ON LARGE FAMILIES OF AUTOMORPHIC L-FUNCTIONS ON GL_2

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ABSTRACT OF THE DISSERTATION

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The goal of this dissertation is analytical investigation of large families of automorphic L-functions on GL_2. With a view towards potential applications in Analytic Number Theory, we investigate families of holomorphic modular forms, but additionally averaged over the nebentypus - characters and over the levels. We establish orthogonality in such a family in the limited range in the form of large sieve type inequality. Further we investigate non-vanishing at the central point of the corresponding L-functions and give a bound for the sixth moment for $\Gamma_1(q)$-family, consistent with Lindelöf hypothesis on average.

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Dedication

за Ирину и Гордану,

с љубављу и захвалношћу

Table of Contents

Chapter 1

Introduction and preliminaries

1.1 Automorphic forms

The main object of study in this thesis are holomorphic automorphic forms on GL_2 with respect to congruence subgroups of $SL_2(\mathbb{Z})$ and L-functions associated to them.

Classical picture is the following: the group $SL_2(\mathbb{R})$ acts on the hyperbolic upper-half plane $\mathbb{H} = \{\mathfrak{Im}(z) > 0\}$ by Möbius transformations. If Γ is an arbitrary discrete subgroup of $SL_2(\mathbb{R})$, a point $s \in \mathbb{R} \cup \{\infty\}$ is a *cusp* of Γ if there exists a parabolic element of Γ fixing s. Then Γ acts on $\mathbb{H}^*$, the union of $\mathbb{H}$ and all the cusps of Γ. Subgroups for which $X(\Gamma) = \Gamma \backslash \mathbb{H}^*$ is a compact Riemann surface are called Fuchsian groups of the fist kind. Such subgroups have a finite covolume and at most a finite number of Γ-inequivalent cusps.

The most important example of such groups is the Hecke congruence subgroup - a discrete subgroup of $SL_2(\mathbb{R})$ defined by

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{q} \right\}.$$

Let χ be a Dirichlet character modulo q. Then a function $f : \mathbb{H} \to \mathbb{C}$ is a $\Gamma_0(q)$-automorphic form of weight k and nebentypus χ if:

- $f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z), \qquad$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q)$;
- f is holomorphic in $\mathbb{H}$ and
- f is holomorphic at every cusp of $\Gamma_0(q)$.

We denote the space of all such functions by $M_k(\Gamma_0(q), \chi)$. A function $f \in M_k(\Gamma_0(q), \chi)$

is a *cusp form* if it vanishes at every cusp of $\Gamma_0(q)$, and the subspace of such forms is denoted by $S_k(\Gamma_0(q),\chi)$.

Geometrically, we can interpret holomorphic Γ-automorphic forms of even weight $2k$ and trivial character as follows: let $\Omega = T^*(X(\Gamma))$ be the cotangent bundle to $X(\Gamma)$ and $\Omega^{\otimes k}$ its kth tensor power. Then automorphic forms of weight $2k$ are the holomorphic sections of $\Omega^{\otimes k}$, or in other words, every meromorphic differential on $X(\Gamma)$ of degree k pulls back (with respect to the natural projection $\mathbb{H} \to X(\Gamma)$) to a meromorphic differential $f(z)(dz)^k$ on $\mathbb{H}$, where f is an automorphic form of the given type.

Denote $j(g,z) = cz+d$, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, and

$$K = SO(2) = \left\{ r(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in [0,2\pi) \right\}.$$

Then every automorphic form $f \in M_k(\Gamma_0(q),\chi)$ can be lifted to a function on $SL_2(\mathbb{R})$ defined by:

$$\phi_{f,\infty}(g) = j(g,i)^{-k} f(g(i)), \qquad g \in SL_2(\mathbb{R}),$$

which has the following properties:

- $\phi_{f,\infty}(\gamma g) = \chi(\gamma)\phi_{f,\infty}(g)$, for all $\gamma \in \Gamma_0(q)$, where $\chi(\gamma) := \chi(d_\gamma)$;
- $\phi_{f,\infty}(g\, r(\theta)) = e^{-ik\theta}\phi_{f,\infty}(g)$, for all $r(\theta) \in K$;
- $\phi_{f,\infty}(g)$ satisfies the growth condition:
$$\phi_{f,\infty}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} r(\theta) \right) \ll y^N, \quad \text{for some} \quad N, \quad \text{as} \quad y \to +\infty;$$
- $\Delta\phi_{f,\infty} = -\frac{k}{2}\left(\frac{k}{2}-1\right)\phi_{f,\infty}$, where Δ is the Casimir operator of $SL_2(\mathbb{R})$ given in (x,y,θ)- coordinates by
$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - y\frac{\partial^2}{\partial x \partial\theta};$$
- if f is a cusp form, then $\phi_{f,\infty}$ is *cuspidal*, meaning:
$$\int_0^1 \phi_{f,\infty}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx = 0, \quad \text{for all} \quad g.$$

The modern context for automorphic forms is provided by the theory of automorphic representations of reductive groups over adeles over number fields. Here we will just briefly describe the lift of classical automorphic forms to the adelic space.

Let $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ denote the ring of adeles of $\mathbb{Q}$ i.e. the restricted direct product of $\mathbb{Q}_\infty = \mathbb{R}$ and $\mathbb{Q}_p$, $p = 2, 3, 5, \ldots$ with respect to open compact subrings $(\mathbb{Z}_p)_{p=2,3,\ldots}$. The image of $\mathbb{Q}$ in $\mathbb{A}$ is a discrete subgroup since $\mathbb{Q} \cap (-1, 1) \times \prod_{p<\infty} \mathbb{Z}_p = \{0\}$.

The group $GL_2(\mathbb{A})$ is the restricted direct product of $GL_2(\mathbb{R})$ and $GL_2(\mathbb{Q}_p)$, $p = 2, 3, 5, \ldots$ with respect to $(GL_2(\mathbb{Z}_p))_p$. Now, if we denote

$$K_{0,p}(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p) \ : c \in q\mathbb{Z}_p \right\}$$

and then $K_0(q) = \prod_{p<\infty} K_{0,p}(q)$, the following holds:

$$\mathbb{A}^\times = \mathbb{Q}^\times \, \mathbb{R}^\times_{>0} \prod_{p<\infty} \mathbb{Q}_p,$$

$$GL_2(\mathbb{A}) = GL_2(\mathbb{Q}) \, GL_2^+(\mathbb{R}) \, K_0(q) \qquad \text{and}$$

$$GL_2(\mathbb{Q}) \cap \, GL_2^+(\mathbb{R}) \, K_0(q) \ = \Gamma_0(q). \tag{1.1}$$

Next, each Dirichlet character $\chi \pmod q$ canonically determines a character of the idele class group $\mathbb{Q}^\times \backslash \mathbb{A}^\times$: let χ_p be the composition of the original χ and the natural projection $\mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}/q\mathbb{Z})^\times$. Then $\prod_{p<\infty} \chi_p$ is a character of $\mathbb{A}^\times$ trivial on $\mathbb{Q}^\times \, \mathbb{R}^\times_{>0}$, which we will also denote χ. Moreover, it induces the character of $K_0(q)$ in the usual manner.

Any $g \in GL_2(\mathbb{A})$ can be decomposed according to (1.1) as $g = \gamma g_\infty k_0$, and then for any $f \in S_k(\Gamma_0(q), \chi)$ we can define a function ϕ_f on $GL_2(\mathbb{A})$ by

$$\phi_f(g) = \phi_f(\gamma g_\infty k_0) := \, j(g_\infty, i)^{-k} f(g_\infty(i)) \chi(k_0).$$

This gives a realization of $S_k(\Gamma_0(q), \chi)$ in the space of functions ϕ on $GL_2(\mathbb{A})$ satisfying the following conditions:

- $\phi(\gamma g) = \phi(g)$, for all $\gamma \in GL_2(\mathbb{Q})$;

- $\phi(g \, k_0) = \phi(g) \, \chi(k_0)$, for all $k_0 \in K_0(q)$;

- $\phi(g\,r(\theta)) = e^{-ik\theta}\phi(g)$;
- the restriction of ϕ on $GL_2^+(\mathbb{R})$ satisfies:
$$\Delta\phi = -\frac{k}{2}\left(\frac{k}{2}-1\right)\phi\,;$$
- $\phi(g\,z) = \chi(z)\phi(g)$ for all $z \in Z(\mathbb{A}) \simeq \mathbb{A}^\times$ – the center of $GL_2(\mathbb{A})$;
- for every $c > 0$ and every compact $\Omega \subseteq GL_2(\mathbb{A})$, there exists N such that
$$\phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) \ll |a|^N$$
for all $g \in \Omega$ and $a \in \mathbb{A}^\times,\ |a| > c$;
- ϕ is cuspidal:
$$\int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0, \qquad \text{for almost all} \quad g.$$

The quotient $\mathbb{Q}\backslash\mathbb{A} \cong N(\mathbb{Q})\backslash N(\mathbb{A})$ is compact and hence the last integral is defined for almost all g.

For our fixed unitary idele class character $\chi : Z(\mathbb{Q})\backslash Z(\mathbb{A}) \to \mathbb{C}^*$, let $L^2(\chi)$ be the Hilbert space of functions ϕ on $GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})$ such that $\phi(zg) = \chi(z)\phi(g)$ and
$$\int_{GL_2(\mathbb{Q})Z(\mathbb{A})\backslash GL_2(\mathbb{A})} |\phi(g)|^2 dg < \infty,$$
where dg is a $GL_2(\mathbb{A})$-invariant measure on $GL_2(\mathbb{Q})Z(\mathbb{A})\backslash GL_2(\mathbb{A})$. Also, let $L_0^2(\chi) \subseteq L^2(\chi)$ denote the closed, $GL_2(\mathbb{A})$-invariant subspace of cuspidal functions.

The group $GL_2(\mathbb{A})$ acts unitarily on $L^2(\chi)$ by right translations ρ_χ:
$$(\rho(g)\phi)(x) = \phi(xg),$$
and let $\rho_{\chi,0}$ denote the restriction of ρ_χ to $L_0^2(\chi)$. Now a fundamental theorem of Gelfand and Piatetski-Shapiro (cf. [9]) gives that $\rho_{\chi,0}$ is completely reducible, that is
$$\rho_{\chi,0} \xrightarrow{\sim} \bigoplus m(\pi)\,\pi,$$
where $m(\pi)$ is the multiplicity of π and is finite, while π ranges over inequivalent irreducible unitary representations of $GL_2(\mathbb{A})$. Moreover the multiplicity one theorem of Jacquet-Langlands, which is a consequence of the uniqueness of the Whittaker models, gives here $m(\pi) = 1$ for any cuspidal irreducible unitary representation π of $GL_2(\mathbb{A})$.

1.2 Congruence subgroups: $\Gamma_0(q)$ vs. $\Gamma_1(q)$

The principal congruence subgroup of level q is

$$\Gamma(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,:\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod q \right\}.$$

Since it is the kernel of the natural projection $SL_2(\mathbb{Z}) \twoheadrightarrow SL_2(\mathbb{Z}/q\mathbb{Z})$, it is a normal subgroup of $SL_2(\mathbb{Z})$ and hence

$$SL_2(\mathbb{Z})/\Gamma(q) \cong SL_2(\mathbb{Z}/q\mathbb{Z}),$$

which gives that $[SL_2(\mathbb{Z}) : \Gamma(q)] = q^3 \prod_{p|q} \left(1 - \frac{1}{p^2}\right)$.

A *congruence subgroup* $\Gamma \subseteq SL_2(\mathbb{Z})$ of level q is any subgroup containing $\Gamma(q)$. One such group is already introduced - the Hecke congruence subgroup $\Gamma_0(q)$. The other one that will be important in this thesis is

$$\Gamma_1(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,:\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod q \right\}.$$

These groups satisfy

$$\Gamma(q) \lhd \Gamma_1(q) \lhd \Gamma_0(q) \subseteq SL_2(\mathbb{Z}),$$

where the accompanying epimorphisms

$$\Gamma_1(q) \twoheadrightarrow \mathbb{Z}/q\mathbb{Z}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b \pmod q,$$

$$\Gamma_0(q) \twoheadrightarrow (\mathbb{Z}/q\mathbb{Z})^*, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \pmod q,$$

respectively show that

$$\Gamma_1(q)/\Gamma(q) \cong \mathbb{Z}/q\mathbb{Z}, \quad [\Gamma_1(q) : \Gamma(q)] = q, \quad \text{and}$$

$$\Gamma_0(q)/\Gamma_1(q) \cong (\mathbb{Z}/q\mathbb{Z})^*, \quad [\Gamma_0(q) : \Gamma_1(q)] = \phi(q),$$

hence in particular $[SL_2(\mathbb{Z}) : \Gamma_0(q)] = q \prod_{p|q} \left(1 + \frac{1}{p}\right)$.

We will recall also the moduli space interpretation for both $\Gamma_0(q)$ and $\Gamma_1(q)$, that puts them in more arithmetic perspective. We will call a pair (E, C) with E an elliptic curve over $\mathbb{C}$ and $C < E$ a cyclic subgroup of order q - an enhanced elliptic curve for $\Gamma_0(q)$. Two such pairs are equivalent $(E_1, C_1) \sim (E_2, C_2)$ if some isomorphism $E_1 \to E_2$ maps C_1 to C_2.

Similarly, an enhanced elliptic curve for $\Gamma_1(q)$ is a pair (E, P) where E is a $\mathbb{C}$-elliptic curve and $P \in E$ a point of order q. Two pairs are equivalent $(E_1, P_1) \sim (E_2, P_2)$ if some isomorphism $E_1 \to E_2$ maps P_1 to P_2.

For $\tau \in \mathbb{H}$ we denote the corresponding lattice $\Lambda_\tau = \tau\mathbb{Z} \oplus \mathbb{Z}$, and by $\langle P \rangle$ we denote the group generated by the point P. Then for the modular curves $Y_0(q) = \Gamma_0(q)\backslash\mathbb{H}$ and $Y_1(q) = \Gamma_1(q)\backslash\mathbb{H}$ we have the following isomorphisms with the corresponding moduli spaces of complex elliptic curves with q-torsion data:

$$\{\text{enhanced elliptic curves for } \Gamma_0(q)\}/\sim \quad \cong Y_0(q),$$

$$\text{class of } \left(\mathbb{C}/\Lambda_\tau,\ \langle 1/q + \Lambda_\tau \rangle\right) \mapsto \Gamma_0(q)\tau,$$

and

$$\{\text{enhanced elliptic curves for } \Gamma_1(q)\}/\sim \quad \cong Y_1(q)$$

$$\text{class of } \left(\mathbb{C}/\Lambda_\tau,\ 1/q + \Lambda_\tau\right) \mapsto \Gamma_1(q)\tau.$$

Finally, the following decomposition of the space of holomorphic $\Gamma_1(q)$-automorphic forms into χ-eigenspaces, that is the spaces of $\Gamma_0(q)$-automorphic forms with characters will be the starting point for several calculations:

$$S_k(\Gamma_1(q)) = \bigoplus_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^k}} S_k(\Gamma_0(q), \chi). \tag{1.2}$$

Here it is important to emphasize the parity condition $\chi(-1) = (-1)^k$, since otherwise the corresponding χ-eigenspace is empty.

1.3 Hecke operators and Hecke eigenforms

For any integer k, every $\beta \in GL_2^+(\mathbb{Q})$ induces the operator $[\beta]_k$ on functions $f : \mathbb{H} \to \mathbb{C}$ given by $f[\beta]_k(z) = \frac{(\det\beta)^{k/2}}{j(\beta,z)^k} f(\beta z)$. For any two congruence subgroups Γ_1 and Γ_2 of $SL_2(\mathbb{Z})$ and $\alpha \in GL_2^+(\mathbb{Q})$, Γ_1 acts on the double coset $\Gamma_1\alpha\Gamma_2$ by the left multiplication.

There is a finite number of orbits and let $\Gamma_1\alpha\Gamma_2 = \bigsqcup \Gamma_1\beta_j$ for some representatives β_j. Then the *double coset operator* of weight k is defined by

$$f[\Gamma_1\alpha\Gamma_2]_k = \sum_j f[\beta_j]_k$$

and it maps $[\Gamma_1\alpha\Gamma_2]_k : M_k(\Gamma_1) \to M_k(\Gamma_2)$, also preserving the cuspidality condition. It can also be interpreted as the homomorphism between the divisor groups of the corresponding modular curves

$$[\Gamma_1\alpha\Gamma_2]_k : \mathrm{Div}(X(\Gamma_2)) \to \mathrm{Div}(X(\Gamma_1)),$$

where the map is defined as a $\mathbb{Z}$-linear extension of

$$X(\Gamma_2) \to \mathrm{Div}(X(\Gamma_1)), \qquad \Gamma_2\, z \mapsto \sum_j \Gamma_1\beta_j(z).$$

Now specializing to the situation $\Gamma_1 = \Gamma_2 = \Gamma_1(q)$, we get an action of the group $\Gamma_0(q)$ on $M_k(\Gamma_1(q))$ via double coset operators: for $\alpha \in \Gamma_0(q)$ and $f \in M_k(\Gamma_1(q))$, $f[\Gamma_1(q)\alpha\Gamma_1(q)]_k = f[\alpha]_k \in M_k(\Gamma_1(q))$. The subgroup $\Gamma_1(q) \lhd \Gamma_0(q)$ acts trivially, and therefore this action can be seen as the action of the quotient $(\mathbb{Z}/q\mathbb{Z})^*$ on $M_k(\Gamma_1(q))$ via so called *diamond operators*:

$$\langle d \rangle : M_k(\Gamma_1(q)) \to M_k(\Gamma_1(q)),$$

where $\langle d \rangle f = f[\alpha]_k$ for any $\alpha \in \Gamma_0(q)$ with the lower right entry $\equiv d \pmod q$. Now the spaces $M_k(\Gamma_0(q), \chi)$ for Dirichlet characters $\chi \pmod q$ can be characterized as the χ-eigenspaces of the diamond operators:

$$M_k(\Gamma_0(q), \chi) = \{f \in M_k(\Gamma_1(q)) : \langle d \rangle f = \chi(d) f, \text{ for all } d \in (\mathbb{Z}/q\mathbb{Z})^*\}.$$

The choice $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$, for p prime, leads to the classical Hecke operators: $T_p = [\Gamma_1(q) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \Gamma_1(q)]_k$, while taking $\alpha = \left(\begin{smallmatrix} 0 & -1 \\ q & 0 \end{smallmatrix}\right)$ gives the Fricke involution W.

Explicitly, the operator T_p for $p \nmid q$ on $M_k(\Gamma_1(q))$ is given by:

$$T_p f = \sum_{j=0}^{p-1} f\left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\right]_k + f\left[\begin{pmatrix} m & n \\ q & p \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right]_k,$$

where $mp - nq = 1$.

Adelically, Hecke operators can be seen as follows. For $p < \infty$ a function f on $GL_2(\mathbb{Q}_p)$ is *smooth* if there exists an open compact subgroup $K < GL_2(\mathbb{Q}_p)$ such that $f(k_1 g k_2) = f(g)$ for all $k_1, k_2 \in K$. Let $\mathcal{C}_c^\infty(GL_2(\mathbb{Q}_p))$ denote the convolution algebra of smooth, compactly supported functions on $GL_2(\mathbb{Q}_p)$. Then for any irreducible unitary representation (π, V_π) of $GL_2(\mathbb{Q}_p)$ and any $f \in \mathcal{C}_c^\infty(GL_2(\mathbb{Q}_p))$, one defines an (operator on V_π)-valued integral

$$\pi(f) := \int_{GL_2(\mathbb{Q}_p)} f(g)\pi(g)dg,$$

where dg is a fixed choice of Haar measure.

Now denoting $K_p = GL_2(\mathbb{Z}_p)$, let $\widetilde{T}_p \in \mathcal{C}_c^\infty(GL_2(\mathbb{Q}_p))$ be defined as

$$\widetilde{T}_p := \text{meas}(K_p)^{-1} \times \left(\text{characteristic function of } K_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_p\right).$$

Then, recalling that ρ_χ denotes the action of $GL_2(\mathbb{A})$ on $L^2(\chi)$ by right translations, if we denote for $f \in S_k(\Gamma_0(q), \chi)$ the corresponding function on the adelic space $GL_2(\mathbb{A})$ with ϕ_f as above, the action of the classical Hecke operators T_p for $p \nmid q$ can be recognized in the language of the correspondence $f \leftrightarrow \phi_f$ as follows:

$$\rho_\chi(\widetilde{T}_p)\phi_f = \text{const} \times \phi_{T_p f}.$$

The operators T_p for different primes commute with each other and a *Hecke eigenform* is a simultaneous eigenfunction of all operators T_p, $p \nmid q$.

Every $f \in S_k(\Gamma_0(q), \chi)$ admits a Fourier expansion at the cusp ∞ of the form

$$f(z) = \sum_{n \geq 1} \lambda_f(n) n^{(k-1)/2} e(nz). \tag{1.3}$$

If moreover $f \in \mathcal{H}_\chi$ i.e. f is a Hecke eigenfunction and we normalize it so that $\lambda_f(1) = 1$, the Fourier coefficients are precisely the corresponding Hecke eigenvalues:

$$T_p f = \lambda_f(p) f, \quad \text{for } (p, q) = 1,$$

and in the normalization of (1.3), Deligne's theorem gives the bound $|\lambda_f(p)| \leq 2$.

The Hecke eigenforms $f \in S_k(\Gamma_0(q), \chi)$ correspond to irreducible representations of $GL_2(\mathbb{A})$ in the following way: let V_f be the closed subspace of $L_0^2(\chi)$ generated by ϕ_f under the $GL_2(\mathbb{A})$-action. Then the restriction of ρ_χ on V_f is an irreducible representation of $GL_2(\mathbb{A})$.

1.4 Petersson's formulae

First we introduce the notation for weighted summation over the orthogonal basis $\mathcal{H}_\chi$ for $S_k(\Gamma_0(g),\chi)$ (recall that prime level q and odd weight $k \geq 3$ are fixed in our case, so in particular we can choose $\mathcal{H}_\chi$ consisting only of primitive eigenforms):

$$\sum_{f\in\mathcal{H}_\chi}^{h} \alpha_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f\in\mathcal{H}_\chi} \frac{\alpha_f}{\|f\|^2}. \tag{1.4}$$

For any positive integers m,n Petersson's formula for $S_k(\Gamma_0(q),\chi)$ is the following trace formula:

$$\sum_{f\in\mathcal{H}_\chi}^{h} \lambda_f(n)\overline{\lambda_f}(m) = \delta(m,n) + 2\pi i^{-k} \sum_{\substack{c\geq 1\\ c\equiv 0(\mathrm{mod} q)}} \frac{1}{c} S_\chi(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \tag{1.5}$$

where the normalization of Fourier coefficients is as in (1.3), $\delta(m,n)$ is Kronecker's diagonal symbol and $S_\chi(m,n;c)$ is the Kloosterman sum:

$$S_\chi(m,n;c) = \sum_{a\bar{a}\equiv 1(\mathrm{mod}\ c)} \bar{\chi}(a) e\left(\frac{ma+n\bar{a}}{c}\right). \tag{1.6}$$

For the proof of Petersson's formula we refer to [14], chapter 14. Here we just emphasize that two main ingredients in its derivation are the Hilbert space structure on $S_k(\Gamma_0(q),\chi)$ given by Petersson's inner product:

$$\langle f_1, f_2\rangle = \int_{\Gamma_0(q)\backslash\mathbb{H}} f_1(z)\overline{f_2(z)} y^k d\mu(z),$$

and the Bruhat decomposition

$$\Gamma_0(q) = \Gamma_\infty \bigsqcup \left(\bigsqcup_{\substack{c>0\\ c\equiv 0\ (q)}} \ \bigsqcup_{\substack{d \mod c\\ (d,c)=1}} \Gamma_\infty \begin{pmatrix} a & b\\ c & d\end{pmatrix} \Gamma_\infty \right),$$

where for given $c,d,(c,d)=1$, a and b are any two integers satisfying $ad-bc=1$ and $\Gamma_\infty = \left\{\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right) |\ n\in\mathbb{Z}\right\} \cong \mathbb{Z}$ is the stabilizer of the cusp at infinity. The presence of Γ_∞ enables application of the Poisson summation formula, while the whole decomposition enables access to the information about the group $\Gamma_0(q)$ and creates Kloosterman sums on the right hand side of (1.5).

Since we are interested in $\Gamma_1(q)$-family, we will need various averages of (1.5) with respect to $\chi \,(\mathrm{mod}\, q), \chi(-1) = (-1)^k$. To facilitate the notation for averaging, we will use the operator $\mathcal{K}$ (cf. [15]) defined by:

$$\mathcal{K}f = i^{-k}f + i^k\bar{f} = 2\mathfrak{Re}(i^{-k}f). \tag{1.7}$$

Often we will exploit the orthogonality of characters in the following form:

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=(-1)^k}} \chi(l) = \frac{1}{\phi(q)} \sum_{\chi \mod q} (\chi(l) + \chi(-l)(-1)^k) = \begin{cases} 1, & \text{if } \; l \equiv_q 1 \\ (-1)^k, & \text{if}\, l \equiv_q -1 \\ 0, & \text{otherwise} \end{cases} \tag{1.8}$$

which is all together equal $\delta(l \equiv_q \varepsilon)\mathcal{K}e\left(\frac{1}{6} + \frac{\varepsilon k}{4}\right)$, where $\varepsilon \in \{1, -1\}$ and $\delta(P) = 1$ if P is true and 0 otherwise.

Next, we will encounter also the following averaging for $(l, q) = 1$:

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=(-1)^k}} \chi(l)\tau(\chi) = e\left(\frac{\bar{l}}{q}\right) + (-1)^k e\left(-\frac{\bar{l}}{q}\right) = \begin{cases} 2\cos(2\pi\bar{l}/q), & \text{if } k \text{ is even;} \\ 2i\sin(2\pi\bar{l}/q), & \text{if } k \text{ is odd.} \end{cases} \tag{1.9}$$

Similarly, by opening the Kloosterman sums, we find:

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=(-1)^k}} \chi(l)S_\chi(m, n; c) = i^k\mathcal{K} \sideset{}{^*}\sum_{\substack{a\,(c) \\ a \equiv l\,(q)}} e\left(\frac{ma + n\bar{a}}{c}\right) \tag{1.10}$$

and hence we have the following lemma (which we state for general k and q):

Lemma 1.1. *For all $\chi \mod q$, $\chi(-1) = (-1)^k$, let $\mathcal{H}_\chi$ be any orthogonal basis of $S_k(\Gamma_0(q), \chi)$ and let $\varepsilon \in \{1, -1\}$. Then for any integer l, $(l, q) = 1$ and any positive integers m, n we have:*

$$\begin{aligned} \frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=(-1)^k}} \chi(l) \sideset{}{^h}\sum_{f \in \mathcal{H}_\chi} \lambda_f(n)\overline{\lambda_f}(m) \quad &= \quad \delta(m, n)\delta(l \equiv_q \varepsilon)\mathcal{K}e\left(\frac{1}{6} + \frac{\varepsilon k}{4}\right) + \\ &+ 2\pi \sum_{\substack{c \geq 1 \\ c \equiv 0 (mod q)}} \frac{1}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \mathcal{K} \sideset{}{^*}\sum_{\substack{a\,(c) \\ a \equiv l\,(q)}} e\left(\frac{ma + n\bar{a}}{c}\right). \end{aligned} \tag{1.11}$$

Note that for $l = 1$, (1.11) reduces to Petersson's formula for the trace of orthogonal basis of $S_k(\Gamma_1(q))$, (cf. [15] (2.20)).

1.4.1 J-Bessel functions

On the right hand side of Petersson's formula appear classical J-Bessel functions which are given by the power series

$$J_{k-1}(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!(k-1+l)!}(x/2)^{k-1+2l}.$$

In this thesis, k will be fixed, and we will use often the following basic asymptotic bounds for J_{k-1}:

$$J_{k-1}(x) \ll x^{k-1}, \qquad \text{for} \quad x < 1, \tag{1.12}$$

and

$$J_{k-1}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \left(\cos\left(x - \frac{\pi}{2}(k-1) - \frac{\pi}{4}\right) + O(x^{-1})\right), \qquad \text{for} \quad x \gg 1. \tag{1.13}$$

In the treatment by Fourier analysis, we need the information about the phase of J_{k-1} as provided by the following expression (see [33], p. 206) :

$$J_{k-1}(2\pi x) = \frac{1}{\pi\sqrt{x}} \mathfrak{Re}\left(W(2\pi x) e\left(x - \frac{k}{4} + \frac{1}{8}\right)\right) \tag{1.14}$$

where W is essentially "flat":

$$W(2\pi x) = \frac{1}{\Gamma(k-1/2)} \int_0^{\infty} e^{-u} u^{k-3/2} \left(1 + \frac{u}{4\pi x} i\right)^{k-3/2} du.$$

The only thing we need to know about W is that for any $\nu \geq 0$:

$$x^{\nu} W^{(\nu)}(x) \ll_{k,\nu} 1.$$

Further, we will need the following formula ([29], formulas 6.36 - 6.39): for any $\alpha, \beta > 0$ we have:

$$\mathcal{K} \int_0^{\infty} J_{k-1}(4\pi\sqrt{\alpha\beta} x) e\left((\alpha+\beta)x + \frac{\gamma}{x}\right) \frac{dx}{x} =$$

$$= \begin{cases} 2\pi J_{k-1}(4\pi\sqrt{\alpha\gamma}) J_{k-1}(4\pi\sqrt{\beta\gamma}), & \text{for} \quad \gamma > 0; \\ 0, & \text{for} \quad \gamma \leq 0. \end{cases} \tag{1.15}$$

1.5 Motivation: families of $L-$functions, conductors, orthogonality, large sieves and harmonic analysis on GL_2

1.5.1 The analytic conductor

Discussion in this section will use the notion of the *analytic conductor*, introduced by Iwaniec and Sarnak in [18]. Let f be any arithmetic (automorphic) object which has associated L-function

$$L(f,s)=\prod_p (1-\alpha_1(p)p^{-s})^{-1}\dots(1-\alpha_d(p)p^{-s})^{-1}$$

of degree d, a gamma factor $\gamma(f,s) = \pi^{-ds/2}\prod_{j=1}^d \Gamma\left(\frac{s+\kappa_j}{2}\right)$ and an integer $q_f \geq 1$ such that the completed L-function $\Lambda(f,s) = q_f^{s/2}\gamma(f,s)L(f,s)$ satisfies the functional equation:

$$\Lambda(f,s)=\varepsilon_f\overline{\Lambda(f,1-\overline{s})}$$

where $\varepsilon_f \in \mathbb{C}$, $|\varepsilon_f| = 1$ is the "root number". Then the analytic conductor of the object f is the following quantity:

$$\mathfrak{q}(f)=q_f\prod_{j=1}^d(|\kappa_j|+3),$$

which measures analytic "complexity" of the given object, especially in relation to the harmonic analysis in the family to which f naturally belongs. For example, the analytic conductor of $f\in S_k(\Gamma_0(q),\chi)$ is $\mathfrak{q}(f)\asymp qk^2$.

We also mention a more general notion (cf. [14], chapter 5), including the t-aspect as well, which will be used in a comparison in chapter 3:

$$\mathfrak{q}(f,s)=q_f\prod_{j=1}^d(|s+\kappa_j|+3).$$

1.5.2 Large sieves and harmonic analysis in families

The family of Dirichlet characters modulo q has $\varphi(q)$ elements – harmonics of conductor q. It is a very natural family, with orthogonality coming from "algebraic reasons" – the orthogonality of characters of the finite group $(\mathbb{Z}/q\mathbb{Z})^*$.

But in applications in analytic number theory, much more powerful is the large sieve:

$$\sum_{q\leq Q}\frac{q}{\varphi(q)}\sum_{\chi \bmod q}^{*}\left|\sum_{N<n\leq 2N}a_n\chi(n)\right|^2\leq (Q^2+N)\|\mathbf{a}\|^2,$$

itself a manifestation of the orthogonality of primitive Dirichlet characters modulo q, $q\leq Q$ – hence a family of cardinality $\asymp Q^2$ but conductors still $\asymp Q$. Although we stated it in the "arithmetic form", its proof is purely analytic, being a corollary to the following inequality:

$$\sum_{r}\left|\sum_{n\sim N}a_n e(\alpha_r\, n)\right|^2\leq (\delta^{-1}+N)\|\mathbf{a}\|^2,$$

for any set $\{\alpha_r\}$ of δ-spaced points on the circle $\mathbb{R}/\mathbb{Z}$. Nevertheless, it is the main ingredient in the proof of the Bombieri-Vinogradov theorem:

$$\sum_{q\leq Q}\max_{(a,q)=1}\left|\psi(x;q,a)-\frac{x}{\varphi(q)}\right|\ll x(\log x)^{-A}$$

where $Q=x^{1/2}(\log x)^{-B}$, B depends on A and $\psi(x;q,a)$ counts prime numbers in the arithmetic progression $a \pmod q$ with von Mangoldt weights. The exponent $1/2$ in Q is called the "level of distribution", and measures the uniformity Q up to which prime numbers are equidistributed in arithmetic progressions modulo $q\ll Q$. This is of huge importance in all applications, however, it is still far from the truth, which is conjecturally $1-\varepsilon$ by Elliott-Halberstam conjecture. The exponent $1/2$ therefore is not a natural boundary, it is imposed as a limitation coming from the GL_1 large sieve.

Therefore, there is a natural desire to search for more and more powerful families of harmonics which could be potentially applied to various problems in analytic number theory. One such program is developed in early eighties by Iwaniec and collaborators, where the role of arithmetic harmonics is played by Fourier coefficients of GL_2 cusp forms. We refer to [14], §7.7 for an overview of results on large sieves for GL_2 families.

Here we illustrate the power of GL_2-large sieve estimates by the following application on the distribution of primes in arithmetic progressions obtained by Bombieri-Friedlander-Iwaniec in [2]: for fixed $a>0$, any $\varepsilon>0$ and any $A>0$ we have

$$\sum_{\substack{q<x^{4/7-\varepsilon}\\(q,a)=1}}\lambda(q)\left(\psi(x;q,a)-\frac{x}{\varphi(q)}\right)\ll x(\log x)^{-A},$$

where λ is a well-factorable function found in applications in sieve theory. In particular, this gives the level of distribution $4/7$ (going beyond the classical $1/2$) with some restrictions, but still enough for some applications.

The proof of that result essentially depends on estimates for sums of Kloosterman sums, which themselves are corollaries of the following large sieve type estimates from [5] (theorems 6 and 7 there):

$$\sum_{q\leq Q} \sum_{\substack{\lambda_j \text{expeptional} \\ \text{for } \Gamma_0(q)}} X^{4i\kappa_j} \left| \sum_{n\sim N} a_n \rho_j(n) \right|^2 \ll (Q+N+NX)(QN)^{\varepsilon} \|\mathbf{a}\|^2,$$

$$\sum_{q\leq Q} \sum_{\substack{\lambda_j \text{expeptional} \\ \text{for } \Gamma_0(q)}} X^{4i\kappa_j} \left| \sum_{n\leq N} \rho_j(n) \right|^2 \ll (Q+N+N^{1/2}X)(QN)^{\varepsilon} N,$$

where $\lambda_j = \frac{1}{4} + \kappa_j^2$ are eigenvalues belonging to the exceptional spectrum of $\Gamma_0(q)$ and $\rho_j(n)$ are the Fourier coefficients of the corresponding Maaß forms at the cusp ∞. These can be seen as a motivation for the last chapter in this thesis, since the key feature that gives the power in the last two large sieve estimates is the averaging over the levels q on GL_2.

In the chapter 4 we investigate such averaging over the levels for holomorphic modular forms. Important feature is that the dimension of space $S_k(\Gamma_0(q))$ is $\asymp q$ (for fixed k), which can be interpreted also as the multiplicity of the corresponding eigenvalue, and contrasted with the multiplicity of Maaß forms which are all conjecturally simple. Thus, for holomorphic modular forms we have a lot of harmonics with small conductor. This can be enlarged even more, by averaging over the spaces $S_k(\Gamma_0(q),\chi)$, $\chi \mod q$, thus obtaining a family of $\asymp Q^3$ harmonics of conductors $\asymp Q$ which is "richer" than the GL_1 family of Dirichlet characters $\chi \mod q$, $q \asymp Q$ which has "only" $\asymp Q^2$ harmonics of conductor $\asymp Q$.

Further motivation for averaging over the levels only is that the level aspect appeared so far to have the most significant arithmetic applications – for example the subconvexity results in the level aspect on GL_2 have the most interesting consequences.

At the end, we mention that one could dream of even larger families with controlled size of the conductor. Although $GL_n(\mathbb{R})$ has no discrete series if $n > 2$, if we fix a compact subset S_∞ of the unitary dual $\widehat{GL_n(\mathbb{R})}$ containing an open set of tempered representations, then the set $S_1(q)$ of cuspidal automorphic representations π of conductor q and with $\pi_\infty \in S_\infty$ should have cardinality $\asymp q^n$. But in higher rank the problem so far is inaccessible due to the lack of workable trace formula, although some partial results are obtained via Rankin-Selberg L-functions and lattice reduction theory in [8] and [31] respectively.

1.5.3 $\Gamma_1(q)$-large sieve

Now we recall the version of the large sieve inequality for $\Gamma_1(q)$-modular forms obtained in [15]. Let $\mathcal{B}_\chi$ denote any orthogonal basis (not necessarily Hecke) for $S_k(\Gamma_0(q),\chi)$, $k \geq 3$ and let $\mathcal{B}_\chi \ni f(z) = \sum_{n\geq 1} \psi_f(n) n^{\frac{k-1}{2}} e(nz)$ denote the Fourier expansion at ∞. Authors in [15] derive several asymptotic-large sieve-type inequalities, among which the following one - for the prime level is the most simple:

Theorem 1.2 (Iwaniec, Li)**.** *Let q be prime, $N \geq q$, $T = N/q$ and $1 \leq H \leq T$. Then for any complex vector $\mathbf{a} = (a_n)_{n\sim N}$ and any $\varepsilon > 0$ we have*

$$\frac{2}{\varphi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f\in\mathcal{B}_\chi} \left|\sum a_n \psi_f(n)\right|^2 =$$

$$= \frac{1}{q} \sum_{\substack{1\leq t\leq T \\ (t,q)=1}} \left(\frac{2\pi}{t}\right)^2 \sum_{1\leq h\leq H} |\mathcal{P}_{h,t}(\mathbf{a})|^2 + O\left(N^\varepsilon \left(\frac{N}{q^2} + \sqrt{\frac{N}{qH}}\right)\right) \|\mathbf{a}\|^2, \quad (1.16)$$

where

$$\mathcal{P}_{h,t}(\mathbf{a}) = \sum_{n\sim N} a_n S(h\overline{q}, n; t) J_{k-1}\left(\frac{4\pi}{t}\sqrt{\frac{hn}{q}}\right).$$

The forms $\mathcal{P}_{h,t}(\mathbf{a})$ could be considered as the "dual" forms for the linear forms $\sum a_n \psi_f(n)$ in the Fourier coefficients of automorphic forms. One corollary is that for q prime and $N \geq q$ we have:

$$\frac{2}{\varphi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f\in\mathcal{B}_\chi} \left|\sum a_n \psi_f(n)\right|^2 \ll N^\varepsilon \left(\frac{N}{q^2} + \sqrt{\frac{N}{q}}\right) \|\mathbf{a}\|^2. \quad (1.17)$$

But with (1.16) being asymptotic, one can show even more ([15] Cor. 13.1), namely that there is always a vector $\mathbf{a}_0$ such that the left hand side is $\gg \sqrt{\frac{N}{q}}\|\mathbf{a}_0\|^2$.

This means that in the range $q \le N \le q^2$ we do not have perfect orthogonality. The number of harmonics involved (the cardinality of a basis for $\Gamma_1(q)$-cusp forms) is $\asymp q^2$ and comparing with classical GL_1-large sieves and $\Gamma_0(q)$-large sieve, one would be expecting "perfect orthogonality" for all N up to q^2.

At the same time, this provides additional motivation to study the family of $\Gamma_1(q)$-automorphic forms: although the number of elements in the family is larger, the family still lacks the perfect orthogonality, and hence potentially it may be relatively harder to work with from the purely analytic point of view. Therefore, one aim of this thesis is to investigate the analytic difficulties connected with this family.

In chapters 2 and 3 we will test orthogonality of this family but for special "test vectors" (while in the large sieve setting one works with completely general complex vectors). This will give the possibility for additional cancelation, since certainly one does not expect that all interesting sequences arising in applications show the same bias towards the Fourier coefficients of $\Gamma_1(q)$-automorphic forms.

1.6 Fixing the setting and notation

We are interested in the level aspect and therefore because of simplicity and to avoid complications with oldforms in Petersson formula, in chapters 2 and 3 we will assume that q is prime and $k \ge 3$ odd (of course $k < 12$ and even also "avoids" oldforms). In this situation $S_k^{\text{new}}(\Gamma_0(q), \chi) = S_k(\Gamma_0(q), \chi)$ and we can choose an orthogonal basis $\mathcal{H}_\chi = \{f\}$ entirely consisting of primitive Hecke-eigenforms.

We will often through the thesis use the symbol

$$\sideset{}{^-}\sum_{\chi \mod q}$$

to represent averaging over all *odd* characters modulo q.

The symbol $\int_{(\sigma)}$ means integration along the line $\sigma + it$, $-\infty < t < \infty$.

The symbol $f \asymp g$ means that $c^{-1} f(x) \le g(x) \le c f(x)$ for some $c > 1$ and for all $x > 1$.

As usual, $e(x) = e^{2\pi i x}$. In all occurrences like $e\left(\frac{\bar{a}}{q}\right)$, $\bar{a}$ is the multiplicative inverse modulo q i.e. $a\,\bar{a} \equiv 1 \pmod{q}$.

Chapter 2

Nonvanishing of the family of $\Gamma_1(q)$-automorphic L-functions at the central point

2.1 Introduction

If there are no obvious reasons for an L-function to vanish at $s = \frac{1}{2}$, for example because of the sign of the functional equation, we expect generically that the central value is different from 0. There are numerous investigations of the non-vanishing of the central values of L-functions in families, all of them starting with [17] which gives intricate connection between such GL_2 nonvanishing problem and the problem on GL_1 concerning elimination of the Landau-Siegel zero.

In this chapter we consider the problem of nonvanishing of central values of automorphic L-functions for $\Gamma_1(q)$- family. For special features of this family and motivation we refer to section 2.3.

We prove the following:

Theorem 2.1. *Let $\mathcal{H}(q,\chi)$ denotes basis of Hecke eigenforms in $S_k(\Gamma_0(q),\chi)$ for k fixed, odd and let $\mathcal{A}_{\Gamma_1(q)}$ denotes the following averaging:*

$$\mathcal{A}_{\Gamma_1(q)}[\alpha_f] = \frac{2}{\varphi(q)} \sum_{\substack{\chi\,(q)\\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f\in\mathcal{H}(q,\chi)} \alpha_f.$$

Then, as $q \to \infty$ through primes, we have

$$\underline{\lim}_{q\to\infty} \frac{\mathcal{A}_{\Gamma_1(q)}[\mathbf{1}(L(f,1/2)\neq 0)]}{\mathcal{A}_{\Gamma_1(q)}[1]} \geq 0.2318,$$

where bold $\mathbf{1}$ denotes the indicator function.

We remark that harmonic weights here can be removed by a standard technique (cf. [17], [22]).

2.2 L-functions and approximate functional equations

To every $f \in \mathcal{H}_\chi \subset S_k(\Gamma_0(q),\chi)$ is attached an L-function with Euler product:

$$L(f,s) = \sum_{n\geq 1} \lambda_f(n) n^{-s} = \prod_p (1-\lambda_f(p)p^{-s} + \chi(p)p^{-2s})^{-1}. \tag{2.1}$$

This equation is equivalent with

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \chi(d)\lambda_f\left(\frac{mn}{d^2}\right), \tag{2.2}$$

or by Möbius inversion, with

$$\lambda_f(mn) = \sum_{d|(m,n)} \mu(d)\chi(d)\lambda_f\left(\frac{m}{d}\right)\lambda_f\left(\frac{n}{d}\right),$$

for all $m,n \geq 1$. In particular, we have the exact factorization for any $n \geq 1$:

$$\lambda_f(nq) = \lambda_f(n)\lambda_f(q). \tag{2.3}$$

Further to each such f there is the associated dual form $\bar{f}$ given by:

$$\bar{f}(z) := Kf(z) = \overline{f(-\overline{z})}. \tag{2.4}$$

The Fourier coefficients of $\bar{f}$ are $\{\overline{\lambda_f(n)}\}$ and they satisfy ([14] Lemma 14.10):

$$\overline{\lambda_f}(n) = \overline{\chi}(n)\lambda_f(n), \quad \text{for all} \quad (n,q)=1. \tag{2.5}$$

Moreover each Hecke eigenfunction $f \in \mathcal{H}_\chi$ is by the Multiplicity one theorem also an eigenfunction of the operator KW, hence

$$(KW)f = \eta f$$

where $\eta \in \mathbb{C}$, $|\eta| = 1$ is the corresponding eigenvalue. In the case of interest in this thesis, that is for k odd, q prime, all characters χ appearing in the decomposition (1.2) are primitive, and in that case the following formula holds (see [14] Proposition 14.15):

$$\eta = \frac{\tau(\overline{\chi})\lambda_f(q)}{\sqrt{q}}. \tag{2.6}$$

where $\tau(\overline{\chi}) = \sum_{x(\mathrm{mod} q)} \overline{\chi}(x) e\left(\frac{x}{q}\right)$ is the Gauss sum.

The Dirichlet series (2.1) is absolutely convergent for $\mathfrak{Re}(s) > 1$ but admits analytic continuation to all of $\mathbb{C}$. It satisfies the functional equation

$$\Lambda(f, s) = i^k \overline{\eta} \Lambda(\bar{f}, 1-s) \tag{2.7}$$

where

$$\Lambda(f, s) = \left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma\left(s + \frac{k-1}{2}\right) L(f, s) \tag{2.8}$$

is the corresponding completed L-function.

2.2.1 Approximate functional equations

Our first task is to express $L(f, \frac{1}{2})$ as a rapidly convergent series. The procedure is standard, see for example [14], Theorem 5.3. Choose a function $G(s)$ which is holomorphic and bounded in $|\mathfrak{Re}(s)| < 3$ and such that $G(-s) = G(s)$, $G(0) = 1$. Denoting with (σ) complex integration on vertical lines with real part σ, we start with

$$I(f) := \frac{1}{2\pi i} \int_{(2)} \Lambda(f, \frac{1}{2} + s) G(s) \frac{ds}{s}.$$

Applying Cauchy's theorem and the functional equation (2.7) we get

$$\Lambda(f, \frac{1}{2}) = I(f) + i^k \overline{\eta} I(\overline{f}).$$

On the other hand, integrating termwise gives

$$L(f, 1/2) = \sum_{n \geq 1} \frac{\lambda_f(n)}{\sqrt{n}} V\left(\frac{2\pi n}{\sqrt{q}}\right) + i^k \overline{\eta} \sum_{n \geq 1} \frac{\overline{\lambda_f(n)}}{\sqrt{n}} V\left(\frac{2\pi n}{\sqrt{q}}\right), \tag{2.9}$$

where

$$V(y) = \frac{1}{2\pi i} \int_{(2)} y^{-s} \frac{\Gamma(\frac{k}{2} + s)}{\Gamma(\frac{k}{2})} G(s) \frac{ds}{s}. \tag{2.10}$$

Similarly, integrating $\Lambda(f, \frac{1}{2}+s)\Lambda(\overline{f}, \frac{1}{2}+s)G^2(s)s^{-1}$ we obtain the following formula:

$$|L(f, 1/2)|^2 = 2 \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \frac{\lambda_f(n_1)\overline{\lambda_f}(n_2)}{\sqrt{n_1 n_2}} W\left(\frac{4\pi^2 n_1 n_2}{q}\right), \tag{2.11}$$

where

$$W(y) = \frac{1}{2\pi i} \int_{(2)} y^{-s} \frac{\Gamma^2(\frac{k}{2} + s)}{\Gamma^2(\frac{k}{2})} G^2(s) \frac{ds}{s}. \tag{2.12}$$

One particular admissible choice for the function $G(s)$ can be a polynomial which additionally vanishes at all the poles of $\Gamma\left(s+\frac{k}{2}\right)$ in the strip $|\mathfrak{Re}(s)| \leq A$ for any chosen $A > 0$. Then using Stirling's formula

$$\Gamma(s) = \left(\frac{2\pi}{s}\right)^{1/2} \left(\frac{s}{e}\right)^{s} \left(1 + O\left(\frac{1}{|s|}\right)\right), \qquad \text{valid in the angle} \qquad |\arg s| \leq \pi - \varepsilon,$$

and moving the contour of integration in (2.10) and (2.12) to the left or to the right respectively, one obtains the following asymptotic bounds:

$$V(y), \quad W(y) \quad = 1 + O(y^{A}), \qquad y \to 0, \tag{2.13}$$

$$V(y), \quad W(y) \quad \ll y^{-A}, \qquad y \gg 1. \tag{2.14}$$

2.3 The mollification and the mollified moments

In analytic investigations of L-functions (their size, special values, distribution of zeros etc.) inside the critical strip - the most prominent tool is calculation of various moments of $L-$functions running in some family.

The classical method for investigation of the non-vanishing of $L-$functions at the critical point is the comparison of the first and the second moment by an application of the Cauchy-Schwarz inequality. If this is done for the "pure, un-weighted" $L-$functions, there is inevitable loss of some power of the logarithm of the conductor in the final proportion of non-vanishing, which is due to the variation of size of L-functions along the family. For the first example of such calculation in the GL_2-setting we refer to [6].

In order to obtain the positive proportion an additional device is needed: one has to "save" the logarithmic factor that is, to control the abnormally large oscillations at the central point. Such a device is the introduction of the mollifier - a factor whose purpose is to "mollify" these problematic extreme values. In the analytic number theory, this is first used by Bohr-Landau [1] in their pioneering work on density theorems for the Riemann-zeta function. Much more delicate application of this mollification technique is due to Selberg [30] where he established that a positive proportion of the zeros of the Riemann's $\zeta(s)$ are on the critical line.

Although the behavior of L-functions inside the critical strip is not well understood, heuristically one expects that the mollifier should behave as $L(f,s)^{-1}$ or more precisely as a Dirichlet polynomial approximating it in some sense. There are several difficulties related to the mollification for this particular family. Various optimality results obtained in calculations so far show that the mollifier has to reflect the fact that we are dealing with the L-function of degree 2, as opposed to "classical" mollifiers for the Riemann zeta and Dirichlet L-functions (see in particular [17], where particular attention was given to the optimization of the proportion of nonvanishing because of the connection to the Landau-Siegel zeros).

Next, inside of the critical strip L-function is given by

$$L(f,s) = L_U + \varepsilon_f L_V, \tag{2.15}$$

where L_U and L_V are essentially two Dirichlet polynomials of lengths U and V respectively, where $UV \sim$ conductor of f, and ε_f is the root number from the functional equation. Now strategy can be different. Sometimes the root number is not well understood or controlled analytically (e.g. the family of elliptic curves) and then one is forced to take asymmetric representation, say with $U \sim$ conductor, but then this limits further treatment because the power of harmonic analysis in families decreases and eventually vanishes with longer and longer sums. On the other hand, in our situation we can take $U \sim V$, but still face possible loss of efficiency of the mollifier. In the results on GL_2 L-functions so far, only L-functions with trivial nebentypus were considered where the central value is real and $\varepsilon = \pm 1$. This means that central values in the family "oscillate" only in one direction, while for $\Gamma_1(q)$-family central values are complex and root number ε is distributed along the unit circle, and hence "oscillations" are now in all directions making them more difficult to capture. This is reflected in (2.15) by the fact that ε_f oscillates independently from the arguments of the sums L_U and L_V (cf. [25] where the similar situation is discussed for the family of Dirichlet characters mod q). Thus since the mollifier is one Dirichlet polynomial it can not efficiently capture this more complicated behavior. In [25] there was an attempt to remedy this by introducing a twisted mollifier of the form $M_1 + \varepsilon_f M_2$, but still this appears to have some effect only for higher derivatives.

The first choice of the mollifier could be

$$M_{\text{prelim}}(f) = \sum_{m \leq M} \frac{\lambda_f(m)}{\sqrt{m}} x_m,$$

with $(x_m)_{m \leq M}$ a real vector at our disposal. Already this choice would lead to a positive (and significant) proportion of nonvanishing.

On the other hand, if we denote for $f \in S_k(\Gamma_0(q), \chi)$:

$$L(f, s)^{-1} = \sum_{n \geq 1} \frac{\mu_f(n)}{n^s}, \qquad \mathfrak{Re}(s) > 1,$$

then from the Euler product (2.1) we infer that

$$\mu_f(n) = \begin{cases} \mu(m)\lambda_f(m)\chi(l), & \text{if} \quad n = ml^2, \quad ml \quad \text{squarefree}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.16}$$

Hence, each space $S_k(\Gamma_0(q), \chi)$ requires slightly different mollifier (in l variable) and we will use this second choice in order to increase precision and test the effect of additional averaging over $\chi \mod q$.

Therefore, for $f \in S_k(\Gamma_0(q), \chi)$ the "perfect" mollifier would be

$$M(f) = \sum\sum_{\substack{m,\, l \\ ml^2 \leq M}} \frac{\mu(m)\lambda_f(m)\chi(l)}{\sqrt{m}\, l}\, \mu^2(ml)\, x_{ml^2}.$$

However, the factor $\mu^2(ml)$ has little effect and for notational simplicity we omit it. Also, since the Möbius is already included in the mollifier and the very shape of it takes care of degree 2 situation as discussed above, we can from the start take

$$x_{ml^2} = \psi(ml^2), \tag{2.17}$$

for some smooth function ψ at our disposal. Moreover we will take eventually

$$\psi(x) = P\left(\frac{\log M/x}{\log M}\right),$$

where P is a polynomial with $P(0) = 0$.

Hence, our final choice of the mollifier is the following:

$$M(f) = \sum\sum_{\substack{m,\, l \\ ml^2 \leq M}} \frac{\mu(m)\lambda_f(m)\chi(l)}{\sqrt{m}\, l}\, \psi(ml^2), \tag{2.18}$$

where $M = q^{\Delta}, \quad 0 < \Delta < 1$ is to be chosen later.

Now, for this choice of the mollifier we consider the first and second mollified moments which now depend on the function ψ:

$$\mathcal{L}(\psi) = \frac{2}{\varphi(q)} \sum_{\chi}^{-} \sum_{f \in \mathcal{H}_\chi}^{h} M(f) L(f, 1/2), \tag{2.19}$$

$$\mathcal{Q}(\psi) = \frac{2}{\varphi(q)} \sum_{\chi}^{-} \sum_{f \in \mathcal{H}_\chi}^{h} |M(f) L(f, 1/2)|^2, \tag{2.20}$$

where $\sum_{\chi}^{-}$ denotes the summation over characters $\chi (\mathrm{mod}\ q)$ for which $\chi(-1) = (-1)^k = -1$ (they are all primitive since q is a prime).

Because of the weights attached in spectral averaging (introduced because of Petersson's formula) and $\frac{2}{\varphi(q)}$ – in character averaging, the mollified moments (2.19) and (2.20) behave roughly as the expectations of LM and $|LM|^2$ along the family and hence if we denote with $\mathcal{N}$ the percentage of non-vanishing central values in such "harmonic" average, Cauchy-Schwarz inequality gives:

$$\mathcal{N} \geq \frac{\mathcal{L}(\psi)^2}{\mathcal{Q}(\psi)}. \tag{2.21}$$

Now, after asymptotic evaluations of $\mathcal{L}(\psi)$ and $\mathcal{Q}(\psi)$, the mollification mechanism transfers the problem to the maximization of the quotient (2.21) with respect to the function ψ which is at our disposal.

2.4 The first mollified moment $\mathcal{L}(\psi)$

We start with

$$\mathcal{L}(\psi) = \frac{2}{\varphi(q)} \sum_{\chi}^{-} \sum_{f \in \mathcal{H}_\chi}^{h} L(f, 1/2) \sum_{\substack{m,\, l \\ ml^2 \leq M}} \sum \frac{\mu(m) \lambda_f(m) \chi(l)}{\sqrt{m}\, l} \psi(ml^2) =$$

$$= \sum_{\substack{m,\, l \\ ml^2 \leq M}} \sum \frac{\mu(m)}{\sqrt{m}\, l} \psi(ml^2) \mathcal{A}(m, l),$$

where

$$\mathcal{A}(m, l) = \frac{2}{\varphi(q)} \sum_{\chi}^{-} \sum_{f \in \mathcal{H}_\chi}^{h} \chi(l) \lambda_f(m) L(f, 1/2) \tag{2.22}$$

is the first twisted moment.

Substituting (2.9) and (2.6) in (2.22), using (2.3) and taking into account that $\overline{\tau}(\overline{\chi}) = -\tau(\chi)$ for odd characters, we get:

$$\mathcal{A}(m,l) = \sum_{n\geq 1} \frac{1}{\sqrt{n}} V\left(\frac{2\pi n}{\sqrt{q}}\right) \frac{2}{\varphi(q)} \sum_{\chi}^{-} \sum_{f\in\mathcal{H}_\chi}^{h} \chi(l)\lambda_f(m)\lambda_f(n) - \\ - i^k \sum_{n\geq 1} \frac{1}{\sqrt{nq}} V\left(\frac{2\pi n}{\sqrt{q}}\right) \frac{2}{\varphi(q)} \sum_{\chi}^{-} \sum_{f\in\mathcal{H}_\chi}^{h} \chi(l)\tau(\chi)\lambda_f(m)\overline{\lambda_f}(nq). \quad (2.23)$$

Since the contribution of the terms for $n \gg q^{\frac{1}{2}+\varepsilon}$ is negligible because of the rapid decay of V, in the significant range we have that $(n,q) = 1$ and hence we can apply (2.5) and substitute in the first line $\lambda_f(n)$ with $\chi(n)\overline{\lambda_f}(n)$ (with the negligible error):

$$\mathcal{A}(m,l) = \sum_{\substack{n\geq 1\\(n,q)=1}} \frac{1}{\sqrt{n}} V\left(\frac{2\pi n}{\sqrt{q}}\right) \frac{2}{\varphi(q)} \sum_{\chi}^{-} \chi(ln) \sum_{f\in\mathcal{H}_\chi}^{h} \lambda_f(m)\overline{\lambda_f}(n) - \\ - i^k \sum_{n\geq 1} \frac{1}{\sqrt{nq}} V\left(\frac{2\pi n}{\sqrt{q}}\right) \frac{2}{\varphi(q)} \sum_{\chi}^{-} \chi(l)\tau(\chi) \sum_{f\in\mathcal{H}_\chi}^{h} \lambda_f(m)\overline{\lambda_f}(nq) + O(q^{-2010}). \quad (2.24)$$

Now, for the first line we use the formula (1.11), while for the second line we use Petersson's formula (1.5) and then (1.9) for the diagonal part, while in the off-diagonal part we keep Kloosterman sums:

$$\begin{aligned} \mathcal{A}(m,l) = & \sum_{\substack{n\geq 1\\(n,q)=1}} \frac{1}{\sqrt{n}} V\left(\frac{2\pi n}{\sqrt{q}}\right) \delta(m,n) \left(\delta(ln \equiv_q 1) - \delta(ln \equiv_q -1)\right) + \\ & + 2\pi \sum_{\substack{n\geq 1\\(n,q)=1}} \frac{1}{\sqrt{n}} V\left(\frac{2\pi n}{\sqrt{q}}\right) \sum_{r\geq 1} \frac{1}{qr} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{qr}\right) \mathcal{K} \sum_{\substack{a\ (qr)\\ a\equiv ln\ (q)}}^{*} e\left(\frac{na+m\bar{a}}{qr}\right) \\ & - 2i^{k+1} \sum_{n\geq 1} \frac{\delta(m,nq)}{\sqrt{nq}} V\left(\frac{2\pi n}{\sqrt{q}}\right) \sin(2\pi\bar{l}/q) \\ & - 2\pi \sum_{\substack{n\geq 1\\(n,q)=1}} \frac{1}{\sqrt{n}} V\left(\frac{2\pi n}{\sqrt{q}}\right) \frac{2}{\varphi(q)} \sum_{\chi}^{-} \frac{\chi(l)\tau(\chi)}{\sqrt{q}} \sum_{r\geq 1} \frac{1}{qr} S_\chi(qn,m;qr) J_{k-1}\left(\frac{4\pi\sqrt{mnq}}{qr}\right) \\ & + O(q^{-2010}). \end{aligned} \quad (2.25)$$

Since $m \leq M = q^{\Delta} < q$ the third line is void and similarly in the first line $ln \equiv_q -1$ can not occur in the significant range $n \ll q^{1/2+\varepsilon}$, while $ln \equiv_q 1$ implies $l = n = 1$ and then using the asymptotic (2.13) we replace $V(2\pi/\sqrt{q})$ with 1, making negligible error.

Next, using (2.13), (2.14) for V, bound (1.12) for J_{k-1} and Weil's bound (2.35) for Kloosterman sums in the fourth line, while estimating exponential sums in the second line trivially, we show that the second and the fourth contribute only to the error term:

$$\text{second line} \ll \frac{1}{q} \sum_{n \leq q^{1/2+\varepsilon}} n^{-1/2} \sum_{r \geq 1} \left(\frac{\sqrt{mn}}{qr} \right)^2 \ll \frac{M q^{\varepsilon}}{q^{9/4}}, \tag{2.26}$$

$$\text{fourth line} \ll \sum_{n \leq q^{1/2+\varepsilon}} n^{-1/2} \sum_{r \geq 1} \frac{1}{qr} \tau(qr)(qn, m, qr)^{1/2} (qr)^{1/2} \left(\frac{\sqrt{mnq}}{qr} \right)^2 \ll \frac{M q^{\varepsilon}}{q^{3/4}}. \tag{2.27}$$

Together we have:

$$\mathcal{A}(m, l) = \delta(m, 1)\, \delta(l, 1) \quad + O\left(\frac{M q^{\varepsilon}}{q^{3/4}} \right). \tag{2.28}$$

Of course here one can use the approximate functional equation in the asymmetric form, but it turns out that this error term will match the one obtained for the second twisted moment.

Finally, using only that $\psi \ll 1$, we conclude:

$$\mathcal{L}(\psi) = \psi(1) + O\left(\frac{M^{3/2} q^{\varepsilon}}{q^{3/4}} \right).$$

2.5 The second mollified moment $\mathcal{Q}(\psi)$

Substituting (2.11) and (2.18) in (2.20) we get:

$$\mathcal{Q}(\psi) = 2 \sum_{\substack{m_1,\, l_1 \\ m_1 l_1^2 \leq M}} \sum \sum_{\substack{m_2,\, l_2 \\ m_2 l_2^2 \leq M}} \sum \frac{\mu(m_1)\mu(m_2)}{\sqrt{m_1 m_2}\, l_1 l_2}\, \psi(m_1 l_1^2)\, \psi(m_2 l_2^2)\, \mathcal{B}(m_1, m_2; l_1, l_2), \tag{2.29}$$

where $\mathcal{B}$ is the second twisted moment:

$$\begin{aligned} \mathcal{B}(m_1, m_2; l_1, l_2) = &\sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \frac{1}{\sqrt{n_1 n_2}} W\left(\frac{4\pi^2 n_1 n_2}{q} \right) \\ &\cdot \frac{2}{\varphi(q)} \sum_{\chi}^{-} \chi(l_1) \overline{\chi}(l_2) \sum_{f \in \mathcal{H}_\chi}^{h} \lambda_f(m_1) \lambda_f(n_1) \overline{\lambda_f}(m_2) \overline{\lambda_f}(n_2) \end{aligned} \tag{2.30}$$

Now we use the Hecke relations (2.2), introduce new variables d_1 and d_2 to denote the common divisors in the corresponding Hecke relations and change $n_i \to d_i n_i, \quad i = 1, 2$ to get:

$$\mathcal{B}(m_1, m_2; l_1, l_2) = \sum_{d_1 e_1 = m_1} \sum_{d_2 e_2 = m_2} \frac{1}{\sqrt{d_1 d_2}} \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \frac{1}{\sqrt{n_1 n_2}} W\left(\frac{4\pi^2 d_1 d_2 n_1 n_2}{q} \right)$$

$$\cdot \frac{2}{\varphi(q)} \sum_{\chi}^{-} \chi(l_1 d_1) \overline{\chi}(l_2 d_2) \sum_{f \in \mathcal{H}_\chi}^{h} \lambda_f(e_1 n_1) \overline{\lambda_f}(e_2 n_2)$$

Next, we apply Petersson's formula (1.5) to the inside spectral sums:

$$\mathcal{B}(m_1, m_2; l_1, l_2) = \mathcal{B}_{\text{diag}}(m_1, m_2; l_1, l_2) + \mathcal{B}_{\text{off}}(m_1, m_2; l_1, l_2), \tag{2.31}$$

where

$$\begin{aligned} \mathcal{B}_{\text{diag}}(m_1, m_2; l_1, l_2) = & \sum_{d_1 e_1 = m_1} \sum_{d_2 e_2 = m_2} \frac{1}{\sqrt{d_1 d_2}} \\ & \cdot \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \frac{\delta(e_1 n_1, e_2 n_2)}{\sqrt{n_1 n_2}} W\left(\frac{4\pi^2 d_1 d_2 n_1 n_2}{q}\right) \frac{2}{\varphi(q)} \sum_{\chi}^{-} \chi(l_1 d_1) \overline{\chi}(l_2 d_2) \end{aligned} \tag{2.32}$$

denotes the "diagonal contribution" and

$$\begin{aligned} \mathcal{B}_{\text{off}}(m_1, m_2; l_1, l_2) = 2\pi i^{-k} & \sum_{d_1 e_1 = m_1} \sum_{d_2 e_2 = m_2} \frac{1}{\sqrt{d_1 d_2}} \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \frac{1}{\sqrt{n_1 n_2}} W\left(\frac{4\pi^2 d_1 d_2 n_1 n_2}{q}\right) \cdot \\ \cdot \frac{2}{\varphi(q)} \sum_{\chi}^{-} \chi(l_1 d_1) \overline{\chi}(l_2 d_2) & \sum_{\substack{c \geq 1 \\ c \equiv 0 (q)}} \frac{1}{c} S_\chi(e_2 n_2, e_1 n_1; c) J_{k-1}\left(\frac{4\pi\sqrt{e_1 e_2 n_1 n_2}}{c}\right) \end{aligned} \tag{2.33}$$

denotes the "off-diagonal contribution" to the second twisted moment.

2.5.1 Bounding the off-diagonal contribution

We start the treatment of (2.33) by introducing new variable r, where $c = rq$ and dividing the summation over r into two parts according to $r \leq q$ or $r > q$. Denote the resulting decomposition as:

$$\mathcal{B}_{\text{off}}(m_1, m_2; l_1, l_2) = \mathcal{B}_{\text{off}}^{I}(m_1, m_2; l_1, l_2) + \mathcal{B}_{\text{off}}^{II}(m_1, m_2; l_1, l_2). \tag{2.34}$$

Treatment of $\mathcal{B}_{\text{off}}^{II}(m_1, m_2; l_1, l_2)$: In the range $r > q$ we will use Weil's bound for (twisted) Kloosterman sums:

$$|S_\chi(m, n; c)| \leq \tau(c)(m, n, c)^{1/2} c^{1/2}, \tag{2.35}$$

the bounds (2.14) for $W(x)$ and $J_{k-1}(x) \ll x^2$ (since $k \geq 3$), while the sum $\sum_\chi^-$ we estimate trivially:

$$\mathcal{B}_{\text{off}}^{II}(m_1, m_2; l_1, l_2) \ll$$

$$\sum_{d_1e_1=m_1} \sum_{d_2e_2=m_2} \frac{1}{\sqrt{d_1d_2}} \sum_{n_1n_2 \ll (q/d_1d_2)^{1+\varepsilon}} \frac{1}{\sqrt{n_1n_2}} \sum_{r \geq q} \frac{\tau(qr)(e_1n_1, e_2n_2, qr)^{1/2}(qr)^{1/2}}{qr} \frac{e_1e_2n_1n_2}{q^2r^2}$$

$$\ll \frac{M^2 q^\varepsilon}{q^{5/2}}. \quad (2.36)$$

Treatment of $\mathcal{B}_{\text{off}}^{I}(m_1, m_2; l_1, l_2)$: Here we exploit the orthogonality of Dirichlet characters mod q via (1.11); since $l_1d_1, l_2d_2 \leq M$, we have that $(l_1l_2d_1d_2, q) = 1$ and hence $\chi(l_1d_1)\overline{\chi}(l_2d_2) = \chi(l_1d_1\bar{l}_2\bar{d}_2)$ so that by (1.10):

$$\mathcal{B}_{\text{off}}^{I}(m_1, m_2; l_1, l_2) = 2\pi \sum_{d_1e_1=m_1} \sum_{d_2e_2=m_2} \frac{1}{\sqrt{d_1d_2}} \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \frac{1}{\sqrt{n_1n_2}} W\left(\frac{4\pi^2 d_1d_2n_1n_2}{q}\right) \cdot$$

$$\cdot \sum_{1 \leq r \leq q} \frac{1}{qr} J_{k-1}\left(\frac{4\pi\sqrt{e_1e_2n_1n_2}}{qr}\right) \mathcal{K} \sideset{}{^*}\sum_{\substack{a(qr) \\ a \equiv l_1d_1\bar{d}_2\bar{l}_2\,(q)}} e\left(\frac{e_2n_2a + e_1n_1\bar{a}}{qr}\right). \quad (2.37)$$

Now using (2.14), (1.12) and trivially estimating the last exponential sum we obtain:

$$\mathcal{B}_{\text{off}}^{I}(m_1, m_2; l_1, l_2) \ll \frac{M^2 q^\varepsilon}{q^{3/2}}.$$

Taking together we get an estimate:

$$\mathcal{B}_{\text{off}}(m_1, m_2; l_1, l_2) \ll \frac{M^2 q^\varepsilon}{q^{3/2}}. \quad (2.38)$$

2.5.2 Asymptotic evaluation of $\mathcal{Q}(\psi)$

We start the treatment of (2.32) using the orthogonality relation (1.8): the choice of the length of our mollifier $M = q^\Delta$ will be in the range $\Delta < 1$, so again since $l_1d_1, l_2d_2 < M$ the innermost sum in (2.32) gives non-zero contribution only for $l_1d_1 \equiv_q l_2d_2$ or $l_1d_1 \equiv_q -l_2d_2$. Moreover, again because of the length of the mollifier only the first case is possible and then it becomes the equality $l_1d_1 = l_2d_2$, so $\mathcal{B}_{\text{diag}}(m_1, m_2; l_1, l_2)$ is equal to:

$$\sum_{d_1e_1=m_1} \sum_{d_2e_2=m_2} \frac{\delta(l_1d_1, l_2d_2)}{\sqrt{d_1d_2}} \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \frac{\delta(e_1n_1, e_2n_2)}{\sqrt{n_1n_2}} W\left(\frac{4\pi^2 d_1d_2n_1n_2}{q}\right).$$

Here we introduce $c = (e_1, e_2)$, change $e_i \to ce_i$, $i = 1, 2$, where now $(e_1, e_2) = 1$ and then change $n_1 = e_2 n$, $n_2 = e_1 n$ to get:

$$\mathcal{B}_{\text{diag}}(m_1, m_2; l_1, l_2) = \sum_{\substack{cd_1e_1=m_1 \\ cd_2e_2=m_2 \\ (e_1,e_2)=1}} \frac{\delta(l_1d_1, l_2d_2)}{\sqrt{d_1d_2e_1e_2}} \sum_{n\geq 1} \frac{1}{n} W\left(\frac{4\pi^2 d_1d_2e_1e_2n^2}{q}\right). \tag{2.39}$$

Replacing this in (2.29) we arrive to:

$$\mathcal{Q}_{\text{main}}(\psi) = 2 \sum_{\substack{c,d_1,d_2,e_1,e_2,l_1,l_2,n \\ cd_1e_1l_1^2 \leq M \\ cd_2e_2l_2^2 \leq M \\ (e_1,e_2)=1}} \delta(l_1d_1, l_2d_2) \frac{\mu(cd_1e_1)\mu(cd_2e_2)}{cd_1d_2e_1e_2l_1l_2n}$$

$$\cdot \psi(cd_1e_1l_1^2)\ \psi(cd_2e_2l_2^2) W\left(\frac{4\pi^2 d_1d_2e_1e_2n^2}{q}\right).$$

Here let $b = (d_1, d_2)$; then we replace $d_i \to bd_i$, $i = 1, 2$ with now $(d_1, d_2) = 1$ getting from the delta symbol $l_1 = d_2 l$ and $l_2 = d_1 l$ where l is a new variable:

$$\mathcal{Q}_{\text{main}}(\psi) =$$

$$2 \sum_{\substack{b,c,d_1,d_2,e_1,e_2,l,n \\ bcd_1e_1d_2^2l^2 \leq M \\ bcd_2e_2d_1^2l^2 \leq M \\ (e_1,e_2)=1 \\ (d_1,d_2)=1}} \frac{\mu(bcd_1e_1)\mu(bcd_2e_2)}{b^2cd_1^2d_2^2e_1e_2l^2n} \psi(bcd_1e_1d_2^2l^2)\ \psi(bcd_2e_2d_1^2l^2) W\left(\frac{4\pi^2 b^2 d_1d_2e_1e_2n^2}{q}\right). \tag{2.40}$$

We are going to evaluate this sum using contour integration. First, our choice for the function ψ is

$$\psi(x) = P\left(\frac{\log(M/x)}{\log M}\right),$$

for some polynomial P with $P(0) = 0$. This choice is convenient because of the following Perron type formula, that can be found for example in [24], Lemma 2.1: for a polynomial $P(x) = \sum_k a_k x^k$ such that $P(0) = 0$, let

$$\widehat{P_M}(s) = \sum_k \frac{k! a_k (\log M)^{-k}}{s^k}; \tag{2.41}$$

then if M is not an integer we have

$$\delta(m < M)\, P\left(\frac{\log M/m}{\log M}\right) = \frac{1}{2\pi i} \int_{(3)} \frac{M^s}{m^s} \widehat{P_M}(s) \frac{ds}{s}, \tag{2.42}$$

where as usual, $\delta(m < M) = 1$ if $m < M$ and 0 otherwise. The proof is standard contour integration far to the right and far to the left.

Using this and the contour integral (2.12) for W we get:

$$\mathcal{Q}_{\text{main}}(\psi) = \frac{2}{(2\pi i)^3} \int_{(3)} \int_{(3)} \int_{(3)} \sum_{\substack{b,c,d_1,d_2,e_1,e_2,l,n \\ (e_1,e_2)=1 \\ (d_1,d_2)=1}} \frac{\mu(bcd_1e_1)\mu(bcd_2e_2)}{b^2cd_1^2d_2^2e_1e_2l^2n}$$
$$\cdot \left(\frac{M}{bcd_1e_1d_2^2l^2}\right)^{s_1} \widehat{P_M}(s_1) \left(\frac{M}{bcd_2e_2d_1^2l^2}\right)^{s_2} \widehat{P_M}(s_2)$$
$$\cdot \left(\frac{q}{4\pi^2b^2d_1d_2e_1e_2n^2}\right)^t \frac{\Gamma^2(t+k/2)}{\Gamma^2(k/2)} G^2(t) \frac{ds_1}{s_1}\frac{ds_2}{s_2}\frac{dt}{t} \quad (2.43)$$

$$= \frac{2}{(2\pi i)^3} \int_{(3)} \int_{(3)} \int_{(3)} A(s_1,s_2,t) M^{s_1+s_2} \left(\frac{q}{4\pi^2}\right)^t \widehat{P_M}(s_1)\widehat{P_M}(s_2) \frac{\Gamma^2(t+k/2)}{\Gamma^2(k/2)} G^2(t) \frac{ds_1}{s_1}\frac{ds_2}{s_2}\frac{dt}{t}$$

where $A(s_1,s_2,t)$ is the arithmetic sum:

$$A(s_1,s_2,t) = \sum_{\substack{b,c,d_1,d_2,e_1,e_2,l,n \\ (e_1,e_2)=1 \\ (d_1,d_2)=1}}$$
$$\cdot \frac{\mu(bcd_1e_1)\mu(bcd_2e_2)}{b^{2+s_1+s_2+2t}c^{1+s_1+s_2}d_1^{2+s_1+2s_2+t}d_2^{2+2s_1+s_2+t}e_1^{1+s_1+t}e_2^{1+s_2+t}l^{2+2s_1+2s_2}n^{1+2t}}. \quad (2.44)$$

The l and n sums are immediate:

$$A(s_1,s_2,t) = \zeta(1+2t)\zeta(2+2s_1+2s_2)$$
$$\cdot \sum_{\substack{b,c,d_1,d_2,e_1,e_2 \\ (e_1,e_2)=1 \\ (d_1,d_2)=1}} \frac{\mu(bcd_1e_1)\mu(bcd_2e_2)}{b^{2+s_1+s_2+2t}c^{1+s_1+s_2}d_1^{2+s_1+2s_2+t}d_2^{2+2s_1+s_2+t}e_1^{1+s_1+t}e_2^{1+s_2+t}}. \quad (2.45)$$

We will remove coprimality conditions $(d_1,d_2)=1$ and $(e_1,e_2)=1$ by Möbius inversion:

$$\sum_{(d_1,d_2)=1} f(d_1,d_2) = \sum_{d_1,\, d_2} f(d_1,d_2) \sum_{\alpha|(d_1,d_2)} \mu(\alpha) = \sum_{\alpha} \mu(\alpha) \sum_{d_1,\, d_2} f(\alpha d_1, \alpha d_2),$$

obtaining:

$$A(s_1,s_2,t) = \zeta(1+2t)\zeta(2+2s_1+2s_2) \sum_{\alpha,\beta,b,c,d_1,d_2,e_1,e_2}$$
$$\cdot \frac{\mu(\alpha)\mu(\beta)\mu(\alpha\beta bcd_1e_1)\mu(\alpha\beta bcd_2e_2)}{\alpha^{4+3s_1+3s_2+2t}\beta^{2+s_1+s_2+2t}b^{2+s_1+s_2+2t}c^{1+s_1+s_2}d_1^{2+s_1+2s_2+t}d_2^{2+2s_1+s_2+t}e_1^{1+s_1+t}e_2^{1+s_2+t}}.$$
$$(2.46)$$

First we simplify the sum slightly: β and b appear with the same exponent and one factor $\mu(\beta)$. Hence $\sum_\beta \sum_b$ reduces to 1:

$$A(s_1, s_2, t) = \zeta(1+2t)\zeta(2+2s_1+2s_2) \sum_{\alpha,c,d_1,d_2,e_1,e_2} \cdot \frac{\mu(\alpha)\mu(\alpha c d_1 e_1)\mu(\alpha c d_2 e_2)}{\alpha^{4+3s_1+3s_2+2t} c^{1+s_1+s_2} d_1^{2+s_1+2s_2+t} d_2^{2+2s_1+s_2+t} e_1^{1+s_1+t} e_2^{1+s_2+t}} \quad (2.47)$$

$$= \zeta(1+2t)\zeta(2+2s_1+2s_2) \sum_{\alpha,\, c} \frac{\mu(\alpha)\mu^2(\alpha c)}{\alpha^{4+3s_1+3s_2+2t} c^{1+s_1+s_2}}$$

$$\cdot \sum_{\substack{d_1,e_1 \\ (d_1e_1,\alpha c)=1}} \frac{\mu(d_1e_1)}{d_1^{2+s_1+2s_2+t} e_1^{1+s_1+t}} \sum_{\substack{d_2,e_2 \\ (d_2e_2,\alpha c)=1}} \frac{\mu(d_2e_2)}{d_2^{2+2s_1+s_2+t} e_2^{1+s_2+t}}.$$

Each of the last two sums has the same form:

$$\sum_{\substack{d,e \\ (de,a)=1}} \frac{\mu(de)}{d^{2+u} e^{1+v}} = \sum_{(d,a)=1} \frac{\mu(d)}{d^{2+u}} \sum_{(e,ad)=1} \frac{\mu(e)}{e^{1+v}} = \sum_{(d,a)=1} \frac{\mu(d)}{d^{2+u}} \prod_{p \nmid ad} \left(1 - \frac{1}{p^{1+v}}\right)$$

$$= \prod_{p\nmid a} \left(1 - \frac{1}{p^{1+v}}\right) \sum_{(d,a)=1} \frac{\mu(d)}{d^{2+u}} \prod_{p|d} \frac{1}{1-p^{-1-v}} = \prod_{p\nmid a} \left(1 - \frac{1}{p^{1+v}}\right) \sum_{(d,a)=1} \prod_{p|d} \frac{-1}{p^{2+u}} \frac{1}{1-p^{-1-v}}$$

$$= \prod_{p\nmid a} \left(1 - \frac{1}{p^{1+v}}\right) \prod_{p\nmid a} \left(1 - \frac{1}{p^{2+u}(1-p^{-1-v})}\right) = \zeta^{-1}(1+v)\, \eta(u,v) \prod_{p|a} \frac{1}{1-p^{-1-v}-p^{-2-u}},$$

where η is the following Euler product:

$$\eta(u,v) = \prod_p \left(1 - \frac{1}{p^{2+u}(1-p^{-1-v})}\right). \quad (2.48)$$

Substituting this in A we get:

$$A(s_1, s_2, t) = \frac{\zeta(1+2t)\zeta(2+2s_1+2s_2)\eta(s_1+2s_2+t, s_1+t)\eta(2s_1+s_2+t, s_2+t)}{\zeta(1+s_1+t)\zeta(1+s_2+t)}$$

$$\cdot \sum_{\alpha,\, c} \frac{\mu(\alpha)\mu^2(\alpha c)}{\alpha^{4+3s_1+3s_2+2t} c^{1+s_1+s_2}} \prod_{p|\alpha c} \frac{1}{1-p^{-1-s_1-t}-p^{-2-s_1-2s_2-t}} \; \frac{1}{1-p^{-1-s_2-t}-p^{-2-2s_1-s_2-t}}.$$

In the second line we replace $\alpha c \to c$, and since for square-free c

$$\sum_{\alpha|c} \frac{\mu(\alpha)}{\alpha^{3+2s_1+2s_2+2t}} = \prod_{p|c} \left(1 - \frac{1}{p^{3+2s_1+2s_2+2t}}\right),$$

we have that the second line is equal to

$$\sum_c \frac{\mu^2(c)}{c^{1+s_1+s_2}} \prod_{p|c} \frac{1-p^{-3-2s_1-2s_2-2t}}{(1-p^{-1-s_1-t}-p^{-2-s_1-2s_2-t})(1-p^{-1-s_2-t}-p^{-2-2s_1-s_2-t})}$$

$$= \prod_p \left(1 + \frac{1}{p^{1+s_1+s_2}} \frac{1 - p^{-3-2s_1-2s_2-2t}}{(1 - p^{-1-s_1-t} - p^{-2-s_1-2s_2-t})(1 - p^{-1-s_2-t} - p^{-2-2s_1-s_2-t})}\right)$$

$$= \omega(s_1, s_2, t) \prod_p \left(1 + \frac{1}{p^{1+s_1+s_2}}\right) = \omega(s_1, s_2, t) \frac{\zeta(1 + s_1 + s_2)}{\zeta(2 + 2s_1 + 2s_2)},$$

where

$$\omega(s_1, s_2, t) = \tag{2.49}$$

$$\prod_p \left(1 + \frac{1}{p^{1+s_1+s_2} + 1} \left[\frac{1 - p^{-3-2s_1-2s_2-2t}}{(1 - p^{-1-s_1-t} - p^{-2-s_1-2s_2-t})(1 - p^{-1-s_2-t} - p^{-2-2s_1-s_2-t})} - 1\right]\right)$$

is an Euler product absolutely convergent to the left of $s_1 = s_2 = t = 0$.

Returning to A, this gives:

$$A(s_1, s_2, t) = \frac{\zeta(1 + 2t)\zeta(1 + s_1 + s_2)}{\zeta(1 + s_1 + t)\zeta(1 + s_2 + t)} \eta(s_1+2s_2+t, s_1+t)\eta(2s_1+s_2+t, s_2+t)\omega(s_1, s_2, t)$$

where η and ω are given by (2.48) and (2.49) respectively. We denote them together as

$$\Upsilon(s_1, s_2, t) = \eta(s_1 + 2s_2 + t, s_1 + t)\eta(2s_1 + s_2 + t, s_2 + t)\omega(s_1, s_2, t). \tag{2.50}$$

We note here that

$$\Upsilon(0, 0, 0) = \eta^2(0, 0)\omega(0, 0, 0) = \prod_p \left(1 + \frac{1}{(p-1)^2(p+1)}\right) = 1.438.... \tag{2.51}$$

Finally we obtain the main term in the following form:

$$\mathcal{Q}_{\text{main}}(\psi) = \frac{2}{(2\pi i)^3} \int_{(3)} \int_{(3)} \int_{(3)} M^{s_1+s_2} \left(\frac{q}{4\pi^2}\right)^t \frac{\Gamma^2(t + k/2)}{\Gamma^2(k/2)} G^2(t)$$
$$\cdot \widehat{P_M}(s_1)\widehat{P_M}(s_2) \frac{\zeta(1 + 2t)\zeta(1 + s_1 + s_2)}{\zeta(1 + s_1 + t)\zeta(1 + s_2 + t)} \Upsilon(s_1, s_2, t) \frac{ds_1}{s_1} \frac{ds_2}{s_2} \frac{dt}{t}, \tag{2.52}$$

and together with (2.38), using $\psi \ll 1$, we have:

$$\mathcal{Q}(\psi) = \mathcal{Q}_{\text{main}}(\psi) + O\left(\frac{M^3 q^\varepsilon}{q^{3/2}}\right).$$

2.6 Calculation of the residues and choice of the mollifier

Since we switched from ψ to a polynomial P, the mollified moments depend now on P and final formulas are:

$$\mathcal{L}(P) = P(1) + O\left(\frac{M^{3/2} q^\varepsilon}{q^{3/4}}\right), \qquad \mathcal{Q}(P) = \mathcal{Q}_{\text{main}}(P) + O\left(\frac{M^3 q^\varepsilon}{q^{3/2}}\right),$$

where $M = q^{\Delta}$, and we see that any $\Delta < \frac{1}{2}$ is admissible in order the above formulas to be asymptotic.

To proceed further, we evaluate $\mathcal{Q}_{\mathrm{main}}(P)$ given by the integral (2.52) using contour integration. The goal is to evaluate the residues at the pole $s_1 = s_2 = t = 0$ and then to show that the contribution arising from shifting contours to the left of 0 is smaller (by one factor of $\log q$ is enough).

Calculation of the residues at $s_1 = s_2 = t = 0$: In calculating the residues, various powers of $\log q$ and $\log M$ will appear. We are interested only in the higher order terms and rest can be ignored. Therefore, since the factor $(4\pi^2)^{-t}\frac{\Gamma^2(t+k/2)}{\Gamma^2(k/2)}G^2(t)\Upsilon(s_1, s_2, t)$ has neither zeros nor poles at $s_1 = s_2 = t = 0$ and does not depend on q or M, its derivatives will contribute only lower order terms and hence, from the start we can replace it by its value at $(0, 0, 0)$ which is $\Upsilon(0, 0, 0)$, see (2.51).

From the same reason, we may replace everywhere $\zeta(1 + z)$ with $\frac{1}{z}$, arriving at the following main term:

$$\Upsilon(0, 0, 0)\mathrm{Res}_{(0,0,0)} \frac{M^{s_1+s_2} q^t \widehat{P_M}(s_1)\widehat{P_M}(s_2)(s_1 + t)(s_2 + t)}{s_1 s_2 (s_1 + s_2) t^2}.$$

Now we write

$$\frac{(s_1 + t)(s_2 + t)}{s_1 s_2 (s_1 + s_2) t^2} = \frac{1}{s_1 s_2 t} + \frac{1}{t^2(s_1 + s_2)} + \frac{1}{s_1 s_2 (s_1 + s_2)},$$

and denote the corresponding residues by R^I, R^{II}, R^{III} respectively. The function in R^{III} is holomorphic in t and will be treated later by shifting the contour, giving a smaller order contribution.

The R^I has a simple pole at $t = 0$ and hence

$$R^I = \Upsilon(0, 0, 0)\mathrm{Res}_{(0,0)} \frac{M^{s_1+s_2}\widehat{P_M}(s_1)\widehat{P_M}(s_2)}{s_1 s_2}.$$

These residues can be computed easily but we just cite Lemma 9.1 from [24]:

$$\mathrm{Res}_{s=0} \frac{M^s \widehat{P_M}(s)}{s} = P(1),$$

giving

$$R^I = \Upsilon(0, 0, 0)P(1)^2.$$

Next:

$$R^{II} = \Upsilon(0,0,0)\,\mathrm{Res}_{t=0}\frac{q^t}{t^2}\,\mathrm{Res}_{s_1=s_2=0}\frac{M^{s_1+s_2}\widehat{P_M}(s_1)\widehat{P_M}(s_2)}{(s_1+s_2)}.$$

We evaluate the second residue by using Lemma 9.4 from [24] (the factors $s_1 s_2$ are omitted in the statement there):

$$\mathrm{Res}_{s_1=s_2=0}\frac{M^{s_1+s_2}\widehat{P_M}(s_1)\widehat{Q_M}(s_2)}{s_1 s_2(s_1+s_2)} = (\log M)\int_0^1 P(x)Q(x)dx.$$

Now, recalling the notation (2.41) we have that $s\,\widehat{P_M}(s) = \frac{1}{\log M}\widehat{P'_M}(s)$ so

$$R^{II} = \Upsilon(0,0,0)(\log q)\frac{1}{(\log M)^2}\mathrm{Res}_{s_1=s_2=0}\frac{M^{s_1+s_2}\widehat{P'_M}(s_1)\widehat{P'_M}(s_2)}{s_1 s_2(s_1+s_2)} =$$

$$= \Upsilon(0,0,0)\frac{\log q}{\log M}\int_0^1 P'(x)^2 dx.$$

Shifting the contours: First in (2.52) we shift the s_1, s_2 and t contours to $\mathfrak{Re}(s_1) = \mathfrak{Re}(s_1) = \mathfrak{Re}(t) = 1/2$ passing no poles. Next shift the t contour to $\mathfrak{Re} = -1/2+\varepsilon$ for some small $0 < \varepsilon < \frac{1}{2} - \Delta$. On that line $|M^{s_1+s_2}q^t| = Mq^{-1/2+\varepsilon} = q^{\Delta+\varepsilon-1/2}$ and the other terms do not depend on M, q and hence the contribution of that t-contour is negligible, leaving only the contribution from the pole $t = 0$. The same reasoning, shifting even just slightly to the left of 0, shows that R^{III} is negligible.

Up to the constants, irrelevant and lower order terms we are left with

$$(\log q)\int_{(1/2)}\int_{(1/2)} M^{s_1+s_2}\widehat{P_M}(s_1)\widehat{P_M}(s_2)\frac{\zeta(1+s_1+s_2)}{\zeta(1+s_1)\zeta(1+s_2)}\frac{ds_1}{s_1}\frac{ds_2}{s_2}.$$

Here we shift s_1 and s_2 contours to the "prime number theorem contour", namely slightly to the left of $\mathfrak{Re}(s_i) = 0$ but still to the right of all zeros of $\zeta(1+s_i)$ so that in the process no new poles are introduced. More precisely, one can take the new position to be

$$\mathfrak{Re}(s_i) = -\frac{c}{\log(|\mathfrak{Im}(s_i)|+3)},$$

for some constant $c > 0$ from the zero-free region theorem for ζ. The integrals on the new contour are $\ll e^{-c_1(\log M)^{1/2010}}$ which is enough to overcome the $\log q$ factor.

Finally, the only thing left is the pole $s_1 + s_2 = 0$, i.e. $s_2 = -s_1$ where s_1 runs over the new contour. But in that case, only dependance on M or q is in two $\widehat{P_M}$ factors and since $P(0) = 0$, they contribute the factor $(\log M)^{-2}$ or any smaller power, and hence again in total, this is $\ll (\log q)^{-1}$.

The conclusion: We have proved that

$$\mathcal{Q}_{\text{main}}(P) = \big(1 + O((\log q)^{-1})\big)\, \Upsilon(0,0,0) \left[P(1)^2 + \frac{\log q}{\log M} \int_0^1 P'(x)^2 dx \right].$$

Therefore,

$$\frac{\mathcal{L}_{\text{main}}(P)^2}{\mathcal{Q}_{\text{main}}(P)} \sim \Upsilon(0,0,0)^{-1} \frac{P(1)^2}{P(1)^2 + \Delta^{-1} \int_0^1 P'(x)^2 dx}, \quad \text{as} \quad q \to \infty$$

and in order to maximize this quantity we need to minimize the quotient

$$\mathcal{I}(P) = \frac{\int_0^1 P'(x)^2 dx}{P(1)^2},$$

where $P(0) = 0$.

In order to find the extremal polynomial, we first assume that it exists and then we form the variation equation

$$\frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} \ \mathcal{I}(P + \varepsilon f) = 0,$$

which must be satisfied for all admissible (hence, satisfying $f(0) = 0$) perturbation functions $f(x)$. A short calculation leads to the condition:

$$f(1)P(1)P'(1) - f(1)\int_0^1 P'(x)^2 dx = P(1)\int_0^1 f(x)P''(x)dx,$$

which clearly can be satisfied for all admissible f only if $P''(x) = 0$, and that further leads to the extremal solution $P(x) = x$.

Therefore, the extremum is attained for $P(x) = x$ giving $\mathcal{N} > \Upsilon(0,0,0)^{-1}\frac{1}{1+\Delta^{-1}}$. Substituting our admissible $\Delta = 1/2$ and (2.51) we obtain 23.18% of non-vanishing central values in $\Gamma_1(q)$-family of automorphic L-functions, in harmonic average.

Remark 2.2. The constant $\Upsilon(0,0,0)^{-1}$ can be removed or at least improved by taking perfect mollifier (with $\mu^2(ml)$-factors included) and hence essentially, our result is $\frac{1}{1+\Delta^{-1}}$ which is $\frac{1}{3}$ for a mollifier of length $q^{1/2}$. This is in complete agreement with the percentage obtained in [16] for GL_1 family of the L-functions for Dirichlet characters $\chi \bmod q$. This is hardly surprising, because both these families should have *unitary* underlying symmetry group.

Remark 2.3. We note that even if hypothetically, one can perform perfect off-diagonal harmonic analysis and take maximal possible length q of a mollifier, i.e. $\Delta = 1$ (note that $|L|^2$ is represented essentially by a Dirichlet polynomial of that length), mollification would give only 50% of nonvanishing of central values in harmonic average. Therefore the family of $\Gamma_1(q)$-modular forms still posses the same barrier as the family of $\Gamma_0(q)$-modular forms, which was critical in [17] for the problem of Landau-Siegel zero. Of course, because central values in $\Gamma_1(q)$-family are no longer positive, such an application was not possible anyway.

Chapter 3

The sixth moment of the family of $\Gamma_1(q)$-automorphic L-functions

3.1 Introduction

So far, understanding of L-functions predominantly has come from their embedding into natural families and then trying to extract the required information from "the information on average" by various devices, such are mollification or amplification. Hence, understanding of moments of L-functions in families become vital for various applications in analytic number theory. On the other hand, one can study moments in their own - viewed as an approximation towards the Generalized Lindelöf Hypothesis. Third motivation comes from the Random Matrix Theory models for L-functions ([19], [20], [21], [3]) where calculation of moments provides the testing ground for checking RMT predictions, and by that hopefully increases our understanding of the underlying symmetries.

In this chapter we will bound the sixth moment of the $\Gamma_1(q)$-family. Some analogous situations which also provide the motivation for this chapter are as follows:

- direct analog on GL_1 is the following bound for the sixth moment of Dirichlet L-functions given by Huxley in [10]:

$$\sum_{q\leq Q} \ \sideset{}{^*}\sum_{\chi \bmod q} |L(1/2,\chi)|^6 \ll Q^2 \log^9 Q\,;$$

- the state of art on GL_1 is the following asymptotic from [4]: for Ψ smooth and compactly supported on $[1,2]$, and $\Phi(t)$ of rapid decay as $|t|\to\infty$ the following holds

$$\sum_{q} \ \sideset{}{^*}\sum_{\chi\,(q)} \Psi(q/Q) \int_{-\infty}^{\infty} \Phi(t)|L(1/2+it,\chi)|^6 dt$$

$$\sim 42\, a_3 Q^2 \frac{\log^9 Q}{9!} \int_0^\infty \Psi(x) x dx \int_{-\infty}^\infty \Phi(t) |\Gamma(1/4 + it/2)|^6 dt,$$

where a_3 is some explicit constant; we note it as an example of the use of additional tiny averaging;

- on GL_2 the following result of Kowalski, Michel and Vanderkam from [23] is directly related: the averaging is performed over the set $S_2(q)$ of primitive Hecke eigenforms of weight 2 relative to the subgroup $\Gamma_0(q)$, where q is also a prime, and the result is an asymptotic with the power saving in the error term - for all $\varepsilon > 0$,

$$\sum_{f \in S_2(q)}^{h} L(f, 1/2)^4 = P(\log q) + O_\varepsilon \left(q^{-1/12+\varepsilon}\right),$$

  where P is a polynomial of degree 6;

- recently M. Young in [34] proved the following: let (u_j) be an orthonormal basis of Hecke-Maass cusp forms for $SL_2(\mathbb{Z})$ with corresponding Laplace eigenvalues $\frac{1}{4} + t_j^2$; then the following bound for the sixth moment at the special points $\frac{1}{2} + it_j$ holds:

$$\sum_{t_j \leq T} |L(u_j, \frac{1}{2} + it_j)|^6 \ll T^{2+\varepsilon};$$

  at these points the conductor drops (in effect replacing T by $T^{1/2}$) which analytically has the same effect as enlarging the family of holomorphic $\Gamma_0(q)$-automorphic forms to the family of $\Gamma_1(q)$-forms (which means that the family is enlarged from $\asymp q$ elements to $\asymp q^2$ elements); in both cases $\frac{\log(\text{conductor})}{\log(|\text{family}|)}$ drops by the factor 2.

We will prove the following similar bound, consistent with the Lindelöf Hypothesis on average:

Theorem 3.1. *Let q be a prime. Then we have:*

$$\frac{2}{\varphi(q)} \sum_{\chi(q)}^{-} \sum_{f \in \mathcal{H}_\chi}^{h} |L(f, 1/2)|^6 \ll q^\varepsilon. \tag{3.1}$$

3.2 Approximate functional equation and reductions

There are several choices for representation of $|L|^6$ inside the critical strip. We will start with an expression for L^3 and then pass to $|L^3|^2 = |L|^6$ by Cauchy-Schwarz inequality.

Therefore, starting with Dirichlet series representation (2.1) of $L(f,s)$ for $f \in \mathcal{H}_\chi$ and $\mathfrak{Re}(s) > 1$, and using Hecke relations (2.2) one obtains:

$$L(f,s)^3 = L(f,s)^2 L(f,s) = \sum_{d\geq 1} \frac{\chi(d)}{d^{2s}} \sum_{n\geq 1} \frac{\lambda_f(n)\tau(n)}{n^s} \sum_{m\geq 1} \frac{\lambda_f(m)}{m^s} =$$

$$= \sum_{d\geq 1} \frac{\chi(d)}{d^{2s}} \sum_{n\geq 1} \sum_{m\geq 1} \frac{\tau(n)}{(mn)^s} \sum_{c|(m,n)} \chi(c)\lambda_f\left(\frac{mn}{c^2}\right) =$$

$$= \sum_{d\geq 1} \frac{\chi(d)}{d^{2s}} \sum_{c\geq 1} \sum_{n_1\geq 1} \sum_{m_1\geq 1} \frac{\tau(cn_1)\chi(c)\lambda_f(m_1 n_1)}{(m_1 n_1 c^2)^s}.$$

Now we use the recursion formula for divisor function

$$\tau(cn) = \sum_{a|(c,n)} \mu(a)\tau(c/a)\tau(n/a),$$

and replace c with ac_1 and n_1 with an_2 to get:

$$L(f,s)^3 = \sum_{a\geq 1} \frac{\mu(a)}{a^{2s}} \left(\sum_{d\geq 1} \frac{\chi(d)}{d^{2s}} \sum_{c_1\geq 1} \frac{\tau(c_1)\chi(c_1)}{c_1^{2s}} \right) \left(\sum_{m_1\geq 1} \sum_{n_2\geq 1} \frac{\tau(n_2)\lambda_f(am_1 n_2)}{(am_1 n_2)^s} \right) =$$

$$= \sum_{a\geq 1} \frac{\mu(a)}{a^{2s}} \sum_{b\geq 1} \frac{\chi(b)\tau_3(b)}{b^{2s}} \sum_{n\geq 1} \frac{\lambda_f(an)\tau_3(n)}{(an)^s}. \tag{3.2}$$

Next, using the same notation and the same auxiliary function G as in 2.2, starting from

$$\mathcal{I}(f) = \frac{1}{2\pi i} \int_{(2)} \Lambda(f, \frac{1}{2} + s)^3 G^3(s) \frac{ds}{s} \tag{3.3}$$

moving the contour of integration and using the functional equation (2.7) we get:

$$\Lambda(f, 1/2)^3 = \mathcal{I}(f) + (i^k \bar{\eta})^3 \mathcal{I}(\bar{f})$$

Now introducing

$$U(y) = \frac{1}{2\pi i} \int_{(2)} y^{-s} \frac{\Gamma^3(\frac{k}{2} + s)}{\Gamma^3(\frac{k}{2})} G^3(s) \frac{ds}{s}, \tag{3.4}$$

by termwise integration in (3.3) we arrive at:

$$L(f,1/2)^3 = \sum_{a\geq 1} \sum_{b\geq 1} \sum_{n\geq 1} \frac{\mu(a)\chi(b)\tau_3(b)\lambda_f(an)\tau_3(n)}{\sqrt{a^3 b^2 n}} U\left(\frac{(2\pi)^3 a^3 b^2 n}{q^{3/2}}\right) +$$

$$+ (i^k \bar{\eta})^3 \sum_{a\geq 1} \sum_{b\geq 1} \sum_{n\geq 1} \frac{\mu(a)\chi(b)\tau_3(b)\overline{\lambda_f}(an)\tau_3(n)}{\sqrt{a^3 b^2 n}} U\left(\frac{(2\pi)^3 a^3 b^2 n}{q^{3/2}}\right). \tag{3.5}$$

Similarly as in 2.2 one obtains

$$U(y) \ll (1+y)^{-A},$$

$$U(y) = 1 + O(y^A), \quad \text{for} \quad y \to 0,$$

meaning that the terms in (3.5) for $a^3b^2n \gg q^{3/2+\varepsilon}$ are negligible (contributing only to an error $\ll q^{-2010}$).

Further, it is enough to estimate each of the dual sums separately, and moreover, each of them can be divided into dyadic segments. Hence, by fixing a smooth function Ψ_1 supported in the interval $[1,2]$ and with each of its derivatives absolutely bounded, we see that another test function $\Psi(x) := \Psi_1(x)U(\frac{xX}{q^{3/2+\varepsilon}})$ has the same properties for all $X \leq q^{3/2+\varepsilon}$ (with derivatives now bounded only in terms of k) .

By this, our problem reduces to bounding for any $X \ll q^{3/2+\varepsilon}$ the following smoothed sums:

$$\mathcal{S}(X) = \frac{2}{\varphi(q)} \sum_{\chi(q)}^{-} \sum_{f \in \mathcal{H}_\chi}^{h} |\mathcal{S}_f(X)|^2. \tag{3.6}$$

where

$$\mathcal{S}_f(X) = \sum_{a \geq 1} \sum_{b \geq 1} \sum_{n \geq 1} \frac{\mu(a)\chi(b)\tau_3(b)\lambda_f(an)\tau_3(n)}{\sqrt{a^3b^2n}} \Psi\left(\frac{(2\pi)^3a^3b^2n}{X}\right).$$

Now an application of Cauchy-Scwharz gives

$$|\mathcal{S}_f(X)|^2 \leq \sum_{\substack{a,b \\ ab<\sqrt{X}}} \frac{\tau_3^2(b)}{ab} \sum_{\substack{a,b \\ ab<\sqrt{X}}} \frac{1}{ab} \left| \sum_{n \geq 1} \frac{\lambda_f(an)\tau_3(n)}{\sqrt{an}} \Psi\left(\frac{an}{X/(8\pi^3a^2b^2)}\right) \right|^2,$$

and denoting $Y = \frac{X}{(2\pi)^3a^2b^2}$, the bound for the sixth moment reduces to showing that

$$\frac{2}{\varphi(q)} \sum_{\chi(q)}^{-} \sum_{f \in \mathcal{H}_\chi}^{h} \left| \sum_{n \geq 1} \lambda_f(an) \frac{\tau_3(n)}{\sqrt{an}} \Psi\left(\frac{an}{Y}\right) \right|^2 \ll q^\varepsilon. \tag{3.7}$$

3.3 A summation formula for the twisted divisor function

In the next section we will need a summation formula for the divisor function $\tau_3(n)$, which appears in (3.7). Analogous formula for τ_2 is seen as part of the more general framework - summation formulas for GL_2 L-function coefficients, and as such it has found many applications in GL_2 L-functions theory, most notably in the proofs of subconvexity results (for example, [7] is one of the early instances where such formula is utilized in the context of subconvexity). All formulas of that type are called Voronoi formulas, by their most classical appearance in [32]. General GL_2-Voronoi formulas encode automorphy of the given Fourier coefficients, but in the case of τ_2 for their proof one only needs two variable Poisson summation formula (cf. Theorem 4.10 in [14]).

In the case which we need here, τ_3 "belongs" to GL_3 and such GL_n summation formulas are provided recently in [26] and [27] (for the general survey of the results, see [28]). The summation formula for τ_k is obtained earlier by Ivić in [11], and we will quote here some of his results that we will need.

Let ψ be a smooth compactly supported function on $(0, \infty)$, and

$$\tilde{\psi}(s) = \int_0^\infty \psi(x) x^s \frac{dx}{x}$$

its Mellin transform - an entire function of rapid decay. Then, we are interested in the summation formula for

$$\sum_{n \geq 1} \tau_k(n) e\left(\frac{dn}{c}\right) \psi(n), \tag{3.8}$$

where $k \geq 3$ and $(c, d) = 1$. In order to perform the required summation one can use the corresponding generating Dirichlet series, in this case:

$$E_k\left(s, \frac{d}{c}\right) = \sum_{n \geq 1} \frac{\tau_k(n) e(dn/c)}{n^s}, \quad \text{for} \quad \mathfrak{Re}(s) > 1, \tag{3.9}$$

which could be considered a higher dimensional relative of the classical Estermann zeta-function ($k = 2$ case). It can be easily transformed to

$$E_k\left(s, \frac{d}{c}\right) = c^{-ks} \sum_{a_1=1}^{c} \cdots \sum_{a_k=1}^{c} e\left(\frac{da_1 \ldots a_k}{c}\right) \zeta(s, a_1/c) \ldots \zeta(s, a_k/c) \tag{3.10}$$

where

$$\zeta(s, \alpha) = \sum_{n \geq 0} (n + \alpha)^{-s} \quad \text{for} \quad 0 < \alpha \leq 1, \, \mathfrak{Re}(s) > 1$$

is the Hurwitz zeta-function, from which then $E_k\left(s, \frac{d}{c}\right)$ inherits meromorphic continuation to the whole complex plane and suitable functional equation. Its only singularity is a pole of order k at $s = 1$ (for the precise functional equation we refer to Lemma 1 in

[11]). This analytic data suffice for the derivation of the summation formula for (3.8) via Mellin inversion. To state it, we need the following notation:

$$A_k^{\pm}\left(n,\frac{d}{c}\right)=\frac{1}{2}\sum_{n_1\dots n_k=n}\sum_{a_1=1}^{c}\cdots\sum_{a_k=1}^{c}$$
$$\left(e\left(\frac{n_1a_1+\ldots+n_ka_k+da_1\ldots a_k}{c}\right)\pm e\left(\frac{n_1a_1+\ldots+n_ka_k-da_1\ldots a_k}{c}\right)\right) \tag{3.11}$$

are two complete exponential sums bounded by

$$|A_k^{\pm}\left(n,\frac{d}{c}\right)|\leq c^k\tau_k(n),$$

and for $x>0$ and $0<\sigma<\frac{1}{2}-\frac{1}{r}$ let

$$U_k(x)=\frac{1}{2\pi i}\int_{(\sigma)}x^{-s}\frac{\Gamma^k(\frac{s}{2})}{\Gamma^k(\frac{1-s}{2})}ds,\qquad V_k(x)=\frac{1}{2\pi i}\int_{(\sigma)}x^{-s}\frac{\Gamma^k(\frac{1+s}{2})}{\Gamma^k(\frac{2-s}{2})}ds. \tag{3.12}$$

Then, for any $(c,d)=1$ and any $\psi\in\mathcal{C}_0^\infty(0,\infty)$ we have the following summation formula ([11], Theorem 2):

$$\sum_{n\geq1}\tau_k(n)e\left(\frac{dn}{c}\right)\psi(n)=\frac{1}{2\pi i}\int_{(2)}\tilde{\psi}(s)E_k\left(s,\frac{d}{c}\right)ds=$$
$$=\mathrm{res}_{s=1}\tilde{\psi}(s)E_k\left(s,\frac{d}{c}\right)+\frac{\pi^{k/2}}{c^k}\sum_{n\geq1}A_k^{+}\left(n,\frac{d}{c}\right)\int_0^\infty\psi(x)U_k\left(\frac{\pi^kn}{c^k}x\right)dx+$$
$$+i^{3k}\frac{\pi^{k/2}}{c^k}\sum_{n\geq1}A_k^{-}\left(n,\frac{d}{c}\right)\int_0^\infty\psi(x)V_k\left(\frac{\pi^kn}{c^k}x\right)dx. \tag{3.13}$$

Furthermore, the asymptotic behavior of $U_3(x)$ and $V_3(x)$ for $x\gg1$ and for any fixed integer $K\geq1$ is given by:

$$U_3(x)=\sum_{j=1}^{K}\frac{1}{x^{j/3}}\left(C_{1,j}\cos(6x^{1/3})+C_{2,j}\sin(6x^{1/3})\right)+O(x^{-(K+1)/3}), \tag{3.14}$$

$$V_3(x)=\sum_{j=1}^{K}\frac{1}{x^{j/3}}\left(C_{3,j}\cos(6x^{1/3})+C_{4,j}\sin(6x^{1/3})\right)+O(x^{-(K+1)/3}), \tag{3.15}$$

where $C_{1,j},C_{2,j},C_{3,j},C_{4,j}$ are some absolute constants.

Finally, in the next section we will apply (3.13) for $k=3$ and we will need more precise description of the Laurent expansion of $E_3(s,d/c)$ at its triple pole $s=1$. Again, that could be derived from (3.10) starting with the Laurent expansion of the Hurwitz zeta function:

$$\zeta(s,\alpha)=\frac{1}{s-1}+\sum_{n=0}^{\infty}\gamma_n(\alpha)(s-1)^n,$$

where, for example:

$$\gamma_0(\alpha) = \lim_{K\to\infty} \left[\sum_{k=0}^{K} \frac{1}{k+\alpha} - \log(K+\alpha) \right].$$

Denoting $\sigma_1 = \sigma_1(a_1, a_2, a_3) = \gamma_0(a_1/c) + \gamma_0(a_2/c) + \gamma_0(a_3/c)$ and $\sigma_2 = \sigma_2(a_1, a_2, a_3) = \gamma_0(a_1/c)\gamma_0(a_2/c) + \gamma_0(a_1/c)\gamma_0(a_3/c) + \gamma_0(a_2/c)\gamma_0(a_3/c)$ and using the Taylor expansion

$$c^{-3s} = \frac{1}{c^3} - 3\frac{\log c}{c^3}(s-1) + \frac{9}{2}\frac{(\log c)^2}{c^3}(s-1)^2 + \ldots$$

we get

$$c^{-3s}\zeta\left(s, \frac{a_1}{c}\right)\zeta\left(s, \frac{a_2}{c}\right)\zeta\left(s, \frac{a_3}{c}\right) =$$

$$= \frac{1}{(s-1)^3}c^{-3} + \frac{1}{(s-1)^2}c^{-3}(\sigma_1 - 3\log c) + \frac{1}{(s-1)}c^{-3}\left(\frac{9}{2}(\log c)^2 - 3\sigma_1 \log c + \sigma_2\right) + \ldots. \tag{3.16}$$

To get coefficients in the Laurent expansion

$$E_3\left(s, \frac{d}{c}\right) = \frac{C_{-3}}{(s-1)^3} + \frac{C_{-2}}{(s-1)^2} + \frac{C_{-1}}{(s-1)^1} + C_0 + \ldots \tag{3.17}$$

one need to average (3.16) over a_1, a_2, a_3 modulo c:

$$C_{-3} = \frac{1}{c^3}\sum_{a_1=1}^{c}\sum_{a_2=1}^{c}\sum_{a_3=1}^{c} e\left(\frac{da_1a_2a_3}{c}\right) = \frac{1}{c^2}\sum_{a_1=1}^{c}\sum_{a_2=1}^{c} \delta(c\,|\,a_1a_2),$$

$$C_{-2} = \frac{1}{c^3}\sum_{a_1=1}^{c}\sum_{a_2=1}^{c}\sum_{a_3=1}^{c} e\left(\frac{da_1a_2a_3}{c}\right)\left(\gamma_0\left(\frac{a_1}{c}\right) + \gamma_0\left(\frac{a_2}{c}\right) + \gamma_0\left(\frac{a_3}{c}\right) - 3\log c\right) =$$

$$= \frac{1}{c^2}\sum_{a_1=1}^{c}\sum_{a_2=1}^{c} \delta(c\,|\,a_1a_2)\left(3\gamma_0\left(\frac{a_1}{c}\right) - 3\log c\right),$$

and similarly using symmetry between a_1, a_2 and a_3,

$$C_{-1} = \frac{1}{c^3}\sum_{a_1=1}^{c}\sum_{a_2=1}^{c} \delta(c\,|\,a_1a_2)\left(\frac{9}{2}(\log c)^2 - 9\gamma_0\left(\frac{a_1}{c}\right)\log c + 3\gamma_0\left(\frac{a_1}{c}\right)\gamma_0\left(\frac{a_2}{c}\right)\right).$$

In particular, the crucial property that we will use is that the coefficients C_{-3}, C_{-2} and C_{-1} do not depend on d. Moreover, using $\gamma_0(\alpha) \ll 1$, it is easy to show that

$$C_{-3},\, C_{-2},\, C_{-1} \ll c^{-1+\varepsilon}. \tag{3.18}$$

3.4 An application of the asymptotic $\Gamma_1(q)$-large sieve

The reduced problem (3.7) is in the form suitable for an application of the large sieve, and for the family of $\Gamma_1(q)$-automorphic forms on GL_2 such large sieve inequality is provided by Iwaniec-Li in [15]. However, direct application of the large sieve and subsequent use of Weil's bound for Kloosterman sums does not give enough saving. On the other hand, Iwaniec-Li large sieve is *asymptotic*, and it leaves freedom to exploit further special properties of the test vector (which is in our case τ_3-divisor function).

Therefore, denoting the test-vector $\vec{\alpha} = (\alpha_i)$ with $\alpha_{an} = \frac{\tau_3(n)}{\sqrt{an}} \Psi\left(\frac{an}{Y}\right)$ and zero otherwise, an application of (1.16) gives the following expression for the left hand side of (3.7):

$$\frac{2}{\varphi(q)} \sum_{\chi(q)}^{-} \sum_{f \in \mathcal{H}_\chi}^{h} \left| \sum_n \lambda_f(an) \alpha_{an} \right|^2 =$$

$$= \frac{1}{q} \sum_{\substack{1 \le t \le T \\ (t,q)=1}} \left(\frac{2\pi}{t}\right)^2 \sum_{1 \le h \le H} |\mathcal{P}_{h,t}(\vec{\alpha})|^2 + O\left(q^\varepsilon \left(\frac{Y}{q^2} + \sqrt{\frac{Y}{qH}}\right)\right) \|\vec{\alpha}\|^2, \tag{3.19}$$

where $Y \ll \frac{q^{3/2+\varepsilon}}{a^2b^2}$ and $\|\vec{\alpha}\|^2 \ll a^{-1}(Y/a)^\varepsilon \ll q^\varepsilon$. Hence, the error term in (3.19) is admissible as long as $H \gg \frac{q^{1/2-\varepsilon/2}}{a^2b^2}$. Moreover, $T = Y/q$ and therefore we are left to bound the main term in (3.19) for the following choice of the parameters:

$$T = \frac{q^{1/2+\varepsilon}}{a^2b^2}, \qquad H = \frac{q^{1/2-\varepsilon/2}}{a^2b^2}, \qquad Y \ll \frac{q^{3/2+\varepsilon}}{a^2b^2}. \tag{3.20}$$

Here $\varepsilon > 0$ in the all three exponents is the same and fixed through the end of the argument.

Now, after opening Kloosterman sums,

$$\mathcal{P}_{h,t}(\vec{\alpha}) = \sum_n \frac{\tau_3(n)}{\sqrt{an}} \Psi\left(\frac{an}{Y}\right) S(h\bar{q}, an; t) J_{k-1}\left(\frac{4\pi}{t}\sqrt{\frac{han}{q}}\right) =$$

$$= \sum_{d(t)}^{*} e\left(\frac{h\bar{q}\bar{d}}{t}\right) \sum_n \tau_3(n) e\left(\frac{adn}{t}\right) \frac{1}{\sqrt{an}} J_{k-1}\left(\frac{4\pi}{t}\sqrt{\frac{han}{q}}\right) \Psi\left(\frac{an}{Y}\right),$$

we see that the inner sum can be treated by the formula (3.13). In order to do so, we first need to reduce $\frac{a}{t}$ to coprime $a_1 = \frac{a}{(a,t)}$ and $t_1 = \frac{t}{(a,t)}$. Then we apply (3.13) for

$$\sum_n \tau_3(n) e\left(\frac{a_1 d}{t_1} n\right) \psi(n),$$

where the test function is given by

$$\psi(x) = (ax)^{-1/2} J_{k-1}\left(\frac{4\pi}{t}\sqrt{\frac{hax}{q}}\right)\Psi\left(\frac{ax}{Y}\right). \tag{3.21}$$

First task is to analyze the weight functions appearing on the right hand side in (3.13). Both U_3 and V_3 have the same asymptotic behavior, so we consider only:

$$\int_0^\infty J_{k-1}\left(\frac{4\pi}{t}\sqrt{\frac{hax}{q}}\right) U_3\left(\frac{\pi^3 nx}{t_1^3}\right)\Psi\left(\frac{ax}{Y}\right)(ax)^{-1/2}dx.$$

Now we substitute the expression (1.14) for J-Bessel function, and the asymptotic expansion (3.14) for U_3. In the later, it is enough to consider only the leading order term, since all others behave identically and are smaller. We get two oscillatory integrals $\int e(\phi(x))\ldots$ with the phase functions:

$$\phi_\pm(x) = \pm\frac{2}{t}\left(\frac{ha}{q}\right)^{1/2} x^{1/2} + \frac{3n^{1/3}}{t_1}x^{1/3}.$$

These phase functions have stationary point x_0 (solution to $\phi'(x_0) = 0$) given by

$$x_0 = n^2\left(\frac{t}{t_1}\right)^6\left(\frac{q}{ha}\right)^3$$

and unless x_0 is close to the the support of $\psi(x)$ (which is $\sim Y/a$ because of Ψ) the contribution of the corresponding n-term to the right hand side of (3.13) is negligible, by partial integration enough number of times. That means that only significant terms are given by the condition:

$$x_0 = n^2\left(\frac{t}{t_1}\right)^6\left(\frac{q}{ha}\right)^3 \asymp \frac{Y}{a} \ll \frac{q^{3/2+\varepsilon}}{a^3b^2}.$$

But since $h \leq H$ and recalling (3.20) we get:

$$n^2 \ll \left(\frac{t_1}{t}\right)^6\left(\frac{Ha}{q}\right)^3\frac{q^{3/2+\varepsilon}}{a^3b^2} \ll \frac{1}{a^6b^8q^{\varepsilon/2}}.$$

This means that again here - for the small n, by partial integration many times, we get that the contribution of the n-terms in this range is $\ll q^{-2010}$ which means that the both sums on the right hand side of (3.13) contribute to the $\mathcal{P}_{h,t}$ forms in (3.19) negligible amount.

Therefore, we are left only with the contribution from $\mathrm{res}_{s=1}\tilde{\psi}(s)E_3\left(s, \frac{a_1 d}{t_1}\right)$, that is:

$$\mathcal{P}_{h,t}(\vec{\alpha}) = \sum_{d(t)}^{*} e\left(\frac{h\bar{q}\bar{d}}{t}\right) \mathrm{res}_{s=1}\tilde{\psi}(s)E_3\left(s, \frac{a_1 d}{t_1}\right) + O(q^{-2010}),$$

where ψ is given by (3.21).

The Mellin transform $\tilde{\psi}(s)$ is holomorphic at $s = 1$ and:

$$\tilde{\psi}(s) = \tilde{\psi}(1) + \tilde{\psi}'(1)(s-1) + \frac{1}{2}\tilde{\psi}''(1)(s-1)^2 + \dots$$

where for $\nu = 0, 1, 2$:

$$\tilde{\psi}^{(\nu)}(1) = \int_0^\infty \psi(x)(\log x)^\nu dx = \int_0^\infty \frac{(\log x)^\nu}{\sqrt{ax}} J_{k-1}\left(\frac{4\pi}{t}\sqrt{\frac{hax}{q}}\right)\Psi\left(\frac{ax}{Y}\right)dx.$$

Then recalling the notation (3.17) for the coefficients of Laurent expansion we get:

$$\mathrm{res}_{s=1}\tilde{\psi}(s)E_3\left(s, \frac{a_1 d}{t_1}\right) = C_{-1}\tilde{\psi}(1) + C_{-2}\tilde{\psi}'(1) + \frac{1}{2}C_{-3}\tilde{\psi}''(1),$$

where by (3.18):

$$C_{-3}, C_{-2}, C_{-1} \ll t_1^{-1+\varepsilon}. \tag{3.22}$$

Since C-coefficients do not depend on d we have:

$$\mathcal{P}_{h,t}(\vec{\alpha}) = r_t(h)\int_0^\infty \frac{C_{-1} + C_{-2}\log x + \frac{1}{2}C_{-3}(\log x)^2}{\sqrt{ax}} J_{k-1}\left(\frac{4\pi}{t}\sqrt{\frac{hax}{q}}\right)\Psi\left(\frac{ax}{Y}\right)dx$$
$$+O(q^{-2010}).$$

The main term is the Ramanujan sum times the sum of three oscillatory integrals that all have the same phase and the same amplitudes up to the logarithmic factors which are admissible (the factor q^ε will take care of them). Using (3.22), substituting the expression (1.14) for J-function, using that $t_1 = \frac{t}{(t,a)} \geq \frac{t}{a}$, changing $ax \to x$, and replacing Ψ with another smooth test function Ψ_2 with the same properties, we get the bound:

$$|\mathcal{P}_{h,t}(\vec{\alpha})| \ll q^\varepsilon \frac{|r_t(h)|}{t\sqrt{Y}}\left(t\sqrt{\frac{q}{hY}}\right)^{1/2}\left|\int_0^\infty e\left(\pm\frac{2}{t}\sqrt{\frac{hx}{q}}\right)\Psi_2\left(\frac{x}{Y}\right)dx\right|.$$

Now substituting this in (3.19), we arrive at:

$$\frac{1}{q}\sum_{\substack{1\leq t\leq T\\(t,q)=1}}\left(\frac{2\pi}{t}\right)^2\sum_{1\leq h\leq H}|\mathcal{P}_{h,t}(\vec{\alpha})|^2 \ll$$

$$\ll \frac{q^{\varepsilon}}{q^{1/2}Y^{3/2}} \sum_{1\le t\le T} \sum_{1\le h\le H} \frac{|r_t(h)|^2}{t^3h^{1/2}} \left| \int_0^{\infty} e\left(\pm\frac{2}{t}\sqrt{\frac{hx}{q}}\right) \Psi_2\left(\frac{x}{Y}\right) dx \right|^2 .$$

Next recalling the ranges of relevant parameters (3.20), we divide summation into dyadic boxes $(t,h) \in [\mathcal{T}, 2\mathcal{T}] \times [\mathcal{H}, 2\mathcal{H}]$, total number of which again being absorbed into q^{ε}, and then divide boxes into two sets $\mathcal{B}_1 \sqcup \mathcal{B}_2$, members of $\mathcal{B}_1$ being all the boxes satisfying:

$$Y \gg \frac{\mathcal{T}^2 q^{1+\varepsilon}}{\mathcal{H}}.$$

For them, partial integration of the oscillatory integral many times shows that their total contribution is negligible ($\ll q^{-2010}$).

In the second case, after bounding the integral trivially by Y, we find that the contribution of a box in $\mathcal{B}_2$ is

$$\ll \frac{q^{\varepsilon}Y^{1/2}}{q^{1/2}} \sum_{t\sim\mathcal{T}} \sum_{h\sim\mathcal{H}} \frac{|r_t(h)|^2}{t^3h^{1/2}} \ll q^{\varepsilon} \frac{\mathcal{T}}{\mathcal{H}^{1/2}} \frac{\mathcal{T}\mathcal{H}}{\mathcal{T}^3\mathcal{H}^{1/2}} \ll \frac{q^{\varepsilon}}{\mathcal{T}} \ll q^{\varepsilon},$$

and hence the proof is complete.

Chapter 4

A larger GL_2 large sieve in the level aspect

4.1 Introduction

The starting point and motivation for this section is the phenomenon described in 1.5.3 - the "loss" of orthogonality in the range $q \leq N \leq q^2$. The question is: can we recover the orthogonality by enlarging the family, but keeping the analytic conductors essentially fixed? The analytic conductor of $f \in S_k(\Gamma_1(q))$ is $\asymp qk^2$ and hence averaging in the k-aspect is not suitable for this purpose, but on the other hand we can do averaging $q \sim Q$ in the level aspect and still to have conductor of size $\asymp Q$.

Let $\mathcal{B}_\chi(q)$ denotes any orthogonal basis for $S_k(\Gamma_0(q), \chi)$ and let $(\psi_f(n))_{n\geq 1}$ denote the Fourier coefficients of basis element f normalized as in 1.5.3.

For the purpose of averaging, let $\Phi(x)$ be a $\mathcal{C}^\infty$ function compactly supported on $(\frac{1}{2}, \frac{5}{2})$ and such that $0 \leq \Phi(x) \leq 1$ and $\Phi \equiv 1$ on $[1, 2]$. Also, denote the total mass in summation by $\widetilde{Q} = \sum_{q\geq 1} \Phi(q/Q)$.

Then, for any complex vector $\mathbf{a} = (a_n)_{N<n\leq 2N}$ we are interested in bounding the following large sieve-type average:

$$\mathfrak{B}(\mathbf{a}) = \frac{1}{\widetilde{Q}} \sum_q \Phi\left(\frac{q}{Q}\right) \frac{2}{\varphi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f\in\mathcal{B}_\chi(q)} |\sum_n a_n\psi_f(n)|^2 =$$

$$= \sum_{n\sim N} \sum_{m\sim N} a_n\overline{a}_m \Delta(m, n), \tag{4.1}$$

where $\Delta(m, n)$ is the trace of this enlarged family:

$$\Delta(m, n) = \frac{1}{\widetilde{Q}} \sum_q \Phi\left(\frac{q}{Q}\right) \frac{2}{\varphi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f\in\mathcal{B}_\chi(q)} \psi_f(n)\overline{\psi_f}(m). \tag{4.2}$$

Note that the total number of harmonics on the left hand side of (4.1) is $\asymp Q^3$ and they all have the analytic conductor $\asymp Q$. This is a potentially very interesting feature of the given GL_2-family, since the analogous GL_1-family of primitive Dirichlet characters mod q, $q \leq Q$ has only $\asymp Q^2$ members of the conductor $\ll Q$. Therefore one should expect the following bound for (4.1):

$$\ll \left(1 + \frac{N}{Q^3}\right) \|\mathbf{a}\|^2.$$

i.e. one should expect "perfect orthogonality" of the Fourier coefficients $(\psi_f(n))_{1 \leq n \leq N}$ of automorphic forms in our family up to $N = Q^3$. However we will treat only the limited range $1 \leq N \leq Q^{2-\delta}$, for arbitrary small $\delta > 0$:

Theorem 4.1. *With the notation as above, for any $\varepsilon > 0$, any vector $\mathbf{a} = (a_n)_{n \sim N}$ of complex numbers with N in the range $1 \leq N \leq Q^{2-\delta}$, we have*

$$\mathfrak{B}(\mathbf{a}) \ll Q^{\varepsilon} \|\mathbf{a}\|^2,$$

where the implied constant depends on k, δ and ε.

Extension beyond this, i.e. to the range $Q^2 \ll N \ll Q^3$ is still open, and having in mind (1.5.3) it is not even clear whether perfect orthogonality holds there. Still it could be probable, just on account of previously mentioned large number of harmonics involved - so that they could "wash out" all irregularities or biases. However, in this range one meets extreme difficulties that we do not know how to resolve at present.

Moreover, situation about averaging over the levels on GL_2 is more delicate than on GL_1 also in one additional aspect: isolation and handling of primitive characters on GL_1 was fairly easy, while isolation of the contribution of newforms in averaging over the levels on GL_2 appears to be extremely difficult task. On the other hand, again because of "big" averaging, it might happen that oldforms have no effect to the large sieve bound and such was the case in [15], for the fixed prime level.

4.2 Preparations and reductions: the tail

We emphasize again that we work in the range $1 \leq N \leq Q^{2-\delta}$. Using Lemma 1.1 one derives (or cf. Lemma 2.1. in [15] where the averaging for the fixed level is performed):

$$\Delta(m, n) = \delta(m, n) + \sigma(m, n)$$

where (recall the notation (1.7) for the $\mathcal{K}$-operator)

$$\sigma(m,n)=\frac{1}{\tilde{\tilde{Q}}}\mathcal{K}\sum_{q}\Phi\left(\frac{q}{Q}\right)\sum_{s\geq 1}\sum_{t\geq 1}\frac{2\pi}{qst}V_{qs}(m,n;t)e\left(\frac{m+n}{qst}\right)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{qst}\right) \tag{4.3}$$

and $V_d(m,n;t)$ is the following exponential sum:

$$V_d(m,n;t)=\sum_{\substack{x(\operatorname{mod} t)\\(x(x+d),t)=1}}e\left(\frac{m\bar{x}-n\overline{x+d}}{t}\right).$$

To treat further $\sigma(m,n)$ we divide

$$\sigma(m,n)=\sigma^{\sharp}(m,n)+\sigma^{\flat}(m,n)$$

where $\sigma^{\sharp}(m,n):=\sum_{t\leq N/Q}$ and $\sigma^{\flat}(m,n):=\sum_{t>N/Q}$, and write

$$\mathfrak{B}(\mathbf{a})=\|\mathbf{a}\|^2+\mathfrak{B}^{\sharp}(\mathbf{a})+\mathfrak{B}^{\flat}(\mathbf{a})$$

for the corresponding bilinear forms. The reason for this is that for $t>N/Q$, the analytic factors $e\left(\frac{m+n}{qst}\right)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{qst}\right)$ do not oscillate, and hence only oscillation comes from $V_{qs}(m,n;t)$ where m and n are already separated - and therefore in this range the classical large sieve can finish the job. Namely, using (1.12) for $k\geq 3$ and the power series expansion for $e\cdot J_{k-1}$, we obtain:

$$\mathfrak{B}^{\flat}(\mathbf{a})\ll\frac{1}{\tilde{\tilde{Q}}}\sum_{q}\Phi\left(\frac{q}{Q}\right)\sum_{s\geq 1}\sum_{t>N/Q}\frac{1}{qst}\left|\sum_{n\sim N}\sum_{m\sim N}a_n\bar{a}_mV_{qs}(m,n;t)\right|\left(\frac{N}{qst}\right)^2.$$

Substituting the exponential sum V_{qs} and applying Cauchy-Schwarz while estimating trivially the summations over q and s, this is further

$$\ll\frac{N^2}{Q^3}\sum_{t>N/Q}\frac{1}{t^3}\sum_{x\,(t)}{}^{*}\left|\sum_{n\sim N}a_ne\left(\frac{nx}{t}\right)\right|^2\ll\|\mathbf{a}\|^2,$$

where the last inequality follows from dyadic subdivision and the classical large sieve.

Therefore, from now on, we only need to concentrate on $\mathfrak{B}^{\sharp}(\mathbf{a})$.

4.3 Preparations and reductions: q-summation

Of course, in the treatment of $\sigma^\sharp(m,n)$ we will exploit additional summation over levels q:

$$\sigma^\sharp(m,n) = \\ = \frac{1}{\widetilde{Q}}\mathcal{K}\sum_{s\geq 1}\sum_{1\leq t\leq N/Q}\frac{2\pi}{st}\sum_{\alpha \bmod t} V_{\alpha s}(m,n;t)\sum_{\substack{q\geq 1\\ q\equiv\alpha(t)}}\frac{1}{q}\Phi\left(\frac{q}{Q}\right)e\left(\frac{m+n}{qst}\right)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{qst}\right). \tag{4.4}$$

By the Poisson summation formula we have:

$$\sigma^\sharp(m,n) = \frac{1}{\widetilde{Q}}\mathcal{K}\sum_{s\geq 1}\frac{2\pi}{s}\sum_{1\leq t\leq N/Q}\frac{1}{t^2}\cdot \\ \cdot\sum_{h\in\mathbb{Z}}\sum_{\alpha(t)} V_{\alpha s}(m,n;t)e\left(\frac{h\alpha}{t}\right)\int_0^\infty \Phi\left(\frac{x}{Q}\right)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{stx}\right)e\left(\frac{m+n}{stx}-\frac{h}{t}x\right)\frac{dx}{x}. \tag{4.5}$$

In order to analyze the last integral, we use the formula (1.14) for the J-Bessel function. It gives two oscillatory integrals with the phase functions:

$$\phi_\pm(x) = \frac{m\pm 2\sqrt{mn}+n}{stx} - \frac{h}{t}x.$$

For $h\neq 0$, none of them has a stationary point (since $N<Q^{2-\delta}$ and hence the linear term dominates) and moreover in that case integration by parts A times gives:

$$\int_0^\infty \Phi\left(\frac{x}{Q}\right)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{stx}\right)e\left(\frac{m+n}{stx}-\frac{h}{t}x\right)\frac{dx}{x} \ll \left(\frac{t}{Q|h|}\right)^A.$$

Since $t\leq N/Q \ll Q^{1-\delta}$ taking A sufficiently large makes the contribution of all the terms with $h\neq 0$ negligible. Therefore we are left only with $h=0$ contribution:

$$\sigma^\sharp(m,n) = \frac{1}{\widetilde{Q}}\mathcal{K}\sum_{s\geq 1}\frac{2\pi}{s}\sum_{1\leq t\leq N/Q}\frac{1}{t^2}\cdot \\ \cdot\sum_{\alpha(t)} V_{\alpha s}(m,n;t)\int_0^\infty \Phi\left(\frac{x}{Q}\right)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{stx}\right)e\left(\frac{m+n}{stx}\right)\frac{dx}{x} + \sigma_{err}(m,n), \tag{4.6}$$

where $\sigma_{err}(m,n)\ll q^{-2010}$ by the repeated partial integration.

By the direct computation (or see Lemma 3.1 in [15]) one shows that the resulting exponential sum is equal to:

$$\sum_{\alpha(t)} V_{\alpha s}(m,n;t) = \sum_{\substack{\beta(t)\\ \beta s\equiv 0(t)}} S(\beta,m;t)S(\beta,n;t) = \sum_{\gamma \bmod (s,t)} S\left(\frac{t}{(s,t)}\gamma,m;t\right) S\left(\frac{t}{(s,t)}\gamma,n;t\right).$$

Introducing a new variable $v=(s,t)$ and changing s and t with sv and tv respectively, where now $(s,t)=1$, we arrive at:

$$\sigma^\sharp(m,n) = \frac{2\pi}{\widetilde{Q}}\mathcal{K} \sum_{1\le v\le N/Q}\ \sum_{1\le t\le N/(vQ)}\ \sum_{\substack{s\ge 1\\ (s,t)=1}} \frac{1}{st^2v^3} \sum_{\gamma\,(v)} S(t\gamma,m;tv)S(t\gamma,n;tv)\cdot$$
$$\cdot \int_0^\infty \Phi\left(\frac{x}{Q}\right) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{stv^2x}\right) e\left(\frac{m+n}{stv^2x}\right)\frac{dx}{x} + \sigma_{err}(m,n). \quad (4.7)$$

4.4 Preparations and reductions: s-summation

Next task is to separate variables m and n in the oscillatory integral in (4.7). That can be done by Poisson summation in the s variable and the situation is almost identical as in [15], section 3, so we will follow it closely.

Because of technical convenience in further analysis, we first insert a factor $\eta(s)$ in summation over s, for $\eta\in\mathcal{C}^\infty(\mathbb{R}_{>0})$, $\eta(s)=0$ for $0<s<1/4$, $0\le\eta(s)\le 1$ for $1/4\le s\le 1/2$, $\eta(s)=1$ if $s>1/2$, which is of course redundant.

Now, Φ has compact support and $k\ge 3$, so in particular $J_{k-1}(x)\ll x^2$, and therefore the convergence in (4.7) is absolute and we can freely exchange summations and integration. In particular we get by Poisson summation (we can move operator $\mathcal{K}$ inside since all the terms before are real and $\mathcal{K}$ is $\mathbb{R}$-linear):

$$s\text{-sum} = \mathcal{K} \sum_{\substack{s\ge 1\\ (s,t)=1}} \frac{\eta(s)}{s} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{stv^2x}\right) e\left(\frac{m+n}{stv^2x}\right) =$$
$$= \frac{1}{t}\mathcal{K}\sum_{h\in\mathbb{Z}} r_t(h)\int_0^\infty \frac{\eta(s)}{s} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{stv^2x}\right) e\left(\frac{m+n}{stv^2x}+\frac{h}{t}s\right) ds, \quad (4.8)$$

where $r_t(h)=\sum_{d|(t,h)}\mu(t/d)d$ is the Ramanujan sum.

Again using the expression (1.14), for each h we have two oscillatory integrals with phase functions

$$\phi_\pm(s) = \frac{m\pm 2\sqrt{mn}+n}{stv^2x} + \frac{h}{t}s.$$

Since $s \geq 1/4$ because of the support of η and $x \asymp Q$ because of the support of Φ and $t \leq N/Q$ in $\sigma^\sharp$, we see that for $|h| \geq H := (N/Q)^{1+\varepsilon}$ none of the oscillatory integrals has a stationary point and moreover by partial integration enough number of times we get that the contribution of all terms $|h| > H$ to the total sum is negligible. This truncation of the dual sum can be done by a smooth cut-off factor $\omega(|h|/H)$ where $\omega \in \mathcal{C}^\infty(\mathbb{R}_{\geq 0})$, $\omega(x) = 1$ on $0 \leq x \leq 1$ and $\omega(x) = 0$ for $x > 2$. Hence, continuing from line (4.8):

$$s\text{-sum} = \frac{1}{t}\mathcal{K}\sum_h r_t(h)\omega\left(\frac{|h|}{H}\right)\int_0^\infty \frac{\eta(s)}{s}\dots ds + O(q^{-2010}).$$

Now we replace back η by 1 i.e. write $\eta = 1 - (1-\eta)$ where $1-\eta$ is supported on $[0, 1/2]$:

$$s\text{-sum} = \frac{1}{t}\sum_h r_t(h)\omega\left(\frac{|h|}{H}\right)\mathcal{K}\int_0^\infty \frac{1}{s}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{stv^2x}\right)e\left(\frac{m+n}{stv^2x}+\frac{h}{t}s\right)ds -$$

$$-\frac{1}{t}\mathcal{K}\sum_h r_t(h)\omega\left(\frac{|h|}{H}\right)\int_0^{1/2}\frac{1-\eta(s)}{s}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{stv^2x}\right)e\left(\frac{m+n}{stv^2x}+\frac{h}{t}s\right)ds + O(q^{-2010}).$$

Now, the integral in the fist line can be computed by the formula (1.15) and is equal to:

$$2\pi J_{k-1}\left(\frac{4\pi}{tv}\sqrt{\frac{mh}{x}}\right)J_{k-1}\left(\frac{4\pi}{tv}\sqrt{\frac{nh}{x}}\right)$$

for $h > 0$, and equal 0 otherwise.

On the other hand, the contribution of the second line can be treated and asymptotically computed (again with an error $\ll q^{-2010}$) precisely as in [15], equations (3.9) - (3.14). Therefore, we get:

$$\begin{aligned} s\text{-sum} = {} & \frac{2\pi}{t}\sum_{h\geq 1}\omega\left(\frac{h}{H}\right)r_t(h)J_{k-1}\left(\frac{4\pi}{tv}\sqrt{\frac{mh}{x}}\right)J_{k-1}\left(\frac{4\pi}{tv}\sqrt{\frac{nh}{x}}\right) - \\ & -\frac{2\pi}{t}\int_0^\infty \omega\left(\frac{h}{H}\right)J_{k-1}\left(\frac{4\pi}{tv}\sqrt{\frac{mh}{x}}\right)J_{k-1}\left(\frac{4\pi}{tv}\sqrt{\frac{nh}{x}}\right)dh \quad + O(q^{-2010}). \end{aligned} \tag{4.9}$$

Substituting this in (4.7) we get:

$$\sigma^\sharp(m,n) = \sigma^\sharp_+(m,n) - \sigma^\sharp_-(m,n) + \sigma'_{err}(m,n)$$

where

$$\sigma_+^\sharp(m,n) = \frac{4\pi^2}{\widetilde{Q}} \sum_{1\le v\le N/Q} \sum_{1\le t\le N/(vQ)} \frac{1}{t^3v^3} \sum_{\gamma\,(v)} S(t\gamma,m;tv)S(t\gamma,n;tv)\cdot$$
$$\cdot \sum_{h\ge 1} \omega\left(\frac{h}{H}\right) r_t(h) \int_0^\infty \Phi\left(\frac{x}{Q}\right) J_{k-1}\left(\frac{4\pi}{tv}\sqrt{\frac{mh}{x}}\right) J_{k-1}\left(\frac{4\pi}{tv}\sqrt{\frac{nh}{x}}\right)\frac{dx}{x}, \quad (4.10)$$

$$\sigma_-^\sharp(m,n) = \frac{4\pi^2}{\widetilde{Q}} \sum_{1\le v\le N/Q} \sum_{1\le t\le N/(vQ)} \frac{1}{t^3v^3} \sum_{\gamma\,(v)} S(t\gamma,m;tv)S(t\gamma,n;tv)\cdot$$
$$\cdot \int_0^\infty \omega\left(\frac{h}{H}\right) \int_0^\infty \Phi\left(\frac{x}{Q}\right) J_{k-1}\left(\frac{4\pi}{tv}\sqrt{\frac{mh}{x}}\right) J_{k-1}\left(\frac{4\pi}{tv}\sqrt{\frac{nh}{x}}\right)\frac{dx}{x}dh \quad (4.11)$$

and

$$\sigma'_{err}(m,n) \ll q^{-2010}.$$

Accordingly, bilinear form $\mathfrak{B}^\sharp(\mathbf{a})$ splits further as

$$\mathfrak{B}^\sharp(\mathbf{a}) = \mathfrak{B}_+^\sharp(\mathbf{a}) - \mathfrak{B}_-^\sharp(\mathbf{a}) + \mathfrak{B}_{err}^\sharp(\mathbf{a}).$$

Since we are interested in an upper bound and the contribution $-\mathfrak{B}_-^\sharp(\mathbf{a})$ is negative, it can be ignored (recall that it arose after and because of the introduction of auxiliary function η and therefore the outcome is not surprising).

4.5 Final bounds

We are left with the problem of bounding

$$\mathfrak{B}_+^\sharp(\mathbf{a}) = \sum_{n\sim N}\sum_{m\sim N} a_n\overline{a}_m\, \sigma_+^\sharp(m,n).$$

First we consider the exponential sum $\sum_{\gamma\,(v)} S(t\gamma,m;tv)S(t\gamma,n;tv)$ appearing in (4.10). For it we recall the twisted multiplicativity of Kloosterman sums: for $(q,r)=1$ the following holds

$$S(m,n;qr) = S(m\bar{q},n\bar{q};r)S(m\bar{r},n\bar{r};q) \quad (4.12)$$

where $\bar{q},\bar{r}$ are multiplicative inverses of q,r modulo r,q respectively.

Lemma 4.2. *For m, n, t, v integers and $t, v > 0$ we have:*

$$\sum_{\gamma\,(mod\ v)} S(t\gamma, m; tv)S(t\gamma, n; tv) = \delta\binom{v_1|m}{v_1|n}\, vv_1^2\, r_{t_1}(m)\, r_{t_1}(n) r_v\left(\frac{n}{v_1} - \frac{m}{v_1}\right) \tag{4.13}$$

where $t = t_1v_1$ is the factorization of t such that $(t_1, v) = 1$ and $v_1|v^\infty$ (meaning: for all primes $p|v_1 \implies p|v$) and $\delta = 1$ only if both conditions are satisfied, and 0 otherwise.

Proof. We factor the modulus of Kloosterman sums as $tv = t_1 \cdot v_1 v$, where now $(t, v_1v) = 1$ so we can use the property (4.12):

$$\sum_{\gamma\,(\mathrm{mod}\ v)} S(t\gamma, m; t_1v_1v)S(t\gamma, n; t_1v_1v) =$$

$$= \sum_{\gamma\,(\mathrm{mod}\ v)} S(t\gamma\bar{t}_1, m\bar{t}_1; v_1v)\, S(t\gamma\overline{v_1v}, m\overline{v_1v}; t_1)\, S(t\gamma\bar{t}_1, n\bar{t}_1; v_1v)\, S(t\gamma\overline{v_1v}, n\overline{v_1v}; t_1) =$$

$$= S(0, m\overline{v_1v}; t_1)\, S(0, n\overline{v_1v}; t_1) \sum_{\gamma\,(\mathrm{mod}\ v)} S(v_1\gamma, m\bar{t}_1; v_1v)\, S(v_1\gamma, n\bar{t}_1; v_1v) =$$

$$= S(0, m; t_1)\, S(0, n; t_1) \sum_{\gamma\,(v)} \sum_{\alpha\,(v_1v)}^{*} e\left(\frac{v_1\gamma\alpha + m\overline{t_1\alpha}}{v_1v}\right) \sum_{\beta\,(v_1v)}^{*} e\left(\frac{v_1\gamma\beta + n\overline{t_1\beta}}{v_1v}\right) =$$

$$= r_{t_1}(m)\, r_{t_1}(n) \sum_{\alpha\,(v_1v)}^{*} e\left(\frac{m\overline{t_1\alpha}}{v_1v}\right) \sum_{\beta\,(v_1v)}^{*} e\left(\frac{n\overline{t_1\beta}}{v_1v}\right) \sum_{\gamma\,(v)} e\left(\frac{\gamma(\alpha+\beta)}{v}\right). \tag{4.14}$$

Now t_1 can be absorbed into α and β and the innermost sum is then equal to

$$v\,\delta(t_1(\alpha+\beta) \equiv_v 0) = v\,\delta(\beta \equiv_v -\alpha). \tag{4.15}$$

Next, since $v_1|v^\infty$, $\alpha \in (\mathbb{Z}/v_1v\mathbb{Z})^*$ can be written as $\alpha = \vartheta + vx'$ where $\vartheta \in (\mathbb{Z}/v\mathbb{Z})^*$ and x' runs modulo v_1 and then $\overline{\alpha} = \overline{\vartheta + vx'} = \bar{\vartheta} + vx$ for some uniquely determined x (mod v_1). Similarly, because of (4.15) we can write $\bar{\beta} = -\bar{\vartheta} + vy$, for y (mod v_1) and hence, continuing from (4.14) we get:

$$= v\, r_{t_1}(m)\, r_{t_1}(n) \sum_{\vartheta\,(v)}^{*} \sum_{x\,(v_1)} e\left(\frac{m(\bar{\vartheta} + vx)}{v_1v}\right) \sum_{y\,(v_1)} e\left(\frac{n(-\bar{\vartheta} + vy)}{v_1v}\right) =$$

$$= vv_1^2\, r_{t_1}(m)\, r_{t_1}(n) \delta\binom{v_1|m}{v_1|n} \sum_{\vartheta\,(v)}^{*} e\left(\frac{n/v_1 - m/v_1}{v}\vartheta\right).$$

$\square$

Substituting (4.13) into (4.10) we obtain:

$$\mathfrak{B}_{+}^{\sharp}(\mathbf{a}) = \frac{4\pi^2}{\widetilde{Q}} \sum_{1\le t\le N/Q} \frac{1}{t^3} \sum_{h\ge 1} \omega\left(\frac{h}{H}\right) r_t(h) \cdot$$
$$\cdot \sum_{1\le v\le N/(Qt)} \frac{1}{v^2} v_1^2 \sideset{}{^*}\sum_{\alpha\,(v)} \int_0^{\infty} \Phi\left(\frac{x}{Q}\right) \left| \sum_{n\sim N} a_n \delta(v_1|n) r_{t_1}(n) e\left(\frac{\alpha n/v_1}{v}\right) J_{k-1}\left(\frac{4\pi}{tv}\sqrt{\frac{nh}{x}}\right)\right|^2 \frac{dx}{x}, \tag{4.16}$$

where we caution that $v_1 = v_1(t, v) \,|\, t$ depends on the pair t, v appearing in front of it in the formula.

It is reminiscent to the Iwaniec-Li asymptotic large sieve formula (1.16), although, apart from being more complicated, in our case (4.16) contains on the right hand side negative contribution as well (coming from negative Ramanujan sums $r_t(h)$), reflecting intricate nature of this family of harmonics obtained by averaging over the levels.

However, summation in h-variable already came from the Poisson formula, as the dual variable to s, and hence further treatment by harmonic analysis would be "involutive". Luckily, in the range of interest in this thesis, i.e. $Q \ll N \ll Q^{2-\delta}$, enough saving comes from the averaging in the second line in (4.16), while in the fist line we replace $r_t(h)$ with its absolute value. Hence the strategy is to "freeze" t and h and bound the inner summation.

We remark that in the "typical" case, $v_1 = 1$, the linear form on the right hand side of (4.16) becomes:

$$\sum_{n\sim N} a_n r_t(n) e\left(\frac{\alpha n}{v}\right) J_{k-1}\left(\frac{4\pi}{tv}\sqrt{\frac{nh}{x}}\right),$$

and hence, it is close to the classical GL_1 large sieve, but contaminated with J_{k-1}-function. Such situation is very common in deriving various GL_2 large sieve type inequalities (cf. e.g. [5]) and the proof of all of these ultimately relies on reduction to the GL_1 case.

An appropriate auxiliary large sieve is given by the following:

Lemma 4.3. *For any vector $(a_n)_{n\sim N}$ of complex numbers and any $L, N \ge 1$ and $X > 0$*

we have:

$$\int_X^{2X} \sum_{L<l\le 2L} \sideset{}{^*}\sum_{\alpha\,(l)} \left| \sum_{N<n\le 2N} a_n\, e\left(\frac{\alpha n}{l}\right) J_{k-1}\left(\frac{4\pi}{l}\sqrt{\frac{n}{x}}\right)\right|^2 \frac{dx}{x} \ll L^2\left(X + L\sqrt{\frac{X}{N}}\right)\|\mathbf{a}\|^2, \tag{4.17}$$

where the implied constant depends only on k.

Proof. Change of variables from the large sieve inequality obtained by Iwaniec-Li in [15], lemma 9.1. □

Now we can return to (4.16) and start with the following arrangement:

$$\mathfrak{B}^{\sharp}_{+}(\mathbf{a}) \ll \frac{1}{\widetilde{\widetilde{Q}}} \sum_{1\le t\le N/Q} \frac{1}{t^3} \sum_{h\ge 1} \omega\left(\frac{h}{H}\right) |r_t(h)| \sum_{\substack{t_1 v_1 = t\\ (t_1,v_1)=1}} \mathfrak{B}_{t_1,v_1}(\mathbf{a}), \tag{4.18}$$

where in the form $\mathfrak{B}_{t_1,v_1}(\mathbf{a})$ the pair t_1 and v_1 is now *fixed* and the v-summation can be enlarged to include all v (not necessarily only v such that $v_1|v^\infty$; this will not be a loss because of the nature of classical large sieve):

$$\mathfrak{B}_{t_1,v_1}(\mathbf{a}) = \sum_{1\le v\le N/(Qt)} \frac{v_1^2}{v^2} \sideset{}{^*}\sum_{\alpha\,(v)}$$

$$\int_0^\infty \Phi\left(\frac{x}{Q}\right) \left|\sum_{n\sim N} a_n \delta(v_1|n) r_{t_1}(n) e\left(\frac{\alpha n/v_1}{v}\right) J_{k-1}\left(\frac{4\pi}{tv}\sqrt{\frac{nh}{x}}\right)\right|^2 \frac{dx}{x}. \tag{4.19}$$

Next, we reduce further by dividing summation over v into dyadic intervals:

$$\mathfrak{B}_{t_1,v_1}(\mathbf{a}) \ll v_1^2 \sum_{V=2^\kappa\le N/(Qt)} \frac{1}{V^2} \mathfrak{B}^V_{t_1,v_1}(\mathbf{a}), \tag{4.20}$$

where

$$\mathfrak{B}^V_{t_1,v_1}(\mathbf{a}) = \sum_{v\sim V} \sideset{}{^*}\sum_{\alpha\,(v)} \int_0^\infty \Phi\left(\frac{x}{Q}\right) \left|\sum_{n\sim N} a_n \delta(v_1|n) r_{t_1}(n) e\left(\frac{\alpha n/v_1}{v}\right) J_{k-1}\left(\frac{4\pi}{tv}\sqrt{\frac{nh}{x}}\right)\right|^2 \frac{dx}{x}. \tag{4.21}$$

This form is very close to (4.17) except for the Ramanujan sum, which we need to control more precisely. First, since $(v_1,t_1)=1$ we have that $r_{t_1}(n) = r_{t_1}(n/v_1)$. Second, if we denote $t_2 = (t_1, n)$ we have:

$$|r_{t_1}(n)| = \left|\sum_{d|t_2} d\,\mu(t_1/d)\right| \le \left|\sum_{d|t_2} d\right| \le \tau(t_2)t_2.$$

Recall that t_1 is fixed and hence we have a finite number of possibilities for $t_2|t_1$. Divide summation over n according to the common divisor t_2: $\sum_{n\sim N} = \sum_{t_2|t_1}\sum_{(n,t_1)=t_2}$. Then by Cauchy-Schwarz:

$$\left|\sum_{n\sim N}\right|^2 \leq \tau(t_1)\sum_{t_2|t_1}\left|\sum_{(n,t_1)=t_2}\right|^2$$

and we split (4.21) accordingly:

$$\mathfrak{B}^{V}_{t_1,v_1}(\mathbf{a}) \leq \tau(t_1)\sum_{t_2|t_1}\sum_{v\sim V}\sideset{}{^*}\sum_{\alpha\,(v)}$$
$$\int_0^\infty \Phi\left(\frac{x}{Q}\right)\left|\sum_{\substack{n\sim N\\(n,t_1)=t_2}} a_n\delta(v_1|n)r_{t_1}(n)e\left(\frac{\alpha n/v_1}{v}\right)J_{k-1}\left(\frac{4\pi}{tv}\sqrt{\frac{nh}{x}}\right)\right|^2\frac{dx}{x}. \quad (4.22)$$

Now writing

$$r_{t_1}(n) = \frac{r_{t_1}(n)}{\tau(t_2)t_2}\,\tau(t_2)t_2,$$

since the absolute value of the fraction is ≤ 1, we can absorb it into the test-vector $\mathbf{a} = (a_n)$, obtaining:

$$\mathfrak{B}^{V}_{t_1,v_1}(\mathbf{a}) \leq \tau(t_1)\sum_{t_2|t_1}\tau(t_2)^2\,t_2^2\cdot$$
$$\cdot\sum_{v\sim V}\sideset{}{^*}\sum_{\alpha\,(v)}\int_0^\infty \Phi\left(\frac{x}{Q}\right)\left|\sum_{\substack{n\sim N\\(n,t_1)=t_2}} a_n\delta(v_1|n)e\left(\frac{\alpha n/v_1}{v}\right)J_{k-1}\left(\frac{4\pi}{tv}\sqrt{\frac{nh}{x}}\right)\right|^2\frac{dx}{x}. \quad (4.23)$$

Final preparation of the second line (recall that now v_1, t_1, t_2 are fixed): first, since $t_2|n$, in the numerator of the exponential write $\alpha\,\frac{n}{v_1} = \alpha t_2\,\frac{n}{v_1 t_2}$ and then since $(t_2, v) = 1$ absorb t_2 into α by change of variables. Also, we write the argument of the Bessel function in the following manner:

$$\sum_{v\sim V}\sideset{}{^*}\sum_{\alpha\,(v)}\int_0^\infty \Phi\left(\frac{x}{Q}\right)\left|\sum_{\substack{n\sim N\\(n,t_1)=t_2\\n\equiv 0\,(v_1)}} a_n e\left(\frac{\alpha\, n/(v_1t_2)}{v}\right)J_{k-1}\left(\frac{4\pi}{v}\sqrt{\frac{n/(v_1t_2)}{v_1t_1^2x/(ht_2)}}\right)\right|^2\frac{dx}{x}.$$

On this sum we can now apply (4.17), where the length of the inner summation is $N/(v_1t_2)$ and after the change of variables, $X = \frac{v_1t_1^2Q}{ht_2}$. Hence the previous line is bounded by

$$\ll V^2 \, \frac{v_1 t_1^2 Q}{h t_2} \sum_{\substack{n\sim N\\ (n,t_1)=t_2\\ n\equiv 0\,(v_1)}} |a_n|^2,$$

since in the given range of parameters (recall that $N \ll Q^{2-\delta}$, for δ anything > 0) we have that $\sqrt{X} \gg \frac{V}{\sqrt{N}}$ and therefore the first term on the right hand side of (4.17) dominates in our situation.

Substituting back, we get respectively:

$$\mathfrak{B}^V_{t_1,v_1}(\mathbf{a}) \ll \frac{V^2 Q v_1 t_1^2}{h} \, \tau(t_1) \sum_{t_2|t_1} \tau(t_2)^2 \, t_2 \sum_{\substack{n\sim N\\ (n,t_1)=t_2\\ n\equiv 0\,(v_1)}} |a_n|^2,$$

$$\mathfrak{B}_{t_1,v_1}(\mathbf{a}) \ll \frac{Q t^2 v_1}{h} \, \log(N/Q)\, \tau(t_1) \sum_{t_2|t_1} \tau(t_2)^2 \, t_2 \sum_{\substack{n\sim N\\ (n,t_1)=t_2\\ n\equiv 0\,(v_1)}} |a_n|^2,$$

$$\mathfrak{B}^\sharp_+(\mathbf{a}) \ll \log(N/Q) \sum_{1\le t\le N/Q} \frac{1}{t} \sum_{h\ge 1} \omega\left(\frac{h}{H}\right) \frac{|r_t(h)|}{h} \sum_{\substack{t_1 v_1 = t\\ (t_1,v_1)=1}} \tau(t_1) \sum_{t_2|t_1} \tau(t_2)^2 \, v_1 t_2 \sum_{\substack{n\sim N\\ (n,t_1)=t_2\\ n\equiv 0\,(v_1)}} |a_n|^2.$$

Now in the above formula each $|a_n|^2$ is weighted by at most the following quantity:

$$\ll \log(N/Q) \sum_{v_1 t_2|n} v_1 t_2 \, \tau(t_2)^2 \sum_{\substack{1\le t\le N/Q\\ t\equiv 0\,(v_1 t_2)}} \frac{\tau(t/v_1)}{t} \sum_{h\ge 1} \omega\left(\frac{h}{H}\right) \frac{|r_t(h)|}{h} \ll$$

$$\ll \log^2(N/Q) \sum_{v_1 t_2|n} \tau(t_2)^2 \sum_{\substack{1\le t\le N/Q\\ t\equiv 0\,(v_1 t_2)}} \frac{v_1 t_2 \, \tau(t/v_1)\tau(t)^2}{t} \ll q^\varepsilon,$$

implying the following

Proposition 4.4. *For any vector of complex numbers* $\mathbf{a} = (a_n)_{n\sim N}$ *and any* $\varepsilon > 0$ *we have:*

$$\mathfrak{B}^\sharp_+(\mathbf{a}) = \sum_{n\sim N} \sum_{m\sim N} a_n \overline{a}_m \, \sigma^\sharp_+(m,n) \ll Q^\varepsilon \|\mathbf{a}\|^2, \qquad (4.24)$$

where the implied constant depends only on k (the weight of modular forms).

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