Okay here's our family from one (R1 points to black tower of one) that went to two (R1 points to towers of two) |
| 354. |  | Milin: | That went to four. |
| 355. |  | R1: | That went to four (R1 moves towers of three) |
| 356. | 00:20:04 | Milin: | Then went to (under his breath) fifty. |
| 357. |  | R1: | Okay, if I build, (R1 puts hands on hips). I'll build if you tell me what to build. How does it go? From three to four? (R1 points to towers of three). What do we do? You build the family for the first one. I'm just so confused at these family thats not gonna have enough money ( $R 1$ moves towers of three again) |
| 358. |  | Milin: | Just put a black, blue black and another any kind of cube. |
| 359. |  | R1: | What do you think? You have to tell me exactly what to do. Black blue (R1 builds tower) |

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| 360. | 00:20:31 | Milin: | Black and blue (Milin stretches) |
| :---: | :---: | :---: | :---: |
| 361. |  | R1: | Okay black/blue/black/blue okay (R1 puts finished tower in place) |
| 362. |  | Milin: | Or a black/blue/black/black (Milin stretches and yawns) |
| 363. |  | R1: | Okay, or black/blue/black/black, okay (R1 build tower and puts it in place) |
| 364. | 00:20:49 | Milin: | And that's that family |
| 365. |  | R1: | Now how many is that? |
| 366. |  | Milin: | Four (Milin puts his hand over a tower) |
| 367. |  | R1: | That's ones that go with this |
| 368. |  | Milin: | Uh huh |
| 369. |  | R1: | Okay (Milin stretches) now these four would turn into how many in the next family? (R1 points across row of towers of five) |
| 370. | 00:21:03 | Milin: | Um, eight |
| 371. |  | R1: | And those eight would turn to how many in this family? (R1 points to the row of towers of six) |
| 372. |  | Milin: | Sixteen (Milin laughs) |
| 373. |  | R1: | You think? |
| 374. |  | Milin: | Um (Milin shrugs his shoulders) |
| 375. |  | R1: | Can you think of any that wouldn't work from eight to sixteen? |
| 376. | 00:21:19 | Milin: | Uh... not yet (Milin scratches over his eye) |
| 377. |  | R1: | Not yet? |
| 378. |  | Milin: | See |
| 379. |  | R1: | And so these would all work. Wouldn't they (R1 points to |


|  |  |  | towers of three for which towers of four were built) |
| :---: | :---: | :---: | :---: |
| 380. |  | Milin: | Uh huh |
| 381. |  | R1: | What about this one? (Milin is playing with his mike again). Is it gonna come to the next family? (R1 points to the next tower of three) What would it become? |
| 382. |  | Milin: | (R1 starts to build another tower with black/ black/ blue/blue) but on fours you have to add (R1 pus tower she just built into place) another two all plain |
| 383. |  | R1: | Um |
| 384. |  | Milin: | One color |
| 385. |  | R1: | Um |
| 386. |  | Milin: | And that would |
| 387. |  | R1: | What do you mean? Show me on something |
| 388. |  | Milin: | You see, on this one you just put another blue on (Milin points to the tower of three blues) and on this one you just put another black on (Milin picks up the tower of three blacks) |
| 389. | 00:21:55 | R1: | Yeah, what else could you do to this one? (R1 points to the tower with three blacks) |
| 390. |  | Milin: | See |
| 391. |  | R1: | Okay, here this this family, right here I'm concerned about. (R1 pick up tower of three with black/black/blue) He went to him, and what else did you say he should do? |
| 392. |  | Milin: | Put a black instead of this blue. (Milin points to black/ black/ blue/blue tower of four) |
| 393. | 00:22:06 | R1: | Okay |
| 394. |  | Milin: | Oh man (Milin makes a face) |
| 395. |  | R1: | So it would be two blacks (R1 starts to build another tower) a blue (Milin sighs) from here to here. (R1 points to tower of three changing to tower of four) this is really interesting. |


$\left.$|  |  |  | You said it was gonna go from eight (R1 points to row of <br> towers of three) to sixteen. (Rl points to row of towers of <br> four) |
| :--- | :--- | :--- | :--- |
| 396. | $00: 22: 23$ | Milin: | (Picks up red and yellow tower of six). Sixteen |
| 397. |  | R1: | With their family |
| 398. |  | Milin: | To thirty-two | | 399. |  | Milin: | From thirty-two to fifty (Milin laughs) |
| :--- | :--- | :--- | :--- |
| 400. |  | R1: | Because it somehow breaks down? | \right\rvert\, | Milin: |
| :--- |
| (Milin nods his head) yes, do you give anybody else this |
| problem? |


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| 413. |  | R2: | With the towers of six the last time, and then she went to four, and then she did some work with five |
| 414. |  | R1: | Uh huh |
| 415. | 00:23:33 | R2: | This time, but I think also Michelle was working on this and Jeff |
| 416. |  | R1: | Uh huh |
| 417. |  | R2: | And I think it would be a good idea if you talked to each other. Do you ever talk to each other about these problems? |
| 418. | 00:23:46 | Milin: | Uh uh (Milin shakes his head no and laughs) |
| 419. |  | R2: | I think that, I think Stephanie might really want to tell you what she is doing. (Rl keeps on saying yeah while Dr. Maher is talking) She's ready to talk about it. |
| 420. |  | R1: | Okay, what, what Milin is saying is that when he built up his towers, he came up with this strategy for these smaller ones |
| 421. |  | Milin: | Uh huh |
| 422. | 00:24:02 | R1: | Can you, can you explain, because Dr. Maher wasn't here when you first came in. |
| 423. |  | R2: | You know what, Mrs. O'Brien has an idea |
| 424. |  | R1: | Uh huh |
| 425. |  | R2: | I think it's a good one we ought to consider. |
| 426. | 00:24:10 | R1: | Uh huh |
| 427. |  | R2: | I think we ought to come back and bring them all together |
| 428. |  | R3: | Just let them share |
| 429. |  | R1: | And share, but, but, but this doesn't mean that they can't talk before (R1 points to Milin) |
| 430. |  | R2: | If they want to, but it would be sort of nice though, even if they don't have time., because it doesn't happen (R1 saying |


|  |  |  | yeah as Dr. Maher talks) |
| :---: | :---: | :---: | :---: |
| 431. |  | R1: | Yeah |
| 432. | 00:24:25 | R2: | If we could bring them together to talk about the different ways they're thinking about it because you said something that's very important that everyone's ideas are a little bit different. (Milin has been playing with his red and yellow towers up to this point during the exchange. At this point he knocks down some of the black and blue towers, puts them back and leans on his elbow) and I don't know. I wish 1 saw what you did Milin, but if you could even write about your ideas so that, um, I could read the way you're doing it, and your thinking. Do you think you could write about it. |
| 433. |  | Milin: | Um (Milin nods his head) |
| 434. |  | R2: | That would help me a lot |
| 435. |  | R1: | It really would help me a lot, especially your you're your sort of ideas about how it works a certain way (R1 points to Milin's paper) |
| 436. | 00:24:55 | Milin: | But this |
| 437. |  | R1: | Up to here |
| 438. |  | Milin: | This might not do it, but it might. (Milin points to his paper) |
| 439. |  | R1: | It might. |
| 440. | 00:25:03 | Milin: | I might be wrong or something |
| 441. |  | R1: | Yeah, and you might need to be thinking a little bit about it (Milin picks up some blue cubes, then one of his red and yellow towers). In terms of seeing whether what your, what your, what your strategy is up to here is gonna keep going on or not. Before we leave, because I do want you to go back, maybe and talk to them, is our other problem was the (R1 picks up Milin's original paper). Uh (R1 looks towards $R 2$ ). Milin made up his own problem last time. |
| 442. |  | R2: | Oh |
| 443. |  | R1: | He was very concerned as to what would happen if you had |


|  |  |  | three colors, and he was interested to think about (R1 turns to Milin) what they were three high. |
| :---: | :---: | :---: | :---: |
| 444. | 00:25:35 | Milin: | Yeah |
| 445. |  | R1: | And uh, but even if they were one high and you had three colors how many would there be? |
| 446. |  | Milin: | Three |
| 447. |  | R1: | Yeah, can, I remember we did that last time. (R1 looks at paper) didn't we? |
| 448. |  | Milin: | Uh huh |
| 449. |  | R1: | Uh, maybe we could write that down here (R1 points to paper in front of Milin) because I want to keep these here. ( $R 1$ hands Milin the pen). If there were, if there were towers of one high (Milin starts to write on the right side of the paper) let's say this is two colors. You have that up here (R1 points to paper) |
| 450. | 00:25:57 | Milin: | Up to here (Milin points to paper) is two colors, but |
| 451. |  | R1: | Okay |
| 452. |  | Milin: | I'll just draw this line (Milin draws a horizontal line) |
| 453. |  | R1: | Yeah, okay |
| 454. | 00:26:04 | Milin: | And make it for |
| 455. |  | R1: | Three colors |
| 456. |  | Milin: | Three colors (Milin writes three colors) |
| 457. |  | R1: | Okay, now underneath it (R1 points to paper) maybe we could keep it in the same rows. If there was, if there were towers that were one high |
| 458. | 00:26:14 | Milin: | One? (Milin writes a 1 with a line underneath in the center of the paper) |
| 459. |  | R1: | Let's put that over here. Over here (R1 points o paper). You see where it says one? |


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| 460. |  | Milin: | For this one? |
| 461. |  | R1: | Uh huh |
| 462. |  | Milin: | Cause its gonna be eight in three colors, so |
| 463. |  | R1: | Yeah I know it, but still if there if there if the if the towers only have one block in them how many would there be? (R1 takes the pen from Milin and writes one on the paper then gives the pen back) |
| 464. | 00:26:29 | Milin: | Three |
| 465. |  | R1: | Okay that and put the three underneath (R1 points to where she wants Milin to write the three) |
| 466. |  | Milin: | I'll put a three here. (Milin writes underneath where he had written 1) |
| 467. |  | R1: | Okay, that's fine, alright. Okay, can you tell me what they'd be with your colors? It would have been a uh, a white and (R1 takes white cube) |
| 468. | 00:26:42 | Milin: | Just take one of each cube (Milin reaches for cubes) |
| 469. |  | R1: | A white and a yellow |
| 470. |  | Milin: | White, yellow (Milin pulls the cubes apart with his teeth) |
| 471. |  | R2: | How many, how many did Milin think? I'm curious. |
| 472. |  | R1: | He said it'd be three if they were one high. |
| 473. |  | R2: | No, three high, what did he think |
| 474. |  | R1: | Uh, for three high? What did you think Milin? |
| 475. | 00:26:56 | Milin: | Seventeen or fifteen |
| 476. |  | R1: | Wasn't it, where'd you have it written down? (R1 looks through papers that were on the table) |
| 477. |  | R2: | Seventeen or fifteen? |
| 478. |  | R1: | I just saw it. I just lost your paper. (Milin looks on the floor) |


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| 479. |  | R2: | On the back |
| 480. |  | Milin: | No its |
| 481. |  | R1: | Oh no, its this. (R1 picks up one of the papers from the table). What did you say, at home (R1 points to the paper) |
| 482. |  | Milin: | I (Milin picks up another paper from the table). That's the real answer, but (Milin points to the paper in RI's hand) |
| 483. |  | R1: | Oh that's the real answer, uh (R1 puts her paper down) |
| 484. | 00:27:15 | Milin: | But this I thought was either fifteen or seventeen, but (Milin shrugs). It came to be twenty five. (Milin shows R1 the paper in his hand) |
| 485. |  | R1: | Oh, came to be twenty-five. |
| 486. |  | R2: | What did you do when you were at home? |
| 487. | 00:27:26 | R1: | Okay, we're gonna move these, cause we're done with them for right now. (Milin stretches and yawns. R1 pushes blue and black towers away). Okay there were these three. How many if they were two high? |
| 488. |  | Milin: | Two high would be six. |
| 489. |  | R1: | Why? |
| 490. |  | Milin: | One thing is, put a yellow, a yellow and anther yellow. (Milin puts a yellow on each tower of one. The bell rings) |
| 491. | 00:27:50 | Milin: | I have to get back to class. |
| 492. |  | R1: | And then what do you need to begin with? |
| 493. |  | Milin: | Then-a white (puts a white cube on top of a while cube)... |
| 494. |  | R1: | Yeah. |
| 495. |  | Milin: | ...a red and a white (puts a white cube on top of a red cube)... |
| 496. |  | R1: | Mmhmm.. |

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| 497. | 00:28:07 | Milin: | (suddenly) There would be 9. |
| :---: | :---: | :---: | :---: |
| 498. |  | R1: | Why? |
| 499. |  | Milin: | Red and a white, and then there...We did this in here before... |
| 500. |  | R1: | Yeah, I'm just trying to remember- |
| 501. |  | Milin: | ...and... |
| 502. |  | R1: | Because it's been so long-and then a yellow and a white (puts a white cube on top of a yellow cube, and gives them to Milin). |
| 503. |  | Milin: | ...this...(lines up the three towers of $2 s$ with white on top) |
| 504. | 00:28:25 | R1: | And then... |
| 505. |  | Milin: | A red (grabs a red cube), a red and a red (puts the red cube on another red cube). |
| 506. |  | R1: | Mmhmm... |
| 507. |  | Milin: | A red and a-a yellow and red (puts a red cube on top of a yellow cube). |
| 508. |  | R1: | No, the red is on the bottom. Yeah, yeah, yeah. |
| 509. | 00:28:42 | Milin: | And a white and a red. |
| 510. |  | R1: | No. Oh, okay. Yeah, you're absolutely right. I'm absolutely wrong. Okay. Uh, okay. |
| 511. |  | Milin: | (puts a red cube on top of a white cube. lines up the three towers of $2 s$ with red on top) It can't be 6 because- |
| 512. |  | R1: | Why can't it be 6 ? |
| 513. |  | Milin: | 'Cause there's 3 colors. |
| 514. |  | R1: | Mmhmm... |
| 515. |  | Milin: | You have to do something different. |
| 516. |  | R1: | Mmhmm... |


| 517. | $00: 28: 59$ | Milin: | It was 6, that would be only for-uh-2 colors, I think. No, <br> 2 colors was 4. |
| :--- | :--- | :--- | :--- |
| 518. |  | R1: | Mmhmm... |
| 519. |  | Milin: | For some reasons... |
| 520. |  | R1: | Yeah |
| 521. |  | Milin: | ...this is turning out bad. | | 522. |
| :--- |


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| 535. |  | Milin: | (Does not use pen) 9...I mean 27, hehe, but that's not going to work out on this one. |
| 536. |  | R1: | You don't think? |
| 537. |  | Milin: | It's not going to be in this kind of a pattern (points to list located at top of sheet containing the number of towers with 2 colors for different heights). |
| 538. |  | R1: | You don't think? Why not? |
| 539. |  | Milin: | One thing is it can't be by this (points to list again) because this one was timesed by 2 , but now there's something else. So, it's different. |
| 540. | 00:30:35 | R1: | And so it's different. It worked from 1 to 2. |
| 541. |  | Milin: | Mmhmm. |
| 542. |  | R1: | Yeah. |
| 543. |  | Milin: | But 2 to 3 (shakes head no)... |
| 544. |  | R1: | You don't think there'd be 3 (points to tower with red cube on top of yellow cube) to go with that one? |
| 545. |  | Milin: | Uh, you put that...another one (whispered)... put that there (whispered)... white |
| 546. | 00:30:54 | R1: | Okay, so what would be yellow? Uh, what did you just think? Yellow, red, and...White, (builds tower with yellow, red. white, from bottom to top) Okay, what else could it be? (builds lower with red cube on top of yellow cube) |
| 547. |  | Milin: | Another red on top of that (points to the lower with red cube on top of yellow cube). |
| 548. |  | R1: | A yellow, red, and... |
| 549. | 00:31:10 | Milin: | Red. |
| 550. |  | R1: | ...red (puts red cube on top of the tower). Anything else? |
| 551. |  | Milin: | Yellow, red, and yellow...But someplace it breaks up like 32, hehe... |


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| :--- | :--- | :--- | :--- | :--- |
| 552. | $00: 31: 23$ | R1: | (builds tower with yellow, red, yellow, from bottom to top) <br> Just...you think, you still think it's going to break up between <br> the 2 and the 3... |  |
| 553. |  | Milin: | Mmm. |  |
| 554. |  | R1: | ...to just give you 25? |  |
| 555. |  | Milin: | Mmm. |  |


|  |  |  | sort of from here to the next one. These...this family strategy <br> I think is a really kind of good one. You had some other <br> strategies? |
| :--- | :--- | :--- | :--- |
| 571. |  | Milin: | Uh huh, like staircases. |
| 572. |  | R1: | Yeah. Anything that you can remember? |


| 588. |  | R1: | Yeah, he came in with some really interesting ones. Yeah. |
| :---: | :---: | :---: | :---: |
| 589. |  | Milin: | Mmhmm. |
| 590. |  | R1: | Would that be interesting if we got sort of a group of you that have all been thinking about it together? |
| 591. |  | R2: | Good idea you came up with, Milin, that I think, you know, you know, if Mrs. Barnes wouldn't mind during math if you all came and talked and maybe shared with the rest of class. |
| 592. | 00:34:09 | Milin: | Okay (quietly). |
| 593. |  | R1: | Okay. Now what I'd like you to do because then this is...Stephanie may be thinking about some of the same things too...is to think about this, build them if you need to, but write me, or write for yourself. Do, do you keep a journal? I mean, do you do a math journal or any of those kinds of things, Milin? |
| 594. |  | Milin: | Uh just a daily journal in reading class. |
| 595. |  | R1: | Yeah. Okay, but maybe you just need some writing like you've been doing here (holds up sheet). You've done a really, a really, a good job of keeping some of these records. I'm going to.... |
| 596. |  | Milin: | But, I think I did something wrong on umm. from 32 to go to 6 . I think I did something wrong. |
| 597. | 00:34:58 | R1: | Why? |
| 598. |  | Milin: | Mmm, I don't think that pattern would break down like.. |
| 599. |  | R1: | You really don't? You... |
| 600. |  | R2: | (off-camera) Ah-ha |
| 601. |  | R1: | If it didn't break how many should there be? |
| 602. |  | Milin: | Uh, 64 |
| 603. |  | R2: | (off- camera) "Well I, you know I think you're ready to talk to Stephanie. |
| 604. | 00:35:16 | R1: | Yeah, I think so too. |


| 605. |  | R2: | (off-camera) And tell her what the problem is, so maybe the <br> two of you can come up with something to share with us <br> next time, Milin. |
| :--- | :--- | :--- | :--- |
| 606. |  | Milin: | Mmhmm. |
| 607. |  | R2: | (off-camera) Okay? |
| 608. |  | R1: | The other thing is that I'd like you to share with Stephanie <br> is-Stephanie hasn't been working on this problem |
| 609. |  | R2: | (off-camera) Yeah, I'd like her to work on it. |
| 610. |  | Milin: | This is a new one. And if we're going to come back and talk <br> together, it's only fair that she's had a chance to try some of <br> these too. Is that right? |
| 611. |  | R1: | Mnhmm <br> And so your questions-You were interested in checking <br> out with the towers of six? |
| 612. |  | Milin: | Uh-uh (disagrees). |
| 613. | $00: 35: 50$ | Milin: | My guesses are 50, if I was right the first time, or 64. |
| 614. | $00: 36: 34$ | R1: | Oh! It's different for the family. Your families had hats |


|  |  |  | instead of bottoms for this one, don't they? Yeah. So it's slightly... |
| :---: | :---: | :---: | :---: |
| 621. |  | Milin: | Yeah, but they can have other kinds of hats too. |
| 622. |  | R1: | Could they? |
| 623. |  | Milin: | So... |
| 624. |  | R1: | Yeah, yeah- Okay, and so this came from 3 (points to the lowers of $2 s$ with 3colors), and it turned into 9 . |
| 625. |  | Milin: | And nine came from that. And for nine, who knows? |
| 626. |  | R1: | Well, yeah. You were saying though that, uh, that one of them turned out to have...three hats. It was this one here (pulls out tower with red cube on top of yellow cube). And it came over, and it had a yellow hat (holds onto tower with yellow/ red/ white, from bottom to top), and it had a red hat (holds onto tower with yellow. red. red, from bottom to top). Could it have had another one? |
| 627. |  | Milin: | White. |
| 628. |  | R1: | Really, what would it have looked like? |
| 629. |  | Milin: | A red, yellow, and a white on top of it. |
| 630. |  | R1: | Or a yellow, red, and a white. Is that right? |
| 631. |  | Milin: | Yeah. |
| 632. |  | R1: | And which would it have been? A yellow, a red... |
| 633. |  | Milin: | And then another white. |
| 634. | 00:38:24 | R1: | ...and then a white (builds lower with yellow, red. white, from bottom to top). So it came...it, it ended up with 3. Yeah. |
| 635. |  | Milin: | See, but that's only if you have a bottom, but you might have more or less. |
| 636. |  | R1: | Yeah, you might. So, we're leaving you with a real question there. Do you think you and Stephanie can work on that one some together? |


| 637. |  | Milin: | Mmhmm. |
| :---: | :---: | :---: | :---: |
| 638. |  | R1: | Okay, what do you need to take with you? |
| 639. |  | Milin: | Umm. |
| 640. |  | R1: | Anything? |
| 641. |  | Milin: | Not really. |
| 642. |  | R1: | You have Unifix cubes in your room, in case you need 'em to prove to each other things? |
| 643. |  | Milin: | No. |
| 644. |  | R2: | (off-camera) Let him take them. |
| 645. |  | R1: | Let's, shall we, let me, let's pack up your, uh, your 3 colors because if you're going to be talking to Stephanie, you're going to have to show her what you're talking about. Okay? (starts packing Unifix cubes into bag) |
| 646. |  | Milin: | Mmhmm. |
| 647. |  | R1: | Okay, and, umm... |
| 648. |  | Milin: | Then I'm going to need the 6 color ones, of course. |
| 649. |  | R1: | Well, everything you brought. Can you just do with these yellows, and reds, and whites? Is that right? |
| 650. |  | Milin: | Mmhmm. |
| 651. |  | R1: | Okay, uh, and is this the paper you need to take? (holds up sheet) |
| 652. |  | Milin: | Uh, yeah. That's the question one. |
| 653. |  | R1: | Yeah, uh, it's towers of six with two colors, and towers of three with... |
| 654. | 00:39:35 | Milin: | With a question. |
| 655. |  | R1: | ...with three colors. Okay? Milin, I thank you so much for coming back today, |


|  |  | and cutting your lunch so early. That was really... |
| :---: | :---: | :---: |
| 656. | Milin: | Well, we were going to go out in a minute or two probably, so...(puts sheet into bag) |
| 657. | R1: | Yeah, yeah. You're going to put that down in there? |
| 658. | Milin: | It's not like I missed my lunch. |
| 659. | R1: | You got it! Yeah. Okay, what can we do here? \{continues packing bag) |
| 660. | Milin: | I put tape on it. |
| 661. | R1: | Mmhmm. |
| 662. | Milin: | That's the only thing I could think of. |
| 663. | R1: | Sure. Yeah, because it was just so full. Yeah. Okay, now, I'm going...Do you mind if I just keep this one? (holds up sheet) This was your answer from last time. Okay, let me give it to Amy 'cause she keeps all my papers so that I don't lose 'em. I'm, I'm not near as good as you is at keeping all this stuff, together. Okay? |
| 664. | Milin: | (bag bulges open) Oh, no. |
| 665. | R1: | You may have to ask...Oh, look. Do you know what we have? |
| 666. | R3: | (walks up to table with tape, Milin laughs) Super tape. Let's try this. |
| 667. | R1: | Milin will never get back in. |
| 668. | R3: | Now, let's see. Do we need a scissor to cut this? |
| 669. | R2 | (off-camera) No. It's tear-through. |
| 670. | R3: | Oh...thank you. I love the way she does that. And I think if we do it maybe this way it would be better? |
| 671. | R1: | Yeah, don't you think? |
| 672. | R3: | I think so. Let's make it nice and tight (puts tape on bag)... |

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|  |  |  | tight. That ought to hold. |
| :--- | :--- | :--- | :--- |
| 673. |  | R1: | Here's you package. |
| 674. |  | R3: | Oh, you're stuck with that [the microphone] again, huh? |
| 675. |  | R1: | Uh, me too. Okay, thank you so much, hon'. You have a <br> good weekend. |
| 676. |  | R3: | Take care, Milin. |
| 677. |  | Milin: | You too. |

### 10.8 TRANSCRIPT - "GANG OF FOUR" SMALL GROUP ASSESSMENT ON 3/10/92

| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 1. | 00:02:40 | R2: | And Stephanie did try to work on towers of six and I asked all of you if you- |
| 2. |  | Milin: | So did I. |
| 3. |  | R2: | You did too? If you were building towers of six, how many would there be? |
| 4. |  | Jeff: | I don't know |
| 5. |  | Michelle: | I did some but I didn't- |
| 6. |  | R2: | But do you know how many? |
| 7. |  | Stephanie: | Yeah. |
| 8. |  | Milin: | Probably 64. |
| 9. |  | R2: | Why do you think 64? |
| 10. |  | Milin: | Well, because there was a pattern. |
| 11. |  | R2: | What's that? |
| 12. |  | Milin: | You just times them by two |
| 13. |  | R2: | Times what by two? |
| 14. | 00:03:01 | Milin: | The towers by two, because one is two, and then we figured out two is two, and then, I mean four, and then |
| 15. |  | Jeff: | You are not making much sense! |
| 16. |  | Michelle: | See, if you had only one block up and two colors, then you would have two towers, and we figured out that the other day that you keep on doing... |
| 17. |  | Jeff: | Everything is opposite! |
| 18. |  | Michelle: | ...like two times two would be four and then the three |
| 19. | 00:03:27 | R2: | So four would be for what? |
| 20. |  | Stephanie: | All you have to do- |


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| 21. |  | Michelle: | ...four for, there would be four towers for two high. |
| 22. |  | R2: | Okay. |
| 23. |  | Jeff: | It's always opposite though. |
| 24. |  | R2: | Okay well, let me hear what Michelle is saying. |
| 25. |  | Michelle: | And then for this three high, you would have eight towers and then for four high you would have twelve towers and then you keep on doing it like that. |
| 26. | 00:03:45 | R2: | Do you agree with that? |
| 27. |  | Stephanie: | Well. What it is is- |
| 28. |  | Jeff: | I don't know what they are talking about. |
| 29. |  | Michelle: | Well five high would be twenty five |
| 30. |  | R2: | Okay, lets get a piece of paper and write down what you are saying and see if you all agree. I think Jeff hasn't been with us for a while and he doesn't know what we are talking about. But let's take one at a time. Let's just agree as we are moving along. Go ahead, Michelle. |
| 31. |  | Michelle: | If you had one high see there is red and blue then you would have two and then if you had... |
| 32. |  | R2: | Okay, write that down. Two. Did you agree with that? |
| 33. |  | Jeff: | Yeah. |
| 34. |  | R2: | Do you know what she is talking about? |
| 35. |  | Jeff: | There is one red and one blue so there is only one way to do it so its two. |
| 36. |  | R2: | One way you can do it and so its two.. |
| 37. |  | Jeff: | Yeah. If you have to make towers of one and there is only two colors |


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| 38. |  | Milin: | He doesn't agree on doing that. |
| 39. | 00:04:30 | R2: | Okay, let's go on. |
| 40. |  | Michelle: | If you had two towers that would be four, because you have- |
| 41. |  | Jeff: | Yeah I agree with that. Okay. |
| 42. |  | Michelle: | See you would just times it see two times two |
| 43. |  | R2: | Okay just hold on okay write the four down. Look I don't ...Can you explain to me why from two you would get to four? Milin tell me why. |
| 44. |  | Milin: | For each one of them you could add one no two more for on because there is a black I mean a blue and a red |
| 45. |  | Jeff: | What she is doing... |
| 46. | 00:05:02 | R2: | Let her finish. Okay. |
| 47. |  | Milin: | See for that you just put one more for red you put a black on top and a red on top I mean blue on top instead of black and on blue you put a blue on top and a red on top. You keep on doing that. |
| 48. |  | R2: | Do you understand what he is talking about? |
| 49. |  | Stephanie: | Uh uh! |
| 50. |  | R2: | You all understand what he is talking about? |
| 51. |  | Jeff: | Yeah. |
| 52. | 00:05:26 | R2: | Alright. So so we agree four. What happens if you are building towers three high what did you say it would be? |
| 53. |  | Michelle: | It would be eight. |
| 54. |  | Milin: | It would be eight |
| 55. |  | R2: | Write eight down. Can you give me an argument; |


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|  |  |  | you don't have to do it. Why we jumped from four to eight? |
| 56. |  | Michelle: | There's- |
| 57. |  | R2: | Shhh. That's what Jeff wants to know |
| 58. |  | Michelle: | There's there's- |
| 59. |  | R2: | Go slow. Its Jeff you are convincing not me. |
| 60. |  | Michelle: | There is two blue. There is two here. |
| 61. |  | Jeff: | I know that. |
| 62. |  | Michelle: | And then we went to four so it would have to be times two times two equals four and four times two would equal eight. |
| 63. |  | R2: | That doesn't help Jeff understand. He just knows that we are multiplying two times two |
| 64. |  | Milin: | I know. I know |
| 65. |  | Stephanie: | Alright. |
| 66. |  | Jeff: | If this... |
| 67. |  | R2: | Okay. One at a time. |
| 68. |  | Jeff: | If this was like a pattern it would go two four six in between and then eight. |
| 69. |  | R2: | Yeah, that's what he is saying. |
| 70. |  | Milin: | No! no! |
| 71. |  | Stephanie: | But that's not the pattern we are working on. |
| 72. |  | R2: | Go ahead Stephanie. |
| 73. |  | Stephanie: | The pattern that we saw was this. For one block at a time we found two. |
| 74. |  | Jeff: | We already got two and four towers. |


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| 75. |  | Milin: | Two, four, six- |
| 76. |  | Stephanie: | I know four and then eight alright. Two four and then eight. |
| 77. |  | R2: | Why eight? That's what Jeffery asked about. |
| 78. |  | Milin: | I know. |
| 79. |  | R2: | Go ahead. Let Milin persuade Jeff. |
| 80. |  | Milin: | If you do that you just have to add for each one of those you have to add |
| 81. |  | R2: | Each one of what? These four? |
| 82. |  | Milin: | Yeah. You have to add one more color for each one |
| 83. | 00:06:32 | R2: | Which way are you adding it? Where are you putting that one more color, Milin? |
| 84. |  | Milin: | No two more colors for each one. See- |
| 85. |  | R2: | So this one with red on the bottom and blue on the top. |
| 86. |  | Milin: | You could put another blue or another red. |
| 87. |  | R2: | You agree with that? You can put a blue or red on top and that- |
| 88. |  | Milin: | And that will be two and then on this you could put another red or blue on top that will be four. |
| 89. |  | Jeff: | That is the same right there. |
| 90. | 00:06:56 | R2: | No, this is blue red |
| 91. |  | Jeff: | No. Here look. It's blue oh okay, okay. |
| 92. |  | Milin: | See. Now you see. |
| 93. |  | R2: | Could you find what Milin is saying and now here you could put- |


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| 94. |  | Milin: | A red or a blue and same thing here |
| 95. |  | R2: | Do you understand that? |
| 96. |  | Jeff: | Yeah. |
| 97. |  | R2: | So do you see how you get eight? |
| 98. |  | Jeff: | Yeah. |
| 99. |  | R2: | Do you agree with that? |
| 100. |  | Michelle: | Because there is two here and two more |
| 101. |  | Jeff: | But you know you have to wait you are bound to get eight if you have two colors you are you are going to get different things. Do another one for blue and she already has it. She will. Do it. |
| 102. | 00:07:27 | Milin: | Do it for four that will be sixteen. |
| 103. |  | Jeff: | No. look she already used- |
| 104. |  | R2: | Let's get another piece of paper. Would you give me another piece of paper please? |
| 105. |  | R2: | Go ahead Jeff. Show us what you- |
| 106. |  | Jeff: | She has eight blocks with only with still only two colors. |
| 107. |  | R2: | Eight blocks with two colors. Let's see. |
| 108. |  | Jeff: | She has like this red, and she kept on alternating blue red blue red and blue red...(inaudible) and so she got eight, and then she did the same thing up here |
| 109. |  | Michelle: | I didn't do the same thing above |
| 110. |  | Jeff: | Yeah you kept alternating |
| 111. | 00:08:20 | R2: | Well why don't you do that? And see what happens. |
| 112. | 00:08:24 | R2: | That's what he thought he saw you do but, that's |


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|  |  |  | interesting may be you didn't know you were doing <br> that Michelle but look blue red blue red blue red blue <br> red. He saw you alternating them on the bottom. |
| 113. |  | Jeff: | Now look you got red and blue. |
| 114. | $00: 08: 37$ | R2: | So you are saying that all of these are alternating and <br> these are opposite alternating look this is blue red <br> blue red blue red |
| 115. |  | Jeff: | This is the same thing right here red and blue red and <br> blue. So you have to cross off this one. And now red <br> and blue red and blue so you have to cross off that <br> one because there is another one right there and then, <br> over here you have that one right there- |
| 116. |  | Milin: | But Jeff, Jeff, Jeff- |
| 117. |  | Jeff: | But the thing is- |
| 118. |  | R2: | R2: |
| 119. |  | Milin: | Listen to Milin. Listen to Milin. |
| This was for three so you could add two for each one |  |  |  |
| of the three. |  |  |  |


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|  |  |  | ago or someone was helping me I forget who it was. |


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| 146. | 00:10:56 | R2: | Any ideas Stephanie how to show from four to eight? |
| 147. |  | Milin: | I do. |
| 148. |  | Stephanie: | Alright. |
| 149. |  | R2: | How about you Jeff? |
| 150. |  | Jeff: | I don't- |
| 151. |  | R2: | If what Milin said you understood and he got four. |
| 152. |  | Jeff: | I understand that but- |
| 153. |  | R2: | Okay now, if this is what he got for four and you understood what he just talked about. Can you write down these four and use his idea to see if you can build eight? |
| 154. |  | R2: | Why don't you write these down what he has here? These four. Start with these four because that is what Milin said to start with. These four. |
| 155. |  | R2: | Well you didn't leave space if you are going to build them up. |
| 156. |  | Jeff: | I will go down. |
| 157. |  | R2: | Okay. That's fine. Does it matter? |
| 158. |  | Jeff: | No. |
| 159. | 00:11:59 | R2: | Okay. What was his idea? |
| 160. |  | Jeff: | You got two reds and two blues and the opposite (inaudible) make this blue. |
| 161. |  | R2: | Now hold on, he said, That is not what Milin said |
| 162. |  | Jeff: | What did he say? |
| 163. |  | R2: | Let's wait till he is finished thinking a minute and ask him. Because I think that is the key to it, to know what Milin said and see if that makes sense. |


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| 164. |  | Milin: | Thing is that you have to keep on adding two- |
| 165. |  | R2: | Milin let's talk about this one. He had this right? |
| 166. |  | Milin: | Okay, then you just- |
| 167. | 00:12:41 | R2: | Now he said from here add a blue. Is that what you said? |
| 168. |  | Milin: | No, you have to keep on adding on top. |
| 169. |  | Jeff: | Doesn't matter. |
| 170. |  | R2: | He wants to add on the bottom. |
| 171. |  | Milin: | I see. For this you have to have a bottom down here otherwise it will be different. |
| 172. |  | R2: | He put a bottom B here |
| 173. |  | Jeff: | It would be the same. You see if you put a B up there it would be the same as just as if you put a B down there. It would be blue/blue/red. And then if you crossed that off it would be- put a B up there it would be blue/blue/red. |
| 174. |  | Milin: | Yeah, but- |
| 175. |  | R2: | No. Hold on. Okay, time out for me. I am getting very confused because all of you are talking and you have all different ideas. And I think it would help me if we got one idea on the table at a time. Now the one idea that we on the table that I wish we would explore before we hear the new ideas, is this one here. Now I would like all of you to consider what Milin said here. Do you all see that? Get another piece of paper and write this down. Maybe write it in the middle so we can build them both ways and see if there is a difference. You don't have to cross off what you did. |
| 176. |  | Milin: | Na this is- |
| 177. | 00:13:49 | R2: | But what you have here, leave some space. You have on the bottom red-blue-red blue and on the top blue-red-red-blue |


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| 178. |  | Milin: | Say if you put this right- |
| 179. |  | R2: | Hold on a minute now let everyone get this down. You might want to separate them too. Might be a good idea to do them just the way Milin or Michelle wrote it. I only want to see four down there because we are looking at Milin's strategy. That is not what Milin did. You made a chart. Milin didn't do that. I am interested... okay, Milin actually drew pictures or Michelle drew these pictures right. Did you do this Michelle or Milin? |
| 180. |  | Michelle: | Uhm. |
| 181. |  | R2: | You draw pictures of what these towers are going to look like. See that's really not quite the same. That's interesting. You are all- |
| 182. |  | Jeff: | But that's what we did. That's what she did- |
| 183. |  | R2: | Well- |
| 184. |  | Milin: | So you could do it any way but- |
| 185. |  | R2: | Okay. Now |
| 186. |  | Milin: | They are just put together |
| 187. | 00:14:37 | R2: | Okay. Now what I think Milin is asking all of us to do is to imagine in front of us can you all see in front of you the towers of two that are these colors. Can you all imagine that in your mind? Can you all see the first one red on the bottom blue on the top. Do you see that in your mind? In the middle of the table there can you see it? |
| 188. |  | Jeff: | Yeah. |
| 189. |  | R2: | The other one blue on the bottom red on the top. Okay I see these four towers. Now Milin is calling our attention to this first tower right, red on the bottom and blue on the top and what is he asking us to do with it? |
| 190. |  | Milin: | Put another blue and then make another thing exactly- |


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| 191. |  | R2: | Alright. Put another blue. Now, can you draw a picture of what that tower looks like? Now of three. This is a tower of three. He is putting another blue. He happens to choose to put it top or bottom Milin? Next question we ask- |
| 192. |  | Milin: | See, you put a blue here or you could put a red there. And this one, you could put this way. You could put a red instead of a blue. |
| 193. |  | R2: | Okay, Milin can you show us in the middle here what you just did with that one tower. Thank you. |
| 194. |  | Milin: | See, from these towers right, We are only using this one- |
| 195. |  | R2: | This one tower right here, right. |
| 196. |  | Milin: | See so, I put the blue here the red on top of it so its like this and then I added one more that'd be red but then I did like this blue then I put red back on top of it and then I put blue because there is only two colors. |
| 197. |  | R2: | So, what you are telling me here if I could make my picture if I were doing what Milin asked me to do where we had a blue and a red what he is telling me to do is he is saying from this tower I am going to put blue on the top- |
| 198. |  | Milin: | Or a red. |
| 199. |  | R2: | Or from this tower I am going to put a red on the top. |
| 200. |  | Milin: | Yeah. |
| 201. |  | R2: | Is that what you are telling me to do? So from this tower we get these two. |
| 202. | 00:16:31 | Milin: | Yeah. |
| 203. |  | R2: | Is that what- |
| 204. |  | Milin: | And for each one you keep on doing that. And for six you get sixty four. |


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| 205. | 00:16:51 | R2: | Does that make any sense? |
| 206. |  | Jeff: | Yeah |
| 207. |  | Milin: | It follows the pattern to five why can't it follow the pattern to six? |
| 208. |  | R2: | I guess, what I am confused about Jeff is that you took this one with blue and red and you only put blue on the top and you have only done, you have only made this one. Milin is telling you that you can also make this one that you could have- |
| 209. |  | Jeff: | But I made that. |
| 210. |  | R2: | Where is it? Red on the bottom blue- |
| 211. |  | Jeff: | Red |
| 212. |  | R2: | From this one you could have put two things on top. You only put one. From this one you- |
| 213. |  | Jeff: | Okay, I understand. |
| 214. |  | R2: | Does that make sense what he is talking about? And he is saying that from these four- |
| 215. |  | Milin: | You could make eight. |
| 216. |  | R2: | So, tell me now convince Jeff why it is going to be eight why it is going to be double. |
| 217. | 00::17:35 | Jeff: | I am convinced |
| 218. |  | Michelle: | I already figured it out like this |
| 219. |  | R2: | What is different about the way you did it Michelle? |
| 220. |  | Michelle: | Well, I just, I just I didn’t do what Milin did I just made them out any I didn't find any that were the same. |
| 221. |  | R2: | Okay, well that's not what Milin did he did something very different. How about you Stephanie? |


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| 222. |  | Stephanie: | Well I found it like this. I drew my lines and then I went red/red/red/blue/blue/blue/blue/red/ blue/red/ blue/ blue/ blue/blue/red/red/red/blue/red/blue/red/blue/red/blue/r ed. |
| 223. |  | R2: | Is yours different than the way Milin did it? |
| 224. |  | Milin: | Yes. |
| 225. |  | Stephanie: | Well, yeah. |
| 226. |  | R2: | In what way? |
| 227. |  | Stephanie: | He built his towers up like this he went red/blue/ red/ blue/ red/blue and so on- |
| 228. |  | R2: | I didn't see him do that. |
| 229. |  | Stephanie: | I was sure it was like that $\left\{I^{\prime} l l\right.$ show you like that, then\} |
| 230. |  | R2: | That is not what he did. He started with red and blue. And from this red |
| 231. |  | Milin: | I put a red- |
| 232. |  | Stephanie: | He put a- |
| 233. |  | R2: | Put a red on top |
| 234. |  | Milin: | And a blue on top |
| 235. |  | R2: | He put a red on top and a blue on top so he got blue/red red/red from the blue- |
| 236. |  | Milin: | I did the same thing. |
| 237. | 00:18:43 | R2: | A red on top and so he's got his red/blue and then he put a blue on top he got blue/blue. |
| 238. |  | Stephanie: | Like, yeah, but that's what he is like, that's what is different from mine I just like took the things and went- I just took one and went - |


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| 239. |  | Milin: | And kept on- |
| 240. |  | Stephanie: | Here is one red/red/red, blue/blue/blue and then I go like red/blue/blue, blue/red/blue- |
| 241. |  | R2: | So, what I am hearing you say is that you're just... |
| 242. | 00:19:11 | Milin: | Guessing! |
| 243. |  | R2: | ...you believe there is eight. But you say guessing. Now, why does that sound like guessing? |
| 244. |  | Milin: | Because what if you could make more? |
| 245. |  | Stephanie: | Okay, this is the three high right? And you're convinced you can make eight? |
| 246. |  | Milin: | Yep! |
| 247. |  | Stephanie: | I'm convinced I can make eight. |
| 248. | 00:19:28 | R2: | Yeah, but you haven't- he's proved to me from the four you can only make eight you can get two from this one, two from this one, two from this one and two from this one. |
| 249. |  | Milin: | But could you convince her? |
| 250. |  | Stephanie: | Convince who? Michelle? Him? |
| 251. |  | Milin: | No. Her. |
| 252. |  | Stephanie: | Her? Yeah. All right. I've done this before. Okay. |
| 253. |  | R2: | Take another piece of paper if you want to because it sounds like your approach is a little bit different |
| 254. |  | Stephanie: | Alright. |
| 255. |  | R2: | You've got to convince me there are eight and only eight, and no more or fewer. |
| 256. |  | Milin: | Whew, You do draw big! |
| 257. |  | R2: | Now Jeff this might be a little different. Let's see |


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|  |  |  | what's going on here. |
| 258. |  | Stephanie: | All right, first you have without any blues, which is red/red/red. |
| 259. |  | R2: | Okay, no blues. |
| 260. |  | Stephanie: | Then you have with one blue - |
| 261. |  | R2: | Okay |
| 262. |  | Stephanie: | Blue/red/red or red/blue/red or red/red/blue. |
| 263. | 00:20:46 | R2: | Anything else? |
| 264. |  | Michelle: | And you would do the same pattern for- |
| 265. |  | Stephanie: | No, not with the blue, not with one blue - |
| 266. |  | Michelle: | You would do it, you would do it with one red and two blues? |
| 267. |  | Jeff: | You would alternate- |
| 268. |  | Michelle: | You would do it the other way around. |
| 269. |  | R2: | That's not what she is doing. Let her finish. That's what you would do. You would alternate. Let's see what Stephanie does. Maybe she's not going to do that. |
| 270. |  | Stephanie: | Well, there's no, there's no more of these because if you had to go down another one you'd have to have another block on the bottom. But then you have with three blues - well, not with three blues. I'll go like this. |
| 271. |  | R2: | You buy that, that's all there is of those? |
| 272. |  | Milin, Jeff: | Yep, yeah |
| 273. |  | Stephanie: | But then you have with three blues - well, not with three blues. I'll go like this. |
| 274. |  | R2: | You have no blues and now you have exactly one |


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| 275. |  |  | blue. |
| Stephanie: | Now you have exactly two blues. Sran <br> yeah that's what I did last time I was here. I did actually, <br> exactly two blues. |  |  |
| 276. | $00: 21: 33$ | R2: | Okay. Let's see. |


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| 291. |  | Stephanie: | All right, so show me another two blues. With them <br> stuck together, because that's what I am doing. |
| 292. |  | Milin: | In that case here. |
| 293. | 00:22:44 | R2: | Okay, so now what are you doing, Stephanie? |
| 294. |  | Michelle: | What if you just had two blues and they weren't stuck <br> together, you could - |
| 295. |  | Stephanie: | But that's what I'm doing. I'm doing the blues stuck <br> together. |
| 296. |  | R2: | Okay. |
| 297. |  | Stephanie: | Then we have three blues, which you can only make <br> one of. Then you want two blues stuck apart- not <br> stuck apart - took apart. |
| 298. |  | R2: | Stephanie: |
| 299. |  | Separated? |  |
| Yeah, separated. And you can go blue/red/blue right |  |  |  |
| here- |  |  |  |


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| 306. | $00: 24: 05$ | Jeff: | I have a question. Do you have to make a pattern? Sran |
| 307. |  | Michelle: | No. |
| 308. |  | Jeff: | So, then why is everybody going by a pattern? |


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| 324. |  | Michelle: | 'cause there's 'cause there's only two colors more so you know you can't make more. |
| 325. |  | Milin: | Yeah And there's red I mean blue/red/red And you can't make any more in this one, so you go on to the next one. |
| 326. |  | Stephanie: | All right, and these - |
| 327. |  | Jeff: | How do you know you can't make any more from that? |
| 328. |  | Milin: | Because there's not any more colors. |
| 329. |  | Stephanie: | Look, okay, start here. Sorry. Start here - okay, you have the three together. The one, one blue, right? You have the one blue. How could I build another one blue? |
| 330. |  | Jeff: | You can't. |
| 331. |  | Stephanie: | All right, so I have convinced you that there's no more one blue? |
| 332. |  | Jeff: | Yeah. |
| 333. |  | Stephanie: | All right. |
| 334. |  | Michelle: | But if you didn't have that pattern, it would be harder to convince you. |
| 335. | 00:26:01 | Stephanie: | If I went I will put this one blue over here and that blue will be on another piece of paper- |
| 336. |  | Jeff: | Yeah but- |
| 337. |  | Stephanie: | How will - |
| 338. |  | Jeff: | You can make a blue different from what you did if you go like this- |
| 339. |  | Michelle: | That's if you have four. |
| 340. |  | Jeff: | If you go like this, you can go red/red/blue or you can go blue/red/red |


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| 341. |  | Stephanie: | That's what I have. |
| 342. | 00:26:20 | Jeff: | No they are all different. You can do red/red- |
| 343. |  | Stephanie: | What I am saying is this is one blue. This is one blue. |
| 344. |  | Jeff: | Yeah, there's still all different with one blue. |
| 345. |  | Stephanie: | Yeah- |
| 346. |  | Milin: | No, but only on the bottom |
| 347. |  | Stephanie: | But I have those three. Look blue/re/red, red/blue/red, red/red/blue. But then how am I supposed to make another one once that blue got down to the last block? |
| 348. |  | Jeff: | Okay. |
| 349. |  | Stephanie: | Okay, so I've convinced you that there's no more one blue? |
| 350. |  | Jeff: | Yeah. |
| 351. |  | Stephanie: | All right, now we move on |
| 352. |  | Michelle: | Then you have to go to two blue. |
| 353. | 00:26:49 | Stephanie: | Two blue. Here's one - right? Two blue - we have one, blue/blue/red, then we have red/blue/blue. How am I supposed to make another one? |
| 354. |  | Jeff: | Blue/red/blue. |
| 355. |  | Stephanie: | No, this is together. Milin gave me that same argument. |
| 356. |  | Michelle: | She means, she means together - |
| 357. |  | Jeff: | But the thing is does it matter that they are together? |
| 358. |  | Michelle: | No, she means stuck together. |
| 359. | 00:27:05 | Stephanie: | Stuck together, that means like - |


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| 360. |  | Jeff: | I know. |
| 361. |  | Michelle: | Okay, so can I make any more of that kind? |
| 362. |  | Jeff: | No. |
| 363. |  | Michelle: | Then you have to move to three, which you can make <br> one. |
| 364. |  | Stephanie: | All right, yeah, you can only make one and then you <br> can make the three with out blue with the three red. |
| 365. |  | Michelle: | And then you can make two split apart. |
| 366. |  | Stephanie: | Two split apart, which you can only make one of, and <br> then you can find that you can - you can find the <br> opposites right in this same group. All right, so I've <br> convinced you that there's only eight? |
| 377. |  | R2: | Jeff: |


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|  |  |  | you can make |
| 379. |  | Michelle: | You can put two colors here, two colors there, two colors - and keep on going. |
| 380. |  | Jeff: | Yeah, you can keep on doing two colors for each one. And that's two, four, six, eight, ten, twelve, fourteen, and sixteen. |
| 381. | 00:28:09 | R2: | And so that's the towers of? |
| 382. |  | Jeff: | Four. |
| 383. |  | Milin: | My guess is- |
| 384. |  | R2: | Four. |
| 385. |  | Milin: | Sixteen but- |
| 386. |  | Jeff: | We already got sixteen. |
| 387. |  | Milin: | Why did she say in the beginning the whole thing twelve |
| 388. | 00:28:17 | Jeff: | This, that you get-listen |
| 389. |  | Michelle: | It's like, it's like- |
| 390. |  | R2: | Why did you say twelve Michelle? |
| 391. |  | Jeff: | It can be either a Red or a blue, red or blue, red or blue- |
| 392. |  | Milin: | Jeff, Jeff, Jeff. I know that pattern. But I want to know why she said twelve before- |
| 393. |  | Stephanie: | Yeah Michelle, why did you? |
| 394. |  | Jeff: | She was guessing not making patterns. |
| 395. |  | R2: | Is that true Michelle? |
| 396. |  | R2: | Poor Michelle. It's okay. |
| 397. |  | R2: | You think what twelve or sixteen? |


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| 398. |  | Michelle: | Sixteen. |
| 399. |  | R2: | Michelle thinks sixteen. |
| 400. | 00:28:43 | R2: | Now, Now You made towers of five in class and what did you get? |
| 401. |  | Stephanie: | Thirty-two. |
| 402. |  | Milin: | Thirty-two. |
| 403. |  | R2: | Does that work the same way? |
| 404. |  | Stephanie: | Yeah. |
| 405. |  | Jeff: | They're all multiples of two. |
| 406. |  | Michelle: | If you get towers of four - |
| 407. |  | Stephanie: | The hard part is to make the pattern. Like, from now, we know how to just - oh, you could give us a problem like how many in ten and we could just go - |
| 408. |  | R2: | How many in ten, and you'd know the answer. |
| 409. |  | Stephanie: | Yeah, I know the answer. I figured it out. It's 1,024! |
| 410. |  | R2: | 1,024. |
| 411. |  | R1: | Are you sure? |
| 412. |  | Stephanie: | Uh-huh. |
| 413. |  | Milin: | Uh-huh. |
| 414. |  | Jeff: | Don't try to convince her. |
| 415. | 00:29:13 | R2: | Try to convince me. |
| 416. |  | Jeff: | No. |
| 417. |  | Milin: | Okay, okay, okay |
| 418. |  | Jeff: | No. No. |


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| 419. |  | R2: | You could do that later. However you were saying you know the answer but... |
| 420. |  | Stephanie: | But the problem is, the hard part is you could just give us a problem and we could go well we will go uh twenty two times 2- |
| 421. |  | Michelle: | See how we are doing we keep on adding what you have already. For here you add two more for here you add another four so for here and for the sixteen one- |
| 422. |  | Jeff: | You sure? Because look you have thirty- |
| 423. |  | Stephanie: | You are dividing- |
| 424. |  | Jeff: | I'm not dividing |
| 425. | 00:29:49 | Stephanie: | You are timesing. No you don't times. See that's the same thing I did. I just counted ahead I counted ahead five or six and I just oh I can just multiply it by that and that would give me the same answer but it didn't work. |
| 426. |  | Jeff: | Oh okay, it didn't work. |
| 427. |  | Stephanie: | And you have to figure out what's in between that. |
| 428. |  | Jeff: | Okay. |
| 429. |  | R1 | What did you find? |
| 430. |  | R2 | What did you mean by- |
| 431. |  | Stephanie: | In between that yeah. |
| 432. |  | R2 | Show them a little bit. |
| 433. |  | Stephanie: | You wanted me to figure out ten right? In order to figure out ten I was only up to five so what I had to do was I had to go and I had to say what was six what's seven, what's eight, what's nine and times that times the last one I had. |


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| 434. | 00:30:26 | R2: | Let's let's take a look at what you did here. This is this is what Stephanie did guys. You want to do it yourself for a minute. When I asked Stephanie how many for tower of ten. What Stephanie- why don't you say what to did to get one thousand and twenty four. Then let's talk about this one- |
| 435. |  | Milin: | Yeah. She's right |
| 436. |  | Stephanie: | Okay. |
| 437. |  | R2: | Let's just, can you just listen to Stephanie? |
| 438. |  | Stephanie: | I was up to-I was up to number five so I took the number- |
| 439. |  | R2: | Sixty four is the top- |
| 440. |  | Stephanie: | I was up to number six- |
| 441. | 00:30:54 | R2: | Why don't you write that? Okay. Towers of six. |
| 442. |  | Stephanie: | Okay. Now I was up to number six- |
| 443. |  | R2: | You agree with that? |
| 444. |  | Jeff: | Yeah. |
| 445. |  | R2: | Okay. |
| 446. |  | Stephanie: | So, first of all I tried multiplying it times eight 'cause I figured well, all I have to do is six plus four times that times two that's eight so times sixty four times eight. |
| 447. |  | Jeff: | What are you saying? |
| 448. |  | Milin: | She did it wrong. |
| 449. |  | R2: | No no no let's hear what she is said. Let's hear her thinking. |
| 450. |  | Stephanie: | First I thought well I don't want to go ahead I don't wanna have to multiply seven, eight, nine and ten. Seven, eight, and nine before I can get ten. So, I |


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|  |  |  | figured six plus four equals ten. But since I am timesing times two, I multiply four times two get eight and then just multiply sixty four times eight. |
| 451. | 00:31:16 | Milin: | But she was wrong. |
| 452. |  | Stephanie: | Yeah. |
| 453. |  | Jeff: | Way wrong- |
| 454. |  | Michelle: | And then, and then, no, she was right here, she only times by two so she was right. |
| 455. |  | Milin: | Then keep on timesing by two. |
| 456. |  | Stephanie: | Then I did one hundred twenty eight times two, two fifty six, five twelve, and then- |
| 457. |  | Milin: | You get your answer. |
| 458. | 00:31:54 | R2: | Except that, this is where I am very very interested in what she did- how come she got something - she got five twelve |
| 459. |  | Jeff: | And you already got five twelve over here. |
| 460. |  | R2: | Is that so, is that so very wrong? |
| 461. |  | Jeff: | And then you could have timesed this by two- |
| 462. |  | Michelle: | No, because |
| 463. |  | Milin: | No, that's the same thing. |
| 464. |  | Jeff: | But you could had by times two and could have had a lot easier than going times/times/times/times |
| 465. |  | R2: | So, in other words could this have worked? That's my question. |
| 466. |  | Jeff: | Could have. |
| 467. |  | R2: | When will this work? Why didn't the eight work? Why did you have to keep- |


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| 468. |  | Stephanie: | Well, I just thought of something. I am wondering if this will work. This eight is number eight okay. This is number eight right? |
| 469. |  | Jeff: | Yeah. You had it right |
| 470. |  | Stephanie: | This would be answer for number eight. |
| 471. |  | Jeff: | It's just that you didn't follow the pattern you just took a guess and then if you filled it out - |
| 472. | 00:32:34 | R2: | Okay. So what you are suggesting is multiplying by eight didn't work it gave to five twelve which was- |
| 473. |  | Jeff: | It gave you to number eight. |
| 474. |  | R2: | To number nine. To number eight? |
| 475. |  | Jeff: | So if you- |
| 476. |  | R2: | To number nine. Five twelve give you number eight or number nine? |
| 477. |  | Michelle: | This pattern would have worked. |
| 478. |  | Milin: | It would have worked. |
| 479. |  | R2: | Let's get another piece of paper and see what happened here because this is interesting. |
| 480. |  | Milin: | Her pattern would work but then she has to times it by two after she gets her number. She has to times it by two after she gets her number. |
| 481. | 00:32:58 | R2: | You know what I am thinking? I am thinking may be what we should do is- I want you to- I don't want to throw away Stephanie's idea here, Okay. Because what Stephanie has here and her idea, once she got to her towers of nine right- she said there were five twelve. That by each time multiplying by two. But- |
| 482. |  | Milin: | But then you have- |
| 483. |  | R2: | Now hold on a minute, But - |


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| 484. |  | Michelle: | You multiply- this would work because if you multiply that times two you would get one thousand twenty four. |
| 485. |  | R2: | Right. But why- why didn't multiplying by eight work when she had towers of six? |
| 486. |  | Michelle: | 'cause she wasn't so sure about going- |
| 487. |  | R2: | Alright. But why- how could she be sure? In other words if eight didn't work you understand my challenge to you? |
| 488. |  | Milin: | Yeah. |
| 489. | 00:33:47 | R2: | All you mathematicians here. My challenge to you is I don't want to throw out this idea because- you know if Stephanie has something here she would save you a lot of work in the future right? If she has a good idea here. You understand the problem here? And I think what we will do- I want to be sure I don't know if Mrs. Barnes is gone I want to be sure that your teacher understands what's going on here so to sort of push you to think about this so that next time I come, maybe you could invent another way. If I say towers of- |
| 490. |  | Milin: | Eighty. |
| 491. |  | R2: | -Eighty |
| 492. |  | Jeff: | No. I am Sorry- |
| 493. |  | R2: | Now and I say I will give you a calculator but you have to know what to do with your calculator right? |
| 494. |  | Stephanie: | There is a problem. You have to go all the way from ten to eighty. |
| 495. |  | R2: | Well my question is, let's not worry about the big problem for a moment. Let's try to do it with the simple problem. Suppose you didn't know that towers of six was sixty four and towers of sevenwhat did you say that was? What do you have there? |
| 496. |  | Milin: | Towers of seven- |


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| 497. | 00:34:52 | R2: | One twenty eight- is that what you have Milin? |
| 498. |  | Milin: | Yeah. I think. |
| 499. |  | R2: | Suppose you didn't know that. How could you jump from towers of six to towers of ten without going through all those steps and why? |
| 500. |  | Milin: | Well- |
| 501. |  | R2: | Isn't that a nice challenging question? I have one more question to ask. Well let me put that one aside for a moment because that one is going to take some time. When we come back we are going to talk about that. |
| 502. |  | Michelle: | Yes. |
| 503. |  | Jeff: | Uhg! |
| 504. |  | R2: | You can bring your calculator Jeff. Fair enough? |
| 505. |  | Jeff: | Yeah. |
| 506. | 00:35:16 | R2: | Okay. Now look you said this one was like shirts and pants and I would like you to say if you agree this is like shirts and pants- |
| 507. |  | Jeff: | I agree. |
| 508. |  | R2: | But why? |
| 509. |  | Jeff: | Because- |
| 510. |  | Michelle: | But if you kept on going up- |
| 511. |  | Jeff: | You have to- |
| 512. |  | Michelle: | you'd have to have- |
| 513. |  | R2: | Okay. One at a time. Let's hear Jeff. |
| 514. |  | Jeff: | It's the same pattern. |


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| 515. |  | R2: | In what way? Tell me what it has to do with shirts and pants. |
| 516. |  | Jeff: | 'cause with the shirts you have to keep on alternating the shirts with the pants and keep on alternating the pants with the shirts- |
| 517. | 00:35:40 | R2: | I'm not so sure I follow what you are saying. |
| 518. |  | Jeff: | Neither do I. |
| 519. |  | R2: | Stephanie is working on two to the tenth. |
| 520. |  | Stephanie: | I might have it here. I am thinking if I multiply the last number I got which is one thousand and twenty four times eighty that I got- |
| 521. |  | Michelle: | You would get the answer probably. |
| 522. |  | Stephanie: | -Eighty one thousand nine hundred and twenty but I am not sure if I am right or not. I am going to have to draw the- |
| 523. |  | Milin: | Unh unh! |
| 524. |  | Michelle: | Or may be you have to- |
| 525. |  | Milin: | No! |
| 526. |  | Michelle: | - times it by seventy because you already did ten. |
| 527. |  | Stephanie: | I am not- |
| 528. |  | Milin: | No. but times by eight because you would have to have eight more- |
| 529. |  | Jeff: | You guys are losing me here. |
| 530. | 00:36:12 | R2: | Me too. I am lost too. |
| 531. |  | Stephanie: | You wouldn't times it by eight 'cause we times it by eight when we were on eight. we times by eighty when we are on eighty. |
| 532. |  | Jeff: | True. |


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| 533. |  | R2: | But we are- I don't understand- |
| 534. |  | Milin: | Uhn uhn how did she times it by eight when it was on eight? |
| 535. |  | Stephanie: | When I said well, there's four- |
| 536. |  | Milin: | Eight times eight twenty - sixty four, how can that be? |
| 537. |  | Stephanie: | Actually you would multiply it by one thousand and six hundred. |
| 538. |  | R2: | Can we- can we call time out for a minute? |
| 539. |  | Jeff: | What are you guys talking about? |
| 540. | 00:36:42 | R2: | Yeah. I am a little lost and Jeff is lost so I don't know how Michelle is doing here and you two can continue this when we leave and work this out. However- |
| 541. |  | Michelle: | Finish your fight |
| 542. |  | R2: | I don't really want you to solve the problem of towers of eighty. I want you to solve the problem of towers of ten. |
| 543. |  | Jeff: | We did that (In Unison) |
| 544. |  | Michelle: | We did that |
| 545. |  | Milin: | We did that |
| 546. |  | R2: | But hold on. But you have to pretend you only know the answer for towers of six. |
| 547. |  | Michelle: | Just keep on building- |
| 548. |  | Milin: | I already did that. I already did that. I already did that |
| 549. |  | Stephanie: | You want us to figure it out the way I tried to figure it out the first time. |
| 550. | 00:37:15 | R2: | Right. With only multiplying by one number. |


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|  |  |  |  |
| 551. |  | Stephanie: | Okay |
| 552. |  | R2: | And convince me that number makes sense to multiply it by. |
| 553. |  | Milin: | This it? |
| 554. |  | R2: | You understand? |
| 555. |  | Milin: | This? |
| 556. |  | Jeff: | All you did- |
| 557. |  | R2: | Okay. Hold on. Now time out. |
| 558. |  | Jeff: | You didn't know times two times two would help you. |
| 559. |  | Milin: | Yes I did. |
| 560. | 00:37:32 | R2: | Well you sort of know it but I want to save all those intermediate steps because if you had to go to build towers of eighty- lets' see when you had to build towers of two how many times - of three high - how many times did you multiply by two? |
| 561. |  | Milin: | She is right. You should really- |
| 562. |  | R2: | When you had to build towers of three how many times did you multiply by two? |
| 563. |  | Milin: | Uhn. Times two. Three. Four times two. |
| 564. |  | R2: | I said by two. |
| 565. |  | Milin: | Four. Same thing. |
| 566. |  | R2: | Okay. Two times two that's one time you multiplied by two. You got four. Then you multiplied by two again- |
| 567. |  | Milin: | Oh! Eight. |
| 568. | 00:38:18 | R2: | Right? And that gave you eight. So, how many times |


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| :---: | :---: | :---: | :---: |
|  |  |  | did you multiply by two? |
| 569. |  | Stephanie: | You multiplied it the amount of times- |
| 570. |  | R2: | Or twice. You multiplied it once. This is two times two once right? And then you multiply it by two again. right? Two times two let me write this two times two gave you four that was one time and then you multiplied by two again another time and you got eight. So you multiplied it twice to build towers of three. Is that right? |
| 571. |  | Jeff: | Yeah. |
| 572. |  | Milin: | No. |
| 573. |  | R2: | No? |
| 574. |  | Milin: | Because if you get towers of two it would be much easier. |
| 575. |  | Jeff: | Yeah but the thing is it's right- |
| 576. |  | R2: | I think we have run out of time. |
| 577. |  | Jeff: | Yeah we did. |
| 578. |  | R2: | Would you come back? |
| 579. |  | Jeff: | Yeah. |
| 580. |  | R2: | Would you come back? When we come back another time. |
| 581. |  | Stephanie: | Sure. |
| 582. | 00:38:57 | R2: | Next question is I want to know what this has to do with shirts and pants. |
| 583. |  | Milin: | Shirts and pants? |
| 584. |  | Stephanie: | Oh no! |
| 585. |  | Jeff: | I have no idea. I didn't think of it. |


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| 586. |  | R2: | So you could talk about it before and share that. |

# 10.9 TRANSCRIPT - "GUESS MY TOWER" GROUP WORK ON 

 2/26/93| 1. | 00:03:38 | R1: | And so what I'm going to do is to give everybody a copy of this problem. And while I'm giving it out maybe you can start reading the first part of it first. You'll notice that there's a 1 and 2, they both go with the same problem. So let's read it and see what it's going to ask us to do. And you remember that what we're really interested in is getting your very best ideas about what we're doing. Mrs. O'Brien do you want to see our problem? |
| :---: | :---: | :---: | :---: |
| 2. |  | Teacher: | Love to. (R1 hands her a copy of the problem.) Thank you! |
| 3. | 00:04:25 | R1: | Did everybody get one? I want us to read right now just down through number 1 . Not the first number 1 where it says, cubes are exactly the same color.", but the number 1 underneath it, which is really your first question. And then as soon as you've all had a chance to read it for yourself, let's read it together. (The children begin to read the problem.) When you get finished, when you get down to that blank space, if you look up, then I can tell when you're ready for me to go? (The children read the problem and look up as they finish reading.) <br> Who will read it for me? Somebody? Brian? |
| 4. |  | Brian: | Okay. Should I read the top part first? |
| 5. |  | R1: | Sure. |
| 6. |  | Brian: | (Begins reading.) You have been invited to participate in a TV Quiz show and have the opportunity to win a vacation to Disneyworld. The game is played by choosing one of the four possibilities for winning and then picking a tower out of a covered box. If the tower you pick matches your choice, you win. You are told that the box contains all possible towers that are three tall that can be built when you select from cubes of two colors, red and yellow. <br> You are given the following possibilities for a winning tower: <br> 5. All cubes are exactly the same color; <br> 6. There is only one red cube; <br> 7. Exactly two cubes are red; <br> 8. At least two cubes are yellow. |


|  |  |  | 1. Which choice would you make and why would this choice be better than any of the others? |
| :---: | :---: | :---: | :---: |
| 7. |  | R1: | Okay. Ah, can somebody tell me what that means? What does that mean that we're supposed to do? Yes, Milin? |
| 8. | 00:06:26 | Milin: | Like um, when it says that, it means like, which one would probably have, like, the most possible ways that you could do it. |
| 9. |  | R1: | Well, may be so. But I hear what you're sort of jumping to. Your job is to choose one of those four things. Is that right? And it's as though, you know, all of us are going to be participants in this contest. Maybe all of us are going to win! And so your job is to choose one of those things as if you were in the contest. After you've chosen one of those possibilities, then what would you do if you were a contestant? Say I've chosen number 3, okay? I said, "Okay, I'm choosing number 3." Then what would I do? Would I be through? Would I have won? |
| 10. | 00:07:24 | Student: | No. |
| 11. |  | R1: | What would I do next? |
| 12. |  | Milin: | We would have to look in the box to see. |
| 13. |  | R1: | Would I have to look in the box? |
| 14. |  | Student: | You take one out with your hand. |
| 15. |  | R1: | Yeah, if looked in the box, I'd be a dumb-dumb if I didn't choose one of them; I mean one of the things I had chosen. Wouldn't that be? So what makes it a game? |
| 16. | 00:07:45 | Student: | You have to guess. |
| 17. |  | R1: | You have to guess. |
| 18. |  | Student: | Conjecture. |
| 19. |  | R1: | What no, I've chosen my choice, the end. This is the end. This box with the things, can I see them? |


| 20. |  | Student: | No. |
| :---: | :---: | :---: | :---: |
| 21. | 00:07:59 | R1: | No. that's really what I'm getting at. So I'm going to walk up to the box after I've chosen one of these four things, okay? Which is what Milin was saying a minute ago. I'm going to reach in the box because, if you remember those mystery boxes that we had one time, it's like that. I'm going to reach in and what am I going to get? |
| 22. |  | Student: | Towers. |
| 23. |  | R1: | I'm going to get one tower, okay? Do I know what it is? |
| 24. |  | Students: | No. |
| 25. |  | R1: | Uh-uh (agreement) because it's in a box, all right? And going to pull it out. Now, how would I win? |
| 26. |  | Brian: | You'd have to get the tower that's the one you picked. |
| 27. |  | R1: | Yeah, you say that. Stephanie? I saw a hand. |
| 28. | 00:08:41 | Stephanie: | I was going to say what he said, that you have to get the tower that you picked. If you picked number 3 you have to get that kind of tower. |
| 29. |  | R1: | Uh-huh. Yeah, you have to get that kind of tower. So if chose number 3, then have to get a tower that matches that choice, is that right? Or if I chose number 1, then I'd have to have gotten one that matches number 1. Does that make sense to you all? |
| 30. |  | Students: | Yeah. |
| 31. | 00:09:04 | R1: | Okay. Now your job, as Milin was saying a while ago, is to figure out the choice that you would make. Maybe all the choices are just the same, but to think and make and if that's the case, that's what you might say, "It doesn't make any difference, my lucky number's four.", you know. But you're supposed to figure out which choice you would make you and your partner together. You may have different ideas so you can certainly do that. But you're to figure out which |


|  |  |  | choice would you make and why you think that this choice is better than any of the others, okay? And before we go any further, you can do anything you want to figure that out. You'll note we have stuff around like Unifix cubes; you've got your markers. I want your choice and your reasons for your choice, whatever those reasons are, whether they're pictures or things, or whatever. Can we use the pink paper to do that on? Okay? And so come up with your choice and your argument for your choice. |
| :---: | :---: | :---: | :---: |
| 32. | 00:10:26 | Milin: | At least two cubes are yellow. |
| 33. |  | Michelle: | I put that because if one cube is yellow then You would still get that and that it says at least two cubes yellow. |
| 34. |  | Milin: | Yeah but look at this that means there is only one red cube right? So that means that there is only three ways that you could get that. (the bell rings) so if you want to pick... |
| 35. |  | Michelle: | There is two ways you can- see at least two cubes are yellow and there is only one red is the same thing outside? |
| 36. |  | R3: | I am sorry to interrupt you two. Milin would you mind coming sitting over here so the camera can see you. (Milin moves to the opposite side of the table) |
| 37. |  | Michelle: | You could either pick two or four cause it is the same thing |
| 38. |  | Milin: | No |
| 39. | 00:11:28 | Michelle: | Yeah you could yeah because at least two cubes |
| 40. |  | Milin: | Yeah but that could mean three cubes are yellow. |
| 41. |  | Michelle: | No. That's true it could mean three cubes are yellow but then that would be that all the cubes are yellow but then it would be that all the cubes are the same color |
| 42. | 00:11:45 | Milin: | Well that would be at least you could - this is the highest possible way that you could win |


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| 43. |  | Michelle: | That's true because- but I don't know if they would count as three as all of them 'cause this one then you would pick this one, all the cubes are the same color |
| 44. |  | Milin: | So, this would be a lot better than this |
| 45. | 00:12:11 | Michelle: | I know but what if they count that as only two cubes. |
| 46. |  | Milin: | It says at least |
| 47. |  | Michelle: | Ok |
| 48. |  | Milin: | Number three says that exactly two cubes are red. |
| 49. |  | Michelle: | I know |
| 50. |  | Milin: | Exactly and at least are two different things. Where is my pencil? (looks for his pencil) |
| 51. |  | Michelle: | So we will pick number four. Why would we pick it? |
| 52. | 00:12:44 | Milin: | Because that is the highest possible way! |
| 53. |  | Michelle: | But we have to explain why it is the highest possible way. |
| 54. |  | Milin: | That'd be easy, this has only two ways, this has three ways nah this- yeah this has three ways, this has- |
| 55. |  | Michelle: | I don't get what you are talking about Milin. |
| 56. |  | Milin: | Uh... exactly two red- exactly two- |
| 57. | 00:13:18 | Michelle: | This says exactly two cubes are yellow but what if one cube is yellow? |
| 58. |  | Milin: | Ohm!!! |
| 59. |  | Michelle: | I think we would pick this because you can win if it has two or three. |
| 60. |  | Milin: | Yeah. |
| 61. |  | Michelle: | So we would put write four down (Milin writes 4) let's write four down. And write because... how could we put it in words? Because you could win if its two |


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| :---: | :---: | :---: | :---: |
|  |  |  | yellow or three yellow (both write down the explanation) |
| 62. |  | Milin: | Okay now what? I remember something... |
| 63. | 00:14:28 | Michelle: | You want to draw a picture now. Cause all you would have to do is use this is yellow this is red |
| 64. |  | Milin: | Or we could use a different two colors. |
| 65. |  | Michelle: | That's true. At least two cubes yellow that could mean that you could pick one with two cubes that are one color and one cube the other. Can't tell the difference in the colors I used. |
| 66. |  | Milin: | I am not that sure with mine either. Keep this, get rid of this (picks up a different color marker) not green black. That and one could be like black here- |
| 67. |  | Michelle: | I don't know if that is the real way that we could do it because- what if that only means that two cubes are yellow. |
| 68. |  | Milin: | It says that at least and exactly. Remember? |
| 69. |  | Michelle: | Okay. Assuming that you won, you can play again for the Grand Prize which means you can take a friend to Disney World. But now your box has all possible towers that are four tall built by selecting from the two colors yellow and red. You are to select from the same four possibilities for a winning tower. Which choice would you make this time and why would this choice be better than any of the others? (Michelle reads the second problem statement and Milin continues to make his towers on the pink sheet) |
| 70. |  | Milin: | Uh Assuming that you won- (Milin reads the statement silently) okay. |
| 71. |  | Michelle: | So now which one we choose the same one? |
| 72. |  | Milin: | let's find out how much they are all together, okay. |
| 73. |  | Michelle: | How much what? |
| 74. |  | Milin: | How many buildings there will be? |


| 75. |  | Michelle: | They are four wait- fole tall, uhg four tall and we still have to use these four possibilities. (Michelle points to the four possibilities from the first question that had to be used with the second question as well) I would pick the same one because two of them be yellow three of them can be yellow or four of them can be yellow. I see the same thing. |
| :---: | :---: | :---: | :---: |
| 76. |  | Milin: | Yeah, so do I. |
| 77. |  | Michelle: | Cause if it is two red you have it, if it is three red you have it, and if it is no red you have it cause if it is four yellow you can have it, if it is three yellow, two yellow |
| 78. |  | Milin: | Cause it is the same thing yellow. (Both write their explanations for the second question) uh! (Milin finishes his writing an he looks around) |
| 79. |  | Michelle: | I have said that there's two- I said that four again it says that two cubes are yellow it means- |
| 80. |  | Milin: | Oops! say one because you can't get more (fixes his answer) hope we have something better than what we had last time. |
| 81. |  | Michelle: | But what if it is exactly one yellow |
| 82. | 00:19:06 | Milin: | That's the only thing that would be wrong with it cause it would be one yellow |
| 83. |  | Michelle: | Cause all the other ways we would have it. I like it the way we have it but- two cubes are red there is only one red cube, all cubes are exactly the same color- See Milin, exactly two cubes are red. You look at that with- no. For exactly two cubes are red for this one since you are four tall it would be the same thing as this be the same thing as this, and be the same thing as that. |
| 84. |  | R1: | (comes to the table) |
| 85. |  | Michelle: | For this possibility does that mean- |
| 86. |  | R1: | You have already gone to number 2 but I am not the least bit convinced about number 1 the first one. |


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| 87. |  | Michelle: | Does that mean that you can have three yellow I mean, two yellow or more? Like can you have two yellow and then three yellow cause it says at least 2 yellow |
| 88. |  | R1: | What do you think? |
| 89. |  | Milin: | Yeah |
| 90. |  | R1: | Why? |
| 91. |  | Milin: | Because it says at least two yellow. |
| 92. |  | R1: | What does that mean? |
| 93. |  | Milin: | That means you have to have two yellows or more. |
| 94. |  | R1: | Uh huh and? |
| 95. |  | Milin: | That- |
| 96. |  | R1: | I mean what does that mean in relation to the problem? You've said you think that that gives it to you. |
| 97. | 00:20:44 | Milin: | Well, that's the most ways you could have it. |
| 98. |  | R1: | You may know that but I don't know that. |
| 99. |  | Milin: | I wrote it down like this - the cubes are exactly the same color there would only be two |
| 100. |  | R1: | Oh I see. Is that what it means up there? |
| 101. | 00:20:59 | Milin: | Yeah. That's why I put all these numbers up in there. |
| 102. |  | R1: | Did you explain that to Michelle? Does she agree with you? |
| 103. |  | Michelle: | Well, I didn't understand why he put the numbers there |
| 104. |  | R1: | Okay, well, then you've got on your pink paper, Milin, before you can go any further, this is a joint thing you have got to prove to Michelle what that two means and why that has something to do with your conclusion down here. |

\(\left.\begin{array}{|l|l|l|}\hline 105. \& \& Milin: <br>
\hline 106. \& \& Michelle: <br>
\hline Okay. <br>
\hline Oh. I get it. That means It has two possibilities one of <br>

the possibilities is that it can all be red and the -\end{array}\right]\)| Milin: |
| :--- |
| 108. |


|  |  |  | so that means this is one possibility and this is another possibility so that's for- this is for number one. |
| :---: | :---: | :---: | :---: |
| 122. |  | Milin: | Number two that has three possibilities cause there is only one, red cube |
| 123. | 00:23:24 | Michelle: | I don't get that. Oh, yeah one could be on the bottom one could be on the top and one could be in the center. |
| 124. |  | Milin: | That's it. |
| 125. |  | Michelle: | But it doesn't make a difference if it is on the bottom cause if we just turn it around it is on the top |
| 126. |  | Milin: | No! |
| 127. |  | Michelle: | So if you got one that was on the top wouldn't make a difference cause if you just turn it around it would be on the bottom. |
| 128. |  | Milin: | Yeah but. Then put different then, okay |
| 129. |  | Michelle: | Yeah okay. |
| 130. | 00:24:02 | Milin: | Red, red, red (Milin fills in three red cubes; bottom one in the first tower, middle in the second and the top in the third tower) Okay I'll just use this instead (fills in the remaining cubes with the second marker) |
| 131. | 00:24:50 | Michelle: | Am I right on this? |
| 132. |  | Milin: | Yeah. |
| 133. |  | Michelle: | Great. So now we have to for exactly two cubes red. I still think there is only two possible for this cause if it is on the top or the bottom it is the same thing all you have to do is turn it around (Michelle is talking about three towers with one red) |
| 134. |  | Milin: | But if you use Unifix cubes they wouldn't look the same |
| 135. |  | Michelle: | That's true but what if we just used blocks. But it doesn't say we use Unifix cubes in this it just says we use cubes. |

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| 136. |  | Milin: | Yeah you've right. |
| :--- | :--- | :--- | :--- |
| 137. |  | Michelle: | So I think there would only be two cause if we turn it <br> around it would be the same thing |
| 138. | 00:25:48 | Milin: | But there is one on the top and one on the bottom. |
| 139. |  | Michelle: | If one is on the top we turn it around then it is on the <br> bottom |
| 140. |  | Milin: | But that would be flipping something backwards. |
| 141. |  | Michelle: | But then again if it is just on the sheet of paper like <br> sheets of paper its drawn on sheets of paper then it <br> would be wrong because, no, it wouldn't because if <br> you just if you turn one around like- I don't know how <br> to explain it- like one you just turned around and what <br> if you picked it up looked at it and then turned around <br> what would the difference be? |
| 142. |  | Milin: | That the red is on the top one is- |
| 143. |  | Michelle: | Cause when you are looking at it you could be turning <br> it around when you are looking at it. |
| 144. | $00: 26: 50$ | Milin: | But One is on the top and one is on the bottom, it's <br> like |
| 151. |  | $00: 27: 31$ | R1: |


| 153. |  | Michelle: | What if you picked out a block and only one of them were red and it was on the top |
| :---: | :---: | :---: | :---: |
| 154. |  | R1: | And? |
| 155. |  | Michelle: | And it was on the top and what if you picked it out like this would it count same thing as the bottom cause if you turned it around it would be on the bottom. |
| 156. | 00:27:56 | R1: | For which of your possibilities? |
| 157. |  | Michelle: | For this one to see how many possibilities are for that? |
| 158. | 00:28:04 | R1: | Oh, so you are saying if you picked it out this way or if you picked it out this way would it be two different possibilities? |
| 159. |  | Milin: | Yeah because it is like- |
| 160. |  | Michelle: | Cause what if you just turned it around |
| 161. |  | Milin: | -one is on the top and one is on the bottom so. |
| 162. |  | R1: | So does that mean you are more likely to pick out that one because you might pick it out one way or you might pick it out the other way? What do you think? |
| 163. |  | Milin: | Probably will but I don't know but, you could just flip it over like she says. |
| 164. |  | Michelle: | Cause if - Cause if you just turned it around like if, if you held it in your hand like this and no one saw you turn it around then it would be on the bottom then |
| 165. |  | R1: | So what are you saying? |
| 166. |  | Michelle: | If you were like if you picked it out would it be the same thing as if it were on the bottom would it be the same thing being on the top cause if you turned it around it would be on the top? |
| 167. | 00:29:11 | R1: | Okay. What do you mean by the same thing? Would it be the same tower? |
| 168. |  | Milin: | Yeah would it be the same tower could we count that |

\(\left.$$
\begin{array}{|l|l|l|l|}\hline & & & \text { as } \\
\hline \text { 169. } & & \text { Michelle: } & \text { No , it would look different } \\
\hline \text { 170. } & & \text { Milin: } & \text { Yeah! It would look different. (he stresses the point) } \\
\hline \text { 171. } & & \text { R1: } & \text { Okay, now what is it that you are trying to get? } \\
\hline \text { 172. } & & \text { Michelle: } & \text { Trying to get- you explain it, I can't do it } \\
\hline \text { 173. } & & \text { Milin: } & \begin{array}{l}\text { Yeah I am hearing what you saying what I want to } \\
\text { know what it is it that you are after? For your task } \\
\text { what you are trying to do is to convince me which of } \\
\text { those possibilities is the best one to do. }\end{array}
$$ <br>
\hline 174. \& 00:29:46 \& Mould this be the same thing as this (Milin builds <br>
two three tall towers one with a red on the top and <br>

another with a red on the bottom)\end{array}\right\}\)| 175. |
| :--- |


|  |  |  | color we are gonna have two possibilities cause we <br> have all red or all yellow(holds up both solid towers). |
| :--- | :--- | :--- | :--- |
| 185. |  | R1: | Uh huh. |
| 186. |  | Michelle: | But for number two we just did this because if you <br> took it out this way this is one possibility (red on <br> bottom) this is another possibility(red on top) another <br> possibility is- |
| 187. | 00:30:54 | Milin: | (bell rings)This is going to take a long time. |
| 188. |  | Michelle: | -if you went like this (makes a tower with red in the <br> middle) |
| 189. |  | Milin: | And these are the only three possibilities cause |


|  |  |  | opposite tower with one red on the top) one yellow could be on the bottom-I think we should separate them that looks too confusing (Michelle wants Milin not to put them on top and takes the tower and puts it to one side) and one can be in the middle and that is three there. |
| :---: | :---: | :---: | :---: |
| 200. |  | Milin: | That's only three...but |
| 201. |  | Michelle: | That's all we can get. |
| 202. |  | Milin: | Okay, four. |
| 203. |  | Michelle: | Now we are on number four at least two cubes yellow. |
| 204. | 00:32:23 | R1: | Could I have everybody's attention? Somebody came to me, I think it was with a question, and asked for a clarification that's a really an important one because it makes a difference in terms of winning the game. I think, I'm not sure but I think it makes a difference. And it's not really absolutely clear in terms of this, and so I may get sued, I may get to go to court on this one, uh but it says, you're told that the box contains all possible, you're told that the box contains all possible towers that are three tall. They can be built when you select from cubes of two colors, red and yellow. What I think is, this happens with lots of quiz shows, let me tell you! There are these loopholes you know. Do you think there can be more than one that looks just the same? In the box? |
| 205. |  | Michelle: | Yeah. |
| 206. |  | R1: | That are just the same? That means you might have two that are yellow? Would that make a difference? |
| 207. |  | Michelle: | Yes. |
| 208. | 00:33:39 | R1: | It would or it wouldn't? Would it make any difference? |
| 209. |  | Michelle: | Maybe this TV show is different. |
| 210. |  | R1: | Stephanie, you were going to say something? |
| 211. |  | Stephanie: | Yeah! It makes a big difference because it gives you more choices. I mean, if you pick the whole and you |


|  |  |  | wanted all cubes are exactly the same color yellow and <br> they had three all cubes are exactly the same color <br> yellow that increases your, um your chance of winning <br> with that color a lot. |
| :--- | :--- | :--- | :--- |
| 212. | 00:34:10 | R1: | You mean if I have a hundred and two that are all <br> yellow and then one is all red, that would make a <br> difference? |
| 213. |  | Stephanie: | That increases your chances with the, with getting <br> yellow. |
| 214. |  | R1: | Is that a way that if I was the quiz man that I could <br> really maybe cheat? |
| 216. | Rtephanie: | Uh hm. |  |
| 217. |  | Stephanie: | I could, couldn't I? What would make it fair? |
| 218. |  | R1: | Michere's only one of each color. | | If there's only one of, not each color, but one of each |
| :--- |
| Possibility. |


|  |  |  | towers? Okay? You all go ahead. You're doing wonderfully. |
| :---: | :---: | :---: | :---: |
| 224. |  | Milin: | Okay. |
| 225. |  | Michelle: | Now we are number four at least two cubes are yellow. So two yellow cubes, wait there was two things for each one cause if there was one thing for each one, or two things for each one, it would be the same thing |
| 226. | 00:35:57 | Milin: | Why don't we just use this, this, this and this (Milin picks up the three towers with one red and solid yellow tower) |
| 227. |  | Michelle: | No because that's explaining the other ones and we have to have one explaining number four. That's one (Michelle builds one tower with red on the bottom) That's one (she builds one with red in the center) |
| 228. |  | Milin: | This is one (Milin builds the third tower with red on the top) |
| 229. |  | Michelle: | And one more. These are extra (she makes the solid tower) three, two- |
| 230. | 00:36:29 | Milin: | I guess this one (he picks up the group of four towers) |
| 231. |  | Michelle: | -Three four.(Michelle counts all the groups) It has to be most possibilities we should use that. |
| 232. |  | Milin: | Yeah, has more possibilities. |
| 233. |  | Michelle: | But, see if they said no duplicates of the same tower you would have to take this away, this away, and this away (Michelle takes the three towers with one red and one solid yellow tower out and they are left with eight distinct towers) |
| 234. |  | Milin: | Alright, these towers, which one had duplicates? So, put these together (Milin puts the two solid towers together) this would be for red or yellows (Milin holds three towers with one red cube and two yellow cubes) this would be for yellows (Milin picks up the solid yellow tower) this would be for reds (Milin picks up the three towers with two red cubes and one yellow |


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| :---: | :---: | :---: | :---: |
|  |  |  | cube). |
| 235. |  | Michelle: | I don't understand what you are saying. This is for exactly one red (Michelle is holding three towers with one red cube) this is for same color (she picks up two solid towers) and this is for (holding the three towers with two red cubes) For exactly two red cubes? |
| 236. | 00:37:35 | Milin: | Yeah. |
| 237. |  | Michelle: | Okay. And this is for number four (Michelle puts the solid yellow tower with three towers with one red cube). |
| 238. |  | Milin: | Yeah. (Milin moves the solid yellow to the other side and puts the solid red next to it) Just, Here |
| 239. | 00:37:48 | Michelle: | Why would you put that in if that was for number four? (Michelle is talking about the red tower and moves it away). |
| 240. |  | Milin: | These are all for number four (He picks up the four towers) okay. |
| 241. |  | Michelle: | Okay. This is for number one (Michelle takes two solids) this is for number two (she takes three with one red) This is for number three (she takes three with two reds) and this is for number four (she puts the solid yellow with three with one red cube). |
| 242. |  | Milin: | This is for number four. Since there can't be any duplicates that means that these would |
| 243. |  | Michelle: | That means if I draw, (Michelle groups all eight) that would be in the box |
| 244. |  | Milin: | Yeah, but- |
| 245. |  | Michelle: | It would be, because you can't have duplicates and there is no other duplicate of this or anything else so, that's what would be in the box. |
| 246. |  | Milin: | Okay. |
| 247. |  | Michelle: | So, what we've been explaining does not help us |


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| 248. |  | Milin: | Yeah. |  |  |
| 249. | $00: 38: 56$ | R1: | What do you mean? |  |  |


| 267. | 00:39:46 | Michelle: | This was for number four (She is holding four towers), this was for number two no this was for number one (holds two solid towers), this was for number two (holds three towers with one red cube) and this one is for number three ( three with two red cubes). |
| :---: | :---: | :---: | :---: |
| 268. |  | R1: | Uh huh |
| 269. |  | Milin: | But then we have to get this one out because of the duplicates we have to have because it has duplicates (Milin is talking about the four towers for number four). |
| 270. |  | Michelle: | We had to take that out because it's duplicate and this is same as that, that is same as that (Michelle shows how these were four duplicate towers) |
| 271. |  | R1: | Well help me to understand when you said; are these the towers that were in the box? (R1 points to all the towers on the table) are these the towers? |
| 272. | 00:40:14 | Milin: | These are the only towers that could be in the box. (Milin points to the eight towers) |
| 273. |  | Michelle: | These (Michelle puts them together) |
| 274. |  | R1: | I don't understand the towers that are in had been in the box you just told me there were eight is that right? |
| 275. |  | Michelle: | Yeah they were these (Michelle puts the eight together) |
| 276. |  | R1: | Okay, alright and so that was one thing |
| 277. |  | Michelle: | These were from before- |
| 278. |  | R1: | What is before? |
| 279. |  | Michelle: | When we were trying to see how many possibilities- |
| 280. |  | R1: | Explain, so explain your solutions? |
| 281. |  | Michelle: | Yeah. Like how many towers we can make for each like number one number two number three and number four. |

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| 282. |  | R1: | Yeah I understand that and okay and so it bothers you? What bothers you? Something is bothering you, what's bothering you? |
| :---: | :---: | :---: | :---: |
| 283. | 00:41:06 | Milin: | Nothing |
| 284. |  | R1: | Why not because it seems to me that you were really concerned because there were eight in the box and here you said you had these four (R1 picks up the four including one yellow tower and three with one red cube), and here can you build some more (R1 makes a solid red tower), you had you had this set, leave the ones in the box alone. And then you had this set You had this one and a yellow one can you make it?. Need some more (Points to the yellow cubes) are these for number four? (Points to four towers) |
| 285. |  | Milin: | Yeah |
| 286. |  | Michelle: | These are for number one (Michelle makes the yellow tower and puts it with the red tower). |
| 287. |  | R1: | Yeah and this is for number two |
| 288. |  | Michelle: | And |
| 289. |  | Milin: | Need another red (Milin is building a tower two red on the top) |
| 290. |  | Michelle: | For number three exactly two cubes red |
| 291. |  | Milin: | This (Michelle puts the tower with two reds on the bottom) |
| 292. |  | R1: | Apart |
| 293. |  | Milin: | And this (Milin build one with red in the middle) |
| 294. |  | R1: | Okay, now how many do you have here. |
| 295. | 00:42:40 | Milin: | We have twelve. |
| 296. |  | R1: | Okay now does that mean that there have to be twelve in the box? |


| 297. | Milin: | No, we have because we have duplicates |
| :---: | :---: | :---: |
| 298. | Michelle: | There can't be 12 because there are duplicates |
| 299. | R1: | No I don't mean that so much, oh okay tell me what you mean |
| 300. | Milin: | These duplicates These are duplicates with these so one of them has to go |
| 301. | R1: | Why? Why do they have to go here? |
| 302. | Milin: | Cause these are duplicates |
| 303. | R1: | I understand that, but why is it that you can't build them here for your explanation is this, are these towers that are in the box? |
| 304. | Milin: | No- |
| 305. | R1: | What are these? |
| 306. | Milin: | those are the towers (He takes the duplicates out and points to the remaining explanation set) |
| 307. | R1: | I thought those were the towers in the box (R1 points to the set of eight towers lying flat on the table) |
| 308. | Michelle: | Those are the towers in the box. We are explaining now Milin. |
| 309. | R1: | These are the explanation. Why is the explanation different than the towers in the box? |
| 310. | Milin: | Because these were duplicates with these |
| 311. | R1: | I don't like that I don't understand what you mean by duplicates |
| 312. | Milin: | Like one red cube right thats in all these but one red cube is also in all these |
| 313. | R1: | Why? |
| 314. | Milin: | Because and also |

\(\left.$$
\begin{array}{|l|l|l|}\hline \text { 315. } & & \text { R1: } \\
\hline \text { 316. } & & \begin{array}{l}\text { Answer me why? Why is it one red cube is in these } \\
\text { and one red cube is also in these? }\end{array} \\
\hline \text { Michelle: } & \begin{array}{l}\text { Because for number four it said at least two yellow } \\
\text { cubes are in each so there's two yellow cubes here two } \\
\text { yellow cubes, two yellow cubes and there is three } \\
\text { cause it says at least so this was for number four and } \\
\text { then there are the same because the reds are the same } \\
\text { because there are two yellows here and two yellows } \\
\text { here and one red here and one red here }\end{array}
$$ <br>

\hline 317. \& \& Milin:\end{array} $$
\begin{array}{l}\text { These have to go }\end{array}
$$\right\}\)| 318. |
| :--- |

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| 331. |  | Michelle: | And for number three there is two red cubes so there is two red cubes here, two red cubes here and two red cubes here |
| :---: | :---: | :---: | :---: |
| 332. |  | R1: | Uh huh |
| 333. |  | Michelle: | And for number two there is one red cube there is one red cube here, one red cube here, one red cube here |
| 334. |  | R1: | Uh huh |
| 335. |  | Michelle: | And for number one they are exactly the same color and these two are same color |
| 336. |  | R1: | Yeah, I really like what you have done here because it really helps me to see what you mean by each of those choices but it seems to me that Milin you are really still concerned with the differences between this and that and Michelle you are not so concerned |
| 337. |  | Michelle: | Because we are just explaining we are just using these to explain what possibilities like for number one there is two because you can get two towers |
| 338. | 00:46:34 | R1: | Okay, are these both of two in the box? |
| 339. |  | Milin: | Yes. |
| 340. |  | Michelle: | These two are both in the box. For number two there is only one red cube and there is only one red cube in this and they are in here too |
| 341. |  | R1: | So how many opportunities in the box that you might pull? |
| 342. |  | Milin: | For which number? |
| 343. |  | R1: | We go for number which one is this? Number 2? |
| 344. |  | Milin: | Three |
| 345. |  | R1: | Are they all there? Yeah. What about for number one? |
| 346. |  | Milin: | That's it (Picks up the two solid towers)two. |


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| 347. |  | R1: | Oh I see and what about for these other? |
| 348. |  | Michelle: | See This is for number three which is two red and they are right there and for number four- |
| 349. |  | Milin: | You used all these already |
| 350. |  | Michelle: | we used these but we couldn't we they were already used for- |
| 351. |  | R1: | Does that matter? |
| 352. |  | Milin: | No, does not matter because someone is probably gonna like |
| 353. | 00:47:40 | R1: | Cause you are only gonna reach in a box once aren't you? |
| 354. |  | Milin: | Yeah. |
| 355. |  | R1: | Okay and you are gonna reach into the box once and you gonna do what? |
| 356. |  | Milin: | Like gonna pick one if it has two or more yellows then we win. |
| 357. |  | R1: | If you had chosen number four? |
| 358. |  | Milin: | Yeah. |
| 359. |  | R1: | That's right. Okay. |
| 360. |  | Milin: | If you choose any of these four you will win |
| 361. |  | R1: | Now so that is- |
| 362. |  | Milin: | Like a fifty -fifty chance |
| 363. |  | R1: | Yeah, yeah, yeah. And so this is your description of what number four is. |
| 364. |  | Milin: | Uh huh |
| 365. |  | R1: | And you have a, you say you have a - Why is it that you have a fifty-fifty chance? |


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| 366. |  | Milin: | Because there are only- |
| 367. |  | Michelle: | Because out of number four, I don't think we have a fifty-fifty chance, here this is like half of what is in the box and this is the rest but these are only parts of numbers this is number three and these two together are number one but if you pick if you pick this out and picks number four he would win too so I would pick number four because- |
| 368. | 00:48:53 | Milin: | It has more. |
| 369. |  | Michelle: | If you picked any one of these you would win. |
| 370. |  | R1: | I uh I hear what you are saying why is it you said that you thought it there would give you a fifty-fifty chance? |
| 371. |  | Milin: | Cause it is like half of it would be the fifty we won't have a chance and this half would be the fifty we would have a chance |
| 372. |  | Michelle: | But what if he picks number three and then you only have these |
| 373. |  | Milin: | Then you won't have a fifty-fifty chance. |
| 374. |  | R1: | Is there any way you could tell me what chance you would have? |
| 375. |  | Milin: | I don't know |
| 376. |  | R1: | More than fifty-Fifty? Less than fifty-fifty? |
| 377. |  | Milin: | less |
| 378. |  | Michelle: | Less than 50 |
| 379. |  | R1: | Why? |
| 380. |  | Michelle: | There is four in half of it. |
| 381. |  | R1: | Hmm and that's is that what makes it fifty-fifty? |
| 382. |  | Michelle: | Uh huh. And this is only three so. |


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| 383. |  | R1: | And so because there were four in this it was fiftyfifty? <br> And then because, where's those three over here, it's these three |
| 384. |  | Michelle: | Yeah |
| 385. |  | R1: | it would be that. What about these guys? |
| 386. |  | Michelle: | These two these are from number one but since we were explaining just before this was from number four but you said we could not have any of the same |
| 387. |  | R1: | Are you gonna choose two of these or just one of these things? |
| 388. |  | Michelle: | One |
| 389. |  | R1: | So after you have made your choice then you could have chosen some for instance if I actually pull this one out of the box this one is there more than one of the possibilities is that it would have won for me? |
| 390. |  | Michelle: | You could have won with number one or number four. |
| 391. |  | R1: | Uh. What about with this one? |
| 392. |  | Michelle: | You could have only win with number one |
| 393. |  | R1: | What if I had chosen this one? |
| 394. | 00:50:58 | Michelle: | You could win with number four or number two |
| 395. |  | R1: | But you have to choose first don't you? |
| 396. |  | Michelle: | Uh hum. |
| 397. |  | R1: | Yeah so you don't get to pull it and then choose which one you want that would be, that would be neat won't it? Okay are you saying Michelle that you are happy with, I mean you are satisfied with what this is |
| 398. |  | Michelle: | Uh huh |
| 399. |  | R1: | - and you are satisfied with what that is even though it is kind of confusing? |

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| 400. |  | Milin: | Uh huh |
| :--- | :--- | :--- | :--- |
| 401. |  | R1: | Are you still agreeing with which one is the correct <br> with the best choice? |
| 402. |  | Milin: | I do number four |
| 403. |  | Michelle: | Uh huh |
| 404. |  | R1: | If you do number four and if you should win it and go <br> to Disney World. |
| 405. |  | Michelle: | Well we won't be- <br> 406. <br> 00:51:44 |
| Milin: | The only way that we won't win is if we pick one of <br> these but the only way that we won't win for number <br> one is if we pick any of these. |  |  |
| 407. |  | Michelle: | Oh okay I see so you are pretty sure of number four. <br> Okay, you've done, you have convinced me. You have <br> also convinced me nd helped me to see the difference <br> here uh did you finish drawing everything? |
| reason too. |  |  |  |


| 415. |  | Milin: | Three would be- |
| :--- | :--- | :--- | :--- |
| 416. |  | Michelle: | Number three is the two red one. |
| 417. |  | Milin: | Two reds and- |
| 418. |  | Michelle: | I need that. Thank you. |
| 419. | $00: 53: 57$ | Milin: | Now I know how to do it. There are only eight here <br> right? So one times two is - There'd be sixteen for <br> next. <br> We are going to have to make sixteen. |
| 420. | $00: 54: 12$ | Michelle: | We are going to have to explain different for the next <br> on e because it is four high so it is going to be <br> different. Number four I am going to need little bit <br> more room, number four, so for number four these are <br> examples for there is an example for number four so- <br> can I have that one again. I am sick of taking the cap <br> off. |
| 421. |  | Milin: | Number four that's let's see |
| 422. |  | Michelle: | I will give it to you in a second. Now we have to do <br> the box. I need the orange. I'm done with everything <br> except the box. |
| 429. |  | Milin: | This is gonna be crazy. <br> Take these apart now. We have to take these apart <br> now. |
| 425. |  | Milin: | Michelle: |
| 423. |  | I will just use these two colors |  |
| I have got it down, could I have the brown? |  |  |  |


| 431. |  | Milin: | Should we draw everything in the box first? See what we have to chose from? |
| :---: | :---: | :---: | :---: |
| 432. |  | Michelle: | What we have to chose from draw It first? Yeah, Okay |
| 433. |  | Milin: | Yeah, so we know what we already have |
| 434. |  | Michelle: | Then we just take it from that bring it down to the box |
| 435. |  | Milin: | So we know what we already have |
| 436. |  | Michelle: | How will we explain? |
| 437. | 00:01:43 | Milin: | Well at least we will know what we will be working with. Where is my pen? |
| 438. |  | Michelle: | You can use these |
| 439. |  | Milin: | Where's my pen? Let's see |
| 440. |  | Michelle: | It's Four tall this time? |
| 441. |  | R1: | You are working towards four Is it going to be different you think or is it going to be the same what do you think? |
| 442. |  | Michelle: | I think it is going to be the same thing |
| 443. |  | Milin: | Same choice but different thing cause it is going to be sixteen |
| 444. |  | Michelle: | Now it is going to be four high there is going to be more choices now but I am going to put the same thing because I think you'll still get fifty -fifty chance |
| 445. |  | R1: | So you gonna have to figure that out? |
| 446. |  | Michelle: | Uh huh. So, first these blocks are in my way. Is this in here or? Did you just take this out of here? Oh you did? it in here no it is already over there goes in here. |
| 447. |  | Milin: | No, but it is already in here |
| 448. |  | Michelle: | Where? Yes it is. It is in here too. Exactly two cubes are red oh wait! all the cubes are the same color. The |


|  |  |  | first one for number one is the same thing it's just the other ones that are different. I need the black. There is only one red cube. One red- 4.31 |
| :---: | :---: | :---: | :---: |
| 449. | 00:04:41 | Milin: | Which ones are - use these |
| 450. |  | R1: | I heard you say there was going to be fifty-fifty chance? |
| 451. |  | Michelle: | Well there is- for this one there is the same amount for number one but for number two, three, and four there are going to be more than what we had last time |
| 452. |  | R1: | Why? |
| 453. |  | Michelle: | Because we have more colors- no, not more colors they are higher. |
| 454. |  | Milin: | Towers |
| 455. |  | Michelle: | Towers are higher |
| 456. |  | R1: | Uh huh |
| 457. |  | Michelle: | So if it's only one red cube it can be exactly on the top exactly on the bottom or one of them in the center. So that is going to make a difference for that one and then for two red cubes it can be in the center, there or there I am not sure about it yet |
| 458. |  | R1: | Yeah I see you are still working on that. |
| 459. | 00:06:36 | Michelle: | Milin what are you on? Number one or number two for drawing? |
| 460. |  | Milin: | Drawing? Just drawing the box. |
| 461. |  | Michelle: | I thought we were drawing the explanation right now |
| 462. |  | Milin: | I'm just drawing the box first |
| 463. |  | Michelle: | But we don't know everything that's in the box |
| 464. |  | Milin: | Well so far- |
| 465. |  | Michelle: | Because there may be duplicates or something we |


$\left.$|  |  |  | don't know what is exactly in the box yet we have to <br> see what we have first. |
| :--- | :--- | :--- | :--- |
| 466. |  | Milin: | I am just using the staircase. |
| 467. |  | Michelle: | The staircase. |
| 468. |  | Milin: | Those, those, okay there is our box. |
| 469. |  | Michelle: | One duplicate |
| 470. |  | Milin: | Where |
| 471. |  | Milin: | Milin: | | Nuplicate. (crosses out) |
| :--- |
| here, two on top here another duplicate two orange |
| here two orange here wait, wait see that's why I |
| wanted to explain it first so we don't get it all wrong. |
| Think that's all duplicates I am not sure yet | \right\rvert\, | I think sixteen |
| :--- |
| 472. |


| 483. |  | R2 | Why do you think sixteen? |
| :--- | :--- | :--- | :--- |
| 484. |  | Milin: | Well it works like this, before, I don't know, I can't <br> remember that, before we found out that for blocks of <br> two you have to multiply two from the first tower <br> that's two times two so second one is four then two <br> times four is eight times one more would be times two <br> would be sixteen. |
| 485. |  | R2 | What do you think Michelle is that reasonable? |
| 486. |  | Michelle: | I guess. |
| 487. | $00: 12: 30$ | $\mathbf{R 2}$ | Not sure huh? You have to convince Michelle. Is <br> there a way you could convince her that that works? |
| 488. |  | Milin: | I Could build blocks. |
| 489. |  | R2 | That might help her you know, to show her how that <br> works. I didn't understand How many can you make <br> when they are one high? |
| 490. |  | Milin: | Two |
| 499. |  | $00: 12: 51$ | R2 |


|  |  |  | what did you do to it? No the one with the bottom yellow what did you do to it? |
| :---: | :---: | :---: | :---: |
| 500. |  | Milin: | I put another yellow on top of it |
| 501. |  | R2 | You put another yellow on top of this and got this I could see that. what else |
| 502. | 00:13:44 | Milin: | I put a red on top of that one and a yellow on top of this one |
| 503. |  | R2 | You put a red on top of this one. oh so on this yellow one you put |
| 504. |  | Michelle: | A yellow and a red |
| 505. |  | R2 | Oh and on this |
| 506. | 00:14:01 | Michelle: | A yellow and a red too |
| 507. |  | R2 | Oh. So? |
| 508. | 00:14:08 | Milin: | Number three we already found out that would be eight. |
| 509. |  | R2 | How is it going to be eight? Do you see how it is going to be eight Michelle can you tell me? |
| 510. |  | Milin: | Well it was - |
| 511. |  | R2 | Tell me what Milin is doing here though what would you now Milin? |
| 512. |  | Milin: | With this I would put a another red |
| 513. | 00:14:30 | R2 | Lets see on the red one you would put another red |
| 514. |  | Milin: | On yellow one- |
| 515. |  | R2 | Lets worry about the red on the red one you would put a red. Okay lets leave it here |
| 516. |  | R2 | What are you doing Michelle on the red one he put a red what are you doing with that red one? |
| 517. |  | Michelle: | We could add a yellow one to that |


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| :---: | :---: | :---: | :---: |
| 518. |  | R2 | This red one if you are making it three put a red on it |
| 519. |  | Michelle: | Yeah and put a yellow on top of it |
| 520. |  | R2 | Oh why are you doing that? Why does that work? |
| 521. |  | Michelle: | I don't know |
| 522. |  | Milin: | Just works on top of that we could instead of that we can put yellow in between. |
| 523. |  | Michelle: | You could put yellow on the bottom too. |
| 524. |  | R2 | Now wait wait. Where does that come from where does that one come from? I am getting confused |
| 525. | 00:15:29 | Milin: | Okay, This one comes from this |
| 526. |  | Michelle: | This one comes from that you can put a red on top of it. |
| 527. |  | R2 | Oh |
| 528. |  | Milin: | These two And this one |
| 529. |  | R2 | No These go with this |
| 530. |  | Milin: | This and this comes from this and this comes from this and this comes from this. |
| 531. |  | Michelle: | And these come from this |
| 532. |  | R2 | Okay, but how come this one has two and this only has one? |
| 533. |  | Milin: | Well there would be two for this one also. |
| 534. |  | R2 | What is the other one for this one? I don't understand |
| 535. |  | Milin: | Put yellow on here |
| 536. | 00:16:11 | R2 | See what he did? |
| 537. |  | Michelle: | No |

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| 538. |  | R2 | Michelle doesn't know what you did |
| :--- | :--- | :--- | :--- |
| 539. |  | Milin: | Take these two off Okay? There is only two colors in <br> all. These two are same thing as this right and if you <br> put this to make it to a three you put this on this and to <br> make this one a three you put a red on this and that <br> wouldn't be a duplicate. |
| 540. | 00:16:37 | R2 | So you are telling me when you have this one. To <br> make it one higher you can make it one higher by <br> putting a red on it or by putting a yellow on it? |
| 541. |  | Milin: | Yes. |
| 542. |  | R2 | Does that make sense? |
| 543. |  | R2 | Milin: | | Okay, now I see how you got these. But you only |
| :--- |
| have four but you told me there is going to be more |
| than four. |, | Yeah. |
| :--- |
| 544. |

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| 555. | R2 | Right I know that |
| :---: | :---: | :---: |
| 556. | Milin: | Every single time you have two off of each one by putting different color. |
| 557. | Michelle: | I think you times it by the height no wait, that didn't work. No, never mind. |
| 558. | Milin: | See these two you multiply that by two you get these two or you could- |
| 559. | Michelle: | You multiply by two because there is two colors of them and you could add two colors on top of each one. |
| 560. | R2 | Oh I see, so in other words if I start with this yellow by making it higher the only way I can make it higher was either to put a red that's one choice- |
| 561. | Michelle: | Yeah |
| 562. | R2 | -or a yellow that's a second choice- |
| 563. | Michelle: | Yeah |
| 564. | R2 | -so that's how I get two from this one |
| 565. | Michelle: | Yeah |
| 566. | R2 | and then I get two from this one |
| 567. | Michelle: | Yeah and all the other we got eight out of that so because you could add the, this one is hard to explain. |
| 568. | R2 | Let me understand this one let me see now this one here I am a little confused here which is the one that came from this |
| 569. | Milin: | This and this |
| 570. | R2 | But that doesn't make sense to me cause this one has yellow on the bottom and a red on the next one right? |
| 571. | Michelle: | it is confusing |
| 572. | R2 | Does it make sense to you? You should have a yellow |

\(\left.$$
\begin{array}{|l|l|l|}\hline & & \\
\hline \text { 573. } & & \text { and a red. } \\
\hline \text { 574. } & & \text { R2 } \\
\hline \text { 575. } & & \text { Michelle: } \\
\hline \text { Oh that makes sense. } & \begin{array}{l}\text { This this should be like this yellow/red/and then } \\
\text { yellow. Cause then you will have from the yellow and } \\
\text { the red and you already added the red }\end{array}
$$ <br>
\hline And you have two yellows you added the yellow and <br>
you added the red cause those are the two colors. <br>
From this you added a red and a yellow and From this <br>

you added a red and a yellow.\end{array}\right\}\)| 577. |
| :--- |


| 587. |  | R2 |
| :--- | :--- | :--- |
| 588. |  | Silin: |
| 589. |  | She should. |
| 590. |  | Milin: | | We could ask her to come over here and look. We |
| :--- |
| could invite her over. Stephanie you want to come this? I don't know |
| over here and Matt. |, | If shembers this from last year |
| :--- |
| 591. |


| Line | Time |  | Transcript |
| :--- | :--- | :--- | :--- |
| 605. |  |  | That's neat! I wonder if you could make it into a tree <br> that would be a nice way to do it too. It will be really <br> nice to do a tree who would like to make a tree for <br> this? Cause I like you to share it with the rest of the <br> class. |
| 606. |  | Stephanie | I will do it I guess |


| 622. |  | Michelle: | I think we did build a tree. |
| :---: | :---: | :---: | :---: |
| 623. |  | Milin: | No we didn't build a tree its like um sooner or later we just got into groups Stephanie, you |
| 624. |  | Michelle: | Yeah like they were like out of the classes they pulled me- |
| 625. |  | Matt | Its' like this- |
| 626. |  | Michelle: | -Stephanie, Milin and Matt Aaron uh Jeff. |
| 627. |  | Matt | These are the parents- |
| 628. |  | R2 | Okay. |
| 629. |  | Matt | -children, their children |
| 630. |  | R2 | Oh! Show me. That's neat. These are the parents, okay. |
| 631. | 1:22:19 | Milin: | Children get bigger! |
| 632. |  | R2 | Like a family tree. That's very nice! |
| 633. |  | All | Children, and their children, and their children's children and children's children's children. |
| 634. |  | R2 | I have a problem like that for you to do very famous problem I will bring it in next time. Okay. That's very nice. |
| 635. |  | Matt | Their children, their children |
| 636. |  | R2 | Okay, now what I really hope what you would do is if R1 would let us |
| 637. |  | Michelle: | Why don't we just explain it like this and they can figure out the answer for the next one. (Michelle is talking about towers up to three tall) |
| 638. |  | Milin: | Yeah! |
| 639. | 1:22:45 | R2: | Why don't you do four make them do five. |


| 640. |  | Milin: | That would be good |
| :---: | :---: | :---: | :---: |
| 641. |  | Matt: | Yes! |
| 642. |  | Michelle: | We will do up to four and they will do five |
| 643. |  | R2 | That sounds fair what do you think? |
| 644. |  | Stephanie | Yeah |
| 645. |  | Milin: | Yes, yes |
| 646. |  | R2 | You can get ready to explain |
| 647. |  | Michelle: | Can four of us explain this? |
| 648. |  | R2 | Sure, but I would like Matt to take leadership in this because it is a new thing for Matt that's reasonable and I think you all can help him with that then. |
| 649. |  | Stephanie | Yeah Matty |
| 650. |  | Matt | Yes we will |
| 651. |  | R2 | Okay I will leave you to fill the answer I think so. I will leave you for a minute I am gonna see if R1 is going to let us share some of these ideas. |
| 652. | 1:23:14 | Michelle: | Okay. Let's leave it like this. This adds on to this right? |
| 653. |  | Milin: | Yeah. Let's keep it in separate- |
| 654. |  | Michelle: | We are going to have to spread it out cause this is getting a little bit big. Why don't we start from the beginning and split it out. |
| 655. |  | Milin: | Yeah |
| 656. |  | Michelle: | Good idea |
| 657. |  | Matt | This is the mom |
| 658. |  | Michelle: | So we would have more room |
| 659. |  | Milin: | Let's just start From there |


| 660. |  | Michelle: | Matt has been playing with these things for too long. |
| :---: | :---: | :---: | :---: |
| 661. |  | Stephanie | Yeah (unintelligible) |
| 662. |  | Matt | That's the mommy that's the dad. Oh that's what we were missing the four. |
| 663. |  | Stephanie | No we have - no we don't |
| 664. |  | Michelle: | We have to go up to, we have to- |
| 665. |  | Stephanie | See I told you |
| 666. |  | Milin: | We are just going up to four |
| 667. |  | Michelle: | And they have to figure out up to five |
| 668. |  | Milin: | Yup. W already did that in fourth grade so they should know how. |
| 669. |  | Stephanie | I tried to explain to him but we kept goofing up. |
| 670. |  | Michelle: | Goofing up! Go language |
| 671. |  | Matt | One of the children died |
| 672. |  | Milin: | Laughs |
| 673. |  | Michelle: | You real nice Matt. |
| 674. | 1:24:24 | Stephanie | We didn't have as many combinations we had- we didn't have all the combinations that they had |
| 675. |  | Milin: | Here |
| 676. | 1:24:47 | Michelle: | Don't put it back there put it here so there is enough room |
| 677. |  | Milin: | How? You can't even see which one is which. |
| 678. |  | Matt | Maybe we will make- |
| 679. |  | Michelle: | Oh where does this go? Three there and that goes there. Oh this is confusing! This is four high here |


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| :---: | :---: | :---: | :---: |
| 680. |  | Milin: | Hey I got this one (towers fall and he laughs) |
| 681. |  | Michelle: | Nice move Mil! This is confusing okay, wait(Takes down all the towers) spread it out more |
| 682. |  | Milin: | Yeah |
| 683. |  | Michelle: | Really apart |
| 684. |  | Milin: | Uh! Spread it out his far |
| 685. |  | Michelle: | I don't think so Milin we don't have enough room on the table |
| 686. |  | Milin: | So? Use somebody else's table Matt and Stephanie aren't using theirs |
| 687. |  | Michelle: | Yeah sure pull the table right over here. (compares towers) No. where is the one that goes with this? |
| 688. |  | Milin: | That one |
| 689. |  | Michelle: | There we go! This goes there. That goes there. Where is the other one? |
| 690. | 1:25:50 | Milin: | Here |
| 691. |  | Michelle: | With this? |
| 692. |  | R1: | What are you doing? |
| 693. | 01:26:20 | Michelle: | We are trying to make a family tree. That one goes there. |
| 694. |  | Milin: | Get this out of the way |
| 695. |  | Michelle: | Yellow- red- yellow. Okay. |
| 696. |  | Matt | Put them close together |
| 697. |  | Michelle: | He likes playing with these things as if they are real people |
| 698. |  | Milin: | Hey! Make them far this how we are supposed to |
| 699. |  | Michelle: | Uh this one goes here |


| 700. |  | Milin: | Matt why would you start again from here? |
| :--- | :--- | :--- | :--- |
| 701. |  | Michelle: | Okay, this one, this one wait, this is confusing this one <br> goes here, um- |
| 702. |  | Milin: | This one goes here |
| 703. |  | Michelle: | this one goes with this one um |
| 704. |  | Milin: | Here it goes. Oh! Laughs as the towers fall |
| 705. |  | Michelle: | You are making everything fall today Milin |
| 706. |  | Michelle: | One this goes here two yellow one red one yellow we <br> didn't make that one yet. Two yellow one red one |
| 707. | $1: 27: 00$ | Michelle: | Oh. |
| yellow and then make here one yellow one red one |  |  |  |
| yellow one red |  |  |  |


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| :---: | :---: | :---: | :---: |
| 720. |  | Milin: | That goes uh that doesn't go there |
| 721. | 1:28:12 | Michelle: | This one goes here we already have that one um this one uh |
| 722. |  | R1: | What about this guy |
| 723. |  | Michelle: | It can - this one goes here with this one and then we need one with two |
| 724. |  | Milin: | Didn't we have- |
| 725. |  | Michelle: | This one goes here then we need one |
| 726. |  | Milin: | Yellow red |
| 727. |  | Michelle: | We need more |
| 728. |  | Milin: | Here |
| 729. |  | Michelle: | Uh and here is |
| 730. |  | Milin: | $1,2,3,4 \ldots \ldots .15,16$ we made sixteen! |
| 731. |  | R1: | I don't understand this one. |
| 732. |  | Milin: | Yeah I don't understand this one either |
| 733. |  | Michelle: | Oops! It's wrong. This goes here |
| 734. |  | R1: | That better yeah! |
| 735. |  | Milin: | We did it |
| 736. |  | R1: | What's the point? |
| 737. |  | Michelle: | We have no idea |
| 738. |  | Milin: | This is the box |
| 739. |  | Michelle: | She wanted us to make this so the- |
| 740. |  | Milin: | The kids get to try to find out |
| 741. |  | Michelle: | How many there are for five |


| 742. |  | Milin: | Yup |
| :---: | :---: | :---: | :---: |
| 743. |  | R1: | Oh! I see |
| 744. |  | Milin: | Except for |
| 745. |  | Michelle: | We are supposed to explain it to the class and Stephanie and Matt. |
| 746. |  | Milin: | Yeah |
| 747. |  | R1: | What do you mean? |
| 748. |  | Milin: | Cause Stephanie knew about this |
| 749. |  | Michelle: | See me, him, Stephanie and Jeff knew about this last year |
| 750. |  | R1: | Except this piece over here |
| 751. |  | Michelle: | I forgot about it |
| 752. |  | R1: | And you forgot? (R1 is asking Michelle) |
| 753. |  | Milin: | Yeah |
| 754. |  | Michelle: | Forgot about it. |
| 755. |  | Milin: | Stephanie couldn't get it for Matt because Matt couldn't like get it. |
| 756. |  | R1: | Cause yeah |
| 757. |  | Milin: | And then um the other person um one told Stephanie and Matt to come over here and then |
| 758. |  | R1: | Okay. I also want to know about your game. What did you all decide for the fours? Don't mess up these. |
| 759. |  | Milin: | Oh let's see |
| 760. |  | R1: | Under no circumstances mess up anything |
| 761. | 1:30:32 | Michelle: | Any duplicates in there? |


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| 762. |  | R1: | Okay so you are saying for number one |
| 763. |  | Michelle: | There are sixteen things in the next box. These are the sixteen that are in there |
| 764. |  | R1: | Okay. And so for number one |
| 765. |  | Michelle: | For number one this would be it and this would be it. |
| 766. |  | R1: | Okay, those two |
| 767. |  | Michelle: | Number two there is only one red cube, this, this |
| 768. |  | R1: | How many would there be? |
| 769. |  | Michelle: | Uh I think four. There is this, this, this and |
| 770. |  | R1: | Why? You don't have to show them to me. If you can convince me you don't have to show them to me. What would they look like? |
| 771. |  | Michelle: | Oh! See there's one red on the bottom, there is one red on the bottom one red on top of that one red on top of that |
| 772. |  | Milin: | Right here, right here this one |
| 773. |  | Michelle: | See we go from this to this |
| 774. |  | Milin: | These two |
| 775. |  | Michelle: | To this then to this these four. Then there is two red. Right here one two, three, four, five, six. Six. |
| 776. |  | R1: | So there is six there is two and two |
| 777. |  | Michelle: | Yeah and for number four one, two, three, four, five,oops six, seven, eight, nine, ten. We have better possibility is number four |
| 778. |  | R1: | Just count that one one more time |
| 779. |  | Both | one, two, three, four, five, six, seven, eight, nine, ten eleven |
| 780. |  | R1: | Okay, how many for number one? |


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| :---: | :---: | :---: | :---: |
| 781. |  | Michelle: | Two |
| 782. |  | R1: | For number two |
| 783. |  | Michelle: | Four |
| 784. |  | R1: | How many for number three? |
| 785. |  | Michelle: | Six and eleven |
| 786. |  | R1: | Okay. If I chose number four which ones would I loose? |
| 787. |  | Michelle: | This one |
| 788. |  | Milin: | One |
| 789. |  | Michelle: | This one |
| 790. |  | Milin: | Two |
| 791. |  | Michelle: | This one |
| 792. |  | Milin: | Three |
| 793. |  | Both | Four |
| 794. |  | Both | Five. |
| 795. | 1:32:59 | Milin: | Five of them |
| 796. |  | Michelle: | Yeah five |
| 797. |  | R1: | Does that make sense |
| 798. |  | Michelle: | Yeah. |
| 799. |  | Milin: | Not that much but |
| 800. |  | Michelle: | You would pick number four and there is only five you would miss |
| 801. |  | R1: | Okay. You are ---------- with number four then |
| 802. |  | Michelle: | Yeah |


| 803. |  | Milin: | Four, four, four |
| :---: | :---: | :---: | :---: |
| 804. |  | R1: | Okay if I am gonna ask you which you rather play the four game or the three game? |
| 805. |  | Both | Four game, four game |
| 806. |  | Milin: | Because you have a better chance |
| 807. |  | Michelle: | Because you have more than a fifty-fifty chance |
| 808. |  | Milin: | Yeah you have a seventy thirty chance |
| 809. |  | Michelle: | More than a fifty-fifty chance |
| 810. | 1:33:41 | R1: | What do you think if I had another big box and it had towers of five in it? |
| 811. |  | Milin: | No, no, no, we are not doing that- |
| 812. |  | Michelle: | She said |
| 813. |  | R1: | No what I am asking you is do you think, what do you think about the chances? would you go for number one if you had- |
| 814. |  | Milin: | I think Four, four, double that. |
| 815. |  | Michelle: | Four again, I think I would pick four again. |
| 816. | 01:34:02 | R1: | Why? |
| 817. |  | Milin: | Because this one would be in with it, this one, this one, every single one but it is going to be like about doubled |
| 818. |  | R1: | Why? |
| 819. |  | Milin: | Because, you know there is eleven here that you could win with right? |
| 820. |  | R1: | Uh huh. |
| 821. |  | Milin: | But then- |


| 822. | 01:34:20 | R1: | How many are there for towers of five? |
| :---: | :---: | :---: | :---: |
| 823. |  | Milin: | Thirty six but don't tell anybody. |
| 824. |  | R1: | Thirty six? |
| 825. |  | Milin: | Uh huh (shakes his head) but don't tell anybody about that. |
| 826. |  | R1: | Is that right Michelle? |
| 827. |  | Michelle: | Shakes her head in agreement. |
| 828. |  | Milin: | We found that out last year in class |
| 829. |  | R1: | Can you tell me how you figured that out? |
| 830. |  | Milin: | Well, first time we figured it out we were struggling |
| 831. |  | R1: | I understand. |
| 832. |  | Milin: | For two hours to make |
| 833. |  | R1: | Yeah. |
| 834. |  | Milin: | Stuff out of five, five, five |
| 835. |  | R1: | Uh huh |
| 836. |  | Milin: | I don't know that's about it. |
| 837. |  | R1: | Okay, What I want to know is how did you do it from here? (R1 points to the towers four tall on the table) |
| 838. | 01:34:50 | Milin: | From here? |
| 839. |  | R1: | I mean could you make towers of five now? |
| 840. |  | Milin: | Yeah but |
| 841. |  | R1: | How? With your family here is there some way you could make towers of five? |
| 842. | 01:35:00 | Milin: | Yeah. |
| 843. |  | Michelle: | O yeah you could just add on (Michelle points to the |


|  |  |  | last row of the family tree with four tall towers in a sweeping motion) |
| :---: | :---: | :---: | :---: |
| 844. |  | Milin: | Yeah. |
| 845. |  | Michelle: | Add on and on |
| 846. |  | Milin: | Yeah, yeah. |
| 847. |  | R1: | What do you mean add on and on? For instance just show me how you would start. |
| 848. | 01:35:07 | Milin: | Two for that one. Two for like for this one we would put like this (Milin stands a tower five tall with red bottom and four yellows on top next to the four tall tower with red bottom and three yellows on the top). Or |
| 849. |  | R1: | Yeah |
| 850. |  | Milin: | You could put |
| 851. |  | R1: | Here's a bottom and a top |
| 852. | 01:35:20 | Milin: | Three, you could make these two off of this one (he has made the five tall with red added on top of the four tall tower with red bottom and three yellows) |
| 853. |  | R1: | Okay. |
| 854. |  | Michelle: | You would add a |
| 855. |  | R1: | You have made these off of this one? |
| 856. |  | Milin: | Yeah. |
| 857. |  | R1: | Okay then I want to know- |
| 858. |  | Milin: | And we could do it for all of them. (Milin points to the rest of the four tall towers in front of him). |
| 859. |  | R1: | Okay then tell me how much would that be? |
| 860. |  | Milin: | Thirty six! |
| 861. |  | R1: | Show me. |


| 862. |  | Milin: | Show you? |
| :---: | :---: | :---: | :---: |
| 863. |  | Michelle: | Show you? |
| 864. |  | Milin: | Show you? |
| 865. |  | R1: | No, no, just count them. |
| 866. | 01:35:41 | Milin: | Okay |
| 867. |  | Michelle: | So there would be two |
| 868. |  | Milin: /Michelle: | Two, four, six, eight, ten, twelve, fourteen, sixteen, eighteen, twenty, twenty-two, twenty-four, twenty-six, twenty-eight, thirty, thirty-two |
| 869. |  | Milin: | Thirty two? |
| 870. |  | Michelle: | Laughs. Six- wait where did the pencil go? I mean- |
| 871. |  | Milin: | One, Two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen. Sixteen times two oh yeah thirty-two. |
| 872. |  | Michelle: | (She is using paper and pencil) thirty-two. Because sixteen and sixteen is thirty two. |
| 873. | 01:36:25 | Milin: | Yeah. |
| 874. |  | R1: | You your idea was right. |
| 875. |  | Michelle: | You calculated wrong. |
| 876. |  | Milin: | Yeah. |
| 877. |  | R1: | But you need to change your answer then right? |
| 878. |  | Milin: | Yeah. |
| 879. |  | R1: | Okay then, so you are right. Make this, the two that were over here so that I can remember what you said. Okay, now so what you are saying is that all eleven that were winners before |
| 880. |  | Milin: | Yeah |


| 881. |  | R1: | Are still winners. |
| :---: | :---: | :---: | :---: |
| 882. |  | Milin: | Plus like um like this, this |
| 883. |  | R1: | Tell me which ones |
| 884. |  | Milin: | This one is going to be a winner also (Milin points to red/yellow/red/red) |
| 885. |  | R1: | This one? |
| 886. |  | Milin: | Yeah because you will put a |
| 887. |  | Michelle: | If you pick number four it won't be |
| 888. |  | R1: | Yeah because what you are gonna- |
| 889. |  | Milin: | Yeah it would because you are going to put a red and a yellow on top of that. |
| 890. |  | Michelle: | Oh! Oh cause you could put a yellow on top of it. |
| 891. |  | R1: | Okay. So that would make another winner. Would it make two more winners or just one? |
| 892. |  | Milin: | One. |
| 893. | 01:37:02 | Michelle: | One. |
| 894. |  | R1: | Okay. Okay. And so this one would be how many winners coming from here? |
| 895. |  | Milin: | Two |
| 896. |  | R1: | This would, leave that out, |
| 897. |  | Michelle: | No wait one. |
| 898. |  | R1: | without building them |
| 899. |  | Michelle: | There would be two winners here cause there is already- |
| 900. |  | R1: | Okay, there is two winners here |

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| 901. |  | Milin: | Yeah. |
| :---: | :---: | :---: | :---: |
| 902. |  | R1: | Two winners here |
| 903. |  | Michelle: | One winner here |
| 904. |  | R1: | Can somebody keep some record? |
| 905. |  | Michelle: | Okay, two winners here, two winners here, one winner there |
| 906. |  | Milin: | Two winners |
| 907. |  | R1: | Wait |
| 908. | 01:37:39 | Milin: | One winner here. |
| 909. |  | R1: | Wait just a minute, how many here? |
| 910. |  | Milin: | Two |
| 911. |  | R1: | Okay did you get that down? |
| 912. |  | Michelle: | Uh huh. |
| 913. |  | Milin: | One |
| 914. |  | Michelle: | Yeah |
| 915. | 01:37:47 | Milin: | Two, two |
| 916. |  | Michelle: | Zero |
| 917. |  | Milin: | Zero, no, no, one, one |
| 918. |  | Michelle: | Zero, cause you would add one yellow |
| 919. |  | Milin: | Oh yeah, yeah. This one would be one. |
| 920. |  | Michelle: | One |
| 921. |  | Milin:/Mic helle: | two |
| 922. |  | Michelle: | One |
| 923. |  | Milin: | Two, two, two, two, two |


| 924. |  | Michelle: | Two, wait |
| :---: | :---: | :---: | :---: |
| 925. |  | R1: | How many we got? |
| 926. |  | Michelle: | One, two, three, four |
| 927. |  | R1: | No, count 'em, count 'em. Don't count them by one. |
| 928. |  | Michelle: | I am missing a number cause I only have $1,2,3, . .14$, 15 |
| 929. |  | R1: | Oh what about this one? (rl is holding a solid red tower) |
| 930. |  | Michelle: | Oh yeah. That's one. |
| 931. |  | R1: | So it gave you zero you might put a zero down. So that you make sure- |
| 932. |  | Michelle: | 1,2, 3, 4four 8 ten wait |
| 933. |  | Milin: | Nah, give me that I'll just use the calculator |
| 934. |  | R1: | Don't knock down your |
| 935. |  | Michelle: | Wait. 2, 4, 6, 8, 10 |
| 936. |  | Milin: | $1,2,3, . .10,11$. |
| 937. |  | R1: | You didn't count that zero did you? |
| 938. | 01:39:05 | Milin: | Eleven times two twenty two, 23, 24, 25, 26 |
| 939. |  | R1: | So you think there'd be twenty six? Let's try, let's check it again. Counting you can can say two |
| 940. |  | Both | $2,4,6,8,0,12,14,16,18,20,22,23,24,25,26$. |
| 941. | 01:39:30 | R1: | Write that down before you forget it. okay, so there is twenty six winners. How many losers? |
| 942. |  | Michelle: | Um. |
| 943. |  | Milin: | Wait (Uses his watch calculator) |


| Line | Time |  | Transcript |
| :--- | :--- | :--- | :--- |
| 944. |  | Michelle: | Out of thirty two, twenty six (Michelle is using paper <br> and pencil) |
| 945. |  | Milin: | Six. |
| 946. |  | R1: | Okay, let's see if we can find them. One of them <br> would be here you say? |
| 947. |  | Milin: | No. |
| 948. |  | Michelle: | No, cause there is two already there. One of them <br> would be here (r/y/r/r) |
| 949. |  | R1: | Oh one of them would be off this guy. (R1 points to <br> r/y/r/r) |
| 950. |  | Milin: | Yeah |
| 951. |  | R1: | One of them would be here (R1 points to r/r/y/r) |


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| 964. |  | Milin: | Yeah |
| 965. |  | Michelle: | I think so. |
| 966. |  | R1: | For three how many winners were there? |
| 967. |  | Michelle: | 1,2, 3, 4 (Michelle counts) |
| 968. |  | R1: | Okay and how many losers? |
| 969. |  | Michelle: | Four |
| 970. |  | R1: | Can you write that down? For three |
| 971. |  | Michelle: | For three |
| 972. |  | R1: | There were four winners |
| 973. |  | Michelle: | Four winners |
| 974. |  | R1: | And four losers. For five |
| 975. |  | Milin: | For five, twenty six |
| 976. |  | R1: | You told me there were eleven winners |
| 977. |  | Milin: | For four there were eleven. |
| 978. |  | R1: | For four there were eleven winners and how many losers? |
| 979. |  | Michelle: | All together there were sixteen, five losers. |
| 980. |  | Milin: | Five |
| 981. |  | R1: | Okay so there were four losers for three and five losers for four. |
| 982. |  | Michelle: | Five. |
| 983. |  | Milin: | For five there were |
| 984. |  | Michelle: | Twenty six minus |
| 985. |  | Milin: | Six losers |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 986. |  | R1: | What do you get for six? |
| 987. |  | Milin: | Uh, six |
| 988. |  | Michelle: | Twenty six and twenty six |
| 989. |  | R1: | What do you- |
| 990. |  | Milin: | Thirty-two, sixty-two no sixty-four. |
| 991. |  | R1: | There is sixty-four towers? |
| 992. |  | Michelle: | Fifty two |
| 993. |  | R1: | How many towers were there in all? |
| 994. |  | Milin: | Winner? |
| 995. |  | R1: | No, any towers for six. |
| 996. |  | Michelle: | Fifty two, cause twenty six- |
| 997. |  | Milin: | Hey but it is not twenty six it's thirty two |
| 998. |  | Michelle: | Oh! Dart I am wrong. So it's thirty two, thirty two sixty four |
| 999. |  | Milin: | What time are we leaving? |
| 1000. |  | Michelle: | Two o'clock |
| 1001. | 01:42:17 | R1: | We just did. But how many losers do you think there'd be for six? |
| 1002. |  | Michelle: | Oh. |
| 1003. |  | Milin: | Well there's gonna be more winners |
| 1004. |  | Michelle: | Oh! Seven losers. |
| 1005. |  | R1: | Why? |
| 1006. |  | Michelle: | Because every time- |
| 1007. |  | Milin: | Yeah (Milin stands up) |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 1008. |  | Michelle: | It's one less four losers, |
| 1009. |  | Both | five losers, six losers |
| 1010. |  | Milin: | Seven losers. |
| 1011. |  | Michelle: | Eight losers, nine losers, ten losers |
| 1012. |  | R1: | So okay if there is seven losers |
| 1013. |  | Michelle: | How many winners? There's- |
| 1014. |  | Milin: | Which one is it? sixty four (uses his calculator watch) |
| 1015. |  | Michelle: | Sixty four minus |
| 1016. |  | Milin: | Sixty |
| 1017. |  | Michelle: | Sixty four minus seven |
| 1018. |  | Milin: | Fifty-seven |
| 1019. |  | Michelle: | Fifty-seven, fifty-seven winners |
| 1020. | 01:43:08 | R1: | You think that's gonna be working? |
| 1021. |  | Both | Yeah. |
| 1022. |  | Michelle: | I don't know why. |
| 1023. |  | Milin: | I like the pattern before if we got this each thing and now we are finding out pattern of this |
| 1024. |  | Michelle: | Pattern of how may losers. Yeah so you want it for five for every height you add another loser |
| 1025. |  | Milin: | Yeah. For every height you add one more loser. |
| 1026. |  | R1: | Can you explain that to Dr. Maher? |
| 1027. |  | Both | Sure. |
| 1028. |  | R1: | Dr. Maher we have a new theory that I am not absolutely sure we believe or |
| 1029. |  | R2 | Dr Kelly would you |

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| 1030. |  | Milin: |
| :--- | :--- | :--- |
| 1031. |  | Laughs |
| 1032. |  | R2: | | Got to wake him up to get him |
| :--- |
| 1033. |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 1047. |  | Milin: | Three high? That ' d be this (Milin pulls out his paper for three high)t. |
| 1048. |  | R2 | There are eight choices. Write out eight choices. Oh some of those choices have we are talking about yellows. No yellows so write no yellows |
| 1049. |  | Milin: | What? |
| 1050. |  | R2 | Just make a new chart for me. No yellow is your choices. Let's make a column for no yellows |
| 1051. |  | Milin: | No yellows? Oh okay. |
| 1052. |  | R2 | There are no yellows, or there may be is going to be one yellow exactly lets make a new column |
| 1053. |  | Milin: | No yellow |
| 1054. |  | R2 | There is going to be one yellow next so put exactly one yellow right? Now we are going to do exactly two yellow, right? Or there is going to be exactly three yellow, right? |
| 1055. |  | Milin: | Yeah. |
| 1056. |  | R2 | You told me there are going to be how many all together? This is going to be how many? |
| 1057. |  | Milin: | Eight. |
| 1058. |  | R2 | How many are there with no yellow? |
| 1059. | 01:46:00 | Milin: | No yellow? |
| 1060. |  | R2 | How many towers? |
| 1061. |  | Milin: | There - |
| 1062. |  | R2 | Close your eyes and imagine how many do you have with no yellows? |
| 1063. |  | Milin: | One |
| 1064. |  | Michelle: | One |


| 1065. | R2 | Which one? |
| :---: | :---: | :---: |
| 1066. | Milin: | All red. |
| 1067. | Michelle: | One with all red. |
| 1068. | R2 | So that one could also be all red. ( $R 2$ points to the column heading with no yellows) right? So let's just write one. You can even write exactly which one it is. Okay, how many with exactly one yellow? |
| 1069. | Milin: | Michelle? |
| 1070. | R2 | Close your eyes and imagine. |
| 1071. | Milin: | Huh! |
| 1072. | Michelle: | Two. |
| 1073. | Milin: | Two |
| 1074. | Michelle: | Three. |
| 1075. | Milin: | Three? |
| 1076. | R2 | Now you're confusing me. |
| 1077. | Milin: | Three. |
| 1078. | R2 | Why is it three? |
| 1079. | Milin: | 1,2,3. (Milin points to the three tall towers in front of them) |
| 1080. | Michelle: | One could be on the top one could be on the bottom an one could be in the center. |
| 1081. | Milin: | Yeah. |
| 1082. | R2 | No other possibilities |
| 1083. | Michelle: | Na huh! |
| 1084. | R2 | You are absolutely sure of that? |
| 1085. | Michelle: | Yeah |


| 1086. |  | R2 | What about with two yellows? |
| :---: | :---: | :---: | :---: |
| 1087. | 01:46:48 | Milin: | Two yellows? Three. |
| 1088. |  | R2 | Why three? |
| 1089. |  | Milin: | Okay, one could be on the top one could be on the bottom and one- |
| 1090. |  | Michelle: | Two could be on the top two could be on the bottom and |
| 1091. |  | Milin: | Oops! |
| 1092. |  | Michelle: | And one could be on the top and one could be on the bottom. |
| 1093. |  | Milin: | Yeah. |
| 1094. |  | Michelle: | Like two top two bottom and one-(Michelle picks up the three towers one ta time) |
| 1095. |  | R2 | Oh that's where the red changes position. Okay, how many of those? |
| 1096. |  | Milin: | Three. |
| 1097. |  | R2 | How many for three yellow? |
| 1098. |  | Milin: | One |
| 1099. |  | R2 | I believe you now. I see eight and you have convinced me you have considered all possibilities. |
| 1100. |  | Milin: | Uh huh. |
| 1101. | 01:47:16 | R2 | You showed it to me the other way. Okay so now you are telling me fifty- fifty change of at least two are yellows. Which ones of these would be at least two yellow? |
| 1102. |  | Milin: | This and this |
| 1103. |  | R2 | Oh! Okay. So that's how many? |


| 1104. |  | Michelle: |
| :--- | :--- | :--- |
| 1105. | There are four |  |
| 1106. | Milin: | Yeah |
| 1107. |  | R2 |
| 1108. | Okay so four out of how many/ | Eight. That's fifty-fifty chance |
| 1109. | R2 | I believe you. Go on. Do you believe it Dr. Kelly? |
| 1110. |  | Milin: | | (laughs) If he heard it. |
| :--- |
| 1111. |


|  |  |  | you going to leave the three down here? |
| :---: | :---: | :---: | :---: |
| 1124. |  | Michelle: | Or we could put a marker |
| 1125. |  | R2 | Okay so this is four this is three this is two this is one and this is no. So what goes in there? |
| 1126. |  | Milin: | Some body has to hold this. (hands the pen to Dr. Maher) |
| 1127. |  | R2 | You can have this. |
| 1128. |  | Milin: | I'd rather use this. Okay, No yellow |
| 1129. |  | R2 | This would show better if you use this dark color. |
| 1130. | 01:49:13 | Michelle: | I'll use that if you don't want |
| 1131. |  | R2 | Okay. |
| 1132. |  | Milin: | This is a look at this see |
| 1133. |  | R2 | Why did you put a three there? Oh okay. |
| 1134. |  | Milin: | One yellow. |
| 1135. |  | R2 | Okay Michelle help me here for no yellow you said there's one |
| 1136. |  | Milin: | Yeah because they are all red it's the only one. |
| 1137. |  | Michelle: | Oh so for number one no yellow here there's one. |
| 1138. |  | R2 | Okay. |
| 1139. | 01:49:41 | Michelle: | For one yellow there's |
| 1140. |  | R2 | Close your eyes and tell me how many? |
| 1141. |  | Milin: | 1,2,3,4 (Counts fro the towers on the table) |
| 1142. |  | R2 | What were you imagining when you were telling me? |
| 1143. |  | Michelle: | One would be on the top, one would be on the bottom |
| 1144. |  | Milin: | Four yeah four |

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| 1145. |  | Michelle: | One would be right on top of the bottom and one would be right under the top. |
| :---: | :---: | :---: | :---: |
| 1146. |  | R2 | Fair enough. |
| 1147. | 01:50:09 | Milin: | Um, two yellows that'd be two yellows, ten |
| 1148. |  | Michelle: | 1,2, |
| 1149. |  | Milin: | Ten. Ten. |
| 1150. |  | Michelle: | Three |
| 1151. |  | Milin: | Na, nah |
| 1152. |  | Michelle: | Four |
| 1153. |  | Milin: | Four. |
| 1154. |  | Michelle: | $1,2,3,4$. (Michelle counts from the towers on the table) |
| 1155. |  | Milin: | Four. |
| 1156. |  | Michelle: | Five. |
| 1157. |  | Milin: | Five? |
| 1158. | 01:50:31 | Michelle: | 1,2,3, 4, 5 |
| 1159. |  | Milin: | Five. Four yellows that would be one and for three yellows that'd be one, three yellows three |
| 1160. |  | Michelle: | For four yellows one. |
| 1161. |  | Milin: | Three |
| 1162. |  | Michelle: | Where are you getting three from? |
| 1163. |  | Milin: | Nah, six um, five. |
| 1164. |  | Michelle: | Uh uh. |
| 1165. |  | Milin: | Laughs |


| 1166. |  | Michelle: | Where did the five come from? |
| :---: | :---: | :---: | :---: |
| 1167. |  | R2 | You have to show me where you got the five from? |
| 1168. |  | Milin: | Okay |
| 1169. |  | Michelle: | Where did you get the five? |
| 1170. |  | Milin: | Well because we said there will be eleven winners. Six, eleven |
| 1171. |  | R2 | But that's assuming these are right |
| 1172. |  | Milin: | Cause you already counted. |
| 1173. |  | R2 | The only one that you convinced me are the first two. |
| 1174. |  | Milin: | Okay. |
| 1175. |  | R2 | When you wrote exactly two yellow five you didn't convince me of that the four yellow one I think I can be convinced of that. |
| 1176. |  | Milin: | Okay. There are four yellows we already convinced that. |
| 1177. | 01:51:43 | R2 | How did you convince me of that? |
| 1178. |  | Milin: | One on the bottom one on the second- |
| 1179. |  | Michelle: | One on the bottom one on top of that one on top of that one on top of that. |
| 1180. |  | R2 | Okay you've convinced me |
| 1181. |  | Milin: | Two yellow. Two on the bottom |
| 1182. |  | Michelle: | One |
| 1183. |  | Milin: | Two in the middle |
| 1184. |  | Michelle: | Two, three |
| 1185. |  | Milin: | Two on the top |


| 1186. |  | Michelle: | Four |
| :---: | :---: | :---: | :---: |
| 1187. |  | Milin: | One like this and one like this. (uses his hands to express) uh, two on the bottom |
| 1188. |  | Michelle: | I just made everything fall. |
| 1189. |  | Milin: | Two on the bottom, two on the top uh, two in the middle um, what else? Two in the bottom, top, middle uh |
| 1190. |  | Michelle: | You can have two on the top two on the bottom, one in the middle one on the top one in middle one on the bottom, two in the middle |
| 1191. | 01:52:59 | Milin: | Count. (he gets up to count the towers they already had made on the table) take all these fives out |
| 1192. |  | Michelle: | Here's one, two |
| 1193. |  | Milin: | Hey put them back. Where do they go? Just go like this, $1,2,3,4,5,6$ |
| 1194. |  | Michelle: | Six |
| 1195. | 01:53:26 | Milin: | Six. |
| 1196. |  | R2 | Change your mind? |
| 1197. |  | Milin: | Yeah, six. |
| 1198. |  | R2 | Where did the other one come from? |
| 1199. |  | Michelle: | We didn't count this one (Michelle picks up $y / y / r / r$ ) |
| 1200. |  | R2 | Why is there a four for two yellows does that make sense? |
| 1201. |  | Milin: | Uh |
| 1202. |  | R2 | For exactly two yellow. You found four for exactly one yellow I am wondering why you found four for exactly three yellow does that make sense. Could you expect that to happen? |
| 1203. |  | Milin: | No, but it happened. |


| 1204. | R2 | Think about that for a minute. Think that should make sense. If that should be the same. |
| :---: | :---: | :---: |
| 1205. | Milin: | Yeah we ran out of time. Now we are going to have to wait all the way- |
| 1206. | Dr. Kelly | Did you come up with a rule here? |
| 1207. | Michelle: | Yeah. For every higher one there would be one more loser. |
| 1208. | Dr. Kelly | Uh huh. |
| 1209. | Michelle: | Cause we figured out how many winners and losers there'd be and for every one there was one more loser. Every one higher up there was one more loser. |
| 1210. | Dr. Kelly | Dr. Maher there is an interesting rule proposed here and I like to hear how it is explained. Tell us what you got there Michelle. It is predicting the number of losers if you if you are going to take at least two yellows. How does that work? |
| 1211. | Michelle: | Uh |
| 1212. | Milin: | I'll tell you for, for three there is four losers |
| 1213. | Michelle: | See we picked number four |
| 1214. | Milin: | For four there's five losers and for five there's six losers. Each on e had like one more each time |
| 1215. | R2 | One loser each time. Now here since you convinced me that's true. This last one was at least two yellow. |
| 1216. | Milin: | Yeah. |
| 1217. | R2 | So where's at least two yellow. That not two yellow that not one yellow. At least two yellow would be, this is two , this is four, this is three so how many did you get here? |
| 1218. | Milin: | For where |
| 1219. | Michelle: | Eleven |


| 1220. | Milin: | Yeah, eleven. |
| :---: | :---: | :---: |
| 1221. | R2 | So you got eleven? |
| 1222. | Michelle: | And five losers |
| 1223. | Milin: | Five losers |
| 1224. | R2 | So does it work? |
| 1225. | Milin: | Yeah |
| 1226. | R2 | So that was five losers gee so that is what you are predicting? |
| 1227. | Both | Uh huh |
| 1228. | R2 | So you are predicting that when you are building them six high you get how many? |
| 1229. | Milin: | All right (goes to his watch) |
| 1230. | R2 | What are you predicting here? Three, four, and five |
| 1231. | Milin: | You'll have nah, how many will we have thirty two |
| 1232. | Dr. Kelly | How many losers will you have? |
| 1233. | R2 | You are making towers six high how many losers will you predict if you continue this way? |
| 1234. | Milin: | Seven. |
| 1235. | Michelle: | Forty-two, fifty-two |
| 1236. | Dr. Kelly | How many losers? |
| 1237. | Milin: | Seven |
| 1238. | Michelle: | Seven. I don't know how many winners |
| 1239. | R2 | I am sure you could figure that out |
| 1240. | Milin: | Fifty-seven. We already figured that out |

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| 1241. | R2 | That's very interesting! Let's stop for a minute now. <br> Thank you for sharing that. |
| :--- | :--- | :--- | :--- |

10.10TRANSCRIPT - PRESENTATION ON 2/26/93

| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 1. | 00:05:00 | R2: | Hold on a second. And all come around this table because I want you to hear what she's doing here. This is really neat. |
| 2. |  | Stephanie: | I have... |
| 3. |  | R2: | Hold on a minute, Stephanie. Just hold on one second. |
| 4. |  | Stephanie: | Okay, I have |
| 5. |  | R2: | Wait |
| 6. |  | Student: | Hold on (laughter) |
| 7. |  | Teacher: | Can you take this seat? |
| 8. |  | Milin: | Yeah |
| 9. |  | R2: | Roger can tell us when he's done. Not yet, okay, go ahead. |
| 10. |  | Michelle: | We already know what you're talking about, okay explain. |
| 11. |  | R2: | Now make sure you guys understand what she's saying. |
| 12. |  | Stephanie: | Alright, I have one red, okay?And I have a yellow, and from each of these, you can make two because all you you have to do is you add on a r..., you can add on a red to the red or a yellow to the red. And for the yellow, you can add on a red to the yellow and a yellow to the yellow, okay? |
| 13. |  | Michelle: | So you don't have to look for duplicates. |
| 14. |  | Stephanie: | Then each one of these has two. Like, okay if this is a family tree, thi... the mother, the parents |
| 15. |  |  | Laughter |
| 16. |  | Stephanie: | Have kids and then, six kids, okay, well actually, no eight kids. Then they have eight kids, and each one of them has two kids. And this one, you can add one red one yellow, one yellow one red |


| Line | Time | Name | Transcript Manjit K. Sran |
| :---: | :---: | :---: | :---: |
| 17. |  | Student: | One red one yellow |
| 18. |  | Milin: | One red one yellow |
| 19. |  | Stephanie: | One red, one yellow |
| 20. |  | Student: | And you keep on going on and on and on |
| 21. |  | Stephanie: | Because each one of those... Devon! |
| 22. |  |  | Matt, you wanted to say something? |
| 23. |  | Stephanie: | And then here you can do the exact same thing |
| 24. |  | Milin: | You had a lot of fun with them, didn't you |
| 25. | 01:32:00 | R2: | Does that make sense, Amy? |
| 26. |  | Amy: | Yeah. |
| 27. |  | R2: | Now what did you guys do differently? |
| 28. |  | Student: | We got as much as we can with sixes |
| 29. |  | Student: | Right |
| 30. |  | R2: | And checking, checking for doubles ... |
| 31. |  | Student: | We got one and then we turned it upside down to make two |
| 32. |  | Student: | When we made one, We would make another. |
| 33. |  | R2: | Just by turning it upside down? |
| 34. |  | Student: | Yeah |
| 35. |  | R2: | So in other words, once you made all of them, you turned it upside down and made the other one and then you checked for duplicates? |
| 36. | 01:55:00 | Student: | Mm hm. |
| 37. |  | R2: | Did you get duplicates that way? |
| 38. |  | Student: | Yeah |


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| 39. |  | Student: | Some |
| 40. |  | Student: | We did sometimes |
| 41. |  | R2: | What did you think of their method? |
| 42. |  | Student: | You don't have to check for duplicates. |
| 43. | 02:04:00 | Stephanie: | That's what me and Dana used to do. We used to do that |
| 44. |  | R2: | This is how you used to do it? |
| 45. |  | Stephanie: | Me and my friend Dana, like the first time we ever did this problem, and we would line it out on the desk and then we went back and forth to check for doubles. |
| 46. |  | R2: | Do you think this is an easier way? |
| 47. |  | Stephanie: | Yeah |
| 48. |  | Student: | That's an easier way. |
| 49. |  | R2: | You think that's an easier way. Okay, now, how many of you are convinced that you would take choice number four, at least two cubes are yellow? Why do you think that's the case? Why do you think that's the way to go? Matt. |
| 50. | 02:30:00 | Matt | With number four there are are more there is more choices and if there's more choices, you're liable to get more to win. There's more of a chance of winning. |
| 51. |  | R2: | Okay, does anybody know what the chances are of picking at least two cubes yellow? Some one told me before. |
| 52. |  | Student: | For, for three? |
| 53. |  | R2: | Yeah, for towers three high. |
| 54. |  | Milin: | Yeah, um fifty-fifty. |
| 55. |  | Students: | fifty-fifty |
| 56. |  | R2: | You all agree with that? |


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| 57. |  | Students: | Yeah |
| 58. |  | R2: | Okay, what about what happens when you go to four high? |
| 59. | 02:57:00 | Student: | It's eleven sixteenths |
| 60. |  | Student: | It's eleven... |
| 61. |  | Student: | It's like sixty percent, something like that. |
| 62. |  | R2: | Okay, so are your chances better or worse when you get to the... |
| 63. | 03:03:00 | Students: | Better. |
| 64. |  | Student : | There's five blocks... |
| 65. |  | R2: | Chances are better? |
| 66. |  | Student: | No worse |
| 67. |  | Student: | Better when you get, um... |
| 68. |  | Student: | As you go higher you get |
| 69. |  | Student: | You get |
| 70. |  | Student: | Better |
| 71. | 03:10:00 | R2: | You get better. Okay, why do you think that happens? |
| 72. |  | Milin: | There are chances, but |
| 73. |  | Student: | Because there's more, because there's more blocks |
| 74. |  | Student: | They make more blocks... |
| 75. |  | Student: | There's more possible answers |
| 76. |  | R2: | But, there are more possible red? |
| 77. |  | Student: | As you go higher, one, there's only one answer that will stop you and... |

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