MULTIVARIATE AUTOREGRESSIVE MODULAR PROCESSES

by

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ABSTRACT OF THE DISSERTATION

Multivariate Autoregressive Modular Processes

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This thesis defines a new class of vector-valued stochastic processes, called *MARM* (*Multivariate Autoregressive Modular*) Processes. It describes the construction of two flavors of MARM processes, MARM^+ and MARM^-, studies the statistics of MARM processes (transition structure and second order statistics), and devises MARM-based fitting and forecasting algorithms providing point estimators and confidence intervals. The key advantage of MARM processes is their ability to fit a strong statistical signature consisting of empirical first-order and second-order statistics simultaneously. More precisely, MARM processes exactly fit arbitrary multi-dimensional empirical histograms and approximately fit the leading empirical autocorrelations and cross-correlations functions. This ability appears to make the MARM modeling methodology unique in its goal of fitting a model to such a class of strong statistical signatures. Furthermore, the thesis proposes practical MARM modeling and forecasting methodologies of considerable generality, suitable for implementation on a computer. We demonstrate the efficacy of these methodologies with an example of a three-dimension time series vector, using a software environment, called *MultiArmLab*, which supports MARM modeling and forecasting.

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1. Introduction

This thesis is concerned with fitting multivariate time series models to multivariate empirical data sequences. Multivariate (multidimensional) time series is a vector of time series over a common time domain. Such time series abound in applications, including economic forecasting, budgetary analysis, stock market analysis, process and quality control in manufacturing, biology, medicine, management, social sciences, etc. The study of multivariate time series is primarily motivated by forecasting and the need to model and generate autocorrelated/cross-correlated sequences of variates as input to Monte Carlo simulations.

1.1. Objective and Main Work

This thesis extends the class of *ARM* (*Auto-Regressive Modular*) univariate processes to a class of multivariate processes, called *MARM* (*Multivariate Auto-Regressive Modular*). MARM processes constitute a class of versatile stochastic processes, designed to fit a strong *statistical signature*[1] of empirical vector-valued data. While the ARM modeling methodology aims to capture a strong statistical signature of *univariate* empirical data by simultaneously fitting the empirical marginal distribution and leading empirical autocorrelations, the MARM modeling methodology aims to construct high-fidelity multivariate models from empirical *multivariate* time series, by simultaneously fitting first-order statistics (the multi-dimensional empirical histogram) and second-order statistics (empirical autocorrelations and cross-correlations).

[1] A *statistical signature* is a set of empirical or theoretic statistics, usually in model fitting context, so that a stronger statistical signature contains a weaker one. Statistical signatures can be partially ordered under inclusion. For example, statistical signatures might contain first-order statistics (the common mean, variance and/or the marginal distribution, etc.), second-order statistics (autocorrelation and/or cross-correlation functions etc.), or even higher-order statistics.

Let $\{X_k\}_{k=0}^{\infty}$ be a continuous-space (possibly multivariate), real-valued discrete-time time series defined on a probability space $(\Omega, \mathcal{F}, P)$, where Ω is the sample space, $\mathcal{F}$ is the sigma algebra, and P is the probability measure. The realizations of $\{X_k\}_{k=0}^{\infty}$ are denoted by $\{x_k\}_{k=0}^{\infty}$, where $x_k = X_k(\omega)$, for any fixed $\omega \in \Omega$. In the case that $X_k = (X_k^{(1)}, X_k^{(2)}, \cdots, X_k^{(N)}) = \vec{X}_k$ is multivariate, the corresponding random sequence is denoted by $\{\vec{X}_k\}_{k=0}^{\infty}$, and its realizations are denoted by $\vec{x}_k = (x_k^{(1)}, x_k^{(2)}, \cdots, x_k^{(N)})$, where $x_k^{(j)} = X_k^{(j)}(\omega)$, for any fixed $\omega \in \Omega$. For integer $1 \le m, n \le N$, $j \ge 0$ and $j + \tau \ge 0$, the correlation functions of $X_j^{(m)}$ and $X_{j+\tau}^{(n)}$ are given by $\rho_{m,n}(j,\tau) = \dfrac{\mathbf{Cov}\left[X_j^{(m)}, X_{j+\tau}^{(n)}\right]}{\sigma\left[X_j^{(m)}\right]\sigma\left[X_{j+\tau}^{(n)}\right]}$, where τ is an arbitrary lag (positive or negative) and $\sigma[Y]$ is the standard deviation of random variable Y (assumed finite).

In this thesis, we will use the following goodness-of-fit criteria, which generalize those in [Melamed (1993, 1999), Jagerman and Melamed (1992a)]:

(a) The multidimensional distribution of the model should match its empirical counterpart (multidimensional histogram).

(b) The leading autocorrelations and cross-correlations of $\{\vec{X}_k\}_{k=0}^{\infty}$ should approximate their empirical counterparts.

(c) The sample paths generated by the Monte Carlo simulation of the model should visually "resemble" the empirical data.

Requirement (a) and (b) represent quantitative goodness-of-fit criteria for capturing first-order and second-order statistics. Those are well-defined requirements, which allow numerical goodness-of-fit comparison across model candidates. In contract, requirement (c) is qualitative

and cannot be defined with mathematical rigor; rather, it is left to human judgment, which is subjective and cognitive. However, modelers routinely attempt to validate their models by visual inspection that ensures "path similarity", perceived by multiple observers. Consequently, requirement (c) is included as a heuristic goodness-of-fit criterion.

Compared with the univariate ARM process, MARM processes are more complicated due to their dimensionality and the added temporal dependence across the components in addition to temporal dependence within components. Like ARM [Melamed (1999)], a MARM process consists of a *background process* and a *foreground process*, and a transformation that maps the former to the latter. Broadly speaking, the background process is in effect a hidden Markov chain [Rabiner (1989), Cappé et al. (2005)] whose state space has the *unit circle* topology (here, it means a circle with circumference 1), while the foreground process is obtained by a memoryless transformation. The background/ foreground ensures that the ARM/MARM model can be fitted exactly to any empirical histogram (in the case of MARM, to the multidimensional empirical histogram), while simultaneously approximating the leading empirical autocorrelations (and in the case of MARM also the leading cross-correlations).

The fitting procedures of ARM/MARM processes to empirical data are computationally hard, since the corresponding parameter state space is multidimensional [Brandt and Williams (2007)]. Fortunately, the transition structure of ARM/MARM processes can be derived in closed form and quickly evaluated by a computer through a search followed by gradient-driven optimization. The fitting procedure (called *GSLO* for *Global Search / Local Optimization*) [Jelenkovic and Melamed (1995a, 1995b)] yields a set of candidate models, and the modeler then can decide which one to select. Finally, the transition structure is used to compute forecasts in the form of convex combinations of conditional expectations, and confidence intervals obtained from convex combinations of transition densities.

1.2 Assumptions and Outline

In this thesis, we shall assume that the probability law governing the underlying empirical data and the fitted model are stationary. A univariate time series $\{X_k\}_{k=0}^{\infty}$ is said to be *strictly stationary* if the joint distribution of any $(X_i, \ldots, X_j)$ equals any of its translated versions, $(X_{i+h}, \ldots, X_{j+h})$, where i, j, h are positive integers. A multidimensional time series $\{\vec{X}_k\}_{k=0}^{\infty}$ requires strict stationarity to hold both within and across time series. In the rest of this thesis, we need only require that a multivariate time series $\{\vec{X}_k\}_{k=0}^{\infty}$ be stationary in the sense that the following conditions hold:

1. The distribution of $\{\vec{X}_k\}_{k=0}^{\infty}$ is time invariant
2. The covariance matrix of $\{\vec{X}_k\}_{k=0}^{\infty}$ is *covariance stationary*, namely, its elements satisfy the relation $\mathbf{Cov}\left[X_j^{(m)},\ X_{j+\tau}^{(n)}\right] = \mathbf{Cov}\left[X_k^{(m)}, X_{k+\tau}^{(n)}\right]$, so that covariances depend on time only through the time lag τ.

The assumptions above are designed to yield a strong but parsimonious statistical signature. Thus, only one (common) marginal distribution needs to be estimated, and all correlation functions depend only on the lag. Furthermore, we point out that we may assume stationarity, since empirical data can usually be preprocessed to remove nonstationary components, such as deterministic trends and seasonality, so as to produce stochastic processes that are approximately covariance-stationary. For example, exponential growth components can be removed by taking the difference of the logarithms of the data, and a finite amount of differencing removes

polynomial time components from the data. From now on, we will discuss the modeling methodology in the stationary regime only.

The rest of the thesis is organized as follows. Chapter 2 provides a literature review on the related work on ARM processes. Chapter 3 defines MARM processes and outlines some of the important properties. Chapter 4 presents construction of MARM processes. A detailed exploration of the transition structure and autocorrelation and cross-correlation functions for MARM processes is provided in Chapter 5 and Chapter 6. Chapter 7 describes CSLO algorithm for fitting MARM processes to empirical data. Chapter 8 provides the fundamentals of MARM forecasting methodologies, including the forecasting statistics and choice of parameters. Chapter 9 specializes the general class of MARM processes to a practical sub-class suitable for modeling of empirical vector-valued time series. Chapter 10 and 11 present the empirical MARM fitting and forecasting methodologies which are practical specializations of the general ones to a subclass of empirically-based MARM processes described in Chapter 9. Chapter 12 provides a numerical example of the application of the MARM modeling and forecasting methodologies. Finally, Chapter 13 concludes the thesis by summarizing its contributions and outlining future research work.

2. Related Work and Literature Review

The purpose of time series analysis is usually twofold: to understand and model the stochastic mechanism that gives rise to observed (empirical) time series, and to forecast future values of the time series based on an observed history. Models are used to encapsulate a postulated system structure that "explains" the ensuing behavior of observed time series, and to forecast future behavior.

In practice, many systems have both deterministic and stochastic components. The deterministic components are usually easy to capture faithfully. However, stochastic components are often difficult to model and analyze. Because of this difficulty, the typical tendency in modeling is to oversimplify assumptions and often overlook dependence for the sake of analytical or numerical tractability. Occasionally, this oversimplification is acceptable, for example, when no additional knowledge is available, or one wants to capture qualitative properties only and does not insist on quantitative accuracy. However, when quantitative accuracy is important (as it usually is), oversimplified assumptions (*under-modeling*) give rise to poor models. Of course, modeling unimportant aspects (*over-modeling*) is also undesirable, since by the Principle of Parsimony we are only interested in the simplest possible model that would yield sufficient forecasting accuracy.

In the real world, many random phenomena exhibit dependences, such as *temporal dependence*, *spatial dependence*, etc. Temporal dependence is quite common in real-life phenomena, such as burstiness in telecommunications traffic [Fendick et al. (1989), Livny et al. (1993), Patuwo et al. (1993)], clustering of extreme ocean climate events and evolution patterns in ecology [Aoki (1987)]. Spatial dependence is a measure of the degree of association between measured values at various spatial points. In particular, temporal dependence in traffic flows offered to queueing

system motivated input analysis methods that can capture such dependence, see [Fendick et al. (1989), Livny et al. (1993)]. The autocorrelation and cross-correlation functions are convenient measures of linear temporal dependence in and across stochastic processes, respectively, and are frequently used by engineers as proxies for general temporal dependence [Bendat and Piersol (1986)].

To capture both quantitative and qualitative goodness-of-fit properties of a (univariate) time series model, [Melamed (1991, 1993, 1999), Jagerman and Melamed (1992a)] recommend the adoption of the following criteria for candidate models of a given empirical (univariate) data sequence:

a. The marginal distribution of a candidate model should match its empirical counterpart.
b. The autocorrelation function of a candidate model should approximate its empirical counterpart.
c. Sample paths generated by a Monte Carlo simulation of a candidate model should "resemble" the empirical data.

Based on the above three requirements, Jagerman and Melamed (1992a) proposed the class of *TES* (*Transform-Expand-Sample*) processes and the TES modeling methodology. TES is a versatile class of stochastic sequences consisting of marginally uniform autoregressive schemes with modulo-1 reduction, followed by various transformations. TES meets the three aforementioned goodness-of-fit criteria simultaneously in a systematic way. First, TES guarantees an exact fit to arbitrary marginals (in particular, TES matches any empirical density). Secondly, TES affords considerable modeling flexibility in approximating empirical autocorrelations, including functional forms which are monotone, oscillatory, alternating and so on. Thirdly, TES processes span a wide qualitative range of sample paths, including cyclical as

well as non-cyclical behavior. Altogether, TES is a nonlinear autoregressive scheme, yielding either Markovian or non-Markovian processes. TES processes further enjoy important computational advantages. TES sequences are easy to generate on a computer, and their generation time complexity is relatively small compared to that of the underlying pseudo-number generator, while their space complexity is negligible. Further, TES autocorrelations can be computed from accurate and fast (near real-time) analytical formulas without requiring simulation. A detailed discussion of TES processes and modeling methodology may be found in [Melamed (1993, 1997), Jagerman and Melamed (1992a, 1992b, 1994a, 1994b, 1995)]. A discretized variant of TES, called *QTES* (*Quantized TES*), has been developed in [Melamed et al. (1996)] and used in [Klaoudatos et al. (1999)].

The class of *ARM* (*AutoRegressive Modular*) processes generalizes the class of TES processes in the sense that ARM admits general so-called *innovation sequences* (see Section 4.1 in Chapter 4), whereas TES requires them to be *iid* (*independent and identically distributed*). Still, ARM processes, like TES processes, meet the three aforementioned goodness-of-fit criteria, even as ARM enhances modeling flexibility beyond TES. For a detailed discussion of ARM processes and modeling methodology, see [Melamed (1999)].

Effective ARM (including TES) modeling requires computer support. To this end, [Jelenkovic and Melamed (1995b)] devised a TES fitting algorithm combining brute-force search and steepest-descent optimization. Given an empirical sequence, the algorithm searches for a TES model that fits the empirical histogram exactly and simultaneously approximates the leading empirical autocorrelations by minimizing the sum of weighted squared deviations of the theoretical autocorrelations from their empirical counterparts. This fitting algorithm was later encapsulated in an interactive visual modeling environment called TEStool [Hill and Melamed (1995)].

3. Preliminaries

In this chapter, we provide the necessary preliminaries, including notation and known results, to be used later in the construction of MARM processes.

3.1. Notation

Throughout the thesis, we are going to use the following notation. The bilateral Laplace transform of a function $f(x)$ is defined as $\tilde{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{-sx}\, dx$. The cumulative distribution function (cdf or distribution for short) and probability density function (pdf) of a random variable Z are denoted by F_Z and f_Z respectively, and the corresponding mean and standard deviation are denoted by μ_Z and σ_Z . $Z \sim F$ means that random variable Z has distribution F. The indicator function of a set A is denoted by $1_A(x)$. The imaginary number $\sqrt{-1}$ is denoted by i. For any vector $\vec{v} = (v_1, \ldots, v_n)$, its sub-vector $\vec{v}^{(k)}$, $1 \le k \le n$, is given by $\vec{v}^{(k)} = (v_1, \ldots, v_k)$. The vector of integration is defined as $d\vec{v}^{(k)} = (dv_1, \ldots, dv_k)$. The juxtaposition operation applied to any two vectors, $\vec{x} = (x_1, \ldots, x_k)$ and $\vec{y} = (y_1, \ldots, y_n)$, yields the vector $(\vec{x}, \vec{y})$, defined by $[\vec{x}, \vec{y}] = (x_1, \ldots, x_k, y_1, \ldots, y_n)$.

Throughout this thesis, we assume that the random sequence $\{\vec{X}_j\}_{j=0}^{\infty}$ is stationary, so the autocorrelation function and cross-correlation function only depend on the lag τ. Accordingly, the autocorrelation function for a univariate random sequence $\{X_j^{(n)}\}_{j=0}^{\infty}$ with finite second moments is defined by

$$\rho_{n,n}(j,\tau) = \frac{\mathbf{E}\left[X_j^{(n)} X_{j+\tau}^{(n)}\right] - \mathbf{E}\left[X_j^{(n)}\right]\mathbf{E}\left[X_{j+\tau}^{(n)}\right]}{\sigma^2_{X^{(n)}}}, \tag{3.1}$$

and the cross-correlation function of the random sequences $\{X_j^{(m)}\}_{j=0}^{\infty}$ and $\{X_j^{(n)}\}_{j=0}^{\infty}$ is defined by

$$\rho_{m,n}(j,\tau) = \frac{\mathbf{E}\left[X_j^{(m)} X_{j+\tau}^{(n)}\right] - \mathbf{E}\left[X_j^{(m)}\right]\mathbf{E}\left[X_{j+\tau}^{(n)}\right]}{\sigma_{X^{(m)}}\,\sigma_{X^{(n)}}}. \tag{3.2}$$

When the correlation function depends only on the lag, we omit the first argument in Eqs. (3.1) and (3.2). The conditional density function of a random variable Y given a random variable X is denoted by

$$f_{Y|X}(y \mid x) = \frac{\partial}{\partial y} P\{Y \le y \mid X = x\}. \tag{3.3}$$

3.2. Inversion Method

The *Inversion Method* is a standard technique of long standing for transforming a uniform random variable to one with an arbitrary prescribed distribution F. It has been widely used in the field of simulation, and plays an important role in the MARM modeling methodology.

Lemma 3.1 (Inversion Method)

Let F be an arbitrary cumulative distribution function and $U \sim \text{Unif}(0,1)$, then the random variable $X = F^{-1}(U)$ satisfies $X \sim F$. □

For a proof, see [Bratley et al. (1987), Law and Kelton (1991)]. The Inversion Method provides a very simple way of converting a marginally uniform sequence $\{U_n\}$ into a sequence $\{X_n\}$ with an arbitrary marginal distribution F by simply setting for each n,

$$X_n = F^{-1}(U_n) \tag{3.4}$$

A transformation of the form (3.4) is called an *inversion transformation*. It always exists because F, being a cdf, is non-decreasing, and therefore can always be inverted; although the inversion may not be unique (unless F is strictly increasing), all choices of inversions produce the same effect when applying the Inversion Method.

3.3. Step-Function Inversion

A *step-function inversion* is a special case of the Inversion Method when the underlying density is a step function of the form

$$f(y) = \sum_{i=1}^{I} 1_{[l_i, r_i)}(y) \frac{p_i}{w_i}, \qquad -\infty < y < \infty \tag{3.5}$$

where I is the number of the steps, each of the form $[l_i, r_i)$, $w_i = r_i - l_i > 0$ is the width of step i, and p_i is the probability of step i. The corresponding cdf is the piecewise linear function

$$F(y) = \sum_{j=1}^{J} 1_{[l_j, r_j)}(y) \left[C_{j-1} + (y - l_j) \frac{p_j}{w_j} \right], \quad -\infty < y < \infty \tag{3.6}$$

where $\{C_j\}$ is the cdf of the probability density $\{p_j\}_{j=1}^{J}$, i.e., $C_j = \sum_{i=1}^{j} p_i$, $1 \le j \le J$, with $C_0 = 1$ and $C_J = 1$. The inverse function of F, is the piecewise-linear function

$$F^{-1}(x) = \sum_{j=1}^{J} 1_{[C_{j-1}, C_j)}(x) \left[l_j + (x - C_{j-1}) \frac{w_j}{p_j} \right], \quad 0 \le x \le 1. \tag{3.7}$$

Eq. (3.7) is just a special case of the Inversion Method for piecewise-linear inversions. The MARM modeling methodology uses inversions of form (3.7) to construct candidate models with prescribed step-function densities (typically estimated from empirical data).

In practice, modelers often estimate a step-function density of an empirical sample via an empirical histogram. That is, for an empirical sequence $\vec{y} = \{y_j\}_{j=0}^{J-1}$, the corresponding empirical density $f(y)$ in Eq. (3.5) is often estimated by *histogram* pdf of the form

$$\hat{h}_{\vec{y}}(y) = \sum_{i=1}^{I} 1_{[l_i, r_i)}(y) \frac{\hat{p}_i}{w_i}, \qquad -\infty < y < \infty \tag{3.8}$$

where $\hat{p}_i$ is obtained from some histogram of the empirical sequence as the probability estimate of cell i (relative frequency of observations that fell into that cell). Here and elsewhere, hats indicate sample-based estimated quantities, as opposed to modeler-supplied parameters. Accordingly, the corresponding *histogram cdf*, $\hat{H}_{\vec{y}}$, and *histogram* inversion, $\hat{H}_{\vec{y}}^{-1}$, are obtained from Eqs. (3.6) and (3.7), respectively, by substituting p_j with $\hat{p}_j$.

The merit of the histogram inversion approach in practical modeling is its broad generality, since no assumptions are made on the actual form of the true marginal density. However, to be a good approximation of the true marginal density, sufficient data must be used.

3.4. Construction of Multivariate Random Variables

The MARM modeling methodology makes use of the Inversion Method to construct multivariate random variables, and to this end, one applies it to conditional distributions. We next illustrate this construction for the N-dimensional case. Let $\vec{X} = (X^{(1)}, ..., X^{(N)})$ be a random vector of dimension N, with the standard representation for its joint density,

$$f_{\vec{X}}(\vec{x}) = f_{X^{(1)}}(x)\, f_{X^{(2)}|X^{(1)}}(x^{(2)} \mid x^{(1)}) \cdots f_{X^{(N)}|\, X^{(1)},\, ...,\, X^{(N-1)}}(x^{(N)} \mid x^{(1)}, ..., x^{(N-1)}), \tag{3.9}$$

where $\vec{x} = (x^{(1)}, ..., x^{(N)})$.

Algorithm 3.1 (Random Vector Generating Algorithm)

Input: A joint pdf $f_{\vec{X}}(\vec{x})$, and a random vector $\vec{W} = (W^{(1)}, \ldots, W^{(N)})$ of iid Unif[0,1) components, $W^{(k)}$, $k = 1, ..., N$. From the pdf, we compute all marginal and conditional cdf, which are also assumed given.

Output: A random vector $\vec{X} = (X^{(1)}, ..., X^{(N)})$ with the prescribed joint pdf $f_{\vec{X}}(\vec{x})$.

Algorithm:

1. For $k = 1$, define

$$X^{(1)} = F^{-1}_{X^{(1)}}(W^{(1)}) \tag{3.10}$$

2. For $k > 1$, the definition of $X^{(k)}$ proceeds recursively. Assume that $X^{(1)}, \ldots, X^{(k-1)}$ has already been defined, and define

$$X^{(k)} = F^{-1}_{X^{(k)}|X^{(1)},\ldots,X^{(k-1)}}(W^{(k)}), \tag{3.11}$$

Lemma 3.2

The random vector $\vec{X} = (X^{(1)}, ..., X^{(N)})$ defined by RVGA has the joint pdf $f_{\vec{X}}(\vec{x})$.

Proof:

By construction, $X^{(1)} \sim F_{X^{(1)}}$ as a consequence of the Inversion Method in Lemma 3.1. For $1 < k \leq N$, we similarly have $X^{(k)} \sim F_{X^{(k)}|\vec{X}^{(k-1)}}$ by virtue of the Inversion Method applied to conditional distributions. Thus, the joint pdf of $\vec{X} = (X^{(1)}, ..., X^{(N)})$ is given by (3.9), as required. □

We point out that from a simulation viewpoint, RVGA provides the basis for computer generation of realizations $\vec{x} = (x^{(1)}, ..., x^{(N)})$ from the joint pdf $f_{\vec{X}}(\vec{x})$, via replacing all random variables in RVGA by their realizations.

3.5. Stitching Transformations

A random walk on a *unit circle* (circle with unity circumference, details to be discussed in section 4.1) usually produces visual "discontinuities" when crossing the circle origin in the clockwise or counter-clockwise direction. Here the term "discontinuities" is used figuratively, referring to transitions from large fractions to small ones and vice versa, when the random walk crosses with the origin via the modulo-1 operation in each direction.

In some scenarios, such sample "discontinuities" could have a valid modeling interpretation. For example, crossing point 0 could model a catastrophe when properly defined, such as in a cyclical economic model. However, in most cases, this "discontinuities" in the sample path are undesirable, since they have no counterparts in empirical data. Monotone transformations (including the frequently used Inversion Method) preserve such "discontinuities". To satisfy the goodness-of-fit criteria discussed in Chapter 1, some uniformity-preserving "smoothing" transformations need to be applied in order to "remove" such "discontinuities" while still allowing us to fit arbitrary empirical marginal distributions via the Inversion Method. To this end, one may use the so-called *stitching transformations* [cf. Jagerman and Melamed (1992a), Melamed (1993)].

Formally, a stitching transformation S_ξ maps the interval [0, 1] to itself, and is determined by a *stitching parameter* ξ in the range [0, 1]. For a given ξ, S_ξ is defined by

$$S_\xi(u) = \begin{cases} u / \xi, & 0 \le u \le \xi \\ (1-u) / (1-\xi), & \xi \le u \le 1 \end{cases} \tag{3.12}$$

The term "stitching" is motivated by the fact that S_ξ is continuous on the unit circle as function of ξ in (0,1). Note that for the trivial cases ($\xi = 0$ or 1), we have $S_0(u) = 1 - u$ (the antithetic transformation) and $S_1(u) = u$ (the identity transformation), where no "smoothing" takes place since they are not continuous at the edge point 0 and 1 on the unit circle.

The primary usage of stitching transformations is that they all preserve uniformity, in addition to their "smoothing" effect, which is guaranteed by the following lemma.

Lemma 3.3

If $U \sim \text{Unif}(0, 1)$, then $S_\xi(U) \sim \text{Unif}(0, 1)$, for all $0 \le \xi \le 1$. □

For a proof, see [Melamed (1991)]. It follows from Lemma 3.1 and Lemma 3.3 that a composition distortion of the form

$$D(u) = F_X^{-1}(S_\xi(u)), \quad u \in [0,1) \tag{3.13}$$

employing stitching and inversion transformations in succession on a random variable uniformly distributed in [0,1), will allow us to fit arbitrary empirical marginal distributions, and simultaneously provide sample path "smoothing" for ξ in (0,1). The MARM modeling methodology, to be described in the next chapter, utilizes distortions of the form (3.13).

4. Construction of Multivariate Autoregressive Modular Processes

A MARM process is constructed using two components: a background process which is an auxiliary hidden Markov chain, and a foreground process which is our target time series sequence. We shall refer to the aforementioned construction as foreground/background scheme. The main advantage of the background/foreground paradigm to be presented is its broad modeling flexibility, as will be later demonstrated in this thesis.

In this chapter, we describe the construction of two flavors of MARM processes, MARM$^+$ and MARM$^-$. For notational clarity, we will append the plus or minus symbols to objects associated with the respective MARM process, and omit those symbols when they are common to both flavors of MARM processes.

4.1. MARM Background Processes

The MARM modeling methodology uses *Modulo-1 arithmetic*, which is the operation of taking the fractional part of a real number. Letting $\lfloor x \rfloor = \max\{\text{integer } n : n \le x\}$ be the *floor operator*, the *modulo-1* (*fractional part*) *operator* $< \cdot >$ is defined for any real x by $< x >= x - \lfloor x \rfloor$. Note that the fractional part always lies in the interval [0, 1), even for negative numbers. For example, $< 1.1 >= 0.1$, but $< -1.1 >= 0.9$. Other useful properties of moduo-1 addition are

$$\langle a \pm b \rangle = \langle \langle a \rangle \pm \langle b \rangle \rangle \tag{4.1}$$

and

$$\langle a + b \rangle = \langle c \rangle \;\; \Leftrightarrow \;\; \langle a \rangle = \langle c - b \rangle . \tag{4.2}$$

A geometric representation of modulo-1 arithmetic can be found in [Melamed (1993)], where the interval [0, 1) (the range of fractional parts) has been topologically transformed into a circle in a distance-preserving manner. This circle is referred as *unit circle* (circumference is unity, not the radius), and its origin (the fraction 0) is arbitrary selected, say, on the bottom of the unit circle. A number $0 < y < 1$ is represented by a point on the unit circle at distance y from the origin (the distance is measured by the length of the arc connecting the origin clockwise to y). For any real y (not necessary in [0, 1)), y is represented on the unit circle with the same position as its fractional part $\langle y \rangle$. For a geometric description of modulo-1 arithmetic, see [Melamed (1993)].

We next proceed to construct the MARM background process. Hereafter, all random variables will be defined over some common probability space $(\Omega, \mathcal{F}, P)$. Let U_0 be a uniformly distributed random variable over the interval $[0, 1)$, that is, $U_0 \sim \text{Unif}[0, 1)$. An *innovation sequence* is any random sequence, $\{V_n\}_{n=0}^{\infty}$, independent of U_0. An important class of innovations consists of iid sequences. Let $S_{m,n}^{(V)}$ be the partial sum of innovation random variables, given by

$$S_{m,n}^{(V)} = \begin{cases} 0\,, & \text{if} \quad m > n \\ \sum_{j=m}^{n} V_j\,, & \text{if} \quad m \le n \end{cases} \tag{4.3}$$

4.1.1 MARM$^+$ Background Processes

A MARM$^+$ background sequence, $\{U_n^+\}_{n=0}^{\infty}$, is defined recursively by

$$U_n^+ = \begin{cases} U_0, & n = 0 \\ \left\langle U_{n-1}^+ + V_n \right\rangle, & n > 0 \end{cases} \tag{4.4}$$

The construction of $\{U_n^+\}_{n=0}^{\infty}$ has a simple and intuitive geometric interpretation: the modulo-1 arithmetic gives rise to Markovian walks on the unit circle (a circle with unit circumference), where each point on the circle corresponds to a fraction in the range [0, 1) [cf. Melamed (1993)].

Using Eq. (4.1), we can equivalently represent U_n^+ in (4.4) as

$$U_n^+ = \left\langle U_0 + S_{0,n}^{(V)} \right\rangle = \left\langle U_0 + \left\langle S_{0,n}^{(V)} \right\rangle \right\rangle, \ n \geq 0 \tag{4.5}$$

The motivation for defining MARM background processes stems from the simple fact that uniformity is closed under modulu-1 addition of independent random variable. This fact is due to the following lemma, called General Iterated Uniformity in [Melamed (1993)] (see also [Feller (1971) p. 64]).

Lemma 4.1 (General Iterated Uniformity)

Let $U \sim \text{Unif}[0, 1)$ and let further $X = \left\langle U + V \right\rangle$, where V is an arbitrarily distributed random variable, independent of U. Then, $X \sim \text{Unif}[0,1)$.

Proof.

Refer to [Melamed (1993, 1999)]. □

An important consequence of this lemma and Eq. (4.4) is given in the following corollary.

Corollary 4.1

MARM$^+$ background sequences are Markovian, and their marginal distributions are stationary and uniform on [0,1) regardless of the probability law of the innovation sequences $\{V_n\}$ selected.

Proof.

See [Melamed (1999)]. □

4.1.2 MARM$^-$ Background Processes

A MARM$^-$ background sequence, $\{U_n^-\}_{n=0}^{\infty}$, is defined recursively by

$$U_n^- = \begin{cases} U_n^+, & n \text{ even} \\ 1 - U_n^+, & n \text{ odd} \end{cases} \tag{4.6}$$

Corollary 4.2

MARM$^-$ background sequences are non-stationary Markovian, and their marginal distributions are uniform on [0,1) regardless of the probability law of the innovation sequences $\{V_n\}$ selected.

Proof.

Follows from Eq. (4.6) and Corollary 4.1. □

4.2. MARM Stitched Background Processes

Lemma 4.1 ensures that a MARM background sequence $\{U_n\}_{n=0}^{\infty}$ is a sequence of random variables, uniformly distributed in [0,1), which constitute a Markovian Walk on the unit circle.

From the discussion in Section 3.5, such a process usually has visual discontinuities in its sample paths stemming from the underlying modulo-1 arithmetic. As per the forgoing discussion, stitching transformations will be needed to "smooth" MARM sample paths of the background process, while preserving the uniformity of the marginal distribution. The MARM modeling methodology utilizes stitched MARM background processes, namely $\{S_\xi(U_n)\}_{n=0}^{\infty}$ (where $\xi \in [0,1]$ is the stitching parameter) to generate MARM foreground processes, which will be discussed in detail in the next section.

4.3. MARM Foreground Processes

A *MARM foreground process* is an N -dimensional stochastic process $\{\vec{X}_n\}_{n=0}^{\infty}$ with some state space $S = S^{(1)} \times \cdots \times S^{(N)} \subset \mathbb{R}^N$. It is constructed from a MARM background process via a memoryless transformation, called a *distortion* [cf. Melamed (1993, 1999)]. There are many kinds of distortions that can be applied to the background processes, which partly gives the reason for the versatile modeling flexibility of MARM modeling methodology. In this section, we are interested in two special distortions based on the stitching transformations and the Inversion Method.

Generally speaking, the construction of the MARM foreground sequence, $\{\vec{X}_j\}_{j\geq 0}$, follows the random vector generating algorithm (RVGA) described in Section 3.4. For $j = 0, 1, \ldots$, RVGA generates the corresponding $\vec{X}_j$ through distortions including stitching transformations and Inversion Method. We shall be concerned with distortion functions $D^{(k)}(u, \vec{w}^{(k-1)})$ and vector valued distortions of the form $\vec{D}^{(k)} = (D^{(1)}, \ldots, D^{(k)})$.

The construction is specified by the following algorithm.

Algorithm 4.1 (MARM Foreground Process Construction Algorithm)

Input: A MARM background sequence $\{U_j\}_{j=0}^{\infty}$ (either $\{U_j^+\}_{j=0}^{\infty}$ or $\{U_j^-\}_{j=0}^{\infty}$); a stitching parameter ξ; iid random variables $W_j^{(k)} \sim \text{Unif}[0,1)$, $1 \le k \le N$, $j \ge 0$, which are all independent of $\{U_j\}_{j=0}^{\infty}$; a joint pdf $f_{\vec{X}}(\vec{x})$, from which we compute all marginal and conditional cdfs, as well as their inverses to derive the distortions of the form.

Output: A vector of distortions, $\vec{D}^{(k)} = (D^{(1)}, \ldots, D^{(k)})$, and a MARM foreground processes $\{\vec{X}_j\}_{j=0}^{\infty}$ with the prescribed joint pdf $f_{\vec{X}}(\vec{x})$.

Algorithm: For each $j \ge 0$, apply the following steps:

Step 1: For $k = 1$, apply the stitching transformation to U_j, yielding the corresponding stitched MARM background random variable, $S_\xi(U_j)$. Define $D^{(1)}$ and $\{X_j^{(1)}\}$ by

$$D^{(1)}(u) = F^{-1}_{X^{(1)}}(S_\xi(u)),\ u \in [0,1) \tag{4.7}$$

$$X_j^{(1)} = D^{(1)}(U_j),\ j \ge 0. \tag{4.8}$$

Step 2: For $1 < k \le N$, $D^{(k)}$ and $\{X_j^{(k)}\}$ are defined recursively. Assuming that $\vec{X}_j^{(k-1)} = (X_j^{(1)}, \ldots, X_j^{(k-1)})$ has already been defined, let $D^{(k)}$ and $\{X_j^{(k)}\}$ be defined by

$$D^{(k)}(w, \vec{x}^{(k-1)}) = F^{-1}_{X^{(k)}|\vec{X}^{(k-1)}}(w \mid \vec{x}^{(k-1)})),\ w \in [0,1),\ \vec{x}^{(k-1)} \in S^{(1)} \times \cdots \times S^{(k-1)} \tag{4.9}$$

$$X_j^{(k)} = D^{(k)}(W_j^{(k-1)}, \vec{X}_j^{(k-1)}),\ j \ge 0 \tag{4.10}$$

□

The background/foreground structure of MARM processes is shown schematically in Figure 4.1, where BP stands for the MARM background process, SBP stands for the stitched MARM background process, and FP stands for the MARM foreground process.

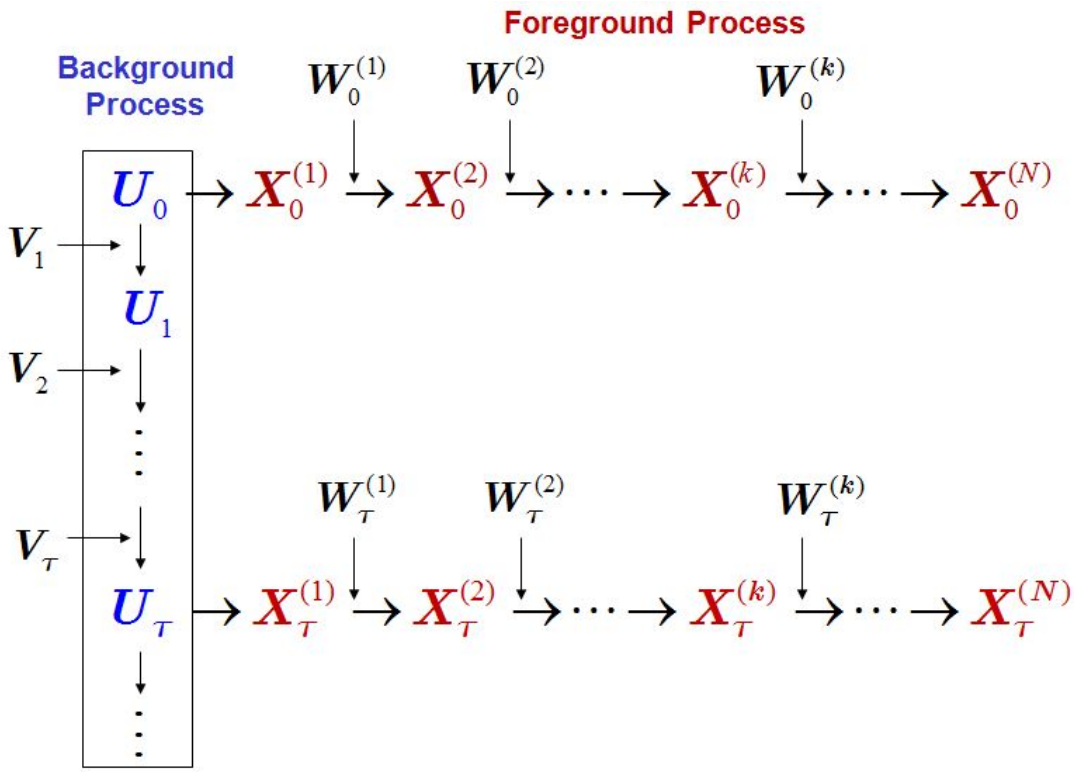

Figure 4.1 Background/Foreground scheme for constructing MARM processes

The arrows in Figure 4.1 denote two kinds of "transitions: The horizontal ones correspond to transformations from $\vec{X}_j^{(n)}$ to $X_j^{(n+1)}$ for $j \geq 0$ and $n \geq 1$, while the vertical ones correspond to background process transitions from U_i to U_{i+1} for any $i \geq 0$.

4.4. Computer Generation of MARM Processes

Computer generation of MARM foreground processes $\{\vec{X}_j\} = \{X_j^{(k)}\}$, $1 \le k \le N$, $j \ge 0$, follows Algorithm 4.1, except that random variables are replaced by their realizations. Lemmas 3.2, 3.3, and 4.1 then jointly ensure that the resultant MARM foreground processes have the prescribed marginal distribution (a prescribed empirical distribution is a special case). To complete the generation procedures, we need to generate realizations of U_0, the innovation variates, $\{V_n\}$, and the auxiliary variates $\{W_j^{(k)}\}$. All these are obtained, using some random number generator, namely, a computer-generated random sequence $\{Z_n\}_{n\ge 0}$, which is iid $\mathrm{Unif}[0,1)$. For example, the random number stream, $\{Z_n\}_{n\ge 1}$, may be used as follows for $1 \le k \le N$, $j \ge 0$:

1. $U_0 \leftarrow Z_0$
2. $V_j \leftarrow F_V^{-1}(Z_{jN})$
3. $W_j^{(k)} \leftarrow Z_{jN+k}$

5. The Transition Structure of MARM Processes

In this chapter, we will derive the transition structure of MARM processes, which will later be used to compute the autocorrelations/cross-correlations. For any MARM background process $\{U_n\}$ and $\tau \geq 1$, we shall use in the sequel the relationship

$$f_{U_j|U_{j+\tau}}(u_j \mid u_{j+\tau}) = f_{U_{j+\tau}|U_j}(u_{j+\tau} \mid u_j)\,\frac{f_{U_j}(u_j)}{f_{U_{j+\tau}}(u_{j+\tau})} = f_{U_{j+\tau}|U_j}(u_{j+\tau} \mid u_j) \qquad (5.1)$$

which readily follows from the fact that the marginal density of MARM background process is uniform over $[0,1)$.

Lemma 5.1

For any function $g(x)$ and integer ν, $\tilde{g}(i2\pi\nu)$ and $\tilde{g}(-i2\pi\nu)$ are complex conjugate pair.

Proof.

Follows from the representations

$$\tilde{g}(i2\pi\nu) = \int g(x)\,e^{-i2\pi\nu x}\,dx = \int g(x)\big[\cos(2\pi\nu x) - i\sin(2\pi\nu x)\big]dx$$

$$\tilde{g}(-i2\pi\nu) = \int g(x)\,e^{i2\pi\nu x}\,dx = \int g(x)\big[\cos(2\pi\nu x) + i\sin(2\pi\nu x)\big]dx\,. \qquad \square$$

Theorem 5.1

Let $0 \leq u_j < 1$, $j \geq 0$, and $x^{(k)} \in S^{(k)}$, $1 \leq k \leq N$.

(a) For $\tau = 0$ and $1 \leq k \leq N$,

$$f_{X_j^{(k)}|U_j}(x^{(k)} \mid u_j) =$$

$$\begin{cases} 1_{\left\{x^{(1)}=F^{-1}_{X^{(1)}}(S_\xi(u_j))\right\}}(x^{(1)}), & k=1 \\ \int_{S^{(2)}} \cdots \int_{S^{(k-1)}} \dfrac{f_{\vec{X}^{(k)}}\left(F^{-1}_{X^{(1)}}(S_\xi(u_j)), y^{(2)}, \ldots y^{(k-1)}, x^{(k)}\right)}{f_{X^{(1)}}\left(F^{-1}_{X^{(1)}}(S_\xi(u_j))\right)} \, dy^{(2)} \ldots dy^{(k-1)}, & k>1 \end{cases} \tag{5.2}$$

(b) For $\tau \geq 0$ and $k=1$,

$$\begin{aligned} f_{X^{(1)}_{j+\tau}|U_j}(x^{(1)} \mid u_j) = & \\ & f_{U_{j+\tau}|U_j}\left(\xi F_{X^{(1)}}(x^{(1)}) \mid u_j\right) \xi \, f_{X^{(1)}}(x^{(1)}) + \\ & f_{U_{j+\tau}|U_j}\left(1-(1-\xi) F_{X^{(1)}}(x^{(1)}) \mid u_j\right)(1-\xi) \, f_{X^{(1)}}(x^{(1)}) \end{aligned} \tag{5.3}$$

Proof.

The first line of part (a) follows immediately since $X_j^{(1)} = F^{-1}_{X^{(1)}}(S_\xi(U_j))$ by construction of MARM foreground processes in Eq. (4.8).

To prove the second line of part (a), note that by construction of MARM processes for $k>1$,

$$f_{X_j^{(2)},\ldots,X_j^{(k)}|X_j^{(1)},\, U_j} = f_{X^{(2)},\ldots,X^{(k)}|X^{(1)}} \, .$$

Consequently, using this fact and the first line of part (a), we have

$$\begin{aligned} f_{\vec{X}_j^{(k)}|U_j}(\vec{x}^{(k)} \mid u_j) &= f_{X_j^{(1)}|U_j}\left(x^{(1)} \mid u_j\right) f_{X_j^{(2)},\ldots,X_j^{(k)}|X_j^{(1)},U_j}\left(x^{(2)},\ldots,x^{(k)} \mid x^{(1)}, u_j\right) \\ &= f_{X_j^{(1)}|U_j}\left(x^{(1)} \mid u_j\right) f_{X^{(2)},\ldots,X^{(k)}|\, X^{(1)}}\left(x^{(2)},\ldots,x^{(k)} \mid x^{(1)}\right) \\ &= 1_{\left\{x^{(1)}=F^{-1}_{X^{(1)}}\left(S_\xi(u_j)\right)\right\}}(x^{(1)}) \frac{f_{\vec{X}^{(k)}}(\vec{x}^{(k)})}{f_{X^{(1)}}(x^{(1)})} \end{aligned}$$

Integrating the above equation with respect to $\vec{x}^{(k-1)}$ now completes the proof Eq. (5.2).

To prove part (b), write

$$
\begin{aligned}
&F_{X^{(1)}_{j+\tau}|U_j}(x^{(1)} \mid u_j) \\
&\quad = \Pr\{X^{(1)}_{j+\tau} \le x^{(1)} \mid U_j = u_j\} \\
&\quad = \Pr\{F^{-1}_{X^{(1)}}(S_\xi(U_{j+\tau})) \le x^{(1)} \mid U_j = u_j\} \\
&\quad = \Pr\{S_\xi(U_{j+\tau}) \le F_{X^{(1)}}(x^{(1)}) \mid U_j = u_j\} \\
&\quad = \Pr\{U_{j+\tau} \le \xi\, F_{X^{(1)}}(x^{(1)}) \mid U_j = u_j\} + \\
&\qquad \Pr\{U_{j+\tau} \ge 1-(1-\xi) F_{X^{(1)}}(x^{(1)}) \mid U_j = u_j\} \\
&\quad = F_{U_{j+\tau}|U_j}\left(\xi\, F_{X^{(1)}}(x^{(1)}) \mid u_j\right) + 1 - F_{U_{j+\tau}|U_j}\left(1-(1-\xi)\, F_{X^{(1)}}(x^{(1)}) \mid u_j\right)
\end{aligned}
\tag{5.4}
$$

Differentiating Eq. (5.4) with respect to $x^{(1)}$ thus yields Eq. (5.3). □

Corollary 5.1

Let $0 \le u_j < 1$, $j \ge 0$, and $x^{(k)} \in S^{(k)}$, $1 \le k \le N$.

(a) For $\tau = 0$ and $1 \le k \le N$,

$$
F_{X^{(k)}_j|U_j}(x^{(k)} \mid u_j) =
\begin{cases}
1_{\left\{x^{(1)} \ge F^{-1}_{X^{(1)}}\left(S_\xi(u_j)\right)\right\}}(x^{(1)}), & k = 1 \\
\displaystyle\int_{-\infty}^{x^{(k)}} \int_{S^{(k-1)}} \cdots \int_{S^{(2)}} \frac{f_{\vec{X}^{(k)}}(F^{-1}_{X^{(1)}}(S_\xi(u_j)), y^{(2)}, \ldots, y^{(k)})}{f_{X^{(1)}}(F^{-1}_{X^{(1)}}(S_\xi(u_j)))}\, dy^{(2)} \ldots dy^{(k-1)}\, dy^{(k)}, & k > 1
\end{cases}
\tag{5.5}
$$

(b) For $\tau \ge 0$ and $k = 1$,

$$
\begin{aligned}
&F_{X^{(1)}_{j+\tau}|U_j}\left(x^{(1)} \mid u_j\right) = \\
&\qquad F_{U_{j+\tau}|U_j}\left(\xi\, F_{X^{(1)}}(x^{(1)}) \mid u_j\right) + 1 - F_{U_{j+\tau}|U_j}\left(1-(1-\xi)\, F_{X^{(1)}}(x^{(1)}) \mid u_j\right)
\end{aligned}
\tag{5.6}
$$

Proof.

The first line of Part (a) is immediate from the first line of Eq.(5.2), while the second line of part (a) follows by integrating the corresponding densities in part (a) of Theorem 5.1. Finally, part (b) has already been proven in part (b) of Theorem 5.1. □

In the sequel, we shall employ the following notation. An alternative representation of the vector distortion operator $\vec{D}^{(k)} = (D^{(1)}, \ldots, D^{(k)})$ from Algorithm 4.1 is

$$D^{(1)}(u) = F^{-1}_{X^{(1)}}(S_{\xi}(u)) \tag{5.7}$$

$$D^{(k)}(u, \vec{w}^{(k-1)}) = F^{-1}_{X^{(k)}|\vec{X}^{(k-1)}}(w^{(k-1)} \mid \vec{D}^{(k-1)}(u, \vec{w}^{(k-2)})), \quad 1 < k \leq N \tag{5.8}$$

Finally, the family of functions $\mathcal{D}^{(k)}(u)$ is given by

$$\mathcal{D}^{(k)}(u) = \begin{cases} D^{(1)}(u), & k = 1 \\ \int_0^1 \cdots \int_0^1 D^{(k)}(u, \vec{w}^{(k-1)})\, d\vec{w}^{(k-1)}, & 1 < k \leq N \end{cases} \tag{5.9}$$

where $D^{(1)}(u)$ and $D^{(k)}(u, \vec{w}^{(k-1)})$ are the distortion functions defined in Eqs. (5.7) and (5.8), respectively.

Corollary 5.2

For $0 \leq u_j < 1$, $j \geq 0$, and $x^{(k)} \in S^{(k)}$, $1 \leq k \leq N$,

$$\mathbf{E}[X_j^{(k)} \mid U_j = u_j] = \mathcal{D}^{(k)}(u_j) \tag{5.10}$$

Proof.

Follows from Algorithm 4.1 and representation in Eq. (5.9). □

Lemma 5.2

For $\tau > 0$ and $1 \le k \le N$,

$$\tilde{\mathcal{D}}^{(k)}(0) = \mu_{X^{(k)}}. \tag{5.11}$$

Proof.

$$\begin{aligned}
\tilde{\mathcal{D}}^{(k)}(0) &= \int_0^1 \mathcal{D}^{(k)}(u)\, du \\
&= \int_0^1 \cdots \int_0^1 D^{(k)}(u, w^{(1)}, \ldots, w^{(k-1)})\, du\, dw^{(1)} \cdots dw^{(k-1)} \\
&= \int_{S^{(1)}} \cdots \int_{S^{(k)}} x^{(k)} f_{X^{(1)}}(x^{(1)})\, f_{X^{(2)}|X^{(1)}}(x^{(2)} \mid x^{(1)}) \cdots f_{\vec{X}^{(k)}|\vec{X}^{(k-1)}}(\vec{x}^{(k)} \mid \vec{x}^{(k-1)})\, d\vec{x}^{(k)} \qquad (5.12) \\
&= \int_{S^{(1)}} \cdots \int_{S^{(k)}} x^{(k)} f_{X^{(k)}|\vec{X}^{(k-1)}}(x^{(k)} \mid \vec{x}^{(k-1)})\, f_{\vec{X}^{(k-1)}}(\vec{x}^{(k-1)})\, d\vec{x}^{(k)} \\
&= \mathbf{E}\left[\mathbf{E}\left[X^{(k)} \mid \vec{X}^{(k-1)}\right]\right] = \mu_{X^{(k)}}
\end{aligned}$$

where the second equality is given by Eq. (5.9). □

We point out that since the case $\tau = 0$ has already been treated above, we will only treat the case $\tau > 0$ in the rest of this chapter.

5.1. Transition Structure of MARM$^+$ Processes

The following theorem recalls a fundamental representation for the transition density function, $f_{U^+_{j+\tau}|U^+_j}(v \mid u)$, of MARM$^+$ background processes.

Theorem 5.2

$\{U_n^+\}$ is a stationary Markov process with stationary transition density $f_{U_{j+\tau}^+|U_j^+}(v \mid u)$, given by

$$f_{U_{j+\tau}^+|U_j^+}(v \mid u) = \begin{cases} 1_{\{u\}}(v), & \tau = 0\,,\ 0 \le v,\ u < 1 \\ 1 + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\, e^{i2\pi\nu(v-u)}\right], & \tau > 0,\ \ 0 \le v,\ u < 1 \\ 1 + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+\tau+1,j}}(i2\pi\nu)\, e^{i2\pi\nu(u-v)}\right], & \tau < 0,\ \ 0 \le v,\ u < 1 \\ 0\ , & \text{otherwise} \end{cases} \tag{5.13}$$

Proof.

We first prove the representation

$$f_{U_{j+\tau}^+|U_j^+}(v \mid u) = \begin{cases} 1_{\{u\}}(v), & \tau = 0\,,\ 0 \le v,\ u < 1 \\ \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\, e^{i2\pi\nu(v-u)}\ , & \tau > 0,\ \ 0 \le v,\ u < 1 \\ \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+\tau+1,j}}(i2\pi\nu)\, e^{i2\pi\nu(u-v)}\ , & \tau < 0,\ \ 0 \le v,\ u < 1 \\ 0\ , & \text{otherwise} \end{cases} \tag{5.14}$$

For $\tau = 0$, the first line of Eq. (5.14) is trivial. For $\tau > 0$ and $0 \le v,\ u < 1$, the proof appears in Theorem 1 in [Melamed (1999)]. For $\tau < 0$ and $0 \le v,\ u < 1$, it follows from Eq. (5.1) and the case for positive τ above. Eq. (5.13) now follows from Eq. (5.14) with the aid of Lemma 5.1 and the fact that $\tilde{f}_{S_{j+1,\,j+\tau}}(0) = \tilde{f}_{S_{j+\tau+1,\,j}}(0) = 1$. □

We next proceed to exhibit the transition structure for the foreground process $\{\vec{X}_{\tau}^{+}\}$. In this section we shall use the shorthand notation $d\vec{x}^{(k)} = dx^{(1)} \cdots dx^{(k)}$ $(k \geq 1)$ for any vector variable $\vec{x}^{(k)} = (x^{(1)}, \cdots, x^{(k)})$.

Theorem 5.3

Let $\tau > 0$, $0 \leq u_j < 1$, $j \geq 0$, and $\vec{x}^{(k)} \in S^{(1)} \times \cdots \times S^{(k)}$, $x^{(k)} \in S^{(k)}$, $k \geq 1$.

(a) $f_{\vec{X}_{j+\tau}^{(k)+} | U_j^{+}}(\vec{x}^{(k)} \mid u_j) =$

$$f_{\vec{X}^{(k)+}}(\vec{x}^{(k)}) \left\{ 1 + 2 \sum_{\nu=1}^{\infty} \mathbf{Re} \left[\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu u_j} \times \left(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{-i2\pi\nu(1-\xi) F_{X^{(1)}}(x^{(1)})} \right) \right] \right\} \tag{5.15}$$

(b) $f_{\vec{X}_{j-\tau}^{(k)+} | U_j^{+}}(\vec{x}^{(k)} \mid u_j) =$

$$f_{\vec{X}^{(k)+}}(\vec{x}^{(k)}) \left\{ 1 + 2 \sum_{\nu=1}^{\infty} \mathbf{Re} \left[\tilde{f}_{S_{j-\tau+1,\, j}}(i2\pi\nu)\, e^{i2\pi\nu u_j} \times \left(\xi\, e^{-i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{i2\pi\nu(1-\xi) F_{X^{(1)}}(x^{(1)})} \right) \right] \right\} \tag{5.16}$$

(c) $f_{X_{j+\tau}^{(k)+} | U_j^{+}}(x^{(k)} \mid u_j) =$

$$f_{X^{(k)+}}(x^{(k)}) + 2 \sum_{\nu=1}^{\infty} \mathbf{Re} \left[\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu u_j} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{x}^{(k)}) \times \left(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{-i2\pi\nu(1-\xi) F_{X^{(1)}}(x^{(1)})} \right) d\vec{x}^{(k-1)} \right] \tag{5.17}$$

(d) $f_{X_{j-\tau}^{(k)+} \mid U_j^+}(x^{(k)} \mid u_j) =$

$$f_{X^{(k)+}}(x^{(k)}) + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)e^{i2\pi\nu u_j} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{x}^{(k)}) \times \left(\xi\, e^{-i2\pi\nu\,\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(x^{(1)})}\right) d\vec{x}^{(k-1)}\right] \tag{5.18}$$

Proof.

To prove part (a), we treat the cases $k = 1$ and $k > 1$ separately, since the proof of the latter depends on that of the former.

Eq. (5.15) for $k = 1$ follows from Eq. (5.3) in view of Theorem 5.2 and the fact that $e^{i2\pi\nu} = 1$. For $1 < k \leq N$, we have by construction of MARM$^+$ processes,

$$\begin{aligned} f_{\vec{X}_{j+\tau}^{(k)+} \mid U_j^+}(\vec{x}^{(k)} \mid u_j) &= f_{X_{j+\tau}^{(1)+} \mid U_j^+}(x^{(1)} \mid u_j)\, f_{(X_{j+\tau}^{(2)+},\ldots,X_{j+\tau}^{(k)+}) \mid X_{j+\tau}^{(1)+}, U_j^+}\left((x^{(2)},\ldots,x^{(k)}) \mid x^{(1)}, u_j\right) \\ &= f_{X_{j+\tau}^{(1)+} \mid U_j^+}(x^{(1)} \mid u_j)\, f_{(X^{(2)},\ldots,X^{(k)}) \mid X^{(1)}}\left((x^{(2)},\ldots,x^{(k)}) \mid x^{(1)}\right) \\ &= f_{X_{j+\tau}^{(1)+} \mid U_j^+}(x^{(1)} \mid u_j)\, \frac{f_{(X^{(1)},X^{(2)},\ldots,X^{(k)})}(x^{(1)}, x^{(2)},\ldots,x^{(k)})}{f_{X^{(1)}}(x^{(1)})} \end{aligned} \tag{5.19}$$

Eq. (5.15) for $1 < k \leq N$ follows by substituting the representation of $f_{X_{j+\tau}^{(1)+} \mid U_j^+}(x^{(1)} \mid u_j)$ for $k = 1$ in Eq. (5.15) into Eq. (5.19).

The proof of Part (b) is similar to that of part (a), but with negative τ. Eq. (5.16) for $k = 1$ follows from Eq. (5.3) in view of Theorem 5.2 and the fact that $e^{i2\pi\nu} = 1$. For $1 < k \leq N$, we have by construction of MARM$^+$ processes,

$$f_{\vec{X}_{j-\tau}^{(k)+}|U_j^+}(\vec{x}^{(k)} \mid u_j) = f_{X_{j-\tau}^{(1)+}|U_j^+}(x^{(1)} \mid u_j)\, f_{(X_{j-\tau}^{(2)+},\ldots,X_{j-\tau}^{(k)+})|X_{j-\tau}^{(1)+},U_j^+}\left((x^{(2)},\ldots,x^{(k)}) \mid x^{(1)}, u_j\right)$$

$$= f_{X_{j-\tau}^{(1)+}|U_j^+}(x^{(1)} \mid u_j)\, f_{(X^{(2)},\ldots,X^{(k)})|X^{(1)}}\left((x^{(2)},\ldots,x^{(k)}) \mid x^{(1)}\right)$$

$$= f_{X_{j-\tau}^{(1)+}|U_j^+}(x^{(1)} \mid u_j)\, \frac{f_{(X^{(1)},X^{(2)},\ldots,X^{(k)})}(x^{(1)},x^{(2)},\ldots,x^{(k)})}{f_{X^{(1)}}(x^{(1)})} \tag{5.20}$$

Eq. (5.16) for $1 < k \leq N$ follows by substituting the representation of $f_{X_{j-\tau}^{(1)+}|U_j^+}(x^{(1)} \mid u_j)$ for $k = 1$ in Eq. (5.16) into Eq. (5.20).

To prove part (c), write

$$f_{X_{j+\tau}^{(k)+}|U_j^+}(x^{(k)} \mid u_j) = \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\vec{X}_{j+\tau}^{(k)+}|U_j^+}(\vec{x}^{(k)} \mid u_j)\; d\vec{x}^{(k-1)}\,. \tag{5.21}$$

Eq. (5.17) readily follows by substituting Eq. (5.15) into Eq. (5.21).

Finally, to prove part (d), write

$$f_{X_{j-\tau}^{(k)+}|U_j^+}(x^{(k)} \mid u_j) = \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\vec{X}_{j-\tau}^{(k)+}|U_j^+}(\vec{x}^{(k)} \mid u_j)\; d\vec{x}^{(k-1)}\,. \tag{5.22}$$

Eq. (5.18) readily follows by substituting Eq. (5.16) into Eq. (5.22). □

Corollary 5.3

For $k = 1$ in Theorem 5.3, we have the following formulas as special cases:

$$f_{X_{j+\tau}^{(1)+}|U_j^+}(x \mid u) = f_{X^{(1)}}(x) + 2 f_{X^{(1)}}(x) \sum_{\nu=1}^{\infty} \mathrm{Re}\Big[\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu u} \times \left(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(x)} + (1-\xi)\, e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(x)}\right)\Big] \tag{5.23}$$

$$f_{X_{j-\tau}^{(1)+}|U_j^+}(x \mid u) = f_{X^{(1)}}(x) + 2 f_{X^{(1)}}(x) \sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j-\tau+1,\, j}}(i2\pi\nu)\, e^{i2\pi\nu u} \times \left(\xi\, e^{-i2\pi\nu\, \xi F_{X^{(1)}}(x)} + (1-\xi)\, e^{i2\pi\nu\,(1-\xi)\, F_{X^{(1)}}(x)} \right) \right] \tag{5.24}$$

□

Proposition 5.1

For $\tau > 0$, $0 \le u_j < 1$, $j \ge 0$, and $x^{(k)} \in S^{(k)}$, $k \ge 1$,

$$F_{X_{j+\tau}^{(k)+}|U_j^+}(x^{(k)} \mid u_j) = F_{X^{(k)+}}(x^{(k)}) + 2 \sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu\, u_j} \int_{-\infty}^{x^{(k)}} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{y}^{(k)}) \times \left(\xi\, e^{i2\pi\nu\, \xi F_{X^{(1)}}(y^{(1)})} + (1-\xi)\, e^{-i2\pi\nu(1-\xi) F_{X^{(1)}}(y^{(1)})} \right) d\vec{y}^{(k)} \right] \tag{5.25}$$

$$F_{X_{j-\tau}^{(k)+}|U_j^+}(x^{(k)} \mid u_j) = F_{X^{(k)+}}(x^{(k)}) + 2 \sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j-\tau+1,\, j}}(i2\pi\nu)\, e^{i2\pi\nu\, u_j} \int_{-\infty}^{x^{(k)}} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{y}^{(k)}) \times \left(\xi\, e^{-i2\pi\nu\, \xi F_{X^{(1)}}(y^{(1)})} + (1-\xi)\, e^{i2\pi\nu(1-\xi) F_{X^{(1)}}(y^{(1)})} \right) d\vec{y}^{(k)} \right] \tag{5.26}$$

Proof.

Eqs. (5.25) and (5.26) follow by integrating Eqs. (5.17) and (5.18) in Theorem 5.3 respectively. □

Corollary 5.4

For $k = 1$ in Proposition 5.1, we have the following formulas as special cases:

$$F_{X_{j+\tau}^{(1)+}|U_j^+}(x \mid u) = F_{X^{(1)+}}(x) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\Bigg[\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu\, u} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \Big(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(y)} + (1-\xi)\, e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(y)}\Big) dy\Bigg] \tag{5.27}$$

$$F_{X_{j-\tau}^{(1)+}|U_j^+}(x \mid u) = F_{X^{(1)+}}(x) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\Bigg[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\, e^{i2\pi\nu\, u} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \Big(\xi\, e^{-i2\pi\nu\,\xi F_{X^{(1)}}(y)} + (1-\xi)\, e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(y)}\Big) dy\Bigg] \tag{5.28}$$

□

Proposition 5.2

For $\tau > 0$, $0 \le u_j < 1$, $j \ge 0$, and $k \ge 1$,

$$\mathbf{E}\Big[X_{j+\tau}^{(k)+} \mid U_j^+ = u_j\Big] = \mu_{X^{(k)+}} + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\Big[\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu\, u_j}\, \tilde{\mathcal{D}}^{(k)}(-i2\pi\nu)\Big] \tag{5.29}$$

$$\mathbf{E}\Big[X_{j-\tau}^{(k)+} \mid U_j^+ = u_j\Big] = \mu_{X^{(k)+}} + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\Big[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\, e^{i2\pi\nu\, u_j}\, \tilde{\mathcal{D}}^{(k)}(i2\pi\nu)\Big] \tag{5.30}$$

Proof.

For Eq. (5.29),

$$\begin{aligned}
\mathbf{E}\Big[X_{j+\tau}^{(k)+} \mid U_j^+ = u_j\Big] &= \mathbf{E}\Big[\mathbf{E}\Big[X_{j+\tau}^{(k)+} \mid U_{j+\tau}^+ = v, \vec{W}_{j+\tau}^{(k-1)} = \vec{w}_{j+\tau}^{(k-1)}\Big]\Big] \\
&= \int_0^1 \cdots \int_0^1 D^{(k)}(v, \vec{w}_{j+\tau}^{(k-1)})\; f_{U_{j+\tau}^+|U_j^+}(v \mid u_j)\, f_{\vec{W}_{j+\tau}^{(k-1)}}(\vec{w}_{\tau}^{(k-1)})\, dv\, d\vec{w}_{j+\tau}^{(k-1)} \\
&= \int_0^1 \cdots \int_0^1 D^{(k)}(v, \vec{w}_{\tau}^{(k-1)})\; f_{U_{j+\tau}^+|U_j^+}(v \mid u_j)\, dv\, d\vec{w}_{j+\tau}^{(k-1)} \\
&= \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu\, u_j}\; \tilde{\mathcal{D}}^{(k)}(-i2\pi\nu)
\end{aligned}$$

where the multiple integrals on the right-hand side above consists of n integrals. Here, the third equation uses the fact that $f_{\vec{W}_{j+\tau}^{(k-1)}}(\vec{w}_{j+\tau}^{(k-1)}) = 1$ in the multiple integral above, and the last

equation follows from Eq. (5.13). Eq. (5.29) now follows with the aid of Lemma 5.1 and Lemma 5.2.

In a similar vein, Eq. (5.30) follows by noting

$$\begin{aligned}
\mathbf{E}\Big[X_{j-\tau}^{(k)+} \mid U_j^+ = u_j\Big] &= \mathbf{E}\Big[\mathbf{E}\Big[X_{j-\tau}^{(k)+} \mid U_{j-\tau}^+ = v, \vec{W}_{j-\tau}^{(k-1)} = \vec{w}_{j-\tau}^{(k-1)}\Big]\Big] \\
&= \int_0^1 \cdots \int_0^1 D^{(k)}(v, \vec{w}_{j-\tau}^{(k-1)})\, f_{U_{j-\tau}^+|U_j^+}(v \mid u_j)\, f_{\vec{W}_{j-\tau}^{(n-1)}}(\vec{w}_\tau^{(k-1)})\, dv\, d\vec{w}_{j-\tau}^{(k-1)} \\
&= \int_0^1 \cdots \int_0^1 D^{(k)}(v, \vec{w}_{j-\tau}^{(k-1)})\; f_{U_{j-\tau}^+|U_j^+}(v \mid u_j)\, dv\, d\vec{w}_{j-\tau}^{(k-1)} \\
&= \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\, e^{i2\pi\nu\, u_j}\; \tilde{\mathcal{D}}^{(k)}(i2\pi\nu)
\end{aligned}$$

Eq. (5.30) now follows with the aid of Lemma 5.1 and Lemma 5.2. □

5.2. Transition Structure of MARM⁻ Processes

The following theorem recalls a fundamental representation for the transition density function, $f_{U_{j+\tau}^-|U_j^-}(v \mid u)$, of MARM⁻ background processes.

Theorem 5.4

$\{U_n^-\}$ is a stationary Markovian with non-stationary transition density $f_{U_{j+\tau}^-|U_j^-}(v \mid u)$, where for $\tau > 0$ and $0 \le u, v < 1$,

$$f_{U_{j+\tau}^{-}|U_j^{-}}(v \mid u) = \begin{cases} 1+2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\ e^{i2\pi\nu(v-u)}\right], & j \text{ even, } \tau \text{ even} \\ 1+2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\ e^{i2\pi\nu(-v-u)}\right], & j \text{ even, } \tau \text{ odd} \\ 1+2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\ e^{i2\pi\nu(-v+u)}\right], & j \text{ odd, } \tau \text{ even} \\ 1+2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\ e^{i2\pi\nu(v+u)}\right], & j \text{ odd, } \tau \text{ odd} \end{cases} \tag{5.31}$$

$$f_{U_{j-\tau}^{-}|U_j^{-}}(v \mid u) = \begin{cases} 1+2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\ e^{i2\pi\nu(u-v)}\right], & j \text{ even, } \tau \text{ even} \\ 1+2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\ e^{i2\pi\nu(u+v)}\right], & j \text{ even, } \tau \text{ odd} \\ 1+2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\ e^{i2\pi\nu(-u+v)}\right], & j \text{ odd, } \tau \text{ even} \\ 1+2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\ e^{i2\pi\nu(-u-v)}\right], & j \text{ odd, } \tau \text{ odd} \end{cases} \tag{5.32}$$

Proof.

To prove Eq. (5.31), we use Theorem 2 in [Melamed (1999)] to write

$$f_{U_{j+\tau}^{-}|U_j^{-}}(v \mid u) = \begin{cases} \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\ e^{i2\pi\nu(v-u)}, & j \text{ even, } \tau \text{ even} \\ \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\ e^{i2\pi\nu(-v-u)}, & j \text{ even, } \tau \text{ odd} \\ \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\ e^{i2\pi\nu(-v+u)}, & j \text{ odd, } \tau \text{ even} \\ \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\ e^{i2\pi\nu(v+u)}, & j \text{ odd, } \tau \text{ odd} \end{cases} \tag{5.33}$$

Eq. (5.31) now follows from Eq. (5.33) with the aid of Lemma 5.1 and by noticing the fact that

$\tilde{f}_{S_{j+1,\, j+\tau}}(0) = \tilde{f}_{S_{j-\tau+1,\, j}}(0) = 1$.

Eq. (5.32) is proved in a similar way. It follows from Eq.(5.1), Theorem 5.2 and Eq.(5.31) that

$$f_{U_{j-\tau}^{-}|U_j^{-}}(v \mid u) = \begin{cases} \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\, e^{i2\pi\nu(u-v)}, & j \text{ even}, \tau \text{ even} \\ \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\, e^{i2\pi\nu(u+v)}, & j \text{ even}, \tau \text{ odd} \\ \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\, e^{i2\pi\nu(-u+v)}, & j \text{ odd}, \tau \text{ even} \\ \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\, e^{i2\pi\nu(-u-v)}, & j \text{ odd}, \tau \text{ odd} \end{cases} \tag{5.34}$$

Eq. (5.32) now follows from Eq. (5.34) with the aid of Lemma 5.1 and the fact that $\tilde{f}_{S_{j+1,\, j+\tau}}(0) = \tilde{f}_{S_{j+\tau+1,\, j}}(0) = 1$. □

Theorem 5.5

Let $\tau > 0$, $0 \le u_j < 1$, $j \ge 0$, and $\vec{x}^{(k)} \in S^{(1)} \times \cdots \times S^{(k)}$, $x^{(k)} \in S^{(k)}$, $k \ge 1$.

(a) $f_{\vec{X}^{(k)-}_{j+\tau}|U^-_j}(\vec{x}^{(k)} \mid u_j) =$

$$
\begin{cases}
f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\,e^{-i2\pi\nu\,u_j}\times\right.\right. \\
\qquad\left.\left.\left(\xi\,e^{i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})}+(1-\xi)\,e^{-i2\pi\nu(1-\xi)\,F_{X^{(1)}}(x^{(1)})}\right)\right]\right\}, \quad j \text{ even, } \tau \text{ even} \\
f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\,e^{-i2\pi\nu\,u_j}\times\right.\right. \\
\qquad\left.\left.\left(\xi\,e^{-i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})}+(1-\xi)\,e^{i2\pi\nu(1-\xi)\,F_{X^{(1)}}(x^{(1)})}\right)\right]\right\}, j \text{ even, } \tau \text{ odd} \\
f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\,e^{i2\pi\nu\,u_j}\times\right.\right. \\
\qquad\left.\left.\left(\xi\,e^{-i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})}+(1-\xi)\,e^{i2\pi\nu(1-\xi)\,F_{X^{(1)}}(x^{(1)})}\right)\right]\right\}, \quad j \text{ odd, } \tau \text{ even} \\
f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\,e^{i2\pi\nu\,u_j}\times\right.\right. \\
\qquad\left.\left.\left(\xi\,e^{i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})}+(1-\xi)\,e^{-i2\pi\nu(1-\xi)\,F_{X^{(1)}}(x^{(1)})}\right)\right]\right\}, \quad j \text{ odd, } \tau \text{ odd}
\end{cases}
\tag{5.35}
$$

(b) $f_{\vec{X}^{(k)}_{j-\tau}|U_j^-}(\vec{x}^{(k)} \mid u_j) =$

$$
\begin{cases}
f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\,e^{i2\pi\nu\,u_j}\times\right.\right. \\
\qquad\left.\left.\left(\xi\,e^{-i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})}+(1-\xi)\,e^{i2\pi\nu(1-\xi)\,F_{X^{(1)}}(x^{(1)})}\right)\right]\right\}, \quad j \text{ even, } \tau \text{ even} \\
f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\,e^{i2\pi\nu\,u_j}\times\right.\right. \\
\qquad\left.\left.\left(\xi\,e^{i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})}+(1-\xi)\,e^{-i2\pi\nu(1-\xi)\,F_{X^{(1)}}(x^{(1)})}\right)\right]\right\}, j \text{ even, } \tau \text{ odd} \\
f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\,e^{-i2\pi\nu\,u_j}\times\right.\right. \\
\qquad\left.\left.\left(\xi\,e^{i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})}+(1-\xi)\,e^{-i2\pi\nu(1-\xi)\,F_{X^{(1)}}(x^{(1)})}\right)\right]\right\}, \quad j \text{ odd, } \tau \text{ even} \\
f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\,e^{-i2\pi\nu\,u_j}\times\right.\right. \\
\qquad\left.\left.\left(\xi\,e^{-i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})}+(1-\xi)\,e^{i2\pi\nu(1-\xi)\,F_{X^{(1)}}(x^{(1)})}\right)\right]\right\}, \quad j \text{ odd, } \tau \text{ odd}
\end{cases}
\tag{5.36}
$$

(c) $f_{X_{j+\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) =$

$$
\begin{cases}
f_{X^{(k)-}}(x^{(k)}) + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\Bigg[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu\, u_j} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{x}^{(k)}) \times \\
\qquad \left(\xi\, e^{i2\pi\nu\,\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{-i2\pi\nu(1-\xi) F_{X^{(1)}}(x^{(1)})}\right) d\vec{x}^{(k-1)}\Bigg], \ j \text{ even}, \ \tau \text{ even} \\
f_{X^{(k)-}}(x^{(k)}) + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\Bigg[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu\, u_j} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{x}^{(k)}) \times \\
\qquad \left(\xi\, e^{-i2\pi\nu\,\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{i2\pi\nu(1-\xi) F_{X^{(1)}}(x^{(1)})}\right) d\vec{x}^{(k-1)}\Bigg], \ j \text{ even}, \ \tau \text{ odd} \\
f_{X^{(k)-}}(x^{(k)}) + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\Bigg[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\, e^{i2\pi\nu\, u_j} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{x}^{(k)}) \times \\
\qquad \left(\xi\, e^{-i2\pi\nu\,\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{i2\pi\nu(1-\xi) F_{X^{(1)}}(x^{(1)})}\right) d\vec{x}^{(k-1)}\Bigg], \ j \text{ odd}, \ \tau \text{ even} \\
f_{X^{(k)-}}(x^{(k)}) + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\Bigg[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\, e^{i2\pi\nu\, u_j} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{x}^{(k)}) \times \\
\qquad \left(\xi\, e^{i2\pi\nu\,\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{-i2\pi\nu(1-\xi) F_{X^{(1)}}(x^{(1)})}\right) d\vec{x}^{(k-1)}\Bigg], \ j \text{ odd}, \ \tau \text{ odd}
\end{cases}
\tag{5.37}
$$

(d) $f_{X_{j-\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) =$

$$\begin{cases} f_{X^{(k)-}}(x^{(k)}) + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\, e^{i2\pi\nu u_j} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{x}^{(k)}) \times \right. \\ \qquad \left. \left(\xi\, e^{-i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{i2\pi\nu(1-\xi) F_{X^{(1)}}(x^{(1)})}\right) d\vec{x}^{(k-1)}\right], \ j \text{ even}, \ \tau \text{ even} \\ f_{X^{(k)-}}(x^{(k)}) + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\, e^{i2\pi\nu u_j} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{x}^{(k)}) \times \right. \\ \qquad \left. \left(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{-i2\pi\nu(1-\xi) F_{X^{(1)}}(x^{(1)})}\right) d\vec{x}^{(k-1)}\right], \ j \text{ even}, \ \tau \text{ odd} \\ f_{X^{(k)-}}(x^{(k)}) + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\, e^{-i2\pi\nu u_j} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{x}^{(k)}) \times \right. \\ \qquad \left. \left(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{-i2\pi\nu(1-\xi) F_{X^{(1)}}(x^{(1)})}\right) d\vec{x}^{(k-1)}\right], \ j \text{ odd}, \ \tau \text{ even} \\ f_{X^{(k)-}}(x^{(k)}) + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\, e^{-i2\pi\nu u_j} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{x}^{(k)}) \times \right. \\ \qquad \left. \left(\xi\, e^{-i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{i2\pi\nu(1-\xi) F_{X^{(1)}}(x^{(1)})}\right) d\vec{x}^{(k-1)}\right], \ j \text{ odd}, \ \tau \text{ odd} \end{cases} \tag{5.38}$$

Proof.

Similar to the proof of Theorem 5.3 with MARM$^-$ random variables replacing their MARM$^+$ counterparts. □

Corollary 5.5

For $k = 1$ in Theorem 5.5, we have the following formulas as special cases:

$$f_{X^{(1)-}_{j+\tau}|U^-_j}(x \mid u_j) =$$

$$\begin{cases} f_{X^{(1)-}}(x)\left\{1 + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu u_j} \times \left(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(x)} + (1-\xi)\, e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(x)}\right)\right]\right\}, & j \text{ even, } \tau \text{ even} \\ f_{X^{(1)-}}(x)\left\{1 + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu u_j} \times \left(\xi\, e^{-i2\pi\nu\xi F_{X^{(1)}}(x)} + (1-\xi)\, e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(x)}\right)\right]\right\}, & j \text{ even, } \tau \text{ odd} \\ f_{X^{(1)-}}(x)\left\{1 + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\, e^{i2\pi\nu u_j} \times \left(\xi\, e^{-i2\pi\nu\xi F_{X^{(1)}}(x)} + (1-\xi)\, e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(x)}\right)\right]\right\}, & j \text{ odd, } \tau \text{ even} \\ f_{X^{(1)-}}(x)\left\{1 + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\, e^{i2\pi\nu u_j} \times \left(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(x)} + (1-\xi)\, e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(x)}\right)\right]\right\}, & j \text{ odd, } \tau \text{ odd} \end{cases} \tag{5.39}$$

$$f_{X^{(1)-}_{j-\tau}|U^{-}_{j}}(x \mid u_j) =$$

$$\begin{cases} f_{X^{(1)-}}(x)\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\,e^{i2\pi\nu u_j}\times \left(\xi\, e^{-i2\pi\nu\xi F_{X^{(1)}}(x)}+(1-\xi)\,e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(x)}\right)\right]\right\}, & j \text{ even, } \tau \text{ even} \\ f_{X^{(1)-}}(x)\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\,e^{i2\pi\nu u_j}\times \left(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(x)}+(1-\xi)\,e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(x)}\right)\right]\right\}, & j \text{ even, } \tau \text{ odd} \\ f_{X^{(1)-}}(x)\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\,e^{-i2\pi\nu u_j}\times \left(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(x)}+(1-\xi)\,e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(x)}\right)\right]\right\}, & j \text{ odd, } \tau \text{ even} \\ f_{X^{(1)-}}(x)\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\,e^{-i2\pi\nu u_j}\times \left(\xi\, e^{-i2\pi\nu\xi F_{X^{(1)}}(x)}+(1-\xi)\,e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(x)}\right)\right]\right\}, & j \text{ odd, } \tau \text{ odd} \end{cases} \tag{5.40}$$

□

Proposition 5.3

For $\tau > 0$, $0 \le u_j < 1$, $j \ge 0$, and $x^{(k)} \in S^{(k)}$, $k \ge 1$,

$$F_{X_{j+\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) =$$

$$\begin{cases} F_{X^{(k)-}}(x^{(k)}) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)e^{-i2\pi\nu u_j}\int_{-\infty}^{x^{(k)}} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{y}^{(k)}) \times \right. \\ \qquad \left. \left(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(y^{(1)})} + (1-\xi)\, e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(y^{(1)})}\right) d\vec{y}^{(k)}\right], \ j \text{ even}, \ \tau \text{ even} \\ F_{X^{(k)-}}(x^{(k)}) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)e^{-i2\pi\nu u_j}\int_{-\infty}^{x^{(k)}} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{y}^{(k)}) \times \right. \\ \qquad \left. \left(\xi\, e^{-i2\pi\nu\xi F_{X^{(1)}}(y^{(1)})} + (1-\xi)\, e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(y^{(1)})}\right) d\vec{y}^{(k)}\right], \ j \text{ even}, \ \tau \text{ odd} \\ F_{X^{(k)-}}(x^{(k)}) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)e^{i2\pi\nu u_j}\int_{-\infty}^{x^{(k)}} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{y}^{(k)}) \times \right. \\ \qquad \left. \left(\xi\, e^{-i2\pi\nu\xi F_{X^{(1)}}(y^{(1)})} + (1-\xi)\, e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(y^{(1)})}\right) d\vec{y}^{(k)}\right], \ j \text{ odd}, \ \tau \text{ even} \\ F_{X^{(k)-}}(x^{(k)}) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)e^{i2\pi\nu u_j}\int_{-\infty}^{x^{(k)}} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{y}^{(k)}) \times \right. \\ \qquad \left. \left(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(y^{(1)})} + (1-\xi)\, e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(y^{(1)})}\right) d\vec{y}^{(k)}\right], \ j \text{ odd}, \ \tau \text{ odd} \end{cases} \tag{5.41}$$

$$
F_{X_{j-\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) =
$$

$$
\begin{cases}
F_{X^{(k)-}}(x^{(k)}) + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)e^{i2\pi\nu\, u_j}\int_{-\infty}^{x^{(k)}} \int\limits_{S^{(1)}} \cdots \int\limits_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{y}^{(k)})\times \right. \\
\qquad \left. \left(\xi\, e^{-i2\pi\nu\,\xi F_{X^{(1)}}(y^{(1)})} + (1-\xi)\, e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(y^{(1)})}\right) d\vec{y}^{(k)}\right],\ j \text{ even},\ \tau \text{ even} \\
F_{X^{(k)-}}(x^{(k)}) + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)e^{i2\pi\nu\, u_j}\int_{-\infty}^{x^{(k)}} \int\limits_{S^{(1)}} \cdots \int\limits_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{y}^{(k)})\times \right. \\
\qquad \left. \left(\xi\, e^{i2\pi\nu\,\xi F_{X^{(1)}}(y^{(1)})} + (1-\xi)\, e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(y^{(1)})}\right) d\vec{y}^{(k)}\right],\ j \text{ even},\ \tau \text{ odd} \\
F_{X^{(k)-}}(x^{(k)}) + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)e^{-i2\pi\nu\, u_j}\int_{-\infty}^{x^{(k)}} \int\limits_{S^{(1)}} \cdots \int\limits_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{y}^{(k)})\times \right. \\
\qquad \left. \left(\xi\, e^{i2\pi\nu\,\xi F_{X^{(1)}}(y^{(1)})} + (1-\xi)\, e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(y^{(1)})}\right) d\vec{y}^{(k)}\right],\ j \text{ odd},\ \tau \text{ even} \\
F_{X^{(k)-}}(x^{(k)}) + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)e^{-i2\pi\nu\, u_j}\int_{-\infty}^{x^{(k)}} \int\limits_{S^{(1)}} \cdots \int\limits_{S^{(k-1)}} f_{\vec{X}^{(k)}}(\vec{y}^{(k)})\times \right. \\
\qquad \left. \left(\xi\, e^{-i2\pi\nu\,\xi F_{X^{(1)}}(y^{(1)})} + (1-\xi)\, e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(y^{(1)})}\right) d\vec{y}^{(k)}\right],\ j \text{ odd},\ \tau \text{ odd}
\end{cases}
\tag{5.42}
$$

Proof.

Eqs. (5.41) and (5.42) follow by integrating Eqs. (5.37) and (5.38), respectively, in Theorem 5.5.

□

Corollary 5.6

For $k = 1$ in Proposition 5.3, we have the following formulas as special cases:

$$F_{X_{j+\tau}^{(1)-}|U_j^-}(x \mid u_j) =$$

$$\begin{cases} F_{X^{(1)-}}(x) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu\, u_j} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \right. \\ \qquad \left. \left(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(y)} + (1-\xi)\, e^{-i2\pi\nu(1-\xi) F_{X^{(1)}}(y)}\right) dy\right], \ j \text{ even}, \ \tau \text{ even} \\ F_{X^{(1)-}}(x) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu\, u_j} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \right. \\ \qquad \left. \left(\xi\, e^{-i2\pi\nu\xi F_{X^{(1)}}(y)} + (1-\xi)\, e^{i2\pi\nu(1-\xi) F_{X^{(1)}}(y)}\right) dy\right], \ j \text{ even}, \ \tau \text{ odd} \\ F_{X^{(1)-}}(x) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)\, e^{i2\pi\nu\, u_j} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \right. \\ \qquad \left. \left(\xi\, e^{-i2\pi\nu\xi F_{X^{(1)}}(y)} + (1-\xi)\, e^{i2\pi\nu(1-\xi) F_{X^{(1)}}(y)}\right) dy\right], \ j \text{ odd}, \ \tau \text{ even} \\ F_{X^{(1)-}}(x) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)\, e^{i2\pi\nu\, u_j} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \right. \\ \qquad \left. \left(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(y)} + (1-\xi)\, e^{-i2\pi\nu(1-\xi) F_{X^{(1)}}(y)}\right) dy\right], \ j \text{ odd}, \ \tau \text{ odd} \end{cases} \tag{5.43}$$

$$F_{X_{j-\tau}^{(1)-}|U_j^-}(x \mid u_j) =$$

$$\begin{cases} F_{X^{(1)-}}(x) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\Big[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\, e^{i2\pi\nu u_j} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \\ \qquad \Big(\xi\, e^{-i2\pi\nu\xi F_{X^{(1)}}(y)} + (1-\xi)\, e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(y)}\Big) dy\Big], \ j \text{ even}, \ \tau \text{ even} \\ F_{X^{(1)-}}(x) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\Big[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\, e^{i2\pi\nu u_j} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \\ \qquad \Big(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(y)} + (1-\xi)\, e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(y)}\Big) dy\Big], \ j \text{ even}, \ \tau \text{ odd} \\ F_{X^{(1)-}}(x) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\Big[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\, e^{-i2\pi\nu u_j} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \\ \qquad \Big(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(y)} + (1-\xi)\, e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(y)}\Big) dy\Big], \ j \text{ odd}, \ \tau \text{ even} \\ F_{X^{(1)-}}(x) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\Big[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\, e^{-i2\pi\nu u_j} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \\ \qquad \Big(\xi\, e^{-i2\pi\nu\xi F_{X^{(1)}}(y)} + (1-\xi)\, e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(y)}\Big) dy\Big], \ j \text{ odd}, \ \tau \text{ odd} \end{cases} \tag{5.44}$$

□

Proposition 5.4

For $\tau > 0$, $0 \le u_j < 1$, $j \ge 0$, and $k \ge 1$,

$$\mathbf{E}\Big[X_{j+\tau}^{(k)-} \mid U_j^- = u_j\Big] =$$

$$\begin{cases} \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\Big[\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu u_j}\, \tilde{\mathcal{D}}^{(k)}(-i2\pi\nu)\Big], \ j \text{ even}, \ \tau \text{ even} \\ \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\Big[\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu u_j}\, \tilde{\mathcal{D}}^{(k)}(i2\pi\nu)\Big], \ j \text{ even}, \ \tau \text{ odd} \\ \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\Big[\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)\, e^{i2\pi\nu u_j}\, \tilde{\mathcal{D}}^{(k)}(i2\pi\nu)\Big], \ j \text{ odd}, \ \tau \text{ even} \\ \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\Big[\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)\, e^{i2\pi\nu u_j}\, \tilde{\mathcal{D}}^{(k)}(-i2\pi\nu)\Big], \ j \text{ odd}, \ \tau \text{ odd} \end{cases} \tag{5.45}$$

$$\mathbf{E}\left[X_{j-\tau}^{(k)-} \mid U_j^- = u_j\right] =$$

$$\begin{cases} \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)e^{i2\pi\nu u_j}\,\tilde{\mathcal{D}}^{(k)}(i2\pi\nu)\right], & j \text{ even, } \tau \text{ even} \\ \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)e^{i2\pi\nu u_j}\,\tilde{\mathcal{D}}^{(k)}(-i2\pi\nu)\right], & j \text{ even, } \tau \text{ odd} \\ \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\,e^{-i2\pi\nu u_j}\,\tilde{\mathcal{D}}^{(k)}(-i2\pi\nu)\right], & j \text{ odd, } \tau \text{ even} \\ \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\,e^{-i2\pi\nu u_j}\,\tilde{\mathcal{D}}^{(k)}(i2\pi\nu)\right], & j \text{ odd, } \tau \text{ odd} \end{cases} \tag{5.46}$$

Proof.

Similar to the proof of Proposition 5.2 with MARM$^-$ random variables replacing their MARM$^+$ counterparts. □

6. The Autocorrelation and Cross-Correlation Structures of Foreground MARM Processes

To derive the autocorrelation and cross-correlation structure of MARM processes, we need to derive the expectations of products of lagged foreground random variables of the form

$$\mathbf{E}\left[X_j^{(m)} X_{j+\tau}^{(n)}\right] = \int_{S^{(m)}} \int_{S^{(n)}} x_j^{(m)}\, x_{j+\tau}^{(n)}\, f_{X_j^{(m)}, X_{j+\tau}^{(n)}}(x_j^{(m)}, x_{j+\tau}^{(n)})\, dx_j^{(m)}\, dx_{j+\tau}^{(n)}\,, \qquad (6.1)$$

for $1 \le m \le n \le N$, $\tau > 0$, $j \ge 0$, as an ingredient of the correlation functions, $\rho_{m,n}(j,\tau)$ of MARM processes $\{X_j^{(m)}\}_{j=0}^{\infty}$ and $\{X_j^{(n)}\}_{j=0}^{\infty}$. Recall that cases of the form $m = n$ correspond to autocorrelation functions, while cases of the form $m \ne n$ correspond to cross-correlation functions. To this end, we derive conditional versions of these expectations and then uncondition them, recalling the random-vector generating algorithm (Algorithm 3.1) in Section 3.4 and the MARM foreground-process generating algorithm (Algorithm 4.1) in Section 4.3.

The following lemma provides a simplified representation of the joint density of pairs of MARM random variables.

Lemma 6.1

For $\tau \ge 0$, $0 \le u_j < 1$, $j \ge 0$, and $x^{(m)} \in S^{(m)}$, $x^{(n)} \in S^{(n)}$, $1 \le m < n \le N$,

$$f_{X_j^{(m)}, X_{j+\tau}^{(n)} | U_j}(x_j^{(m)}, x_{j+\tau}^{(n)} \mid u_j) = f_{X_j^{(m)} | U_j}(x_j^{(m)} \mid u_j)\, f_{X_{j+\tau}^{(n)} | U_j}(x_{j+\tau}^{(n)} \mid u_j) \qquad (6.2)$$

Proof.

By definition,

$$f_{X_k^{(m)}, X_{j+\tau}^{(n)} | U_j}(x_j^{(m)}, x_{j+\tau}^{(n)} \mid u_j) = f_{X_j^{(m)} | U_j}(x_j^{(m)} \mid u_j)\, f_{X_{j+\tau}^{(n)} | U_j, X_j^{(m)}}(x_{j+\tau}^{(n)} \mid u_j, x_j^{(m)})\,.$$

The lemma now follows since $f_{X_{j+\tau}^{(n)}|U_j, X_j^{(m)}}(x_{j+\tau}^{(n)} \mid u_j, x_j^{(m)}) = f_{X_{j+\tau}^{(n)}|U_j}(x_{j+\tau}^{(n)} \mid u_j)$ by construction of MARM processes. □

6.1. Correlation Structure of MARM$^+$ Processes

In this section we derive general formulas for the autocorrelation f and cross-correlation functions of MARM$^+$ processes. The following theorem generalizes the distortions [Jagerman and Melamed (1992b)] from the ARM$^+$ processes to MARM$^+$ processes.

The next theorem exhibits computational formulas for correlation functions of MARM$^+$ processes.

Theorem 6.1

For $1 \le m \le n \le N$, $\tau > 0$ and $j \ge 0$, the correlation functions $\rho_{m,n}^{+}(j,\tau)$ are given by

$$\rho_{m,n}^{+}(j,\tau) = \frac{2}{\sigma_{X^{(m)}}\,\sigma_{X^{(n)}}} \sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\,\tilde{\mathcal{D}}^{(m)}(i2\pi\nu)\,\tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)\right] \tag{6.3}$$

In particular, for $m = n$, Eq. (6.3) reduces to

$$\rho_{n,n}^{+}(j,\tau) = \frac{2}{\sigma_{X^{(n)}}^{2}} \sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\right] \left|\tilde{\mathcal{D}}^{(n)}(i2\pi\nu)\right|^{2} \tag{6.4}$$

Proof.

We first show that for $1 \le m \le n \le N$,

$$\mathbf{E}\left[X_j^{(m)+} X_{j+\tau}^{(n)+}\right] = \mu_{X^{(m)}}\,\mu_{X^{(n)}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\,\tilde{\mathcal{D}}^{(m)}(i2\pi\nu)\,\tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)] \tag{6.5}$$

and in particular, for $m = n$, Eq. (6.5) reduces to

$$\mathbf{E}\left[X_j^{(n)+} X_{j+\tau}^{(n)+}\right] = \mu_{X^{(n)}}^2 + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)\right]\left|\tilde{\mathcal{D}}^{(n)}(i2\pi\nu)\right|^2. \tag{6.6}$$

We prove Eq. (6.5) in three cases.

For $m = n = 1$, Eq. (6.5) follows from Theorem 3 in [Melamed (1999)] by noting that $\mathcal{D}^{(1)}$ is equivalent to the distortion D therein (see also Theorem 3 in [Jagerman and Melamed (1992a)]).

For $m = 1$ and $2 \le n \le N$, we use Eqs. (5.7) and (5.8) to write

$$\mathbf{E}\left[X_j^{(1)+} X_{j+\tau}^{(n)+} \mid U_j^+ = u, U_{j+\tau}^+ = v, \vec{W}_{j+\tau}^{(n-1)} = \vec{w}_{j+\tau}^{(n-1)}\right] = D^{(1)}(u)\, D^{(n)}(v, \vec{w}_{j+\tau}^{(n-1)}).$$

Consequently,

$$\begin{aligned}
\mathbf{E}\left[X_j^{(1)+} X_{j+\tau}^{(n)+}\right] &= \mathbf{E}\left[\mathbf{E}\left[X_j^{(1)+} X_{j+\tau}^{(n)+} \mid U_j^+ = u, U_{j+\tau}^+ = v, \vec{W}_{j+\tau}^{(n-1)} = \vec{w}_{j+\tau}^{(n-1)}\right]\right] \\
&= \int_0^1 \cdots \int_0^1 D^{(1)}(u)\, D^{(n)}(v, \vec{w}_{j+\tau}^{(n-1)})\, f_{U_j^+}(u)\, f_{U_{j+\tau}^+ | U_j^+}(v \mid u)\, f_{\vec{W}_\tau^{(n-1)}}(\vec{w}_\tau^{(n-1)})\, du\, dv\, d\vec{w}_\tau^{(n-1)} \\
&= \int_0^1 \cdots \int_0^1 D^{(1)}(u)\, D^{(n)}(v, \vec{w}_\tau^{(n-1)})\, f_{U_{j+\tau}^+ | U_j^+}(v \mid u)\, du\, dv\, d\vec{w}_\tau^{(n-1)} \\
&= \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)\, \tilde{\mathcal{D}}^{(1)}(i2\pi\nu)\, \tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)
\end{aligned}$$

where the multiple integrals on the right-hand side above consists of $n+1$ integrals. Here, the third equation uses the fact that $f_{U_j^+}(u) = 1$ and $f_{\vec{W}_\tau^{(n-1)}}(\vec{w}_\tau^{(n-1)}) = 1$ in the multiple integral above, and the last equation follows from Eq. (5.13), Eq. (5.9). Eq. (6.5) now follows with the aid of Lemmas 5.1 and 5.2 by noting that $\tilde{f}_{S_{j+1,\, j+\tau}}(0)\, \tilde{\mathcal{D}}^{(1)}(0)\, \tilde{\mathcal{D}}^{(n)}(0) = \mu_{X^{(1)}}\, \mu_{X^{(n)}}$.

For $2 \le m \le n \le N$ note that

$$\mathbf{E}\Big[X_j^{(m)+}X_{j+\tau}^{(n)+} \mid U_j^+ = u, U_{j+\tau}^+ = v, \vec{W}_j^{(m-1)} = \vec{w}_j^{(m-1)}, \vec{W}_\tau^{(n-1)} = \vec{w}_\tau^{(n-1)}\Big]$$
$$= D^{(m)}(u, \vec{w}_j^{(m-1)})\, D^{(n)}(v, \vec{w}_{j+\tau}^{(n-1)})$$

Consequently,

$$\mathbf{E}\Big[X_j^{(m)+}X_{j+\tau}^{(n)+}\Big] = \mathbf{E}\Big[\mathbf{E}\Big[X_j^{(m)+}X_{j+\tau}^{(n)+} \mid U_j^+ = u, U_{j+\tau}^+ = v, \vec{W}_j^{(m-1)} = \vec{w}_j^{(m-1)}, \vec{W}_{j+\tau}^{(n-1)} = \vec{w}_{j+\tau}^{(n-1)}\Big]\Big]$$
$$= \int_0^1 \cdots \int_0^1 D^{(m)}(u, \vec{w}_j^{(m-1)})\, D^{(n)}(v, \vec{w}_{j+\tau}^{(n-1)})\, f_{U_j^+}(u)\, f_{U_{j+\tau}^+|U_j^+}(v \mid u) \times$$
$$f_{\vec{W}_j^{(m-1)}}(\vec{w}_j^{(m-1)}) f_{\vec{W}_{j+\tau}^{(n-1)}}(\vec{w}_{j+\tau}^{(n-1)})\, du\, dv\, d\vec{w}_j^{(m-1)}\, d\vec{w}_{j+\tau}^{(n-1)}$$

where the right-hand side above consists of $m+n$ integrals. Evaluating the multiple integrals above yields

$$\mathbf{E}\Big[X_j^{(m)+}X_{j+\tau}^{(n)+}\Big] = \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\, \tilde{\mathcal{D}}^{(m)}(i2\pi\nu)\, \tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)$$

where we again make use of Eqs. (5.13) and (5.9), and the facts that $f_{U_j^+}(u) = 1$, $f_{\vec{W}_j^{(m-1)}}(\vec{w}_j^{(m-1)}) = 1$ and $f_{\vec{W}_{j+\tau}^{(n-1)}}(\vec{w}_{j+\tau}^{(n-1)}) = 1$ in the multiple integral above. Eq. (6.5) now follows by applying Lemma 5.1 to the representation above and noting that $\tilde{f}_{S_{j+1,j+\tau}}(0)\, \tilde{\mathcal{D}}^{(m)}(0)\, \tilde{\mathcal{D}}^{(n)}(0) = \mu_{X^{(m)}}\, \mu_{X^{(n)}}$ with the aid of Lemma 5.2.

Next, Eq. (6.6) follows from Eq. (6.5) by observing that $\tilde{\mathcal{D}}^{(k)}(i2\pi\nu)$ and $\tilde{\mathcal{D}}^{(k)}(-i2\pi\nu)$ are complex conjugates by Lemma 5.1. Finally, Eqs. (6.3) and (6.4) follow readily by substituting Eqs. (6.5) and (6.6) into Eqs. (3.2) and (3.1), respectively. □

Note that when the innovation sequence, $\{V_n\}$, is iid, then the correlation functions $\rho_{m,n}^+(j,\tau) = \rho_{m,n}^+(\tau)$ are stationary and depend only on the lag τ.

6.2. Correlation Structure of MARM^{-} Processes

The next theorem uses exhibits computational formulas for correlation functions of MARM^{-} processes.

Theorem 6.2

For $1 \le m \le n \le N$, $\tau > 0$ and $j \ge 0$, the correlation functions $\rho^{-}_{m,n}(j,\tau)$ are given by

$$\rho^{-}_{m,n}(j,\tau) = \begin{cases} \dfrac{2}{\sigma_{X^{(m)}}\,\sigma_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\,\tilde{\mathcal{D}}^{(m)}(i2\pi\nu)\,\tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)\right], & j \text{ even, } \tau \text{ even} \\ \dfrac{2}{\sigma_{X^{(m)}}\,\sigma_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\,\tilde{\mathcal{D}}^{(m)}(i2\pi\nu)\,\tilde{\mathcal{D}}^{(n)}(i2\pi\nu)\right], & j \text{ even, } \tau \text{ odd} \\ \dfrac{2}{\sigma_{X^{(m)}}\,\sigma_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\,\tilde{\mathcal{D}}^{(m)}(-i2\pi\nu)\,\tilde{\mathcal{D}}^{(n)}(i2\pi\nu)\right], & j \text{ odd, } \tau \text{ even} \\ \dfrac{2}{\sigma_{X^{(m)}}\,\sigma_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\,\tilde{\mathcal{D}}^{(m)}(-i2\pi\nu)\,\tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)\right], & j \text{ odd, } \tau \text{ odd} \end{cases} \tag{6.7}$$

In particular, for $m = n$, Eq. (6.7) reduces to

$$\rho^{-}_{n,n}(j,\tau) = \begin{cases} \dfrac{2}{\sigma^2_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\right] \left|\tilde{\mathcal{D}}^{(n)}(i2\pi\nu)\right|^2, & j \text{ even, } \tau \text{ even} \\ \dfrac{2}{\sigma^2_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\left(\tilde{\mathcal{D}}^{(n)}(i2\pi\nu)\right)^2\right], & j \text{ even, } \tau \text{ odd} \\ \dfrac{2}{\sigma^2_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\right] \left|\tilde{\mathcal{D}}^{(n)}(i2\pi\nu)\right|^2, & j \text{ odd, } \tau \text{ even} \\ \dfrac{2}{\sigma^2_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\left(\tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)\right)^2\right], & j \text{ odd, } \tau \text{ odd} \end{cases} \tag{6.8}$$

Proof.

Similar to the proof of Theorem 6.1 with MARM^{-} random variables replacing their MARM^{+} counterparts. □

7. MARM Fitting Methodologies

In this chapter we formulate the general MARM fitting methodology, and then describe in some detail a practical empirical version of it in Chapter 10. The general MARM fitting problem is the following optimization problem.

Problem 7.1 (General MARM Fitting Problem)

Given a multivariate distribution $\hat{F}_{\vec{X}}$ and a set of correlation functions $\hat{\rho}_{m,n}(\tau)$, find an iid innovation probability law, $f_{\{V_n^*\}}$, and a stitching parameter ξ^* such that

$$(f_{\{V_n^*\}},\ \xi^*) = \underset{(f_{\{V_n\}},\ \xi)}{\arg\min}\ \{g(f_{\{V_n\}},\ \xi)\}$$

where $g(f_{\{V_n\}},\ \xi)$ is an objective function of the form

$$g(f_{\{V_n\}},\xi) = \sum_{m=1}^{N}\sum_{n=m}^{N}\sum_{\tau=1}^{S(m,n)} a_{m,n}(\tau)\left[\rho_{m,n}(\tau) - \hat{\rho}_{m,n}(\tau)\right]^2, \tag{7.1}$$

$S(m,n)$ is the maximal correlation lag for the (m,n) pair, and $0 \le a_{m,n}(\tau) \le 1$ are weight coefficients. □

The general MARM fitting problem aims to satisfy the first two goodness-of-fit criteria in Section 1.1 (the third one is just a subjective judgment).

1. It automatically satisfies the first criterion, because every MARM process constructed from a multivariate distribution has the latter's marginal distributions by construction. This assertion follows from the following facts. First, by the Lemma 4.1 (General Iterated Uniformity), all MARM background processes are Markovian, and their

marginal distribution is uniform on [0, 1) regardless of the probability law of the innovations $\{V_n\}$ selected; furthermore, stitching transformations preserve uniformity by Lemma 3.3. And second, the Inversion Method permits us, in principle, to transform any uniform random variable U to others with arbitrary distribution F through $F^{-1}(U)$. Thus, any MARM foreground process obtained by applying the Inversion Method to a stitched background process is guaranteed to have the (given) multivariate distribution, regardless of the innovation sequence $\{V_n\}$ and stitching parameter ξ selected.

2. Minimizing the objective function in Eq. (7.1) satisfies the second criterion.

Our solution approach to Problem 7.1 is the *CSLO* (*Comprehensive Search Local Optimization*), algorithm, which generalizes the GSLO algorithm in [Jelenkovic and Melamed (1995a, 1995b)] from ARM processes to MARM processes. As the name suggests, CSLO consists of two sequential stages:

1. **Comprehensive Search Stage.** This stage performs a comprehensive search at some prescribed granularity over pairs $(f_{\{V_n\}}, \xi)$, which results in a set of B "best candidate models" (those with the smallest values of the associated objective function). This set provides $|B|$ "promising" initial models as input to the local optimization of the next stage.
2. **Local Optimization Stage.** This stage performs optimization on each initial model by minimizing its objective function, using the steepest descent method. This optimization requires the computation of the derivatives of the objective function with respect to the aforementioned search parameters.

8. The MARM Forecasting Methodology

In this section, we describe the MARM forecasting methodology which obtains of point estimators and confidence intervals of foreground MARM processes. More specifically, for each $1 \le k \le N$, time index $j \ge 0$ and lag $\tau > 0$, and given the observed history $\{\vec{X}_i^{(k)} : i \le j\}$,

- The point estimator of $X_{j+\tau}^{(k)}$ given $\vec{X}_j^{(k)}$ is an estimator of $\mathbf{E}[X_{j+\tau}^{(k)} \mid \vec{X}_j^{(k)}]$.
- The $1-\alpha$ confidence interval for $X_{j+\tau}^{(k)}$ is computed from an estimator of $F_{X_{j+\tau}^{(k)}|\vec{X}_j^{(k)}}$

To this end, the MARM forecasting methodology makes use estimators of conditional densities of the form $f_{X_{j\pm\tau}^{(k)}|\vec{X}_j^{(k)}}$ and $f_{X_{j\pm\tau}^{(k)}|U_j}$, $1 \le k \le N$. Note that the densities $f_{X_{j-\tau}^{(k)}|\vec{X}_j^{(k)}}$ and $f_{X_{j-\tau}^{(k)}|U_j}$ pertain to time-reversed MARM-related processes [Kelly (1979)].

8.1. Point Estimators and Confidence Intervals for MARM Foreground Processes

In this section, we describe the development of point estimators and confidence intervals for foreground processes using estimated densities, denoted by $\hat{f}_{X_{j+\tau}^{(k)}|\vec{X}_j^{(k)}}$, $1 \le k \le N$. Note that for $\tau \ne 0$ the construction of MARM processes ensures that

$$\hat{f}_{X_{j+\tau}^{(k)}|\vec{X}_j^{(k)}} = \hat{f}_{X_{j+\tau}^{(k)}|X_j^{(1)}}, \; 1 \le k \le N. \tag{8.1}$$

To this end, we need to supplement conditioning on $\vec{X}_j^{(k)}$ by conditioning on U_j as an intermediary step. However, while $X_j^{(1)}$ and U_j are related by Eq. (4.8), the former does not determine uniquely the latter. The problem stems from the fact that the stitching transformation

(3.12) is not one-to-one; more specifically, for a given stitched value, $F_{X^{(1)}}(x_j^{(1)})$, there are generally two corresponding background values, $u_j^{(1)}$ and $u_j^{(2)}$, which solve Eq. (3.12), namely,

$$u_j^{(1)} = \xi\, F_{X^{(1)}}(x_j^{(1)}), \qquad u_j^{(2)} = 1-(1-\xi)\, F_{X^{(1)}}(x_j^{(1)})\,. \tag{8.2}$$

Thus, we need additional probabilistic information pertaining to the two solutions above. To this end, we postulate a simple discrete conditional distribution for the two solutions, say,

$$\mathbf{Pr}\{U_j = u_j^{(1)} \mid X_j^{(1)} = x_j^{(1)}\} = p^{(k)} \text{ and } \mathbf{Pr}\{U_j = u_j^{(2)} \mid X_j^{(1)} = x_j^{(1)}\} = 1-p^{(k)}, \tag{8.3}$$

where the $0 \le p^{(k)} \le 1$, $1 \le k \le N$, are user-selected parameters to be referred to as the *mixing parameter* in the k-th dimension, and their selection is described in Section 8.2. Accordingly, in view of Eq. (8.1), the MARM forecasting methodology uses a convex (probabilistic) combination of transition densities of the form

$$\hat{f}_{X_{j\pm\tau}^{(k)}|X_j^{(1)}}(y \mid x) = p^{(k)}\, f_{X_{j\pm\tau}^{(k)}|U_j}(y \mid u_j^{(1)}) + (1-p^{(k)})\, f_{X_{j\pm\tau}^{(k)}|U_j}(y \mid u_j^{(2)}) \tag{8.4}$$

where the densities $f_{X_{j\pm\tau}^{(k)}|U_j}(y \mid u)$ are given in Eqs. (5.17), (5.18), (5.37) and(5.38).

Let $\hat{\mathbf{E}}[X_{j\pm\tau}^{(k)} \mid \vec{X}_j^{(k)}]$ be the conditional expectation induced by the conditional density $\hat{f}_{X_{j\pm\tau}^{(k)}|\vec{X}_j^{(k)}}(y \mid x)$ from Eq. (8.4). The following proposition provides the structure of the requisite point estimators.

Proposition 8.1

For $\tau > 0$, $j \ge 0$ and $\vec{x}^{(k)} \in S^{(1)} \times \cdots \times S^{(k)}$, $1 \le k \le N$,

$$\begin{aligned} &\hat{\mathbf{E}}[X_{j\pm\tau}^{(k)} \mid \vec{X}_j^{(k)} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j\pm\tau}^{(k)+} \mid X_j^{(1)+} = x_j^{(1)}] = \\ &\qquad p^{(k)}\, \mathbf{E}[X_{j\pm\tau}^{(k)} \mid U_j = u_j^{(1)}] + (1-p^{(k)})\, \mathbf{E}[X_{j\pm\tau}^{(k)} \mid U_j = u_j^{(2)}] \end{aligned} \tag{8.5}$$

Proof.

Immediate from Eqs. (8.1) and (8.4). □

The next two propositions present computational formulas for the point estimators $\hat{\mathbf{E}}[X_{j\pm\tau}^{(k)+} \mid \vec{X}_j^{(k)+}]$ and $\hat{\mathbf{E}}[X_{j\pm\tau}^{(k)-} \mid \vec{X}_j^{(k)-}]$.

Proposition 8.2

For $\tau > 0$, $j \geq 0$ and $\vec{x}^{(k)} \in S^{(1)} \times \cdots \times S^{(k)}$, $1 \leq k \leq N$,

$$\hat{\mathbf{E}}[X_{j+\tau}^{(k)+} \mid \vec{X}_j^{(k)+} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j+\tau}^{(k)+} \mid X_j^{(1)+} = x_j^{(1)}] =$$

$$\mu_{X^{(k)}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)\left(p^{(k)} e^{-i2\pi\nu u_j^{(1)}} + (1-p^{(k)})\, e^{-i2\pi\nu u_j^{(2)}}\right)\times\right.$$

$$\left.\int_{S^{(k)}} \cdots \int_{S^{(1)}} x^{(k)}\, f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(x^{(1)})}\right) d\vec{x}^{(k)}\right] \tag{8.6}$$

$$\hat{\mathbf{E}}[X_{j-\tau}^{(k)+} \mid \vec{X}_j^{(k)+} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j-\tau}^{(k)+} \mid X_j^{(1)+} = x_j^{(1)}] =$$

$$\mu_{X^{(k)}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\, j}}(i2\pi\nu)\left(p^{(k)} e^{i2\pi\nu u_j^{(1)}} + (1-p^{(k)}) e^{i2\pi\nu u_j^{(2)}}\right)\times\right.$$

$$\left.\int_{S^{(k)}} \cdots \int_{S^{(1)}} x^{(k)} f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left(\xi\, e^{-i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(x^{(1)})}\right) d\vec{x}^{(k)}\right] \tag{8.7}$$

Proof.

It follows from Eqs. (8.5) and (5.17) that

$$\hat{\mathbf{E}}[X_{j+\tau}^{(k)+} \mid \vec{X}_j^{(k)+}] = \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu) \int_{S^{(k)}} \cdots \int_{S^{(1)}} x^{(k)}\, f_{\vec{X}^{(k)}}(\vec{x}^{(k)}) \times$$

$$\left\{ p^{(k)} \left(\xi\, e^{i2\pi\nu\left[\xi F_{X^{(1)}}(x^{(1)}) - u_j^{(1)}\right]} + (1-\xi)\, e^{-i2\pi\nu\left[(1-\xi) F_{X^{(1)}}(x^{(1)}) + u_j^{(1)}\right]} \right) + \right.$$

$$\left. (1-p^{(k)}) \left(\xi\, e^{i2\pi\nu\left[\xi F_{X^{(1)}}(x^{(1)}) - u_j^{(2)}\right]} + (1-\xi)\, e^{-i2\pi\nu\left[(1-\xi) F_{X^{(1)}}(x^{(1)}) + u_j^{(2)}\right]} \right) \right\} d\vec{x}^{(k)} \tag{8.8}$$

Eq. (8.6) is the reduced form of Eq.(8.8) with the aid of Lemma 5.1 and Lemma 5.2. Eq.(8.7) is proved in a similar way. □

Proposition 8.3

Let $\tau > 0$, $j \geq 0$ and $\vec{x}^{(k)} \in S^{(1)} \times \cdots \times S^{(k)}$, $1 \leq k \leq N$.

(a.1) For j even and τ even,

$$\hat{\mathbf{E}}[X_{j+\tau}^{(k)-} \mid \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j+\tau}^{(k)-} \mid X_j^{(1)-} = x_j^{(1)}] =$$

$$\mu_{X^{(k)}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu) \left(p^{(k)} e^{-i2\pi\nu u_j^{(1)}} + (1-p^{(k)})\, e^{-i2\pi\nu\, u_j^{(2)}} \right) \times \right.$$

$$\left. \int_{S^{(k)}} \cdots \int_{S^{(1)}} x^{(k)} f_{\vec{X}^{(k)}}(\vec{x}^{(k)}) \left(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{-i2\pi\nu(1-\xi) F_{X^{(1)}}(x^{(1)})} \right) d\vec{x}^{(k)} \right] \tag{8.9}$$

(a.2) For j even and τ odd,

$$\hat{\mathbf{E}}[X_{j+\tau}^{(k)-} \mid \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j+\tau}^{(k)-} \mid X_j^{(1)-} = x_j^{(1)}] =$$

$$\mu_{X^{(k)}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu) \left(p^{(k)} e^{-i2\pi\nu u_j^{(1)}} + (1-p^{(k)})\, e^{-i2\pi\nu u_j^{(2)}} \right) \times \right.$$

$$\left. \int_{S^{(k)}} \cdots \int_{S^{(1)}} x^{(k)} f_{\vec{X}^{(k)}}(\vec{x}^{(k)}) \left(\xi\, e^{-i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{i2\pi\nu(1-\xi) F_{X^{(1)}}(x^{(1)})} \right) d\vec{x}^{(k)} \right] \tag{8.10}$$

(a.3) For j odd and τ even,

$$\hat{\mathbf{E}}[X_{j+\tau}^{(k)-} \mid \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j+\tau}^{(k)-} \mid X_j^{(1)-} = x_j^{(1)}] =$$

$$\mu_{X^{(k)}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\Bigg[\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)\Big(p^{(k)}\, e^{i2\pi\nu u_j^{(1)}} + (1-p^{(k)})\, e^{i2\pi\nu u_j^{(2)}}\Big)\times$$

$$\int_{S^{(k)}} \cdots \int_{S^{(1)}} x^{(k)} f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\Big(\xi\, e^{-i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(x^{(1)})}\Big) d\vec{x}^{(k)}\Bigg] \tag{8.11}$$

(a.4) For j odd and τ odd,

$$\hat{\mathbf{E}}[X_{j+\tau}^{(k)-} \mid \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j+\tau}^{(k)-} \mid X_j^{(1)-} = x_j^{(1)}] =$$

$$\mu_{X^{(k)}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\Bigg[\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)\Big(p^{(k)} e^{i2\pi\nu u_j^{(1)}} + (1-p^{(k)})\, e^{i2\pi\nu u_j^{(2)}}\Big)\times$$

$$\int_{S^{(k)}} \cdots \int_{S^{(1)}} x^{(k)} f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\Big(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(x^{(1)})}\Big) d\vec{x}^{(k)}\Bigg] \tag{8.12}$$

(b.1) For j even and τ even,

$$\hat{\mathbf{E}}[X_{j-\tau}^{(k)-} \mid \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j-\tau}^{(k)-} \mid X_j^{(1)-} = x_j^{(1)}] =$$

$$\mu_{X^{(k)}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\Bigg[\tilde{f}_{S_{j-\tau+1,\, j}}(i2\pi\nu)\Big(p^{(k)} e^{i2\pi\nu u_j^{(1)}} + (1-p^{(k)})\, e^{i2\pi\nu\, u_j^{(2)}}\Big)\times$$

$$\int_{S^{(k)}} \cdots \int_{S^{(1)}} x^{(k)} f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\Big(\xi\, e^{-i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(x^{(1)})}\Big) d\vec{x}^{(k)}\Bigg] \tag{8.13}$$

(b.2) For j even and τ odd,

$$\hat{\mathbf{E}}[X_{j-\tau}^{(k)-} \mid \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j-\tau}^{(k)-} \mid X_j^{(1)-} = x_j^{(1)}] =$$

$$\mu_{X^{(k)}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\left(p^{(k)}e^{i2\pi\nu u_j^{(1)}} + (1-p^{(k)})\,e^{i2\pi\nu u_j^{(2)}}\right)\times\right.$$

$$\left.\int_{S^{(k)}}\cdots\int_{S^{(1)}} x^{(k)} f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(x^{(1)})}\right)d\vec{x}^{(k)}\right] \tag{8.14}$$

(b.3) For j odd and τ even,

$$\hat{\mathbf{E}}[X_{j-\tau}^{(k)-} \mid \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j-\tau}^{(k)-} \mid X_j^{(1)-} = x_j^{(1)}] =$$

$$\mu_{X^{(k)}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\left(p^{(k)}e^{-i2\pi\nu u_j^{(1)}} + (1-p^{(k)})e^{-i2\pi\nu u_j^{(2)}}\right)\times\right.$$

$$\left.\int_{S^{(k)}}\cdots\int_{S^{(1)}} x^{(k)}\, f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(x^{(1)})}\right)d\vec{x}^{(k)}\right] \tag{8.15}$$

(b.4) For j odd and τ odd,

$$\hat{\mathbf{E}}[X_{j-\tau}^{(k)-} \mid \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j-\tau}^{(k)-} \mid X_j^{(1)-} = x_j^{(1)}] =$$

$$\mu_{X^{(k)}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\left(p^{(k)}e^{-i2\pi\nu u_j^{(1)}} + (1-p^{(k)})e^{-i2\pi\nu u_j^{(2)}}\right)\times\right.$$

$$\left.\int_{S^{(k)}}\cdots\int_{S^{(1)}} x^{(k)} f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left(\xi\, e^{-i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(x^{(1)})}\right)d\vec{x}^{(k)}\right] \tag{8.16}$$

Proof.

Similar to the proof in Proposition 8.2. □

For a prescribed confidence level α, a $1-\alpha$ confidence interval for $X_{j+\tau}^{(k)}$ about its point estimator, $\hat{\mathbf{E}}[X_{j+\tau}^{(k)} \mid \vec{X}_j^{(k)}]$, could be obtained in several ways. One approach is to use equal-width two-sided confidence interval of the form

$$\left[\hat{\mathbf{E}}[X_{j+\tau}^{(k)} \mid \vec{X}_j^{(k)}]-\gamma,\ \hat{\mathbf{E}}[X_{j+\tau}^{(k)} \mid \vec{X}_j^{(k)}]+\gamma\right]=\left[\hat{\mathbf{E}}[X_{j+\tau}^{(k)} \mid X_j^{(1)}]-\gamma,\ \hat{\mathbf{E}}[X_{j+\tau}^{(k)} \mid X_j^{(1)}]+\gamma\right],$$

where $\mathbf{Pr}\left\{\left|X_{j+\tau}^{(k)}-\hat{\mathbf{E}}[X_{j+\tau}^{(k)} \mid \vec{X}_j^{(k)}]\right|>\gamma\right\}=\mathbf{Pr}\left\{\left|X_{j+\tau}^{(k)}-\hat{\mathbf{E}}[X_{j+\tau}^{(k)} \mid X_j^{(1)}]\right|>\gamma\right\}=\alpha$.

Another approach is to use a symmetric confidence interval, about the point estimator. In this thesis, we shall adopt the first approach and compute the requisite $1-\alpha$ two-sided equal-width confidence interval via a line search utilizing the estimated cdf $\hat{F}_{X_{j+\tau}^{(k)}|X_j^{(1)}}(y \mid x)$, which can be readily computed from Eq. (8.4).

8.2. Selection of the Mixing Parameter

The selection of the mixing parameter takes advantage of time reversal [Kelly (1979), Jagerman and Melamed (1995)]. This approach exploits the fact that the Markov property of background MARM processes is preserved under time reversal, and the transition density of the reversed process can be readily computed, as shown in Chapter 5.

We can construct estimated time-reversed conditional expectations of the form

$$\hat{\mathbf{E}}\left[X_{j-\tau}^{(k)} \mid \vec{X}_j^{(k)}=\vec{x}\right]=p^{(k)}\,\mathbf{E}\left[X_{j-\tau}^{(k)} \mid U_j=u_j^{(1)}\right]+(1-p^{(k)})\,\mathbf{E}\left[X_{j-\tau}^{(k)} \mid U_j=u_j^{(2)}\right]. \quad (8.17)$$

(see Proposition 8.1). To obtain a mixing parameter $p^{(k)}$ for each dimension k, $1 \le k \le N$, and each time index j, we minimize the sum of the squared deviations of the estimator of the

conditional expectation $\hat{\mathbf{E}}\left[X_{j-\tau}^{(k)} \mid \vec{X}_j^{(k)} = \vec{\hat{x}}_j^{(k)}\right] = \hat{\mathbf{E}}\left[X_{j-\tau}^{(k)} \mid X_j^{(1)} = \hat{x}_j^{(1)}\right]$ from the observed history data $\{\vec{\hat{x}}_j^{(k)}\}$. That is, we minimize the following objective function

$$g_j(p^{(k)}) = \sum_{\tau=1}^{N} w_\tau^{(k)} \left[p^{(k)}\, \mathbf{E}[X_{j-\tau}^{(k)} \mid U_j = u_j^{(1)}] + (1 - p^{(k)})\, \mathbf{E}[X_{j-\tau}^{(k)} \mid U_j = u_j^{(2)}] - \hat{x}_{j-\tau}^{(k)} \right]^2 \quad (8.18)$$

where the $w_\tau^{(k)}$ are weights and $\hat{x}_{j-\tau}^{(k)}$ is the observed value of $X_{j-\tau}^{(k)}$. The minimizer, $p^{(k)}$, of Eq. (8.18) is given by the following proposition.

Proposition 8.4

For $1 \le k \le N$, $\tau > 0$ and $j \ge \tau$,

$$p^{(k)} = \frac{\sum_{\tau=1}^{N} w_\tau^{(k)} \left(\mathbf{E}[X_{j-\tau}^{(k)} \mid U_j = u_j^{(1)}] - \mathbf{E}[X_{j-\tau}^{(k)} \mid U_j = u_j^{(2)}]\right)\left(\hat{x}_{j-\tau}^{(k)} - \mathbf{E}[X_{j-\tau}^{(k)} \mid U_j = u_j^{(2)}]\right)}{\sum_{\tau=1}^{N} w_\tau^{(k)} \left(\mathbf{E}[X_{j-\tau}^{(k)} \mid U_j = u_j^{(1)}] - \mathbf{E}[X_{j-\tau}^{(k)} \mid U_j = u_j^{(2)}]\right)^2} \quad (8.19)$$

where $u_j^{(1)}$ and $u_j^{(2)}$ are given by Eq. (8.2).

Proof.

Standard minimization by differentiating $g_j(p^{(k)})$ with respect to $p^{(k)}$, setting the derivative to zero and then solving for $p^{(k)}$. □

The minimizer obtained in Eq. (8.19) is a nominal mixing parameter in that the denominator may vanish or the ratio may not always be a legitimate probability value. Thus, the *actual mixing parameter*, $\hat{p}^{(k)}$, is selected as follows:

1. If the denominator of $p^{(k)}$ in Eq. (8.19) vanishes, then select any probability value of $\hat{p}^{(k)}$, say, $\hat{p}^{(k)} = 1$.

2. Otherwise, if $\boldsymbol{p}^{(k)}$ in Eq. (8.19) is negative, then set $\hat{\boldsymbol{p}}^{(k)} = 0$; and if it exceeds 1, then set $\hat{\boldsymbol{p}}^{(k)} = 1$.

3. In all other cases, set $\hat{\boldsymbol{p}}^{(k)}$ to $\boldsymbol{p}^{(k)}$ of Eq. (8.19).

9. Empirically-Based MARM Processes

In this chapter, we specialize the general class of MARM processes to a practical sub-class suitable for modeling of empirical vector-valued time series. Specifically, we shall consider MARM processes with iid step-function innovation densities and hyper-step distortions based on an empirical multi-dimensional histogram.

9.1. Preliminaries

Recall that the state space of an *N*-dimensional MARM process $\{\vec{X}_n\}_{n=0}^{\infty}$ is the Cartesian product $S = S^{(1)} \times \cdots \times S^{(N)}$. For simplicity we assume that the constituent state spaces are $S^{(k)} = [l^{(k)}, r^{(k)}]$, $k = 1, 2, \cdots, N$. For each $k = 1, 2, \cdots, N$, $S^{(k)}$ is partitioned as $S^{(k)} = \bigcup_{j=1}^{I_k} S_j^{(k)}$, where

$$S_j^{(k)} = \begin{cases} [l_j^{(k)},\ r_j^{(k)}), & 1 \le j < I_k \\ [l_j^{(k)},\ r_j^{(k)}], & j = I_k \end{cases}$$

and each $S_j^{(k)}$ will be referred to as a *step*, and I_k is the corresponding number of steps in $S^{(k)}$. We denote the width of step $S_j^{(k)}$ by $d_i^{(k)} = r_i^{(k)} - l_i^{(k)}$. In a similar vein, a *hyper-step* of dimension k, $1 \le k \le N$, is a is a hyper-cube of the form,

$$\eta_k(i_1, \cdots, i_k) = S_{i_1}^{(1)} \times \cdots \times S_{i_k}^{(k)}, \quad 1 \le i_j \le I_j,\ 1 \le j \le k, \tag{9.1}$$

and its volume is denoted by $V(i_1, \ldots, i_k) = d_{i_1}^{(1)} \times d_{i_2}^{(2)} \times \cdots \times d_{i_k}^{(k)}$.

Let $p(i_1,\cdots,i_N)$, $1 \le i_k \le I_k$, $1 \le k \le N$, be an N-dimensional probability mass function, defined over all hyper-cubes of dimension N. For every $1 \le i_k \le I_k$, $1 \le k \le N$, define the *k-marginal mass functions*

$$M(i_1,\cdots,i_k) = \begin{cases} \sum_{i_{k+1}=1}^{I_{k+1}} \cdots \sum_{i_N=1}^{I_N} p(i_1,\ldots,i_N), & 1 \le k < N \\ p(i_1,\ldots,i_N), & k = N \end{cases} \tag{9.2}$$

and the *k-cumulative mass functions*

$$\begin{aligned} Q(i_1) &= \sum_{j=1}^{i_1} M(j), \quad k = 1 \\ Q_{i_1,\cdots,i_{k-1}}(i_k) &= \sum_{j=1}^{i_k} M(i_1,\ldots,i_{k-1},j), \quad 1 < k \le N \end{aligned} \tag{9.3}$$

Further, for $1 \le i_k \le I_k$, $2 \le k \le N$, define two more auxiliary functions to be used later

$$Z_L(i_1,\ldots,i_k) = \begin{cases} 0, & \text{if } M(i_1,\ldots,i_{k-1}) = 0 \\ Q_{i_1,\ldots,i_{k-1}}(i_k - 1) \,/\, M(i_1,\ldots,i_{k-1}), & \text{otherwise} \end{cases} \tag{9.4}$$

$$Z_R(i_1,\ldots,i_k) = \begin{cases} 0, & \text{if } M(i_1,\ldots,i_{k-1}) = 0 \\ Q_{i_1,\ldots,i_{k-1}}(i_k) \,/\, M(i_1,\ldots,i_{k-1}), & \text{otherwise} \end{cases} \tag{9.5}$$

Finally, The following family of sets will be used frequently in indicator functions for $1 \le i_k \le I_k$, $1 \le k \le N$,

$$\begin{aligned} C(i_1,\ldots,i_k) &= \begin{cases} [\xi Q(i_1 - 1), \xi Q(i_1)) \cup [1-(1-\xi)Q(i_1),\ 1-(1-\xi)Q(i_1 - 1)), & k = 1 \\ C(i_1,\ldots,i_{k-1}) \times [L(i_1,\ldots,i_k),\ R(i_1,\ldots,i_k)), & 1 < k \le N \end{cases} \\ &= \begin{cases} [\xi Q(i_1 - 1), \xi Q(i_1)) \cup [1-(1-\xi)Q(i_1),\ 1-(1-\xi)Q(i_1 - 1)), & k = 1 \\ C(i_1,\ldots,i_{k-1}) \times \left[\dfrac{Q_{i_1,\ldots,i_{k-1}}(i_k - 1)}{M(i_1,\ldots,i_{k-1})},\ \dfrac{Q_{i_1,\ldots,i_{k-1}}(i_k)}{M(i_1,\ldots,i_{k-1})} \right), & 1 < k \le N \end{cases} \end{aligned} \tag{9.6}$$

By construction, the stationary joint pdf of a MARM process $\{\vec{X}_n\}_{n=0}^{\infty}$ is the hyper-step density

$$f_{\vec{X}}(\vec{x}) = \sum_{i_1=1}^{I_1} \cdots \sum_{i_N=1}^{I_N} 1_{\eta_N(i_1,\cdots,i_N)}(\vec{x}) \frac{p(i_1,\ldots,i_N)}{V(i_1,\ldots,i_N)}, \quad \vec{x} \in S \,. \tag{9.7}$$

From the above hyper-step density, we can readily compute all lower-dimension joint pdfs,

$$\begin{aligned} f_{\vec{X}^{(k)}}(\vec{x}^{(k)}) &= \int_{S^{(k+1)}} \cdots \int_{S^{(N)}} \sum_{i_1=1}^{I_1} \cdots \sum_{i_N=1}^{I_N} 1_{\eta_N(i_1,\cdots,i_N)}(\vec{x}) \frac{p(i_1,\ldots,i_N)}{V(i_1,\ldots,i_N)} dx^{(k+1)} \cdots dx^{(N)} \\ &= \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} 1_{\eta_k(i_1,\cdots,i_k)}(\vec{x}^{(k)}) \frac{M(i_1,\ldots,i_k)}{d_{i_1}^{(1)} \times \cdots \times d_{i_k}^{(k)}}, \quad 1 \le k < N \end{aligned} \tag{9.8}$$

Lemma 9.1

Let $u \in [0,1)$, $x^{(k)} \in S^{(k)}$, and $\vec{x}^{(k)} \in S^{(1)} \times \cdots \times S^{(k)}$, $1 \le k \le N$.

(a) For $k = 1$,

$$f_{X^{(1)}}(x^{(1)}) = \sum_{j=1}^{I_1} 1_{S_j^{(1)}}(x^{(1)}) \frac{M(j)}{d_j^{(1)}} \tag{9.9}$$

$$F_{X^{(1)}}(x^{(1)}) = \sum_{j=1}^{I_1} 1_{S_j^{(1)}}(x^{(1)}) \left[Q(j-1) + (x^{(1)} - l_j^{(1)}) M(j) \,/\, d_j^{(1)} \right] \tag{9.10}$$

$$F_{X^{(1)}}^{-1}(S_\xi(u)) = \sum_{j=1}^{I_1} 1_{C(j)}(u) \left[l_j^{(1)} + (u - Q(j-1)) d_j^{(1)} \,/\, M(j) \right] \tag{9.11}$$

(b) For $1 < k \le N$,

$$\begin{aligned} & f_{X^{(k)}|\vec{X}^{(k-1)}}(x^{(k)} \mid \vec{x}^{(k-1)}) = \\ & \sum_{i_1=1}^{I_1} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} 1_{\eta_{k-1}(i_1,\cdots,i_{k-1})}(\vec{x}^{(k-1)}) \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \frac{M(i_1,\ldots,i_{k-1},j)}{M(i_1,\ldots,i_{k-1})\, d_j^{(k)}} \end{aligned} \tag{9.12}$$

$$F_{X^{(k)}|\vec{X}^{(k)}}(x^{(k)} \mid \vec{x}^{(k-1)}) = \sum_{i_1=1}^{I_1} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} 1_{\eta_{k-1}(i_1,\cdots,i_{k-1})}(\vec{x}^{(k-1)}) \sum_{j=1}^{I_k} \left[Z_L(i_1,\ldots,i_{k-1},j) + (x^{(k)} - l_j^{(k)}) \frac{M(i_1,\ldots,i_{k-1},j)}{M(i_1,\ldots,i_{k-1})\, d_j^{(k)}} \right] \tag{9.13}$$

$$F^{-1}_{X^{(k)}|\vec{X}^{(k-1)}}(u \mid \vec{x}^{(k-1)}) = \sum_{i_1=1}^{I_1} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} 1_{\eta_{k-1}(i_1,\cdots,i_k)}(\vec{x}^{(k-1)}) \times \tag{9.14}$$

$$\sum_{j=1}^{I_k} 1_{[Z_L(i_1,\ldots,i_{k-1},j),\, Z_R(i_1,\ldots,i_{k-1},j))}(u) \left[l_j^{(k)} + [u - Z_L(i_1,\ldots,i_{k-1},j)] \frac{M(i_1,\ldots,i_{k-1})\, d_j^{(k)}}{M(i_1,\ldots,i_{k-1},j)} \right]$$

Proof.

Follows by the definition of hyper-step distributions in Eq. (9.7). □

In practice, a multi-dimensional step-function density is often estimated from an empirical sample via an empirical multi-dimensional histogram. That is, for an empirical sequence $\{\vec{y}_j\}_{j=0}^{J-1}$, where each observation $\vec{y}_j$ is an N-dimensional vector, let $\hat{H}$ be a corresponding empirical multi-dimensional histogram. For $1 \le i_k \le I_k$, $1 \le k \le N$, let $n(i_1,\ldots,i_N)$ be the number of vector observations from $\{\vec{y}_j\}_{j=0}^{J-1}$ that fall into hyper-cube $\eta_N(i_1,\ldots,i_N)$, and let the corresponding *relative frequency function* be denoted by $\hat{p}(i_1,\ldots,i_N) = \dfrac{n(i_1,\ldots,i_N)}{J}$. The corresponding empirical density $f_{\vec{X}}$ in Eq. (9.7) is then estimated by a *hyper-histogram* pdf of the form

$$\hat{h}(\vec{x}) = \sum_{i_1=1}^{I_1} \cdots \sum_{i_N=1}^{I_N} 1_{\eta_N(i_1,\cdots,i_N)}(\vec{x}) \frac{\hat{p}(i_1,\ldots,i_N)}{V(i_1,\ldots,i_N)}, \quad \vec{x} \in S. \tag{9.15}$$

Accordingly, the corresponding formulas in Lemma 9.1 are then obtained by substituting $\hat{p}(i_1,\ldots,i_N)$ for $p(i_1,\ldots,i_N)$. In particular, the *marginal relative frequencies* $\hat{p}_i^{(k)}$ of cell $S_i^{(k)}$ be given by $\hat{p}_i^{(k)} = n_i^{(k)} / \sum_{j=1}^{I_k} n_j^{(k)}$, where $n_i^{(k)}$ is the number of scalar observations from $\{\vec{y}_j\}_{j=0}^{J-1}$ that fall into cell $S_i^{(k)}$, and the corresponding *marginal cumulative frequencies* are given by $\hat{Q}_i^{(k)} = \sum_{j=1}^{i} \hat{p}_j^{(k)}$. Thus, these definitions reduce to the univariate case as described in Section 3.3. Finally, we mention that for practical fitting of MARM models to empirical data, we shall use step-function innovation densities of the form (10.2) with Laplace transform given in Eq. (10.12) to be discussed later.

Proposition 9.1

For $0 \le u_j < 1, j \ge 0$ and $x^{(k)} \in S^{(k)}$, $1 \le k \le N$,

$$f_{X_j^{(k)}|U_j}(x^{(k)} \mid u_j) = \begin{cases} 1_{\{x^{(1)} = F^{-1}_{X^{(1)}}(S_\xi(u_j))\}}(x^{(1)}), & k = 1 \\ \sum_{i_1=1}^{I_1} 1_{\eta_1(i_1)}(F^{-1}_{X^{(1)}}(S_\xi(u_j))) \sum_{i_k=1}^{I_k} 1_{S_{i_k}^{(k)}}(x^{(k)}) \sum_{i_2=1}^{I_2} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} \dfrac{M(i_1,\cdots,i_k)}{M(i_1)\, d_{i_k}^{(k)}}, & k > 1 \end{cases} \tag{9.16}$$

where $F^{-1}_{X^{(1)}}(S_\xi(u))$ is given by Eq. (9.11).

Proof.

From Eq. (5.2) in Theorem 5.1,

$$f_{X_j^{(k)}|U_j}(x^{(k)} \mid u_j) = \begin{cases} 1_{\{x^{(1)} = F^{-1}_{X^{(1)}}(S_\xi(u_j))\}}(x^{(1)}), & k = 1 \\ \int_{S^{(k-1)}} \cdots \int_{S^{(2)}} \dfrac{f_{\vec{X}^{(k)}}\left(F^{-1}_{X^{(1)}}(S_\xi(u_j)), y^{(2)}, \ldots y^{(k-1)}, x^{(k)}\right)}{f_{X^{(1)}}\left(F^{-1}_{X^{(1)}}(S_\xi(u_j))\right)}\, dy^{(2)} \ldots dy^{(k-1)}, & k > 1 \end{cases}$$

The result now follows by evaluating the above integration and substituting there $f_{X^{(1)}}(x^{(1)})$ and $f_{\vec{X}^{(k)}}(\vec{X}^{(k)})$ with the aid of Eqs. (9.9) and (9.8), respectively. □

Proposition 9.2

For $0 \le u_j < 1, j \ge 0$ and $x^{(k)} \in S^{(k)}$, $1 \le k \le N$,

$$F_{X_j^{(k)}|U_j}(x^{(k)} \mid u_j) = \begin{cases} 1_{\left\{x^{(1)} \ge F^{-1}_{X^{(1)}}\left(S_\xi(u_j)\right)\right\}}(x^{(1)}), & k = 1 \\ \sum_{i_1=1}^{I_1} 1_{\eta_1(i_1)}\left(F^{-1}_{X^{(1)}}(S_\xi(u_j))\right) \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \sum_{i_2=1}^{I_2} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} & \\ \qquad \left(\frac{M(i_1,\cdots,i_{k-1},j)}{M(i_1)\, d_j^{(k)}}\left(x^{(k)} - l_j^{(k)}\right) + \sum_{i_k=1}^{j-1} \frac{M(i_1,\cdots,i_k)}{M(i_1)}\right), & k > 1 \end{cases} \tag{9.17}$$

where $F^{-1}_{X^{(1)}}(S_\xi(u))$ is given by Eq. (9.11).

Proof.

From Eq. (5.5) in Corollary 5.1,

$$F_{X_j^{(k)}|\,U_j}(x^{(k)} \mid u_j) = \begin{cases} 1_{\left\{x^{(1)} \ge F^{-1}_{X^{(1)}}\left(S_\xi(u_j)\right)\right\}}(x^{(1)}), & k = 1 \\ \int_{-\infty}^{x^{(k)}} \int_{S^{(k-1)}} \cdots \int_{S^{(2)}} \frac{f_{\vec{X}^{(k)}}(F^{-1}_{X^{(1)}}(S_\xi(u_j)), y^{(2)}, \ldots, y^{(k)})}{f_{X^{(1)}}(F^{-1}_{X^{(1)}}(S_\xi(u_j)))}\, dy^{(2)} \ldots dy^{(k-1)}\, dy^{(k)}, & k > 1 \end{cases}$$

The result now follows by evaluating the above integration and substituting there $f_{X^{(1)}}(x^{(1)})$ and $f_{\vec{X}^{(k)}}(\vec{X}^{(k)})$ with the aid of Eqs. (9.9) and (9.8), respectively. □

Lemma 9.2

For $1 \le j \le I_1$ and $0 \le \xi \le 1$,

$$\tilde{1}_{C(j)}(s) = \frac{e^{-s\xi Q(j)} - e^{-s\xi Q(j-1)} + e^{s(1-\xi)Q(j-1)} - e^{s(1-\xi)Q(j)}}{-s} \tag{9.18}$$

Proof.

Evaluating the Laplace transform in Eq. (9.18) with the aid of Eq. (9.6) yields

$$\tilde{1}_{C(j)}(s) = \int_0^1 1_{C(j)}(u)\, e^{-su}\, du = \int_{\xi Q(j-1)}^{\xi Q(j)} e^{-su}\, du + \int_{1-(1-\xi)Q(j)}^{1-(1-\xi)Q(j-1)} e^{-su}\, du .$$

Eq. (9.18) follows by evaluating the two integrals above. □

In this section we use the shorthand notation $d\vec{w}_\tau^{(k)} = dw_\tau^{(1)} \cdots dw_\tau^{(k)}$, and for any stationary process $X^{(k)} = \{X_j^{(k)}\}_{j=0}^{\infty}$, we denote the common expectation and standard deviation by $\mu_{X^{(k)}}$ and $\sigma_{X^{(k)}}$, respectively.

Before proceeding with deriving computational formulas for the requisite autocorrelations and cross-correlations, we introduce the following three auxiliary functions, $\mathcal{J}_1(i_1,\ldots,i_m)$, and $\mathcal{J}_2(i_1,\ldots,i_m)$, $1 \le i_j \le I_j$, $1 \le j \le m,\ 1 < m \le N$, given by

$$\mathcal{J}_1(i_1,\ldots,i_m) = \int_0^1 \cdots \int_0^1 1_{[Z_L(i_1,i_2),Z_R(i_1,i_2))\times\cdots\times[Z_L(i_1,\ldots,i_{m-1}),Z_R(i_1,\ldots,i_{m-1}))}(\vec{w}^{(m-2)})\, d\vec{w}^{(m-2)},\ m > 2 \tag{9.19}$$

$$\mathcal{J}_2(i_1,\ldots,i_m) = \int_0^1 1_{\left[Z_L(i_1,\ldots,i_m),Z_R(i_1,\ldots,i_m)\right)}(w^{(m-1)})\times \left(l_{i_m}^{(m)} + \left(w^{(m-1)} - Z_L(i_1,\ldots,i_m)\right)\frac{M(i_1,\ldots,i_{m-1})\,d_{i_m}^{(m)}}{M(i_1,\ldots,i_m)}\right)dw^{(m-1)} \tag{9.20}$$

The following lemma provides alternative representations for $\mathcal{J}_1$ and $\mathcal{J}_2$.

Lemma 9.3

(a) For $2 < m \le N$, $1 \le i_k \le I_k$, $1 \le k \le m$,

$$\mathcal{J}_1(i_1,\ldots,i_m) = \begin{cases} 0, & \text{if at least one of } M(i_1,\ldots,i_k) \text{ is 0, } 1 \le k < m \\ M(i_1,\ldots,i_{m-1}) \,/\, M(i_1), & \text{otherwise} \end{cases} . \tag{9.21}$$

(b) For $1 < m \le N$, $1 \le i_k \le I_k$, $1 \le k \le m$,

$$\mathcal{J}_2(i_1,\ldots,i_m) = \begin{cases} 0, & \text{if at least one of } M(i_1,\ldots,i_k) \text{ is 0, } 1 \le k \le m \\ \left(l_{i_m}^{(m)} + d_{i_m}^{(m)} \,/\, 2\right) M(i_1,\ldots,i_m) \,/\, M(i_1,\ldots,i_{m-1}), & \text{otherwise} \end{cases} \tag{9.22}$$

Proof.

To prove part (a), we can rewrite the multiple integral in Eq.(9.19), noting the independence of the integrals, as a product of single integrals, yielding

$$\begin{aligned}\mathcal{J}_1(i_1,\ldots,i_m) &= \int_{Z_L(i_1,i_2)}^{Z_R(i_1,i_2)} dw^{(1)} \times\cdots\times \int_{Z_L(i_1,\ldots,i_{m-1})}^{Z_R(i_1,\ldots,i_{m-1})} dw^{(m-2)} \\ &= \left[Z_R(i_1,i_2) - Z_L(i_1,i_2)\right]\times\cdots\times\left[Z_R(i_1,\ldots,i_{m-1}) - Z_L(i_1,\ldots,i_{m-1})\right]\end{aligned}$$

Eq. (9.21) follows from the representation above with the aid of Eqs. (9.3), (9.4) and (9.5).

Finally, to prove part (b), the independence of the integrals in Eq. (9.20) allows us to write

$$\mathcal{J}_2(i_1,\ldots,i_m) =$$
$$= \int_{Z_L(i_1,\ldots,i_m)}^{Z_R(i_1,\ldots,i_m)} \left[l_{i_m}^{(m)} + \left(w^{(m-1)} - Z_L(i_1,\ldots,i_m) \right) \frac{M(i_1,\ldots,i_{m-1})\, d_{i_m}^{(m)}}{M(i_1,\ldots,i_m)} \right] dw^{(m-1)}$$

Eq. (9.22) follows by evaluating the integrals above with the aid of Eqs. (9.3), (9.4) and (9.5). □

We are now in a position to exhibit the computational representations of $\tilde{\mathcal{D}}^{(k)}(i2\pi\nu)$ and $\mathcal{D}^{(k)}(u)$ for the hyper-step distribution.

Lemma 9.4

(a) For $k = 1$, $\nu \neq 0$,

$$\begin{aligned}
&\tilde{\mathcal{D}}^{(1)}(i2\pi\nu) \\
&= \sum_{j=1}^{I_1} \left\{ \frac{l_j^{(1)} \left[e^{-i2\pi\nu\xi Q(j-1)} - e^{i2\pi\nu(1-\xi)Q(j-1)} \right]}{i2\pi\nu} + \frac{r_j^{(1)} \left[e^{i2\pi\nu(1-\xi)Q(j)} - e^{-i2\pi\nu\xi Q(j)} \right]}{i2\pi\nu} \right\} \\
&+ \sum_{j=1}^{I_1} \frac{d_j^{(1)}}{M(j)} \left\{ \frac{e^{-i2\pi\nu\xi Q(j)} - e^{-i2\pi\nu\xi Q(j-1)}}{\xi\,(2\pi\nu)^2} + \frac{e^{i2\pi\nu(1-\xi)Q(j)} - e^{i2\pi\nu(1-\xi)Q(j-1)}}{(1-\xi)\,(2\pi\nu)^2} \right\}
\end{aligned} \tag{9.23}$$

Furthermore, letting $\tilde{\mathcal{D}}^{(1)}(i2\pi\nu) = a_{\xi,\nu} + ib_{\xi,\nu}$, $\nu \neq 0$, where $a_{\xi,\nu}$ and $b_{\xi,\nu}$ are real, one has the representations

$$\begin{aligned}
a_{\xi,\nu} &= \sum_{j=1}^{I_1} \frac{r_j^{(1)}[\sin(2\pi\,\nu\,\xi\,Q(j)) + \sin(2\pi\nu(1-\xi)\,Q(j))]}{2\pi\nu} \\
&- \sum_{j=1}^{I_1} \frac{l_j^{(1)}[\sin(2\pi\,\nu\,\xi\,Q(j-1)) + \sin(2\pi\,\nu\,(1-\xi)\,Q(j-1))]}{2\pi\nu} \\
&+ \sum_{j=1}^{I_1} \frac{d_j^{(1)}}{M(j)} \left[\frac{\cos(2\pi\,\nu\,\xi\,Q(j)) - \cos(2\pi\,\nu\,\xi\,Q(j-1))}{\xi(2\pi\nu)^2} \right. \\
&\qquad \left. + \frac{\cos(2\pi\,\nu\,(1-\xi)\,Q(j)) - \cos(2\pi\,\nu\,(1-\xi)\,Q(j-1))}{(1-\xi)(2\pi\nu)^2} \right]
\end{aligned} \tag{9.24}$$

$$
\begin{aligned}
b_{\xi,\nu} = & \sum_{j=1}^{I_1} \frac{r_j^{(1)}[\cos(2\pi\,\nu\,\xi\,Q(j)) - \cos(2\pi\,\nu\,(1-\xi)\,Q(j))]}{2\pi\nu} \\
& - \sum_{j=1}^{I_1} \frac{l_j^{(1)}[\cos(2\pi\,\nu\,\xi\,Q(j-1)) - \cos(2\pi\,\nu\,(1-\xi)\,Q(j-1))]}{2\pi\nu} \\
& - \sum_{j=1}^{I_1} \frac{d_j^{(1)}}{M(j)} \left[\frac{\sin(2\pi\,\nu\,\xi\,Q(j)) - \sin(2\pi\,\nu\,\xi\,Q(j-1))}{\xi(2\pi\nu)^2} \right. \\
& \left. - \frac{\sin(2\pi\,\nu\,(1-\xi)\,Q(j)) - \sin(2\pi\,\nu\,(1-\xi)\,Q(j-1))}{(1-\xi)(2\pi\nu)^2} \right]
\end{aligned}
\tag{9.25}
$$

(b) For $1 < k \le N$ and $\nu \ne 0$

$$
\begin{aligned}
\tilde{\mathcal{D}}^{(k)}(i2\pi\nu) = & \frac{1}{-i2\pi\nu} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} \left\{ \frac{M(i_1,\ldots,i_k)}{M(i_1)} \left(l_{i_k}^{(k)} + d_{i_k}^{(k)} / 2 \right) \right. \\
& \left. \left[e^{-i2\pi\nu\xi Q(i_1)} - e^{-i2\pi\nu\xi Q(i_1-1)} + e^{i2\pi\nu(1-\xi)Q(i_1-1)} - e^{i2\pi\nu(1-\xi)Q(i_1)} \right] \right\}
\end{aligned}
\tag{9.26}
$$

Furthermore, letting $\tilde{\mathcal{D}}^{(k)}(i2\pi\nu) = a_{\xi,\nu} + ib_{\xi,\nu}$, $\nu \ne 0$, where $a_{\xi,\nu}$ and $b_{\xi,\nu}$ are real, one has the representations

$$
\begin{aligned}
a_{\xi,\nu} = & \frac{1}{2\pi\nu} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} \left\{ \frac{M(i_1,\ldots,i_k)}{M(i_1)} \left(l_{i_k}^{(k)} + d_{i_k}^{(k)} / 2 \right) \times \right. \\
& \left[\sin(2\pi\,\nu\,\xi\,Q(i_1)) - \sin(2\pi\,\nu\,\xi\,Q(i_1-1)) - \right. \\
& \left. \left. \sin(2\pi\,\nu\,(1-\xi)Q(i_1-1)) + \sin(2\pi\,\nu\,(1-\xi)\,Q(i_1)) \right] \right\}
\end{aligned}
\tag{9.27}
$$

$$
\begin{aligned}
b_{\xi,\nu} = & \frac{1}{2\pi\nu} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} \left\{ \frac{M(i_1,\ldots,i_k)}{M(i_1)} \left(l_{i_k}^{(k)} + d_{i_k}^{(k)} / 2 \right) \times \right. \\
& \left[\cos(2\pi\,\nu\,\xi\,Q(i_1)) - \cos(2\pi\,\nu\,\xi\,Q(i_1-1)) + \right. \\
& \left. \left. \cos(2\pi\,\nu\,(1-\xi)\,Q(i_1-1)) - \cos(2\pi\,\nu\,(1-\xi)\,Q(i_1)) \right] \right\}
\end{aligned}
\tag{9.28}
$$

Proof.

For $k = 1$, the proof of part (a) is identical to the one in Theorem 3 of [Melamed (1999)] by noting that $\mathcal{D}^{(1)}$ is equivalent to the distortion D_ξ in [Jagerman and Melamed (1992b)].

To prove part (b) for $1 < k \le N$, we have by definition of $\tilde{\mathcal{D}}^{(k)}(i2\pi\nu)$,

$$\tilde{\mathcal{D}}^{(k)}(i2\pi\nu) = \int_0^1 \cdots \int_0^1 D^{(k)}(u, \vec{w}^{(k-1)})\, e^{-i2\pi\nu u}\, du\, d\vec{w}^{(k-1)}$$

$$= \int_0^1 \cdots \int_0^1 e^{-i2\pi\nu u} \left\{ \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} 1_{C(i_1,\ldots,i_k)}([u, \vec{w}^{(k-1)}]) \times \right. \tag{9.29}$$

$$\left. \left[l_{i_k}^{(k)} + \left(w^{(k-1)} - L(i_1,\ldots,i_k) \right) \frac{M(i_1,\ldots,i_{k-1})\, d_{i_k}^{(k)}}{M(i_1,\ldots,i_k)} \right] \right\} du\, dw^{(1)} \cdots dw^{(k-1)}$$

where $C(i_1,\ldots,i_k)$ is defined in (9.6). Interchanging integration and summation above, and noting the independence of the integrals, we rewrite the right-hand side above as the sum of products of $\mathcal{I}_1(i_1,\ldots,i_m)$, $\mathcal{I}_2(i_1,\ldots,i_m)$, as follows

$$\tilde{\mathcal{D}}^{(k)}(i2\pi\nu) = \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} \tilde{1}_{C(i_1)}(i2\pi\nu)\, \mathcal{I}_1(i_1,\ldots,i_k)\, \mathcal{I}_2(i_1,\ldots,i_k)$$

Finally, substituting the representations of $\mathcal{I}_1$ and $\mathcal{I}_2$ from Lemma 9.3 into the above equation yields part (b). □

Lemma 9.5

For $0 \le u < 1$ and $1 \le k \le N$,

$$\mathcal{D}^{(k)}(u) = \begin{cases} \sum_{j=1}^{I_1} 1_{C(j)}(u) \left[l_j^{(1)} + (u - Q(j-1)) d_j^{(1)} \,/\, M(j) \right], & k = 1 \\ \sum_{i_1=1}^{I_1} 1_{C(i_1)}(u) \sum_{i_2=1}^{I_2} \cdots \sum_{i_k=1}^{I_k} \frac{M(i_1,\ldots,i_k)}{M(i_1)} \left(l_{i_k}^{(k)} + d_{i_k}^{(k)} \,/\, 2 \right), & 1 < k \le N \end{cases} \tag{9.30}$$

Proof.

By definition, $\mathcal{D}^{(1)}(u) = D^{(1)}(u)$, thus the first line of Eq.(9.30) follows from Eq. (5.7) by substituting there Eq. (9.11).

For $1 < k \le N$, we have by definition of $\mathcal{D}^{(k)}(u)$,

$$\begin{aligned}\mathcal{D}^{(k)}(u) &= \int_0^1 \cdots \int_0^1 D^{(k)}(u, \vec{w}^{(k-1)})\, d\vec{w}^{(k-1)} \\ &= \int_0^1 \cdots \int_0^1 \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} 1_{C(i_1,\ldots,i_k)}([u, \vec{w}^{(k-1)}]) \times \\ &\qquad \left[l_{i_k}^{(k)} + \left(w^{(k-1)} - L(i_1,\ldots,i_k) \right) \frac{M(i_1,\ldots,i_{k-1})\, d_{i_k}^{(k)}}{M(i_1,\ldots,i_k)} \right] dw^{(1)} \cdots dw^{(k-1)}\end{aligned} \tag{9.31}$$

where $C(i_1,\ldots,i_k)$ is defined in (9.6). Interchanging integration and summation above, and noting the independence of the integrals, we rewrite the right-hand side above as the sum of products of $\mathcal{I}_1(i_1,\ldots,i_m)$, $\mathcal{I}_2(i_1,\ldots,i_m)$, as follows

$$\mathcal{D}^{(k)}(u) = \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} 1_{C(i_1)}(u)\, \mathcal{I}_1(i_1,\ldots,i_k)\, \mathcal{I}_2(i_1,\ldots,i_k)$$

Finally, substituting the representations of $\mathcal{I}_1$ and $\mathcal{I}_2$ from Lemma 9.3 into the above equation yields the second line of Eq. (9.30) □

Proposition 9.3

For $0 \le u_j < 1$, $j \ge 0$, and $x^{(k)} \in S^{(k)}$, $1 \le k \le N$,

$$\mathbf{E}[X_j^{(k)} \mid U_j = u_j] = \begin{cases} \sum_{j=1}^{I_1} 1_{C(j)}(u_j) \left[l_j^{(1)} + (u_j - Q(j-1)) d_j^{(1)} / M(j) \right], & k = 1 \\ \sum_{i_1=1}^{I_1} 1_{C(i_1)}(u_j) \sum_{i_2=1}^{I_2} \cdots \sum_{i_k=1}^{I_k} \frac{M(i_1,\ldots,i_k)}{M(i_1)} \left(l_{i_k}^{(k)} + d_{i_k}^{(k)} / 2 \right), & k > 1 \end{cases} \tag{9.32}$$

Proof.

Follows from Eq. (5.10) and Lemma 9.5. □

9.2. Transition Structure of Empirically-Based MARM Processes

In this section we exhibit the transition structure of empirically-based MARM processes by specializing Theorem 5.3 and 5.5.

9.2.1 Transition Functions of Empirically-Based MARM$^+$ Processes

This section exhibits the transition structure for empirically-based MARM$^+$ foreground process $\{\vec{X}_j^+\}$.

Proposition 9.4

Let $\{\vec{X}_\tau^+\}$ be a MARM$^+$ process with hyper-step pdf defined by Eq. (9.7). Then for $\tau > 0$, $0 \le u_j < 1$, $j \ge 0$, and $\vec{x}^{(k)} \in S^{(1)} \times \cdots \times S^{(k)}$, $x^{(k)} \in S^{(k)}$, $k \ge 1$, Theorem 5.3 becomes

(a) $f_{\vec{X}_{j+\tau}^{(k)+}|U_j^+}(\vec{x}^{(k)} \mid u_j) =$

$$f_{\vec{X}^{(k)+}}(\vec{x}^{(k)})\left\{1 + 2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)\,e^{-i2\pi\nu u_j} \times \left(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})} + (1-\xi)\, e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(x^{(1)})}\right)\right]\right\} \tag{9.33}$$

(b) $f_{\vec{X}_{j-\tau}^{(k)+}|U_j^+}(\vec{x}^{(k)} \mid u_j) =$

$$f_{\vec{X}^{(k)+}}(\vec{x}^{(k)})\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\; e^{i2\pi\nu u_j}\times \left(\xi\, e^{-i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})}+(1-\xi)\, e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(x^{(1)})}\right)\right]\right\} \tag{9.34}$$

where $f_{\vec{X}^{(k)}}(\vec{x}^{(k)})$ and $F_{X^{(1)}}(x)$ are given by Eqs. (9.8) and (9.10), respectively.

(c) $f_{X_{j+\tau}^{(k)+}|U_j^+}(x^{(k)} \mid u_j) =$

$$\begin{aligned}&\sum_{i_1=1}^{I_1}\cdots\sum_{i_k=1}^{I_k}1_{S_{i_k}^{(k)}}(x^{(k)})M(i_1,\cdots,i_k)\,/\,d_{i_k}^{(k)}+\\&2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\frac{\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)}{i2\pi\nu}e^{-i2\pi\nu u_j}\sum_{i_1=1}^{I_1}\cdots\sum_{i_k=1}^{I_k}1_{S_{i_k}^{(k)}}(x^{(k)})\frac{M(i_1,\cdots,i_k)}{M(i_1)\,d_{i_k}^{(k)}}\times\right.\\&\left.\left(e^{i2\pi\nu\xi Q(i_1)}-e^{i2\pi\nu\xi\,Q(i_1-1)}-e^{-i2\pi\nu(1-\xi)\,Q(i_1)}+e^{-i2\pi\nu(1-\xi)Q(i_1-1)}\right)\right]\end{aligned} \tag{9.35}$$

(d) $f_{X_{j-\tau}^{(k)+}|U_j^+}(x^{(k)} \mid u_j) =$

$$\begin{aligned}&\sum_{i_1=1}^{I_1}\cdots\sum_{i_k=1}^{I_k}1_{S_{i_k}^{(k)}}(x^{(k)})M(i_1,\cdots,i_k)\,/\,d_{i_k}^{(k)}+\\&2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\frac{\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)}{-i2\pi\nu}e^{i2\pi\nu\,u_j}\sum_{i_1=1}^{I_1}\cdots\sum_{i_k=1}^{I_k}1_{S_{i_k}^{(k)}}(x^{(k)})\frac{M(i_1,\cdots,i_k)}{M(i_1)d_{i_k}^{(k)}}\times\right.\\&\left.\left(e^{-i2\pi\nu\xi Q(i_1)}-e^{-i2\pi\nu\xi\,Q(i_1-1)}-e^{i2\pi\nu(1-\xi)\,Q(i_1)}+e^{i2\pi\nu(1-\xi)Q(i_1-1)}\right)\right]\end{aligned} \tag{9.36}$$

Proof.

Eqs. (9.33) and (9.34) follows from Eqs. (5.15) and (5.16), Eq. (9.35) follows from integrating Eq.(9.33) and Eq. (9.36) follows from integrating Eq. (9.34). □

Corollary 9.1

For $k=1$ in Proposition 9.1, we have the following formulas as special cases:

$$f_{X^{(1)+}_{j+\tau}|U^+_j}(x \mid u) = f_{X^{(1)}}(x) + 2f_{X^{(1)}}(x)\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)\,e^{-i2\pi\nu u}\times \left(\xi\,e^{i2\pi\nu\xi F_{X^{(1)}}(x)} + (1-\xi)\,e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(x)}\right)\right] \tag{9.37}$$

$$f_{X^{(1)+}_{j-\tau}|U^+_j}(x \mid u) = f_{X^{(1)}}(x) + 2f_{X^{(1)}}(x)\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\,e^{i2\pi\nu u}\times \left(\xi\,e^{-i2\pi\nu\,\xi F_{X^{(1)}}(x)} + (1-\xi)\,e^{i2\pi\nu(1-\xi)\,F_{X^{(1)}}(x)}\right)\right] \tag{9.38}$$

where $f_{X^{(1)}}(x)$ and $F_{X^{(1)}}(x)$ are given respectively by Eqs. (9.9) and (9.10) in Lemma 9.1. □

Proposition 9.5

Let $\{\vec{X}^+_\tau\}$ be a MARM$^+$ process with hyper-step pdf defined by Eq.(9.7). Then for $\tau > 0$, $0 \le u_j < 1$, $j \ge 0$, and $k \ge 1$, Proposition 5.1 becomes

$$\begin{aligned}
&F_{X^{(k)+}_{j+\tau}|U^+_j}(x^{(k)} \mid u_j) = \\
&\sum_{i_1=1}^{I_1}\cdots\sum_{i_{k-1}=1}^{I_{k-1}}\sum_{j=1}^{I_k}1_{S_j^{(k)}}(x^{(k)})\left(\sum_{i_k=1}^{j-1}M(i_1,\cdots,i_k) + \frac{M(i_1,\cdots,i_{k-1},j)}{d_j^{(k)}}\left(x^{(k)} - l_j^{(k)}\right)\right) + \\
&2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\frac{\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)}{i2\pi\nu}e^{-i2\pi\nu u_j}\sum_{i_1=1}^{I_1}\left(e^{i2\pi\nu\xi Q(i_1)} - e^{i2\pi\nu\,\xi\,Q(i_1-1)} - e^{-i2\pi\nu\,(1-\xi)\,Q(i_1)} + e^{-i2\pi\nu(1-\xi)Q(i_1-1)}\right)\times\right. \\
&\left.\sum_{i_2=1}^{I_2}\cdots\sum_{i_{k-1}=1}^{I_{k-1}}\sum_{j=1}^{I_k}1_{S_j^{(k)}}(x^{(k)})\left(\sum_{i_k=1}^{j-1}\frac{M(i_1,\cdots,i_k)}{M(i_1)} + \frac{M(i_1,\cdots,i_{k-1},j)}{M(i_1)\,d_j^{(k)}}\left(x^{(k)} - l_j^{(k)}\right)\right)\right]
\end{aligned} \tag{9.39}$$

$$F_{X_{j-\tau}^{(k)+}|U_j^+}(x^{(k)} \mid u_j) =$$

$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \left(\sum_{i_k=1}^{j-1} M(i_1, \cdots, i_k) + \frac{M(i_1, \cdots, i_{k-1}, j)}{d_j^{(k)}} \left(x^{(k)} - l_j^{(k)} \right) \right) +$$

$$2 \sum_{\nu=1}^{\infty} \mathrm{Re} \left[\frac{\tilde{f}_{S_{j-\tau+1,\, j}}(i2\pi\nu)}{-i2\pi\nu} e^{i2\pi\nu u_j} \sum_{i_1=1}^{I_1} \left(e^{-i2\pi\nu\xi Q(i_1)} - e^{-i2\pi\nu\, \xi\, Q(i_1-1)} - \right. \right. \tag{9.40}$$

$$\left. e^{i2\pi\nu\,(1-\xi)\,Q(i_1)} + e^{i2\pi\nu(1-\xi)Q(i_1-1)} \right) \times$$

$$\left. \sum_{i_2=1}^{I_2} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \left(\sum_{i_k=1}^{j-1} \frac{M(i_1, \cdots, i_k)}{M(i_1)} + \frac{M(i_1, \cdots, i_{k-1}, j)}{M(i_1)\, d_j^{(k)}} \left(x^{(k)} - l_j^{(k)} \right) \right) \right]$$

Proof.

Follow readily by integrating Eqs. (9.35) and (9.36). □

Corollary 9.2

For $k = 1$ in Proposition 9.2, we have the following formulas as special cases:

$$F_{X_{j+\tau}^{(1)+}|U_j^+}(x \mid u) = F_{X^{(1)+}}(x) + 2 \sum_{\nu=1}^{\infty} \mathrm{Re} \left[\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu u} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \right.$$

$$\left. \left(\xi\, e^{i2\pi\nu\xi F_{X^{(1)}}(y)} + (1-\xi)\, e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(y)} \right) dy \right] \tag{9.41}$$

$$F_{X_{j-\tau}^{(1)+}|U_j^+}(x \mid u) = F_{X^{(1)+}}(x) + 2 \sum_{\nu=1}^{\infty} \mathrm{Re} \left[\tilde{f}_{S_{j-\tau+1,\, j}}(i2\pi\nu)\, e^{i2\pi\nu u} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \right.$$

$$\left. \left(\xi\, e^{-i2\pi\nu\, \xi F_{X^{(1)}}(y)} + (1-\xi)\, e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(y)} \right) dy \right] \tag{9.42}$$

where $f_{X^{(1)}}(x)$ and $F_{X^{(1)}}(x)$ are given respectively by Eqs. (9.9) and (9.10) in Lemma 9.1. □

Proposition 9.6

For $\tau > 0$, $0 \le u_j \le 1$, $j \ge 0$, and $k \ge 1$,

$$\mathbf{E}[X_{j+\tau}^{(k)+} \mid U_j^+ = u_j] = \mu_{X^{(k)+}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu u_j}\, \tilde{\mathcal{D}}^{(k)}(-i2\pi\nu)\right] \tag{9.43}$$

$$\mathbf{E}[X_{j-\tau}^{(k)+} \mid U_j^+ = u_j] = \mu_{X^{(k)+}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\, e^{i2\pi\nu u_j}\, \tilde{\mathcal{D}}^{(k)}(i2\pi\nu)\right] \tag{9.44}$$

where $\tilde{\mathcal{D}}^{(k)}(i2\pi\nu)$ and $\tilde{\mathcal{D}}^{(k)}(-i2\pi\nu)$ are given respectively by Eq. (9.23) and (9.26).

Proof.

Follow from Eqs. (5.29) and (5.30). □

9.2.2 Transition Functions of Empirically-Based MARM$^-$ Processes

This section exhibits the transition structure for empirically-based MARM$^-$ foreground process $\{\vec{X}_j^-\}$.

Proposition 9.7

Let $\{\vec{X}_\tau^-\}$ be a MARM$^-$ process with hyper-step pdf defined by Eq. (9.7). Then for $\tau > 0$, $0 \le u_j < 1$, $j \ge 0$, and $\vec{x}^{(k)} \in S^{(1)} \times \cdots \times S^{(k)}$, $x^{(k)} \in S^{(k)}$, $k \ge 1$, Theorem 5.5 becomes

(a) $f_{\vec{X}^{(k)}_{j+\tau}|U_j^-}(\vec{x}^{(k)} \mid u_j) =$

$$
\begin{cases}
f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)\,e^{-i2\pi\nu\,u_j}\times\right.\right. \\
\qquad\left.\left.\left(\xi\,e^{i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})}+(1-\xi)\,e^{-i2\pi\nu(1-\xi)\,F_{X^{(1)}}(x^{(1)})}\right)\right]\right\}, \quad j \text{ even, } \tau \text{ even} \\
f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)\,e^{-i2\pi\nu\,u_j}\times\right.\right. \\
\qquad\left.\left.\left(\xi\,e^{-i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})}+(1-\xi)\,e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(x^{(1)})}\right)\right]\right\}, j \text{ even, } \tau \text{ odd} \\
f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)\,e^{i2\pi\nu\,u_j}\times\right.\right. \\
\qquad\left.\left.\left(\xi\,e^{-i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})}+(1-\xi)\,e^{i2\pi\nu(1-\xi)\,F_{X^{(1)}}(x^{(1)})}\right)\right]\right\}, \quad j \text{ odd, } \tau \text{ even} \\
f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)\,e^{i2\pi\nu\,u_j}\times\right.\right. \\
\qquad\left.\left.\left(\xi\,e^{i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})}+(1-\xi)\,e^{-i2\pi\nu(1-\xi)\,F_{X^{(1)}}(x^{(1)})}\right)\right]\right\}, \quad j \text{ odd, } \tau \text{ odd}
\end{cases}
\tag{9.45}
$$

(b) $f_{\vec{X}_{j-\tau}^{(k)-}|U_j^-}(\vec{x}^{(k)} \mid u_j) =$

$$\begin{cases} f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\,e^{i2\pi\nu\,u_j}\times \left(\xi\,e^{-i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})}+(1-\xi)\,e^{i2\pi\nu(1-\xi)\,F_{X^{(1)}}(x^{(1)})}\right)\right]\right\}, & j \text{ even, } \tau \text{ even} \\ f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\,e^{i2\pi\nu\,u_j}\times \left(\xi\,e^{i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})}+(1-\xi)\,e^{-i2\pi\nu(1-\xi)\,F_{X^{(1)}}(x^{(1)})}\right)\right]\right\}, & j \text{ even, } \tau \text{ odd} \\ f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\,e^{-i2\pi\nu\,u_j}\times \left(\xi\,e^{i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})}+(1-\xi)\,e^{-i2\pi\nu(1-\xi)\,F_{X^{(1)}}(x^{(1)})}\right)\right]\right\}, & j \text{ odd, } \tau \text{ even} \\ f_{\vec{X}^{(k)}}(\vec{x}^{(k)})\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\,e^{-i2\pi\nu\,u_j}\times \left(\xi\,e^{-i2\pi\nu\xi F_{X^{(1)}}(x^{(1)})}+(1-\xi)\,e^{i2\pi\nu(1-\xi)\,F_{X^{(1)}}(x^{(1)})}\right)\right]\right\}, & j \text{ odd, } \tau \text{ odd} \end{cases} \tag{9.46}$$

(c.1) For j even, τ even,

$$\begin{aligned} f_{X_{j+\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) = & \\ & \sum_{i_1=1}^{I_1}\cdots\sum_{i_k=1}^{I_k} 1_{S_{i_k}^{(k)}}(x^{(k)})M(i_1,\cdots,i_k)\,/\,d_{i_k}^{(k)} + \\ & 2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\frac{\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)}{i2\pi\nu}\,e^{-i2\pi\nu\,u_j}\sum_{i_1=1}^{I_1}\cdots\sum_{i_k=1}^{I_k}1_{S_{i_k}^{(k)}}(x^{(k)})\frac{M(i_1,\cdots,i_k)}{M(i_1)\,d_{i_k}^{(k)}}\times \right. \\ & \left. \left(e^{i2\pi\nu\xi Q(i_1)}-e^{i2\pi\nu\,\xi\,Q(i_1-1)}-e^{-i2\pi\nu(1-\xi)\,Q(i_1)}+e^{-i2\pi\nu(1-\xi)Q(i_1-1)}\right)\right] \end{aligned} \tag{9.47}$$

(c.2) For j even, τ odd,

$$f_{X_{j+\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) =$$
$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} 1_{S_{i_k}^{(k)}}(x^{(k)}) M(i_1, \cdots, i_k) \,/\, d_{i_k}^{(k)} +$$
$$2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\frac{\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)}{-i2\pi\nu} e^{-i2\pi\nu\, u_j} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} 1_{S_{i_k}^{(k)}}(x^{(k)}) \frac{M(i_1, \cdots, i_k)}{M(i_1)\, d_{i_k}^{(k)}} \times \right.$$
$$\left. \left(e^{-i2\pi\nu\xi Q(i_1)} - e^{-i2\pi\nu\xi\, Q(i_1-1)} - e^{i2\pi\nu(1-\xi)\, Q(i_1)} + e^{i2\pi\nu(1-\xi) Q(i_1-1)}\right)\right] \tag{9.48}$$

(c.3) For j odd, τ even,

$$f_{X_{j+\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) =$$
$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} 1_{S_{i_k}^{(k)}}(x^{(k)}) M(i_1, \cdots, i_k) \,/\, d_{i_k}^{(k)} +$$
$$2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\frac{\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)}{-i2\pi\nu} e^{i2\pi\nu\, u_j} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} 1_{S_{i_k}^{(k)}}(x^{(k)}) \frac{M(i_1, \cdots, i_k)}{M(i_1)\, d_{i_k}^{(k)}} \times \right. \tag{9.49}$$
$$\left. \left(e^{-i2\pi\nu\xi Q(i_1)} - e^{-i2\pi\nu\xi\, Q(i_1-1)} - e^{i2\pi\nu\,(1-\xi)\, Q(i_1)} + e^{i2\pi\nu(1-\xi) Q(i_1-1)}\right)\right]$$

(c.4) For j odd, τ odd,

$$f_{X_{j+\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) =$$
$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} 1_{S_{i_k}^{(k)}}(x^{(k)}) M(i_1, \cdots, i_k) \,/\, d_{i_k}^{(k)} +$$
$$2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\frac{\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)}{i2\pi\nu} e^{i2\pi\nu\, u_j} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} 1_{S_{i_k}^{(k)}}(x^{(k)}) \frac{M(i_1, \cdots, i_k)}{M(i_1)\, d_{i_k}^{(k)}} \times \right. \tag{9.50}$$
$$\left. \left(e^{i2\pi\nu\xi Q(i_1)} - e^{i2\pi\nu\,\xi\, Q(i_1-1)} - e^{-i2\pi\nu(1-\xi)\, Q(i_1)} + e^{-i2\pi\nu(1-\xi) Q(i_1-1)}\right)\right]$$

(d.1) For j even, τ even,

$$
\begin{aligned}
&f_{X_{j-\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) = \\
&\quad \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} 1_{S_{i_k}^{(k)}}(x^{(k)}) M(i_1, \cdots, i_k) \,/\, d_{i_k}^{(k)} + \\
&\quad 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\frac{\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)}{-i2\pi\nu}\, e^{i2\pi\nu\, u_j} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} 1_{S_{i_k}^{(k)}}(x^{(k)}) \frac{M(i_1, \cdots, i_k)}{M(i_1)\, d_{i_k}^{(k)}} \times \right. \\
&\qquad \left. \left(e^{-i2\pi\nu\xi Q(i_1)} - e^{-i2\pi\nu\,\xi\, Q(i_1-1)} - e^{i2\pi\nu(1-\xi)\,Q(i_1)} + e^{i2\pi\nu(1-\xi)Q(i_1-1)} \right) \right]
\end{aligned}
\tag{9.51}
$$

(d.2) For j even, τ odd,

$$
\begin{aligned}
&f_{X_{j-\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) = \\
&\quad \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} 1_{S_{i_k}^{(k)}}(x^{(k)}) M(i_1, \cdots, i_k) \,/\, d_{i_k}^{(k)} + \\
&\quad 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\frac{\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)}{i2\pi\nu}\, e^{i2\pi\nu\, u_j} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} 1_{S_{i_k}^{(k)}}(x^{(k)}) \frac{M(i_1, \cdots, i_k)}{M(i_1)\, d_{i_k}^{(k)}} \times \right. \\
&\qquad \left. \left(e^{i2\pi\nu\xi Q(i_1)} - e^{i2\pi\nu\xi\, Q(i_1-1)} - e^{-i2\pi\nu(1-\xi)\,Q(i_1)} + e^{-i2\pi\nu(1-\xi)Q(i_1-1)} \right) \right]
\end{aligned}
\tag{9.52}
$$

(d.3) For j odd, τ even,

$$
\begin{aligned}
&f_{X_{j-\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) = \\
&\quad \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} 1_{S_{i_k}^{(k)}}(x^{(k)}) M(i_1, \cdots, i_k) \,/\, d_{i_k}^{(k)} + \\
&\quad 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\frac{\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)}{i2\pi\nu}\, e^{-i2\pi\nu\, u_j} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} 1_{S_{i_k}^{(k)}}(x^{(k)}) \frac{M(i_1, \cdots, i_k)}{M(i_1)\, d_{i_k}^{(k)}} \times \right. \\
&\qquad \left. \left(e^{i2\pi\nu\xi Q(i_1)} - e^{i2\pi\nu\xi\, Q(i_1-1)} - e^{-i2\pi\nu\,(1-\xi)\,Q(i_1)} + e^{-i2\pi\nu(1-\xi)Q(i_1-1)} \right) \right]
\end{aligned}
\tag{9.53}
$$

(d.4) For j odd, τ odd,

$$f_{X_{j+\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) =$$

$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} 1_{S_{i_k}^{(k)}}(x^{(k)}) M(i_1, \cdots, i_k) \,/\, d_{i_k}^{(k)} +$$

$$2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\frac{\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)}{-i2\pi\nu} e^{-i2\pi\nu\, u_j} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} 1_{S_{i_k}^{(k)}}(x^{(k)}) \frac{M(i_1, \cdots, i_k)}{M(i_1)\, d_{i_k}^{(k)}} \times \right.$$

$$\left. \left(e^{-i2\pi\nu\xi Q(i_1)} - e^{-i2\pi\nu\,\xi\, Q(i_1-1)} - e^{i2\pi\nu(1-\xi)Q(i_1)} + e^{i2\pi\nu(1-\xi)Q(i_1-1)}\right)\right] \tag{9.54}$$

Proof.

Eqs. (9.45) and (9.46) follow from Eqs. (5.35) and (5.36) respectively. Eqs. (9.47) - (9.50) follow from Eq. (5.37) while Eqs. (9.51) - (9.54) follow from Eq. (5.38). □

Corollary 9.3

For $k = 1$ in Proposition 9.3, we have the following formulas as special cases:

$$f_{X_{j+\tau}^{(1)-}|U_j^-}(x \mid u_j) =$$

$$\begin{cases} f_{X^{(1)-}}(x)\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\,e^{-i2\pi\nu\,u_j}\times \left(\xi\,e^{i2\pi\nu\xi F_{X^{(1)}}(x)}+(1-\xi)\,e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(x)}\right)\right]\right\}, \; j \text{ even}, \; \tau \text{ even} \\ f_{X^{(1)-}}(x)\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\,e^{-i2\pi\nu\,u_j}\times \left(\xi\,e^{-i2\pi\nu\xi F_{X^{(1)}}(x)}+(1-\xi)\,e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(x)}\right)\right]\right\}, \; j \text{ even}, \; \tau \text{ odd} \\ f_{X^{(1)-}}(x)\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\,e^{i2\pi\nu\,u_j}\times \left(\xi\,e^{-i2\pi\nu\xi F_{X^{(1)}}(x)}+(1-\xi)\,e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(x)}\right)\right]\right\}, \; j \text{ odd}, \; \tau \text{ even} \\ f_{X^{(1)-}}(x)\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\,e^{i2\pi\nu\,u_j}\times \left(\xi\,e^{i2\pi\nu\xi F_{X^{(1)}}(x)}+(1-\xi)\,e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(x)}\right)\right]\right\}, \; j \text{ odd}, \; \tau \text{ odd} \end{cases} \tag{9.55}$$

$$f_{X_{j-\tau}^{(1)-}|U_j^-}(x \mid u_j) =$$

$$\begin{cases} f_{X^{(1)-}}(x)\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\,e^{i2\pi\nu\,u_j}\times \left(\xi\,e^{-i2\pi\nu\xi F_{X^{(1)}}(x)}+(1-\xi)\,e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(x)}\right)\right]\right\}, & j \text{ even, } \tau \text{ even} \\ f_{X^{(1)-}}(x)\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\,e^{i2\pi\nu\,u_j}\times \left(\xi\,e^{i2\pi\nu\xi F_{X^{(1)}}(x)}+(1-\xi)\,e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(x)}\right)\right]\right\}, & j \text{ even, } \tau \text{ odd} \\ f_{X^{(1)-}}(x)\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\,e^{-i2\pi\nu\,u_j}\times \left(\xi\,e^{i2\pi\nu\xi F_{X^{(1)}}(x)}+(1-\xi)\,e^{-i2\pi\nu(1-\xi)F_{X^{(1)}}(x)}\right)\right]\right\}, & j \text{ odd, } \tau \text{ even} \\ f_{X^{(1)-}}(x)\left\{1+2\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\,e^{-i2\pi\nu\,u_j}\times \left(\xi\,e^{-i2\pi\nu\xi F_{X^{(1)}}(x)}+(1-\xi)\,e^{i2\pi\nu(1-\xi)F_{X^{(1)}}(x)}\right)\right]\right\}, & j \text{ odd, } \tau \text{ odd} \end{cases} \tag{9.56}$$

where $f_{X^{(1)}}(x)$ and $F_{X^{(1)}}(x)$ are given by Eqs. (9.9) and (9.10) separately in Lemma 9.1. □

Proposition 9.8

Let $\{\vec{X}_\tau^-\}$ be a MARM$^-$ process with hyper-step pdf defined by Eq.(9.7). Then for $\tau > 0$,

$0 \le u_j < 1$, $j \ge 0$, and $x^{(k)} \in S^{(k)}$, $k \ge 1$, Proposition 5.3 becomes

(a.1) For j even, τ even,

$$F_{X_{j+\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) =$$

$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \left(\sum_{i_k=1}^{j-1} M(i_1, \cdots, i_k) + \frac{M(i_1, \cdots, i_{k-1}, j)}{d_j^{(k)}} \left(x^{(k)} - l_j^{(k)} \right) \right) +$$

$$2 \sum_{\nu=1}^{\infty} \mathbf{Re} \left[\frac{\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)}{i2\pi\nu} e^{-i2\pi\nu u_j} \sum_{i_1=1}^{I_1} \left(e^{i2\pi\nu\xi Q(i_1)} - e^{i2\pi\nu\xi Q(i_1-1)} - \right. \right. \tag{9.57}$$

$$\left. e^{-i2\pi\nu(1-\xi)Q(i_1)} + e^{-i2\pi\nu(1-\xi)Q(i_1-1)} \right) \times$$

$$\left. \sum_{i_2=1}^{I_2} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \left(\sum_{i_k=1}^{j-1} \frac{M(i_1, \cdots, i_k)}{M(i_1)} + \frac{M(i_1, \cdots, i_{k-1}, j)}{M(i_1) d_j^{(k)}} \left(x^{(k)} - l_j^{(k)} \right) \right) \right]$$

(a.2) For j even, τ odd,

$$F_{X_{j+\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) =$$

$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \left(\sum_{i_k=1}^{j-1} M(i_1, \cdots, i_k) + \frac{M(i_1, \cdots, i_{k-1}, j)}{d_j^{(k)}} \left(x^{(k)} - l_j^{(k)} \right) \right) +$$

$$2 \sum_{\nu=1}^{\infty} \mathbf{Re} \left[\frac{\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)}{-i2\pi\nu} e^{-i2\pi\nu u_j} \sum_{i_1=1}^{I_1} \left(e^{-i2\pi\nu\xi Q(i_1)} - e^{-i2\pi\nu\xi Q(i_1-1)} - \right. \right. \tag{9.58}$$

$$\left. e^{i2\pi\nu(1-\xi)Q(i_1)} + e^{i2\pi\nu(1-\xi)Q(i_1-1)} \right) \times$$

$$\left. \sum_{i_2=1}^{I_2} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \left(\sum_{i_k=1}^{j-1} \frac{M(i_1, \cdots, i_k)}{M(i_1)} + \frac{M(i_1, \cdots, i_{k-1}, j)}{M(i_1) d_j^{(k)}} \left(x^{(k)} - l_j^{(k)} \right) \right) \right]$$

(a.3) For j odd, τ even,

$$F_{X_{j+\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) =$$

$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \left(\sum_{i_k=1}^{j-1} M(i_1, \cdots, i_k) + \frac{M(i_1, \cdots, i_{k-1}, j)}{d_j^{(k)}} \left(x^{(k)} - l_j^{(k)} \right) \right) +$$

$$2 \sum_{\nu=1}^{\infty} \mathbf{Re} \left[\frac{\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)}{-i2\pi\nu} e^{i2\pi\nu u_j} \sum_{i_1=1}^{I_1} \left(e^{-i2\pi\nu\xi Q(i_1)} - e^{-i2\pi\nu\xi Q(i_1-1)} - \right. \right. \tag{9.59}$$

$$\left. e^{i2\pi\nu(1-\xi)Q(i_1)} + e^{i2\pi\nu(1-\xi)Q(i_1-1)} \right) \times$$

$$\left. \sum_{i_2=1}^{I_2} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \left(\sum_{i_k=1}^{j-1} \frac{M(i_1, \cdots, i_k)}{M(i_1)} + \frac{M(i_1, \cdots, i_{k-1}, j)}{M(i_1) d_j^{(k)}} \left(x^{(k)} - l_j^{(k)} \right) \right) \right]$$

(a.4) For j odd, τ odd,

$$F_{X_{j+\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) =$$

$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \left(\sum_{i_k=1}^{j-1} M(i_1, \cdots, i_k) + \frac{M(i_1, \cdots, i_{k-1}, j)}{d_j^{(k)}} \left(x^{(k)} - l_j^{(k)} \right) \right) +$$

$$2 \sum_{\nu=1}^{\infty} \mathbf{Re} \left[\frac{\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)}{i2\pi\nu} e^{i2\pi\nu u_j} \sum_{i_1=1}^{I_1} \left(e^{i2\pi\nu\xi Q(i_1)} - e^{i2\pi\nu\xi Q(i_1-1)} - \right. \right. \tag{9.60}$$

$$\left. e^{-i2\pi\nu(1-\xi)Q(i_1)} + e^{-i2\pi\nu(1-\xi)Q(i_1-1)} \right) \times$$

$$\left. \sum_{i_2=1}^{I_2} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \left(\sum_{i_k=1}^{j-1} \frac{M(i_1, \cdots, i_k)}{M(i_1)} + \frac{M(i_1, \cdots, i_{k-1}, j)}{M(i_1) d_j^{(k)}} \left(x^{(k)} - l_j^{(k)} \right) \right) \right]$$

(b.1) For j even, τ even,

$$F_{X_{j-\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) =$$

$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \left(\sum_{i_k=1}^{j-1} M(i_1, \cdots, i_k) + \frac{M(i_1, \cdots, i_{k-1}, j)}{d_j^{(k)}} \left(x^{(k)} - l_j^{(k)} \right) \right) +$$

$$2 \sum_{\nu=1}^{\infty} \mathbf{Re} \left[\frac{\tilde{f}_{S_{j-\tau+1,\, j}}(i2\pi\nu)}{-i2\pi\nu} e^{i2\pi\nu u_j} \sum_{i_1=1}^{I_1} \left(e^{-i2\pi\nu\xi Q(i_1)} - e^{-i2\pi\nu\xi Q(i_1-1)} - \right. \right. \tag{9.61}$$

$$\left. e^{i2\pi\nu(1-\xi)Q(i_1)} + e^{i2\pi\nu(1-\xi)Q(i_1-1)} \right) \times$$

$$\left. \sum_{i_2=1}^{I_2} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \left(\sum_{i_k=1}^{j-1} \frac{M(i_1, \cdots, i_k)}{M(i_1)} + \frac{M(i_1, \cdots, i_{k-1}, j)}{M(i_1) d_j^{(k)}} \left(x^{(k)} - l_j^{(k)} \right) \right) \right]$$

(b.2) For j even, τ odd,

$$F_{X_{j-\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) =$$

$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \left(\sum_{i_k=1}^{j-1} M(i_1, \cdots, i_k) + \frac{M(i_1, \cdots, i_{k-1}, j)}{d_j^{(k)}} \left(x^{(k)} - l_j^{(k)} \right) \right) +$$

$$2 \sum_{\nu=1}^{\infty} \mathbf{Re} \left[\frac{\tilde{f}_{S_{j-\tau+1,\, j}}(i2\pi\nu)}{i2\pi\nu} e^{i2\pi\nu u_j} \sum_{i_1=1}^{I_1} \left(e^{i2\pi\nu\xi Q(i_1)} - e^{i2\pi\nu\xi Q(i_1-1)} - \right. \right. \tag{9.62}$$

$$\left. e^{-i2\pi\nu(1-\xi)Q(i_1)} + e^{-i2\pi\nu(1-\xi)Q(i_1-1)} \right) \times$$

$$\left. \sum_{i_2=1}^{I_2} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \left(\sum_{i_k=1}^{j-1} \frac{M(i_1, \cdots, i_k)}{M(i_1)} + \frac{M(i_1, \cdots, i_{k-1}, j)}{M(i_1) d_j^{(k)}} \left(x^{(k)} - l_j^{(k)} \right) \right) \right]$$

(b.3) For j odd, τ even,

$$F_{X_{j-\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) =$$

$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \left(\sum_{i_k=1}^{j-1} M(i_1, \cdots, i_k) + \frac{M(i_1, \cdots, i_{k-1}, j)}{d_j^{(k)}} \left(x^{(k)} - l_j^{(k)} \right) \right) +$$

$$2 \sum_{\nu=1}^{\infty} \mathbf{Re} \left[\frac{\tilde{f}_{S_{j-\tau+1,\, j}}(i2\pi\nu)}{i2\pi\nu} e^{-i2\pi\nu u_j} \sum_{i_1=1}^{I_1} \left(e^{i2\pi\nu\xi Q(i_1)} - e^{i2\pi\nu\xi Q(i_1-1)} - \right. \right. \tag{9.63}$$

$$\left. e^{-i2\pi\nu(1-\xi)Q(i_1)} + e^{-i2\pi\nu(1-\xi)Q(i_1-1)} \right) \times$$

$$\left. \sum_{i_2=1}^{I_2} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \left(\sum_{i_k=1}^{j-1} \frac{M(i_1, \cdots, i_k)}{M(i_1)} + \frac{M(i_1, \cdots, i_{k-1}, j)}{M(i_1)\, d_j^{(k)}} \left(x^{(k)} - l_j^{(k)} \right) \right) \right]$$

(b.4) For j odd, τ odd

$$F_{X_{j-\tau}^{(k)-}|U_j^-}(x^{(k)} \mid u_j) =$$

$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \left(\sum_{i_k=1}^{j-1} M(i_1, \cdots, i_k) + \frac{M(i_1, \cdots, i_{k-1}, j)}{d_j^{(k)}} \left(x^{(k)} - l_j^{(k)} \right) \right) +$$

$$2 \sum_{\nu=1}^{\infty} \mathbf{Re} \left[\frac{\tilde{f}_{S_{j-\tau+1,\, j}}(i2\pi\nu)}{-i2\pi\nu} e^{-i2\pi\nu u_j} \sum_{i_1=1}^{I_1} \left(e^{-i2\pi\nu\xi Q(i_1)} - e^{-i2\pi\nu\xi Q(i_1-1)} - \right. \right. \tag{9.64}$$

$$\left. e^{i2\pi\nu(1-\xi)Q(i_1)} + e^{i2\pi\nu(1-\xi)Q(i_1-1)} \right) \times$$

$$\left. \sum_{i_2=1}^{I_2} \cdots \sum_{i_{k-1}=1}^{I_{k-1}} \sum_{j=1}^{I_k} 1_{S_j^{(k)}}(x^{(k)}) \left(\sum_{i_k=1}^{j-1} \frac{M(i_1, \cdots, i_k)}{M(i_1)} + \frac{M(i_1, \cdots, i_{k-1}, j)}{M(i_1)\, d_j^{(k)}} \left(x^{(k)} - l_j^{(k)} \right) \right) \right]$$

Proof.

Eq. (9.57) - (9.60) follow from integrating Eqs. (9.47) - (9.50) while Eqs. (9.61) - (9.64) follow from integrating Eqs. (9.51) - (9.54), respectively. □

Corollary 9.4

For $k = 1$ in Proposition 9.4, we have the following formulas as special cases:

$$F_{X_{j+\tau}^{(1)-}|U_j^-}(x \mid u_j) =$$

$$\begin{cases} F_{X^{(1)-}}(x) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu\, u_j} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \right. \\ \qquad \left. \left(\xi\, e^{i2\pi\nu\, \xi F_{X^{(1)}}(y)} + (1-\xi)\, e^{-i2\pi\nu(1-\xi) F_{X^{(1)}}(y)}\right) dy\right], \ j \text{ even}, \ \tau \text{ even} \\ F_{X^{(1)-}}(x) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu\, u_j} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \right. \\ \qquad \left. \left(\xi\, e^{-i2\pi\nu\, \xi F_{X^{(1)}}(y)} + (1-\xi)\, e^{i2\pi\nu(1-\xi) F_{X^{(1)}}(y)}\right) dy\right], \ j \text{ even}, \ \tau \text{ odd} \\ F_{X^{(1)-}}(x) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)\, e^{i2\pi\nu\, u_j} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \right. \\ \qquad \left. \left(\xi\, e^{i2\pi\nu\, \xi F_{X^{(1)}}(y)} + (1-\xi)\, e^{-i2\pi\nu(1-\xi) F_{X^{(1)}}(y)}\right) dy\right], \ j \text{ odd}, \ \tau \text{ even} \\ F_{X^{(1)-}}(x) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)\, e^{i2\pi\nu\, u_j} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \right. \\ \qquad \left. \left(\xi\, e^{i2\pi\nu\, \xi F_{X^{(1)}}(y)} + (1-\xi)\, e^{-i2\pi\nu(1-\xi) F_{X^{(1)}}(y)}\right) dy\right], \ j \text{ odd}, \ \tau \text{ odd} \end{cases} \tag{9.65}$$

$$F_{X_{j-\tau}^{(1)-}|U_j^-}(x \mid u_j) =$$

$$\begin{cases} F_{X^{(1)-}}(x) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\, e^{i2\pi\nu u_j} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \right. \\ \qquad \left. \left(\xi\, e^{-i2\pi\nu\,\xi F_{X^{(1)}}(x)} + (1-\xi)\, e^{i2\pi\nu(1-\xi) F_{X^{(1)}}(x)}\right) dy\right], \ j \text{ even}, \ \tau \text{ even} \\ F_{X^{(1)-}}(x) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\, e^{i2\pi\nu u_j} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \right. \\ \qquad \left. \left(\xi\, e^{i2\pi\nu\,\xi F_{X^{(1)}}(x)} + (1-\xi)\, e^{-i2\pi\nu(1-\xi) F_{X^{(1)}}(x)}\right) dy\right], \ j \text{ even}, \ \tau \text{ odd} \\ F_{X^{(1)-}}(x) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\, e^{-i2\pi\nu u_j} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \right. \\ \qquad \left. \left(\xi\, e^{i2\pi\nu\,\xi F_{X^{(1)}}(x)} + (1-\xi)\, e^{-i2\pi\nu(1-\xi) F_{X^{(1)}}(x)}\right) dy\right], \ j \text{ odd}, \ \tau \text{ even} \\ F_{X^{(1)-}}(x) + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\, e^{-i2\pi\nu u_j} \int_{-\infty}^{x} f_{X^{(1)}}(y) \times \right. \\ \qquad \left. \left(\xi\, e^{-i2\pi\nu\,\xi F_{X^{(1)}}(x)} + (1-\xi)\, e^{i2\pi\nu(1-\xi) F_{X^{(1)}}(x)}\right) dy\right], \ j \text{ odd}, \ \tau \text{ odd} \end{cases} \tag{9.66}$$

where $f_{X^{(1)}}(x)$ and $F_{X^{(1)}}(x)$ are given respectively by Eqs. (9.9) and (9.10) in Lemma 9.1. □

Proposition 9.9

For $\tau > 0$, $0 \le u_j < 1$, $j \ge 0$, and $x^{(k)} \in S^{(k)}$, $k \ge 1$,

$$\mathbf{E}[X_{j+\tau}^{(k)-} \mid U_j^- = u_j] =$$

$$\begin{cases} \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu u_j}\, \tilde{\mathcal{D}}^{(k)}(-i2\pi\nu)\right], \ j \text{ even}, \ \tau \text{ even} \\ \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\, e^{-i2\pi\nu u_j}\, \tilde{\mathcal{D}}^{(k)}(i2\pi\nu)\right], \ j \text{ even}, \ \tau \text{ odd} \\ \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\, e^{i2\pi\nu u_j}\, \tilde{\mathcal{D}}^{(k)}(i2\pi\nu)\right], \ j \text{ odd}, \ \tau \text{ even} \\ \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\, e^{i2\pi\nu u_j}\, \tilde{\mathcal{D}}^{(k)}(-i2\pi\nu)\right], \ j \text{ odd}, \ \tau \text{ odd} \end{cases} \tag{9.67}$$

$$\mathbf{E}[X_{j-\tau}^{(k)-} \mid U_j^- = u_j] =$$

$$\begin{cases} \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)e^{i2\pi\nu u_j}\,\tilde{\mathcal{D}}^{(k)}(i2\pi\nu)\right], & j \text{ even, } \tau \text{ even} \\ \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)e^{i2\pi\nu u_j}\,\tilde{\mathcal{D}}^{(k)}(-i2\pi\nu)\right], & j \text{ even, } \tau \text{ odd} \\ \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\,e^{-i2\pi\nu u_j}\,\tilde{\mathcal{D}}^{(k)}(-i2\pi\nu)\right], & j \text{ odd, } \tau \text{ even} \\ \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)\,e^{-i2\pi\nu u_j}\,\tilde{\mathcal{D}}^{(k)}(i2\pi\nu)\right], & j \text{ odd, } \tau \text{ odd} \end{cases} \tag{9.68}$$

Proof.

Follow respectively from Eqs. (5.45) and (5.46). □

9.3. Correlation Structure of Empirically-Based MARM Processes

In this section we exhibit the correlation structure of empirically-based MARM processes by specializing Theorem 6.1 and 6.2.

9.3.1 Correlation Functions of Empirically-Based MARM$^+$ Processes

This section exhibits the correlation functions of empirically-based MARM$^+$ foreground process $\{\vec{X}_j^+\}$.

Proposition 9.10

For $1 \le m \le n \le N$, $\tau > 0$ and $j \ge 0$, the correlation functions $\rho_{m,n}^+(j,\tau)$ are given by

$$\rho^{+}_{m,n}(j,\tau) = \frac{2}{\sigma_{X^{(m)}}\,\sigma_{X^{(n)}}} \sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\,\tilde{\mathcal{D}}^{(m)}(i2\pi\nu)\,\tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)\right] \quad (9.69)$$

where the $\tilde{\mathcal{D}}^{(n)}(i2\pi\nu)$ are given by Eqs. (9.23) and (9.26).

In particular, for $m = n$, Eq. (9.69) reduces to

$$\rho^{+}_{n,n}(j,\tau) = \frac{2}{\sigma^{2}_{X^{(n)}}} \sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\right] \left|\tilde{\mathcal{D}}^{(n)}(i2\pi\nu)\right|^{2} \quad (9.70)$$

Proof.

Follows directly from Theorem 6.1 and Lemma 9.4. □

9.3.2 Correlation Functions of Empirically-Based MARM^{-} Processes

This section exhibits the correlation functions of empirically-based MARM^{-} foreground process $\{\vec{X}^{-}_{j}\}$.

Proposition 9.11

For $1 \le m \le n \le N$, $\tau > 0$ and $j \ge 0$, the correlation functions $\rho^{-}_{m,n}(\tau)$ are given by

$$\rho^{-}_{m,n}(j,\tau) = \begin{cases} \dfrac{2}{\sigma_{X^{(m)}}\sigma_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\,\tilde{\mathcal{D}}^{(m)}(i2\pi\nu)\,\tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)\right], & j \text{ even, } \tau \text{ even} \\ \dfrac{2}{\sigma_{X^{(m)}}\sigma_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\,\tilde{\mathcal{D}}^{(m)}(i2\pi\nu)\,\tilde{\mathcal{D}}^{(n)}(i2\pi\nu)\right], & j \text{ even, } \tau \text{ odd} \\ \dfrac{2}{\sigma_{X^{(m)}}\sigma_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\,\tilde{\mathcal{D}}^{(m)}(-i2\pi\nu)\,\tilde{\mathcal{D}}^{(n)}(i2\pi\nu)\right], & j \text{ odd, } \tau \text{ even} \\ \dfrac{2}{\sigma_{X^{(m)}}\sigma_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\,\tilde{\mathcal{D}}^{(m)}(-i2\pi\nu)\,\tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)\right], & j \text{ odd, } \tau \text{ odd} \end{cases} \tag{9.71}$$

where the $\tilde{\mathcal{D}}^{(n)}(i2\pi\nu)$ are given by Eqs. (9.23) and (9.26).

In particular, for $m = n$, Eq. (9.71) reduces to

$$\rho^{-}_{n,n}(j,\tau) = \begin{cases} \dfrac{2}{\sigma^2_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\right] \left|\tilde{\mathcal{D}}^{(n)}(i2\pi\nu)\right|^2, & j \text{ even, } \tau \text{ even} \\ \dfrac{2}{\sigma^2_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\left(\tilde{\mathcal{D}}^{(n)}(i2\pi\nu)\right)^2\right], & j \text{ even, } \tau \text{ odd} \\ \dfrac{2}{\sigma^2_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\right] \left|\tilde{\mathcal{D}}^{(n)}(i2\pi\nu)\right|^2, & j \text{ odd, } \tau \text{ even} \\ \dfrac{2}{\sigma^2_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu)\left(\tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)\right)^2\right], & j \text{ odd, } \tau \text{ odd} \end{cases} \tag{9.72}$$

Proof.

Follows from Theorem 6.2 and Lemma 9.4 directly. □

10. The Empirical MARM Fitting Methodology

This chapter presents the empirical MARM fitting methodology which is a practical specialization of the general MARM fitting methodology of Chapter 7 to a subclass of empirically-based MARM processes described in Chapter 9 with iid step-function innovation densities.

Given an empirical multivariate time series $\{\vec{x}_j\}_{j=0}^{J-1}$, let $\hat{H}$ be an associated empirical multi-dimensional histogram, and for $1 \le m \le n \le N$, let the associated empirical correlation functions be estimated by [Shumway and Stoffer (2007)]

$$\hat{\rho}_{m,n}(\tau) = \frac{\frac{1}{J-\tau}\sum_{i=1}^{J-\tau} x_i^{(m)} x_{i+\tau}^{(n)} - \frac{1}{(J-\tau)^2}\sum_{i=1}^{J-\tau} x_i^{(m)} \sum_{i=1}^{J-\tau} x_{i+\tau}^{(n)}}{\sqrt{\frac{1}{J-\tau}\sum_{i=1}^{J-\tau}\left(x_i^{(m)}\right)^2 - \left[\frac{1}{J-\tau}\sum_{i=1}^{J-\tau} x_i^{(m)}\right]^2}\sqrt{\frac{1}{J-\tau}\sum_{i=1}^{J-\tau}\left(x_{i+\tau}^{(n)}\right)^2 - \left[\frac{1}{J-\tau}\sum_{i=1}^{J-\tau} x_{i+\tau}^{(n)}\right]^2}} \tag{10.1}$$

The choice of the weighting coefficients, $a_{m,n}(\tau)$, can be made in a number of ways. We describe here two choices.

1. $a_{m,n}(\tau) = \alpha^{\tau}$ for some $0 < \alpha < 1$. This choice corresponds to a geometric decrease in the lag.
2. $a_{m,n}(\tau) = |\hat{\rho}_{m,n}(\tau)|$. In this case, the absolute value of the correlation function is viewed as an indicator of how hard it is to forecast a variate of lag τ, since the magnitude of a correlation coefficient is a measure of linear association between the underlying random variables.

Both choices are reasonable and easy to implement, and we will usually adopt the second one in our experiments.

10.1. The Empirical CSLO Algorithm

Following [Jelenkovic and Melamed (1995a, 1995b)], we next introduce an empirical MARM fitting methodology, which will require a trade-off between the generality of the innovation-density search parameter and its practical computation. To this end, we shall restrict the search parameters, $f_{\{V_n\}}$ and ξ as described in the sequel.

We first outline how to suitably restrict the functional form of innovation densities in the search space.

I. First, we assume for simplicity that the innovation sequence, $\{V_n\}$, is iid. Consequently, the search space becomes the set of all pairs of the form (f_V, ξ), where f_V is the marginal density of the innovation process, $\{V_n\}$.

II. Second, we restrict the f_V to be step functions, since those are particularly simple, and yet they can approximate any other density function arbitrarily closely.

III. Third, recall that Eq. (4.1) implies for a MARM background sequence, the representation $U_n = \langle U_{n-1} + V_n \rangle = \langle U_{n-1} + \langle V_n \rangle \rangle$, which in turn implies that only innovation random variables of the form $\langle V_n \rangle$ need be considered. Thus we may further restrict consideration to step function densities f_V with support on [0, 1). In fact, due to the modulo-1 arithmetic properties, any support interval of length one will do. Accordingly, we choose the interval [-0.5, 0.5) as a convenient case.

IV. Fourth, we restrict f_V to have a fixed number K of steps of equal-length $1/K$, namely, f_V is of the form

$$f_V(x) = \sum_{n=1}^{K} \frac{P_n}{1/K} 1_{[-0.5+(n-1)/K,\ -0.5+n/K)}(x), \tag{10.2}$$

where the vector $\vec{\mathbf{P}} = (P_1, P_2, \ldots, P_K)$ is a probability mass function

V. Finally, we select a quantum $q = 1/Q$, where the integer $Q > 0$ is the number of quanta, and restrict the probability vectors $\vec{\mathbf{P}} = (P_1, P_2, \ldots, P_K)$ to a search space $\mathcal{P}_K$ consisting of probabilities which are integer multiples of the quantum q, namely,

$$\mathcal{P}_K = \left\{ (n_1 q,\ n_2 q, \ldots, n_K q) :\ n_i \text{ are nonnegative intergers such that } \sum_{i=1}^{K} n_i = Q \right\} \tag{10.3}$$

The corresponding restriction on ξ is rather simple. Let L be a moderate-size integer (for example, $L = 10$), and define the search space of ξ as the set

$$\mathcal{S}_L = \left\{ \frac{n}{L} :\ n = 0, 1, \ldots, L \right\}. \tag{10.4}$$

We are now in a position to define the search space of our empirical MARM fitting problem as

$$\mathcal{G}_K = \{ (\vec{\mathbf{P}}, \xi) : \vec{\mathbf{P}} \in \mathcal{P}_K,\ \xi \in \mathcal{S}_L \}. \tag{10.5}$$

The empirical MARM fitting problem can now be stated as follows.

Problem 10.1 (Empirical MARM Fitting Problem)

Given an empirical multivariate time series $\{\vec{x}_j\}_{j=0}^{J}$ and fixed positive integers K, Q, and L, find optimal parameters $(\vec{\mathbf{P}}^*, \xi^*) \in \mathcal{G}_K$ such that

$$(\vec{\mathbf{P}}^*, \xi^*) = \underset{(\vec{\mathbf{P}},\ \xi) \in \mathcal{G}_K}{\arg\min} \ \{ g_K(\vec{\mathbf{P}},\ \xi) \} \tag{10.6}$$

where $g_K(\vec{\mathbf{P}}, \xi)$ is an objective function defined by

$$g_K(\vec{\mathbf{P}}, \xi) = \sum_{m=1}^{N} \sum_{n=m}^{N} \sum_{\tau=1}^{S(m,n)} a_{m,n}(\tau) \left[\rho_{m,n}(\tau) - \hat{\rho}_{m,n}(\tau) \right]^2, \qquad (\vec{\mathbf{P}},\ \xi) \in \mathcal{G}_K, \qquad (10.7)$$

$\hat{\rho}_{m,n}(\tau)$ are empirical autocorrelation/cross-correlation functions, $S(m,n)$ is the maximal correlation lag for the (m,n) pair, and $0 \le a_{m,n}(\tau) \le 1$ are weight coefficients. □

Problem 10.1 reduces the general search space of Problem 7.1 to a finite-dimension search space with two nice properties: the search space is finite, and there exist analytical formulas for fast and stable computation of the objective function (using the formulas developed in Chapter 9) and its partial derivatives with respect to every optimization parameter (to be developed in Section 10.2). Consequently, Problem 10.1 can be practically solved using existing optimization algorithms, as discussed in detail in Section 10.1.2.

We now proceed to describe the two-stage CSLO algorithm in detail: a comprehensive search of $\mathcal{G}_K$ that identifies a set of "best" MARM models with respect to the objective function $g_K(\vec{\mathbf{P}}, \xi)$, followed by a Steepest Descent optimization of this objective function with the aforementioned set of "best" models as starting points.

10.1.1 Stage I: Comprehensive Search

Given an empirical multivariate time series $\{\vec{x}_j\}_{j=0}^{J-1}$, the first stage of CSLO proceeds as follows.

1. Estimate the joint probability density function $f_{\vec{X}}(\vec{x})$ of $\{\vec{X}_j\}_{j=0}^{\infty}$ by the empirical multi-dimensional histogram pdf $\hat{h}(\vec{x})$ (see Section 9.1).

2. Compute from $f_{\vec{X}}(\vec{x})$ all marginal and conditional density functions of the form $f_{\vec{X}^{(k)}}(\vec{x}^{(k)})$, and $f_{X^{(k)}|\vec{X}^{(k-1)}}(x^{(k)} \mid \vec{x}^{(k-1)})$ (see Section 9.1).

3. Compute the empirical correlation matrix, with elements $\hat{\rho}_{m,n}(\tau)$ via Eq. (10.1).

4. Select positive integers as follows: K (the number of steps of the candidate step-function innovation densities), $N_P = Q + 1$ (the number of distinct probability values assigned to a step, where Q number of probability quanta) can assume; and $N_\xi = L + 1$ ((the number of distinct values assigned to the stitching parameter, ξ). These parameters determine the search space $\mathcal{G}_K$ of Eq. (10.5).

5. For a prescribed integer, $B > 0$, compute the set of B "best models" over $\mathcal{G}_K$ by evaluating the objective function (10.7) for each point $(\vec{\mathbf{P}}, \xi)$ in $\mathcal{G}_K$, and keeping a running set of B candidate models with the smallest objective function values.

Stage I implies that the cardinality of $\mathcal{G}_K$, is given by [Jelenkovic and Melamed (1995)]

$$N_{\text{tot}} = N_\xi \binom{N_P + K - 2}{K - 1}. \tag{10.8}$$

Thus, for practical computations, the parameters N_P, N_ξ and K must be moderate, since N_{tot} grows very quickly in them.

The output of this stage is the set of B "best models", and this set is the input of the second stage.

10.1.2 Stage II: Local Optimization

Stage II further improves the fitting results of Stage I by searching for a local minimum of the objective function for each of the B "best" candidate models as a starting point. The local optimization is implemented via a Steepest-Descent nonlinear programming algorithm, and the B optimal models obtained are reordered in ascending objective-function values. Any of those models can be used as the output of the CSLO algorithm, though the first one (with the smallest objective-function value) is typically the preferred choice.

The local optimization in the stage II is implemented following the procedure outlined in [Jelenkovic and Melamed (1995)] using the *Zoutendijk Feasible Direction Method* [Bazaraa et al. (1993)], summarized below. This method is iterative, such that at each iteration it generates an improving feasible direction and then proceeds to optimize along that direction. Let $\nabla g_K(\vec{\mathbf{x}})$ denote the gradient of g_K of Eq. (10.7), evaluated at $\vec{\mathbf{x}} \in \mathcal{G}_K$. A direction in $\mathcal{G}_K$ is any real vector $\vec{\mathbf{d}} = (d_1, \ldots, d_{K+1})$. Given a feasible point $\vec{\mathbf{x}} = (\vec{\mathbf{P}}, \xi)$ in the feasible space, $\mathcal{G}_K$, a vector-nonzero direction $\vec{\mathbf{d}}$ is called a *feasible direction* at $\vec{\mathbf{x}} \in \mathcal{G}_K$ if there exists a $\delta > 0$ such that $\vec{\mathbf{x}} + \lambda\vec{\mathbf{d}} \in \mathcal{G}_K$ for all $\lambda \in (0, \delta)$, where λ is referred to as a *step length*. Furthermore, $\vec{\mathbf{d}}$ is called a descent direction at $\vec{\mathbf{x}} \in \mathcal{G}_K$ if, in addition, $\nabla g_K(\vec{\mathbf{x}})\,\vec{\mathbf{d}}^t < 0$ where t denotes here the transpose operator. Note that if a vector $\vec{\mathbf{d}}$ is a descent direction, then any vector $\lambda\vec{\mathbf{d}}$ ($\lambda > 0$) is also a descent direction. But because minimizing $\nabla g_K(\vec{\mathbf{x}})\,\vec{\mathbf{d}}^t$ yields $-\infty$ as its optimal objective value, a constraint that bounds the vector $\vec{\mathbf{d}}$ must be introduced. Such a restriction is usually referred to as a *normalization constraint*, and a common constraint is $\|\vec{\mathbf{d}}\|_\infty \leq 1$. Thus,

the optimal feasible direction $\vec{\mathbf{d}}^*$, subject to the normalization constraint, is a solution of the following optimization problem

Direction Optimization

For a given $\vec{\mathbf{x}} \in \mathcal{G}_K$,

Minimize $\nabla g_K(\vec{\mathbf{x}})\,\vec{\mathbf{d}}^t$

Subject to $\vec{\mathbf{d}}$ is feasible (feasibility constraint) (10.9)

$-1 \le d_j \le 1,\ 1 \le j \le K+1$ (normalization constraint) □

Once the optimal feasible direction $\vec{\mathbf{d}}^*$ is found, an optimal step length λ^* is determined through a line search, formulated as the following optimization problem.

Step Optimization

For a given $\vec{\mathbf{x}} \in \mathcal{G}_K$ and a $\vec{\mathbf{d}}^*$ obtained from Direction Optimization,

Minimize $g_K(\vec{\mathbf{x}} + \lambda\vec{\mathbf{d}}^*)$

Subject to $0 \le \lambda \le \lambda_{\max}(\vec{\mathbf{x}})$ (10.10)

where $\lambda_{\max}(\vec{\mathbf{x}})$ is the maximal feasible step length □

The line search in the Step Optimization problem is conducted on the finite interval $(0, \lambda_{\max}(\vec{\mathbf{x}}))$, so it is possible, in principle, to approximate the global minimum well. However, $\lambda_{\max}(\vec{\mathbf{x}})$ is based on the solution of the Direction Optimization problem, which is a linear approximation of the objective function using its first derivative only. In the theory of nonlinear

programming, this approximation is known to result in the so-called zigzagging effect [Bazaraa el at (1993)]. One widely-used approximation of the step length is the so-called *Armijo's Rule* [Bazaraa el at (1993)], which is described briefly as follows. Let $\theta(\lambda) = g_K(\vec{\mathbf{x}} + \lambda\vec{\mathbf{d}})$, $0 \le \lambda \le \lambda_{\mathbf{max}}(\vec{\mathbf{x}})$, and let $0 < \varepsilon < 1$ and $\alpha > 1$ be two parameters which keep the approximated step length from being too small or too large, respectively as explained below (typical values are $\varepsilon = 0.2$ and $\alpha = 2$.) Define further $\hat{\theta}(\lambda) = \theta(0) + \varepsilon\,\lambda\,\theta'(0)$ as the first-order approximation of $\theta(\lambda)$ at 0. A step length λ is called *acceptable*, provided that $\theta(\lambda) \le \hat{\theta}(\lambda)$. However, to prevent λ from being too small, Armijo's Rule also requires that $\theta(\alpha\lambda) > \hat{\theta}(\alpha\lambda)$, which constrains λ to lie in some acceptable range. In practice, Armijo's Rule initially sets $\lambda = \lambda_{\mathbf{max}}(\vec{\mathbf{x}})$, and while $\theta(\lambda) > \hat{\theta}(\lambda)$, it set $\lambda \leftarrow \lambda / \alpha$ repeatedly, and stops with $\lambda = \lambda^*$ once $\theta(\lambda^*) \le \hat{\theta}(\lambda^*)$.

The Direction Optimization problem has a closed form solution due to the relatively simple linear constraints involved [Jelenkovic and Melamed (1995b)]. Since $P_K = 1 - \sum_{n=1}^{K-1} P_n$, we can reduce the problem dimension. The reduced parameter space becomes

$$\mathcal{H}_{K-1} = \{(P_1,\ldots,P_{K-1},\xi) : P_n \ge 0,\ 1 \le n \le K-1,\ \textstyle\sum_{n=1}^{K-1} P_n \le 1,\ \xi \in \mathcal{S}_L\},$$

and the reduced objective function h_{K-1} over $\mathcal{H}_{K-1}$, is given by

$$h_{K-1}(P_1,\ldots,P_{K-1},\xi) = g_K(P_1,\ldots,P_{K-1},\ 1 - \textstyle\sum_{n=1}^{K-1} P_n, \xi), \quad (P_1,\ldots,P_{K-1},\xi) \in \mathcal{H}_{K-1}$$

In the corresponding reduced Direction Optimization problem, $\mathcal{G}_K$ is replaced by $\mathcal{H}_{K-1}$, g_K is replaced by h_{K-1}, and direction vectors are of the form $\vec{\mathbf{d}} = (d_1,\ldots,d_K)$. Assuming that the

constraints (except for the feasibility constraint) are satisfied for the new problem, it is obvious that the minimum of $\nabla h_{K-1}(\vec{\mathbf{y}})\,\vec{\mathbf{d}}^t$ is attained for

$$d_j = -\text{sign}\left(\frac{\partial}{\partial y_j} h_{K-1}(\vec{\mathbf{y}})\right), \quad 1 \le j \le K, \tag{10.11}$$

where $\text{sign}(z)$ is $+1$ or -1 according as z is non-negative or negative, respectively. The optimization's goal is to change the d_j in such a way that the feasibility constraint is satisfied while the objective function $\nabla h_{K-1}(\vec{\mathbf{y}})\,\vec{\mathbf{d}}^t$ is increased as little as possible. To this end, one first fixes the boundary constraints for each coordinate. for example, for $n = 1,\ldots,K-1$, if $P_n = 0$ and $d_n = -1$, then set $d_n \leftarrow 0$, and if $P_n = 1$ and $d_n = 1$, then set $d_n \leftarrow 0$. Similar actions are taken for ξ as well. Finally, the only constraint left is $\sum_{n=1}^{K-1} P_n \le 1$, so that an infeasible direction will be generated if $\sum_{n=1}^{K-1} P_n = 1$ and $\sum_{n=1}^{K-1} d_n > 0$.

The best feasible direction $\vec{\mathbf{d}}$ is obtained when $\sum_{n=1}^{K-1} d_n = 0$ and the increase of $\nabla h_{K-1}(\vec{\mathbf{y}})\,\vec{\mathbf{d}}^t$ is kept to a minimum. To this end, let $J_{K-1}(\vec{\mathbf{d}}) = \{j : d_j = 1 \text{ or } (d_j = 0 \text{ and } P_j = 1)\}$ be the set of indexes j for which d_j can be decreased without violating the normalization constraint. Then $\nabla h_{K-1}(\vec{\mathbf{y}})\,\vec{\mathbf{d}}^t$ increases the least by choosing the index j^*, such that $-\frac{\partial}{\partial P_{j^*}} h_{K-1}(\vec{\mathbf{y}})$ is minimized over $J_{K-1}(\vec{\mathbf{d}})$. For such an index j^*, decrease d_{j^*} just enough to obtain $\sum_{j=1}^{K-1} d_j = 0$ while still keeping $d_{j^*} \ge -1$; if this is not possible, then set $d_{j^*} \leftarrow -1$, remove j^* from $J_{K-1}(\vec{\mathbf{d}})$ and repeat. The optimal direction $\vec{\mathbf{d}}$ is obtained when the corresponding sum of d_j vanishes.

Once the optimal step length λ^* is obtained, set $\vec{\mathbf{x}} \leftarrow \vec{\mathbf{x}} + \lambda^* \vec{\mathbf{d}}^*$ at each iteration. Zoutendijk Feasible Direction Method terminates when the optimal value of $\nabla g_K(\mathbf{x})\vec{\mathbf{d}}^t$ falls below a prescribed threshold. Thus, it is essential to solve the Direction Optimization and Step Optimization problems efficiently,

10.2. Partial Derivatives of the Objective Function

This section derives the partial derivatives of the objective function, $g_K(\vec{\mathbf{P}}, \xi)$, in Eq. (10.7) with respect to the search parameters, $(\vec{\mathbf{P}}, \xi)$, needed by the Steepest-Descent optimization algorithm. To this end, we shall make use of the following three lemmas.

Lemma 10.1

For the step function innovation (10.2) with probability vector $\vec{\mathbf{P}} = (P_1, P_2, \ldots, P_K)$, we have for $\nu \neq 0$,

$$\tilde{f}_V(i2\pi\nu) = \sum_{j=1}^{K} P_j \frac{e^{i\pi\nu(K-2j+1)/K} \sin(\pi\nu / K)}{\pi\nu / K}, \tag{10.12}$$

and for $k = 1, \ldots, K$

$$\frac{\partial}{\partial P_k} \tilde{f}_V^{\tau}(i2\pi\nu) = \tau \left(\sum_{j=1}^{K} P_j \, e^{i\pi\nu(K-2j+1)/K} \right)^{\tau-1} \left[\frac{\sin(\pi\nu / K)}{\pi\nu / K} \right]^{\tau} e^{i\pi\nu(K-2k+1)/K} \tag{10.13}$$

Proof.

Eq. (10.12) follows from Proposition 2 in [Jagerman and Melamed (1992b)]. Next, differentiating the function $\tilde{f}_V^{\tau}(i2\pi\nu)$ with respect to P_k with the aid of Eq. (10.12) yields

$$\frac{\partial}{\partial P_k}\tilde{f}_V^{\tau}(i2\pi\nu) = \tau\,\tilde{f}_V^{\tau-1}(i2\pi\nu)\frac{\partial}{\partial P_k}\tilde{f}_V(i2\pi\nu)$$

$$= \tau\,\tilde{f}_V^{\tau-1}(i2\pi\nu)\frac{e^{i\pi\nu\,(K-2k+1)/K}\,\sin(\pi\nu\,/\,K)}{\pi\nu\,/\,K}$$

Eq. (10.13) now follows by substituting Eq. (10.12) into the above equation. □

Lemma 10.2

For $1 \le j \le I_1$

$$\frac{\partial}{\partial \xi}\tilde{1}_{C(j)}(s) = Q(j)\,e^{-s\xi Q(j)}\left[1 - Q(j)\,e^{sQ(j)}\right] - Q(j-1)\,e^{-s\xi\,Q(j-1)}\left[1 - Q(j-1)\,e^{sQ(j-1)}\right] \tag{10.14}$$

Proof.

Eq. (10.14) follows by differentiating Eq. (9.18) with respect to ξ. □

Lemma 10.3

$$\begin{aligned}\frac{\partial}{\partial \xi}\tilde{\mathcal{D}}^{(1)}(s) = &\sum_{j=1}^{I_1}\left\{l_j^{(1)}\,Q(j-1)\,e^{-s\xi Q(j-1)}\left[e^{sQ(j-1)} - 1\right] - r_j^{(1)}\,Q(j)\,e^{-s\xi Q(j)}\left[e^{sQ(j)} - 1\right]\right\}\\ &- \sum_{j=1}^{I_1}\frac{d_j^{(1)}}{M(j)\,s^2}\left\{\frac{s}{\xi}\left[Q(j-1)\,e^{-s\xi Q(j-1)} - Q(j)\,e^{-s\xi Q(j)}\right]\right.\\ &\qquad + \frac{s}{1-\xi}\left[Q(j-1)\,e^{s(1-\xi)Q(j-1)} - Q(j)\,e^{s(1-\xi)Q(j)}\right]\\ &\qquad \left. - \frac{e^{-s\xi Q(j)} - e^{-s\xi Q(j-1)}}{\xi^2} - \frac{e^{s(1-\xi)Q(j)} - e^{s(1-\xi)Q(j-1)}}{(1-\xi)^2}\right\}\end{aligned} \tag{10.15}$$

$$\frac{\partial}{\partial \xi}\tilde{\mathcal{D}}^{(n)}(s) = \sum_{i_1=1}^{I_1}\cdots\sum_{i_n=1}^{I_n}\left\{\frac{M(i_1,\ldots,i_n)}{M(i_1)}\left(l_{i_n}^{(n)} + d_{i_k}^{(n)}\,/\,2\right)\times\right.$$

$$\left[Q(i_1)\,e^{-s\xi Q(i_1)} - Q(i_1-1)\,e^{-s\xi Q(i_1-1)} + \right. \tag{10.16}$$

$$\left.\left. Q(i_1-1)\,e^{s(1-\xi)Q(i_1-1)} - Q(i_1)\,e^{s(1-\xi)Q(i_1)}\right]\right\}$$

where the $\tilde{\mathcal{D}}^{(n)}$ are defined by Eqs. (9.23) and (9.26).

Proof.

Eqs. (10.15) and (10.16) follow by differentiating $\tilde{\mathcal{D}}^{(k)}(s)$ in Eqs. (9.23) and (9.26), respectively, with respect to ξ with the aid of Eq. (10.14). □

10.2.1 Partial Derivatives for MARM$^+$ Processes

The next two propositions exhibit the partial derivatives $\frac{\partial}{\partial P_k}\rho_{m,n}^{+}(j,\tau)$ and $\frac{\partial}{\partial \xi}\rho_{m,n}^{+}(j,\tau)$.

Proposition 10.1

For $\tau \geq 1$, $j \geq 0$, $1 \leq k \leq K$ and $1 \leq m \leq n \leq N$,

$$\frac{\partial}{\partial P_k}\rho_{m,n}^{+}(j,\tau) = \frac{2\tau}{\sigma_{X^{(m)}}\sigma_{X^{(n)}}}\sum_{\nu=1}^{\infty}\left[\frac{\sin(\pi\nu\,/\,K)}{\pi\nu\,/\,K}\right]^{\tau}\mathrm{Re}\left[e^{i\pi\nu(K-2k+1)/K}\times\right.$$

$$\left.\left(\sum_{j=1}^{K}P_j\,e^{i\pi\nu(K-2j+1)/K}\right)^{\tau-1}\tilde{\mathcal{D}}^{(m)}(i2\pi\nu)\,\tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)\right] \tag{10.17}$$

where the $\tilde{D}^{(m)}(i2\pi\nu)$ are given by Eqs. (9.23) and (9.26).

Proof.

Eq. (10.17) follows by taking the partial derivatives of the cross correlation functions in Eq.(6.3) with respect to P_k with the aid of Eq. (10.13). □

Proposition 10.2

For $\tau \geq 1$, $j \geq 0$, $0 \leq \xi \leq 1$ and $1 \leq m \leq n \leq N$,

$$\frac{\partial}{\partial \xi} \rho^{+}_{m,n}(j,\tau) = \frac{2}{\sigma_{X^{(m)}} \sigma_{X^{(n)}}} \sum_{\nu=1}^{\infty} \mathrm{Re}\left[\left(\sum_{k=1}^{K} P_k \frac{e^{i\pi\nu(K-2k+1)/K} \sin(\pi\nu / K)}{\pi\nu / K}\right)^{\tau} \times \left\{\tilde{\mathcal{D}}^{(n)}(-i2\pi\nu) \frac{\partial}{\partial \xi} \tilde{\mathcal{D}}^{(m)}(i2\pi\nu) + \tilde{\mathcal{D}}^{(m)}(i2\pi\nu) \frac{\partial}{\partial \xi} \tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)\right\}\right] \tag{10.18}$$

where the $\tilde{\mathcal{D}}^{(m)}(i2\pi\nu)$ are given by Eqs. (9.23) and (9.26), while the $\frac{\partial}{\partial \xi} \tilde{\mathcal{D}}^{(m)}(i2\pi\nu)$ are given by Eqs. (10.15) and (10.16).

Proof.

Eq. (10.18) follows by taking the partial derivatives of the cross correlation functions in Eq.(6.3) with respect to ξ. □

10.2.2 Partial Derivatives for MARM^{-} Processes

The following propositions exhibit the requisite partial derivatives, $\frac{\partial}{\partial P_k} \rho^{-}_{m,n}(j,\tau)$ and $\frac{\partial}{\partial \xi} \rho^{-}_{m,n}(j,\tau)$, for MARM^{-} processes.

Proposition 10.3

For $\tau \geq 1$, $j \geq 0$, $1 \leq k \leq K$ and $1 \leq m \leq n \leq N$,

$$\frac{\partial}{\partial P_k} \rho^{-}_{m,n}(j,\tau) =$$

$$\begin{cases} \dfrac{2\tau}{\sigma_{X^{(m)}}\sigma_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \left(\frac{\sin(\pi\nu / K)}{\pi\nu / K} \right)^{\tau} \mathrm{Re}\left[e^{i\pi\nu(K-2k+1)/K} \left(\sum_{j=1}^{K} P_j \, e^{i\pi\nu(K-2j+1)/K} \right)^{\tau-1} \times \right. \\ \qquad \left. \tilde{\mathcal{D}}^{(m)}(i2\pi\nu)\, \tilde{\mathcal{D}}^{(n)}(-i2\pi\nu) \right], \qquad j \text{ even, } \tau \text{ even} \\ \dfrac{2\tau}{\sigma_{X^{(m)}}\sigma_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \left(\frac{\sin(\pi\nu / K)}{\pi\nu / K} \right)^{\tau} \mathrm{Re}\left[e^{i\pi\nu(K-2k+1)/K} \left(\sum_{j=1}^{K} P_j \, e^{i\pi\nu(K-2j+1)/K} \right)^{\tau-1} \times \right. \\ \qquad \left. \tilde{\mathcal{D}}^{(m)}(i2\pi\nu)\, \tilde{\mathcal{D}}^{(n)}(i2\pi\nu) \right], \qquad j \text{ even, } \tau \text{ odd} \\ \dfrac{2\tau}{\sigma_{X^{(m)}}\sigma_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \left(\frac{\sin(\pi\nu / K)}{\pi\nu / K} \right)^{\tau} \mathrm{Re}\left[e^{i\pi\nu(K-2k+1)/K} \left(\sum_{j=1}^{K} P_j \, e^{i\pi\nu(K-2j+1)/K} \right)^{\tau-1} \times \right. \\ \qquad \left. \tilde{\mathcal{D}}^{(m)}(-i2\pi\nu)\, \tilde{\mathcal{D}}^{(n)}(i2\pi\nu) \right], \qquad j \text{ odd, } \tau \text{ even} \\ \dfrac{2\tau}{\sigma_{X^{(m)}}\sigma_{X^{(n)}}} \displaystyle\sum_{\nu=1}^{\infty} \left(\frac{\sin(\pi\nu / K)}{\pi\nu / K} \right)^{\tau} \mathrm{Re}\left[e^{i\pi\nu(K-2k+1)/K} \left(\sum_{j=1}^{K} P_j \, e^{i\pi\nu(K-2j+1)/K} \right)^{\tau-1} \times \right. \\ \qquad \left. \tilde{\mathcal{D}}^{(m)}(-i2\pi\nu)\, \tilde{\mathcal{D}}^{(n)}(-i2\pi\nu) \right], \qquad j \text{ odd, } \tau \text{ odd} \end{cases} \tag{10.19}$$

where the $\tilde{D}^{(m)}(i2\pi\nu)$ are given by Eqs. (9.23) and (9.26).

Proof.

Eq. (10.19) follows by taking the partial derivatives of the cross correlation functions in Eq. (6.7) with respect to P_k with the aid of Eq. (10.13). □

Proposition 10.4

For $\tau \geq 1$, $j \geq 0$, $0 \leq \xi \leq 1$ and $1 \leq m \leq n \leq N$,

$$\frac{\partial}{\partial \xi}\rho^{-}_{m,n}(j,\tau)=$$

$$\begin{cases} \dfrac{2}{\sigma_{X^{(m)}}\sigma_{X^{(n)}}}\displaystyle\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\left(\sum_{k=1}^{K}P_k\,\frac{e^{i\pi\nu(K-2k+1)/K}\,\sin(\pi\nu\,/\,K)}{\pi\nu\,/\,K}\right)^{\tau}\times\right. \\ \qquad\left.\left\{\tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)\dfrac{\partial}{\partial\xi}\tilde{\mathcal{D}}^{(m)}(i2\pi\nu)+\tilde{\mathcal{D}}^{(m)}(i2\pi\nu)\dfrac{\partial}{\partial\xi}\tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)\right\}\right],\ j \text{ even},\ \tau \text{ even} \\ \dfrac{2}{\sigma_{X^{(m)}}\sigma_{X^{(n)}}}\displaystyle\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\left(\sum_{k=1}^{K}P_k\,\frac{e^{i\pi\nu(K-2k+1)/K}\,\sin(\pi\nu\,/\,K)}{\pi\nu\,/\,K}\right)^{\tau}\times\right. \\ \qquad\left.\left\{\tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)\dfrac{\partial}{\partial\xi}\tilde{\mathcal{D}}^{(m)}(i2\pi\nu)+\tilde{\mathcal{D}}^{(m)}(i2\pi\nu)\dfrac{\partial}{\partial\xi}\tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)\right\}\right],\ j \text{ even},\ \tau \text{ odd} \\ \dfrac{2}{\sigma_{X^{(m)}}\sigma_{X^{(n)}}}\displaystyle\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\left(\sum_{k=1}^{K}P_k\,\frac{e^{i\pi\nu(K-2k+1)/K}\,\sin(\pi\nu\,/\,K)}{\pi\nu\,/\,K}\right)^{\tau}\times\right. \\ \qquad\left.\left\{\tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)\dfrac{\partial}{\partial\xi}\tilde{\mathcal{D}}^{(m)}(i2\pi\nu)+\tilde{\mathcal{D}}^{(m)}(i2\pi\nu)\dfrac{\partial}{\partial\xi}\tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)\right\}\right],\ j \text{ odd},\ \tau \text{ even} \\ \dfrac{2}{\sigma_{X^{(m)}}\sigma_{X^{(n)}}}\displaystyle\sum_{\nu=1}^{\infty}\mathbf{Re}\left[\left(\sum_{k=1}^{K}P_k\,\frac{e^{i\pi\nu(K-2k+1)/K}\,\sin(\pi\nu\,/\,K)}{\pi\nu\,/\,K}\right)^{\tau}\times\right. \\ \qquad\left.\left\{\tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)\dfrac{\partial}{\partial\xi}\tilde{\mathcal{D}}^{(m)}(i2\pi\nu)+\tilde{\mathcal{D}}^{(m)}(i2\pi\nu)\dfrac{\partial}{\partial\xi}\tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)\right\}\right],\ j \text{ odd},\ \tau \text{ odd} \end{cases} \tag{10.20}$$

where the $\tilde{\mathcal{D}}^{(m)}(i2\pi\nu)$ are given by Eqs. (9.23) and (9.26) , while the $\dfrac{\partial}{\partial\xi}\tilde{\mathcal{D}}^{(m)}(i2\pi\nu)$ are given by Eqs. (10.15) and (10.16).

Proof.

Eq. (10.20) follows by taking the partial derivatives of the cross correlation functions in Eq. (6.7) with respect to ξ. □

11. The Empirical MARM Forecasting Methodology

This chapter presents the empirical MARM forecasting methodology which is a practical specialization of the general MARM forecasting methodology of Chapter 8 to a subclass of empirically-based MARM processes described in Chapter 9 with iid step-function innovation densities. Specifically, for each $1 \le k \le N$, time index $j \ge 0$ and lag $\tau > 0$, we shall exhibit computational formulas for the point estimators

$$\hat{\mathbf{E}}[X_{j\pm\tau}^{(k)} \mid \vec{X}_j^{(k)}] = \hat{\mathbf{E}}[X_{j\pm\tau}^{(k)} \mid X_j^{(1)}]$$

induced by the conditional density $\hat{f}_{X_{j\pm\tau}^{(k)}|X_j^{(1)}}(y \mid x)$ from Eq. (8.4), and confidence intervals of the form

$$\left[\hat{\mathbf{E}}[X_{j+\tau}^{(k)} \mid \vec{X}_j^{(k)}] - \gamma,\ \hat{\mathbf{E}}[X_{j+\tau}^{(k)} \mid \vec{X}_j^{(k)}] + \gamma\right] = \left[\hat{\mathbf{E}}[X_{j+\tau}^{(k)} \mid X_j^{(1)}] - \gamma,\ \hat{\mathbf{E}}[X_{j+\tau}^{(k)} \mid X_j^{(1)}] + \gamma\right],$$

where $\mathbf{Pr}\left\{\left|X_{j+\tau}^{(k)} - \hat{\mathbf{E}}[X_{j+\tau}^{(k)} \mid \vec{X}_j^{(k)}]\right| > \gamma\right\} = \alpha$.

11.1. Point Estimation for Empirically-Based MARM Processes

In this section, we provide computable formulas of the requisite point estimators $\hat{\mathbf{E}}[X_{j\pm\tau}^{(k)+} \mid X_j^{(1)+}]$ and $\hat{\mathbf{E}}[X_{j\pm\tau}^{(k)-} \mid X_j^{(1)-}]$ for empirically-based MARM processes.

Proposition 11.1

For $\tau > 0$, $j \ge 0$ and $\vec{x}^{(k)} \in S^{(1)} \times \cdots \times S^{(k)}$, $1 \le k \le N$,

$$\hat{\mathbf{E}}[X_{j+\tau}^{(k)+} \mid \vec{X}_j^{(k)+} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j+\tau}^{(k)+} \mid X_j^{(1)+} = x_j^{(1)}] =$$

$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} \frac{M(i_1,\cdots,i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right)}{2} +$$

$$\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\frac{\tilde{f}_{S_{j+1,\,j+\tau}}(i2\pi\nu)}{i2\pi\nu} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} \frac{M(i_1,\cdots,i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right)}{M(i_1)} \times \right. \tag{11.1}$$

$$\left(p^{(k)} e^{-i2\pi\nu u_j^{(1)}} + \left(1 - p^{(k)}\right) e^{-i2\pi\nu u_j^{(2)}}\right) \times$$

$$\left.\left(e^{i2\pi\nu\xi Q(i_1)} - e^{i2\pi\nu\,\xi\,Q(i_1-1)} - e^{-i2\pi\nu(1-\xi)\,Q(i_1)} + e^{-i2\pi\nu(1-\xi)Q(i_1-1)}\right)\right]$$

$$\hat{\mathbf{E}}[X_{j-\tau}^{(k)+} \mid \vec{X}_j^{(k)+} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j-\tau}^{(k)+} \mid X_j^{(1)+} = x_j^{(1)}] =$$

$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} \frac{M(i_1,\cdots,i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right)}{2} +$$

$$\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\frac{\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)}{i2\pi\nu} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} \frac{M(i_1,\cdots,i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right)}{M(i_1)} \times \right. \tag{11.2}$$

$$\left(p^{(k)} e^{i2\pi\nu u_j^{(1)}} + \left(1 - p^{(k)}\right) e^{i2\pi\nu u_j^{(2)}}\right) \times$$

$$\left.\left(e^{-i2\pi\nu\xi Q(i_1)} - e^{-i2\pi\nu\,\xi\,Q(i_1-1)} - e^{i2\pi\nu(1-\xi)\,Q(i_1)} + e^{i2\pi\nu(1-\xi)Q(i_1-1)}\right)\right]$$

where

$$u_j^{(1)} = u_j^{(1)}(x_j^{(1)}) = \xi \sum_{i_1=1}^{I_1} 1_{S_{i_1}^{(1)}}(x_j^{(1)}) \left[Q(i_1 - 1) + (x_j^{(1)} - l_{i_1}^{(1)}) M(i_1) \,/\, d_{i_1}^{(1)}\right] \tag{11.3}$$

$$u_j^{(2)} = u_j^{(2)}(x_j^{(1)}) = 1 - (1 - \xi) \sum_{i_1=1}^{I_1} 1_{S_{i_1}^{(1)}}(x_j^{(1)}) \left[Q(i_1 - 1) + (x_j^{(1)} - l_{i_1}^{(1)}) M(i_1) \,/\, d_{i_1}^{(1)}\right] \tag{11.4}$$

Proof.

Eqs. (11.1) and (11.2) follow by substituting Eqs. (9.8) into Eqs. (8.6) and (8.7), respectively, while Eqs. (11.3) and (11.4) follow by substituting Eq.(9.10) into each equation in Eq. (8.2). □

Proposition 11.2

Let $\tau > 0$, $j \geq 0$ and $\vec{x}^{(k)} \in S^{(1)} \times \cdots \times S^{(k)}$, $1 \leq k \leq N$.

(a.1) For j even, τ even,

$$\begin{aligned}
&\hat{\mathbf{E}}[X_{j+\tau}^{(k)-} \mid \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j+\tau}^{(k)-} \mid X_j^{(1)-} = x_j^{(1)}] = \\
&\quad \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} M(i_1, \cdots, i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right) / 2 + \\
&\quad \sum_{\nu=1}^{\infty} \mathbf{Re}\left[\frac{\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)}{i2\pi\nu} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} \frac{M(i_1, \cdots, i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right)}{M(i_1)} \times \right. \\
&\qquad \left(p^{(k)} e^{-i2\pi\nu u_j^{(1)}} + \left(1 - p^{(k)}\right) e^{-i2\pi\nu u_j^{(2)}}\right) \times \\
&\qquad \left.\left(e^{i2\pi\nu\xi Q(i_1)} - e^{i2\pi\nu\,\xi\, Q(i_1-1)} - e^{-i2\pi\nu(1-\xi)Q(i_1)} + e^{-i2\pi\nu(1-\xi)Q(i_1-1)}\right)\right]
\end{aligned} \tag{11.5}$$

(a.2) For j even, τ odd,

$$\begin{aligned}
&\hat{\mathbf{E}}[X_{j+\tau}^{(k)-} \mid \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j+\tau}^{(k)-} \mid X_j^{(1)-} = x_j^{(1)}] = \\
&\quad \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} M(i_1, \cdots, i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right) / 2 + \\
&\quad \sum_{\nu=1}^{\infty} \mathbf{Re}\left[\frac{\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)}{i2\pi\nu} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} \frac{M(i_1, \cdots, i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right)}{M(i_1)} \times \right. \\
&\qquad \left(p^{(k)} e^{-i2\pi\nu u_j^{(1)}} + \left(1 - p^{(k)}\right) e^{-i2\pi\nu u_j^{(2)}}\right) \times \\
&\qquad \left.\left(e^{-i2\pi\nu\xi Q(i_1)} - e^{-i2\pi\nu\,\xi\, Q(i_1-1)} - e^{i2\pi\nu(1-\xi)Q(i_1)} + e^{i2\pi\nu(1-\xi)Q(i_1-1)}\right)\right]
\end{aligned} \tag{11.6}$$

(a.3) For j odd, τ even,

$$\hat{\mathbf{E}}[X_{j+\tau}^{(k)-} \mid \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j+\tau}^{(k)-} \mid X_j^{(1)-} = x_j^{(1)}] =$$

$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} M(i_1, \cdots, i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right) / 2 +$$

$$\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\frac{\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)}{i2\pi\nu} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} \frac{M(i_1, \cdots, i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right)}{M(i_1)} \times \right. \tag{11.7}$$

$$\left(p^{(k)} e^{i2\pi\nu u_j^{(1)}} + \left(1 - p^{(k)}\right) e^{i2\pi\nu u_j^{(2)}}\right) \times$$

$$\left.\left(e^{-i2\pi\nu\xi Q(i_1)} - e^{-i2\pi\nu\,\xi\,Q(i_1-1)} - e^{i2\pi\nu(1-\xi)Q(i_1)} + e^{i2\pi\nu(1-\xi)Q(i_1-1)}\right)\right]$$

(a.4) For j odd, τ odd,

$$\hat{\mathbf{E}}[X_{j+\tau}^{(k)-} \mid \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j+\tau}^{(k)-} \mid X_j^{(1)-} = x_j^{(1)}] =$$

$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} M(i_1, \cdots, i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right) / 2 +$$

$$\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\frac{\tilde{f}_{S_{j+1,\, j+\tau}}(i2\pi\nu)}{i2\pi\nu} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} \frac{M(i_1, \cdots, i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right)}{M(i_1)} \times \right. \tag{11.8}$$

$$\left(p^{(k)} e^{i2\pi\nu u_j^{(1)}} + \left(1 - p^{(k)}\right) e^{i2\pi\nu u_j^{(2)}}\right) \times$$

$$\left.\left(e^{i2\pi\nu\xi Q(i_1)} - e^{i2\pi\nu\,\xi\,Q(i_1-1)} - e^{-i2\pi\nu(1-\xi)Q(i_1)} + e^{-i2\pi\nu(1-\xi)Q(i_1-1)}\right)\right]$$

(b.1) For j even, τ even,

$$\hat{\mathbf{E}}[X_{j-\tau}^{(k)-} \mid \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j-\tau}^{(k)-} \mid X_j^{(1)-} = x_j^{(1)}] =$$

$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} M(i_1, \cdots, i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right) / 2 +$$

$$\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\frac{\tilde{f}_{S_{j-\tau+1,\, j}}(i2\pi\nu)}{i2\pi\nu} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} \frac{M(i_1, \cdots, i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right)}{M(i_1)} \times \right. \tag{11.9}$$

$$\left(p^{(k)} e^{i2\pi\nu u_j^{(1)}} + \left(1 - p^{(k)}\right) e^{i2\pi\nu u_j^{(2)}}\right) \times$$

$$\left.\left(e^{-i2\pi\nu\xi Q(i_1)} - e^{-i2\pi\nu\,\xi\, Q(i_1-1)} - e^{i2\pi\nu(1-\xi)\,Q(i_1)} + e^{i2\pi\nu(1-\xi)Q(i_1-1)}\right)\right]$$

(b.2) For j even, τ odd,

$$\hat{\mathbf{E}}[X_{j-\tau}^{(k)-} \mid \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j-\tau}^{(k)-} \mid X_j^{(1)-} = x_j^{(1)}] =$$

$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} M(i_1, \cdots, i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right) / 2 +$$

$$\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\frac{\tilde{f}_{S_{j-\tau+1,\, j}}(i2\pi\nu)}{i2\pi\nu} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} \frac{M(i_1, \cdots, i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right)}{M(i_1)} \times \right. \tag{11.10}$$

$$\left(p^{(k)} e^{i2\pi\nu u_j^{(1)}} + \left(1 - p^{(k)}\right) e^{i2\pi\nu u_j^{(2)}}\right) \times$$

$$\left.\left(e^{i2\pi\nu\xi Q(i_1)} - e^{i2\pi\nu\,\xi\, Q(i_1-1)} - e^{-i2\pi\nu(1-\xi)\,Q(i_1)} + e^{-i2\pi\nu(1-\xi)Q(i_1-1)}\right)\right]$$

(b.3) For j odd, τ even,

$$\hat{\mathbf{E}}[X_{j-\tau}^{(k)-} \mid \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j-\tau}^{(k)-} \mid X_j^{(1)-} = x_j^{(1)}] =$$

$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} M(i_1, \cdots, i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right) / 2 +$$

$$\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\frac{\tilde{f}_{S_{j-\tau+1,\, j}}(i2\pi\nu)}{i2\pi\nu} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} \frac{M(i_1, \cdots, i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right)}{M(i_1)} \times \right. \tag{11.11}$$

$$\left(p^{(k)} e^{-i2\pi\nu u_j^{(1)}} + \left(1 - p^{(k)}\right) e^{-i2\pi\nu u_j^{(2)}}\right) \times$$

$$\left.\left(e^{i2\pi\nu\xi Q(i_1)} - e^{i2\pi\nu\,\xi\, Q(i_1-1)} - e^{-i2\pi\nu(1-\xi)\,Q(i_1)} + e^{-i2\pi\nu(1-\xi)Q(i_1-1)}\right)\right]$$

(b.4) For j odd, τ odd,

$$\hat{\mathbf{E}}[X_{j-\tau}^{(k)-} \mid \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] = \hat{\mathbf{E}}[X_{j-\tau}^{(k)-} \mid X_j^{(1)-} = x_j^{(1)}] =$$

$$\sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} M(i_1, \cdots, i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right) / 2 +$$

$$\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\frac{\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu)}{i2\pi\nu} \sum_{i_1=1}^{I_1} \cdots \sum_{i_k=1}^{I_k} \frac{M(i_1, \cdots, i_k)\left(l_{i_k}^{(k)} + r_{i_k}^{(k)}\right)}{M(i_1)} \times \right. \tag{11.12}$$

$$\left(p^{(k)} e^{-i2\pi\nu u_j^{(1)}} + \left(1 - p^{(k)}\right) e^{-i2\pi\nu u_j^{(2)}}\right) \times$$

$$\left.\left(e^{-i2\pi\nu\xi Q(i_1)} - e^{-i2\pi\nu\,\xi\,Q(i_1-1)} - e^{i2\pi\nu(1-\xi)Q(i_1)} + e^{i2\pi\nu(1-\xi)Q(i_1-1)}\right)\right]$$

where $u_j^{(1)} = u_j^{(1)}(x_j^{(1)})$ and $u_j^{(2)} = u_j^{(2)}(x_j^{(1)})$ are given by Eqs. (11.3) and (11.4).

Proof.

Similar to the proof of Proposition 11.1. □

11.2. Selection of the Mixing Parameter for Empirically-Based MARM Processes

Selection of each mixing parameter $p^{(k)}$ $(1 \le k \le N)$ is implemented in the way discussed in Section 8.2. In particular, $p^{(k)}$ is selected by Proposition 8.4, where $\mathbf{E}\left[X_{j-\tau}^{(k)} \mid U_j = u_j^{(1)}\right]$ and $\mathbf{E}\left[X_{j-\tau}^{(k)} \mid U_j = u_j^{(2)}\right]$ are given by

$$\mathbf{E}[X_{j-\tau}^{(k)+} \mid U_j^{+} = u_j] = \mu_{X^{(k)+}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,\,j}}(i2\pi\nu) e^{i2\pi\nu\,u_j} \tilde{\mathfrak{D}}^{(k)}(i2\pi\nu)\right] \tag{11.13}$$

$$\mathbf{E}[X_{j-\tau}^{(k)-} \mid U_j^- = u_j] =$$

$$\begin{cases} \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)e^{i2\pi\nu u_j}\,\tilde{\mathcal{D}}^{(k)}(i2\pi\nu)\right], & j \text{ even, } \tau \text{ even} \\ \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)e^{i2\pi\nu u_j}\,\tilde{\mathcal{D}}^{(k)}(-i2\pi\nu)\right], & j \text{ even, } \tau \text{ odd} \\ \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\,e^{-i2\pi\nu u_j}\,\tilde{\mathcal{D}}^{(k)}(-i2\pi\nu)\right], & j \text{ odd, } \tau \text{ even} \\ \mu_{X^{(k)-}} + 2\sum_{\nu=1}^{\infty} \mathbf{Re}\left[\tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu)\,e^{-i2\pi\nu u_j}\,\tilde{\mathcal{D}}^{(k)}(i2\pi\nu)\right], & j \text{ odd, } \tau \text{ odd} \end{cases} \tag{11.14}$$

and the $\tilde{\mathcal{D}}^{(k)}(i2\pi\nu)$ are given by Lemma 9.4.

12. A MARM Fitting and Forecasting Example

In this chapter, we apply the empirical MARM fitting and forecasting methodologies described in Chapter 10 and 11 to an empirical three-dimensional time series. The data consist of the daily adjusted closing values of the following three U.S. major stock indexes:

(1) Standard & Poor 500 Index (S&P)

(2) NASDAQ Composite (NASDAQ)

(3) Dow Jones Industrial Average (DJIA)

We used a data set of historical values covering the period 1/2/2004 – 7/1/2008 (1132 values each), obtained from [38], so that for each day, the data consisted of three values, one from each stock index above. A software tool, called *MultiArmLab*, was written in C# to implement the fitting and forecasting algorithms, with both graphical and textual output displays. The tool included a visual user interface to facilitate data and parameter entry. The figures in the sequel are screenshots from *MultiArmLab* displaying fitting and forecasting graphics.

12.1. MARM Fitting

To perform MARM fitting, the following input data were entered into *MultiArmLab*:

- The empirical time series vector of the three stock indexes
- CSGO as the fitting method and its parameters
- The absolute values of the empirical autocorrelations and cross-correlations as weights of the objective function
- The number of autocorrelation and cross-correlations lags to be fitted (set to 100 each)
- The number of cells in the marginal empirical histograms (set to 10), to be used in the specification of the hyper-cubes of the empirical hyper histogram

- The number of innovation steps (set to 100)
- The search space for the stitching parameter (set to 10 equidistant values in the interval [0,1])
- The fitting scope was set to encompass both MARM^{+} and MARM^{-} processes

We next describe the output of the fitting runs. Figure 12.1 displays information on the best model fitted by *MultiArmLab* to the empirical indexes data, using the MARM fitting methodology described in chapter 10. This model turned out to be a MARM^{+} model with a single-step innovation density (depicted in the graph), objective function value of 0.5786, and a stitching parameter value of 0.9.

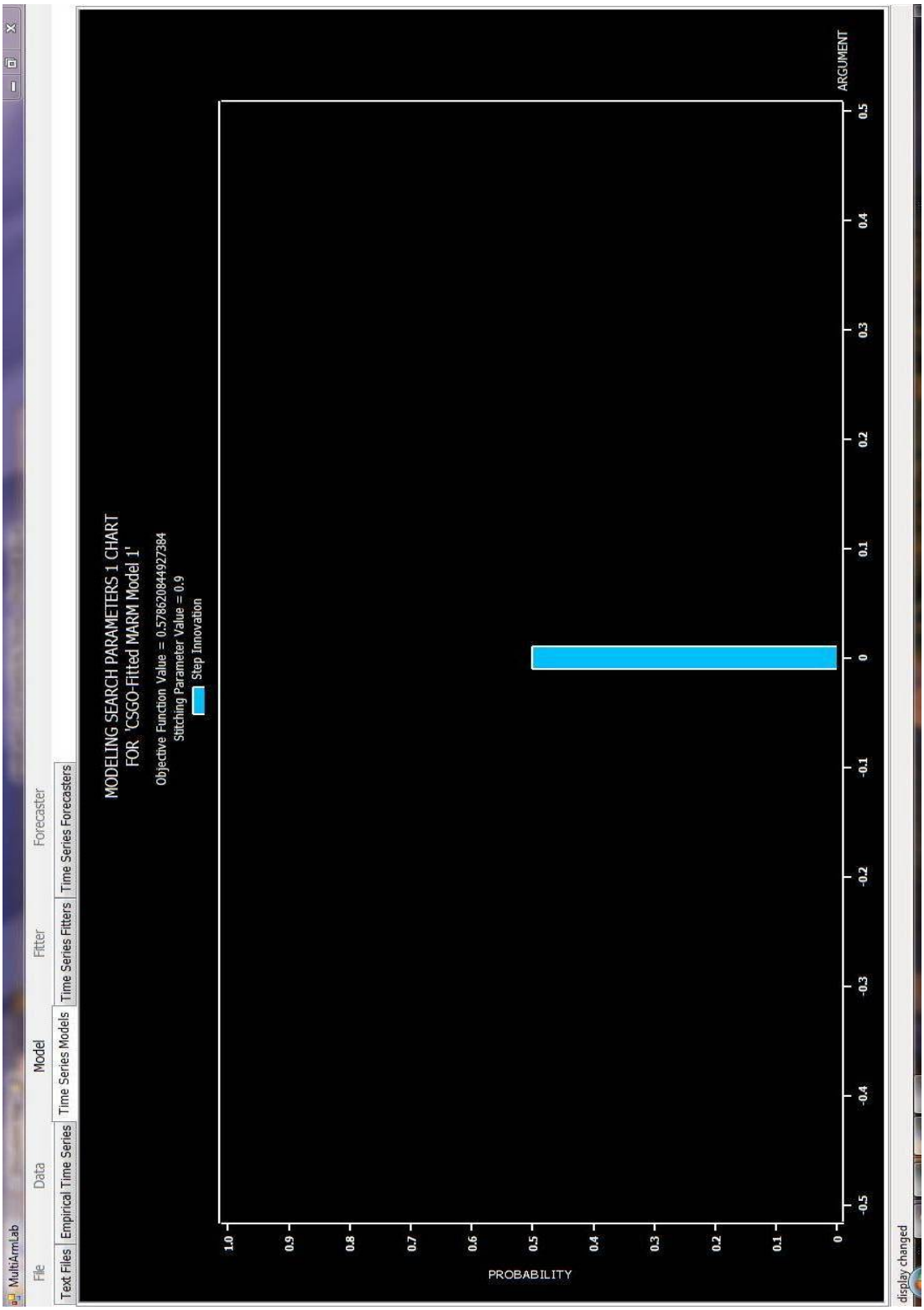

Figure 12.1 Modeling Search Parameters for the best fitted MARM model

Figures 12.2 -- 12.4 display the marginal statistics of each dimension of the best fitted model found by the *MultiArmLab* run. The screen images in Figures 12.2 – 12.4 consist of four panels with various statistics providing visual information on the fit quality as follows:

1. The upper-left panel depicts an individual empirical time series and a simulated one of the fitted model, one for each dimension of the underlying vector-valued time series (in our case a financial index). The initial values of corresponding pairs of empirical and simulated time series were arranged to coincide.
2. The upper-right panel depicts an empirical histogram of the corresponding financial index with a compatible histogram of the simulated histogram superimposed on it.
3. The lower-left panel depicts the empirical autocorrelation function of the corresponding financial index data, as well as its simulated and theoretical counterparts.
4. The lower-right panel depicts the empirical spectral density function of the corresponding financial index data, as well as its simulated and theoretical counterparts.

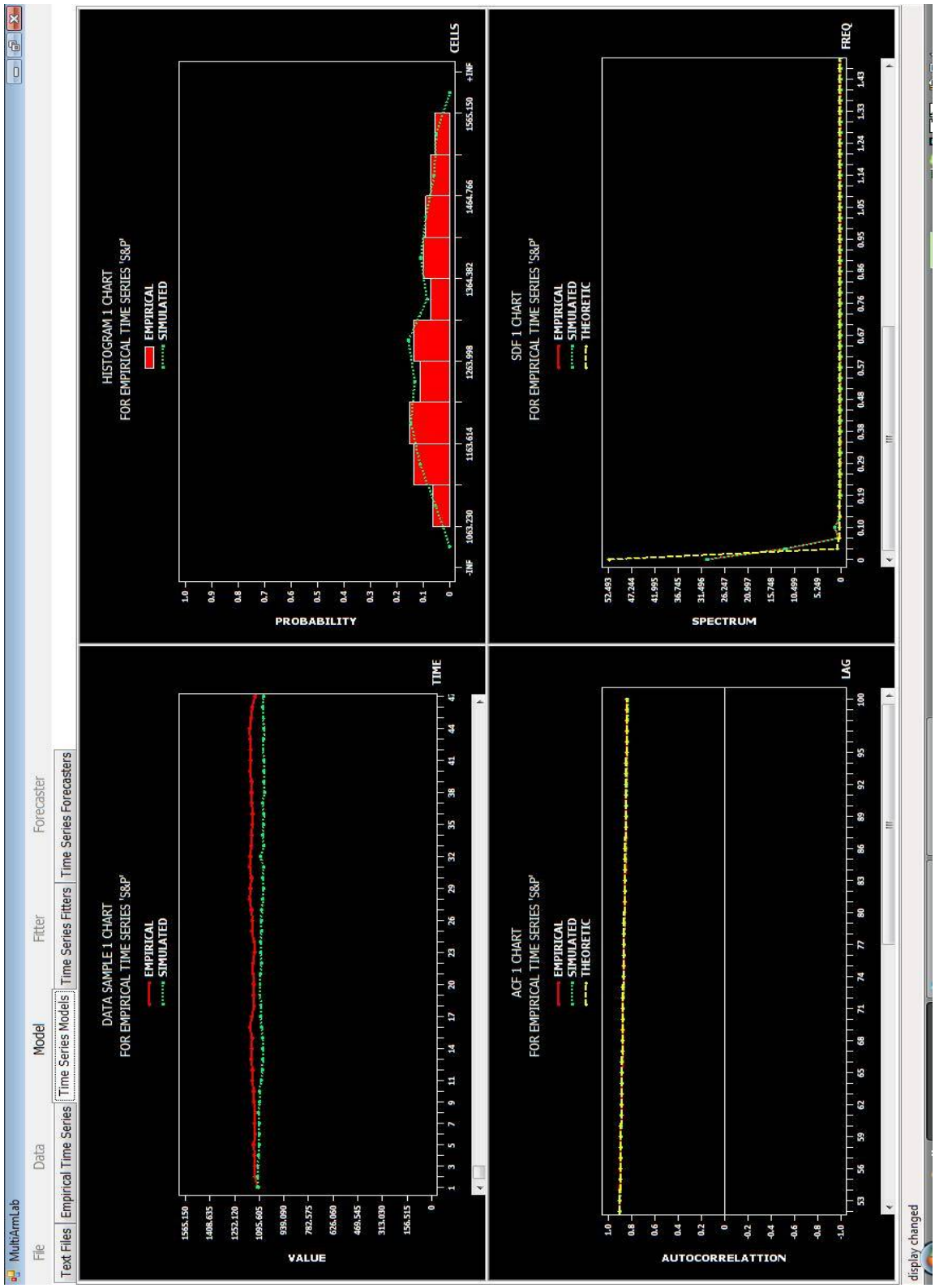

Figure 12.2 Statistics of the best MARM model fitted to a sample of S&P data

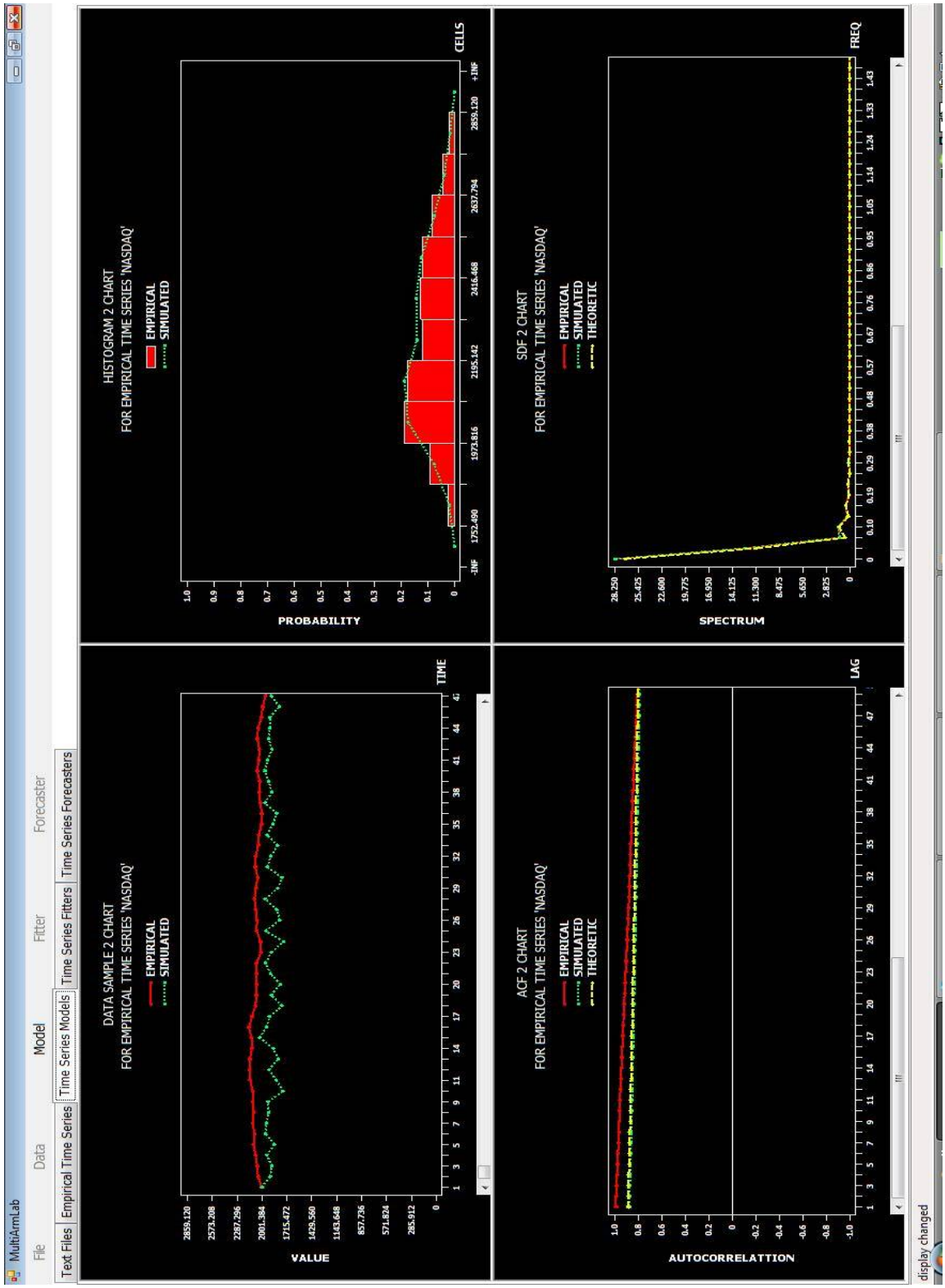

Figure 12.3 Statistics of the best MARM model fitted to a sample of NASDAQ data

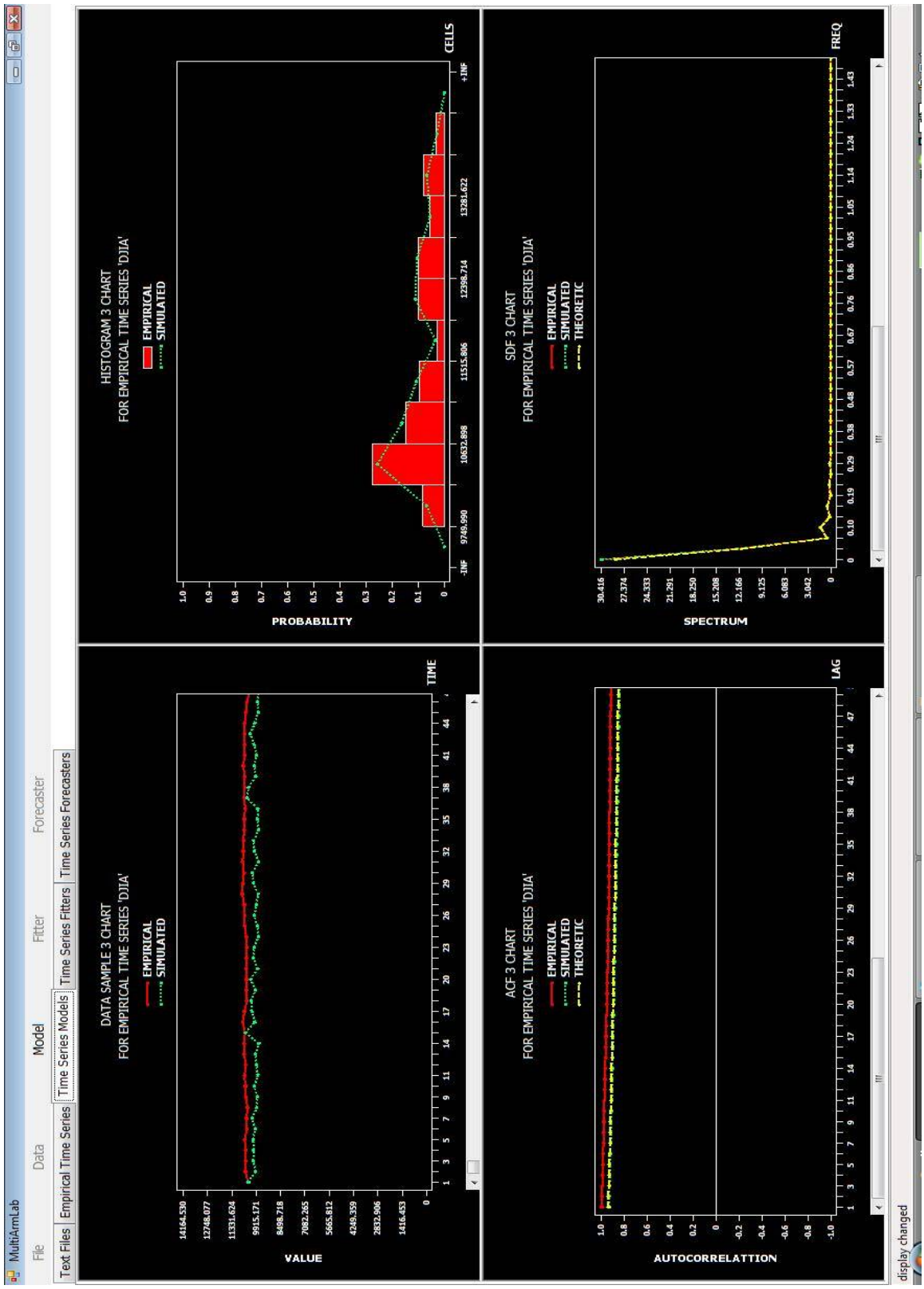

Figure 12.4 Statistics of the best MARM model fitted to a sample of DJIA data

The screen images in Figures 12.5 – 12.7 depict all empirical cross-correlation functions for each pair of financial indexes, as well as their simulated and theoretical counterparts.

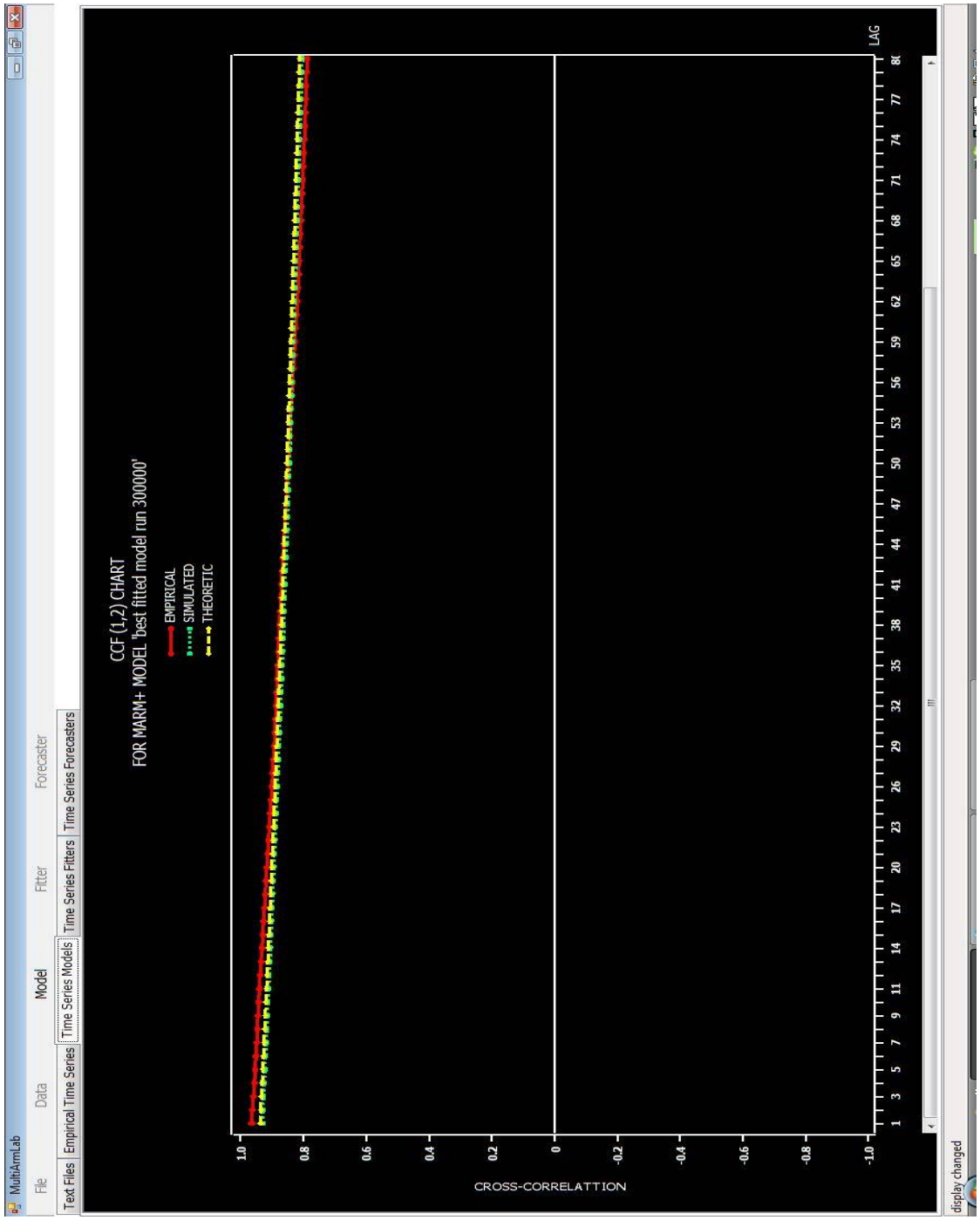

Figure 12.5 Cross-correlation functions of the best-fitted MARM model for the S&P and NASDAQ indexes

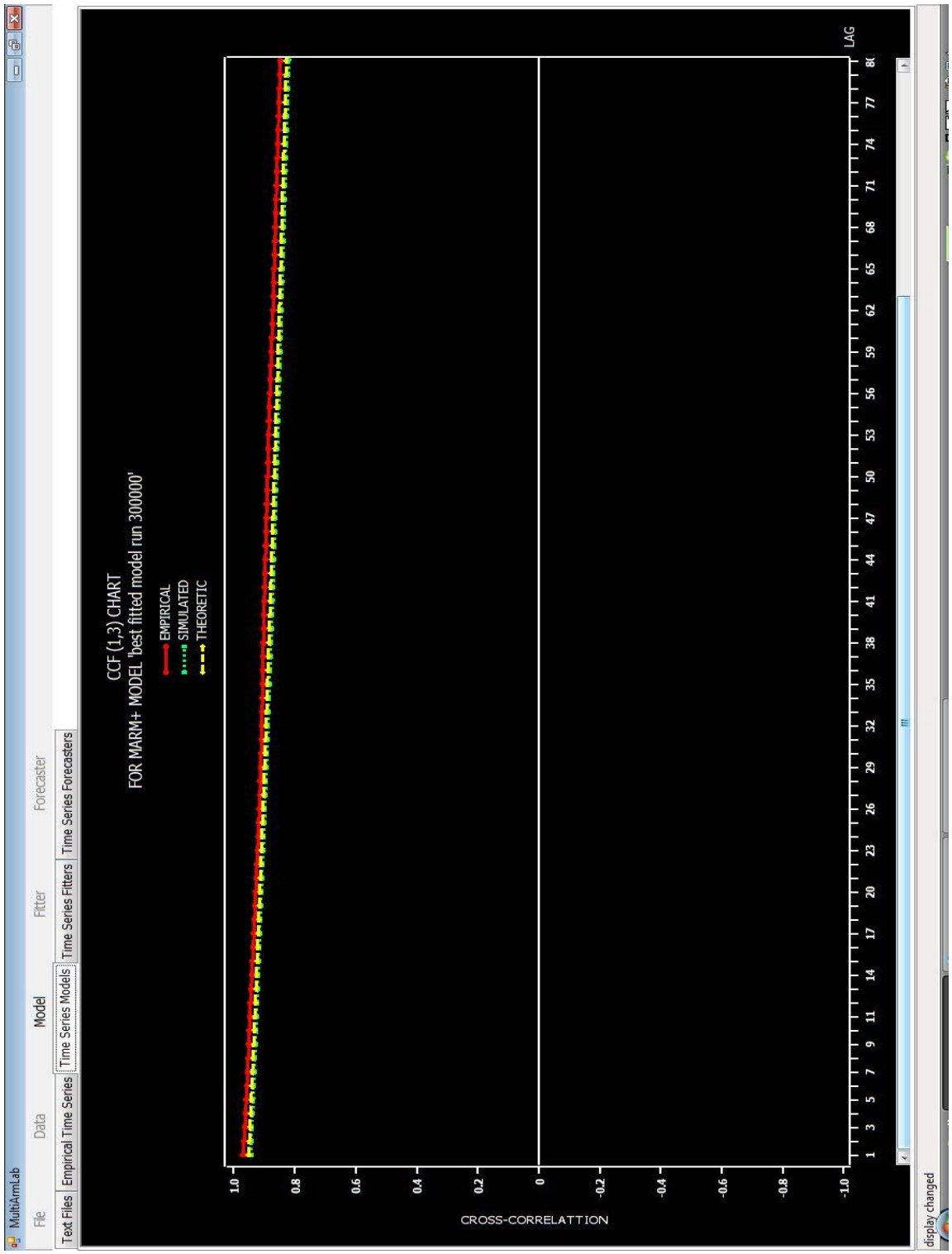

Figure 12.6 Cross-correlation functions of the best-fitted MARM model for the S&P and DJIA indexes

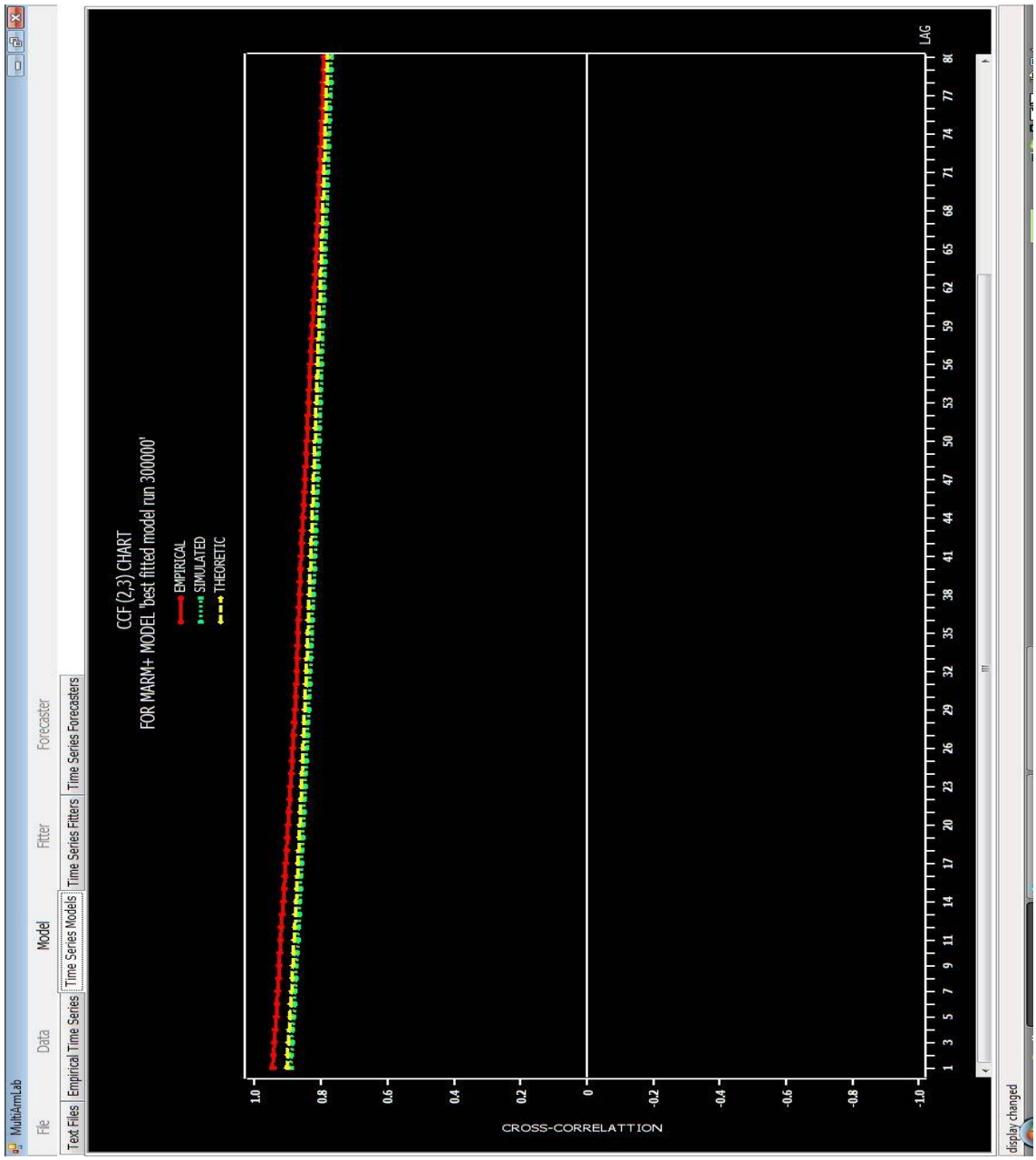

Figure 12.7 Cross-correlation functions of the best-fitted MARM model for the NASDAQ and DJIA indexes

An inspection of Figures 12.2 – 12.7 reveals that the best fitted MARM$^+$ model provides a good fit to the empirical data in accordance with requirements (a) – (c) in Chapter 1.

12.2. MARM Forecasting

Next, the best fitted MARM$^+$ model from Section 12.1 was exercised to compute point forecasts (conditional expectations) using *MultiArmLab*. The *MultiArmLab* forecasting procedure first fits a MARM model to an empirical vector-valued data sample, and then proceeds to compute the requisite forecasts, which may be both in-sample or out of sample. To perform MARM forecasting, the following input data were entered into *MultiArmLab*:

- The empirical time series vector of the three stock indexes
- A set of fitting parameters, including a template MARM fitting model, which contained the parameters of the requisite best fitted model to be used in the forecasting. In this case, that model flavor was set to MARM$^+$.

We next describe the output of the forecasting runs. Figures 12.8 - 12.10 display various forecasting-related statistics for multiple lags for each of the empirical sample data. The screen images in Figures 12.8 – 12.10 consist of four panels with various statistics pertaining to point forecasts as follows:

1. The upper-left panel depicts the tail-end of the empirical data sample of each individual index with forward forecasts superimposed on it. Each empirical value "sprouts" a branch consisting of a sequence of 3 forward forecasts, computed by optimizing the backward fitting of the previous 5 empirical values (if available).
2. The upper-right panel depicts the tail-end of the empirical data sample of each individual index and the corresponding sequence of lag-1 forward forecasts. The deviation of a forecast value from the (empirical) true value can be gauged by the vertical distances along the two time series.
3. The lower-left panel depicts the sequence of the aforementioned deviation values.

4. The lower-right panel depicts a histogram of the aforementioned deviation values.

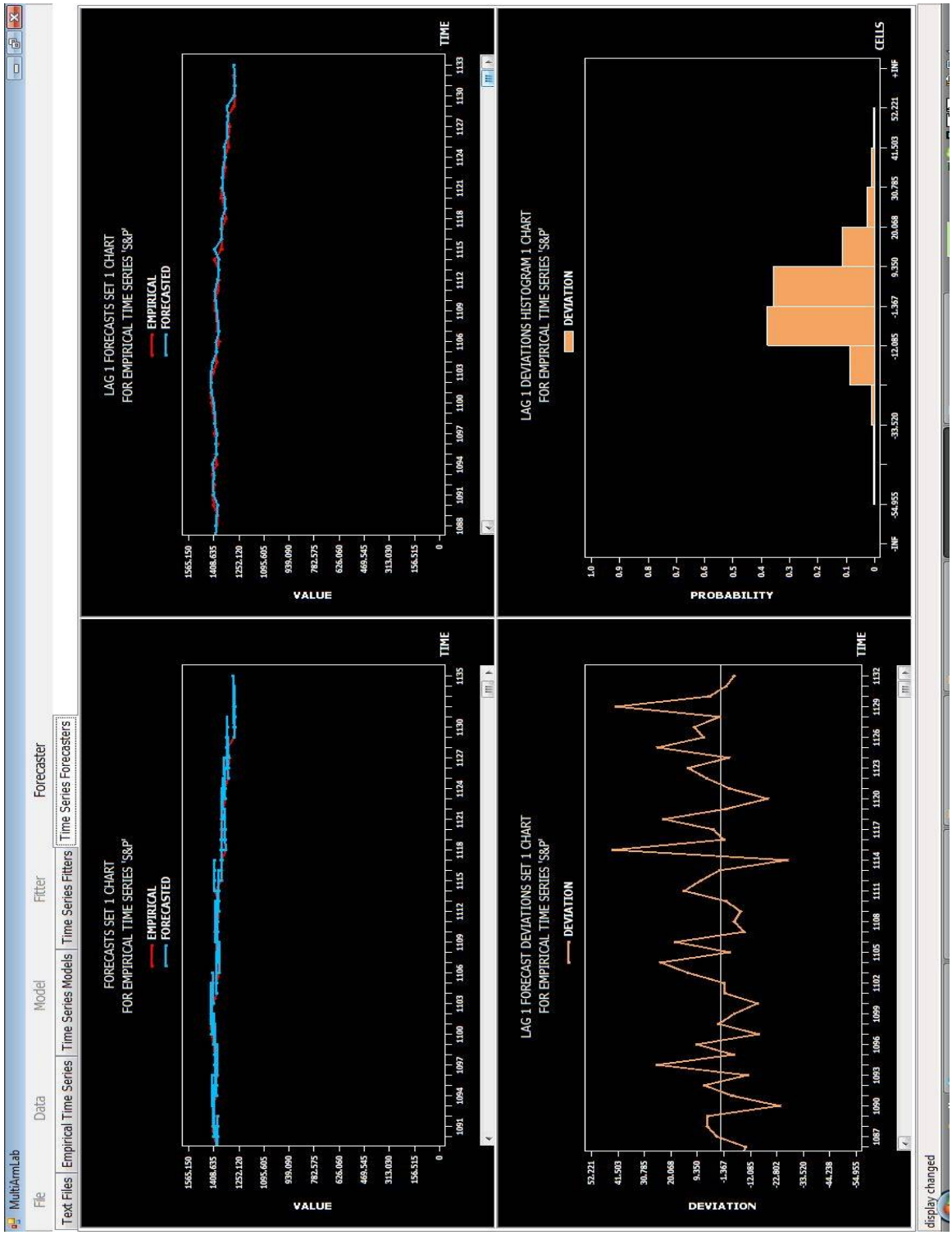

Figure 12.8 Multiple-lag forecasts and deviation statistics of the best MARM^{+} model fitted to a sample of S&P data

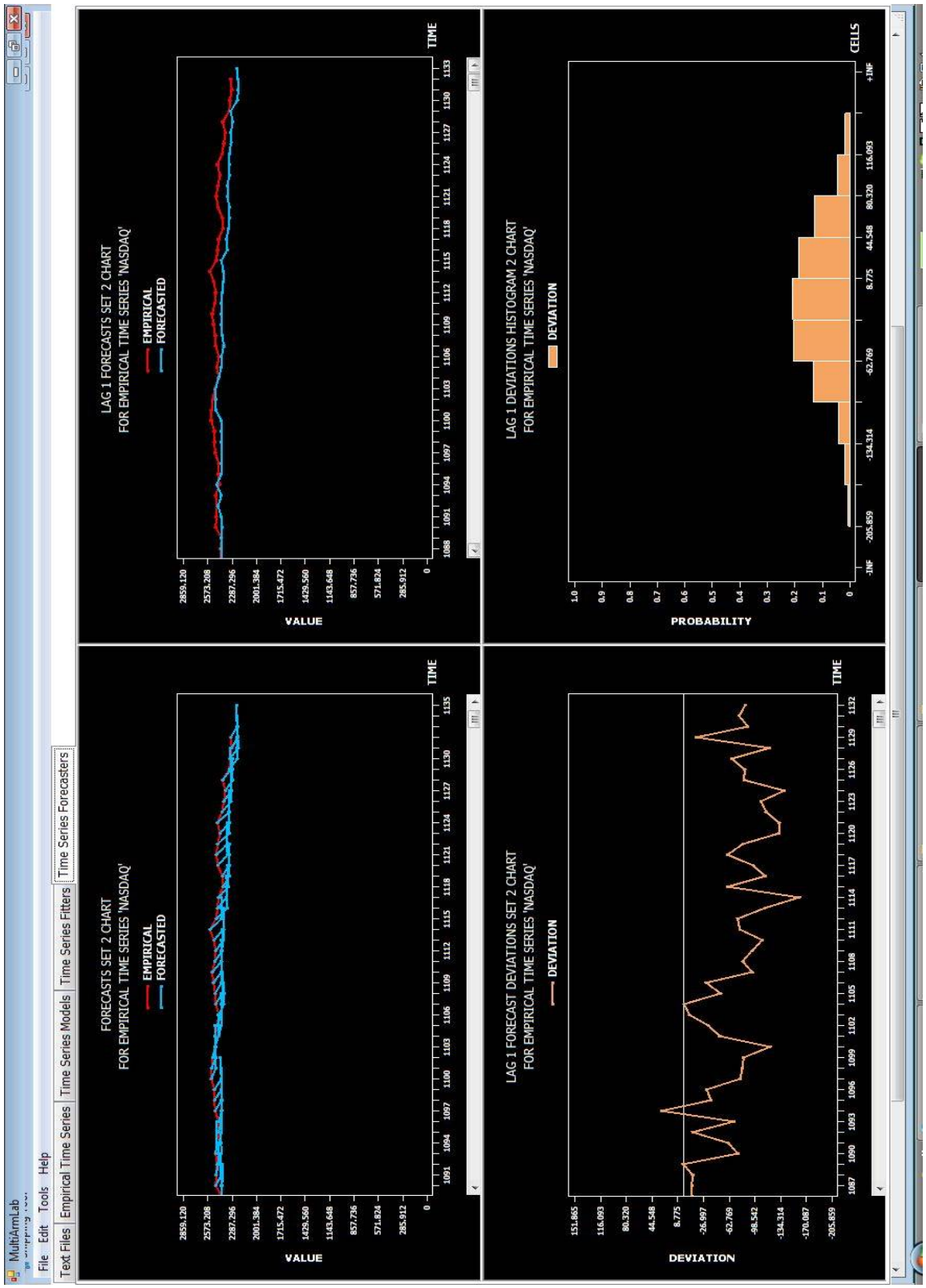

Figure 12.9 Multiple-lag forecasts and deviation statistics of the best MARM^{+} model fitted to a sample of NASDAQ data

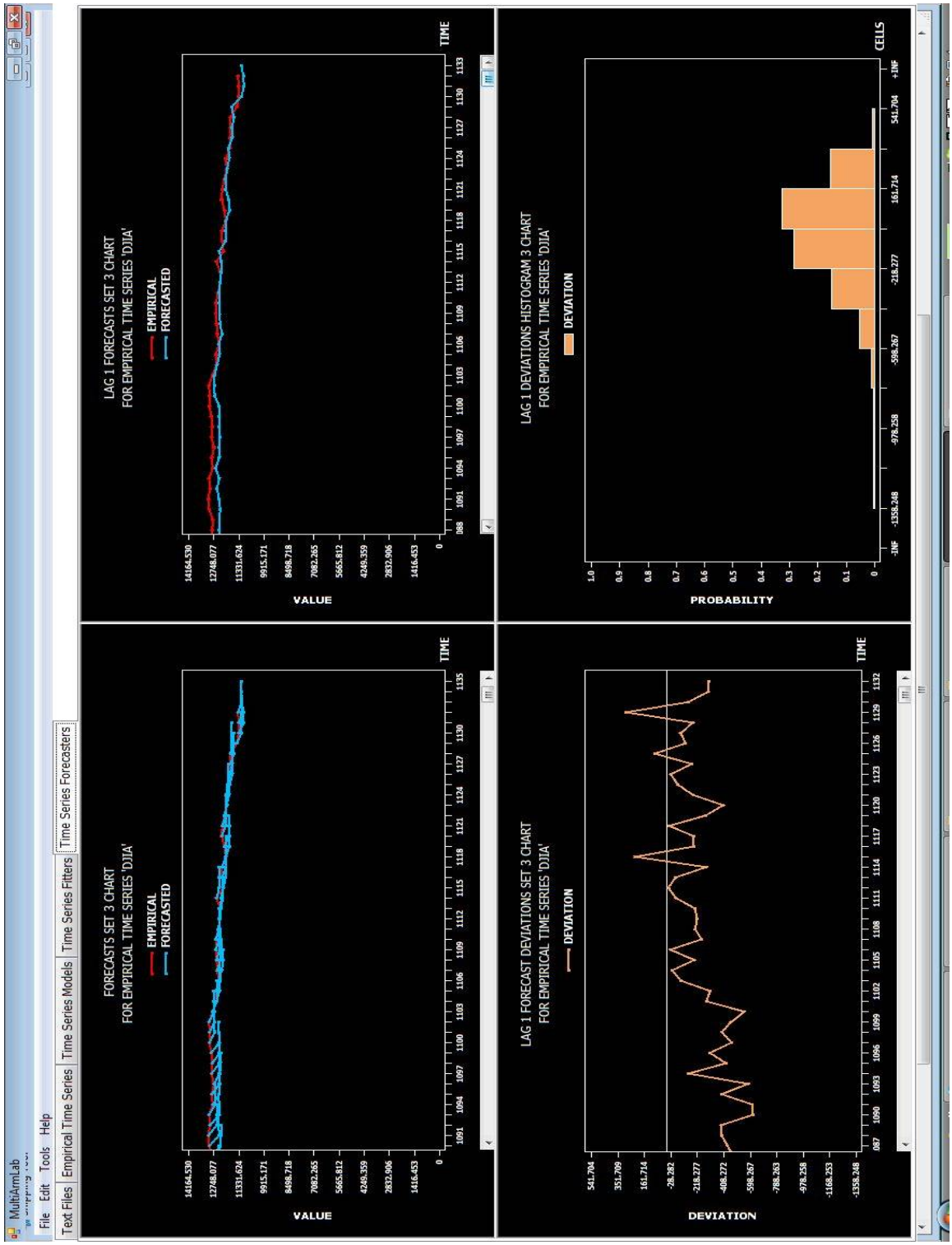

Figure 12.10 Multiple-lag forecasts and deviation statistics of the best MARM^{+} model fitted to a sample of DJIA data

An inspection of Figures 12.8 - 12.10 shows that the forecasts are quite good. In particular, inspection of the last panels shows that the deviations are arranged in a tight range around the origin, supporting this observation.

Finally, the best fitted $MARM^+$ model was exercised to compute forecast distributions (conditional distributions and densities as well as confidence intervals for the point estimates), as described in Chapter 11. Figures 12.11 - 12.13 depict these statistics for the last empirical sample of each individual index data. Note that these forward forecast values constitute out-of-sample predictions on the future values of each index. We next, discuss the forecasting results for each of the empirical time series (financial indexes). Figures 12.11 – 12.13 each displays two panels with forecast information as follows:

1. The upper panel depicts the empirical index time series with forward and backward forecasts superimposed on it for the last empirical value. Shown are the sequence of 3 forward forecasts and 5 backward forecasts, as well as the 95% confidence interval for each of them.
2. The lower panel depicts the density of the corresponding forward lag-1 forecast, conditioned on the last empirical value. Accordingly, the confidence interval for the lag-1 forward forecast was obtained by a simple search over the values of the cdf corresponding to the displayed pdf.

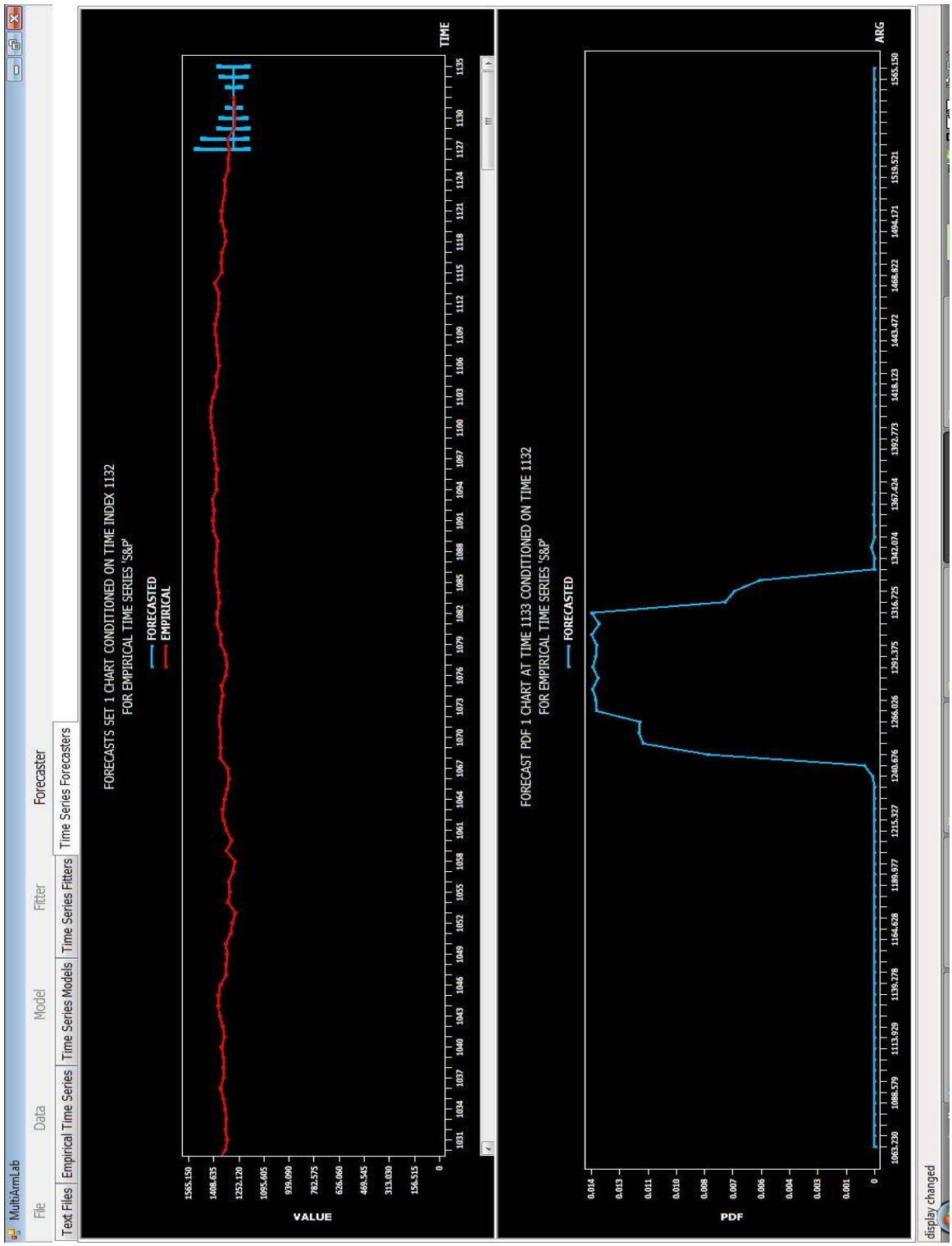

Figure 12.11 S&P forecasting statistics of the best-fitted MARM$^+$ model, conditioned on the value at index 1132 of the S&P data

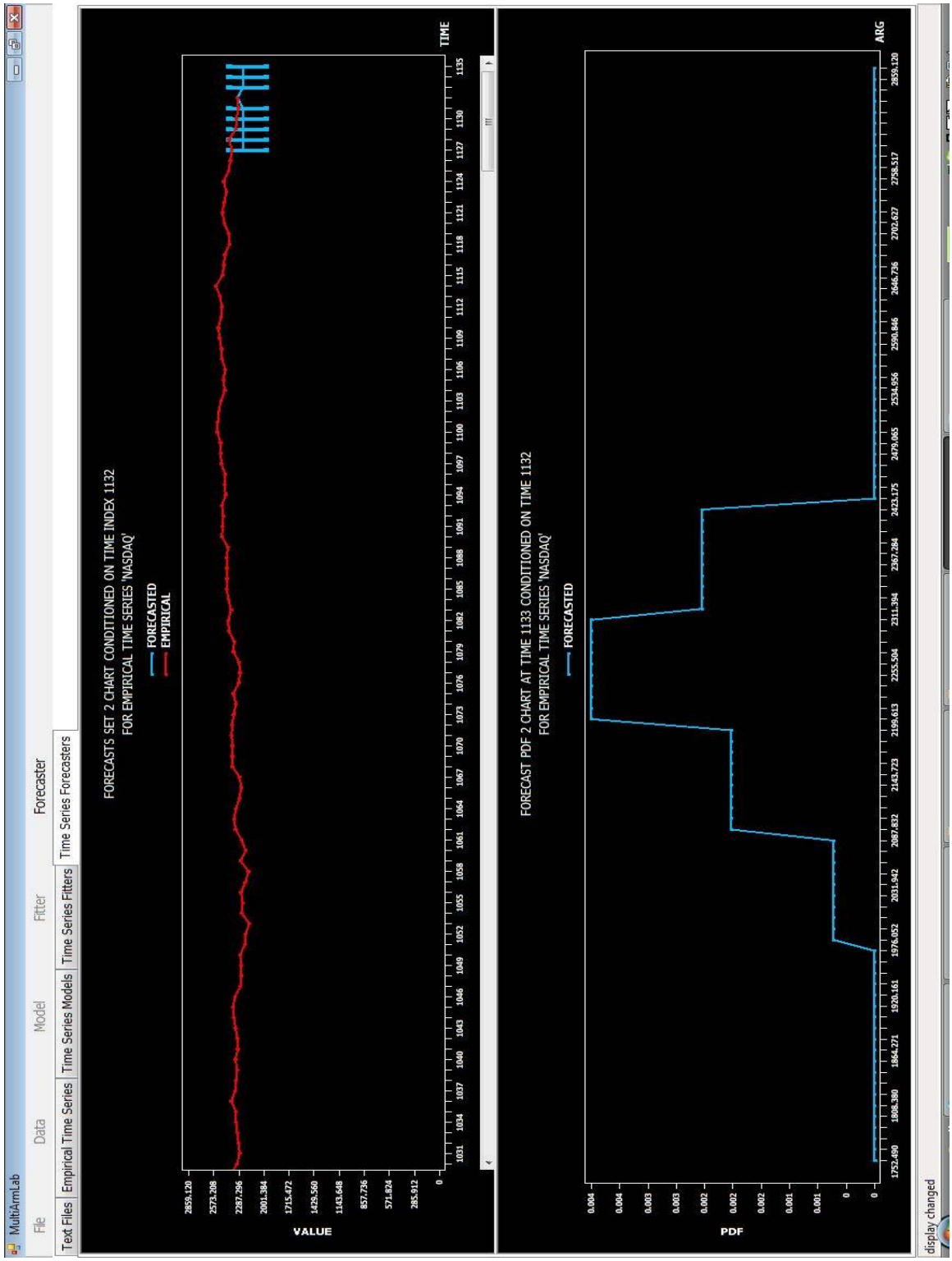

Figure 12.12 NASDAQ forecasting statistics of the best-fitted MARM^{+} model, conditioned on the value at index 1132 of the S&P data

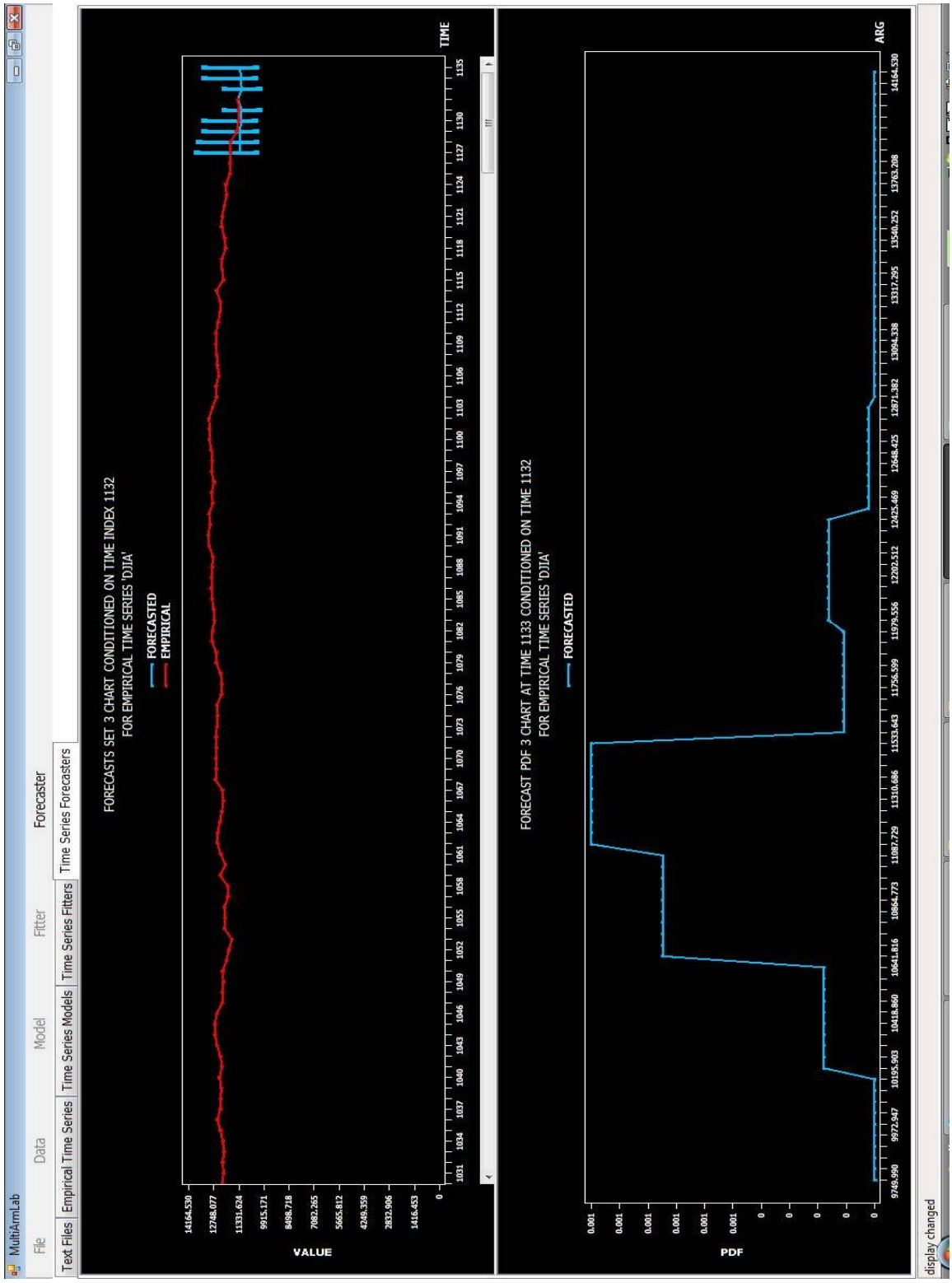

Figure 12.13 DJIA forecasting statistics of the best-fitted MARM^{+} model, conditioned on the value at index 1132 of the S&P data

Table 12.1 summarizes the forecasting related values from Figure 12.11 – 12.13; the lag-1 true value of each stock index was obtained from [38].

Table 12.1 True values versus forecast statistics for the best-fitted MARM$^+$ model, conditioned on the value at index 1132 of the S&P data

	S&P	***NASDAQ***	***DJIA***
Lag-1 True Value	1,261.52	2,251.46	11,215.51
Lag-1 Point Estimate	1,284.89	2,241.25	11,174.13
95% Confidence Interval	[1,246.66, 1,323.17]	[1,973.99, 2,395.88]	[10,192.47, 12,155.80]

It can be seen that the point forecasts are close to their true counterparts in the sense that the forecasting relative errors are ranged approximately from 5% to 8%. The respective confidence intervals are reasonably tight.

12.3. Convergence of Forecasting Statistics

The Fourier expansions used in the MARM processes formulas (e.g., transition densities, conditional expectations, autocorrelations and cross-correlations) contain infinite summations. When coding these formulas into the MARM fitting and forecasting algorithms, one obviously has to truncate the infinite sums to finite ones, so the resulting values are approximations of the true values. It is, therefore, of interest to study the convergence of the approximated values as function of an increasing expansion length.

Table 12.2 illustrates the convergence of point estimates and corresponding 95% confidence intervals, conditioned on the last empirical value for S&P stock index, with respect to expansion length. The lag-1 true value for which forecasts were computed was 1261.52.

Table 12.2 S&P forecast statistics for index 1133, conditioned on the value at index 1132 of the S&P data, generated the best-fitted MARM$^+$ model as function of expansion length

Expansion Length	***Point Estimate***	***95% Confidence Interval***
100	1,284.93	[1,243.85, 1,326.01]
1,000	1,284.89	[1,246.66, 1,323.17]
100,000	1,284.89	[1,246.63, 1,323.14]

Table 12.3 illustrates the convergence of point estimates and corresponding 95% confidence intervals, conditioned on the last empirical value for NASDAQ stock index, with respect to expansion length. The lag-1 true value for which forecasts were computed was 2,251.46.

Table 12.3 NASDAQ forecast statistics for index 1133, conditioned on the value at index 1132 of the S&P data, generated the best-fitted MARM$^+$ model as function of expansion length

Expansion Length	***Point Estimate***	***95% Confidence Interval***
100	2,241.27	[1,973.99, 2,395.88]
1,000	2,241.25	[1,973.99, 2,395.88]
100,000	2,241.25	[1,973.99, 2,395.88]

Table 12.4 illustrates the convergence of point estimates and corresponding 95% confidence intervals, conditioned on the last empirical value for DJIA stock index, with respect to expansion length. The lag-1 true value for which forecasts were computed was 11,215.51.

Table 12.4 DJIA forecast statistics for index 1133, conditioned on the value at index 1132 of the S&P data, generated the best-fitted MARM$^+$ model as function of expansion length

Expansion Length	***Point Estimate***	***95% Confidence Interval***
100	11,169.23	[10,197.89, 12,140.56]
1,000	11,174.13	[10,192.47, 12,155.80]
100,000	11,174.02	[10,192.38, 12,155.67]

The corresponding visual effects on the attendant conditional densities are shown in Figures 12.14 – 12.22.

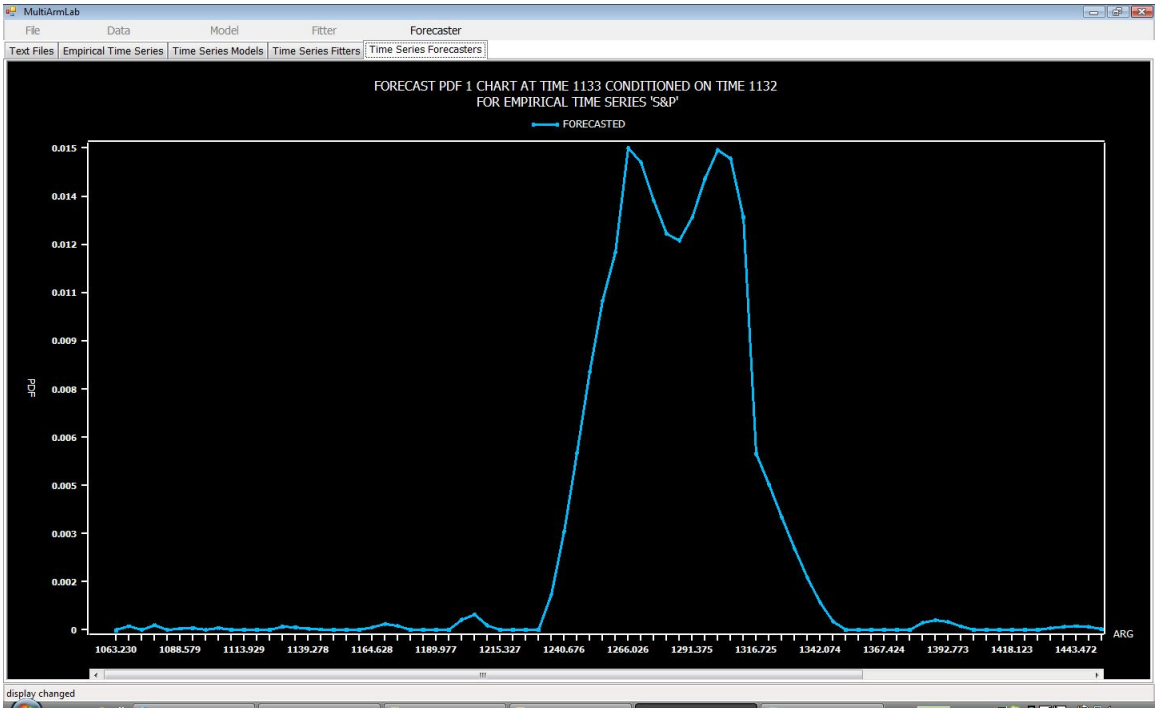

Figure 12.14 S&P density forecast at index 1133, conditioned on index 1132 of the S&P data, generated by the best-fitted MARM^{+} model with expansion length 100

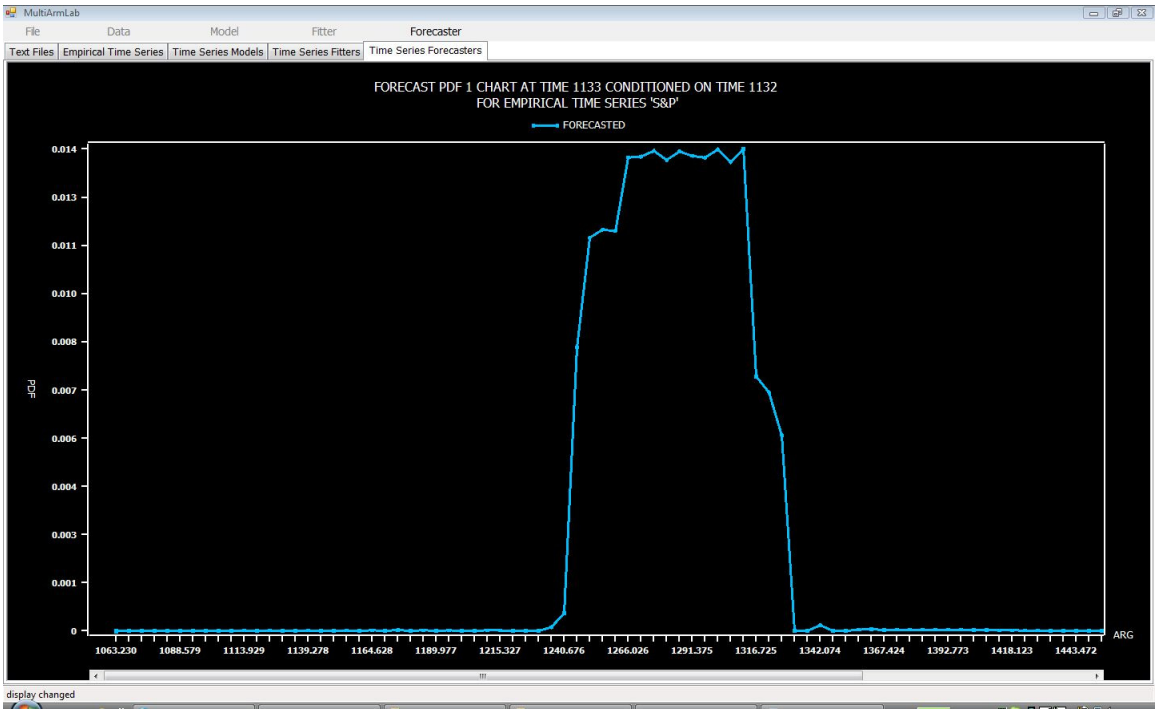

Figure 12.15 S&P density forecast at index 1133, conditioned on index 1132 of the S&P data, generated by the best-fitted MARM^{+} model with expansion length 1,000

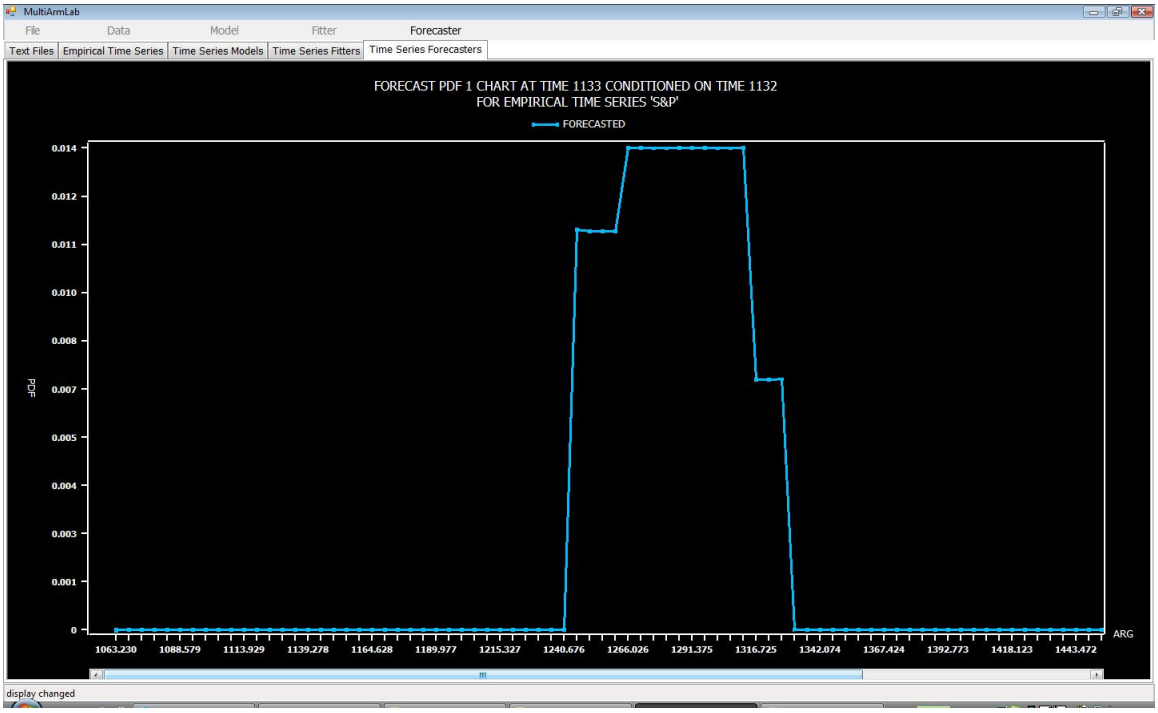

Figure 12.16 S&P density forecast at index 1133, conditioned on index 1132 of the S&P data, generated by the best-fitted MARM^{+} model with expansion length 100,000

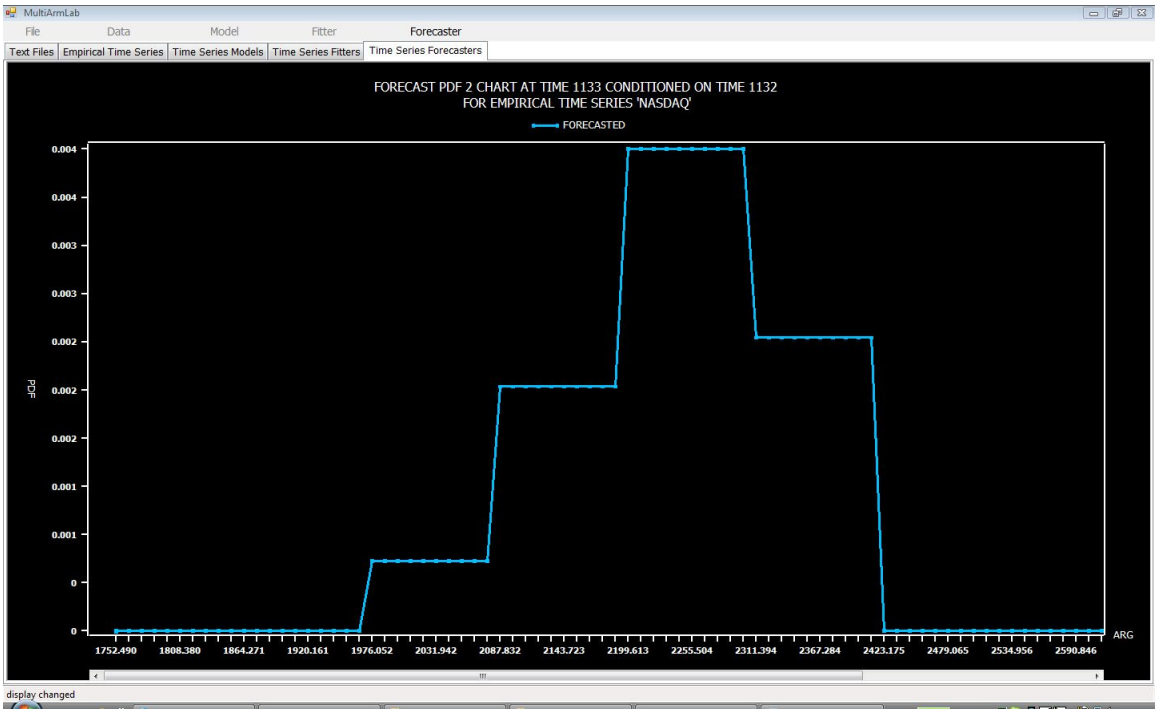

Figure 12.17 NASDAQ density forecast at index 1133, conditioned on index 1132 of the S&P data, generated by the best-fitted MARM^{+} model with expansion length 100

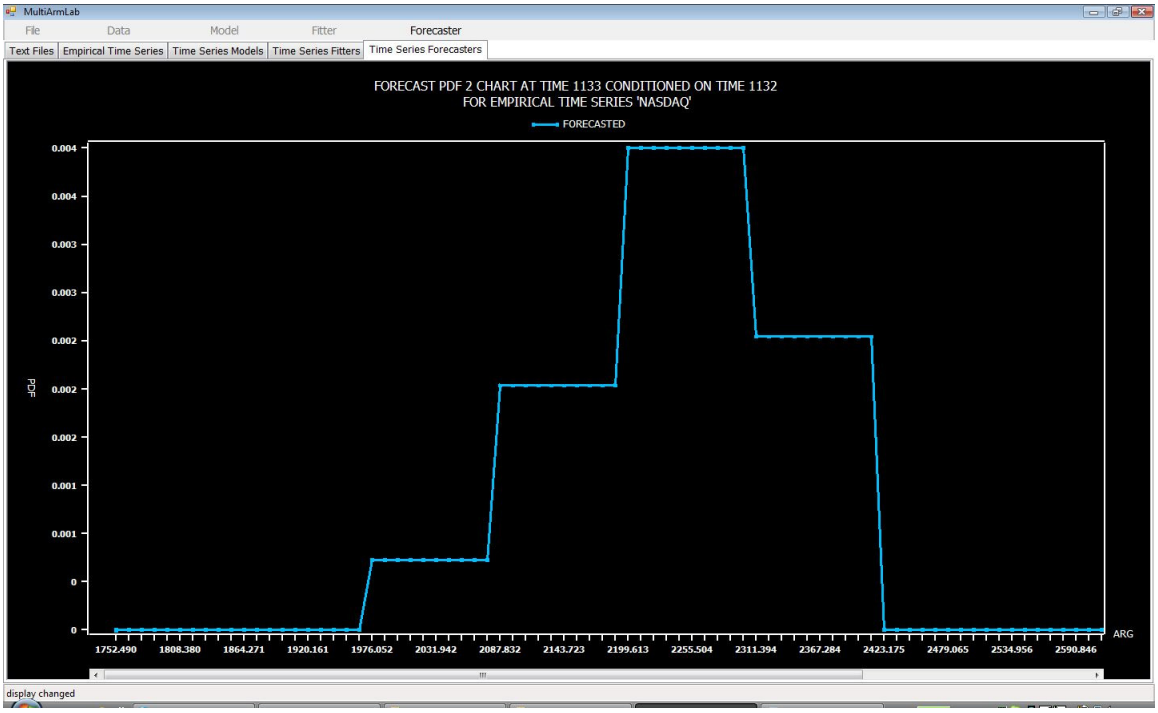

Figure 12.18 NASDAQ density forecast at index 1133, conditioned on index 1132 of the S&P data, generated by the best-fitted MARM^{+} model with expansion length 1,000

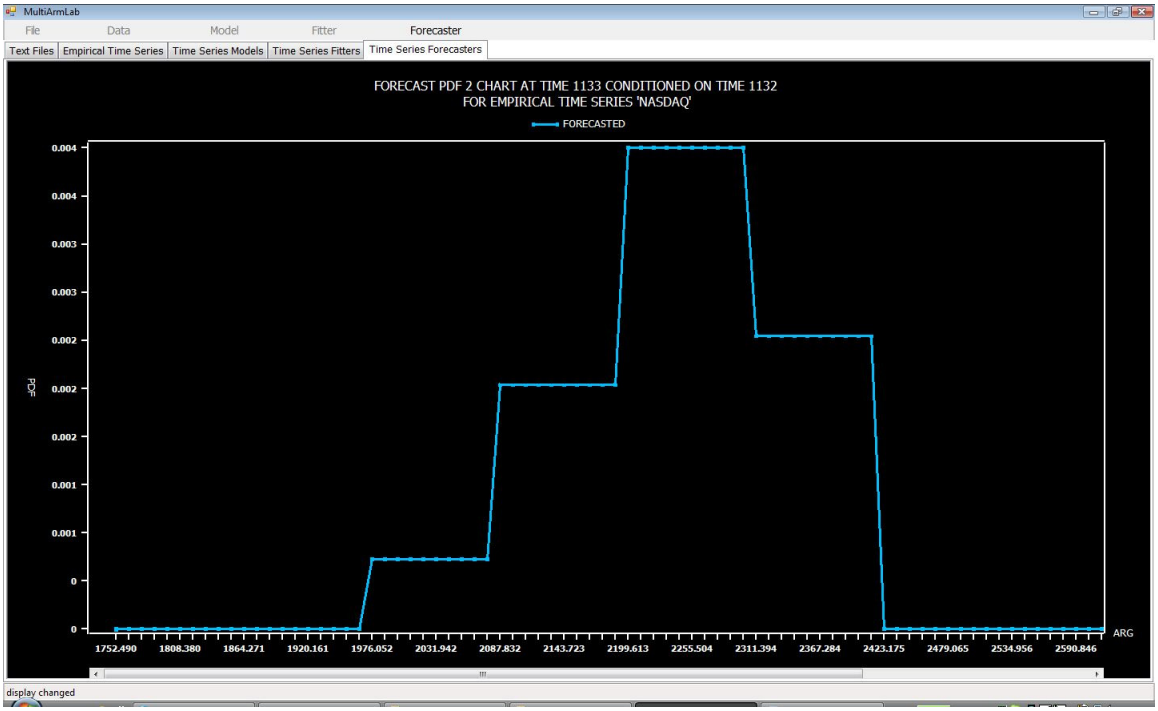

Figure 12.19 NASDAQ density forecast at index 1133, conditioned on index 1132 of the S&P data, generated by the best-fitted MARM^{+} model with expansion length 100,000

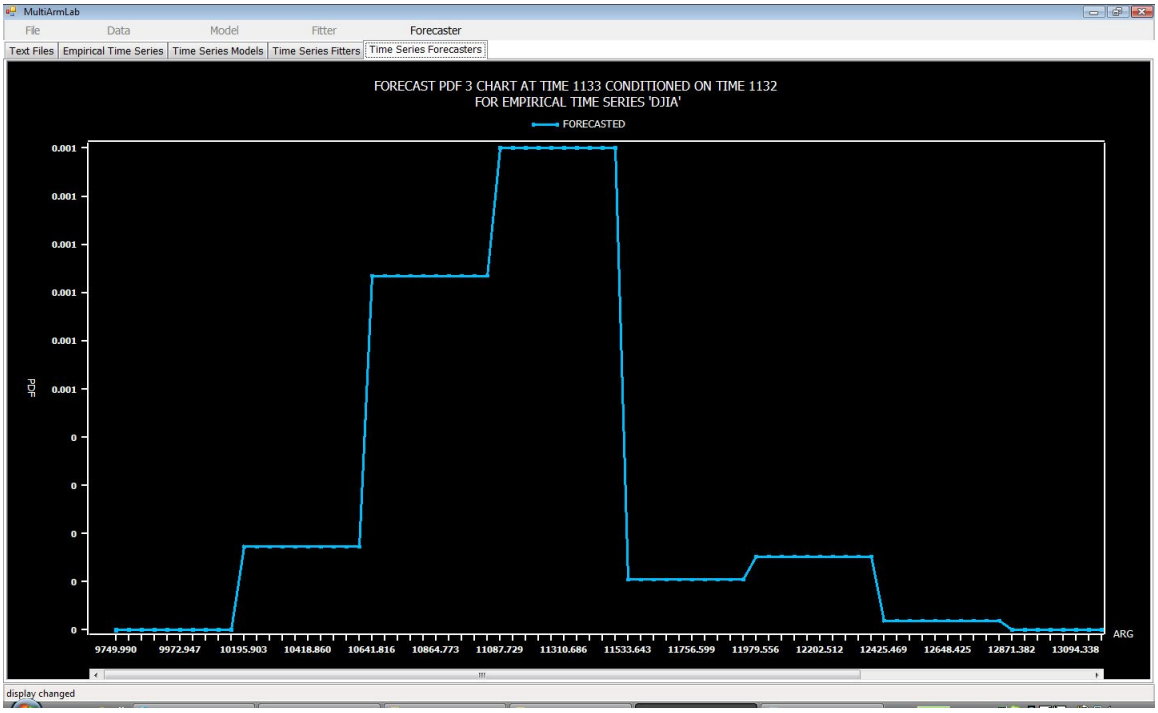

Figure 12.20 DJIA density forecast at index 1133, conditioned on index 1132 of the S&P data, generated by the best-fitted MARM$^+$ model with expansion length 100

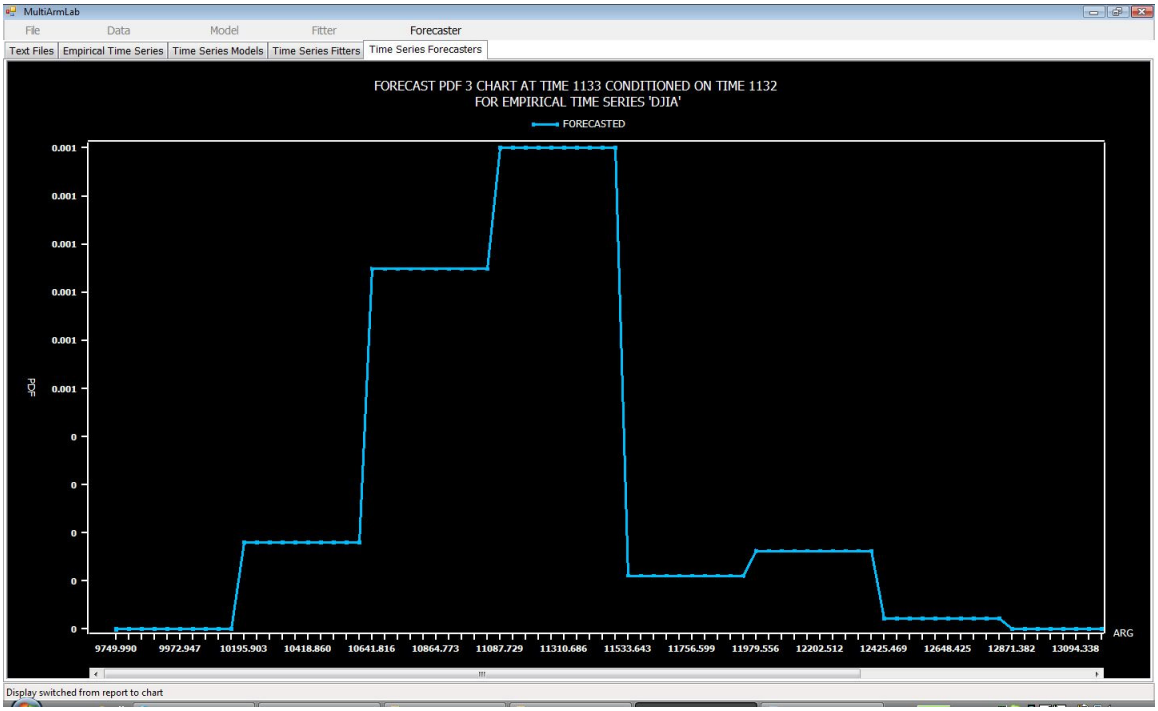

Figure 12.21 DJIA density forecast at index 1133, conditioned on index 1132 of the S&P data, generated by the best-fitted MARM$^+$ model with expansion length 1,000

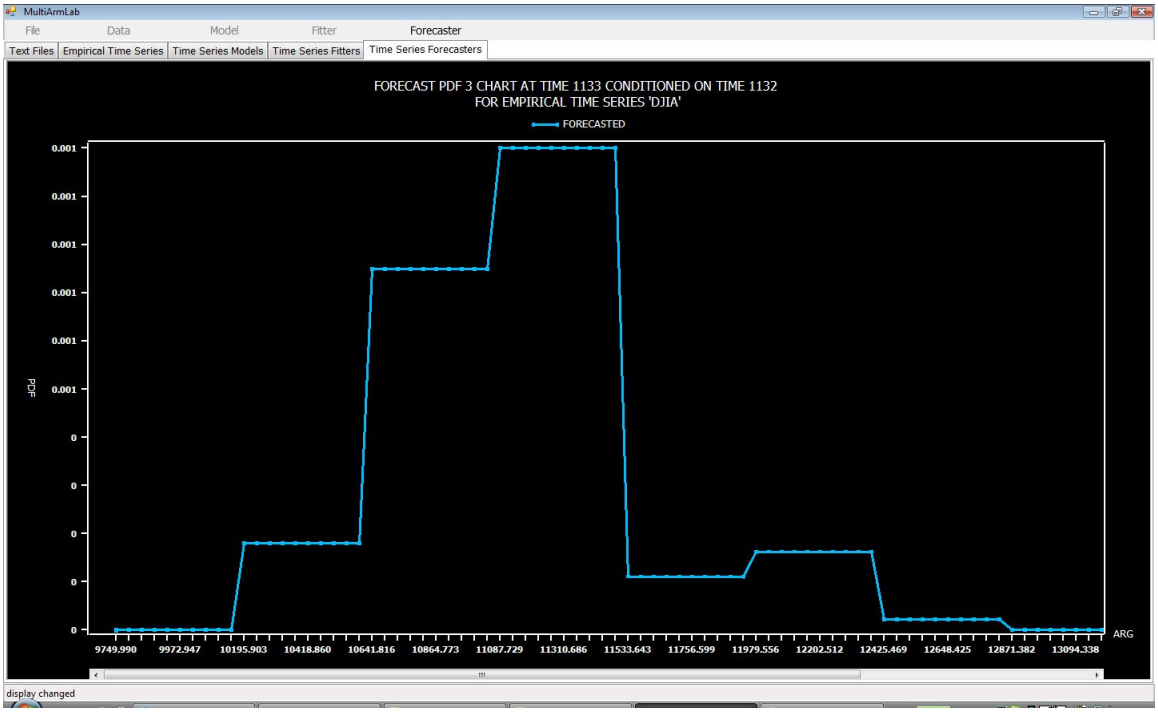

Figure 12.22 DJIA density forecast at index 1133, conditioned on index 1132 of the S&P data, generated by the best-fitted MARM^{+} model with expansion length 100,000

We found that the effect of expansion length on the convergence depends on the dimension within the empirical data vectors. Specifically, the expansion length affects more the first dimension, whereas the higher dimensions are affected far less. This phenomenon is a consequence of the MARM process definition, since the integration operations involved in the construction of marginal MARM processes of dimension higher than one dilute that effect. Tables 12.2 -12.4 as well as Figures 12.14 – 12.22 illustrate that an expansion length of 1000 is adequate for computing forecasting statistics, since the aforementioned statistics appear to already have adequately converged for all practical purposes. It may be noted that the shape of the conditional pdf becomes progressively smoother when increasing the expansion length.

13. Conclusion

This thesis defined a new class of vector-valued stochastic processes, called MARM. It described the construction of two flavors of MARM processes, $MARM^+$ and $MARM^-$, thereby extending previous work on ARM processes (one-dimensional MARM processes) in Melamed (1999). It studied the statistics of MARM processes (transition structure and second order statistics), and devised MARM-based fitting and forecasting algorithms providing point estimators and confidence intervals. The thesis illustrates how the MARM modeling methodology produces high-fidelity multivariate models from empirical vector-valued time series, by simultaneously fitting first-order statistics (the multi-dimensional empirical histogram) and second-order statistics (empirical autocorrelations and cross-correlations).

The key advantage of MARM processes is its ability to fit a strong statistical signature consisting of empirical first-order and second-order statistics simultaneously. More precisely, MARM processes exactly fit an arbitrary multi-dimensional empirical histogram and approximately fit the leading terms of autocorrelations and cross-correlations functions. This ability appears to make the MARM modeling methodology unique in its goal of fitting a model to such a class of strong statistical signatures. Furthermore, the specialized MARM modeling and forecasting methodologies, utilizing iid step-function innovation densities and hyper-step distortions, as proposed by this thesis, constitute practical methodologies, suitable for implementation on a computer. We demonstrated the efficacy of these methodologies with an example of a three-dimension time series vector, using a software environment, called *MultiArmLab*, which supports MARM modeling and forecasting.

MARM processes have a number of shortcomings. First, MARM processes are not invariant under permutations of the underlying empirical time series components of in the empirical time series vector. More specifically, the definition of MARM processes depends on the ordering of individual empirical time series in the empirical time series vector in the sense that the ordering selected affects both fitting and forecasting results. Unfortunately, the construction of MARM processes does not allow us to eliminate this dependence on ordering. However, we may explore and select an ordering that provides the best MARM model among several ordering candidates.

Secondly, the reliance on a hyper-histogram requires a large amount of vector-valued empirical data, and these data requirements grow exponentially in the underlying data dimensionality. Consequently, the requisite amount of data may not be available.

The innovation density and distortion discussed in this thesis are just two possible implementations of the general MARM modeling and forecasting methodologies. Thus, this work could be extended to additional variants of MARM fitting and forecasting methodologies. Such extensions can be the subject of future work.

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