SLIDING MODE CONTROL FOR SYSTEMS WITH SLOW AND FAST MODES

by

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ABSTRACT OF THE DISSERTATION

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This dissertation addresses the problems of sliding mode control for systems with slow and fast dynamics. Sliding mode control is a type of variable structure control, where sliding surfaces or manifolds are designed such that system trajectories exhibit desirable properties when confined to these manifolds. A system using a sliding mode control strategy can display a robust performance against parametric and exogenous disturbances under the matching condition (Drazenovic's condition). This property is of extreme importance in practice where most systems are affected by parametric uncertainties and external disturbances.

First, we investigate a high gain output feedback sliding mode control problem for sampled-data systems with an unknown external disturbance. It is well-known that under high gain output feedback, a regular system can be brought into a singularly perturbed form with slow and fast dynamics. An output feedback based sliding surface is designed using some standard techniques for continuous-time systems. Next, we construct a discrete-time output feedback sliding mode control law for the sliding surface. The main challenge in this work is the appearance of the external disturbance in the control law. A remedy is to approximate the disturbance by system information of the previous time sampling period. The synthesized control law is able to provide promising results with high robustness against the external disturbance, which is demonstrated by the bounds of the sliding mode and state variables. These characteristics are further

improved by a method which takes into account system information of two previous time instants in order to better approximate the disturbance. The stability and robustness of the closed-loop system under the proposed control laws are analyzed by studying a transformed singularly perturbed discrete-time system.

The second topic of the thesis is to study sliding mode control for singularly perturbed systems which exhibit slow and fast dynamics. A state feedback control law is designed for either slow or fast modes. Then, the system under that state feedback control law is put into a triangular form. In the new coordinates, a sliding surface is constructed for the remaining modes using Utkin and Young's method. The sliding mode control law is synthesized by a control method which is an improved version of the unit control method by Utkin. Lastly, the proposed composite control law consisting of the state feedback control law and sliding mode control is realized. It is shown that stability and disturbance rejection are achieved. Our results show much improvement when compared to the other works available in the literature on the same problem.

The problem of sliding mode control for singularly perturbed systems is also addressed by the Lyapunov approaches. First, a state feedback composite control is designed to stabilize the system. Then, Lyapunov functions based on the state feedback control law and the system dynamics are employed in an effort to synthesize a sliding surface. Two sliding surfaces and two sliding mode controllers are proposed in this direction. Theoretical and simulation results show the effectiveness of the proposed methods. Like composite approaches, the Lyapunov ones provide asymptotic stability and disturbance rejection.

We also study singularly perturbed discrete-time systems with parametric uncertainty. Proceeding along the same lines as in the continuous-time case, we propose two approaches to construct a composite control law: a state feedback controller to stabilize either slow or fast modes and a sliding mode controller designed for the remaining modes. It is shown that the closed-loop system under the proposed control laws is asymptotically stable provided the perturbation parameter is small enough.

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Chapter 1

Introduction

1.1 Sliding Mode Control

Variable structure systems (VSS) are special structure systems that have been extensively studied for decades. The basic philosophy of the variable structure approach is that the structure of the system varies under certain conditions from one to another member of a set of admissible continuous time functions (Utkin, 1977). A VSS can inherit combined useful properties from the structures. In addition, it can be endowed with special properties which are not present in any of the structures (Utkin, 1977).

Sliding mode control is a type of variable structure control, where sliding surfaces or manifolds are designed such that system trajectories exhibit desirable properties as confined to these manifolds. A system using a sliding mode control strategy can display a robust performance against parametric uncertainties and exogenous disturbances. This property is of extreme importance in practice where most of plants are heavily affected by parametric and external disturbances.

Consider a general VSS described by

$$\dot{x}(t) = f(x(t), t, u(t)) \tag{1.1}$$

where $x(t) \in R^n$, and $u(t) \in R^m$. Each component of control is assumed to act in discontinuous fashions based on some appropriate conditions,

$$u_i(t) = \begin{cases} u_i^+(x,t) & \text{if} \quad s_i(x) > 0 \\ u_i^-(x,t) & \text{if} \quad s_i(x) < 0 \end{cases}, \quad i = 1, ..., m \tag{1.2}$$

where $s_i(x)$ plays the role of a sliding surface.

Since differential equations (1.1), (1.2) have discontinuous right hand sides, they do not meet the classical requirements on the existence and uniqueness of solutions. A formal technique called the equivalent control method was introduced by Utkin (1977) to analyze the equivalent dynamics of the closed-loop system. In this approach, a control called the equivalent control is obtained by solving $\dot{s}(t) = 0$, namely

$$\frac{ds}{dt} = \frac{\partial s}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial s}{\partial x} f(x, u^{eq}, t) = 0. \tag{1.3}$$

The system dynamics on the sliding surface is studied by substituting the equivalent control u^{eq} in equation (1.1). Note that the equivalent control u^{eq} is not physically realizable due to the unknown disturbances. Furthermore, the equivalent system dynamics is not exactly but very close to the sliding dynamics (Utkin, 1977).

The existence of the sliding mode is described by the following conditions (Utkin, 1978)

- Sliding condition (sufficient, local)

$$\lim_{s \to 0^+} \dot{s} < 0, \quad \lim_{s \to 0^-} \dot{s} > 0 \tag{1.4}$$

- Reaching condition (sufficient, global)

$$\dot{s} < -\sigma \text{sgn}(s), \tag{1.5}$$

  where $\sigma > 0$ is a parameter to be designed.

- Control magnitude constraint (necessary)

$$u_{min} \leq u^{eq} \leq u_{max} \tag{1.6}$$

In sliding mode control, a sliding surface is first constructed to meet existence conditions of the sliding mode. Then, a discontinuous control law is sought to drive the system state to the sliding surface in a finite time and stay thereafter on that surface.

We now present some fundamental designs of sliding mode control for regular linear systems. Consider a linear system

$$\dot{x}(t) = Ax(t) + Bu(t) + Df(t) \tag{1.7}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control, $f(t) \in R^r$ is the unknown but bounded exogenous disturbance $\|f\| \leq M$, with $m \leq p < n$. It is assumed that B is of full rank, (A, B) is controllable and the following matching condition (Drazenovic, 1969) is satisfied

$$\text{rank}[B|D] = \text{rank}D \tag{1.8}$$

In other words, D can be written as $D = BG$.

There are several systematic methods for constructing sliding surfaces, such that Utkin and Young's method (Utkin and Young, 1979), Lyapunov method (Gutman and Leitmann, 1976; Gutman, 1979; Su *et al.*, 1996). Utkin and Young's method and Lyapunov method to be presented below will be employed in solving our problem in the following chapters.

Since B has full rank, there exists a nonsingular transformation T such that

$$TB = \begin{bmatrix} 0_{(n-m)\times m} \\ B_0 \end{bmatrix}, \quad \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = Tx(t),$$

which bring (1.7) into the normal form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ B_0 G \end{bmatrix} f(t) \tag{1.9}$$

where $x_1(t) \in R^{n-m}$, $x_2(t) \in R^m$. Note that B_0 is an $m \times m$ matrix and it is nonsingular because B is of full rank.

Regard $x_2(t)$ as a control input to the first subsystem of (1.9)

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) \tag{1.10}$$

and construct a state feedback gain for (1.10) as

$$x_2(t) = -Kx_1(t) \tag{1.11}$$

Hence, the sliding surface in the (x_1, x_2) coordinate can be chosen as

$$[K \quad I_{m\times m}] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \tag{1.12}$$

or

$$s(t) = Cx(t) = [K \quad I_{m\times m}]Tx(t) = 0 \tag{1.13}$$

in the original coordinates. Once the sliding surface coefficient matrix C is designed, one can proceed to construct the sliding mode control law. Taking the derivative of (1.13) with respect to t, we have

$$\dot{s}(t) = C\dot{x}(t) = CAx(t) + CBu(t) + CDf(t) \tag{1.14}$$

It is important to emphasize that the matrix CB in (1.14) is an $m \times m$ nonsingular matrix equal to B_0 since

$$CB = [K \quad I]TB = [K \quad I]\begin{bmatrix} 0 \\ B_0 \end{bmatrix} = B_0. \tag{1.15}$$

A sliding mode control law can be designed by using the unit control method or the signum method (Utkin, 1984). From (1.15), a unit control law can be chosen as (Utkin, 1984)

$$u(t) = -(CB)^{-1}CAx - (CB)^{-1}(\gamma + \sigma)\frac{s}{\|s\|} \tag{1.16}$$

where

$$\gamma = \|CD\|M \tag{1.17}$$

is a value that helps to tackle disturbances. It can be shown that (1.16) satisfies the vector form of the reaching condition, that is

$$s^T\dot{s} = -\sigma\|s\| - \gamma\|s\| + s^TCDd(t) < -\gamma\|s\| \tag{1.18}$$

where γ is chosen as in (1.17) and σ is a design parameter for adjusting the reaching time. One can find the finite reaching time by considering the Lyapunov function

$$V = s^Ts \tag{1.19}$$

Taking its derivative, we have

$$\dot{V} < -2\gamma\|s\| = -2\gamma\sqrt{V} \tag{1.20}$$

This yields

$$\frac{dV}{\sqrt{V}} < -2\gamma dt. \tag{1.21}$$

Hence,

$$\sqrt{V(t)} - \sqrt{V(0)} < -t\gamma. \tag{1.22}$$

Let $t = T_r$ be the time to reach the sliding mode ($V(T_r) = 0$). Thus, the reaching time is bounded as

$$T_r < \frac{\sqrt{V(0)}}{\gamma} \tag{1.23}$$

A sliding surface can be designed using the Lyapunov approach (Su *et al.*, 1996). Since (A, B) is controllable, there exists a stabilizing feedback gain K such that $A_s = A+BK$ is asymptotically stable (Su *et al.*, 1996). Hence, there exists a positive-definite matrix P, which is the solution to the Lyapunov equation

$$PA_s + A_s^T P = -Q, \quad Q > 0. \tag{1.24}$$

A sliding surface is chosen as

$$s(t) = HB^T Px(t) = 0. \tag{1.25}$$

where H is an $m \times m$ nonsingular matrix. It was proven that the system (1.7) with the sliding mode on the sliding surface (1.25) is asymptotically stable (Su *et al.*, 1996). The idea of employing Lyapunov's second method to construct a sliding surface can be extended to nonlinear systems (Su *et al.*, 1996).

1.2 Singularly Perturbed Systems

Singularly perturbed systems are systems that possess small time constant, or similar "parasitic" parameters which usually are neglected due to simplified modeling. When taking into account those small quantities, the order of the model is increased and the computation needed for control design can be expensive and even ill-conditioned. However, if one uses a simplified model to design a control strategy, the desired performance may not be achieved or the system can be unstable. As a result, singular perturbation methods have been developed for years to address the stability and robustness of those systems. For an extensive study, we refer to (Kokotovic *et al.*, 1986; Gajic and Lim, 2001).

Consider a linear singularly perturbed system without control

$$\begin{aligned} \dot{x}(t) &= A_{11}x(t) + A_{12}z(t), \quad x(t_0) = x_0 \\ \epsilon\dot{z}(t) &= A_{21}x(t) + A_{22}x(t), \quad z(t_0) = z_0, \end{aligned} \tag{1.26}$$

where $x(t)$ and $z(t)$ are respectively slow and fast state variables and ϵ is a small positive parameter.

To analyze the system (1.26), one common way is to use the Chang transformation to transform (1.26) into a block-diagonal system where the slow and fast dynamics are completely decoupled (Chang, 1972). The Chang transformation is represented by

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} I_1 & \epsilon H \\ -L & I_2 - \epsilon LH \end{bmatrix} \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} = T^{-1} \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} \tag{1.27}$$

and its inverse transformation is given by

$$\begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} I_1 - \epsilon HL & -\epsilon H \\ L & I_2 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = T \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}. \tag{1.28}$$

where L and H matrices satisfy algebraic equations

$$A_{21} - A_{22}L + \epsilon LA_{11} - \epsilon LA_{21}L = 0 \tag{1.29}$$

and

$$\epsilon(A_{11} - A_{12}L)H - H(A_{22} + \epsilon LA_{12}) + A_{12} = 0. \tag{1.30}$$

Matrices L and H can be found using several methods. For example, the Newton method is presented in (Grodt and Gajic, 1988). The resulting decoupled form is

$$\begin{bmatrix} \dot{\xi}(t) \\ \dot{\eta}(t) \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12}L & 0 \\ 0 & A_{22} + \epsilon LA_{12} \end{bmatrix} \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix}. \tag{1.31}$$

Now we present a short summary on the design of state feedback control for deterministic linear continuous time singularly perturbed systems. Consider the following controlled system

$$\begin{aligned} \dot{x}(t) &= A_{11}x(t) + A_{12}z(t) + B_1u(t), \quad x(t_0) = x_0 \\ \epsilon\dot{z}(t) &= A_{21}x(t) + A_{22}z(t) + B_2u(t), \quad z(t_0) = z_0. \end{aligned} \tag{1.32}$$

where $x(t) \in R^{n_1}$, $z(t) \in R^{n_2}$, and $u(t) \in R^m$. The system (1.33) is approximately decomposed into an n_1 dimensional slow subsystem and an n_2 fast subsystem by setting $\epsilon = 0$ in (1.32). The slow subsystem is

$$\begin{aligned} \dot{x}_s(t) &= A_sx_s(t) + B_su_s(t), \quad x_s(t_0) = x_0 \\ z_s(t) &= -A_{22}^{-1}(A_{21}x_s(t) + B_2u_s(t), \end{aligned} \tag{1.33}$$

where

$$A_s = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad B_s = B_1 - A_{12}A_{22}^{-1}B_2 \tag{1.34}$$

and the vectors $x_s(t)$, $z_s(t)$, and $u_s(t)$ denote the slow parts of $x(t)$, $z(t)$ and $u(t)$. The fast subsystem is

$$\epsilon\dot{z}_f(t) = A_{22}z_f(t) + B_2u_f(t), \quad z_f(t_0) = z_0 - z_s(t_0), \tag{1.35}$$

where $z_f(t) = z(t) - z_s(t)$, and $u_f(t) = u(t) - u_s(t)$ describe the fast parts of the corresponding variables $z(t)$ and $u(t)$. A composite control law consists of slow and fast parts as

$$u(t) = u_s(t) + u_f(t) \tag{1.36}$$

where $u_s(t) = G_0x_s(t)$, and $u_f(t) = G_2z_f(t)$ are independently constructed for the slow and fast subsystems (1.33) and (1.35). G_0 and G_2 can be designed by using classic control theory with an assumption that (A_s, B_s) and (A_{22}, B_2) is controllable. Nonetheless, a realizable control law must be presented in terms of the actual system states $x(t)$ and $z(t)$. Replacing $x_s(t)$ by $x(t)$ and $z_f(t)$ by $z(t) - z_f(t)$ bring the composite control (1.34) into the realizable feedback form as follows.

$$u(t) = G_0x(t) + G_2[z(t) + A_{22}^{-1}(A_{21}x(t) + B_2G_0x(t))] = G_1x(t) + G_2z(t) \tag{1.37}$$

where

$$G_1 = (I_1 + G_2A_{22}^{-1}B_2)G_0 + G_2A_{22}^{-1}A_{21}. \tag{1.38}$$

The discrete-time version of singularly perturbed systems is described in (Litkouhi and Khalil, 1985). Consider the difference equation

$$\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \begin{bmatrix} (I_1 + \epsilon A_{11}) & \epsilon A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + \begin{bmatrix} \epsilon B_1 \\ B_2 \end{bmatrix} u[k] \tag{1.39}$$

where $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$, $u \in R^m$, $\epsilon > 0$ is a small parameter and $\det[I_2 - A_{22}] \neq 0$. As in the continuous version, the slow and fast parts of equation (1.39) without control can be separated by a decoupling transformation (Litkouhi and Khalil, 1985). A control law which consists of slow and fast components is in the form

$$u[k] = u_s[k] + u_f[k]$$

where $u_f[k]$ decays exponentially. To investigate the slow subsystem, we neglect $u_f[k]$. The resulting equations are given by

$$\bar{x}_1[k+1] = (I_1 + A_{11})\bar{x}_1[k] + \epsilon A_{12}\bar{x}_2[k] + \epsilon B_1 u_s[k] \tag{1.40}$$

$$\bar{x}_2[k] = A_{21}\bar{x}_1[k] + A_{22}\bar{x}_2[k] + B_2 u_s[k]. \tag{1.41}$$

From (1.40) and (1.41), the slow subsystem is defined by

$$\bar{x}_s[k+1] = (I_1 + \epsilon A_s)\bar{x}_s[k] + \epsilon B_s u_s[k] \tag{1.42}$$

where

$$x_s = \bar{x}_1, \tag{1.43}$$

$$A_s = A_{11} + A_{12}(I_2 - A_{22})^{-1}A_{21}, \tag{1.44}$$

$$B_s = B_1 + A_{12}(I_2 - A_{22})^{-1}B_2. \tag{1.45}$$

If the pair (A_s, B_s) is stabilizable in the continuous sense, i.e., every eigenvalue of A_s which lies in the closed right-half complex plane is controllable (Litkouhi and Khalil, 1985), then a state feedback control law for $u_s[k]$ is designed as $u_s[k] = F_s x_s[k]$ where F_s is chosen such that

$$\mathrm{Re}\{\lambda(A_s + B_s F_s)\} < 0. \tag{1.46}$$

With this choice, the closed-loop slow subsystem system is asymptotically stable. This shows that the actual design problem for the discrete-time slow subsystem is a continuous one (Litkouhi and Khalil, 1985).

The fast subsystem is defined by assuming that the slow variables are constant during the fast transient, i.e., $\bar{x}[k+1] = \bar{x}[k]$, and $u_s[k+1] = u_s[k]$. From (1.41) and (1.39), the fast subsystem is given by

$$x_f[k+1] = A_f x_f[k] + B_f u_f[k] \tag{1.47}$$

where $x_f = x_2 - \bar{x}_2$, $A_f = A_{22}$, and $B_f = B_2$. If the pair (A_f, B_f) is stabilizable in the discrete-time sense, i.e., every eigenvalue of A_f which lies outside or on the unit circle is controllable (Litkouhi and Khalil, 1985), then a state feedback control law for $u_f[k]$ is given by $u_f[k] = F_f x_f[k]$ where F_f is chosen such that

$$|\lambda(A_f + B_f F_f)| < 1. \tag{1.48}$$

As a result, the closed-loop fast subsystem is asymptotically stable. A composite control law is taken as the sum of the slow and fast control components

$$\begin{aligned} u[k] &= F_s x_s[k] + F_f x_f[k] = F_s x_s[k] + F_f(x_2[k] - \bar{x}_2[k]) \\ &= [F_s - F_f(I_2 - A_f)^{-1}(A_{21} + B_f F_s)]x_1[k] + F_f x_2[k]. \end{aligned} \tag{1.49}$$

It was shown that under the composite control law (1.49), the closed-loop full-order system is asymptotically stable for sufficiently small ϵ. Like in the continuous case, a stabilizing state feedback control law can be synthesized from slow and fast controllers that are designed independently.

1.3 Literature on Relevant Works

It is pointed out that for a regular system, a sliding mode control strategy can reject disturbances and produce a robust performance. In systems with slow and fast modes, little work has been devoted to the study of sliding mode control (Yue and Xu, 1996; Su, 1999).

Yue and Xu (1996) studied a singularly perturbed system as follows

$$\begin{aligned} \dot{x}(t) &= A_{11}x(t) + A_{12}z(t) + B_1u(t) + B_1 f(t,x,z), \\ \epsilon\dot{z}(t) &= A_{21}x(t) + A_{22}z(t) + B_2u(t) + B_2 g(t,x,z), \end{aligned} \tag{1.50}$$

where $x(t) \in R^{n_1}$, $z(t) \in R^{n_2}$, and $u(t) \in R^m$. $f(t,x,z), \quad g(t,x,z) : R_+ \times R^{n_1} \times R^{n_2} \to R$ denote the parameter uncertainties and external disturbances. $0 < \epsilon < 1$ represents the singular perturbation parameter. Furthermore, the disturbances $f(t,x,z)$, $g(t,x,z)$ are assumed to satisfy the following inequalities:

$$\begin{aligned} |f(t,x,z)\| &\leq \rho_1(x,z) = a_0 + a_1\|x\| + a_2\|z\| \\ \|g(t,x,z\| &\leq \rho_2(x,z) = b_0 + b_1\|x\| + b_2\|z\|. \end{aligned}$$

In addition, they satisfy

$$\|f(t,x,z) - g(t,x,z)\| \leq \alpha\|x\| + \beta\|z\|.$$

In their approach, a designed control law includes two continuous time state feedback terms and a switching term. The objective of the two continuous-time terms is to stabilize the system as no disturbances are taken into account. Specifically, the control law is in the form

$$u = Kx + K_0\eta + w \tag{1.51}$$

where K and K_0 are designed such that $A_s + B_sK$ and $A_{22} + B_2K_0$ are stable, and w is a switching term to be defined. Here, η is a new state variable given by

$$\eta = z + A_{22}^{-1}(A_{21} + B_2K)x. \tag{1.52}$$

To choose w, Yue and Xu (1996) considered a Lyapunov function candidate as follows

$$V = x^T P_1 x + \epsilon\eta^T P_2\eta$$

where P_1, P_2 are positive definite solutions to the following Lyapunov equations

$$(A_s + B_sK)^T P_1 + P_1(A_s + B_sK) = -Q_1$$
$$(A_{22} + B_2K_0)^T P_2 + P_2(A_{22} + B_2K_0) = -Q_2.$$

A control law is chosen as

$$u = -Kx - K_0\eta - (\hat{b}_0 + \hat{b}_1^0\|x\| + \hat{b}_2\|\eta\|)\mathrm{sgn}(B_1^T P_1 x + B_2^T P_2\eta) \tag{1.53}$$

where $\hat{b}_0$, $\hat{b}_1^0$, and $\hat{b}_2$ are acquired from the definition of disturbances $f(t, x, z)$ and $g(t, x, z)$ and matrices A_{21}, A_{22}, K. With this control law, the trajectories x and η ultimately satisfy (Yue and Xu, 1996)

$$\|x\| \le O(\epsilon), \quad \|\eta\| \le O(\epsilon).$$

Yue and Xu (1996) also proved the existence of the sliding motion for the sliding surface $s = B_1^T P_1 x + B_2^T P_2\eta$ provided some condition are satisfied. They employed Lyapunov functions to construct the sliding function and the sliding mode control law. Although, their approach deals with disturbances and provides some certain robust characteristics, they only guarantee that the trajectories of the system stay in an $O(\epsilon)$ boundary of the origin. Furthermore, it is somewhat complicated to compute some parameters for their control law.

Heck (1991) studied a singularly perturbed system in the form of (1.50) without any disturbances. The full-order system is separated into slow and fast subsystems (1.33), (1.35). A sliding-mode controller is constructed for each subsystem. Specifically, a slow sliding surface can be chosen as

$$s_s(x_s) = S_s x_s = 0$$

and a fast sliding surface is taken as

$$s_f(z_f) = S_f z_f = 0.$$

Corresponding slow and fast control laws are designed for these sliding surfaces. A composition of these control law is then implemented for the full-order system. Stability analysis was carried out by using the equivalent method of Utkin (1977). It was shown that the reduced-order subsystems approximate the full-order system with accuracy $O(\epsilon)$. If reaching conditions are satisfied for the reduced-order models and an additional condition is met, the reaching conditions are satisfied for the full-order system (Heck, 1991). One draw back of Heck's approach is that the boundedness of the derivative of the slow sliding mode control must be supposed. Furthermore, parameter uncertainties and external disturbances were not taken into consideration. As a result, Heck's scheme is limited in real applications.

Li *et al.* (1995a) also considered a singularly perturbed system in the form of (1.32). Similarly to Heck's approach, the full-order system is first decomposed into slow and fast subsystems. Then, slow and fast sliding mode controllers are designed for the subsystems individually. The composite control consists of two terms

$$u = u_{eq} + \Delta u$$

where u_{eq} is the equivalent control for the full-order system and Δu is the switching term which is the regulating control moment for the full-order system. A sigmoid function is exploited to eliminate the chattering phenomenon. Although the switching surface of the full-order system is decided by the fast switching surface of the reduced-order system, when reverting back to the original coordinates, the slow sliding control still exists in the composite one. Like Heck's method, their approach does not address uncertainties and external disturbances.

Su (1999) studied the problem of sliding surface design for the system (1.32). Like (Heck, 1991; Li *et al.*, 1995a), the full-order system is separated into slow and fast subsystems. Then, stabilizing state feedback controllers are constructed individually for each subsystem, leading to a composite control law (1.36). The closed-loop system is transformed into an exact slow and an exact fast subsystems by using the Chang transformation (Chang, 1972; Kokotovic *et al.*, 1986). The exact subsystems in the new coordinates are

$$\dot{\xi} = T_s\xi$$
$$\epsilon\dot{\eta} = T_f\eta.$$

There exist positive definite matrices P_s and P_f such that

$$P_sT_s + T_s^TP_s = Q_s, \quad Q_s > 0$$
$$P_fT_f + T_f^TP_f = -Q_f, \quad Q_f > 0.$$

Then, the sliding surface for the singularly perturbed system can be chosen as

$$s(x,z) = \begin{bmatrix} B_1 \\ B_2/\epsilon \end{bmatrix}^T J^T \begin{bmatrix} P_s & 0 \\ 0 & \epsilon P_f \end{bmatrix} J \begin{bmatrix} x \\ z \end{bmatrix} = 0.$$

It was shown that (Su, 1999) if the sliding motion is achieved, the system is asymptotically stable. However, a control strategy has not been provided to realize the sliding motion.

There have been several approaches to deal with the problem of sliding mode control for singularly perturbed systems. Many of them (Heck, 1991; Li *et al.*, 1995a; Su, 1999) do not address parameter uncertainties and external disturbances which are inherent in many practical plants. Heck (1991); Li *et al.* (1995a) exploited the decomposition of the full-order system into slow and fast subsystems in an effort to construct sliding surfaces and sliding mode controls for each subsystems. Meanwhile, Su (1999) only took advantage of the structure of the closed-loop system under the composite control law, by which a sliding surface is designed. Although Yue and Xu (1996) dealt with parameter uncertainties and external disturbances, their method is not able to reject disturbances completely. Instead, the trajectories of the system are ensured in an $O(\epsilon)$

bound of the origin. One common feature of the above approaches is that Lyapunov functions are employed to construct sliding surfaces and sliding mode controls.

1.4 Contributions of the Dissertation

The contributions of the dissertation are summarized in the following:

- Two methods for design of an output feedback sliding mode control law for sampled-data systems are proposed in Chapter 2. The first method employs the one-step predictor ahead technique to approximate the external disturbance. The second method offers better way to approximate the disturbance by using system information from two previous time instants. The main feature of the proposed methods is that the control law is high gain, which leads to singular perturbation behavior of the closed-loop system. The characteristics of the closed-loop systems are much better than the other works available in the control literature. In addition, they capture the same level as in the state feedback case. The results in this topic have been presented at 2009 American Control Conference, 2010 American Control Conference, and published in IEEE Transactions on Automatic Control.

- The development of two composite design methods for a singularly perturbed system with external disturbances is used to design a sliding mode controller. A state feedback control law for either slow or fast modes combines with a sliding mode control law to constitute a composite control law. The closed-loop system under the proposed methods is asymptotically stable and robust against external disturbances in the sliding mode. This contribution has been accepted for publication in Dynamics of Continuous, Discrete and Impulsive Systems journal.

- Two approaches based on the Lyapunov equations are used to design a sliding mode controller for singularly perturbed systems with external disturbances. The results obtained are comparable to those of the composite control approaches; namely, disturbance rejection and asymptotic stability in the sliding mode are attained. These features show the advantages of the proposed methods when compared to other works available in the literature. This contribution is submitted for journal publication.

- The formulation and analysis of a sliding mode control problem for a discrete-time singularly perturbed systems is investigated. Two composite approaches are proposed to deal with the stability and robustness of a system with parametric uncertainties. A state feedback control law is constructed for either slow or fast modes. The remaining modes are handled by a sliding mode control law. A composite control law combining the two components provides the system with asymptotic stability and robustness against modeling uncertainties.

1.5 Organization of the Dissertation

Throughout the dissertation, we deal with different issues of sliding mode control for singularly perturbed systems: discrete- and continuous-time domains. Each chapter presents unified techniques or tools to address specific problems.

In Chapter 2, we formulate and develop an output feedback control strategy for sampled-data systems. Stability and robustness will be analyzed by using singular perturbation techniques. Furthermore, it will be theoretically shown that the same characteristics as in the state feedback case are maintained. At the end of the chapter, the numerical simulation of an aircraft model illustrates the advantages of the proposed method.

Chapter 3 presents two composite control approaches for the sliding mode control for singularly perturbed systems with external disturbances. We show how to design a sliding surface and a corresponding control law to effectively stabilize the system and reject the external disturbances. Two numerical examples at the end of the chapter demonstrate the effectiveness of the proposed methods.

The same problem is addressed in Chapter 4 by different approaches. We employ Lyapunov functions and the Chang transformation to construct a sliding surface and a sliding mode control law to stabilize the closed-loop system and reject external disturbances.

The techniques of Chapter 3 are applied to singularly perturbed discrete-time systems with parametric uncertainties in Chapter 5. Two composite sliding mode control strategies are presented to deal with the stability and robustness of the closed-loop system. A numerical example of a discrete-time model of a steam power system is provided to illustrate the efficacy of the methods.

Finally, Chapter 6 presents an overview of the results of the dissertation which is followed by some future directions.

Chapter 2

Output Feedback Sliding Mode Control for Sampled-Data Systems

2.1 Introduction

The problem of output feedback sliding mode control with disturbances has been extensively studied for years (Zak and Hui, 1993; EL-Khazali and DeCarlo, 1995; Heck *et al.*, 1995; Edwards and Spurgeon, 1995, 1998). It was pointed out that the problem of choosing the desired poles of the sliding mode dynamics can be approached by using the classical "squared-down" techniques (MacFarlane and Karcanias, 1976). In order to attain a global attraction to the sliding surface, Heck *et al.* (1995) established numerical methods that adjust the switching gain to compensate for the unknown state and disturbance variables. Edwards and Spurgeon (1995) proposed a procedure to construct a sliding surface based on the output information by taking advantage of the fact that the invariant zeros of a system appear in the dynamics of the sliding motion. The remaining eigenvalues of the sliding mode dynamics can be chosen appropriately in the framework of a static output feedback pole placement problem for a subsystem (Zak and Hui, 1993; Edwards and Spurgeon, 1995).

In this chapter, we consider the output feedback sliding mode control for sampled-data linear systems. It is well-known that the exact continual sliding motion cannot be achieved in the discrete-time case due to the sample/hold effect (Milosavljevic, 1985). Specifically, the system trajectory only travels in a neighborhood of the sliding surface forming a boundary layer (Su *et al.*, 2000). Several approaches were proposed to address the problem of discrete-time output feedback sliding mode control (Lai *et al.*, 2007; Xu and Abidi, 2008a,b). Some of them are devoted to sliding mode control of sampled-data systems. Xu and Abidi (2008a) employed a disturbance observer and a state observer with an integral sliding surface to address the output tracking problem for sampled-data systems. Under their proposed control approach, the stability of the closed-loop system is guaranteed and the effect of external disturbances is reduced (Xu and Abidi, 2008b).

The deadbeat control strategy proposed in (Oloomi and Sawan, 1997) is able to decouple external disturbances with an $O(\epsilon)$ accuracy, where ϵ is the sampling period. In (Su *et al.*, 2000), an one-step delayed disturbance approximation approach has been shown to be effective in dealing with disturbances that exhibit certain continuity characteristics. We shall exploit the continuity attribute of the state variables and the disturbances by using similar ideas to deal with the similar estimation problem encountered above.

We will present two dynamic output feedback discrete-time control strategies that take into account the disturbance compensation as in (Su *et al.*, 2000). In the first approach, the estimation of the disturbance is based on its previous time instant value, while the second approach employs two previous time instant values. Both methods possess high gain output feedback control. It is pointed out that with high gain output feedback control, the system exhibits the two-time scale behavior (Litkouhi and Khalil, 1985; Gajic and Lim, 2001; Oloomi and Sawan, 1997; Young *et al.*, 1977). By using singular perturbation analysis, we will study the stability of the closed-loop system and the accuracy of the sliding mode. Specifically, we will show that the state trajectory will be remained in an $O(\epsilon^2)$ boundary layer of the sliding surface in the first approach and an $O(\epsilon^3)$ vicinity of the sliding surface in the second method. While the first approach shares with (Xu and Abidi, 2008a) some characteristics such as the bound of sliding motion and the ultimate bound of state variables, the second one presents stronger results. In addition to a controller for implementation, the method in (Xu and Abidi, 2008a) is performed with two observers for state and disturbance estimation, while we employ no observers. On the other hand, the method of Xu and Abidi (2008a) applies to systems of relative degree higher than one, while our methods are limited to systems with relative degree one.

Throughout the dissertation, $\lambda\{A\}$ denotes the spectrum of matrix A, while I_m

stands for an identity matrix of order m. A vector function $f(t,\epsilon) \in R^n$ is said to be $O(\epsilon)$ over an interval $[t_1, t_2]$ Kokotovic *et al.* (1986) if there exists positive constants k and ϵ^* such that

$$\|f(t,\epsilon)\| \leq k\epsilon \quad \forall \epsilon \in [0,\epsilon^*], \quad \forall t \in [t_1, t_2]$$

where $\|.\|$ is the Euclidean norm. Moreover, it is said to be $O(1)$ over $[t_1, t_2]$ if

$$\|f(t,\epsilon)\| \leq k, \quad \forall t \in [t_1, t_2].$$

2.2 Problem Formulation

We consider a linear system described by

$$\begin{aligned} \dot{x}_0(t) &= A_0 x_0(t) + B_0 u(t) + D_0 f(t) \\ y(t) &= C_0 x_0(t) \end{aligned} \tag{2.1}$$

where $x_0(t) \in R^n$ is the system state, $u(t) \in R^m$ is the control, $y(t) \in R^p$ is the system output, $f(t) \in R^r$ is the unknown but bounded exogenous disturbance, with $m \leq p < n$. The system matrices A_0, B_0, C_0, D_0 are constant of appropriate dimensions with magnitudes $O(1)$. The following assumptions are made:

1. The system (2.1) has relative degree 1.
2. Matrices B_0 and C_0 have full rank.
3. (A_0, B_0, C_0) is controllable and observable (EL-Khazali and DeCarlo, 1995).
4. The invariant zeros of system (2.1) are stable.

In addition, D_0 satisfies the matching condition (Drazenovic, 1969)

$$\mathrm{rank}([B_0|D_0]) = \mathrm{rank}(B_0) \tag{2.2}$$

In other words, there exists a matrix K such that

$$D_0 = B_0 K. \tag{2.3}$$

The sliding surface under consideration is

$$s(t) = Hy(t) = HC_0 x_0(t) = 0, \tag{2.4}$$

where H is a full rank $m \times p$ matrix, designed to render stable sliding dynamics. It is shown that the eigenvalues of the sliding mode dynamics include the invariant zeros of the system (2.1) (Edwards and Spurgeon, 1998). One can place the remaining eigenvalues of the zero dynamics of the sliding surface (2.4) if the Davison-Kimura condition (Davison and Wang, 1975) is satisfied (Edwards and Spurgeon, 1998). In the case the Davison-Kimura condition is not satisfied, a dynamic compensator is constructed to produce a tractable structure for the sliding surface design (Edwards and Spurgeon, 1998). Refer to (EL-Khazali and DeCarlo, 1995; Zak and Hui, 1993; Edwards and Spurgeon, 1998) for design of H. Note that HC_0B_0 is nonsingular. Our objective is to construct a discrete-time sliding mode controller given output sliding surface (2.4).

2.3 Discrete-time Regular Form

In this section, we will use several similarity transformations to facilitate system design and analysis. Since $\text{rank}(B_0) = m$, there exists a coordinate transformation T_0 such that

$$B = T_0 B_0 = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}.$$

where B_2 is a nonsingular square matrix of dimension m. The new variables are defined as

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = T_0 x_0.$$

The similarity transformation T_0 brings the original system (2.1) into the regular form (Utkin and Young, 1979)

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) \\ \dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) + D_2f(t), \end{aligned} \tag{2.5}$$

where $D_2 = B_2K$. The sliding surface (2.4) is now described as

$$s(t) = HC_0x_0(t) = HC_0T_0^{-1}x(t) = Cx(t) = C_1x_1(t) + C_2x_2(t) = 0, \tag{2.6}$$

where

$$HC_0T_0^{-1} = C.$$

The zero dynamics of the sliding mode is represented by the eigenvalues of matrix $A_c = A_{11} - A_{12}C_2^{-1}C_1$. Note that H has been chosen in (2.4) to render a sliding surface coefficient matric C such that C_2 is invertible and A_c is stable (Utkin and Young, 1979).

Sampling the continuous-time system (2.5) with the sampling period ϵ results in the following discrete-time model

$$x[k+1] = \Phi x[k] + \Gamma u[k] + d[k] \tag{2.7}$$

where

$$\begin{aligned} \Phi &= e^{A\epsilon}, \quad \Gamma = \int_0^{\epsilon} e^{A\tau} d\tau B, \\ d[k] &= \int_0^{\epsilon} e^{A\tau} BKf((k+1)\epsilon - \tau)d\tau. \end{aligned} \tag{2.8}$$

The system matrices, Φ and Γ, of the sampled-data system (2.7) can be reformulated by taking the Taylor series expansion as

$$\Phi = e^{A\epsilon} = I + \epsilon A + \frac{\epsilon^2}{2!}A^2 + \cdots = I + \epsilon(A + \epsilon \Delta A) = O(1) \tag{2.9}$$

and

$$\Gamma = \int_0^{\epsilon} e^{A\tau} d\tau B = \epsilon(B + \epsilon\Delta B) = O(\epsilon), \tag{2.10}$$

where

$$\Delta A = \frac{1}{2!}A^2 + \frac{\epsilon}{3!}A^3 + \cdots = O(1) \tag{2.11}$$

and

$$\Delta B = \frac{1}{2!}AB + \frac{\epsilon}{3!}A^2 B + \cdots = \begin{bmatrix} \Delta B_1 \\ \Delta B_2 \end{bmatrix} = O(1), \tag{2.12}$$

where the dimensions of ΔB_1 and ΔB_2 are $(n-m) \times m$ and $m \times m$, respectively. Furthermore, since $B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$, Γ can be written as

$$\Gamma = \begin{bmatrix} \epsilon^2 \Delta B_1 \\ \epsilon \bar{B}_2 \end{bmatrix}, \tag{2.13}$$

where

$$\bar{B}_2 = B_2 + \epsilon \Delta B_2. \tag{2.14}$$

Due to the sampling process, the disturbance in the sampled-data system (2.7) exhibits unmatched components, as demonstrated by the following lemma, which is an enhanced version of Abidi *et al.* (2007).

Lemma 2.1. *If the first and second derivatives of the disturbance $f(t)$ in (2.1) are both well defined and bounded, then*

$$\begin{aligned} &d[k] = \int_0^{\epsilon} e^{A\tau} BKf((k+1)\epsilon - \tau)d\tau = \Gamma K f[k] + \frac{\epsilon}{2}\Gamma K v[k] + \epsilon^3 \Delta d[k], \\ &d[k] - d[k-1] = O(\epsilon^2), \\ &d[k] - 2d[k-1] + d[k-2] = O(\epsilon^3), \end{aligned} \tag{2.15}$$

where $v(t) = df(t)/dt$ and $\Delta d[k]$ is a bounded uncertainty.

Proof. Consider $0 \le \tau < \epsilon$ and express $f((k+1)\epsilon - \tau)$ as

$$\begin{aligned} f((k+1)\epsilon - \tau) &= f[k] + \int_{k\epsilon}^{(k+1)\epsilon-\tau} v(\beta)d\beta \\ &= f[k] + \int_{k\epsilon}^{(k+1)\epsilon-\tau} (v[k] + \int_{k\epsilon}^{\beta} \dot{v}(\sigma)d\sigma)d\beta \\ &= f[k] + v[k](\epsilon - \tau) + \int_{k\epsilon}^{(k+1)\epsilon-\tau} \int_{k\epsilon}^{\beta} \dot{v}(\sigma)d\sigma d\beta \end{aligned} \tag{2.16}$$

with $k\epsilon \le \sigma \le \beta < (k+1)\epsilon - \tau$. Substituting (2.16) into the expression of $d[k]$ yields

$$\begin{aligned} d[k] &= \int_0^{\epsilon} e^{A\tau} BKf((k+1)\epsilon - \tau)d\tau \\ &= \int_0^{\epsilon} e^{A\tau} BKf[k]d\tau + \int_0^{\epsilon} e^{A\tau} BKv[k](\epsilon - \tau)d\tau \\ &+ \int_0^{\epsilon} e^{A\tau} BK \int_{k\epsilon}^{(k+1)\epsilon-\tau} \int_{k\epsilon}^{\beta} \dot{v}(\sigma)d\sigma d\beta d\tau. \end{aligned} \tag{2.17}$$

Following the same steps as in Lemma 1 in Abidi *et al.* (2007), we obtain

$$\int_0^{\epsilon} e^{A\tau} BKf[k]d\tau = \Gamma K f[k], \tag{2.18}$$

$$\int_0^{\epsilon} e^{A\tau} BKv[k](\epsilon - \tau)d\tau = \frac{\epsilon}{2}\Gamma K v[k] + \hat{M} v[k]\epsilon^3, \tag{2.19}$$

where $\hat{M}$ is a constant matrix. Assume the second derivative of $f(t)$ is bounded by W, namely $\|\dot{v}(t)\| \le W$. Then, we have

$$\begin{aligned} &\| \int_{k\epsilon}^{(k+1)\epsilon-\tau} \int_{k\epsilon}^{\beta} \dot{v}(\sigma)d\sigma d\beta \| \le \| \int_{k\epsilon}^{(k+1)\epsilon-\tau} \int_{k\epsilon}^{\beta} W d\sigma d\beta \| \\ &\le W(\epsilon - \tau)^2 < W\epsilon^2. \end{aligned} \tag{2.20}$$

Hence,

$$\begin{aligned}&\|\int_0^{\epsilon} e^{A\tau}BK\int_{k\epsilon}^{(k+1)\epsilon-\tau}\int_{k\epsilon}^{\beta}\dot{v}(\sigma)d\sigma d\beta d\tau\|\\ &<\|\int_0^{\epsilon} e^{A\tau}BKd\tau\|W\epsilon^2=\|\Gamma K\|W\epsilon^2.\end{aligned}$$

This implies

$$\int_0^{\epsilon} e^{A\tau}BK\int_{k\epsilon}^{(k+1)\epsilon-\tau}\int_{k\epsilon}^{\beta}\dot{v}(\sigma)d\sigma d\beta d\tau=O(\epsilon^3). \tag{2.21}$$

From (2.17), (2.18), 2.19) and (2.21) with noting the boundedness of $v(k)$, we obtain the expression of $d[k]$ in (2.15). The rest of the proof is similar to that of Lemma 1 in Abidi *et al.* (2007). □

Remark 2.1. *The difference between our lemma and Lemma 1 in Abidi* et al. *(2007) is that the first and second derivatives of the disturbance $f(t)$ need to be bounded. Note that if this requirement is not satisfied, equations (48) and (49) in Abidi* et al. *(2007) will not hold. Our lemma is also stronger than Lemma 1 in Abidi* et al. *(2007) in the fact that the smoothness of $f(t)$ is not required.*

This lemma implies that the mismatched part of the disturbance $d(k)$ is $O(\epsilon^3)$. Such an $O(\epsilon^3)$ disturbance mismatch places an ultima performance that a sampled-data control law can ever achieve at each sampling instant.

To facilitate discrete-time sliding mode control design, we put (2.7) into a discrete-time regular form, which is similar to (2.5). To this end, we employ the following transformation for (2.7)

$$T_1=\begin{bmatrix} I_{n-m} & -\epsilon\Delta B_1\bar{B}_2^{-1}\\ 0 & I_m\end{bmatrix} \tag{2.22}$$

with new variables

$$z=\begin{bmatrix} z_1\\ z_2\end{bmatrix}=T_1\begin{bmatrix} x_1\\ x_2\end{bmatrix}. \tag{2.23}$$

The inverse transformation of T_1 is

$$T_1^{-1}=\begin{bmatrix} I_{n-m} & \epsilon\Delta B_1\bar{B}_2^{-1}\\ 0 & I_m\end{bmatrix}. \tag{2.24}$$

The $m \times m$ matrix $\bar{B}_2$ is an ϵ perturbed version of the nonsingular matrix B_2 as seen in (2.14), hence it is nonsingular so that the transformation matrices T_1 and T_1^{-1} both exist.

Under the transformation T_1, the discrete-time system in its regular form is given by

$$z(k+1) = \bar{\Phi} z(k) + \bar{\Gamma} u(k) + d_1(k), \tag{2.25}$$

in which the new system matrix $\bar{\Phi}$ still keeps it original form as an ϵ-perturbed identity matrix as in (2.9)

$$\bar{\Phi} = T_1 \Phi T_1^{-1} = T_1(I + \epsilon A + \epsilon^2 \Delta A) T_1^{-1} = I + \epsilon \bar{A} \tag{2.26}$$

$$\bar{A} = T_1(A + \epsilon \Delta A) T_1^{-1} = \begin{bmatrix} A_{11} + \epsilon \Delta \bar{A}_{11} & A_{12} + \epsilon \Delta \bar{A}_{12} \\ A_{21} + \epsilon \Delta A_{21} & A_{22} + \Delta \bar{A}_{22} \end{bmatrix}, \tag{2.27}$$

while the first $n - m$ rows of the new control coefficient matrix $\bar{\Gamma}$ are nullified

$$\bar{\Gamma} = T_1 \Gamma = \begin{bmatrix} 0 \\ \epsilon \bar{B}_2 \end{bmatrix}. \tag{2.28}$$

Moreover, all the matched portion in the original disturbance vector $d(k)$ is now transferred to the bottom m row of the new disturbance vector $d_1(k)$, leaving only the $O(\epsilon^3)$ mismatched portion in the first $n - m$ rows

$$\begin{aligned} d_1[k] &= T_1 d[k] = T_1(\Gamma K f[k] + \frac{1}{2}\Gamma K v[k] + \epsilon^3 \Delta d[k]) \\ &= \begin{bmatrix} 0 \\ \epsilon \bar{B}_2 \end{bmatrix} (K f[k] + \frac{1}{2} K v[k]) + \epsilon^3 T_1 \Delta d = \begin{bmatrix} d_{11}[k] \\ d_{12}[k] \end{bmatrix} \end{aligned} \tag{2.29}$$

where

$$d_{11}[k] = O(\epsilon^3) \tag{2.30}$$

$$d_{12}[k] = \epsilon \bar{B}_2 (K f[k] + \frac{1}{2} K v[k]) + O(\epsilon^3) = O(\epsilon). \tag{2.31}$$

The sliding surface vector in the new coordinates is now described as

$$s[k] = C T_1^{-1} z(k) = C_1 z_1[k] + \bar{C}_2 z_2[k] \tag{2.32}$$

where the $m \times m$ matrix $\bar{C}_2$ is an ϵ perturbed version of the original nonsingular matrix in (2.6):

$$\bar{C}_2 = C_2 - \epsilon C_1 \Delta B_1 \bar{B}_2^{-1}. \tag{2.33}$$

Therefore, $\bar{C}_2$ is nonsingular if ϵ is small enough.

2.4 One-Step Delayed Disturbance Approximation Approach

2.4.1 Output Feedback Control Design

In this section, we develop a control strategy that forces the state to reach the sliding surface (2.4) in a finite time. Applying the transformation

$$T_2 = \begin{bmatrix} I_{n-m} & 0 \\ C_1 & \bar{C}_2 \end{bmatrix} \tag{2.34}$$

with state variables

$$\begin{bmatrix} z_1[k] \\ s[k] \end{bmatrix} = T_2 \begin{bmatrix} z_1[k] \\ z_2[k] \end{bmatrix} \tag{2.35}$$

recasts the sampled-data system (2.25) into

$$\begin{aligned} z_1[k+1] &= A_s z_1[k] + \epsilon \bar{A}_{12} \bar{C}_2^{-1} s[k] + d_{11}[k] \\ s[k+1] &= \epsilon \Omega_1 z_1[k] + (I_m + \epsilon \Omega_2) s[k] + \epsilon \bar{C}_2 \bar{B}_2 u[k] + C_1 d_{11}[k] + \bar{C}_2 d_{12}[k], \end{aligned} \tag{2.36}$$

where

$$A_s = I_{n-m} + \epsilon(\bar{A}_{11} - \bar{A}_{12} \bar{C}_2^{-1} C_1) \tag{2.37}$$

$$\Omega_1 = (\bar{C}_1 \bar{A}_{11} + \bar{C}_2 \bar{A}_{21}) - (\bar{C}_1 \bar{A}_{12} + \bar{C}_2 \bar{A}_{22}) \bar{C}_2^{-1} \bar{C}_1 \tag{2.38}$$

$$\Omega_2 = (\bar{C}_1 \bar{A}_{12} + \bar{C}_2 \bar{A}_{22}) \bar{C}_2^{-1}. \tag{2.39}$$

Equation (2.32) reveals the purpose of the coordinate transformation T_2. It is convenient for control design and analysis to bring the sliding vector $s(k)$ directly into the system dynamics (2.36) so that the control design objective can also be regarded as an output regulation problem ($s[k] = 0$). When the discrete-time sliding mode sets in, the dynamics of system (2.36) is reduced to that of subsystem $z_1[k]$, where the sliding mode dynamics is determined by the subsystem matrix A_s. In view of (2.33), the inverse of $\bar{C}_2$ can be written as

$$\bar{C}_2^{-1} = C_2^{-1} + \epsilon \Delta C_{2i}. \tag{2.40}$$

where ΔC_{2i} is a constant matrix. From (2.37) and (2.40), we have

$$A_s = I_{n-m} + \epsilon(A_c + \epsilon \Delta A_c). \tag{2.41}$$

where

$$A_c = A_{11} - A_{12} C_2^{-1} C_1 \tag{2.42}$$

and

$$\Delta A_c = \Delta \bar{A}_{11} - (\Delta \bar{A}_{12} \bar{C}_2^{-1} + A_{12} \Delta C_{2i}) C_1. \tag{2.43}$$

Note that A_c contains the zero dynamics of the original continuous-time sliding motion in (2.6) (Utkin and Young, 1979). The matrices C_1 and C_2 have been designed properly such that all the eigenvalues of A_c have negative real parts. In view of (2.41), the discrete-time sliding mode dynamics matrix A_s is composed of an identity matrix with an additive ϵ perturbation of A_c. It will be shown later that all eigenvalues of A_s will lie in the unit circle. Therefore, the system (2.36) is stable in the discrete-time sliding mode.

Our goal now is to find a control law to attain sliding mode (2.4) at each sampling instant. Rewrite the $s[k]$ dynamics in (2.36) as

$$s[k+1] = (I_m + \epsilon \Omega_2) s[k] + \epsilon \bar{C}_2 \bar{B}_2 u[k] + g[k], \tag{2.44}$$

where $g[k]$ contains the state variables in $z_1[k]$ and the portion of disturbances lying in the control range space

$$g[k] = \epsilon \Omega_1 z_1[k] + d_2[k] \tag{2.45}$$

with

$$d_2[k] = C_1 d_{11}[k] + \bar{C}_2 d_{12}[k] = O(\epsilon). \tag{2.46}$$

By solving $s[k+1] = 0$, we obtain the discrete-time equivalent control law (Utkin and Drakunov, 1989) as follows

$$u^{eq}[k] = -\frac{1}{\epsilon} (\bar{C}_2 \bar{B}_2)^{-1} ((I_m + \epsilon \Omega_2) s[k] + g[k]). \tag{2.47}$$

This control law $u^{eq}[k]$ theoretically keeps the system state on the sliding surface with a full knowledge of the disturbance $g[k]$ at each sampling instant . Since $g[k]$ contains the unmeasurable state variables in $z_1[k]$ and disturbances $d_2[k]$, it is unknown to the controller at time step k, hence the equivalent control law (2.47) is not realizable.

Nevertheless, equation (2.44) reveals that we are able to approximate $g[k]$ numerically by $g[k-1]$

$$g[k-1] = s[k] - (I_m + \epsilon\Omega_2)s[k-1] - \epsilon\bar{C}_2\bar{B}_2 u[k-1]. \tag{2.48}$$

Use of $g[k-1]$ in place of $g[k]$ in $u^{eq}[k]$ leads to a realizable control law with additional dynamics in $u[k]$

$$\begin{aligned} u[k] &= -\frac{1}{\epsilon}(\bar{C}_2\bar{B}_2)^{-1}((I_m + \epsilon\Omega_2)s[k] + g[k-1]) \\ &= -\frac{1}{\epsilon}(\bar{C}_2\bar{B}_2)^{-1}((2I_m + \epsilon\Omega_2)s[k] - (I_m + \epsilon\Omega_2)s[k-1]) + u[k-1]. \end{aligned} \tag{2.49}$$

A similar technique was employed by Su *et al.* (2000) for state feedback discrete-time sliding mode control, where only the external disturbances are to be approximated. In this paper, however, the unknown term $g[k]$ under consideration includes both the external disturbances and the unmeasured state variables.

Remark 2.2. *The proposed control contains no switching actions; hence, no chattering phenomenon will take place. On the other hand, it is observed that control law (2.49) is not able to completely compensate disturbance $g[k]$. However, by taking into account the past information, control law (2.49) still provides the closed-loop system with certain characteristics to reduce the influence of external disturbances. This will be confirmed by stability and accuracy analyses in this Section.*

2.4.2 Stability Analysis

The discrete-time equivalent control law (2.47) is also known as a deadbeat control law that brings the current state to the vicinity of the sliding surface in one sampling period ϵ. It brings about a gigantic control action leading to a fast-time system behavior in the state vector $s[k]$ of (2.36). It was also pointed out in (Oloomi and Sawan, 1997; Young *et al.*, 1977; Popescu and Gajic, 1999) that such a deadbeat high gain control leads to singular perturbation behaviors for sampled-data systems. In this section, we will employ the singular perturbation methodology to analyze the stability of the closed-loop system driven by the proposed sliding mode control law (2.49).

The involvement of the past disturbance $g[k-1]$ in the control law (2.49) produces additional m order dynamics for the closed-loop system. Following a similar way as in (Su *et al.*, 2000), we study an augmented dynamic system of $z_1[k]$, $s[k]$, and $u[k]$. From (2.48) and (2.49), we have

$$\begin{aligned} u[k+1] = &-(\bar{C}_2\bar{B}_2)^{-1}(2I_m+\epsilon\Omega_2)\Omega_1 z_1[k] - \frac{1}{\epsilon}(\bar{C}_2\bar{B}_2)^{-1}(I_m+\epsilon\Omega_2)^2 s[k] \\ &- \frac{1}{\epsilon}(\bar{C}_2\bar{B}_2)^{-1}(I_m+\epsilon\Omega_2)\epsilon\bar{C}_2\bar{B}_0 u[k] - (\bar{C}_2\bar{B}_2)^{-1}(2I_m+\epsilon\Omega_2)d_2[k]. \end{aligned} \tag{2.50}$$

Note that the presence of $1/\epsilon$ complicates the analysis of the augmented system. To overcome that obstacle, we introduce a new variable

$$\gamma[k] = \epsilon\bar{C}_2\bar{B}_2 u[k]. \tag{2.51}$$

Thus, the new augmented system of $z_1[k], \xi[k]$, and $v[k]$ is described by

$$\begin{bmatrix} z_1[k+1] \\ s[k+1] \\ \gamma[k+1] \end{bmatrix} = A_{aug}\begin{bmatrix} z_1[k] \\ s[k] \\ \gamma[k] \end{bmatrix} + \begin{bmatrix} d_{11}[k] \\ d_2[k] \\ -(2I_m+\epsilon\Omega_2)d_2[k] \end{bmatrix}, \tag{2.52}$$

where

$$A_{aug} = \begin{bmatrix} A_s & \epsilon\bar{A}_{12}\bar{C}_2^{-1} & 0 \\ \epsilon\Omega_1 & (I_m+\epsilon\Omega_2) & I_m \\ -\epsilon(2I_m+\epsilon\Omega_2)\Omega_1 & -(I_m+\epsilon\Omega_2)^2 & -(I_m+\epsilon\Omega_2) \end{bmatrix}. \tag{2.53}$$

The above augmented system is both a weakly coupled system and a singularly perturbed discrete-time system (Litkouhi and Khalil, 1985; Gajic and Lim, 2001; Gajic and Shen, 1989; Gajic *et al.*, 2009) where the slow dynamics is represented by $z_1[k]$, and fast dynamics is represented by $s[k]$ and $\gamma[k]$. The slow and fast components affect each other by a regular $O(\epsilon)$ perturbation (or by weak couplings). To see this, partition the augmented matrix A_{aug} as

$$A_{aug} = \begin{bmatrix} A_s & O(\epsilon) \\ O(\epsilon) & A_f \end{bmatrix}$$

and

$$A_f = \begin{bmatrix} (I_m + \epsilon\Omega_2) & I_m \\ -(I_m + \epsilon\Omega_2)^2 & -(I_m + \epsilon\Omega_2) \end{bmatrix}.$$

It is then natural to expect (2.52) to be separated into distinct slow and fast subsystems by using a decoupling transformation (Litkouhi and Khalil, 1985) provided that $I_{2m} - A_f$ is nonsingular. The following lemma shows that all eigenvalues of A_f are zero, hence the condition for the existence of a decoupling transformation is satisfied (Litkouhi and Khalil, 1985).

Lemma 2.2. *The $2m \times 2m$ matrix A_f possesses $2m$ zero eigenvalues.*

Proof. Since

$$A_f^2 = 0, \tag{2.54}$$

it is strait forward to see that all eigenvalues of A_f are zeros.

□

To separate the dynamics of (2.52) into reduced-order subsystems, we employ a decoupling transformation (Litkouhi and Khalil, 1985) as follows

$$T_3 = \begin{bmatrix} I_{n-m} - \epsilon ML & -\epsilon M \\ L & I_{2m} \end{bmatrix}, \tag{2.55}$$

and its inverse is given by

$$T_3^{-1} = \begin{bmatrix} I_{n-m} & \epsilon M \\ -L & I_{2m} - \epsilon LM \end{bmatrix}. \tag{2.56}$$

The matrices $L \in R^{2m \times (n-m)}$ and $M \in R^{(n-m) \times 2m}$ satisfy the following algebraic equations

$$0 = \epsilon N_2 + LA_s - A_f L - \epsilon L N_1 L \tag{2.57}$$

$$0 = N_1 + A_s M - M A_f - \epsilon N_1 L M - \epsilon M L N_1, \tag{2.58}$$

where $N_1 = [\bar{A}_{12} C_2^{-1} \quad 0_{(n-m) \times m}]$ and $N_2 = \begin{bmatrix} \Omega_1 \\ -(2I_m + \epsilon\Omega_2)\Omega_1 \end{bmatrix}$. Setting $\epsilon = 0$ in two equations (2.57) and (2.58) yields

$$\begin{aligned} (I_{2m} - A_f)L_0 &= 0 \\ M_0(I_{2m} - A_f) &= -N_1. \end{aligned}$$

Since $(I_{2m} - A_f)$ is nonsingular due to Lemma 2.2, $L_0 = 0$ and $M_0 = -N_1(I_{2m} - A_f)^{-1}$. This implies

$$L = O(\epsilon) \tag{2.59}$$

and

$$M = O(1). \tag{2.60}$$

The transformation T_3 puts the augmented system dynamics into the fully decoupled two-time scale formulation. Let

$$\begin{bmatrix} w[k] \\ \eta[k] \end{bmatrix} = T_3 \begin{bmatrix} z_1[k] \\ s[k] \\ \gamma[k] \end{bmatrix} \tag{2.61}$$

where $w[k] \in R^{n-m}$ contains the slow modes due to the sliding mode dynamics and $\eta[k] \in R^{2m}$ contains the fast modes due to the dead-beat control and the one-step delayed disturbance approximation.

The transformed system is given by

$$\begin{bmatrix} w[k+1] \\ \eta[k+1] \end{bmatrix} = \begin{bmatrix} \Phi_s & 0 \\ 0 & \Phi_f \end{bmatrix} \begin{bmatrix} w[k] \\ \eta[k] \end{bmatrix} + d_3[k], \tag{2.62}$$

where

$$\Phi_s = A_s - \epsilon N_1 L, \tag{2.63}$$

$$\Phi_f = A_f + \epsilon L N_1, \tag{2.64}$$

and

$$d_3[k] = T_3 \begin{bmatrix} d_{11}[k] \\ d_2[k] \\ -(2I_m + \epsilon\Omega_2)d_2[k] \end{bmatrix} = \begin{bmatrix} O(\epsilon^2) \\ d_2[k] + O(\epsilon^3) \\ -(2I_m + \epsilon\Omega_2)d_2[k] + O(\epsilon^3) \end{bmatrix}. \tag{2.65}$$

Since $L = O(\epsilon)$ and $M = O(1)$ from (2.59) and (2.60), we have

$$\begin{bmatrix} w[k+1] \\ \eta[k+1] \end{bmatrix} = \begin{bmatrix} A_s + O(\epsilon^2) & 0 \\ 0 & A_f + O(\epsilon^2) \end{bmatrix} \begin{bmatrix} w[k] \\ \eta[k] \end{bmatrix} + d_3[k] \tag{2.66}$$

The stability of the closed-loop system is decided by the eigenvalues of Φ_s and Φ_f. It is seen that

$$\lambda\{\Phi_s\} = 1 + \epsilon\lambda\{A_c + \epsilon\Delta A_c - N_1 L\}. \tag{2.67}$$

Since A_c contains stable eigenvalues of the zero dynamics of the original continuous-time sliding motion on the sliding surface (2.4) and $N_1 = O(\epsilon)$, there exist a small ϵ such that the eigenvalues of $(A_c + \epsilon\Delta A_c - N_1 L)$ have negative real parts. Therefore, the eigenvalues of Φ_s lie in the unit circle for sufficiently small ϵ.

On the other hand, let q be an eigenvector corresponding to an eigenvalue of Φ_f. We have

$$\lambda^2\{\Phi_f\}\|q\| = \|\lambda^2\{\Phi_f\}q\| = \|\Phi_f^2 q\| \leq \|\Phi_f^2\|\|q\|. \tag{2.68}$$

This implies

$$\lambda^2\{\Phi_f\} \leq \|\Phi_f^2\|.$$

Since $A_f^2 = 0$ and $N_1 = O(\epsilon)$, it is obvious that

$$\Phi_f^2 = A_f^2 + \epsilon A_f L N_1 + \epsilon L N_1 A_f + \epsilon^2 (L N_1)^2 = O(\epsilon^2). \tag{2.69}$$

Hence, $\lambda\{\Phi_f\} = O(\epsilon)$ and the eigenvalues of Φ_f lie in the unit circle for sufficiently small ϵ.

Remark 2.3. *The eigenvalues of the closed-loop system can be dissected into two groups: the slow and the fast ones. Those which lie in an $O(\epsilon)$ neighborhood of the unit circle represent slow modes and those which locate in an $O(\epsilon)$ neighborhood of the origin represent fast modes. Therefore, the closed-loop system (2.52) is asymptotically stable for sufficiently small ϵ.*

We summarize the above discussion in the following theorem.

Theorem 2.1. *The discrete-time output feedback sliding mode control law (2.49) renders the sampled-data system (2.7) asymptotic stability.*

Remark 2.4. *The structure of the augmented system of $z_1[k]$, $s[k]$, and $u[k]$ is difficult to study. This is remedied by using variable $\gamma[k]$. As a result, the augmented dynamics* *is brought into a representative of two time scale behaviors. Hence, the singular perturbation methodology plays a pivotal role in analyzing the stability of the closed-loop system.*

2.4.3 Accuracy Analysis

In this subsection, we deal with the accuracy issues of the sliding mode and the state variables when the closed-loop system is under the influence of external disturbances. It is of interest to show that the closed-loop system exhibits good robustness against external disturbances under the high gain control law (2.49). From (2.41) and (2.66), we have

$$w[k+1] = (A_s + O(\epsilon^2))w[k] + O(\epsilon^2) = (I_{n-m} + \epsilon A_c + O(\epsilon^2))w[k] + O(\epsilon^2). \quad (2.70)$$

At steady state, $w[k+1] \approx w[k]$. Hence, it follows that

$$w[k] = O(\epsilon). \quad (2.71)$$

Similarly, we can obtain that $\eta[k] = O(\epsilon)$. However, we will exploit the special structures in the matrix A_f and the corresponding disturbance vector in $d_3[k]$ for a better insight of $\eta[k]$. First of all, it can be easily seen that $A_f^2 = O$. Consequently, it takes two sampling periods for the sliding mode controller to actually drive the state into a vicinity of the origin. This is shown in the following relationship of $\eta[k+2]$ and $\eta[k]$

$$\begin{aligned}
\eta[k+2] &= (A_f^2 + O(\epsilon^2))\eta(k) + (A_f + O(\epsilon^2)) \begin{bmatrix} d_2[k+1] + O(\epsilon^2) \\ -(2I_m + \epsilon\Omega_2)d_2[k+1] + O(\epsilon^2) \end{bmatrix} \\
&+ \begin{bmatrix} d_2[k] + O(\epsilon^2) \\ -(2I_m + \epsilon\Omega_2)d_2[k] + O(\epsilon^2) \end{bmatrix} \\
&= O(\epsilon^2)\eta[k] + \begin{bmatrix} -d_2[k+1] + d_2[k] + O(\epsilon^2) \\ O(\epsilon) \end{bmatrix}. \quad (2.72)
\end{aligned}$$

Also note that

$$
\begin{aligned}
d_2[k+1] - d_2[k] &= C_1(d_{11}[k+1] - d_{11}[k]) + \bar{C}_2(d_{12}[k+1] - d_{12}[k]) \\
&= \epsilon \bar{C}_2 \bar{B}_2 K(f[k+1] - f[k] + 1/2(v[k+1] - v[k]) + O(\epsilon^3) = O(\epsilon^2)
\end{aligned} \tag{2.73}
$$

since $\|f[k+1] - f[k]\| = \| \int_0^\epsilon v(t)dt\| = O(\epsilon)$ and $\|v[k+1] - v[k]\| = \| \int_0^\epsilon \frac{dv(t)}{dt} dt\| = O(\epsilon)$. Thus, after two sampling periods, we have

$$
\eta[k] = \begin{bmatrix} \eta_1[k] \\ \eta_2[k] \end{bmatrix} = \begin{bmatrix} O(\epsilon^2) \\ O(\epsilon) \end{bmatrix}. \tag{2.74}
$$

The fast mode $\eta(k)$ contains two parts $\eta_1[k]$ and $\eta_2[k]$, both of which are vectors in R^m. With $L = O(\epsilon)$ and $M = O(1)$ in (2.59) and (2.60), the inverse transformation (2.56) renders

$$
z_1[k] = w[k] + \epsilon M \eta[k] \tag{2.75}
$$

and

$$
\begin{bmatrix} s[k] \\ \gamma[k] \end{bmatrix} = -Lw[k] + \eta[k]. \tag{2.76}
$$

Equations (2.74) and (2.76) imply that after two sampling periods, the sliding motion and the bound of $\gamma[k]$ are achieved with the accuracy of $O(\epsilon)$ as long as $w[k] = O(1)$. Furthermore, when $w[k] = O(\epsilon)$ at steady state, the accuracy of sliding motion is

$$
s[k] = O(\epsilon^2). \tag{2.77}
$$

From (2.75) and (2.77), the ultimate bound of the original state variables is given by

$$
x_0[k] = T_0^{-1} T_1^{-1} T_2^{-1} \begin{bmatrix} z_1[k] \\ s[k] \end{bmatrix} = O(\epsilon). \tag{2.78}
$$

Also in view of (2.51) and (2.76), the ultimate bound of control input is

$$
u[k] = O(1). \tag{2.79}
$$

Remark 2.5. *By taking into account the past information of the disturbance, the proposed control law (2.49) guarantees that under the influence of external disturbances,* *the sliding motion is achieved at the accuracy of $O(\epsilon^2)$, the system state variables and the control input possess ultimate bounds of $O(\epsilon)$ and $O(1)$ respectively.*

In between consecutive samples, the system trajectory may deviate from the sliding surface under the control law (2.49), being a piecewise constant function $u(t) = u(k\epsilon)$. The following lemma shows the bounds on the state variables and the accuracy of the sliding motion in an inter-sample interval.

Lemma 2.3. *If the disturbance $f(t)$ is differentiable and bounded, then the discrete-time sliding mode output feedback control law (2.49) leads to $s[k] = O(\epsilon^2)$ for $k \geq 2$, and that $x[k]$ approaches $O(\epsilon)$ asymptotically. Furthermore, the magnitudes of the sliding variables and the state variables during the sampling period, $k\epsilon \leq t \leq (k+1)\epsilon$, are $s(t) = O(\epsilon^2)$ and $x_0(t) = O(\epsilon)$ respectively.*

Proof. We use the same steps as in (Su *et al.*, 2000) to prove the lemma. Let $0 \leq \tau \leq \epsilon$. Integrating both sides of (2.1) yields

$$x_0(k\epsilon + \tau) - x_0(k\epsilon) = \int_{k\epsilon}^{k\epsilon+\tau} (A_0 x_0(t) + D_0 f(t))dt + \tau B_0 u[k]. \tag{2.80}$$

If the disturbance $f(t)$ is differentiable and bounded, the integration in (2.74) can be approximated by using Euler's method

$$\int_{k\epsilon}^{k\epsilon+\tau} (A_0 x_0(t) + D_0 f(t))dt = \tau(A_0 x_0[k] + D_0 f[k]) + O(\tau^2) \tag{2.81}$$

Define new variables

$$\begin{aligned} e_1 &= \int_{k\epsilon}^{k\epsilon+\tau} (A_0 x_0(t) + D_0 f(t))dt - \tau(A_0 x_0[k] + D_0 f[k]) = O(\tau^2) \\ e_2 &= \int_{k\epsilon}^{(k+1)\epsilon} (A_0 x_0(t) + D_0 f(t))dt - \epsilon(A_0 x_0[k] + D_0 f[k]) = O(\epsilon^2). \end{aligned}$$

The state variable vector is computed as

$$x_0(k\epsilon + \tau) = x_0[k] + e_1 + \frac{\tau}{\epsilon}(x_0[k+1] - x_0[k] - e_2). \tag{2.82}$$

This implies that the state vector is bounded within $O(\epsilon)$. The accuracy of the sliding motion can be proved in the same way. $\square$

The above results are summarized in the following theorem.

Theorem 2.2. *If the exogenous disturbance $f(t)$ is smooth and bounded, then the sampled-data output feedback control*

$$u[k] = -\frac{1}{\epsilon}(\bar{C}_2\bar{B}_2)^{-1}((2I_m + \epsilon\Omega_2)Hy[k] - (I_m + \epsilon\Omega_2)Hy[k-1]) + u[k-1]$$

produces sliding motion on the sliding surface $s(t)$ after two sampling periods. Furthermore, $s(t) = O(\epsilon^2)$, $x_0(t) = O(\epsilon)$ and $u(t) = O(1)$ at steady state.

2.5 Two-Step Delayed Disturbance Approximation Approach

2.5.1 Output Feedback Control Design

The last statement in Lemma 2.1 shows that the disturbance $d[k]$ can be approximated by $2d[k-1] - d[k-2]$. Based on this observation, we approximate the disturbance $g[k]$ by $2g[k-1] - g[k-2]$. Note that $g[k-1]$ and $g[k-2]$ can be computed by (2.48). Hence, replacing $g[k]$ by $2g[k-1] - g[k-2]$ in (2.47), we obtain a new control law

$$\begin{aligned} u[k] = &-\frac{1}{\epsilon}(\bar{C}_2\bar{B}_2)^{-1}((3I_m + \epsilon\Omega_2)s[k] \\ &- (3I_m + 2\epsilon\Omega_2)s[k-1] + (I_m + \epsilon\Omega_2)s[k-2]) + 2u[k-1] - u[k-2]. \end{aligned} \tag{2.83}$$

Like in the one-step delayed disturbance approximation approach, the proposed control law (2.83) contains no switching actions. On the other hand, the incorporation of the past system information $s[k-1]$, $s[k-2]$, $u[k-1]$ and $u[k-2]$ leads to additional dynamics. Also note that $\frac{1}{\epsilon}$ in the coefficient term suggests that it is a high gain linear control law, which induces the singular perturbation phenomenon in the overall closed-loop system. In the next section, we will study the characteristics of the closed-loop system which is represented by an augmented linear one.

2.5.2 Stability Analysis

The discrete-time control law (2.83) is different from the corresponding one in the one-step delayed disturbance approximation approach in the fact that two previous time instants are taken into account. Like the control law (2.50), this control action is a high gain in nature due to the inverse of ϵ. Following a similar derivation in the one-step delayed disturbance approximation approach, we will employ the singular perturbation methodology to analyze the stability of the closed-loop system driven by the proposed sliding mode control law (2.83).

The involvement of the past quantities at time instants $[k-1]$ and $[k-2]$ in the control law (2.83) induces additional dynamics for the closed-loop system. Hence, we study an augmented dynamic system of $z_1[k]$, $s[k]$, $s[k-1]$, $u[k]$ and $u[k-1]$.

From (2.48) and (2.83), we have

$$\begin{aligned} u[k+1] = &-(\bar{C}_2\bar{B}_2)^{-1}((3I_m+\epsilon\Omega_2)\Omega_1 z_1[k] + (2I_m+\epsilon\Omega_2)\Omega_2 s[k] \\ &+ (I_m+\epsilon\Omega_2)s[k-1]) - (I_m+\epsilon\Omega_2)u[k] - u[k-1] - (\bar{C}_2\bar{B}_2)^{-1}(3I_m+\epsilon\Omega_2)d_2[k]. \end{aligned} \tag{2.84}$$

In order to make it easier for analysis, we introduce the following new variables:

$$s_1[k-1] = s[k-1], \tag{2.85}$$

$$\gamma[k] = \epsilon\bar{C}_2\bar{B}_2 u[k], \tag{2.86}$$

and

$$\gamma_1[k] = \epsilon\bar{C}_2\bar{B}_2 u[k-1]. \tag{2.87}$$

Thus, the new augmented system is described by

$$\begin{bmatrix} z_1[k+1] \\ s[k+1] \\ s_1[k+1] \\ \gamma[k+1] \\ \gamma_1[k+1] \end{bmatrix} = A_{aug_2} \begin{bmatrix} z_1[k] \\ s[k] \\ s_1[k] \\ \gamma[k] \\ \gamma_1[k] \end{bmatrix} + \begin{bmatrix} d_{11}[k] \\ d_2[k] \\ 0 \\ -(3I_m+T\Omega_2)d_2[k] \\ 0 \end{bmatrix}, \tag{2.88}$$

where

$$A_{aug_2} = \begin{bmatrix} A_s & \epsilon P_1 \\ \epsilon P_2 & A_{f_2} \end{bmatrix}. \tag{2.89}$$

with

$$A_{f_2} = \begin{bmatrix} (I_m + \epsilon\Omega_2) & 0 & I_m & 0 \\ I_m & 0 & 0 & 0 \\ -\epsilon(2I_m + \epsilon\Omega_2)\Omega_2 & -(I_m + \epsilon\Omega_2) & -(I_m + \epsilon\Omega_2) & -I_m \\ 0 & 0 & I_m & 0 \end{bmatrix}, \tag{2.90}$$

$$P_1 = [\bar{A}_{12}C_2^{-1} \quad 0_{(n-m)\times m} \quad 0_{(n-m)\times m} \quad 0_{(n-m)\times m}], \tag{2.91}$$

and

$$P_2 = \begin{bmatrix} \Omega_1 \\ 0 \\ -(3I_m + T\Omega_2)\Omega_1 \\ 0 \end{bmatrix}. \tag{2.92}$$

The above augmented system is a singularly perturbed discrete-time system (Litkouhi and Khalil, 1985) where the slow dynamics is represented by $z_1[k]$, and the fast dynamics is represented by $s[k]$, $s_1[k]$, $\gamma[k]$, and $\gamma_1[k]$. On the other hand, the slow and fast components affect each other by a regular $O(\epsilon)$ perturbation (or by weak couplings), and hence the system also has the weakly coupled structure (Gajic and Lim, 2001; Gajic and Shen, 1989; Gajic *et al.*, 2009).

According to (Litkouhi and Khalil, 1985), the system (2.88) can be separated into distinct slow and fast subsystems by using a decoupling transformation provided that $I_{4m} - A_{f_2}$ is nonsingular. Since $A_{f_2}^4 = 0$, Lemma 2.2 implies that $\lambda\{A_{f_2}^4\} = 0$. The eigenvalues of $A_{f_2}^4$ satisfy the following equation

$$\det(\lambda I_{4m} - A_{f_2}^4) = \det(\lambda I_{4m} - A_{f_2})\det(\lambda I_{4m} + A_{f_2})\det(\lambda I_{4m} + A_{f_2}^2) = 0. \tag{2.93}$$

This implies the eigenvalues of A_f lie in the set of the eigenvalues of A_f^4. Thus, all eigenvalues of A_f are zero. In other words, $I_{4m} - A_f$ is nonsingular. Hence, the following decoupling transformation can be employed to separate the dynamics of (2.88) into reduced-order subsystems (Litkouhi and Khalil, 1985)

$$T_4 = \begin{bmatrix} I_{n-m} - \epsilon M_2 L_2 & -\epsilon M_2 \\ L_2 & I_{4m} \end{bmatrix},$$

$$T_4^{-1} = \begin{bmatrix} I_{n-m} & \epsilon M_2 \\ -L_2 & I_{4m} - \epsilon L_2 M_2 \end{bmatrix}. \tag{2.94}$$

The matrices $L_2 \in R^{4m\times(n-m)}$ and $M_2 \in R^{(n-m)\times 4m}$ satisfy the following algebraic equations

$$0 = \epsilon P_2 + L_2 A_s - A_{f_2} L_2 - \epsilon L_2 P_1 L_2, \tag{2.95}$$

$$0 = P_1 + A_s M_2 - M A_{f_2} - \epsilon P_1 L_2 M_2 - \epsilon M_2 L_2 P_1. \tag{2.96}$$

Note that A_{f_2}, L_2, and M_2 are functions of ϵ. Setting $\epsilon = 0$ in equations (2.95) and (2.96) yields

$$(I_{4m} - A_{f_2}(0))L_2(0) = 0, \tag{2.97}$$

and

$$M_2(0)(I_{4m} - A_{f_2}(0)) = -P_1. \tag{2.98}$$

It follows that $L_2(0) = 0$, and hence

$$L_2 = O(\epsilon). \tag{2.99}$$

On the other hand, we have

$$\begin{aligned} M_2(0) &= -P_1(I_{4m} - A_{f_2}(0))^{-1} = -[\bar{A}_{12}C_2^{-1} \quad 0_{(n-m)\times m} \quad 0_{(n-m)\times m} \quad 0_{(n-m)\times m}] \\ &\times \begin{bmatrix} I_m & -I_m & I_m & -I_m \\ I_m & 0 & I_m & -I_m \\ -I_m & 0 & 0 & 0 \\ I_m & 0 & 0 & I_m \end{bmatrix} = \bar{A}_{12}C_2^{-1} \begin{bmatrix} -I_m & I_m & -I_m & I_m \end{bmatrix}. \end{aligned}$$

Thus, we obtain

$$M_2 = \bar{A}_{12}C_2^{-1} \begin{bmatrix} -I_m & I_m & -I_m & I_m \end{bmatrix} + O(\epsilon). \tag{2.100}$$

The transformation P_3 puts the augmented system dynamics into the fully decoupled two-time scale forms. Let

$$\begin{bmatrix} w[k] \\ \eta[k] \end{bmatrix} = T_4 \begin{bmatrix} z_1[k] \\ s[k] \\ \gamma[k] \end{bmatrix} \tag{2.101}$$

where $w[k] \in R^{n-m}$ and $\eta[k] \in R^{2m}$. The transformed system is given by

$$\begin{bmatrix} w[k+1] \\ \eta[k+1] \end{bmatrix} = \begin{bmatrix} \Phi_{s_2} & 0 \\ 0 & \Phi_{f_2} \end{bmatrix} \begin{bmatrix} w[k] \\ \eta[k] \end{bmatrix} + d_3[k], \tag{2.102}$$

where $\Phi_{s_2} = A_s - \epsilon P_1 L_2$, $\Phi_{f_2} = A_f + \epsilon L_2 P_1$, and

$$d_3[k] = T_4 \begin{bmatrix} d_{11}[k] \\ d_2[k] \\ 0 \\ -(3I_m + \epsilon\Omega_2)d_2[k] \\ 0 \end{bmatrix} = \begin{bmatrix} O(\epsilon^2) \\ d_2[k] + O(\epsilon^3) \\ O(\epsilon^3) \\ -(3I_m + \epsilon\Omega_2)d_2[k] + O(\epsilon^3) \\ O(\epsilon^3) \end{bmatrix}. \tag{2.103}$$

The stability of the closed-loop system is decided by the eigenvalues of Φ_s and Φ_f. It can be seen that

$$\lambda\{\Phi_{s_2}\} = 1 + \epsilon\lambda\{A_c + \epsilon\Delta A_c - P_1 L_2\}. \tag{2.104}$$

Since A_c contains stable eigenvalues of the zero dynamics of the original continuous-time sliding motion in (2.4) and $P_1 = O(\epsilon)$, there exist a small ϵ such that the eigenvalues of $(A_c + \epsilon\Delta A_c - P_1 L_2)$ have negative real parts. Therefore, the eigenvalues of Φ_{s_2} lie in the unit circle for a sufficiently small ϵ.

On the other hand, let q be an eigenvector corresponding to an eigenvalue of Φ_{f_2}. We have

$$(\lambda\{\Phi_{f_2}\})^4\|q\| = \|(\lambda\{\Phi_{f_2}\})^4 q\| = \|\Phi_{f_2}^4 q\| \le \|\Phi_{f_2}^4\|\|q\|. \tag{2.105}$$

This implies

$$(\lambda\{\Phi_{f_2}\})^4 \le \|\Phi_{f_2}^4\|. \tag{2.106}$$

Since $A_{f_2}^4 = 0$ and $P_1 = O(\epsilon)$, we have

$$\begin{aligned} \Phi_{f_2}^4 &= A_{f_2}^4 + \epsilon A_{f_2}^2(A_{f_2}L_2P_1 + L_2P_1A_{f_2} + \epsilon(L_2P_1)^2) \\ &\quad + \epsilon(A_{f_2}L_2P_1 + L_2P_1A_{f_2} + \epsilon(L_2P_1)^2)A_{f_2}^2 + \epsilon^4(L_2P_1)^2 = O(\epsilon^2). \end{aligned} \tag{2.107}$$

Hence, it follows from (2.107) that $\lambda\{\Phi_{f_2}\} = O(\sqrt{\epsilon})$ and the eigenvalues of Φ_{f_2} lie in the unit circle for a sufficiently small ϵ.

Remark 2.6. *The eigenvalues of the closed-loop system can be dissected into two groups: the slow and fast ones. Those which lie in an $O(\epsilon)$ neighborhood of the unit circle represent slow modes and those which locate in an $O(\sqrt{\epsilon})$ neighborhood of the origin represent fast modes.*

We summarize the above discussion in the following theorem.

Theorem 2.3. *In the absence of external disturbances, the discrete-time output feedback sliding mode control law (2.83) renders the sampled-data system (2.5) asymptotically stable for the sampling period ϵ being small enough .*

Remark 2.7. *It is seen that the closed-loop system possesses similar properties as in the one-step delayed disturbance approximation approach. However, the fast modes in this approach are characterized by eigenvalues which lie in an $O(\sqrt{\epsilon})$ boundary layer of the origin while the fast eigenvalues in the one-step delayed disturbance approximation approach lie in a smaller boundary layer of the origin.*

Remark 2.8. *The control laws in both methods can induce high gain efforts if the state variables are far from the sliding surface. We can employ a saturation control method in (Bartolini* et al.*, 1995) to handle undesired high gain phenomena.*

2.5.3 Accuracy Analysis

In this subsection, we will investigate the accuracy issue of the sliding motion and the bound of the state variables when the closed-loop system is under the influence of external disturbances.

From (2.46), (2.94), (2.99) and (2.100), we have

$$\begin{bmatrix} z_1[k+4] \\ s[k+4] \\ s_1[k+4] \\ \gamma[k+4] \\ \gamma_1[k+4] \end{bmatrix} = A_{aug_2}^4 \begin{bmatrix} z_1[k] \\ s[k] \\ s_1[k] \\ \gamma[k] \\ \gamma_1[k] \end{bmatrix} + d_4[k] \tag{2.108}$$

where

$$d_4[k] = \begin{bmatrix} d_{41}[k] \\ d_{42}[k] \\ d_{43}[k] \\ d_{44}[k] \\ d_{45}[k] \end{bmatrix} \tag{2.109}$$

with

$$d_{41}[k] = \epsilon \bar{A}_{12}\bar{C}_2^{-1}(d_2[k+2] - d_2[k+1]) + O(\epsilon^3),$$

$$d_{42}[k] = d_2[k+3] - 2d_2[k+2] + d_2[k+1] + O(\epsilon^3),$$

$$d_{43}[k] = d_2[k+2] - 2d_2[k+1] + d_2[k] + O(\epsilon^3),$$

$$\begin{aligned} d_{44}[k] =& -(3I_m + \epsilon\Omega_2)d_2[k+3] + (3I_m + 2\epsilon\Omega_2)d_2[k+2] \\ & -(I_m + \epsilon\Omega_2)d_2[k+1] + O(\epsilon^3), \end{aligned}$$

and

$$\begin{aligned} d_{45}[k] =& -(3I_m + \epsilon\Omega_2)d_2[k+2] + (3I_m + 2\epsilon\Omega_2)d_2[k+1] \\ & -(I_m + \epsilon\Omega_2)d_2[k] + O(\epsilon^3). \end{aligned}$$

Applying the transformation (2.96) to (2.108) and using of Lemma 2.1 with $d_2[k]$ playing the role of $d[k]$, we have

$$\begin{bmatrix} w[k+4] \\ \eta[k+4] \end{bmatrix} = \begin{bmatrix} \Phi_{s_2}^4 & 0 \\ 0 & \Phi_{f_2}^4 \end{bmatrix} \begin{bmatrix} w[k] \\ \eta[k] \end{bmatrix} + \begin{bmatrix} O(\epsilon^3) \\ O(\epsilon^3) \\ O(\epsilon^3) \\ O(\epsilon) \\ O(\epsilon) \end{bmatrix}. \tag{2.110}$$

At steady state, $w[k+4] \approx w[k]$. Hence, it follows that

$$(I_{n-m} - \Phi_{s_2}^4)w[k] = O(\epsilon^3).$$

Since $I_{n-m} - \Phi_{s_2}^4 = -4\epsilon A_c + O(\epsilon^2)$ and $\lambda\{A_c\} \neq 0$, for small enough ϵ we attain

$$w[k] = O(\epsilon^2). \tag{2.111}$$

Similarly, $\eta[k+4] \approx \eta[k]$ at steady state. From (2.107) and (2.110), we have

$$(I_{4m} - O(\epsilon^2))\eta[k] = \begin{bmatrix} O(\epsilon^3) \\ O(\epsilon^3) \\ O(\epsilon) \\ O(\epsilon) \end{bmatrix}.$$

Hence,

$$\eta[k] = \begin{bmatrix} O(\epsilon^3) \\ O(\epsilon^3) \\ O(\epsilon) \\ O(\epsilon) \end{bmatrix}. \tag{2.112}$$

Transforming (2.111) and (2.112) back to the coordinates (2.94) renders

$$\begin{bmatrix} z_1[k] \\ s[k] \\ s_1[k] \\ \gamma[k] \\ \gamma_1[k] \end{bmatrix} = T_4^{-1} \begin{bmatrix} w[k] \\ \eta[k] \end{bmatrix} = \begin{bmatrix} O(\epsilon^2) \\ O(\epsilon^3) \\ O(\epsilon^3) \\ O(\epsilon) \\ O(\epsilon) \end{bmatrix}. \tag{2.113}$$

This shows that the accuracy of the sliding mode obtained is of the order of $O(\epsilon^3)$. The ultimate bound of the original state variables is given by

$$x_0[k] = T_0^{-1}T_1^{-1}T_2^{-1} \begin{bmatrix} z_1[k] \\ s[k] \end{bmatrix} = O(\epsilon^2). \tag{2.114}$$

By the similar arguments of Lemma 2.3, we arrive at the following theorem which shows the ultimate bounds of the state variables and the accuracy of the sliding motion.

Theorem 2.4. *If the exogenous disturbance $f(t)$ is bounded and smooth, then the sampled-data output feedback control law (2.83) produces a sliding motion on the sliding surface $s(t)$. Furthermore, $s(t) = O(\epsilon^3)$ and $x_0(t) = O(\epsilon^2)$ at steady state.*

Remark 2.9. *In view of (2.36), if $s[k] = 0$ at some time instants, the dynamics of $z_1[k]$ is still affected by an $O(\epsilon^3)$ disturbance action. Therefore, the best bound of $z_1[k]$ is $O(\epsilon^2)$, and hence, the ultimate bound of the state variables are at best $O(\epsilon^2)$.*

Remark 2.10. *The proposed control strategy ensures that the state variables evolve in an $O(\epsilon^3)$ boundary layer of the sliding surface. Also, the state variables are kept in an ultimate bound of $O(\epsilon^2)$. These results are stronger than those in the one-step delayed disturbance approximation approach.*

2.6 Numerical Example

We use the L-1011 aircraft model in (Heck *et al.*, 1995) to illustrate the proposed approach with system matrices given by:

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.154 & -0.0042 & 1.54 & 0 & -0.744 & -0.032 \\ 0 & 0.249 & -1 & -5.2 & 0 & 0.337 & -1.12 \\ 0.0386 & -0.996 & -0.0003 & -2.117 & 0 & 0.02 & 0 \\ 0 & 0.5 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -25 \end{bmatrix}, \tag{2.115}$$

$$B_0 = D_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 25 \end{bmatrix}^T, \tag{2.116}$$

$$C_0 = \begin{bmatrix} 0 & -0.154 & -0.0042 & 1.54 & 0 & -0.744 & -0.032 \\ 0 & 0.249 & -1 & -5.2 & 0 & 0.337 & -1.12 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{2.117}$$

The coefficient matrix of the sliding surface $s(t) = Hy(t) = 0$ was given by Heck *et al.* (1995)

$$H = \begin{bmatrix} -0.0067 & 0.0167 & 0.0033 & 0 \\ 0.0167 & -0.0333 & 0 & 0.0333 \end{bmatrix}. \tag{2.118}$$

Consider the external disturbance vector

$$f(t) = \begin{bmatrix} 1 + \sin(0.5t) \\ 0.8\sin(t) \end{bmatrix}. \tag{2.119}$$

The sampling period is $\epsilon = 0.01$ second. The initial condition is $x(0) = [1 \quad -2 \quad 2 \quad -4 \quad 3 \quad 4 \quad -1]^T$. For comparison, we plot the numerical results for the one-step delayed disturbance approximation approach and the two-step delayed disturbance approximation approach.

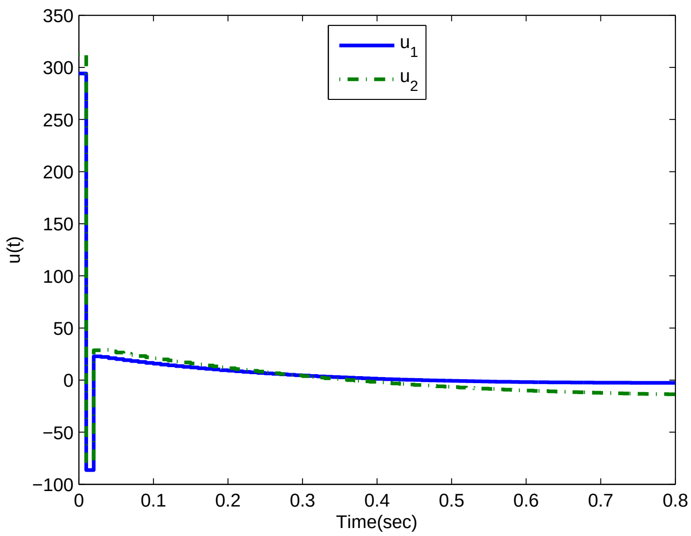

Figure 2.1: Evolution of the control law for the one-step delayed disturbance approximation approach

Fig. 2.4 indicates that the control law of the two-step delayed disturbance approximation approach possesses high gain in three sampling periods. The control law of the one-step delayed disturbance approximation approach experiences a high gain in two sampling periods in Fig. 2.1. On the other hand, it is observed from Fig. 2.2, Fig. 2.3, Fig. 2.5, Fig. 2.6 that the ultimate bounds of state variables and the accuracy of the sliding motion for the two-step delayed disturbance approximation approach are better than those for the one-step delayed disturbance approximation approach. This agrees with the analysis in the previous section.

2.7 Conclusions

We have investigated the output feedback sliding mode control problem for sampled-data systems with external disturbances. By some suitable linear transformations and changes of variables, the closed-loop system under the high gain control law (2.49) as

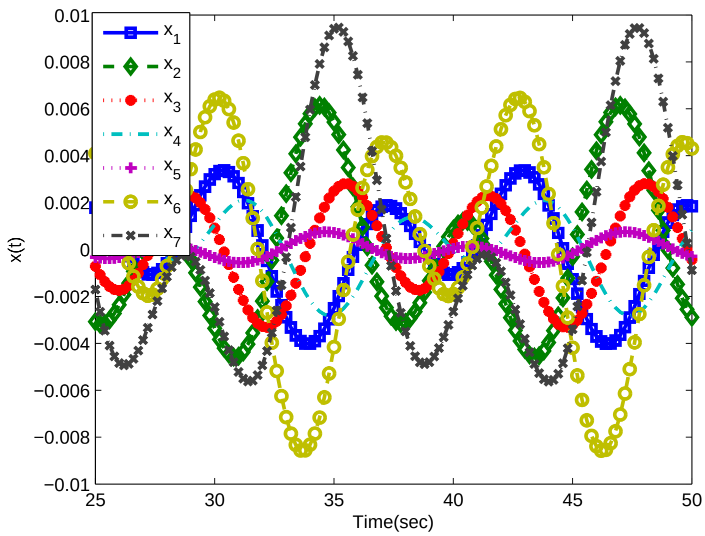

Figure 2.2: $O(\epsilon)$ bounds of the state variables for the one-step delayed disturbance approximation approach

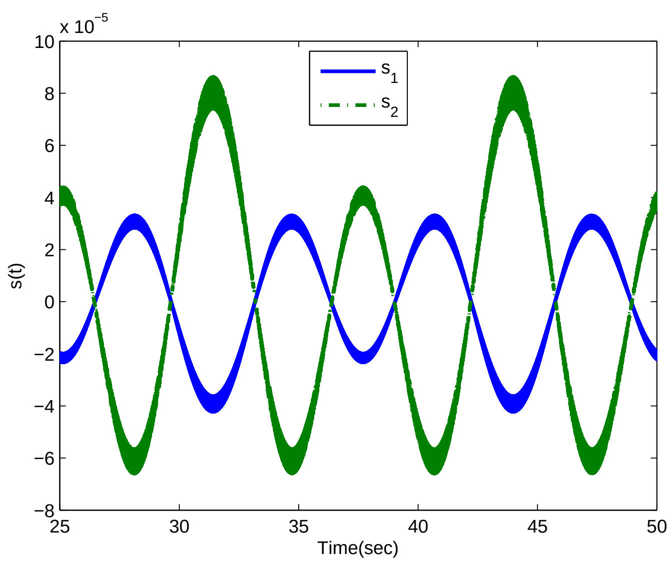

Figure 2.3: $O(\epsilon^2)$ accuracy of the sliding motion for the one-step delayed disturbance approximation approach

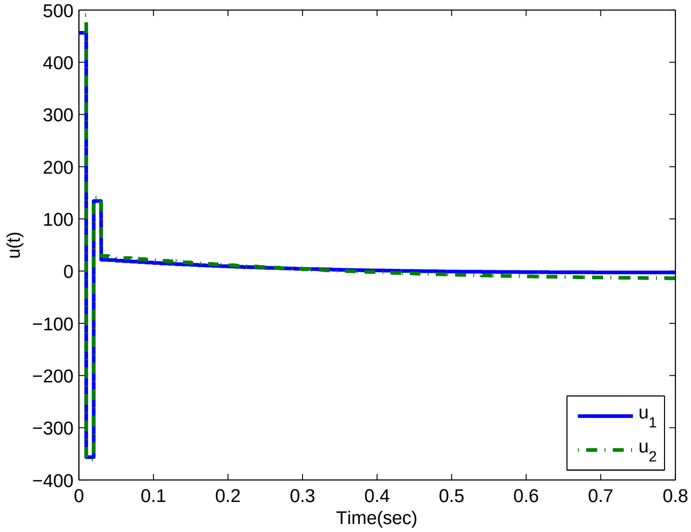

Figure 2.4: Evolution of the control law for the two-step delayed disturbance approximation approach

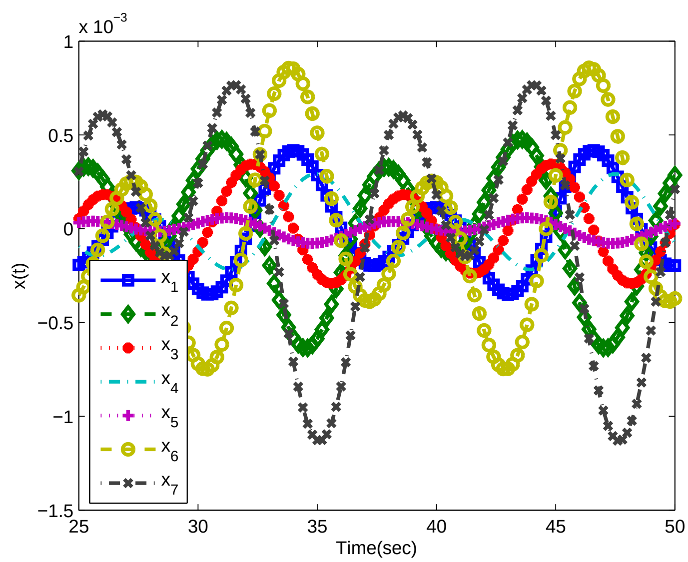

Figure 2.5: $O(\epsilon^2)$ bounds of the state variables for the two-step delayed disturbance approximation approach

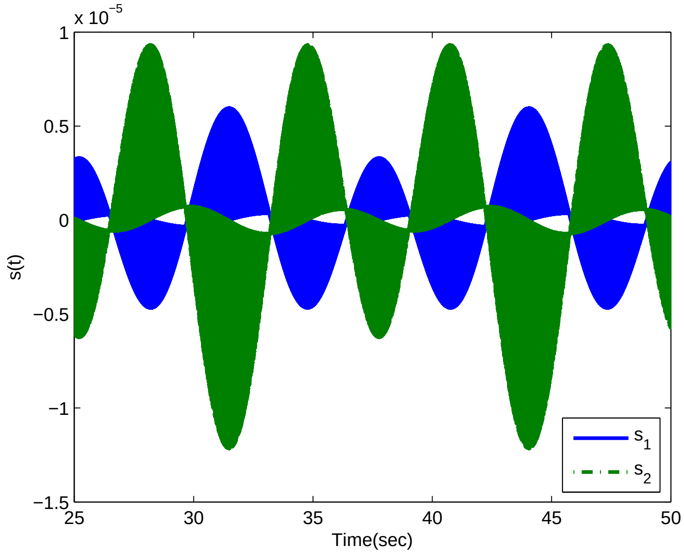

Figure 2.6: $O(\epsilon^3)$ accuracy of the sliding motion for the two-step delayed disturbance approximation approach

well as the control law (2.83) is shaped into a two-time scale representative. This paves way to the framework of discrete-time singular perturbation analysis, by which the eigenvalues of the closed-loop system are clustered into two groups: the slow and fast eigenvalues. For a small enough sampling period, the stability of the closed-loop system is guaranteed in the absence of external disturbances.

The idea of approximating disturbances by the past information equips the control law (2.49) and the control law (2.83) with an ability to maintain the system state in a respective $O(\epsilon^2)$ and $O(\epsilon^3)$ boundary layer of the sliding surface. Also, the ultimate bounds of the state variables are $O(\epsilon^2)$ and $O(\epsilon^3)$ for the two approaches. In other words, under two proposed control laws, the closed-loop system exhibits good robustness against exogenous disturbances. It is also shown that the second approach produces much better characteristics than the first one. As an illustration, the numerical example of an aircraft attitude output feedback control problem has been provided to show the efficiency of the proposed methods.

Chapter 3

Sliding Mode Control for Singularly Perturbed Linear Continuous-Time Systems: Composite Control Approaches

3.1 Introduction

In systems with slow and fast modes, several works have addressed the study of sliding mode control (Heck, 1991; Li *et al.*, 1995a; Yue and Xu, 1996; Su, 1999; Innocenti *et al.*, 2003; Ahmed *et al.*, 2004; Fridman, 2001, 2002a,b; Alvarez-Gallegos and Silva-Navarro, 1997; Soto-Cota *et al.*, 2006). In general, a singularly perturbed system is decomposed into slow and fast subsystems for which a state feedback control law is synthesized. Then, the results are combined in a composite feedback control law (Kokotovic *et al.*, 1986). Sliding mode control, a powerful tool dealing with uncertainty and disturbances, has been utilized for decades (Utkin, 1977, 1978, 1992). However, it is not straitforward to synthesize a sliding mode control law for singularly perturbed systems due to the complication of different time-scale behavior and the discontinuous nature of switching actions. We will review some papers in the literature regarding sliding mode control for singularly perturbed systems.

Various attempts to apply the knowledge of sliding mode control for singularly perturbed systems have been realized in several papers (Heck, 1991; Li *et al.*, 1995a; Yue and Xu, 1996; Su, 1999; Innocenti *et al.*, 2003; Ahmed *et al.*, 2004; Fridman, 2001, 2002a,b; Alvarez-Gallegos and Silva-Navarro, 1997; Soto-Cota *et al.*, 2006). Heck (1991) proposed a composite sliding mode control method derived from two slow and fast sliding mode control laws in an effort to stabilize a class of linear time-invariant systems. A similar approach was investigated by Li *et al.* (1995a). Alvarez-Gallegos and Silva-Navarro (1997) investigated a more general sliding mode control method for a class of nonlinear singularly perturbed systems. Meanwhile, Innocenti *et al.* (2003) considered a class of linear singularly perturbed systems in which the control input enters the system dynamics via fast variables. Fridman (2001, 2002a,b) investigated some issues of stability and chattering analysis for a class of singularly perturbed relay systems, in which the fast modes are represented by the dynamics of fast actuators. Soto-Cota *et al.* (2006) studied synchronous generator systems in which the control input only appears in the slow dynamics equation. Different from the above works, Su (1999) employed the Lyapunov function method to design a sliding surface which contains both slow and fast variables. Yue and Xu (1996) took into the account external disturbances and proposed a composite sliding mode control law in an effort to reduce the influence of external disturbances.

Heck (1991) studied a singularly perturbed system without external disturbances:

$$\begin{aligned} \dot{x}(t) &= A_{11}x(t) + A_{12}z(t) + B_1u(t), \quad x(t_0) = x_0 \\ \epsilon\dot{z}(t) &= A_{21}x(t) + A_{22}z(t) + B_2u(t), \quad z(t_0) = z_0, \end{aligned} \tag{3.1}$$

where $x(t) \in R^{n_1}$, $z(t) \in R^{n_2}$, $u(t) \in R^m$, and ϵ is a small positive parameter. It has been assumed in (Heck, 1991) that A_{22} is stable. This assumption is stronger than the condition that A_{22} is invertible (a standard assumption in singular perturbations (Kokotovic *et al.*, 1986)).

The full-order system is decoupled into reduced-order subsystems. The slow subsystem is represented by

$$\begin{aligned} \dot{x}_s(t) &= A_0x_s(t) + B_0u_s(t) \\ z_s(t) &= -A_{22}^{-1}(A_{21}x_s(t) + B_2u_s(t)) \end{aligned} \tag{3.2}$$

where $A_0 = (A_{11} - A_{12}A_{22}^{-1}A_{21})$ and $B_0 = B_1 - A_{12}A_{22}^{-1}B_2$. The fast dynamic model is given in terms of a stretched time variable $\tau = (t - t_0)/\epsilon$ and a new state variable $z_f = z - z_s$:

$$\frac{dx(\tau)}{d\tau} = \epsilon(A_{11}x(\tau) + A_{12}z(\tau) + B_1u) = O(\epsilon), \quad x(0) = x_0, \tag{3.3}$$

$$\frac{dz_f(\tau)}{d\tau} = A_{22}z_f(\tau) + B_2u_f - \frac{dz_s}{dx(\tau)}\frac{dx(\tau)}{d\tau}, \quad z_f(0) = z_0 - z_s(0) \tag{3.4}$$

where $\tilde{z}(\tau) = z(\epsilon\tau + t_0)$, $\tilde{x}(\tau) = x(\epsilon\tau + t_0)$, and $u_f = u - u_s$. If $\frac{dz_s)}{dx(\tau)}$ is $O(1)$, equation (3.4) can be approximated by the following reduced-order model:

$$\frac{d\hat{z}_f}{d\tau} = A_{22}\hat{z}_f + B_2 u_f, \quad \hat{z}_f(0) = z_0 - h(x_0) \tag{3.5}$$

where $\hat{z}_f$ is an approximation for $z_f = z - h(x)$ during the initial boundary layer. For the slow subsystem (3.2), a sliding surface is chosen as

$$s_s = C_s x_s = 0. \tag{3.6}$$

The control law is designed in the form:

$$u_{si}(x_s) = \begin{cases} u_{si}^+(x_s), & \text{if} \quad s_{si}(x_s) > 0 \\ u_{si}^-(x_s), & \text{if} \quad s_{si}(x_s) \leq 0 \end{cases} \tag{3.7}$$

where $s_{si} = C_{si}x_s$ is the ith linear switching function, u_{si}^+ and u_{si}^- are smooth functions to be defined later. The equivalent control for the slow subsystem during sliding is given by Utkin (1984)

$$u_{es} = -(C_s B_0)^{-1} C_s A_0 x_s. \tag{3.8}$$

For the fast subsystem (3.5), a sliding surface is chosen as

$$s_f = C_f \hat{z}_f = 0. \tag{3.9}$$

The control is chosen in the form:

$$u_{fi}(\hat{z}_f) = \begin{cases} u_{fi}^+(\hat{z}_f), & \text{if} \quad s_{fi}(\hat{z}_f) > 0 \\ u_{fi}^-(\hat{z}_f), & \text{if} \quad s_{fi}(\hat{z}_f) \leq 0 \end{cases} \tag{3.10}$$

where $s_{fi} = C_{fi}\hat{z}_f$ is the ith linear switching function, u_{fi}^+ and u_{fi}^- are smooth functions to be defined later. The equivalent control for the fast subsystem during sliding is given by Utkin (1984)

$$u_{ef} = -(C_f B_2)^{-1} C_f A_{22}\hat{z}_f. \tag{3.11}$$

The control law for the full-order system is a composite of the slow and fast controls as given by

$$u = u_s(x_s) + u_f(z_f) \tag{3.12}$$

where u_s is defined in (3.7) and u_f is defined in (3.10).

The structure of the slow control law (3.7) and the fast control one (3.10) can be chosen by three methods (Heck, 1991). For a system

$$\dot{x} = Ax + Bu \tag{3.13}$$

the control law is given (Utkin, 1977)

$$u_i = (-\alpha \sum_{i=1}^{m} |x_m| - \delta)\mathrm{sgn}(s_i) \tag{3.14}$$

where u_i is the ith component of the control, s_i is the ith row vector of $s(t)$. A control law proposed by DeCarlo *et al.* (1988) can be used for the slow subsystem

$$u = -(SB)^{-1}SAx - \delta(SB)^{-1}\mathrm{SGN}(Sx) \tag{3.15}$$

where $\mathrm{SGN}(y)$ is a vector-valued function with the ith component equal to $\mathrm{sgn}(y_i)$. The composite control law can be given by

$$u = -\delta\mathrm{SGN}(K_x x + K_z z) \tag{3.16}$$

where K_x, K_z are to be determined. The main contribution of Heck is that the problem of sliding mode control for the full order system (3.1) is addressed by two reduced-order problems for the slow and fast subsystems. The drawback of Heck's method is the assumption on $\frac{dz_s)}{dx(\tau)}$ to validate the fast model; hence, the generality of the method is limited. Furthermore, external disturbances were not brought into the picture.

Li *et al.* (1995a) also considered a singularly perturbed system in the form of (3.1). Similarly to Heck's approach, the full-order system is first decomposed into slow and fast subsystems. Then, slow and fast sliding surfaces are designed for the subsystems separately. The only difference is that Li *et al.* (1995a) proposed a non switching control strategy. The control law for the slow subsystem is given by

$$u_s = u_{es} + \Delta u_s \tag{3.17}$$

where u_{es} is defined in (3.8) and

$$\Delta u_s = -(C_s B_0)^{-1}\delta_s \mathrm{sig}(C_s x_s) * \mathrm{sgn}(C_s x_s) \tag{3.18}$$

where " $*$ " stands for the component-wise multiplication, δ_s is a positive scalar, and $\mathrm{sig}(C_s x_s)$ is a vector function whose ith component is given by

$$\mathrm{sig}(C_{s_i} x_s) = \frac{1 - e^{-|C_{s_i} x_s|}}{1 + e^{-|C_{s_i} x_s|}} \geq 0. \tag{3.19}$$

Similarly, the fast sliding mode control law is given by

$$u_f = u_{ef} + \Delta u_f \tag{3.20}$$

where

$$u_{ef} = -(C_f B_2)^{-1} C_f A_{22} z_f$$

is equivalent control and

$$\Delta u_f = -(C_f B_2)^{-1} \delta_s \mathrm{sig}(C_f \eta) * \mathrm{sgn}(C_f z_f)$$

The composite control is constructed from slow and fast control laws:

$$u = u_s(x) + u_f(z_f)$$

where u_s and u_f are defined in (3.17) and (3.20). The control method proposed by Li *et al.* (1995a) employs sigmoid functions to reduce the chattering phenomenon. Like (Heck, 1991), they did not consider the problem of external disturbances beside the closed-loop stability issue.

Following the same direction as in (Heck, 1991; Li *et al.*, 1995a), Innocenti *et al.* (2003) studied a class of singularly perturbed systems:

$$\begin{aligned} \dot{x}(t) &= A_{11}x(t) + A_{12}z(t), \quad x(t_0) = x_0 \\ \epsilon \dot{z}(t) &= A_{21}x(t) + A_{22}z(t) + B_2 u(t), \quad z(t_0) = z_0, \end{aligned} \tag{3.21}$$

where $x(t) \in R^{n_1}$, $z(t) \in R^{n_2}$, $u(t) \in R^m$, ϵ is a small positive parameter, A_{22} is nonsingular, B_2 is of full rank, and (A_{22}, B_2) is controllable. It is seen that system (3.21) is less general than system (3.1) when the control input only affects the system through the dynamics of $z(t)$. Like in (Heck, 1991), they employed standard techniques of singular perturbations to decompose system (3.21) into slow and fast subsystems.

The slow model is given by

$$\begin{aligned} \dot{x}_s(t) &= A_0 x_s + B_0 u_s(x_s) \\ z_s &= -A_{22}^{-1}(A_{21}x_s + B_2 u_s(x_s)) \end{aligned} \tag{3.22}$$

where $A_0 = (A_{11} - A_{12}A_{22}^{-1}A_{21})$ and $B_0 = -A_{12}A_{22}^{-1}B_2$. The slow sliding surface is chosen as

$$s_s = C_s x_s = 0.$$

The slow control law is given by

$$u_s(x_s) = -(C_s B_0)^{-1}(K_s s_s + Q_s \rho_s(s_s) + C_s A_0 x_s) \tag{3.23}$$

where $K_s \in R^{r\times r}$, $Q_s \in R^{r\times r}$ are positive definite diagonal matrices and $\rho_s(\sigma) : R^r \to R^r$ is a vector function, whose ith component is given by

$$\rho_{s_i}(\sigma_i) = \frac{\sigma_i}{|\sigma_i| + \delta_{s_i}} \tag{3.24}$$

and δ_{s_i} is a small positive quantity. The fast reduced model obtained is the same as in (Heck, 1991; Li *et al.*, 1995a):

$$\frac{dz_f}{d\tau} = A_{22}z_f + B_2 u_f \tag{3.25}$$

where $z_f = z - z_s$ is the fast component of z and $u_f = u - u_s$ is the fast control. If the fast sliding surface is chosen as

$$s_f = C_f z_f = 0,$$

the fast sliding control law is given by

$$u_f(z_f) = -(C_f B_2)^{-1}(K_f s_f + Q_f \rho_f(s_f) + C_f A_{22} z_f). \tag{3.26}$$

The composite control is synthesized from slow control (3.23) and fast control (3.26)

$$u_c = u_s(x) + u_f(z_f).$$

To study the stability of dynamics of subsystems, Innocenti *et al.* (2003) employed a state space decomposition to construct a quadratic Lyapunov function and a procedure in (Kokotovic *et al.*, 1986). Under their proposed scheme, the closed-loop system is proved to be globally stable with sufficiently small ϵ. Different from Heck (1991); Li *et al.* (1995a), Innocenti *et al.* (2003) utilized a continuous-time control law instead of discontinuous one. Hence, it helps avoid chattering phenomena. While external disturbances are also not considered, the class of systems under consideration is less general than in (Heck, 1991; Li *et al.*, 1995a).

Unlike the above approaches, where no disturbances are taken into account, Yue and Xu (1996) studied a singularly perturbed system of the form

$$\begin{aligned} \dot{x}(t) &= A_{11}x(t) + A_{12}z(t) + B_1u(t) + B_1f(t,x,z), \\ \epsilon\dot{z}(t) &= A_{21}x(t) + A_{22}z(t) + B_2u(t) + B_2g(t,x,z), \end{aligned} \tag{3.27}$$

where $x(t) \in R^{n_1}$, $z(t) \in R^{n_2}$, and $u(t) \in R^m$. $0 < \epsilon \ll 1$ represents the singular perturbation parameter. $f(x,z,t)$, $\quad g(x,z,t) : R_+ \times R^{n_1} \times R^{n_2} \to R$ denote the parameter uncertainties and external disturbances. Furthermore, the disturbances $f(x,z,t)$, $g(x,z,t)$ are assumed to satisfy the following inequalities:

$$|f(x,z,t)\| \leq \rho_1(x,z) = a_0 + a_1\|x\| + a_2\|z\|$$

$$\|g(x,z,t)\| \leq \rho_2(x,z) = b_0 + b_1\|x\| + b_2\|z\|.$$

In addition, they satisfy

$$\|f(x,z,t) - g(x,z,t)\| \leq \alpha\|x\| + \beta\|z\|.$$

In their approach, a designed control law includes two continuous time state feedback terms and a switching term. The objective of the two continuous-time terms is to stabilize the system as no disturbances are taken into account. Specifically, the control law is in the form

$$u = Kx + K_0\eta + w \tag{3.28}$$

where K and K_0 are designed such that $A_s + B_sK$ and $A_{22} + B_2K_0$ are stable, and w is a switching term to be defined. Here, η is a new state variable given by

$$\eta = z + A_{22}^{-1}(A_{21} + B_2K)x. \tag{3.29}$$

The new system is given by

$$\dot{x} = A_{11}^0 x + (A_{12} + B_1 K_0)\eta + B_1 w + B_1 f$$
$$\epsilon\dot{\eta} = (A_{22} + B_2 K_0)\eta + B_2 w + B_2 g + O(\epsilon).$$

To choose w, Yue and Xu considered a Lyapunov function candidate as follows

$$V = x^T P_1 x + \epsilon \eta^T P_2 \eta$$

where P_1, P_2 are positive definite solutions to the following Lyapunov equations

$$(A_s + B_s K)^T P_1 + P_1 (A_s + B_s K) = -Q_1$$
$$(A_{22} + B_2 K_0)^T P_2 + P_2 (A_{22} + B_2 K_0) = -Q_2.$$

The control law is chosen as

$$u = -Kx - K_0\eta - (\hat{b}_0 + \hat{b}_1^0\|x\| + \hat{b}_2\|\eta\|)\mathrm{sgn}(B_1^T P_1 x + B_2^T P_2 \eta) \tag{3.30}$$

where $\hat{b}_0$, $\hat{b}_1^0$, and $\hat{b}_2$ are acquired from the definition of disturbances $f(t,x,z)$, $g(t,x,z)$, and matrices A_{21}, A_{22}, K. With this control law, the system (3.27) is uniformly practically stable for $\epsilon \in (0, \epsilon^*]$ and the trajectories x and η ultimately satisfy (Yue and Xu, 1996)

$$\|x\| \leq O(\epsilon), \quad \|\eta\| \leq O(\epsilon).$$

Yue and Xu also proved the existence of the sliding motion for the sliding surface $s = B_1^T P_1 x + B_2^T P_2 \eta$ provided some condition are satisfied (Yue and Xu, 1996). Although, their approach deals with disturbances and provides some certain robust characteristics, it only guarantees the trajectories of the system travel in a $O(\epsilon)$ boundary layer of the origin. Furthermore, it is complicated to compute some parameters for the control law.

Su (1999) studied the problem of sliding surface design for the system (3.1). Like in (Heck, 1991; Li *et al.*, 1995a; Innocenti *et al.*, 2003), the full-order system is separated into slow and fast subsystems. Then, stabilizing state feedback controls are constructed individually for each subsystem, leading to a composite control law

$$u = u_s + u_f = K_1 x + K_2 z. \tag{3.31}$$

Then, the closed-loop system is written as

$$\begin{bmatrix} \dot{x} \\ \epsilon \dot{z} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \tag{3.32}$$

where

$$T_{11} = A_{11} + B_1 K_1,$$

$$T_{12} = A_{12} + B_1 K_2,$$

$$T_{21} = A_{21} + B_2 K_1,$$

$$T_{22} = A_{22} + B_2 K_2.$$

The closed-loop system is transformed into an exact slow and an exact fast subsystem by using the Chang transformation (Chang, 1972; Kokotovic *et al.*, 1986):

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} I_{n_1} - \epsilon HL & -\epsilon H \\ L & I_{n_2} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = J \begin{bmatrix} \xi \\ \eta \end{bmatrix}. \tag{3.33}$$

The exact subsystems in the new coordinates are

$$\begin{aligned} \dot{\xi} &= (T_{11} - T_{12})\xi = T_s \xi \\ \epsilon \dot{\eta} &= (T_{22} + \epsilon L T_{12})\eta = T_f \eta. \end{aligned} \tag{3.34}$$

There exist positive definite matrices P_s and P_f such that

$$\begin{aligned} P_s T_s + T_s^T P_s &= -Q_s, \quad Q_s > 0 \\ P_f T_f + T_f^T P_f &= -Q_f, \quad Q_f > 0. \end{aligned} \tag{3.35}$$

Then, the sliding surface for the singularly perturbed system can be chosen as

$$s(x,z) = \begin{bmatrix} B_1 \\ B_2/\epsilon \end{bmatrix}^T J^T \begin{bmatrix} P_s & 0 \\ 0 & \epsilon P_f \end{bmatrix} J \begin{bmatrix} x \\ z \end{bmatrix} = 0. \tag{3.36}$$

It was shown that (Su, 1999) if the sliding motion is achieved, the system is asymptotically stable. Like in (Yue and Xu, 1996), only one sliding surface is designed for the full order system. However, a control strategy has not been provided to realize the sliding motion.

In this chapter, we address the problem of sliding mode control for a singularly perturbed system with the external disturbance. While several papers in the literature only address the stability of the closed-loop system (Heck, 1991; Li *et al.*, 1995a; Innocenti *et al.*, 2003), we consider both closed-loop stability and disturbance rejection. In our method, a state feedback control law is firstly established to stabilize either slow or fast dynamics. Then, a sliding mode control law is designed for the remaining dynamics of the system to ensure stability and disturbance rejection. Putting the two controls together produces a composite control law that makes the closed-loop system asymptotically stable. The advantage of the proposed method over others in the literature is that external disturbances are completely excluded.

3.2 Problem Formulation

Consider a singularly perturbed system:

$$\begin{bmatrix} \dot{x}(t) \\ \epsilon \dot{z}(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + Bu(t) + Df(t), \tag{3.37}$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},$$

where $x(t) \in R^{n_1}$ and $z(t) \in R^{n_2}$ are the slow time and fast time state variables, $u(t) \in R^m$ is the control input, ϵ is a small positive parameter. Matrix A_{22} is invertible, that is $\text{rank}(A_{22}) = n_2$. Matrices A, B, D are constant and of appropriate dimension. Furthermore, $f(t) \in R^r$ is an *unknown but bounded exogenous disturbance* with $\|f(t)\| \leq h$.

Disturbance rejection and parameter variation invariance will be achieved if the matching condition of Drazenovic (1969) is satisfied:

$$\text{rank}[B|D] = \text{rank}B. \tag{3.38}$$

Due to this invariance condition, there exists an $m \times r$ matrix G such that

$$D = BG. \tag{3.39}$$

Hence, $D_1 = B_1 G$ and $D_2 = B_2 G$.

Our objective is to find a sliding mode control law to achieve both system stability and disturbance rejection.

3.3 Main Results

In this section, we will present two sliding mode control strategies for the singularly perturbed system (3.37). The two control methods share a similar procedure:

- A state feedback control law is designed to maintain either the fast or slow modes asymptotically stable.
- A discontinuous sliding mode control law for the remaining modes is established to reject disturbances.
- The results are synthesized in a composite control law to ensure the stability and robustness of the whole system.

Before proceeding to the main results, we need the following assumption.

Assumption 3.1. (A_0, B_0) *and* (A_{22}, B_2) *are controllable.*

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad B_0 = B_1 - A_{22}^{-1}B_2.$$

This assumption allows us to construct state feedback control laws separately for the slow and fast subsystems provided the original full-order system is controllable.

3.3.1 Dominating Slow Dynamics Approach

In this approach, a linear state feedback control law is designed to place eigenvalues of the fast subsystem into appropriate positions, and then a sliding mode control law is used for the slow subsystem to exhibit the desired slow time system performance. Although the original singularly perturbed system can be decoupled into two time-scales and into two lower dimensional state vectors, $\xi(t)$ and $z(t)$, the similar type of decomposition does not hold for the control law and the disturbance. As it can be seen in the slow and fast subsystems, the control law $u(t)$ is decomposed only in the time scale, $u_s(t)$ and $u_f(t)$, but not in its dimension. In other words, both $u_s(t)$ and $u_f(t)$ are still m-vectors as its composite version $u(t) = u_s(t) + u_f(t)$. The same holds for the disturbance $f(t)$. Therefore, we allege that when it comes to the subject of disturbance rejection, we can only have one subsystem that enters the sliding mode unless an appropriate dimensional decomposition in the control law can be achieved.

With Assumption 3.1, a continuous fast-time state feedback control law is chosen as

$$u_f(t) = K_f z(t) \tag{3.40}$$

such that $A_{22} + B_2 K_f$ is asymptotically stable. The gain matrix K_f can be chosen appropriately via the eigenvalue placement technique since (A_{22}, B_2) is controllable. Then, the system under the composite control law $u(t) = u_s(t) + u_f(t)$ is defined as

$$\begin{bmatrix} \dot{x}(t) \\ \epsilon \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & (A_{12} + B_1 K_f) \\ A_{21} & (A_{22} + B_2 K_f) \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} (u_s(t) + Gf(t)). \tag{3.41}$$

By the change of variables

$$\xi(t) = x(t) - \epsilon M(\epsilon) z(t), \tag{3.42}$$

the system (3.41) is transformed into a lower triangular form (sensor form) (Kokotovic *et al.*, 1986)

$$\begin{bmatrix} \dot{\xi}(t) \\ \epsilon \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ A_{21} & A_{22} + B_2 K_f + \epsilon A_{21} M \end{bmatrix} \begin{bmatrix} \xi(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B_s \\ B_2 \end{bmatrix} (u_s(t) + Gf(t)) \tag{3.43}$$

where

$$A_s = A_{11} - M A_{21}, \tag{3.44}$$

$$B_s = B_1 - M B_2 \tag{3.45}$$

and M is the solution to the following algebraic equation (Kokotovic *et al.*, 1986)

$$A_{12} + B_1 K_f - M(A_{22} + B_2 K_f) + \epsilon A_{11} M - \epsilon M A_{21} M = 0. \tag{3.46}$$

Matrix M can be found either using the fixed-point iterations or the Newton method (Grodt and Gajic, 1988; Gajic and Lim, 2001).

The system formulation (3.43) is related with its original form (3.37) via state feedback (3.40) and the similarity transformation. Therefore, the controllability of the slow subsystem pair (A_s, B_s) is intact (Chen, 1998). The design objective of the slow-time control law $u_s(t)$ is to stabilize the slow subsystem and simultaneously reject the disturbance $f(t)$ by applying the sliding mode control technique. We choose a sliding surface for the dominating slow dynamics using the method of (Utkin and Young, 1979) as

$$s_s(t) = C_s \xi(t) \tag{3.47}$$

If $m < n_1$, there exists a transformation $T_s \in R^{n_1 \times n_1}$ for the slow subsystem of (3.43) such that (Utkin and Young, 1979)

$$T_s B_s = \begin{bmatrix} 0 \\ B_{s_0} \end{bmatrix} \tag{3.48}$$

Under this transformation, the slow subsystem of (3.43) becomes

$$\begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{bmatrix} = \begin{bmatrix} A_{s_{11}} & A_{s_{12}} \\ A_{s_{21}} & A_{s_{22}} \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_{s_0} \end{bmatrix} (u_s(t) + Gf(t)) \tag{3.49}$$

where

$$\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = T_s \xi(t).$$

The variable $\xi_2(t)$ should be regarded as a control input to the dynamic equation of $\xi_1(t)$. According to (Utkin and Young, 1979), the controllability of (A_s, B_s) implies the controllability of $(A_{s_{11}}, A_{s_{12}})$. As a result, we can find a gain matrix K_1 such that $A_{s_{11}} - A_{s_{12}} K_1$ is stable. Then, a sliding surface can be chosen as

$$s_s(t) = K_1 \xi_1(t) + \xi_2(t) = 0 \tag{3.50}$$

Representing the sliding surface in $\xi(t)$ coordinates, we obtain

$$s_s(t) = [K_1 \quad I_{n_1}] T_s \xi(t) = C_s \xi(t) = 0 \tag{3.51}$$

where $C_s = [K_1 \quad I_{n_1}] T_s$.

We are now in a position to design a sliding control law for the sliding surface (3.51) as C_s is chosen. Taking the derivative of the sliding surface (3.51), we have

$$\dot{s}_s(t) = C_s A_s \xi(t) + C_s B_s u_s(t) + C_s B_s G f(t). \tag{3.52}$$

According to (Utkin and Young, 1979), $C_s B_s$ is nonsingular if B_s has full rank. Assume that B_s has full rank. Based on the control method of (Utkin, 1978), we choose a sliding mode control law $u_s(t)$ as

$$u_s(t) = -(C_s B_s)^{-1}(C_s A_s \xi(t) - S_1 s_s(t) + (\gamma_1 + \sigma_1)\frac{s_s(t)}{\|s_s(t)\|}) \tag{3.53}$$

where $\gamma_1 = \|C_s B_s G\| h$, σ_1 is a positive parameter and matrix S_1 is asymptotically stable. The reaching condition is satisfied since

$$\begin{aligned} s_s^T(t)\dot{s}_s(t) = & -\frac{1}{2}s_s^T(t)P_1 s_s(t) - \sigma_1\|s_s(t)\| - \gamma_1\|s_s(t)\| \\ & + s_s^T(t)C_s B_s G f(t) < -\sigma_1\|s_s(t)\| \end{aligned} \tag{3.54}$$

where

$$P_1 = -S_1^T - S_1 > 0. \tag{3.55}$$

This means that $u_s(t)$ is able to drive the slow variable $\xi(t)$ to reach the sliding surface $s_s(t)$ in a finite time and reject the disturbance $f(t)$. In the following, we will estimate the interval of the reaching time.

Choose a Lyapunov function

$$V(t) = s_s^T(t)s_s(t). \tag{3.56}$$

We have

$$-\lambda_{max}\{P_1\}V(t) - 2(\sigma_1 + 2\gamma_1)\sqrt{V(t)} \leq \dot{V}(t) \leq -\lambda_{min}\{P_1\}V(t) - 2\sigma_1\sqrt{V(t)}. \tag{3.57}$$

Let τ_1 be the time needed to reach the sliding mode ($V(\tau_1) = 0$). Taking the derivative of (3.57), we have

$$\begin{aligned} & \frac{2}{\lambda_{max}\{P_1\}}\ln(\frac{\lambda_{max}\{P_1\}\sqrt{V(0)} + 2\sigma_1 + 4\gamma_1}{2\sigma_1 + 4\gamma_1}) \leq \tau_1 \leq \\ & \qquad \frac{2}{\lambda_{min}\{P_1\}}\ln(\frac{\lambda_{min}\{P_1\}\sqrt{V(0)} + 2\sigma_1}{2\sigma_1}). \end{aligned} \tag{3.58}$$

In other words, the reaching time of the sliding mode lies in the interval

$$\frac{2}{\lambda_{max}\{P_1\}}\ln(\frac{\lambda_{max}\{P_1\}\sqrt{s_s^T(0)s_s(0)}+2\sigma_1+4\gamma_1}{2\sigma_1+4\gamma_1}) \leq \tau_1$$
$$\leq \frac{2}{\lambda_{min}\{P_1\}}\ln(\frac{\lambda_{min}\{P_1\}\sqrt{s_s^T(0)s_s(0)}+2\sigma_1}{2\sigma_1}). \tag{3.59}$$

Remark 3.1. *The sliding mode control law (3.53) offers a flexibility in adjusting the reaching time. Inequalities (3.59) show that choosing appropriate candidates for S_1 (P_1), σ_1, and γ_1 affects the reaching time. Since σ_1 and γ_1 constitute the magnitude of the control effort in sliding mode, their large values are undesired. Hence, we only need to pick up a suitable value of S_1 (P_1) to obtain a fast reaching time.*

From (3.40) and (3.53), the composite control is given by

$$u(t) = K_f z(t) - (C_s B_s)^{-1}(C_s A_s \xi(t) - S_1 s_s(t) - (\gamma_1+\sigma_1)\frac{s_s(t)}{\|s(t)\|}). \tag{3.60}$$

In terms of the original state variables, the control law is rewritten as

$$u(t) = K_f z(t) - (C_s B_s)^{-1}(C_s A_s(x(t)-\epsilon M z(t)) - S_1 C_s(x(t)$$
$$-\epsilon M z(t)) - (\gamma_1+\sigma_1)\frac{C_s(x(t)-\epsilon M z(t))}{\|C_s(x(t)-\epsilon M z(t))\|}). \tag{3.61}$$

When the sliding mode is achieved, one can use the equivalent control method (Utkin, 1977) to study the dynamics of the closed-loop system. The stability of the closed-loop system is guaranteed by the following theorem.

Theorem 3.1. *There exists $\epsilon^* > 0$ such that, in the sliding mode, the closed-loop system is asymptotically stable for $\epsilon \in (0, \epsilon^*]$ and invariant to the external disturbance $f(t)$.*

Proof. To study the dynamics of the closed-loop system under the control law (3.60) or (3.61), we employ the equivalent control method of (Utkin, 1977). The equivalent control of the sliding motion is defined by solving $\dot{s}_s(t) = 0$. This yields

$$u_s^{eq}(t) = -(C_s B_s)^{-1}(C_s A_s \xi(t) + C_s B_s G f(t)). \tag{3.62}$$

Substituting the equivalent control into (3.43) results in the following equivalent dynamics

$$\begin{bmatrix} \dot{\xi}(t) \\ \epsilon \dot{z}(t) \end{bmatrix} = \Phi_1 \begin{bmatrix} \xi(t) \\ z(t) \end{bmatrix} \tag{3.63}$$

where

$$\Phi_1 = \begin{bmatrix} A_s - B_s(C_sB_s)^{-1}C_sA_s & 0 \\ A_{21} - B_2(C_sB_s)^{-1}C_sA_s & A_{22} + B_2K_f + \epsilon A_{21}M \end{bmatrix}.$$

The dynamics of the system (3.63) is defined by the eigenvalues of $A_s - B_s(C_sB_s)^{-1}C_sA_s$ and $A_{22}+B_2K_f+\epsilon A_{21}M$. Matrix $A_s - B_s(C_sB_s)^{-1}C_sA_s$ contains m zeros corresponding to the sliding motion and $n_1 - m$ stable eigenvalues. Since matrix $A_{22} + B_2K_f$ is asymptotically stable, there exists a small $\epsilon^* \geq 0$ such that for all $\epsilon \in [0, \epsilon^*]$ the eigenvalues of $A_{22} + B_2K_f + \epsilon A_{21}M$ have negative real parts. As a result, the closed-loop system is asymptotically stable according to (Utkin, 1977).

□

Remark 3.2. *One can only study the stability of the slow dynamics by applying the equivalent control to the dynamics equation (3.49). However, according to (3.43), the whole transformed system is still affected by u_s. Hence, when proving stability of the closed-loop system, we still need to consider the influence of u_s on the whole system.*

3.3.2 Dominating Fast Dynamics Approach

This approach presents a composite control law that consists of slow state feedback control and fast sliding mode control. According to Assumption 3.1, (A_0, B_0) is controllable. Thus, there exists a gain matrix K_s such that state feedback control

$$u_s(t) = K_s x(t) \tag{3.64}$$

renders matrix $A_0 + B_0K_s$ asymptotically stable. The system under the control law $u(t) = u_s(t) + u_f(t)$ is described as

$$\begin{bmatrix} \dot{x}(t) \\ \epsilon\dot{z}(t) \end{bmatrix} = \begin{bmatrix} A_{11} + B_1K_s & A_{12} \\ A_{21} + B_2K_s & A_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} (u_f(t) + Gf(t)). \tag{3.65}$$

Introducing the change of variables

$$\eta(t) = z(t) + L(\epsilon)x(t) \tag{3.66}$$

the system (3.65) is brought into the actuator form (Kokotovic *et al.*, 1986)

$$\begin{bmatrix} \dot{x}(t) \\ \epsilon\dot{\eta}(t) \end{bmatrix} = \begin{bmatrix} A_{11} + B_1 K_s - A_{12} L & A_{12} \\ 0 & A_f \end{bmatrix} \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_f \end{bmatrix} (u_f(t) + Gf(t)) \tag{3.67}$$

where

$$A_f = A_{22} + \epsilon L A_{12} \tag{3.68}$$

$$B_f = B_2 + \epsilon L B_1 \tag{3.69}$$

and L is the solution to the following algebraic equation (Kokotovic *et al.*, 1986)

$$A_{21} + B_2 K_s - A_{22} L + \epsilon L (A_{11} + B_1 K_s) - \epsilon L A_{12} L = 0. \tag{3.70}$$

This equation can be solved using either the fixed-point iterations or the Newton method (Grodt and Gajic, 1988). Since $L = A_{22}^{-1}(A_{21} + B_2 K_s) + O(\epsilon)$ (Kokotovic *et al.*, 1986), we have

$$A_{11} + B_1 K_s - A_{12} L = A_0 + B_0 K_s + O(\epsilon) \tag{3.71}$$

Because $A_0 + B_0 K$ is asymptotically stable, there exists a small $\epsilon^* > 0$ such that for all $\epsilon \in [0, \epsilon^*]$, $A_{11} + B_1 K_s - A_{12} L$ is asymptotically stable.

We are now in a position to construct a sliding mode control law for the fast subsystem in (3.67). Employing the same technique as in the dominating slow dynamics approach, we choose a sliding surface via the method of (Utkin and Young, 1979) as:

$$s_f(t) = C_f \eta(t) \tag{3.72}$$

If $m < n_2$, there exists a transformation $T_f \in R^{n_2 \times n_2}$ for the fast subsystem of (3.67) such that Utkin and Young (1979)

$$T_f B_f = \begin{bmatrix} 0 \\ B_{f_0} \end{bmatrix}. \tag{3.73}$$

Under this transformation, the fast subsystem of (3.67) becomes

$$\begin{bmatrix} \dot{\eta}_1(t) \\ \dot{\eta}_2(t) \end{bmatrix} = \begin{bmatrix} A_{f_{11}} & A_{f_{12}} \\ A_{f_{21}} & A_{f_{22}} \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_{f_0} \end{bmatrix} (u_f(t) + Gf(t)). \tag{3.74}$$

Consider $\eta_2(t)$ as a control input to the dynamic equation of $\eta_1(t)$. Since $(A_{f_{21}}, A_{f_{22}})$ is controllable, we can find a gain matrix K_2 such that $A_{s_{21}} - A_{s_{22}}K_2$ is asymptotically stable. Then, a sliding surface can be chosen as

$$s_f(t) = K_2\eta_1(t) + \eta_2(t) = 0. \tag{3.75}$$

Representing this sliding surface in the previous coordinates, we obtain

$$s_f(t) = [K_2 \quad I_{n_2}]T_f\eta(t) = C_f\eta(t) = 0 \tag{3.76}$$

where $C_f = [K_2 \quad I_{n_2}]T_f$.

Taking the derivative of the sliding surface (3.76) with respect to t, we have

$$\epsilon\dot{s_f}(t) = C_fA_f\eta(t) + C_fB_fu_f(t) + C_fB_fGf(t). \tag{3.77}$$

With the assumption that the disturbance $f(t)$ is bounded, and B_f has full rank, a control law $u_f(t)$ can be chosen as

$$u_f(t) = -(C_fB_f)^{-1}(C_fA_f\eta(t) - S_2s_f(t) + (\gamma_2 + \sigma_2)\frac{s_f(t)}{\|s_f(t)\|}) \tag{3.78}$$

where $\gamma_2 = \|C_fB_fG\|h$, σ_2 is a positive parameter, and matrix S_2 is asymptotically stable. The reaching condition is satisfied since

$$\epsilon s_f^T(t)\dot{s_f}(t) = -\frac{1}{2}s_f^T(t)P_2s_f(t) - \sigma_2\|s_f(t)\| - \gamma_2\|s_f(t)\| + s_f^TC_fB_fGf(t) < -\sigma_2\|s_f(t)\| \tag{3.79}$$

where

$$P_2 = -S_2^T - S_2 > 0. \tag{3.80}$$

Similarly to (3.59), the reaching time of the sliding mode satisfies

$$\begin{aligned}\frac{2\epsilon}{\lambda_{max}\{P_2\}}\ln(\frac{\lambda_{max}\{P_2\}\sqrt{s_2^T(0)s_2(0)} + 2\sigma_2 + 4\gamma_2}{2\sigma_2 + 4\gamma_2}) &\leq \tau_2 \\ \leq \frac{2\epsilon}{\lambda_{min}\{P_2\}}\ln(\frac{\lambda_{min}\{P_2\}\sqrt{s_2^T(0)s_2(0)} + 2\sigma_2}{2\sigma_2}).\end{aligned} \tag{3.81}$$

Like in the first approach, the reaching time of the sliding mode can be monitored by choosing a suitable value of P_2 (S_2) without affecting the magnitude of the control effort during sliding.

The composite control law is described in terms of the slow state feedback law and the fast sliding mode control law as follows

$$u(t) = K_s x(t) - (C_f B_f)^{-1}(C_f A_f \eta(t) - S_2 s_f(t) - (\gamma_2 + \sigma_2)\frac{s_f(t)}{\|s_f(t)\|}). \tag{3.82}$$

In the original coordinates, the control law (3.82) is

$$\begin{aligned} u(t) =& K_s x(t) - (C_f B_f)^{-1}(C_f A_f (z(t) + Lx(t)) - S_2 C_f (z(t) + Lx(t)) - \\ & (\gamma_2 + \sigma_2)\frac{C_f(z(t) + Lx(t))}{\|C_f(z(t) + Lx(t))\|}). \end{aligned} \tag{3.83}$$

The stability of the closed-loop system under the control law (3.83) is proved in the following theorem.

Theorem 3.2. *Assume (A, B) is controllable. Then there exists $\epsilon^* > 0$ such that in the sliding mode, the closed-loop system is asymptotically stable for $\epsilon \in (0, \epsilon^*]$ and invariant to the external disturbance $f(t)$.*

Proof. Like the dominating slow dynamics approach, we employ the equivalent control method to study the dynamics of the closed-loop system. The equivalent control of the sliding motion of (3.75) is defined by solving $\dot{s}_f(t) = 0$ as follows

$$u_f^{eq}(t) = -(C_f B_f)^{-1} C_f A_f \eta(t) - Gf(t). \tag{3.84}$$

Hence, the equivalent dynamics of the closed-loop system under the equivalent control law (3.84) is given by

$$\begin{bmatrix} \dot{x}(t) \\ \epsilon\dot{\eta}(t) \end{bmatrix} = \Phi_2 \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix} \tag{3.85}$$

where

$$\Phi_2 = \begin{bmatrix} A_{11} + B_1 K_s - A_{12} L & A_{12} - B_1 (C_f B_f)^{-1} C_f A_f \\ 0 & A_f - B_f (C_f B_f)^{-1} C_f A_f \end{bmatrix}.$$

The dynamics of the system (3.85) is specified by the eigenvalues of $A_{11} + B_1 K_s - A_{12} L$ and $\frac{1}{\epsilon}(A_f - B_f (C_f B_f)^{-1} C_f A_f)$. The eigenvalues of matrix $A_f - B_f (C_f B_f)^{-1} C_f A_f$ include m zeros corresponding to the sliding motion and $n_1 - m$ asymptotically stable eigenvalues. In addition, $A_{11} + B_1 K_s - A_{12} L$ is asymptotically stable. As a result, the dynamics of the system (3.85) is represented by $n_1 + n_2 - m$ stable eigenvalues and m zeros. According to (Utkin, 1977), the closed-loop system is asymptotically stable. This implies that the composite control law (3.82) is able to reject external disturbances. $\square$

Remark 3.3. *Inequalities of (3.59) imply that if the initial values of the sliding function $s_s(t)$, σ, and the minimum eigenvalue of P_1 are $O(1)$, then the reaching time is $O(1)$. Meanwhile, inequalities of (3.81) show that the reaching time of the dominating fast dynamics approach is $O(\epsilon)$ if the corresponding initial values of the sliding function $s_f(t)$, σ_2, and the minimum eigenvalue of P_2 are $O(1)$. However, depending on specific systems and appropriate choice of parameters, the reaching time of the dominating slow dynamics design can be made faster than that of the dominating fast dynamics design.*

Remark 3.4. *The applicability of each method depends on the structure of the system under consideration. For example, if the number of the slow variables is larger than the dimension of the control input, the dominating slow dynamics design can be employed. Also, if the dimension of the control input is smaller than the number of the fast variables, the dominating fast dynamics approach can be chosen.*

3.4 Numerical Examples

In this section, we use two examples to demonstrate the efficiency of our methods. All simulations results were implemented using Matlab/Simulink with the ode45 solver.

3.4.1 Example 1

Consider a longitudinal model of an F8 aircraft (Kokotovic *et al.*, 1986) with system matrices given by

$$A_{11} = \begin{bmatrix} -0.195378 & -0.676469 \\ 1.478265 & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} -0.917160 & 0.109033 \\ 0 & 0 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} -0.051601 & 0 \\ 0.013579 & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} -0.367954 & 0.43804 \\ -2.102596 & -0.214640 \end{bmatrix},$$

$$B_1 = D_1 = \begin{bmatrix} -0.023109 \\ -16.945030 \end{bmatrix}, B_2 = D_2 = \begin{bmatrix} -0.048184 \\ -3.810954 \end{bmatrix},$$

and $\epsilon = 0.0336$. The initial condition is

$$\begin{bmatrix} x(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -4 \\ 1 \end{bmatrix}$$

The external disturbance is given by

$$f(t) = 2\sin(3t).$$

a) The dominating slow dynamics approach:

A state feedback control law for the fast subsystem is taken as

$$u_f(t) = K_f z(t) = [2.099531 \quad 1.132588]z(t)$$

M of (3.10) is found using the Newton method (Grodt and Gajic, 1988):

$$M = \begin{bmatrix} 0.632643 & 0.054246 \\ -5.467584 & 3.772558 \end{bmatrix}.$$

Using the change of variables (3.42), we obtain a new system in the form of system (3.43) with

$$A_s = \begin{bmatrix} -0.16347 & -0.676469 \\ 1.144905 & 0 \end{bmatrix}, B_s = \begin{bmatrix} 0.214104 \\ -2.831434 \end{bmatrix}.$$

The slow subsystem of (3.43) is transformed into the normal form (3.50) via the transformation

$$T_s = \begin{bmatrix} 2.831434 & 0.214104 \\ 0 & 1 \end{bmatrix}.$$

According to (3.52), the sliding surface is chosen as

$$s_s(t) = [-2.831434 \quad 0.785896]\xi(t) = C_s\xi(t) = C_s(x + \epsilon M z(t)).$$

The slow sliding mode control is computed from (3.57) as

$$u_s(t) = [0.481251 \quad 0.676469]\xi(t)(t) + s_s(t) + 3\text{sign}(s_s(t)).$$

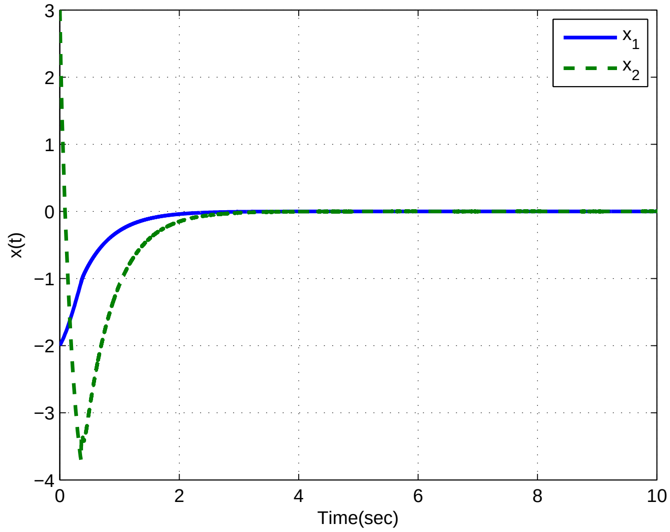

Figure 3.1: Evolution of the slow state variables for the dominating slow dynamics approach

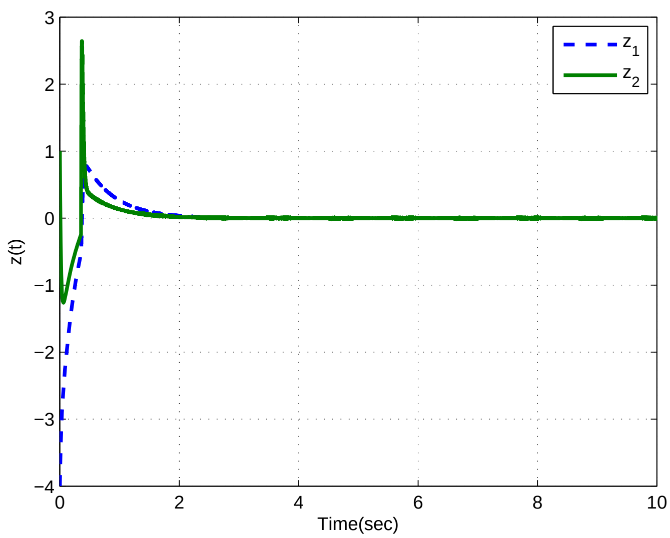

Figure 3.2: Evolution of the fast state variables for the dominating slow dynamics approach

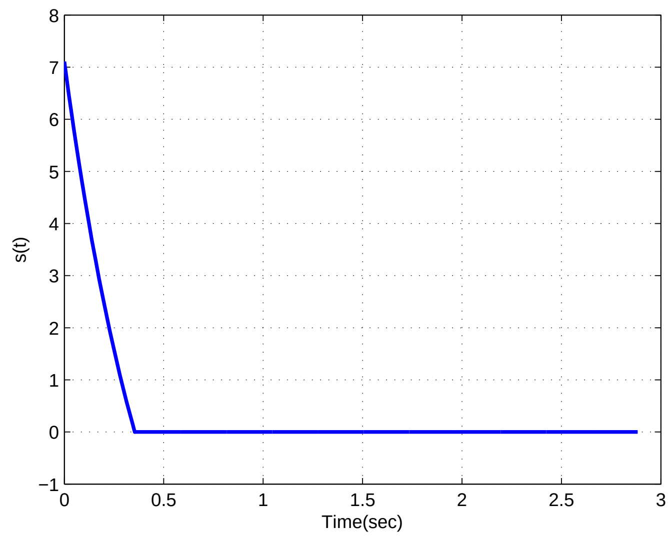

Figure 3.3: Sliding function evolution for the dominating slow dynamics approach

Simulation results of the dominating slow dynamics approach are shown in Fig. 3.1–3.4

With $\gamma = 5.662868$, $\sigma = 2.831434$ and $S_1 = 5.662868$, the reaching time lies in the following interval which is calculated from (3.59)

$$0.31236s < \tau_1 < 0.73914s.$$

In Fig. 3.3, the reaching time is about 0.3545 seconds, which satisfies the above interval. Fig. 3.1 and Fig. 3.2 show the evolution of the state variables for the dominating slow dynamics design. There are jumping phenomena when the switching action begins. In the sliding mode, the behaviors of the state variables exhibit nearly smooth curves. This demonstrates that the impact of the external disturbance is rejected. In the sliding mode, the magnitude of the control is about 5 (Fig. 3.4).

b) The dominating fast dynamics approach:

To make $A_0 + B_0K_s$ stable, we choose $K_s = [-0.585158 \quad 0.296061]$. Using the Newton method (Grodt and Gajic, 1988) to find L of equations (3.70), we get

$$L = \begin{bmatrix} -1.074737 & 0.530216 \\ -0.400169 & 0.227065 \end{bmatrix}.$$

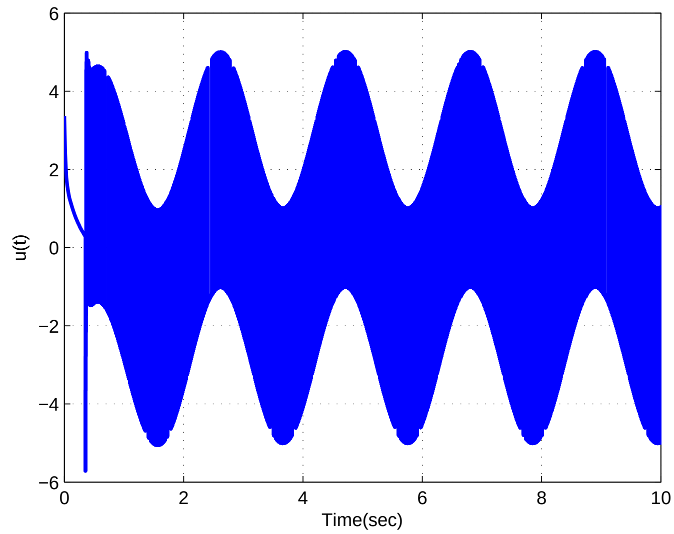

Figure 3.4: Evolution of the composite control law for the dominating slow dynamics approach

As a result, we have the following parameters:

$$A_f = \begin{bmatrix} -0.334834 & 0.434103 \\ -2.090264 & -0.216106 \end{bmatrix}, B_f = \begin{bmatrix} -0.349230 \\ -3.939923 \end{bmatrix}.$$

According to (3.76), the sliding surface can be chosen as

$$s_f(t) = [3.939923 \quad 0.650770]\eta(t) = [3.939923 \quad 0.650770](z(t) + Lx(t)).$$

The fast sliding mode controller is computed from (3.78) as

$$u_f(t) = [-0.680090 \quad 0.398408]\eta(t) - s_f(t) - 3\text{sign}(s_f(t)).$$

The simulation results of the closed-loop system under the composite control $u(t) = u_s(t) + u_f(t)$ are plotted in Fig. 3.5–3.8.

The evolution of the state variables are reflected in Fig. 3.5 and Fig. 3.6 where the impact of the external disturbance is rejected. Evaluating (3.83) with $\gamma_2 = 7.879846$, $\sigma_2 = 3.939923$, $S_2 = 7.879846$ yields

$$0.000953s < \tau_2 < 0.003960s.$$

Fig. 3.7 shows that the reaching time of the sliding mode is about 0.001534 seconds, which satisfies the calculation above. Furthermore, the reaching time of the dominating fast dynamics design is much smaller than that of the dominating slow dynamics

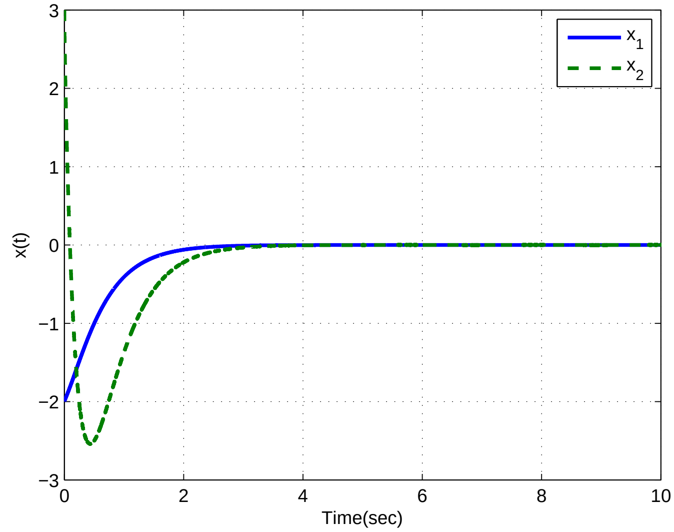

Figure 3.5: Evolution of the slow state variables for the dominating fast dynamics approach

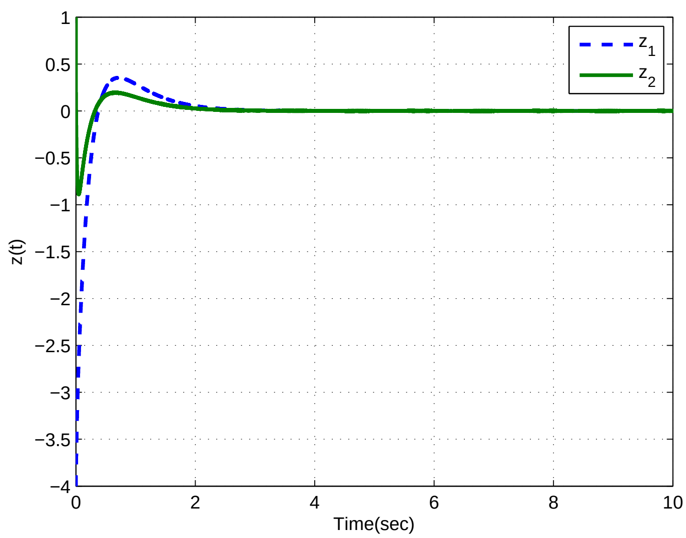

Figure 3.6: Evolution of the fast state variables for the dominating fast dynamics approach

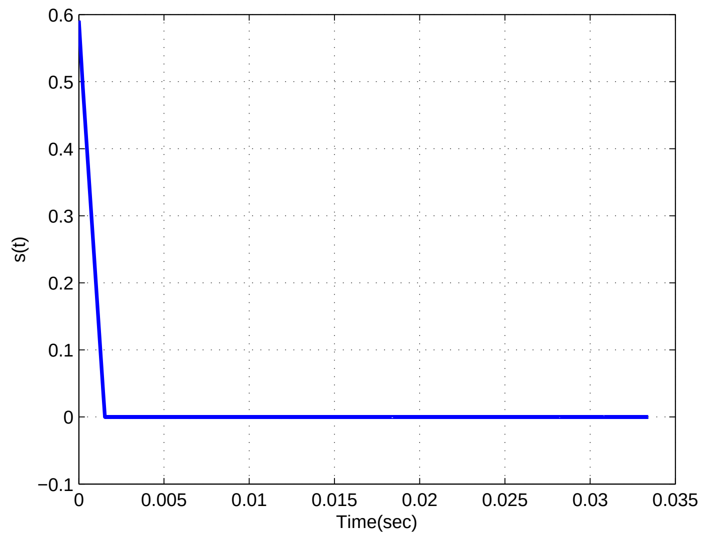

Figure 3.7: Evolution of the sliding function for the dominating fast dynamics approach

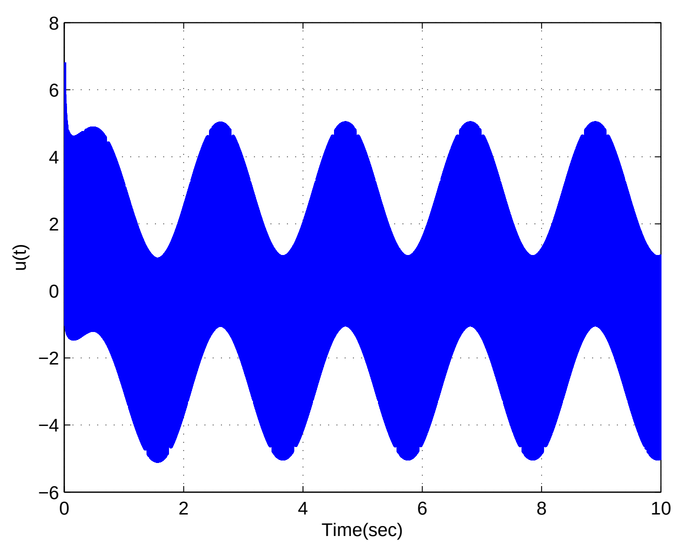

Figure 3.8: Evolution of the composite control law for the dominating fast dynamics approach

approach. The switching control law is depicted in Fig. 3.8 where the magnitude of the control in the sliding mode is about 5. Like in the dominating slow dynamics design, the switching action in the sliding mode plays an important role in rejecting the external disturbance.

3.4.2 Example 2

Consider a magnetic control system (Su, 1999) with system matrices given by

$$
\begin{aligned}
A_{11} &= \begin{bmatrix} 0 & 0.4 \\ 0 & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 & 0 \\ 0.345 & 0 \end{bmatrix}, \\
A_{21} &= \begin{bmatrix} 0 & -0.524 \\ 0 & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} -0.465 & 0.262 \\ 0 & -1 \end{bmatrix}, \\
B_1 &= D_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B_2 = D_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\end{aligned}
$$

and $\epsilon = 0.1$. The initial condition is

$$
\begin{bmatrix} x(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -4 \\ 1 \end{bmatrix}
$$

The external disturbance is

$$
f(t) = 2\sin(3t).
$$

a) The dominating slow dynamics approach:

A state feedback control law for the fast subsystem is taken as

$$
u_f(t) = K_f z(t) = [-3.134447 \quad -1.535000] z(t)
$$

M of (3.10) is found using the Newton method (Grodt and Gajic, 1988):

$$
M = \begin{bmatrix} 0.020354 & 0.002862 \\ -0.448947 & -0.046835 \end{bmatrix}.
$$

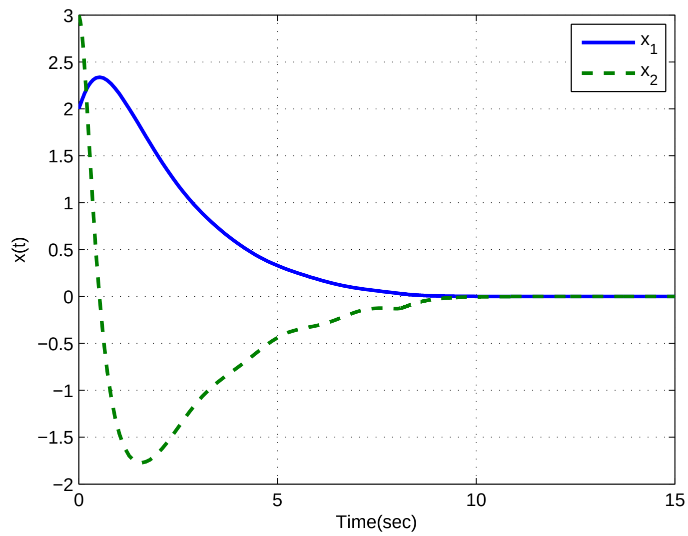

Figure 3.9: Evolution of the slow state variables for the dominating slow dynamics approach

Using the change of variables (3.42), we obtain a new system in the form of system (3.43) with

$$A_s = \begin{bmatrix} 0 & 0.410666 \\ 0 & -0.235248 \end{bmatrix}, B_s = \begin{bmatrix} -0.002862 \\ 0.046835 \end{bmatrix}.$$

The slow subsystem of (3.43) is transformed into the normal form (3.50) via the transformation

$$T_s = \begin{bmatrix} -0.046835 & -0.002862 \\ 0 & 1 \end{bmatrix}.$$

According to (3.52), the sliding surface is chosen as

$$s_s(t) = [4.683471 \quad 1.286239]\xi(t) = C_s\xi(t) = C_s(x + \epsilon M z(t)).$$

The slow sliding mode control is computed from (3.57) as

$$u_s(t) = [0 \quad -34.605849]\xi(t)(t) - 10s_s(t) - 3\text{sign}(s_s(t)).$$

Simulation results of the dominating slow dynamics approach are shown in Fig. 3.9–3.12.

With $\gamma = 0.093669$, $\sigma = 0.046835$ and $S_1 = 0.936694$, the reaching time lies in the following interval which is calculated from (3.59)

$$7.043089s < \tau_1 < 10.415470s.$$

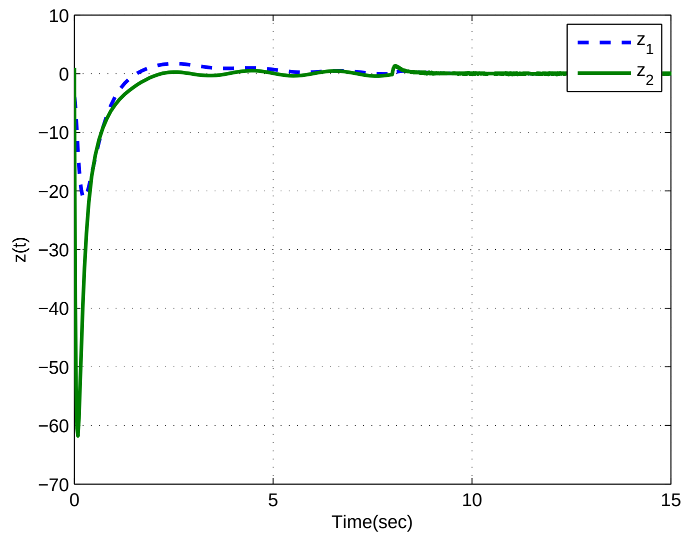

Figure 3.10: Evolution of the fast state variables for the dominating slow dynamics approach

In Fig. 3.11, the reaching time is about 8s, which satisfies the above interval. Fig. 3.9 and Fig. 3.10 show the evolution of the state variables when using the dominating slow dynamics design. There are jumping phenomena when the switching action begins. In the sliding mode, the behaviors of the state variables exhibit nearly smooth curves. This demonstrates that the impact of the external disturbance is rejected. In the sliding mode, the magnitude of the control is about 5 (Fig. 3.12). If we raise the values of σ, or γ, or S_1, the reaching time will be faster but the control magnitude in the sliding mode and the control overshoot will be much higher.

b) The dominating fast dynamics approach:

To make $A_0 + B_0 K_s$ stable, we choose $K_s = [-25.721872 \quad -13.433123]$. Using the Newton method (Grodt and Gajic, 1988) to find L from equation (3.70), we get

$$L = \begin{bmatrix} 68.576717 & 6.844279 \\ 59.905785 & 14.448607 \end{bmatrix}.$$

As a result, we have the following parameters:

$$A_f = \begin{bmatrix} -0.228872 & 0.262000 \\ 0.498477 & -1 \end{bmatrix}, B_f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

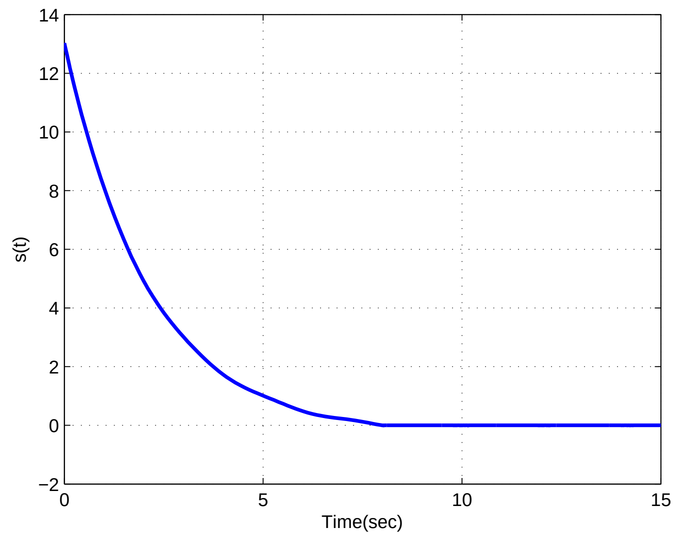

Figure 3.11: Sliding function evolution for the dominating slow dynamics approach

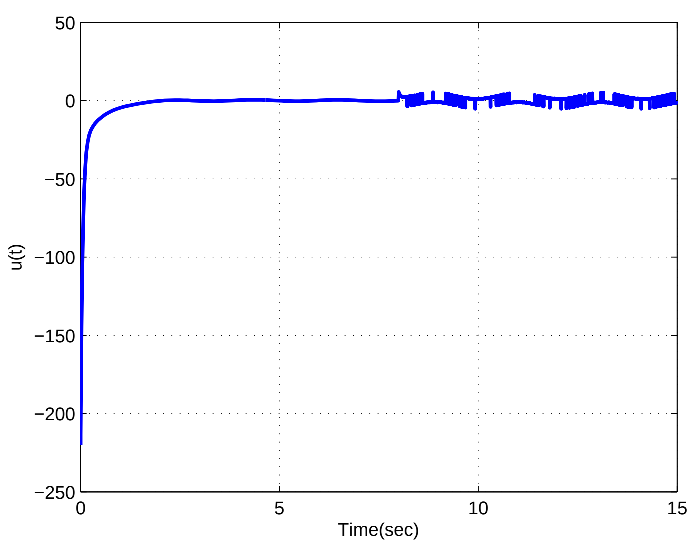

Figure 3.12: Evolution of the composite control law for the dominating slow dynamics approach

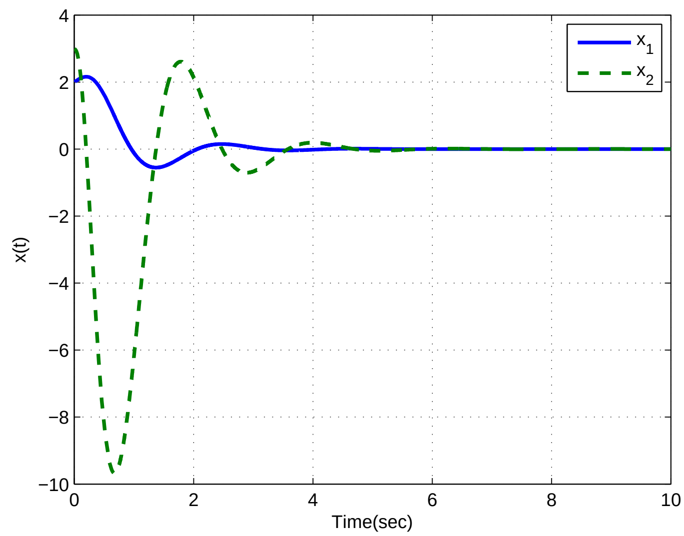

Figure 3.13: Evolution of the slow state variables for the dominating fast dynamics approach

According to (3.76), the sliding surface can be chosen as

$$s_f(t) = [3 \quad 1]\eta(t) = [3 \quad 1](z(t) + Lx(t)).$$

The fast sliding mode controller is computed from (3.78) as

$$u_f(t) = [0.188140 \quad 0.214000]\eta(t) - s_f(t) - 3\text{sign}(s_f(t)).$$

The simulation results of the closed-loop system under the composite control $u(t) = u_s(t) + u_f(t)$ are plotted in Fig. 3.13–3.16.

The evolution of the state variables are reflected in Fig. 3.13 and Fig. 3.16. It can be seen that the impact of the external disturbance is rejected. Evaluating (3.83) with $\gamma_2 = 2$, $\sigma_2 = 1$, $S_2 = 2$ yields

$$0.483663s < \tau_2 < 0.643970s.$$

Fig. 3.15 shows that the sliding mode reaching time is about 0.6s, which satisfies the calculation above. Furthermore, the reaching time of the dominating fast dynamics design is much smaller than that of the dominating slow dynamics approach. The switching control law is depicted in Fig. 3.16 where the magnitude of the control in the sliding mode is about 5.

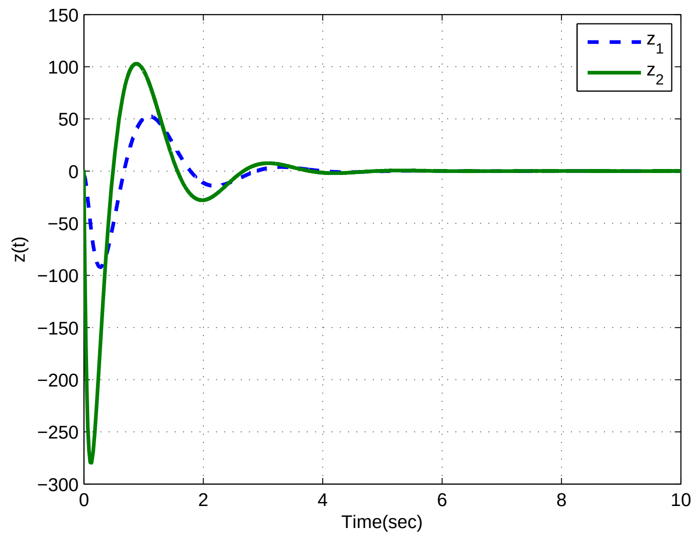

Figure 3.14: Evolution of the fast state variables for the dominating fast dynamics approach

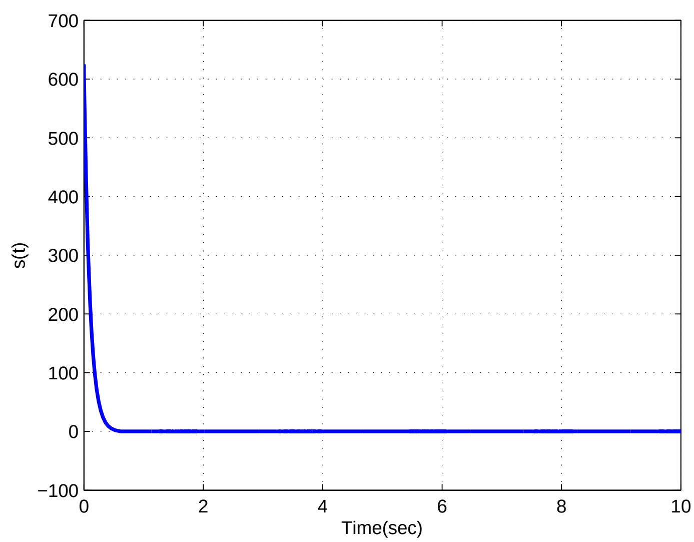

Figure 3.15: Evolution of the sliding function for the dominating fast dynamics approach

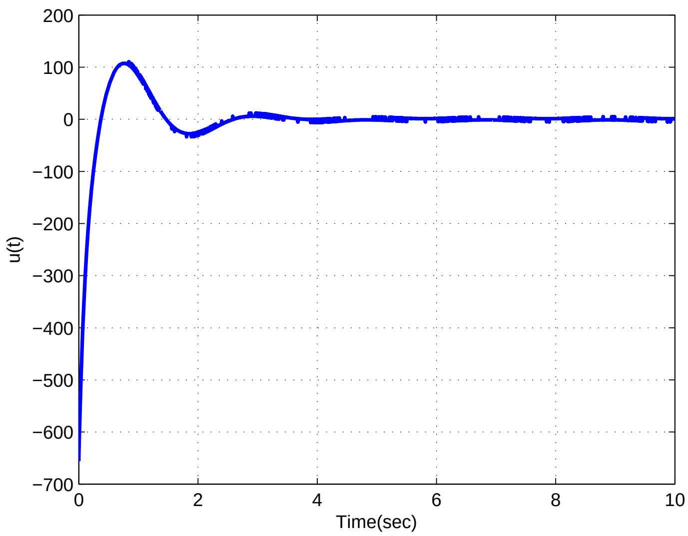

Figure 3.16: Evolution of the composite control law for the dominating fast dynamics approach

3.5 Conclusions

We have presented two sliding mode control approaches for singularly perturbed continuous-time systems. The two proposed methods share a common mechanism: the continuous state feedback and the discontinuous sliding mode design. With the state feedback control ensuring the stability for one set of (slow or fast) modes, the sliding mode control is demonstrated to render the sliding mode such that stability for the other modes can also be guaranteed and that disturbance rejection for the entire system is achieved. The two control laws are combined to construct a composite control law. It has been shown that the closed-loop system under the proposed approaches displays asymptotic stability and robustness gainst external disturbances. The efficiency of the two approaches has been illustrated in the numerical examples.

Chapter 4

Sliding Mode Control for Singularly Perturbed Linear Continuous-Time Systems: Lyapunov Approaches

4.1 Introduction

Chapter 3 presents two composite sliding mode control strategies, that combine state feedback control of either slow or fast modes and sliding mode control of the remaining modes. In this chapter, sliding surfaces are constructed based on the Lyapunov equations of the slow and fast subsystems. Two sliding mode control approaches are proposed to address external disturbances and stabilize the singularly perturbed system in the sliding mode. Numerical examples will be provided to illustrate the efficiency of the proposed methods.

4.2 Problem Formulation

Consider a singularly perturbed system:

$$\begin{bmatrix} \dot{x}(t) \\ \epsilon\dot{z}(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + Bu(t) + Df(t) \tag{4.1}$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},$$

$x(t) \in R^{n_1}$ and $z(t) \in R^{n_2}$ are the slow time and fast time state variables, $u(t) \in R^m$ is the control input, ϵ is a small positive parameter, and A_{22} is assumed to be invertible. Furthermore, $f(t) \in R^r$ is an unknown but bounded exogenous disturbance with $\|f(t)\| \leq h$.

External disturbance rejection will be achieved if the matching condition (Drazenovic, 1969) is satisfied:

$$\operatorname{rank}[B|D] = \operatorname{rank}B. \tag{4.2}$$

Due to this invariance condition, there exists a matrix G of dimension $m \times r$ such that (Drazenovic, 1969)

$$D = BG. \tag{4.3}$$

Hence, $D_1 = B_1G$ and $D_2 = B_2G$.

We will find a discontinuous control law to achieve stability and disturbance rejection by employing the Lyapunov approach of (Su *et al.*, 1996).

4.3 Main Results

In this section, we employ singular perturbation techniques (Kokotovic *et al.*, 1986) to decompose the singularly perturbed system (4.1) into reduced-order subsystems. To that end, two Lyapunov equations are constructed from the two subsystems. Based on the Lyapunov approach (Su *et al.*, 1996), two sliding surfaces are designed from the two Lyapunov equations. The construction of the corresponding control laws is similar to that in Chapter 3.

If we do not take into account the external disturbance $f(t)$, then the singularly perturbed system (4.1) can be decomposed into reduced-order slow and fast subsystems using the standard technique of (Kokotovic *et al.*, 1986). The slow subsystem is given by

$$\begin{aligned} \dot{x}_s(t) &= A_0 x_s(t) + B_0 u_s(t) \\ z_s(t) &= -A_{22}^{-1}(A_{21}x_s(t) + B_2 u_s(t)) \end{aligned} \tag{4.4}$$

and the fast subsystem is described as

$$\epsilon \dot{z}_f(t) = A_{22} z_f(t) + B_2 u_f(t) \tag{4.5}$$

where $z_f(t) = z(t) - z_s(t)$ and $u_f(t) = u(t) - u_s(t)$.

According to (3.1), (A_0, B_0) is controllable; hence, there exists state feedback $u_s(t) = K_0 x_s(t)$ to stabilize the slow subsystem, that is $A_0 + B_0K_0$ is stable (Kokotovic *et al.*, 1986). Similarly, due to controllability of the pair (A_{22}, B_2), there exists state feedback $u_f(t) = K_2 z_f(t)$ such that $A_{22} + B_2 K_2$ is asymptotically stable (Kokotovic *et al.*, 1986).

Then, we can choose a control law of the form

$$u(t) = K_1 x(t) + K_2 z(t) + v(t) \tag{4.6}$$

where $K_1 = (I_{n_1} + K_2 A_{22}^{-1} B_2) K_0 + K_2 A_{22}^{-1} A_{21}$ (Kokotovic *et al.*, 1986). Hence, under the control law (4.6), the closed-loop system is given by

$$\begin{bmatrix} \dot{x}(t) \\ \epsilon \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A_{11} + B_1 K_1 & A_{12} + B_1 K_2 \\ A_{21} + B_2 K_1 & A_{22} + B_2 K_2 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} v(t) + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} f(t). \tag{4.7}$$

The system (4.7) can be decomposed into exact slow and exact fast subsystems by using the Chang transformation (Chang, 1972):

$$\begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} I_{n_1} - \epsilon H L & -\epsilon H \\ L & I_{n_2} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = P \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \tag{4.8}$$

where H and L are solutions of the following algebraic equations

$$T_{21} - T_{22} L + \epsilon L T_{11} - \epsilon L T_{12} L = 0 \tag{4.9}$$

and

$$\epsilon (T_{11} - T_{12} L) H - H (A_{22} + \epsilon L T_{12}) + T_{12} = 0 \tag{4.10}$$

where

$$T_{11} = A_{11} + B_1 K_1, T_{12} = A_{12} + B_1 K_2,$$

$$T_{21} = A_{21} + B_2 K_1, T_{22} = A_{22} + B_2 K_2.$$

H and L can be efficiently found by using either the Newton method or the fixed point iterations (Grodt and Gajic, 1988; Gajic and Lim, 2001), or the eigenvector method (Kecman *et al.*, 1999). The exact reduced-order subsystems are given in the new coordinates by

$$\begin{bmatrix} \dot{\xi}(t) \\ \epsilon \dot{\eta}(t) \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & A_f \end{bmatrix} \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} B_s \\ B_f \end{bmatrix} v(t) + \begin{bmatrix} D_s \\ D_f \end{bmatrix} f(t) \tag{4.11}$$

where

$$A_s = T_{11} - T_{12}L, \tag{4.12}$$

$$A_f = T_{22} + \epsilon L T_{12}, \tag{4.13}$$

$$B_s = (I_{n_1} - \epsilon HL)B_1 - HB_2, \tag{4.14}$$

$$B_f = \epsilon L B_1 + B_2, \tag{4.15}$$

$$D_s = B_s G, \tag{4.16}$$

$$D_f = B_f G. \tag{4.17}$$

Note that A_s and A_f are asymptotically stable (Kokotovic *et al.*, 1986), which means that given positive definite matrices Q_s and Q_f, there exist positive definite matrices P_s and P_f that satisfy the algebraic Lyapunov equations (Gajic and Qureshi, 2008)

$$P_s A_s + A_s^T P_s = -Q_s \tag{4.18}$$

$$P_f A_f + A_f^T P_f = -Q_f. \tag{4.19}$$

We will employ matrices P_s and P_f to design the sliding surfaces.

4.3.1 Dominating Slow Dynamics Approach

A sliding surface can be chosen as

$$s_1(t) = B_s^T P_s \xi(t) + \epsilon B_f^T P_f \eta(t) = 0 \tag{4.20}$$

Choose a Lyapunov function candidate as

$$V_1(t) = \xi^T(t) P_s \xi(t) + \epsilon^2 \eta^T(t) P_f \eta(t) \tag{4.21}$$

Taking its derivative with respect to t leads to

$$\begin{aligned}\dot{V}_1(t) = &-\xi^T(t) Q_s \xi(t) - \epsilon \eta^T(t) Q_f \eta(t) + (\xi^T(t) P_s B_s + \\ &\epsilon \eta^T P_f B_f) v(t) + (\xi^T(t) P_s B_s + \epsilon \eta^T P_f B_f) G f(t)\end{aligned} \tag{4.22}$$

When the sliding mode is achieved, namely $s_1(t) = 0$, we have

$$\dot{V}_1(t) = -\xi^T(t) Q_s \xi(t) - \epsilon \eta^T(t) Q_f \eta(t) < 0. \tag{4.23}$$

This implies the asymptotic stability of the system on the sliding surface $s_1(t)$.

The sliding surface (4.20) can be rewritten in the original coordinates as

$$\begin{aligned} s_1(t) = &(B_s^T P_s(I_{n_1} - \epsilon HL) + \epsilon B_f^T P_f L)x(t) + \\ &\epsilon(B_f^T P_f - B_s^T P_s H)z(t) = R_1 x(t) + \epsilon R_2 z(t) = 0 \end{aligned} \tag{4.24}$$

We now design a control law for the sliding surface (4.20), which is similar to the control laws in Chapter 3. Taking the derivative of the sliding function (4.20), we have

$$\begin{aligned} \dot{s}_1(t) = &(R_1 A_{11} + R_2 A_{21})x(t) + (R_1 A_{12} + R_2 A_{22})z(t) + \\ &(R_1 B_1 + R_2 B_2)u(t) + (R_1 B_1 + R_2 B_2)Gf(t) \end{aligned} \tag{4.25}$$

A control law can be chosen as

$$\begin{aligned} u_1(t) = &-(R_1 B_1 + R_2 B_2)^{-1}((R_1 A_{11} + R_2 A_{21})x(t) + \\ &(R_1 A_{12} + R_2 A_{22})z(t) - A_{p_1} s_1(t) + \\ &(\gamma_1 + \sigma_1)s_1(t)/\|s_1(t)\|) \end{aligned} \tag{4.26}$$

where $\gamma_1 = \|(R_1 B_1 + R_2 B_2)G\|h$, σ_1 is a positive parameter and A_{p_1} is asymptotically stable. We will study the reaching condition and reaching time of the sliding mode when using the control law (4.26). Consider the Lyapunov function

$$V_2(t) = s_1^T(t)s_1(t). \tag{4.27}$$

From (4.25) and (4.26), we have

$$\begin{aligned} \dot{V}_2(t) = &-s_1^T(t)W_1 s_1(t) - 2\sigma_1\|s_1(t)\| - 2\gamma_1\|s_1(t)\| + \\ &2s_1^T(t)(R_1 B_1 + R_2 B_2)Gf(t) \end{aligned} \tag{4.28}$$

where

$$W_1 = -(A_{p_1} + A_{p_1}^T). \tag{4.29}$$

Since A_{p_1} is asymptotically stable, matrix W_1 is positive definite. From (4.28), we have

$$\begin{aligned} -\lambda_{max}\{W_1\}s^T(t)s(t) - (2\sigma_1 + 4\gamma_1)\|s(t)\| \le \dot{V}_2(t) \le \\ -\lambda_{min}\{W_1\}s^T(t)s(t) - 2\sigma_1\|s(t)\| < 0. \end{aligned} \tag{4.30}$$

Hence, the reaching condition is satisfied.

Let τ_1 be the time needed to reach the sliding mode ($V_2(\tau_1) = 0$). Then, integrating (4.30) yields

$$\frac{2}{\lambda_{max}\{W_1\}}\ln(\frac{\lambda_{max}\{W_1\}\sqrt{V_2(0)} + 2\sigma_1 + 4\gamma_1}{2\sigma_1 + 4\gamma_1}) \leq \tau_1 \leq \frac{2}{\lambda_{min}\{W_1\}}\ln(\frac{\lambda_{min}\{W_1\}\sqrt{V_2(0)} + 2\sigma_1}{2\sigma_1}), \tag{4.31}$$

or

$$\frac{2}{\lambda_{max}\{W_1\}}\ln(\frac{\lambda_{max}\{W_1\}\sqrt{s_1^T(0)s_1(0)} + 2\sigma_1 + 4\gamma_1}{2\sigma_1 + 4\gamma_1}) \leq \tau_1 \leq \frac{2}{\lambda_{min}\{W_1\}}\ln(\frac{\lambda_{min}\{W_1\}\sqrt{s_1^T(0)s_1(0)} + 2\sigma_1}{2\sigma_1}). \tag{4.32}$$

These inequalities show that the reaching time τ_1 can be made small by choosing proper values of W_1 and σ_1.

The previous presentation can be summarized in the following theorem.

Theorem 4.1. *The system is asymptotically stable in the sliding mode of the sliding surface (4.30).*

4.3.2 Dominating Fast Dynamics Approach

In (Su, 1999), a method to design a sliding surface for the singularly perturbed system was proposed. However, a switching control has not been provided yet. In this section, we will construct a sliding mode controller for that sliding surface.

The sliding surface under consideration is given by Su (1999):

$$s_2(t) = B_s^T P_s \xi(t) + B_f^T P_f \eta(t) = 0. \tag{4.33}$$

Remark 4.1. *The difference between the sliding surface (4.20) and the sliding surface (4.33) lies in the presence of the fast variables. The quantities which contain the fast variables are $O(\epsilon)$ and $O(1)$ in (4.20) and (4.33) respectively.*

Choose a Lyapunov function candidate as

$$V_3(t) = \xi^T(t) P_s \xi(t) + \epsilon \eta^T(t) P_f \eta(t). \tag{4.34}$$

Taking the derivative with respect to t leads to

$$\begin{aligned}\dot{V}_3(t) = &-\xi^T(t)Q_s\xi(t) - \eta^T(t)Q_f\eta(t) + (\xi^T(t)P_sB_s+\\ &\eta^T P_f B_f)v(t) + (\xi^T(t)P_sB_s + \eta^T P_f B_f)Gf(t).\end{aligned} \tag{4.35}$$

When the sliding mode is achieved, namely $s_2(t) = 0$, we have

$$\dot{V}_3(t) = -\xi^T(t)Q_s\xi(t) - \eta^T(t)Q_f\eta(t) < 0. \tag{4.36}$$

This implies the asymptotic stability of the system.

The sliding surface (4.33) is rewritten in the original coordinates as

$$\begin{aligned}&s_2(t) = (B_s^T P_s(I_{n_1} - \epsilon HL) + B_f^T P_f L)x(t)+\\ &(-\epsilon B_s^T P_s H + B_f^T P_f)z(t) = S_1x(t) + S_2z(t) = 0.\end{aligned} \tag{4.37}$$

Taking the derivative of the sliding function (4.37), we have

$$\begin{aligned}&\epsilon\dot{s}_2(t) = (\epsilon S_1A_{11} + S_2A_{21})x(t) + (\epsilon S_1A_{12} + S_2A_{22})z(t)\\ &+ (\epsilon S_1B_1 + S_2B_2)u(t) + (\epsilon S_1B_1 + S_2B_2)Gf(t).\end{aligned} \tag{4.38}$$

Similar to Chapter 3, a control law can be chosen as

$$\begin{aligned}u_2(t) = &-(\epsilon S_1B_1 + S_2B_2)^{-1}((\epsilon S_1A_{11} + S_2A_{21})x(t)+\\ &(\epsilon S_1A_{12} + S_2A_{22})z(t) - A_{p_2}s_2(t)+\\ &(\gamma_2 + \sigma_2)s_2(t)/\|s_2(t)\|).\end{aligned} \tag{4.39}$$

where $\gamma_2 = \|(\epsilon S_1B_1 + S_2B_2)G\|h$, σ_2 is a positive parameter, and A_{p_2} is asymptotically stable. We will study the reaching condition and the reaching time of the sliding mode under the control law (4.39). Consider the Lyapunov function

$$V_4(t) = s_2^T(t)s_2(t). \tag{4.40}$$

From (4.38) and (4.39), we have

$$\begin{aligned}\dot{V}_4(t) = &\frac{1}{\epsilon}(-s_2^T(t)W_2s_2(t) - 2\sigma_2\|s_2(t)\| - \gamma_2\|s_2(t)\|+\\ &2s_2^T(t)(\epsilon S_1B_1 + S_2B_2)Gf(t))\end{aligned} \tag{4.41}$$

where

$$W_2 = -(A_{p_2} + A_{p_2}^T). \tag{4.42}$$

Note that matrix W_2 is positive definite because A_{p_2} is asymptotically stable. From (4.41), we have

$$\begin{aligned} -\lambda_{max}\{W_2\}s^T(t)s(t) - (\sigma_2 + 2\gamma_2)\|s(t)\| \leq \epsilon\dot{V}_4(t) \leq \\ -\lambda_{min}\{W_2\}s^T(t)s(t) - \sigma_2\|s(t)\|. \end{aligned} \tag{4.43}$$

Hence, the reaching condition is satisfied.

Let τ_2 be the time needed to reach the sliding mode ($V_4(t = \tau_2) = 0$). Integrating (4.43) results in

$$\begin{aligned} \frac{2\epsilon}{\lambda_{max}\{W_2\}}\ln(\frac{\lambda_{max}\{W_2\}\sqrt{V_4(0)} + 2\sigma_2 + 4\gamma_2}{2\sigma_2 + 4\gamma_2}) \leq \tau_2 \leq \\ \frac{2\epsilon}{\lambda_{min}\{W_2\}}\ln(\frac{\lambda_{min}\{W_2\}\sqrt{V_4(0)} + 2\sigma_2}{2\sigma_2}), \end{aligned} \tag{4.44}$$

or

$$\begin{aligned} \frac{2\epsilon}{\lambda_{max}\{W_2\}}\ln(\frac{\lambda_{max}\{W_2\}\sqrt{s_2^T(0)s_2(0)} + 2\sigma_2 + 4\gamma_2}{2\sigma_2 + 4\gamma_2}) \leq \tau_2 \leq \\ \frac{2\epsilon}{\lambda_{min}\{W_2\}}\ln(\frac{\lambda_{min}\{W_2\}\sqrt{s_2^T(0)s_2(0)} + 2\sigma_2}{2\sigma_2}), \end{aligned} \tag{4.45}$$

Like in the dominating slow dynamics design, the reaching time τ_2 can be adjusted by choosing appropriate values of W_2 and σ_2.

Remark 4.2. *Inequalities of (4.45) imply that if the initial conditions of the sliding surface (4.43) are $O(1)$, then the reaching time is $O(\epsilon)$. While in the dominating slow dynamics design, the reaching time of the sliding surface (4.20) can be $O(1)$ with $O(1)$ initial conditions of the sliding function.*

The previous derivations are summarized in the following theorem.

Theorem 4.2. *If the sliding mode of the sliding surface (4.33) is attained, the system will be asymptotically stable.*

4.4 Numerical Examples

In this section, we use the two models in Chapter 3 to demonstrate the efficiency of our methods.

4.4.1 Example 1

We continue to use the aircraft model presented in Chapter 3.

The slow subsystem (4.4) is stabilized by the slow control $u_s(t) = K_0 x_s(t)$ where

$$K_0 = [-2.926472 \quad 0.460080].$$

The fast subsystem (4.5) is stabilized by the fast control $u_f(t) = K_2 z_f(t)$ where

$$K_f = [11.895539 \quad 2.320739].$$

The parameters of the exact reduced-order subsystems (4.8) are given by

$$A_s = \begin{bmatrix} -5.770548 & 0.101252 \\ 56.307006 & -8.140588 \end{bmatrix},$$

$$A_f = \begin{bmatrix} -6.598444 & -0.835219 \\ -48.100451 & -9.172767 \end{bmatrix},$$

$$B_s = \begin{bmatrix} 0.095399 \\ -1.117245 \end{bmatrix},$$

$$B_f = \begin{bmatrix} -0.539225 \\ -3.862600 \end{bmatrix}.$$

We choose $Q_s = Q_f = I$, then the solutions of (4.18) and (4.19) are given by

$$P_s = \begin{bmatrix} 2.854552 & 0.283665 \\ 0.283665 & 0.064949 \end{bmatrix},$$

$$P_f = \begin{bmatrix} 3.766945 & -0.506357 \\ -0.506357 & 0.100615 \end{bmatrix}.$$

The dominating slow dynamics sliding surface is chosen as

$$s_1(t) = [0.036986 \quad -0.056156]x(t) + 0.0336[-0.375811 \quad 0.114168]z(t) = 0.$$

The sliding mode control for this sliding surface is given by

$$u_1(t) = [0.129836 \quad 0.046877]x(t) + [0.254232 \quad 0.346792]z(t) - 3\mathrm{sgn}(s_1(t)) - 5s_1(t).$$

With $\sigma = 0.533728$ and $\gamma = 1.067456$, the reaching time is estimated by (4.32)

$$0.064584s < \tau < 0.248415s.$$

Fig. 4.3 shows that the reaching time is about 0.1 second. The dominating fast dynamics sliding surface is chosen as

$$s_2(t) = [0.289852 \quad -0.129459]x(t) + [-0.085471 \quad -0.107876]z(t) = 0.$$

The sliding mode control for this sliding surface is given by

$$u_2(t) = [0.011024 \quad 0.013481]x(t) + [-0.510192 \quad 0.027058]z(t) - 3\mathrm{sgn}(s_2(t)) - 5s_2(t).$$

With $\sigma_2 = 0.488711$ and $\gamma_2 = 0.977422$, the reaching time is estimated by (4.45)

$$0.007569s < \tau < 0.021193s.$$

Fig. 4.7 shows that the reaching time is about 0.02 second. According to Theorem 4.1 and Theorem 4.2, the reaching time is dependent on the initial values of the sliding function and ϵ. This example show that the reaching time of the second approach is bigger than in the first one because the initial value of the sliding function $s_1(t)$ is much smaller than $s_2(t)$.

4.4.2 Example 2

Consider the magnetic tape control system (Su, 1999).

The slow subsystem of (4.1) is stabilized by the slow control input $u_s(t) = K_0 x_s(t)$ with

$$K_0 = [-6.43047 \quad -5.71656].$$

The fast subsystem of (4.1) is stabilized by the fast control input $u_f(t) = K_2 z_f(t)$ with

$$K_f = [-14.85200 \quad -3.53500].$$

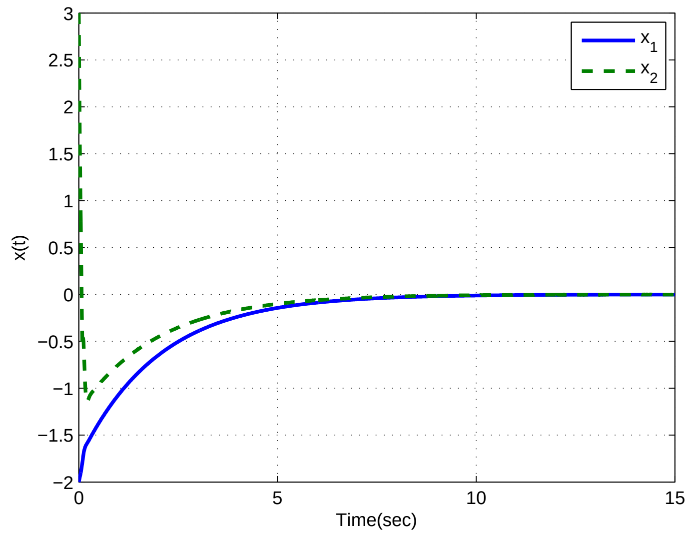

Figure 4.1: Evolution of the slow state variables for the dominating slow dynamics approach

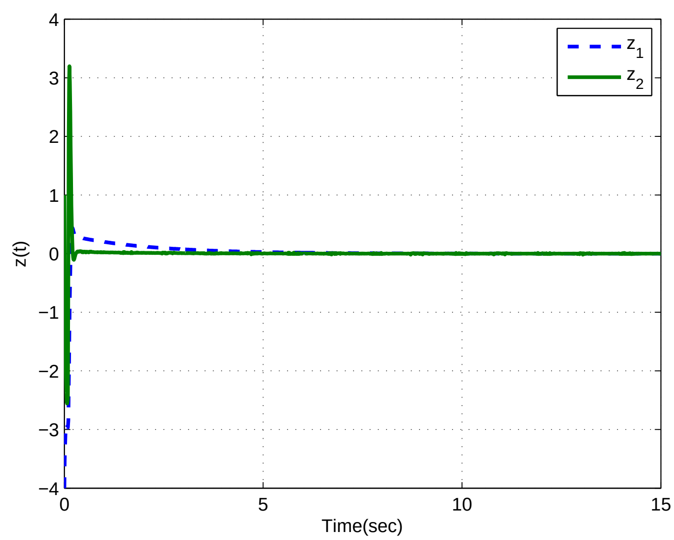

Figure 4.2: Evolution of the fast state variables for the dominating slow dynamics approach

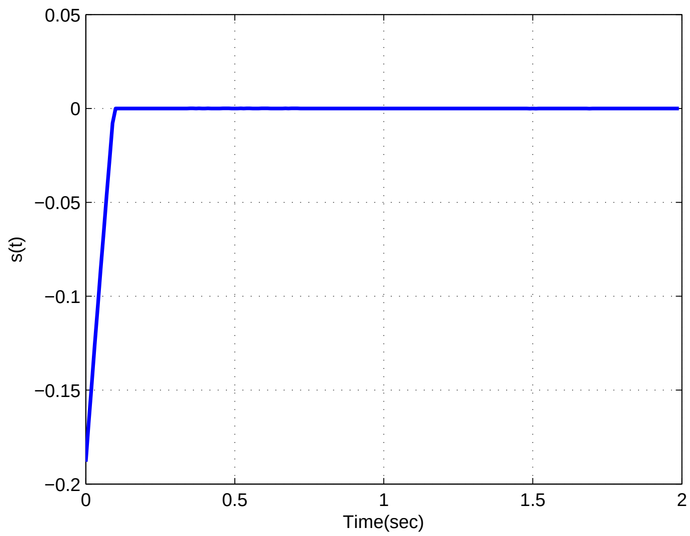

Figure 4.3: Sliding function evolution for the dominating slow dynamics approach

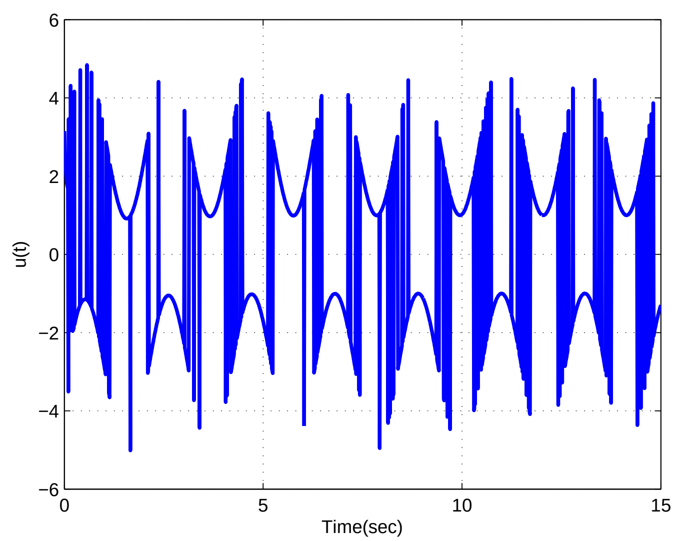

Figure 4.4: Evolution of the sliding mode control law for the dominating slow dynamics approach

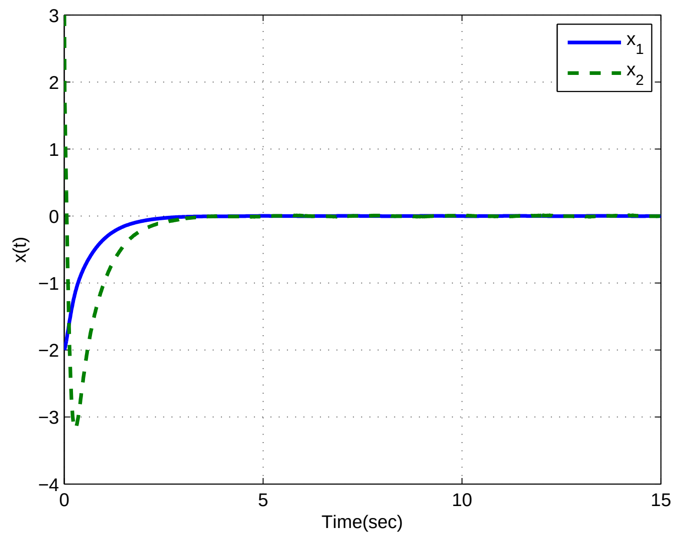

Figure 4.5: Evolution of the slow state variables for the dominating fast dynamics approach

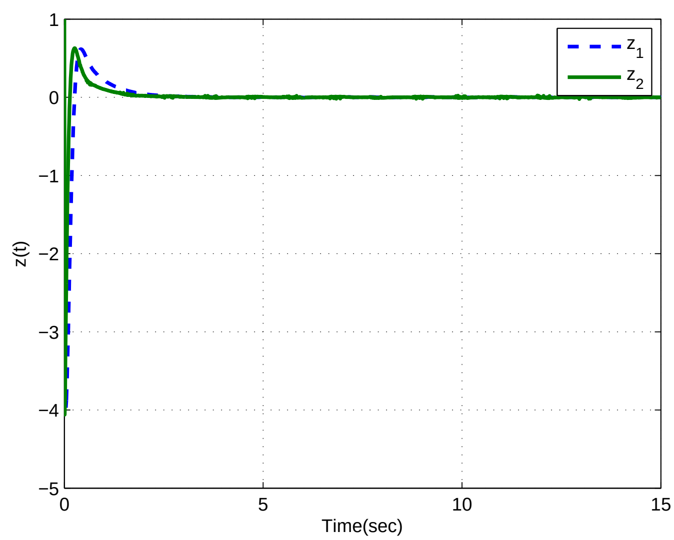

Figure 4.6: Evolution of the fast state variables for the dominating fast dynamics approach

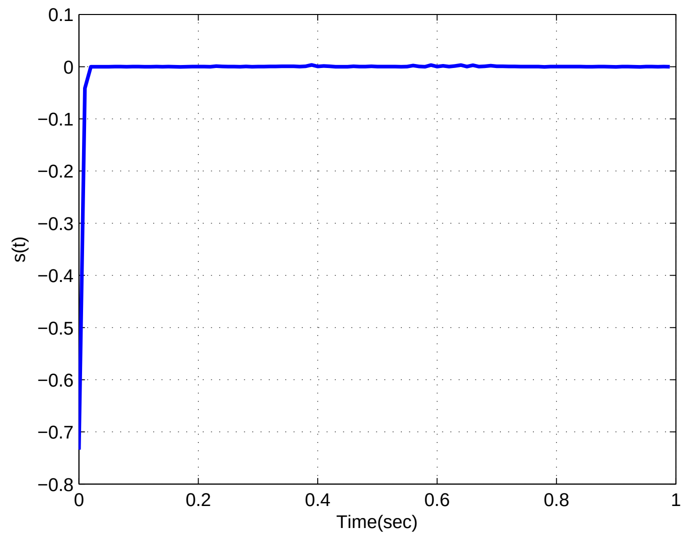

Figure 4.7: Sliding function evolution for the dominating fast dynamics approach

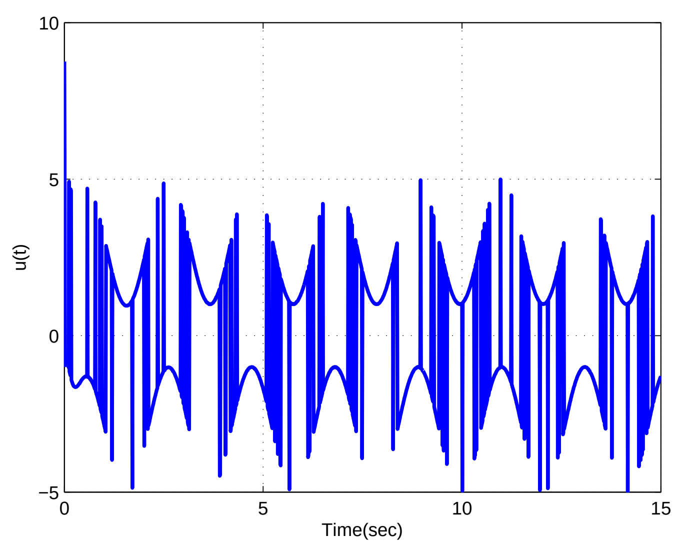

Figure 4.8: Evolution of the sliding mode control law for the dominating fast dynamics approach

The parameters of the exact reduced-order systems (4.8) are given by

$$A_s = \begin{bmatrix} 0 & 0.4 \\ -1.44170 & -1.67648 \end{bmatrix}, B_s = \begin{bmatrix} -0.00073 \\ 0.02036 \end{bmatrix},$$
$$A_f = \begin{bmatrix} -0.29735 & 0.26200 \\ -14.70872 & -4.53500 \end{bmatrix}, B_f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We choose $Q_s = Q_f = I$, then the solutions of (4.18) and (4.19) are given by

$$P_s = \begin{bmatrix} 2.82675 & 0.34681 \\ 0.34681 & 0.38099 \end{bmatrix},$$

$$P_f = \begin{bmatrix} 4.81557 & -0.06336 \\ -0.06336 & 0.10659 \end{bmatrix}.$$

The dominating slow dynamics sliding surface is chosen as

$$\begin{aligned} s_1(t) =& [0.03019 \quad 0.02226]x(t) + \\ & 0.1[-0.060856 \quad 0.10674]z(t) = 0. \end{aligned}$$

The sliding mode control law for this sliding surface is given by

$$\begin{aligned} u_1(t) =& [0 \quad -0.41187]x(t) + [-0.33706 \quad 1.14937]z(t) - \\ & 5s_1(t) - 3\text{sgn}(s_1(t)). \end{aligned}$$

With $\sigma = 0.10674$ and $\gamma = 0.21348$, the reaching time lies in the following interval calculated from (4.32)

$$0.07606 \leq \tau_1 \leq 0.35273.$$

Fig. 4.11 shows that the reaching time is around 0.15s.

The dominating fast dynamics sliding surface is chosen as

$$s_2(t) = [0.24689 \quad 0.14359]x(t) + [-0.06311 \quad 0.10661]z(t) = 0.$$

The sliding mode control for this sliding surface is given by

$$\begin{aligned} u_2(t) =& [0 \quad -0.40282]x(t) + [-0.32173 \quad 1.15509]z(t) - \\ & 10s_2(t) - 3\text{sgn}(s_2(t)). \end{aligned}$$

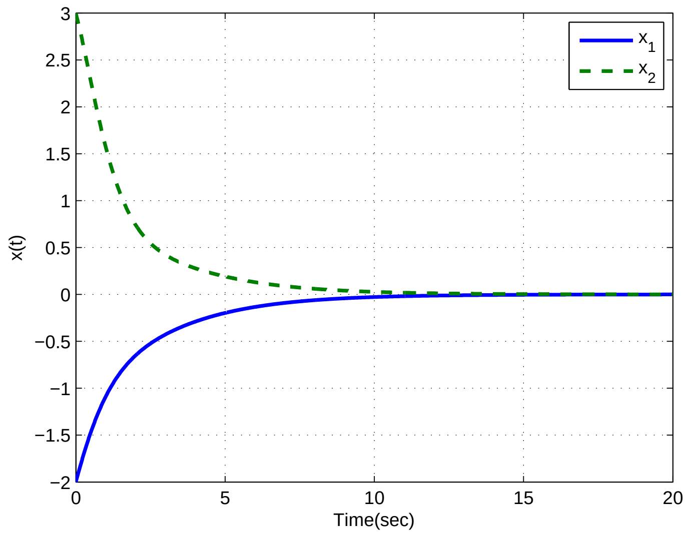

Figure 4.9: Evolution of the slow state variables for the dominating slow dynamics approach

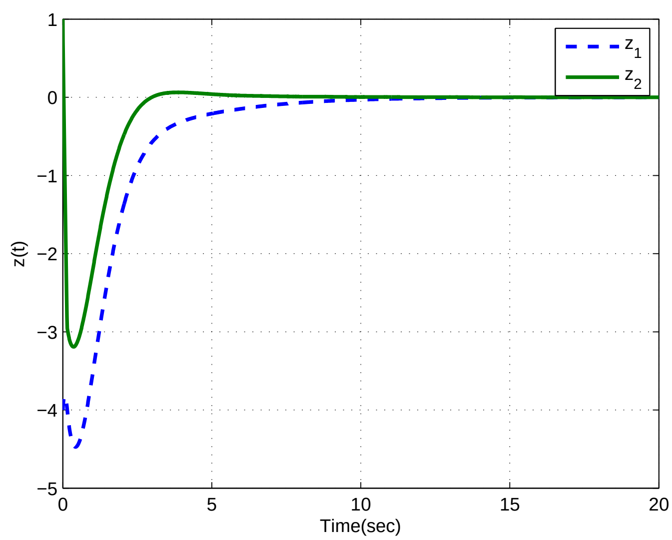

Figure 4.10: Evolution of the fast state variables for the dominating slow dynamics approach

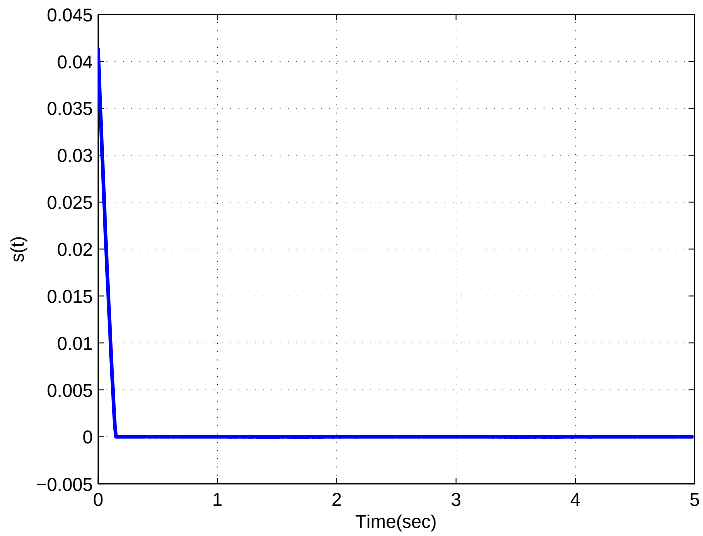

Figure 4.11: Sliding function evolution for the dominating slow dynamics approach

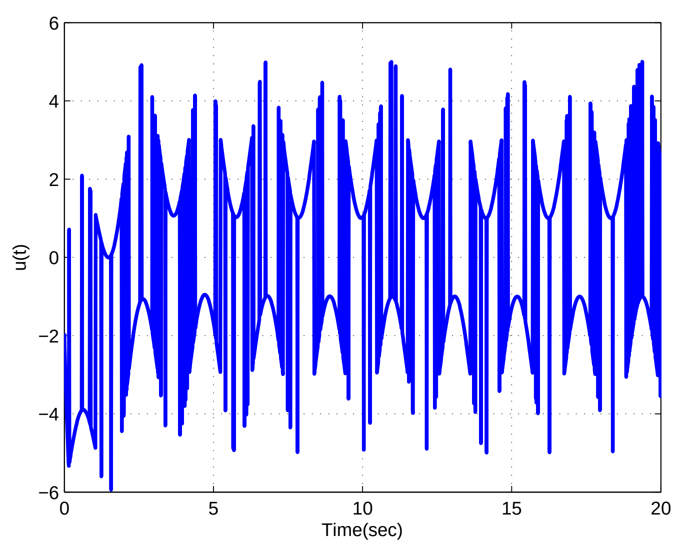

Figure 4.12: The evolution of the sliding mode control law for the dominating slow dynamics approach

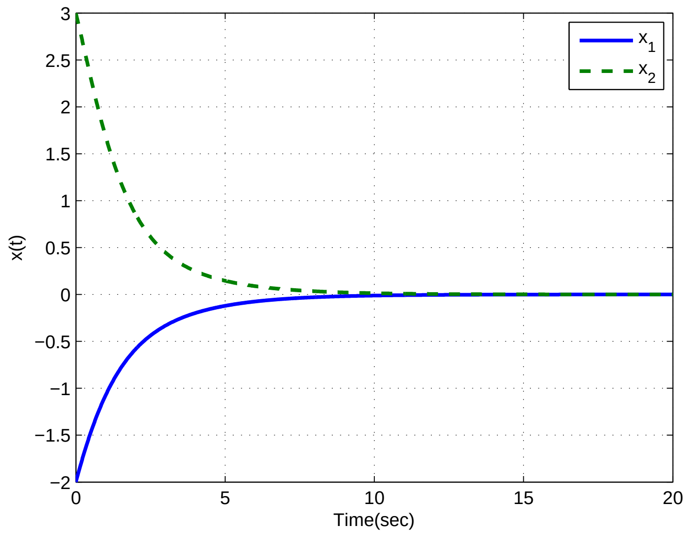

Figure 4.13: Evolution of the slow state variables for the dominating fast dynamics approach

We choose $\sigma_2 = 0.10661$ and $\gamma_2 = 0.21322$. The reaching time is in the interval which is calculated from (4.44)

$$0.04362 \leq \tau \leq 0.1291.$$

Fig. 4.15 shows that the reaching time of the dominating fast dynamics approach is around 0.07s.

The evolution of the state variables for the dominating slow dynamics design is depicted in Fig. 4.9 and Fig. 4.10. Is is observed that the influence of the external disturbance is rejected in the sliding mode.

Fig. 4.13 and Fig. 4.14 show the evolution of the state variables for the dominating fast dynamics design, in which the impact of the external disturbance is rejected. Both sliding mode control laws experience chattering phenomena in the sliding mode, which help stabilize the system and reject the external disturbance.

4.5 Conclusions

Two sliding mode methods have been developed to deal with the stability and disturbance rejection problems. The main ideas of the two approaches are based on Lyapunov functions together with the Chang transformation. Our methods have been proved to

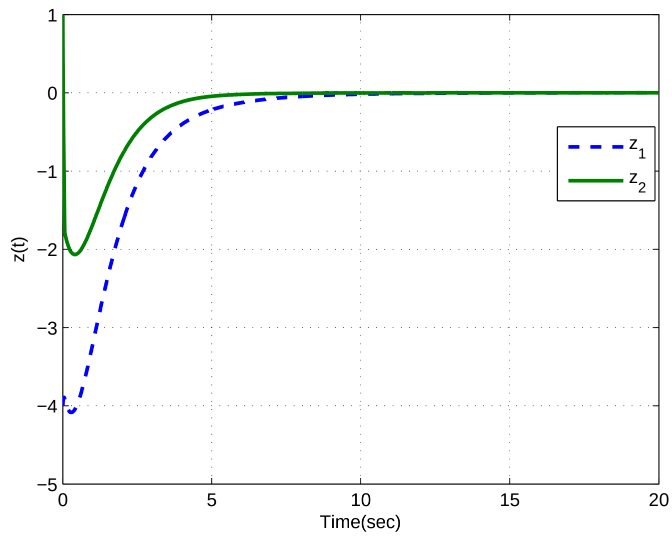

Figure 4.14: Evolution of the fast state variables for the dominating fast dynamics approach

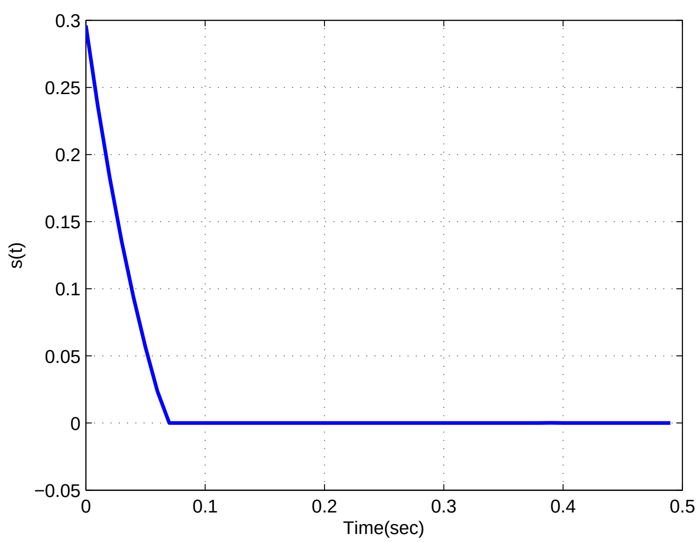

Figure 4.15: Sliding function evolution for the dominating fast dynamics approach

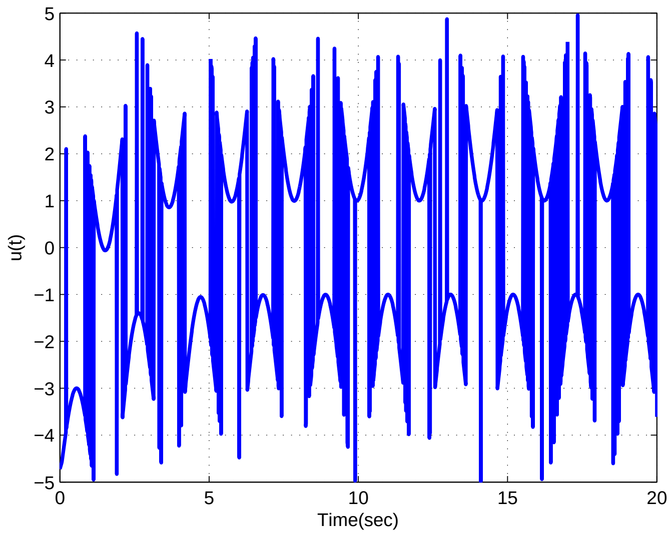

Figure 4.16: Evolution of the sliding mode control law for the dominating fast dynamics approach

provide the asymptotic stability of the closed-loop system and disturbance rejection. This is a distinctive advantage of our methods when compared to other approaches in the literature. The two examples illustrate the effectiveness of the proposed methods.

Chapter 5

Sliding Mode Control for Singularly Perturbed Linear Discrete-Time Systems

5.1 Introduction

As digital controllers have been widely employed for years, the studies on discrete-time sliding mode control have drawn a considerable amount of attention from the control community (Milosavljevic, 1985; Su *et al.*, 2000; Utkin and Drakunov, 1989). The work by Milosavljevic (1985) shows that it is impossible to exactly achieve the sliding motion due to the nature of the sampling process. Instead, a quasi-sliding mode is obtained within a boundary layer of the sliding surface. Su *et al.* (2000) proposed a non switching discrete-time control strategy that obtains an $O(\tau^2)$ thickness of the boundary layer where τ is the sampling period.

The theory of singular perturbation approach has been well studied for the last 50 years. See (Kokotovic *et al.*, 1986; Gajic and Lim, 2001) for an extensive list of references. There have been various works in an attempt to extend singular perturbation ideas to discrete-time systems (Comstock and Hsiao, 1976; Hoppensteadt and Miranker, 1977; Blankenship, 1981; Litkouhi and Khalil, 1984, 1985; Gajic and Lim, 2001). However, there has been sofar only one paper studying sliding mode control for singularly perturbed discrete-time systems (Li *et al.*, 1995b). Li *et al.* (1995b) addressed sliding mode control for a class of discrete-time singularly perturbed systems whose mathematical form was adopted in (Litkouhi and Khalil, 1984, 1985) and references therein. They made good use of the same approach as they did for singularly perturbed continuous-time systems (Li *et al.*, 1995a). A proposed composite control strategy consists of an equivalent control law and a fast sliding mode control law to stabilize the closed-loop system. Nonetheless, parametric uncertainties and external disturbances were not taken into account yet. In this chapter, we propose two novel composite control schemes, that are counterparts of the corresponding approaches for singularly perturbed continuous-time systems.

5.2 Problem Formulation

We study a singularly perturbed discrete-time system, that was formulated and studied by Litkouhi and Khalil (Litkouhi and Khalil, 1985; Gajic and Lim, 2001):

$$\begin{aligned} x[k+1] &= (I_1 + \epsilon A_{11})x[k] + \epsilon A_{12}z[k] + \epsilon B_1 u[k] + \epsilon D_1 f_1[k] \\ z[k+1] &= A_{21}x[k] + A_{22}z[k] + B_2 u[k] + D_2 f_2[k] \end{aligned} \tag{5.1}$$

where $x[k] \in R^{n_1}$ and $z[k] \in R^{n_2}$ are the slow time and fast time state variables, $u[k] \in R^m$ is the control input, ϵ is a small positive parameter. It is assumed that $(I_{n_2} - A_{22})$ is invertible. Furthermore, $f_1[k] \in R^r$ and $f_2[k] \in R^r$ are unknown parametric uncertainty functions which are given by

$$f_1[k] = \Delta A_{11}x[k] + \Delta A_{12}z[k] \tag{5.2}$$

and

$$f_2[k] = \Delta A_{21}x[k] + \Delta A_{22}z[k]. \tag{5.3}$$

Assume that ΔA_{11}, ΔA_{12}, ΔA_{21}, and ΔA_{22} are $O(\epsilon)$. Disturbances rejection and parameter variations invariance will be achieved if the matching condition of Drazenovic (1969) is satisfied:

$$\text{rank}[B|D] = \text{rank}B. \tag{5.4}$$

Due to this invariance condition, there exists a $m \times r$ matrix G such that

$$D = BG. \tag{5.5}$$

Hence, $D_1 = B_1 G$ and $D_2 = B_2 G$.

Like in the continuous case, we will find a composite control to deal with the slow and fast modes as well as the parametric uncertainties

$$u[k] = u_s[k] + u_f[k] \tag{5.6}$$

5.3 Main Results

5.3.1 Dominating Slow Dynamics Approach

In this subsection, a state feedback control law is constructed for the fast subsystem, and a sliding mode control law is established for the slow dynamics. We choose a matrix K_f such that the state feedback

$$u_f[k] = K_f z[k] \tag{5.7}$$

stabilizes the fast modes, namely the eigenvalues of $(A_{22} + B_2 K_f)$ lie in the unit circle. The original system (5.1) under the composite control law $u[k] = u_s[k] + u_f[k]$, is described as follows

$$\begin{aligned} x[k+1] &= (I_{n_1} + A_{11})x[k] + \epsilon(A_{12} + B_1 K_f)z[k] + \epsilon B_1 u_s[k] + \epsilon B_1 G f_1[k] \\ z[k+1] &= A_{21}x[k] + (A_{22} + B_2 K_f)z[k] + B_2 u_s[k] + B_2 G f_2[k]. \end{aligned} \tag{5.8}$$

We will transform system (5.8) into a triangular system by the change of state variables:

$$\xi[k] = x[k] - \epsilon M z[k] \tag{5.9}$$

where M is the solution of the following algebraic equation

$$(A_{12} + B_1 K_f) + \epsilon A_{11} M - \epsilon M A_{21} M - M(A_{22} + B_2 K_f - I_{n_2}) = 0. \tag{5.10}$$

With this change of variables, we have a new system

$$\begin{aligned} \xi[k+1] &= (I_{n_1} + \epsilon A_s)\xi[k] + \epsilon B_s u_s[k] + \epsilon B_s G f_1[k] \\ z[k+1] &= A_{21}\xi[k] + (A_{22} + B_2 K_f + \epsilon A_{21} M)z[k] + B_2 u_s[k] + B_2 G f_2[k]. \end{aligned} \tag{5.11}$$

where

$$A_s = A_{11} - M A_{21}, \tag{5.12}$$

and

$$B_s = B_1 - M B_2. \tag{5.13}$$

The parametric uncertainties are described in terms of the new variables as follow

$$f_1[k] = \Delta A_{11}\xi[k] + (\epsilon \Delta A_{11} M + \Delta A_{12})z[k] \tag{5.14}$$

and

$$f_2[k] = \Delta A_{21}\xi[k] + (\epsilon\Delta A_{21}M + \Delta A_{22})z[k] \tag{5.15}$$

We need the following assumptions.

Assumption 5.1. (A_s, B_s) *is controllable.*

Assumption 5.2. B_s *has full rank.*

If B_s is of full rank, there exists a transformation P_1 such that

$$P_1 B_s = \begin{bmatrix} 0 \\ B_{s_2} \end{bmatrix} \tag{5.16}$$

With the new variables

$$\begin{bmatrix} \xi_1[k] \\ \xi_2[k] \end{bmatrix} = P_1\xi[k], \tag{5.17}$$

the dynamics of $\xi[k]$ is written in the new coordinates as

$$\begin{bmatrix} \xi_1[k+1] \\ \xi_2[k+1] \end{bmatrix} = \begin{bmatrix} I_{m-n_1} + \epsilon A_{s_{11}} & \epsilon A_{s_{12}} \\ A_{s_{21}} & I_m + A_{s_{11}} \end{bmatrix} \begin{bmatrix} \xi_1[k] \\ \xi_2[k] \end{bmatrix} + \begin{bmatrix} 0 \\ B_{s_2} \end{bmatrix} G f_1[k]. \tag{5.18}$$

Since (A_s, B_s) is controllable, $(I_{m-n_1} + \epsilon A_{s_{11}}, \epsilon A_{s_{12}})$ is controllable. There exists a matrix K_1 such that the eigenvalues of $(I_{m-n_1} + \epsilon A_{s_{11}} - \epsilon A_{s_{12}} K_1)$ lie in the unit circle. Hence, a sliding surface is chosen as

$$s_s[k] = K_1\xi_1[k] + \xi_2[k] = [K_1 \quad I_m]P_1\xi[k] = C_s\xi[k] = 0. \tag{5.19}$$

Set $s_s[k+1] = 0$ to find a control law

$$s_s[k+1] = C_s(I_{n_1} + \epsilon A_s)\xi[k] + \epsilon C_s B_s u_s[k] + \epsilon C_s B_s G f_1[k] = 0. \tag{5.20}$$

This leads to

$$u_s[k] = -(\epsilon C_s B_s)^{-1} C_s(I_{n_1} + \epsilon A_s)\xi[k] - G f_1[k]. \tag{5.21}$$

The magnitude of this control law is high gain when the state variables are far from the sliding surface. Setting

$$s_s[k+1] = C_s(I_{n_1} + \epsilon A_s)\xi[k] + \epsilon C_s B_s u_s[k] + \epsilon C_s B_s G f_1[k] = 0. \tag{5.22}$$

Solving (5.22) for $u_s[k]$, we have

$$u_s[k] = -(\epsilon C_s B_s)^{-1} C_s (I_{n_1} + \epsilon A_s)\xi[k] - Gf_1[k]. \tag{5.23}$$

Since $Gf_1[k]$ is unknown at time step k, we approximate it by its past value $Gf_1[k-1]$ which is calculated by

$$Gf_1[k-1] = (\epsilon C_s B_s)^{-1} C_s (I_{n_1} + \epsilon A_s \xi[k-1]) - u_s[k-1]. \tag{5.24}$$

The realizable control is given by

$$u_s[k] = -(\epsilon C_s B_s)^{-1} (I_{n_1} + \epsilon A_s)\xi[k] - Gf_1[k-1]. \tag{5.25}$$

Under the composite control (5.6), the sliding surface is given by

$$\begin{aligned} s_s[k+1] = \epsilon C_s B_s G(f_1[k] - f_1[k-1]) = \epsilon C_s B_s G(\Delta A11(x[k] - x[k-1]) \\ + \Delta A_{12}(z[k] - z[k-1])). \end{aligned} \tag{5.26}$$

The system matrix of the augmented system of $\xi[k], u_s[k], z[k]$ is given by

$$\Phi_1 = \Phi_{11} + \Phi_{12} = \Phi_{11} + O(\epsilon). \tag{5.27}$$

where

$$\Phi_{11} = \begin{bmatrix} I_{n_1} + \epsilon A_s & \epsilon B_s & 0 \\ -(\epsilon C_s B_s)^{-1} C_s (I_{n_1} + \epsilon A_s)^2 & -(\epsilon C_s B_s)^{-1} \epsilon C_s (I_{n_1} + \epsilon A_s) B_s & 0 \\ A_{21} & B_2 & A_{22} + B_2 K_f \end{bmatrix}. \tag{5.28}$$

and

$$\Phi_{12} = \begin{bmatrix} \epsilon B_s G \Delta A_{11} & 0 & \epsilon B_s G(\Delta A_{12} + \epsilon \Delta A_{11} M) \\ 0 & 0 & 0 \\ B_2 G \Delta A_{21} & 0 & \epsilon A_{21} M + B_2 G(\epsilon \Delta A_{21} M + \Delta A_{22}) \end{bmatrix} = O(\epsilon). \tag{5.29}$$

According to (Kato, 1995), we have

$$\lambda\{\Phi_1\} = \lambda\{\Phi_{11}\} + O(\epsilon^{1/p}) \tag{5.30}$$

where p is a positive integer $(0 < p \leq n)$. The eigenvalues of matrix Φ_{11} include those of matrix $A_{22} + B_2K_f$ and Φ_s where

$$\Phi_s = \begin{bmatrix} (I_{n_1} + \epsilon A_s) & \epsilon B_s \\ -(\epsilon C_s B_s)^{-1} C_s (I_{n_1} + \epsilon A_s)^2 & -(\epsilon C_s B_s)^{-1} \epsilon C_s (I_{n_1} + \epsilon A_s) B_s \end{bmatrix}. \tag{5.31}$$

Since $A_{22} + B_2K_f$ is stable, it is enough to show that the latter matrix is stable too. We have

$$\begin{aligned} \det[\lambda I_{n_1} - \Phi_s] &= \det \begin{bmatrix} \lambda I_{n_1} - (I_{n_1} + \epsilon A_s) & -\epsilon B_s \\ (\epsilon C_s B_s)^{-1} C_s (I_{n_1} + \epsilon A_s)^2 & \lambda I_m + (\epsilon C_s B_s)^{-1} \epsilon C_s (I_{n_1} + \epsilon A_s) B_s \end{bmatrix} \\ &= \det \begin{bmatrix} \lambda I_{n_1} - (I_{n_1} + \epsilon A_s) & -\epsilon B_s \\ (\epsilon C_s B_s)^{-1} C_s (I_{n_1} + \epsilon A_s)\lambda & \lambda I_m \end{bmatrix} \\ &= \lambda^m \det \begin{bmatrix} \lambda I_{n_1} - (I_{n_1} + \epsilon A_s) & -\epsilon B_s \\ (\epsilon C_s B_s)^{-1} C_s (I_{n_1} + \epsilon A_s) & I_m \end{bmatrix} \\ &= \lambda^m \det \left[\lambda I_{n_1} - (I_{n_1} + \epsilon A_s)(I_{n_1} - B_s (C_s B_s)^{-1} C_s) \right] = 0. \end{aligned} \tag{5.32}$$

It means the eigenvalues of the matrix Φ_s includes 2m zeros and $n_1 - m$ stable eigenvalues which present the zero dynamics of the sliding mode. Therefore, matrix Φ_0 is stable. Hence, (5.30) implies that there exists a small enough ϵ such that Φ_1 is stable. As a result, the closed-loop system under the composite control law (5.6) is stable. Moreover, the sliding mode is asymptotically achieved due to (5.26).

The above derivations are summarized in the following theorem.

Theorem 5.1. *There exists $\epsilon^* > 0$ such that, under the control law (5.6), the closed-loop system is asymptotically stable for $\epsilon \in (0, \epsilon^*]$.*

5.3.2 Dominating Fast Dynamics Approach

In this subsection, a slow state feedback control law is developed, and a sliding mode control law is constructed to deal with disturbances in the fast dynamics. We choose the slow state feedback control law

$$u_s[k] = K_s x[k] \tag{5.33}$$

such that the eigenvalues of matrix $I_{n_1} + \epsilon(A_0 + B_0 K_s)$ lie in the unit circle where

$$A_0 = A_{11} - A_{12}(I_{n_2} - A_{22})^{-1} A_{21}, \tag{5.34}$$

and

$$B_0 = B_1 - A_{12} A_{22}^{-1} B_2. \tag{5.35}$$

As a result, the system under the composite control law becomes

$$\begin{aligned} x[k+1] &= (I_{n_1} + \epsilon(A_{11} + B_1 K_s))x[k] + \epsilon A_{12} z[k] + \epsilon B_1 u_f[k] + \epsilon B_1 G f_1[k] \\ z[k+1] &= (A_{21} + A_{22} K_s)x[k] + A_{22} z[k] + B_2 u_s[k] + B_2 G f_2[k]. \end{aligned} \tag{5.36}$$

We will transform system (5.36) into a triangular system by the change of state variables:

$$\eta[k] = z[k] + Lx[k] \tag{5.37}$$

where L is the solution of the following algebraic equation

$$A_{21} + B_2 K_s - (A_{22} - I_{n_2})L + \epsilon L(A_{11} + B_1 K_s) - \epsilon L A_{12} L = 0. \tag{5.38}$$

In the new variables, (5.36) is written as

$$\begin{aligned} x[k+1] &= (I_{n_1} + \epsilon(A_{11} + B_1 K_s - A_{12} L))x[k] + \epsilon A_{12} \eta[k] + \epsilon B_1 u_f[k] + \epsilon B_1 G f_1[k] \\ \eta[k+1] &= A_f \eta[k] + B_f u_f[k] + B_f G f_2[k]. \end{aligned} \tag{5.39}$$

where

$$A_f = A_{22} + \epsilon L A_{12}, \tag{5.40}$$

and

$$B_f = B_2 + \epsilon L B_1. \tag{5.41}$$

The parametric uncertainties are given in terms of the new variables by

$$f_1[k] = (\Delta A_{11} + \Delta A_{12} L)x[k] + \Delta A_{12} \eta[k] \tag{5.42}$$

and

$$f_2[k] = (\Delta A_{21} + \Delta A_{22} L)x(k) + \Delta A_{22} \eta[k] \tag{5.43}$$

It can be observed from (5.38) that for sufficiently small ϵ, we have

$$L(0) = -(I_{n_2} - A_{22})^{-1}(A_{21} + A_{22}K_s). \tag{5.44}$$

which implies

$$I_{n_1} + \epsilon(A_{11} + B_1K_s - A_{12}L) = I_{n_1} + \epsilon(A_0 + B_0K_s) + O(\epsilon^2). \tag{5.45}$$

Since $I_{n_1}+\epsilon(A_0+B_0K_s)$ is stable, there exists a small enough ϵ such that the eigenvalues of $I_{n_1} + \epsilon(A_{11} + B_1K_s - A_{12}L)$ lie in the unit circle. We will design a sliding mode control law for the fast modes of (5.39) with the following assumptions.

Assumption 5.3. (A_f, B_f) *is controllable.*

Assumption 5.4. B_f *has full rank.*

If B_f is of full rank, there exists a transformation P_2 such that

$$P_2B_f = \begin{bmatrix} 0 \\ B_{f_2} \end{bmatrix} \tag{5.46}$$

With the new variables

$$\begin{bmatrix} \eta_1[k] \\ \eta_2[k] \end{bmatrix} = P_2\eta[k], \tag{5.47}$$

the dynamics of $\eta[k]$ is written in the new coordinates as

$$\begin{bmatrix} \eta_1[k+1] \\ \eta_2[k+1] \end{bmatrix} = \begin{bmatrix} A_{f_{11}} & A_{f_{12}} \\ A_{f_{21}} & A_{f_{11}} \end{bmatrix} \begin{bmatrix} \eta_1[k] \\ \eta_2[k] \end{bmatrix} + \begin{bmatrix} 0 \\ B_{f_2} \end{bmatrix} u_f[k] + \begin{bmatrix} 0 \\ B_{f_2} \end{bmatrix} Gf[k] \tag{5.48}$$

Since (A_f, B_f) is controllable, $(A_{f_{11}}, A_{f_{12}})$ is controllable. There exists a matrix K_2 such that the eigenvalues of $(A_{f_{11}} - A_{f_{12}}K_2)$ lie in the unit circle. Hence, a sliding surface is chosen as

$$s_f[k] = K_2\eta_1[k] + \eta_2[k] = [K_2 \quad I_m]P_2^{-1}\eta[k] = C_f\eta[k] = 0. \tag{5.49}$$

Set $s_f[k+1] = 0$ to find a control law

$$s_f[k+1] = C_fA_f\eta[k] + C_fB_fu_f[k] + C_fB_fGf_2[k] = 0. \tag{5.50}$$

This leads to the equivalent control

$$u_f[k] = -(C_f B_f)^{-1} C_f A_f \eta[k] - (C_f B_f)^{-1} G f_2[k]. \tag{5.51}$$

Since $Gf_2[k]$ is unknown, we estimate it by $Gf_2[k-1]$ through the following formula

$$Gf_2[k-1] = s_f[k] - C_f A_f \eta[k-1] - C_f B_f u_f[k-1]. \tag{5.52}$$

Hence, the realizable control law is given by

$$u_f[k] = -(C_f B_f)^{-1} C_f A_f (\eta[k] - \eta[k-1] - (C_f B_f)^{-1} s_f[k] + u_f[k-1]. \tag{5.53}$$

The composite control law is given by

$$u[k] = K_s x[k] + u_f[k]. \tag{5.54}$$

Under the composite control law, the sliding surface is given by

$$\begin{aligned} s_f[k+1] &= C_f B_f G(f_2[k] - f_2[k-1]) \\ &= \epsilon C_f B_f G(\Delta A_{21}(x[k] - x[k-1]) + \Delta A_{22}(z[k] - z[k-1])). \end{aligned} \tag{5.55}$$

The system matrix of the augmented system of $x[k], \eta[k], u_f[k]$ is given by

$$\Phi_2 = \Phi_{21} + \Phi_{22} = \Phi_{21} + O(\epsilon) \tag{5.56}$$

where

$$\Phi_{21} = \begin{bmatrix} I_{n_1} + \epsilon(A_{11} + B_1 K_s - A_{12} L) & 0 & 0 \\ 0 & A_f & B_f \\ 0 & -(C_f B_f)^{-1} C_f A_f^2 & -(C_f B_f)^{-1} C_f A_f B_f \end{bmatrix} \tag{5.57}$$

and

$$\Phi_{22} = \begin{bmatrix} \epsilon B_1 G(\Delta A_{11} + \Delta A_{12} L) & \epsilon(A_{12} + B_1 G \Delta A_{12}) & \epsilon B_1 \\ B_f G(\Delta A_{21} + \Delta A_{22} L) & B_f G \Delta A_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} = O(\epsilon). \tag{5.58}$$

According to Kato (1995), we have

$$\lambda\{\Phi_2\} = \lambda\{\Phi_{21}\} + O(\epsilon^{1/p}). \tag{5.59}$$

The eigenvalues of matrix Φ_0 include those of matrix $I_{n_1} + \epsilon(A_{11} + B_1 K_s - A_{12}L)$ and Φ_f where

$$\Phi_f = \begin{bmatrix} A_f & B_f \\ -(C_f B_f)^{-1} C_f A_f^2 & -(C_f B_f)^{-1} C_f A_f B_f \end{bmatrix}. \tag{5.60}$$

Note that $I_{n_1} + \epsilon(A_{11} + B_1 K_s - A_{12}L)$ is stable for a small enough ϵ. Hence, it is enough to show that the latter matrix is stable. We have

$$\begin{aligned} \det[\lambda I_{n_2} - \Phi_f] &= \det \begin{bmatrix} \lambda I_{n_2} - A_f & -B_f \\ (C_f B_f)^{-1} C_f A_f^2 & \lambda I_m + (C_f B_f)^{-1} C_f A_f B_f \end{bmatrix} \\ &= \det \begin{bmatrix} \lambda I_{n_2} - A_f & -B_f \\ (C_f B_f)^{-1} C_f A_f \lambda & \lambda I_m \end{bmatrix} = \lambda^m \det \begin{bmatrix} \lambda I_{n_2} - A_f & -B_f \\ (C_f B_f)^{-1} C_f A_f & I_m \end{bmatrix} \\ &= \lambda^m \det \left[\lambda I_{n_2} - (A_f - B_f (C_f B_f)^{-1} C_f A_f) \right] = 0. \end{aligned} \tag{5.61}$$

It means the eigenvalues of the matrix Φ_s includes 2m zeros and $n_2 - m$ stable eigenvalues which present the zero dynamics of the sliding mode. Therefore, matrix Φ_{21} is stable. Hence, (5.59) implies that there exists a small enough ϵ such that Φ_2 is stable. Therefore, the closed-loop system is asymptotically stable under the composite control law (5.54). In addition, (5.55) implies the sliding motion is asymptotically achieved.

The above results are summarized in the following theorem.

Theorem 5.2. *There exists $\epsilon^* > 0$ such that, under the control law (5.54), the closed-loop system is asymptotically stable for $\epsilon \in (0, \epsilon^*]$.*

5.4 Numerical Example

Consider a discrete-time model of a steam power system (Mahmoud, 1982; Li *et al.*, 1995b)

$$\begin{aligned} x[k+1] &= (I_2 + \epsilon A_{11}) x[k] + \epsilon A_{12} z[k] + \epsilon B_1 u[k] + \epsilon D_1 f_1[k] \\ z[k+1] &= A_{21} x[k] + A_{22} z[k] + B_2 u[k] + D_2 f_2[k] \end{aligned} \tag{5.62}$$

where

$$
A_{11} = \begin{bmatrix} -0.31481 & 0.18889 \\ -0.11111 & -0.41111 \end{bmatrix}, A_{12} = \begin{bmatrix} 0.14074 & 0.055556 & 0.14074 \\ -0.01852 & 0.17037 & 0.41111 \end{bmatrix},
$$

$$
A_{21} = \begin{bmatrix} -0.00600 & 0.46800 \\ -0.71500 & -0.02200 \\ -0.14800 & -0.00300 \end{bmatrix}, A_{22} = \begin{bmatrix} 0.24700 & 0.01400 & 0.04800 \\ -0.02110 & 0.24000 & -0.02400 \\ -0.00400 & 0.09000 & 0.02600 \end{bmatrix},
$$

$$
B_1 = D_1 = \begin{bmatrix} 0.03630 \\ 0.45185 \end{bmatrix}, B_2 = D_2 = \begin{bmatrix} 0.03600 \\ 0.56200 \\ 0.11500 \end{bmatrix},
$$

and $\epsilon = 0.27$. The initial condition is

$$
\begin{bmatrix} x[0] \\ z[0] \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \\ -4 \\ -2 \end{bmatrix}
$$

The parametric uncertainties are

$$
\begin{aligned}
f_1[k] &= ([0.135 \quad 0.054]x[k] + [-0.108 \quad 0.054 \quad 0.054]z[k]), \\
f_2[k] &= ([0.081 \quad -0.054]x[k] + [0.054 \quad 0.054 \quad -0.108]z[k]).
\end{aligned}
$$

We will employ two proposed approaches to construct the sliding mode control laws for the system defined in (5.62).

a) Dominating Slow Dynamics Design

A state feedback control law for the fast subsystem is taken as

$$
u_f[k] = K_f z[k] = [1 \quad -1 \quad 1]z[k]
$$

M of (5.10) is found by using the Newton method (Grodt and Gajic, 1988):

$$
M = \begin{bmatrix} 0.44640 & 0.42081 \\ 1.74865 & 3.10481 \\ 2.63817 & 6.46556 \end{bmatrix}.
$$

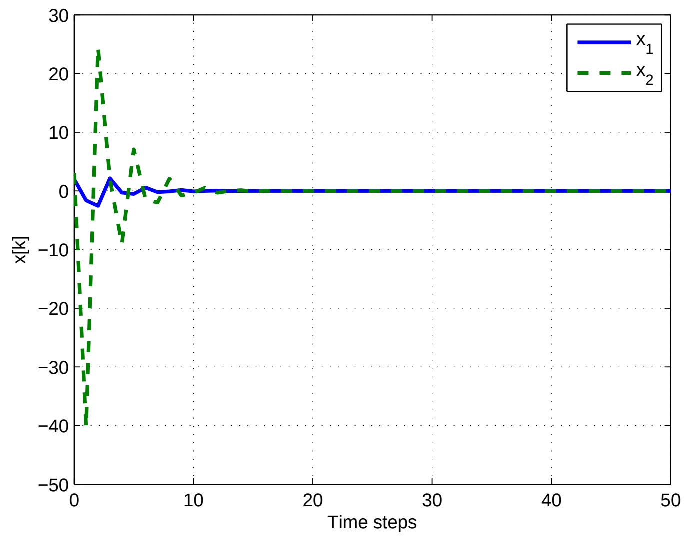

Figure 5.1: Evolution of the slow state variables for the dominating slow dynamics approach.

Using the change of variables (5.9), we obtain a new system in the form of system (5.11) where

$$A_s = \begin{bmatrix} -0.35951 & 0.30820 \\ -0.07892 & -0.12692 \end{bmatrix}, B_s = \begin{bmatrix} 0.07924 \\ 0.44411 \end{bmatrix}.$$

The slow subsystem of (5.11) is transformed into the normal form (5.18) by the transformation

$$T_s = \begin{bmatrix} -0.44411 & 0.07924 \\ 0 & 1 \end{bmatrix}.$$

The sliding surface is chosen as

$$s_s[k] = [-30 \quad 1]T_s\xi[k] = [13.32336 \quad -1.37715]\xi[k] = C_s\xi[k] = C_s(x[k] + \epsilon Mz[k]).$$

The slow sliding mode control is computed from (5.22) as

$$u_s[k] = [-211.68179 \quad 13.33023]\xi[k] + [100.57068 \quad -1.84539]\xi[k-1] + u_s[k-1].$$

The effectiveness of the composite control $u[k] = u_s[k] + u_f[k]$ is illustrated in Fig. 5.1–5.6.

It is seen in Fig. 5.1 that the slow state variables reach the steady state in around 10 steps.

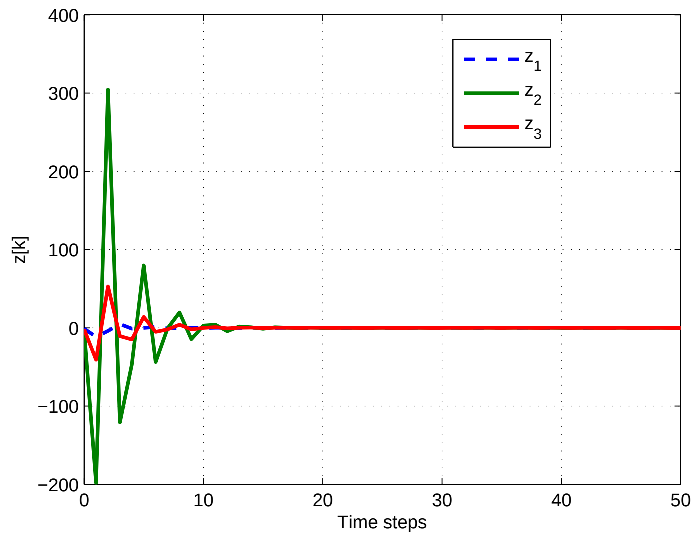

Figure 5.2: Evolution of the fast state variables for the dominating slow dynamics approach.

Fig. 5.2 shows behavior of the fast variable in which the convergence time is about 10 steps .

Fig. 5.3 reveals the reaching time of sliding mode is about 10 steps. After that, the sliding function asymptotically reaches the origin.

Fig. 5.4 shows the evolution of the composite control law. Some peaks appear at the beginning due to the nature of high gain control and the uncertainty. The largest magnitude of the control is about 620. After that, the control converges to a neighborhood of the origin.

b) Dominating Fast Dynamics Approach

First, we design a state feedback control law $u_s[k] = K_s x[k]$ to stabilize the slow subsystem of (5.62). Choose $K_s = [1 \quad 1]$. Using the Newton method (Grodt and Gajic, 1988) to find L of equations (5.38), we get

$$L = \begin{bmatrix} 0.44640 & 0.42081 \\ 1.74865 & 3.10481 \\ 2.63817 & 6.46556 \end{bmatrix}.$$

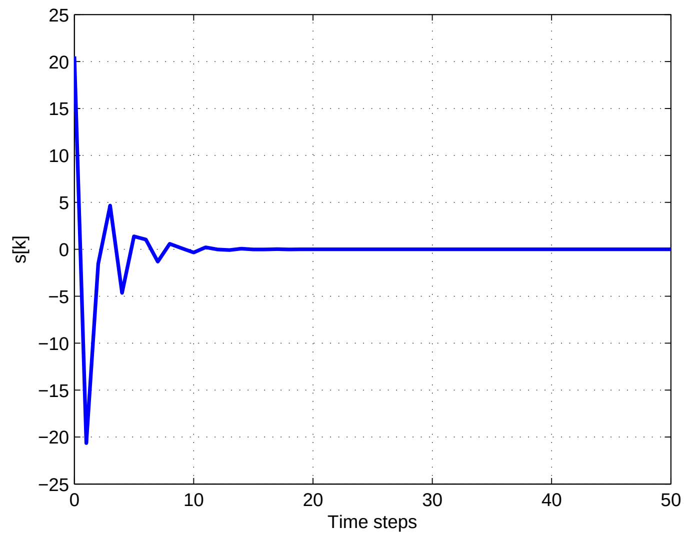

Figure 5.3: Sliding function evolution for the dominating slow dynamics approach.

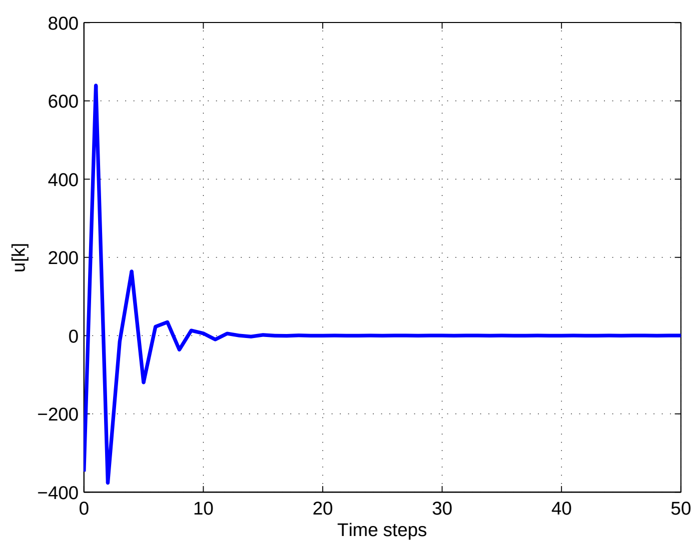

Figure 5.4: Evolution of the composite control law for the dominating slow dominating approach.

As a result, we have the following parameters:

$$A_f = \begin{bmatrix} 0.26186 & 0.04005 & 0.11167 \\ 0.02982 & 0.40905 & 0.38708 \\ 0.06392 & 0.42699 & 0.84393 \end{bmatrix}, B_f = \begin{bmatrix} 0.09171 \\ 0.95792 \\ 0.92965 \end{bmatrix}.$$

Choose the transformation $T = \begin{bmatrix} -0.95792 & 0.09171 & 0 \\ -0.92965 & 0 & 0.09171 \\ 0 & 0 & 1 \end{bmatrix}$. Then, the sliding surface is chosen as

$$\begin{aligned} s_f[k] &= [1 \quad -1 \quad 1]T\eta[k] = [0.90138 \quad 0.09171 \quad 0.81657]\eta[k] \\ &= C_f\eta[k] = C_f(z[k] + Lx[k]). \end{aligned}$$

The fast sliding mode control is computed from (5.53) as

$$\begin{aligned} u_f[k] =& [-1.28257 \quad -0.55289 \quad -1.76610]\eta[k] \\ &+ [0.31299 \quad 0.45424 \quad 0.88774]\eta[k-1] + u_f[k-1]. \end{aligned}$$

The simulation results of the closed-loop system under the composite control $u[k] = u_s[k] + u_f[k]$ are plotted in Fig. 5.5–5.8.

Fig. 5.5 shows the evolution of the slow state variables. These slow variables asymptotically reach the origin at about 30 steps.

It is seen in Fig. 5.7 that the reaching time of sliding mode is achieved at about step 6, that is much faster than that of the sliding surface using the dominating slow dynamics design.

The largest magnitude of the control of the dominating fast dynamics design is about 41 which is much smaller than high gain magnitudes of the control law for the dominating slow dynamics approach in Fig. 5.4.

It is observed that the dominating fast dynamics design offers a faster reaching time than the dominating slow dynamics design.

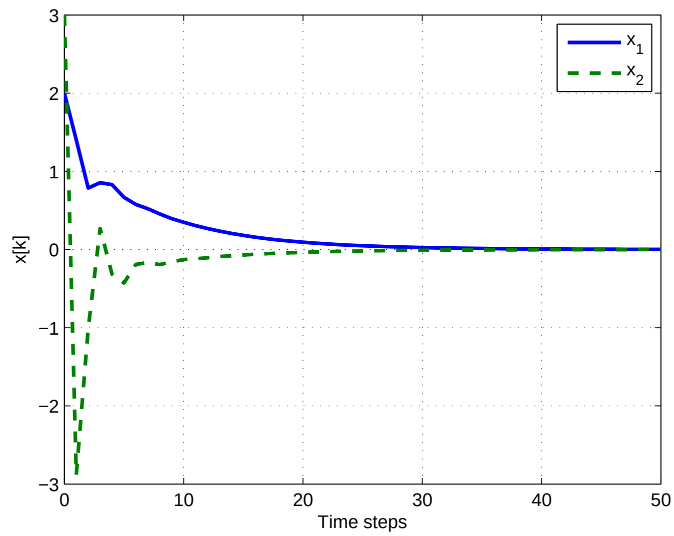

Figure 5.5: Evolution of the slow state variables for the dominating fast dynamics approach.

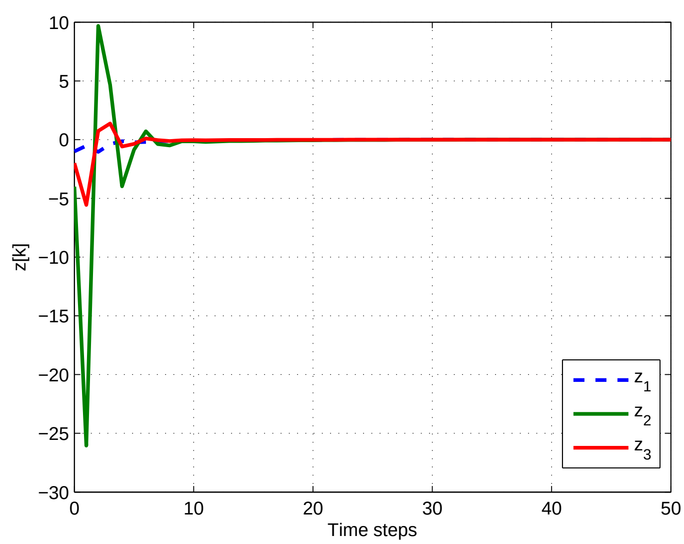

Figure 5.6: Evolution of the fast state variable for the dominating fast dynamics approach.

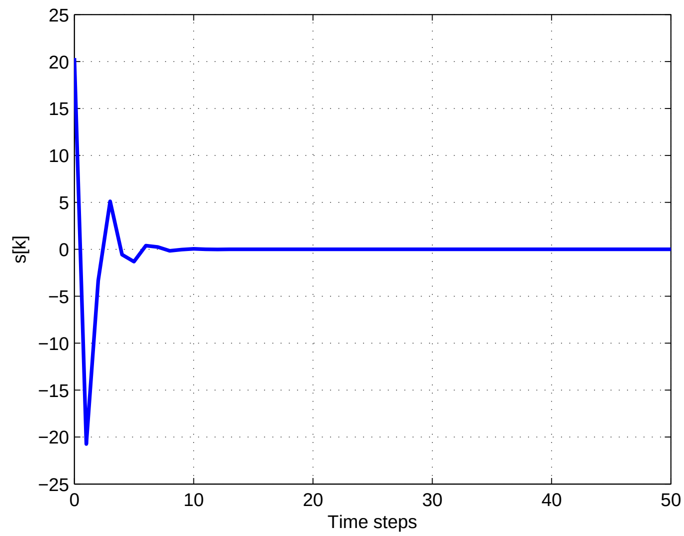

Figure 5.7: Sliding function evolution for the dominating fast dynamics approach.

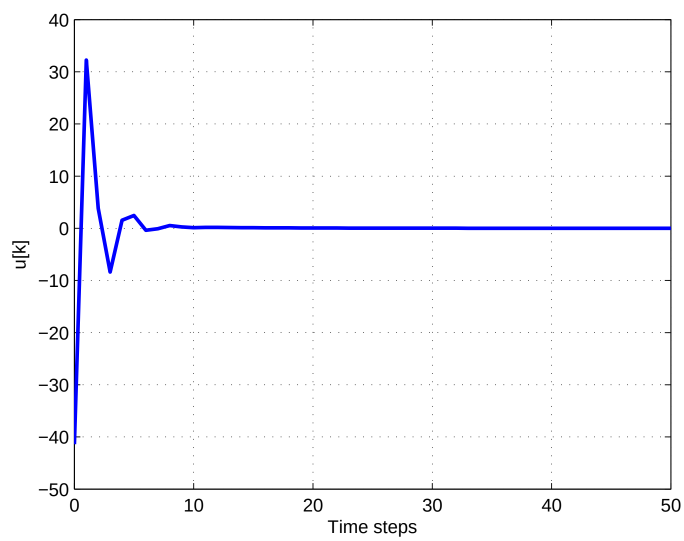

Figure 5.8: Evolution of the composite control law for the dominating fast dynamics approach.

5.5 Conclusions

Two discrete-time sliding mode control strategies have been presented. Like the continuous counterparts, the control laws consist of two components: state feedback control and sliding mode control. The state feedback component is designed such that either the fast or slow subsystem is stable. The sliding mode control component is constructed to alleviate the influence of parametric uncertainties by using the time delayed disturbance approximation technique (Su *et al.*, 2000). The numerical example has demonstrated that the dominating fast dynamics design produces smaller control efforts and faster sliding mode than the dominating slow dynamics design.

Chapter 6

Conclusions

6.1 Conclusions

The dissertation addresses some issues of sliding mode control for singularly perturbed systems. The main objective is to attenuate external disturbances in the case of output feedback control for sampled-data systems or reject exogenous disturbances and parametric uncertainties for singularly perturbed continuous- and discrete-time systems using sliding mode control. In the following, we will summarize the contributions of the dissertation.

Within the context of high gain output feedback, a dynamical system exhibits two-time scale behavior. This leads to employing perturbation techniques to study the stability and robustness of the closed-loop system. The problem is concerned with the digital implementation of sliding mode of a given sliding surface which is constructed for the continuous time system. While many works in the literature used observers to estimate state variables and disturbances, our methods only employ output information. As a result, the structure of our controller is less complicated. In our approaches, we employ the one-step delayed disturbance approximation and the two-step delayed disturbance approximation techniques, in which unmeasured state variables are seen as disturbances. As pointed out in Chapter 2, the closed-loop system is asymptotically stable and exhibits strong robustness against external disturbances. In the one-step delayed disturbance approximation method, the accuracy of the sliding mode is $O(\epsilon^2)$ and the bounds of the state variables are achieved within $O(\epsilon)$. These quantities are at the same order as in the state feedback case. The two-step delayed disturbance approximation approach provides better results with $O(\epsilon^3)$ quasi-sliding motion and $O(\epsilon^2)$ bounds of the state variables.

Most of works in the literature of sliding mode control for singularly perturbed systems did not deal with external disturbances, while others which addressed external disturbances did not offer complete disturbance rejection. We study a more general problem where both external disturbance rejection and stability are taken into account. In this respect, we have proposed unified composite control methods to stabilize singularly perturbed systems and reject external disturbances. First, a state feedback control law is constructed for either slow or fast modes. Then, a sliding mode controller is designed for the remaining modes in a transformed system (a triangular form). The analysis of stability and disturbance rejection is realized by Utkin's method. The results of our work are proved to be much better than the other works available in the control literature.

In the same light of the problem, Lyapunov functions are employed to tackle the disturbance rejection and stability. In this direction, a state feedback control law is first constructed to stabilize the system and then a decoupled system is studied, which leads to the design of a sliding surface and a sliding mode control law. Like composite control approaches, the Lyapunov methods yield the stability and disturbance rejection of the closed-loop system.

The last contribution of the dissertation is presented in Chapter 5. In that chapter, the study of sliding mode control for singularly perturbed continuous time system is extended to the discrete time case in which parametric uncertainties are taken into consideration. A composite control law includes a state feedback component which stabilizes either slow or fast modes and a sliding mode one which deals with the remaining modes. By employing the one step delayed estimation, we have showed that the closed-loop system is asymptotically stable provided the perturbation parameter is small enough.

6.2 Future Work

The results of this dissertation on singularly perturbed systems can be applied to practical problems in chemical industry, wireless communications, mechanical systems, power systems, systems biology, where multi-time scale phenomena appear and disturbances need to be addressed. In addition, the methodology can be applied to weakly-coupled systems.

Based on the results obtained, some potential works will be considered for implementation in the future to real physical systems. Most of the existing works on output feedback sliding mode control pose an assumption on the relative degree of the system under consideration. The future work could focus on the problem of output feedback sliding mode control for systems with an arbitrary relative degree. In addition, investigating disturbance attenuation using sliding mode control for sampled-data nonlinear systems will be very challenging since unlike the linear case, a discrete-time nonlinear model is not generally exactly computed from a continuous-time nonlinear system. In this case, efforts to study the approximative model will be carried out in order to find a possible sliding mode control law. This direction is promising in finding applications to many practical nonlinear systems.

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Publication List

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1. Thang Nguyen, and Zoran Gajic. Solving the differential Riccati equation: a Lyapunov equation approach, *IEEE Transactions on Automatic Control*, Jan 2010, pp. 191-194.
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