SLIDING MODE CONTROL OF CONTINUOUS-TIME WEAKLY COUPLED SYSTEMS

by

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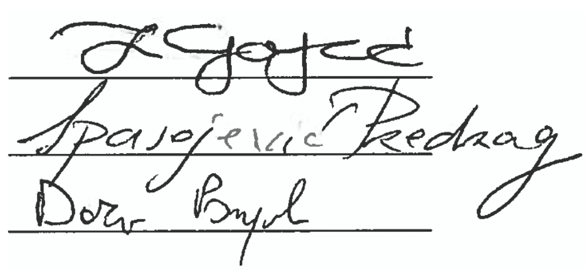

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ABSTRACT OF THE THESIS

Sliding Mode Control of Continuous-time Weakly Coupled Systems

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Sliding mode control is a form of variable structure control which is a powerful tool to cope with external disturbances and uncertainty. There are many applications of sliding mode control of weakly coupled system to absorption columns, catalytic crackers, chemical plants, chemical reactors, helicopters, satellites, flexible beams, cold-rolling mills, power systems, electrical circuits, computer/communication networks, etc. In this thesis, the problem of sliding mode control for systems, which are composed of two weakly coupled subsystems, is firstly addressed.

This thesis presents three methods to study continuous-time linear weakly coupled systems using sliding mode control. First one is Utkin and Young's sliding mode control method for each subsystem using a decoupling transformation technique. Next one is a composite control approach composed of two controllers, which are a state feedback controller and a sliding mode controller. The last one is a sliding mode control technique using the Lyapunov approach. These methods provide controls which make the systems asymptotic stabile with a robust performance against parametric uncertainties and exogenous disturbances.

In this thesis, we demonstrate the effectiveness of the proposed methods through theoretical and simulation results.

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Chapter 1

Introduction

1.1 Continuous-Time Sliding Mode Control

Continuous-time sliding mode control has been recognized as a robust control approach, which yields to reject matched disturbances and system uncertainties. The design of sliding mode control is achieved in two steps. Firstly, a sliding surface is described which ensures the system to remain on a plane after reaching it from any initial conditions in a finite time. Secondly, discontinuous control is designed to render a sliding mode.

Consider the following single input linear system (Sinha, 2007).

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{1.1}$$

If $x_2(t) = -\lambda x_1(t)$, where $\lambda > 0$, then $x_1(t)$ and $x_2(t)$ are asymptotically stable because (1.1) yields $\dot{x}_1(t) = -\lambda x_1(t)$. Define a line as follows

$$s(t) = x_2(t) + \lambda x_1(t), \ \lambda > 0 \tag{1.2}$$

Figure 1.1 shows a sliding line in the state space. The control objectives are to design

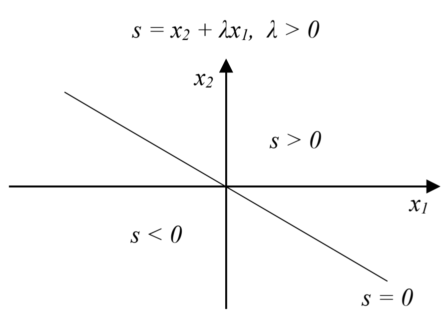

Figure 1.1: A sliding line $s(t) = 0$

$u(t)$ to ensure that the system reaches the sliding line from any initial condition in a finite time and stay on the line after reaching it. The conditions to achieve these objectives are called reaching and sliding conditions. The reaching condition provides that the system state reaches the sliding surface in a finite time, whereas the sliding condition facilitates that the system state slides on the sliding line towards the origin.

The reaching condition is described as (Utkin, 1977; Young, 1978)

$$\dot{s}(t) = -\delta \text{sgn}(s(t)),\ \ \delta > 0 \tag{1.3}$$

where the signum function $\text{sgn}(s(t))$ is defined as follows

$$\text{sgn}(s(t)) = \begin{cases} +1 \text{if } s(t) > 0 \\ 0 \text{if } s(t) = 0 \\ -1 \text{if } s(t) < 0 \end{cases} \tag{1.4}$$

which yields to the following condition (Sinha, 2007, Chap. 6)

$$\dot{s}(t)s(t) < 0 \tag{1.5}$$

The equivalent control $u_{eq}(x(t))$ is obtained when the system remains on the sliding mode, that is $\dot{s}(t) = 0$. From (1.2), we have

$$\dot{s}(t) = \alpha x_1(t) + (\beta + \lambda)x_2(t),\ \ \lambda > 0 \tag{1.6}$$

For $\dot{s}(t) = 0$, it follows

$$u_{eq}(t) = -\alpha x_1(t) - (\beta + \lambda)x_2(t),\ \ \lambda > 0 \tag{1.7}$$

Therefore, the control law to satisfy the reaching condition (1.3) is

$$u(t) = u_{eq}(t) - \delta \text{sgn}(s(t)),\ \ \delta > 0 \tag{1.8}$$

The sliding condition (Young, 1978)

$$\lim_{s(t)\to 0^+} \dot{s}(t) < 0,\ \ \lim_{s(t)\to 0^-} \dot{s}(t) > 0 \tag{1.9}$$

is sufficient and local.

1.1.1 Constructing Sliding Surfaces of MIMO system

Utkin and Young's Method

Consider a continuous-time linear system which is given by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.10}$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, and A, B are constant matrices of appropriate dimensions, and B has full rank.

There exists a similarity transformation defined by (Utkin and Young, 1978)

$$q(t) = Hx(t) \tag{1.11}$$

with

$$H = \begin{bmatrix} N & B \end{bmatrix}^T \tag{1.12}$$

and columns of the $n \times (n-m)$ matrix N composed of basis vectors in the null space of B^T, which puts (1.10) into the form

$$\dot{q}(t) = \bar{A}q(t) + \bar{B}u(t) \tag{1.13}$$

with $\bar{A} = HAH^{-1}$ and $\bar{B} = HB = \begin{bmatrix} 0 \\ \bar{B}_r \end{bmatrix}$. Equation (1.13) is decomposed as follows

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{B}_r \end{bmatrix} u(t) \tag{1.14}$$

where $q_1(t) \in \mathbf{R}^{n-m}$, $q_2(t) \in \mathbf{R}^m$, and $\bar{B}_r$ is an $m \times m$ nonsingular matrix.

Equation (1.14) yields

$$\dot{q}_1(t) = \bar{A}_{11}q_1(t) + \bar{A}_{12}q_2(t) \tag{1.15}$$

and

$$\dot{q}_2(t) = \bar{A}_{21}q_1(t) + \bar{A}_{22}q_2(t) + \bar{B}_r u(t) \tag{1.16}$$

$q_2(t)$ is treated as a control input to the system (1.15) and a state feedback gain K, which makes the system stable, is defined by

$$q_2(t) = -Kq_1(t). \tag{1.17}$$

For the system (1.15), Utkin and Yound (1978) have shown that $(\bar{A}_{11}, \bar{A}_{12})$ is controllable if and only if (A, B) is controllable (See also Chen, 1999).

On the sliding surface, the system trajectory in the $(q_1(t), q_2(t))$ coordinates is expressed as

$$\begin{bmatrix} K & I_m \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = 0 \tag{1.18}$$

or

$$s(t) = Gx(t) = \begin{bmatrix} K & I_m \end{bmatrix} Hx(t) = 0 \tag{1.19}$$

in the original coordinates.

Lyapunov Method

Consider a continuous-time nonlinear system in the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{1.20}$$

Lyapunov's second method to make the system (1.10) or (1.20) asymptotically stable requires the following assumptions (Su *et al.*, 1996).

(i) There exists a stablizing feedback $Kx(t)$ for (1.10) or $k(x(t))$ for (1.20) such that the autonomous systems

$$\dot{x}(t) = Ax(t) + BKx(t) \tag{1.21}$$

$$\dot{x}(t) = f(x(t)) + g(x(t))k(x(t)) \tag{1.22}$$

are asymptotically stable.

(ii) A Lyapunov function $V(x(t))$ exists and satisfies $\dot{V}(x(t)) < 0$.

(iii) $\dot{V}(x(t)) \neq 0$ in the state trajectory, except at the origin.

If $\dot{x}(t) = f(x(t))$ is stable with a Lyapunov function $\dot{V}(x(t)) < 0$, then

$$\dot{V}(x(t)) = (\frac{\partial V(x(t))}{\partial x(t)})^T f(x(t)) < 0 \tag{1.23}$$

Equation (1.23) yields

$$\dot{V}(x(t)) = (\frac{\partial V(x(t))}{\partial x(t)})^T f(x) + (\frac{\partial V(x(t))}{\partial x(t)})^T g(x(t))u(t) < 0 \tag{1.24}$$

when $(\frac{\partial V(x(t))}{\partial x(t)})^T g(x(t)) = 0$. From (1.20) and (1.24), the sliding surfaces can be described as

$$s(x(t)) = g^T(x(t))(\frac{\partial V(x(t))}{\partial x(t)}) = 0 \tag{1.25}$$

where $s(x(t))$ is a $m \times 1$ vector function. On the sliding surface (1.25), the nonlinear system (1.20) becomes asymptotically stable.

If $\dot{x}(t) = f(x(t))$ is unstable with a Lyapunov function $V(x(t))$, a feedback gain $k(x(t))$ which stabilizes the system (1.20) exists. Letting system input as $u(t) = k(x(t)) + v(t)$, eqaution (1.20) yields

$$\dot{x}(t) = f(x(t)) + g(x(t))(kx(t) + u(t)) = f_s(x(t)) + g(x(t))u(t) \tag{1.26}$$

where the corresponding autonomous dynamics $f_s(x(t)) = f(x(t)) + g(x(t))ku(t)$ is stable. By choosing a Lyapunov function $W(x(t)) > 0$, which satisfies $(\frac{\partial W(x(t))}{\partial x(t)})^T f_s(x) < 0$, the nonlinear sliding surfaces of equation (1.20) can be expressed as

$$s(x(t)) = g^T(x(t))(\frac{\partial W(x(t))}{\partial x(t)}) = 0 \tag{1.27}$$

1.1.2 Variable Structure Control Law Design

Three major types of discontinuous (switching) control exist: variable structure type, signum function, and unit control.

Variable Structure Type

Consider a single-input system (Utkin, 1977)

$$\begin{aligned} \dot{x}_i(t) &= \dot{x}_{i+1}(t),\ i = 1, \dots, n-1 \\ \dot{x}_n(t) &= -\sum_{i=1}^{n} a_i(t)x_i(t) + u(t) \end{aligned} \tag{1.28}$$

where a_i are time-varying parameters and $u(t)$ is $m \times 1$ vector. This system has the sliding surface as

$$s(t) = x_n(t) + c_{n-1}(t)x_{n-1}(t) + \cdots + c_1(t)x_1(t) = 0 \tag{1.29}$$

The control law is described as

$$u(t) = -\sum_{i=1}^{n-1} \Psi_i x_i(t) - \delta \mathrm{sgn}(s(t)) \tag{1.30}$$

where δ is a small positive number and the feedback gains are

$$\Psi_i = \{ \begin{matrix} \alpha_i \text{ if } x_i s(t) > 0 \\ \beta_i \text{ if } x_i s(t) < 0 \end{matrix} \tag{1.31}$$

Signum Function

Consider a multi input system with a disturbance $d(t)$

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t) \tag{1.32}$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, $d(t) \in \mathbf{R}^l$ and A, B, E are constant matrices of appropriate dimensions, B and E have full rank. The sliding mode of (1.32) can be described as

$$s(t) = Gx(t) = 0 \tag{1.33}$$

where G is a $m \times n$ matrix. The sliding variable dynamics controls can be chosen by considering

$$\dot{s}(t) = G\dot{x}(t) = GAx(t) + GBu(t) + GEd(t) \tag{1.34}$$

Introduce a new variable $\bar{s}(t)$ to decouple control input

$$\bar{s}(t) = (GB)^{-1} s(t) \tag{1.35}$$

From equations (1.34) and (1.35), the new sliding dynamics is obtained

$$\dot{\bar{s}}(t) = (GB)^{-1} GAx(t) + u(t) + (GB)^{-1} GEd(t) \tag{1.36}$$

$$= \begin{bmatrix} f_1(x(t)) + u_1(t) + d_1(t) \\ f_2(x(t)) + u_2(t) + d_2(t) \\ \vdots \\ f_m(x(t)) + u_m(t) + d_m(t) \end{bmatrix} \tag{1.37}$$

such that each control law can be designed separately as (Su, 2009)

$$\begin{aligned} u_1(t) &= -f_1(x(t)) - (d_1(t)+\sigma_1)\text{sgn}(\bar{s}_1(t)) \\ u_2(t) &= -f_2(x(t)) - (d_2(t)+\sigma_2)\text{sgn}(\bar{s}_2(t)) \\ &\vdots \\ u_m(t) &= -f_m(x(t)) - (d_m(t)+\sigma_m)\text{sgn}(\bar{s}_m(t)) \end{aligned} \tag{1.38}$$

Unit Control

Consider the same system as on (1.32)

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t) \tag{1.39}$$

with the same sliding variable dynamics such as

$$\dot{s}(t) = G\dot{x}(t) = GAx(t) + GBu(t) + GEd(t) \tag{1.40}$$

The control law which satisfies the reaching condition directly can be chosen as

$$u(t) = -(GB)^{-1}GAx(t) - (GB)^{-1}(\gamma+\sigma)(\frac{s(t)}{||s(t)||}) \tag{1.41}$$

where

$$\gamma = ||GE||d_{max} \tag{1.42}$$

1.1.3 The Invariance Condition for Linear Systems with Exogenous Disturbances

Consider a multi input system with a disturbance $d(t)$ (Drazenovic, 1969)

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t) \tag{1.43}$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, $d(t) \in \mathbf{R}^l$ and A, B, E are constant matrices of appropriate dimensions, B and E have full rank. The sliding mode of (1.32) can be described as

$$s(t) = Gx(t) = 0, \tag{1.44}$$

where $G(t)$ is a $m \times n$ matrix. Equation (1.32) is invariant to $d(t)$ in the sliding mode if and only if

$$\text{rank}\begin{bmatrix} B & | & E \end{bmatrix} = \text{rank}\begin{bmatrix} B \end{bmatrix} \tag{1.45}$$

1.2 Continuous-Time Weakly Coupled Systems

Linear weakly coupled systems have been studied in different set-ups by many researchers since Kokotovic and his coworkers introduced them in 1969 (Delacour, 1978; Khalil, 1978; Sezer and Siljak, 1986; Petrovic and Gajić, 1988; Gajić and Shen, 1989; Shen and Gajić, 1990; Su and Gajić, 1991; Qureshi, 1992; Aganovic *et al.*, 1996; Gajić and Borno, 2000; Mukaidani, 2006; Kim and Lim, 2007; Prljaca and Gajić, 2007; Mukaidani, 2007*a*; Mukaidani, 2007*b*; Sagara *et al.*, 2008; Mukaidani, 2009). Traditionally, solutions of weakly coupled systems were obtained in terms of Taylor series and power series expansions with respect to a small weak coupling parameter ϵ (Kokotovic *et al.*, 1969; Delacour, 1978). In 1989, Gajić and Shen, under certain conditions, introduced a decoupling transformation which exactly decompose weakly coupled linear systems composed of two subsystems into independent two reduced-order subsystems. In Qureshi (1992), another version of the transformation was obtained.

The linear weakly coupled system composed of two subsystem is defined by (Kokotovic *et al.*, 1969)

$$\begin{aligned}\frac{dx_1(t)}{dt} &= A_1x_1(t) + \epsilon A_2x_2(t) + B_1u_1(t) + \epsilon B_2u_2(t)\\ \frac{dx_2(t)}{dt} &= \epsilon A_3x_1(t) + A_4x_2(t) + \epsilon B_3u_1(t) + B_4u_2(t)\end{aligned} \tag{1.46}$$

where ϵ is a small weak coupling parameter and $x_i(t) \in \mathbf{R}^{n_i}$ are state space variables and, $u_i(t) \in \mathbf{R}^{m_i}$ are subsystem controls. Two standard assumptions for weakly coupled linear system exist (Gajić *et al.*, 2009, pp. 98-100).

Assumption 1.2.1. *Matrices A_i, $i = 1,2,3,4$, are constant and $O(1)$. In addition, magnitudes of all system eigenvalues are $O(1)$, that is, $|\lambda_j| = O(1)$, $j = 1,2,\ldots,n$, which implies that the matrices A_1, A_4 are nonsingular with* $\det\{A_1\} = O(1)$ *and* $\det\{A_4\} = O(1)$.

Assumption 1.2.2. *Matrices A_1 and A_4 have no common eigenvalues.*

1.2.1 Decoupling Transformation of Gajic and Shen

Consider a linear weakly coupled system (Gajić and Shen, 1989; see also Gajić *et al.*, 2009, Chap. 5)

$$\frac{dx_1(t)}{dt} = A_1x_1(t) + \epsilon A_2x_2(t) + B_1x_1(t) + \epsilon B_2x_2(t) \tag{1.47}$$

$$\frac{dx_2(t)}{dt} = \epsilon A_3x_1(t) + A_4x_2(t) + \epsilon B_3x_1(t) + B_4x_2(t) \tag{1.48}$$

where $x_1(t) \in \mathbf{R}^{n_1}$, $x_2(t) \in \mathbf{R}^{n_2}$, $n_1 + n_2 = n$, are subsystem states, $u_i(t) \in \mathbf{R}^{m_i}$, $i = 1, 2$, are subsystem controls, and ϵ is a small coupling parameter. Introducing new variables η_1 and a matrix L_1 as follows

$$x_1(t) = \eta_1(t) + \epsilon L_1x_2(t) \tag{1.49}$$

transforms (1.47) into

$$\dot{\eta}_1(t) = A_{10}\eta_1(t) + \epsilon\Phi_1(L_1)x_2(t) + B_{10}u_1(t) + \epsilon B_{20}u_2(t) \tag{1.50}$$

where

$$\begin{aligned} A_{10} &= A_1 - \epsilon^2 L_1A_3 \\ B_{10} &= B_1 - \epsilon^2 L_1B_3 \\ B_{20} &= B_2 - L_1B_4 \end{aligned} \tag{1.51}$$

and

$$\Phi_1(L_1) = A_1L_1 - L_1A_4 + A_2 - \epsilon^2 L_1A_3L_1 \tag{1.52}$$

If L_1 is chosen such that $\Phi_1(L) = 0$, (1.50) is completely decoupled subsystem

$$\dot{\eta}_1(t) = A_{10}\eta_1(t) + B_{10}u_1(t) + \epsilon B_{20}u_2(t) \tag{1.53}$$

Introducing another change of variables as follows

$$\eta_2(t) = x_2(t) + \epsilon H_1\eta_1(t) \tag{1.54}$$

we have from (1.48) and (1.53)

$$\dot{\eta}_2(t) = \epsilon\Phi_1(H_1)\eta_1(t) + A_{40}\eta_2(t) + \epsilon B_{30}u_1(t) + B_{40}u_2(t) \tag{1.55}$$

where

$$A_{40} = A_4 + \epsilon^2 A_3 L_1$$
$$B_{30} = B_3 + H_1 B_{10} \tag{1.56}$$
$$B_{40} = B_4 + \epsilon^2 H_1 B_{20}$$

and

$$\Phi_2(H_1) = H_1 A_{10} - A_{40} H_1 + A_3 \tag{1.57}$$

Assuming that matrix H_1 can be chosen such that $\Phi_2(H_1) = 0$, (1.55) represents another decoupled subsystem

$$\dot{\eta}_2(t) = A_{40}\eta_2(t) + \epsilon B_{30} u_1(t) + B_{40} u_2(t) \tag{1.58}$$

The original system (1.47)-(1.48) is transformed into the decoupled subsystems using the similarity transformation

$$\begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = \begin{bmatrix} I_{n_1} & -\epsilon L_1 \\ \epsilon H_1 & I_{n_2} - \epsilon^2 H_1 L_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathbf{T}_1 \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{1.59}$$

where

$$\mathbf{T}_1^{-1} = \begin{bmatrix} I_{n_1} - \epsilon^2 L_1 H_1 & \epsilon L_1 \\ -\epsilon H_1 & I_{n_2} \end{bmatrix}. \tag{1.60}$$

1.2.2 Decoupling Transformation of Qureshi

The difficulty of the decoupling transformation of Gajić and Shen is that computation must be done sequentially. Introducing the change of variables to overcome this difficulty (Qureshi, 1992; see also Gajić *et al.*, 2009, Chap. 5)

$$\begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = \begin{bmatrix} I_{n_1} & -\epsilon L_2(t) \\ \epsilon H_2(t) & I_{n_2} - \epsilon^2 H_2(t) L_2(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathbf{T}_2 \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{1.61}$$

where

$$\mathbf{T}_2^{-1} = \begin{bmatrix} I_{n_1} - \epsilon^2 L_2(t) M(t) H_2(t) & \epsilon L_2(t) M(t) \\ -\epsilon M(t) H_2(t) & M(t) \end{bmatrix} \tag{1.62}$$

with $M(t) = (I_{n_1} - \epsilon^2 H_2(t) L_2(t))^{-1}$, the original system (1.46) is transformed into

$$\dot{\eta}_1(t) = (A_1(t) - \epsilon^2 L_2(t) A_3(t))\eta_1(t) + B_{10} u_1(t) + \epsilon B_{20} u_2(t)$$
$$\dot{\eta}_2(t) = (A_4(t) - \epsilon^2 H_2(t) A_2(t))\eta_2(t) + \epsilon B_{30} u_1(t) + B_{40} u_2(t) \tag{1.63}$$

where matrices $L_2(t)$ and $H_2(t)$ are obtained from

$$\begin{aligned}\Phi_3(L_2(t), \dot{L_2}(t)) = \dot{L_2}(t) - A_1(t)L_2(t) + L_2(t)A_4(t)\\ - A_2(t) + \epsilon^2 L_2(t)A_3(t)L_2(t) = 0\\ \Phi_4(H_2(t), \dot{H_2}(t)) = \dot{H_2}(t) - A_4(t)H_2(t) + H_2(t)A_1(t)\\ A_3(t) + \epsilon^2 H_2(t)A_2(t)H_2(t) = 0\end{aligned} \tag{1.64}$$

with

$$\begin{aligned}B_{10} &= B_1 - \epsilon^2 L_1 B_3\\ B_{20} &= B_2 - L_2 B_4\\ B_{30} &= B_3 - H_2 B_1\\ B_{40} &= B_4 - \epsilon^2 H_1 B_2\end{aligned} \tag{1.65}$$

Note that equations for $L_2(t)$ and $H_2(t)$ are independent of each other.

1.2.3 Decoupling Transformation for N Weakly Coupled Subsystems

Consider a continuous-time systems consisting of n states represented by Gajić and Borno, 2000; see also Gajić *et al.*, 2009, Chap. 5)

$$\frac{dx(t)}{dt} = Ax(t) \tag{1.66}$$

where $x(t)$ is n-dimentional state vector partitioned consistently with N subsystems as $x(t) = \begin{bmatrix} x_1^T(t) & x_2^T(t) & \dots & x_1^T(t)\end{bmatrix}^T$, $x_i(t) \in \mathbf{R}_i^n$, and constant matrix A is

$$A = \begin{bmatrix} A_{11} & \epsilon A_{12} & \dots & \epsilon A_{1N}\\ \epsilon A_{21} & A_{22} & \dots & \dots\\ \dots & \dots & \dots & \dots\\ \epsilon A_{N1} & \epsilon A_{N2} & \dots & A_{NN}\end{bmatrix} \tag{1.67}$$

The similar Assumptions as Assumption 1.2.1 and 1.2.2 of are imposed for N weakly coupled linear system (Gajić *et al.*, 2009, pp. 108-111).

Assumption 1.2.3. *All matrices A_{ij} are constant and $O(1)$, and magnitudes of all system eigenvalues are $O(1)$, that is, $|\lambda_j| = O(1)$, $j = 1, 2, \dots, n$, which implies that the matrices A_{ii}, $j = 1, 2, \dots, N$ are nonsingular with* $\det\{A_{ii}\} = O(1)$.

Assumption 1.2.4. *Matrices A_{jj} and A_{ii} have no eigenvalues in common for every $i, j, i \neq j$.*

The corresponding similarity transformation matrix is given by

$$\eta(t) = \Gamma x(t) \tag{1.68}$$

where

$$\Gamma(\epsilon) = \begin{bmatrix} I & \epsilon L_{12} & \dots & \epsilon L_{1N} \\ \epsilon L_{21} & I & \dots & \epsilon L_{2N} \\ \dots & \dots & \dots & \dots \\ \epsilon L_{N1} & \dots & \epsilon L_{N(N-1)} & I \end{bmatrix} \tag{1.69}$$

The original system (1.66) is decoupled into

$$\dot{\eta}_i(t) = \Omega_i \eta_i(t), \; i = 1, 2, \dots, N \tag{1.70}$$

with

$$\Omega_i = A_{ii} + \epsilon^2 \sum_{j=1, j \neq i}^{N} L_{ij} A_{ji}, \; j = 1, 2, \dots, N \tag{1.71}$$

where L_{ij} satisfies

$$\begin{aligned} \Omega_{ij}(L_{ij}, \epsilon) = L_{ij} A_{jj} - A_{ii} L_{ij} + A_{ij} + \epsilon (\sum_{k=1, k \neq i,j}^{N} L_{ik} A_{ki}) \\ - \epsilon^2 (\sum_{k=1, k \neq i}^{N} L_{ik} A_{ki}) L_{ij} = 0, \\ i, j = \forall 1, 2, \dots, N, \; i \neq j \end{aligned} \tag{1.72}$$

These equations can be solved iteratively by starting with

$$L_{ij}^{(0)} A_{jj} - A_{ii} L_{ij}^{(0)} + A_{ij} = 0 \tag{1.73}$$

and performing the following iteration

$$\begin{aligned} L_{ij}^{(m+1)} A_{jj} - A_{ii} L_{ij}^{(m+1)} + A_{ij} + \epsilon (\sum_{k=1, k \neq i,j}^{N} L_{ik}^{(m)} A_{ki}) \\ - \epsilon^2 (\sum_{k=1, k \neq i}^{N} L_{ik}^{(m)} A_{ki}) L_{ij}^{(m)} = 0, \\ i, j = 1, 2, \dots, N, \; i \neq j; \; m = 0, 1, 2, \dots \end{aligned} \tag{1.74}$$

This algorithm converges with the rate of $O(\epsilon)$, that

$$||L_{ij}^{(m)} - L_{ij}^{(0)}|| = O(\epsilon^i), m = 0, 1, 2, \ldots \tag{1.75}$$

Other methods, like the Newton method (Gajić *et al.*, 2009), can be used to solve (1.72).

1.3 Thesis Contribution

The primary contributions of this work are two fold: the introduction of basic concepts of sliding modes along with the sliding mode control and methodologies to solve continuous-time linear weakly coupled systems using sliding mode control. This work is the first study to find sliding mode controls for continuous-time linear weakly coupled systems.

We present three methods in this thesis. First one is Utkin and Young's sliding mode control method for each subsystem using transformation of Gajić and Shen (1989). This work provides the sliding surfaces and sliding mode controls for each reduced-order subsystems. Next one is a composite control approach in the sense that one control is used to stabilize one subsystem by a state feedback control law and the other control is found by sliding mode control law for the other subsystem. The last one is the Lyapunov approach to find sliding mode control laws. This approach does not require a partition of the system into the canonical forms. These approaches make the system asymptotically stable as well as robust against the external disturbances and uncertainty.

Chapter 2

Continuous-Time Sliding Mode Control of Weakly Coupled Systems: Utkin and Young's Approach

2.1 Introduction

Three main procedures could be developed for sliding mode control of weakly coupled systems. First one is to decompose the full-order system into subsystems by using the transformation technique. Next is to find a sliding surface and design a control law for each subsystem. The final step is to implement composite sliding mode control combining control variables from each subsystem.

In this chapter, we address the problem of sliding mode control of a weakly coupled system with external disturbance. The first procedure is followed by using the decoupling transformation of Gajić and Shen (1989) and the next step is obtained by using the result of Utkin and Young's approach (1978) with the discontinuous control law. The inverse transformation of Gajić and Shen (1989) is used on the last step.

2.2 Problem Formulation

Consider a continuous-time weakly coupled system represented by

$$\begin{aligned}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} &= \begin{bmatrix}A_1 & \epsilon A_2\\ \epsilon A_3 & A_4\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}B_1 & \epsilon B_2\\ \epsilon B_3 & B_4\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix} + Ed(t)\\ &= A\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + B\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix} + Ed(t)\end{aligned} \tag{2.1}$$

where $x_1(t) \in \mathbf{R}^{n_1}$, $x_2(t) \in \mathbf{R}^{n_2}$, $n_1 + n_2 = n$, are state variables, $u_i(t) \in \mathbf{R}^{m_i}$, $i = 1, 2$, are control inputs, and $d(t) \in \mathbf{R}^l$ is the disturbance. ϵ is a small weak coupling parameter. It is assumed that matrices A_1, A_4 are constant and $O(1)$. In addition, magnitudes of all system eigenvalues are $O(1)$, that is, $|\lambda_j| = O(1)$, $j = 1, 2, \ldots, n$, which implies that the matrices A_1, A_4 are nonsingular with $\det\{A_1\} = O(1)$ and $\det\{A_4\} = O(1)$. It is also assumed that matrices A_1 and A_4 have no common eigenvalues (see Assumption 1.2.1). A, B, E are constant matrices of appropriate dimensions. Furthermore, B and E have full rank.

The system (2.1) is invariant to $d(t)$ if and only if the matching condition is satisfied (Drazenovic, 1969)

$$\text{rank}\begin{bmatrix} B & | & E \end{bmatrix} = \text{rank}\begin{bmatrix} B \end{bmatrix} \tag{2.2}$$

which means there exists a $m \times l$ matrix D such that

$$E = BD \tag{2.3}$$

The main objective is to find a sliding surface using Utkin and Young's approach with a discontinuous control law to achieve system stability and disturbance rejection.

2.3 Result

Apply decoupling transformation of Gajić and Shen (1989)

$$\begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = \begin{bmatrix} I_{n_1} & -\epsilon L_1 \\ \epsilon H_1 & I_{n_2} - \epsilon^2 H_1 L_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{2.4}$$

to system (2.1). The system is completely decoupled

$$\begin{aligned} \begin{bmatrix} \dot{\eta}_1(t) \\ \dot{\eta}_2(t) \end{bmatrix} &= \begin{bmatrix} A_{10} & 0 \\ 0 & A_{40} \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} + \begin{bmatrix} B_{10} & \epsilon B_{20} \\ \epsilon B_{30} & B_{40} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \mathbf{T} E d(t) \\ &= \begin{bmatrix} A_{10} & 0 \\ 0 & A_{40} \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} + \begin{bmatrix} B_{10} & \epsilon B_{20} \\ \epsilon B_{30} & B_{40} \end{bmatrix} \left(\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} d(t) \right) \end{aligned} \tag{2.5}$$

with

$$\begin{aligned} A_{10} &= A_1 - \epsilon^2 L_1 A_3 \\ A_{40} &= A_4 + \epsilon^2 A_3 L_1 \\ B_{10} &= B_1 - \epsilon^2 L_1 B_3 \\ B_{20} &= B_2 - L_1 B_4 \\ B_{30} &= B_3 + H_1 B_{10} \\ B_{40} &= B_4 + \epsilon^2 H_1 B_{20} \end{aligned} \tag{2.6}$$

where L_1 and H_1 are solutions of the following algebraic equations

$$\begin{aligned} A_1 L_1 - L_1 A_4 + A_2 - \epsilon^2 L_1 A_3 L_1 = 0 \\ H_1 A_{10} - A_{40} H_1 + A_3 = 0 \end{aligned} \tag{2.7}$$

Numerical solutions for L_1 and H_1 can be obtained by using the fixed point iterations (Petrovic and Gajić, 1988)

$$A_1 L_1^{(i+1)} - L_1^{(i+1)} A_4 + A_2 - \epsilon^2 L_1^{(i)} A_3 L_1^{(i)} = 0, \; i = 0, 1, \ldots, N-1 \tag{2.8}$$

$$H_1^{(N)} A_{10}^{(N)} - A_{40}^{(N)} H_1^{(N)} + A_3 = 0 \tag{2.9}$$

with

$$\begin{aligned} A_{10}^{(N)} &= A_1 - \epsilon^2 L_1^{(N)} A_3 \\ A_{40}^{(N)} &= A_4 + \epsilon^2 A_3 L_1^{(N)} \end{aligned} \tag{2.10}$$

where $L_1^{(0)}$ and $H_1^{(0)}$ are obtained from following equations

$$\begin{aligned} A_1 L_1^{(0)} - L_1^{(0)} A_4 + A_2 = 0 \\ H_1^{(0)} A_1 - A_4 H_1^{(0)} + A_3 = 0 \end{aligned} \tag{2.11}$$

Using the results, it can be shown that

$$\begin{aligned} L_1 &= L_1^{(N)} + O(\epsilon^{2N}) \\ H_1 &= H_1^{(N)} + O(\epsilon^{2N}) \end{aligned} \tag{2.12}$$

Other methods, like Newton method (Gajić *et al.*, 2009, Chap. 5) or the eigenvector method (Kecman and Tomasevic, 2006), can be used to solve (2.7).

After the system is completely decoupled, two sliding surfaces can be defined for each subsystem. For the first subsystem of (2.5)

$$\begin{aligned}\dot{\eta}_1(t) &= A_{10}\eta_1(t) + \begin{bmatrix} B_{10} & \epsilon B_{20} \end{bmatrix} \left(\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} d(t) \right) \\ &= A_{10}\eta_1(t) + B_{10}(u_1(t) + D_1 d(t)) + O(\epsilon)\end{aligned} \tag{2.13}$$

if $m_1 < n_1$, there exists a nonsingular similarity transformation (Utkin and Young, 1978), $T_1 = \begin{bmatrix} N \\ B_{10} \end{bmatrix}$ which yields

$$\begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{bmatrix} = \begin{bmatrix} \bar{A}_{1_{11}} & \bar{A}_{1_{12}} \\ \bar{A}_{1_{21}} & \bar{A}_{1_{22}} \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{B}_1 \end{bmatrix} (u(t) + D_1 d(t)) + O(\epsilon) \tag{2.14}$$

where $\xi_1(t) \in \mathbf{R}^{n_1 - m_1}$, $\xi_2(t) \in \mathbf{R}^{m_1}$, and $\bar{B}_1$ is an $m_1 \times m_1$ nonsingular matrix. We can find a state feedback gain matrix K_1 such that $\bar{A}_{1_{11}} - K_1 \bar{A}_{1_{12}}$ is asymptotically stable.

On the sliding surface using Utkin and Young's method (Utkin and Young, 1978), the system trajectory in the $(\xi_1(t), \xi_2(t))$ coordinates is expressed as

$$\begin{bmatrix} K_1 & I_m \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = 0 \tag{2.15}$$

or

$$s_1(t) = G_1\eta_1(t) = \begin{bmatrix} K_1 & I_m \end{bmatrix} T_1\eta_1(t) = 0 \tag{2.16}$$

in the original coordinates.

Apply the same procedure for the second subsystem of (2.5)

$$\begin{aligned}\dot{\eta}_2(t) &= A_{40}\eta_2(t) + \begin{bmatrix} \epsilon B_{30} & B_{40} \end{bmatrix} \left(\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + D_2 d(t) \right) \\ &= A_{40}\eta_1(t) + B_{40}(u_2(t) + D_2 d(t)) + O(\epsilon)\end{aligned} \tag{2.17}$$

If $m_2 < n_2$, there exists a nonsingular similarity transformation $T_2 = \begin{bmatrix} N \\ B_{40} \end{bmatrix}$ which yields

$$\begin{bmatrix} \dot{\zeta}_1(t) \\ \dot{\zeta}_2(t) \end{bmatrix} = \begin{bmatrix} \bar{A}_{4_{11}} & \bar{A}_{4_{12}} \\ \bar{A}_{4_{21}} & \bar{A}_{4_{22}} \end{bmatrix} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{B}_4 \end{bmatrix} (u(t) + D_2 d(t)) + O(\epsilon) \tag{2.18}$$

where $\zeta_1(t) \in \mathbf{R}^{n_2-m_2}$, $\zeta_2(t) \in \mathbf{R}^{m_2}$, and $\bar{B}_4$ is an $m_2 \times m_2$ nonsingular matrix. There exists K_2 such that $\bar{A}_{4_{11}} - K_2\bar{A}_{4_{12}}$ is asymptotically stable. According to Utkin and Young (1978), the system trajectory in the $(\zeta_1(t), \zeta_2(t))$ coordinates is expressed as

$$\begin{bmatrix} K_2 & I_m \end{bmatrix} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} = 0 \tag{2.19}$$

or

$$s_2(t) = G_2\eta_2(t) = \begin{bmatrix} K_2 & I_m \end{bmatrix} T_2\eta_2(t) = 0 \tag{2.20}$$

in the original coordinates.

Starting with $\dot{s}_1(t) = 0$ and $\dot{s}_2(t) = 0$, we design sliding mode control laws for the sliding surfaces (2.16) and (2.20).

$$\begin{aligned} \dot{s}_1(t) &= 0 = G_1\dot{\eta}_1(t) = G_1A_{10}\eta_1(t) + G_1B_{10}(u_1(t) + D_1d(t)) + O(\epsilon) \\ \dot{s}_2(t) &= 0 = G_2\dot{\eta}_2(t) = G_2A_{40}\eta_2(t) + G_2B_{40}(u_2(t) + D_2d(t)) + O(\epsilon) \end{aligned} \tag{2.21}$$

Applying the inverses to each subsystem (2.21), $u_1(t)$ and $u_2(t)$ can be given by

$$\begin{aligned} u_1(t) &= -(G_1B_{10})^{-1}G_1A_{10}\eta_1(t) - (G_1B_{10})^{-1}(\gamma_1 + \sigma_1)\frac{s_1(t)}{||s_1(t)||} + O(\epsilon) \\ u_2(t) &= -(G_2B_{40})^{-1}G_2A_{40}\eta_2(t) - (G_2B_{40})^{-1}(\gamma_2 + \sigma_2)\frac{s_2(t)}{||s_2(t)||} + O(\epsilon) \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} \gamma_1 &= ||G_1B_{10}D_1||d_{max} \\ \gamma_2 &= ||G_2B_{40}D_2||d_{max} \end{aligned} \tag{2.23}$$

is required to overcome the disturbance $d(t)$. It can be easily seen that (2.22) satisfies the vector type reaching condition (1.41)

$$\begin{aligned} s_1^T(t)\dot{s}_1(t) &= -\sigma_1||s_1(t)|| - \gamma_1||s_1(t)|| + s_1(t)^TG_1B_{10}D_1d(t) + O(\epsilon) \\ s_2^T(t)\dot{s}_2(t) &= -\sigma_2||s_2(t)|| - \gamma_2||s_2(t)|| + s_2(t)^TG_2B_{40}D_2d(t) + O(\epsilon) \end{aligned} \tag{2.24}$$

with γ_1 and γ_2 chosen as in (2.23). Other methods, like variable structure type (Utkin and Young, 1978) or signum function (Su, 2009), can be used to design the control law.

Lemma 2.3.1. *The reaching time of sliding mode control* (2.22) *satisfies*

$$\begin{aligned} \frac{\sqrt{s_1^T(0)s_1(0)}}{\sigma_1 + 2\gamma_1} &< \tau < \frac{\sqrt{s_1^T(0)s_1(0)}}{\sigma_1} \\ \frac{\sqrt{s_2^T(0)s_2(0)}}{\sigma_2 + 2\gamma_2} &< \tau < \frac{\sqrt{s_2^T(0)s_2(0)}}{\sigma_2} \end{aligned} \tag{2.25}$$

Proof. Choose a Lyapunov function for $s_1(t)$.

$$V(t) = s_1^T(t)s_1(t) \tag{2.26}$$

The derivative of the Lyapunov function is given by

$$\begin{aligned} \dot{V}_1(t) &= 2s_1^T(t)\dot{s}_1(t) \\ &= -2\sigma_1||s_1(t)|| - 2\gamma_1||s_1(t)|| + 2s_1(t)^T G_1 B_{10} D_1 d(t) + O(\epsilon) \end{aligned} \tag{2.27}$$

(2.26) and (2.27) yield

$$-2(\sigma_1 + 2\gamma_1)\sqrt{V_1(t)} < \dot{V}_1(t) < -2\sigma_1\sqrt{V_1(t)} \tag{2.28}$$

or

$$-2(\sigma_1 + 2\gamma_1) < \frac{\dot{V}_1(t)}{\sqrt{V_1(t)}} < -2\sigma_1 \tag{2.29}$$

Hence

$$-2(\sigma_1 + 2\gamma_1)t < \sqrt{V_1(t)} - \sqrt{V_1(0)} < -2\sigma_1 t \tag{2.30}$$

Let τ be the time needed to reach the sliding mode ($V_1(\tau) = 0$). Then, the reaching time satisfies

$$\frac{\sqrt{V_1(0)}}{\sigma_1 + 2\gamma_1} < \tau < \frac{\sqrt{V_1(0)}}{\sigma_1} \tag{2.31}$$

which is finite. The similar proof can be used for estimating the reaching time for $s_2(t)$ □

From (2.22), the control law is given by

$$\begin{aligned} u_1(t) &= -(G_1B_{10})^{-1}G_1A_{10}(x_1(t) - \epsilon L_1 x_2(t)) \\ &\qquad - (G_1B_{10})^{-1}(\gamma_1 + \sigma_1)\frac{s_1(t)}{||s_1(t)||} + O(\epsilon) \\ u_2(t) &= -(G_2B_{40})^{-1}G_2A_{40}(\epsilon H_1 x_1(t) - \epsilon^2 H_1 L_1 x_2(t) \\ &\qquad - (G_2B_{40})^{-1}(\gamma_2 + \sigma_2)\frac{s_2(t)}{||s_2(t)||} + O(\epsilon) \end{aligned} \tag{2.32}$$

in the original coordinates.

In (2.13) and (2.17), each $O(\epsilon)$ is affected by the input of the other subsystem. To find more accurate result, $O(\epsilon)$ can be used in (2.13) and (2.17).

2.4 Example

To illustrate the proposed method, we consider the system with problem matrices given by

$$A = \begin{bmatrix} A_1 & \epsilon A_2 \\ \epsilon A_3 & A_4 \end{bmatrix}$$
$$B = \begin{bmatrix} B_1 & \epsilon B_2 \\ \epsilon B_3 & B_4 \end{bmatrix} \tag{2.33}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -5 & -3 \end{bmatrix}, \; A_2 = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} -1 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix}, \; A_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -4 \end{bmatrix} \tag{2.34}$$

and

$$B_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \; B_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \; B_4 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \tag{2.35}$$

The system can be decoupled with $n_1 = 3$ and $n_2 = 3$. The initial states are chosen as

$$x_1(0) = \begin{bmatrix} -3 \\ 5 \\ -10 \end{bmatrix} \text{ and } x_2(0) = \begin{bmatrix} -10 \\ 2 \\ -5 \end{bmatrix} \tag{2.36}$$

and the external disturbance is

$$d(t) = \sin(t) \tag{2.37}$$

We simulated the system when small coupling parameter ϵ is 0.01 and 0.1. For Drazenovic's invariance condition, we put

$$E = \begin{bmatrix} B1 \\ \epsilon B3 \end{bmatrix} \times 10^{-3} \tag{2.38}$$

2.4.1 Case $\epsilon = 0.01$

The nonsingular decoupling transformation $\mathbf{T}$ of (2.4) and L_1 and H_1 of (2.7) are found using the Newton method with the accuracy of $O(10^{-10})$.

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & -0.0299 & 0.0527 & 0.0188 \\ 0 & 1 & 0 & -0.0076 & -0.0967 & -0.0326 \\ 0 & 0 & 1 & 0.0553 & 0.1007 & 0.0139 \\ -0.1198 & -0.0506 & -0.0114 & 1.0033 & -0.0026 & -0.0008 \\ 0.0240 & -0.0427 & -0.0064 & -0.0007 & 1.0048 & 0.0018 \\ 0.0293 & 0.0356 & -0.0136 & -0.0019 & -0.0033 & 0.9992 \end{bmatrix} \tag{2.39}$$

The systems is decoupled into two subsystems

$$\dot{\eta}_1(t) = \begin{bmatrix} 0.0010 & 0.9982 & 0.0006 \\ -0.0012 & 0.0021 & 0.9983 \\ -2.9994 & -5.0010 & -2.9982 \end{bmatrix} \eta_1(t) + \begin{bmatrix} 3.0010 \\ -1.0012 \\ 1.0006 \end{bmatrix} \left(u_1(t) + 10^{-3}d(t)\right) + O(\epsilon) \tag{2.40}$$

and

$$\dot{\eta}_2(t) = \begin{bmatrix} -0.0007 & 1.0015 & 0.0007 \\ -0.0004 & -0.0035 & 0.9990 \\ -2.0009 & -3.0035 & -4.0008 \end{bmatrix} \eta_2(t) + \begin{bmatrix} -6.8915 \\ 7.7832 \\ -8.4237 \end{bmatrix} \left(u_2(t) + 10^{-3}d(t)\right) + O(\epsilon) \tag{2.41}$$

From (2.16) and (2.20), we can make the sliding surfaces such as

$$s_1(t) = G_1\eta_1(t) = \begin{bmatrix} 1.3435 & -15.6873 & -8.7232 \end{bmatrix} \eta_1(t) = 0 \tag{2.42}$$

and

$$s_2(t) = G_2\eta_2(t) = \begin{bmatrix} -54.7511 & -49.7828 & -10.2291 \end{bmatrix} \eta_2(t) = 0 \tag{2.43}$$

From (2.22), the sliding mode controls $u_1(t)$ and $u_2(t)$ can be given by

$$\begin{aligned} u_1(t) &= \begin{bmatrix} -2.3783 & -4.0812 & -0.9531 \end{bmatrix} \eta_1(t) - 0.0918\frac{s_1(t)}{||s_1(t)||} \\ u_2(t) &= \begin{bmatrix} -1.4620 & 1.7048 & 0.6302 \end{bmatrix} \eta_2(t) - 0.0721\frac{s_2(t)}{||s_2(t)||} \end{aligned} \tag{2.44}$$

To compare the decoupled systems with the original system, we put σ_i, $i = 1, 2$, in (2.22) equal to 1.

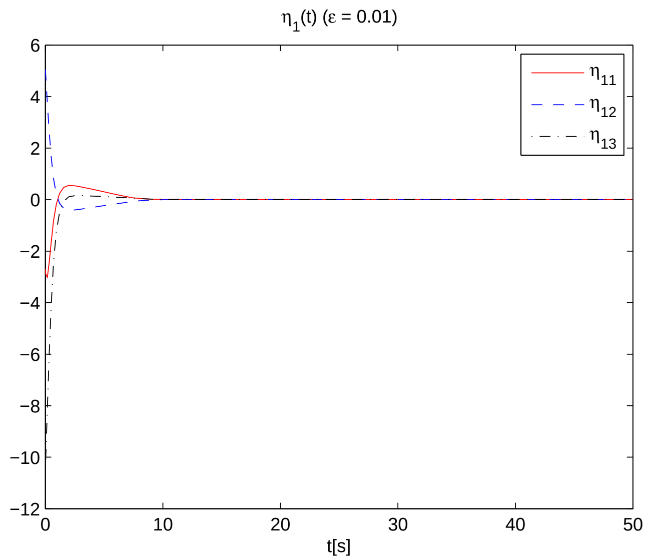

Figure 2.1: $\eta_1(t)$ of the reduced-order subsystem with ϵ=0.01

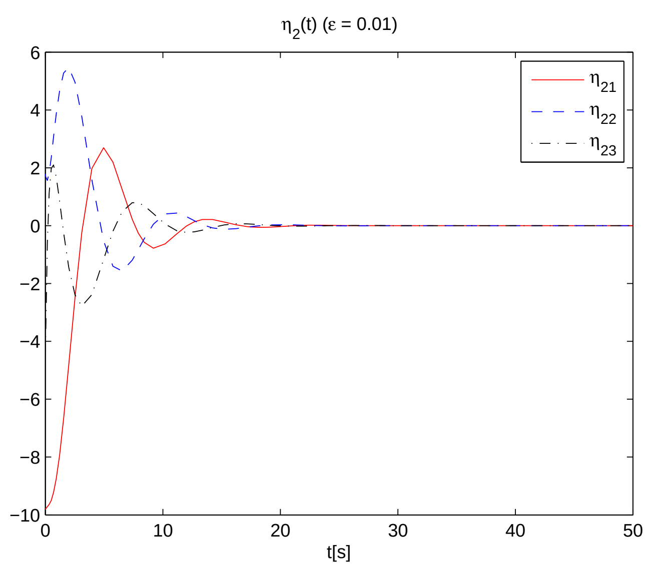

Figure 2.2: $\eta_2(t)$ of the reduced-order subsystem with ϵ=0.01

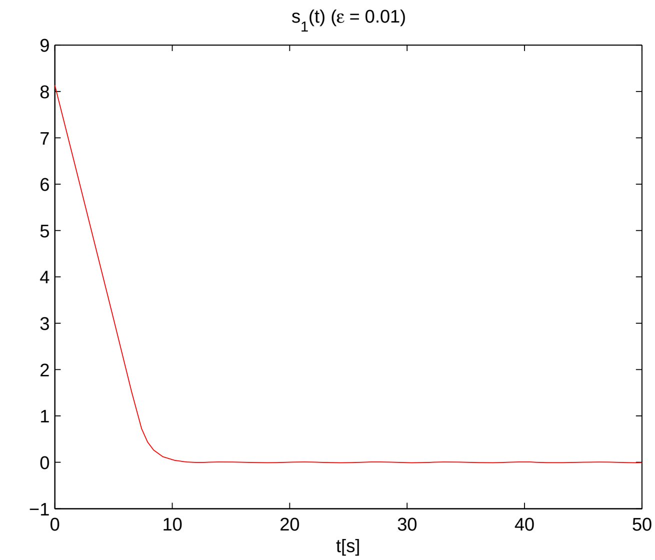

Figure 2.3: $s_1(t)$ of the reduced-order subsystem with ϵ=0.01

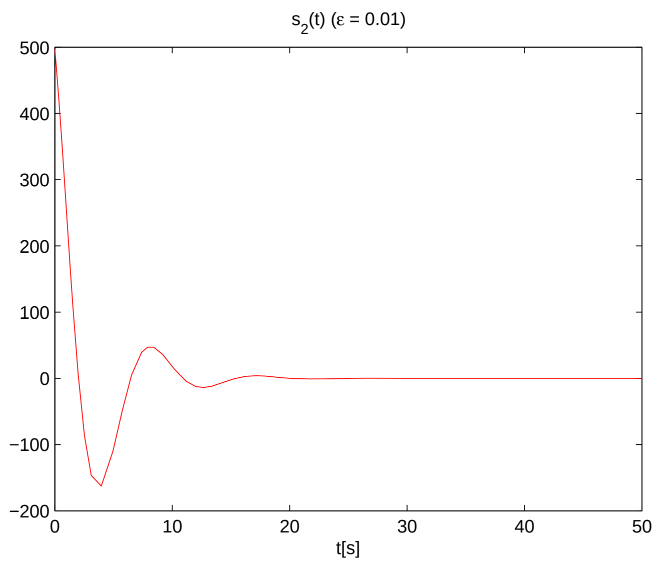

Figure 2.4: $s_2(t)$ of the reduced-order subsystem with ϵ=0.01

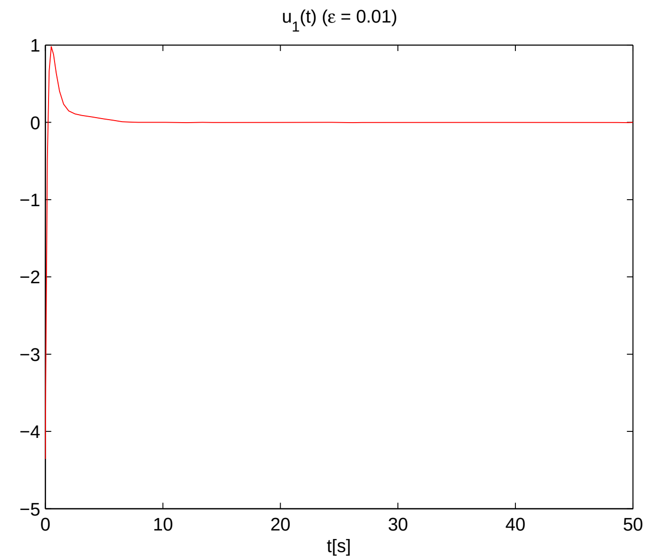

Figure 2.5: $u_1(t)$ of the reduced-order subsystem with ϵ=0.01

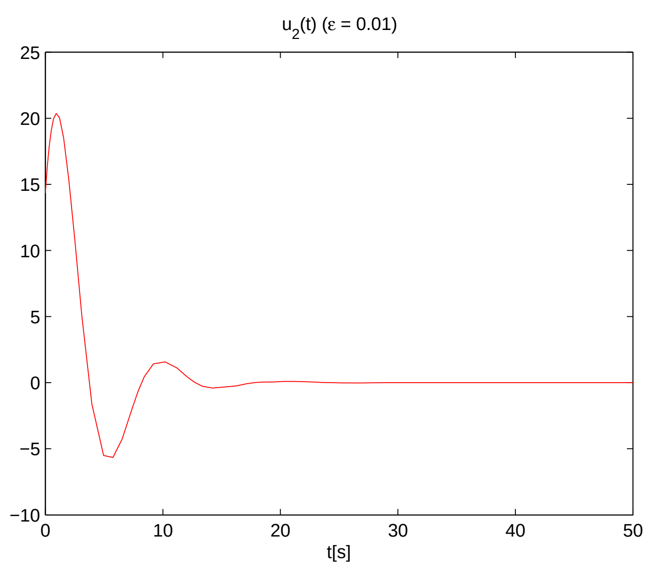

Figure 2.6: $u_2(t)$ of the reduced-order subsystem with ϵ=0.01

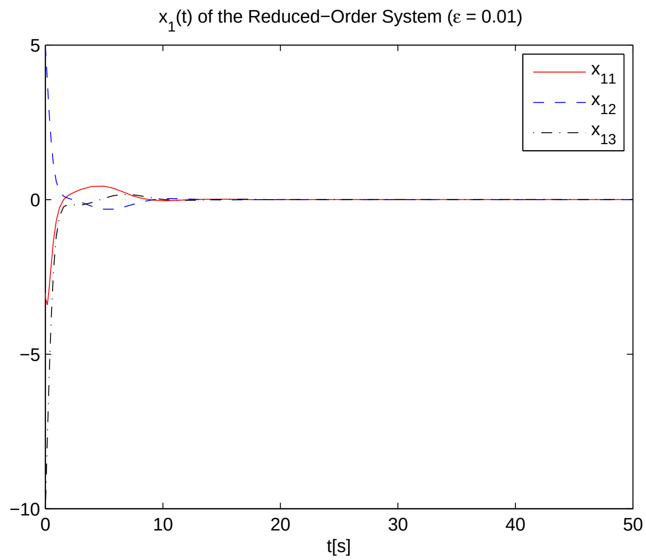

Figure 2.7: $x_1(t)$ of the reduced-order system with ϵ=0.01

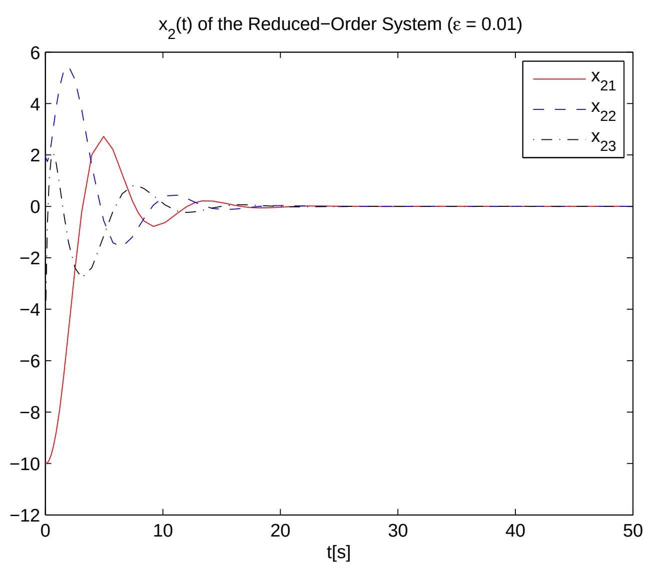

Figure 2.8: $x_2(t)$ of the reduced-order system with ϵ=0.01

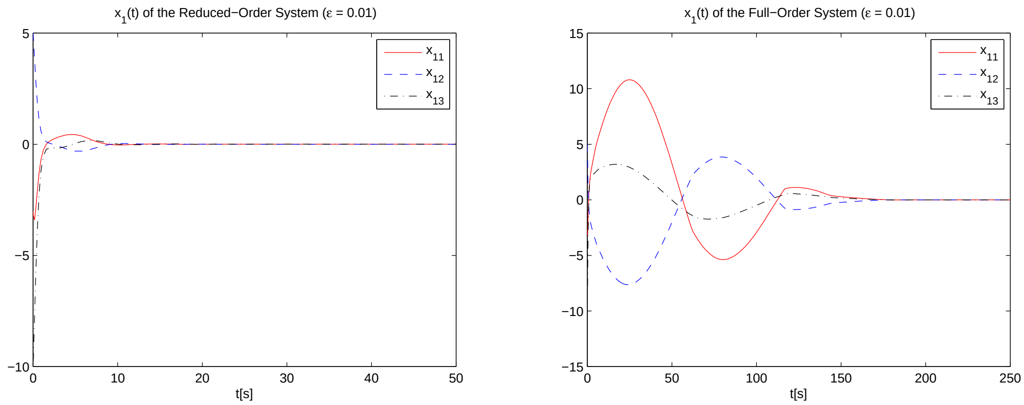

Figure 2.9: Comparison of $x_1(t)$ of the reduced-order and full-order systems

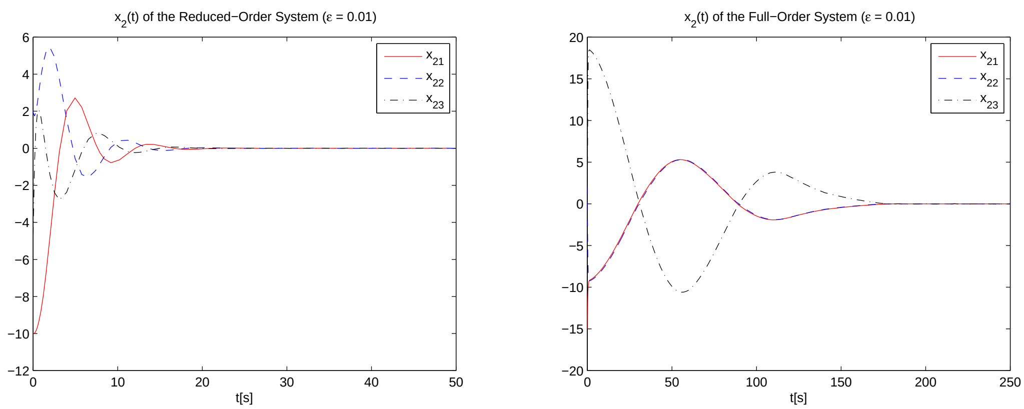

Figure 2.10: Comparison of $x_2(t)$ of the reduced-order and full-order systems

The simulation results are presented in Figures 2.1-2.6 in the new coordinates. Figures 2.1 and 2.2 show the evolution of the state variables. The chattering phenomena of the state variables in the sliding mode are found due to the effect of the switching control law. We can make two sliding surfaces (one surface for each subsystem) using Utkin and Young's method (1978). Figures 2.5 and 2.6 indicate the sliding mode control law using unit control.

Figures 2.7 and 2.8 show the state variables in the original coordinates after applying $\mathbf{T}^{-1}$. Figures 2.9 - 2.12 present comparisons of the state variables and chattering phenomena of the reduced-order system and the full-order system. The trajectories of the state variables of decoupled system take less time to converge as well as have small chattering phenomena compared with the full-order system.

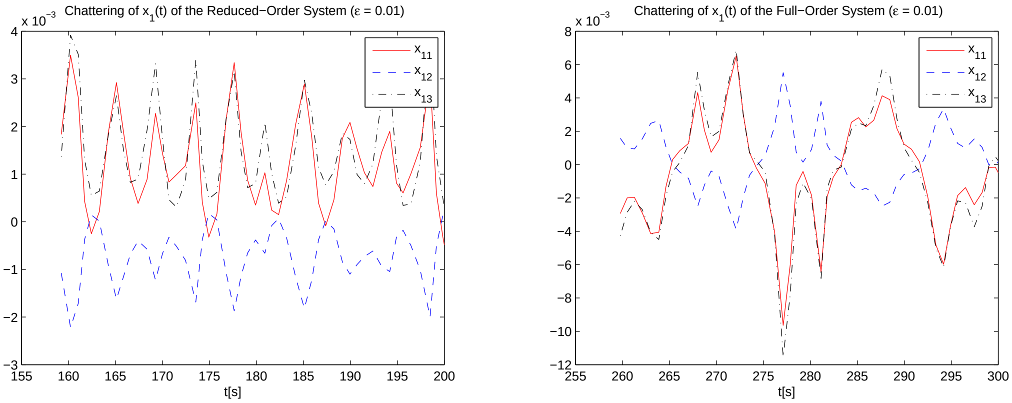

Figure 2.11: Chattering of $x_1(t)$ with ϵ=0.01

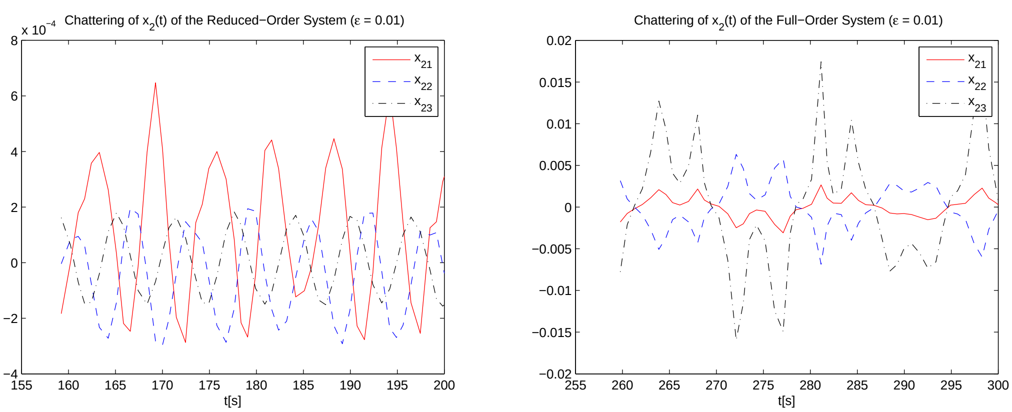

Figure 2.12: Chattering of $x_2(t)$ with ϵ=0.01

2.4.2 Case $\epsilon = 0.1$

The nonsingular decoupling transformation $\mathbf{T}$ of (2.4) and L_1 and H_1 of (2.7) are found using the Newton method with $O(10^{-10})$ accuracy.

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & -0.0910 & 0.7340 & 0.2354 \\ 0 & 1 & 0 & -0.3807 & -1.5277 & -0.4781 \\ 0 & 0 & 1 & 1.3270 & 2.3774 & 0.4982 \\ -2.7596 & -1.5714 & -0.3972 & 1.3221 & -0.5692 & -0.0961 \\ 0.6352 & -0.5223 & -0.3325 & -0.3002 & 1.4737 & 0.2336 \\ 1.2499 & 1.5445 & 0.4270 & -0.1350 & -0.4270 & 0.7685 \end{bmatrix} \tag{2.45}$$

The systems is decoupled into two subsystems

$$\dot{\eta}_1(t) = \begin{bmatrix} 0.1060 & 0.8115 & 0.1114 \\ -0.1625 & 0.2772 & 0.7136 \\ -2.8452 & -5.2599 & -2.5299 \end{bmatrix} \eta_1(t) + \begin{bmatrix} 3.1060 \\ -1.1625 \\ 1.1548 \end{bmatrix} \Big(u_1(t) + 10^{-3} d(t) \Big) + O(\epsilon) \tag{2.46}$$

and

$$\dot{\eta}_2(t) = \begin{bmatrix} -0.0657 & 1.1412 & 0.0693 \\ -0.1997 & -0.6167 & 0.8310 \\ -2.2944 & -3.7016 & -4.1710 \end{bmatrix} \eta_2(t) + \begin{bmatrix} 1.9737 \\ -2.4976 \\ 3.0804 \end{bmatrix} \Big(u_2(t) + 10^{-3} d(t) \Big) + O(\epsilon) \tag{2.47}$$

From (2.16) and (2.20), we can make the sliding surfaces such as

$$s_1(t) = G_1\eta_1(t) = \begin{bmatrix} -5.0470 & -34.1787 & -10.1525 \end{bmatrix} \eta_1(t) = 0 \tag{2.48}$$

and

$$s_2(t) = G_2\eta_2(t) = \begin{bmatrix} -75.7295 & -96.6329 & -23.4569 \end{bmatrix} \eta_2(t) = 0 \tag{2.49}$$

From (2.22), the sliding mode controls $u_1(t)$ and $u_2(t)$ can be given by

$$\begin{aligned} u_1(t) &= \begin{bmatrix} -2.7492 & -3.2297 & -0.0596 \end{bmatrix} \eta_1(t) - 0.0821 \frac{s_1(t)}{||s_1(t)||} \\ u_2(t) &= \begin{bmatrix} -3.9798 & -3.0575 & -0.6259 \end{bmatrix} \eta_2(t) - 0.0646 \frac{s_2(t)}{||s_2(t)||} \end{aligned} \tag{2.50}$$

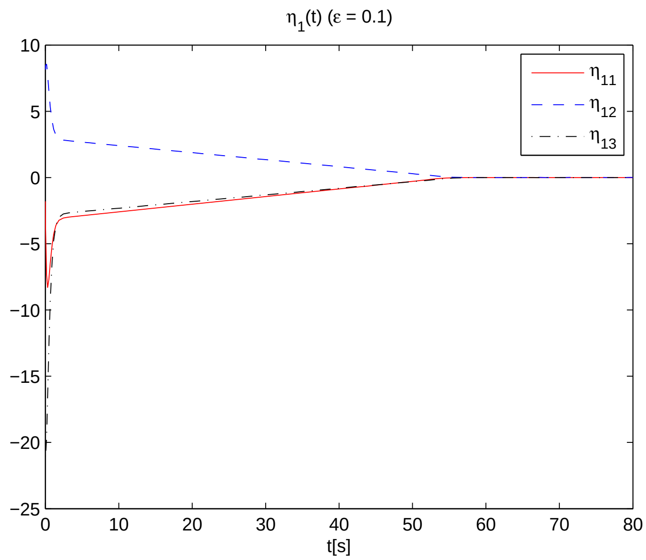

Figure 2.13: $\eta_1(t)$ of the reduced-order system with ϵ=0.1

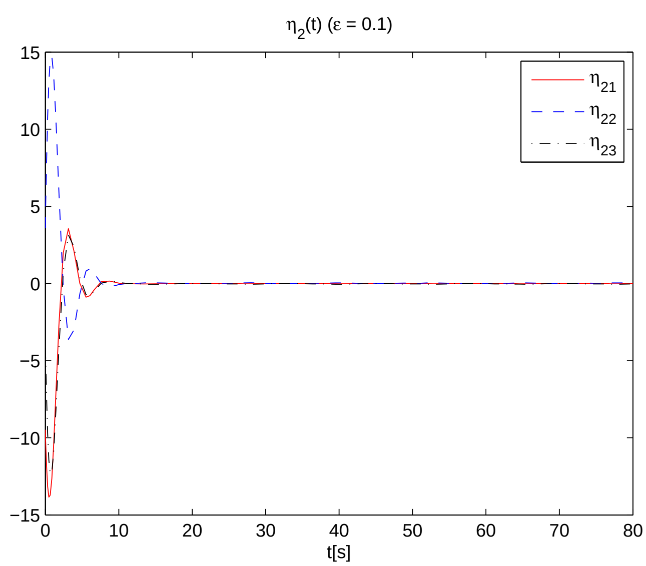

Figure 2.14: $\eta_2(t)$ of the reduced-order system with ϵ=0.1

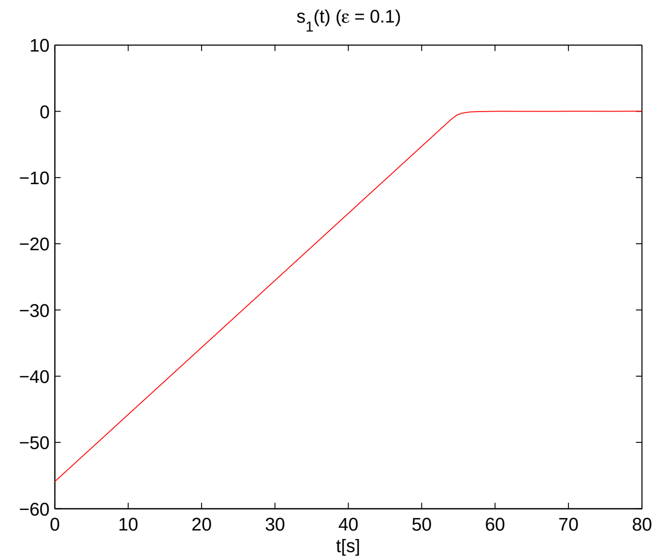

Figure 2.15: $s_1(t)$ of the reduced-order system with ϵ=0.1

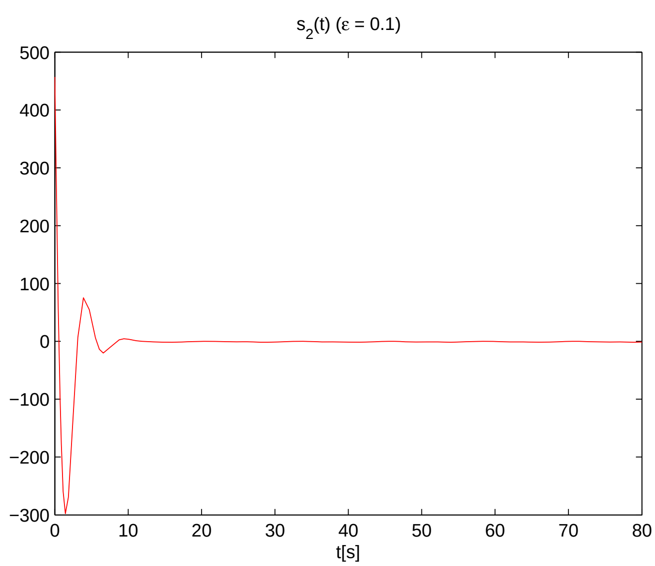

Figure 2.16: $s_2(t)$ of the reduced-order system with ϵ=0.1

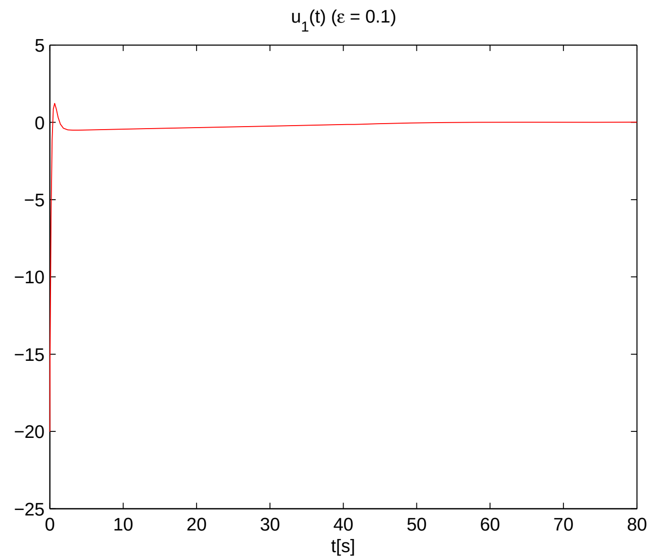

Figure 2.17: $u_1(t)$ of the reduced-order system with ϵ=0.1

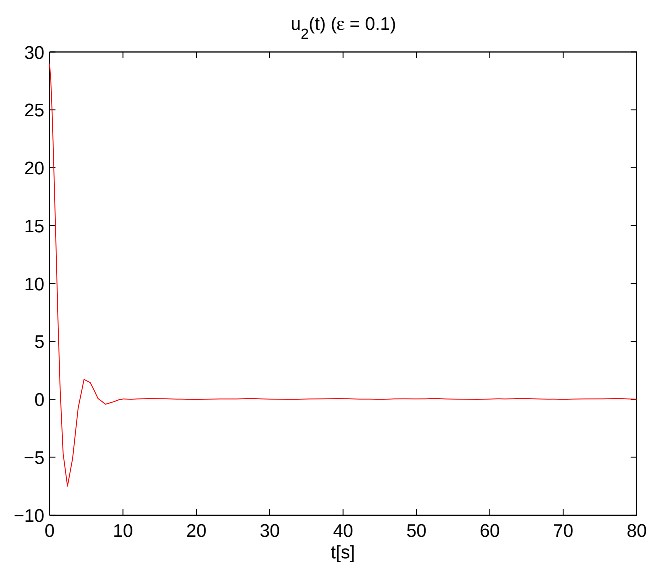

Figure 2.18: $u_2(t)$ of the reduced-order system with ϵ=0.1

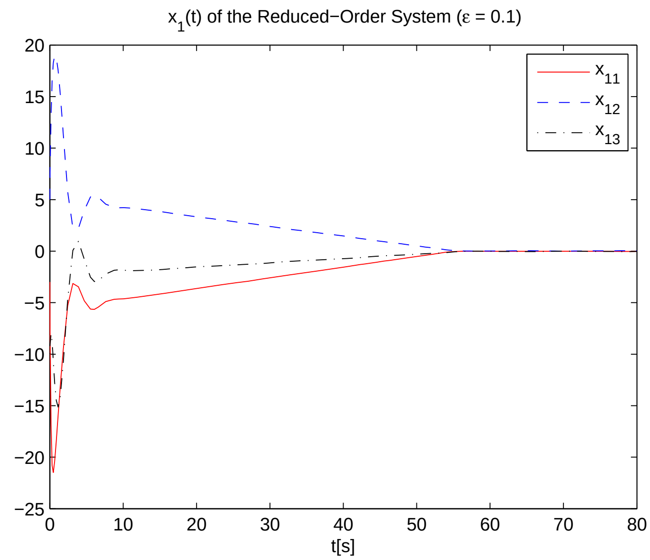

Figure 2.19: $x_1(t)$ of the reduced-order system with ϵ=0.1

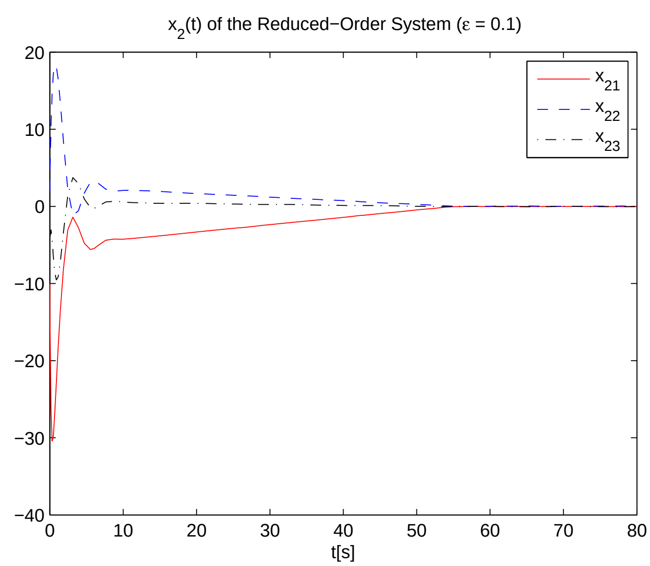

Figure 2.20: $x_2(t)$ of the reduced-order system with ϵ=0.1

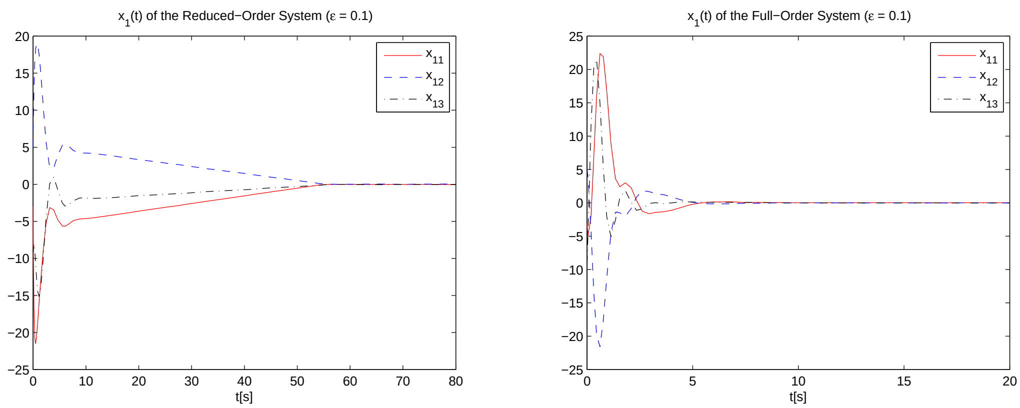

Figure 2.21: Comparison of $x_1(t)$ of the reduced-order and full-order systems

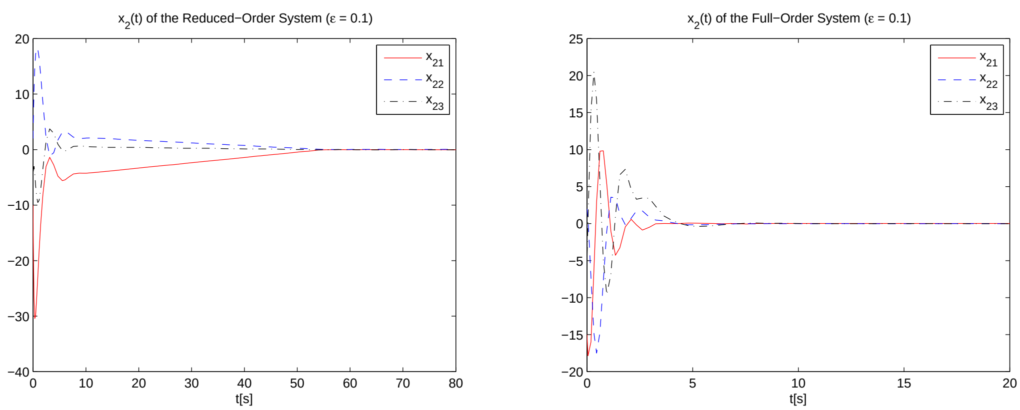

Figure 2.22: Comparison of $x_2(t)$ of the reduced-order and full-order systems

The simulation results are presented in Figures 2.13 - 2.18 in the new coordinates. Figures 2.13 and 2.14 show the evolution of the state variables. The chattering phenomena of the state variables in the sliding mode are found due to the effect of the switching control law. We can make two sliding surfaces (one surface for each subsystem) using Utkin and Young's method (1978). Figures 2.17 and 2.18 indicate the sliding mode control law using unit control.

Figures 2.19 and 2.20 show the state variables in the original coordinates after applying $\mathbf{T}^{-1}$. We can see that the chattering phenomena are smaller in the reduced-order system compared with the full-order system in Figures 2.23 and 2.24. The trajectories of the variables in the reduced-order systems are different from those in the full-order systems but we still control the systems via the sliding mode technique using sliding mode controllers designed for the reduced-order systems.

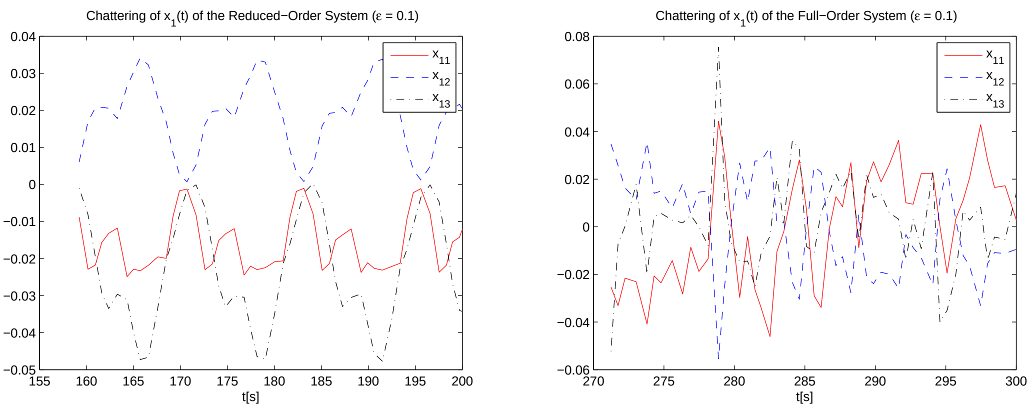

Figure 2.23: Chattering of $x_1(t)$ with ϵ=0.1

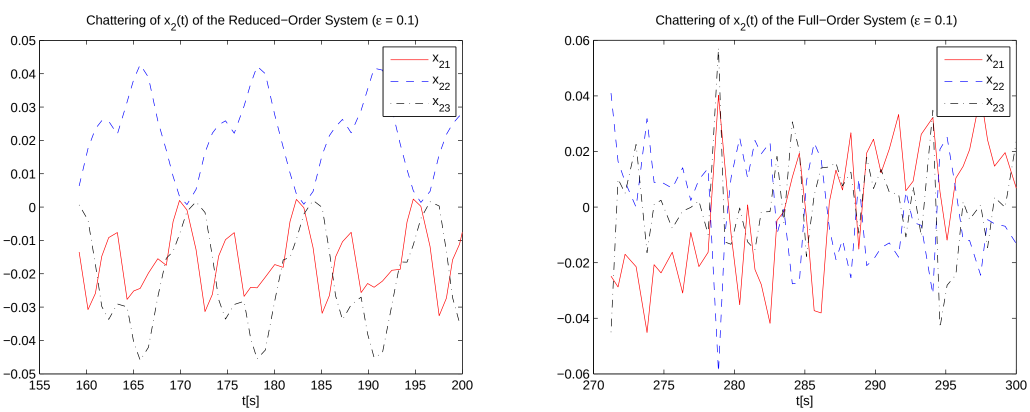

Figure 2.24: Chattering of $x_2(t)$ with ϵ=0.1

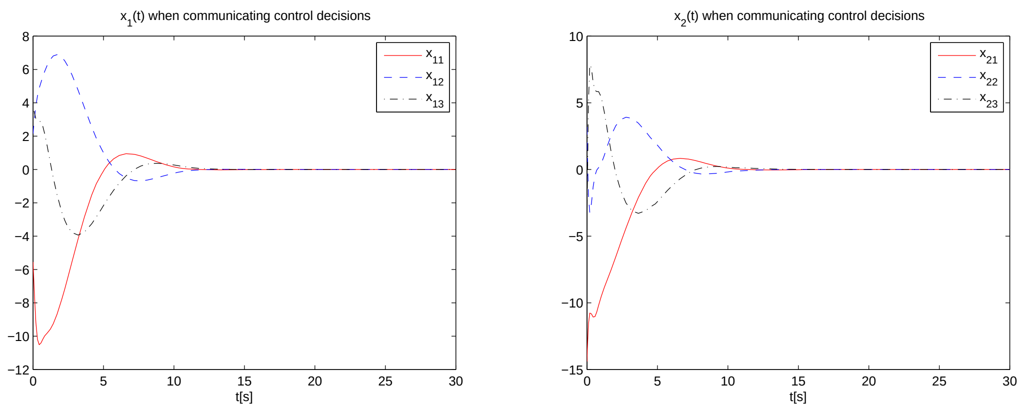

Figure 2.25: $x(t)$ of the reduced-order system staying when control agencies communicate their decisions to each other

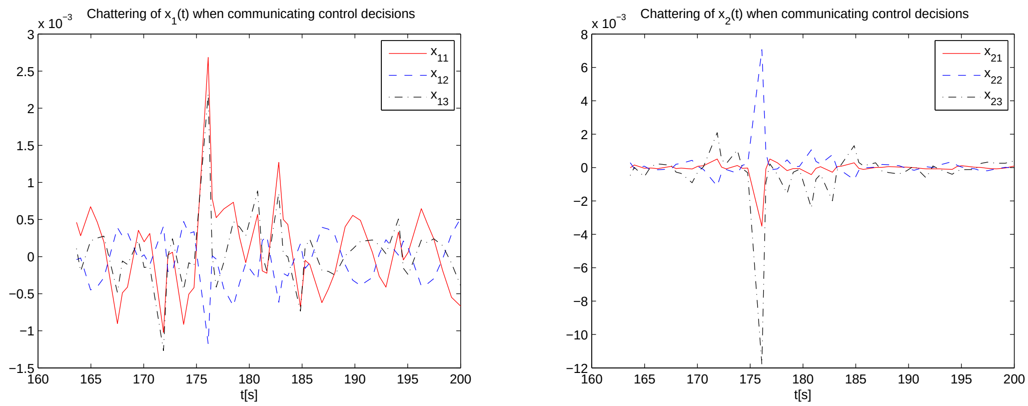

Figure 2.26: Chattering of $x(t)$ of the reduced-order system when control agencies communicate their decisions to each other

In the reduced-order system, $O(\epsilon)$ is composed of the input and disturbance of another system. If control agencies communicates their decisions to each other, the more efficient result is found. Figure 2.25 depicts the state variables when control agencies communicate their decisions to each other. The trajectories of Figure 2.25 become similar with those of the full-order system while the reduced-order system has small chattering phenomena (Figure 2.26).

2.5 Conclusion

In this chapter, the continuous-time linear weakly coupled systems are solved using sliding mode control after the decomposition of Gajić and Shen. With the accuracy of $O(\epsilon)$, the sliding surfaces and sliding mode controls are found for each reduced-order subsystem. An advantage of sling mode control of reduced-order systems to full-order systems is that it is of a lower order and thus has less time to find design parameters. Furthermore, due to the split into two independent subsystems, each input is assigned for each subsystem. For the more accurate result, $O(\epsilon)$, which is affected by the input of the other subsystem, can be considered.

Chapter 3

Continuous-Time Sliding Mode Control of Weakly Coupled Systems: Composite Control Approach

3.1 Introduction

In the previous chapter, two sliding surfaces are designed for weakly coupled linear systems composed of two subsystems using Utkin and Young's approach. In this chapter, the composite control approach is considered for a weakly coupled linear system composed of two subsystems with an external disturbance. For this approach, only one discontinuous sliding mode control law is designed while a state feedback law is chosen to maintain the other subsystem asymptotically stable. This approach includes two steps. Firstly, a state feedback control law is established to make the first subsystem stable. Secondly, a sliding surface with a control law is designed for the remaining dynamics of the system to ensure stability and disturbance rejection. For the second step, we used the decoupling procedure of Gajić and Shen (1989) and Utkin and Young's approach (1978). The composite control composed of two controls makes the whole system asymptotically stable.

3.2 Problem Formulation

Consider a continuous-time weakly coupled system represented by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & \epsilon A_2 \\ \epsilon A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 & \epsilon B_2 \\ \epsilon B_3 & B_4 \end{bmatrix} u(t) + Ed(t) \tag{3.1}$$

$$= A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + B \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + Ed(t) \tag{3.2}$$

where $x_1(t) \in \mathbf{R}^{n_1}$, $x_2(t) \in \mathbf{R}^{n_2}$, $n_1 + n_2 = n$, are state variables, $u_i(t) \in \mathbf{R}^{m_i}$, $i = 1, 2$, are control inputs, and $d(t) \in \mathbf{R}^l$ is the disturbance. ϵ is a small weak coupling parameter. It is assumed that matrices A_1, A_4 are constant and $O(1)$. In addition, magnitudes of all system eigenvalues are $O(1)$, that is, $|\lambda_j| = O(1)$, $j = 1, 2, \ldots, n$, which implies that the matrices A_1, A_4 are nonsingular with $\det\{A_1\} = O(1)$ and $\det\{A_4\} = O(1)$. It is also assumed that matrices A_1 and A_4 have no common eigenvalues (see Assumption 1.2.1). A, B, E are constant matrices of appropriate dimensions. Furthermore, B and E have full rank.

The system (3.7) is invariant to $d(t)$ if and only if the matching condition is satisfied (Drazenovic, 1969)

$$\text{rank}\begin{bmatrix} B & | & E \end{bmatrix} = \text{rank}\begin{bmatrix} B \end{bmatrix} \tag{3.3}$$

which means there exists a $m \times l$ matrix D such that

$$E = BD \tag{3.4}$$

3.3 Result

We apply the composite control approach in the sense that $u_{2C}(t)$ is used to stabilize the second subsystem and $u_{1C}(t)$ is used to provide sliding mode control with the global control given by

$$u(t) = \begin{bmatrix} u_{1C}(t) \\ u_{2C}(t) \end{bmatrix} \tag{3.5}$$

A state feedback control law is chosen as

$$u_{2C}(t) = K_2 x_2(t) \tag{3.6}$$

which makes the second system asymptotically stable. The system under the composite control law becomes

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + \epsilon(A_2 + B_2 K_2) x_2(t) + B_1 u_{1C}(t) + B_1 D_1 d(t) \\ \dot{x}_2(t) &= \epsilon A_3 x_1(t) + (A_4 + B_4 K_2) x_2(t) + \epsilon B_3 u_{1C}(t) + \epsilon B_3 D_1 d(t) \end{aligned} \tag{3.7}$$

By the change of variables

$$\eta_1(t) = x_1(t) - \epsilon L_1 x_2(t) \tag{3.8}$$

the systems (3.7) is transformed into a lower triangular form

$$\begin{bmatrix} \dot{\eta}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \bar{A}_1 & 0 \\ \epsilon A_3 & A_4 + B_4K_2 + \epsilon^2 A_3 L_1 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \epsilon B_3 \end{bmatrix} u_{1C}(t) + \begin{bmatrix} \bar{B}_1 \\ \epsilon B_3 \end{bmatrix} D_1 d(t) \tag{3.9}$$

where

$$\begin{aligned} \bar{A}_1 &= A_1 - \epsilon^2 L_1 A_3 \\ \bar{B}_1 &= B_1 - \epsilon^2 L_1 B_3 \end{aligned} \tag{3.10}$$

and L_1 is the solution of the following equation (Gajić and Shen, 1989)

$$A_1 L_1 - L_1(A_4 + B_4 K_2) + A_2 + B_2 K_2 - \epsilon^2 L_1 A_3 L_1 = 0 \tag{3.11}$$

Numerical solutions for L_1 can be obtained by using the fixed point iterations (Petrovic and Gajić, 1988), Newton method (Gajić *et al.*, 2009, Chap. 5), or the eigenvector method (Kecman and Tomasevic, 2006).

For the first subsystem

$$\dot{\eta}_1(t) = \bar{A}_1 \eta_1(t) + \bar{B}_1(u_{1C}(t) + D_1 d(t)) \tag{3.12}$$

if $m_1 < n_1$, there exists a nonsingular similarity transformation (Utkin and Young, 1978), $T_1 = \begin{bmatrix} N \\ \bar{B}_1 \end{bmatrix}$ which yields

$$\begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{bmatrix} = \begin{bmatrix} \bar{A}_{1_{11}} & \bar{A}_{1_{12}} \\ \bar{A}_{1_{21}} & \bar{A}_{1_{22}} \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{B}_{r1} \end{bmatrix} u_{1C}(t) \tag{3.13}$$

where $\xi_1(t) \in \mathbf{R}^{n_1 - m_1}$, $\xi_2(t) \in \mathbf{R}^{m_1}$, and $\bar{B}_1$ is an $m_1 \times m_1$ nonsingular matrix. We can find a state feedback gain matrix K_1 such that $\bar{A}_{1_{11}} - K_1 \bar{A}_{1_{12}}$ is asymptotically stable.

On the sliding surface, the system trajectory in the $(\xi_1(t), \xi_2(t))$ coordinates is expressed as

$$\begin{bmatrix} K_1 & I_{m_1} \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = 0 \tag{3.14}$$

or

$$s_1(t) = G_1 \eta_1(t) = \begin{bmatrix} K_1 & I_{m_1} \end{bmatrix} T_1 \eta_1(t) = 0 \tag{3.15}$$

in the original coordinates.

Starting with $\dot{s}_1(t) = 0$, we design a sliding control law of the sliding surface (3.15), that is

$$\dot{s}_1(t) = 0 = G_1\dot{\eta}_1(t) = G_1\bar{A}_1\eta_1(t) + G_1\bar{B}_1u_{1C}(t) + G_1\bar{B}_1D_1d(t) \tag{3.16}$$

From (3.16), sliding mode control $u_{1C}(t)$ can be described as

$$u_{1C}(t) = -(G_1\bar{B}_1)^{-1}G_1\bar{A}_1\eta_1(t) - (G_1\bar{B}_1)^{-1}(\gamma_1 + \sigma_1)\frac{s_1(t)}{||s_1(t)||} \tag{3.17}$$

where

$$\gamma_1 = ||G_1B_1D_1||d_{max} \tag{3.18}$$

is required to overcome the disturbance $d(t)$. It can be shown that (3.17) satisfies the vector type reaching condition (1.41)

$$s_1^T(t)\dot{s}_1(t) = -\sigma_1||s_1(t)|| - \gamma_1||s_1(t)|| + s_1(t)^TG_1\bar{B}_1D_1d(t) \tag{3.19}$$

with γ_1 chosen as in (3.41).

Lemma 3.3.1. *The reaching time of sliding mode control* (3.17) *satisfies*

$$\frac{\sqrt{s_1^T(0)s_1(0)}}{\sigma_1 + 2\gamma_1} < \tau < \frac{\sqrt{s_1^T(0)s_1(0)}}{\sigma_1} \tag{3.20}$$

Proof. Choose a Lyapunov function

$$V(t) = s_1^T(t)s_1(t) \tag{3.21}$$

The derivative of the Lyapunov function is given by

$$\dot{V}(t) = 2s_1^T(t)\dot{s}_1(t) = -2\sigma_1||s_1(t)|| - 2\gamma_1||s_1(t)|| + 2s_1(t)^TG_1\bar{B}_1D_1d(t) \tag{3.22}$$

which yields

$$-2(\sigma_1 + 2\gamma_1)\sqrt{V(t)} < \dot{V}(t) < -2\sigma_1\sqrt{V(t)} \tag{3.23}$$

or

$$-2(\sigma_1 + 2\gamma_1) < \frac{\dot{V}(t)}{\sqrt{V(t)}} < -2\sigma_1 \tag{3.24}$$

Hence

$$-2(\sigma_1 + 2\gamma_1)t < \sqrt{V(t)} - \sqrt{V(0)} < -2\sigma_1 t \tag{3.25}$$

Let τ be the time needed to reach the sliding mode ($V(\tau) = 0$). Then, the reaching time satisfies

$$\frac{\sqrt{V(0)}}{\sigma_1 + 2\gamma_1} < \tau < \frac{\sqrt{V(0)}}{\sigma_1} \tag{3.26}$$

which is finite. □

From (3.6) and (3.17), the composite control law $u(t) = u_{1C}(t) + u_{2C}(t)$ is given by

$$u(t) = \begin{bmatrix} -(G_1\bar{B}_1)^{-1}G_1\bar{A}_1\eta_1(t) - (G_1\bar{B}_1)^{-1}(\gamma_1 + \sigma_1)\frac{s_1(t)}{||s_1(t)||} \\ K_2x_2(t) \end{bmatrix} \tag{3.27}$$

or

$$u(t) = \begin{bmatrix} -(G_1\bar{B}_1)^{-1}G_1\bar{A}_1(x_1(t) - \epsilon L_1x_2(t)) - (G_1\bar{B}_1)^{-1}(\gamma_1 + \sigma_1)\frac{s_1(t)}{||s_1(t)||} \\ K_2x_2(t) \end{bmatrix} \tag{3.28}$$

in the original coordinates.

We also start with $u_{1C}(t)$ for the composite control approach in the sense that $u_{1C}(t)$ is used to stabilize the first subsystems and $u_{2C}(t)$ is used to provide sliding mode control with the global control given by

$$u(t) = \begin{bmatrix} u_{1C}(t) \\ u_{2C}(t) \end{bmatrix} \tag{3.29}$$

A state feedback control law is chosen as

$$u_{1C}(t) = K_1x_1(t) \tag{3.30}$$

which makes the first system asymptotically stable.

The system under the composite control law is defined as

$$\begin{aligned} \dot{x}_1(t) &= (A_1 + B_1K_1)x_1(t) + \epsilon A_2x_2(t) + \epsilon B_2u_{2C}(t) + \epsilon B_2D_2d(t) \\ \dot{x}_2(t) &= \epsilon(A_3 + B_3K_1)x_1(t) + A_4x_2(t) + B_4u_{2C}(t) + B_4D_2d(t) \end{aligned} \tag{3.31}$$

By the change of variables

$$\eta_2(t) = -\epsilon L_2x_1(t) + x_2(t) \tag{3.32}$$

the systems (3.31) is transformed into a upper triangular form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{\eta}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 + B_1K_1 + \epsilon^2A_2L_2 & \epsilon A_2 \\ 0 & \bar{A}_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ \eta_2(t) \end{bmatrix} + \begin{bmatrix} \epsilon B_2 \\ \bar{B}_4 \end{bmatrix} u_{2C}(t) + \begin{bmatrix} \epsilon B_2 \\ \bar{B}_4 \end{bmatrix} D_2d(t) \tag{3.33}$$

where

$$\bar{A}_4 = A_4 - \epsilon^2 L_2 A_2$$
$$\bar{B}_4 = B_4 - \epsilon^2 L_2 B_2 \tag{3.34}$$

and L_2 is the solution of the following equation (Gajić and Shen, 1989)

$$A_4 L_2 - L_2(A_1 + B_1 K_1) + A_3 + B_3 K_1 - \epsilon^2 L_2 A_2 L_2 = 0 \tag{3.35}$$

For the second subsystem

$$\dot{\eta}_2(t) = \bar{A}_4 \eta_2(t) + \bar{B}_4(u_{2C}(t) + D_2 d(t)) \tag{3.36}$$

if $m_2 < n_2$, there exists a nonsingular similarity transformation (Utkin and Young, 1978) such as $T_2 = \begin{bmatrix} N \\ \bar{B}_2 \end{bmatrix}$ which yields

$$\begin{bmatrix} \dot{\zeta}_1(t) \\ \dot{\zeta}_2(t) \end{bmatrix} = \begin{bmatrix} \bar{A}_{2_{11}} & \bar{A}_{2_{12}} \\ \bar{A}_{2_{21}} & \bar{A}_{2_{22}} \end{bmatrix} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{B}_{r2} \end{bmatrix} u_{2C}(t) \tag{3.37}$$

where $\zeta_1(t) \in \mathbf{R}^{n_2 - m_2}$, $\zeta_2(t) \in \mathbf{R}^{m_2}$, and $\bar{B}_1$ is an $m_2 \times m_2$ nonsingular matrix. We can find a state feedback gain matrix K_2 such that $\bar{A}_{2_{11}} - K_2 \bar{A}_{2_{12}}$ is asymptotically stable.

On the sliding surface using Utkin and Young's method (Utkin and Young, 1978), the system trajectory in the $(\zeta_1(t), \zeta_2(t))$ coordinates is expressed as

$$\begin{bmatrix} K_2 & I_{m_2} \end{bmatrix} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} = 0 \tag{3.38}$$

or

$$s_2(t) = G_2 \eta_2(t) = \begin{bmatrix} K_2 & I_{m_2} \end{bmatrix} T_2 \eta_2(t) = 0 \tag{3.39}$$

in the original coordinates.

The sliding mode control $u_{2C}(t)$ can be described as

$$u_{2C}(t) = -(G_2 \bar{B}_2)^{-1} G_2 \bar{A}_2 \eta_2(t) - (G_2 \bar{B}_2)^{-1} (\gamma_2 + \sigma_2) \frac{s_2(t)}{||s_2(t)||} \tag{3.40}$$

where

$$\gamma_2 = ||G_2 \bar{B}_4 D_2|| d_{max} \tag{3.41}$$

is required to overcome the disturbance $d(t)$. From (3.30) and (3.40), the composite control law $u(t) = u_{1C}(t) + u_{2C}(t)$ is given by

$$u(t) = \begin{bmatrix} K_1 x_1(t) \\ -(G_2\bar{B}_2)^{-1}G_2\bar{A}_2\eta_2(t) - (G_2\bar{B}_2)^{-1}(\gamma_2+\sigma_2)\frac{s_2(t)}{||s_2(t)||} \end{bmatrix} \tag{3.42}$$

or

$$u(t) = \begin{bmatrix} K_1 x_1(t) \\ -(G_2\bar{B}_2)^{-1}G_2\bar{A}_2(-\epsilon L_2 x_1(t) + x_2(t)) - (G_2\bar{B}_2)^{-1}(\gamma_2+\sigma_2)\frac{s_2(t)}{||s_2(t)||} \end{bmatrix} \tag{3.43}$$

in the original coordinates.

3.4 Example

To illustrate the proposed method, we consider an industrially important reactor considered in Arkun and Ramakrishnan (1983). The system matrices A, B, and E are determined by

$$\begin{aligned} A &= \begin{bmatrix} -16.11 & -0.39 & 27.2 & 0 & 0 \\ 0.01 & -16.99 & 0 & 0 & 12.47 \\ 15.11 & 0 & -53.6 & -16.57 & 71.78 \\ -53.36 & 0 & 0 & -107.2 & 232.11 \\ 2.27 & 69.1 & 0 & 0 & -102.99 \end{bmatrix} \\ B^T &= \begin{bmatrix} 11.12 & -3.61 & -21.91 & -53.6 & 69.1 \\ -12.6 & 3.36 & 0 & 0 & 0 \end{bmatrix} \\ E^T &= \begin{bmatrix} 11.12 & -3.61 & -21.91 & -53.6 & 69.1 \\ -12.6 & 3.36 & 0 & 0 & 0 \end{bmatrix} \times 10^{-3} \end{aligned} \tag{3.44}$$

The eigenvalues of the matrix A are -129, -82, -74, -7.7, and -2.8. The small parameter is chosen as $\epsilon = 0.1$, which is roughly the ratio of 7.7 and 74 (Gajić *et al.*, 2009, Chap.

4). The initial condition is

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 10 \\ -10 \\ 5 \\ 7 \\ -15 \end{bmatrix} \tag{3.45}$$

The external disturbance is

$$d(t) = 2\sin(t) \tag{3.46}$$

The matrices A and B are partitioned with $n_1 = 3$ and $n_2 = 2$.

A state feedback control law to stabilize the second subsystem is taken as

$$u_{2C}(t) = \begin{bmatrix} 0.1 & 0 \end{bmatrix} x_2(t) \tag{3.47}$$

L_1 of (3.11) is found using the Newton method with $O(10^{-10})$ accuracy

$$L_1 = \begin{bmatrix} -1.0096 & -3.9455 \\ -0.0322 & -1.3774 \\ 3.7093 & 4.6965 \end{bmatrix} \tag{3.48}$$

Using the change of variables of (3.8), the first subsystem is defined as

$$\dot{\eta}_1(t) = \begin{bmatrix} -20.6015 & 26.8731 & 27.2000 \\ 0.1511 & -7.4720 & 0 \\ 33.8366 & -32.4526 & -53.6000 \end{bmatrix} \eta_1(t) + \begin{bmatrix} 32.9718 \\ 5.7356 \\ -34.4809 \end{bmatrix} (u_{1C}(t) + D_2 d(t)) \tag{3.49}$$

Using a similarity transformation (Utkin and Young, 1978)

$$T_1 = \begin{bmatrix} -0.1194 & 0.9916 & 0.0508 \\ 0.7176 & 0.0508 & 0.6946 \\ 32.9718 & 5.7356 & -34.4809 \end{bmatrix} \tag{3.50}$$

which yields

$$\begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{bmatrix} = \begin{bmatrix} -12.98 & -1.66 & 0.12 \\ -5.55 & -6.23 & 0.38 \\ 2305.29 & 682.40 & -62.47 \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2308.97 \end{bmatrix} u_{1C}(t) \tag{3.51}$$

where $\xi_1(t) \in \mathbf{R}^2$ and $\xi_2(t) \in \mathbf{R}^1$. The sliding surface (3.15) can be chosen as

$$\begin{aligned} s_1(t) &= \begin{bmatrix} 32.1327 & -30.5918 & -41.3261 \end{bmatrix} \eta_1(t) \\ &= \begin{bmatrix} 32.1327 & -30.5918 & -41.3261 \end{bmatrix} (x_1(t) - \epsilon L_1 x_2(t)) \end{aligned} \tag{3.52}$$

and sliding mode control $u_{1C}(t)$ (3.17) can be described as

$$u_{1C}(t) = \begin{bmatrix} 0.8943 & -1.0538 & -1.3379 \end{bmatrix} \eta_1(t) - 0.0024 \frac{s_1(t)}{||s_1(t)||} \tag{3.53}$$

Figure 3.1 and figure 3.2 present the evolution of the state variables. The chattering phenomena are found in both figures due to the effect of the switching control law. Figure 3.3 shows the evolution of the sliding function. We can check the reaching time is around 75 seconds which satisfies Lemma 3.3.1 such that

$$63.56s < \tau < 91.08s \tag{3.54}$$

Figure 3.4 is the sliding mode control law for the first subsystem.

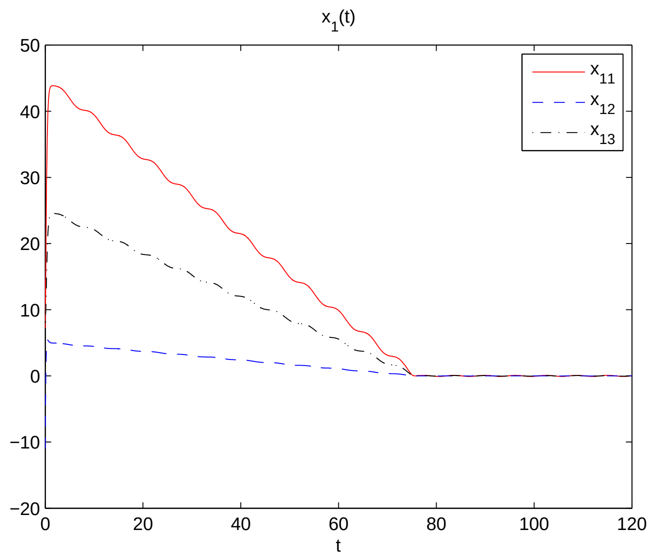

Figure 3.1: $x_1(t)$ state variable

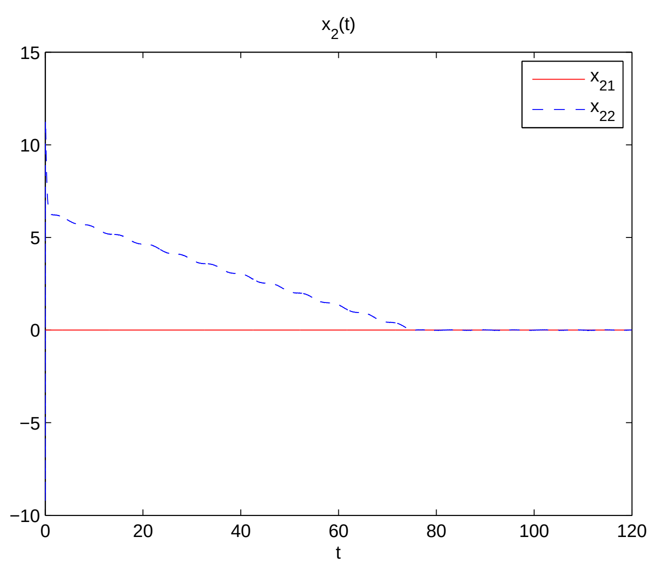

Figure 3.2: $x_2(t)$ state variable

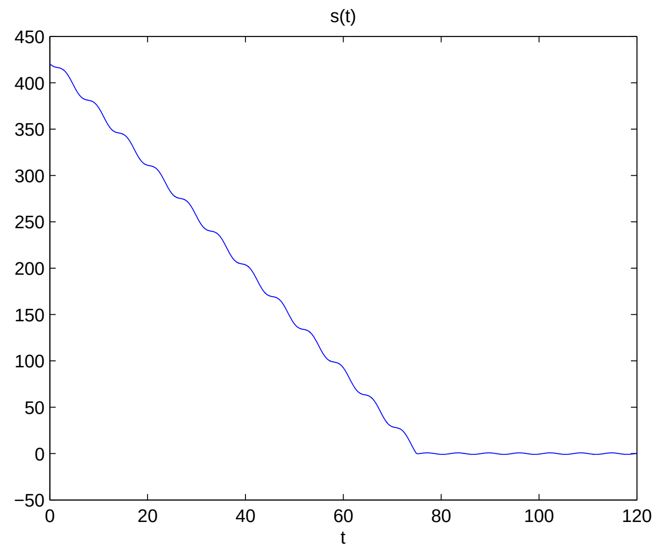

Figure 3.3: Sliding surface for the first subsystem

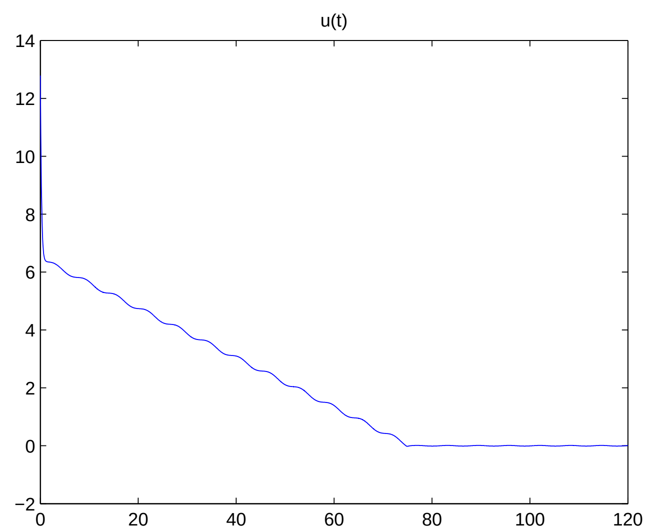

Figure 3.4: Sliding mode control for the first subsystem

We can also start with a state feedback control law to stabilize the first subsystem, taken as

$$u_{1C}(t) = \begin{bmatrix} -0.9126 & -0.8385 & 2.8105 \end{bmatrix} x_1(t) \tag{3.55}$$

L_2 of (3.35) is found using the Newton method with $O(10^{-10})$ accuracy

$$L_2 = \begin{bmatrix} -47.8563 & -30.3972 & 20.4838 \\ -26.2794 & -26.7637 & -16.5370 \end{bmatrix} \tag{3.56}$$

Using the change of variables of (3.32), the second subsystem is defined as

$$\dot{\eta}_2(t) = \begin{bmatrix} -73.2583 & 122.9825 \\ -27.4018 & 49.0868 \end{bmatrix} \eta_2(t) + \begin{bmatrix} -50.0854 \\ -24.1194 \end{bmatrix} (u_{2C}(t) + D_2 d(t)) \tag{3.57}$$

Using a similarity transformation (Utkin and Young, 1978)

$$T_2 = \begin{bmatrix} -0.4339 & 0.9010 \\ -50.0854 & -24.1194 \end{bmatrix} \tag{3.58}$$

which yields

$$\begin{bmatrix} \dot{\zeta}_1(t) \\ \dot{\zeta}_2(t) \end{bmatrix} = \begin{bmatrix} -11.3082 & -0.0437 \\ -8495.0827 & -12.8634 \end{bmatrix} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 3090.2978 \end{bmatrix} u_{2C}(t) \tag{3.59}$$

where $\xi_1(t) \in \mathbf{R}^1$ and $\xi_2(t) \in \mathbf{R}^1$. The sliding surface (3.39) can be chosen as

$$\begin{aligned} s_2(t) &= \begin{bmatrix} -68.0232 & 13.1294 \end{bmatrix} \eta_2(t) \\ &= \begin{bmatrix} -68.0232 & 13.1294 \end{bmatrix} (-\epsilon L_2 x_1(t) + x_2(t)) \end{aligned} \tag{3.60}$$

and sliding mode control $u_{2C}(t)$ (3.40) can be described as

$$u_{2C}(t) = \begin{bmatrix} -1.4961 & 2.4985 \end{bmatrix} \eta_2(t) - 0.0023 \frac{s_2(t)}{||s_2(t)||} \tag{3.61}$$

Figures 3.5 - 3.6 present the evolution of the state variables. The chattering phenomena are found in both figures due to the effect of the switching control law. Figure 3.3 shows the evolution of the sliding function. We can check that the reaching time is around 95 seconds which satisfies Lemma 3.3.1 since

$$82.28s < \tau < 108.91s \tag{3.62}$$

Figure 3.8 is the sliding mode control law for the second subsystem.

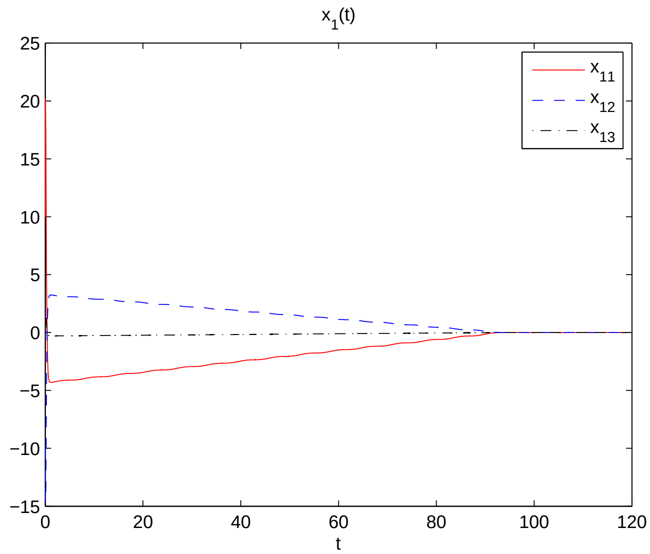

Figure 3.5: $x_1(t)$ state variable

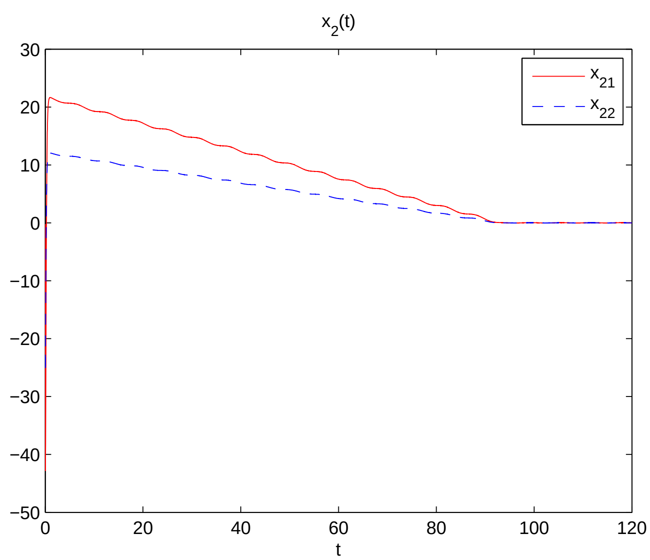

Figure 3.6: $x_2(t)$ state variable

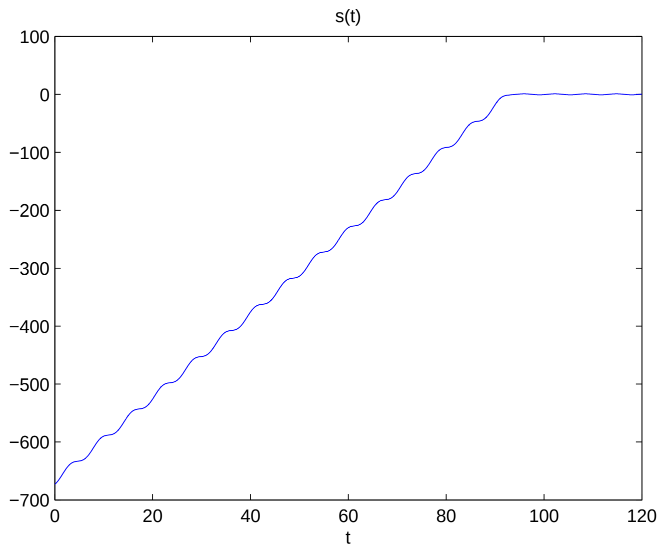

Figure 3.7: Sliding surface for the second subsystem

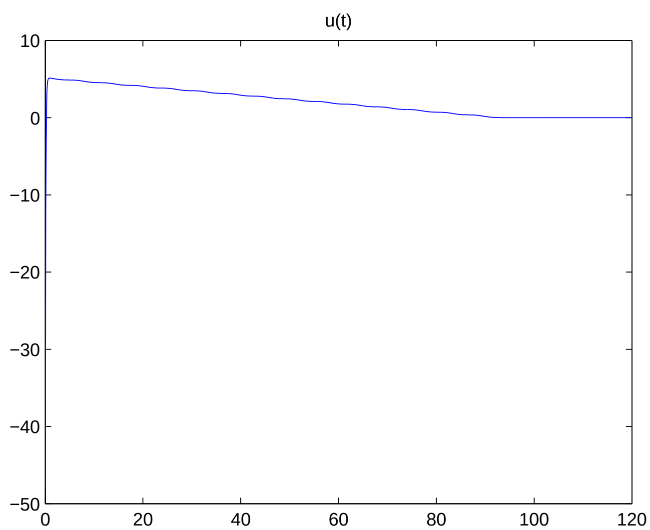

Figure 3.8: Sliding mode control for the second subsystem

3.5 Conclusion

We have presented composite control composed of two controls for each subsystem. The first subsystem is asymptotically stabilized by a state feedback control law, and then, the second system is controlled by sliding mode control law. Putting these two control laws together produces a composite control law that makes the closed-loop system asymptotically stable. We also consider another composite control law that is composed of a state feedback control law, which maintains the second subsystem stable, and a sliding mode control law, which makes the first subsystem stable. This approach makes the system asymptotically stable as well as robust against the external disturbances.

Chapter 4

Continuous-Time Sliding Mode Control of Weakly Coupled Systems: Lyapunov Approach

4.1 Introduction

In the previous chapters, Utkin and Young's approach is used to design the sliding surfaces. In this chapter, we design sliding surfaces on which the weakly coupled system is asymptotically stable using the Lyapunov approach. After the weakly coupled linear system, composed of two subsystems is decoupled, the Lyapunov functions for both subsystems are found and synthesized into one single Lyapunov function for the whole system

4.2 Problem Formulation

Consider a continuous-time weakly coupled system represented by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & \epsilon A_2 \\ \epsilon A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 & \epsilon B_2 \\ \epsilon B_3 & B_4 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \tag{4.1}$$

$$= A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + B \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \tag{4.2}$$

where $x_1(t) \in \mathbf{R}^{n_1}$, $x_2(t) \in \mathbf{R}^{n_2}$, $n_1 + n_2 = n$, are state variables, $u_i(t) \in \mathbf{R}^{m_i}$, $i = 1, 2$, are control inputs, and ϵ is a small weak coupling parameter. It is assumed that matrices A_1, A_4 are constant and $O(1)$. In addition, magnitudes of all system eigenvalues are $O(1)$, that is, $|\lambda_j| = O(1)$, $j = 1, 2, \dots, n$, which implies that the matrices A_1, A_4 are nonsingular with $\det\{A_1\} = O(1)$ and $\det\{A_4\} = O(1)$. It is also assumed that matrices A_1 and A_4 have no common eigenvalues (see Assumption 1.2.1). Furthermore, A and B are constant matrices of appropriate dimensions. Matrix B has full rank.

The main objective is to find a sliding surface using the Lyapunov approach with a discontinuous control law to achieve asymptotic stability.

4.3 Result

We can design a decentralized control in the original coordinates such as

$$u = \begin{bmatrix} u_{1C}(t) \\ u_{2C}(t) \end{bmatrix} = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = K \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{4.3}$$

The closed loop system is written as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} J_1 & \epsilon J_2 \\ \epsilon J_3 & J_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{4.4}$$

where $J_1 = A_1 + B_1 K_1$, $J_2 = A_2 + B_2 K_2$, $J_3 = A_3 + B_3 K_1$, and $J_4 = A_4 + B_4 K_2$. Apply the non-singular decoupling transformation of Gajić and Shen (1989) as

$$\begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = \begin{bmatrix} I_{n_1} & -\epsilon L \\ \epsilon H & I_{n_2} - \epsilon^2 HL \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{4.5}$$

(4.4) where L satisfies $J_1 L - L J_4 + J_2 - \epsilon^2 L J_3 L = 0$ and H satisfies $H(J_1 - \epsilon^2 L J_3) - (J_4 + \epsilon^2 J_3 L)H + J_3 = 0$. The decoupled subsystems in the new coordinate are

$$\begin{aligned} \dot{\eta}_1 &= (J_1 - \epsilon^2 L J_3)\eta_1 = M_1 \eta_1 \\ \dot{\eta}_2 &= (J_4 + \epsilon^2 J_3 L)\eta_2 = M_2 \eta_2 \end{aligned} \tag{4.6}$$

Both subsystems (4.6) are asymptotically stable. There exist positive definite matrices P_1 and P_2 that satisfies the Lyapunov equations where Q_1 and Q_2 are positive definite matrices.

$$\begin{aligned} P_1 M_1 + M_1^T P_1 &= -Q_1, \; Q_1 > 0 \\ P_2 M_2 + M_2^T P_2 &= -Q_2, \; Q_2 > 0 \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} M_1 &= J_1 - \epsilon^2 L J_3 \\ M_2 &= J_4 + \epsilon^2 J_3 L \end{aligned} \tag{4.8}$$

Note that $M_i = f(K_i)$, $i = 1, 2$.

With P_1 and P_2, the sliding surface of weakly coupled system can be described as

$$s(x_1(t), x_2(t)) = \begin{bmatrix} B_1 & \epsilon B_2 \\ \epsilon B_3 & B_4 \end{bmatrix}^T \mathbf{T}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \mathbf{T} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 0 \tag{4.9}$$

Proof. Introduce a dummy control variable which stabilizes the weakly coupled system

$$u = K \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - K \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + u \tag{4.10}$$

and substitute into (4.1) to yield

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} J_1 & \epsilon J_2 \\ \epsilon J_3 & J_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 & \epsilon B_2 \\ \epsilon B_3 & B_4 \end{bmatrix} (-K \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}) \tag{4.11}$$

where

$$\begin{bmatrix} J_1 & \epsilon J_2 \\ \epsilon J_3 & J_4 \end{bmatrix} = \begin{bmatrix} A_1 & \epsilon A_2 \\ \epsilon A_3 & A_4 \end{bmatrix} + \begin{bmatrix} B_1 & \epsilon B_2 \\ \epsilon B_3 & B_4 \end{bmatrix} K \tag{4.12}$$

is an asymptotically stable matrix. Choose a Lyapunov function as

$$V(x_1(t), x_2(t)) = \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} \tag{4.13}$$

$$= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}^T \mathbf{T}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \mathbf{T} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{4.14}$$

It follows that

$$\begin{aligned} \dot{V}(x_1(t), x_2(t)) = & \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}^T \begin{bmatrix} J_1 & \epsilon J_2 \\ \epsilon J_3 & J_4 \end{bmatrix}^T \mathbf{T}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \mathbf{T} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ & + \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}^T \mathbf{T}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \mathbf{T} \begin{bmatrix} J_1 & \epsilon J_2 \\ \epsilon J_3 & J_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ & + 2\Big(- K \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \Big) \begin{bmatrix} B_1 & \epsilon B_2 \\ \epsilon B_3 & B_4 \end{bmatrix}^T \mathbf{T}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \mathbf{T} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned} \tag{4.15}$$

In sliding mode (4.9), $\dot{V}(x_1(t), x_2(t))$ can be expressed as

$$\dot{V}(x_1(t), x_2(t)) = - \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix}^T \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} < 0 \tag{4.16}$$

□

We do not need actually to apply the stabilizing control (4.3) in the variable structure control law. The feedback gain K is useful for yielding a Lyapunov function only. According to Drazenovic (1969), all controls and disturbances entering the systems through the control channels B can be nullified in the sliding mode. By choosing the sliding surface (4.9), asymptotic stability with the associate Lyapunov functions is realized in the sliding mode instead of being enforced directly through the decentralized continuous control (Su, 1999).

4.4 Example

The design of a chemical reactor example considered in Patnaik *et al.* (1980) is represented by the state space model of the form

$$A = \begin{bmatrix} -4.02 & 5.12 & 0 & 0 & -2.08 & 0 & 0 & 0 & 0.87 \\ -0.35 & 0.99 & 0 & 0 & -2.34 & 0 & 0 & 0 & 0.97 \\ -7.91 & 15.41 & -4.07 & 0 & -6.45 & 0 & 0 & 0 & 2.68 \\ -21.82 & 35.61 & -0.34 & -3.87 & -17.8 & 0 & 0 & 0 & 7.39 \\ -60.2 & 98.19 & -7.91 & 0.34 & -53.01 & 0 & 0 & 0 & 20.4 \\ 0 & 0 & 0 & 0 & 94 & -147.2 & 0 & 53.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 94 & -147.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12.8 & 0 & -31.6 & 0 \\ 0 & 0 & 0 & 0 & 12.8 & 0 & 0 & 18.8 & -31.6 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 0.010 & 0.003 & 0.009 & 0.024 & 0.068 & 0 & 0 & 0 & 0 \\ -0.011 & -0.021 & -0.059 & -0.162 & -0.445 & 0 & 0 & 0 & 0 \\ -0.151 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.17}$$

with

$$x(0)^T = \begin{bmatrix} 2 & 1 & 2 & 1 & -1 & 15 & 50 & -7 & -3 \end{bmatrix} \tag{4.18}$$

The system can be decoupled with $n_1 = 5$ and $n_2 = 4$. Using the formula for an estimate of a small coupling parameter ϵ, as suggested by Shen and Gajić (1990), we have obtained $\epsilon = 0.47 = 94/200.4$. Eigenvalues of A are -147.2, -153.12, -56.04, -37.54, -15.55, -0.3047, -4.66, -3.30, and -3.86. The composite control (4.3) places the eigenvalues at -153.46, -147.20, -124.94, -32.27 ± 5.86i, -9.33, -5.77 ± 3.07i, -0.14, by letting eigenvalues of J_1 as -124.44, -11.14, -9.26, -7.92, and -0.78 and eigenvalues of J_4 as -152.82, -147.20, -31.60, and -25.98. The closed loop systems (4.4) is written as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = J \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{4.19}$$

with

$$J = \begin{bmatrix} 125.2 & -57.7 & -190.8 & 31.8 & -10.1 & 0 & 0 & 0 & 0.9 \\ 402.0 & -193.0 & -597.2 & 100.2 & -28.9 & 0 & 0 & 0 & 1.0 \\ 48.2 & -19.1 & -74.8 & 8.8 & -3.2 & 0 & 0 & 0 & 2.7 \\ 122.1 & -54.2 & -179.4 & 17.7 & -8.1 & 0 & 0 & 0 & 7.4 \\ 364.8 & -162.5 & -544.0 & 67.1 & -28.6 & 0 & 0 & 0 & 20.4 \\ 0 & 0 & 0 & 0 & 94.0 & -147.2 & 0 & 53.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 94.0 & -147.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12.8 & 0 & -31.6 & 0 \\ 0 & 0 & 0 & 0 & 12.8 & 0 & 0 & 18.8 & -31.6 \end{bmatrix} \tag{4.20}$$

Applying the decoupling transformation of Gajić and Shen (1989) with the accuracy of $O(10^{-10})$ as

$$\mathbf{T} = \begin{bmatrix} I_{n_1} & -\epsilon L \\ \epsilon H & I_{n_2} - \epsilon^2 HL \end{bmatrix} \tag{4.21}$$

with

$$L = \begin{bmatrix} -0.0028 & 0 & 0.0450 & 0.2282 \\ 0.0435 & 0 & 0.7130 & 1.1714 \\ -0.0200 & 0 & -0.1993 & -0.1609 \\ -0.0285 & 0 & -0.2629 & -0.2537 \\ -0.0837 & 0 & -0.7570 & -0.6502 \end{bmatrix} \tag{4.22}$$

and

$$H = \begin{bmatrix} 17.0131 & -7.6799 & -25.3479 & 3.6489 & -2.2973 \\ 84.5500 & -38.1612 & -125.9357 & 18.1831 & -6.1422 \\ -4.0636 & 1.8936 & 6.0577 & -1.1417 & -0.1615 \\ -2.6498 & 1.3231 & 3.9678 & -1.1197 & -0.6225 \end{bmatrix} \tag{4.23}$$

Solving the Lyapunov equations with $Q_1 = I$ and $Q_2 = I$ renders

$$P_1 = \begin{bmatrix} 5.0217 & 4.9624 & 2.0308 & 0.3265 & -3.0537 \\ 4.9624 & 5.9727 & 1.6543 & -0.2313 & -4.4569 \\ 2.0308 & 1.6543 & 0.9465 & 0.3228 & -0.7973 \\ 0.3265 & -0.2313 & 0.3228 & 0.9621 & 2.2452 \\ -3.0537 & -4.4569 & -0.7973 & 2.2452 & 9.2902 \end{bmatrix} \tag{4.24}$$

$$P_2 = \begin{bmatrix} 0.0039 & 0.0014 & 0.0016 & -0.0019 \\ 0.0014 & 0.004 & 0.0009 & -0.0009 \\ 0.0016 & 0.0009 & 0.0165 & 0.0031 \\ -0.0019 & -0.0009 & 0.0031 & 0.0154 \end{bmatrix}$$

The sliding surface can be chosen as

$$s(x_1(t), x_2(t)) = B^T \mathbf{T}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \mathbf{T} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 0 \tag{4.25}$$

with B from (4.17), $\mathbf{T}$ from (4.21), and $\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$ from (4.24).

The simulation results are presented in Figures 4.1 - 4.6 in $(\eta_1(t), \eta_2(t))$ coordinates. Figures 4.1 and 4.2 show the evolutions of state variables in the new coordinates. Figures 4.3 and 4.4 depict sliding surfaces and figures 4.5 and 4.6 show sliding mode controls in the new coordinates.

Figures 4.7 and 4.8 show state variables in the original coordinates. Figures 4.9 and 4.10 present chattering phenomena which are small because we did not consider any disturbances. If disturbances exist, we could handle them through the control channels B using Drazenovic's invariance condition.

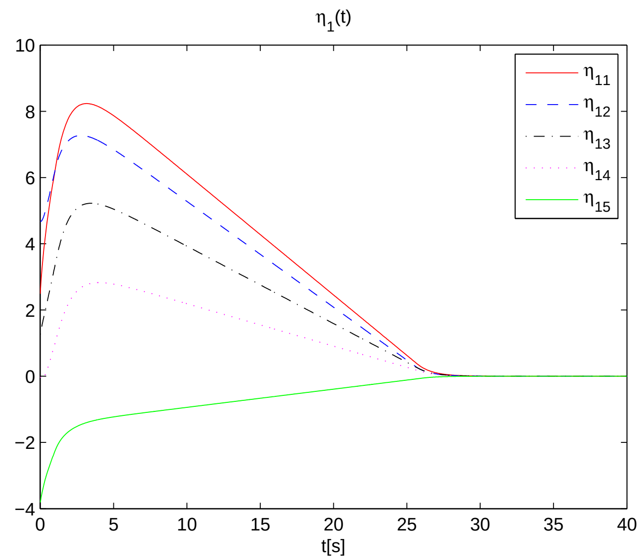

Figure 4.1: $\eta_1(t)$ state variables

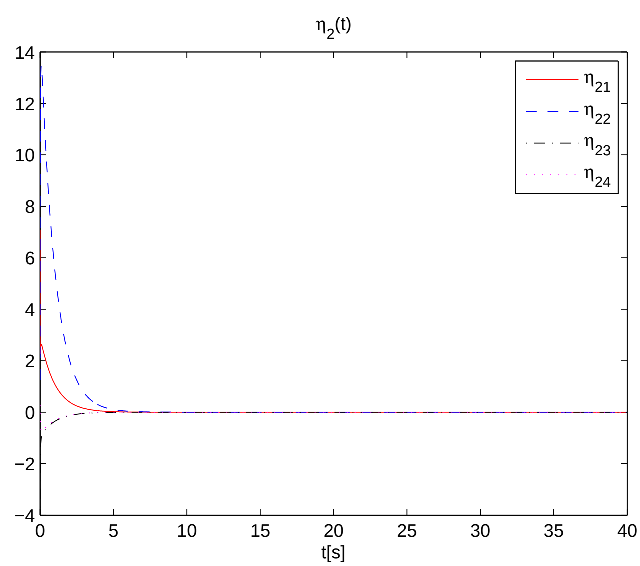

Figure 4.2: $\eta_2(t)$ state variables

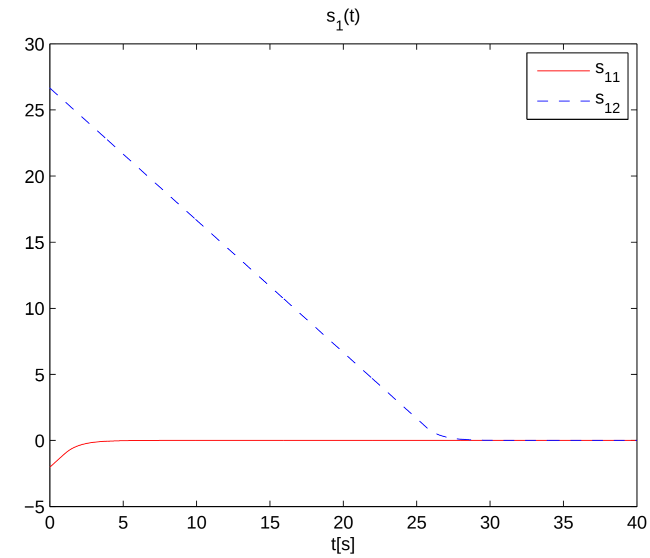

Figure 4.3: Sliding surfaces for the first subsystem

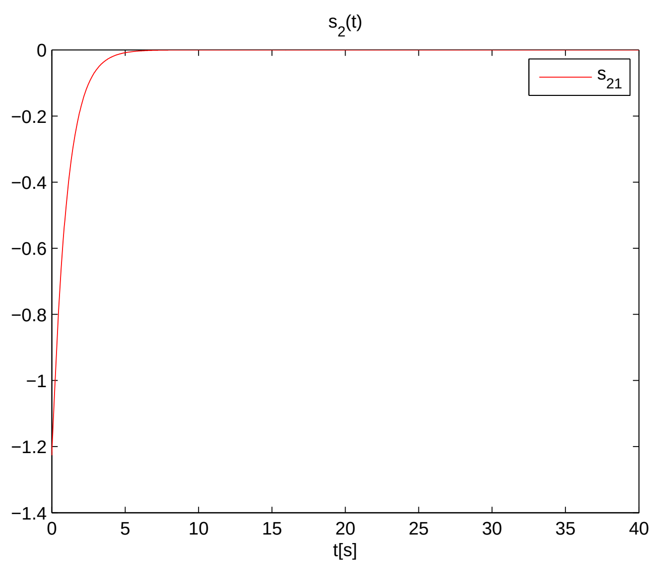

Figure 4.4: Sliding surface for the second subsystem

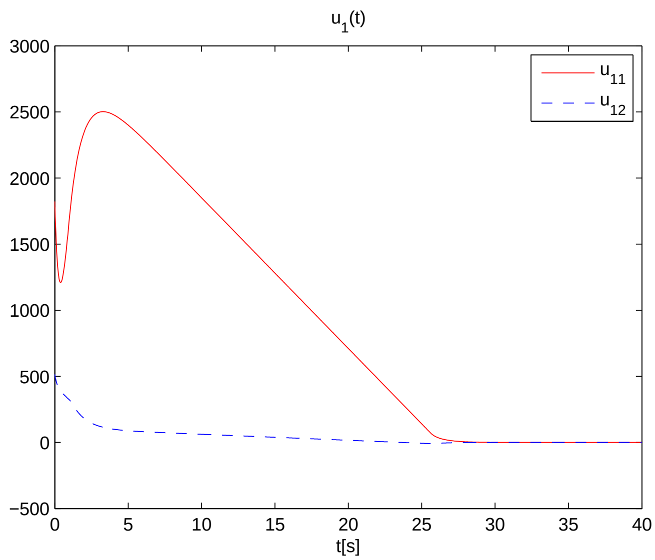

Figure 4.5: Sliding mode controls for the first subsystem

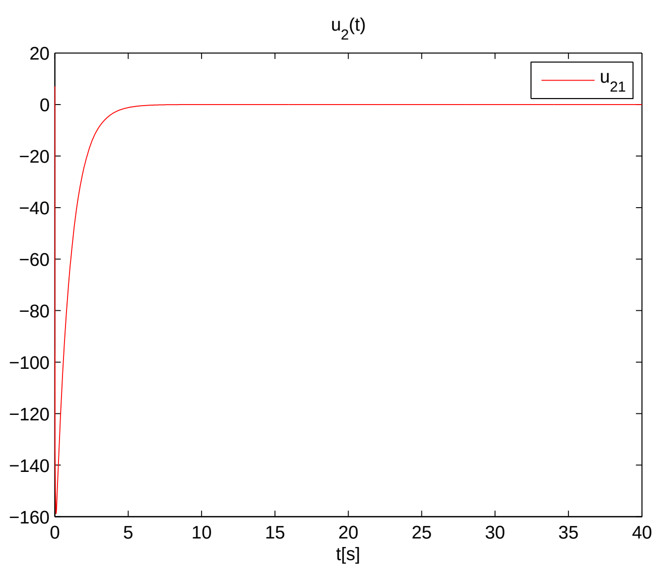

Figure 4.6: Sliding mode control for the second subsystem

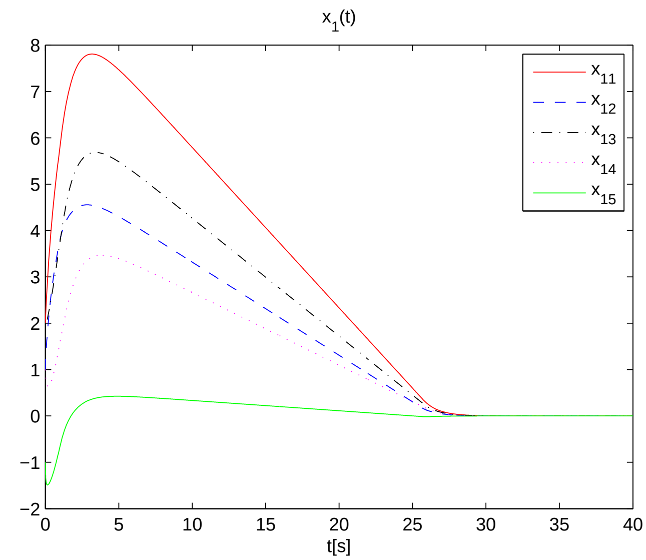

Figure 4.7: $x_1(t)$ state variables

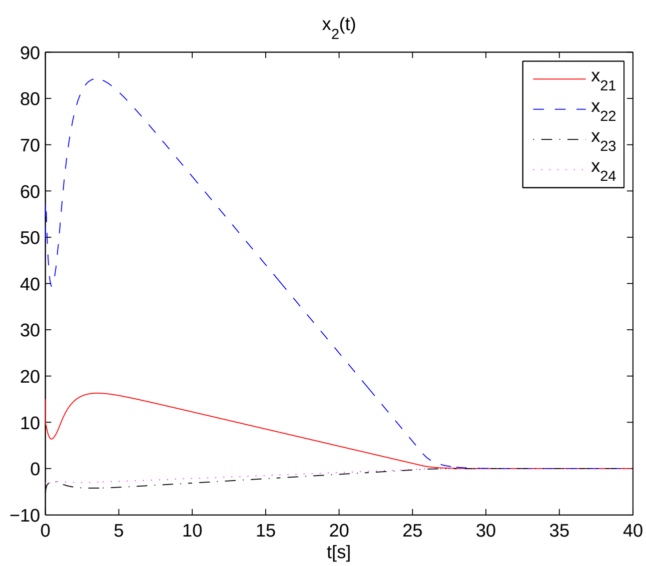

Figure 4.8: $x_2(t)$ state variables

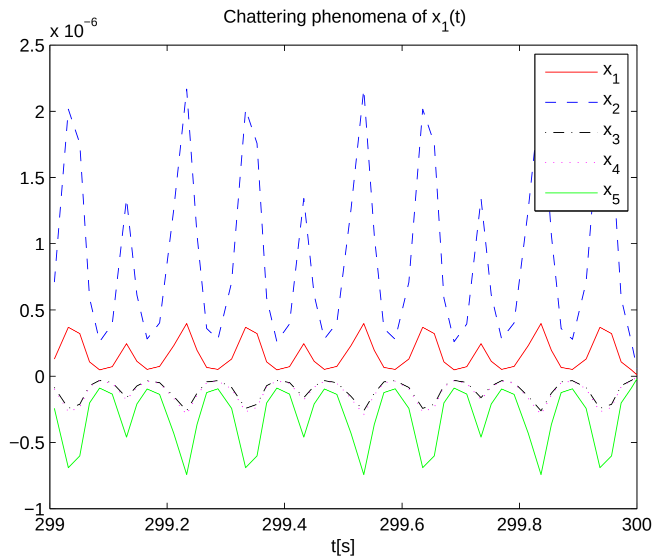

Figure 4.9: Chattering of $x_1(t)$ state variables

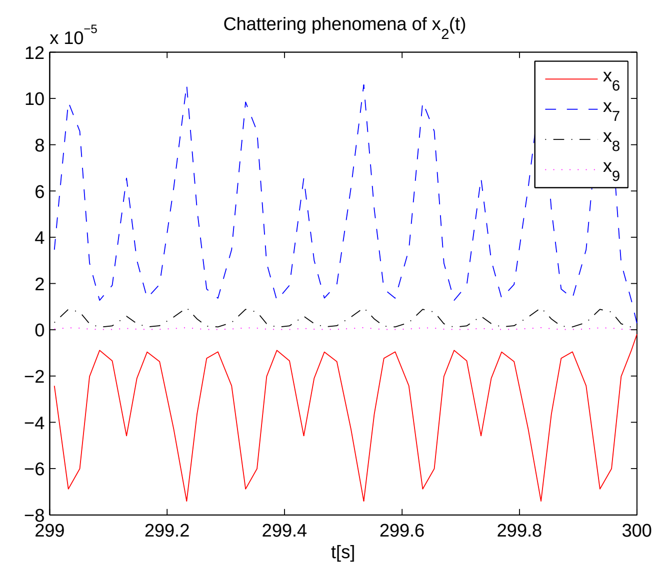

Figure 4.10: Chattering of $x_2(t)$ state variables

4.5 Conclusion

In this chapter, we use the Lyapunov approach for the sliding surface design. This approach is composed of two steps. First one is to decompose the system into two stabilized reduced-order subsystems and the next step is to find Lyapunov functions for both subsystems and synthesize into one single Lyapunov function for the whole system. We do not need actually the decentralized control which is for yielding a Lyapunov function only. Sliding mode control law, like unit control law or signum function control law, should be found in the sliding mode.

Chapter 5

Conclusions and Future Work

5.1 Conclusions

The study of sliding mode control for continuous-time weakly coupled systems is firstly introduced in this thesis. Using the decoupling transformation of weakly coupled systems, we can apply the sliding mode control to the systems. Three methods - Utkin and Young's sliding mode control method for each subsystem, composite control approach composed of a state feedback control law and a sliding mode control law, and sliding mode control using the Lyapunov approach - are presented. All three methods provide controls that make the systems asymptotically stable and robust against parametric uncertainties and exogenous disturbances which can deteriorate the performance of dynamic systems. We have demonstrated that full-order weakly coupled systems can be successfully controlled via the sliding mode technique with the sliding mode controllers designed on the subsystem levels.

5.2 Future Work

We can extend the results of this thesis to the systems composed of N weakly coupled subsystems using the decoupling transformation for N weakly coupled subsystems. The N weakly coupled systems is decoupled into N subsystems completely and each sliding surface can be designed for each subsystem. Also, the Lyapunov approach in Chapter 4 can be extended to the nonlinear, time-varying, or delayed weakly coupled systems. Furthermore, the sliding mode of a deterministic weakly coupled system could be developed with feedback of estimated states, and the optimal sliding Gaussian control of the weakly coupled system could be found for stochastic systems.

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